Cyclotomic q–Schur algebras

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1 Introduction

Recently, we [**9**] and, independently, Du and Scott [**11**], defined an analogue of the q–Schur algebra [**6**, **7**] for an Iwahori–Hecke algebra of type **B**. In this paper we study an analogue of the q–Schur algebra for an arbitrary Ariki–Koike algebra.

The Ariki–Koike algebra $\mathscr{H}$ is a cyclotomic algebra of type $G(r,1,n)$ [**2**], and it becomes the Iwahori–Hecke algebra of type **A** or **B** when $r=1$ or 2 respectively. By working over a ring R which contains a primitive rth root of unity, and by specializing the parameters appropriately, the Ariki–Koike algebra turns into the group algebra $R(C_r \wr \mathfrak{S}_n)$ of the wreath product of the cyclic group C_r of order r with the symmetric group $\mathfrak{S}_n$ of degree n.

For each multicomposition λ of n, we construct a right ideal M^λ of $\mathscr{H}$ (see Definition 3.8). The cyclotomic q–Schur algebra is then defined to be $\mathscr{S}(\Lambda_r) = \operatorname{End}_{\mathscr{H}}\left(\oplus_\lambda M^\lambda\right)$. (Under the

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specialization above where $\mathscr{H} \cong R(C_r \wr \mathfrak{S}_n)$, the module M^λ becomes a module induced from a subgroup of the form $(C_r \times \cdots \times C_r) \rtimes \mathfrak{S}_\lambda$.)

In this paper we construct a cellular basis for the cyclotomic q–Schur algebra. As a consequence we obtain a Weyl module W^λ for each multipartition λ of n. We show that W^λ has simple head F^λ and that the set $\{F^\lambda\}$, as λ ranges over the multipartitions of n, is a complete set of non–isomorphic irreducible $\mathscr{S}(\Lambda_r)$–modules. Using the cellular structure of $\mathscr{S}(\Lambda_r)$, it is now easy to see that the cyclotomic q–Schur algebra is quasi–hereditary.

In order to prove these results about the cyclotomic q–Schur algebra, we need to examine the ideals M^λ in some detail. Using the cellular structure of the Ariki–Koike algebra $\mathscr{H}$ (cf. [**10**, **13**]), we obtain a basis of M^λ and a special series of submodules of M^λ, known as a Specht series. From the Specht series of M^λ we construct the cellular basis of $\mathscr{S}(\Lambda_r)$.

2 The Ariki–Koike algebra

Throughout this paper, r and n will be fixed positive integers with $r \geq 1$ and $n \geq 1$.

Let R be a commutative ring with 1 and let $q, Q_1, \dots, Q_r$ be elements of R with q invertible. The Ariki–Koike algebra $\mathscr{H}$ is the associative unital R–algebra with generators $T_0, T_1, \dots, T_{n-1}$ subject to the following relations

$$\begin{array}{rcll}
(T_0 - Q_1)\cdots(T_0 - Q_r) &=& 0 & \\
T_0T_1T_0T_1 &=& T_1T_0T_1T_0 & \\
(T_i + 1)(T_i - q) &=& 0 & \text{for } 1 \leq i \leq n-1\\
T_{i+1}T_iT_{i+1} &=& T_iT_{i+1}T_i & \text{for } 1 \leq i \leq n-2\\
T_iT_j &=& T_jT_i & \text{for } 0 \leq i < j-1 \leq n-2.
\end{array}$$

Suppose that $\pi \in R$ is a primitive rth root of 1 and that $q = 1$ and $Q_k = \pi^k$ for $k = 1, 2, \dots, r$. Then it follows from the definition that $\mathscr{H}$ is isomorphic to $R(C_r \wr \mathfrak{S}_n)$.

For each finite set X, let $\mathfrak{S}(X)$ be the group of permutations of X; thus, the symmetric group $\mathfrak{S}_n = \mathfrak{S}(\{1, 2, \dots, n\})$. Let $s_i = (i, i+1)$ for $1 \leq i < n$; then $s_1, s_2, \dots, s_{n-1}$ are the standard Coxeter generators of $\mathfrak{S}_n$. If $w \in \mathfrak{S}_n$ then a word $w = s_{i_1} \dots s_{i_k}$ for w is a reduced expression for w if k is minimal; in this case we say that w has length k and write $\ell(w) = k$. Given a reduced expression $s_{i_1} \dots s_{i_k}$ for $w \in \mathfrak{S}_n$, let $T_w = T_{i_1} \dots T_{i_k}$; the relations in $\mathscr{H}$ ensure that T_w is independent of the choice of reduced expression. We denote by $\mathscr{H}(\mathfrak{S}_n)$ the subalgebra of $\mathscr{H}$ generated by $T_1, T_2, \dots, T_{n-1}$. Then $\mathscr{H}(\mathfrak{S}_n)$ has basis $\{\, T_w \mid w \in \mathfrak{S}_n \,\}$, and it is isomorphic to an Iwahori–Hecke algebra of type **A**.

Define elements $L_m = q^{1-m}T_{m-1} \dots T_1T_0T_1 \dots T_{m-1}$ for $m = 1, 2, \dots, n$; these are analogues of the q–Murphy operators of the Iwahori–Hecke algebras of type **A** [**5**, **15**]. Easy calculations using the relations in $\mathscr{H}$ (cf. [**5**, (2.1), (2.2)]) show that we have the following results.

(2.1) Suppose that $1 \leq i \leq n-1$ and $1 \leq m \leq n$. Then

1. L_i and L_m commute.

2. T_i and L_m commute if $i \neq m-1, m$.
3. T_i commutes with L_iL_{i+1} and $L_i + L_{i+1}$.
4. If $a \in R$ and $i \neq m$ then T_i commutes with $(L_1 - a)(L_2 - a)\dots(L_m - a)$.

Using the elements T_w and L_m defined above, Ariki and Koike gave a basis for $\mathscr{H}$ as follows.

(2.2) Theorem (Ariki–Koike [1, (3.10)]) *The algebra $\mathscr{H}$ is a free R–module with basis*

$$\{\, L_1^{c_1}L_2^{c_2}\dots L_n^{c_n}T_w \mid w \in \mathfrak{S}_n \quad \text{and} \quad 0 \le c_m \le r-1 \quad \text{for} \quad m = 1, 2, \dots, n \,\}.$$

In particular, $\mathscr{H}$ is free of rank $r^n n!$

(2.3) Let $*$ be the R–linear antiautomorphism of $\mathscr{H}$ determined by $T_i^* = T_i$ for all i with $0 \le i \le n-1$. Then $T_w^* = T_{w^{-1}}$ and $L_m^* = L_m$ for all $w \in \mathfrak{S}_n$ and $m = 1, 2, \dots, n$.

We therefore have the following result.

(2.4) $\{\, T_wL_1^{c_1}L_2^{c_2}\dots L_n^{c_n} \mid w \in \mathfrak{S}_n \quad \text{and} \quad 0 \le c_m \le r-1 \quad \text{for} \quad m = 1, 2, \dots, n \,\}$ is a basis of $\mathscr{H}$.

3 A cellular basis of $\mathscr{H}$

In their paper [12], which introduced the concept of cellular algebras, Graham and Lehrer gave a cellular basis of $\mathscr{H}$, using the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_n)$. We require a different cellular basis, namely one similar to the basis of $\mathscr{H}(\mathfrak{S}_n)$ introduced by Murphy [16]. Although the construction of the cellular basis of $\mathscr{H}$ in this section is similar to that in [10, 13], we are obliged to keep track of new information concerned with the cellular basis (see Corollary 3.24 below).

Consider the R–submodule of $\mathscr{H}$ which is spanned by

$$\{\, L_1^{c_1}L_2^{c_2}\dots L_n^{c_n} \mid 0 \le c_i \le r-1 \quad \text{for} \quad 1 \le i \le n \,\}.$$

When $q = 1$, this is a subalgebra of $\mathscr{H}$, but one of the main difficulties of working with the Ariki–Koike algebra is that it is not a subalgebra in general. (To see this, consider L_2^2 when $r = 2$.) We shall need certain elements $u_{\mathbf{a}}^+$ of this R–submodule of $\mathscr{H}$, and we introduce these now.

(3.1) Definition *Suppose that $\mathbf{a} = (a_1, \dots, a_r)$ is an r–tuple of integers a_i such that $0 \le a_i \le n$ for all i. Let $u_{\mathbf{a}}^+ = u_{\mathbf{a},1}u_{\mathbf{a},2}\dots u_{\mathbf{a},r}$ where*

$$u_{\mathbf{a},k} = \prod_{m=1}^{a_k}(L_m - Q_k) \quad \textit{for} \quad 1 \le k \le r.$$

(3.2) Remarks 1. Suppose that every a_k is non–zero. Then $(L_1 - Q_k)$ is a factor of each $u_{\mathbf{a},k}$; so $u^+_{\mathbf{a}}$ has a factor

$$\prod_{k=1}^{r}(L_1 - Q_k) = \prod_{k=1}^{r}(T_0 - Q_k) = 0.$$

Therefore, $u^+_{\mathbf{a}}$ is zero in this case.

2. Rearranging the order of $a_1, a_2, \ldots, a_r$ amounts just to reordering the parameters $Q_1, Q_2, \ldots, Q_r$. For example, if we define $u^-_{\mathbf{a}} = u^+_{(a_r, a_{r-1}, \ldots, a_1)}$ then $u^-_{\mathbf{a}}$ is obtained from $u^+_{\mathbf{a}}$ by replacing Q_k by Q_{r-k+1} for $1 \le k \le r$.
3. In practice, we shall use $u^+_{\mathbf{a}}$ only for r–tuples $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ such that $0 = a_1 \le a_2 \le \ldots \le a_r \le n$. Our last two remarks show that there is no loss in doing this, and that we could equally well work with the elements $u^-_{\mathbf{a}}$ defined in Remark (ii).

(3.3) Example Suppose that $r = 4$, $n \ge 5$ and $\mathbf{a} = (0, 2, 4, 5)$. Then

$$\begin{aligned} u^+_{\mathbf{a}} &= (L_1 - Q_2)(L_2 - Q_2) \\ &\quad \times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3) \\ &\quad \times (L_1 - Q_4)(L_2 - Q_4)(L_3 - Q_4)(L_4 - Q_4)(L_5 - Q_4). \end{aligned}$$

Our first lemma relates $u^+_{\mathbf{a}}$ and $u^+_{\mathbf{b}}$ when $\mathbf{b}$ is obtained from $\mathbf{a}$ by increasing a single part by one.

(3.4) Lemma *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ *and assume that* $1 \le k \le r$ *and* $a_k + 1 \le n$. *Let* $\mathbf{b} = (a_1, \ldots, a_{k-1}, a_k + 1, a_{k+1}, \ldots, a_r)$. *Then, for some* $h_1, h_2 \in \mathscr{H}(\mathfrak{S}_n)$, *we have* $u^+_{\mathbf{a}} T_{a_k} T_{a_k - 1} \ldots T_1 T_0 = u^+_{\mathbf{a}} h_1 + u^+_{\mathbf{b}} h_2$.

PROOF: The definition of $u^+_{\mathbf{b}}$ gives $u^+_{\mathbf{b}} = u^+_{\mathbf{a}}(L_{a_k+1} - Q_k)$. Hence,

$$u^+_{\mathbf{a}} T_{a_k} T_{a_k-1} \ldots T_1 T_0 T_1 \ldots T_{a_{k-1}} T_{a_k} = q^{a_k} u^+_{\mathbf{a}} L_{a_k+1} = q^{a_k} Q_k u^+_{\mathbf{a}} + q^{a_k} u^+_{\mathbf{b}}.$$

The desired result follows by postmultiplying by $T^{-1}_{a_k} \ldots T^{-1}_1$. □

We next turn our attention to the subalgebra $\mathscr{H}(\mathfrak{S}_n)$ of $\mathscr{H}$. Here we shall need the notation and combinatorics of multipartitions.

A **composition** $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a finite sequence of non–negative integers; we denote by $|\alpha|$ the sum of this sequence. A **multicomposition** of n is an ordered r–tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of compositions $\lambda^{(k)}$ such that $\sum_{k=1}^{r} |\lambda^{(k)}| = n$. We call $\lambda^{(k)}$ the kth **component** of λ. A **partition** is a composition whose parts are non–increasing; a multicomposition is a **multipartition** if all its components are partitions.

For each composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ with $|\alpha| = m$ we have a Young subgroup $\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots$ of $\mathfrak{S}_m$. Similarly, to each multicomposition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of n we associate the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \mathfrak{S}_{\lambda^{(2)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ of $\mathfrak{S}_n$.

We are now in a position to define certain key elements m_λ of $\mathscr{H}$. The element m_λ depends upon the Young subgroup $\mathfrak{S}_\lambda$ and also involves one of the elements $u^+_{\mathbf{a}}$ defined above.

(3.5) Definition *Suppose that* λ *is a multicomposition of* n *and define* $\mathbf{a} = (a_1, a_2, \dots, a_r)$ *by* $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$. *Let* $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$ *and set* $m_\lambda = u_{\mathbf{a}}^+ x_\lambda$.

(3.6) Example Suppose that $\lambda = \big((2,1),(1^2),(2)\big)$. Then $\mathfrak{S}_\lambda = \mathfrak{S}_2\times\mathfrak{S}_1\times\mathfrak{S}_1\times\mathfrak{S}_1\times\mathfrak{S}_2$, $\mathbf{a} = (0,3,5)$, and

$$\begin{aligned} m_\lambda \;=\;& (L_1-Q_2)(L_2-Q_2)(L_3-Q_2)\\ &\times\,(L_1-Q_3)(L_2-Q_3)(L_3-Q_3)(L_4-Q_3)(L_5-Q_3)\\ &\times\,(1+T_1)(1+T_6). \end{aligned}$$

(3.7) Remark If $\alpha = (|\lambda^{(1)}|, |\lambda^{(2)}|, \dots, |\lambda^{(r)}|)$ then all of the elements in $\mathscr{H}(\mathfrak{S}_\alpha)$ commute with $u_{\mathbf{a}}^+$ by (2.1)(iv). In particular, $m_\lambda = u_{\mathbf{a}}^+ x_\lambda = x_\lambda u_{\mathbf{a}}^+$. Hence, $m_\lambda^* = m_\lambda$, where $*$ is the antiautomorphism of (2.3).

The $\mathscr{H}$–modules which will be our main concern in this paper are the right ideals generated by the m_λ, as λ varies over the multicompositions of n. The cyclotomic q–Schur algebra will be built from endomorphisms between such right ideals.

