The (Q, q)–Schur Algebra

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Abstract

In this paper we use the Hecke algebra of type $\mathbf{B}$ to define a new algebra $\mathcal{S}$ which is an analogue of the q–Schur algebra. We construct Weyl modules for $\mathcal{S}$ and obtain, as factor modules, a family of irreducible $\mathcal{S}$–modules over any field.

1 Introduction

The ordinary Schur algebra is of key importance in the study of the representation theory of general linear groups in the describing characteristic, and it provides a link between the general linear groups and the symmetric groups. In [8, 9] we introduced the q–Schur algebra, and demonstrated its usefulness in the representation theory of $GL_n(q)$ over a field of non–describing characteristic. In [5] it was shown that the q–Schur algebra is given as the dual of a homogeneous part of quantum–GL_n, or alternatively as the factor of quantum–GL_n or the

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corresponding quantum enveloping algebra modulo the kernel of its action on quantum tensor space (compare [1]). In particular, the representations of q–Schur algebras are precisely the homogeneous polynomial representations of quantum–GL_n in a fixed degree (compare [4]).

The construction of the q–Schur algebra involves the Hecke algebra of type **A**; in this paper we use the Hecke algebra of type **B** to build an algebra which we call the (Q,q)–Schur algebra. Others have devised a version of a Schur algebra of type **B** [12, 13, 15], but ours is a larger algebra. Applications to the representation theory of finite symplectic groups in the non–describing characteristic case have already been provided [16, 6], and we expect that further applications will ensue, using our larger and more complicated algebra.

Hecke algebras of type **B** have been studied in [10, 11, 18]. We begin by recalling and extending some of the notation and results which we used in those papers. The remainder of the paper is then devoted to introducing the (Q,q)–Schur algebra and investigating its main properties. In particular we construct a generic basis of the (Q,q)–Schur algebra and we define (Q,q)–Weyl modules. The Weyl modules are labelled by bipartitions and they have unique maximal submodules. The corresponding factor modules are pairwise non–isomorphic irreducible representations of the (Q,q)–Schur algebra. We show that the decomposition matrix which describes the composition multiplicities of these irreducible modules in the (Q,q)–Weyl modules is unitriangular, and we construct a "semistandard basis" for each (Q,q)–Weyl module.

In a forthcoming paper we shall construct a cellular basis of the (Q,q)-Schur algebra $\mathcal{S}$. As a consequence, every irreducible representation of $\mathcal{S}$ is isomorphic to one of the irreducible representations which we construct here and, in addition, $\mathcal{S}$ is quasi–hereditary. Furthermore, we generalize our results to construct Schur algebras of the Ariki–Koike algebras.

After this manuscript was submitted for publication, we received a preprint by Jie Du and Leonard Scott entitled "The q–Schur^2 algebra" where they independently construct an algebra which is Morita equivalent to the one considered here; furthermore they show that it is quasi–hereditary.

2 The Hecke algebra of type B

Let W_r be the group $C_2 \wr \mathfrak{S}_r$, where $\mathfrak{S}_r$ is the symmetric group of degree r. Then W_r is generated by elements $s_0, s_1, \dots, s_{r-1}$ which satisfy the following relations:

$$\begin{array}{rcll} s_i^2 &=& 1 & \text{for } 0 \le i \le r-1 \\ s_i s_j &=& s_j s_i & \text{if } 1 \le i+1 < j \le r-1 \\ s_i s_{i+1} s_i &=& s_{i+1} s_i s_{i+1} & \text{if } 1 \le i \le r-2 \\ s_0 s_1 s_0 s_1 &=& s_1 s_0 s_1 s_0. & \end{array} \tag{2.1}$$

Let $\boldsymbol{r}^+ = \{1, 2, \dots, r\}$ and $\boldsymbol{r}^- = \{-1, -2, \dots, -r\}$. We identify W_r with a subgroup of the symmetric group on $\boldsymbol{r}^\pm = \boldsymbol{r}^+ \cup \boldsymbol{r}^-$ by letting

$$\begin{aligned} s_0 &= (1, -1) \\ s_i &= (i, i+1)(-i, -i-1) \qquad \text{for } 1 \le i \le r-1. \end{aligned}$$

It is easy to see that W_r acts transitively on $\boldsymbol{r}^{\pm}$. The subgroup of W_r generated by $s_1,\ldots,s_{r-1}$ is the symmetric group $\mathfrak{S}_r$ of degree r.

The concepts of a **reduced expression** for an element of W_r and the **length** $\ell(w)$ of $w\in W_r$ are defined in the usual way. It is useful to note the following well–known method for calculating $\ell(w)$. Consider the $\mathbb{Q}W_r$–module V whose basis is $\{e_i\mid i\in\boldsymbol{r}^+\}$ and where, for $i\in\boldsymbol{r}^+$ and $w\in W_r$, we have

$$e_iw=\begin{cases} e_{iw} & \text{if } iw>0\\ -e_{-iw} & \text{if } iw<0.\end{cases}$$

The **roots** for W_r are

$$\pm e_i \text{ and } \pm e_i\pm e_j \quad (i,j\in\boldsymbol{r}^+, i\neq j).$$

The **positive roots** are

$$e_i,\ e_i+e_j,\ e_j-e_i \quad (i,j\in\boldsymbol{r}^+, i<j),$$

and all other roots are **negative roots**. The length of w is equal to the number of positive roots changed to negative roots by w.

