Representations of Hecke algebras and finite groups of Lie type

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1 Introduction

The aim of this article is to lead the reader on a journey through the representation theory of finite groups of Lie type and Hecke algebras. We will present some basic results obtained in recent years, explain the ideas behind them, and give lots of examples; proofs are usually omitted but we provide explicit references to an extensive bibliography.

Let G be a finite group of Lie type. Examples are $G = \mathrm{GL}_n(q)$, where q is a prime power, or any finite classical group. Hecke algebras arise naturally as endomorphism algebras of certain induced modules for G. This shows their importance for studying representations of G. By a result of Iwahori [80], they can be obtained as deformations of group algebras of finite Coxeter groups. Here, the standard example is the symmetric group $\mathfrak{S}_n$.

The group G is defined over a finite field, whose characteristic we call the defining characteristic of G. The case of representations over an algebraically closed field of characteristic 0 is relatively well-understood. The irreducible representations are classified by Lusztig [93], and Lusztig's subsequent work on character sheaves has provided the framework in which the problem of computing the corresponding character values can be attacked; see Shoji [115].

Therefore, the emphasis in this article will be on modular representations over a field of positive characteristic, which is different from the defining characteristic of G. The overall aim of the theory, as we present it here, is to find a classification of the simple modules and to determine properties of the simple modules such as their degrees, their Brauer characters, projective covers etc.

On our journey, we will see our group G comes along with a number of rich additional structures (like 'generic' objects and 'Hom-functors'). It is the interplay between these structures which is responsible for most of the progress made in recent years. It should also be pointed out that explicit computations played an important role in 'guessing' the correct form of a result or providing hints for further research.

Let us now describe in somewhat more detail the contents of this article.

Chapter 2 introduces two basic principles for studying modular representations of G: Harish-Chandra philosophy and Brauer's theory of blocks. The Harish-Chandra philosophy, i.e., the idea of inducing representations from subgroups and studying their endomorphism algebras, has been successfully applied to the classification problem. It may look surprising that, as far as the formulation of the statements is concerned, this theory bears many similarities with that in characteristic 0, but a number of new features do arise. The main results are stated in Corollary 2.4 (classification of irreducible representations of G by certain triples) and Theorem 2.9 (structure of endomorphism algebras). The unknown quantities in this set-up are the representations which are not covered by the induction process; these are the *cuspidal representations*. We do not yet have a satisfactory theory for them; however, explicit results have been obtained in various examples, see Section 4.7.

A basic part of Brauer's theory is the idea of relating modular representations with those in characteristic 0 via the process of modular reduction. In this way it is possible to define a partition of the representations in characteristic 0 into blocks. In the case of groups of Lie type the right approach to this question seems to be via generalized Harish-Chandra theories of generic groups. We describe this set-up in Section 2.2 and then state the main results in Section 2.3.

Chapter 3 deals with Hecke algebras. Recall that these algebras arise as endomorphism algebras of induced cuspidal representations. Up to possible twists and extensions by automorphisms, they can also be described abstractly as deformations of group algebras of finite Weyl groups. The connections between the representations of these 'Iwahori-Hecke algebras' and those of G will be explained in Section 3.1 and, in a refined form, in Section 3.3. The remaining sections give a survey on the known results about Iwahori-Hecke algebras. Conjecture 3.12 formulates in precise terms a conjectural 'generic' behavior of modular representations of Iwahori-Hecke algebras (originally due to James in type A). The most striking recent result is Ariki's solution of the Lascoux-Leclerc-Thibon conjecture, which we explain in Section 3.6.

In Chapter 4, we come back to Brauer's theory of modular representations. Note that knowing the characters of the modular representations of G is equivalent to knowing the characters in characteristic 0 plus the decomposition matrix determined by the process of modular reduction. Thus, the determination of decomposition matrices (or, at least, certain properties of them) is of central importance. All the known examples indicate that the decomposition matrices we are looking at (i.e., the non-defining characteristic case) have a lower uni-triangular shape. The results in this chapter can be seen to support this observation. We introduce the notion of basic sets in Section 4.1. In Section 4.2 we formulate some general conjectures, in particular on a Jordan decomposition of Brauer characters. The remaining sections give a survey on the present knowledge about ℓ-modular decomposition matrices of groups of Lie type, where ℓ is different from the defining characteristic. The general linear groups are considered in Section 4.3, without any restriction on ℓ. Section 4.4 treats the other classical groups under some restrictions on ℓ. We give a short survey on the exceptional groups in Section 4.6. Our final Section 4.7 presents the known results on the distribution of the irreducible Brauer characters into Harish-Chandra series.

2 Harish-Chandra philosophy

2.1 Finite groups with BN-pairs

Many important results of Harish-Chandra theory can be formulated and proved within the framework of groups with split BN-pairs. Thus let G be a finite group with a split BN-pair of characteristic p, satisfying the commutator relations (cf. [16, §§ 65,69]). Without further notice we shall use the letters B and N to denote the Borel subgroup and the monomial subgroup of G, respectively. We also fix an integral domain R and work in the category **mod**-RG of finitely generated right RG-modules.

Harish-Chandra induction and restriction. We begin by introducing the functors *Harish-Chandra induction*, and *Harish-Chandra restriction.* There is one pair of such functors for each Levi subgroup of G.

Let L be a Levi subgroup of G and P a parabolic subgroup with Levi complement L, i.e., $P = UL$, where U is the largest normal p-subgroup of P, and $U \cap L = \{1\}$. Then the functor

$$R_{L,P}^{G} : \mathbf{mod}\text{-}RL \to \mathbf{mod}\text{-}RG, \quad M \mapsto \mathrm{Ind}_P^G(\mathrm{Infl}_L^P(M)),$$

is called *Harish-Chandra induction (from L to G)*, and the functor

$$^{*}R_{L,P}^{G} : \mathbf{mod}\text{-}RG \to \mathbf{mod}\text{-}RL, \quad M \mapsto \mathrm{Inv}_U(M),$$

is called *Harish-Chandra restriction (from G to L).* Here, Ind_P^G is the usual induction functor from P to G, and Infl_L^P denotes inflation from L to P via the epimorphism $P \to L$. Furthermore, $\mathrm{Inv}_U(M)$ denotes the set of U-invariants of the RG-module M, considered as RL-module.

Harish-Chandra restriction from G to L is right adjoint, and, in case p is invertible in R, also left adjoint, to Harish-Chandra induction from L to G. The collection of functors R_L^G and their adjoints satisfy a transitivity property and a Mackey formula. For more details we refer to Dipper and Fleischmann [24], where these functors are investigated in a much more general context.

The following result is of fundamental importance in Harish-Chandra theory.

Theorem 2.1. (Deligne, Dipper/Du, Howlett/Lehrer) *Suppose that p is invertible in R. Let P and Q be two parabolic subgroups of G such that L is a Levi complement in P and in Q. Then the two functors $R_{L,P}^G$ and $R_{L,Q}^G$ are naturally isomorphic. The same is then true for their adjoint functors ${}^*R_{L,P}^G$ and ${}^*R_{L,Q}^G$.*

A proof of this theorem in the special case that R is a field of characteristic 0, was given by Deligne (see Lusztig and Spaltenstein [95]). In the general case it was proved independently by Dipper and Du [22] and by Howlett and Lehrer [79].

In the applications we have in mind, the assumption of Theorem 2.1 is always satisfied. In these cases we may, and will, omit the subscript P from the notation for Harish-Chandra induction and restriction.

Let us consider a simple example which shows that the assumption on the characteristic of R cannot be omitted. Let $G = \mathrm{SL}_2(p)$, where p is odd. For B and N we take the upper triangular matrices and the monomial matrices in G, respectively. Let $R = k$ denote the prime field of characteristic p. Then the group L of diagonal matrices in G is isomorphic to the multiplicative group of k. Let M denote the 1-dimensional kL-module affording this k-representation of L. Then if B' denotes the subgroup of lower triangular matrices in G, the two kG-modules $R_{L,B}^G(M) = \mathrm{Ind}_B^G(\mathrm{Infl}_L^B(M))$ and $R_{L,B'}^G(M) = \mathrm{Ind}_{B'}^G(\mathrm{Infl}_L^{B'}(M))$ are not isomorphic. To see this, observe that $B' = s^{-1}Bs$ where s is an element in $N \setminus L$. Hence $\mathrm{Ind}_{B'}^G(\mathrm{Infl}_L^{B'}(M)) \cong \mathrm{Ind}_B^G(\mathrm{Infl}_L^B(M^s))$. But $\mathrm{Infl}_L^B(M^s)$ is not isomorphic to $\mathrm{Infl}_L^B(M)$ and so, by Green correspondence, the induced modules are not isomorphic either.

Definition 2.2. Suppose that p is invertible in R. An RG-module M is called *cuspidal*, if ${}^*R_L^G(M) = \{0\}$ for all Levi subgroups L properly contained in G.

The notion of cuspidal modules was introduced by Harish-Chandra in [68, Section 3] in the case that R is a field of characteristic 0, and by Dipper and Fleischmann in general [24]. We remark that this notion only makes sense if p is invertible in R. For example, if $R = k$ is a field of characteristic p, then the unipotent radical of every parabolic subgroup of G has non-trivial fixed points on every non-zero kG-module M, in other words ${}^*R_{L,P}^G(M) \neq \{0\}$ for all Levi subgroups L and all parabolic subgroups P with Levi complement L.

To illustrate the importance of the concept of cuspidal modules let us assume for the moment that $R = k$ is a field of characteristic $\ell \neq p$. If M is a kG-module, let $P(M)$ denote its projective cover. Then a simple kG-module M is not cuspidal if and only if there is a proper Levi subgroup L of G and a projective kL-module Q such that $P(M)$ is a direct summand of $R_L^G(Q)$.

Harish-Chandra series. In this section we let k denote a field of characteristic distinct from p. The following theorem gives some structural information about modules which are Harish-Chandra induced from simple cuspidal modules.

Theorem 2.3. (Geck/Hiss/Malle, Geck/Hiss) *Let L be a Levi subgroup of G. Suppose that X is a simple cuspidal kL-module, and that Y is an indecomposable direct summand of $R_L^G(X)$. Then*

(a) *Y has a unique simple quotient and a unique simple submodule which are isomorphic to each other.*

If L' is another Levi subgroup, X' a cuspidal simple kL'-module and Y' an indecomposable direct summand of $R_{L'}^G(X')$, then

(b) *Y is isomorphic to Y' if and only if the simple quotient of Y is isomorphic to the simple quotient of Y'.*

(c) *If Y and Y' are isomorphic, then the pairs (L, X) and (L', X') are conjugate in G.*

Statement (c) follows from [72, Theorem 5.8]. Proofs of (a) (without the statement on the socles) and (b) are given in [54, Theorem 2.4], based on Dipper's theory of Hom-functors [19]. The full version of (a) and an alternative proof of (b) is established in [51, Theorem 2.9], on the basis of the arguments of Sawada, Green and Cabanes, see [12]. Dipper has now shown that the above theorem holds in the general set-up of Hom-functors, see [20, Theorem 1.8, Lemma 2.2].

As a corollary we obtain a classification of the simple kG-modules.

Corollary 2.4. (Geck/Hiss/Malle [54, Theorem 2.4]) *There is a bijection between the set of isomorphism classes of simple kG-modules and the set of equivalence classes of triples (L, X, ψ), where L is a Levi subgroup of G, X is a simple cuspidal kL-module and ψ a simple $\mathrm{End}_{kG}(R_L^G(X))$-module (where the pair (L, X) has to be taken modulo conjugation in G and ψ modulo isomorphism).*

This classification allows to partition the set of isomorphism classes of simple kG-modules.

Definition 2.5. (Dipper/Du [22], Hiss [72]) Let M be a simple kG-module labelled, via Corollary 2.4, by the triple (L, X, ψ). Then L and X are called *semisimple vertex* and *semisimple source of M*, respectively.

The *Harish-Chandra series* corresponding to L and X is the set of isomorphism classes of simple kG-modules labelled by the triples (L, X, ψ), where ψ runs through the simple $\mathrm{End}_{kG}(R_L^G(X))$-modules. Thus two simple kG-modules are in the same Harish-Chandra series if and only if they have a common semisimple vertex and a common semisimple source.

In Section 4.7 we shall give examples in classical groups where the distribution of the simple kG-modules into Harish-Chandra series is completely known.

There are alternative characterizations of Harish-Chandra series.

Remark 2.6. (Dipper/Du [22], Hiss [72]) (a) Let $\mathcal{K}$ denote the set of pairs

$$\{(L, X) \mid L \text{ Levi subgroup of } G, X \text{ a simple } kL\text{-module (up to isomorphism)}\}.$$

Define a partial order on $\mathcal{K}$ by $(L, X) \leq (L', X')$ if and only if $L \leq L'$ and X is isomorphic to a composition factor of ${}^*R_L^{L'}(X')$.

Then a pair $(L, X) \in \mathcal{K}$ is minimal with respect to this partial ordering, if and only if X is a cuspidal simple kL-module. In this case, the Harish-Chandra series corresponding to (L, X) is the set of simple kG-modules M such that $(L, X) \leq (G, M)$.

If M is a simple kG-module, there is a unique (up to conjugacy and isomorphism) minimal pair (L, X) with $(L, X) \leq (G, M)$.

(b) Let (L, X) be a minimal pair in $\mathcal{K}$. Then the set of isomorphism classes of simple kG-modules in the head of $R_L^G(X)$ equals the set of isomorphism classes of simple kG-modules in the socle of $R_L^G(X)$. This set of simple kG-modules equals the Harish-Chandra series corresponding to (L, X).

(c) Let (L, X) be a minimal pair in $\mathcal{K}$ and let M be a simple kG-module not lying in the Harish-Chandra series corresponding to (L, X).

If M is a composition factor of $R_L^G(X)$, the semisimple vertex of M' strictly contains L. If $P(M)$ is a direct summand of $R_L^G(P(X))$, then the semisimple vertex of M is strictly contained in L.

It may in particular happen that, in the notation of the above remark, some composition factor of $R_L^G(X)$ is cuspidal, even is L is strictly contained in G. Simple kG-modules which do not arise in this way are called supercuspidal.

Definition 2.7. (Vigneras [117]) A simple kG-module is called *supercuspidal* if it is not contained as a composition factor in a kG-module which is Harish-Chandra induced from a proper Levi subgroup.

Remark 2.8. (a) Let M be a simple kG-module. Then M is supercuspidal if and only if $P(M)$ is cuspidal. (This follows from the adjointness of Harish-Chandra induction and restriction, see [73].)

(b) We may also order the set $\mathcal{K}$ of Remark 2.6(a) as follows. Define $(L, X) \leq' (L, X')$ if and only if $L \leq L'$ and X' is isomorphic to a composition factor of $R_L^{L'}(X)$. Then a pair $(L, X) \in \mathcal{K}$ is minimal with respect to $\leq'$, if and only if X is a supercuspidal simple kL-module.

Endomorphism algebras. Again let $R = k$ be a field of characteristic not dividing p. We next turn to the question of finding the structure of the endomorphism algebras $\mathrm{End}_{kG}(R_L^G(X))$, where L is a Levi subgroup of G and X is a simple cuspidal kL-module. We put $\mathcal{H}(L, X) := \mathrm{End}_{kG}(R_L^G(X))$.

Let $W(L) := (N_G(L) \cap N)L/L$ denote the relative Weyl group of L and let $W(L, X)$ be the stabilizer of X in $W(L)$. Then, by a result of Dipper and Fleischmann [24], the dimension of $\mathcal{H}(L, X)$ equals the order of $W(L, X)$.

Theorem 2.9. (Howlett/Lehrer, Geck/Hiss/Malle) *The group $W(L, X)$ has a semidirect product decomposition*

$$W(L, X) = R(L, X)C(L, X)$$

with normal subgroup $R(L,X)$ and complement $C(L,X)$ such that $R(L,X)$ is a Coxeter group and the following hold:

(a) *$\mathcal{H}(L,X)$ is a $C(L,X)$ graded k-algebra, i.e., there are subspaces $\mathcal{H}_x$ of $\mathcal{H}(L,X)$, $x \in C(L,X)$, such that*

$$\mathcal{H}(L,X) = \bigoplus_{x \in C(L,X)} \mathcal{H}_x,$$

$\mathcal{H}_1$ is a subalgebra of $\mathcal{H}(L,X)$ and $\mathcal{H}_x\mathcal{H}_y = \mathcal{H}_{xy}$ for all $x,y \in C(L,X)$.

(b) *$\mathcal{H}_1$ is an Iwahori-Hecke algebra associated to the Coxeter group $R(L,X)$, with multiplication possibly twisted by a 2-cocycle.*

(c) *$\mathcal{H}(L,X)$ is a symmetric k-algebra.*

For the definition of an Iwahori-Hecke algebra associated to a Coxeter system, we refer the reader to Section 3.2.

Special cases of Theorem 2.9 for $\mathrm{char}(k) = 0$ were proved by Iwahori, Harish-Chandra, Lusztig and Kilmoyer. The general case was settled by Howlett and Lehrer in characteristic 0 [78] and, building on their work, by Geck, Hiss and Malle [54, Section 3] in positive characteristic. The fact that $\mathcal{H}(L,X)$ is symmetric was proved in [51].

If $R_L^G(X)$ has an indecomposable direct summand with multiplicity 1, or if X can be extended to its inertia subgroup in $(N_G(L) \cap N)L$, then the 2-cocycle appearing in Theorem 2.9(c) is trivial (see [54, Corollary 3.13]). In this case we call $\mathcal{H}(L,X)$ an extended Iwahori-Hecke algebra associated to $W(L,X)$.

