On quotients of Hom-functors and representations of finite general linear groups II

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Abstract

This is a second paper on quotients of Hom-functors and their applications to the representation theory of finite general linear groups in non-describing characteristic. After some general result on quotients of Hom-functors and their connection to Harish-Chandra theory these contructions are used to obtain a full classification of the ℓ-modular irreducible representations of $GL_n(q)$ for some prime power q which is not divisible by the prime ℓ and to explain some facts on their Harish-Chandra series and decomposition numbers.

0 Introduction

This is a second paper dealing with representations of finite general linear groups G in non-describing characteristic and quotients of Hom-functors in fulfilment of an anouncement in [9], promising new proofs for the classification of the irreducible representations of G.

In the first three sections the general theory of those functors is further developed. In particular some errors of [9] are corrected. The relation between the functors for different projective resolutions is investigated as well as the image of the different right inverses of those.

Our main applications of these functors concern representations of finite reductive groups G in nondescribing characteristic, (they have however been successfully applied to p-adic groups as well, see [46]). Quotients of Hom-functors arise here especially in connection with Harish-Chandra theory, which is used to subdivide the irreducible G-modules into Harish-Chandra series in terms of Hecke algebras associated with reflection groups. Partial results in this direction were obtained first in [7], [8] (for general linear groups) and in [15], [16] for arbitrary finite reductive groups. In [28] Geck, Hiss and Malle established then

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the general case: They showed that Harish-Chandra induced irreducible cuspidal modules in finite reductive groups in nondescribing characteristic always satisfy Hypothesis 1.1 and the results of [9] apply. In particular the Harish-Chandra series can be described in terms of the irreducible representations of endomorphism rings of these modules. In section two we give details on these results in the general setting of Hom-functors and Harish-Chandra theory as developed in [13]. Geck, Hiss and Malle showed too that those endomorphism rings are Hecke algebras associated with certain (abelian extensions) of reflection groups extending Howlett-Lehrer theory [36]. However, since there is no complete classification of the irreducible cuspidal representations of finite reductive groups G, the Harish-Chandra series of G are still not determined.

About twenty five years ago Green and Sawada utilized different functors, connecting representations of G-modules with those of the endomorphism ring of a certain induced G-module, to obtain a classification of the irreducible G-modules in the describing characteristic case. These methods have been reinvestigated recently by Cabanes in [3]. Subsequently Geck and Hiss showed in [27], that they can be applied in the nondescribing characteristic case as well and they produced an alternate method to obtain the Harish-Chandra series in terms of Hecke algebras. Our approach here and the Green-Sawada-Cabanes approach use different assumptions: We need Hypothesis 1.1, whereas there the assumption is needed, that the endomorphism ring in question is a selfinjective algebra. The proofs in both approaches are very similar. In addition, if both conditions hold, the corresponding functors coincide on a large class of modules including those irreducible G-modules, which are not taken to the zero module by the functors. More details on these functors will be given in an upcoming paper by Volker Schubert.

There is a further question besides the classification of the irreducible G-modules, which can be studied using quotients of Hom-functors, namely the calculation of decomposition numbers. To do this, we need an integral theory describing the transition from characteristic zero to characteristic ℓ representations of G using quotients of Hom-functors between module categories of $\mathcal{O}$-orders over integral domains $\mathcal{O}$. Results on these from [9] are revised (and corrected in part) in section one. The key theorems here are [9, 4.10] and Theorem 1.30 of this paper. The applications are twofold: First we are able to describe part of the ℓ-modular decomposition matrix of G in terms of decomposition matrices of Hecke algebras. Secondly we use 1.30 to enlarge Hecke algebras to q-Schur algebras (which are not selfinjective in general even over fields) to apply the theory of quotients of Hom-functors a second time.

All this works especially well, when G is a finite general linear group. Thus the second part of the paper, starting with section four, is devoted to the representation theory of $GL_n(q)$. As a consequence we get a classification of the irreducible ℓ-modular representations of $G = GL_n(q)$ for primes ℓ not dividing q and information on decomposition numbers in terms of Hecke- and q-Schur algebras. Most of this was done before in [7],[8],[38], [39] and [40], however the method and the proofs here are new and more conceptual.

At the end of section six we recover and generalize a result of Takeuchi [45] who constructed q-tensor space directly as subquotient of the group algebra of G and showed, that it satisfies Schur-Weyl duality. The result follows from our investigation in section three, where we demonstrate, that this is a general result on quotients of Hom-functors with respect to selfinjective endomorphism rings.

The representation theory of finite general linear groups in nondescribing characteristic is an especially well behaved application of our general set-up. Hiss and Gruber used similar methods to classify the irreducible ℓ-modular representations of classical groups in [31] for so called linear primes ℓ (roughly a third of all primes) and reduce the computation of the ℓ-modular decomposition matrix of these groups to those of q-Schur algebras.

In [18] Gruber and the author apply the general theory of this paper to representations of arbitrary finite groups G with split BN-pair. Generalised q-Schur algebras are constructed and used, to produce many irreducible ℓ-modular G-modules for nondescribing primes ℓ, and it is shown that their decomposition matrices are part of the ℓ-modular decomposition matrix of G. The results however are not as complete in general as in the case of general linear or classical groups with respect to linear primes. We neither get a classification of all irreducible G-modules nor do we obtain a complete description of the decomposition matrix of G in terms of generalised q-Schur algebras.

For an application of quotients of Hom-functors to p-adic groups and some general results on the functors we refer to a recent preprint of M. F. Vigneras [46].

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1 Quotients of Hom-functors

In general we shall take over notation and main definitions from [9]. However there will be exceptions.

We begin with the basic setup of [9] and supplement a few further results which will be needed later on. Thus R is a commutative noetherian ring, and T is a semiperfect R-algebra which is finitely generated as R-module. All occurring modules are, if not stated otherwise, finitely generated. The category of finitely generated right T-modules is denoted by mod_T, of finitely generated left T-modules by ${}_T\mathrm{mod}$. Let $M \in \operatorname{mod}_T$, and let $\beta : P \to M$ be a projective presentation of M. Thus $P \in \operatorname{mod}_T$ is projective, and β is an epimorphism. Let $\mathcal{E} = \operatorname{End}_T(P)$. We take $\mathcal{E}_\beta = \{\phi \in \mathcal{E} \mid \phi(\ker\beta) \subseteq \ker\beta\}$. The endomorphism ring of M is denoted by $\mathcal{H}$. Obviously $J_\beta = \{\psi \in \mathcal{E} \mid \operatorname{im}\psi \leq \ker\beta\}$ is an ideal of $\mathcal{E}_\beta$ and $\mathcal{E}_\beta/J_\beta \cong \mathcal{H}$ as R-algebra canonically, (comp. [9, 2.1]). Our basic hypothesis is now the following:

Hypothesis 1.1 *For $M \in \operatorname{mod}_T$ we say that the projective presentation $\beta : P \to M$ of M satisfies* (1.1), *if $\mathcal{E}_\beta = \mathcal{E}$.*

Then J_β is an ideal of $\mathcal{E}$, and we identify $\mathcal{H}$ and $\mathcal{E}/J_\beta$ by the canonical isomorphism induced by β. We have now a functor

$$H = H^\beta = H_M^\beta : \operatorname{mod}_T \to \operatorname{mod}_{\mathcal{H}}$$

which takes the T-module V to the $\mathcal{H}$-module $\mathrm{Hom}_T(P,V)/\mathrm{Hom}_T(P,V)J_\beta$. On maps H^β is defined in the obvious way.

The *P-torsion submodule* $t_P(V)$ is the unique maximal submodule X of $V \in \mathrm{mod}_T$ with respect to the property that $\mathrm{Hom}_T(P,X) = (0)$. The *kernel* $\mathbf{ker}P$ of P is the full subcategory of mod_T, whose objects are T-modules V with $\mathrm{Hom}_T(P,V) = (0)$. So $V \in \mathbf{ker}P$ precisely if $t_P(V) = V$. We have a functor $A_P : \mathrm{mod}_T \to \mathrm{mod}_T$ taking $V \in \mathrm{mod}_T$ to $V/t_P(V)$. We say that V is *P-torsionless* if $t_P(V) = (0)$. So $A_P(V)$ is the maximal P-torsionless factor module of V. We define the functor $\hat{H}_M = \hat{H} : \mathrm{mod}_\mathcal{H} \to \mathrm{mod}_T$ to be the composite functor $A_P \circ (- \otimes_\mathcal{H} M)$. For $V \in \mathrm{mod}_T$ the *trace $\tau_P(V)$ of P in V* is the T-submodule of V generated by the images of all homomorphisms $\phi : P \to V$.

It was shown in [9, 2.16] that $\hat{H}$ is a right inverse of the functor H. Thus in particular, if T is a finite dimensional algebra over some field, $\mathfrak{H}$ is a complete set of non-isomorphic irreducible $\mathcal{H}$-modules, and $\mathfrak{T}_M$ a complete set of non-isomorphic irreducible T-modules which are not taken to the zero module under the functor H, then H induces a bijection between $\mathfrak{T}_M$ and $\mathfrak{H}$ [9, 2.28].

For an arbitrary module V the *Jacobson radical* $\mathrm{Jac}(V)$ of V is the intersection of all maximal submodules of V, the head $\mathrm{hd}(V)$ is the factor module $V/\mathrm{Jac}(V)$. Thus $\mathrm{hd}(V)$ is the largest semisimple factor module of V. Similarly the socle $\mathrm{soc}(V)$ of V is the largest semisimple submodule of V. For a ring S the *Jacobson radical* $\mathrm{Jac}(S)$ of S is defined to be the radical of the regular representation S_S of S.

More precisely Results [9, 2.28,2.32, and 2.20ii] say:

Theorem 1.2 *Suppose that $\beta : P \to M$ satisfies* 1.1. *Then*

(i) *If M is P-torsionless and X is a right ideal of $\mathcal{H}$. Then*

$$\hat{H}(X) = XM \cong A_P(X \otimes_\mathcal{H} M).$$

(ii) *If R is a field then $\mathfrak{T}_M$ is a complete set of nonisomorphic irreducible constituents of* $\mathrm{hd}(M)$. *Then every indecomposable direct summand of M has simple head and factoring out the Jacobson radical induces a bijection between the isomorphism classes of indecomposable direct summands of M and the elements of $\mathfrak{T}_M$.*

It was remarked in [9, 2.4] that the functor $H = H_M$ is independent of the choice of the projective presentation $\beta : P \to M$. This however is not correct. Here are some precise statements about what happens, if we change the projective presentation of M to another one, for which Hypothesis 1.1 as well holds. Obviously it is enough to consider the minimal projective cover $\beta_1 : P_1 \to M$ of M. Note that by general theory P_1 splits from the kernel of β, that is $\ker\beta = \ker\beta_1 \oplus P_2$, $P = P_1 \oplus P_2$ and $\beta = \beta_1 \oplus 0$ in the following short exact sequence:

$$0 \longrightarrow \ker\beta_1 \oplus P_2 \longrightarrow P_1 \oplus P_2 \xrightarrow{\beta_1 \oplus 0} M \longrightarrow 0. \tag{1.3}$$

We can write down elements of P as column vectors with two components, the first one from P_1 and the second one from P_2, and accordingly the endomorphisms of P by 2×2-matrices with entries in the appropriate Hom-spaces:

$$\mathcal{E}_\beta = \begin{pmatrix} \mathcal{E}_1 & \mathrm{Hom}_T(P_2,P_1) \\ \mathrm{Hom}_T(P_1,P_2) & \mathcal{E}_2 \end{pmatrix}, \tag{1.4}$$

setting $\mathcal{E}_i = \mathcal{E}_{\beta_i}$ for $i = 1, 2$. For a ring S and S-modules V_1, V_2 and for $U \leq V_1$ we define $\mathrm{Hom}_S(V_1, V_2)U$ to be the submodule of V_2 spanned by all images $\mathrm{im}\,\phi$ for restrictions ϕ to U of homomorphisms from V_1 to V_2. We calculate:

$$\begin{pmatrix} \mathcal{E}_1 & \mathrm{Hom}_T(P_2, P_1) \\ \mathrm{Hom}_T(P_1, P_2) & \mathcal{E}_2 \end{pmatrix} \begin{pmatrix} \ker\beta_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1 \ker\beta_1 + \mathrm{Hom}_T(P_2, P_1)P_2 \\ \mathrm{Hom}_T(P_1, P_2)\ker\beta_1 + \mathcal{E}_2 P_2 \end{pmatrix} \subseteq \begin{pmatrix} \ker\beta_1 + \tau_{P_2}(P_1) \\ P_2 \end{pmatrix}.$$

Thus β satisfies 1.1 if and only if $\tau_{P_2}(P_1) \subseteq \ker\beta_1$ or equivalently

$$\mathrm{Hom}_T(P_2, P_1) = \mathrm{Hom}_T(P_2, \ker\beta_1).$$

Assume 1.1 holds. Then

$$J_\beta = \begin{pmatrix} J_1 & \mathrm{Hom}_T(P_2, \ker\beta_1) \\ \mathrm{Hom}_T(P_1, P_2) & \mathcal{E}_2 \end{pmatrix}.$$

We denote the Hom-set $\mathrm{Hom}_T(P_i, P_j)$ by $\mathcal{E}_{i,j}$ for $i, j = 1, 2$ and $i \neq j$. Thus in particular $\mathrm{Hom}_T(P_2, \ker\beta_1) = \mathcal{E}_{2,1}$. Homomorphisms from P into a T-module V are written as row vectors $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_i \in \mathrm{Hom}_T(P_i, V)$ for $i = 1, 2$. We have:

$$\begin{aligned} &\mathrm{Hom}_T(P, V)J_\beta \\ &= (\mathrm{Hom}_T(P_1, V),\quad \mathrm{Hom}_T(P_2, V)) \begin{pmatrix} J_1 & \mathcal{E}_{2,1} \\ \mathcal{E}_{1,2} & \mathcal{E}_2 \end{pmatrix} \\ &= \big(\mathrm{Hom}_T(P_1, V)J_1 + \mathrm{Hom}_T(P_2, V)\mathcal{E}_{1,2},\quad \mathrm{Hom}_T(P_2, V)\big), \end{aligned}$$

setting $J_1 = J_{\beta_1}$.

We have shown:

Lemma 1.5 *Let $\beta = (\beta_1, 0) : P_1 \oplus P_2 \to M$ be given as above. Then β satisfies* 1.1 *if and only if*

$$\tau_{P_2}(P_1) \subseteq \ker\beta_1.$$

Moreover for $V \in \mathrm{mod}_T$ we have

$$H^\beta(V) \cong H^{\beta_1}(V)/(\mathrm{Hom}_T(P_2, V)\mathcal{E}_{1,2}).$$

Thus $H^\beta = H^{\beta_1}$ if and only if every homomorphism from P_1 to P_2 factors through a linear combination of endomorphisms of P_1 whose image is contained in $\ker\beta_1$, that is $H^{\beta_1}(P_2) = (0)$. If this condition does not hold, then H^β is a proper quotient of H^{β_1}.

Note that $\tau_{P_2}(P_1) \subseteq \ker\beta_1$ is equivalent to $\mathrm{Hom}_T(P_2, M) = (0)$ or, if R is a field, that no composition factor of the head of P_2 occurs as composition factor of M. Thus we have:

Lemma 1.6 *Let $\beta = (\beta_1, 0)$ as above. Then β satisfies* 1.1 *if and only if* $\mathrm{Hom}_T(P_2, M) = (0)$. *In this case M is P-torsionless if and only if it is P_1-torsionless.*

Of course the last condition in 1.5 holds in particular if $\mathrm{Hom}_T(P_1, P_2) = (0)$ and hence we have:

Corollary 1.7 *Suppose that P_2 is a projective T-module and assume that* $\mathrm{Hom}_T(P_1, P_2) = (0)$ *and that $\tau_{P_2}(P_1) \subseteq \ker \beta_1$. Then $\beta = \beta_1 \oplus 0 : P_1 \oplus P_2 \to M$ satisfies* 1.1, *and $H^\beta = H^{\beta_1}$.*

We are now ready to sharpen results [9, 2.28 and 2.32] in special cases. A ring $\mathcal{H}$ is called *selfinjective* if the regular module $\mathcal{H}_\mathcal{H}$ is injective.

Theorem 1.8 *Suppose $\beta : P \to M$ satisfies* 1.1. *Assume that R is a field, and that M is P-torsionless. If $\mathcal{H}$ is selfinjective, then every element of $\mathfrak{H}$ is isomorphic to some minimal right ideal X of $\mathcal{H}$ and the corresponding element $\hat{H}(X)$ of $\mathfrak{T}_M$ is isomorphic to XM. Moreover $\mathfrak{T}_M$ is up to isomorphism a complete set of the irreducible constituents of* $\mathrm{soc}(M)$ *as well as of* $\mathrm{hd}(M)$, *every indecomposable direct summand of M has simple socle and simple head, and taking socles respectively heads induce bijections between the isomorphism classes of indecomposable direct summands of M and the elements of $\mathfrak{T}_M$. Socle and head of the indecomposable direct summands of M are isomorphic if in addition $\mathcal{H}$ is a symmetric algebra.*

Proof The first part of the theorem follows immediately from 1.2. By the same result $\mathfrak{T}_M$ is up to isomorphism precisely the set of irreducible constituents of $\mathrm{hd}(M)$, hence of the head of the projective cover P_1 of M. Since M is P-torsionless by assumption it is P_1-torsionless by 1.6. Thus every irreducible submodule of M is isomorphic to an irreducible constituent of $\mathrm{hd}(P_1)$, hence to an element of $\mathfrak{T}_M$. Let S be an element of $\mathfrak{T}_M$. By [9, 2.27 and 2.28] $S = \hat{H}(X)$ for some irreducible $\mathcal{H}$-module X which we may assume to be a minimal right ideal of $\mathcal{H}$ since $\mathcal{H}$ is selfinjective by assumption. By Theorem 1.2 we have $S = XM$ and is therefore contained in the socle of M and therefore $\mathfrak{T}_M$ is up to isomorphisms a complete set of irreducible constituents of $\mathrm{soc}(M)$. The last assertion of the theorem follows now easily from the fact, that the head and the socle of an indecomposable projective $\mathcal{H}$-modules are isomorphic if $\mathcal{H}$ is a symmetric algebra. □

As remarked in the introduction, using the Green-Sawada-Cabanes approach one can obtain a similar theorem replacing our Hypothesis 1.1 by the assumption that $\mathcal{H}$ is a symmetric finite dimensional algebra over some field and that every composition factor of the head of M is also a composition factor of the socle and vice versa. For details see [27, 2.8-2.9].

We turn now to the right inverse $\hat{H} = \hat{H}^\beta = \hat{H}^\beta_M$ of H^β. This as well depends on the choice of P in general since A_P does. Here are some results relating $\hat{H}^\beta_M$ and $\hat{H}^{\beta_1}_M$ for the minimal projective cover P_1 of M. Being the composite functors of $- \otimes_\mathcal{H} M$ with A_P respectively A_{P_1} they are distinguished only by the latter functors. Obviously for $V \in \mathrm{mod}_T$ the module $A_{P_1}(V)$ is an epimorphic image of $A_P(V)$, since $\mathbf{ker}P \subseteq \mathbf{ker}P_1$ and therefore $t_P(V) \leq t_{P_1}(V)$. We have shown:

Lemma 1.9 *Let $X \in \mathrm{mod}_{\mathcal{H}}$. Then there is a natural epimorphism from $\hat{H}_M^{(P)}(X)$ onto $\hat{H}_M^{(P_1)}(X)$.*

The kernel of the map in 1.9 is obviously $A_{P_1}(\hat{H}_M^{(P)}(X))$.

We shall prove now some auxiliary results for later use. First we want to deduce that the ideal J_{β_1} of the endomorphism ring $\mathcal{E}_1 = \mathrm{End}_T(P_1)$ is contained in the Jacobson radical $\mathrm{Jac}(\mathcal{E}_1)$ of $\mathcal{E}_1$. For this it suffices for example to assume that T is noetherian. In all our applications T has this property.

Lemma 1.10 *Suppose that T is noetherian. Then*

$$J_1 \subseteq \mathrm{Jac}(\mathcal{E}_1).$$

Proof Since T is noetherian and P_1 is finitely generated every surjective endomorphism of P_1 is actually an isomorphism, see e.g. [6, 5.8]. From this one concludes immediately that $\{\phi \in \mathcal{E}_1 \,|\, \mathrm{im}\,\phi \subseteq V\}$ is a maximal right ideal of $\mathcal{E}_1$ for a maximal submodule V of P_1 and one gets every maximal right ideal of $\mathcal{E}_1$ in this way. As a consequence

$$\{\phi \in \mathcal{E}_1 \,|\, \mathrm{im}\,\phi \subseteq \mathrm{Jac}(P_1)\} = \mathrm{Jac}(\mathcal{E}_1).$$

Since $\beta_1 : P_1 \to M$ is a minimal projective cover and T is semiperfect,

$$\ker \beta_1 \subseteq P_1 \mathrm{Jac}(T) \subseteq \mathrm{Jac}(P_1),$$

by general theory, (see e.g. [6, 6.21 and 5.6]). The claim follows now immediately from the definition of J_1. □

We assume from now on that T is noetherian. Lemma 1.10 implies the following useful criterion to determine if a given T-module is mapped to the zero module or not. We keep the notation of 1.5 and denote the class of all T-modules which are taken to isomorphic $\mathcal{H}$-modules by the functors H^β and H^{β_1} by $\mathfrak{M}^\beta$ and the class of all T-modules which are taken to the same $\mathcal{H}$-module under all Hom-functors H^β by $\mathfrak{M}$.

Corollary 1.11 *Let $V \in \mathfrak{M}^\beta$. Then*

$$H^\beta(V) = (0) \Leftrightarrow \mathrm{Hom}_T(P_1, V) = (0).$$

Proof This follows immediately from 1.10 and Nakayamas Lemma ([5, 5.7]). □

Remark 1.12 *Lemma 1.5 and Corollary 1.7 provide criterions for $\mathfrak{M}^\beta$ to be mod_T. In the applications which we are mostly interested in, we have often that $H^\beta = H^{\beta_1}$ or we apply 1.11 to T-modules of which we know by some other reason that it belongs to $\mathfrak{M}$. Later we shall determine under special assumptions large classes of modules in $\mathfrak{M}$ (see 1.25).*

Having a left inverse the functors $\hat{H}^{\beta}_{M}$ and $\hat{H}^{\beta_1}_{M}$ are injective on objects. In particular they take only the zero object to zero. Moreover, $\mathcal{H}$-modules which are decomposable are taken to decomposable T-modules by these functors. However they do not preserve indecomposability in general.

Lemma 1.13 *Let $X \in \operatorname{mod}_{\mathcal{H}}$ be indecomposable, and let V be one of the T-modules $X \otimes_{\mathcal{H}} M$, $X \otimes_{\mathcal{E}} P$, $\hat{H}^{\beta}_{M}(X)$ or $A_P(X \otimes_{\mathcal{E}} P)$. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be a decomposition of V into a direct sum of indecomposable T-modules. Then there is an index $i \in \{1, \ldots, k\}$ such that the following holds (viewing X if needed as $\mathcal{E}$-module via the epimorphism $\mathcal{E} \to \mathcal{H}$):*

(i) $H_M(V_i) = X$ *and* $H_M(V_j) = (0)$ *for* $i \neq j \in \{1, \ldots, k\}$.
(ii) $V_j \not\cong V_i$ *for* $i \neq j \in \{1, \ldots, k\}$.
(iii) $\operatorname{Hom}_T(P_1, V_i) \neq (0)$ *and* $\operatorname{Hom}_T(P_1, V_j) = (0)$ *for* $i \neq j \in \{1, \ldots, k\}$.

Proof Since H_M is a left inverse of each of the four functors which appear as a choice in the definition of V from X by [9, 2.16], the $\mathcal{H}$-module $H_M(V) \cong X$ is indecomposable by assumption. Since H_M preserves direct sums there is precisely one direct summand V_i which is taken by H_M to X, and all other direct summands $V_j, j \neq i$ are taken to the zero module. This proves part one. The second part of the lemma follows immediately and the third one using Result 1.11 for the first two choices of V, since they are taken back to X under the functor H^{β_1}.

For the remaining two choices for V we observe that the functor A_P preserves direct sums and the result follows easily using the projectivity of P_1. □

For the next lemma we assume that T is a finite dimensional algebra over some field (or more generally an artin algebra). Since P_1 is the minimal projective cover of M, these two modules have the same head. The previous lemma implies now immediately:

Corollary 1.14 *In the situation of Lemma 1.13 suppose that $H^{\beta} = H^{\beta_1}$, where T is a finite dimensional algebra over some field. Then no composition factor of the head of P_1 hence of M occurs as composition factor of V_j for $i \neq j \in \{1, \ldots, k\}$.*

There is a further application of [9, 2.21] in connection with Result 1.13:

Corollary 1.15 *Keep the notation of Lemma 1.13 and suppose that M is P-torsionless. Let X be an indecomposable right ideal of $\mathcal{H}$. Then*

$$\hat{H}^{\beta}_{M}(X) = XM$$

is indecomposable.

Proof By [9, 2.20] we have that $\hat{H}^{\beta}_{M}(X) = XM$, and by [9, 2.21] H^{β}_{M} induces an isomorphism between the endomorphism rings of X and XM. Since X is indecomposable, its endomorphism ring is local, hence the endomorphism ring of XM is local as well and XM is indecomposable as desired. □

In the applications we have in mind, $\mathcal{H}$ is a selfinjective finite dimensional algebra over some field, and hence every irreducible $\mathcal{H}$-module is isomorphic to a minimal right ideal of $\mathcal{H}$. Note too that this result applies in particular to principal projective indecomposable $\mathcal{H}$-modules of $\mathcal{H}$. These are principal right ideals $X = x\mathcal{H}$ generated by an idempotent, and for those one has obviously

$$XM = xM \cong x \otimes_{\mathcal{H}} M = X \otimes_{\mathcal{H}} M, \tag{1.16}$$

such that in this case one gets $\hat{H}^{\beta}_{M}(X)$ already as tensor product $X \otimes_{\mathcal{H}} M$ provided, M is P-torsionless.

The following lemma is a refinement of Corollary 1.15:

Lemma 1.17 *Let X be an indecomposable direct summand of an ideal of $\mathcal{H}$ and suppose that M is P-torsionless. Then*

$$\hat{H}^{\beta}_{M}(X) = XM = A_P(X \otimes_{\mathcal{H}} M)$$

is indecomposable.

Proof The claim follows as in 1.15 observing that $- \otimes_{\mathcal{H}} M$ and t_P preserve direct sums. $\square$

We now turn to the more specialized situation described in [9, section4]. So let $\mathcal{O}$ be a discrete complete valuation ring with quotient field K and residue field F. Let T_K be a semisimple K-algebra and let $T_{\mathcal{O}}$ be an $\mathcal{O}$-order in T_K. We say $V_{\mathcal{O}}$ is an *irreducible $T_{\mathcal{O}}$-lattice* if $V_K = K \otimes_{\mathcal{O}} V_{\mathcal{O}}$ is an irreducible T_K-module.

Recall that an T_F-module V is *liftable*, if $V = F \otimes_{\mathcal{O}} U$ for some $T_{\mathcal{O}}$-lattice U. Moreover, if $U_{\mathcal{O}}, V_{\mathcal{O}}$ are $T_{\mathcal{O}}$-lattices, then

$$\operatorname{Hom}_{T_K}(U_K, V_K) \cong K \otimes_{\mathcal{O}} \operatorname{Hom}_{T_{\mathcal{O}}}(U_{\mathcal{O}}, V_{\mathcal{O}}) \tag{1.18}$$

canonically. Similarly we have an homomorphism of F-spaces

$$F \otimes_{\mathcal{O}} \operatorname{Hom}_{T_{\mathcal{O}}}(U_{\mathcal{O}}, V_{\mathcal{O}}) \to \operatorname{Hom}_{T_F}(U_F, V_F) \tag{1.19}$$

which is injective but not necessarily surjective. The homomorphisms in the image of that map are called *liftable*, and similarly $\operatorname{Hom}_{T_F}(U_F, V_F)$ is said to be *liftable*, if this homomorphism is bijective.

We take $M_{\mathcal{O}} \in \operatorname{mod}_{T_{\mathcal{O}}}$ to be $\mathcal{O}$-free and assume that the projective presentation $\beta : P_{\mathcal{O}} \to M_{\mathcal{O}}$ satisfies (1.1). We let $\mathcal{H}_{\mathcal{O}}$ be $\operatorname{End}_{T_{\mathcal{O}}}(M_{\mathcal{O}})$. Similarly we indicate for other objects that they are defined over $\mathcal{O}$, by adding a subscript $\mathcal{O}$. For $R \in \{K, \mathcal{O}, F\}$ and an object $X_{\mathcal{O}}$ we abbreviate $R \otimes_{\mathcal{O}} X_{\mathcal{O}}$ by X_R. By [9, 4.7] 1.1 (for $R = \mathcal{O}$) is equivalent to $\ker \beta_K$ having no irreducible constituent with M_K in common. Moreover if this holds, the decomposition matrix of $\mathcal{H}_{\mathcal{O}}$ is a submatrix of the decomposition matrix of $T_{\mathcal{O}}$, [9, 4.10].