The simple roots

$$\alpha_0=e_1,\ \alpha_1=e_2-e_1,\ \alpha_2=e_3-e_2,\ \ldots,\ \alpha_{r-1}=e_r-e_{r-1}$$

form a basis Δ_0 of V, and every positive root is a non–negative linear combination of simple roots. Moreover, for $i=0,1,\ldots,r-1$, s_i acts on V as the reflection in the hyperplane orthogonal to α_i. If α is a positive root we sometimes write s_α for the reflection in the hyperplane orthogonal to α; in particular, $s_i=s_{\alpha_i}$. Note that $s_\alpha\in W_r$ for all positive roots α.

Let R be a commutative ring with 1, and let q and Q be invertible elements of R. The **Hecke algebra** $\mathcal{H}_{R,q,Q}(W_r)$ **of type** B_r is defined to be the free R–module with basis $\{T_w\mid w\in W_r\}$ and multiplication defined as below. If e is the identity element of W then T_e is the multiplicative identity for $\mathcal{H}_{R,q,Q}(W_r)$ and for $\rho\in R$ we abbreviate ρT_e as ρ. We often write T_{s_i} as T_i for $0\le i\le r-1$ and $\mathcal{H}_{R,q,Q}(W_r)$ as $\mathcal{H}(W_r)$ or as $\mathcal{H}$. Then

(i) if $w=v_1v_2\cdots v_l$ is a reduced expression for $w\in W_r$ where each v_i belongs to $\{s_0,s_1,\ldots,s_{r-1}\}$, then $T_w=T_{v_1}T_{v_2}\cdots T_{v_l}$;
(ii) $(T_i)^2=q+(q-1)T_i$ for $1\le i\le r-1$;
(iii) $(T_0)^2=Q+(Q-1)T_0$.

Let $\mathcal{H}(\mathfrak{S}_r)$ denote the subalgebra of $\mathcal{H}$ generated by $T_1,T_2,\ldots,T_{r-1}$.

For each pair of integers i,j in $\boldsymbol{r}^+$ define $s_{i,j}\in W_r$ by

$$s_{i,j}=\begin{cases} s_is_{i+1}\ldots s_{j-1} & \text{if } i\le j\\ s_{i-1}s_{i-2}\ldots s_j & \text{if } i>j\end{cases}$$

For example, $s_{1,r}$ is the permutation of $\boldsymbol{r}^{\pm}$ which is given in its cycle decomposition as

$$(r, r-1, \dots, 2, 1)(-r, -r+1, \dots, -2, -1).$$

For $0 \le a \le r$, let $w_{a,r-a} = (s_{1,r})^a$. As in [10, 2.4–2.9] we have the following.

2.2 *Let $i, j \in \boldsymbol{r}^+$ and $0 \le a \le r$. Then*

(i) $\ell(s_{i,j}) = |j - i|$.
(ii) $\ell(w_{a,r-a}) = a(r-a)$.
(iii) $s_{a+1,1}s_{a+2,2}\cdots s_{r,r-a}$ *gives a reduced expression for* $w_{a,r-a}$.

For typographical reasons, let $T_{i,j} = T_{s_{i,j}}$ and $h_{a,r-a} = T_{w_{a,r-a}}$.

Definition 2.3 *For $0 \le a \le r$, let the elements u_a^+ and u_a^- of $\mathcal{H}$ be given by*

$$u_a^+ = \prod_{i=1}^{a}(q^{i-1} + T_{i,1}T_0T_{1,i})$$
$$u_a^- = \prod_{i=1}^{a}(Qq^{i-1} - T_{i,1}T_0T_{1,i})$$

It is easy to prove the following (compare [10, 3.3 and 3.4]).

2.4 *Let $0 \le a \le r$.*

(i) *If $0 \le b \le r-1$ with $b \ne a$, then u_a^+ and u_a^- commute with T_b.*
(ii) *If $a > 0$ then $u_a^+T_0 = Qu_a^+$ and $u_a^-T_0 = -u_a^-$.*

An a-**bicomposition** of r is an ordered pair $(\lambda^{(1)}, \lambda^{(2)})$ of compositions, where $\lambda^{(1)}$ is a composition of a and $\lambda^{(2)}$ is a composition of $r-a$; if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are partitions, then $(\lambda^{(1)}, \lambda^{(2)})$ is an a-**bipartition** of r.

Let $\Lambda_2(n, r)$ denote the set of bicompositions $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of r with the property that the sum of the number of parts of $\lambda^{(1)}$ and the number of parts of $\lambda^{(2)}$ is n. (Note that we allow a composition to have zero parts.)

We order the bipartitions of r by letting all a–bipartitions precede all b–bipartitions if $a > b$, and by saying that the a–bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ precedes the a–bipartition $\mu = (\mu^{(1)}, \mu^{(2)})$ if $\lambda^{(1)}$ precedes $\mu^{(1)}$ lexicographically or $\lambda^{(1)} = \mu^{(1)}$ and $\lambda^{(2)}$ precedes $\mu^{(2)}$ lexicographically. We call this the **lexicographic order** on bipartitions.

Suppose that $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ and $\mu = (\mu^{(1)}, \mu^{(2)})$ are bicompositions of r. We say that λ and μ are **associated** if $\mu^{(i)}$ can be obtained from $\lambda^{(i)}$, $i = 1, 2$, by reordering the parts.