In Section 4.7 we shall give some examples for the types and parameters of Iwahori-Hecke algebras occurring in groups of Lie type as endomorphism algebras of Harish-Chandra induced cuspidal modules.

2.2 Generic finite reductive groups

In the case of finite groups of Lie type the Harish-Chandra theory introduced in the previous section has a surprising generalization to a whole family of such theories, at least in the case where the ground field has characteristic 0. These provide a powerful tool to determine the distribution into blocks of the irreducible characters. The most convenient framework for explaining the theory and to prove results is that of generic groups. This notion captures much of the generic behavior of finite reductive groups in a series.

Generalized Harish-Chandra theories for groups of Lie type. Let $\mathbf{G}$ be a connected reductive algebraic group over the algebraic closure of a finite field of characteristic p. Assume that $\mathbf{G}$ is already defined over a finite field and let $F : \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius morphism. The finite group of Lie type $\mathbf{G}^F$ then has a split BN-pair of characteristic p satisfying the commutator relations. Thus the Harish-Chandra theory described in Section 2.1 applies to the ordinary and modular representations of $\mathbf{G}^F$. In the particular case considered here the Harish-Chandra theory for ordinary characters can be regarded as a special case of a whole family of generalized Harish-Chandra theories, as will be described in this section.

Let $\mathbf{T} \leq \mathbf{G}$ be an F-stable maximal torus of $\mathbf{G}$. Let X, Y be the character and the cocharacter group of $\mathbf{T}$ with respect to some Borel subgroup, $R \subset X$ the set of roots and $R^\vee \subset Y$ the set of coroots. Then F acts as $q\phi$ on $Y_\mathbb{R} := Y \otimes_\mathbb{Z} \mathbb{R}$, where q is a power of p and ϕ is an automorphism of finite order. A different choice of Borel subgroup containing $\mathbf{T}$ changes ϕ by an element of the Weyl group W of $\mathbf{G}$ with respect to $\mathbf{T}$. Hence ϕ is uniquely determined as automorphism of $Y_\mathbb{R}$ up to an element of W. Thus to $(\mathbf{G}, \mathbf{T}, F)$ is naturally associated the quintuple

$$(X, R, Y, R^\vee, W\phi)\,.$$

Here, X, Y are free $\mathbb{Z}$-modules of equal finite rank, endowed with a duality $X \times Y \to \mathbb{Z}$, $(x, y) \mapsto \langle x, y\rangle$, $R \subset X$ and $R^\vee \subset Y$ are root systems with a bijection $R \to R^\vee$, $\alpha \mapsto \alpha^\vee$, such that $\langle \alpha, \alpha^\vee\rangle = 2$. Moreover, W is the Weyl group of the root system $R^\vee$ in $Y_\mathbb{R}$ and ϕ is an automorphism of Y of finite order stabilizing $R^\vee$. A quintuple $(X, R, Y, R^\vee, W\phi)$ satisfying these properties is called a *generic finite reductive group*, or *generic group* for short.

Let $\mathbb{G} = (X, R, Y, R^\vee, W\phi)$ be a generic finite reductive group. Then for any choice of a prime number p, $\mathbb{G}$ determines a pair $(\mathbf{G}, \mathbf{T})$ as above, up to inner automorphisms of $\mathbf{G}$ induced by $\mathbf{T}$. Moreover, the additional choice of a power q of p determines a triple $(\mathbf{G}, \mathbf{T}, F)$ as above. In this way $\mathbb{G}$ gives rise to a whole series of groups $\mathbb{G}(q) := \mathbf{G}^F$ of groups of Lie type.

A *generic Levi subgroup* of $\mathbb{G} = (X, R, Y, R^\vee, W_\mathbb{G}\phi)$ is by definition a generic finite reductive group of the form $\mathbb{L} = (X, R', Y, R'^\vee, W_\mathbb{L} w\phi)$, where $R'^\vee$ is a $w\phi$-stable parabolic subsystem of $R^\vee$ with Weyl group $W_\mathbb{L}$. A *generic torus* is a generic finite reductive group with $R = R^\vee = \emptyset$. Thus a generic torus of $\mathbb{G}$ has the form $(X', Y', w\phi)$ where Y' is a $w\phi$-stable direct summand of Y and X' is its dual. For any choice of p and q (and hence of $(\mathbf{G}, \mathbf{T}, F)$) there exists a natural bijection between the $W_\mathbb{G}$-classes of generic Levi subgroups of $\mathbb{G}$ and the $\mathbf{G}^F$-classes of F-stable Levi subgroups of $\mathbf{G}$, and similarly between the $W_\mathbb{G}$-classes of generic tori of $\mathbb{G}$ and the $\mathbf{G}^F$-classes of F-stable tori of $\mathbf{G}$ ([7, Theorem 2.1]). This result allows us to switch freely between the language of generic finite reductive groups and that of actual groups of Lie type.

For any generic finite reductive group $\mathbb{G}$ there exists a polynomial $|\mathbb{G}|(x) \in \mathbb{Z}[x]$ such that for any choice of a prime power q the order of the corresponding finite group $\mathbb{G}(q) = \mathbf{G}^F$ is given by $|\mathbb{G}(q)| = |\mathbb{G}|(q)$. Moreover, $|\mathbb{G}|$ has the form

$$|\mathbb{G}|(x) = x^N \prod_d \Phi_d^{a_d}$$

for suitable integers $a_d \geq 0$, where Φ_d denotes the dth cyclotomic polynomial over $\mathbb{Q}$. For any d such that Φ_d divides $|\mathbb{G}|$ we will define a d-Harish-Chandra theory as follows. An F-stable torus $\mathbf{S} \leq \mathbf{G}$ is called a *Φ_d-torus* if the order polynomial $|\mathbb{S}|(x)$ is a power of Φ_d. Here $\mathbb{S}$ denotes the generic finite reductive group corresponding to the F-stable torus $\mathbf{S}$ of $\mathbf{G}$. The centralizers of Φ_d-tori of $\mathbf{G}$ are called *d-split Levi subgroups* (they are F-stable Levi subgroups of $\mathbf{G}$ not necessarily contained in an F-stable parabolic subgroup). These notions were introduced by Broué and Malle [7]. The set of F-fixed points of d-split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ will replace the set of standard Levi subgroups in the set-up of Section 2.1.

For any F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ Deligne and Lusztig defined functors

$$R_{\mathbf{L}}^{\mathbf{G}}:\mathbb{Z}\mathrm{Irr}(\mathbf{L}^F)\to\mathbb{Z}\mathrm{Irr}(\mathbf{G}^F),\quad {}^*R_{\mathbf{L}}^{\mathbf{G}}:\mathbb{Z}\mathrm{Irr}(\mathbf{G}^F)\to\mathbb{Z}\mathrm{Irr}(\mathbf{L}^F),$$

between the character groups of $\mathbf{L}^F$ and $\mathbf{G}^F$, adjoined to each other with respect to the usual scalar product of characters. In the case that $\mathbf{L}$ is contained in an F-stable parabolic subgroup of $\mathbf{G}$ this is just ordinary Harish-Chandra induction and restriction. In the general case $R_{\mathbf{L}}^{\mathbf{G}}$ is defined as an alternating trace on ℓ-adic cohomology groups of a certain algebraic variety attached to $\mathbf{L}$ and $\mathbf{G}$ (see for example [15]). (In fact, the definition also depends on the choice of a parabolic subgroup containing $\mathbf{L}$, but in the situation considered below, both functors are independent of this choice.) We note that $R_{\mathbf{L}}^{\mathbf{G}}$ takes characters to generalized characters. This Lusztig induction will replace ordinary Harish-Chandra induction in the set-up of the previous section.

The following generalization of Definition 2.5 first appeared in [9]:

Definition 2.10. An irreducible character $\gamma\in\mathrm{Irr}(\mathbf{G}^F)$ is called *d-cuspidal* if ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\gamma)=0$ for all d-split Levi subgroups $\mathbf{L}$ properly contained in $\mathbf{G}$.

An irreducible character $\gamma\in\mathrm{Irr}(\mathbf{G}^F)$ is called *unipotent* if there exists an F-stable maximal torus $\mathbf{S}\le\mathbf{G}$ such that γ is a constituent of $R_{\mathbf{S}}^{\mathbf{G}}(1)$. Let $\mathrm{Uch}(\mathbf{G}^F)$ denote the set of unipotent characters of $\mathbf{G}^F$. It was shown by Lusztig that there exists a set $\mathrm{Uch}(\mathbb{G})$ and a map

$$\mathrm{Deg}:\mathrm{Uch}(\mathbb{G})\to\mathbb{Q}[x],\quad \gamma\mapsto\mathrm{Deg}(\gamma),$$

such that for any choice of p and q (and hence of $\mathbf{G}$ and F) there is a bijection $\psi_q^{\mathbb{G}}:\mathrm{Uch}(\mathbb{G})\to\mathrm{Uch}(\mathbf{G}^F)$ such that $\psi_q^{\mathbb{G}}(\gamma)$ has degree $\psi_q^{\mathbb{G}}(\gamma)(1)=\mathrm{Deg}(\gamma)(q)$. Furthermore, it follows from results of Shoji that the maps $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ are generic, i.e., there exist maps

$$R_{\mathbb{L}}^{\mathbb{G}}:\mathbb{Z}\mathrm{Uch}(\mathbb{L})\to\mathbb{Z}\mathrm{Uch}(\mathbb{G}),\quad {}^*R_{\mathbb{L}}^{\mathbb{G}}:\mathbb{Z}\mathrm{Uch}(\mathbb{G})\to\mathbb{Z}\mathrm{Uch}(\mathbb{L}),$$

satisfying $\psi_q^{\mathbb{G}}\circ R_{\mathbb{L}}^{\mathbb{G}}=R_{\mathbf{L}}^{\mathbf{G}}\circ\psi_q^{\mathbb{L}}$ for all q (see [9, Ths. 1.26 and 1.33]). In this sense, the set of unipotent characters and the functors $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ are generic for a series of groups of Lie type.

Therefore, and because the unipotent characters play a prominent role in Lusztig's classification of the irreducible characters of $\mathbf{G}^F$ (see Section 2.3), we will now restrict our attention to the subset $\mathrm{Uch}(\mathbf{G}^F)\subset\mathrm{Irr}(\mathbf{G}^F)$, respectively to its generic version $\mathrm{Uch}(\mathbb{G})$. In this case we have the following characterization of d-cuspidality (see [9, Prop. 2.9]):

Proposition 2.11. *A unipotent character $\gamma\in\mathrm{Uch}(\mathbb{G})$ is d-cuspidal if and only if $\mathrm{Deg}(\gamma)_d=|\mathbb{G}_{\mathrm{ss}}|_d$.*

Here, we write f_d for the Φ_d-part of $f \in \mathbb{Q}[x]$. Also, $\mathbb{G}_{ss}$ denotes the semisimple quotient of $\mathbb{G}$, i.e., the generic finite reductive group attached to $\mathbf{G}/\mathrm{Rad}(\mathbf{G})$.

A pair $(\mathbb{M}, \mu)$ consisting of a d-split Levi subgroup $\mathbb{M}$ of $\mathbb{G}$ and a unipotent character $\mu \in \mathrm{Uch}(\mathbb{M})$ is called *d-split*. It is called *d-cuspidal* if moreover μ is d-cuspidal. We introduce the following relation on the set of d-split pairs:

Definition 2.12. Let $(\mathbb{M}_1, \mu_1), (\mathbb{M}_2, \mu_2)$ be d-split. We say that $(\mathbb{M}_1, \mu_1) \leq_d (\mathbb{M}_2, \mu_2)$ if $\mathbb{M}_1$ is a d-split Levi subgroup of $\mathbb{M}_2$ and μ_2 occurs in $R_{\mathbb{M}_1}^{\mathbb{M}_2}(\mu_1)$.

For a d-cuspidal pair $(\mathbb{L}, \lambda)$ of $\mathbb{G}$ we write $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))$ for the set of $\gamma \in \mathrm{Uch}(\mathbb{G})$ such that $(\mathbb{L}, \lambda) \leq_d (\mathbb{G}, \gamma)$. We call $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))$ the *d-Harish Chandra series above* $(\mathbb{L}, \lambda)$ because of the following fundamental result, which shows that for any d we obtain a generalized Harish-Chandra theory:

Theorem 2.13. (Broué/Malle/Michel) *Let $\mathbb{G}$ be a generic finite reductive group and $d \geq 1$ such that Φ_d divides $|\mathbb{G}|$.*

(a) *(Disjointness) The sets* $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))$ *(where $(\mathbb{L}, \lambda)$ runs over a system of representatives of the $W_{\mathbb{G}}$-conjugacy classes of d-cuspidal pairs) form a partition of* $\mathrm{Uch}(\mathbb{G})$.

(b) *(Transitivity) Let $(\mathbb{L}, \lambda)$ be d-cuspidal and $(\mathbb{M}, \mu)$ be d-split such that $(\mathbb{L}, \lambda) \leq_d (\mathbb{M}, \mu)$ and $(\mathbb{M}, \mu) \leq_d (\mathbb{G}, \gamma)$. Then $(\mathbb{L}, \lambda) \leq_d (\mathbb{G}, \gamma)$.*

This was shown in special cases by Schewe [113] and Fong/Srinivasan [38]. The general statement is given in [9, Ths. 3.2 and 3.11]. Note that part (b) is far from obvious since $R_{\mathbf{L}}^{\mathbf{G}}$ does not necessarily take characters to actual characters. For $d = 1$ we recover the ordinary Harish-Chandra theory for characters.

The statement of Theorem 2.13 has been checked case by case. In the ordinary case $d = 1$, Harish-Chandra series may be indexed by irreducible modules of endomorphism algebras of induced cuspidal modules. A similar statement is conjectured to hold in the d-case [8]. This should also provide a more conceptual proof of the preceding theorem. At present, only some consequences of this deep conjecture have been verified. To formulate these results we need to introduce one further invariant. Let $(\mathbf{L}, \lambda)$ be d-cuspidal. Then $W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_{\mathbf{G}}(\mathbf{L}, \lambda)/\mathbf{L}$ is called the *relative Weyl group of* $(\mathbf{L}, \lambda)$ *in* $\mathbf{G}$.

Theorem 2.14. (Broué/Malle/Michel) *For each d there exists a collection of isometries*

$$I_{(\mathbb{L},\lambda)}^{\mathbb{M}} : \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L}, \lambda)) \to \mathbb{Z}\mathrm{Uch}(\mathbb{M}, (\mathbb{L}, \lambda)),$$

such that for all $\mathbb{M}$ and all $(\mathbb{L}, \lambda)$ we have

$$R_{\mathbb{M}}^{\mathbb{G}} \circ I_{(\mathbb{L},\lambda)}^{\mathbb{M}} = I_{(\mathbb{L},\lambda)}^{\mathbb{G}} \circ \mathrm{Ind}_{W_{\mathbb{M}}(\mathbb{L},\lambda)}^{W_{\mathbb{G}}(\mathbb{L},\lambda)}.$$

Here $\mathbb{M}$ runs over the d-split Levi subgroups of $\mathbb{G}$ and $(\mathbb{L}, \lambda)$ over the set of d-cuspidal pairs of $\mathbb{M}$.

This result was proved in [9, Theorem 3.2]. It seems to indicate that the constituents of $R_{\mathbb{L}}^{\mathbb{G}}(\lambda)$ can be parameterized by the irreducible characters of an algebra which is a deformation of the group algebra of the relative Weyl group $W_{\mathbb{G}}(\mathbb{L},\lambda)$. It can be checked that all relative Weyl groups occurring in groups of Lie type are complex reflection groups. Suitable deformations of group algebras of complex reflection groups, called cyclotomic Hecke algebras, have been introduced by Broué and Malle [8]. We cannot present the details here.

Generic blocks. The importance of d-Harish-Chandra series lies in their connection with ℓ-blocks of finite groups of Lie type for primes ℓ not dividing q.

We fix $\mathbb{G}$ and a choice of a prime power q, hence a pair $(\mathbf{G},F)$. A prime ℓ not dividing q is called *large for* $\mathbb{G}$ if ℓ does not divide the order of $W\langle\phi\rangle$. If ℓ is large there exists a unique d such that $\ell|\Phi_d(q)$ and $\Phi_d\big||\mathbb{G}|$.

Theorem 2.15. (Broué/Malle/Michel) *Let ℓ be large for $\mathbb{G}$, and assume that $\ell|\Phi_d(q)$. Then the partition of* $\mathrm{Uch}(\mathbf{G}^F)$ *into ℓ-blocks coincides with the image under $\psi_q^{\mathbb{G}}$ of the partition of* $\mathrm{Uch}(\mathbb{G})$ *into d-Harish-Chandra series. In particular, the distribution of unipotent characters of $\mathbf{G}^F$ into ℓ-blocks is generic.*

This is proved in [9, Theorem 5.24]. We write $b_\ell(\mathbf{L},\lambda)$ for the unipotent ℓ-block of $\mathbf{G}^F$ indexed by $(\mathbb{L},\lambda)$. The unipotent blocks and their defect groups can be described more precisely. For this we need:

Definition 2.16. For $\gamma\in\mathrm{Uch}(\mathbb{G})$ we denote by $S_d(\gamma)$ the set of Φ_d-tori of $\mathbb{G}$ contained in a maximal torus $\mathbb{S}$ of $\mathbb{G}$ such that ${}^*R_{\mathbb{S}}^{\mathbb{G}}(\gamma)\neq 0$. The maximal elements of $S_d(\gamma)$ are called the *Φ_d-defect tori of γ*.