In [9, 4.7] it was also shown that 1.1 holds for $\beta_R = 1_R \otimes_{\mathcal{O}} \beta$ for all choices of $R \in \{K, \mathcal{O}, F\}$. The three resulting Hom-functors and their right inverses are distinguished by a suffix R, but usually we drop the suffix M_R, if no ambiguities arise.

We adopt the notation introduced at the beginning of the section but add a further index R, where R is as usual one of the rings $F, \mathcal{O}$ or K. In particular $\beta_{1,R} : P_{1,R} \to M_R$ is a projective cover of M_R for $R = \mathcal{O}$ or F, and $\beta_R = \beta_{1,R} \oplus 0$ for all choices of R.

Lemma 1.20 *Let $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$ be a projective presentation of the $T_{\mathcal{O}}$-lattice $M_{\mathcal{O}}$. Then M_R is P_R-torsionless for $R = K$ and $\mathcal{O}$.*

Proof This follows immediately from the fact that β_K is an epimorphism and that $\mathrm{Hom}_{T_K}(P_K, M_K) \cong K \otimes_{\mathcal{O}} \mathrm{Hom}_{T_{\mathcal{O}}}(P_{\mathcal{O}}, M_{\mathcal{O}})$. □

Remark 1.21 *Note that the Hom-functor $H^{\beta}_{\mathcal{O}}$ does not take $T_{\mathcal{O}}$-lattices to $\mathcal{H}_{\mathcal{O}}$-lattices in general, since $\mathrm{Hom}_{T_{\mathcal{O}}}(P_{\mathcal{O}}, V_{\mathcal{O}})J_{\beta_{\mathcal{O}}}$ may be a sublattice of $\mathrm{Hom}_{T_{\mathcal{O}}}(P_{\mathcal{O}}, V_{\mathcal{O}})$ such that the corresponding factor module has $\mathcal{O}$-torsion, that is $\mathrm{Hom}_{T_{\mathcal{O}}}(P_{\mathcal{O}}, V_{\mathcal{O}})J_{\beta_{\mathcal{O}}}$ is not pure in $\mathrm{Hom}_{T_{\mathcal{O}}}(P_{\mathcal{O}}, V_{\mathcal{O}})$. As a consequence, whereas $H^{\beta}_R(V_R) = R \otimes_{\mathcal{O}} H^{\beta}_{\mathcal{O}}(V_{\mathcal{O}})$ is still true for $R = K$ and $R = F$, the dimension of $H^{\beta}_F(V_F)$ may be larger as that of $H^{\beta}_K(V_K)$, since the $\mathcal{O}$-torsion part of $H^{\beta}_R(V_R)$ is killed by tensoring with K but survives, if we tensor with F. In the next theorem we investigate what happens, if $V_{\mathcal{O}}$ is taken to a lattice by the Hom-functor H^{β_1}.*

The next theorem tells us that for many applications we may neglect the fact, that different projective presentations produce different Hom-functors.

Theorem 1.22 *Suppose that $\beta_{1,\mathcal{O}} : P_{1,\mathcal{O}} \to M_{\mathcal{O}}$ satisfies 1.1. Let $P_{2,\mathcal{O}}$ be a projective $T_{\mathcal{O}}$-lattice. Then*

$$\beta_R = \beta_{1,R} \oplus 0 : P_{1,R} \oplus P_{2,R} \to M_R$$

satisfies 1.1 for all choices of R if and only if the image of any homomorphism from $P_{2,R'}$ to $P_{1,R'}$ is in $\ker \beta_{R'}$, for just one ring R' in $\{K, \mathcal{O}, F\}$. Morover, if 1.1 is satisfied by β_R and $X_{\mathcal{O}} = H^{\beta_1}_{\mathcal{O}}(V)$ is an $\mathcal{H}_{\mathcal{O}}$-lattice, where V is a $T_{\mathcal{O}}$-lattice, then

$$H^{\beta}_R(V) \cong H^{\beta_1}_R(V) \cong R \otimes_{\mathcal{O}} X$$

for all choices of R.

Proof By [9, 4.7] and 1.5 it suffices to show for the first part of the theorem that the condition for $R' = F$ implies the condition for $R' = \mathcal{O}$. But if there is a homomorphism from $P_{2,\mathcal{O}}$ to $P_{1,\mathcal{O}}$, whose image is not contained in $\ker \beta_{1,\mathcal{O}}$ we get a nonzero homomorphism from $P_{2,\mathcal{O}}$ to $M_{\mathcal{O}}$, and hence from $P_{2,F}$ to M_F as well. But this lifts to an endomorphism of $P_F = P_{1,F} \oplus P_{2,F}$ which does not leave $\ker \beta_F$ invariant, a contradiction.

To prove the second part of the theorem we observe that $H^{\beta}_K = H^{\beta_1}_K$, since

$$P_K = \ker(\beta_K \oplus M_K) = P_{1,K} \oplus P_{2,K},$$

and M_K has no irreducible composition factor with $P_{2,K}$ in common, since $P_{2,K}$ is contained in $\ker \beta_K$. Thus every homomorphism $P_{1,K} \to P_{2,K}$ factors through $J_{1,K} = J_{\beta_{1,K}}$ and we are done by 1.5.

Thus we assume that $X = H^{\beta_1}_{\mathcal{O}}(V)$ is free as $\mathcal{O}$-module. As we have seen in 1.5, the functor $H^{\beta}_{\mathcal{O}}$ takes V to an epimorphic image $\hat{X}$ of X. Obviously $KX \cong H^{\beta_1}_K(KV)$ and similarly $K\hat{X} \cong H^{\beta}_K(KV)$, hence by our argument above $KX \cong K\hat{X}$. Since tensoring by

K is exact, we conclude that the kernel of the epimorphism τ from X onto $\hat{X}$, that is τ is an isomorphism.

If $\sigma_F : P_{1,F} \to P_{2,F}$ and $\rho_F : P_{2,F} \to V_F = F \otimes_{\mathcal{O}} V$ are given, we can lift them to get maps $\sigma_{\mathcal{O}}$ and $\rho_{\mathcal{O}}$ for the corresponding $T_{\mathcal{O}}$-lattices. But by the above and 1.5 we can express $\rho_{\mathcal{O}}\sigma_{\mathcal{O}}$ as a linear combination of the composite of maps $\phi_{\mathcal{O}} \in J_{1,\mathcal{O}}$ and $\psi_{\mathcal{O}} : P_{1,\mathcal{O}} \to V$ and we have the following commutative diagram (indicating by the sum symbol that we have to take linear combinations of the composite maps):

(1.23)
$$\begin{tikzcd}
& P_{1,\mathcal{O}} \arrow[r, "\sigma_{\mathcal{O}}"] \arrow[dl] \arrow[dd, "\sum\phi_{\mathcal{O}}"'] & P_{2,\mathcal{O}} \arrow[dd, "\rho_{\mathcal{O}}"] \\
\ker_{1,\mathcal{O}} \arrow[dr] & & \\
& P_{1,\mathcal{O}} \arrow[r, "\sum\psi_{\mathcal{O}}"'] & V,
\end{tikzcd}$$

hence $\rho_F\sigma_F$ is a linear combination of the corresponding composite maps $\psi_F\phi_F$. Using 1.5 one checks now easily that

$$H_F^{\beta}(V_F) \cong H_F^{\beta_1}(V_F) \cong X_F.$$

□

For the right inverses of the Hom-functors we have a similar result:

Theorem 1.24 *Assume that $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$ satisfies 1.1 and let $\beta_{1,\mathcal{O}} : P_{1,\mathcal{O}} \to M_{\mathcal{O}}$ be the projective cover of $M_{\mathcal{O}}$. Let $X_{\mathcal{O}}$ be an $\mathcal{H}_{\mathcal{O}}$-lattice and $V_{\mathcal{O}} = X_{\mathcal{O}} \otimes_{\mathcal{H}_{\mathcal{O}}} M_{\mathcal{O}}$. Then the following holds:*

(i) *There is a natural isomorphism between $K \otimes_{\mathcal{O}} V_{\mathcal{O}}$ and $X_K \otimes_{\mathcal{H}_K} M_K = V_K$.*

(ii) *V_K is $P_{1,K}$ und P_K-torsionless, and*

$$\mathrm{Hom}_{T_K}(P_{1,K}, V_K J_{1,K}) = (0) = \mathrm{Hom}_{T_K}(P_K, V_K J_{1,K}).$$

(iii) $\mathrm{Hom}_{T_{\mathcal{O}}}(P_{1,\mathcal{O}}, V_{\mathcal{O}})J_{1,\mathcal{O}} = (0) = \mathrm{Hom}_{T_{\mathcal{O}}}(P_{\mathcal{O}}, V_{\mathcal{O}})J_{1,\mathcal{O}}$.

(iv) *The $\mathcal{O}$-torsion submodule of $V_{\mathcal{O}}$ is precisely the $P_{1,\mathcal{O}}$-torsion submodule $t_{P_{1,\mathcal{O}}}(V_{\mathcal{O}})$ of $V_{\mathcal{O}}$ and coincides with $t_{P_{\mathcal{O}}}(V_{\mathcal{O}})$.*

(v) *$\hat{H}_R^{\beta}(X_R) = \hat{H}_R^{\beta_1}(X_R) = V_R/t_{P_{1,R}}(V_R)$ for all choices of R.*

Proof Part one of the theorem is easy and left to the reader. To prove part two we may assume that X_K is an irreducible $\mathcal{H}_K$-module, since $\mathcal{H}_K$ is semisimple. But then $X_K = e\mathcal{H}_K$ for some primitive idempotent e of $\mathcal{H}_K$ and $V_K \cong eM_K$ by [9, 2.20 (ii)]. Now part two follows from 1.20 and the fact that there are no homomorphisms from $\ker \beta_{1,K}$ respectively from $\ker \beta_K$ to M_K. Part three follows now immediately.

To prove part four let U be the $\mathcal{O}$-torsion submodule of $V_{\mathcal{O}}$. Then obviously $Y = \mathrm{Hom}_{T_{\mathcal{O}}}(P_{1,\mathcal{O})}, U)$ is an $\mathcal{O}$-torsion module as well. Suppose $Y \neq (0)$. Then by 1.11 the proper epimorphic image $H^{\beta_1}_{\mathcal{O}}(U)$ of Y is $\mathcal{O}$-torsion and a submodule of

$$H^{\beta_1}_{\mathcal{O}}(V) = H^{\beta_1}_{\mathcal{O}} \hat{H}^{\beta_1}_{\mathcal{O}}(X) = X,$$

which in turn is $\mathcal{O}$-torsionfree by assumption. But this is a contradiction. Thus $Y = (0)$.

$$K \otimes_{\mathcal{O}} (V_{\mathcal{O}}/U) \cong K \otimes_{\mathcal{O}} V_{\mathcal{O}} = V_K$$

which is $P_{1,K}$- and P_K-torsionless by part three, hence $V_{\mathcal{O}}/U$ is $P_{1,\mathcal{O}}$- and $P_{\mathcal{O}}$-torsionless, as desired. Now part four follows from [1, section 5]. Part five for $R = \mathcal{O}$ or K is an obvious consequence. For $R = F$ we observe first that homomorphisms from projective modules are always liftable, and hence $F \otimes_{\mathcal{O}} U$ is $P_{1,F}$- and P_F-torsionless (but not necessarily equal to $t_{P_{1,F}}(F \otimes_{\mathcal{O}} V_{\mathcal{O}})$ or to $t_{P_F}(F \otimes_{\mathcal{O}} V_{\mathcal{O}})$). However, let $P_F = P_{1,F} \oplus P_{2,F}$ as in the beginning of this section. But now $P_{2,K}$ and V_K do not have a composition factor in common, since $P_{2,K}$ belongs to $\ker \beta_K$. Thus $V_{\mathcal{O}}/U$ is in $\mathbf{ker}(P_{2,\mathcal{O}})$, and the same holds if we tensor by F (note that U splits from $V_{\mathcal{O}}$ as $\mathcal{O}$-space). From this, part five follows for $F = R$ easily. □

We remark that in part five of the theorem $\hat{H}^{\beta}_F(X_F) = (F \otimes_{\mathcal{O}} V)/t_{P_F}(F \otimes_{\mathcal{O}} V)$ is not in general equal to $F \otimes_{\mathcal{O}} \hat{H}^{\beta}_{\mathcal{O}}(X_{\mathcal{O}})$, since the latter might still have a submodule which is in $\mathbf{ker}(P_F)$. All what we claim here is, that the functors for β and β_1 send also on the F-level X_F to the same T_F-module.

As remarked in 1.21 we cannot expect that the functor $H^{\beta}_{\mathcal{O}}$ takes lattices to lattices. But there are large classes of lattices where this happens, and where we can hence apply Theorem 1.22.

Remark 1.25 *Let R as above. Take the class $\mathfrak{I}_M(\hat{H})_R$ of all T_R-modules V_R which arise in the following way: Let $\hat{H}_{\mathcal{O}}$ be any of the four right inverses of the Hom-functor $H^{\beta}_{\mathcal{O}}$ for any projective presentation $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$, satisfying* 1.1*, and take $V_R = R \otimes_{\mathcal{O}} U_{\mathcal{O}}$, where $U_{\mathcal{O}}$ is any sublattice or $\mathcal{O}$-free epimorphic image of $\hat{H}_{\mathcal{O}}(X_{\mathcal{O}})$, and $X_{\mathcal{O}}$ is any $\mathcal{H}_{\mathcal{O}}$-lattice. Then all Hom-functors derived from projective presentations of $M_{\mathcal{O}}$ satisfying* 1.1 *coincide on $\mathfrak{I}_M(\hat{H})_R$. Thus $\mathfrak{I}_M(\hat{H})_R \subseteq \mathfrak{M}_R$. (Recall that $\mathfrak{M}_R$ consists of all T_R-modules which are taken to the same $\mathcal{H}_R$-modules by all Hom-functors H^{β}).*

Proof We observe that by part three of the Theorem 1.24

$$\mathrm{Hom}_{T_{\mathcal{O}}}(P_{1,\mathcal{O}}, X_{\mathcal{O}} \otimes M_{\mathcal{O}}) J_{1,\mathcal{O}} = (0), \tag{1.26}$$

hence the same holds, if $X_{\mathcal{O}} \otimes M_{\mathcal{O}}$ is replaced by a sublattice or epimorphic image (regardless, if it is $\mathcal{O}$-free or not). Since all $T_{\mathcal{O}}$-lattices $Y = \hat{H}_{\mathcal{O}}(X_{\mathcal{O}})$ arise in this way too, we have the same result if we replace $X_{\mathcal{O}} \otimes M_{\mathcal{O}}$ by sublattices or epimorphic images of Y in 1.26, in particular for all members of $\mathfrak{I}_M(\hat{H})_{\mathcal{O}}$. By [9, 4.12] $H^{\beta_1}\mathcal{O}(V_{\mathcal{O}})$ is free as $\mathcal{O}$-module for $V_{\mathcal{O}} \in \mathfrak{I}_M(\hat{H})_{\mathcal{O}}$ and the result follows from 1.22. □

Remark 1.27 *Note that indeed* [9, 4.12] *describes an even wider class of $T_{\mathcal{O}}$-lattices to which we may apply Theorem* 1.22. *Note too, that in particular all sublattices $V_{\mathcal{O}}$ of $M_{\mathcal{O}}$ belong to this class by similar arguments, since $\mathcal{M}_{\mathcal{O}}$ is $P_{\mathcal{O}}$ torsionless by* 1.20 *and hence there can be no non trivial homomorphisms from* $\ker \beta_R$ *to V_R for $R = K$ and hence for $R = \mathcal{O}$.*

In [9, 4.17] there is another mistake which I would like to straighten out here. Let X, Y be pure right ideals of $\mathcal{H}_{\mathcal{O}}$, thus they have $\mathcal{O}$-complements in $\mathcal{H}_{\mathcal{O}}$. The purification of a sublattice U of $M_{\mathcal{O}}$ is defined to be $\sqrt{U} = KU \cap M_{\mathcal{O}}$. We know by 1.20 that $M_{\mathcal{O}}$ is $P_{\mathcal{O}}$-torsionless. By [9, 2.20 and 4.14]

$$H_R(XM) \cong H_R(R \otimes_{\mathcal{O}} \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}) \cong X_R, \tag{1.28}$$

where R is one of the rings $F, \mathcal{O}$, or K, and a similar statement holds for Y_R. Note that the lattices XM_R and YM_R belong to $\mathfrak{M}_R$ by Remark 1.25, hence we have not to specify the projective presentation of $M_{\mathcal{O}}$ which we use.

In [9, 2.21] it was shown that the functor $H_R = H_{M_R}$ induces an isomorphism, also denoted by H_R:

$$H_R : \mathrm{Hom}_{T_R}(X_R M_R, Y_R M_R) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}_R}(X_R, Y_R). \tag{1.29}$$

In [9, 4.17] this result was extended to the purification of the corresponding sublattices of $M_{\mathcal{O}}$. However the result needs an additional assumption, and the hypotheses in [9, 4.17], that $M_{\mathcal{O}}$ is $P_{\mathcal{O}}$-torsionless can be dropped in view of Result 1.20:

Theorem 1.30 *Suppose that $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$ satisfies* 1.1. *Let $X = X_{\mathcal{O}}$ and $Y = Y_{\mathcal{O}}$ be pure right ideals of $\mathcal{H}_{\mathcal{O}} = \mathrm{End}_{T_{\mathcal{O}}}(M_{\mathcal{O}})$. Suppose every homomorphism from $X_{\mathcal{O}}$ to $\mathcal{H}_{\mathcal{O}}$ is given as λ_h for some $h \in \mathcal{H}_{\mathcal{O}}$, where $\lambda_h : X_{\mathcal{O}} \to \mathcal{H}_{\mathcal{O}}$ is left multiplication by h. Then the associated Hom-functor H_R induces an isomorphism*

$$H_R : \mathrm{Hom}_{T_R}(R \otimes_{\mathcal{O}} \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}, R \otimes_{\mathcal{O}} \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}_R}(X_R, Y_R), \tag{1.31}$$

for $R = \mathcal{O}$ and K. If in addition M_F is P_F-torsionfree and $\mathrm{Hom}_{\mathcal{H}_F}(X_F, Y_F)$ *is liftable then Formula* 1.31 *holds for $R = F$ as well.*

Proof By [9, 2.22 ii)] the homomorphism

$$\begin{aligned} H_{\mathcal{O}} : \mathrm{Hom}_{T_{\mathcal{O}}}(\sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}, \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}}) \\ \to \mathrm{Hom}_{\mathcal{H}_{\mathcal{O}}}(H_{\mathcal{O}}(\sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}), H_{\mathcal{O}}(\sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}})) \\ = \mathrm{Hom}_{\mathcal{H}_{\mathcal{O}}}(X_{\mathcal{O}}, Y_{\mathcal{O}}) \end{aligned} \tag{1.32}$$

is injective. Let $\phi : X_{\mathcal{O}} \to Y_{\mathcal{O}}$ be $\mathcal{H}_{\mathcal{O}}$-linear. We use the natural embedding $i_{\mathcal{O}} : Y_{\mathcal{O}} \to \mathcal{H}_{\mathcal{O}}$ to find $h \in \mathcal{H}_{\mathcal{O}}$ such that $i_{\mathcal{O}}\phi = \lambda_h$, where λ_h is left multiplication by h. By [9, 2.21] $\phi = \lambda_h = \mathcal{H}_{\mathcal{O}}(\psi)$ for some $T_{\mathcal{O}}$-linear map $\psi : X_{\mathcal{O}} M_{\mathcal{O}} \to Y_{\mathcal{O}} M_{\mathcal{O}}$. We denote by ψ_K an extension of $1 \otimes_{\mathcal{O}} \psi : X_K M_K \to Y_K M_K$ to an endomorphism of M_K. Obviously we can choose ψ_K to be the left multiplication λ_h by $h \in \mathcal{H}_{\mathcal{O}} \subseteq \mathcal{H}_K$. Let $m \in \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}$, that is

$m \in M_{\mathcal{O}}$ and $\wp^k m \in X_{\mathcal{O}} M_{\mathcal{O}}$ for some natural number k, where $\wp$ is a generator of the unique maximal ideal of the complete discrete valuation ring $\mathcal{O}$. So

$$\wp^k(\psi_K(m)) = \psi_K(\wp^k m) = h\wp^k m = \wp^k hm$$

is an element of $Y_{\mathcal{O}} M_{\mathcal{O}}$. On the other hand

$$\psi_K(m) = \lambda_h(m) = hm \in M_{\mathcal{O}},$$

and hence $\psi_K(m) \in M_{\mathcal{O}} \cap Y_K M_K = \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}}$. We have shown that the restriction of ψ_K to $\sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}$ extends the $T_{\mathcal{O}}$-linear map $\psi : X_{\mathcal{O}} M_{\mathcal{O}} \to Y_{\mathcal{O}} M_{\mathcal{O}}$ to a $T_{\mathcal{O}}$-linear map from $\sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}$ to $\sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}}$. This shows that $H_{\mathcal{O}}$ in 1.32 is surjective, and hence an isomorphism. Thus our claim holds for $R = \mathcal{O}$ and obviously for $R = K$ as well.

Now assume that M_F is P_F-torsionless, and that $\mathrm{Hom}_{\mathcal{H}_F}(X_F, Y_F)$ is liftable. Let $\overline{\phi} : X_F \to Y_F$ be $\mathcal{H}_F$-linear. By assumption $\overline{\phi} = 1_F \otimes_{\mathcal{O}} \lambda_h = \lambda_{\overline{h}}$ for some $h \in \mathcal{H}_{\mathcal{O}}$ setting $\overline{h} = 1_F \otimes_{\mathcal{O}} h \in \mathcal{H}_F$. We set $\psi = H_{\mathcal{O}}^{-1}(\lambda_h)$ and conclude that $H_F(1_F \otimes_{\mathcal{O}} \psi) = \overline{\phi}$. Thus

$$\begin{aligned}(1.33)\quad H_F : \mathrm{Hom}_{T_F}(F \otimes_{\mathcal{O}} \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}, F \otimes_{\mathcal{O}} \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}}) \\ \to \mathrm{Hom}_{\mathcal{H}_F}(H_F(F \otimes_{\mathcal{O}} \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}), H_F(F \otimes_{\mathcal{O}} \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}})) \\ = \mathrm{Hom}_{\mathcal{H}_F}(X_F, Y_F)\end{aligned}$$

is surjective by [9, 4.12, ii]. By [9, 2.22 ii] this map has to be injective too, finishing the proof of the theorem. □

Remark 1.34 *In the situation of the theorem we have indeed shown: The functor H_F induces an epimorphism*

$$H_F : \mathrm{Hom}_{T_F}(F \otimes_{\mathcal{O}} \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}, F \otimes_{\mathcal{O}} \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}}) \to \mathrm{Hom}_{\mathcal{H}_F}(X_F, Y_F),$$

provided $\mathrm{Hom}_{H_F}(X_F, Y_F)$ is liftable. We used the hypothesis, that M_F is P_F-torsionless only to ensure, that this map is injective too. There are however other sufficient conditions for this to hold. For instance, our proof shows also that the restriction of H_F to the subspace $F \otimes_{\mathcal{O}} \mathrm{Hom}_{T_{\mathcal{O}}}(\sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}, \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}})$ of $\mathrm{Hom}_{T_F}(F \otimes_{\mathcal{O}} \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}, F \otimes_{\mathcal{O}} \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}})$ is injective. Thus, if $\mathrm{Hom}_{T_F}(F \otimes_{\mathcal{O}} \sqrt{X_{\mathcal{O}} M_{\mathcal{O}}}, F \otimes_{\mathcal{O}} \sqrt{Y_{\mathcal{O}} M_{\mathcal{O}}})$ is liftable, H_F is an isomorphism.

Recall that an arbitrary ring S is selfinjective if the regular representation S_S is an injective S-module. If so every homomorphism from a right ideal X into S can be extended to an endomorphism of S and hence is left multiplication by some element of S. In general $\mathcal{H}_{\mathcal{O}}$ is not selfinjective, but in all our applications $\mathcal{H}_F$ is not only selfinjective, but even stronger a *symmetric* algebra. That means that there is a symmetric nondegenerate bilinear form $<\ ,\ >$ on $\mathcal{H}_F$ which is associative that is $< xh\, ,\, y >=< x\, ,\, hy >$ for all $h, x, y \in \mathcal{H}_F$. In our applications this form can be realized over $\mathcal{O}$: We have an associative bilinear form $<\ ,\ >: \mathcal{H}_{\mathcal{O}} \times \mathcal{H}_{\mathcal{O}} \to \mathcal{O}$ such that its Gram determinant with respect to an $\mathcal{O}$-basis of $\mathcal{H}_{\mathcal{O}}$ is a unit in $\mathcal{O}$ which induces the form on $\mathcal{H}_F$. If such a form on $\mathcal{H}_R$ is given, we define the subset $X^{\perp}$ *orthogonal* to the subset X of $\mathcal{H}_R$ to be $\{y \in \mathcal{H}_R \mid < y\, ,\, x >= 0\ \forall x \in X\}$,

($R \in \{K, \mathcal{O}, F\}$). We shall show now, that the existence of such a form on $\mathcal{H}_\mathcal{O}$ is sufficient for the assumption in Theorem 1.30 to hold. We first need an auxiliary result on left annihilators which generalizes well known facts on bilinear forms on algebras over fields. For convenience we include a proof.

For a ring S and a subset X of S we denote the left annihilator of X in S by $l(X)$. Similarly we define the right annihilator $r(X)$ of X.

Lemma 1.35 *Let*

$$< \ , \ >: \mathcal{H}_\mathcal{O} \times \mathcal{H}_\mathcal{O} \to \mathcal{O} : (x,y) \mapsto < x\, , \, y > \in \mathcal{O}$$

be an associative bilinear form whose Gram determinant is a unit in $\mathcal{O}$. Let X be a right ideal of $\mathcal{H}_\mathcal{O}$. Then the left annihilator $l(X)$ of X is a pure left ideal of $\mathcal{H}_\mathcal{O}$ of $\mathcal{O}$-rank $\mathrm{rk}(\mathcal{H}_\mathcal{O}) - \mathrm{rk}(X)$. *Moreover $l(X) = l(\sqrt{X_\mathcal{O}}) = X^\perp$. If in addition X is pure in $\mathcal{H}_\mathcal{O}$, then*

$$l(X_F) = F \otimes_\mathcal{O} l(X_\mathcal{O}).$$

In particular $\dim_F(l(X_F)) = \dim_F(\mathcal{H}_F) - \dim_F(X_F)$.

Proof It is easy to check that $l(X) = l(\sqrt{X_\mathcal{O}})$, and that $l(X)$ is a pure left ideal of $\mathcal{H}_\mathcal{O}$.

Let X be pure in $\mathcal{H}_\mathcal{O}$. Then $X_R \leq \mathcal{H}_R$ for $R = F$ or K. We have obviously induced forms on $\mathcal{H}_R$ which again are denoted by $< \ , \ >$. Since the Gram determinant of $< \ , \ >$ over $\mathcal{O}$ is a unit these forms are nondegenerate. Obviously we have also $\mathcal{H}_\mathcal{O}^\perp = (0)$. Now for $R \in \{K, \mathcal{O}, F\}$ and $y \in \mathcal{H}_R$ we have

$$yX_R = (0) \Leftrightarrow < yX_R\, , \mathcal{H}_R >= (0) \Leftrightarrow < y\, , X_R\mathcal{H}_R >= (0) \Leftrightarrow < y\, , X_R >= (0) \Leftrightarrow y \in X_R^\perp,$$

and hence $l(X_R) = X_R^\perp$.

The form $< \ , \ >$ induces an homomorphism from $\mathcal{H}_R$ into the dual space $\mathcal{H}_R^* = \mathrm{Hom}_R(\mathcal{H}_R, R)$ sending $h \in \mathcal{H}_R$ to

$$f_h : \mathcal{H}_R \to R : x \mapsto < x\, , \, h > \in R,$$

and one checks easily that this map is $\mathcal{H}_R$-linear on the left. Moreover it is an isomorphism, its matrix with respect to a dual pair of bases of $\mathcal{H}_R$ and $\mathcal{H}_R^*$ given as Gram matrix of $< \ , \ >$ with respect to the given basis of $\mathcal{H}_R$. Thus it is an isomorphism of left $\mathcal{H}_R$-modules. The embedding of $X_R = R \otimes_R X$ into $\mathcal{H}_R$ induces an left $\mathcal{H}_R$- module epimorphism from $\mathcal{H}_R^*$ onto $X_R^* = \mathrm{Hom}_R(X_R, R)$, given by restricting linear maps from $\mathcal{H}_r$ to X_R. One checks easily that the kernel of the composite map

$$\mathcal{H}_R \to \mathcal{H}_R^* \to X_R^*$$

is $X_R^\perp = l(X_R)$. Observe that $R \otimes_\mathcal{O} l(X) \subseteq l(X_R)$. The lemma now follows comparing dimensions. □

Since the induced form on $\mathcal{H}_R$ for $R = K$ or F is nondegenerate, $\mathcal{H}_R$ is a Frobenius algebra. Such algebras are in particular selfinjective, (see e.g. [6, 9.9]), and hence every homomorphism from X_R to $\mathcal{H}_R$ is given by left multiplication by elements of $\mathcal{H}_R$. In the situation of the previous lemma we can indeed lift this property to the $\mathcal{O}$-level:

Theorem 1.36 *Let $<\ ,\ >$ be an associative bilinear form on $\mathcal{H}_{\mathcal{O}}$ whose Gram determinant is a unit. Let X be a pure right ideal of $\mathcal{H}_{\mathcal{O}}$. Then*

$$\mathrm{Hom}_{\mathcal{H}_F}(X_F, \mathcal{H}_F) \cong F \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{H}_{\mathcal{O}}}(X_{\mathcal{O}}, \mathcal{H}_{\mathcal{O}})$$

and every homorphism from X to $\mathcal{H}_{\mathcal{O}}$ is given by left multiplication by some element of $\mathcal{H}_{\mathcal{O}}$.