(3.8) Definition *Suppose that* λ *is a multicomposition of* n. *Let* $M^\lambda = m_\lambda\mathscr{H}$.

We leave the proof of the following remarks to the reader.

(3.9) Remarks

1. If the multicomposition μ is obtained from λ by reordering the parts in each component then $M^\mu \cong M^\lambda$.
2. Suppose that $q = 1$ and that $Q_k = \pi^k$, for $k = 1, 2, \dots, r$, where π is a primitive rth root of unity in R. Then $\mathscr{H} \cong R(C_r \wr \mathfrak{S}_n)$. Let $\alpha_k = |\lambda^{(k)}|$ for $1 \le k \le r$. Then M^λ is induced from a module U for the subgroup $\big(C_r^{\alpha_1}\times C_r^{\alpha_2}\times\cdots\times C_r^{\alpha_r}\big)\rtimes\mathfrak{S}_\lambda$, where $C_r^{\alpha_i} = C_r\times\ldots\times C_r$ (α_i factors). The restriction of U to the subgroup $C_r^{\alpha_1}\times\cdots\times C_r^{\alpha_r}$ has the form $U_1^{\otimes\alpha_1}\otimes\cdots\otimes U_r^{\otimes\alpha_r}$ where U_k has rank k for $1\le k\le r$. The restriction of U to $\mathfrak{S}_\lambda$ is the trivial module.

We shall construct a basis of M^λ, and study $\mathscr{H}$–homomorphisms between the various modules M^λ. To this end, we introduce λ–tableaux.

The diagram of a composition $\alpha = (\alpha_1, \alpha_2, \dots)$ is $\{\,(i,j)\mid 1\le i \text{ and } 1\le j\le\alpha_i\,\}$, which we regard as an array of nodes, or boxes, in the plane. The diagram of a multicomposition is the ordered r–tuple of the diagrams of its components.

Let λ be a multicomposition of n. A λ–tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ is obtained from the diagram of λ by replacing each node by one of the integers $1, 2, \dots, n$, allowing no repeats. We call the tableaux $\mathfrak{t}^{(k)}$ the components of $\mathfrak{t}$.

(3.10) Definition

1. *A* λ*–tableau is* row standard *if the entries in each row of each component increase from left to right.*
2. *A* λ*–tableau* $\mathfrak{t}$ *is* standard *if* λ *is a multipartition of* n, $\mathfrak{t}$ *is row standard and the entries in each column of each component of* $\mathfrak{t}$ *increase from top to bottom.*

3. *If λ is a multipartition of n, then let* $\mathrm{Std}(\lambda)$ *be the set of standard λ–tableaux.*

Note, particularly, that we define standard λ–tableaux only when λ is a multipartition whereas row standard tableaux are defined for all multicompositions.

We require partial orders on the set of multicompositions and on the set of row standard tableaux.

If $\mathfrak{t}$ is a row standard λ–tableau and $1 \le m \le n$, then the entries $1, 2, \dots, m$ in $\mathfrak{t}$ occupy the diagram of a multicomposition; let $\mathfrak{t}\downarrow m$ denote this multicomposition. For example, $\mathfrak{t}\downarrow n = \lambda$. We use this notation in our next definition.

(3.11) Definition *Suppose that $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ are multicompositions of n.*

1. *We say that λ* dominates *μ, and write $\lambda \trianglerighteq \mu$, if*

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{i=1}^{j} \lambda_i^{(k)} \ge \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{i=1}^{j} \mu_i^{(k)}$$

   for all k and j with $1 \le k \le r$ and $j \ge 0$. If $\lambda \trianglerighteq \mu$ and $\lambda \ne \mu$ then we write $\lambda \triangleright \mu$.
2. *Suppose that $\mathfrak{s}$ is a row standard λ–tableau and that $\mathfrak{t}$ is a row standard μ–tableau. We say that $\mathfrak{s}$* dominates *$\mathfrak{t}$, and write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $\mathfrak{s}\downarrow m \trianglerighteq \mathfrak{t}\downarrow m$ for all m with $1 \le m \le n$. If $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \ne \mathfrak{t}$ then we write $\mathfrak{s} \triangleright \mathfrak{t}$.*

For example, if $n = r = 2$, then the multipartitions of 2 are ordered by $\big((2),(0)\big) \triangleright \big((1^2),(0)\big) \triangleright \big((1),(1)\big) \triangleright \big((0),(2)\big) \triangleright \big((0),(1^2)\big)$.

Note that if $\mathfrak{s}$ is a row standard λ–tableau and $\mathfrak{t}$ is a row standard μ tableau such that $\mathfrak{s} \trianglerighteq \mathfrak{t}$ then $\lambda \trianglerighteq \mu$.

Our next definition gives another relation between tableaux.

(3.12) Definition *Suppose that $\mathfrak{s}$ is a tableau and that $1 \le j \le n$. We write* $\mathrm{comp}_{\mathfrak{s}}(j) = k$ *if j appears in the kth component $\mathfrak{s}^{(k)}$ of $\mathfrak{s}$.*

Suppose that $\mathfrak{t}$ is another tableau. Then we write $\mathrm{comp}_{\mathfrak{s}} = \mathrm{comp}_{\mathfrak{t}}$ *if* $\mathrm{comp}_{\mathfrak{s}}\big(j\big) = \mathrm{comp}_{\mathfrak{t}}(j)$ *for all j with $1 \le j \le n$. We also write* $\mathrm{comp}_{\mathfrak{s}} \ge \mathrm{comp}_{\mathfrak{t}}$ *if* $\mathrm{comp}_{\mathfrak{s}}(j) \ge \mathrm{comp}_{\mathfrak{t}}(j)$ *for all $1 \le j \le n$; and* $\mathrm{comp}_{\mathfrak{s}} > \mathrm{comp}_{\mathfrak{t}}$ *if* $\mathrm{comp}_{\mathfrak{s}} \ge \mathrm{comp}_{\mathfrak{t}}$ *and* $\mathrm{comp}_{\mathfrak{s}} \ne \mathrm{comp}_{\mathfrak{t}}$.

(3.13) Example Let

$$\mathfrak{s} = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & 6 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 3 & 8 & 9 \\ \hline 2 & 7 \\ \cline{1-2}\end{array}\right), \quad \mathfrak{t} = \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 7 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 6 & 8 \\ \hline 5 & 9 \\ \cline{1-2}\end{array}\right) \quad \text{and} \quad \mathfrak{u} = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 & 7 \\ \hline\end{array}\,,\ \begin{array}{|c|c|}\hline 5 & 9 \\ \hline 8 \\ \cline{1-1}\end{array}\right).$$

Then $\mathfrak{s}$ is row standard, and $\mathfrak{t}$ and $\mathfrak{u}$ are standard. We have $\mathfrak{u} \triangleright \mathfrak{s}$ and $\mathfrak{u} \triangleright \mathfrak{t}$ but $\mathfrak{s}$ and $\mathfrak{t}$ are incomparable in the dominance order. Also, $\mathrm{comp}_{\mathfrak{t}} > \mathrm{comp}_{\mathfrak{u}}$, but there are no other equations or inequalities between $\mathrm{comp}_{\mathfrak{s}}$, $\mathrm{comp}_{\mathfrak{t}}$ and $\mathrm{comp}_{\mathfrak{u}}$.

Let λ be a multicomposition of n. The symmetric group $\mathfrak{S}_n$ acts from the right on the set of λ–tableaux by permuting the entries in each tableau. Let $\mathfrak{t}^\lambda$ be the λ–tableau where $1, 2, \dots, n$ appear in order along the rows of the first component, and then along the rows of the second component, and so on. The row stabilizer of $\mathfrak{t}^\lambda$ is the Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$. For example, if $\lambda = \big((3,2),(1^2),(3)\big)$ then

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}\,,\quad \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline \end{array}\,,\quad \begin{array}{|c|c|c|}\hline 8 & 9 & 10 \\ \hline \end{array} \right).$$

For a row standard λ–tableau $\mathfrak{s}$, let $d(\mathfrak{s})$ be the element of $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$. Then $d(\mathfrak{s})$ is a distinguished right coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$ and we obtain, in this way, a correspondence between the set of row standard λ–tableaux and the set of right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$.

Recall the antiautomorphism $*$ of $\mathscr{H}$ from (2.3).

(3.14) Definition *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. Let $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}$.*

Note that $m_\lambda = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda}$. Also, $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$ (cf. Remark 3.7).

One of the aims of this section is to show that the elements $m_{\mathfrak{s}\mathfrak{t}}$, as $(\mathfrak{s}, \mathfrak{t})$ varies over the ordered pairs of standard tableaux of the same shape, give a cellular basis of $\mathscr{H}$. We shall also establish useful properties of the right ideals M^λ. Initially, though, we concentrate upon the two–sided ideal generated by m_λ.

(3.15) Lemma *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. Let $h \in \mathscr{H}(\mathfrak{S}_n)$. Then $m_{\mathfrak{s}\mathfrak{t}} h$ is a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{v}}$ where each $\mathfrak{v}$ is a row standard λ–tableau.*

PROOF: Suppose that $w \in \mathfrak{S}_n$. Then there exist $y \in \mathfrak{S}_\lambda$ and a distinguished right coset representative d of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$ such that $w = yd$ and $\ell(w) = \ell(y) + \ell(d)$. Hence, $x_\lambda T_w = x_\lambda T_y T_d = q^{\ell(y)} x_\lambda T_d$. If $m_\lambda = u^+_{\mathbf{a}} x_\lambda$ then

$$m_{\mathfrak{s}\mathfrak{t}^\lambda} T_w = T^*_{d(\mathfrak{s})} u^+_{\mathbf{a}} x_\lambda T_w = q^{\ell(y)} T^*_{d(\mathfrak{s})} u^+_{\mathbf{a}} x_\lambda T_d = q^{\ell(y)} m_{\mathfrak{s}\mathfrak{v}}$$

where the tableau $\mathfrak{v} = \mathfrak{t}^\lambda d$ is row standard.

Now, $m_{\mathfrak{s}\mathfrak{t}} h = m_{\mathfrak{s}\mathfrak{t}^\lambda} T_{d(\mathfrak{t})} h$, and this is a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{t}^\lambda} T_w$ with $w \in \mathfrak{S}_n$. Therefore, the required result follows. □

In order to apply a result of Murphy in Proposition 3.18 below, we require a combinatorial lemma which concerns the dominance order on row standard tableaux. A preliminary definition sets the scene.

(3.16) Definition *We say that a tableau $\mathfrak{s}$ is of the* initial kind *(for λ) if* $\mathrm{comp}_{\mathfrak{s}} = \mathrm{comp}_{\mathfrak{t}^\lambda}$.

Note that a μ–tableau $\mathfrak{s}$ can be of the initial kind for λ even though $\mu \neq \lambda$.

(3.17) Lemma *Suppose that $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multicomposition of n, and let $\alpha = (|\lambda^{(1)}|, \dots, |\lambda^{(r)}|)$. Suppose that w is a distinguished right coset representative of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ and that $\mathfrak{s}$ is a row standard λ–tableau of the initial kind. Then the following hold.*

1. *The tableau $\mathfrak{s}w$ is row standard.*
2. *If $\mathfrak{s}$ is standard then $\mathfrak{s}w$ is standard.*
3. *If $\mathfrak{t}$ is a row standard λ–tableau of the initial kind with $\mathfrak{s} \rhd \mathfrak{t}$ then $\mathfrak{s}w \rhd \mathfrak{t}w$.*

PROOF: The fact that w is a distinguished right coset representative of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ implies that whenever x and y are integers such that x and y are in the same component of $\mathfrak{s}$ and $x < y$, then $xw < yw$. Hence, (i) and (ii) are true.

Now assume that $\mathfrak{t}$ is a row standard λ–tableau of the initial kind, and that $\mathfrak{s} \rhd \mathfrak{t}$. Then $d(\mathfrak{s}) \rhd d(\mathfrak{t})$ in the Bruhat–Chevalley order $\rhd$ on $\mathfrak{S}_n$, by [**4**, (1.3)]. (Our notation for the order is such that $1 \trianglerighteq v$ for all $v \in \mathfrak{S}_n$.) Since w is a distinguished right coset representative of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$, we also have $\ell(d(\mathfrak{t})w) = \ell(d(\mathfrak{t})) + \ell(w)$. The well–known "cancellation" property of the Bruhat–Chevalley order now lets us conclude that $d(\mathfrak{s})w \rhd d(\mathfrak{t})w$. But $\mathfrak{s}w$ and $\mathfrak{t}w$ are row standard, so $d(\mathfrak{s})w = d(\mathfrak{s}w)$ and $d(\mathfrak{t})w = d(\mathfrak{t}w)$. By applying [**4**, (1.3)] again, we deduce that $\mathfrak{s}w \rhd \mathfrak{t}w$. □

(3.18) Proposition *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. Then $m_{\mathfrak{s}\mathfrak{t}}$ is a linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableaux for some multipartition μ of n, and*

1. $\mathfrak{u} \trianglerighteq \mathfrak{s}$ *and* $\mathrm{comp}_\mathfrak{u} = \mathrm{comp}_\mathfrak{s}$*; and,*
2. $\mathfrak{v} \trianglerighteq \mathfrak{t}$ *and* $\mathrm{comp}_\mathfrak{v} = \mathrm{comp}_\mathfrak{t}$.

PROOF: When $r = 1$, this is a Theorem of Murphy [**16**, (4.18)]. We deduce the general case from this.

Let $\alpha = (|\lambda^{(1)}|, \dots, |\lambda^{(r)}|)$. We may write $\mathfrak{s} = \mathfrak{s}'w_1$ and $\mathfrak{t} = \mathfrak{t}'w_2$ where $\mathfrak{s}'$ and $\mathfrak{t}'$ are row standard λ–tableau of the initial kind, and w_1 and w_2 are distinguished right coset representatives for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$.

Let $\mathbf{a} = (a_1, \dots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$, as in the definition of m_λ. We have $m_{\mathfrak{s}\mathfrak{t}} = T^*_{w_1} m_{\mathfrak{s}'\mathfrak{t}'} T_{w_2}$ and $m_{\mathfrak{s}'\mathfrak{t}'} = T^*_{d(\mathfrak{s}')} u^+_\mathbf{a} x_\lambda T_{d(\mathfrak{t}')} = u^+_\mathbf{a} T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')}$ since $d(\mathfrak{s}') \in \mathfrak{S}_\alpha$.