The Φ_d-defect tori of a unipotent character can be characterized as follows (see [9, Theorem 4.8]):

Proposition 2.17. *Let $(\mathbb{L},\lambda)$ be a d-cuspidal pair and $\gamma\in\mathrm{Uch}(\mathbb{G},(\mathbb{L},\lambda))$. Then the Φ_d-defect tori of γ are conjugate to* $\mathrm{Rad}(\mathbb{L})_d$.

Here $\mathrm{Rad}(\mathbb{L})$ is the generic finite reductive group associated to the radical of $\mathbf{L}$ (this is in fact a torus), and $\mathrm{Rad}(\mathbb{L})_d$ denotes its maximal Φ_d-subtorus.

For an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ denote by $\mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$ the group of characters (over a splitting field of characteristic 0) of ℓ-power order of the abelian group $\mathbf{L}^F/[\mathbf{L}^F,\mathbf{L}^F]$. Then [9, Theorem 5.24] gives the following sharpening of Theorem 2.15:

Theorem 2.18. (Broué/Malle/Michel) *Let ℓ be large for $\mathbb{G}$, and assume that $\ell|\Phi_d(q)$. Let $(\mathbf{L},\lambda)$ be a d-cuspidal pair of $\mathbf{G}$.*

(a) *The ℓ-block $b_\ell(\mathbf{L},\lambda)$ of $\mathbf{G}^F$ consists of the irreducible constituents of the virtual characters $R_{\mathbf{L}}^{\mathbf{G}}(\theta\lambda)$, where $\theta\in\mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$.*

(b) *The defect groups of $b_\ell(\mathbf{L},\lambda)$ are the Sylow ℓ-subgroups of the groups of F-fixed points of the Φ_d-defect tori of unipotent characters in $b_\ell(\mathbf{L},\lambda)$.*

(c) *The block $b_\ell(\mathbf{L},\lambda)$ is isotypic to the principal ℓ-block of the normalizer of its defect group.*

Broué and Michel [11, Théorème] have shown that part (c) actually holds under the weaker assumption that the Sylow ℓ-subgroup of $\mathbf{G}^F$ is abelian.

2.3 On ℓ-blocks of finite groups of Lie type

The results in the previous section have been generalized by Cabanes and Enguehard [13] in a program to determine all ℓ-blocks of finite groups of Lie type. In order to describe their results we first have to recall Lusztig's classification of the irreducible complex characters of these groups. As before, let $\mathbf{G}$ be a connected reductive algebraic group, defined over a finite field $\mathbb{F}_q$, with corresponding Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$. Let $\mathbf{G}^*$ be a group in duality with $\mathbf{G}$, with Frobenius morphism F^*. We fix an isomorphism between the multiplicative group of the algebraic closure of $\mathbb{F}_q$ and the group $(\mathbb{Q}/\mathbb{Z})_{p'}$, and an embedding of the latter into $\bar{\mathbb{Q}}^\times$. Then the set of $\mathbf{G}^F$-classes of pairs $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, is in bijection with the set of ${\mathbf{G}^*}^{F^*}$-classes of pairs $(\mathbf{T}^*, s)$, where $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ and $s \in {\mathbf{T}^*}^{F^*}$. We will write $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ if $(\mathbf{T}, \theta)$ and $(\mathbf{T}^*, s)$ correspond to each other in this way. This allows to introduce the so-called *Lusztig series*

$$\mathcal{E}(\mathbf{G}^F, s) := \{\chi \in \mathrm{Irr}(\mathbf{G}^F) \mid \langle \chi, R_{\mathbf{T}^*}^{\mathbf{G}}(s)\rangle \neq 0 \text{ for some } \mathbf{T}^* \ni s\}$$

attached to the class of an F-stable semisimple element s of $\mathbf{G}^*$. Lusztig has shown that this defines a partition of the set of irreducible complex characters of $\mathbf{G}^F$:

$$\mathrm{Irr}(\mathbf{G}^F) = \coprod_s \mathcal{E}(\mathbf{G}^F, s)\,.$$

Here s runs over a system of representatives of conjugacy classes of semisimple elements in the dual group ${\mathbf{G}^*}^{F^*}$. Furthermore, there is a bijection

$$\mathcal{E}(\mathbf{G}^F, s) \xrightarrow{\sim} \mathrm{Uch}(C_{\mathbf{G}^*}(s)^{F^*})$$

of the Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ with the set of unipotent characters of the centralizer $C_{\mathbf{G}^*}(s)$ [93,94]. This is called the Jordan decomposition of characters.

Let ℓ be a prime dividing $|\mathbf{G}^F|$ but not dividing q. For a semisimple ℓ-regular element $s \in {\mathbf{G}^*}^{F^*}$ define

$$\mathcal{E}_\ell(\mathbf{G}^F, s) := \bigcup_t \mathcal{E}(\mathbf{G}^F, st)$$

where the union now is over semisimple ℓ-elements t in the centralizer $C_{\mathbf{G}^*}(s)^{F^*}$ of s. The following result of Broué and Michel [10, Théorème 2.2] shows a certain compatibility between Lusztig series and ℓ-blocks:

Theorem 2.19. (Broué/Michel) *Each set $\mathcal{E}_\ell(\mathbf{G}^F, s)$ is a union of ℓ-blocks of $\mathbf{G}^F$.*

The description of arbitrary ℓ-blocks is analogous to that given for unipotent ℓ-blocks in the previous section, although at present some restrictions on the prime ℓ have to be made, and the result is phrased in terms of $R_{\mathbf{L}}^{\mathbf{G}}$, whose decomposition is not yet known in general. Let e be a positive integer. In extension of our above definitions we say that $(\mathbf{L}, \lambda)$ is e-cuspidal if $\mathbf{L}$ is e-split and

$\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ is e-cuspidal. We let $\ll_e$ be the transitive closure of the relation $\leq_e$ on e-split pairs introduced above. As an example for the type of results which can be obtained we cite:

Theorem 2.20. (Cabanes/Enguehard) *Let $(\mathbf{G}, F)$ be as above, and ℓ a prime satisfying the following conditions: ℓ does not divide q, ℓ is good for $\mathbf{G}$, $\ell \neq 2$, $\ell \neq 3$ if $\mathbf{G}^F$ has a component of type ${}^3D_4$. Let e denote the order of q modulo ℓ.*

(a) *There is a natural bijection between the set of ℓ-blocks of $\mathbf{G}^F$ and the set of $\mathbf{G}^F$-classes of e-cuspidal pairs $(\mathbf{L}, \lambda)$ with $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$ for some ℓ-regular s.*

(b) *Let $b_\ell(\mathbf{L}, \lambda)$ be the block corresponding to the e-cuspidal pair $(\mathbf{L}, \lambda)$. Then*

$$b_\ell(\mathbf{L}, \lambda) \cap \bigcup_s \mathcal{E}(\mathbf{G}^F, s) = \{\gamma \in \mathrm{Irr}(\mathbf{G}^F) \mid (\mathbf{L}, \lambda) \ll_e (\mathbf{G}, \gamma)\}$$

where the union runs over ℓ-regular semisimple elements s.

(c) *Let $(\mathbf{L}, \lambda)$ be e-cuspidal. Let $\mathbf{G}(s)$ be a group in duality with $C^\circ_{\mathbf{G}^*}(s)$, $\mathbf{L}(s)$ a Levi subgroup of $\mathbf{G}(s)$ in duality with $C^\circ_{\mathbf{L}^*}(s)$. By the Jordan decomposition of characters $\lambda \in \mathcal{E}(C^\circ_{\mathbf{L}^*}(s)^F, 1)$ corresponds to an element of $\mathcal{E}(\mathbf{L}(s)^F, 1)$, which we also call λ. Then $(\mathbf{L}(s), \lambda)$ is e-cuspidal. The defect group of $b_\ell(\mathbf{L}, \lambda)$ is isomorphic to the defect group of the unipotent block $b_\ell(\mathbf{L}(s), \lambda)$.*

This was proved in [13, Theorem 3.3 and 3.4, and Rem. 3.6]. The ℓ-blocks of classical groups had previously been determined by Fong and Srinivasan [39], and Hiss [70] had obtained partial results in the case of exceptional groups.

Remark 2.21. It is not known whether the relations $\ll_e$ and $\leq_e$ coincide. If so, we would obtain an e-Harish-Chandra theory for arbitrary Lusztig series, generalizing Theorem 2.13. Cabanes and Enguehard [13, (2.1.R)] give a technical condition on the decomposition of $R_{\mathbf{L}}^{\mathbf{G}}$ which would imply the equality of $\ll_e$ and $\leq_e$.

Remark 2.22. Theorem 2.20 compares the defect groups of arbitrary ℓ-blocks to those of unipotent ℓ-blocks of (possibly) smaller groups. The defect groups of unipotent ℓ-blocks for large primes ℓ where described in Theorem 2.18(b). For small ℓ, when the Sylow ℓ-subgroup is no longer abelian, the structure of the defect group is explicitly given in [13, Theorem 3.5].

3 Hecke algebras

In Section 2.1 we described the endomorphism algebra of a Harish-Chandra induced cuspidal representation of a finite group G of Lie type. These endomorphism rings carry a lot of information on representations of G. We shall make this connection now more precise by defining certain functors which tie together representations of G and those of the Hecke algebras in Theorem 2.9. This motivates to study the representations of Hecke algebras abstractly. In subsequent sections we present some results on Iwahori-Hecke algebras associated with real (and in view of Section 2.2 with complex) reflection groups.

3.1 Hom-functors and Hecke algebras

We shall present the theory of Hom-functors only as far as it is needed for the representation theory of finite reductive groups and Hecke algebras. Proofs and details in a more general setting can be found in [19] and [20].

Throughout R is either a field or a complete discrete valuation ring. R-algebras and modules of R-algebras are always assumed to be free of finite rank as R-modules. Let T be an R-algebra, $M \in \mathbf{mod}\text{-}T$, and let $\beta : P \to M$ be a projective presentation of M. Assume henceforth that $\ker \beta$ is invariant under the endomorphism ring $\mathrm{End}_T(P)$ of P. Then $J_\beta = \{\phi \in \mathrm{End}_T(P) \mid \beta\phi = 0\}$ is an ideal of $\mathrm{End}_T(P)$ and $\mathrm{End}_T(P)/J_\beta \cong \mathrm{End}_T(M) =: \mathcal{H}$ canonically.

The functor H^β from $\mathbf{mod}\text{-}T$ to $\mathbf{mod}\text{-}\mathcal{H}$ is defined on objects $V \in \mathbf{mod}\text{-}T$ by

$$H^\beta : V \to \mathrm{Hom}_T(P, V)/\mathrm{Hom}_T(P, V)J_\beta,$$

and on morphisms similarly. It is called *Hom-functor* associated with the projective presentation $\beta : P \to M$ of M.

The *P-torsion submodule* $t_P(V)$ is the unique maximal submodule X of $V \in \mathbf{mod}\text{-}T$ with respect to the property that $\mathrm{Hom}_T(P, X) = \{0\}$. We have a functor $A_P : \mathbf{mod}\text{-}T \to \mathbf{mod}\text{-}T$ taking $V \in \mathbf{mod}\text{-}T$ to $V/t_P(V)$. We say that V is *P-torsionless* if $t_P(V) = (0)$. We define the functor $\hat{H}^\beta : \mathbf{mod}\text{-}\mathcal{H} \to \mathbf{mod}\text{-}T$ to be the composite functor $\hat{H}^\beta = A_P \circ (- \otimes_{\mathcal{H}} M)$. It was shown in [19, 2.16] that $\hat{H}^\beta$ is a right inverse of the functor H^β.

We turn now to the calculation of decomposition numbers. For this let $(K, \mathcal{O}, k)$ be an ℓ-modular system. We let $T_{\mathcal{O}}$ be an $\mathcal{O}$-order in some finite dimensional semisimple K-algebra T_K, where K is the quotient field of $\mathcal{O}$. Let k be the residue field of $\mathcal{O}$. Henceforth, tensoring is always tensoring over $\mathcal{O}$ if not stated otherwise, and objects tensored over $\mathcal{O}$ by K or k are denoted by an index K, k respectively. Thus in particular $T_k = k \otimes T_{\mathcal{O}}$ is a finite dimensional k-algebra. Choosing $T_{\mathcal{O}}$-lattices $M_{\mathcal{O}}$ in the irreducible T_K-modules M_K and recording the composition factors of M_k into a matrix (fixing an ordering of the irreducibles over K and k) yields the *decomposition matrix* of $T_{\mathcal{O}}$.

We take a $T_{\mathcal{O}}$-lattice $M_{\mathcal{O}}$ and let $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$ be a projective presentation of $M_{\mathcal{O}}$. Note that $P_{\mathcal{O}}$ is a $T_{\mathcal{O}}$-lattice and that $\beta_k = 1 \otimes \beta_{\mathcal{O}} : k \otimes P_{\mathcal{O}} \to k \otimes M_{\mathcal{O}}$ is a projective presentation of $M_k = k \otimes M_{\mathcal{O}}$. Moreover, as in the case of group algebras we may lift idempotents, and homomorphism from projective T_k-modules are always liftable. The endomorphism ring of M_R for $R \in \{K, \mathcal{O}, k\}$ is denoted by $\mathcal{H}_R$.

Theorem 3.1. (Dipper) *Suppose that $\ker \beta_{\mathcal{O}}$ is invariant under every endomorphism of $P_{\mathcal{O}}$. Then $\ker \beta_R$ is invariant under $\mathrm{End}_{T_R}(P_R)$ for every choice of $R \in \{K, \mathcal{O}, k\}$. Moreover $\mathrm{End}_{T_R}(M_R) = \mathcal{H}_R \cong R \otimes \mathrm{End}_{T_{\mathcal{O}}}(M_{\mathcal{O}})$, and the decomposition matrix of $\mathcal{H}_{\mathcal{O}}$ is the part of the decomposition matrix of $T_{\mathcal{O}}$ describing the composition multiplicities of the irreducible T_k-modules occurring in the head of M_k in the reductions from K to k of the irreducible constituents of the semisimple module M_K.*

The theorem is an easy consequence of Fittings lemma and was shown first in [19, 4.7 and 4.10]. It is applied in the following situation. For a finite group $G = \mathbf{G}^F$ of Lie type we take a regular cuspidal representation C_K of some Levi subgroup L of G. The term *regular* means here, that C_K occurs as constituent of the *Gelfand-Graev* representation Γ_K^L of L. This is a special representation, which is multiplicity free and is induced from the unipotent radical of any Borel subgroup of L (see e.g. [15] for details). In particular $\Gamma_{\mathcal{O}}^L$ is a projective $\mathcal{O}L$-lattice, where the characteristic of K is now zero and that of k is a prime ℓ different from the describing characteristic p of G.

We recall the notion of Lusztig series introduced in Section 2.3. Thus corresponding to the character of C_K we have a representative s of a semisimple conjugacy class of the dual group L^* of L. Since all irreducible direct summands of Γ_K^L occur with multiplicity one, there is up to isomorphisms precisely one $\mathcal{O}G$-lattice $C_{\mathcal{O}}(s)$ affording this cuspidal representation such that $C_{\mathcal{O}}(s)$ is an epimorphic image of $\Gamma_{\mathcal{O}}^L$. Among other things this implies that $\Gamma_{\mathcal{O}}^L$ contains a projective cover Q of $C_{\mathcal{O}}(s)$ as direct summand, the head of $C_k(s)$ is simple occurring with multiplicity one as composition factor of $C_k(s)$, and $\mathrm{End}_{kL}(C_k(s)) = k$. We Harish-Chandra induce the epimorphism from Q onto $C_{\mathcal{O}}(s)$ to get a surjective map $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}(s)$, where $P_{\mathcal{O}} = R_L^G(Q)$ is a projective $\mathcal{O}G$-lattice and $M_{\mathcal{O}}(s) = R_L^G(C_{\mathcal{O}}(s))$. As in Section 2.1, we set $W(L)$ to be the relative Weyl group of L in G. Note that $W(L)$ acts on the conjugacy classes of semisimple elements of L. For any $t \in L^*$, denote the stabilizer of the L^*-conjugacy class containing t in $W(L^*)$ by $W(L,t)$. Note that, if $R = K$ or if $R = k$ and $C_k(s)$ is irreducible, $W(L,s)$ is isomorphic to the group $W(L, C_R(s))$ of Theorem 2.9. In [24, 4.10] it was shown that $W(L,s)$ is a real reflection group provided that the center of $\mathbf{G}$ is connected.

Theorem 3.2. (Dipper/Fleischmann) *Let s and $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}(s)$ be as above. Denote the ℓ-regular part of s by s'. Suppose that*

$$W(L,s) = W(L,s').$$

Then the kernel of $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}(s)$ is invariant under all endomorphisms of $P_{\mathcal{O}}$. Moreover, the endomorphism ring of $M_{\mathcal{O}}(s)$ is the (extended) Iwahori-Hecke algebra $\mathcal{H}_{\mathcal{O}}(W(L,s))$ associated to $W(L,s)$.

This theorem was proved in [24, 4.6] for connected groups with connected center and [24, Part II, Theorem 1] in full generality. As a consequence of Theorem 3.1 we obtain that the decomposition matrix of $\mathcal{H}_{\mathcal{O}}(W(L,s))$ is part of the ℓ-decomposition matrix of G.