Proof Let $\psi : X_F \to \mathcal{H}_F$ be $\mathcal{H}_F$-linear. Since $\mathcal{H}_F$ is a Frobenius algebra we find $\bar{h} \in \mathcal{H}_F$ such that $\psi = \lambda_{\bar{h}}$. Let $h \in \mathcal{H}_{\mathcal{O}}$ be such that $\bar{h} = 1_F \otimes_{\mathcal{O}} h$. Then $\psi = \lambda_{\bar{h}} = 1_F \otimes_{\mathcal{O}} \lambda_h$, and hence every homomorphism from X_F into $\mathcal{H}_F$ is liftable.

Let $\phi : X \to \mathcal{H}_{\mathcal{O}}$ be $\mathcal{H}_{\mathcal{O}}$-linear, and let $\bar{\phi} = 1_F \otimes_{\mathcal{O}} \phi : X_F \to \mathcal{H}_F$. The extension of ϕ to a homomorphism from X_K to $\mathcal{H}_K$ is again denoted by ϕ. Since $\mathcal{H}_K$ is a Frobenius algebra we find $a \in \mathcal{H}_K$ such that $\phi = \lambda_a$. Since $a \in \mathcal{H}_K$ we find $0 \leq k \in \mathbb{Z}$ such that $\wp^k a = b \in \mathcal{H}_{\mathcal{O}}$, where again $\wp$ is a generator of the unique maximal ideal of the valuation ring $\mathcal{O}$. We may choose a and b such that k is minimal such that $\phi = \lambda_a$ and $\wp^k a = b \in \mathcal{H}_{\mathcal{O}}$.

Suppose $k \neq 0$. Note that $\lambda_b = \lambda_{\wp^k a} = \wp^k \lambda_a = \wp^k \phi$. We have

$$bx = \wp^k ax = \wp^k \lambda_a x = \wp^k \phi x \in \wp^k \mathcal{H}_{\mathcal{O}} \quad \forall x \in X,$$

and hence

$$\bar{b} = 1_F \otimes_{\mathcal{O}} b \in l(X_F) = F \otimes_{\mathcal{O}} l(X),$$

by lemma 1.35. Thus we find $c \in l(X)$ with $\bar{b} = 1_F \otimes_{\mathcal{O}} c$, and hence $d \in \mathcal{H}_{\mathcal{O}}$ such that $b = c + \wp d$. Since b and $\wp d$ differ only by something in the annihilator of X (namely c), we have $\lambda_b = \lambda_{\wp d}$ on X, hence

$$\wp^k \phi = \lambda_b = \lambda_{\wp d} = \wp \lambda_d$$

on X, and therefore

$$\phi = \wp^{1-k} \lambda_d$$

a contradiction to the minimality of k. Thus $k = 0$. But then $a = b \in \mathcal{H}_{\mathcal{O}}$ and $\phi = \lambda_a = \lambda_b$ as desired. □

In the situation of Theorem 1.30 we need later some additional information in the special case, where $Y_R = \mathcal{H}_R$. Here all the Hom-sets in Formulas 1.29 and 1.31 inherit left $\mathcal{H}_R$-module structures by viewing M_R and $\mathcal{H}$ as left $\mathcal{H}$-modules.

Lemma 1.37 *In the situation of* 1.30 *let Y_R be the regular representation $\mathcal{H}_R$ and suppose that H_R is an isomorphism for all choices of R. Then H_R in* 1.29 *and* 1.30 *are isomorphisms of left $\mathcal{H}_R$-modules.*

Proof Obviously we may assume $R = \mathcal{O}$ and omit the index R. Let V be XM or $\sqrt{XM}$, and let $H = H^\beta$. Since V is contained in M, the ideal J_β annihilates $\mathrm{Hom}_T(P, V)$ and (compare 1.27). Thus (embedding $\mathrm{mod}_{\mathcal{H}}$ into $\mathrm{mod}_{\mathcal{E}}$ by the inflation functor) $\mathrm{Hom}_T(P, M) = H(M) = \mathcal{H}$ and $\mathrm{Hom}_T(P, V) = H(V)$.

Let $f : V \to M$ be T-linear, $\alpha \in \mathrm{Hom}_T(P,V)$, $h \in \mathcal{H}$, and $p \in P$. Then

$$(H(hf))(\alpha) : p \to (hf)(\alpha(p)) = h(f \circ \alpha(p)),$$

which is as well the image of p under $(h(H(f)))(\alpha)$, hence

$$H(hf) = hH(f),$$

as desired, since the canonical isomorphism $\mathrm{Hom}_T(P,M) \to \mathcal{H}$ is an $\mathcal{H}$-$\mathcal{E}$-bimodule map. $\square$

The question arises if the situation of 1.1 comes up often. The following remark, which was pointed out to me by Gerhard Hiss, shows that we can produce situation 1.1 in the general setting described in [9, section 4].

Remark 1.38 *Let $P_{\mathcal{O}}$ be a projective $\mathrm{T}_{\mathcal{O}}$-lattice. Let $\mathcal{M}$ be a subset of irreducible T_K-modules occurring as constituent of P_K, and suppose that not every composition factor of P_K is isomorphic to an element of $\mathcal{M}$. Let U be the sum of all submodules of P_K whose constituents belong to $\mathcal{M}$. Thus no constituent of a complement of U in P_K belongs to $\mathcal{M}$. Take $U_{\mathcal{O}} = U \cap P_{\mathcal{O}}$. Then $U_{\mathcal{O}}$ is a pure sublattice of $P_{\mathcal{O}}$ and $U = U_K$. Moreover, setting $M_{\mathcal{O}} = P_{\mathcal{O}}/U_{\mathcal{O}}$ and $\beta : P_{\mathcal{O}} \to M_{\mathcal{O}}$ to be the natural epimorphism, we have constructed a projective presentation of $M_{\mathcal{O}}$ satisfying* 1.1. *Obviously every situation as described in* [9, section 4] *and in Results* 1.20 *through* 1.37 *arises in this way.*

2 Hom-functors and Harish-Chandra theory

We shall give now a first application which is important for finite reductive groups but can be formulated in a more general context.

Let G be a finite group and let F be a field. For a subfactor L of G, given by a pair (P,U) of subgroups of G with $U \trianglelefteq P$ and $L = P/U$, we denote *Harish-Chandra induction* (HC-induction for short) from mod_{FL} to mod_{FG} by R_L^G and its right adjoint functor, *Harish-Chandra truncation* (HC-restriction for short), by T_L^G. Thus R_L^G lifts an FL-module to an FP-module by letting U act trivially and then induces it from P to G, and dually T_L^G restricts a FG-module to P and takes U-fixpoints to yield an FL-module ([15]). If in addition the order of U is invertible in the field F, a condition which we shall usually assume in the following, T_L^G is adjoint on both sides to R_L^G. In [13] those functors were used to generalize Greens theory of vertices and sources from subgroups of G to subfactors.

We mostly adopt the notation introduced in [13] and [10] and refer the reader to those papers for details. For two subfactors L, K of G, given by the pairs (P,U) and (Q,V) of subgroups of G, we define the *shift intersection* to be given by the pair

$$L \mathbin{\tilde{\cap}} K = ((P \cap Q)V, (U \cap Q)V).$$

This is a relation which is neither commutative nor associative, however $L \mathbin{\tilde{\cap}} K \cong K \mathbin{\tilde{\cap}} L$ naturally. How to conjugate subfactors is obvious. We have a Mackey decomposition theorem [13, 1.4]:

Theorem 2.1 *Let L, K be a subfactors of G given by the pairs (P, U), Q, V respectively. Suppose that the orders of U and V are invertible in F. Let M be an FL-module. Then*

$$\mathrm{T}^G_K \circ \mathrm{R}^G_L(M) \cong \bigoplus_{x \in P\backslash G/Q} \mathrm{R}^K_{L^x \tilde{\cap} K} \circ \mathrm{T}^{L^x}_{K \tilde{\cap} L^x}(M^x)$$

where M^x denotes the conjugate module for the conjugate subfactor $L^x = x^{-1}Lx$ of G, and $P\backslash G/Q$ is a set of P-Q-double coset representatives in G.

We remark that in Theorem 2.1 all the intersections are again given by pairs (P', U') of subgroups of G (with $U' \trianglelefteq P$) such that the order of U' is invertible in F.

A system $\mathcal{M}$ of subfactors of G is called a *Mackey system*, if it is closed under shift intersection and conjugation and if it contains G. We say that $\mathcal{M}$ is *p-modular*, p a prime, if all the subgroups U determining the elements of $\mathcal{M}$ are p-regular, that is their order is not divisible by p. Thus if F has characteristic p and $\mathcal{M}$ is p-modular, we may apply 2.1 to subfactors in $\mathcal{M}$. For $L \in \mathcal{M}$ the set

$$\mathcal{M}_L = \{H \tilde{\cap} L \,|\, H \in \mathcal{M}\}$$

defines a Mackey system in L, which is p-modular, if $\mathcal{M}$ is so. Note that $\mathcal{M}_L \subseteq \mathcal{M}$.

Thus fix now $\mathcal{M}$ to be a Mackey system of subfactors of G and assume that it is p-modular, if F has characteristic p. An FG-module M is called *cuspidal* (with respect to $\mathcal{M}$), if $\mathrm{T}^G_L(M) = (0)$ for all subfactors $L \neq G$ in $\mathcal{M}$. Observe that $T^G_L(M) = (0)$, if L is actually a subgroup of G, given as pair $(L, (1))$. Thus, if $\mathcal{M}$ contains a proper subgroup, then FG has no cuspidal modules. If $L \in \mathcal{M}$ and M is an FL-module, we say M is cuspidal (with respect to $\mathcal{M}$), if it is cuspidal with respect to $\mathcal{M}_L$. Of course, even if $\mathcal{M}$ contains a proper subgroup and hence does not have cuspidal representations, the same might be not true for $\mathcal{M}_L$. We point out that the definition of "cuspidal" in [13] and [10] is slightly different excluding subfactors in $\mathcal{M}$ which are actually subgroups of G, hence correspond to a pair $(L, (1))$. The adjusted definition seems to be more consistent and practical. Indeed [13, 4.6] is false under the old definition but holds assuming ours here.

The following theorem relates Harish-Chandra theory and the results of [9]:

Lemma 2.2 *Let $\mathcal{M}$ be a Mackey system for G which is p-modular, if the characteristic of the field F is $p > 0$, and let L be a subfactor of G contained in $\mathcal{M}$. Let $\tilde{M}$ be an irreducible cuspidal FL-module, $\tilde{\beta} : \tilde{X} \to \tilde{M}$ a minimal projective cover of $\tilde{M}$. Then*

$$\mathrm{R}^G_L(\tilde{\beta}) : \mathrm{R}^G_L(\tilde{X}) \to \mathrm{R}^G_L(\tilde{M})$$

satisfies Hypothesis 1.1.

Proof The lemma generalizes a result of M. Geck, G. Hiss and G. Malle in [28, 2.2] on finite reductive groups G, where $\mathcal{M}$ is given as the set of Levi subgroups L considered as subfactors (P, U), where $P = LU$ is a parabolic subgroup of G with unipotent radical U and Levi complement L. The proof there (based on [15, 2.7]) carries easily over to the general case. For convenience we shall give details here.

We set $\beta = \mathrm{R}_L^G(\tilde{\beta}), X = \mathrm{R}_L^G(\tilde{X})$ and $M = R_L^G(\tilde{M})$. We adopt the notation of section one (replacing P there by our module X). Thus $\mathcal{E} = \mathrm{End}_{FG}(X), \mathcal{H} = \mathrm{End}_{FG}(M), \mathcal{E}_\beta = \{\phi \in \mathcal{E} \mid \phi(\ker\beta) \leq \ker\beta\}$ and $J_\beta = \{\psi \in \mathcal{E} \mid \operatorname{im}\psi \leq \ker\beta\}$. We have a map

$$\beta^* : \mathcal{E} \to \mathrm{Hom}_{FG}(X, M) : \phi \mapsto \beta \circ \phi,$$

which is surjective by projectivity of X, and which has kernel J_β. Thus

$$\dim_F \mathrm{Hom}_{FG}(X, M) = \dim_F \mathcal{E} - \dim_F J_\beta.$$

On the other hand

$$\dim_F \mathcal{H} = \dim_F \mathcal{E}_\beta - \dim_F J_\beta,$$

since $\mathcal{H} \cong \mathcal{E}_\beta / J_\beta$ canonically. This implies now that $\mathcal{E}_\beta = \mathcal{E}$ if and only if

$$\dim_F \mathrm{Hom}_{FG}(X, M) = \dim_F \mathrm{Hom}_{FG}(M, M).$$

We now apply adjointness of the functors R_L^G and T_L^G and Mackey decomposition 2.1 to get:

$$\begin{aligned} \mathrm{Hom}_{FG}(M, M) &\cong \bigoplus_{x \in P\backslash G/P} \mathrm{Hom}_{FL}(\tilde{M}, \mathrm{R}^L_{L^x \tilde{\cap} L} \circ \mathrm{T}^{L^x}_{L \tilde{\cap} L^x}(\tilde{M}^x)) \\ &= \bigoplus_{x \in N_G(P,U) \cap P\backslash G/P} \mathrm{Hom}_{FL}(\tilde{M}, \tilde{M}^x), \end{aligned} \tag{2.3}$$

where L is given by the pair (P, U) and $N_G(P, U)$ is defined to be the set $\{x \in G \mid L^x \tilde{\cap} L = L\}$ of elements of G, which "shift" normalize the pair (P, U). The second equation follows from the fact, that $\tilde{M}$ is cuspidal, hence truncation to every proper subfactor of L in $\mathcal{M}_L$ is zero. Similarly we see:

$$\mathrm{Hom}_{FG}(P, M) \cong \bigoplus_{x \in N_G(P,U) \cap P\backslash G/P} \mathrm{Hom}_{FL}(\tilde{P}, \tilde{M}^x). \tag{2.4}$$

Since $\tilde{M}$ is irreducible, $\dim_F \mathrm{Hom}_{FG}(\tilde{M}, \tilde{M}^x)$ is one, if $\tilde{M} \cong \tilde{M}$, and zero otherwise. Since $\tilde{P}$ is the minimal projective cover of $\tilde{M}$, we have similarly $\dim_F \mathrm{Hom}_{FG}(\tilde{P}, \tilde{M}^x)$ is one, if $\tilde{M} \cong \tilde{M}$, and zero otherwise. The lemma follows. □

In the situation of the theorem we have now the functors $H = H_M : \mathrm{mod}_{FG} \to \mathrm{mod}_{\mathcal{H}}$ and $\hat{H} = \hat{H}_M : \mathrm{mod}_{\mathcal{H}} \to \mathrm{mod}_{FG}$ described in section one [9, 2.28] implies the result [28, 2.4] of Geck, Hiss and Malle (again in slightly generalized form):

Corollary 2.5 *The functors H and $\hat{H}$ provide a bijection between the isomorphism classes of the irreducible FG-modules occurring in the head of $M = \mathrm{R}_L^G(\tilde{M})$ and the isomorphism classes of irreducible $\mathcal{H}$-modules.*

Suppose now we have found all irreducible $\mathcal{H}$-modules. Application of the functor $\hat{H}$ will then produce the corresponding irreducible FG- modules. Another way is to exhibit primitive idempotents e in $\mathcal{H}$. Then eM is an indecomposable direct summand of M and has simple head. The irreducible FG-module $eM/\mathrm{Jac}(eM)$ corresponds to the irreducible head of the projective indecomposable $\mathcal{H}$-module $e\mathcal{H}$. All this was shown in [9, 2.32].

We relate this to the notion of semisimple vertices and sources, which was introduced in [13] based on work of Hiss [34]. An irreducible FG-module M is *weakly L-projective* ($L \in \mathcal{M}$), if $T_L^G(M) \neq (0)$, and L is called a *semisimple vertex* of M, if it is minimal with this property. By [13, 4.9] there is then an irreducible cuspidal FL-module (unique up to conjugation by certain group elements) such that M is a direct summand of the head of $R_L^G(N)$. N is called a *semisimple source* of M. If the Mackey system $\mathcal{M}$ is in particular nice ([13, 4.13]), then the isomorphism classes of irreducible FG-modules are partitioned into subsets labelled by semisimple vertices and sources. In this case we get a full set of nonisomorphic irreducible FG-modules with semisimple vertex L and semisimple source N precisely as set of nonisomorphic components of $\mathrm{hd}(M) = \mathrm{hd}(R_L^G(N))$, and hence by exhibiting a full set of nonisomorphic irreducible $\mathcal{H}$-modules.

3 Double centralizer theorems

We continue with our general discussion. We shall see later, that the main result of this section provides the general background for a paper of Takeuchi [45] . We begin with an auxiliary lemma. Thus for the moment let A be a finite dimensional semisimple K-algebra and assume that K is a splitting field for A. For $M \in \mathrm{mod}_A$ and an idempotent e of A we consider the space Me as eAe-module, where the eAe-action is given by restriction. Then the following result holds:

Lemma 3.1 *Assume that $Se \neq (0)$ for every composition factor S of M. Then the restriction of A-endomorphisms of M to the subset Me of M defines an isomorphism*

$$\mathrm{End}_A(M) \cong \mathrm{End}_{eAe}(Me).$$

Proof Obviously the restriction $\phi \mapsto \phi|_{Me}$ is an algebra homomorphism from $\mathrm{End}_A(M)$ into $\mathrm{End}_{eAe}(Me)$. The condition on e implies, that the A-module eA is a generator for M, that is M is an epimorphic image of a direct sum of copies of eA. From this we conclude immediately, that $MeA = M$, and therefore the restriction map above is injective. Moreover acting by primitive central idempotents one sees easily, that we may assume that A is simple. Thus $M = S \oplus S \oplus \cdots \oplus S$, where S is the unique simple A-module. Let the number of direct summands in this sum be m, and let $e = e_1 + \cdots + e_k$ be a decomposition of e into a sum of primitive orthogonal idempotents. Using Schur's lemma we conclude that $\mathrm{End}_A(M)$ is the matrix algebra $M_m(K)$. Note that by the Wedderburn theorems $k \leq \dim_K(S)$. Moreover by Fitting's theorem (see e.g. [41, 1.4]) we have

$$Me \cong \mathrm{Hom}_A(eA, M) = \mathrm{Hom}_A(eA, S) \oplus \cdots \oplus \mathrm{Hom}_A(eA, S) \cong Se \oplus \cdots \oplus Se,$$

as K-space. On the other hand Se is a k-dimensional K-subspace of S, and computing $\dim_K(eAe)$ to be k^2 one concludes that Se is an irreducible eAe-module. Therefore Me is a direct sum of m copies of the irreducible eAe-module Se, and hence

$$\dim_K(\mathrm{End}_{eAe}(Me)) = m^2 = dim_K(\mathrm{End}_A(M)).$$

The lemma follows. □

We return now to the situation of section one and keep the notation introduced there. As usual R is one of the rings $K, \mathcal{O}, F$. In particular, if not stated otherwise we assume that modules for an $\mathcal{O}$-order (e.g. $T_{\mathcal{O}}$, or $\mathcal{H}_{\mathcal{O}}$) are free and finitely generated over $\mathcal{O}$, (thus are *lattices*). We remind the reader of the notion of liftable modules and homomorphisms (see 1.19).

Given a T_R-module V the *annihilator* $\mathrm{ann}_{T_R}(V)$ of V in T_R consists of all elements $t \in T_R$ which annihilate V, that is which satisfy $vt = 0$ for all $v \in V$. Note that $\mathrm{ann}_{T_R}(V)$ is an ideal, and that we have always a canonical injective algebra homomorphism from $T/\mathrm{ann}_{T_R}(V)$ into the *bicentralizer* $\mathrm{End}_{\mathcal{E}}(V)$, where $\mathcal{E} = \mathrm{End}_{T_R}(V)$. The module V is said to be *balanced* or has the *double centralizer property*, if this map is surjective too, that is if $\mathrm{End}_{\mathcal{E}}(V) \cong T_R/\mathrm{ann}_{T_R}(V)$. The ring T is called *balanced*, if every right T-module is balanced. It is shown in [24, III.3.2], that this is equivalent to every left T-module being balanced. Note that finite dimensional semisimple algebras over fields are always balanced. This can be seen easily, using Wedderburn's structural theory and Fitting's lemma. It is also a special case of a result of Nesbitt and Thrall [42], which states that uniserial rings are balanced. For details see [24]. As an consequence we have the following auxiliary result:

Lemma 3.2 *Let $M = M_{\mathcal{O}}$ be a $T_{\mathcal{O}}$-lattice, and write $M_R = R \otimes_{\mathcal{O}} M_{\mathcal{O}}$ for $R = F, K$. Set $\mathcal{H}_{\mathcal{O}} = \mathrm{End}_{\mathrm{T}_{\mathcal{O}}}(M_{\mathcal{O}})$ and $\mathcal{H}_R = R \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}$. Suppose that $\mathrm{End}_{T_F}(M_F)$ is liftable, that is $\mathcal{H}_F = \mathrm{End}_{T_F}(M_F)$. Moreover assume that $\mathrm{End}_{\mathcal{H}_F}(M_F)$ is liftable as well. Then M_R is balanced for every choice of $R \in \{K, \mathcal{O}, F\}$ if and only if $\mathrm{ann}_{T_F}(M_F) = F \otimes_{\mathcal{O}} \mathrm{ann}_{T_{\mathcal{O}}}(M_{\mathcal{O}})$.*

Proof We have

$$\dim_F(\mathrm{End}_{T_F}(M_F)) = \dim_K(\mathrm{End}_{T_K}(M_K))$$
$$\dim_F(\mathrm{End}_{\mathcal{H}_F}(M_F)) = \dim_K(\mathrm{End}_{\mathcal{H}_K}(M_K))$$

by assumption. The annihilator condition in the lemma is equivalent to

$$\dim_F(T_F/\mathrm{ann}_{T_F}(M_F)) = \dim_K(T_K/\mathrm{ann}_{T_K}(M_K)),$$

hence the result for $R = F$ follows immediately comparing dimensions: We need only to observe that $T_F/\mathrm{ann}_{T_F(M_F)}$ is contained in $\mathrm{End}_{\mathcal{H}_F}(M_F)$, and that $\mathrm{End}_{\mathcal{H}_K}(M_K)$ is isomorphic to $T_K/\mathrm{ann}_{T_K}(M_K)$, since T_K is semisimple and hence balanced.

It remains to show the result for $R = \mathcal{O}$. Obviously the assumption that the endomorphism rings over F are liftable implies, that M_R is balanced for $R = F$ if it is balanced for $F = \mathcal{O}$, hence the annihilator condition holds in this case.

Thus conversely suppose that the annihilator condition holds. We have seen that M_F is balanced. Let $f : M \to M$ be $\mathcal{E}$-linear, $\mathcal{E} = \mathcal{E}_{\mathcal{O}}$. Let $\mathfrak{p}$ be a generator of the unique maximal ideal of $\mathcal{O}$. Since $1 \otimes f : K \otimes_{\mathcal{O}} M \to K \otimes_{\mathcal{O}} M$ is an element of $\mathrm{End}_{\mathcal{E}_K}(M_K)$ we find $t \in T_{\mathcal{O}}$ and $0 \leq k \in \mathbb{Z}$ such that $1 \otimes f$ is right multiplication by $\mathfrak{p}^{-k}t$. Suppose $k \neq 0$. Then multiplication by $t = \mathfrak{p}^k f$ maps $M_{\mathcal{O}}$ into $\mathfrak{p}M_{\mathcal{O}}$ and consequently $\bar{t} = t + \mathfrak{p}T_{\mathcal{O}} \in T_F$ annihilates M_F. Our annihilator condition says that we find $r \in \mathrm{ann}_{T_{\mathcal{O}}}(M_{\mathcal{O}})$ satisfying

$\bar{r} = \bar{t}$, hence $t_1 \in T_{\mathcal{O}}$ such that $t = r + \mathfrak{p}t_1$. Since r annihilates $M_{\mathcal{O}}$, t operates precisely as $\mathfrak{p}t_1$ on $M_{\mathcal{O}}$. Thus t_1 operates as $\mathfrak{p}^{k-1}f$ on $M_{\mathcal{O}}$. Continuing in this fashion one gets $t_k \in T_{\mathcal{O}}$ such that the action of f on $M_{\mathcal{O}}$ is multiplication by t_k. This shows that $M_{\mathcal{O}}$ is balanced as desired. □

We take now an $T_{\mathcal{O}}$-lattice $M = M_{\mathcal{O}}$ and let $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$ be a projective presentation of $M_{\mathcal{O}}$ satisfying Hypothesis 1.1. We assume that M is cyclic, that is $M = mT$ for an $m \in M$, and that $P_{\mathcal{O}}$ is direct summand of $T_{\mathcal{O}}$. Thus $P_{\mathcal{O}} = eT_{\mathcal{O}}$ for an idempotent $e \in T_{\mathcal{O}}$, and we may assume that β takes e to m. Thus the endomorphism ring $\mathcal{E}_R = \mathrm{End}_{T_R}(eT_R) = eT_Re$, where we denote the image of e in T_R again by e. Write the T-endomorphism ring of M_R as $\mathcal{H}_R$ for $R \in \{K, \mathcal{O}, F\}$. Note that by [9, 4.8] $\mathrm{End}_{T_F}(M_F)$ is liftable, that is $\mathcal{H}_F = F \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}$, so our notation is consistent. Recall that by [9, 4.7] β_R satisfies 1.1 for all choices of R, and that $\ker \beta_K$ and M_K have no constituent in common. Finally we denote $\ker \beta_R$ by U_R, and remark that indeed $U_R = R \otimes_{\mathcal{O}} U_{\mathcal{O}}$ by [9, 4.7].

We note that in this situation the functor H_R takes $V \in \mathrm{mod}_{T_R}$ to $Ve/VeJ_{\beta_R}e$ by [9, 3.6]. Moreover [9, 2.8 and 3.4] imply the following lemma:

Lemma 3.3 *Let the $T_{\mathcal{O}}$-lattice $M_{\mathcal{O}}$ be as above. Then for $R \in \{K, \mathcal{O}, F\}$ we have*

$$M_Re \cong \mathcal{H}_R$$

as $\mathcal{H}_R$-eT_Re-bimodule, where the right action by eT_Re is given by inflating the right H_R-action to $\mathcal{E}_R = eT_Re$ along the epimorphism $eT_Re \to eT_Re/J_{\beta_R} \cong \mathcal{H}_R$. Thus in particular M_Re is a balanced eT_Re-module.

We apply our lemmas now in the following special situation:

Situation 3.4 *We assume that $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$ satisfies Hypothesis* 1.1. *Note that $M_{\mathcal{O}}$ is $P_{\mathcal{O}}$-torsionless by Lemma* 1.20. *Suppose that $P_R = eT_R$ for an idempotent e of $T_{\mathcal{O}}$.*

We take pure right ideals $X_{1,\mathcal{O}}, \dots, X_{k,\mathcal{O}}$ of $\mathcal{H}_{\mathcal{O}}$. The T_R-module $N_{i,R}$ is defined as

$$N_{i,R} = R \otimes_{\mathcal{O}} \sqrt{X_{i,\mathcal{O}}M_{\mathcal{O}}}.$$

Set

$$X_R = \bigoplus_{i=1}^{k} X_{i,R},$$

and

$$N_R = \bigoplus_{i=1}^{k} N_{i,R}.$$

We denote the endomorphism ring $\mathrm{End}_{\mathcal{H}_{\mathcal{O}}}(X_{\mathcal{O}})$ *by $\mathcal{S} = \mathcal{S}_{\mathcal{O}}$, and we assume that* $\mathrm{End}_{\mathcal{H}_F}(X_F)$ *is liftable, hence we may apply* [9, 4.17] *by* 1.30. *In particular defining the functor $H_R = H_R^{\beta}$*

as in section one we have by Equation 1.28 *that* $H_R(N_R) \cong X_R$, *and* H_R *induces by* 1.30 *an isomorphism*

$$H_R : \mathrm{End}_{T_R}(N_R) \xrightarrow{\sim} \mathrm{End}_{\mathcal{H}_R}(X_R) = \mathcal{S}_R = R \otimes_{\mathcal{O}} \mathcal{S}.$$

Thus $\mathrm{End}_{T_F}(N_F)$ *is liftable as well. Assume that there are pairwise orthogonal idempotents* $e = e_1, e_2, \ldots, e_k$ *and projective presentations* $\beta_{i,R} : e_iT_R \to N_{i,R}$, *which satisfy* 1.1. *Assume in addition that* $e_1 = e$, *thus* $N_{1,\mathcal{O}} = M_{\mathcal{O}}$ *and* $X_{1,\mathcal{O}} = \mathcal{H}_{\mathcal{O}}$.