We may write $T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')}$ as a product of r commuting terms, one for each component of λ; say, $T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')} = x_1 x_2 \dots x_r$, where the kth term x_k involves only elements T_w with $w \in \mathfrak{S}(\{a_k + 1, a_k + 2 \dots, a_{k+1}\})$. For example, $x_1 = T^*_{d(\mathfrak{s}'_1)} x_{\lambda^{(1)}} T_{d(\mathfrak{t}'_1)}$ where $\mathfrak{s}'_1$ is the first component of $\mathfrak{s}'$ and $\mathfrak{t}'_1$ is the first component of $\mathfrak{t}'$. By applying Murphy's result [**16**, (4.18)] to the Hecke algebra $\mathscr{H}(\mathfrak{S}_{\alpha_1})$ we may write x_1 as a linear combination of terms $T^*_{d(\mathfrak{u}'_1)} x_{\mu^{(1)}} T_{d(\mathfrak{v}'_1)}$ where $\mathfrak{u}'_1$ and $\mathfrak{v}'_1$ are standard $\mu^{(1)}$–tableaux for some partition $\mu^{(1)}$ of a_1, and $\mathfrak{u}'_1 \trianglerighteq \mathfrak{s}'_1$ and $\mathfrak{v}'_1 \trianglerighteq \mathfrak{t}'_1$.

We can apply the same technique for the other factors $x_2, \dots, x_r$, to conclude that $T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')}$ is a linear combination of terms of the form $T^*_{d(\mathfrak{u}')} x_\mu T_{d(\mathfrak{v}')}$ where $\mathfrak{u}'$ and $\mathfrak{v}'$ are standard μ–tableaux for some multipartition μ of n, and $\mathfrak{u}' \trianglerighteq \mathfrak{s}'$ and $\mathfrak{v}' \trianglerighteq \mathfrak{t}'$. Also, $\mathfrak{u}'$ and $\mathfrak{v}'$ are of the initial kind, and $|\mu^{(k)}| = |\lambda^{(k)}|$ for $1 \le k \le r$. Therefore, $m_{\mathfrak{s}'\mathfrak{t}'}$ is a linear combination of terms $m_{\mathfrak{u}'\mathfrak{v}'}$ where the sum runs over the same set of pairs $(\mathfrak{u}', \mathfrak{v}')$. Consequently, $m_{\mathfrak{s}\mathfrak{t}}$ is a linear combination of terms $T^*_{w_1} m_{\mathfrak{u}'\mathfrak{v}'} T_{w_2}$.

Now, $T_{d(\mathfrak{v}')} T_{w_2} = T_{d(\mathfrak{v}')w_2} = T_{d(\mathfrak{v}'w_2)}$, so $T^*_{w_1} m_{\mathfrak{u}'\mathfrak{v}'} T_{w_2} = T^*_{w_1} m_{\mathfrak{u}'\mathfrak{v}}$ where $\mathfrak{v} = \mathfrak{v}'w_2$. By Lemma 3.17, since $\mathfrak{v}'$ is of the initial kind, $\mathfrak{v} = \mathfrak{v}'w_2$ is standard, and $\mathfrak{v}'w_2 \trianglerighteq \mathfrak{t}'w_2$; that is, $\mathfrak{v} \trianglerighteq \mathfrak{t}$. Moreover, $\mathrm{comp}_{\mathfrak{v}'w_2} = \mathrm{comp}_{\mathfrak{t}'w_2}$ since $\mathfrak{v}'$ and $\mathfrak{t}'$ are of the initial kind.

Similar remarks applied to $T^*_{w_1} T^*_{d(\mathfrak{u}')}$ now complete the proof of the Proposition. □

(3.19) Corollary *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. If $h \in \mathscr{H}(\mathfrak{S}_n)$ then $m_{\mathfrak{s}\mathfrak{t}}h$ is a linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableaux for some multipartition μ of n, and $\mu \trianglerighteq \lambda$, $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and* $\mathrm{comp}_{\mathfrak{u}} = \mathrm{comp}_{\mathfrak{s}}$.

PROOF: Combine Lemma 3.15 and Proposition 3.18, and recall that $\mathfrak{u} \trianglerighteq \mathfrak{s}$ implies that $\mu \trianglerighteq \lambda$. □

Corollary 3.19 provides the kind of information we need when we multiply $m_{\mathfrak{s}\mathfrak{t}}$ by elements of $\mathscr{H}(\mathfrak{S}_n)$. More complicated is our next proposition, which shows what happens when we multiply $m_{\mathfrak{s}\mathfrak{t}}$ by T_0. It is vital to the proposition that λ is a multipartition, not merely a multicomposition.

(3.20) Proposition *Suppose that λ is a multipartition of n, and that $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ–tableaux. Then $m_{\mathfrak{s}\mathfrak{t}}T_0 = x_1 + x_2$ where*

1. *x_1 is a linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard λ–tableaux, with $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathrm{comp}_{\mathfrak{u}} = \mathrm{comp}_{\mathfrak{s}}$, and*
2. *x_2 is a linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableaux for some multipartition μ of n, with $\mu \rhd \lambda$ and $\mathrm{comp}_{\mathfrak{s}} > \mathrm{comp}_{\mathfrak{u}}$.*

PROOF: Let $\alpha = (|\lambda^{(1)}|, \dots, |\lambda^{(r)}|)$ and let $\mathbf{a} = (a_1, a_2, \dots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$ for $1 \le k \le r$. Write $d(\mathfrak{t}) = yc$ with $y \in \mathfrak{S}_\alpha$ and c a distinguished right coset representative for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$. Then the α–tableau $\mathfrak{t}^\alpha c$ is row standard. Assume that 1 is in row k of $\mathfrak{t}^\alpha c$; thus, $1 \le k \le r$.

Now, $c = (a_k, a_k+1)(a_k-1, a_k)\dots(1,2)c'$, where $\ell(c) = a_k + \ell(c')$ and c' fixes 1. Thus, $T_c = T_{a_k} T_{a_{k-1}} \dots T_1 T_{c'}$ and $T_{c'} T_0 = T_0 T_{c'}$. Let $\mathbf{b} = (a_1, \dots, a_k+1, \dots, a_r)$. Then

$$u^+_{\mathbf{a}} T_c T_0 = u^+_{\mathbf{a}} T_{a_k} T_{a_{k-1}} \dots T_1 T_0 T_{c'} = u^+_{\mathbf{a}} h_1 + u^+_{\mathbf{b}} h_2$$

for some $h_1, h_2 \in \mathfrak{S}_n$ by Lemma 3.4. Since $y \in \mathfrak{S}_\alpha$ and $\mathfrak{t}^\lambda y$ is standard, y fixes $a_k + 1$; therefore, T_y commutes with $u^+_{\mathbf{b}}$. Hence, using Remark 3.7, we have

$$\begin{aligned} m_{\mathfrak{s}\mathfrak{t}} T_0 &= T^*_{d(\mathfrak{s})} u^+_{\mathbf{a}} x_\lambda T_{d(\mathfrak{t})} T_0 = T^*_{d(\mathfrak{s})} x_\lambda u^+_{\mathbf{a}} T_y T_c T_0 = T^*_{d(\mathfrak{s})} x_\lambda T_y u^+_{\mathbf{a}} T_c T_0 \\ &= T^*_{d(\mathfrak{s})} x_\lambda T_y (u^+_{\mathbf{a}} h_1 + u^+_{\mathbf{b}} h_2) = T^*_{d(\mathfrak{s})} x_\lambda (u^+_{\mathbf{a}} T_y h_1 + u^+_{\mathbf{b}} T_y h_2). \end{aligned}$$

The first term, $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathbf{a}}T_y h_1$, is a linear combination of terms of the required form by Corollary 3.19. If $k = 1$ then $u^+_{\mathbf{b}} = 0$ by Remark 3.2(i), and the proof is complete in this case. Assume therefore that $k \geq 2$. We turn out attention to the term $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathbf{b}}T_y h_2$. We now need to digress in order to prove that $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathbf{b}}T_y h_2$ has the required form.

Define the multicomposition $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$ of n by $\nu^{(i)} = \lambda^{(i)}$ for $i \neq k-1, k$, $\nu^{(k-1)} = (\lambda^{(k-1)}_1, \lambda^{(k-1)}_2, \dots, 1)$ and $\nu^{(k)} = (\lambda^{(k)}_1 - 1, \lambda^{(k)}_2, \dots)$. (Thus we reduce the first part of the kth component of λ by 1, and adjoin a new part of size 1 to the end of the $(k-1)$th component of λ.) Note that $\nu \rhd \lambda$.

Let $l = \lambda^{(k)}_1$ be the first part of $\lambda^{(k)}$. The entries in the first row of the kth component of $\mathfrak{t}^\lambda$ are then $a_k+1, a_k+2, \dots, a_k+l$. Let G_1 be the symmetric group on these numbers, and let G_2 be the symmetric group on $a_k+2, \dots, a_k+l$. Let $c_1, c_2, \dots, c_l$ be the distinguished right coset representatives for G_2 in G_1, ordered in terms of increasing length. Then the tableaux $\mathfrak{t}^\nu c_i$ are row standard, and they agree with $\mathfrak{t}^\nu$ except on the last row of the $(k-1)$th component and the first row of the kth component. The last row of the $(k-1)$th component of $\mathfrak{t}^\nu c_i$ contains the single entry $a_k + i$.

We are given that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$ is a row standard λ–tableau. Let $\mathfrak{u}_i = \mathfrak{t}^\nu c_i d(\mathfrak{s})$ for $1 \leq i \leq l$. Each $\mathfrak{u}_i$ agrees with $\mathfrak{s}$ except on the last row of the $(k-1)$th component of $\mathfrak{u}_i$ and on the first row of the kth component of $\mathfrak{u}_i$. Furthermore, since $(a_k+1)d(\mathfrak{s}) < \dots < (a_k+l)d(\mathfrak{s})$, it follows that each $\mathfrak{u}_i$ is row standard.

We also have $\mathrm{comp}_{\mathfrak{u}_i}(j) = \mathrm{comp}_{\mathfrak{s}}(j)$ if $j \notin \{(a_k+1)d(\mathfrak{s}), \dots, (a_k+l)d(\mathfrak{s})\}$ and, for $j \in \{(a_k+1)d(\mathfrak{s}), \dots, (a_k+l)d(\mathfrak{s})\}$, we have $\mathrm{comp}_{\mathfrak{u}_i}(j) = k-1$ or k and $\mathrm{comp}_{\mathfrak{s}}(j) = k$. Therefore, $\mathrm{comp}_{\mathfrak{s}} > \mathrm{comp}_{\mathfrak{u}_i}$.

Now x_λ has a factor

$$\sum_{w \in G_1} T_w = \sum_{i=1}^{l} T^*_{c_i} \sum_{w \in G_2} T_w.$$

Hence, $x_\lambda = \sum_{i=1}^l T^*_{c_i} x_\nu$. Note that $m_\nu = x_\nu u^+_{\mathbf{b}}$. Thus,

$$T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathbf{b}} = T^*_{d(\mathfrak{s})} \sum_{i=1}^{l} T^*_{c_i} x_\nu u^+_{\mathbf{b}} = \sum_{i=1}^{l} T^*_{c_i d(\mathfrak{s})} x_\nu u^+_{\mathbf{b}} = \sum_{i=1}^{l} m_{\mathfrak{u}_i \mathfrak{t}^\nu}.$$

Hence, by Corollary 3.19, $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathbf{b}}T_y h_2$ is a linear combination of the required form.

This completes the proof of the Proposition. □

(3.21) Definition *Suppose that λ is a multicomposition of n.*

1. *Let N^λ be the R–module spanned by*

$$\left\{ m_{\mathfrak{s}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \mu\text{–tableaux for some} \\ \text{multipartition } \mu \text{ of } n \text{ with } \mu \unrhd \lambda \end{array} \right\}.$$

2. *Let $\overline{N^\lambda}$ be the R–module spanned by*

$$\left\{\, m_{\mathfrak{s}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \mu\text{–tableaux for some} \\ \text{multipartition } \mu \text{ of } n \text{ with } \mu \rhd \lambda \end{array} \right\}.$$

We now apply Propositions 3.18 and 3.20 to obtain a sequence of useful results.

(3.22) Proposition *Suppose that λ is a multicomposition of n. Then N^λ and $\overline{N^\lambda}$ are two–sided ideals of $\mathscr{H}$.*

PROOF: Corollary 3.19 shows that N^λ is closed under postmultiplication by $T_1, T_2, \dots, T_{n-1}$ and Proposition 3.20 shows that N^λ is also closed under postmultiplication by T_0. Because $\mathscr{H}$ is generated by $T_0, T_1, \dots, T_{n-1}$, we deduce that N^λ is a right ideal of $\mathscr{H}$. Since $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$, by applying the antiautomorphism $*$ we see that N^λ is also a left ideal of $\mathscr{H}$.

Finally, $\overline{N^\lambda}$ is a two sided ideal of $\mathscr{H}$ since $\overline{N^\lambda} = \sum_{\mu \rhd \lambda} N^\mu$. □

A proof very similar to that of Proposition 3.22 gives our next result.

(3.23) Proposition *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ is a row standard λ–tableau. Let $I_\mathfrak{s}$ be the R–module spanned by*

$$\left\{\, m_{\mathfrak{u}\mathfrak{v}} \;\middle|\; \begin{array}{c} \mathfrak{u} \text{ and } \mathfrak{v} \text{ are standard } \mu\text{–tableaux for some multipartition} \\ \mu \text{ of } n \text{ with } \mu \trianglerighteq \lambda \text{ and } \operatorname{comp}_\mathfrak{s} \geq \operatorname{comp}_\mathfrak{u} \end{array} \right\}.$$

Then $I_\mathfrak{s}$ is a right ideal of $\mathscr{H}$.

Recall that $M^\lambda = m_\lambda \mathscr{H}$.