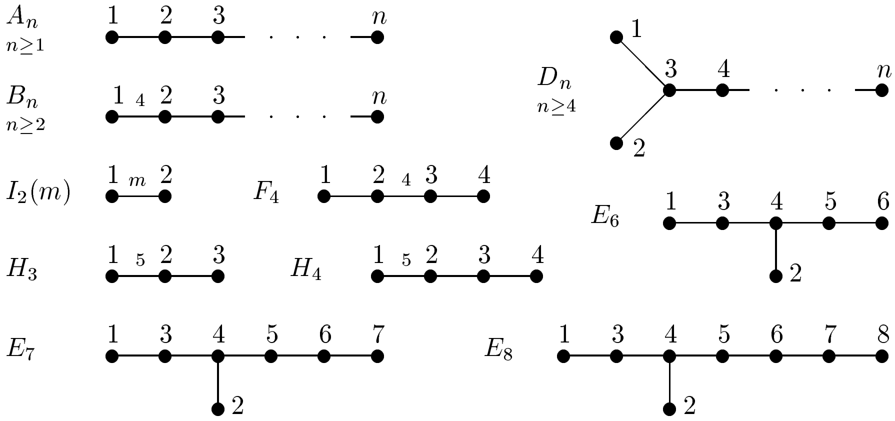
3.2 Iwahori-Hecke algebras

The results of the previous section show that part of the decomposition numbers of finite groups of Lie type can be obtained from those of Iwahori-Hecke algebras. Therefore, we now briefly survey some features of the representation theory of Iwahori-Hecke algebras associated with reflection groups.

Let (W, S) be a finite Coxeter system, i.e., W is a finite group and S is a subset of W such that we have a presentation of W with

$$\begin{array}{ll} \text{generators: } s \in S, & \\ \text{relations:} \quad (st)^{m_{st}} = 1 & \text{for } s, t \in S, \end{array}$$

where (m_{st}) is a matrix of integers with $m_{ss} = 1$ and $m_{ts} = m_{st} > 1$ for $s \neq t$. The matrices (m_{st}) such that the above presentation defines a finite group are classified. It turns out that the irreducible Coxeter groups fall into several infinite series and a number of exceptional types, which are conveniently described by the following diagrams:



(where $m \geq 5$ in type $I_2(m)$).

The vertices are in bijection with S, and two vertices $s \neq t$ are joined by an edge if $m_{st} > 2$ (i.e., s, t do not commute with each other). An edge is labelled by the integer m_{st} unless we have $m_{st} = 3$ in which case we omit this label. For example, the diagram of type A_{n-1} defines the symmetric group on $\{1, \ldots, n\}$, generated by the basic transpositions $(1, 2), \ldots, (n-1, n)$.

An important tool in studying W is the length function $l \colon W \to \mathbb{N}_0$: given $w \in W$ we can write $w = s_1 \cdots s_n$ with $s_i \in S$. If such an expression is of shortest possible length, we call it a *reduced expression* and let $l(w) := n$.

The Iwahori-Hecke algebra associated with (W, S) is defined as a deformation of the group algebra of W over a commutative ring R:

Definition 3.3. Let $q_s \in R$ $(s \in S)$ such that $q_s = q_t$ whenever $s, t \in S$ are conjugate in W. Then the *Iwahori-Hecke algebra* $H = H_R(W; \{q_s \mid s \in S\})$ over R is the associative algebra with 1 defined by the presentation with

$$\begin{array}{ll} \text{generators:} & T_w, w \in W, \\ \text{relations:} & T_w = T_{s_1} \cdots T_{s_n} \quad \text{if } w = s_1 \cdots s_n \text{ with } s_i \in S \text{ is reduced}, \\ & T_s^2 = q_s T_1 + (q_s - 1) T_s \quad \text{for } s \in S. \end{array}$$

It is a non-trivial fact that H is free as an R-module, with a basis given by the elements $\{T_w \mid w \in W\}$ (see [16, §68A]).

Now there is a universal choice for the pair $(R, \{q_s \mid s \in S\})$: we take the ring of Laurent polynomials $A_0 = \mathbb{Z}[u_s^{\pm 1} \mid s \in S]$ in indeterminates u_s such that $u_s = u_t$ whenever $s, t \in S$ are conjugate in W. The Iwahori-Hecke algebra defined with respect to the pair $(A_0, \{u_s \mid s \in S\})$ will be called the *generic Iwahori-Hecke algebra* associated with (W, S); we shall denote it by $\mathcal{H}$. Given any commutative ring R and elements $\{q_s \mid s \in S\} \subseteq R$ as above, we have a canonical isomorphism

$$\mathcal{H}_R := R \otimes_{A_0} \mathcal{H} \cong H_R(W; \{q_s \mid s \in S\}),$$

where R is regarded as an A_0-module via the ring homomorphism $A_0 \to R$, $u_s \mapsto q_s$ $(s \in S)$.

The reason why we took Laurent polynomials rather than just polynomials is the following: first of all, it implies that all basis elements T_w are invertible. Indeed, we have $T_w^{-1} = T_{s_n}^{-1} \cdots T_{s_1}^{-1}$ if $w = s_1 \cdots s_n$ is a reduced expression and

$$T_s^{-1} = u_s^{-1} T_s + (u_s^{-1} - 1) T_1 \qquad \text{for all } s \in S.$$

Secondly, it implies that the Gram matrix of the bilinear form $\mathcal{H} \times \mathcal{H} \to A_0$ defined by

$$(T_w, T_{w'}) \mapsto \text{ coefficient of } T_1 \text{ in } T_w T_{w'},$$

is invertible over A_0, and hence $\mathcal{H}$ is a symmetric algebra.

The generic algebra and its character table. In view of our applications to groups of Lie type, we will now only consider the case where W is a finite Weyl group. Then it is known that $\mathbb{Q}$ is a splitting field for W. In order to obtain a similar statement for $\mathcal{H}$, we consider the ring $A = \mathbb{Z}[u_s^{\pm 1/2} \mid s \in S] \supseteq A_0$ and let K be its field of fractions. It follows from results of Benson/Curtis and Lusztig (see [4], [92] and also [47]) that the algebra $\mathcal{H}_K = K \otimes_{A_0} \mathcal{H}$ is split semisimple. Hence, by Tits' Deformation Theorem (see [16, §68A]), we have in fact $\mathcal{H}_K \cong KW$. In particular, there is a bijection between the irreducible characters of $\mathcal{H}_K$ and those of W. Explicitly, such a bijection is given as follows: since the structure constants of $\mathcal{H}_K$ with respect to the basis $\{T_w \mid w \in W\}$ lie in A_0, a general argument shows that if χ is an irreducible character of $\mathcal{H}_K$ then $\chi(T_w)$ lies in the integral closure of A_0 in K, i.e., we have

$$\chi(T_w) \in \mathbb{Z}[u_s^{1/2} \mid s \in S] \qquad \text{for all } w \in W.$$

Now consider the ring homomorphism $\theta \colon A \to \mathbb{Q}$, $u_s^{1/2} \mapsto 1$ $(s \in S)$. Then Tits' Deformation Theorem states more precisely that if χ is an irreducible character of $\mathcal{H}_K$ then the map $w \mapsto \theta(\chi(T_w))$ $(w \in W)$ defines an irreducible character of W, and this gives the required bijection.

We now wish to define the character table of $\mathcal{H}_K$ as a square matrix of character values $\chi(T_w)$, where χ runs over the irreducible characters of $\mathcal{H}_K$ and

$w \in W$ runs over a set of representatives of the conjugacy classes of W. However, simple examples show that we may have $\chi(T_w) \neq \chi(T_{w'})$ even if $w, w' \in W$ are conjugate. The question of a good choice of representatives is solved as follows.

Theorem 3.4. (Geck/Pfeiffer [58]) *Let $Cl(W)$ be the set of conjugacy classes of W. Let $C \in Cl(W)$ and $w, w' \in C$. Then, if w, w' have minimal length in C, the basis elements T_w and $T_{w'}$ are conjugate in $\mathcal{H}$.*

Thus, if we choose for each $C \in \mathrm{Cl}(W)$ a representative $w_C \in C$ of minimal length, then the matrix of character values

$$(\chi(T_{w_C}))_{\chi,C} \qquad \text{where } \chi \in \mathrm{Irr}(\mathcal{H}_K) \text{ and } C \in \mathrm{Cl}(W),$$

is independent of any choices, and it is called the *character table* of $\mathcal{H}$.

Example 3.5. Let (W, S) be of type $G_2 = I_2(6)$, i.e., W is the dihedral group of order 12, generated by two involutions s, t such that st has order 6. There are 6 conjugacy classes, with representatives of minimal length as given in the following table.

	T_1	T_s	T_t	T_{st}	T_{stst}	T_{ststst}
ind	1	u_s	u_t	$u_s u_t$	$(u_s u_t)^2$	$(u_s u_t)^3$
ε_1	1	u_s	-1	$-u_s$	u_s^2	$-u_s^3$
ε_2	1	-1	u_t	$-u_t$	u_t^2	$-u_t^3$
ε	1	-1	-1	1	1	1
χ_1	2	$u_s - 1$	$u_t - 1$	$\sqrt{u_s u_t}$	$-u_s u_t$	$-2\sqrt{u_s u_t}^3$
χ_2	2	$u_s - 1$	$u_t - 1$	$-\sqrt{u_s u_t}$	$-u_s u_t$	$2\sqrt{u_s u_t}^3$

The rows are indexed by the irreducible characters of $\mathcal{H}_K$. There are 4 characters of degree 1 denoted by $\mathrm{ind}, \varepsilon_1, \varepsilon_2, \varepsilon$ and 2 characters of degree 2 denoted by χ_1, χ_2. We see from the above table that square roots of the parameters u_s, u_t do actually occur in character values.

The character tables of all Iwahori-Hecke algebras of all types are now known in the following sense: for classical types there exist recursive formulas in the spirit of the Murnaghan-Nakajama rule, for the exceptional types the tables have been computed explicitly.

- The table for type A_{n-1} was first computed in 1975 by A.J. Starkey (a Ph.D. student of J.A. Green in Warwick). The results are explained in [14].
- Murnaghan-Nakayama formulas for type A_n were found by A. Ram [110]; subsequently, different proofs of this result were found (see [107]) and generalizations were proved for types B_n and D_n (see [67] and [108], [109]).
- The tables for the exceptional types F_4, E_6 and E_7 were computed in [47] and [48], using CHEVIE [52] and GAP [114].
- The last remaining and biggest table, namely that of type E_8, was determined in Geck/Michel [57], again using CHEVIE and GAP.

The main tool for the computation of character tables of Iwahori-Hecke algebras of exceptional type is a general result about the eigenvalues of the basis elements T_w in the irreducible representations of $\mathcal{H}_K$:

Theorem 3.6. (Geck/Michel [57]) *Let $\rho: \mathcal{H}_K \to K^{d\times d}$ be an irreducible matrix representation and $w \in W$ be an element which has minimal length in its conjugacy class. Then the eigenvalues of $\rho(T_w)$ are of the form*

$$\zeta \prod_{s\in S} u_s^{f_s} \qquad \text{where } \zeta \text{ is a root of unity and } f_s \in \mathbb{Q} \text{ for all } s \in S.$$

Although Theorems 3.4 and 3.6 are formulated in general terms, their proofs depend on a case-by-case analysis. The first theorem depends on a strong form of conjugation in the underlying Weyl group, while the second theorem follows from a result about taking powers of suitable elements in the associated Artin-Tits braid groups. These results in turn were proved by inductive methods for the classical types A_n, B_n, D_n and were verified by explicit computations for the exceptional types using CHEVIE.

Remark 3.7. Theorems 3.4 and 3.6 remain correct for arbitrary finite Coxeter groups, i.e., also for non-Weyl groups, see [52,57]. The character tables of the Iwahori-Hecke algebras of the non-crystallographic types $I_2(m)$, H_3, H_4 are constructed from explicit matrix representations as computed by Kilmoyer/Solomon [16, §67C], Lusztig [92] and Alvis/Lusztig [1], respectively.

3.3 Hom-functors and q-Schur algebras

Having described Hecke algebras and exhibited their basic features we want to introduce a further type of finite dimensional algebras, *q-Schur algebras*, which are derived from Hecke algebras enlarging those, and whose representations carry also information on the representations for finite groups of Lie type. We first explain how this comes about.

Let us keep the notation of Section 3.1. Being an additive functor of abelian categories the Hom-functor H^β introduced there induces homomorphism on Hom-sets and endomorphism algebras. A special case is of particular interest (see [19, 2.21]):

Proposition 3.8. (Dipper) *Let T be an R-algebra, $M \in$ **mod**-T, and $\beta : P \to M$ be a projective presentation of M, such that* $\ker\beta$ *is invariant under all endomorphisms of P. Suppose that M is P-torsionless. Then H^β induces an isomorphism, again denoted by H^β:*

$$H^\beta : \mathrm{Hom}_T(XM, YM) \to \mathrm{Hom}_{\mathcal{H}}(X, Y),$$

where X, Y are right ideals of $\mathcal{H}$.

This can again be applied to the representation theory of a finite group of Lie type G. The idea is roughly as follows: We take right ideals X_i of $\mathcal{H}_{\mathcal{O}}$, where $\mathcal{H}_{\mathcal{O}}$ is one of the Hecke algebras coming up in Theorem 3.2, then we consider the $\mathcal{O}G$-lattice

$$N_{\mathcal{O}} = \bigoplus_i X_i M_{\mathcal{O}}.$$

We construct a projective presentation $\gamma_{\mathcal{O}} : Q_{\mathcal{O}} \to N_{\mathcal{O}}$ such that $\ker \gamma_{\mathcal{O}}$ is invariant under all endomorphisms of $Q_{\mathcal{O}}$. Then we apply Theorem 3.1 again to connect representations of RG with those of the endomorphism ring $\mathcal{S}_R = \mathrm{End}_{RG}(N_R)$. On the other hand Proposition 3.8 yields that $\mathcal{S}_R$ is the endomorphism ring of the direct sum of the X_i as well and can hence be computed in terms of Hecke algebras again. However in 3.8 one needs often to replace $X_i M_{\mathcal{O}}$ by its purification in $M_{\mathcal{O}}$ which yields as well an isomorphism (for details see [19, 4.17] and [20, 1.30]) or even lattices in between ([25]).

The special case of the unipotent characters of G may serve as illustration: We take as Levi subgroup a maximally split torus T and let $M_R = R_T^G(I_R)$, where I_R is the trivial RT-module. Thus M_R is the trivial B-module induced to G, where $T \subseteq B$ is a Borel subgroup of G. The irreducible characters afforded by the constituents of M_K are unipotent characters. This situation was first investigated for $R = K$ by Iwahori in [80] and is a special case of Theorem 2.9. The endomorphism ring $\mathrm{End}_{RG}(M_R)$ is the Iwahori-Hecke algebra $\mathcal{H}_R = \mathcal{H}_R(W)$ associated with the Weyl group W of G, where the parameter values are determined by the field of definition of G. For instance if G is untwisted over the finite field with q elements, all parameters equal q. Note that this is a special case of the situation in Theorem 3.2. The corresponding Hom-functors are denoted by H_R, where as usual $R \in \{K, \mathcal{O}, k\}$.

The Levi subgroups $B \subseteq L \subseteq G$ are parametrized by the subsets J of the set S of basic reflections of the Weyl group. We denote by L_J the Levi subgroup of G corresponding to $J \subseteq S$. Similarly, W_J denotes the corresponding parabolic subgroup of W. For $J \subseteq S$ we set

$$y_J = \sum_{w \in W_J} (-1)^{l(w)} \mathrm{ind}(T_w)^{-1} T_w \in \mathcal{H}_R,$$

where $\mathrm{ind}\colon \mathcal{H}_R \to R$ is the 1-dimensional representation such that $\mathrm{ind}(T_s) = q_s$ for $s \in S$. Moreover we define

$$Y_R = \bigoplus_{J \subseteq S} y_J \mathcal{H}_R.$$

Note that $Y_R \cong R \otimes Y_{\mathcal{O}}$, since $y_J \mathcal{H}_{\mathcal{O}}$ is pure in $\mathcal{H}_{\mathcal{O}}$. Moreover $\mathrm{St}_{J,K} = y_J R_T^{L_J}(I_K)$ is a KL_J-submodule of M_K isomorphic to the Steinberg module of L_J, which occurs with multiplicity one in the Gelfand-Graev representation $\Gamma_{J,K}$ of L_J. Consequently there is a unique $\mathcal{O}L_J$-lattice $\mathrm{St}_{J,\mathcal{O}}$ in $\mathrm{St}_{J,K}$ which is an epimorphic image of $\Gamma_{J,\mathcal{O}}$, which hence contains a projective cover of $\mathrm{St}_{J,\mathcal{O}}$. We set $N_{J,\mathcal{O}} = R_{L_J}^G(\mathrm{St}_{J,\mathcal{O}})$ and

$$N_R = R \otimes \bigoplus_{J \subseteq S} N_{J,\mathcal{O}},$$

for $R \in \{K, \mathcal{O}, k\}$. Similarly we Harish-Chandra induce the projective cover of $\mathrm{St}_{J,\mathcal{O}}$ and take the direct sum over all subsets J of S to obtain a projective presentation $\gamma_{\mathcal{O}} : Q_{\mathcal{O}} \to N_{\mathcal{O}}$. We have the following result:

Theorem 3.9. (Dipper/Gruber) *The kernel of γ_R is invariant under all endomorphisms of Q_R. Moreover $H_R(N_R) \cong Y_R$ and H_R induces an isomorphism from $\mathrm{End}_{RG}(Y_R)$ onto $\mathrm{End}_{\mathcal{H}_R}(Y_R)$.*

As a consequence of Theorem 3.1 the decomposition matrix of $\mathrm{End}_{\mathcal{H}_{\mathcal{O}}}(Y_{\mathcal{O}})$ embeds into the decomposition matrix of G. For a proof (and a much more general statement, dealing with the situation of Theorem 3.2) see [25]. It should be noted too that for $J \subseteq S$ the $\mathcal{O}G$-sublattice $N_{J,\mathcal{O}}$ of $M_{\mathcal{O}}$ contains $y_J M_{\mathcal{O}}$ and is contained in the purification $y_J M_K \cap M_{\mathcal{O}}$ of the latter, but is not equal to one of these in general.