From λ we obtain a corresponding **diagram** $[\lambda]$ which consists of crosses in the plane, as in the example $\lambda = ((4, 3, 1), (3, 2))$, where

$$[\lambda] = \begin{pmatrix} \times & \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & & , & \times & \times & \\ \times & & & & & & & \end{pmatrix}$$

A λ-**bitableau** is obtained from $[\lambda]$ by replacing each cross by one of the numbers from $\boldsymbol{r}^{\pm}$ in such a way that for every $i \in \boldsymbol{r}^{+}$, precisely one of $\{i, -i\}$ is used. If $\mathfrak{t}$ is a λ–bitableau then $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$, where $\mathfrak{t}^{(1)}$ is a $\lambda^{(1)}$–tableau and $\mathfrak{t}^{(2)}$ is a $\lambda^{(2)}$–tableau.

Definition 2.5 *Suppose that* $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2(n, r)$*, and that* $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ *is a* λ*–bitableau. Assume that* i *is one of the entries in* $\mathfrak{t}$*. Let* $\mathrm{row}_{\mathfrak{t}}(i) = j$ *if* i *belongs to row* j *of* $\mathfrak{t}^{(1)}$ *and let* $\mathrm{row}_{\mathfrak{t}}(i) = n + j$ *if* i *belongs to row* j *of* $\mathfrak{t}^{(2)}$*.*

Definition 2.6 *Let* λ *be a bicomposition.*

(i) *A* λ*–bitableau* $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ *is* **row standard** *if the entries increase from left to right in each row of* $\mathfrak{t}^{(1)}$ *and in each row of* $\mathfrak{t}^{(2)}$*, and all entries in* $\mathfrak{t}^{(1)}$ *belong to* $\boldsymbol{r}^{+}$*.*

(ii) *The* λ*–bitableaux* $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ *and* $\mathfrak{s} = (\mathfrak{s}^{(1)}, \mathfrak{s}^{(2)})$ *are* **row equivalent** *if* $|\mathfrak{t}^{(1)}|$ *and* $|\mathfrak{s}^{(1)}|$ *are row equivalent, and* $\mathfrak{t}^{(2)}$ *and* $\mathfrak{s}^{(2)}$ *are row–equivalent. Here the entries of* $|\mathfrak{t}^{(1)}|$ *are the absolute values of the entries of* $\mathfrak{t}^{(1)}$ *and* $|\mathfrak{s}^{(1)}|$ *is defined similarly. A row equivalence class of the* λ*–bitableaux is a* λ–**bitabloid** *and the* λ*–bitabloid containing* $\mathfrak{t}$ *is denoted by* $\{\mathfrak{t}\}$*.*

(iii) *A* λ*–bitableau is* **standard** *if all its entries belong to* $\boldsymbol{r}^{+}$*, and it is row standard, and all the entries increase down each column of* $\mathfrak{t}^{(1)}$ *and each column of* $\mathfrak{t}^{(2)}$*.*

We remark that every λ–bitabloid contains exactly one row standard λ–bitableau. When dealing with λ–bitabloids $\{\mathfrak{t}\}$, we often find it convenient to specify that $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ is row standard; this ensures that all the entries in $\mathfrak{t}^{(1)}$ are positive.

Here is an example of a row standard bitableau:

$$\begin{pmatrix} 2 & 8 & & -5 & -1 & 6 \\ 3 & 4 & , & -7 & 9 & \end{pmatrix}.$$

Next, we wish to specify, for a given $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, several special standard λ–bitableaux. The λ–bitableaux which we wish to define are most easily understood with an example.

Example 2.7 *Suppose that* $\lambda = ((4, 3, 1), (3, 2))$*. We shall define the* λ*–bitableaux* $\mathfrak{t}^{\lambda}$*,* $\hat{\mathfrak{t}}^{\lambda}$*,* $\mathfrak{t}_{\lambda}$ *and* $\hat{\mathfrak{t}}_{\lambda}$ *so that*

$$
\mathfrak{t}^\lambda = \left(\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & & & \end{matrix} \quad , \quad \begin{matrix} 9 & 10 & 11 \\ 12 & 13 & \\ & & \end{matrix} \right),
$$
$$
\hat{\mathfrak{t}}^\lambda = \left(\begin{matrix} 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & \\ 13 & & & \end{matrix} \quad , \quad \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ & & \end{matrix} \right),
$$
$$
\mathfrak{t}_\lambda = \left(\begin{matrix} 1 & 4 & 6 & 8 \\ 2 & 5 & 7 & \\ 3 & & & \end{matrix} \quad , \quad \begin{matrix} 9 & 11 & 13 \\ 10 & 12 & \\ & & \end{matrix} \right),
$$
$$
\hat{\mathfrak{t}}_\lambda = \left(\begin{matrix} 6 & 9 & 11 & 13 \\ 7 & 10 & 12 & \\ 8 & & & \end{matrix} \quad , \quad \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \\ & & \end{matrix} \right).
$$