(3.24) Corollary *Suppose that λ is a multicomposition of n. Then every element of M^λ is a linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableaux for some multipartition μ of n with $\mu \trianglerighteq \lambda$ and $\operatorname{comp}_{\mathfrak{t}^\lambda} \geq \operatorname{comp}_\mathfrak{u}$.*

PROOF: Since $m_\lambda = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda}$, Proposition 3.18 shows that $m_\lambda \in I_{\mathfrak{t}^\lambda}$ (note that λ may not be a multipartition). Therefore, $M^\lambda \subseteq I_{\mathfrak{t}^\lambda}$ and the desired result follows by Proposition 3.23. □

(3.25) Proposition *Suppose that λ is a multipartition of n and that $\mathfrak{t}$ is a standard λ–tableau. Let $h \in \mathscr{H}$. Then for every standard λ–tableau $\mathfrak{v}$ there exists $r_\mathfrak{v} \in R$ such that, for all standard λ–tableau $\mathfrak{s}$, we have*

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} r_\mathfrak{v} m_{\mathfrak{s}\mathfrak{v}} \bmod \overline{N^\lambda}.$$

PROOF: Let U be the R–module spanned by $\{\, m_{\mathfrak{t}^\lambda\mathfrak{t}'} \mid \mathfrak{t}' \text{ is a standard } \lambda\text{–tableau} \,\}$. Note that $\mathfrak{t}^\lambda$ is the unique λ–tableau $\mathfrak{u}$ such that $\mathfrak{u} \trianglerighteq \mathfrak{t}^\lambda$. Therefore, $U + \overline{N^\lambda}$ is closed under postmultiplication by $T_1, T_2, \dots, T_{n-1}$ by Proposition 3.18, and it is closed under postmultiplication by T_0 by Proposition 3.20. Hence $U + \overline{N^\lambda}$ is a right ideal of $\mathscr{H}$. Thus, for each $\mathfrak{v} \in \operatorname{Std}(\lambda)$ there exists $r_\mathfrak{v} \in R$ such that

$$m_{\mathfrak{t}^\lambda\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} r_\mathfrak{v} m_{\mathfrak{t}^\lambda\mathfrak{v}} \bmod \overline{N^\lambda}.$$

Multiply this congruence on the left by $T^*_{d(\mathfrak{s})}$, and use the fact that $\overline{N^\lambda}$ is an ideal, to obtain the congruence of the proposition. □

(3.26) Theorem ([**13**, (1.7)]) *The algebra $\mathscr{H}$ is a free R–module with basis*

$$\mathscr{M} = \left\{\, m_{\mathfrak{s}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \lambda\text{–tableaux for some} \\ \text{multipartition } \lambda \text{ of } n \end{array} \right\}.$$

Moreover, $\mathscr{M}$ is a cellular basis of $\mathscr{H}$.

PROOF: Since $m_\lambda = 1$ when $\lambda = \big((0), \dots, (0), (1^n)\big)$, Proposition 3.25 shows that $\mathscr{M}$ spans $\mathscr{H}$. By Theorem 2.2, $\mathscr{H}$ is free of rank $r^n n!$ Since this is also the cardinality of $\mathscr{M}$, we deduce that $\mathscr{M}$ is a basis of $\mathscr{H}$. Finally, the properties that $\mathscr{M}$ must satisfy in order to be a cellular basis of $\mathscr{H}$ (as given in [**12**, (1.1)]), are covered by Proposition 3.25 and the fact that $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$. □

(3.27) Definition *We call $\mathscr{M}$ the* standard basis *of $\mathscr{H}$.*

We can now apply Graham and Lehrer's theory of cellular algebras [**12**] to describe the representation theory of $\mathscr{H}$.

(3.28) Definition *Suppose that λ is a multipartition of n. Let $z_\lambda = (\overline{N^\lambda} + m_\lambda)/\overline{N^\lambda}$. The* Specht module *$S^\lambda$ is the submodule of $\mathscr{H}/\overline{N^\lambda}$ given by $S^\lambda = z_\lambda \mathscr{H}$.*

Directly from Proposition 3.25 and Theorem 3.26, S^λ is a free R–module with basis $\{\, z_\lambda T_{d(\mathfrak{t})} \mid \mathfrak{t} \text{ a standard } \lambda\text{–tableau} \,\}$.

Define a bilinear form $\langle\ ,\ \rangle$ on the Specht module S^λ by

$$m_\lambda T_{d(\mathfrak{s})} T^*_{d(\mathfrak{t})} m_\lambda \equiv \langle z_\lambda T_{d(\mathfrak{s})}, z_\lambda T_{d(\mathfrak{t})} \rangle m_\lambda \bmod \overline{N^\lambda}.$$

By Proposition 3.25 (and the version of Proposition 3.25 obtained by applying the antiautomorphism $*$), the bilinear form is well defined (cf. [**12**]). Moreover, the bilinear form is symmetric and $\langle uh, v\rangle = \langle u, vh^*\rangle$ for all $u, v \in S^\lambda$ and all $h \in \mathscr{H}$ [**12**, (2.4)]. Consequently, $\operatorname{rad} S^\lambda$, the radical of the bilinear form, is an $\mathscr{H}$–module.

(3.29) Definition *Suppose that λ is a multipartition of n. Let $D^\lambda = S^\lambda/\operatorname{rad} S^\lambda$.*

(3.30) Theorem *Suppose that R is a field. Then the non–zero $\mathscr{H}$ modules in*

$$\{\, D^\lambda \mid \lambda \text{ a multipartition of } n \,\}$$

form a complete set of non–isomorphic irreducible $\mathscr{H}$–modules. Moreover, each irreducible module D^λ is absolutely irreducible.

PROOF: Since $\mathscr{M}$ is a cellular basis of $\mathscr{H}$ the Theorem follows from the general theory of cellular algebras [**12**, (3.4)]. □

The theory in [**12**, (3.6)] also shows that if $D^\mu \neq 0$ and D^μ is a composition factor of S^λ then $\lambda \trianglerighteq \mu$.

When $r = 1$ the partitions μ for which $D^\mu \neq 0$ have been classified in [**4**, (7.6)]. When $r > 1$, if $q = 1$, or if the parameters Q_k are powers of q, the multipartitions μ for which $D^\mu \neq 0$ are classified by the results of [**12**, (5.9)], [**13**, (1.17)] and [**14**, (3.7)]; in type **B**, see also [**8**, (5.3)] and [**10**, (8.19)].

4 A Specht series for M^μ

In the last section, we used various elements m_μ of $\mathscr{H}$ to construct a cellular basis of $\mathscr{H}$. We saw that the right ideal M^μ generated by m_μ is an analogue of an induced module; indeed, if $r = 1$ and $q = 1$ then M^μ is the permutation module of $\mathfrak{S}_n$ on the Young subgroup $\mathfrak{S}_\mu$. We shall use the right ideals M^μ of $\mathscr{H}$ to construct the cyclotomic q–Schur algebra, but, before that, we study the individual modules M^μ in more detail. In particular, we shall give a semistandard basis of M^μ and construct a Specht series for M^μ.

We are going to define a new kind of λ–tableau. This will have entries which are ordered pairs (i, k) where i is a positive integer and $1 \leq k \leq r$. We denote such tableaux by capital letters.

(4.1) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. A λ–tableau S has* type *μ if its entries are ordered pairs (i, k), as above, and for all m and k the number of times (i, k) is an entry in S is $\mu_i^{(k)}$.*

Next, we introduce a function which converts standard λ–tableaux (or, indeed any tableau whose entries are $1, 2, \dots, n$) into a λ–tableau of type μ.

(4.2) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathfrak{s}$ be a standard λ–tableau. Define $\mu(\mathfrak{s})$ to be the λ–tableau obtained from $\mathfrak{s}$ by replacing each entry m in $\mathfrak{s}$ by (i, k) if m is in row i of the kth component of $\mathfrak{t}^\mu$.*

In our examples, we shall always write i_k for the ordered pair (i, k).

(4.3) Example Suppose that $\mu = \big((3,1),(2),(2,1^2)\big)$. Then

$$\mathfrak{t}^\mu = \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 5&6\\\hline\end{array},\ \begin{array}{|c|c|}\hline 7&8\\\hline 9\\\cline{1-1} 10\\\cline{1-1}\end{array} \right) \quad\text{and}\quad \mu(\mathfrak{t}^\mu) = \left(\begin{array}{|c|c|c|}\hline 1_1&1_1&1_1\\\hline 2_1\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1_2&1_2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1_3&1_3\\\hline 2_3\\\cline{1-1} 3_3\\\cline{1-1}\end{array} \right).$$

Suppose that

$$\mathfrak{s}_1 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 6&10\\\hline\end{array},\ \begin{array}{|c|c|}\hline 7&8\\\hline 9\\\cline{1-1}\end{array} \right), \qquad \mathfrak{s}_2 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 5&10\\\hline\end{array},\ \begin{array}{|c|c|}\hline 7&8\\\hline 9\\\cline{1-1}\end{array} \right)$$

and

$$\mathfrak{s}_3 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&5&9\\\hline 7\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 3&10\\\hline\end{array},\ \begin{array}{|c|c|}\hline 4&6\\\hline 8\\\cline{1-1}\end{array} \right).$$

Then

$$\mu(\mathfrak{s}_1) = \mu(\mathfrak{s}_2) = \left(\begin{array}{|c|c|c|c|}\hline 1_1&1_1&1_1&1_2\\\hline 2_1\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1_2&3_3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1_3&1_3\\\hline 2_3\\\cline{1-1}\end{array} \right)$$

and

$$\mu(\mathfrak{s}_3) = \left(\begin{array}{|c|c|c|c|}\hline 1_1&1_1&1_2&2_3\\\hline 1_3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1_1&3_3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2_1&1_2\\\hline 1_3\\\cline{1-1}\end{array} \right).$$

Given two pairs (i_1, k_1) and (i_2, k_2), we write $(i_1, k_1) < (i_2, k_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $i_1 < i_2$. Note that if $\mathfrak{s}$ is a standard λ–tableau, then $\mu(\mathfrak{s})$ is a λ–tableau of type μ whose entries are weakly increasing along rows and down columns. We next single out some of these tableaux $\mu(\mathfrak{s})$.

(4.4) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathtt{S} = (\mathtt{S}^{(1)}, \dots, \mathtt{S}^{(r)})$ be a λ–tableau of type μ. Then $\mathtt{S}$ is* semistandard *if*

1. *the entries in each row of each component $\mathtt{S}^{(k)}$ of $\mathtt{S}$ are non–decreasing; and*
2. *the entries in each column of each component $\mathtt{S}^{(k)}$ of $\mathtt{S}$ are strictly increasing; and*
3. *for each k with $1 \le k \le r$, no entry in $\mathtt{S}^{(k)}$ has the form (i, l) with $l < k$.*

Let $\mathcal{T}_0(\lambda, \mu)$ denote the set of semistandard λ–tableaux of type μ.

For example, $\mu(\mathfrak{s}_1)$ in Example 4.3 is semistandard, but $\mu(\mathfrak{s}_3)$ is not, since the third part of Definition 4.4 is not satisfied.

The point of the third condition in Definition 4.4 is the following result (cf. Corollary 3.24).

(4.5) Suppose that $\mathfrak{s}$ is a standard λ–tableau. Then $\mu(\mathfrak{s})$ satisfies Definition 4.4(iii) if and only if $\operatorname{comp}_{\mathfrak{t}^\mu} \ge \operatorname{comp}_{\mathfrak{s}}$

If λ is a multipartition and $\mathtt{S} \in \mathcal{T}_0(\lambda, \mu)$, then it is clear that there exists a standard λ–tableau $\mathfrak{s}$ with $\mathtt{S} = \mu(\mathfrak{s})$. In order to say more about this, we introduce another definition.

(4.6) Definition *If $\mathfrak{s}$ is a standard λ–tableau and $1 \le m \le n$ then let* $\operatorname{row}_{\mathfrak{s}}(m) = (i,k)$ *if m belongs to row i of the kth component of $\mathfrak{s}$.*

The definition of $\operatorname{row}_{\mathfrak{s}}$ allows us to recover $\mathfrak{s}$ from the function $\operatorname{row}_{\mathfrak{s}}$

Suppose that $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$ and let $\mathfrak{s}$ be a standard λ–tableau such that $\mu(\mathfrak{s}) = \mathsf{S}$. If there exists an integer i such that $\operatorname{row}_{\mathfrak{s}}(i) \ne \operatorname{row}_{\mathfrak{s}}(i+1)$ and i and $i+1$ are in the same row of $\mathfrak{t}^\mu$ then the tableau $\mathfrak{s}_1 = \mathfrak{s}(i,i+1)$ is standard and $\mu(\mathfrak{s}_1) = \mathsf{S}$. Also, $\mathfrak{s} \rhd \mathfrak{s}_1$ if $\operatorname{row}_{\mathfrak{s}}(i) < \operatorname{row}_{\mathfrak{s}}(i+1)$, and $\mathfrak{s}_1 \rhd \mathfrak{s}$ if $\operatorname{row}_{\mathfrak{s}}(i) > \operatorname{row}_{\mathfrak{s}}(i+1)$. Hence we have shown the following.

(4.7) Suppose that $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$. Then there exist standard λ–tableaux $\operatorname{first}(\mathsf{S})$ and $\operatorname{last}(\mathsf{S})$ such that

1. $\mu(\operatorname{first}(\mathsf{S})) = \mu(\operatorname{last}(\mathsf{S})) = \mathsf{S}$; and,
2. if $\mathfrak{s}$ is any standard λ–tableau such that $\mu(\mathfrak{s}) = \mathsf{S}$ then $\operatorname{first}(\mathsf{S}) \unrhd \mathfrak{s} \unrhd \operatorname{last}(\mathsf{S})$.

(4.8) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then $m_{\mathsf{S}\mathfrak{t}}$ is the element of $\mathscr{H}$ given by*

$$m_{\mathsf{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathsf{S}}} m_{\mathfrak{s}\mathfrak{t}}.$$

Note that $m_{\mathsf{S}\mathfrak{t}}$ is a sum of standard basis elements $m_{\mathfrak{s}\mathfrak{t}}$ where $\operatorname{first}(\mathsf{S}) \unrhd \mathfrak{s} \unrhd \operatorname{last}(\mathsf{S})$.