Suppose that $N_{i,\mathcal{O}} = M_{\mathcal{O}}e_iT_{\mathcal{O}}$, *and note that this condition means, that* $N_{i,R}$ *is the* e_iT_R*-trace* $\tau_{i,R}(M_R)$ *in* M_R, *writing* $\tau_{i,R}$ *instead of* τ_{P_R}, *for* $P_R = e_iT_R$. *This is obvious for* $R = \mathcal{O}$ *and* K, *and follows for* $R = F$, *since homomorphisms from projective modules are always liftable and* $N_{i,\mathcal{O}}$ *is pure in* $M_{\mathcal{O}}$. *We define the idempotent* $f = f_R$ *of* T_R *to be the sum* $e_1 + \ldots + e_k$, *(where we denote idempotents in* T_R *for the various rings* R *by the same symbols which is justified by uniqueness of lifting idempotents).*

We define $\gamma_R = \beta_{1,R} + \ldots + \beta_{k,R}$. The last condition $N_{i,\mathcal{O}} = \tau_{i,\mathcal{O}}(M_{\mathcal{O}})$ in 3.4 implies in particular that $\ker\gamma_K$ and N_K have no irreducible constituent in common and hence $\gamma_R : fT_R \to N_R$ also satisfies 1.1. To see this we take an irreducible constituent S from $\ker\gamma_K$. Then S is an irreducible constituent of $\ker\beta_{i,K}$ for some i. Suppose S is constituent of $N_{j,K}$. Then $i \neq j$. But then there is a nontrivial T_K-linear map from e_iT_K into $N_{j,K}$, whose image contains S. Since $\tau_{i,K}(M_K) = N_{i,K}$, this implies that S is contained in $N_{i,K}$, a contradiction to our assumption 3.4.

The situation described in 3.4 seems to be very special, but it is precisely what we need for our applications and for the promised generalization of Takeuchi's result [45]. Note that all modules involved satisfy the requirements of Remark 1.25 and hence belong to $\mathfrak{I}_M(\hat{H})_R$. Thus the T_R-modules $N_{i,R}$ and N_R depend not really on the chosen projective presentation $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$.

For a T_R-module V define

$$V^{\#} = \mathrm{Hom}_{T_R}(V, T_R). \tag{3.5}$$

Note that $V^{\#}$ is canonically a left T_R-module by the left T_R-module structure of T_R. We extend this notation in the obvious way to other modules, for example $\mathcal{H}_R$-modules. We observe that $-^{\#}$ is a contravariant functor from mod_T to ${}_T\mathrm{mod}$, which acts on homomorphisms by composition. More precisely if $\alpha : U \to V$ is a homomorphism in mod_T then $\alpha^{\#}$ takes $f : V \to T$ to $f^{\alpha^{\#}} = f \circ \alpha$. Writing $\alpha^{\#}$ to the right as superscript has the advantage that now

$$(\alpha \circ \beta)^{\#} = \alpha^{\#} \circ \beta^{\#}.$$

This is consistent with our convention to write homomorphisms opposite to the side of the ring action. Since # is involutive it induces bijections on Hom-sets. Therefore we have:

Lemma 3.6 *Let* $V \in \mathrm{mod}_T$. *Then* $\mathrm{End}_T(V) \cong \mathrm{End}_T(V^{\#})$ *canonically.*

We need another auxiliary result:

Lemma 3.7 *Let* $V \in \mathrm{mod}_T$, *and let* f *be an idempotent element of* T. *Then*

$$\mathrm{ann}_{fTf}(Vf) = f(\mathrm{ann}_T(V))f.$$

Proof Obviously $f(\mathrm{ann}_T(V))f \subseteq \mathrm{ann}_{fTf}(Vf)$. Let $x \in \mathrm{ann}_{fTf}(Vf)$, and let $v \in V$. Then $x = fxf$ and $vx = (vf + v(1-f))fxf = 0$. The result follows. □

Theorem 3.8 *Keep the notation introduced in* 3.4. *Then the following holds for every choice of* $R \in \{K, \mathcal{O}, F\}$*:*

(i) *The left* $\mathcal{H}_R$*-module* $M_R f$ *is isomorphic to* $X_R^{\#}$*, and* $\mathrm{End}_{\mathcal{H}_R}(M_R f) \cong \mathcal{S}_R$ *canonically.*
(ii) *The ring* fTf *acts on* $M_R f$ *by right multiplication as* $\mathcal{H}_R$*-endomorphisms such that the canonical map* $fT_R f \to \mathrm{End}_{\mathcal{H}_R}(M_R f)$ *is surjective.*
(iii) *The endomorphism ring of the* $fT_R f$*-module* $M_R f$ *is* $\mathrm{End}_{fT_R f}(M_R f) = \mathcal{H}_R$ *by restriction.*
(iv) *The* $fT_R f$*-module* $M_R f$ *is balanced.*

Proof We may omit indices R for the moment. Obviously Mf is a submodule of the left $\mathcal{H}$-module M. Since the $e_i, i = 1, \ldots, k$ are pairwise orthogonal idempotents,

$$Mf = \bigoplus_{i=1}^{k} Me_i,$$

as $\mathcal{H}$-module. Thus to show i) it suffices to show that $Me_i \cong X_i^{\#}$ for all $i \in \{1, \ldots, k\}$. It is well known that for an idempotent e of the ring T and $V \in \mathrm{mod}_T$ the abelian group Ve is canonically isomorphic to $\mathrm{Hom}_T(eT, V)$. This isomorphism is natural in V, hence it is an isomorphism of left S-modules if V is an S-T-bimodule for some ring S. Thus for $V = M$ we have an isomorphism of left $\mathcal{H}$-modules

$$Me_i \cong \mathrm{Hom}_T(e_i T, M). \tag{3.9}$$

Let $\phi : e_i T \to M$ be T-linear. Now 3.4 states that $\mathrm{im}\,\phi \subseteq N_i \subseteq M$, hence we may view ϕ as element of $\mathrm{Hom}_T(e_i T, N_i)$, and that $\beta_i : e_i T \to N_i$ satisfies 1.1. Thus by [9, 2.8] $\ker \beta_i \leq \ker \phi$, hence ϕ factorizes through N_i, and we have:

$$\mathrm{Hom}_T(e_i T, M) \cong \mathrm{Hom}_T(N_i, M). \tag{3.10}$$

and this isomorphism respects the left $\mathcal{H}$-action on these Hom-sets coming from the left module structure of M too. We apply the functor H and get from 3.4 and 1.37 an isomorphism of left $\mathcal{H}$-modules:

$$\mathrm{Hom}_T(N_i, M) \cong \mathrm{Hom}_{\mathcal{H}}(X_i, \mathcal{H}), \tag{3.11}$$

Equations 3.9, 3.10, 3.11 and 3.5 together with Lemma 3.6 give part i).

Next we prove ii): Note that $N_1 = M$ by assumption 3.4. This shows that

$$\mathrm{ann}_T(N) \subseteq \mathrm{ann}_T(M). \tag{3.12}$$

But all the direct summands N_i of N are contained in M, hence every element of T which annihilates M, annihilates N_i for all $i \in \{1, \ldots, k\}$, and therefore N, and we have equality in Equation 3.12.

The image of fTf in $\mathrm{End}_{\mathcal{H}}(Mf)$ is $fTf/\mathrm{ann}_{fTf}(Nf)$ and by Lemma 3.7 we have

$$\mathrm{ann}_{fTf}(Mf) = f(\mathrm{ann}_T(M))f = f(\mathrm{ann}_T(N))f = \mathrm{ann}_{fTf}(Nf). \tag{3.13}$$

We observe that Lemma 3.3 applies to the T-module N. Thus taking $\mathcal{E} = fTf = \mathrm{End}(fT)$ we see that N is a balanced $\mathcal{E}$-module, and is isomorphic to $\mathcal{S}$ as $\mathcal{S}$-$\mathcal{E}$-bimodule. We conclude that

$$fT_Rf/\mathrm{ann}_{fT_Rf}(M_Rf) = fT_Rf/\mathrm{ann}_{fT_Rf}(N_Rf) \cong \mathcal{S}_R \tag{3.14}$$

for every choice of R. Now ii) follows from i).

To prove iii) for $R = K$ we may use 3.1, observing that an irreducible constituent of M_K cannot be annihilated by f, since we have an epimorphism from fT_K onto $M_K = N_{1,K}$. An alternate proof which works for all choices of R goes as follows: First as left $\mathcal{H}$-module Mf decomposes into the direct sum

$$Mf = \bigoplus_{i=1}^{k} Me_i.$$

But $Me_1 = Me \cong \mathcal{H}$ as left $\mathcal{H}$-modules, hence Mf contains the left regular $\mathcal{H}$-module as direct summand. By [5, 59.4] the left $\mathcal{H}$-module Mf is balanced. By ii) and i) its endomorphism ring is

$$\mathrm{End}_{\mathcal{H}}(Mf) \cong \mathcal{S} \cong fTf/\mathrm{ann}_{fTf}(Mf), \tag{3.15}$$

and therefore

$$\mathrm{End}_{fTf}(Mf) = \mathrm{End}_{fTf/\mathrm{ann}_{fTf}(Mf)}(Mf) = \mathcal{H} \tag{3.16}$$

by the double centralizer property. We check now that all hypotheses of Lemma 3.2 are satisfied for $M_{\mathcal{O}}f$ and part iv) follows. □

We note that part ii) of Theorem 3.8 says in particular, that the endomorphism ring $\mathcal{S}_R$ of X_R (and of N_R) is a subfactor of T_R.

In the applications which we have in mind, the algebra $\mathcal{H}_F$ is often a Frobenius algebra. Note that $\mathcal{H}_K$ is always Frobenius, since it is semisimple, and note too that a Frobenius algebra is in particular selfinjective. So assume that $\mathcal{H}_F$ is selfinjective, and let $\mathcal{H} = \mathcal{H}_R$ for $R = F$ or K. Let I be a right ideal of $\mathcal{H}$. Then, for $h \in \mathcal{H}$, left multiplication by h defines an $\mathcal{H}$-linear $\lambda_h : I \to \mathcal{H}$. Since $\mathcal{H}$ is selfinjective, every homomorphism from I to $\mathcal{H}_{\mathcal{H}}$ lifts to an endomorphism of the right regular module $\mathcal{H}_{\mathcal{H}}$ and hence is left multiplication by an element of $\mathcal{H}$. Thus

$$I^{\#} = \mathrm{Hom}_{\mathcal{H}}(I, \mathcal{H}) \cong \mathcal{H}/l(I), \tag{3.17}$$

where $l(I)$ denotes the left annihilator of I in $\mathcal{H}$.

We apply this to the case, where $I = x\mathcal{H}$ is a principal right ideal and conclude that

$$\mathcal{H}x \to \mathrm{Hom}_{\mathcal{H}}(x\mathcal{H}, \mathcal{H}) : hx \mapsto \lambda_h \tag{3.18}$$

is a well defined isomorphism.

Lemma 3.19 *Let $\mathcal{H}_F$ be self injective, and let $x \in \mathcal{H}_\mathcal{O}$ be such that $\mathcal{H}_\mathcal{O}x$ is pure in the left regular $\mathcal{H}_\mathcal{O}$-module ${}_{\mathcal{H}_\mathcal{O}}\mathcal{H}_\mathcal{O}$. Then the map*

$$\mathcal{H}_R x \to (x\mathcal{H}_R)^{\#} \cong \mathrm{Hom}_{\mathcal{H}_R}(x\mathcal{H}_R, \mathcal{H}_R) : hx \mapsto \lambda_h$$

is a well defined isomorphism for all choices of $R \in \{K, \mathcal{O}, F\}$. So in particular, if in addition $x\mathcal{H}_\mathcal{O}$ is pure in the right regular module $\mathcal{H}_{\mathcal{O}\,\mathcal{H}_\mathcal{O}}$, then $\mathrm{Hom}_{\mathcal{H}_F}(x\mathcal{H}_F, \mathcal{H}_F)$ is liftable.

Proof The only case not yet shown is $R = \mathcal{O}$. Let $\phi : x\mathcal{H}_\mathcal{O} \to \mathcal{H}_\mathcal{O}$ be an $\mathcal{H}_\mathcal{O}$-linear map. Formula 3.18 applied to the case $R = K$ implies that $\phi(x) \in \mathcal{H}_K x \cap \mathcal{H}_\mathcal{O}$ which in turn equals $\mathcal{H}_\mathcal{O}x$, since the latter is pure by assumption. Thus $\phi(x) = hx$ for some $h \in \mathcal{H}_\mathcal{O}$. Clearly now $\phi = \lambda_h$.

$x\mathcal{H}_\mathcal{O}$ being pure in $\mathcal{H}_\mathcal{O}$ means that $x\mathcal{H}_F = F \otimes_\mathcal{O} x\mathcal{H}_\mathcal{O}$ (writing again x for $1 \otimes x \in F \otimes_\mathcal{O} \mathcal{H}_\mathcal{O} \cong \mathcal{H}_F$). The second assertion follows. □

We observe that taking # commutes with taking direct sums. Thus Lemma 3.19 applied to the Situation 3.4 gives immediately the following reformulation of the part i) of Theorem 3.8:

Corollary 3.20 *Keep the notation of* 3.4 *and assume in addition that $\mathcal{H}_F$ is selfinjective. Suppose that X_i is a cyclic module $X_i = x_i\mathcal{H}_\mathcal{O}$ for some $x_i \in \mathcal{H}_\mathcal{O}$ such that both, $\mathcal{H}_\mathcal{O}x_i$ and $x_i\mathcal{H}_\mathcal{O}$ are pure in the left respectively right regular $\mathcal{H}_\mathcal{O}$-module for $1 \le i \le k$. Then*

$$M_R f \cong X_R^{\#} \cong \bigoplus_{i=1}^{k} \mathcal{H}_R x_i$$

for all choices of $R \in \{K, \mathcal{O}, F\}$.

4 Hecke functors for general linear groups

For the moment let G be an arbitrary finite reductive group. Suppose the field of definition of G is the finite field with q elements for some power q of some prime p. We denote the Weyl group of G by W. The system $\mathcal{M}$ of Levi subgroups of G considered as subfactors (P, U) by choosing a parabolic subgroup P of G with unipotent radical U such that $L = P/U$ satisfies the requirements of a Mackey system as defined in the second section. Henceforth the terms as HC-induction, restriction, cuspidal, vertices are always meant with respect to this special Mackey system. Note that for a Levi subgroup L of G the Mackey system $\mathcal{M}_L$ is the Mackey system of Levi subgroups of L. For details we refer to [4]. Although HC-induction and restriction involve the parabolic subgroup P which the Levi subgroup L is a Levi complement of. It is a classical result over fields of characteristic zero (see e.g. [4]), that HC-induction and restriction are independent of the choice of P and the same was shown by Howlett and Lehrer in [37] and independently by Du and the author in [13, 5.2] for arbitrary commutative rings R, in which $q1_R$ is invertible. We take ℓ to be a prime number different from p and a split ℓ-modular system $(K, \mathcal{O}, F)$ for G. Note that q is invertible in any ring $R \in \{K, \mathcal{O}, F\}$ and may be identified with the invertible element

$q1_R$ of R. As a consequence the functors corresponding to conjugate Levi subgroups are naturally equivalent. Thus the irreducible RG-modules ($R = K$ or F) are partitioned according to their semisimple vertices and sources into *Harish Chandra series*, (HC-series for short), (compare [4] for the classical and [34] for the ℓ-modular case).

By [28, 2.4] (compare 2.5) the task to find the irreducible RG-modules $R = F, K$ subdivides into the following two parts:

Procedure 4.1 *To determine the irreducible RG-modules we have to proceed as follows:*

(i) *For any Levi subgroup L find the irreducible cuspidal RL-modules.*
(ii) *For a Levi subgroup L and a cuspidal irreducible RL-module C find the irreducible $\mathcal{H}$-modules, where $\mathcal{H} = \mathrm{End}_{RG}(\mathrm{R}_L^G(C))$.*

For $R = K$ the structure of the endomorphism ring $\mathcal{H}$ was determined by Howlett and Lehrer in [36]. In the special case of general linear groups their result extended (in view of [21, 3.3]) to the case $R = F$ in [7, 3.6] (details will be provided later in this section in Theorem 4.32). The general case was done in [28].

We turn now to finite general linear groups. Thus let $G = GL_n(q)$, where q is a power of some prime p. Let $g \in G$. In [21] the conjugacy class g^G of g was labelled by an (n, ∞)-index:

$$g^G \leftrightarrow I(g) = \left(\begin{array}{ccc|ccc} d_1 & d_2 & \dots & n_1 & n_2 & \dots \\ s_1 & s_2 & \dots & \lambda^{(1)} & \lambda^{(2)} & \dots \end{array} \right), \tag{4.2}$$

where the meaning of the parts of this index is explained below.

Let $g = su$ be the Jordan decomposition of g, that is the decomposition of g into its semisimple part s and the unipotent part $u \in C_G(s)$. Note that this is precisely the decomposition of g into its p-regular part s and p-part u. We may assume that s is given in its canonical rational form. Thus

$$s = (s_1)^{\times n_1} \times (s_2)^{\times n_2} \times \cdots\cdots, \tag{4.3}$$

where $s_1, s_2, \dots$ are pairwise different elements of some large extension field of $GF(q)$ and (s_i) is the companion matrix of the minimum polynomial of s_i over $GF(q)$. In order to make things uniquely defined we shall usually order the s_i according to their degrees d_i downward and within a fixed degree d according to an arbitrary but fixed ordering of the corresponding elements of $GF(q^d)$ on the orbits under the Galois group. For a given i the label d_i in 4.2 is now the degree of the minimum polynomial of s_i over $GF(q)$, that is the rank of the matrix (s_i). Note that $n = n_1d_1 + n_2d_2 + \dots\dots$ and

$$C_G(s) \cong GL_{n_1}(q^{d_1}) \times GL_{n_2}(q^{d_2}) \times \cdots\cdots, \tag{4.4}$$

where $\times$ means again matrix direct sum (which is isomorphic to the direct product) and the right hand side of 4.4 is suitably imbedded into G. We set

$$\boldsymbol{n}_s = \boldsymbol{n} = (n_1, n_2, \dots\dots). \tag{4.5}$$

The G-conjugacy classes of the unipotent part u in $C_G(s)$ is given by the canonical rational form, which is of course the same as the Jordan form. Hence it is described by the distribution of ones above the diagonal, that is a multipartition

$$\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots\dots), \tag{4.6}$$

where $\lambda^{(i)} = (\lambda_1^{(i)}, \dots)$ is a partition of n_i. We abbreviate this by $\boldsymbol{\lambda} \vdash \boldsymbol{n}$ and and we use for multicompositions $\boldsymbol{\mu}$ of n similarly the notation $\boldsymbol{\mu} \models \boldsymbol{n}$. It is obvious that the left half of the index in 4.2 describes the semisimple and the right half the unipotent part of g, and hence those parts of the index are called the semisimple and unipotent part of $I(g)$ respectively.

For short we may label the conjugacy classes of G by the pair $(s^G, \boldsymbol{\lambda})$.

The ℓ-regular classes of G are now precisely given by pairs $(s^G, \boldsymbol{\lambda})$ where $s \in G$ is an ℓ-regular semisimple element, since unipotent elements are p-elements, and therefore especially ℓ-regular. The corresponding index in [21] looks precisely as in 4.2, but is called an (n, ℓ)-*head* index to indicate that its semisimple part is ℓ-regular.

We want to describe the irreducible KG-modules. Their classification was done (in terms of character theory) by J. Green in 1955, [29].

First we use Deligne-Lusztig theory to determine the irreducible cuspidal KG-modules (for details see e.g. [43]). Take a semisimple element $s \in G$ whose minimum polynomial has degree n. It generates together with the center of G a maximal (non split) torus T which is *minisotropic*, that is it is not contained in any proper Levi subgroup of G. The Deligne-Lusztig operator taking characters of T to generalized characters of G is denoted by R_T^G. Now T being abelian is isomorphic to its character group, and we may choose such an isomorphism; (we have to do this in a consistent manner, that means at the same time for G and all its standard Levi subgroups). Denote the image of s by $\hat{s}$. Then $\mathrm{R}_T^G(\hat{s})$ is up to sign the character of an irreducible cuspidal KG-module which we shall denote by C_s. We may label these module by the pair $(s^G, (1))$, where (1) is the only partition of the number 1 or following [21] the head (n, ∞) index

$$C_s \quad \leftrightarrow I(C_s) = \left(\begin{array}{c|c} n & 1 \\ s & (1) \end{array} \right). \tag{4.7}$$

We remark that the G-conjugacy classes of parabolic subgroups are parametrized by compositions of n. A set of representatives of these consists of the set of all parabolic subgroups containing one fixed Borel subgroup B of G. We choose B to be the set of all upper triangular matrices in G. Then the *standard parabolic subgroup* corresponding to the composition μ of n (written as $\mu \models n$) is denoted by P_μ and its Levi subgroup by L_μ. Moreover, if $\mu = (\mu_1, \mu_2, \dots)$

$$L_\mu = GL_{\mu_1}(q) \times GL_{\mu_2} \times \cdots,$$

diagonally embedded into G.

Now let $s \in G$ be an arbitrary semisimple element. We assume it is given in its canonical rational form as in Formula 4.3. Let the composition $\mu = \mu_s$ of n be given by

$$\mu_s = (\underbrace{d_1, d_1, \dots, d_1}_{n_1\text{-times}}, \underbrace{d_2, \dots, d_2}_{n_2\text{-times}}, \dots\dots), \tag{4.8}$$

where the n_i and the d_i are defined in 4.4. Together with the center of G the element s generates a minisotropic maximal torus T_s of $L_s = L_\mu$. Consequently, since L_s is a direct product of general linear groups, the generalized character $\mathrm{R}_{T_s}^{L_S}(\hat{s})$ is up to sign the character afforded by an irreducible cuspidal KL_s-module C_s. Indeed we can decompose C_s into an outer tensor product:

$$C_s \cong C_{s_1}^{\otimes n_1} \otimes C_{s_2}^{\otimes n_2} \otimes \cdots\cdots \tag{4.9}$$

All modules considered are semisimple, hence in order to find the irreducible components of $M_s = \mathrm{R}_{L_s}^G(C_s)$, we have to inspect the endomorphism ring $\mathcal{H}_s = \mathrm{End}_{KG}(M_s)$ and apply Fitting's Lemma or equivalently 2.5 (considering the projective presentation $\beta = id : M_s \to M_s$ of M_s).

Let $\mathfrak{C}_{ss}$ be a set of representatives of conjugate semisimple elements of G e.g. given in the form 4.3. Then the set of irreducible components of M_s is precisely the HC-series with semisimple vertex $L_s = L_\mu$ and semisimple source C_s (and coincides with the Deligne-Lusztig series associated with $(T_s, \hat{s})$ which is defined to be the set of irreducible constituents of $\mathrm{R}_{T_s}^G(\hat{s})$). For general linear groups this series is as well a *geometric conjugacy classes* of G and is denoted by $\mathcal{E}(s^G)$ or simply by $\mathcal{E}(s)$. (This is special for general linear groups. For arbitrary finite groups of Lie type geometric conjugacy classes in general are unions of Deligne-Lusztig series).

The structure of $\mathcal{H}_s$ has been determined by Howlett and Lehrer in [36] to be the Hecke algebra associated with the *ramification group* $W_s = W(L_s, C_s)$, that is the stabilizer of C_s in W factored by the Weyl group of L_s. In the case of general linear groups it is again a Weyl group of type A (comp. [7]), indeed W_s is isomorphic to the Weyl group of $C_G(s)$ (see 4.4). As subgroup of W it is it is given as follows:

$$W_s = \mathfrak{S}_{\{\bar{1},\dots,\overline{n_1}\}} \times \mathfrak{S}_{\{\overline{n_1+1},\dots,\overline{n_1+n_2}\}} \times \cdots\cdots, \tag{4.10}$$

where $\bar{i}$ denotes the i-th subset in the set partition of $\{1,\dots,n\}$ given by $\mu \models n$. Thus, if $\bar{i} = \{b+1,\dots,b+j\}$ and $\overline{i+1} = \{b+j+1,\dots,b+2j\}$ belong to the same subset above, (that is both, i and $i+1$ are between $n_1 + n_2 + \cdots n_a$ and $n_1 + \cdots n_a + n_{a+1}$ for some a), then the corresponding generating transposition of W_s is the permutation in $W = \mathfrak{S}_n$ which swaps $b+x$ and $b+x+j$ for $1 \le x \le j$ and keeps all other numbers fixed.

For a Coxeter group W generated by basic reflections $v_1, \dots, v_r$, a commutative ring R and a unit $q \in R$ the *Hecke algebra* $\mathcal{H}_{R,q}(W)$ over R with respect to q associated with W has R-basis $\{T_w \mid w \in W\}$ and multiplication is defined by the following rule:

$$T_v T_w = \begin{cases} T_{vw} & \text{if } l(vw) = l(w)+1 \\ qT_{vw} + (q-1)T_w & \text{otherwise} \end{cases}, \tag{4.11}$$

for $w \in W$ and $v \in \{v_1, \dots, v_r\}$. Here $l : W \to \mathbb{N}$ denotes the length function which is defined to be the least number k for $w \in W$ such that one can w write as word with k factors in the generators. It can be shown that these relations indeed determine an associative R-algebra.

Theorem 4.12 (R.Howlett, G.Lehrer) *Let $s \in G$ be semisimple. Then*

$$\mathcal{H}_s = \mathrm{End}_{KG}(M_s) \cong \mathcal{H}_{K,q^{d_1}}(\mathfrak{S}_{n_1}) \otimes \mathcal{H}_{K,q^{d_2}}(\mathfrak{S}_{n_2}) \otimes \cdots\cdots,$$

Explicit representations for $\mathcal{H}_{K,q}(W)$ were constructed by P.N. Hoefsmit in [35] for Weyl groups of classical type. For Weyl groups of type A, that is symmetric groups, the representations of the associated Hecke algebras $\mathcal{H}_{R,q}(\mathfrak{S}_r)$ were investigated by G. James and the author in [19] for arbitrary R and q and we shall use the notation developed there. Thus the irreducible $\mathcal{H}_s$-modules are labelled by multipartitions $\boldsymbol{\lambda} \vdash \boldsymbol{n}$, where $\boldsymbol{n} = \boldsymbol{n}_s = (n_1, n_2, \dots)$ and $\boldsymbol{\lambda}$ are given in 4.5 and 4.6. The $\mathcal{H}_s$-module corresponding to $\boldsymbol{\lambda} \vdash \boldsymbol{n}$ is the *Specht module* $S^{\boldsymbol{\lambda}}$ and may be realized as right ideal in $\mathcal{H}_s$. Thus

$$S(s, \boldsymbol{\lambda}) = S^{\boldsymbol{\lambda}} M_s$$

is an irreducible KG-submodule of M_s by Fitting's Lemma. We have:

Theorem 4.13 *Let $G = Gl_n(q)$ and K a splitting field for G of characteristic zero. Then*

$$\{S(s, \boldsymbol{\lambda}) \mid s \in \mathfrak{C}_{ss}, \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$$

is a complete set of nonisomorphic irreducible KG-modules. Here $\mathcal{C}_{ss}$ denotes a set of representatives of semisimple conjugacy classes in G.

By the discussion at the beginning of this section, we may parametrize the irreducible KG-modules as well by (n, ∞)-indices.