The R-algebra $\mathrm{End}_{\mathcal{H}_R}(Y_R)$ is called the *q-Schur algebra* associated with $\mathcal{H}_R$ (of *parabolic type*) and it is denoted by $\mathcal{S}_R = \mathcal{S}_R(W)$ adding the parameters as additional indices if needed.

In [29] Dipper and James introduced q-Schur algebras associated with Weyl groups of type A and observed that their representations are closely related with those of finite general linear groups, (compare Section 4.3). Subsequently it was discovered by Dipper and Donkin that these come up as well as duals of the subcoalgebra of homogeneous parts in the quantum monoid of $(n \times n)$-matrices, for details see [21].

In [51] and [66] Geck, Gruber and Hiss defined q-Schur algebras (somewhat smaller than the q-Schur algebras of parabolic type introduced above) for Weyl groups of types B and D, and used them to relate the decomposition matrices of classical groups with those of general linear groups (see Section 4.4).

Parshall and Wang showed in [106] that q-Schur algebras of type A are quasi hereditary. This is not always true for other types. In [36] Du, Parshall and Scott found that those satisfy the weaker condition to be stratified algebras. For further results see e.g. [62] and [112].

In [37] Du and Scott and independently in [32] Dipper, James, and Mathas constructed quasi hereditary q-Schur algebras of type B, called *q-Schur^2* algebras in [37] and *(Q,q)-Schur algebras* in [32] by using not only q-analogues of right ideals $y_J\mathcal{H}$ coming from parabolic subgroups of W, but from arbitrary reflection subgroups as well.

In [33] Dipper, James, and Mathas generalized this further by constructing *cyclotomic q-Schur algebras* which are associated with Ariki-Koike algebras. These are deformations of the group algebra of certain complex reflection groups, which are wreath products of a cyclic group of order r by the symmetric group $\mathfrak{S}_n$. For $r = 1$ one obtains the Iwahori-Hecke algebra of type A_n and for $r = 2$ of type B_n. Thus the previous results appear as special cases of this construction.

In [63] R. Green used similar ideas to construct q-Schur algebras associated with affine Hecke algebras of type A.

Using the quantum group approach Oehms defined a different generalized q-Schur algebra of type B, called *symplectic q-Schur algebra* in his dissertation [102]. The Brauer algebra acts on tensor space centralizing the action of symplectic and orthogonal groups. A q-deformation of the Brauer algebra is the Birman-Murakami-Wenzl algebra which acts on tensor space as well. Using a generalization of the FRT-construction the symplectic quantized monoid is defined as centralizer coalgebra of this action. This is a bialgebra graded by subcoalgebras of finite rank, whose duals are defined to be the symplectic q-Schur algebras.

3.4 Specializations and decomposition maps

Let $\mathcal{H}$ be a generic Iwahori-Hecke algebra or a suitable q-Schur algebra associated with a finite Weyl group W. Then any specialization $\theta\colon A \to k$ into a field k gives rise to a specialized algebra $\mathcal{H}_k = k \otimes_A \mathcal{H}$. In Section 3.2 we have already encountered this situation in connection with Tits' Deformation Theorem. Now we consider the case where the specialized algebra is not semisimple. Then we have a non-trivial decomposition map between the Grothendieck groups of representations of $\mathcal{H}_K$ and of $\mathcal{H}_k$.

In this section, we consider only the case where all parameters u_s $(s \in S)$ equal a single parameter u, say. In view of our applications to finite groups of Lie type, we are mainly interested in specializations of the following type: let q be a prime power, ℓ be a prime not dividing q, and

$$\theta\colon A \to \bar{\mathbb{F}}_\ell \quad \text{such that} \quad \theta(u) = q \in \mathbb{F}_\ell.$$

By the theory of Hom-functors (see Section 3.1), the associated decomposition matrix of $\mathcal{H}$ is a submatrix of the decomposition matrix of a corresponding finite group of Lie type over $\mathbb{F}_q$. Now there are of course infinitely many choices for ℓ and q. But one can in fact control this situation by factorizing θ into two steps: one from A to a cyclotomic extension of $\mathbb{Q}$ and one from that cyclotomic extension to $\bar{\mathbb{F}}_\ell$.

More precisely, let $e \geq 1$ be minimal such that $\theta(u)^e = 1$. Then we can factorize θ as follows. Let $\zeta_e \in \mathbb{C}$ be a primitive e-th root of unity and $\theta_e\colon A \to \mathbb{C}$ be a ring homomorphism such that $\theta_e(u) = \zeta_e$. Then there exists a ring homomorphism $\theta_{e,\ell}\colon \mathbb{Q}[\zeta_e^{1/2}] \to \bar{\mathbb{F}}_\ell$ such that $\theta = \theta_{e,\ell} \circ \theta_e$.

Theorem 3.10. *Let $\theta\colon A \to \bar{\mathbb{F}}_\ell$ be a specialization as above and D be the corresponding decomposition matrix between the irreducible representations of $\mathcal{H}_K$ and $\bar{\mathbb{F}}_\ell \otimes_{A_0} \mathcal{H}$. Then we have a factorization $D = D_{e,\ell} \cdot D_e$, where D_e and $D_{e,\ell}$ are the decomposition matrices induced by θ_e and $\theta_{e,\ell}$, respectively.*

The article [59] provides a rather general framework in which a result as above can be proved. Thus, the decomposition matrix D_e can be viewed as a first approximation to D, and D itself is given by multiplying D_e with the "adjustment matrix" $D_{e,\ell}$ (a term introduced by James).

Example 3.11. Let $\mathcal{H}$ be an Iwahori-Hecke algebra of type G_2. Then D_e is not the identity matrix only for $e \in \{2, 3, 6\}$. In these cases, we have:

D_e	$e = 2$	$e = 3$	$e = 6$
ind	1 . .	1	1 . . . .
ε_1	1 . .	. 1 . . .	. 1 . . .
ε_2	1 . .	. . 1 . .	. . 1 . .
ε	1 . .	. . . 1 .	. . . 1 .
χ_1	. 1 .	 1	1 . . 1 .
χ_2	. . 1	1 . . 1 .	 1

If $\ell \in \{2, 3\}$ we only have to consider adjustment matrices $D_{e,\ell}$ for $(e, \ell) \in \{(2, 2), (2, 3), (3, 3)\}$; they are given as follows:

$$\begin{pmatrix} 1 & . \\ . & 1 \\ . & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . \\ 2 & . \\ . & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ . & . & 1 & 1 \end{pmatrix}, \qquad \text{respectively.}$$

If $\ell > 3$, then $D_{e,\ell}$ is always the identity matrix of the appropriate size. These results are easily proved by explicit computations using the matrix representations given in [16, §67C].

The above example seems to indicate that most adjustment matrices are in fact identity matrices. A general argument (which is given for Iwahori-Hecke algebras in [44, Prop. 5.5]) shows that this is indeed the case for all sufficiently large primes ℓ. The term "sufficiently large" is specified in the following conjecture, which was first stated by James [84, §4] for Iwahori-Hecke algebras and q-Schur algebras of type A_{n-1} and then formulated for Iwahori-Hecke algebras of any type in [44, (5.6)].

Conjecture 3.12. If $\mathcal{H}$ is the generic Iwahori-Hecke algebra associated with a finite Weyl group, then the adjustment matrix $D_{e,\ell}$ is the identity matrix if ℓ does not divide the order of W.

Probably, in the case of q-Schur algebras and Iwahori-Hecke algebras of type A_{n-1}, this result will even be true under the strengthened condition that $e\ell > n$. The conjecture is established in [44] for blocks of defect 1, where the decomposition matrix is given in terms of a Brauer tree, similarly as for blocks of defect 1 in finite groups.

For explicit results on Iwahori-Hecke algebras of classical type, see Section 3.5. The known results for exceptional types are summarized as follows:

– For type G_2 with equal parameters, see Example 3.11. (The case of unequal parameters is easily treated by similar methods.)

- The matrices D_e are completely determined for types F_4, E_6 and E_7; see [55], [45], [48], and [98]. Partial results on the matrices D_e for type E_8 are obtained by J. Müller [98]. The techniques used include R. Parker's MEATAXE [105] and S. Linton's VECTOR-ENUMERATOR [91]. In [99], the analogous matrices for the non crystallographic types $I_2(m)$, H_3, H_4 are determined.
- The above conjecture is established by explicit computation of all possible decomposition matrices for types F_4 (with equal parameters) and E_6, see [55] and [45], respectively. K. Bremke [5] has considered those cases of unequal parameters in type F_4 which actually occur in groups of Lie type.

In the case of Iwahori-Hecke algebras of exceptional type or of type A_n, B_n, D_n (n odd) the following result can be extracted from the above explicit results and the results in [26,34,104].

Theorem 3.13. *(Geck [49]) If $\mathcal{H}$ is an Iwahori-Hecke algebra of a finite Weyl group (where all parameters are equal), then the matrix D_e always has a lower unitriangular shape, for a natural ordering of the rows and columns given in terms of the relation $\leq_{LR}$ of Kazhdan-Lusztig [86].*

3.5 Iwahori-Hecke and q-Schur algebras of classical type

We now turn to the Iwahori-Hecke algebras of classical type. We begin with the type A case. Thus $W = \mathfrak{S}_n$ is the symmetric group, $\mathcal{H} = \mathcal{H}_R(W;q)$, where R is a commutative ring and q is a unit of R. For fields R the irreducible $\mathcal{H}$-modules were determined by Dipper and James in [26, 7.6] and it was shown that their decomposition matrices are lower unitriangular.

For a partition $\lambda \vdash n$ of n let $\mathfrak{ST}_\lambda$ denote the set of standard λ-tableaux. Murphy introduced in [100] and [101] a basis

$$\{x_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathfrak{ST}_\lambda, \lambda \vdash n\}$$

of $\mathcal{H}$, called *Murphy basis*, which possesses remarkable properties. Indeed it is a cellular basis as defined by Graham and Lehrer in [60], and hence their general results on cellular algebras may be used to recover most of the results in [26] and some of [28]. We apply now the general theory of cellular algebras [60]. For a partition λ of n we define N^λ to be the R-module spanned by

$$\{x_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathfrak{ST}_\mu, \mu \vdash n, \mu \trianglerighteq \lambda\}$$

and $\bar{N}^\lambda$ to be the subspace spanned by those $x_{\mathfrak{st}}$ where $\mu \triangleright \lambda$.

Theorem 3.14. *Let λ be a partition of n. Then N^λ and $\bar{N}^\lambda$ are two sided ideals of $\mathcal{H}$.*

The *Specht* (or *standard*) *module* S^λ is now the submodule of $\mathcal{H}/\bar{N}^\lambda$ generated by $z_\lambda = x_\lambda + \bar{N}^\lambda/\bar{N}^\lambda \in \mathcal{H}/\bar{N}^\lambda$, where $x_\lambda := \sum_{w \in W_\lambda} T_w$. On S^λ we have an $\mathcal{H}$-invariant bilinear form whose radical is denoted by $\mathrm{rad} S^\lambda$. We set $D^\lambda = S^\lambda/\mathrm{rad} S^\lambda$.

Theorem 3.15. *Suppose R is a field. Then the non-zero $\mathcal{H}$-modules in*

$$\{D^\lambda \mid \lambda \vdash n\}$$

is a complete set of irreducible $\mathcal{H}$-modules. If we choose a linear ordering of the partitions of n which is compatible with the dominance order downward then the matrix Δ describing the composition factors of Specht modules is lower unitriangular. If R is a field and $\mathcal{H}$ is semisimple then $(0) \neq S^\lambda = D^\lambda$ for every partition λ of n. Thus

$$\{S^\lambda \mid \lambda \vdash n\}$$

is a complete set of non-isomorphic irreducible $\mathcal{H}$-modules in this case.

It should be pointed out that the Kazhdan-Lusztig basis of $\mathcal{H}$ as defined in [86] is cellular in type A but not in general.

Theorem 3.15 states what can directly be concluded from the fact that $\mathcal{H}$ has a cellular basis. There are some further results which however do not follow directly from properties of a cellular basis, but require special proofs. In particular one knows which of the D^λ are nonzero in Theorem 3.15 by a result of Dipper and James [26, 7.6]. Thus let R be a field. It is known that $\mathcal{H}$ is not semisimple if and only if the parameter $q \in R$ is a primitive eth root of unity for some $1 < e \leq n$ or $q = 1$ and R is of characteristic $\ell \leq n$. We define e to be ℓ in the latter case. A partition λ is called e-regular, if there are at most $e - 1$ repetitions in λ. We then have:

Theorem 3.16. (Dipper/James) *Let e be as above. Then $D^\lambda \neq (0)$ precisely if λ is e-regular.*

The blocks of $\mathcal{H}$ are given by e-cores respectively p-cores in the group case by [28, 4.13] and [85, 4.29], and the primitive central idempotents were exhibited in [28, 5.4] to name a few further results.

In the special case of a field R of characteristic 0, where q is a primitive eth root of unity, an algorithm how to compute the decomposition matrices explicitly will be discussed in more detail in the next section.

We come now to other types of Iwahori-Hecke algebras and q-Schur algebras. In [97] Mathas produced a cellular basis of Ariki-Koike algebras generalizing results of Dipper, James and Murphy [34] on Iwahori-Hecke algebras of type B. Here the labels of the basis elements are standard multi-tableaux whose shapes are multi-partitions, which in turn label the standard and the irreducible modules. The decomposition matrix is again lower unitriangular. The question of actually classifying the irreducible modules is presently still open, although a lot of evidence towards a conjecture concerning this question has been obtained, (see [34, 8.13] for the case of Iwahori-Hecke algebras of type B and [97] for its generalization to Ariki-Koike algebras).

A similar picture evolves for q-Schur algebras and their generalizations. The case of type A was investigated first in [29] and [30] by Dipper and James. They defined q-Weyl modules, labelled by partitions of n and determined a semistandard basis of those, labelled by semistandard tableaux. The q-Weyl modules have unique maximal submodules for fields R and the unique irreducible quotients are precisely the irreducible modules for the q-Schur algebra. The decomposition matrix describing the composition factors of the q-Weyl modules is a square matrix and is unitriangular.

In [35] Du extended the Kazhdan-Lusztig basis for Iwahori-Hecke algebras to q-Schur algebras, and showed that it is cellular. He also found that it is dual to Lusztig's canonical basis of quantum groups.

In his thesis [61] Richard Green extended the Murphy basis of the Iwahori-Hecke algebra of type A to q-Schur algebras using bideterminants and showed that it is cellular.

The cyclotomic q-Schur algebras (and hence the special cases for type A and B of Section 3.3) are cellular ([33]) as well as the symplectic q-Schur algebras of Oehms in [102]. Moreover, in both cases the bilinear form on standard modules are non-zero, and hence the decomposition matrices are square matrices and the algebras are quasi hereditary.

Under a special assumption there is a direct way to investigate Iwahori-Hecke and q-Schur algebras of type B, yielding more complete results. Thus let $W = C_2 \wr \mathfrak{S}_n$ be the Weyl group of type B of rank n. The Iwahori-Hecke algebra $\mathcal{H}$ involves two parameter q and Q. Let

$$f_n(Q,q) = \prod_{i=-(n-1)}^{n-1} (Q + q^i).$$

Theorem 3.17. (Dipper/James) *Let R be a commutative ring, and let q, Q be units of R. Suppose that $f_n(Q,q) \neq 0$. Then $\mathcal{H}$ is Morita equivalent to*

$$\bigoplus_{a=0}^{n} \mathcal{H}_R(\mathfrak{S}_a; q) \otimes \mathcal{H}_R(\mathfrak{S}_{n-a}; q).$$

For details we refer to [31]. As a consequence one obtains:

Theorem 3.18. (Gruber/Hiss) *Let $R, Q, q, \mathcal{H}$ be as above and suppose again that $f_n(Q,q) \neq 0$. Then the (Q,q)-Schur algebra $\mathcal{S}$ is Morita equivalent to the corresponding q-Schur algebra of parabolic type which in turn is Morita equivalent to the direct sum of q-Schur algebras associated with the Iwahori-Hecke algebras $\mathcal{H}_R(\mathfrak{S}_a; q) \otimes \mathcal{H}_R(\mathfrak{S}_{n-a}; q)$, where a runs from 0 to n.*

This theorem was shown in [66] for those choices of parameters occurring in the representation theory of finite classical groups. The condition $f_n(Q,q) \neq 0$ holds in roughly a third of all possible cases for these groups. Gruber and Hiss used their representations to establish Theorem 3.18 and to show even that using only parabolic subgroups of type A produces another Morita equivalent q-Schur algebra. However Theorem 3.18 holds in general as can be seen easily from [32]. Obviously Theorems 3.17 and 3.18 allow to carry results for the type A-case to those for the type B_n-case if $f_n(Q,q) \neq 0$. In particular using 3.16 there is a complete description of the irreducible modules and decomposition matrix of $\mathcal{H}$ in terms of Iwahori-Hecke algebras of type A and that of the corresponding q-Schur algebra of type B in terms of q-Schur algebras of type A. In Section 4.4 this will be applied to representations of classical groups.