(4.9) Example Suppose that $\lambda = \big((4,3),(2,1),(2,1)\big)$, $\mu = \big((3^2,1),(1^2),(2,1^2)\big)$ and

$$\mathsf{S} = \left(\begin{array}{|c|c|c|c|}\hline 1_1 & 1_1 & 1_1 & 2_1 \\ \hline 2_1 & 2_1 & 3_1 \\ \cline{1-3} \end{array}\,,\ \begin{array}{|c|c|}\hline 1_2 & 3_3 \\ \hline 2_2 \\ \cline{1-1} \end{array}\,,\ \begin{array}{|c|c|}\hline 1_3 & 1_3 \\ \hline 2_3 \\ \cline{1-1} \end{array} \right).$$

Then $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$ and for any $\mathfrak{t} \in \operatorname{Std}(\lambda)$ we have $m_{\mathsf{S}\mathfrak{t}} = m_{\mathfrak{s}_1\mathfrak{t}} + m_{\mathfrak{s}_2\mathfrak{t}} + m_{\mathfrak{s}_3\mathfrak{t}}$ where

$$\mathfrak{s}_1 = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} \end{array}\,,\ \begin{array}{|c|c|}\hline 8 & 13 \\ \hline 9 \\ \cline{1-1} \end{array}\,,\ \begin{array}{|c|c|}\hline 10 & 11 \\ \hline 12 \\ \cline{1-1} \end{array} \right), \quad \mathfrak{s}_2 = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 \\ \cline{1-3} \end{array}\,,\ \begin{array}{|c|c|}\hline 8 & 13 \\ \hline 9 \\ \cline{1-1} \end{array}\,,\ \begin{array}{|c|c|}\hline 10 & 11 \\ \hline 12 \\ \cline{1-1} \end{array} \right)$$

and

$$\mathfrak{s}_3 = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & 7 \\ \cline{1-3} \end{array}\,,\ \begin{array}{|c|c|}\hline 8 & 13 \\ \hline 9 \\ \cline{1-1} \end{array}\,,\ \begin{array}{|c|c|}\hline 10 & 11 \\ \hline 12 \\ \cline{1-1} \end{array} \right).$$

Here $\mathfrak{s}_1 = \operatorname{first}(\mathsf{S})$ and $\mathfrak{s}_3 = \operatorname{last}(\mathsf{S})$.

(4.10) Lemma *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then $m_{\mathsf{S}\mathfrak{t}} \in M^\mu$.*

PROOF: Let $\mathbf{a} = (a_1,\dots,a_r)$ where $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$, and $\mathbf{b} = (b_1,\dots,b_r)$ where $b_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$.

Let $\mathfrak{s}_1 = \mathrm{first}(\mathtt{S})$ and let $d = d(\mathfrak{s}_1)$. Then, as in [**4**],

$$\sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathtt{S}}} x_\lambda T_{d(\mathfrak{s})} = \sum_{w\in\mathfrak{S}_\lambda d\mathfrak{S}_\mu} T_w = hT_d x_\mu,$$

where $h = \sum T_v$, the sum running over certain elements $v \in \mathfrak{S}_\lambda$. Since $h \in \mathscr{H}(\mathfrak{S}_\lambda)$, we have $h^* u_{\mathbf{b}}^+ = u_{\mathbf{b}}^+ h^*$; so we obtain

$$m_{\mathtt{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathtt{S}}} m_{\mathfrak{s}\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathtt{S}}} T_{d(\mathfrak{s})}^* x_\lambda u_{\mathbf{b}}^+ T_{d(\mathfrak{t})} = x_\mu T_d^* u_{\mathbf{b}}^+ h^* T_{d(\mathfrak{t})}.$$

Therefore, since $m_\mu = x_\mu u_{\mathbf{a}}^+$, it is sufficient to prove that $T_d^* u_{\mathbf{b}}^+ \in u_{\mathbf{a}}^+\mathscr{H}$.

Assume that $1 \le k \le r$. Recall that $\mathfrak{s}_1 = \mathrm{first}(\mathtt{S})$, $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$ and $b_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$ and note that $a_k \le b_k$ since $\mathtt{S}$ is semistandard. Define $\mathfrak{t}_k$ to be the standard λ–tableau such that $1, 2, \dots, a_k$ occupy the same positions in $\mathfrak{s}_1$ and $\mathfrak{t}_k$; $b_k + 1, \dots, n$ occupy the same positions in $\mathfrak{t}^\lambda$ and $\mathfrak{t}_k$; and that for $a_k + 1 \le i < j \le b_k$ we have $\mathrm{row}_{\mathfrak{t}_k}(i) \le \mathrm{row}_{\mathfrak{t}_k}(j)$ — thus the numbers $a_k+1, \dots, b_k$ are inserted into $\mathfrak{t}_k$ in "row order". The fact that $\mathtt{S}$ is semistandard ensures that $\mathfrak{t}_k$ is well defined. Note that $\mathfrak{t}_1 = \mathfrak{t}^\lambda$ and $\mathfrak{t}_r = \mathfrak{s}_1$.

Now define $w_k \in \mathfrak{S}_n$ inductively by requiring that $\mathfrak{t}_k = \mathfrak{t}^\lambda w_1 w_2 ... w_k$. Then $w_1 = 1$ and $d = w_1 w_2 ... w_r$; moreover, $\ell(d) = \ell(w_1) + ... + \ell(w_r)$. By construction, the element w_k fixes each of the numbers $1, 2, \dots, a_{k-1}$ and it also fixes each of $b_k + 1, b_k + 2, \dots, n$.

For $1 \le k \le r$, let $u_{\mathbf{a},k}$ and $u_{\mathbf{b},k}$ be as defined in Definition 3.1, and let $v_k = \prod_{m=a_k+1}^{b_k} (L_m - Q_k)$. Then $u_{\mathbf{b},k} = u_{\mathbf{a},k} v_k$.

Now, T_{w_k} commutes with $u_{\mathbf{a},l}$ for $l < k$ (since w_k fixes $1, 2, \dots, a_{k-1}$), and T_{w_k} commutes with $u_{\mathbf{b},l}$ for $l \ge k$ (since w_k fixes $b_k + 1, b_k + 2, \dots, n$); see (2.1)(iv). Therefore,

$$\begin{array}{rcc}
T_d^* u_{\mathbf{b}}^+ & = & T_{w_r}^* \dots T_{w_1}^* u_{\mathbf{b},1} u_{\mathbf{b},2} \dots u_{\mathbf{b},r} \\
& = & T_{w_r}^* \dots T_{w_2}^* u_{\mathbf{b},1} u_{\mathbf{b},2} \dots u_{\mathbf{b},r} T_{w_1}^* \\
& = & u_{\mathbf{a},1} T_{w_r}^* \dots T_{w_2}^* u_{\mathbf{b},2} \dots u_{\mathbf{b},r} v_1 T_{w_1}^* \\
& = & u_{\mathbf{a},1} u_{\mathbf{a},2} T_{w_r}^* \dots T_{w_3}^* u_{\mathbf{b},3} \dots u_{\mathbf{b},r} v_2 T_{w_2}^* v_1 T_{w_1}^* \\
& \vdots & \vdots \\
& = & u_{\mathbf{a},1} \dots u_{\mathbf{a},r} v_r T_{w_r}^* v_{r-1} T_{w_{r-1}}^* \dots v_2 T_{w_2}^* v_1 T_{w_1}^*.
\end{array}$$

Thus, $T_d^* u_{\mathbf{b}}^+ \in u_{\mathbf{a}}^+\mathscr{H}$, as required. □

Having shown that $m_{\mathtt{S}\mathfrak{t}} \in M^\mu$, our next aim is to prove that the set of elements of the form $m_{\mathtt{S}\mathfrak{t}}$ give a basis of M^μ.

Let $\mathbf{a} = (a_1, \dots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$ as in the previous proof; then $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Since $u_{\mathbf{a}}^+$ and x_μ commute, we have $M^\mu = m_\mu\mathscr{H} \subseteq u_{\mathbf{a}}^+\mathscr{H} \cap x_\mu\mathscr{H}$. We shall show that we have equality here.

We have seen that $\mathscr{H}$ has a standard basis which consists of elements $m_{\mathfrak{s}\mathfrak{t}}$ with $\mathfrak{s}$ and $\mathfrak{t}$ standard λ–tableaux for some multipartition λ of n. Suppose that $h \in \mathscr{H}$ and let $h =$

$\sum_{\mathfrak{s},\mathfrak{t}} r_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$, with $r_{\mathfrak{s}\mathfrak{t}} \in R$, be the unique expression for h in terms of the standard basis. We say that $m_{\mathfrak{s}\mathfrak{t}}$ is involved in h if $r_{\mathfrak{s}\mathfrak{t}} \neq 0$.

(4.11) Lemma *Suppose that μ is a multipartition of n and that $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Let $h \in x_\mu \mathscr{H} \cap u_{\mathbf{a}}^+ \mathscr{H}$ and suppose that $(\mathfrak{s}, \mathfrak{t})$ is a pair of standard tableaux of the same shape such that*

1. *$m_{\mathfrak{s}\mathfrak{t}}$ is involved in h; and,*
2. *if $(\mathfrak{u}, \mathfrak{v})$ is a pair of standard tableaux of the same shape such that $\mathfrak{s} \trianglerighteq \mathfrak{u}$, $\mathfrak{t} \trianglerighteq \mathfrak{v}$ and $(\mathfrak{s}, \mathfrak{t}) \neq (\mathfrak{u}, \mathfrak{v})$ then $m_{\mathfrak{u}\mathfrak{v}}$ is not involved in h.*

Let $\mathtt{S} = \mu(\mathfrak{s})$. Then $\mathtt{S}$ is semistandard and $\mathfrak{s} = \operatorname{last}(\mathtt{S})$.

PROOF: Since $\mathfrak{s}$ is standard, the entries in $\mathtt{S}$ are non–decreasing down rows and columns.

Suppose that i and $i+1$ belong to the same row of $\mathfrak{t}^\mu$. Then $T_i x_\mu = q x_\mu$, so $T_i h = qh$. Therefore, by [**16**, (4.19)], i and $i+1$ do not belong to the same column of $\mathfrak{s}$. Hence the entries in $\mathtt{S}$ are strictly increasing down columns.

Since $h \in u_{\mathbf{a}}^+ \mathscr{H}$ we have $\operatorname{comp}_{\mathfrak{t}^\mu} \geq \operatorname{comp}_{\mathfrak{s}}$ by Corollary 3.24 (applied to the multi-composition $\big((1^{\alpha_1}), (1^{\alpha_2}), \dots, (1^{\alpha_r})\big)$, where $\alpha = (|\mu^{(1)}|, \dots, |\mu^{(r)}|)$). Hence, by (4.5), $\mathtt{S}$ is semistandard.

Finally, suppose that $\mathfrak{s} \neq \operatorname{last}(\mathtt{S})$. Then there exist integers i and $i+1$ in the same row of $\mathfrak{t}^\mu$ such that the tableau $\mathfrak{s}' = \mathfrak{s}(i, i+1)$ is standard and $\mathfrak{s} \triangleright \mathfrak{s}'$. As in [**16**, (4.19)], $m_{\mathfrak{s}'\mathfrak{t}}$ is involved in $T_i h = qh$, contradicting part (ii) of our hypothesis. Therefore, $\mathfrak{s} = \operatorname{last}(\mathtt{S})$. □

(4.12) Corollary *Suppose that μ is a multicomposition of n and $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Then $u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$ is spanned by*

$$\left\{\, m_{\mathtt{S}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathtt{S} \in \mathcal{T}_0(\lambda, \mu) \text{ and } \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some} \\ \text{multipartition } \lambda \text{ of } n \end{array} \right\}.$$

PROOF: Note that $m_{\mathtt{S}\mathfrak{t}} \in M^\mu$ by Lemma 4.10.

For each multipartition λ of n let

$$\mathcal{T}_l(\lambda, \mu) = \{\, \mathfrak{s} \in \operatorname{Std}(\lambda) \mid \mu(\mathfrak{s}) = \operatorname{last}(\mathtt{S}) \text{ for some } \mathtt{S} \in \mathcal{T}_0(\lambda, \mu) \,\}.$$

By Lemma 4.11 every non–zero element of $u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$ involves a standard basis element $m_{\mathfrak{s}\mathfrak{t}}$ where $\mathfrak{s} \in \mathcal{T}_l(\lambda, \mu)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$ for some multipartition λ.

Now suppose that $h \in u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$ and write h in terms of the standard basis; say, $h = \sum r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}}$ with $r_{\mathfrak{u}\mathfrak{v}} \in R$. Let

$$h' = h - \sum_{\lambda} \sum_{\mathfrak{s} \in \mathcal{T}_l(\lambda, \mu)} \sum_{\mathfrak{t} \in \operatorname{Std}(\lambda)} r_{\mathfrak{s}\mathfrak{t}} m_{\mu(\mathfrak{s})\mathfrak{t}}.$$

Then $h' \in u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$, but h' does not involve any term $m_{\mathfrak{s}\mathfrak{t}}$ for any $\mathfrak{s} \in \mathcal{T}_l(\lambda, \mu)$. Therefore, $h' = 0$, and we have obtained an expression for h as a linear combination of the required form. □

(4.13) Corollary *Suppose that μ is a multicomposition of n and $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Then $M^\mu = u_{\mathbf{a}}^+\mathscr{H} \cap x_\mu\mathscr{H}$.*

PROOF: We have that $M^\mu \subseteq u_{\mathbf{a}}^+\mathscr{H} \cap x_\mu\mathscr{H}$. The inclusion $u_{\mathbf{a}}^+\mathscr{H} \cap x_\mu\mathscr{H} \subseteq M^\mu$ follows from Corollary 4.12 and Lemma 4.10 □

The next theorem for Iwahori–Hecke algebras of type **A** (that is, the case $r = 1$), is due to Murphy [**16**]; for Iwahori–Hecke algebras of type **B** (that is, the case $r = 2$), the result is due to Du and Scott [**11**]

(4.14) Theorem *Suppose that μ is a multicomposition of n. Then M^μ is free as an R–module with basis*

$$\left\{ m_{\mathsf{St}} \;\middle|\; \begin{array}{c} \mathsf{S} \in \mathcal{T}_0(\lambda,\mu) \text{ and } \mathsf{t} \in \operatorname{Std}(\lambda) \text{ for} \\ \text{some multipartition } \lambda \text{ of } n \end{array} \right\}.$$

PROOF: By Lemma 4.10 each element m_{St} belongs to M^μ. Since distinct elements m_{St} involve distinct standard basis elements $m_{\mathfrak{s}\mathsf{t}}$, the elements m_{St} are linearly independent. Finally, Corollaries 4.12 and 4.13 show that the elements m_{St} span M^μ. □

(4.15) Corollary *Suppose that μ is a multicomposition of n. Then there exists a filtration of M^μ,*

$$M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0$$

such that for each i with $1 \le i \le k$ there exists a multipartition λ_i of n with $M_i/M_{i+1} \cong S^{\lambda_i}$. Moreover, if λ is a multipartition of n, then the number of i such that $\lambda = \lambda_i$ is equal to the number of semistandard λ–tableaux of type μ.