Remark 4.14 *In* 4.13 *we made implicitly some choices and it is a priori not clear that all of our constructions are independent of those. As an example we consider $G = GL_2(q)$ and $s = 1 \in G$. Thus the Levi subgroup L_s is the maximal split torus T of invertible diagonal matrices and we have two parabolic subgroups containing T as Levi subgroup, namely the groups B_1 of invertible upper and B_2 of lower triangular matrices. The corresponding Hecke algebra $\mathcal{H} = \mathrm{End}_{KG}(\mathrm{R}_T^G(K_T))$ is the Hecke algebra $\mathcal{H} = \mathcal{H}_{K,q}(\mathfrak{S}_2)$ (which in turn is isomorphic to the group algebra $K\mathfrak{S}_2$), where K_T denotes the trivial KT-module. Now $\mathcal{H}$ has two one dimensional representations, namely the trivial and the alternating one corresponding to the partitions $\lambda = (2)$ and $\mu = (1^2)$ respectively. By* [13, 37] *the HC-induction $R_{T \subset B_i}^G = \mathrm{Ind}_{B_i}^G \mathrm{Inf}_T^{B_i}$ for $i = 1, 2$ does not depend on the choice of the embedding T as Levi subgroup into a parabolic subgroup of G. In particular we have an isomorphism $\phi : \mathrm{R}_{T \subset B_1}^G(K_T) \to \mathrm{R}_{T \subset B_2}^G(K_T)$, hence their endomorphism rings are isomorphic too. Those, however, are given by the Howlett-Lehrer construction* [36]. *It is not immediate that both constructions give precisely the same answer, the trivial and the alternating representation of $\mathcal{H}$ could be interchanged by the isomorphism ϕ. In this special case this cannot happen, because the corresponding irreducible direct summands of $\mathrm{R}_{T \subset B_i}^G(K_T)$ are the trivial and the Steinberg module, which are distinguishable by their degrees. In general one checks easily that there is only one critical choice involved, namely the order of the factors in the representation* 4.3 *of the canonical rational form of s or equivalently the precise choice of the canonical rational form of s and hence the Levi subgroup L_s among the conjugate standard Levi subgroups. Thus one has to show the following: Given standard Levi subgroups L_i of G with $i = 1, 2$ which are conjugate in G (and hence by an element d of W) then the natural transformation B_d from $\mathrm{R}_{L_1}^G$ to $\mathrm{R}_{L_2}^G$ induces an $\mathcal{H}_s$-KG-bimodule isomorphism from $\mathrm{R}_{L_1}^G(C_s)$ onto $\mathrm{R}_{L_2}^G(C_s^d)$, where C_s is defined in* 4.9 *and C_s^d denotes the conjugate module. But this can be checked directly from the construction of the natural transformation B_d in*

[13, 5.8]. *Note that this fact then justifies our convention in* 4.2 *fixing for the corresponding* (n,∞)*-indices the order of the terms and the order of the parts of* s.

For later use we remark that an analogous statement holds, if K *is replaced by* F *(or* $\mathcal{O}$*) and* (n,∞)*-indices by* (n,ℓ)*-indices (compare* [21, 2.5]*).*

It should be pointed out here that we know a lot more about the irreducible KG-modules $S(s,\boldsymbol{\lambda})$, for example their dimensions. G. James even constructed a "standard basis" for them. For details we refer to his paper [40], and to [38, 39].

We now turn to the ℓ-modular representation theory of G. For this we have to consider suitable reductions modulo ℓ of the irreducible KG-modules constructed in Theorem 4.13. We adopt the convention to drop sub- and superscripts whenever there is no danger of ambiguities. Moreover, the ring R of scalars we are working with is often indicated by a subscript R. More specifically, whenever an object is defined over the order $\mathcal{O}$, then an index R means that we tensor the object over $\mathcal{O}$ with R. Usually R is $F, \mathcal{O}$, or K. Finally the symbol "$=_G$" means "equal up to conjugation in G", and we adopt a similar notation for further symbols, as "$\subseteq_G$" etc.

Here are some results on cuspidal representations:

4.15 *Let* C_K *be a cuspidal irreducible* KG*-module. Then the following holds:*

(i) *Any reduction modulo* ℓ *of* C_K *is cuspidal. The converse holds too: The lift to* KG *of a liftable cuspidal* FG*-module is cuspidal.*
(ii) *Let* C_F *be a reduction modulo* ℓ *of* C_K. *Then* C_F *is irreducible. Thus* C_K *has an up to isomorphism a uniquely determined* $\mathcal{O}G$*-lattice* $C_{\mathcal{O}}$.

The first part follows immediately from the definition of cuspidal modules. Part 2 was first proved in [7, 8]. However the proof there requires the full classification of the irreducible FG-modules and is therefore not useful for our treatment here. There is a direct proof in [40] involving the Gelfand-Graev representation of G and none of our forthcoming results here. The reader should therefore consult [40] for details. It should be pointed out that this part does not hold in general for finite groups of Lie type. At the end of section five in 5.23 we shall prove that the converse of part 2 above holds for $G = GL_n(q)$ as well: Every cuspidal irreducible FG-module is liftable and hence lifts to an irreducible cuspidal KG-module. This as well does not hold for other types of finite reductive groups. For example, if G is a finite reductive group with Dynkin diagram G_2 defined over $GF(q)$ such that 5 divides $q+1$ but 25 does not, then G has a 5-modular irreducible cuspidal representation which does not occur in any reduction modulo 5 of any complex irreducible cuspidal representation of G, and hence is in particular not liftable. Moreover G has plenty complex cuspidal irreducible representations which do not remain irreducible upon reduction modulo 5. For details see Appendix C of [32].

For $g \in G$ we denote the ℓ-regular part of g by g'. Thus $g = g'v = vg'$ for some ℓ-element v in $C_G(g)$. The set of ℓ-elements of a subset U of G is denoted by U_ℓ.

The following theorem can be derived from [25] respectively from [21]. Its generalization to arbitrary finite groups of Lie type is due to M. Broué an J. Michel [2]:

Theorem 4.16 *Let* s *be a semisimple* ℓ*-regular element of* G. *Then*

$$\mathcal{E}_\ell(s) = \bigcup_{y \in C_G(s)_\ell} \mathcal{E}(sy),$$

is the set of irreducible KG-modules in a union of ℓ-blocks of G.

The following results concern induced cuspidal representations. Note that HC-induction commutes with base change, that is for $s \in G$ semisimple

$$M_{F,s} = F \otimes_{\mathcal{O}} M_{\mathcal{O},s} \cong \mathrm{R}_{L_s}^G(C_{F,s})$$

canonically, justifying our notation.

Lemma 4.17 *Let $s, t \in G$ be semisimple.*

(i) *Assume that $L_s =_G L_t$. Suppose $s' =_G t'$. Then $C_{F,s} \cong C_{F,t}$, and hence $M_{F,s} \cong M_{F,t}$.*
(ii) *Let s, t be semisimple elements of G such that $s' \neq_G t'$. Then the geometric conjugacy classes $\mathcal{E}_s$ and $\mathcal{E}_t$ belong to disjoint ℓ-blocks of G.*

Proof To see the first part one checks that the characters of $C_{K,s}$ and $C_{K,t}$ coincide on ℓ-regular conjugacy classes of G if and only if s and t have the same ℓ-regular parts, and the claim follows. For details see [21, 3.3] and for a generalization to finite groups of Lie type [33]. The second part of the lemma follows immediately from 4.16, but can also seen by inspecting the linear characters of the center of KG, (comp. [21]). □

Next we need the Gelfand-Graev representation of G and of Levi subgroups. Thus, for $\mu \models n$ let U_μ the unipotent radical of the standard Borel subgroup of L_μ. Thus U_μ consists of all unipotent upper triangular matrices in L_μ. We take a linear character χ of U_μ whose restriction to any root subgroup X_i is nontrivial for any simple root α_i. More precisely, X_i consists of those matrices which have ones on the diagonal and the only nonzero entry occurs in row i and column $i+1$, where $1 \leq i \leq n-1$ is chosen such that X_i is contained in L_μ. If $L = L_\mu$ we have an idempotent

$$e_L = e_\mu = \frac{1}{|U_\mu|} \sum_{u \in U_\mu} \chi(u) u, \tag{4.18}$$

which is contained in $\mathcal{O}L$, since the order of U_μ is invertible in $\mathcal{O}$. By general theory the $\mathcal{O}L$-lattice $\Gamma_{L,\mathcal{O}} = e_L \mathcal{O}L$ is projective and hence up to isomorphisms the only $\mathcal{O}L$-lattice in the KL-module $e_\mu KL = \Gamma_{L,K}$. For R as usual the RL-lattice $\Gamma_{L,R}$ is the *Gelfand-Graev* module of RL. If $L = L_\mu$ we also write $\Gamma_{L,R} = \Gamma_{\mu,R}$.

Here are some properties of the Gelfand-Graev representation $\Gamma_R = \Gamma_{G,R}$:

4.19 *Let s be a semisimple element of G, and let $\boldsymbol{n}_s$ be defined as in* 4.5. *Then the following holds:*

(i) *The geometric conjugacy class $\mathcal{E}(s)$ of G has precisely one irreducible KG-module with Γ_K in common, namely the KG-module $S_K(s, \boldsymbol{\lambda})$ of Theorem* 4.13, *where $\boldsymbol{\lambda} \vdash \boldsymbol{n}$ is given as $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots) = ((1^{n_1}), (1^{n_2}), \dots)$.*
(ii) *The Gelfand-Graev representation Γ_K is multiplicity free.*
(iii) *Every cuspidal irreducible KG-module occurs in Γ_K precisely once as composition factor.*

(iv) *The Hom-set* $\mathrm{Hom}_{RG}(\Gamma_R, M_{R,s})$ *is free of rank one as* R*-module for all choices of* $R \in \{K, \mathcal{O}, F\}$.

Proof The first two parts are well known (see e.g. [4]), and the remaining parts are straightforward consequences hereof. □

Again parts three and four of 4.19 hold for general linear groups only. Part one is true for finite reductive groups with connected center, and part two in general. We define the multipartition $\boldsymbol{\omega}_s \vdash \boldsymbol{n}_s$ to be

$$\boldsymbol{\omega}_s = ((1^{n_1}), (1^{n_2}), \dots\dots). \tag{4.20}$$

We remark that everything what has been done so far for G applies as well to Levi subgroups L of G. We denote the corresponding objects for L by an additional subscript. So for a semisimple element s of G which is (up to conjugation in G) contained in L we have an associated geometric conjugacy class $\mathcal{E}_L(s)$ (taking s to be an element of L). Immediately from the definition of geometric conjugacy classes and transitivity of HC-induction follows that the modules in $\mathcal{E}_G(s)$ are the irreducible constituents of the modules $\mathrm{R}_L^G(N)$ for $N \in \mathcal{E}_L(s)$. Thus

$$\mathrm{R}_L^G(\mathcal{E}_L(s)) = \mathcal{E}_G(s). \tag{4.21}$$

But note that the semisimple element t in L may be not conjugate in L to our given s but in G, and hence $\mathcal{E}_L(s) \cap \mathcal{E}_L(t) = \emptyset$ but $\mathcal{E}_G(s) = R_L^G(\mathcal{E}_L(s)) = R_L^G(\mathcal{E}_L(t)) = \mathcal{E}_G(t)$. We can however describe precisely when this happens: Both s and t are given in its canonical rational form, but t is derived from s by permuting and mixing parts of s in equation (4.1) of equal degree. Here is an example:

$$\begin{array}{rlccccc} s & = \cdots \times (s_i) \times \cdots \times (s_i) \times & (s_i) & \times & (s_{i+1}) & \times (s_{i+1}) \times \cdots \\ t & = \cdots \times (s_i) \times \cdots \times (s_i) \times & (s_{i+1}) & \times & (s_i) & \times (s_{i+1}) \times \cdots, \end{array} \tag{4.22}$$

where in the notation of Equation 4.3 $d_i = d_{i+1}$.

Fix s to be a semisimple element of G given as in 4.3. Set again $M_{R,s} = R \otimes_{\mathcal{O}} M_{\mathcal{O},s}$ with $M_{\mathcal{O},s} = \mathrm{R}_{L_s}^G(C_{\mathcal{O},s})$ and let $L = L_s$.

Lemma 4.23 *There exist a projective presentation*

$$\beta_{\mathcal{O},s} = \beta_{\mathcal{O}} : \mathcal{P}_{\mathcal{O},s} \to M_{\mathcal{O},s}$$

with the following property: Every composition factor of the kernel $\ker \beta_K$ *of* $\beta_K = \beta_{K,s}$ *has the form* $S(s'y, \boldsymbol{\lambda})$ *for some* $\boldsymbol{\lambda} \vdash \boldsymbol{n}_{s'y}$ *with* $1 \neq y \in C_L(s')_\ell$ *such that* $s \neq_L s'y$.

Proof Let $L = L_s$, and denote the Gelfand-Graev representation of $\mathcal{O}L$ by $\Gamma_{\mathcal{O},s}$. We apply Result 4.19: There exists a nonzero homomorphism $\psi_{\mathcal{O}}$ from an indecomposable direct summand $\mathcal{P}_1$ onto $C_{\mathcal{O},s}$ and the kernel of ψ_K contains only summands of the form $S_L(t, \boldsymbol{\omega}_t)$. Being composition factor of the same indecomposable projective $\mathcal{O}G$-module tensored by K as $S_L(s,(1)) = C_{K,s}$ both modules must be in the same ℓ-block of L. Result

4.17 implies that $s' =_L t'$. We apply 4.19 once more to conclude that s and t cannot be conjugate in L. Thus $t = s'y$ has the required form. Now we take $\mathcal{P}_{\mathcal{O},s} = \mathrm{R}_L^G(\mathcal{P}_1)$ and $\beta_{\mathcal{O}} = \mathrm{R}_L^G(\psi_{\mathcal{O}})$ then Equation 4.21 yields the desired result. □

Of course we would like to show that we are in situation [9, 4.7], by showing that $\ker \beta_K$ has no constituent in common with $M_{K,s}$. For this we need in Lemma 4.23 that $s \neq_G s'y$ holds. Unfortunately it can happen that these elements are conjugate in G as Equation 4.22 demonstrates.

Lemma 4.24 *Let $\beta_{\mathcal{O}}$ be defined as above. Let $S(s'y, \boldsymbol{\lambda})$ be an irreducible composition factor of $\ker \beta_K$, such that s and $s'y$ of L are conjugate in G. Then $L_s = L_{s'y}$ and $s = s'\tilde{y}$ for some $y =_G \tilde{y} \in C_L(s')_\ell$. Moreover if s' is given in its canonical rational form as in 4.3*

$$s' = (s'_1)^{\times n_1} \times \cdots \times (s'_k)^{\times n_k},$$

then there exists an index $1 \leq a \leq k$ and ℓ-elements $\{y_1, y_2, \dots\}$ in some extension field $GF(q^{d_a m})$ of $GF(q^{d_a})$ with the following properties, (where d_a is the degree of the minimum polynomial s_a):

(i) *The minimum polynomial of y_i over $GF(q^{d_a})$ has degree m for all i.*
(ii) *The elements of $\{y_1, y_2 \dots\}$ have pairwise different minimum polynomials.*
(iii) *In $C_G(s') = \cdots \times GL_{n_a}(q^{d_a}) \times \cdots$ the companion matrix (y_i) of the minimum polynomials of the y_i occurs with the same positive multiplicity in y and $\tilde{y}$, for $i = 1, 2, \dots$, but in different order.*

Conversely if $s = s\tilde{y}$ and $t = s'y$ are given as above, then $s =_G t$ and $M_{K,s} \cong M_{K,t}$. In this case every irreducible composition factor of the image $M_{K,s}$ under $\beta_{K,s}$ is composition factor of $\ker \beta_{K,s}$ as well.

Proof Immediately from Equation 4.22 and Lemma 4.23 follows that y satisfies the (i), (ii) and (iii). To see the converse we observe that it is enough to show that the KL_s-module $C_{K,t}$ is contained in the kernel of $\psi_{K,s}$ which maps $\mathcal{P}_{1,K,s}$ onto $C_{K,s}$, (defined in the proof of Lemma 4.23 and now given appropriate indices). Note that $C_{K,s}$ and $C_{K,t}$ are not isomorphic, since s and t are not conjugate in $L_s = L_t$. On the other hand by part (i) of Lemma 4.17 we have $C_{F,s} \cong C_{F,t}$, since s and t have equal ℓ-regular parts. Consequently we have an epimorphism from $\mathcal{P}_{1,F,s}$ onto $C_{F,t}$ and by Brauers reciprocity law $C_{K,t}$ is composition factor of $\mathcal{P}_{1,K,s}$ as well. Being different from $C_{K,s}$ it has to be in the kernel of $\psi_{K,s}$. Thus the lemma holds. □

We may visualize the situation in the following example:

$$\begin{array}{rcccccc}
s' & = & \cdots\times & \underbrace{((s'_a)\times\cdots\times(s'_a))}_{m\text{-factors}} & \times\cdots\times & \underbrace{((s'_a)\times\cdots\times(s'_a))}_{m\text{-factors}} & \times\cdots \\
\tilde{y} & = & \cdots\times & (y_1) & \times\cdots\times & (y_2) & \times\cdots \\
y & = & \cdots\times & (y_2) & \times\cdots\times & (y_1) & \times\cdots \\
s'\tilde{y} & = & \cdots\times & (s'_a y_1) & \times\cdots\times & (s'_a y_2) & \times\cdots \\
s'y & = & \cdots\times & (s'_a y_2) & \times\cdots\times & (s'_a y_1) & \times\cdots
\end{array} \quad . \tag{4.25}$$

Note that indeed the minimum polynomial of $s'_a y_i,\ i=1,2$ over $GF(q)$ has degree md_a. The canonical rational form of $s', s'y, s'\tilde{y}$ in 4.25 has to be taken in G, that of $y, \tilde{y}$ in $C_L(s')$.

Definition 4.26 *Let $s \in G$ be semisimple:*

$$s = \cdots \times (s_a)^{n_a} \times \cdots \times (s_b)^{n_b} \times \cdots .$$

Let $s_i = s'_i y_i$ be the decomposition of s_i into its ℓ-regular and ℓ-singular parts and let d_i be the degree of the minimum polynomial of s_i. Then we say that s is ℓ-stable when it satisfies the following condition: Whenever $(s'_a) = (s'_b)$ and $d_a = d_b$ then y_a and y_b have the same minimum polynomial over $GF(q^{d'_a})$, where d'_a is the degree of the minimum polynomial of s'_a over $GF(q)$.

We remark that our definition of ℓ-stable here is precisely the hypothesis in [15, 4.6] for the special case of general linear groups (which are known to coincide with their dual groups). Lemmas 4.23 and 4.24 (in connection with [9, 4.7]) imply now immediately:

Theorem 4.27 *Let $s \in G$ be semisimple. Then the projective presentation*

$$\beta_{R,s} : \mathcal{P}_{R,s} \to M_{R,s}$$

satisfies Hypothesis 1.1 *for all choices $R \in \{K, \mathcal{O}, F\}$ if and only if s is ℓ-stable.*

Immediate from the definition we have:

Lemma 4.28 *Let $s \in G$ be semisimple and ℓ-regular. Then s is ℓ-stable.*

It is obvious how to turn an arbitrary semisimple element into a ℓ-stable one. If for example $s = s'\tilde{y}$ is given as in Lemma 4.24 we may change the direct factor $(s'_a y_2)$ into $(s'_a y_1)$ and do this for all $y_i \in \{y_2, \dots\}$ and for all indices a to get a semisimple element $\tilde{s}$ which is ℓ-stable and satisfies $s' = \tilde{s}'$. Part (i) of Result 4.17 says, that over F our constructions depend only on the ℓ-regular part s' of s and the "shape" of its ℓ-singular part y. We call two semisimple elements s and t of G *ℓ-equivalent*, and we write $s \sim_\ell t$, if $s' =_G t'$ and $L_s =_G L_t$. In this case

$$M_{F,s} \cong M_{F,\tilde{s}} \cong M_{F,\tilde{t}} \cong M_{F,t},$$

where the ℓ-stable elements $\tilde{s}$ and $\tilde{t}$ are constructed as above. Obviously $\sim_\ell$ is an equivalence relation, and each equivalence class contains ℓ-stable elements, since $s \sim_\ell \tilde{s}$.

Now Result 4.17 implies:

Lemma 4.29 *Let $s, t \in G$ be semisimple, and let $t \sim_\ell s$. Then*

$$\begin{array}{rcl} C_{F,s} & \cong & C_{F,t} \\ \mathcal{P}_{F,s} & \cong & P_{F,t} \\ M_{F,s} & \cong & M_{F,t} \end{array}$$

Thus replacing $\beta_{F,s} : P_{F,s} \to M_{F,s}$ by $\beta_{F,\tilde{s}}$ we obtain a projective presentation of $M_{F,s}$ which satisfies Hypothesis 1.1.

We apply [9] to attach to each geometric conjugacy class $\mathcal{E}(s)$ for which $s \in G$ is ℓ-stable, three pairs of functors:

$$\begin{array}{rcllcl} H_s & = & H_{R,s}: & \mathrm{mod}_{RG} & \to & \mathrm{mod}_{\mathcal{H}_{R,s}} \\ \hat{H}_s & = & \hat{H}_{R,s}: & \mathrm{mod}_{\mathcal{H}_{R,s}} & \to & \mathrm{mod}_{RG}, \end{array} \tag{4.30}$$

where $R \in \{K, \mathcal{O}, F\}$. If s is not ℓ-stable, we have to replace in case of $R = F$ the element s by $\tilde{s}$ and get the functors $H_{F,\tilde{s}}, \hat{H}_{F,\tilde{s}}$.

Recall that these functors are defined as follows:

$$\begin{array}{rcl} H_s & = & \mathrm{Hom}_{RG}(\mathcal{P}_s, -)/Hom_{RG}(\mathcal{P}_s, -)J_s \\ \hat{H}_s & = & A_{\mathcal{P}_s} \circ (- \otimes_{\mathcal{H}_s} M_{R,s}). \end{array} \tag{4.31}$$

Here $\mathcal{H}_s = \mathcal{H}_{R,s} = R \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O},s}$ is the endomorphism ring $\mathrm{End}_{RG} M_{R,s}$, if s is ℓ-stable or $R = \mathcal{O}$ or K. Otherwise we again replace s by $\tilde{s}$ and apply Lemma 4.29. Moreover $J_s = J_{R,s}$ is defined as in section 1.

The next result generalizes Theorem 4.12 and was shown in [7, 3.6]:

Theorem 4.32 *Let $s \in G$ be semisimple and ℓ-stable. Then*

$$\mathrm{End}_{RG}(M_{r,s}) = \mathcal{H}_{R,s} \cong \mathcal{H}_{R,q}(W_s) = \mathcal{H}_{R,q^{d_1}}(\mathfrak{S}_{n_1}) \otimes \mathcal{H}_{R,q^{d_2}}(\mathfrak{S}_{n_2}) \otimes \cdots,$$

for $R \in \{K, \mathcal{O}, F\}$, where the d_i are given as degrees of the irreducible factors of the minimum polynomial and the n_i as multiplicities of these factors in the characteristic polynomial of s.

Note that by construction

$$W(L_s, C_s) \leq W(L_{\tilde{s}}, C_{\tilde{s}}). \tag{4.33}$$

Indeed $W(L_s, C_s)$ is a standard Young subgroup of $W(L_{\tilde{s}}, C_{\tilde{s}})$. Thus

$$F \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}G}(M_{\mathcal{O},s}) \leq \mathrm{End}_{F,s}(M_{F,s}) = F \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}G}(M_{\mathcal{O},\tilde{s}}) \cong \mathcal{H}_{F,\tilde{s}},$$

and the inequality is proper, if s is not ℓ-stable.

We shall now construct "standard" ℓ-stable elements. For this we need to know how to find ℓ-elements in finite extensions of $GF(q^i)$. The following results follow easily from elementary facts in finite field theory.

For $0 \leq i \in \mathbb{Z}$ define $e(i)$ to be the smallest natural number k such that

$$1 + q^i + (q^i)^2 + \cdots + (q^i)^{k-1} \equiv 0 \bmod \ell. \tag{4.34}$$

Note that $e(i)$ is the multiplicative order of q^i in the finite field $GF(l)$, unless $q^i \equiv 1 \bmod \ell$, in which case $e(i) = \ell$. In any case $e(i) \leq \ell$. Let $e = e(1)$. If $q \equiv 1 \bmod \ell$, then $q^i \equiv 1 \bmod \ell$ for all $1 \leq i \in \mathbb{Z}$. Otherwise $q^i \equiv 1 \bmod \ell$ if and only if $i = el^m$ for some $0 \leq m \in \mathbb{Z}$. Replacing e by $e(i)$ we obtain

$$e(e(i)\ell^m) = l, \tag{4.35}$$

for all nonnegative integers m. Moreover $GF(q^{ij})$ contains an ℓ-element which is not contained in any proper subfield if and only if $j = e(i)\ell^m$ for some $0 \leq m \in \mathbb{Z}$.

Definition 4.36 *We choose now for each field $GF(q^i)$ and each number $e(i)\ell^m$ for every nonnegative integer m a fixed ℓ-element*

$$y_i(m) \in GF(q^{ie(i)l^m}),$$

which is not contained in any proper subfield. Thus its minimum polynomial over $GF(q^i)$ has degree $e(i)\ell^m$, and its companion matrix is denoted by $(y_i(m))$ (compare 4.3). If s_i is a ℓ-regular element of $GF(q^i)$ whose minimum polynomial over $GF(q)$ has degree i, then $s_i y_i(m)$ generates $GF(q^{ie(i)\ell^m})$ over $GF(q)$, hence its companion matrix $(s_i y_i(m))$ is an $ie(i)\ell^m \times ie(i)\ell^m$-matrix.

A composition μ of k is called $e(i)$-ℓ-adic, if it has the form:

$$\mu = (1, 1, \ldots, 1, e(i), \ldots, e(i), e(i)\ell, \ldots, e(i)\ell, e(i)\ell^2, \ldots).$$

Let s in G be a fixed semisimple ℓ-regular element, and let $\boldsymbol{n} = \boldsymbol{n}_s$ be defined as in Equation 4.5. A multi composition

$$\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(i)}, \ldots) \models \boldsymbol{n}$$

is called $\boldsymbol{e}$-ℓ-adic, if $\mu^{(i)} \models n_i$ is $e(d_i)$-ℓ-adic for all i, where

$$\boldsymbol{e} = \boldsymbol{e}_s = (e(d_1), e(d_2), \ldots).$$

The corresponding parabolic subgroup $W_{\boldsymbol{\mu}}$ of W_s is called $\boldsymbol{e}$-ℓ-parabolic as well as the corresponding parabolic subgroup $P_{\boldsymbol{\mu}}$, Levi subgroup $L_{\boldsymbol{\mu}}$ etc. of $C_G(s)$. For multipartitions we use a similar notation. The set of $\boldsymbol{e}$-ℓ-adic multipartitions of $\boldsymbol{n}_s$ is denoted by $\mathfrak{P}_s^{\mathbf{e},\ell}$

For $\boldsymbol{\lambda} \in \mathfrak{P}_s^{\mathbf{e},\ell}$ we define the element $y_{\boldsymbol{\lambda}} \in C_G(s)_\ell$ as follows: Given

$$C_G(s) = \cdots \times GL_{n_i}(q^{d_i}) \times \cdots$$

and correspondingly

$$\boldsymbol{\lambda} = (\ldots, \lambda^{(i)}, \ldots)$$

with

$$\lambda^{(i)} = (\ldots, \underbrace{e(d_i)\ell^m, \ldots, e(d_i)\ell^m}_{j\text{-times}} \ldots) \vdash n_i,$$

we define $y_{\boldsymbol{\lambda}} = \cdots \times y_{\lambda^{(i)}} \times \cdots$ with

$$y_{\lambda^{(i)}} = \cdots \times \underbrace{(y_{d_i}(m)) \times \cdots \times (y_{d_i}(m))}_{j\text{-times}} \times \cdots.$$

The element

$$s_{\boldsymbol{\lambda}} = s y_{\boldsymbol{\lambda}}$$

is obviously an ℓ-stable semisimple element of G.

Note that two semisimple conjugate elements of G are ℓ-equivalent. The next result can be seen easily:

Lemma 4.37 *Let s be a semisimple ℓ-regular element of G. Then*

$$\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathfrak{P}_s^{\mathbf{e},\ell}\}$$

is a complete set of representatives of ℓ-equivalence classes of semisimple elements in G with semisimple part conjugate to s.

So every semisimple s element of G is ℓ-equivalent to precisely one standard ℓ-stable element defined above. We shall denote it by $\tilde{s}$, (refining our previous definition). Thus

$$\tilde{s} = s'_{\boldsymbol{\lambda}} \tag{4.38}$$

for an $\boldsymbol{\lambda} \in \mathfrak{P}_{s'}^{\mathbf{e},\ell}$.

We define $\mathfrak{C}_{ss}^{\ell}$ to be the subset of the representatives $\mathfrak{C}_{ss}$ of G-classes of semisimple elements consisting of the ℓ-regular elements in $\mathfrak{C}_{ss}$, and we define

$$\mathfrak{S}\mathfrak{T}_{\ell} = \{s_{\boldsymbol{\lambda}} \in G \mid s \in \mathfrak{C}_{ss}^{\ell},\ \boldsymbol{\lambda} \in \mathfrak{P}_s^{\mathbf{e},\ell}\}, \tag{4.39}$$

thus $\mathfrak{S}\mathfrak{T}_{\ell}$ is a set of representatives of ℓ-equivalence classes of elements of G consisting of ℓ-stable elements. Moreover we may assume that $\mathfrak{S}\mathfrak{T}_{\ell} \subseteq \mathfrak{C}_{ss}$.

Theorems [9, 2.28 and 4.10] imply immediately:

Lemma 4.40 *To each geometric conjugacy class $\mathcal{E}(s^G)$ of G we have attached six functors (choosing s as representative in s^G):*

$$\begin{array}{rccc} H_{R,s}: & \mathrm{mod}_{RG} & \to & \mathrm{mod}_{\mathcal{H}_{R,s}} \\ H_{F,\tilde{s}}: & \mathrm{mod}_{FG} & \to & \mathrm{mod}_{\mathcal{H}_{F,\tilde{s}}} \\ \hat{H}_{R,s}: & \mathrm{mod}_{\mathcal{H}_{R,s}} & \to & \mathrm{mod}_{RG} \\ \hat{H}_{F,\tilde{s}}: & \mathrm{mod}_{\mathcal{H}_{F,\tilde{s}}} & \to & \mathrm{mod}_{FG} \end{array},$$

where $R = \mathcal{O}$ or K.