Iwahori-Hecke algebras of type D seem far less accessible. In particular there is no cellular basis known, and only q-Schur algebras of parabolic type have been considered so far. Pallikaros generalized Theorem 3.17 to Iwahori-Hecke algebras of type D in [104]. However, his results are not as complete as in the type B case. In particular, he does not get a complete classification of the irreducible modules in the case of even n. Note, however, that the triangular shape for the decomposition matrix is established in [49], using a completely different approach. There are some more results in special cases, due to Gruber and Hiss in [66], using the fact that Iwahori-Hecke algebras of type D are contained as subalgebras in Iwahori-Hecke algebras of type B for a special choice of the second parameter.

There are many numerical results on decomposition matrices of Iwahori-Hecke and q-Schur algebras, of which we shall mention only two, since they are connected with Conjecture 3.12. Mathas wrote the share package SPECHT in GAP [114] to compute decomposition numbers of Iwahori-Hecke algebras of type A at roots of unity in characteristic zero. In his thesis which is under preparation, Lipp used an implementation in GAP of combinatorial rules [84] for the computation of decomposition numbers of q-Schur algebras of type A to verify the James conjecture in many special cases (compare Section 3.4). In general he shows that the conjecture holds for fields R of small characteristic and based on results of Richards [111] proved a block wise version of James conjecture for blocks of weight ≤ 3.

3.6 The theory of Lascoux-Leclerc-Thibon

Let $\mathcal{H}_n$ be the generic Iwahori-Hecke algebra associated with the symmetric group $\mathfrak{S}_n$. Consider a specialization $\theta: A \to \mathbb{C}$ which takes the parameter u to a primitive e-th root of unity (for some $e \geq 1$), and let D_e be the corresponding decomposition matrix. Lascoux, Leclerc and Thibon [89] conjectured that the problem of computing D_e can be translated to that of computing the canonical basis (in the sense of Kashiwara and Lusztig) of a certain highest weight module for an affine Kac-Moody algebra. This has now been proved by Ariki [3]. (In October 1995, Grojnowski also announced a proof, following the ideas in [64].) Here, we shall only describe the purely algorithmic part of the solution.

For any $n \geq 0$, let $\mathcal{F}_n$ be the $\mathbb{Q}(u)$-vector space with a basis consisting of the partitions $\lambda \vdash n$; by convention, $\mathcal{F}_0$ has as basis the empty partition, $\emptyset$. Let

$$\mathcal{F} := \bigoplus_{n \geq 0} \mathcal{F}_n, \qquad \text{the so-called “Fock space”}.$$

As usual, a partition $\lambda \vdash n$ is represented by a Young diagram, which is a collection of n boxes arranged in left-justified rows, where the number of boxes in each row is given by the parts of λ.

The e-residue of a box which occupies the j-th position in the i-th row say, is the unique $r \in \{0, \ldots, e-1\}$ such that $r \equiv j - i \pmod e$.

Next, if we can remove the right-most box in a row and the result is still a Young tableau (for a partition of $n-1$) we call that box a removable box. (For example, the right-most box in the last row is always removable.)

Finally, if we can add a new box to some row and the result is again a Young tableau (for a partition of $n+1$), we call that row an indent row. (For example, we can always add a box to the first row; note that it is allowed to add a new row consisting of one box to the bottom of the diagram.) The e-residue of such a row is the e-residue of the new box that we can add.

We shall now define linear maps $f_r\colon \mathcal{F} \to \mathcal{F}$ $(0 \leq r \leq e-1)$ such that $f_r(\mathcal{F}_n) \subseteq \mathcal{F}_{n+1}$ for all n, r. Let $\lambda \vdash n$ and $0 \leq r \leq e-1$. Suppose that the i-th row is an indent row of λ whose e-residue is r. Then we let $N(\lambda, r, i)$ be the number of indent rows strictly above i whose e-residue is r, minus the number of removable boxes in rows above i whose e-residue is r. We define

$$f_r(\lambda) = \sum\nolimits_i u^{N(\lambda,r,i)}\, \lambda_{[i]},$$

where the sum is over all indent rows i of λ with e-residue r, and $\lambda_{[i]}$ denotes the partition of $n+1$ obtained by adding a new box to the i-th row. For any $k \geq 1$, we let $[k]_u := (u^k - u^{-k})/(u - u^{-1})$, and define operators $f_r^{(k)}$ by

$$f_r^{(1)} := f_r \quad \text{and} \quad f_r^{(k)} = f_r^{(k-1)}/[k]_u \quad (k \geq 2).$$

Let $\mathcal{K}_n \subseteq \mathcal{F}_n$ be the subspace generated by all $f_\iota^\kappa(\emptyset) := f_{i_n}^{(k_n)} \cdots f_{i_1}^{(k_1)}(\emptyset)$, where $\iota = (i_1, \ldots, i_n)$ and $\kappa = (k_1, \ldots, k_n)$ are any sequences of integers with $0 \leq i_1, \ldots, i_n \leq e-1$ and $k_1, \ldots, k_n \geq 1$.

Lemma 3.19. *For any e-regular partition $\nu \vdash n$, there exist ι, κ as above such that*

$$K_\nu := f_\iota^\kappa(\emptyset) = \nu + \mathbb{Z}[u^{\pm 1}]\text{-linear combination of various } \lambda \vdash n,\ \lambda \lhd \nu.$$

Here, $\lhd$ denotes the usual dominance order. Note that this element depends on the choice of ι, κ, but for any choice, the elements $\{K_\nu\}$ form a basis of $\mathcal{K}_n$.

The proof in [89, §6] also yields an algorithm for finding suitable sequences ι, κ for each ν. Now, using an interpretation of $\mathcal{F}$ as a module for an affine Kac-Moody algebra and applying the theory of canonical bases, it can be shown that there exist unique elements $\{G_\mu\}$ in $\mathcal{K}_n$ (for e-regular $\mu \vdash n$) such that

$$G_\mu = \mu + \sum_{\lambda \vdash n, \lambda \lhd \mu} d_{\lambda\mu}(u)\lambda \quad \text{where } d_{\lambda\mu}(u) \in \mathbb{Z}[u] \text{ and } d_{\lambda\mu}(0) = 0.$$

Lemma 3.20. *Each G_μ is uniquely a linear combination $G_\mu = \sum_\nu \gamma_{\nu\mu}(u) K_\nu$ where $\gamma_{\nu\mu}(u) \in \mathbb{Z}[u, u^{-1}]$ are such that $\gamma_{\nu\mu}(u^{-1}) = \gamma_{\nu\mu}(u)$.*

The proof in [89, §6] shows that the $\gamma_{\nu\mu}(u)$ can be computed by a "triangular algorithm". All this is implemented in Mathas' GAP package SPECHT [114], and in Veigneau's package ACE [116]. Here is an example, for $e = 2$ and $n = 5$:

	$K_{(5)}$	$K_{(41)}$	$K_{(32)}$
κ :	11111	1211	122
ι :	01010	0101	010
(5)	1	.	.
(41)	.	1	.
(32)	1	.	1
(311)	$2u$	.	u
(221)	u^2	.	u^2
(2111)	.	u	.
(11111)	u^2	.	.

$$G_{(32)} = K_{(32)}$$

$$G_{(41)} = K_{(41)}$$

$$G_{(5)} = K_{(5)} - K_{(32)}$$

Now let us finally come to D_e, the decomposition matrix that we are trying to compute. By [26], we know already that D_e has rows labelled by all partitions $\lambda \vdash n$ and columns labelled by the e-regular partitions $\mu \vdash n$. The following result, which is Conjecture 6.9(ii) in [89], shows that D_e is determined by G_μ:

Theorem 3.21. (Ariki [3, Theorem 4.4]) *For any e-regular partition $\mu \vdash n$, let us write $G_\mu = \sum_{\lambda \vdash n} d_{\lambda\mu}(u)\,\lambda$ where $d_{\lambda\mu}(u) \in \mathbb{Z}[u]$. Then we have $D_e = (d_{\lambda\mu}(1))$.*

We remark that Ariki proves in fact a more general result, which also applies to Iwahori-Hecke algebras of type B_n and even to the cyclotomic Hecke algebras associated with the wreath product of $\mathfrak{S}_n$ and a finite cyclic group. Moreover, Leclerc and Thibon [90] have proposed a generalization to the q-Schur algebra.

4 Decomposition matrices

The concept of the decomposition matrix provides a link between the "ordinary" complex representations of a finite group and the "modular" representations over a field of prime characteristic. In the case of finite groups of Lie type, the ordinary character tables are relatively well understood, due to the work of Lusztig and Shoji (see [93,115] and the references there). The situation for modular representations in the defining characteristic is settled by Lusztig's conjecture, if that characteristic is not too small (see [2] and the references there). In this section we consider the case of non-defining characteristic, and give a survey of the present state of knowledge. Most of the results were first proved for the groups $\mathrm{GL}_n(q)$ (see [18,29]). The precise formulation of possible generalizations to other types of groups was in many cases only possible after extensive calculations in exceptional groups of low rank (see [70]).

4.1 Basic sets

The notion of a basic set (as defined below) was originally conceived by R. Parker as a computational tool for the explicit determination of decomposition numbers of sporadic simple groups. Since then, it has also proved useful as a conceptual tool in the modular representation theory of finite groups of Lie type.

First we give the general definitions. Let G be a finite group, ℓ a prime, and $\mathcal{B}$ a union of ℓ-blocks of G. We take the point of view that we know the ordinary irreducible characters of G belonging to $\mathcal{B}$. Let D be the decomposition matrix of $\mathcal{B}$. One of the most basic theoretical results that we have is that the elementary divisors of D are all 1. This leads to the following:

Definition 4.1. A subset of the set of ordinary irreducible characters in $\mathcal{B}$ is called an *ordinary basic set* for $\mathcal{B}$ if the submatrix of D with rows labelled by the elements in that subset has determinant ± 1.

More generally, a set of Brauer characters is called a *basic set* for $\mathcal{B}$ if it is linearly independent and if every irreducible Brauer character belonging to $\mathcal{B}$ is an integral linear combination of the elements of that set. Thus, if we have an ordinary basic set for $\mathcal{B}$, the restrictions of its members to the ℓ'-elements of G yield a basic set for $\mathcal{B}$. In the literature, one sometimes finds a definition where also generalized Brauer characters are allowed to be members of a basic set, but we do insist on having actual characters.

Note that basic sets always exist (e.g., take the irreducible Brauer characters belonging to $\mathcal{B}$), but it is an open question whether or not ordinary basic sets always exist. (K. Lux has checked that ordinary basic sets exist, for example, for all blocks of sporadic simple groups.)

The importance of the existence of an ordinary basic set lies in the following facts. First of all, its cardinality is the number of irreducible Brauer characters in $\mathcal{B}$. Moreover, any ordinary irreducible character in $\mathcal{B}$ outside the basic set can be written—on the ℓ'-elements of G—as an integral linear combination of the elements in the basic set. Thus, in order to know the whole decomposition matrix of $\mathcal{B}$, it is sufficient to determine that part of it which has rows labelled by the elements in the basic set. Note that this is a square matrix of non-negative integers with determinant ± 1.

It has turned out that, under some mild conditions, blocks of finite groups of Lie type always possess ordinary basic sets which have a simple description in terms of Lusztig series. To state these results, we fix the following notation. Let $\mathbf{G}$ be a connected reductive group defined over the finite field $\mathbb{F}_q$, with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. Let $\mathbf{G}^*$ be the dual group over $\mathbb{F}_q$, with Frobenius map F^*. Recall that we have a partition of the irreducible characters of the finite group $\mathbf{G}^F$ into Lusztig series, $\mathcal{E}(\mathbf{G}^F, s)$, one for each F^*-stable class of semisimple elements $s \in \mathbf{G}^*$.

Let ℓ be a prime not dividing q. For each F-stable semisimple element $s \in \mathbf{G}^*$ of order prime to ℓ, we have a corresponding union of ℓ-blocks $\mathcal{E}_\ell(\mathbf{G}^F, s)$; see Theorem 2.19. We have the following general result:

Theorem 4.2. (Geck/Hiss [50,46]) *Assume that ℓ is a good prime for each simple component of G, i.e., $\ell \neq 2$ in types B_n, C_n, D_n, $\ell \neq 2, 3$ in types G_2, F_4, E_6, E_7, and $\ell \neq 2, 3, 5$ in type E_8. Assume, moreover, that ℓ does not divide the* *order of the largest quotient of the group $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ on which F acts trivially. Then, for any F^*-stable semisimple element $s \in \mathbf{G}^*$ of order prime to ℓ, the characters in $\mathcal{E}(\mathbf{G}^F, s)$ form an ordinary basic set for $\mathcal{E}_\ell(\mathbf{G}^F, s)$.*

If the assumption on ℓ or on the center of $\mathbf{G}$ is not satisfied, the assertion of the above theorem does not necessarily hold; for example, take $\mathbf{G}^F = \mathrm{SL}_3(q)$ where $\ell = 3 \mid q-1$ or $\mathbf{G}^F = G_2(q)$ and $\ell = 2$. Nevertheless, we believe that ordinary basic sets for finite groups of Lie type in non-defining characteristic will always exist. Some partial results in this direction are obtained in Part III of [46], where 2-modular Brauer characters of groups of classical type and with a connected center are considered. In particular, it is shown that in this case the number of irreducible Brauer characters in $\mathcal{E}_2(\mathbf{G}^F, s)$ equals the number of unipotent classes of $C_{\mathbf{G}^*}(s)^{*F^*}$.

In [51], we have found the number of irreducible Brauer characters in $\mathcal{B} = \mathcal{E}_\ell(\mathbf{G}^F, 1)$ for exceptional groups $\mathbf{G}$ with a connected center, under some mild restrictions on q. We take this opportunity to correct an error in the tables in [51, (5.5)(c) and (6.6)] which was found by F. Lübeck. The error is caused by a preceding error in [115, (5.1)], where the supporting sets for cuspidal character sheaves in type E_8 are stated incorrectly. (Details of the corrections will appear elsewhere.) The correct numbers of irreducible Brauer characters in $\mathcal{E}_\ell(\mathbf{G}^F, 1)$, for bad primes ℓ, are given as follows:

Type	$\ell = 2$	$\ell = 3$	$\ell = 5$
G_2	9	8	
F_4	28	35	
E_6	27	28	
E_7	64	72	
E_8	131	150	162

(Thus it is not necessary to consider the case of type E_8 with q a power of 3 separately.) We hope to be able to settle the question of the existence of ordinary basic sets in these cases by explicitly computing the necessary parts of the character tables in the framework of CHEVIE [52].

4.2 Triangular shape

Consider again a union $\mathcal{B}$ of ℓ-blocks for some finite group G, and let D be the corresponding decomposition matrix. Let $\mathrm{IBr}_\ell(\mathcal{B})$ be the set of irreducible Brauer characters in $\mathcal{B}$. Assume that there exists an ordinary basic set S for $\mathcal{B}$; see Definition 4.1. Let D_S be the submatrix of D with rows labelled by the elements in S.

Definition 4.3. We say that D has a triangular shape with respect to the ordinary basic set S, if it is possible to order the sets S and $\mathrm{IBr}_\ell(\mathcal{B})$ in such a way that the matrix D_S has a lower triangular shape with 1 along the diagonal.

There are examples where this is not possible; e.g., for $G = \mathrm{SL}_2(8)$ and $\ell = 2$. But there are no counter-examples known for finite groups of Lie type in non-defining characteristic! The advantage of having a triangular shape as above is that we then have a canonical labelling of the irreducible Brauer characters in $\mathcal{B}$ by the ordinary characters in S. Indeed, we have:

Lemma 4.4. *Assume that D has a triangular shape with respect to S. Choose labellings $S = \{\chi_1, \ldots, \chi_m\}$ and $\mathrm{IBr}_\ell(\mathcal{B}) = \{\theta_1, \ldots, \theta_m\}$ such that the corresponding matrix D_S is lower triangular with 1 along the diagonal. Then the bijection*

$$S \leftrightarrow \mathrm{IBr}_\ell(\mathcal{B}), \qquad \chi_i \leftrightarrow \theta_i \quad (1 \leq i \leq m),$$

does not depend on the chosen labellings.

Now consider again a finite group of Lie type $\mathbf{G}^F$ over $\mathbb{F}_q$ as in the previous section, and let ℓ be a prime not dividing q.

As in the case of ordinary characters, the union of unipotent blocks $\mathcal{E}_\ell(\mathbf{G}^F, 1)$ plays a distinguished role in the modular representation theory of the group $\mathbf{G}^F$. A reduction to this case may be possible by an—as yet conjectural—Jordan decomposition of blocks and decomposition numbers. More precisely, assume that $\mathbf{G}$ has a connected center and that $s \in \mathbf{G}^*$ is an F^*-stable semisimple element of order prime to ℓ.

Conjecture 4.5. (Broué [6, Section 2A]/Hiss [70, Section 4.4]) If the ordinary irreducible characters in $\mathcal{E}(\mathbf{G}^F, s)$ and in $\mathcal{E}(C_{\mathbf{G}^*}(s)^{*F^*}, 1)$ are matched via the Jordan decomposition of characters (see Section 2.3), then the decomposition matrix of $\mathcal{E}(\mathbf{G}^F, s)$ equals that of $\mathcal{E}(C_{\mathbf{G}^*}(s)^{*F^*}, 1)$.