PROOF: Choose an ordering $\mathsf{S}_1 > \mathsf{S}_2 > \cdots > \mathsf{S}_k$ on the set of semistandard tableaux of type μ such that $j > i$ if $\lambda_i \rhd \lambda_j$ where $\mathsf{S}_i \in \mathcal{T}_0(\lambda_i,\mu)$ and $\mathsf{S}_j \in \mathcal{T}_0(\lambda_j,\mu)$. Let M_i be the R–submodule of M^μ with basis $\{\, m_{\mathsf{S}_j\mathsf{t}} \mid j \ge i \text{ and } \mathsf{t} \in \operatorname{Std}(\lambda_j) \,\}$. Then

$$M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0,$$

and, by Proposition 3.25 and Theorem 4.14, each M_i is a right ideal of $\mathscr{H}$.

Suppose that $1 \le i \le k$. Then $M_i \cap \overline{N^{\lambda_i}} \subseteq M_{i+1}$. Hence, there is a well defined $\mathscr{H}$–homomorphism θ from S^{λ_i} onto M_i/M_{i+1} such that $\theta(z_{\lambda_i}) = m_{\mathsf{S}_i\mathsf{t}^{\lambda_i}} + M_{i+1}$. Since both S^{λ_i} and M_i/M_{i+1} have rank equal to the number of standard λ_i–tableaux, θ is an isomorphism. The Corollary now follows. □

5 The double annihilator of m_μ

The purpose of this section is to compute the double annihilator of the element m_μ of $\mathscr{H}$ for any multicomposition μ of n. This will enable us to calculate a basis for $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ in the next section.

(5.1) Definition *Suppose S is a subset of $\mathscr{H}$ and define* $\mathrm{r}(S) = \{\, h \in \mathscr{H} \mid Sh = 0 \,\}$ *and* $\mathrm{l}(S) = \{\, h \in \mathscr{H} \mid hS = 0 \,\}$. *The* double annihilator *of S is* $\mathrm{lr}(S) = \mathrm{l}(\mathrm{r}(S))$.

It is elementary that $\mathrm{lr}(S)$ contains the left ideal of $\mathscr{H}$ generated by S. If $\mathscr{H}$ is a quasi–Frobenius algebra, then $\mathrm{lr}(S)$ is equal to the left ideal of $\mathscr{H}$ generated by S [**3**, (61.2)]. Although $\mathscr{H}$ is quasi–Frobenius for $r \le 2$ (see, for example, [**4**, section 2]), we do not know whether it is quasi–Frobenius for $r > 2$. If $\mathscr{H}$ is quasi–Frobenius for all r, then the main result of this section, namely Theorem 5.16, is immediate.

A connection between double annihilators and homomorphisms is provided by the following easy lemma.

(5.2) Lemma *Suppose that $m \in \mathscr{H}$ and that $\mathrm{lr}(m) = \mathscr{H}m$. Let I be a right ideal of $\mathscr{H}$.*

1. *For all $\varphi \in \operatorname{Hom}_{\mathscr{H}}(m\mathscr{H}, I)$ there exists $h_\varphi \in \mathscr{H}$ such that $\varphi(m) = h_\varphi m$.*
2. *$\operatorname{Hom}_{\mathscr{H}}(m\mathscr{H}, I) \cong \mathscr{H}m \cap I$.*

PROOF: For all $x \in \mathrm{r}(m)$ we have $\varphi(m)x = \varphi(mx) = 0$. Therefore, $\varphi(m) \in \mathrm{lr}(m) = \mathscr{H}m$; so $\varphi(m) = h_\varphi m$ for some $h_\varphi \in \mathscr{H}$. The proof of part (ii) is now straightforward. □

The main result of this section is that $\mathrm{lr}(m_\mu) = \mathscr{H}m_\mu$. Because the Iwahori–Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ is quasi–Frobenius, our first step is easy.

(5.3) Lemma *Suppose that μ is a multicomposition of n. Then $\mathrm{lr}(x_\mu) = \mathscr{H}x_\mu$.*

PROOF: Since $\mathscr{H}(\mathfrak{S}_n)$ is a quasi–Frobenius algebra, we have $\mathrm{lr}(x_\mu) \cap \mathscr{H}(\mathfrak{S}_n) = \mathscr{H}(\mathfrak{S}_n)x_\mu$. Now suppose that $z \in \mathrm{lr}(x_\mu)$. By Theorem 2.2, we may write

$$z = \sum_{\mathbf{c} \in \mathscr{C}} L_1^{c_1}, \dots L_n^{c_n} h_{\mathbf{c}}$$

where $\mathscr{C} = \{\, \mathbf{c} = (c_1, \dots, c_r) \mid 0 \le c_j < r \text{ for } 1 \le j \le n \,\}$ and $h_{\mathbf{c}} \in \mathscr{H}(\mathfrak{S}_n)$. Then, for all $y \in \mathrm{r}(x_\mu) \cap \mathscr{H}(\mathfrak{S}_n)$, we have

$$0 = zy = \sum_{\mathbf{c} \in \mathscr{C}} L_1^{c_1}, \dots L_n^{c_n} h_{\mathbf{c}} y.$$

Therefore, by Theorem 2.2, we have $h_{\mathbf{c}} y = 0$ for all $\mathbf{c} \in \mathscr{C}$. Thus, $h_{\mathbf{c}} \in \mathrm{lr}(x_\mu) \cap \mathscr{H}(\mathfrak{S}_n) = \mathscr{H}(\mathfrak{S}_n)x_\mu$, and so $z \in \mathscr{H}x_\mu$. □

For the remainder of this section we fix a multicomposition μ and define $\mathbf{a} = (a_1, \dots, a_r)$ by $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$; so $0 = a_1 \le a_2 \le \dots \le a_r \le n$. Our main task in this section is to prove that $\mathrm{lr}(u_{\mathbf{a}}^+) = \mathscr{H}u_{\mathbf{a}}^+$. To achieve this goal, we use the next result.

(5.4) Lemma *Assume that I is a left ideal of $\mathscr{H}$ and suppose that S is a subset of* $\mathrm{r}(I)$ *such that* $\mathrm{l}(S) \subseteq I$. *Then* $\mathrm{lr}(I) = I$.

PROOF: By assumption $S \subseteq \mathrm{r}(I)$ and $\mathrm{l}(S) \subseteq I$. Also, $I \subseteq \mathrm{lr}(I)$. Therefore, $\mathrm{lr}(I) \subseteq \mathrm{l}(S) \subseteq I \subseteq \mathrm{lr}(I)$; so $\mathrm{lr}(I) = I$ as required. □

Our definition of $u_{\mathbf{a}}^+$ expresses $u_{\mathbf{a}}^+$ as a double product

$$u_{\mathbf{a}}^+ = \prod_{k=1}^{r} \prod_{m=1}^{a_k} (L_m - Q_k).$$

We now introduce notation which allows us to reverse the order of the multiplication here.

(5.5) Definition *Suppose that* $1 \le i \le n$.
1. *Let* $\gamma_i = k$ *if* k *is maximal such that* $a_k < i$. *Let* $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$.
2. *Let* $v_i = \prod_{k=\gamma_i+1}^{r} (L_i - Q_k)$.

Then $u_{\mathbf{a}}^+ = v_1 v_2 \dots v_n$ and all of these factors commute.

(5.6) Definition *Suppose that* $1 \le i \le n$. *Let* $y_i = T_{i-1} \dots T_2 T_1 \prod_{k=1}^{\gamma_i} (L_1 - Q_k)$.

(5.7) Example Suppose that $r = 4$, $n = 5$ and $\mathbf{a} = (0, 2, 4, 5)$. Then

$$\begin{aligned} u_{\mathbf{a}}^+ &= (L_1 - Q_2)(L_2 - Q_2) \\ &\quad \times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3) \\ &\quad \times (L_1 - Q_4)(L_2 - Q_4)(L_3 - Q_4)(L_4 - Q_4)(L_5 - Q_4). \end{aligned}$$

Also $\gamma = (1, 1, 2, 2, 3)$ and, for $1 \le i \le n$, the product of the factors in the ith column of the above array is v_i. We have

$$\begin{aligned} y_1 &= (L_1 - Q_1) \\ y_2 &= T_1(L_1 - Q_1) \\ y_3 &= T_2 T_1 (L_1 - Q_1)(L_1 - Q_2) \\ y_4 &= T_3 T_2 T_1 (L_1 - Q_1)(L_1 - Q_2) \\ y_5 &= T_4 T_3 T_2 T_1 (L_1 - Q_1)(L_1 - Q_2)(L_1 - Q_3). \end{aligned}$$

(5.8) Lemma *Suppose that* $1 \le i \le n$. *Then* $v_1 v_2 \dots v_i y_i = 0$.

PROOF: Assume that k satisfies $\gamma_i + 1 \le k \le r$. Then $v_1 v_2 \dots v_i$ has a factor $(L_1 - Q_k)(L_2 - Q_k)\dots(L_i - Q_k)$. By (2.1)(iv), this factor commutes with $T_{i-1}\dots T_2 T_1$. Hence, $v_1 v_2 \dots v_i T_{i-1}\dots T_2 T_1$ has a right factor $\prod_{k=\gamma_i+1}^r (L_1 - Q_k)$. However,

$$\prod_{k=1}^{r}(L_1 - Q_k) = \prod_{k=1}^{r}(T_0 - Q_k) = 0;$$

so it follows that $v_1 v_2 \dots v_i y_i = 0$ as claimed. □

Since $u_{\mathbf{a}}^+ = v_1 v_2 \dots v_n$ we have the following Corollary.

(5.9) Corollary *Suppose that $1 \le i \le n$. Then $u_{\mathbf{a}}^+ y_i = 0$.*

It turns out that $y_1, y_2, \dots, y_n$ generate $\mathrm{r}(u_{\mathbf{a}}^+)$. More importantly, together with Lemma 5.4 the elements y_i will allow us to prove that $\mathrm{lr}(u_{\mathbf{a}}^+) = \mathscr{H} u_{\mathbf{a}}^+$. For our proof, we require the technical Lemma 5.13 below; for this result we need some preparation.

(5.10) Lemma *Suppose that $1 \le i < n$ and that $a, b \in \{0, 1, \dots, r-1\}$.*

1. *If $a \le b$ then* $L_i^a L_{i+1}^b T_i = T_i L_i^b L_{i+1}^a + (q-1)\sum_{c=1}^{b-a} L_i^{b-c} L_{i+1}^{a+c}.$
2. *If $a > b$ then* $L_i^a L_{i+1}^b T_i = q T_i^{-1} L_i^b L_{i+1}^a - (q-1)\sum_{c=1}^{a-b-1} L_i^{b+c} L_{i+1}^{a-c}.$

PROOF: (i) Assume that $a \le b$. If $b = a$ then T_i commutes with $L_i^a L_{i+1}^a$ by (2.1)(iii), so the result is correct in this case. Now suppose that $b > a$. Since $L_{i+1} T_i = T_i L_i + (q-1) L_{i+1}$ we have

$$L_i^a L_{i+1}^b T_i = L_i^a L_{i+1}^{b-1}\big(T_i L_i + (q-1) L_{i+1}\big),$$

which gives the required result by induction on $b - a$.

(ii) Either argue similarly, or apply the antiautomorphism $*$ to the result of part (i) and rearrange, interchanging a and b. □

We next generalize Lemma 5.10 as follows.

(5.11) Lemma *Suppose that $1 \le i \le n$ and $b_j \in \{0, 1, \dots, r-1\}$ for $j = 1, 2, \dots, n$. Then there exists an integer b and $\epsilon_1, \dots, \epsilon_{i-1} \in \{\pm 1\}$ such that*

$$L_1^{b_1} L_2^{b_2} \dots L_i^{b_i} T_{i-1} \dots T_2 T_1 = x_1 + x_2,$$

where $x_1 = q^b T_{i-1}^{\epsilon_{i-1}} \dots T_2^{\epsilon_2} T_1^{\epsilon_1} L_1^{b_i} L_2^{b_1} L_3^{b_2} \dots L_i^{b_{i-1}}$ and x_2 is a linear combination of terms of the form $T_w L_1^{c_1} L_2^{c_2} \dots L_i^{c_i}$ where

1. *$w \in \mathfrak{S}_n$; and*

2. $c_1,\dots,c_i\in\{0,1,\dots,r-1\}$ *with* $c_1+c_2+\cdots+c_i=b_1+b_2+\cdots+b_i$*; and*
3. *either* $\prod_{j=1}^{i}(c_j+1)<\prod_{j=1}^{i}(b_j+1)$, *or* $c_1,c_2,\dots,c_i$ *is a permutation of* $b_1,b_2,\dots,b_i$ *and* $c_1<b_i$.

Proof: The result is true when $i=1$. (In this case, $x_1=L_1^{b_1}$ and $x_2=0$.) Assume, inductively, that $i<n$ and that the result as stated is true. By Lemma 5.10, there exists $b'\in\{0,1\}$ and $\epsilon_i=\pm1$ such that

$$L_i^{b_i}L_{i+1}^{b_{i+1}}T_i=q^{b'}T_i^{\epsilon_i}L_i^{b_{i+1}}L_{i+1}^{b_i}\pm(q-1)\sum L_i^{d_i}L_{i+1}^{d_{i+1}}$$

where $d_i,d_{i+1}\in\{0,1,\dots,r-1\}$ and $d_i+d_{i+1}=b_i+b_{i+1}$ and either $(d_i+1)(d_{i+1}+1)<(b_i+1)(b_{i+1}+1)$ or $d_i=b_i<b_{i+1}$. Using this result, and (2.1)(ii), we obtain

$$\begin{aligned}L_1^{b_1}L_2^{b_2}\dots L_i^{b_i}T_{i-1}\dots T_2T_1 &= q^{b'}T_i^{\epsilon_i}L_1^{b_1}\dots L_{i-1}^{b_{i-1}}L_i^{b_{i+1}}T_{i-1}\dots T_2T_1L_{i+1}^{b_i}\\ &\quad\pm(q-1)\sum L_1^{b_1}\dots L_{i-1}^{b_{i-1}}L_i^{d_i}T_{i-1}\dots T_2T_1L_{i+1}^{d_{i+1}},\end{aligned}$$

where b', d_i and d_{i+1} are as above. The result now follows from our inductive hypothesis. □

For the statement and proof of our next Lemma, we need some more notation.