Definition 2.8 *Suppose that λ is an a–bicomposition of r.*

(i) *Let $\mathfrak{t}^\lambda = (\mathfrak{t}^{\lambda^{(1)}}, \mathfrak{t}^{\lambda^{(2)}})$ be the standard λ–bitableau in which the numbers $1, 2, \dots, a$ appear in order by rows in $\mathfrak{t}^{\lambda^{(1)}}$ and the numbers $a+1, a+2, \dots, r$ appear in order by rows in $\mathfrak{t}^{\lambda^{(2)}}$.*

(ii) *Let $\hat{\mathfrak{t}}^\lambda = (\hat{\mathfrak{t}}^{\lambda^{(1)}}, \hat{\mathfrak{t}}^{\lambda^{(2)}})$ be the standard λ–bitableau in which the numbers $1, 2, \dots, r-a$ appear in order by rows in $\hat{\mathfrak{t}}^{\lambda^{(2)}}$ and the numbers $r-a+1, r-a+2, \dots, r$ appear in order by rows in $\hat{\mathfrak{t}}^{\lambda^{(1)}}$.*

(iii) *Let $\mathfrak{t}_\lambda = ({\mathfrak{t}_\lambda}^{(1)}, {\mathfrak{t}_\lambda}^{(2)})$ be the standard λ–bitableau in which the numbers $1, 2, \dots, a$ appear in order by columns in ${\mathfrak{t}_\lambda}^{(1)}$ and the numbers $a+1, a+2, \dots, r$ appear in order by columns in ${\mathfrak{t}_\lambda}^{(2)}$.*

(iv) *Let $\hat{\mathfrak{t}}_\lambda = ({\hat{\mathfrak{t}}_\lambda}{}^{(1)}, {\hat{\mathfrak{t}}_\lambda}{}^{(2)})$ be the standard λ–bitableau in which the numbers $1, 2, \dots, r-a$ appear in order by columns in ${\hat{\mathfrak{t}}_\lambda}{}^{(2)}$ and the numbers $r-a+1, r-a+2, \dots, r$ appear in order by columns in ${\hat{\mathfrak{t}}_\lambda}{}^{(1)}$.*

Note that W_r acts on the set of λ–bitableaux. If $\mathfrak{t}$ is a λ–bitableau and $w \in W_r$, then we obtain the λ–bitableau $\mathfrak{t}w$ by replacing each i in $\mathfrak{t}$ by iw. For example, if λ is an a–bicomposition of r, then $\mathfrak{t}^\lambda w_{a,r-a} = \hat{\mathfrak{t}}^\lambda$. It is easy to see that W_r acts transitively on the set of λ–bitableaux.

One easily checks that the action of W_r preserves the row equivalence classes of λ–bitableaux. We therefore have a transitive action of W_r on the set of λ–bitabloids. If λ is an a–bicomposition, then the stabilizer of the bitabloid $\{\mathfrak{t}^\lambda\}$ is the subgroup

$$
W_\lambda = ((\underbrace{C_2 \times \cdots \times C_2}_{a \text{ factors}}) \rtimes \mathfrak{S}_{\lambda^{(1)}}) \times \mathfrak{S}_{\lambda^{(2)}} \tag{2.9}
$$

of W_r. Thus the permutation representation of W_r on the set of λ–bitabloids is equivalent to the representation of W_r on the cosets of W_λ.

A **reflection subgroup** W_J of W_r is a subgroup generated by a set of reflections $\{s_\alpha \mid \alpha \in J\}$ for some subset J of the positive roots. In particular, if $J \subseteq \Delta_0$ then W_J is a **parabolic subgroup** of W_r. If λ is an a–bicomposition then W_λ is a parabolic subgroup of W_r if and only if $\lambda^{(1)} = (a)$; in this case W_λ is a direct product of a Weyl group W_a of type **B** and several Weyl groups of type **A**. In general the reflection group W_λ has precisely one factor of type **B** for each nonzero part of $\lambda^{(1)}$.

Associated with the parabolic subgroup W_J, $J \subseteq \Delta_0$, is the parabolic subalgebra $\mathcal{H}_J = \sum_{w\in W_J} RT_w$ of $\mathcal{H}$. This algebra is isomorphic to the Hecke algebra $\mathcal{H}_{R,Q,q}(W_J)$ of W_J. If W_λ is a reflection subgroup of W_r, which is not a parabolic subgroup, then $\sum_{w\in W_\lambda} RT_w$ is not a subalgebra of $\mathcal{H}$ in general.

Definition 2.10 *Suppose that $\lambda \in \Lambda_2(n,r)$. The elements x_λ, $\hat{x}_\lambda$, y_λ and $\hat{y}_\lambda$ of $\mathcal{H}$ are defined as follows.*

$$\begin{aligned}
x_\lambda &= \sum \{T_w \mid w \in W_r \textit{ and } w \textit{ stabilizes the rows of } \mathfrak{t}^\lambda\}\\
\hat{x}_\lambda &= \sum \{T_w \mid w \in W_r \textit{ and } w \textit{ stabilizes the rows of } \hat{\mathfrak{t}}^\lambda\}\\
y_\lambda &= \sum \{(-q)^{-\ell(w)} T_w \mid w \in W_r \textit{ and } w \textit{ stabilizes the columns of } \mathfrak{t}_\lambda\}\\
\hat{y}_\lambda &= \sum \{(-q)^{-\ell(w)} T_w \mid w \in W_r \textit{ and } w \textit{ stabilizes the columns of } \hat{\mathfrak{t}}_\lambda\}
\end{aligned}$$