Those functors induce bijections between

- *The set of irreducibles in the geometric conjugacy class $\mathcal{E}(s)$ and the set of nonisomorphic irreducible $\mathcal{H}_{K,s}$-modules.*
- *The set of nonisomorphic irreducible composition factors in the head of $M_{F,s}$ and the nonisomorphic irreducible $\mathcal{H}_{F,\tilde{s}}$- modules.*

If in addition s is ℓ-stable, then those functors provide an embedding of the decomposition matrix of the triple

$$F \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O},s} \cong \mathcal{H}_{F,s} \leftarrow \mathcal{H}_{\mathcal{O},s} \to \mathcal{H}_{K,s} \cong K \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O},s},$$

and the decomposition numbers of $\mathcal{O}G$ describing the composition multiplicities of the irreducibleconstituents of $\mathrm{hd}(M_{F,s})$ in reductions modulo ℓ of the elements of $\mathcal{E}(s)$.

In the next section we shall use results in representation theory of Hecke algebras to derive a first classification of the irreducible FG-modules. As pointed out in section 2 this will give us as well the subdivision of those irreducibles according to their semisimple vertices and sources. However, besides the labelling of the irreducible FG-modules we shall obtain more detailed information as for example about decomposition numbers. For the classification of the irreducibles alone and their subdivision in to Harish-Chandra series one needs only Result 4.16, a result of Fong and Srinivasan in [25] saying that the Brauer characters of KG-modules of the form $S_K(s, \boldsymbol{\lambda})$, where $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ and $s \in \mathfrak{C}_{ss}$ is ℓ-regular, are a basis of the space of Brauer characters of G, and the fact that the decomposition matrix of G is unitriangular (Corollary 6.44). All but the last part is known to hold for arbitrary finite reductive groups analogously, (for details and an extensive bibliography we refer to [17]). The unitriangularity of the decomposition matrix is conjectured. For general linear and unitary groups it can be achieved by HC-inducing ordinary Gelfand-Graev characters of all Levi subgroups and cutting by blocks (see [26] for details).

5 The irreducible FG-modules

In the last section we constructed the Hecke functors H_s and $\hat{H}_s$ attached to the geometric conjugacy class indexed by the semisimple element s. Over F we only need to consider representatives s of ℓ-equivalence classes. Our goal in this section is to construct a complete set of irreducible FG-modules. We know from [9], that we may construct irreducible FG-modules by applying the functor $\hat{H}_s$ to irreducible $\mathcal{H}_s$-modules.

We begin by describing briefly some results on representations of $\mathcal{H} = \mathcal{H}_{R,s}(\mathfrak{S}_n)$, for details we refer to [19]. Here for the moment R is an arbitrary commutative domain and q is a unit in R. For a composition $\mu = (\mu_1, \mu_2, \dots) \models n$ we let $W_\mu = \mathfrak{S}_{\{1,2,\dots,\mu_1\}} \times \mathfrak{S}_{\{\mu_1+1,\dots,\mu_1+\mu_2\}} \times \cdots$ be the corresponding standard parabolic subgroup of $W = \mathfrak{S}_n$. In section 4 a basis of $\mathcal{H}$ was presented together with the defining relations for the multiplication (see Equation 4.11). Obviously the R-space spanned by $\{T_w \mid w \in W_\mu\}$ is closed under multiplication and hence a subalgebra $\mathcal{H}_\mu$ of $\mathcal{H}$ which is of course the Hecke algebra associated with W_μ. We define two special elements of $\mathcal{H}_\mu \leq \mathcal{H}$:

$$\text{(5.1)} \qquad \mathfrak{x}_\mu = \sum_{w \in W_\mu} T_w \qquad \mathfrak{y}_\mu = \sum_{w \in W_\mu} (-q)^{l(w)} T_w \quad .$$

Both elements generate one dimensional representations of $\mathcal{H}_\mu$, and we denote the induced representations $\mathfrak{x}_\mu\mathcal{H}$ and $\mathfrak{y}_\mu\mathcal{H}$ by M^μ and $\tilde{M}^\mu$ respectively. Let λ be a composition of n and denote its dual partition by λ'. There is a unique element of shortest length (a distinguished double coset representative) w_λ of W such that $W_\lambda^{w_\lambda} \cap W_{\lambda'} = 1$. We define $\mathfrak{z}_\lambda = \mathfrak{x}_\lambda T_{w_\lambda} \mathfrak{y}_{\lambda'}$ and the $\mathcal{H}_s$-module $S^\lambda = S_R^\lambda = \mathfrak{z}_\lambda \mathcal{H}_{R,s}$. Recall that a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is e - *restricted* for the positive integer e, if $\lambda_{i+1} - \lambda_i < e$ for all indices i, and it is *e-regular* if the dual partition λ' is e-restricted. We summarize:

5.2 *For $\lambda \vdash n$ we call S^λ Specht module.*

- *There is a symmetric associative bilinear form $<\ ,\ >$ on $\mathcal{H}$ whose Gram determinant with respect to any basis of $\mathcal{H}$ is a unit. In particular, for fields R the Hecke algebra*

$\mathcal{H}$ is a symmetric algebra and consequently every irreducible $\mathcal{H}$-modules is isomorphic to a minimal right ideal of $\mathcal{H}$.

- *The Specht modules are free as R-modules and have a basis independent of the choice of R and q. If R is a field and $\mathcal{H}$ is semisimple, then $\{S^\lambda \mid \lambda \vdash n\}$ is a complete set of nonisomorphic irreducible $\mathcal{H}$-modules.*
- *If R is a field then $\mathcal{H}$ is not semisimple precisely if q is an e-th root of unity in R for some $e \leq n$.*

There is an antiautomorphism $*$ of $\mathcal{H}$ induced by $T_w^* = T_{w^{-1}}$. Thus we can define dual modules, and $\mathcal{H}$-invariant bilinear forms.

- *For $\lambda \vdash n$ there is an $\mathcal{H}$-invariant symmetric bilinear form on S^λ, universally defined for all choices of R and q. The radical $S^{\lambda\perp}$ of that form is either S^λ or is the unique maximal submodule of S^λ.*

Let now R be a field of characteristic $\ell \geq 0$, and let q be a primitive e-th root of unity or, if $q = 1$, let $e = \ell$ for $\ell > 0$ and $e = \infty$ otherwise.

- *$S^{\lambda\perp} \neq S^\lambda$ precisely if λ is e- regular.*

Set $D^\lambda = S^\lambda / S^{\lambda\perp}$ if the partition λ of n is e-regular.

- *$\{D^\lambda \mid \lambda \vdash n, \lambda$ is ℓ-regular$\}$ is a complete set of nonisomorphic irreducible $\mathcal{H}$-modules.*
- *The composition factors of S^λ are of the form D^μ for some e-regular partition μ of n satisfying $\lambda \trianglelefteq \mu$, and D^λ has composition multiplicity 1 in S^λ provided λ is e-regular.*

Here $\trianglelefteq$ denotes the usual dominance ordering on the set $\Lambda(n)$ of compositions of n. Thus if $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ are compositions of n we write $\lambda \trianglelefteq \mu$ if $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$ for all j.

The last part of 5.2 asserts in particular, that the decomposition matrix $\Delta = \Delta_{\mathcal{H}}$ of $\mathcal{H}$ is unitriangular, where the partitions respectively the e-regular partitions of n are ordered compatibly with the dominance ordering downward.

Of course all these results carry over to tensor products of Hecke algebras associated with symmetric groups. In particular it carries over to our Hecke algebras $\mathcal{H}_{R,s}(W_s) = \mathcal{H}_s$ of Theorem 4.32. We have to adjust however notation. Partitions have to be replaced by multipartitions $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$, where again s is a semisimple element of G. The dominance ordering of partitions can be extended to multipartitions too: We define that $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash \boldsymbol{n}_s$ if $\lambda^{(i)} \trianglelefteq \mu^{(i)}$ for all i. Note that q^i is a primitive e-th root of unity in F if and only if $e = e(i)$, provided $q^i \not\equiv 1 \bmod \ell$, where $e(i)$ is defined in 4.34.

Thus in the notation of the previous section, setting $\boldsymbol{n} = \boldsymbol{n}_s$ and $\boldsymbol{e} = \boldsymbol{e}_s$ (compare 4.36) the set $\{S_K^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash \boldsymbol{n}\}$ is a complete set of irreducible nonisomorphic $\mathcal{H}_{K,s}$-modules and of course these modules are precisely the Specht modules used in Theorem 4.13. We say that the multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots)$ of $\boldsymbol{n}$ is *$\boldsymbol{e}$-regular*, if $\lambda^{(i)}$ is $e(d_i)$-regular for all i, and we denote the set of multipartitions of $\boldsymbol{n}_s$ by $\mathfrak{P}_s$ and the subset of $\boldsymbol{e}$-regular multipartitions of $\boldsymbol{n}_s$ by $\mathfrak{P}_s^{\mathrm{reg}}$. Thus, if s is ℓ-stable, then $\{D^{\boldsymbol{\mu}} \mid \mu \in \mathfrak{P}_s^{\mathrm{reg}})\}$ is a complete set of nonisomorphic irreducible $\mathcal{H}_{F,s}$-modules and the decomposition matrix $\Delta_s = \Delta_{H_s}$ of $\mathcal{H}_{\mathcal{O},s}$ is unitriangular.

We continue to use the notation introduced in the previous section. Thus henceforth $C_{\mathrm{R},s}$ denotes the cuspidal RL_s-module as defined in 4.7,$R \in \{K, \mathcal{O}, F\}$, where the Levi subgroup L_s was introduced in 4.8. Moreover $M_{R,s} = \mathrm{R}^G_{L_s}(C_{R,s})$.

The next result tells us that we may often ignore the functor $A_{\mathcal{P}_{R,s}}$ in our definition of $\hat{H}_{R,s}$, simplifying calculations:

Theorem 5.3 *Let $s \in G$ be semisimple, and let the projective presentation $\beta = \beta_{R,s}$: $\mathcal{P}_{R,s} \to M_{R,s}$ be defined as in Lemma 4.23 and Theorem 4.27 for $R \in \{K, \mathcal{O}, F\}$. Then $M_{R,s}$ is $\mathcal{P}_{R,s}$-torsionless for $R = K$ and $R = \mathcal{O}$. If in addition s is ℓ-stable, then $M_{F,s}$ is $\mathcal{P}_{F,s}$-torsionless as well.*

Proof For $R = K$ and $R = \mathcal{O}$ the theorem is Lemma 1.20. Thus let s be ℓ-stable. We present a proof which works for all choices of R.

We drop indices, when possible. We have to show that

$$\mathrm{Hom}_{RG}(\mathcal{P}, X) \neq (0)$$

for all nonzero submodules X of M_s. Recall that L_s is the smallest standard Levi subgroup of G containing s. Let $P_s = L_s U_s$ be the corresponding parabolic subgroup. Let $\hat{U}_s = |U_s|^{-1} \sum_{u \in U_s} u$. Recall the definition of the idempotent $e_{\mu_s} = e_{L_s}$ (Formulas 4.18 and 4.8), which generates the Gelfand-Graev representation of L_s, and remember our construction of $\mathcal{P}_1$ and $\mathcal{P} = \mathrm{R}^G_{L_s}(\mathcal{P}_1)$ in Lemma 4.23: Being a principal indecomposable projective L_s-module, $\mathcal{P}_1 = e_s RL_s$ for some primitive idempotent e_s of RL_s which is one of the summands in a decomposition of e_{L_s} into a sum of orthogonal primitive idempotents. Thus $\mathcal{P} = \mathrm{R}^G_{L_s}(\mathcal{P}_1) = e_s \hat{U}_s RG$.

Adjointness of HC-induction and HC-restriction (see e.g. [15, 1.7]) implies:

$$\mathrm{Hom}_{RG}(\mathcal{P}, X) = \mathrm{Hom}_{RG}(\mathrm{R}^G_{L_s}(\mathcal{P}_1), X) = \mathrm{Hom}_{RL_s}(\mathcal{P}_1, \mathrm{T}^G_{L_s}(X)). \tag{5.4}$$

For a subgroup H of G we denote the normalizer of H by $\mathcal{N}_G(H)$. Now we apply Mackey decomposition ([13, 1.4]) to get

$$\mathrm{T}^G_{L_s}(X) \leq \mathrm{T}^G_{L_s}(M_s) = \bigoplus_{d \in \mathcal{D}_{\mu,\mu}} \mathrm{R}^{L_s}_{L_s^d \cap L_s} \mathrm{T}^{L_s^w}_{L_s^d \cap L_s}(C_s^d) = \bigoplus_{d \in \mathcal{N}_G(L_s) \cap \mathcal{D}_{\mu,\mu}} C_s^d. \tag{5.5}$$

where $\mathcal{D}\mu$ denotes the set of distinguished right coset representatives W_μ of W, that is, $d \in \mathcal{D}_\mu$ is the unique shortest element in $W_\mu d$. Moreover for $\lambda, \mu \models n$ let $\mathcal{D}_{\lambda,\mu} = \mathcal{D}_\lambda^{-1} \cap \mathcal{D}_\mu$. Note that the elements of $\mathcal{D}_{\lambda,\mu}$ are distinguished W_λ-W_μ-double coset representatives, that is consist of the shortest elements in each double coset. We also use in Equation 5.5 that $\mathcal{D}_{\lambda,\mu}$ is as well a set of P_λ-P_μ-double coset representatives in G, and for the second part, that C_s is cuspidal and hence its HC-restriction of any of its conjugates to a proper Levi subgroup is zero.

We let $\mathfrak{b}_s$ be the block of $\mathcal{O}L_s$ containing C_s, and let ε_s be the corresponding central idempotent (considered as element of FL_s or $\mathcal{O}L_s \subseteq KL_s$). Let $d \in \mathcal{N}_G(L_s) \cap \mathcal{D}_{\mu,\mu}$ and assume that C_s^d belongs to $\mathfrak{b}_s$. Note that $C_s^d = C_{s^d}$. By Theorem 4.16 the ℓ-regular parts of s and s^d are conjugate in L_s. Since s is ℓ-stable by assumption, $s =_{L_S} s^d$, that is $d \in W_s$, where $W_s \leq W$ is defined as in Formula 4.10. Note that $W_s \subseteq \mathcal{D}_{\mu,\mu}$. We conclude that

$$\bigoplus_{d\in\mathcal{N}_G(L_s)\cap\mathcal{D}_{\mu,\mu}} C_s^d\varepsilon_s = \bigoplus_{d\in W_s} C_s, \tag{5.6}$$

Since $\mathcal{P}_1 = e_sRL_s$, we have by general theory that

$$\mathrm{Hom}_{RL_s}(\mathcal{P}_1, \mathrm{T}_{L_s}^G(X)) \cong X\hat{U}_se_s.$$

Let $0 \neq x \in X$. Let U be the unipotent radical of the fixed Borel subgroup B of G consisting of upper triangular matrices. Thus U is the subgroup of all upper unitriangular matrices in G. Denote the opposite group of lower unitriangular matrices in G by V, and for $w \in W$ define U_w^- to be the subgroup $\{u \in U \mid wu \in V\}$ of U. Since

$$M_s = \mathrm{R}_{L_s}^G(C_s) = \bigoplus_{w\in\mathcal{D}_\mu, u\in U_w^-} C_s \otimes wu$$

and $X \leq M_s$, we may write

$$x = m_\circ \otimes 1 + \sum_{1\neq w\in\mathcal{D}_\mu, u\in U_w^-} m_{w,u} \otimes wu, \tag{5.7}$$

for certain $m_{w,u}$, $m_\circ \in C_s$. We may assume that $m_\circ \neq 0$. To see this note, that we can always multiply by a suitable element of G (of the form $(wu)^{-1}$) to get a nonzero coefficient $m_\circ$. For $R = \mathcal{O}$ we may assume in addition that $m_\circ$ is not contained in $\mathfrak{p}C_{\mathcal{O},s}$, where $\mathfrak{p}$ as in the proof of Lemma 3.2 is the generating element of the maximal ideal of $\mathcal{O}$. Otherwise we may replace x by $\mathfrak{p}^{-k}x$ for some nonnegative integer k to get a suitable element of $X_{\mathcal{O}}$. Set $x_1 = x\hat{U}_s\varepsilon_s$. Then Equations 5.5 and 5.6 imply

$$0 \neq x_1 = m_\circ \otimes 1 + \sum_{1\neq w\in W_s} m_w \otimes w\hat{U}_s \in X, \tag{5.8}$$

for certain $m_w \in C_s$, since $U_w^- \leq U_s$ for $w \in W_s$. To see this one has to use that w is distinguished. Choose $0 \neq \psi_R \in \mathrm{Hom}_{RL_s}(\mathcal{P}_1, C_s)$, and let $m = \psi_R(e_s)$. Then $me_s = m \neq 0$ and we find an element $a \in RL_s$ such that $m_\circ a = m$. We have

$$0 \neq x_1ae_s = m \otimes 1 + \sum_{1\neq w\in W_s} m_w(ae_s)^{w^{-1}} \otimes w\hat{U}_s. \tag{5.9}$$

We have shown that $X\hat{U}_se_s = \mathrm{T}_{L_s}^G(X)e_s \neq (0)$, and the theorem follows from Formula 5.4. □

We remark that in [40, 4.7] a similar but weaker result is shown. Moreover the theorem can be applied to the case where s is not necessarily ℓ-stable. For $R = K$ and $\mathcal{O}$ it holds as stated, for $R = F$ we have to replace s by $\tilde{s}$, (compare to Lemma 4.29).

Recall that $\mathfrak{S}\mathfrak{T}_\ell$ denotes a set of representatives of ℓ-equivalence classes of semisimple elements of G consisting of ℓ-stable elements (see 4.39). We fix $s \in \mathfrak{S}\mathfrak{T}_\ell$. By Theorem 4.27

the projective presentation $\beta_{\mathcal{O},s} : \mathcal{P}_{\mathcal{O},s} \to M_{\mathcal{O},s}$ of $M_{\mathcal{O},s}$ satisfies Hypothesis 1.1 and by 4.40 the functors $H_{F,s}$ and $\hat{H}_{F,s}$ produce a bijection between a complete set of nonisomorphic irreducible $\mathcal{H}_{F,s}$-modules and the nonisomorphic irreducible FG-modules occurring as composition factors in $\mathrm{hd}(M_{F,s})$. Thus with $\boldsymbol{n} = \boldsymbol{n}_s$ (see 4.5) and $\boldsymbol{e} = \boldsymbol{e}_s$ (see 4.36) we define for $\boldsymbol{\lambda} \vdash \boldsymbol{n}$ using 5.2

$$S_R(s, \boldsymbol{\lambda}) = R \otimes_{\mathcal{O}} \sqrt{\hat{H}_{\mathcal{O},s}(S^{\lambda}_{\mathcal{O}})} \tag{5.10}$$

for $R \in \{K, \mathcal{O}, F\}$. If in addition $\boldsymbol{\lambda} \in \mathfrak{P}^{\mathrm{reg}}_s$ we get

$$D(s, \boldsymbol{\lambda}) = \hat{H}_{F,s}(D^{\boldsymbol{\lambda}}). \tag{5.11}$$

It is well known that $\mathcal{H}_R = \mathcal{H}_{R,s}$ is a symmetric R-algebra for fields R (see e.g. [19, 2.3]). Thus by Theorems 5.3 and 1.2 :

$$S_R(s, \boldsymbol{\lambda}) = R \otimes_{\mathcal{O}} \sqrt{S^{\boldsymbol{\lambda}}_{\mathcal{O}} M_{\mathcal{O},s}} = R \otimes_{\mathcal{O}} \sqrt{z_{\boldsymbol{\lambda}} M_{\mathcal{O},s}}, \tag{5.12}$$

where $z_{\boldsymbol{\lambda}} \in \mathcal{H}_{\mathcal{O},s}$ is defined as in the beginning of this section. For $\boldsymbol{\lambda} \in \mathfrak{P}^{\mathrm{reg}}_s$ we have:

$$D(s, \boldsymbol{\lambda}) = D^{\boldsymbol{\lambda}} M_{F,s}. \tag{5.13}$$

for $\boldsymbol{\mu} \vdash \boldsymbol{n}$ and $\boldsymbol{\lambda} \in \mathfrak{P}^{\mathrm{reg}}_s$ we denote the multiplicity of $D^{\boldsymbol{\lambda}}$ as composition factor of $S^{\boldsymbol{\mu}}_F$ by $d_{\boldsymbol{\mu\lambda}}$. The matrix

$$\Delta = \Delta_s = (d_{\boldsymbol{\mu},\boldsymbol{\lambda}})$$

is the *decomposition matrix* of $\mathcal{H}_{\mathcal{O},s}$. We denote the set $\{D^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathfrak{P}^{\mathrm{reg}}_s\}$ of irreducible $\mathcal{H}_{F,s}$-modules by $\mathfrak{H}_s$.

Theorem 5.14 *Let $s \in \mathfrak{S}\mathfrak{T}_\ell$. Then*

$$\mathfrak{T}_s = \{D(s, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathfrak{P}^{reg}_s\} \tag{5.15}$$

is precisely the Harish-Chandra series of irreducible FG-moduls with semisimple HC-vertex L_s and semisimple HC-source $C_{F,s}$. If $s \neq t \in \mathfrak{S}\mathfrak{T}_\ell$ then $\mathfrak{T}_s$ and $\mathfrak{T}_t$ are disjoint. If $\boldsymbol{\mu} \vdash \boldsymbol{n}$, then the multiplicity of $D(s, \boldsymbol{\lambda})$ as composition factor of $S_F(s, \boldsymbol{\mu})$ equals the decomposition number $d_{\boldsymbol{\mu\lambda}}$ of $\mathcal{H}_{\mathcal{O},s}$. Thus Δ_s is part of the ℓ-decomposition matrix of G.

Proof By 4.40 the functors $H_{F,s}$ and $\hat{H}_{F,s}$ produce a bijection between $\mathfrak{T}_S$ and $\mathfrak{H}_s$, and $\mathfrak{T}_s$ is the set of composition factors of $\mathrm{hd}(M_{F,s})$. By Corollary 2.5 (compare 4.1) $\mathfrak{T}_s$ is precisely the set of irreducible FG-modules having semisimple HC-vertex L_s and semisimple source $C_{F,a}$, since $M_{F,s} = \mathrm{R}^G_{L_s}(C_{F,s})$. If $s \neq t \in \mathfrak{S}\mathfrak{T}_\ell$ then especially $s =_\ell t$ by 4.39 , and hence $C_{F,s}$ and $C_{F,t}$ cannot be conjugate in G by the definition of ℓ-equivalence in Lemma 4.29. By [13, 4.16] $\mathfrak{T}_s$ and $\mathfrak{T}_t$ are disjoint. The theorem follows using [9, 4.10], (compare 4.40). □

So far the set

$$\mathfrak{T} = \bigcup_{s \in \mathfrak{S}\mathfrak{T}_\ell} \mathfrak{T}_s \tag{5.16}$$

is a set of nonisomporphic irreducible FG-modules, which is partitioned into HC-series $\mathfrak{T}_s$. We show now, that $\mathfrak{T}$ is indeed a complete set of non-isomorphic irreducible FG-modules.

We modify the (n,∞)-index $I(s)$ attached to the pair $(s,\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \vdash \boldsymbol{n_s}$ in 4.2, by replacing the elements $s_1, s_2, \dots$ in the lower lefthand part of $I(s)$ belonging to some extension field of $GF(q)$ by their ℓ-regular parts $s'_1, s'_2, \dots$ to get an (n,ℓ)-index, that is we have a map

$$(s,\boldsymbol{\lambda}) \mapsto I_\ell(s,\boldsymbol{\lambda}) = \left(\begin{array}{ccc|ccc} d_1 & d_2 & \dots, & n_1 & n_2 & \dots, \\ s'_1 & s'_2 & \dots, & \lambda^{(1)} & \lambda^{(2)} & \dots, \end{array} \right) \tag{5.17}$$

For convenience we give the details of the definition of (n,ℓ)-indices, (see [21, 2.4]):

Definition 5.18 *An* (n,ℓ)-index *is an array*

$$I = \left(\begin{array}{cccc|cccc} d_1 & d_2 & \dots, & d_N & n_1 & n_2 & \dots, & n_N \\ s_1 & s_2 & \dots, & s_N & \lambda^{(1)} & \lambda^{(2)} & \dots, & \lambda^{(N)} \end{array} \right),$$

such that

(i) *the s_i are representatives of orbits under the Galois group of the extension field $GF(q^{d_i})$ of $GF(q)$ as in 4.3. In addition s_i is ℓ-regular and there exists an ℓ-element $y_i \in GF(q^{d_i})$ with $s_i y_i$ of degree d_i over $GF(q)$.*
(ii) *The integers n_i are positive for all i and $\lambda^{(i)} \vdash n_i$.*
(iii) $n_1 d_1 + \dots + n_N d_N = n$

Moreover in I the entries are unordered with respect to $1, \dots, N$, that is, if π is a permutation of $(1, 2, \dots, N)$ the (n,ℓ)-indices

$$\left(\begin{array}{cccc|cccc} d_1 & d_2 & \dots, & d_N & n_1 & n_2 & \dots, & n_N \\ s_1 & s_2 & \dots, & s_N & \lambda^{(1)} & \lambda^{(2)} & \dots, & \lambda^{(N)} \end{array} \right)$$

and

$$\left(\begin{array}{cccc|cccc} d_{1\pi} & d_{2\pi} & \dots, & d_{N\pi} & n_{1\pi} & n_{2\pi} & \dots, & n_{N\pi} \\ s_{1\pi} & s_{2\pi} & \dots, & s_{N\pi} & \lambda^{(1\pi)} & \lambda^{(2\pi)} & \dots, & \lambda^{(N\pi)} \end{array} \right)$$

are identified. An (n,ℓ)-index is called

(i) *a* head, *if the s_i are pairwise different, and s_i has degree d_i over $GF(q)$ for all i,*
(ii) *a* foot, *if $\lambda^{(i)}$ is $e(d_i)$-regular for all i, where $e(d_i)$ is defined in 4.34, and*
(iii) special, *if $s_i = s_j$ implies either $i = j$ or $d_i \neq d_j$.*

We remark that all head indices are special. The indices above are defined for all primes ℓ and $l = \infty$. The latter case produces (n,∞)-indices, and the head (n,∞)-indices parametrize the conjugacy classes of G, (compare 4.2). Similarly, for our prime ℓ, the ℓ-regular conjugacy classes of G are parametrized by head (n,ℓ)-indices by [21, 2.8].

Lemma 5.19 *Let $s \in \mathfrak{S}\mathfrak{T}_\ell$ and $\boldsymbol{\lambda} \in \mathfrak{P}_s^{reg}$. Then $I_s(s,\boldsymbol{\lambda})$, defined in 5.17 is special foot, and the map*

$$(s,\boldsymbol{\lambda}) \mapsto I_\ell(s,\boldsymbol{\lambda})$$

is a bijection between $\{(s,\boldsymbol{\lambda}) \mid s \in \mathfrak{S}\mathfrak{T}_\ell, \boldsymbol{\lambda} \in \mathfrak{P}_s^{reg}\}$ and the set of special foot (n,ℓ)-indices.

Proof Obviously $I_\ell(s,\boldsymbol{\lambda})$ is an (n,ℓ)-index, and it is a foot, since $\boldsymbol{\lambda}$ is $\boldsymbol{e}$-regular, by Definition of $\boldsymbol{e}$-regularity in the beginning of this section. Let $i \neq j$ and suppose $s'_i = s'_j$. Then $s_i \neq s_j$, hence $d_i \neq d_j$ by the construction 4.36 of standard ℓ-stable semisimple elements of G. Thus $I_\ell(s,\boldsymbol{\lambda})$ is a special foot. Conversely 4.36 shows too that every special foot (n,ℓ)-index corresponds to exactly one pair $(s,\boldsymbol{\lambda})$ with $s \in \mathfrak{S}\mathfrak{T}_\ell$ and $\boldsymbol{\lambda} \in \mathfrak{P}_s^{\mathrm{reg}}$. □

Remark 5.20 *Permuting the parts of s corresponds to the analogous permutation in the associated (n,ℓ)-index. As remarked in* 4.14 *our constructions are invariant under such operations justifying* [21, 2.5] *and our convention to fix the order of the parts of s in Formula* 4.3.

It was shown in [21, 2.13] (compare [7, 6.4] and [8, Section 2]) that there is a bijection between the set of special foot and the set of head (n,ℓ)-indices, which in turn label the ℓ-regular conjugacy classes of G as remarked above. Thus we have shown:

Theorem 5.21 (The irreducible FG-modules) *The set*

$$\mathfrak{T} = \{D(s,\boldsymbol{\lambda}) \mid s \in \mathfrak{S}\mathfrak{T}_\ell, \boldsymbol{\lambda} \in \mathfrak{P}_s^{reg}\}$$

is a complete set of nonisomorphic irreducible FG-modules. Moreover

$$\mathfrak{T} = \bigcup_{s \in \mathfrak{S}\mathfrak{T}_\ell} \mathfrak{T}_s$$

is the decomposition of $\mathfrak{T}$ into Harish-Chandra series.