(5.12) Notation Suppose that $0\le i\le n$. Let V_i be the R–module with basis

$$\left\{\,T_wL_1^{b_1}L_2^{b_2}\dots L_n^{b_n}v_1v_2\dots v_i\;\middle|\;\begin{array}{c}w\in\mathfrak{S}_n,\ 0\le b_j<\gamma_j,\text{ for }1\le j\le i,\\ \text{and }0\le b_j<r,\text{ for }i+1\le j\le n\end{array}\right\}.$$

Result (2.4) shows the elements in these sets are indeed linearly independent. Note also that $V_0=\mathscr{H}$ and $V_n=\mathscr{H}u_{\mathbf{a}}^+$.

We shall use (2.4) extensively in the proof of Lemma 5.13.

(5.13) Lemma *Suppose that* $1\le i\le n$ *and that* $z\in V_{i-1}$ *and* $zy_i=0$. *Then* $z\in V_i$.

Proof: Since $z\in V_{i-1}$ we may write z as a linear combination of linearly independent terms

$$T_wL_1^{b_1}\dots L_{i-1}^{b_{i-1}}f(L_i)L_{i+1}^{b_{i+1}}\dots L_n^{b_n}v_1v_2\dots v_{i-1}$$

where $w\in\mathfrak{S}_n$ and $0\le b_j<\gamma_j$ for $1\le j\le i-1$ and $0\le b_j<r$ for $i+1\le j\le n$, and $f(X)$ is a polynomial in $R[X]$ of degree less than r. Write

$$f(X)=g(X)\prod_{k=\gamma_i+1}^{r}(X-Q_k)+h(X),$$

where $g(X)$ and $h(X)$ are polynomials in $R[X]$ such that $\deg g(X) < \gamma_i$ and $\deg h(X) < r-\gamma_i$. We may apply the "Euclidean algorithm" in this way, even though $R[X]$ need not be a Euclidean domain, because the divisor is a monic polynomial.

Note that $\prod_{k=\gamma_i+1}^{r}(L_i-Q_k)=v_i$. As a consequence, we have written z as z_1+z_2 where $z_1\in V_i$ and z_2 is a linear combination of terms

$$T_wL_1^{b_1}\dots L_{i-1}^{b_{i-1}}h(L_i)L_{i+1}^{b_{i+1}}\dots L_n^{b_n}v_1v_2\dots v_{i-1}.$$

We shall prove that $z_2=0$.

Note that $z_1y_i=0$, by Lemma 5.8, since $z_1\in V_i$. Therefore, because $zy_i=0$ by assumption, it follows that $z_2y_i=0$.

For each n–tuple $\mathbf{c}=(c_1,\dots,c_n)$ with $0\le c_i<r$, for all i with $1\le i\le n$, let $U_{\mathbf{c}}=\mathscr{H}(\mathfrak{S}_n)L_1^{c_1},\dots L_n^{c_n}$. Then we may write

$$z_2=\sum_{\mathbf{c}\in\mathscr{C}}z_{\mathbf{c}}\quad\text{where}\quad z_{\mathbf{c}}\in U_{\mathbf{c}}$$

and $\mathscr{C}=\{\,\mathbf{c}=(c_1,\dots,c_n)\,|\,0\le c_j<r \text{ for } 1\le j\le n \text{ and } c_i<r-\gamma_i\,\}$.

Assume that $z_2\neq 0$. Among the $\mathbf{c}$ such that $z_{\mathbf{c}}\neq 0$ choose $\mathbf{d}=(d_1,\dots,d_n)$ such that

(5.14) 1. $d_1+d_2+\dots+d_n$ is maximal; and,
2. $(d_1+1)(d_2+1)\dots(d_n+1)$ is maximal subject to (i); and,
3. d_i is maximal subject to (i) and (ii).

Then $z_{\mathbf{d}}=h_{\mathbf{d}}L_1^{d_1},\dots L_n^{d_n}$ for some non–zero element $h_{\mathbf{d}}$ of $\mathscr{H}(\mathfrak{S}_n)$.

Now, $T_{i-1}\dots T_2T_1$ commutes with each of the elements $L_{i+1},L_{i+2},\dots,L_n$. Therefore, by Lemma 5.11 there exists an invertible element $h\in\mathscr{H}(\mathfrak{S}_n)$ (namely, $h=q^bT_{i-1}^{\epsilon_{i-1}}\dots T_2^{\epsilon_2}T_1^{\epsilon_1}$) such that

$$z_2T_{i-1}\dots T_2T_1=h_{\mathbf{d}}hL_1^{d_i}L_2^{d_1}L_3^{d_2}\dots L_i^{d_{i-1}}L_{i+1}^{d_{i+1}}\dots L_n^{d_n}+\text{ terms in the sets }U_{\mathbf{e}},$$

where $\mathbf{e}=(e_1,\dots,e_n)$ and $e_1+\dots+e_n\le d_1+\dots+d_n$, and either

1. $e_1+\dots+e_n<d_1+\dots+d_n$ (by Lemma 5.11(ii) and (5.14)(i)); or,
2. $(e_1+1)\dots(e_n+1)<(d_1+1)\dots(d_n+1)$ (using Lemma 5.11(iii) and (5.14)(ii)); or,
3. $e_1,\dots,e_n$ is a permutation of $d_1,\dots,d_n$ and $e_1<d_i$ (using Lemma 5.11(iii) and (5.14)(iii)); or,
4. $e_1,\dots,e_n$ is a permutation of $d_1,\dots,d_n$ but $\mathbf{e}\neq(d_i,d_1,d_2,\dots,d_{i-1},d_{i+1},\dots,d_n)$ (consider the term x_1 in Lemma 5.11).

In particular, no $\mathbf{e}$ on the right hand side is equal to $(d_i,d_1,d_2,\dots,d_{i-1},d_{i+1},\dots,d_n)$.

Next we postmultiply our expression for $z_2T_{i-1}\dots T_2T_1$ by $\prod_{m=1}^{\gamma_i}(L_1-Q_m)$ to obtain z_2y_i. Note that if $v_{\mathbf{e}}\in U_{\mathbf{e}}$ and k is any positive integer then $v_{\mathbf{e}}L_1^k$ is a linear combination of terms $v_{\mathbf{f}}\in U_{\mathbf{f}}$ where $\mathbf{f}=(f_1,e_2,\dots,e_n)$ and $f_1<r$. Thus, if $e_1+\dots+e_n\le d_1+\dots+d_n$ and $e_1+k\ge r$ then $v_{\mathbf{e}}L_1^k$ is a linear combination of terms $v_{\mathbf{f}}$ with $f_1+e_2+\dots+e_n<d_1+\dots+d_n+k$. Therefore,

$$z_2y_i=h_{\mathbf{d}}hL_1^{d_i+\gamma_i}L_2^{d_1}L_3^{d_2}\dots L_i^{d_{i-1}}L_{i+1}^{d_{i+1}}\dots L_n^{d_n}+\text{ terms in the sets }U_{\mathbf{f}}$$

where $\mathbf{f} \neq (d_i + \gamma_i, d_1, d_2, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$.

Note that $d_i + \gamma_i < r$ since $\mathbf{d} \in \mathscr{C}$. Therefore, since $h_{\mathbf{d}} \neq 0$ and h is invertible, $z_2 y_i \neq 0$ giving a contradiction. Hence, our assumption that $z_2 \neq 0$ is false. Thus, $z = z_1 \in V_i$ as required. □

(5.15) Corollary *We have* $\operatorname{lr}(u_{\mathbf{a}}^+) = \mathscr{H} u_{\mathbf{a}}^+$.

PROOF: Let $I = \mathscr{H} u_{\mathbf{a}}^+$ and $S = \{y_1, y_1, \dots, y_n\}$. Then Corollary 5.9 shows that $S \subseteq \operatorname{r}(I)$.

Suppose that $z \in \mathscr{H}$ and $z y_i = 0$ for all i with $1 \leq i \leq n$. Since $V_0 = \mathscr{H}$ and $V_n = \mathscr{H} u_{\mathbf{a}}^+ = I$, it follows from Lemma 5.13 that $z \in I$. Thus, $\operatorname{l}(S) \subseteq I$. Lemma 5.4 now implies that $\operatorname{lr}(I) = I$; that is, $\operatorname{lr}(u_{\mathbf{a}}^+) = \mathscr{H} u_{\mathbf{a}}^+$. □

Finally, we can prove the main result of this section.

(5.16) Theorem *Suppose that μ is a multicomposition of n. Then* $\operatorname{lr}(m_\mu) = \mathscr{H} m_\mu$.

PROOF: Write $m_\mu = u_{\mathbf{a}}^+ x_\mu$ as in Definition 3.5. By applying the definitions, Corollary 5.15 and Lemma 5.3, we obtain

$$\mathscr{H} m_\mu \subseteq \operatorname{lr}(m_\mu) \subseteq \operatorname{lr}(u_{\mathbf{a}}^+) \cap \operatorname{lr}(x_\mu) = \mathscr{H} u_{\mathbf{a}}^+ \cap \mathscr{H} x_\mu.$$

However, $(u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H})^* = (m_\mu \mathscr{H})^*$ by Corollary 4.13. Hence, we have equality throughout and $\operatorname{lr}(m_\mu) = \mathscr{H} m_\mu$ as claimed. □

(5.17) Corollary *Suppose that μ and ν are multicompositions of n. Then*

1. *For every element φ in $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ there exists $h_\varphi \in \mathscr{H}$ such that $\varphi(m_\nu) = h_\varphi m_\nu$; in particular, $\varphi(m_\nu) \in M^{\nu *} \cap M^\mu$.*
2. *$\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu) \cong M^{\nu *} \cap M^\mu$.*

PROOF: By Theorem 5.16 we may apply Lemma 5.2(i) with $m = m_\nu$ to obtain that $\varphi(m_\nu) = h_\varphi m_\nu$ for some $h_\varphi \in \mathscr{H}$. Thus, $\varphi(m_\nu) \in M^{\nu *}$. It is clear that $\varphi(m_\nu) \in M^\mu$. Part (ii) follows from Lemma 5.2(ii), where an explicit isomorphism is given by $\varphi \longmapsto \varphi(m_\nu)$. □

6 The cyclotomic q–Schur algebra

In order to make our results as general as possible, let Λ_r be a poset ideal in the set of all multicompositions of n in the sense that if $\mu \in \Lambda_r$ and ν is a multipartition such that $\nu \rhd \mu$ then $\nu \in \Lambda_r$. We let Λ_r^+ be the set of multipartitions in Λ_r.

(6.1) Definition *The* cyclotomic q–Schur algebra *is the endomorphism algebra*

$$\mathscr{S}(\Lambda_r) = \mathrm{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda_r} M^\mu\Big).$$

Thus, $\mathscr{S}(\Lambda_r) \cong \bigoplus_{\mu,\nu\in\Lambda_r} \mathrm{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$.

Suppose that $\varphi \in \mathrm{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$. Then $\varphi(m_\nu h) = \varphi(m_\nu)h$ for all $h \in \mathscr{H}$; thus φ is completely determined by $\varphi(m_\nu)$. Moreover, $\varphi(m_\nu) \in M^{\nu*} \cap M^\mu$ by Corollary 5.17. These remarks motivate us to construct a basis of $M^{\nu*} \cap M^\mu$.

(6.2) Definition *Suppose that μ and ν are multicompositions of n and that λ is a multipartition of n. Assume that $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and that $\mathtt{T} \in \mathcal{T}_0(\lambda,\nu)$. Let*

$$m_{\mathtt{ST}} = \sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{st}}$$

where the sum is over all $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda)$ with $\mu(\mathfrak{s}) = \mathtt{S}$ and $\nu(\mathfrak{t}) = \mathtt{T}$.

Note that $m_{\mathtt{ST}}^* = m_{\mathtt{TS}}$.

(6.3) Proposition *Suppose that μ and ν are multicompositions of n. Then*

$$\Big\{\, m_{\mathtt{ST}} \;\Big|\; \begin{array}{c}\mathtt{S} \in \mathcal{T}_0(\lambda,\mu) \text{ and } \mathtt{T} \in \mathcal{T}_0(\lambda,\nu) \text{ for some}\\ \text{multipartition } \lambda \text{ of } n\end{array} \Big\}$$

is a basis of $M^{\nu*} \cap M^\mu$.

Proof: Since

$$m_{\mathtt{ST}} = \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathtt{S}}} (m_{\mathtt{T}\mathfrak{s}})^* = \sum_{\substack{\mathfrak{t}\in\mathrm{Std}(\lambda)\\ \nu(\mathfrak{t})=\mathtt{T}}} m_{\mathtt{S}\mathfrak{t}},$$

Lemma 4.10 shows that $m_{\mathtt{ST}} \in M^{\nu*} \cap M^\mu$. Moreover, the elements $m_{\mathtt{ST}}$ are linearly independent since they involve distinct elements $m_{\mathfrak{st}}$ of the standard basis of $\mathscr{H}$.

Now suppose that $h \in M^{\nu*} \cap M^\mu$. Since $h \in \mathscr{H}$, we may express h in terms of the standard basis $\mathscr{M}$; say

$$h = \sum_{m_{\mathfrak{st}}\in\mathscr{M}} r_{\mathfrak{st}} m_{\mathfrak{st}}$$

where $r_{\mathfrak{st}} \in R$. Since $h \in M^\mu$, we have $r_{\mathfrak{st}} = r_{\mathfrak{s}'\mathfrak{t}}$ if $\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$, by Theorem 4.14. Similarly, since $h \in M^{\nu*}$, we have $r_{\mathfrak{st}} = r_{\mathfrak{st}'}$ if $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$. Thus, if $\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$ and $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$ then $r_{\mathfrak{st}} = r_{\mathfrak{s}'\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}'}$. Furthermore, $r_{\mathfrak{st}} = 0$ unless both $\mu(\mathfrak{s})$ and $\nu(\mathfrak{t})$ are semistandard. Therefore, h is a linear combination of elements $m_{\mathtt{ST}}$. This completes the proof of the Proposition. □

Proposition 6.3 shows that, in our next Definition, $\varphi_{\mathtt{ST}}$ is a well defined $\mathscr{H}$–homomorphism from M^ν into M^μ.

(6.4) Definition *Suppose that $\mu, \nu \in \Lambda_r$ and $\lambda \in \Lambda_r^+$. Assume that $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T} \in \mathcal{T}_0(\lambda,\nu)$. Define $\varphi_{\mathtt{ST}} \in \operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ by*

$$\varphi_{\mathtt{ST}}(m_\nu h) = m_{\mathtt{ST}} h$$

for all $h \in \mathscr{H}$. Extend $\varphi_{\mathtt{ST}}$ to an element of the cyclotomic q–Schur algebra $\mathscr{S}(\Lambda_r)$ by defining $\varphi_{\mathtt{ST}}$ to be zero on M^κ when $\nu \neq \kappa \in \Lambda_r$.