Example 2.11 *Suppose that $\lambda = ((2,1),(1))$. Then*

$$\mathfrak{t}^\lambda = \begin{pmatrix} \begin{matrix}1 & 2\\ 3 & \end{matrix} & , & 4 \end{pmatrix} \qquad \mathfrak{t}_\lambda = \begin{pmatrix} \begin{matrix}1 & 3\\ 2 & \end{matrix} & , & 4 \end{pmatrix}$$

$$\hat{\mathfrak{t}}^\lambda = \begin{pmatrix} \begin{matrix}2 & 3\\ 4 & \end{matrix} & , & 1 \end{pmatrix} \qquad \hat{\mathfrak{t}}_\lambda = \begin{pmatrix} \begin{matrix}2 & 4\\ 3 & \end{matrix} & , & 1 \end{pmatrix}$$

and

$$\begin{aligned}
x_\lambda &= 1 + T_{(1,2)} & \qquad y_\lambda &= 1 - q^{-1}T_{(1,2)}\\
\hat{x}_\lambda &= 1 + T_{(2,3)} & \qquad \hat{y}_\lambda &= 1 - q^{-1}T_{(2,3)}.
\end{aligned}$$

We remark that all four elements defined in (2.10) belong to $\mathcal{H}(\mathfrak{S}_r)$ and that $\hat{x}_\lambda = x_{\hat{\lambda}}$ and $\hat{y}_\lambda = y_{\hat{\lambda}}$, where $\hat{\lambda}$ is the $(r-a)$–bicomposition $(\lambda^{(2)}, \lambda^{(1)})$.

From (2.4) we obtain the following.

2.12 *Assume that λ is an a–bicomposition. Then*

(i) *x_λ and y_λ commute with u_a^+;*
(ii) *$\hat{x}_\lambda$ and $\hat{y}_\lambda$ commute with u_{r-a}^-.*

As we noted in [10, (2.5)], we also have the following (see the remark after (2.2)).

2.13 *If λ is an a–bicomposition then $x_\lambda h_{a,r-a} = h_{a,r-a}\hat{x}_\lambda$.*

Definition 2.14 *Let π_λ be the element of W_r which is given by $\mathfrak{t}^\lambda\pi_\lambda = \mathfrak{t}_\lambda$ and $\hat{\pi}_\lambda$ be the element of W_r which is given by $\hat{\mathfrak{t}}^\lambda\hat{\pi}_\lambda = \hat{\mathfrak{t}}_\lambda$*

We note that π_λ and $\hat{\pi}_\lambda$ belong to $\mathfrak{S}_r \le W_r$, since the entries of the bitableaux involved are all positive. Moreover $\hat{\pi}_\lambda = \pi_{\hat{\lambda}}$, and $h_{a,r-a}T_{\hat{\pi}_\lambda} = T_{\pi_\lambda}h_{a,r-a}$. Note too that

$$T_{\hat{\pi}_\lambda}u^-_{r-a} = u^-_{r-a}T_{\hat{\pi}_\lambda}$$

by (2.4) since $\hat{\pi}_\lambda \in \mathfrak{S}_{(r-a,a)}$.

Now, [10, (3.11)] and [7, (4.1)] imply the following fundamental result.

Theorem 2.15 *Suppose that λ is an a–bipartition. Then*

$$u^+_a x_\lambda \mathcal{H} u^-_{r-a}\hat{y}_\lambda$$

is a one–dimensional R–module. It is spanned by the vector

$$u^+_a x_\lambda h_{a,r-a}T_{\hat{\pi}_\lambda}u^-_{r-a}\hat{y}_\lambda.$$

Definition 2.16 *Suppose that λ is an a–bipartition. Let*

$$z_\lambda = u^+_a x_\lambda h_{a,r-a}T_{\hat{\pi}_\lambda}u^-_{r-a}\hat{y}_\lambda.$$

Note that, in the light of (2.12) and (2.13), we have several alternative expressions for z_λ; for example,

$$\begin{aligned} z_\lambda &= u^+_a x_\lambda T_{\pi_\lambda}h_{a,r-a}u^-_{r-a}\hat{y}_\lambda \\ &= u^+_a h_{a,r-a}u^-_{r-a}\hat{x}_\lambda T_{\hat{\pi}_\lambda}\hat{y}_\lambda \\ &= u^+_a x_\lambda h_{a,r-a}u^-_{r-a}T_{\hat{\pi}_\lambda}\hat{y}_\lambda. \end{aligned} \tag{2.17}$$

We call the right ideal $z_\lambda\mathcal{H}$ of $\mathcal{H}$ the (Q,q)–**Specht module** S^λ. It is straightforward to show that S^λ is the dual of the module $\tilde{S}^\lambda$ which we introduced in [11, §4]; in particular S^λ has dimension equal to the number of standard λ–bitableaux. Moreover there exists an $\mathcal{H}$–invariant bilinear form on $M^\lambda = u^+_a x_\lambda\mathcal{H}$ and a submodule theorem holds for M^λ (compare Theorem 5.15 and [7, 4.8]). When R is a field, the submodule theorem allows us to construct irreducible $\mathcal{H}$–modules, and so we can recover the main results of [11]. We do not wish to pursue this here; instead, we turn to the construction of the (Q,q)-Schur algebra.