In the case where $C_{\mathbf{G}^*}(s)$ is a Levi $\mathbf{L}^*$ subgroup of $\mathbf{G}^*$, the Jordan decomposition is given by the operator R_L^G and Broué has in fact conjectured that this should define a Morita equivalence between the corresponding unions of blocks. We will come back to these questions below in the discussion of general linear groups and the linear prime case for classical groups.

By Theorem 4.2, the unipotent characters of $\mathbf{G}^F$ form an ordinary basic set for $\mathcal{E}_\ell(\mathbf{G}^F, 1)$ provided that ℓ is a good prime for $\mathbf{G}$ and the center of $\mathbf{G}$ is connected. Suppose we are in this situation. Then it is known that the corresponding decomposition matrix has a triangular shape in the following cases:

- $\mathbf{G}^F = \mathrm{GL}_n(q)$, any ℓ; see Section 4.3 below. In this case, we also have a Jordan decomposition of blocks and decomposition numbers.
- $\mathbf{G}^F$ is a finite classical group, ℓ is a linear prime; see Section 4.4. There is also a Jordan decomposition of blocks and decomposition numbers.
- $\mathbf{G}^F = \mathrm{GU}_n(q)$, any ℓ; see [42].
- $\mathbf{G}^F$ is an exceptional group of low rank (possibly with further conditions on q or ℓ); see [70] for $G_2(q)$, [43] for ${}^3D_4(q)$, and [120] for $F_4(q)$. A discussion of a special case in $E_6(q)$ is contained in [51, §7].

Some partial results are known in the bad prime case. It is shown in [46, Part III] that if $\mathbf{G}^F$ is a finite classical group, q is large and $\ell = 2$, then the decomposition matrix of $\mathcal{E}_2(\mathbf{G}^F, 1)$ also has a triangular shape but only with respect to ordinary basic sets which do not necessarily consist of unipotent characters.

Among all irreducible characters, the cuspidal ones play a special role from the viewpoint of Harish-Chandra theory. It has been shown by Dipper and James [18] that, if $\mathbf{G}^F = \mathrm{GL}_n(q)$, then the ℓ-modular reduction of an ordinary cuspidal irreducible character remains irreducible as a Brauer character, and all cuspidal irreducible Brauer characters arise in this way. This need not be the case for other types of groups: there are examples of cuspidal ordinary characters which do not remain irreducible under ℓ-modular reduction; and there are examples of cuspidal Brauer characters which do not occur in the ℓ-modular reduction of any ordinary cuspidal irreducible character. However, the second named author has formulated in his Ph.D. thesis (1990) the following:

Conjecture 4.6. (Geck) The ℓ-modular reduction of a cuspidal *unipotent* character remains irreducible as a Brauer character.

If this were true, we would then have a connection between the theory of Harish-Chandra series of ordinary characters and of ℓ-modular Brauer characters. The conjecture is known to be true in the following cases:

- $\mathbf{G}^F = \mathrm{GU}_n(q)$, any ℓ; see [53].
- $\mathbf{G}^F$ is a finite classical group, q is large and $\ell = 2$; see [56].
- $\mathbf{G}^F$ is an exceptional group of low rank (possibly with further conditions on q or ℓ); see [70] for $G_2(q)$, [43] for type ${}^3D_4(q)$, and [51, §7] for $E_6(q)$.

The proofs of these results use the fact that the decomposition matrix is known to have a triangular shape in these cases, and a kind of Harish-Chandra theory for generalized Gelfand-Graev representations in order to produce "sufficiently many zeroes" in the decomposition matrix (see [56]).

4.3 Decomposition matrices of general linear groups

For $G = \mathbf{G}^F = \mathrm{GL}_n(q)$ we have a complete classification of the ℓ-modular irreducible representations in terms of q-Schur algebras. The ordinary irreducible characters in geometric conjugacy classes belonging to semisimple ℓ-regular conjugacy classes of G constitute an ordinary basic set for G and the ℓ-modular decomposition matrix of G is unitriangular. Indeed the computation of the ℓ-modular decomposition matrix of G can be completely reduced to the computation of decomposition matrices of q-Schur algebras of type A using the approach of Section 3.3. How this is done will be described briefly below.

Thus let $G = \mathrm{GL}_n(q)$ for some prime power q, and let ℓ be a prime not dividing q. Here $G^* = G$ hence the geometric conjugacy classes are labelled by conjugacy classes of semisimple elements of G. We choose a set $\mathcal{C}$ of representatives of these classes, and for $s \in \mathcal{C}$ we denote the corresponding geometric conjugacy class of G by $\mathcal{E}(G,s)$. For general linear groups geometric conjugacy classes coincide with Harish-Chandra series.

We let $(K,\mathcal{O},k)$ be a split ℓ-modular system for G and fix $s \in \mathcal{C}$. We choose a minimal Levi subgroup $L = L_s$ containing s. Then the corresponding cuspidal $\mathcal{O}L$-lattice $C_{\mathcal{O}}(s)$ is regular, that is it occurs in the Gelfand-Graev representation of L (see Section 3.1).

We concentrate now on the special case where the ℓ-regular part s' of s is the identity 1 of G. This is justified, since the union $\mathcal{E}_\ell(G,s)$ of all ℓ-blocks of G with semisimple part s (compare Theorem 2.19) has the same decomposition matrix as the union $\mathcal{E}_\ell(C_G(s),1)$ of unipotent ℓ-blocks of $C_G(s)$, i.e., Conjecture 4.5 holds for G. There is, however, so far no independent proof of this fact, it follows a posteriori by showing that the decomposition matrices of the same q-Schur algebras determine these decomposition matrices with the same algorithms. Everything what is said below can be generalized to the case of arbitrary $s \in \mathcal{C}$.

Thus s is an ℓ-element of G. Then $W(L_s,1)$ in Theorem 3.2 is the normalizer $W(L_s)$ of L_s in the Weyl group W of G. We have:

Theorem 4.7. (Dipper) *$C_k(s) = k \otimes C_{\mathcal{O}}(s)$ is an irreducible cuspidal kL-module. There exists an ℓ-element $t \in \mathcal{C}$ such that $L = L_t = L_s$ and the subgroup $W(L,t)$ of $W(L)$ equals $W(L)$. Moreover $C_k(t) \cong C_k(s)$ and $W(L,s)$ is a parabolic subgroup of $W(L) = W(L,t)$.*

This result can be used to reduce the computation of the decomposition numbers of the geometric conjugacy class $\mathcal{E}(G,s)$ completely to those of $\mathcal{E}(G,t)$. For details and a proof we refer to [20, Section 4] and [29]. With other words it suffices to calculate decomposition numbers of Harish-Chandra series $\mathcal{E}(G,s)$ satisfying the hypothesis of Theorem 3.2. In particular we may assume that $\mathcal{H}_R(W(L,s)) = R \otimes_{\mathcal{O}} \mathcal{H}(W(L,s))$ for $R \in \{K,\mathcal{O},k\}$, and write $\mathcal{H}_R(s) := \mathcal{H}_R(W(L,s))$. Similarly, we write $\mathcal{S}_R(s) := \mathcal{S}_R(W(L,s))$ for the corresponding q-Schur algebras.

Let $e = \ell$ if ℓ divides $q-1$ and e be the multiplicative order of q modulo ℓ otherwise. The standard and the irreducible $\mathcal{H}_R(s)$-modules are now labelled by multi-partitions respectively regular multi-partitions corresponding to the tensor factors of $\mathcal{H}_R(s)$, where $R = K$ or k. Here *regular* means e_i-regular in the ith part were $e_i = e$ if $d_i = 1$ and $e_i = \ell$ otherwise. We now apply Theorems 3.1 and 3.9 (in their general form [25]) to obtain:

Theorem 4.8. (Dipper/James) *The decomposition matrix Δ_s of the q-Schur algebra $\mathcal{S}_{\mathcal{O}}(s)$ is part of the decomposition matrix of $\mathcal{E}(G,s)$. The columns corresponding to regular multi-partitions are the columns of the decomposition matrix of $\mathcal{H}_{\mathcal{O}}(s)$. If $L_s = \mathrm{GL}_d(q)^m$ for natural numbers m,d with $md = n$, then Δ_s is the complete ℓ-modular decomposition matrix of $\mathcal{E}(G,s)$.*

This was proved in [29] up to the last part, which is a result of James in [83]. Consider now the special case $s = 1$. Then L can be taken as the Borel subgroup of all lower triangular matrices in G and $\mathcal{H}_R(s) = \mathcal{H}_R(\mathfrak{S}_n;q)$. The rows and columns of the decomposition matrix Δ_1 of the corresponding q-Schur algebra are labelled by partitions of n. An easy counting argument (applied to all semisimple ℓ-regular elements $s \in \mathcal{C}$) shows that Δ_1 is the complete decomposition matrix of $\mathcal{E}(G, 1)$, that is, we have a complete list of the irreducible kG-modules in the union of the unipotent blocks. We have thus:

Theorem 4.9. (Dipper/James) *There is a bijection between the irreducible kG-modules and the irreducible $\mathcal{S}_{\mathcal{O}}(s)$-modules, where s runs through a set of representatives of semisimple ℓ-regular conjugacy classes of G. Moreover the ℓ-modular decomposition matrix of G is lower unitriangular.*

The irreducible kG-modules were determined by Dipper in [18] in terms of Hecke algebras and it was shown there that the decomposition matrix of G is lower unitriangular. James derived in [81], [82], and [83] a classification by labels coming from q-Schur algebras (without defining these), as in the theorem. In [27] Dipper and James exhibited their connection: For $s \in \mathcal{C}$ as above and a multi-partition λ occurring as label for some irreducible $\mathcal{S}_k(s)$-module let $D(s, \lambda)$ be the corresponding kG-module. Taking a multi-partition λ of n which is not e-regular there is corresponding to the e-ℓ-adic decomposition $\tilde{\lambda}$ of λ (which is a multi partition) a semisimple ℓ-element s such that $\tilde{\lambda}$ occurs as a label of an irreducible $\mathcal{H}_k(s)$-module. We have (for a proof see [27, 5.1]):

Theorem 4.10. (Dipper/James) *Let λ, s and $\tilde{\lambda}$ be as above. Then*

$$D(1, \lambda) = D(s, \tilde{\lambda}).$$

The theorem looks like Steinberg's tensor product theorem. Using quantum groups, Dipper and Du showed in [23] that indeed there is a tensor product theorem behind this coming from a tensor product theorem for q-Schur algebras. Moreover, the classification of the irreducible kG-modules in terms of Hecke algebras gives the Harish-Chandra series of kG. With other words the irreducible kG-module $D(1, \lambda)$ has semisimple vertex L_s and semisimple source $C_{s,k}$, which is the cuspidal kL_s-module corresponding to $s \in L_s$ (see Definition 2.5. For a proof see [23, 4.3.12]).

Theorem 4.10 connects column labels of Δ_s with those of Δ_1, where $s \in \mathcal{C}$ is an ℓ-element. However there may be entries in the decomposition matrix of $\mathcal{E}(G, s)$ which are in a column not matching any column in Δ_s. As can be seen from Theorem 4.8 this happens only if L_s is not of the form $\mathrm{GL}_d(q)^m$. In this case there is another combinatorial algorithm involving only inverting matrices Δ_s and the Littlewood-Richardson rule to calculate those entries of the decomposition matrix of G. For proofs and details we refer to [29, Section 7] or [27]. We have:

Theorem 4.11. (Dipper/James) *The ℓ-modular decomposition matrix of finite general linear groups for non-describing characteristic ℓ can be computed from the decomposition matrices of q-Schur algebras by an algorithm which involves only the Littlewood-Richardson rule and the inversion of triangular matrices.*

The decomposition numbers of q-Schur algebras for integral parameters q and hence of $\mathrm{GL}_n(q)$ are known up to $n = 10$ by work of James, [84].

4.4 Decomposition matrices of classical groups

In this section we survey the present knowledge on decomposition numbers of classical groups. To simplify the exposition, we shall assume here that our groups arise from algebraic groups with connected center. This excludes, e.g., the symplectic groups or the special orthogonal groups in even dimensions. Many of the results to follow are also valid for such groups. We exclude the general linear groups, since these have been dealt with in the previous section.

The classical groups. Thus a classical group in this section is one of the groups of the following list. Let $\mathbb{F}_q$ denote the field with q elements, where q is a power of a prime p, and let n be a positive integer. We write $G_n(q)$ for any one of the following groups.

(1) The unitary group $\mathrm{GU}_n(q)$ of an n-dimensional vector space over $\mathbb{F}_{q^2}$ with a non-degenerate unitary form.
(2) The special orthogonal group $\mathrm{SO}_n(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of maximal Witt index, where n is odd.
(3) The conformal symplectic group $\mathrm{CSp}_n(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a non-degenerate symplectic form (such that n is even).
(4) The conformal special orthogonal group $\mathrm{CSO}_n^+(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of maximal Witt index, where n is even.
(5) The conformal special orthogonal group $\mathrm{CSO}_n^-(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of Witt index $n/2-1$, where n is even.

Linear and very large primes. Given a non-zero integer a and a prime ℓ not dividing a we write $d := d(a, \ell)$ for the multiplicative order of a modulo ℓ, i.e., d is the smallest positive integer such that ℓ divides $a^d - 1$.

Definition 4.12. Let $G = G_n(q)$ be one of the groups in the list above, and let ℓ be a prime not dividing q. Put

$$e := e(G, \ell) := \begin{cases} d(-q, \ell), & \text{if } G \text{ is a unitary group,} \\ d(q, \ell), & \text{otherwise.} \end{cases}$$

If G is unitary, ℓ is called *linear* for G, if $e(G, \ell)$ is even. In the other cases, ℓ is called *linear* for G, if ℓ and $e(G, \ell)$ are odd. We call ℓ *very large* for G, if $e(G, \ell) > n/2$.

Notice that a linear prime for a unitary group is odd. Also, if ℓ is very large for G, a Sylow ℓ subgroup of G is cyclic.

Decomposition matrices. There is a great amount of information on the ℓ-decomposition matrices of G if ℓ is very large or linear. First of all, the hypotheses of Theorem 4.2 on basic sets of Brauer characters are satisfied in these cases. Thus in order to compute the irreducible Brauer characters of G in a union of blocks $\mathcal{E}_\ell(G, s)$ (see Section 2.3 for the notation), it suffices to find the decomposition numbers of the characters in $\mathcal{E}(G, s)$. The matrix of these decomposition numbers is a square matrix, but we can say a lot more about its entries in case of linear or very large primes. For example, if ℓ is linear, the decomposition matrix of $\mathcal{E}(G, s)$ is equal to the decomposition matrix of a generalized q-Schur algebra (see Section 3.3).

The results of Fong and Srinivasan [40, (9B)] and of Gruber and Hiss [66, Theorem 9.3] allow a reduction to the case of unipotent characters. Namely, let $G = \mathbf{G}^F$ be one of the classical groups above, ℓ a linear or a very large prime for G, and s a non-isolated semisimple ℓ'-element in the dual group G^*, i.e., $C_{\mathbf{G}^*}(s)$ is contained in a proper Levi subgroup of $\mathbf{G}^*$. Then the decomposition matrices of $\mathcal{E}(G, s)$ and of $\mathcal{E}(C_{G^*}(s), 1)$ are the same, i.e., Conjecture 4.5 is true in these cases.

Unfortunately, no a priori proof of this reduction is known. For very large primes it follows from the fact that, for the two cases, corresponding arguments can be given to establish the links on the Brauer trees. For linear primes it follows from the fact that the two q-Schur algebras which determine the decomposition numbers of $\mathcal{E}(G, s)$ and $\mathcal{E}(C_{G^*}(s), 1)$ are isomorphic to each other.

We may thus, for the purpose of this survey, restrict attention to the decomposition numbers for unipotent characters. It turns out that in case of linear and very large primes, the distribution of the unipotent characters into Harish-Chandra series (of ordinary characters) plays an important role.

Theorem 4.13. (Fong/Srinivasan [40], Gruber/Hiss [66, Theorem 9.3]) *Let G be a classical group from the list above and let ℓ be a linear or a very large prime for G. Then the decomposition matrix of the unipotent characters is a block diagonal matrix*

$$\operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_r),$$

where the Δ_i are square matrices corresponding to the Harish-Chandra series of ordinary characters, i.e., each Δ_i gives the decomposition into irreducible Brauer characters of the restricted ordinary characters of a particular Harish-Chandra series of unipotent characters.

This theorem no longer holds in general. An example is provided by the group $\mathrm{GU}_3(q)$ and a prime $\ell \geq 5$ dividing $q + 1$.

We are now going to describe the matrices Δ_i which occur in the preceding theorem. We define the a-invariant of a unipotent character χ to be the largest $r \geq 0$ such x^r divides the generic degree of χ (see the paragraph following Definition 2.10).

Theorem 4.14. (Fong/Srinivasan [40]) *Let G be a classical group from the list above and let ℓ be a very large prime for G. Consider a Harish-Chandra series S of unipotent characters of G. Suppose that the elements of S fall into the ℓ-blocks $B_1, \ldots, B_s$ of G. Then, for $1 \leq i \leq s$, the a-invariants of the unipotent characters in $S \cap B_i$ are all different. Moreover, arranging these characters according to increasing a-invariants, the decomposition matrix of $S \cap B_i$ equals $J(b_i)$ (where*

$b_i = |S \cap B_i|$ and $J(b_i)$ denotes the lower unitriangular Jordan $(b_i \times b_i)$-matrix). With this ordering of the characters in S, the decomposition matrix Δ of S is the block diagonal matrix $\mathrm{diag}(J(b_1), J(b_2), \ldots, J(b_s))$.