Remark 5.22 *The bijection* [21, 2.13] *between head and special foot (n,ℓ)-indices is given by an* $\mathbf{e}$-ℓ-adic *decomposition of the involved multipartitions. The classification in Theorem* 5.21 *was derived in* [7, 6.16] *and* [8, 3.7].

We come now to the promised proof that all irreducible cuspidal FG-modules are liftabel. In 5.21 we have found all irreducible FG-modules in the set $\mathfrak{T}$ and therefore we have to determine which of those are cuspidal. By construction this happens precisely for such $D(s,\boldsymbol{\lambda}$ which satisfy $L_s = G$. But then s is ℓ-regular, the multipartition $\boldsymbol{\lambda}$ consists precisely of one part (1) and $D(s,\boldsymbol{\lambda}) = D(s,(1)) = C_{F,s}$. Thus we have:

Corollary 5.23 *Let $\mathfrak{C}_{ss}^n$ be the subset of all elements s of $\mathfrak{C}_{ss}$ whose minimum polynomial has degree n. Then*

$$\{D(s,(1)) \mid s \in \mathfrak{C}_{ss}^n\}$$

is a complete set of nonisomorphic cuspidal irreducible $FGL_n(q)$-modules. In particular every irreducible cuspidal $FGL_n(q)$-module is liftable.

6 q-Schur algebras

In this section we shall derive a classification of the irreducible FG-modules in terms of head (n,ℓ)-indices. This was first done by James in [40] without employing q-Schur algebras. Here we shall recover this classification using our general set up in the first section.

Let $s \in \mathfrak{C}_{ss}$. Thus by 4.3

$$s = (s_1)^{\times n_1} \times (s_2)^{\times n_2} \times \cdots ,$$

and $n = n_1 d_1 + n_2 d_2 + \cdots$, where d_i is the degree of the minimum polynomial of $s_i \in GF(q^{d_i})$ over $GF(q)$. By 4.8 $L_s = L_\mu$, where $\mu = \mu_s$ is given as

$$\mu_s = (\underbrace{d_1, \dots, d_1}_{n_1\text{-times}}, \underbrace{d_2, \dots, d_2}_{n_2\text{-times}}, \dots\dots).$$

We set

$$\hat{L}_s = GL_{n_1 d_1} \times GL_{n_2 d_2} \times \cdots\cdots . \tag{6.1}$$

To $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots) \models \boldsymbol{n}_s$ we define the composition $\tilde{\lambda}$ of n by

$$\tilde{\lambda} = (d_1\lambda_1^{(1)}, d_1\lambda_2^{(1)}, \dots, d_1\lambda_k^{(1)}, \dots, d_2\lambda_1^{(2)}, \dots, d_i\lambda_j^{(i)}, \dots\dots). \tag{6.2}$$

The next lemma is obvious:

Lemma 6.3 *Let $s \in \mathfrak{C}_{ss}$. Then $L_s \le \hat{L}_s$. The map $\boldsymbol{\lambda} \mapsto \tilde{\lambda} \models n$ for $\boldsymbol{\lambda} \models \boldsymbol{n}_s$ induces a bijection $W_{\boldsymbol{\lambda}} \mapsto W_{\tilde{\lambda}}$ between the set of parabolic subgroups $W_{\boldsymbol{\lambda}}$ of the ramification group W_s, defined in* 4.10 *and the set of all parabolic subgroups of the Weylgroup $W = \mathfrak{S}_n$ of G containing the Weyl group $W(L_s)$ of L_s and being contained in the Weyl group $W(\hat{L}_s)$ of $\hat{L}_s$. Thus*

$$\{L_{\tilde{\lambda}} \mid \boldsymbol{\lambda} \models \boldsymbol{n}_s\} \quad \textit{and} \quad \{P_{\tilde{\lambda}} \mid \boldsymbol{\lambda} \models \boldsymbol{n}_s\}$$

is precisely the set of standard Levi respectively parabolic subgroups placed between L_s and $\hat{L}_s$, and P_s and $\hat{P}_s$ respectively, (extending the hat*-notation to parabolic subgroups in the obvious way).*

We have seen in 4.10 that the ramification group W_s may be identified with a subgroup of W. Moreover the standard parabolic subgroups of W_s are parametrized by $\{\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \models \boldsymbol{n}_s\}$. Obviously we have $W_s \le W(\hat{L}_s) \le \hat{L}_s$.

Theorem 4.32 implies:

Lemma 6.4 *Let $\boldsymbol{\lambda} \models \boldsymbol{n}_s$, and let $R \in \{K, \mathcal{O}, F\}$. Suppose that s is ℓ-stable if $R = F$. Then*

$$\mathrm{End}_{RL_{\tilde{\lambda}}}(\mathrm{R}_{L_s}^{L_{\tilde{\lambda}}} C_{R,s}) \cong \mathcal{H}_{R,s}(W_{\boldsymbol{\lambda}}),$$

where $W_{\boldsymbol{\lambda}}$ is the parabolic subgroup of W_s corresponding to $\boldsymbol{\lambda} \models \boldsymbol{n}_s$, and

$$\mathcal{H}_{R,s}(W_{\boldsymbol{\lambda}}) = \mathcal{H}_{R,q^{d_1}}(W_{\lambda^{(1)}}) \otimes \mathcal{H}_{R,q^{d_2}}(W_{\lambda^{(2)}}) \otimes \cdots\cdots .$$

Here $W_{\lambda^{(i)}}$ is the parabolic subgroup of $\mathfrak{S}_{n_i}$ coresponding to $\lambda^{(i)} \models n_i$.

Note that

$$W_{\boldsymbol{\lambda}} = W_s \cap W(L_{\tilde{\lambda}}) = W_s \cap L_{\tilde{\lambda}} = W_s \cap P_{\tilde{\lambda}}. \tag{6.5}$$

If $R = F$ and s is not ℓ-stable, we have to replace s by $\tilde{s}$ by 4.29. The next two lemmas are easy to prove using transitivity of HC-induction (see [15, 1.12]):

Lemma 6.6 *Let $\mathcal{A} \subseteq \mathcal{H}_{R,s}(W_{\boldsymbol{\lambda}}) \subseteq \mathcal{H}_{R,s}$, and let $\tilde{L} = L_{\tilde{\lambda}}, L = L_s$. Then*

$$\mathcal{A}(\mathrm{R}_L^G C_{R,s}) = \mathrm{R}_{\tilde{L}}^G(\mathcal{A}\mathrm{R}_L^{\tilde{L}} C_{R,s}),$$

as RG-modules.

Lemma 6.7 *Let $\tilde{L}$ be a standard Levi subgroup of G containing $L = L_s$. Let X be a pure sublattice of $\mathrm{R}_L^{\tilde{L}} C_{\mathcal{O},s}$. Then $\mathrm{R}_{\tilde{L}}^G X$ is a pure sublattice of $R_L^G C_{\mathcal{O},s}$.*

We fix now $s \in \mathfrak{S}\mathfrak{T}_\ell$ and hence may omit indices s. Thus $L = L_s$, $\hat{L} = \hat{L}_s$ and $\boldsymbol{n} = \boldsymbol{n}_s$ etc. We fix as well $\boldsymbol{\lambda} \models \boldsymbol{n}$ and set $\tilde{L} = L_{\tilde{\lambda}}$. Note that $W_{\boldsymbol{\lambda}} = W_{\lambda^{(1)}} \times W_{\lambda^{(2)}} \times \cdots$, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots)$ and $\lambda^{(i)} \models n_i$. We set

$$\mathfrak{y}_{\boldsymbol{\lambda}} = \mathfrak{y}_{\lambda^{(1)}} \mathfrak{y}_{\lambda^{(2)}} \cdots \in \mathcal{H}(W_{\boldsymbol{\lambda}}) \subseteq \mathcal{H}, \tag{6.8}$$

where $\mathfrak{y}_{\lambda^{(i)}} \in \mathcal{H}_{R,q^{d_i}}$ is defined as in Formula 5.1, replacing q by q^{d_i}. We define $\mathfrak{x}_{\boldsymbol{\lambda}} \in \mathcal{H}(W_{\boldsymbol{\lambda}})$ analogously.

Observe that $R\mathfrak{y}_{\boldsymbol{\lambda}}$ is an one dimensional $\mathcal{H}(W_{\boldsymbol{\lambda}})$-lattice. Consequently

$$\mathfrak{y}_{\boldsymbol{\lambda}} \mathrm{R}_L^{\tilde{L}} C_{K,s} = St_{\boldsymbol{\lambda},K}$$

is an irreducible $K\tilde{L}$-module. Parts (i) and (iv) of 4.19 applied to $\tilde{L}$ say:

Lemma 6.9 *$St_{\boldsymbol{\lambda},K}$ is the unique irreducible constituent of the Gelfand-Graev representation $\Gamma_{\boldsymbol{\lambda},K}$ of $\tilde{L}$ which is contained in the geometric conjugacy class of $\tilde{L}$ associated with $s \in \tilde{L} \subseteq G$. Moreover this component occurs with multiplicity one as composition factor of $\mathrm{R}_L^{\tilde{L}} C_K$.*

The idempotent $e_{\tilde{L}} \in \mathcal{O}\tilde{L}$, generating the Gelfand-Graev representation $\Gamma_{\boldsymbol{\lambda},\mathcal{O}}$ of $\mathcal{O}\tilde{L}$ was introduced in Formula 4.18. By general arguments the $R\tilde{L}$-module

$$Y_{\boldsymbol{\lambda},R} = \mathrm{R}_L^{\tilde{L}} C_R e_{\tilde{L}} R\tilde{L}, \tag{6.10}$$

for $R \in \{K, \mathcal{O}, F\}$, is the $R\tilde{L}$-submodule of $\mathrm{R}_L^{\tilde{L}} C_R$ generated by the images of all homomorphisms $\Phi : \Gamma_{\boldsymbol{\lambda},R} \to \mathrm{R}_L^{\tilde{L}} C_R$. Thus in the language of [9, 2.15] (compare section 1), $Y_{\boldsymbol{\lambda},R}$ is the Γ-trace $\tau_\Gamma(\mathrm{R}_L^{\tilde{L}} C_R)$, setting momentarily $\Gamma = \Gamma_{\boldsymbol{\lambda},R}$. James showed in [40, 7.22] that the dimensions of $Y_{\boldsymbol{\lambda},F}$ and $Y_{\boldsymbol{\lambda},K}$ coincide. Consequently $Y_{\boldsymbol{\lambda},\mathcal{O}}$ is a pure sublattice of $\mathrm{R}_L^{\tilde{L}} C_{\mathcal{O}}$. Lemma 6.9 implies that $Y_{\boldsymbol{\lambda},K} = St_{\boldsymbol{\lambda},K} = \mathfrak{y}_{\boldsymbol{\lambda}} \mathrm{R}_L^{\tilde{L}} C_K$. We conclude that $Y_{\boldsymbol{\lambda},\mathcal{O}} = \mathfrak{y}_{\boldsymbol{\lambda}} \mathrm{R}_L^{\tilde{L}} C_K \cap \mathrm{R}_L^{\tilde{L}} C_{\mathcal{O}}$ and hence contains $\mathfrak{y}_{\boldsymbol{\lambda}} \mathrm{R}_L^{\tilde{L}} C_{\mathcal{O}}$. We have therefore:

Theorem 6.11 (G. James) *Let $s \in \mathfrak{S}\mathfrak{T}_\ell$ and $\boldsymbol{\lambda} \models \boldsymbol{n}_s$. Then the $\mathcal{O}\tilde{L}$-lattice $Y_{\boldsymbol{\lambda},\mathcal{O}}$ is the purification $\sqrt{\mathfrak{y}_{\boldsymbol{\lambda}}\mathrm{R}_L^{\tilde{L}}C_{\mathcal{O}}}$ of $\mathfrak{y}_{\boldsymbol{\lambda}}\mathrm{R}_L^{\tilde{L}}C_{\mathcal{O}}$ in $\mathrm{R}_L^{\tilde{L}}C_{\mathcal{O}}$.*

Remark 6.12 *Most of what we have done so far can in this section carries over in slightly adapted form to arbitrary finite groups with (BN)-pair. However Theorem* 6.11 *does not. How this happens and what can be done in this more general situation will be investigated in a forthcoming paper of the author and J. Gruber. It also will contain a direct much shorter proof of* 6.11*.*

By Part (iv) of 4.19 there exists an epimorphism $\Psi_{\boldsymbol{\lambda},\mathcal{O}} : \Gamma_{\boldsymbol{\lambda},\mathcal{O}} \to Y_{\boldsymbol{\lambda},\mathcal{O}}$, thus $\Gamma_{\boldsymbol{\lambda},\mathcal{O}}$ contains the minimal projective cover $\beta_{\boldsymbol{\lambda},\mathcal{O}} : \mathcal{P}_{\boldsymbol{\lambda},\mathcal{O}} \to Y_{\boldsymbol{\lambda},\mathcal{O}}$ of $Y_{\boldsymbol{\lambda},\mathcal{O}}$. As in 4.23 we see that $\mathcal{P}_{\boldsymbol{\lambda},\mathcal{O}}$ is indecomposable, hence $\mathcal{P}_{\boldsymbol{\lambda},\mathcal{O}} = f_{\boldsymbol{\lambda},s}\mathcal{O}\tilde{L}$ for some primitive idempotent $f_{\boldsymbol{\lambda},s} \in \mathcal{O}\tilde{L}$. Recall the construction of the composition $\tilde{\lambda}$ of n associated with $\boldsymbol{\lambda} \models \boldsymbol{n}$ in Formula 6.2. The idempotent $\hat{f}_{\boldsymbol{\lambda},s} \in \mathcal{O}P_{\tilde{\lambda}}$ is obtained by multiplying $f_{\boldsymbol{\lambda},s}$ by the idempotent $\hat{U}^{\tilde{\lambda}}$ affording the trivial representation of the unipotent radical $U^{\tilde{\lambda}}$ of the parabolic subgroup $P_{\tilde{\lambda}}$ of G. Similarly $\hat{e}_{\tilde{\lambda}} = e_{\tilde{L}}\hat{U}^{\tilde{\lambda}}$. Note that by construction in Formula 4.18

$$\{\hat{e}_{\tilde{\lambda}} \mid \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$$

is a set of orthogonal primitive idempotents of $\mathcal{O}U$ of the group U of unipotent upper triangular matrices, affording pairwise different linear characters of U.

We set

$$\boldsymbol{P}_{R,s} = \bigoplus_{\boldsymbol{\lambda}\vdash\boldsymbol{n}_s} \mathrm{R}_{\tilde{L}}^{G}\mathcal{P}_{\boldsymbol{\lambda},R}, \quad N_{\boldsymbol{\lambda},R} = \mathrm{R}_{\tilde{L}}^{G}Y_{\boldsymbol{\lambda},R}, \quad \text{and} \quad N_{R,s} = \bigoplus_{\boldsymbol{\lambda}\vdash\boldsymbol{n}_s} N_{\boldsymbol{\lambda},R}. \tag{6.13}$$

A projective presentation of $N_{R,s}$ is given as

$$\boldsymbol{\beta}_{R,s} = \bigoplus_{\boldsymbol{\lambda}\vdash\boldsymbol{n}_s} \mathrm{R}_{L_{\tilde{\lambda}}}^{G}\beta_{\boldsymbol{\lambda},R} : \boldsymbol{P}_{R,s} \to N_{R,s}, \tag{6.14}$$

where the subscript R indicates as usual, that we have tensored the corresponding $\mathcal{O}$-object by $R \in \{K, \mathcal{O}, F\}$. Note that HC-induction commutes with tensoring by R, thus all these objects are well defined.

Obviously

$$\mathrm{R}_{\tilde{L}}^{G}\mathcal{P}_{\boldsymbol{\lambda},R} = \hat{f}_{\boldsymbol{\lambda},s}RG$$

and we may decompose the idempotents $\hat{e}_{\tilde{\lambda}} \in \mathcal{O}P_{\tilde{\lambda}}$ into a sum of orthogonal primitive idempotents and choose $\hat{f}_{\boldsymbol{\lambda},s}$ among those such that

$$\{\hat{f}_{\boldsymbol{\lambda},s} \mid \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$$

is a set of pairwise orthogonal idempotents of $\mathcal{O}G$.

We set

$$\boldsymbol{f}_s = \sum_{\boldsymbol{\lambda}\vdash\boldsymbol{n}_s} \hat{f}_{\boldsymbol{\lambda},s} \in \mathcal{O}G, \tag{6.15}$$

and we conclude that

$$\boldsymbol{P}_{R,s} = \boldsymbol{f}_s RG.$$

Note that for $\boldsymbol{\omega} = ((1^{n_1}), (1^{n_2}), \dots) \vdash \boldsymbol{n}_s$ we have $\mathfrak{y}_{\boldsymbol{\omega}} = 1 \in \mathcal{H}_{R,s}$, hence

$$\hat{f}_{\boldsymbol{\omega},s} RG \cong \mathcal{P}_{R,s} \tag{6.16}$$

where $\mathcal{P}_{R,s}$ is defined as in 4.23. In particular $\mathcal{P}_{\mathcal{O},s}$ is direct summand of $\boldsymbol{P}_{\mathcal{O},s}$. Moreover, for $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ we have

$$N_{\boldsymbol{\lambda},R} \leq M_{R,s}. \tag{6.17}$$

Consequently, for fields R, every simple submodule of $N_{R,s}$ is a submodule of $M_{R,s}$ too. Therefore it occurs in $\mathrm{hd}(\mathcal{P}_{R,s})$ by 5.3, and hence in $\mathrm{hd}(\boldsymbol{P}_{R,s})$. From this we conclude immediately:

Theorem 6.18 *Let $s \in \mathfrak{S}\mathfrak{T}_\ell$. Then $N_{R,s}$ is $\boldsymbol{P}_{R,s}$-torsionless for $R \in \{K, \mathcal{O}, F\}$.*

Theorem 6.19 *Let $s \in \mathfrak{C}_{ss}$ be ℓ-stable. Then*

$$\boldsymbol{\beta}_{R,s} : \boldsymbol{P}_{R,s} \to N_{R,s}$$

satisfies Hypothesis 1.1 *for any choice of $R \in \{K, \mathcal{O}, F\}$.*

Proof Let $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$. All irreducible components of $N_{\boldsymbol{\lambda},K}$ are contained in the geometric conjugacy class $\mathcal{E}(s)$ associated with $s \in \mathfrak{C}_{ss}$: This is immediate from the construction using Formula 4.21. The composition factors of the kernel of $\beta_{\boldsymbol{\lambda},K}$ belong to geometric conjugacy classes of the form $\mathcal{E}(s'y)$ for some $y \in C_G(s')_\ell$ such that $s \neq_{L_{\tilde{\lambda}}} s'y$, by 4.19. Since s is ℓ-stable we have $s \neq_G s'y$ by 4.24. This holds for all $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$, and we conclude, that the irreducible constituents of $\ker \boldsymbol{\beta}_{K,s}$ and $N_{K,s}$ belong to different geometric conjugacy classe. In particular these modules have no composition factor in common and the theorem foolows from [9, 4.7]. □

We apply [9, 2.2 and 2.15] to construct functors connecting representations of G with those of $\mathrm{End}_{RG}(N_{R,s})$. We use now 1.30 to investigate this endomorphism ring in terms of $\mathcal{H}_{R,s}$.

Lemma 6.20 *Let $s \in \mathfrak{C}_{ss}$ and $\boldsymbol{\lambda} \models \boldsymbol{n}_s$. Then $\mathfrak{y}_{\boldsymbol{\lambda}} \mathcal{H}_{s,\mathcal{O}}$ is a pure sublattice of $\mathcal{H}_{s,\mathcal{O}}$.*

Proof The lemma is easily reduced to the case, where $s = 1 \in G$, hence $\mathcal{H}_{R,s} = \mathcal{H}_{R,q}(\mathfrak{S}_n)$, and $\lambda = (n) \vdash n$. Now the lemma follows easily from defintion 5.1. □

We define the q-Schur algebra $\mathcal{S}_{R,s}$ to be

$$\mathcal{S}_{R,s} = \mathrm{End}_{\mathcal{H}_{R,s}}\Big(\bigoplus_{\boldsymbol{\lambda} \vdash \boldsymbol{n}_s} \mathfrak{y}_{\boldsymbol{\lambda}} \mathcal{H}_{R,s} \Big). \tag{6.21}$$

Note that $\mathcal{S}_{R,s} \cong R \otimes_{\mathcal{O}} \mathcal{S}_{s,\mathcal{O}}$ by 6.20. We set

$$E_{R,s} = \bigoplus_{\boldsymbol{\lambda} \vdash \boldsymbol{n}_s} \mathfrak{y}_{\boldsymbol{\lambda}} \mathcal{H}_{R,s}. \tag{6.22}$$

Note that by 5.2 $\mathcal{H}_{\mathcal{O},s}$ admits an associative bilinear form whose Gram determinant is a unit. By Theorem 1.36 the assumptions of 1.30 are satisfied in view of Result 5.3. Using Equation 1.28, functoriality of the Hom-functor $H_{R,s}$ of 4.30, and Theorem 1.30 we get:

Theorem 6.23 *Let $s \in \mathfrak{C}_{ss}$ be ℓ-stable. Then*

$$H_{R,s}(N_{R,s}) = E_{R,s} \tag{6.24}$$

and the functor $H_{R,s}$ induces an R-algebra isomorphism also denoted by $H_{R,s}$:

$$H_{R,s} : \mathrm{End}_{RG}(N_{R,s}) \to \mathrm{End}_{\mathcal{H}_{R,s}}(E_{R,s}) = \mathcal{S}_{R,s}. \tag{6.25}$$

Thus $\mathrm{End}_{RG}(N_{R,s})$ is the q-Schur algebra $\mathcal{S}_{R,s}$. In [22] q-Schur algebras were introduced, related to representations of finite general linear groups, and their representations were investigated in [22] and [23]. In [11] it was shown that they (or rather Morita equivalent algebras) are building blocks of quantum coordinate rings of general linear groups. Indeed here q-Schur were defined as endomorphism ring of a certain action of the Hecke algebra $\mathcal{H} = \mathcal{H}_{q,R}(\mathfrak{S}_r)$ on tensor space $E^{\otimes r}$, with $E = R^n$, such that

$$E^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n,r)} \mathfrak{x}_\lambda \mathcal{H}$$

as $\mathcal{H}$-module. Here $\Lambda(n,r)$ is the set of compositions of r into n parts. For $n = r$ one gets an algebra Morita equivalent to ours here, observing that $\mathfrak{x}_\lambda \mathcal{H}$ and $\mathfrak{y}_\lambda \mathcal{H}$ are conjugate $\mathcal{H}$-modules under an outer automorphism of $\mathcal{H}$.

We describe next some facts about q-Schur algebras using results [23],[11], and [12], and we refer for details and proof to these papers. As in the beginning of section 5 we define for $\lambda \models n$ the element $w_\lambda \in \mathfrak{S}_n$ to be the unique distinguished W_λ-$W_{\lambda'}$-double coset representative with $W_\lambda^{w_\lambda} \cap W_{\lambda'} = (1)$ and extend this notation to multi compositions:

$$w_{\boldsymbol{\lambda}} = w_{\lambda^{(1)}} w_{\lambda^{(2)}} \cdots \in W_s \tag{6.26}$$

for $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots) \models \boldsymbol{n}_s$. Moreover we define

$$\mathfrak{z}_{\boldsymbol{\lambda}} = \mathfrak{z}_{\lambda^{(1)}} \mathfrak{z}_{\lambda^{(2)}} \cdots \in \mathcal{H}_{R,s}. \tag{6.27}$$

Let $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$. We may premultiply $\mathfrak{y}_{\boldsymbol{\lambda}} \mathcal{H}_{R,s}$ by $\mathfrak{x}_{\boldsymbol{\lambda}'} T_{w_{\boldsymbol{\lambda}'}}$, where $\boldsymbol{\lambda}' = (\lambda^{(1)'}, \lambda^{(2)'}, \dots)$. We get so an epimorphism from $\mathfrak{y}_{\boldsymbol{\lambda}} \mathcal{H}_{R,s}$ onto the Specht module $S_R^{\boldsymbol{\lambda}'}$ to the dual weight $\boldsymbol{\lambda}'$ of $\boldsymbol{\lambda}$, (see 5.2). We apply [19, 4.12] and [22, 2.1 and 2.4] to the special case of semisimple Hecke algebras and see that $S_K^{\boldsymbol{\lambda}'}$ occurs exactly once as composition factor of $\mathfrak{y}_{\boldsymbol{\lambda}} \mathcal{H}_{K,s}$ and if $S_K^{\boldsymbol{\mu}'}$ occurs as well with $\boldsymbol{\mu} \vdash \boldsymbol{n}_s$, then $\boldsymbol{\mu}' \trianglerighteq \boldsymbol{\lambda}$. Consequently there is for any commutative ring R

precisely one indecomposable direct summand $\mathcal{Y}_{\boldsymbol{\lambda}',R}$ of $\mathfrak{y}_{\boldsymbol{\lambda}}\mathcal{H}_{R,s}$ such that $\mathfrak{x}_{\boldsymbol{\lambda}'}T_{w_{\boldsymbol{\lambda}'}}\mathcal{Y}_{\boldsymbol{\lambda}',R} \neq (0)$ and then

$$\mathfrak{x}_{\boldsymbol{\lambda}'}T_{w_{\boldsymbol{\lambda}'}}\mathcal{Y}_{\boldsymbol{\lambda}',R} \cong S_R^{\boldsymbol{\lambda}'} \tag{6.28}$$

is an epimorphic image of $\mathcal{Y}_{\boldsymbol{\lambda}',R}$. We call $\mathcal{Y}_{\boldsymbol{\lambda}',R}$ *alternating source* or *Young* module, following [12, 2.1].

Note that $\mathcal{Y}_{\boldsymbol{\lambda},K} = S_K^{\boldsymbol{\lambda}}$, but $\mathcal{Y}_{\boldsymbol{\lambda},R}$ is not a Specht module in general. In particular $\mathcal{Y}_{\boldsymbol{\lambda},K} \ncong K \otimes \mathcal{Y}_{\boldsymbol{\lambda},\mathcal{O}}$, where tensoring is as usual tensoring over $\mathcal{O}$. However, since $\mathcal{O}$ is a complete discrete valuation ring we may lift idempotents from $\mathrm{End}_{\mathcal{H}_{F,s}}(E_{F,s}) \cong F \otimes \mathrm{End}_{\mathcal{H}_{s,\mathcal{O}}}(E_{s,\mathcal{O}})$ to $\mathrm{End}_{\mathcal{H}_{s,\mathcal{O}}}(E_{s,\mathcal{O}})$ and we have therefore:

$$\mathcal{Y}_{\boldsymbol{\lambda},F} \cong F \otimes_{\mathcal{O}} \mathcal{Y}_{\boldsymbol{\lambda},\mathcal{O}}. \tag{6.29}$$

Let $\boldsymbol{\mu} \vdash \boldsymbol{n}_s$ and suppose that $S_K^{\boldsymbol{\mu}}$ occurs as composition factor in $K \otimes \mathcal{Y}_{\boldsymbol{\lambda},\mathcal{O}}$, then $S_K^{\boldsymbol{\mu}}$ occurs as compostion factor of $\mathfrak{y}_{\boldsymbol{\lambda}'}\mathcal{H}_{K,s} = K \otimes \mathfrak{y}_{\boldsymbol{\lambda}'}\mathcal{H}_{s,\mathcal{O}}$ as well. We apply [19, 4.12] observing that Specht modules here are irreducible and selfdual by [19, 4.15], since $\mathcal{H}_{K,s}$ is semisimple, and conclude from [20, 3.1 and 3.5] that $\boldsymbol{\mu}' \trianglerighteq \boldsymbol{\lambda}'$ and hence $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}$. We order the set $\mathfrak{P}_s$ of multipartitions of $\boldsymbol{n}_s$ compatibly with the dominance ordering downward (e.g. lexicographically downward) and let $d_{\boldsymbol{\mu}\boldsymbol{\lambda}}$ to be the multiplicity of $S_K^{\boldsymbol{\mu}}$ as composition factor of $K \otimes \mathcal{Y}_{\boldsymbol{\lambda},\mathcal{O}}$. We define the square matrix $\hat{\Delta}_s$ to be:

$$\hat{\Delta} = \hat{\Delta}_s = (d_{\boldsymbol{\mu}\boldsymbol{\lambda}}). \tag{6.30}$$

We shall see below that our notation here is consistent with our defintion of decomposition numbers of Hecke algebras in the previous section.

Let X be an indecomposable direct summand of $E_{R,s}$ and let $\pi \in \mathcal{S}_{R,s}$ be the projection of $E_{R,s}$ onto X. Thus $\pi\mathcal{S}_{R,s}$ is a projective indecomposable $\mathcal{S}_{R,s}$-module, which has a simple head for fields R. We denote the irreducible $\mathcal{S}_{K,s}$-module corresponding to the indecomposable direct summand $\mathcal{Y}_{\boldsymbol{\lambda},K} \cong S_K^{\boldsymbol{\lambda}}$ of $E_{K,s}$ by $\mathcal{W}_K^{\boldsymbol{\lambda}}$, and call this module *Weyl module* to *highest weight* $\boldsymbol{\lambda}$. Indeed this is the Weyl module to weight $\boldsymbol{\lambda}$ as investigated in [23, 3.8] (using dual weights). It is defined universally, that is for any commutative ring R and any choice of parameters for $\mathcal{H}_{R,s}$, as follows.