3 The $\mathcal{H}$–module M^λ

Definition 3.1 *Suppose that λ is an a–bicomposition . Let M^λ be the right ideal of $\mathcal{H}$ which is given by $M^\lambda = u_a^+ x_\lambda \mathcal{H}$.*

If λ is an a–bicomposition then M^λ contains the (Q,q)-Specht module S^λ, and post-multiplication by $u_{r-a}^- \hat{y}_\lambda$ maps M^λ on to the one–dimensional R–module spanned by the generator z_λ of S^λ, by Theorem 2.15.

An argument similar to that of [9, (1.1)] shows the following.

3.2 *If λ and μ are associated bicompositions of r, then M^λ and M^μ are isomorphic $\mathcal{H}$–modules.*

We next construct a basis of M^λ.

Theorem 3.3 *Assume that λ is an a–bicomposition of r. Then M^λ is a free R–module with basis*

$$\{u_a^+ x_\lambda T_w \mid w \in W_r \textit{ and } \mathfrak{t}^\lambda w \textit{ is row standard}\}.$$

Proof: Note first, that $u_a^+ x_\lambda \mathcal{H}(W_a) = u_a^+ x_\lambda \mathcal{H}(\mathfrak{S}_a)$. This is immediate if $a = 0$ since $\mathcal{H}(W_0) = \mathcal{H}(\mathfrak{S}_0) = \{1\}$. If $a > 0$ then from (2.4), for $h \in \mathcal{H}(\mathfrak{S}_a)$,

$$u_a^+ x_\lambda h T_0 = x_\lambda h u_a^+ T_0 = Q x_\lambda h u_a^+ = Q u_a^+ x_\lambda h.$$

Next, [7, (3.2)(i)] and [10, (3.6)] imply that a basis of $u_a^+ x_\lambda \mathcal{H}(\mathfrak{S}_a)$ is given by

$$\{u_a^+ x_\lambda T_w \mid w \in \mathfrak{S}_a \text{ and } \mathfrak{t}^\lambda w \text{ is row standard}\}.$$

Let $\mathfrak{S}_{\lambda^{(2)}}$ denote the subgroup of $\mathfrak{S}_{\{a+1,\dots,r\}}$ which stabilizes the rows of $\mathfrak{t}^{\lambda^{(2)}}$. Then $W_a \times \mathfrak{S}_{\lambda^{(2)}}$ is a parabolic subgroup of W_r. Let $\mathcal{D}$ denote the set of distinguished right coset representatives of $W_a \times \mathfrak{S}_{\lambda^{(2)}}$ in W_r. We claim that $\mathtt{B}$ is a basis of M^λ, where

$$\mathtt{B} = \{u_a^+ x_\lambda T_w T_d \mid w \in \mathfrak{S}_a, d \in \mathcal{D} \text{ and } \mathfrak{t}^\lambda w \text{ is row standard}\}. \tag{3.4}$$

To see this notice that we can write any element in W_r as $w_1 w_2 d$ where $w_1 \in W_a$, $w_2 \in \mathfrak{S}_{\lambda^{(2)}}$ and $d \in \mathcal{D}$; but

$$u_a^+ x_\lambda T_{w_1} T_{w_2} T_d = q^{\ell(w_2)} u_a^+ x_\lambda T_{w_1} T_d \in u_a^+ x_\lambda \mathcal{H}(\mathfrak{S}_a) T_d,$$

so $\mathtt{B}$ spans M^λ. Secondly, if d and d' are distinct elements of $\mathcal{D}$ then corresponding elements in $\mathtt{B}$ have distinct supports, and hence $\mathtt{B}$ is linearly independent.

Since $T_w T_d = T_{wd}$ for $w \in \mathfrak{S}_a$ and $d \in \mathcal{D}$, the proof of the theorem will be complete if we show that

$$\begin{aligned}(3.5)\quad \{wd \mid w \in \mathfrak{S}_a,\ d \in \mathcal{D} \text{ and } \mathfrak{t}^\lambda w \text{ is row standard}\}&\\ = \{v \in W_r \mid \mathfrak{t}^\lambda v \text{ is row standard}\}.&\end{aligned}$$

Suppose that $w \in \mathfrak{S}_a$, $d \in \mathcal{D}$ and $\mathfrak{t}^\lambda w$ is row standard. Since d is a distinguished coset representative, d sends the positive roots for $W_a \times \mathfrak{S}_{\lambda^{(2)}}$ into positive roots. The positive roots for $W_a \times \mathfrak{S}_{\lambda^{(2)}}$ are

$$e_i \quad (1 \le i \le a), \; e_i + e_j, \; e_j - e_i \quad (1 \le i < j \le a),$$

together with

$$e_j - e_i \quad (i < j \text{ with } i, j \text{ in the same row of } \mathfrak{t}^{\lambda^{(2)}}).$$

If $1 \le i \le a$ then $e_i d$ is a positive root, so $id \in \boldsymbol{r}^+$; thus all the entries in the first component of $\mathfrak{t}^\lambda wd$ must be positive. If $i < j$ and i and j are in the same row of $\mathfrak{t}^\lambda w$ then $(e_j - e_i)d$ is a positive root, so $id < jd$. Therefore, $\mathfrak{t}^\lambda wd$ is row standard (see Definition 2.6).