We finally turn to the case of linear primes. For this purpose we have to describe, in some more detail, the parameterization of the characters inside a Harish-Chandra series of unipotent characters. This is the same for all classical groups and all Harish-Chandra series except for the principal series of the conformal group $\mathrm{CSO}_n^+(q)$. For reasons of simplicity we therefore omit the latter from the following considerations. Then the characters of a Harish-Chandra series of unipotent characters are parameterized by bi-partitions of some positive integer $m \leq n/2$. This is due to the fact that the Hecke algebra corresponding to the Harish-Chandra series (according to Definition 2.5) is the Iwahori-Hecke algebra of type B_m. In the following theorem, δ equals 2 if G is a unitary group, and 1, otherwise. Also, if λ is a partition, we write $|\lambda|$ for the sum of its parts.

Theorem 4.15. (Gruber/Hiss [66, Corollary 8.11]) *Let G be a classical group from the list above and let ℓ be a linear prime for G. Let Δ denote the decomposition matrix of a Harish-Chandra series S of unipotent characters of G (where S is not the principal series if $G = \mathrm{CSO}_n^+(q)$).*

For a non-negative integer j let Λ_j denote the decomposition matrix of the unipotent characters of $\mathrm{GL}_j(q^\delta)$. Then Δ is a block diagonal matrix of the form

$$\Delta = \begin{bmatrix} \Lambda_0 \otimes \Lambda_m & 0 & \cdots & & 0 \\ 0 & \Lambda_1 \otimes \Lambda_{m-1} & & & \\ & & \ddots & & \vdots \\ \vdots & & & & \\ & & & \Lambda_{m-1} \otimes \Lambda_1 & 0 \\ 0 & & \cdots & 0 & \Lambda_m \otimes \Lambda_0 \end{bmatrix}.$$

Here, $\otimes$ denotes the Kronecker product of the matrices.

To be more precise, the elements of S are labelled as $\chi = \chi^{(\mu_1,\mu_2)}$ by bi-partitions (μ_1, μ_2) of m via ordinary Harish-Chandra theory. The Brauer characters occurring as constituents of the elements in S also have a labeling as $\varphi = \varphi^{(\nu_1,\nu_2)}$ by bi-partitions (ν_1, ν_2) of m. The multiplicity of $\varphi^{(\nu_1,\nu_2)}$ in $\chi^{(\mu_1,\mu_2)}$ is 0, if $|\mu_1| \neq |\nu_1|$, and it equals $d_{\mu_1\nu_1}(q^\delta) d_{\mu_2\nu_2}(q^\delta)$, otherwise. Here, $d_{\lambda_1\lambda_2}(q^\delta)$ is the ℓ-decomposition number of $\mathrm{GL}_{|\lambda_1|}(q^\delta)$ which corresponds to the unipotent character parameterized by the partition λ_1 and the Brauer character parameterized by λ_2.

Let us state one corollary of the results described in this section.

Corollary 4.16. *Let G be a classical group from the list above and let ℓ be a linear or a very large prime for G. Then the ordinary irreducible characters of G can be arranged in such a way that the decomposition matrix is lower unitriangular.*

This follows from the theorems given in this section (except for the conformal orthogonal groups $\mathrm{CSO}_n^+(q)$, where it is also true) and from the corresponding fact for the general linear groups, see Theorem 4.10.

We also have an important result on the size of the decomposition numbers.

Theorem 4.17. (Gruber/Hiss [66, Corollary 8.11]) *Let $G = G_n(q)$ be a classical group from the list above and let ℓ be a linear prime for G. Then the ℓ-decomposition numbers of G are bounded above by the order of the Weyl group of G.*

Note that this bound on the decomposition numbers is independent of q and ℓ.

4.5 Decomposition matrices of $\mathrm{Sp}_4(q)$

The determination of decomposition matrices of classical groups for non-linear, not very large primes appears to be a very hard problem. The answer is not even completely known for one of the "smallest" series of classical groups, the 3-dimensional unitary groups $\mathrm{GU}_3(q)$. Geck has determined their decomposition numbers in [41], with one exception in case ℓ divides $q + 1$.

A similar problem, again for the case ℓ divides $q + 1$, has been left open by White, who investigated the decomposition numbers for the symplectic groups $\mathrm{Sp}_4(q)$ [119]. In this case, however, Okuyama and Waki [103] were able to solve White's problem. It should be noted that here, again, the decomposition numbers of the unipotent characters are bounded independently of q and ℓ. In fact, the largest decomposition number of a unipotent character is 2.

In contrast to the case of linear primes the ℓ-modular decomposition numbers do not only depend on the integer $e(G, \ell)$ (see Definition 4.12) but also on the highest power of ℓ dividing $q + 1$.

4.6 Decomposition matrices of exceptional groups

Let us shortly summarize the knowledge on the decomposition matrices (in non-defining characteristic) for the exceptional groups of Lie type. Let G be an exceptional group of Lie type.

Complete decomposition matrices. Complete decomposition matrices are known only if G is a Suzuki group ${}^2B_2(q)$ [96] or a Ree group ${}^2G_2(q)$ [87,71]. Almost complete results are available for Steinberg's triality groups ${}^3D_4(q)$ [43] and for the Chevalley groups $G_2(q)$ [69]. For example, all ℓ-modular decomposition numbers are known for $G_2(q)$ in case ℓ is greater than 3 and does not divide $q + 1$. If $\ell > 3$ divides $q + 1$, the results of [69] allow to compute all but two of the ℓ-modular irreducible characters. The remaining problems in these two series are very similar to those already encountered in the 3-dimensional unitary groups and the 4-dimensional symplectic groups (see Section 4.5). We hope that they can be solved with methods similar to those introduced by Okuyama and Waki in [103].

Brauer trees. If we consider primes ℓ with the property that the Sylow ℓ-subgroups of G are cyclic, the situation is much better. In this case the decomposition matrices are described by Brauer trees. These are known for all exceptional groups of Lie type except for $E_7(q)$ and $E_8(q)$ (and some exceptions for ${}^2E_6(q)$) [75,74].

4.7 Harish-Chandra series of Brauer characters

In order to describe the Harish-Chandra series of Brauer characters, we have to describe the cuspidal irreducible Brauer characters first. For reasons of simplicity, we only consider unipotent characters. As always we let $G = \mathbf{G}^F$ be a finite group of Lie type of characteristic p, and let ℓ be a prime different from p.

The Gelfand-Graev representation of G is projective and has a unique direct summand containing the Steinberg representation as a constituent. We denote by φ_{St} the corresponding irreducible ℓ-modular character of G.

General linear groups. Let $G = \mathrm{GL}_n(q)$, and let e be as in 4.3, i.e., $e = \ell$, if $\ell \mid q-1$, and $e = d(q,\ell)$ is the order of q modulo ℓ, otherwise.

Theorem 4.18. (Dipper/James) *If G has a cuspidal unipotent Brauer character φ, then $\varphi = \varphi_{\mathrm{St}}$. Moreover, φ_{St} is cuspidal if and only if $n = 1$ or $n = e\ell^j$ for some $j \geq 0$.*

If $\lambda = (1^{a_1}, 2^{a_2}, \ldots)$ is a partition of n, we write

$$L_\lambda := \mathrm{GL}_1(q)^{a_1} \times \mathrm{GL}_2(q)^{a_2} \times \cdots \tag{1}$$

for the corresponding standard Levi subgroup of G. If φ is a cuspidal unipotent Brauer character of L_λ, we must have

$$\lambda = (1^{m_{-1}}, e^{m_0}, (e\ell)^{m_1}, (e\ell^2)^{m_2}, \ldots), \tag{2}$$

and

$$\varphi = \varphi_{\lambda,St} := {\varphi_{\mathrm{St}}}^{m_{-1}} \otimes {\varphi_{\mathrm{St}}}^{m_0} \otimes {\varphi_{\mathrm{St}}}^{m_1} \otimes \cdots$$

by Theorem 4.18. Thus the Harish-Chandra series of unipotent Brauer characters of G are in bijection with the partitions of n of the form (2). The question of which unipotent Brauer characters lie in the Harish-Chandra series corresponding to such a partition is answered in the following theorem. By the results of Section 4.3 we have a natural labelling of the unipotent Brauer characters of G by partitions of n. Let φ_μ denote the unipotent Brauer character of G corresponding to the partition μ of n.

Theorem 4.19. (Dipper/Du [23, 4.3.12]) *Let μ be a partition of n with e-ℓ-adic expansion*

$$\mu = \mu^{(-1)} + e\mu^{(0)} + e\ell\mu^{(1)} + \cdots.$$

Then φ_μ lies in the Harish-Chandra series corresponding to the partition λ of the form (2) with $m_i := |\mu_i|$ for $i = -1, 0, 1, \ldots$

Classical groups. We now let $G = G_n(q)$ be a classical group as in 4.4, and we let ℓ be a linear prime for G. In this case the cuspidal unipotent Brauer characters are easy to describe.

Theorem 4.20. (Gruber/Hiss [66, Corollary 8.8]) *The group G has a cuspidal unipotent Brauer character φ if and only if it has an ordinary cuspidal unipotent character χ. In this case φ is liftable to χ. Moreover, χ is an ℓ-defect 0 character of $G/Z(G)$.*

It follows in particular, that G has at most one cuspidal unipotent Brauer character. The Levi subgroups of G are of the form

$$L_{a,\lambda} = G_a(q) \times L_\lambda,$$

where $a \leq n$ is an integer such that $n - a$ is even, $G_a(q)$ is a group of the same type as G, λ is a partition of $(n-a)/2$, and L_λ is the Levi subgroup of $\mathrm{GL}_{(n-a)/2}(q^\delta)$ introduced in (1) with δ as in Theorem 4.15. It is well known that $L_{a,\lambda}$ has an ordinary cuspidal unipotent character only for special values of a and then only for $\lambda = (1^m)$.

Let $0 \leq a_1 < a_2 < \ldots < a_r \leq n$ denote the integers such that $n - a_i$ is even and such that $G_{a_i}(q)$ has a cuspidal unipotent character χ_i. The ordinary unipotent characters in the Harish-Chandra series corresponding to χ_i are labelled by bi-partitions of $n_i = (n - a_i)/2$. (Again we exclude the case of the principal series of $\mathrm{CSO}_n^+(q)$.) Thus the unipotent characters are labelled by the triples $(a_i; \lambda_1, \lambda_2)$ where (λ_1, λ_2) is a bi-partition of n_i and i runs from 1 to r. (It is not difficult to translate this labelling of the unipotent characters to the more common one by symbols or partitions.) In turn, by Corollary 4.16, the unipotent Brauer characters of G are labelled by such triples.

Theorem 4.21. *Let $(a_i; \mu_1, \mu_2)$ be the label of a unipotent Brauer character φ of G, and let $\mu_1^{(j)}$ and $\mu_2^{(j)}$, $j = -1, 0, 1, \ldots$, denote the parts of the e-ℓ-adic expansion of μ_1 and μ_2, respectively. (Here, e is defined as in 4.3 with respect to q^δ.) Then φ lies in the Harish-Chandra series corresponding to $L_{a_i,\lambda}$, where λ is the partition of $(n - a_i)/2$ of the form (2) with $m_j = |\mu_1^{(j)}| + |\mu_2^{(j)}|$, $j = -1, 0, 1, \ldots$*

This theorem was conjectured for the unitary groups in [53, Conjecture 9.2], and proved in the case that φ lies in an ℓ-block with a cyclic defect group [53, Proposition 9.3]. In the general case it follows from the results of [66, Section 8].

The unitary groups. We now give a result which indicates the existence of many cuspidal unipotent Brauer characters which are not liftable to ordinary characters. Let $G = \mathrm{GU}_n(q)$ and let ℓ be an odd prime dividing $q + 1$. The unipotent Brauer characters of G have a natural parameterization by partitions of n (see Section 4.2). Let φ_μ denote the unipotent Brauer character of G corresponding to the partition μ of n.

Theorem 4.22. (Geck/Hiss/Malle, Gruber) *The unipotent Brauer character φ_μ is cuspidal if and only if two consecutive parts of μ differ by at most 1. (This is the same as saying that μ', the partition conjugate to μ, is 2-regular).*

We can also describe the Harish-Chandra series in this case.

Theorem 4.23. (Geck/Hiss/Malle, Gruber) *Let μ be a partition of n. Write $\mu = \mu_1 + 2\mu_2$, where μ_1 and μ_2 are uniquely determined by the fact that μ_1' is 2-regular. Define $m = |\mu_1|$ and $m_j = |\mu_2^{(j)}|$, where the $\mu_2^{(j)}$ are the parts of the ℓ-adic expansion of μ_2, $j = 0, 1, 2, \ldots$.*

Then φ_μ lies in the Harish-Chandra series of $\varphi_{\mu_1} \otimes {\varphi_{\mathrm{St}}}^{m_0} \otimes {\varphi_{\mathrm{St}}}^{m_1} \otimes \cdots$, where ${\varphi_{\mathrm{St}}}^{m_j}$ is the ℓ-modular Steinberg character of $\mathrm{GL}_{\ell^j}(q^2)^{m_j}$, $j = 0, 1, 2, \ldots$

These two results were conjectured in [53, Conjecture 9.6], and proved to be true in [54, Theorem 4.12] for $\ell > n$, and in general by Gruber in [65].

Exceptional groups. The modular Harish-Chandra series and the endomorphism algebras of Harish-Chandra induced cuspidal modules occurring in exceptional groups $G(q)$ were determined by Geck/Hiss/Malle in [54] in all cases where the order $d = d(q, \ell)$ of q modulo ℓ is odd and ℓ does not divide the order of the Weyl group of G, except for $d \in \{3, 5\}$ for $G = E_8(q)$. In order to indicate the nature of the results to expect we reproduce from Table 5.4 in loc. cit. the results pertaining to Harish-Chandra series of $G = E_7(q)$ in the form of a table.

Its first column contains the number $d = d(q, \ell)$, the order of q modulo ℓ.

The second column of the table gives the pairs (L, X), where L is a Levi subgroup of G and X is a cuspidal simple kL-module. The Levi subgroups are denoted by their types. The trivial kL-module (which is cuspidal only if L is the maximally split torus), is denoted by 1, the Steinberg module, if it is cuspidal, by φ_{St}. Further cuspidal modules are denoted by c or c_i, $i = 1, \ldots, 6$.

The third and fourth columns then give the groups $R = R(L, X)$ and $C = C(L, X)$, and the sixth column the endomorphism algebra $\mathcal{H} = \mathcal{H}(L, X) = \mathrm{End}_{kG}(R_L^G(X))$ (see Theorem 2.9). The parameters of these all lie in the finite field $\mathbb{F}_\ell \subseteq k$, i.e., q also denotes its image in $\mathbb{F}_\ell$. There are some parameters $p_i \in {\mathbb{F}_\ell}^* \setminus \{-1\}$, $i = 1, \ldots, 6$, which are not yet known explicitly.

The last column of the table gives the number of simple kG-modules in the Harish-Chandra series corresponding to (L, X), which is equal to the number of simple $\mathcal{H}$-modules by Corollary 2.4.

Table. Modular Harish-Chandra series for $E_7(q)$

d	L	X	R	C	$\mathcal{H}$	#irr.
3	1	1	E_7	1	$\mathcal{H}(E_7;q)$	32
	A_2	φ_{St}	A_5	2	$\mathcal{H}(A_5;q):2$	14
	$A_2\times A_2$	$\varphi_{\mathrm{St}}\times\varphi_{\mathrm{St}}$	$G_2\times A_1$	1	$\mathcal{H}(G_2;q,1)\otimes\mathcal{H}(A_1;1)$	8
	D_4	c	C_3	1	$\mathcal{H}(C_3;q,q)$	4
	E_6	c_1,c_2	A_1	1	$\mathcal{H}(A_1;1)$	$2+2$
	E_6	$c_3,\ldots,c_6$	A_1	1	$\mathcal{H}(A_1;p_i),\ i=1,\ldots,4$	8
						+ 6 cuspidals
5	1	1	E_7	1	$\mathcal{H}(E_7;q)$	54
	A_4	φ_{St}	A_2	2	$\mathcal{H}(A_2;q):2$	6
	D_4	c	C_3	1	$\mathcal{H}(C_3;q,q^4)$	10
	E_6	c_1,c_2	A_1	1	$\mathcal{H}(A_1;q^4)$	$2+2$
						+ 2 cuspidals
7	1	1	E_7	1	$\mathcal{H}(E_7;q)$	58
	D_4	c	C_3	1	$\mathcal{H}(C_3;q,q^4)$	10
	A_6	φ_{St}	A_1	1	$\mathcal{H}(A_1;p_5)$	2
	E_6	c_1,c_2	A_1	1	$\mathcal{H}(A_1;q^2)$	$2+2$
						+ 2 cuspidals
9	1	1	E_7	1	$\mathcal{H}(E_7;q)$	58
	D_4	c	C_3	1	$\mathcal{H}(C_3;q,q^4)$	10
	E_6	c_1,c_2	A_1	1	$\mathcal{H}(A_1;1)$	$2+2$
	E_6	φ_{St}	A_1	1	$\mathcal{H}(A_1;p_6)$	2
						+ 2 cuspidals

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