For $\boldsymbol{\omega} = ((1^{n_1}),(1^{n_2}),\dots) \vdash \boldsymbol{n}_s$ we have $\mathfrak{y}_{\boldsymbol{\omega}} = 1 \in \mathcal{H}_{R,s}$ and so $\mathfrak{y}_{\boldsymbol{\omega}}\mathcal{H}_{R,s}$ is the regular $\mathcal{H}_{R,s}$-module. Let $\pi_{\boldsymbol{\lambda}}$ be the idempotent of $\mathcal{S}_{R,s}$ projecting $E_{R,s}$ onto $\mathfrak{y}_{\boldsymbol{\lambda}}\mathcal{H}_{R,s}$. Then $\mathcal{H}_{R,s} \cong \pi_{\boldsymbol{\omega}}\mathcal{S}_{R,s}\pi_{\boldsymbol{\omega}}$ and so multiplication by $\pi_{\boldsymbol{\omega}}$ on the right is a functor $\Pi_{R,s}$, which takes $\mathcal{S}_{R,s}$-modules to $\mathcal{H}_{R,s}$-modules, (called the q-Schur functor). It is a special case of the constructions in [9] and this type of functor has been thoroughly investigated (see e.g [1] or for the special case of Schur algebras [30]). As endomorphism of $E_{R,s}$, an element $h \in \mathcal{H}_{r,s}$ acts by projecting first $E_{R,s}$ onto $\mathcal{H}_{R,s} = \mathfrak{y}_{\boldsymbol{\omega}}\mathcal{H}_{R,s}$, and then by left multiplication by h denoted again by h abusing notation. Thus h is identified with $h\pi_{\boldsymbol{\omega}} \in \mathcal{S}_{R,s}$. Now the Weyl module $\mathcal{W}_R^{\boldsymbol{\lambda}}$ is the right ideal of $\mathcal{S}_{R,s}$ generated by

$$\hat{z}_{\boldsymbol{\lambda}} = z_{\boldsymbol{\lambda}}\pi_{\boldsymbol{\omega}} = \mathfrak{x}_{\boldsymbol{\lambda}}T_{w_{\boldsymbol{\lambda}}}\mathfrak{y}_{\boldsymbol{\lambda}'}\pi_{\boldsymbol{\omega}}. \tag{6.31}$$

For fields R we have an irreducible $\mathcal{S}_{R,s}$-module $\mathcal{L}_R^{\boldsymbol{\lambda}}$ taking the head of the principal projective $\mathcal{S}_{R,s}$-module generated by the projection of $E_{R,s}$ onto the Young module $\mathcal{Y}_{\boldsymbol{\lambda},R}$.

We summarize some of the main results of [23]:

Theorem 6.32 *Let $s \in \mathfrak{C}_{ss}$ be ℓ-stable. Then*

(i) $\{\mathcal{Y}_{\boldsymbol{\lambda},R} \mid \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$ *is a complete set of nonisomorphic indecomposable direct summands of $E_{R,s}$.*
(ii) $\{\mathcal{W}_K^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$ *is a complete set of nonisomorphic irreducibel $\mathcal{S}_{K,s}$-modules.*
(iii) *The Weyl module $\mathcal{W}_R^{\boldsymbol{\lambda}}$ to highest weight $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ is free as R-module with basis independent of the choice of R and the parameters occurring in $\mathcal{H}_{R,s}$. It is a cyclic $\mathcal{S}_{R,s}$-module generated by a highest weight vector. Moreover, if R is a field, then $\mathcal{W}_R^{\boldsymbol{\lambda}}$ has a unique maximal ideal $J(\mathcal{W}_R^{\boldsymbol{\lambda}})$ and*

$$\mathcal{W}_R^{\boldsymbol{\lambda}}/J(\mathcal{W}_R^{\boldsymbol{\lambda}}) \cong \mathcal{L}_R^{\boldsymbol{\lambda}}.$$

Finally $\mathcal{W}_R^{\boldsymbol{\lambda}} = \mathcal{L}_R^{\boldsymbol{\lambda}}$ for all $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ if and only if $\mathcal{H}_{R,s}$ is semisimple.
(iv) *If R is a field, $\{\mathcal{L}_R^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$ is a complete set of nonisomorphic irreducible $\mathcal{S}_{R,s}$-modules.*
(v) *If R is a field, and $\mathcal{L}_R^{\boldsymbol{\mu}}$ is a composition factor of $\mathcal{W}_R^{\boldsymbol{\lambda}}$, then $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathfrak{P}_s$. Moreover the irreducible module $\mathcal{L}_R^{\boldsymbol{\lambda}}$ has multiplicity one as composition factor of $\mathcal{W}_R^{\boldsymbol{\lambda}}$.*
(vi) *The decomposition matrix of $\mathcal{S}_{s,\mathcal{O}}$, describing the multiplicity of $\mathcal{L}^{\boldsymbol{\mu}}$ as composition factor of $\mathcal{W}_R^{\boldsymbol{\lambda}}$, is $\hat{\Delta}_s$. In particular ths matrix is a lower unitriangular square matrix.*

Note that part vi) of the Theorem follows easily from Fittings lemma and Brauer reciprocity, observing the previous parts.

Remark 6.33 *Our labelling of Weyl and irreducible modules of q-Schur algebras is the one of* [11]. *In* [23] *and related papers the labelling is dual to ours here, and consequently the decomposition matrix of $\mathcal{S}_{s,\mathcal{O}}$ there is upper unitriangular.*

Note that the functor $\Pi_{R,s} : \mathrm{mod}_{\mathcal{S}_{R,s}} \to \mathrm{mod}_{\mathcal{H}_{R,s}}$ has a right inverse, namely

$$\hat{\Pi}_{R,s} = - \otimes_{\mathcal{H}_{R,s}} \pi_{\boldsymbol{\omega}} \mathcal{S}_{R,s}. \tag{6.34}$$

Moreover, as remarked before, it is a special case of our constructions in [9] and the first section, considering the identity map as projective presentation of the $\mathcal{S}_{R,s}$-module $\pi_{\boldsymbol{\omega}}\mathcal{S}_{R,s}$. In particular for fields R, the functor $\Pi_{R,s}$ maps an irreducible module for $\mathcal{S}_{R,s}$ either to the zero or to an irreducible $\mathcal{H}_{r,s}$-module, and the decomposition matrix of $\mathcal{H}_{\mathcal{O},s}$ is part of the decomposition matrix of $\mathcal{S}_{\mathcal{O},s}$ by [9, 4.10]. We have:

Theorem 6.35 *Let $\boldsymbol{\lambda} \in \mathfrak{P}_s$. Then $\Pi_{R,s}(\mathcal{W}_R^{\boldsymbol{\lambda}}) \cong S_R^{\boldsymbol{\lambda}}$. Moreover $\Pi_{R,s}(\mathcal{L}_R^{\boldsymbol{\lambda}})$ is isomorphic to $D_R^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} \in \mathfrak{P}_s^{reg}$ and (0) otherwise. Finally Δ_s can be obtained from $\hat{\Delta}_s$ by eliminating all columns from $\hat{\Delta}_s$ labelled by $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ with $\boldsymbol{\lambda} \notin \mathfrak{P}^{reg}$.*

Proof By 6.31 the Weyl module $\mathcal{W} = \mathcal{W}_R^{\boldsymbol{\lambda}}$ is generated by $\hat{\mathfrak{z}}_{\boldsymbol{\lambda},R}$, hence

$$\Pi_{R,s}(\mathcal{W}_R^{\boldsymbol{\lambda}}) = \mathcal{W}_R^{\boldsymbol{\lambda}} \cdot \pi_{\boldsymbol{\omega}} = \hat{\mathfrak{z}}_{\boldsymbol{\omega}} \mathcal{S}_{R,s} \pi_{\boldsymbol{\omega}} = \mathfrak{z}_{\boldsymbol{\lambda},R} \pi_{\boldsymbol{\omega}} \mathcal{S}_{R,s} \pi_{\boldsymbol{\omega}} = \mathfrak{z}_{\boldsymbol{\lambda},R} \mathcal{H}_{R,s} = S_R^{\boldsymbol{\lambda}}.$$

Now the theorem follows easily from the fact, that the unitriangular decomposition matrix Δ_s of $\mathcal{H}_{\mathcal{O},s}$ is by [9, 4.10] part of the decomposition matrix $\hat{\Delta}_s$ of $\mathcal{S}_{\mathcal{O},s}$, and from [9, 2.30]).

□

We apply now [9] using Theorems 6.19, 6.23, and the projective presentation $\boldsymbol{\beta}_{s,R}$: $\boldsymbol{P}_{R,s} \to N_{R,s}$ of 6.19 to get functors:

$$\begin{aligned} \boldsymbol{S}_{R,s} &: \operatorname{mod}_{RG} \to \operatorname{mod}_{\mathcal{S}_{R,s}} \\ \hat{\boldsymbol{S}}_{R,s} &: \operatorname{mod}_{\mathcal{S}_{s,R}} \to \operatorname{mod}_{RG}, \end{aligned} \tag{6.36}$$

where $R \in \{K, \mathcal{O}, F\}$. Note that $\mathcal{W}_{\boldsymbol{\lambda},\mathcal{O}} N_{\mathcal{O},s}$ is an $\mathcal{O}G$-sublattice of $N_{\mathcal{O},s}$ which might be not pure. By 1.2 and 6.18 however

$$\mathcal{W}_{\mathcal{O}}^{\boldsymbol{\lambda}} N_{\mathcal{O},s} = \hat{\boldsymbol{S}}_{\mathcal{O},s}(\mathcal{W}_{\mathcal{O}}^{\boldsymbol{\lambda}}), \tag{6.37}$$

hence $\hat{\boldsymbol{S}}_{\mathcal{O},s}(\mathcal{W}_{\mathcal{O}}^{\boldsymbol{\lambda}}) \leq N_{\mathcal{O},s}$ and the next definition makes sense.

Definition 6.38 *Let $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$. Then we define G-modules*

(i) $S_R(s, \boldsymbol{\lambda}) = R \otimes_{\mathcal{O}} \sqrt{\hat{\boldsymbol{S}}_{\mathcal{O},s}(\mathcal{W}_{\mathcal{O}}^{\boldsymbol{\lambda}})}$ *for* $R \in \{K, \mathcal{O}, F\}$.

(ii) $D(s, \boldsymbol{\lambda}) = \hat{\boldsymbol{S}}_{F,s}(\mathcal{L}^{\boldsymbol{\lambda}})$.

We have to explain why our notation here and in 5.12 and 5.11 is consistent. By construction the map $H_{R,s}$: $\operatorname{End}_{RG}(N_{R,s}) \to \operatorname{End}_{\mathcal{H}_{R,s}}(E_{R,s})$ respects projections onto direct summands. In particular, it takes the projection $\tilde{\pi}_{\boldsymbol{\omega}}$ of $N_{R,s}$ onto its direct summand $M_{R,s} = N_{\boldsymbol{\omega},R}$ to the projection $\pi_{\boldsymbol{\omega}}$ of $E_{R,s}$ onto its direct summand $\mathfrak{y}_{\boldsymbol{\omega}} \mathcal{H}_{R,s} = \mathcal{H}_{R,s}$. Omitting indices R, s we have:

Lemma 6.39 *The following diagram is commutative:*

$$\begin{tikzcd} \operatorname{mod}_{\mathcal{H}} \arrow[r, "- \otimes_{\mathcal{H}} M"] \arrow[d, "\hat{\Pi}"'] & \operatorname{mod}_{RG} \\ \operatorname{mod}_{\mathcal{S}} \arrow[ur, "- \otimes_{\mathcal{S}} N"'] & \end{tikzcd}$$

Proof We have by 6.34

$$\begin{aligned} (- \otimes_{\mathcal{S}} N) \circ \hat{\Pi} = (- \otimes_{\mathcal{S}} N) \circ (- \otimes_{\mathcal{H}} \pi_{\boldsymbol{\omega}} \mathcal{S}) &= - \otimes_{\mathcal{H}} \pi_{\boldsymbol{\omega}} \mathcal{S} \otimes_{\mathcal{S}} N \\ &= - \otimes_{\mathcal{H}} M, \end{aligned} \tag{6.40}$$

since $\pi_{\boldsymbol{\omega}} \mathcal{S} \otimes_{\mathcal{S}} N = \tilde{\pi}_{\boldsymbol{\omega}} N = M$. □

Further relations between the functors Π, H, $\boldsymbol{S}$ and their various right inverses will be investigated elsewhere. We shall justify now our notation for the irreducible G-modules using the same labels for images under the functors $\hat{\boldsymbol{S}}$ and $\hat{H}$:

Theorem 6.41 *Let $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$. Then*

$$\hat{H}_{R,s}(S_R^{\boldsymbol{\lambda}}) = \hat{\boldsymbol{S}}_{R,s}(\mathcal{W}_R^{\boldsymbol{\lambda}}) = S(s,\boldsymbol{\lambda}),$$

and if in addition $\boldsymbol{\lambda} \in \mathfrak{P}_s^{reg}$, then

$$\hat{H}_{F,s}(D_F^{\boldsymbol{\lambda}}) = \hat{\boldsymbol{S}}_{F,s}(\mathcal{L}_F^{\boldsymbol{\lambda}}) = D(s,\boldsymbol{\lambda}).$$

If $\boldsymbol{\lambda} \notin \mathfrak{P}_s^{reg}$, then $\Pi(\mathcal{L}_F^{\boldsymbol{\lambda}}) = (0)$.

Proof We apply Theorems 6.18, 5.3 and 1.2 to obtain

$$\begin{aligned}\hat{\boldsymbol{S}}_{R,s}(\mathcal{W}_R^{\boldsymbol{\lambda}}) &\cong \mathcal{W}_R^{\boldsymbol{\lambda}}\pi_{\boldsymbol{\omega}} \cdot N_{R,s} = \mathcal{W}_R^{\boldsymbol{\lambda}}\pi_{\boldsymbol{\omega}}\pi_{\boldsymbol{\omega}} \cdot N_{R,s} \cong \mathcal{W}_R^{\boldsymbol{\lambda}}\pi_{\boldsymbol{\omega}} \cdot \tilde{\pi}_{\boldsymbol{\omega}} N_{R,s} \\ &\cong \mathcal{W}_R^{\boldsymbol{\lambda}}\pi_{\boldsymbol{\omega}} \cdot M_{R,s} \cong S_R^{\boldsymbol{\lambda}} \cdot M_{R,s} \cong \hat{H}_{R,s}(S_R^{\boldsymbol{\lambda}}) = S(s,\boldsymbol{\lambda}),\end{aligned}$$

since $\mathcal{W}_R^{\boldsymbol{\lambda}}\pi_{\boldsymbol{\omega}} = S_R^{\boldsymbol{\lambda}} = S(s,\boldsymbol{\lambda})$ by Theorem 6.35. A similar argument applies for the FG-modules $D(s,\boldsymbol{\lambda})$. □

We choose now $s \in \mathfrak{C}_{ss}^{\ell}$, thus s is semisimple of order prime to ℓ, and let $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$. Since s is ℓ-regular, the associated (n,∞)-index 4.2

$$(6.42) \qquad (s,\boldsymbol{\lambda}) \leftrightarrow \quad I_\ell(s,\boldsymbol{\lambda}) = \left(\begin{array}{ccc|ccc} d_1 & d_2 & \dots & n_1 & n_2 & \dots \\ s_1 & s_2 & \dots & \lambda^{(1)} & \lambda^{(2)} & \dots \end{array}\right)$$

is a head (n,ℓ)-index by Definition 5.18, where again $s_i \in GF(q^{d_i})$ is determined by 4.3, $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots)$, and $\boldsymbol{n}_s = (n_1, n_2, \dots)$ Moreover d_i is the degree of the minimum polynomial of s_i over $GF(q)$. Thus we have a bijection between the set $\{(s,\boldsymbol{\lambda}) \mid s \in \mathfrak{C}_{ss}^{\ell}, \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$ and the set of head (n,ℓ)-indices. Recall that the latter parametrizes the ℓ-regular conjugacy classes by [21, 2.8]. Observe further that $D(s,\boldsymbol{\lambda}) \not\cong D(t,\boldsymbol{\mu})$ for $s,t \in \mathfrak{C}_{ss}^{\ell}$, $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ and $\boldsymbol{\mu} \vdash \boldsymbol{n}_t$ with $(s,\boldsymbol{\lambda}) \neq (t,\boldsymbol{\mu})$. This follows from 4.16 for $s \neq t$ and from [9, 2.28] if $s = t$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. Thus the number of elements in $\{D(s,\boldsymbol{\lambda}) \mid s \in \mathfrak{C}_{ss}^{\ell}, \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$ is precisely the number of ℓ-regular conjugacy classes of G, and we have shown (comp. [22, 4.9]):

Theorem 6.43 *The set*

$$\{D(s,\boldsymbol{\lambda}) \mid s \in \mathfrak{C}_{ss}^{\ell}, \boldsymbol{\lambda} \vdash \boldsymbol{n}_s\}$$

is a complete set of non-isomorphic irreducible FG-modules. The FG-module $S_F(s,\boldsymbol{\mu})$ for $s \in \mathfrak{C}_{ss}^{\ell}$ and $\boldsymbol{\mu} \vdash \boldsymbol{n}_s$ is a reduction modulo ℓ of the irreducible KG-module $S_K(s,\boldsymbol{\mu})$ and every composition factor of $S_F(s,\boldsymbol{\mu})$ is of the form $D(s,\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$. The matrix $\hat{\Delta}_s$ defined in 6.30 is the part of the ℓ-modular decomposition matrix of G which describes the composition factors in reductions modulo ℓ of irreducible KG-modules in the geometric conjugacy class $\mathcal{E}(s)$.

As an immediate consequence in view of Result 4.16 we have:

Corollary 6.44 *The ordinary and ℓ-modular irreducible representations of G can be ordered such that the ℓ-modular decomposition matrix $\hat{\Delta}$ of G is lower unitriangular. Here the lower unitriangular matrices $\hat{\Delta}_s$ with $s \in \mathfrak{C}_{ss}^{\ell}$ are positioned down the main diagonal of $\hat{\Delta}$.*

As remarked in 5.22 there is a bijection between the head and special foot (n,ℓ)-indices given by an $\boldsymbol{e}$-ℓ-adic decomposition of multipartitions. It was shown in [21] that the corresponding irreducible FG-modules for corresponding indices coincide. More percisely, if $s \in \mathfrak{C}_{ss}^{\ell}$ and $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ and if $I_\ell(s,\boldsymbol{\lambda})$ is the corresponding head (n,ℓ)-index as defined in 6.42, we find $y \in C_G(s)_\ell$ (constructed by the procedure in Definition 4.36) such that the following holds: $sy \in \mathfrak{S}\mathfrak{T}_\ell$ and there is $\hat{\boldsymbol{\lambda}} \in \mathfrak{P}_{sy}^{\mathrm{reg}}$, given by the $\boldsymbol{e}$-ℓ-adic decomposition of $\boldsymbol{\lambda}$, such that

$$I_\ell(s,\boldsymbol{\lambda}) \mapsto I_\ell(sy,\hat{\boldsymbol{\lambda}}) \tag{6.45}$$

is a bijection between the set of head and the set of special foot (n,ℓ)-indices. Here the special foot (n,ℓ)-index $I_\ell(sy,\hat{\boldsymbol{\lambda}})$ is defined as in 5.17. The following result was shown in [21, 5.1]:

Theorem 6.46 *Let the (n,ℓ)-indices $I_\ell(s,\boldsymbol{\lambda})$ and $I_\ell(sy,\hat{\boldsymbol{\lambda}})$ be defined as in* 6.45. *Then*

$$D(s,\boldsymbol{\lambda}) \cong D(sy,\hat{\boldsymbol{\lambda}}).$$

In the representation theory of algebraic groups in positive characteristic Steinberg's famous tensor product theorem [44] decomposes irreducible rational representations into twisted tensor products. On the level of highest weights, Theorem 6.46 looks precisely like an analogous result. In [14] it was shown, that behind this is indeed a tensor decomposition for the involved FG-modules.

We can now describe the Harish-Chandra series of the irreducible FG-modules in more detail. For $s \in \mathfrak{C}_{ss}^{\ell}$ and $\boldsymbol{\lambda} \vdash \boldsymbol{n}_s$ we determine $y \in C_G(s)_\ell$ as above. Let

$$sy = (t_1)^{\times k_1} \times (t_2)^{\times k_2} \times \cdots , \tag{6.47}$$

where t_i is of degree l_i over $GF(q)$, then

$$L_t = \underbrace{GL_{l_1}(q) \times \cdots \times GL_{l_1}(q)}_{k_1 factors} \times GL_{l_2}(q) \cdots \tag{6.48}$$

by Formula 4.8 and L_t is the semisimple HC-vertex of $D(s,\boldsymbol{\lambda})$ by Theorem 5.21. The outer tensor product

$$\underbrace{D(t_1,(1)) \otimes \cdots D(t_1,(1))}_{k_1 factors} \otimes D(t_2,(1)) \otimes \cdots \tag{6.49}$$

is a semisimple HC-source of $D(s,\boldsymbol{\lambda})$ (replacing (t_i) in indices by t_i). Note that $D(t_i,(1)) = C_{F,t_i}$. Moreover, applying 6.46 once more we have $D(t_i,(1)) = D(s_i,(1^{l_i}))$, where s_i is the ℓ-regular part of t_i (which coincides with one of the parts s_j in Formula 4.3 for s). Thus the semisimple HC-source of $D(s,\boldsymbol{\lambda})$ can also be described as outer tensor product

$$\underbrace{D(s_1,(1^{l_1})) \otimes \cdots \otimes D(s_1,(1^{l_1}))}_{k_1 factors} \otimes D(s_2,(1^{l_2})) \otimes \cdots . \tag{6.50}$$

In Theorems 5.14 and 6.43 we obtained also results on the ℓ-modular decomposition matrix of G. Indeed the information we have so far suffices to calculate all decomposition numbers of G from those of q-Schur algebras by an purley combinatorial algorithm. This was shown in [21] and [22]. For this, however, quotients of Hom-functors do not contribute anything new, hence we present here only a rough sketch of some ideas connected with Hom-functors and refer for details and proofs to the papers mentioned above. First one observes that the statement on decomposition numbers in Theorem 6.43 may be extended to all ℓ-stable elements s in $\mathfrak{C}_{ss}$ by [9, 4.10]:

Theorem 6.51 *Let $s \in \mathfrak{S}\mathfrak{T}_\ell$. Then $\hat{\Delta}_s$ describes the composition factors of the FG-modules $S_F(s, \boldsymbol{\lambda})$ of the form $D(s, \boldsymbol{\mu})$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash \boldsymbol{n}_s$.*

Theorem 6.46 matches up (again by [21]) part the columns of $\hat{\Delta}_{s'}$ and of $\hat{\Delta}_s$. However, in general, $\hat{\Delta}_{s'}$ has more columns than $\hat{\Delta}_s$, and hence there may be composition factors in $S_F(s, \boldsymbol{\lambda})$ which are not of the form $D(s, \boldsymbol{\mu})$, where $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash \boldsymbol{n}_s$. However, it was shown in [40] that this is not the case under the additional hypothesis, that s has an irreducible minimum polynomial, or equivalently that $s = (s_1)^{\times d_1}$ and $d_1 n_1 = n$. From this one derives the general case by an algorithm which involves only the Richardson Littlewood rule and inverting some decomposition matrices of q-Schur algebras. This takes care of all geometric conjugacy classes $\mathcal{E}(s)$ for $s \in \mathfrak{S}\mathfrak{T}_\ell$.

Thus let $s \in \mathfrak{C}_{ss} \backslash \mathfrak{S}\mathfrak{T}_\ell$. Let $\tilde{s}$ be the unique element in $\mathfrak{S}\mathfrak{T}_\ell$ which is ℓ-equivalent to s which exists by Lemma 4.37. Note that $s' = \tilde{s}'$ by construction. Then we apply Lemma 4.29 to conclude that $F \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}G}(N_{s,\mathcal{O}})$ is a parabolic subalgebra of $\mathrm{End}_{FG}(N_{F,s}) \cong \mathrm{End}_{FG}(N_{F,\tilde{s}'})$, extending the constructions of this section to elements $s \in \mathfrak{C}_{ss}$ which are not ℓ-stable. The Littlewood Richardson rule tells us, how Weyl modules decompose after inducing them to larger q-Schur algebras, and this corresponds to taking corresponding linear combinations in the decomposition matrix of $\mathcal{E}(\tilde{s}')$ (of which $\hat{\Delta}_{\tilde{s}'}$ is a part) to get the decomposition matrix of $\mathcal{E}(s)$ yielding thus all decomposition numbers of $\mathcal{O}G$ in terms of the matrices $\hat{\Delta}_s$.

Recall Result 4.16 about the blocks of G. I particular for $s \in \mathfrak{C}^\ell_{ss}$ we introduced the notation $\mathcal{E}_\ell(s)$ to be the sum of blocks with semisimple part s. Since we vary G now we add a supperscript G, that is we write $\mathcal{E}^G_\ell(s) = \mathcal{E}_\ell(s)$. Blocks whose semisimple part is the identity element of G are called *unipotent*. Here is another application of our pervious combinatorial results, which were first observed in [22, 6.2] and yields a *Jordan decomposition* of blocks and decomposition numbers of G (which is for arbitrary finite reductive groups still conjectural, see e.g. [17, 4.5]).

Corollary 6.52 *Let $s \in \mathfrak{C}^\ell_{ss}$. Then the decomposition matrix of the sum $\mathcal{E}^G_\ell(s)$ equals the decomposition matrix of the sum $\mathcal{E}^{C_G(s)}_\ell(1)$ of unipotent blocks o the centralizer $C_G(s)$ of s in G.*

In fact it is conjectured that the block ideals of $\mathcal{E}^G_\ell(s)$ and $\mathcal{E}^{C_G(s)}_\ell(1)$ are Morita equivalent $\mathcal{O}$-algebras.

Finally we want to apply the results of section 3 to generalize Takeuchi's result [45]. Let $s \in \mathfrak{S}\mathfrak{T}_\ell$. In 3.4 applied to 4.27 we take as right ideals of $\mathcal{H}_{\mathcal{O}} = \mathcal{H}_{\mathcal{O},s}$ the ideals $\mathfrak{y}_{\boldsymbol{\lambda}} \mathcal{H}_{\mathcal{O},s}$, and these are pure right ideals by Lemma 6.20. The corresponding $\mathcal{O}G$-lattices are the modules $N_{\boldsymbol{\lambda},\mathcal{O}}$, so X_R of 3.4 becomes $E_{R,s}$ and N_R there is replaced by $N_{R,s}$ here. Furthermore the idempotent f of 3.4 is the idempotent $\boldsymbol{f}_s$ defined in Formula 6.15. Now Formulas 6.10, 6.16 and Theorem 6.11 imply that all assertions of 3.4 are satisfied. Moreover by Theorem 5.2 $\mathcal{H}_{F,s}$ is a symmetric hence self injective algebra. Thus we may apply Theorem 3.8 and Corollary 3.20 to get the following theorem, of which Takeuchi's result [45] is a special case:

Theorem 6.53 *Let $s \in \mathfrak{S}\mathfrak{T}_\ell$, and let $\boldsymbol{f}_s \in \mathcal{O}G$, $E_{R,s}$, $M_{R,s}$, and $N_{R,s}$ for $R \in \{K, \mathcal{O}, F\}$ be defined as above. Then the following holds:*

(i) $M_{R,s}\boldsymbol{f}_s \cong \bigoplus_{\boldsymbol{\lambda} \vdash \boldsymbol{n}_s} \mathcal{H}_{R,s}\mathfrak{y}_{\boldsymbol{\lambda}}$ *as left $\mathcal{H}_{R,s}$-module.*
(ii) $\mathrm{End}_{\mathcal{H}_{R,s}}(M_{R,s}\boldsymbol{f}_s) \cong \mathcal{S}_{R,s}$ *canonically.*
(iii) *The ring $\boldsymbol{f}_sRG\boldsymbol{f}_s$ acts on $M_{R,s}\boldsymbol{f}_s$ by right multiplication as $\mathcal{H}_{R,s}$-endomorphisms such that the canonical map $\boldsymbol{f}_sRG\boldsymbol{f}_s \to \mathrm{End}_{\mathcal{H}_{R,s}}(M_{R,s}\boldsymbol{f}_s)$ is surjective.*
(iv) *The $\boldsymbol{f}_sRG\boldsymbol{f}_s$-module $M_{R,s}\boldsymbol{f}_s$ has the double centralizer property. Thus in particular* $\mathrm{End}_{\boldsymbol{f}_sRG\boldsymbol{f}_s}(M_{R,s}\boldsymbol{f}_s) \cong \mathcal{H}_{R,s}$.

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