Finally, note that the sets on the two sides of equation (3.5) have the same size. This completes the proof of (3.5) and hence of the theorem. □

Definition 3.6 *Let $J \subseteq \Delta_0$. The subalgebra of $\mathcal{H}$ corresponding to the parabolic subgroup W_J of W_r is denoted by $\mathcal{H}_J$. The* **induction functor** *from $\mathcal{H}_J$ to $\mathcal{H}_K$ for $J \subseteq K \subseteq \Delta_0$ is denoted by Ind_J^K. Similarly the* **restriction functor** *from $\mathcal{H}_K$ to $\mathcal{H}_J$ is denoted by Res_J^K.*

Recall that $\mathcal{H}_J$ is free as an R–module with basis $\{T_w \mid w \in W_J\}$. Moreover, $\mathcal{H}$ is free as a left $\mathcal{H}_J$–module with basis $\{T_d \mid d \in \mathcal{D}_J\}$, where $\mathcal{D}_J$ denotes the set of **distinguished right coset representatives** of W_J in W_r which is the set of all elements of W_r which map the roots in J to positive roots. Note too, that $\mathcal{H}(\mathfrak{S}_r) = \mathcal{H}_\Delta$ with $\Delta = \{\alpha_1, \dots, \alpha_r\} = \Delta_0 \setminus \{\alpha_0\}$.

Lemma 3.7 *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be an a–bicomposition, and let*

$$A = \{\alpha_0, \alpha_1, \dots, \alpha_{a-1}\} \cup J_2$$

where J_2 is the subset of $\{\alpha_{a+1}, \alpha_{a+2}, \dots, \alpha_{r-1}\}$ corresponding to $\lambda^{(2)}$. Then $u_a^+ x_\lambda \in \mathcal{H}_A$ and $M^\lambda = \mathrm{Ind}_A^{\Delta_0} u_a^+ x_\lambda \mathcal{H}_A$.

Proof: This follows by general arguments since $\mathcal{H}_A = \mathcal{H}(W_a \times \mathfrak{S}_{\lambda^{(2)}})$. □

We find it convenient to adopt a shorthand for the basis elements of M^λ which appear in Theorem 3.3.

Notation 3.8 *Suppose that λ is an a–bicomposition. Let $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ be a λ–bitableau. We identify the λ–bitabloid $\{\mathfrak{t}\}$ and $u_a^+ x_\lambda T_w$, where $\mathfrak{t}^\lambda w$ is the unique row standard λ–bitableau which is row equivalent to $\mathfrak{t}$.*

This identification gives us an action of $\mathcal{H}$ on the λ–bitabloids. The discussion after Definition 2.8 shows that this is a (Q, q)–analogue of the permutation action of W_r on the cosets of W_λ.

Theorem 3.3 says that M^λ has a basis which consists of the distinct λ–bitabloids. The action of $\mathcal{H}$ on a λ–bitabloid $\{\mathfrak{t}\}$ is determined once we know $\{\mathfrak{t}\}T_i$ for $i = 0, 1, \dots, r-1$. From (3.8) we have the following.

Lemma 3.9 *Suppose that $0 \le i \le r-1$ and that both $\mathfrak{t}$ and $\mathfrak{t}s_i$ are row standard (say $\mathfrak{t}=\mathfrak{t}^\lambda w$). Then*

$$\{\mathfrak{t}\}T_i = \begin{cases} \{\mathfrak{t}s_i\}, & \text{if } \ell(ws_i)=\ell(w)+1,\\ q\{\mathfrak{t}s_i\}+(q-1)\{\mathfrak{t}\}, & \text{if } \ell(ws_i)=\ell(w)-1 \quad \text{and} \quad i>0,\\ Q\{\mathfrak{t}s_0\}+(Q-1)\{\mathfrak{t}\}, & \text{if } \ell(ws_i)=\ell(w)-1 \quad \text{and} \quad i=0. \end{cases}$$

By considering roots, it is straightforward to determine from the tableau $\mathfrak{t}$ which case of the lemma applies.

Lemma 3.10 *Maintain the notation of Lemma 3.9. Then $\ell(ws_i)=\ell(w)+1$ if and only if one of the following holds.*

(i) *$i,\ i+1 \in \mathfrak{t}$ and $\mathrm{row}_\mathfrak{t}(i) < \mathrm{row}_\mathfrak{t}(i+1)$;*
(ii) *$-i,\ -i-1 \in \mathfrak{t}$ and $\mathrm{row}_\mathfrak{t}(-i-1) < \mathrm{row}_\mathfrak{t}(-i)$;*
(iii) *$-i,\ i+1 \in \mathfrak{t}$; or,*
(iv) *$i=0$ and $1 \in \mathfrak{t}^{(2)}$.*

The cases where $\mathfrak{t}s_i$ is not row standard are covered by the next result.

Lemma 3.11 *Suppose that $\mathfrak{t}=(\mathfrak{t}^{(1)},\mathfrak{t}^{(2)})$ is row standard and that $\mathfrak{t}s_i$ is not. Then either*

(i) *$i=0$ and $1\in\mathfrak{t}^{(1)}$ and $\{\mathfrak{t}\}T_0 = Q\{\mathfrak{t}\}$; or*
(ii) *$i>0$ and i and $i+1$ belong to the same row of $\mathfrak{t}$ and $\{\mathfrak{t}\}T_i = q\{\mathfrak{t}\}$; or*
(iii) *$i>0$ and $-i$ and $-i-1$ belong to the same row of $\mathfrak{t}$ and $\{\mathfrak{t}\}T_i = q\{\mathfrak{t}\}$.*

Proof: It is clear that the only cases where $\mathfrak{t}s_i$ is not row standard are covered by the statement of the lemma.