(6.5) Definition *Suppose that $\lambda \in \Lambda_r^+$. Let $\overline{\mathscr{S}(\Lambda_r)_\lambda}$ be the R–submodule of $\mathscr{S}(\Lambda_r)$ spanned by*

$$\left\{ \varphi_{\mathtt{UV}} \;\middle|\; \begin{array}{c} \mathtt{U} \in \mathcal{T}_0(\alpha,\mu),\ \mathtt{V} \in \mathcal{T}_0(\alpha,\nu) \text{ for some} \\ \mu,\nu \in \Lambda_r \text{ and } \alpha \in \Lambda_r^+ \text{ with } \alpha \rhd \lambda \end{array} \right\}.$$

This brings us to one of the main results of our paper. The work in Sections 4 and 5 was aimed at proving part (i) of the Theorem.

(6.6) Theorem (The Semistandard Basis Theorem)

1. *The cyclotomic q–Schur algebra $\mathscr{S}(\Lambda_r)$ is free as an R–module with basis*

$$\left\{ \varphi_{\mathtt{ST}} \;\middle|\; \begin{array}{c} \mathtt{S} \in \mathcal{T}_0(\lambda,\mu),\ \mathtt{T} \in \mathcal{T}_0(\lambda,\nu) \text{ for some} \\ \mu,\nu \in \Lambda_r \text{ and } \lambda \in \Lambda_r^+ \end{array} \right\}.$$

2. *Suppose that $\mu,\nu \in \Lambda_r$ and $\lambda \in \Lambda_r^+$ and let $\varphi \in \mathscr{S}(\Lambda_r)$. Then, for every $\kappa \in \Lambda_r$ and every $\mathtt{T}' \in \mathcal{T}_0(\lambda,\kappa)$ there exists $r_{\mathtt{T}'} \in R$ such that for all $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$, we have*

$$\varphi_{\mathtt{ST}}\varphi \equiv \sum_{\kappa\in\Lambda_r} \sum_{\mathtt{T}'\in\mathcal{T}_0(\lambda,\kappa)} r_{\mathtt{T}'}\varphi_{\mathtt{ST}'} \bmod \overline{\mathscr{S}(\Lambda_r)_\lambda}.$$

PROOF: (i) By Corollary 5.17(ii), $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu) \cong M^{\nu *} \cap M^\mu$, where an isomorphism is given by $\varphi \longmapsto \varphi(m_\nu)$. Hence, by Proposition 6.3 and Definition 6.4, the set $\{\varphi_{\mathtt{ST}}\}$ is a basis of $\mathscr{S}(\Lambda_r)$.

(ii) It is sufficient to consider the case where $\varphi \in \operatorname{Hom}_{\mathscr{H}}(M^\kappa, M^\nu)$ for some $\kappa \in \Lambda_r$. Suppose that $\varphi(m_\kappa) = m_\nu h$ where $h \in \mathscr{H}$. Then, for all $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T} \in \mathcal{T}_0(\lambda,\nu)$, we have

$$\varphi_{\mathtt{ST}}\varphi(m_\kappa) = m_{\mathtt{ST}}h \in M^{\kappa *} \cap M^\mu.$$

By Proposition 6.3, $m_{\mathtt{ST}}h = \sum r_{\mathtt{UV}} m_{\mathtt{UV}}$, where $r_{\mathtt{UV}} \in R$ and the sum is over $\mathtt{U} \in \mathcal{T}_0(\alpha,\mu)$ and $\mathtt{V} \in \mathcal{T}_0(\alpha,\kappa)$ for some $\alpha \in \Lambda_r^+$. By applying Proposition 3.25 we deduce that

$$m_{\mathtt{ST}}h = \sum_{\mathtt{T}'\in\mathcal{T}_0(\lambda,\kappa)} r_{\mathtt{T}'} m_{\mathtt{ST}'} + \sum_{\mathtt{U}',\mathtt{V}'} r_{\mathtt{U}'\mathtt{V}'} m_{\mathtt{U}'\mathtt{V}'}$$

where $r_{\mathtt{T}'}, r_{\mathtt{U}'\mathtt{V}'} \in R$ and the second sum is over $\mathtt{U}' \in \mathcal{T}_0(\alpha,\mu)$ and $\mathtt{V}' \in \mathcal{T}_0(\alpha,\nu)$ for some $\alpha \in \Lambda_r^+$ with $\alpha \rhd \lambda$. Therefore,

$$\varphi_{\mathtt{ST}}\varphi \equiv \sum_{\mathtt{T}'\in\mathcal{T}_0(\lambda,\kappa)} r_{\mathtt{T}'}\varphi_{\mathtt{ST}'} \bmod \overline{\mathscr{S}(\Lambda_r)_\lambda}.$$

This completes the proof. □

(6.7) Definition *We call the basis* $\{\varphi_{\mathtt{ST}}\}$ *the* semistandard basis *of* $\mathscr{S}(\Lambda_r)$.

(6.8) Remark It is straightforward to prove that, up to Morita equivalence, $\mathscr{S}(\Lambda_r)$ depends only upon the set Λ_r^+ (and not on Λ_r).

In order to show that the semistandard basis of $\mathscr{S}(\Lambda_r)$ is a cellular basis we need an appropriate antiautomorphism $*$ for $\mathscr{S}(\Lambda_r)$.

(6.9) Proposition *Let* $*\colon \mathscr{S}(\Lambda_r)\longrightarrow\mathscr{S}(\Lambda_r)$ *be the unique* R*–linear map such that* $\varphi_{\mathtt{ST}}^*=\varphi_{\mathtt{TS}}$ *for all elements* $\varphi_{\mathtt{ST}}$ *in the semistandard basis. Then* $*$ *is an antiautomorphism of* $\mathscr{S}(\Lambda_r)$.

PROOF: Assume that $\mathtt{S}\in\mathcal{T}_0(\lambda,\mu)$ and that $\mathtt{T}\in\mathcal{T}_0(\lambda,\nu)$ for some $\mu,\nu\in\Lambda_r$ and $\lambda\in\Lambda_r^+$. Then

$$\varphi_{\mathtt{ST}}(m_\nu)=m_{\mathtt{ST}}=(m_{\mathtt{TS}})^*=\big(\varphi_{\mathtt{TS}}(m_\mu)\big)^*.$$

The proposition now follows from the next general lemma.

(6.10) Lemma *Suppose that* $\mathscr{H}$ *is an algebra with an antiautomorphism* $*$. *Let* $\{\,m_\mu\mid\mu\in\Lambda\,\}$ *be a set of elements of* $\mathscr{H}$ *such that* $m_\mu^*=m_\mu$ *for all* $\mu\in\Lambda$ *and let*

$$S=\operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda}m_\mu\mathscr{H}\Big).$$

Assume that, for all $\mu,\nu\in\Lambda$, *every* $\mathscr{H}$*–homomorphism from* $m_\nu\mathscr{H}$ *to* $m_\mu\mathscr{H}$ *is given by left multiplication by an element of* $\mathscr{H}$. *Then the following hold.*

1. *For each* $\varphi\in\operatorname{Hom}_{\mathscr{H}}(m_\nu\mathscr{H},m_\mu\mathscr{H})$ *there is a unique* $\varphi^*\in\operatorname{Hom}_{\mathscr{H}}(m_\mu\mathscr{H},m_\nu\mathscr{H})$ *such that* $\varphi^*(m_\mu)=\big(\varphi(m_\nu)\big)^*$.
2. *The map* $*$ *is an antiautomorphism of* $\mathscr{S}(\Lambda_r)$.

PROOF: (i) Suppose that $\varphi\in\operatorname{Hom}_{\mathscr{H}}(m_\nu\mathscr{H},m_\mu\mathscr{H})$. Then $\varphi(m_\nu)=x_1m_\nu=m_\mu y$ for some $x_1,y\in\mathscr{H}$. Define $\varphi^*\in\operatorname{Hom}_{\mathscr{H}}(m_\mu\mathscr{H},m_\nu\mathscr{H})$ by $\varphi^*(m_\mu h)=y^*m_\mu h$ for all $h\in\mathscr{H}$. Then φ^* is a well–defined $\mathscr{H}$–homomorphism; also, it maps into $m_\nu\mathscr{H}$, since $y^*m_\mu=m_\nu x_1^*$. Since $\varphi^*(m_\mu)=\big(\varphi(m_\nu)\big)^*$, the proof of (i) is complete.

(ii) Suppose that $\psi\in\operatorname{Hom}_{\mathscr{H}}(m_\mu\mathscr{H},m_\lambda\mathscr{H})$ for some $\lambda\in\Lambda$. Then $\psi(m_\mu)=x_2m_\mu$ for some $x_2\in\mathscr{H}$. We have,

$$\begin{aligned}(\psi\varphi)^*(m_\lambda)=\big(\psi\varphi(m_\nu)\big)^*&=(x_2x_1m_\nu)^*&&=m_\nu x_1^*x_2^*\\ =\varphi^*(m_\mu)x_2^*&=\varphi^*(m_\mu x_2^*)&&=\varphi^*\psi^*(m_\lambda).\end{aligned}$$

Therefore, $(\psi\varphi)^*=\varphi^*\psi^*$, and it follows that $*$ is an antiautomorphism of $\mathscr{S}(\Lambda_r)$. □

(6.11) Corollary *The* R*–module* $\overline{\mathscr{S}(\Lambda_r)_\lambda}$ *is a two–sided ideal of* $\mathscr{S}(\Lambda_r)$.

PROOF: Theorem 6.6(ii) shows that $\overline{\mathscr{S}(\Lambda_r)_\lambda}$ is a right ideal. By applying the antiautomorphism $*$, we deduce that it is also a left ideal. □

(6.12) Theorem *The semistandard basis of $\mathscr{S}(\Lambda_r)$ is a cellular basis.*

PROOF: This follows at once from Theorem 6.6 and Proposition 6.9. □

We now apply the theory of cellular algebras to the representation theory of $\mathscr{S}(\Lambda_r)$, just as we treated $\mathscr{H}$ in Section 3.

Suppose that λ is a multipartition of n. Let $\mathtt{T}^\lambda = \lambda(\mathfrak{t}^\lambda)$ (see Definition 4.2). Then $\mathtt{T}^\lambda$ is the unique semistandard λ–tableau of type λ (cf. Example 4.3). Define $\varphi_\lambda = \varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}$; then φ_λ is the identity map on M^λ.

(6.13) Definition *Suppose that $\lambda \in \Lambda_r^+$, The* Weyl module *W^λ is the submodule of $\mathscr{S}(\Lambda_r)/\overline{\mathscr{S}(\Lambda_r)_\lambda}$ given by $W^\lambda = \mathscr{S}(\Lambda_r)(\varphi_\lambda + \overline{\mathscr{S}(\Lambda_r)_\lambda})$.*

If $\mathtt{S}$ is a semistandard λ–tableau let $\varphi_\mathtt{S} = \varphi_{\mathtt{S}\mathtt{T}^\lambda}(\varphi_\lambda + \overline{\mathscr{S}(\Lambda_r)_\lambda}) = \varphi_{\mathtt{S}\mathtt{T}^\lambda} + \overline{\mathscr{S}(\Lambda_r)_\lambda}$. Then, from Theorem 6.12, we obtain the following result.

(6.14) Corollary *The Weyl module W^λ is a free R–module with basis*

$$\{\, \varphi_\mathtt{S} \mid \mathtt{S} \in \mathcal{T}_0(\lambda,\mu) \text{ for some } \mu \in \Lambda_r \,\}.$$

Define a bilinear form $\langle\ ,\ \rangle$ on W^λ by requiring that

$$\varphi_{\mathtt{T}^\lambda\mathtt{S}}\varphi_{\mathtt{T}\mathtt{T}^\lambda} \equiv \langle \varphi_\mathtt{S}, \varphi_\mathtt{T}\rangle \varphi_\lambda \bmod \overline{\mathscr{S}(\Lambda_r)_\lambda}$$

for all semistandard λ–tableaux $\mathtt{S}$ and $\mathtt{T}$. This bilinear form is well–defined and symmetric, and also satisfies $\langle su, v\rangle = \langle u, s^*v\rangle$ for all $u, v \in W^\lambda$ and all $s \in \mathscr{S}(\Lambda_r)$ [**12**, (2.4)]. Consequently, $\operatorname{rad} W^\lambda = \{\, u \in W^\lambda \mid \langle u, v\rangle = 0 \text{ for all } v \in W^\lambda \,\}$ is a submodule of W^λ.

(6.15) Definition *Suppose that $\lambda \in \Lambda_r^+$. Let $F^\lambda = W^\lambda / \operatorname{rad} W^\lambda$.*

(6.16) Theorem *Suppose that R is a field. Then*

$$\{\, F^\lambda \mid \lambda \in \Lambda_r^+ \,\}$$

is a complete set of non–isomorphic irreducible $\mathscr{S}(\Lambda_r)$–modules. Moreover, each F^λ is absolutely irreducible.

PROOF: Let $\lambda \in \Lambda_r^+$. From the definition of the bilinear form on W^λ, we have

$$\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda}\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda} \equiv \langle \varphi_{\mathrm{T}^\lambda}, \varphi_{\mathrm{T}^\lambda}\rangle \varphi_\lambda \bmod \overline{\mathscr{S}(\Lambda_r)_\lambda}.$$

However, $\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda}\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda} = \varphi_\lambda$ is the identity on M^λ; so $\langle \varphi_{\mathrm{T}^\lambda}, \varphi_{\mathrm{T}^\lambda}\rangle = 1$. Consequently, F^λ is non–zero. The Theorem now follows from [**12**, (3.4)]. □

If $\lambda, \mu \in \Lambda_r^+$, let $d_{\lambda\mu}$ denote the composition multiplicity of F^μ as a composition factor of W^λ. Then

$$\big(d_{\lambda\mu}\big)_{\lambda,\mu\in\Lambda_r^+}$$

is the decomposition matrix of $\mathscr{S}(\Lambda_r)$. The theory of cellular algebras [**12**, (3.6)] yields the following.

(6.17) Corollary *The decomposition matrix of $\mathscr{S}(\Lambda_r)$ is unitriangular. That is, for $\lambda, \mu \in \Lambda_r^+$, we have $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \trianglerighteq \mu$.*

Finally, Theorem 6.16 combined with [**12**, (3.10)], gives us our last result.

(6.18) Corollary *The cyclotomic q–Schur algebra is quasi–hereditary.*

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