Consider first the case where $i=0$ and $1\in\mathfrak{t}^{(1)}$. We know that $\{\mathfrak{t}\}=u_a^+x_\lambda T_w T_d$, as in (3.5). Moreover, 1 is fixed by d because $1\in\mathfrak{t}^{(1)}$. Therefore, T_0 commutes with T_d, and using (2.4) we get

$$\{\mathfrak{t}\}T_0 = x_\lambda T_w u_a^+ T_0 T_d = Q x_\lambda T_w u_a^+ T_d = Q\{\mathfrak{t}\}.$$

This completes the proof of part (i) of the lemma. The proof of the other parts is similar to that of [7, 3.2(ii)]. □

For future reference, we next collect several results concerning the action of $\mathcal{H}$ on M^λ.

Lemma 3.12 *Suppose that $\mathfrak{t}$ is row standard. Assume that i is a positive integer in $\mathfrak{t}^{(2)}$. Then $\{\mathfrak{t}\}T_{i,1}T_0T_{1,i}$ is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ such that $\mathfrak{s}$ is row standard and*

(i) $-i \in \mathfrak{s}$; *and*
(ii) *for all integers* j *with* $j \neq \pm i$, *we have* $j \in \mathfrak{s}$ *if and only if* $j \in \mathfrak{t}$.

Proof: If $i = 1$ then the claim follows immediately from Lemmas 3.9 and 3.10. Thus let $i > 1$. Using (3.9), (3.10) and (3.11), we see that $\{\mathfrak{t}\}T_{i-1}$ is a linear combination of one or two λ–bitabloids $\{\mathfrak{s}\}$ where $\mathfrak{s}$ is row standard and

(i) $i - 1 \in \mathfrak{s}^{(2)}$; and
(ii) $i \in \mathfrak{s} \Leftrightarrow i - 1 \in \mathfrak{t}$; and
(iii) $j \in \mathfrak{s} \Leftrightarrow j \in \mathfrak{t}$ for $|j| \neq i - 1, i$.

Hence, by induction we may assume that

$$\{\mathfrak{t}\}T_{i,1}T_0T_{1,i-1} = \{\mathfrak{t}\}T_{i-1}T_{i-1,1}T_0T_{1,i-1}$$

is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ such that $\mathfrak{s}$ is row standard and

(i) $-(i - 1) \in \mathfrak{s}^{(2)}$; and
(ii) $i \in \mathfrak{s} \Leftrightarrow i - 1 \in \mathfrak{t}$; and
(iii) $j \in \mathfrak{s} \Leftrightarrow j \in \mathfrak{t}$ for $|j| \neq i - 1, i$.

Therefore, $\{\mathfrak{t}\}T_{i,1}T_0T_{1,i} = \{\mathfrak{t}\}T_{i-1}T_{i-1,1}T_0T_{1,i-1}T_{i-1}$ is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ as in the statement of the lemma. □

Definition 3.13 *Let* M^λ_- *denote the* R*–submodule of* M^λ *spanned by those* λ*–bitabloids* $\{\mathfrak{t}\}$ *where* $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ *and* $\mathfrak{t}^{(2)}$ *contains a negative integer.*

Note that Lemmas 3.9 and 3.11 show that

3.14 M^λ_- *is an* $\mathcal{H}(\mathfrak{S}_r)$*–submodule of* M^λ.

Lemma 3.12 now has the following corollary.

Corollary 3.15 *Suppose that* $\{\mathfrak{t}\}$ *is a* λ*–bitabloid. Assume that* $1, 2, \ldots, b \in \mathfrak{t}^{(2)}$. *Then*

$$\{\mathfrak{t}\}u_b^- \equiv Q^b q^{b(b-1)/2}\{\mathfrak{t}\} \pmod{M^\lambda_-}.$$

Proof: Note that $Q^b q^{b(b-1)/2}$ is the coefficient of the identity in u_b^-. The corollary follows from multiplying out u_b^- and applying Lemma 3.12. □

Lemma 3.16 *Assume that $\{\mathfrak{t}\}$ is a λ–bitabloid and that all the entries in $\mathfrak{t}$ are positive. Suppose that $1 \le i < j \le r$. Then for some integer k*

$$\{\mathfrak{t}\}T_{j,i} = q^k\{\mathfrak{t}s_{j,i}\} + m,$$

where m is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ such that $\mathfrak{s}$ is row standard and $\mathrm{row}_{\mathfrak{s}}(i) > \mathrm{row}_{\mathfrak{t}s_{j,i}}(i) = \mathrm{row}_{\mathfrak{t}}(j)$.

Proof: Note first that $js_{j,i} = i$, so $\mathrm{row}_{\mathfrak{t}s_{j,i}}(i) = \mathrm{row}_{\mathfrak{t}}(j)$. Now,

$$\{\mathfrak{t}\}T_{j-1} = q^{\epsilon}\{\mathfrak{t}s_{j-1}\} + m_1$$

where $\epsilon = 0$ or 1, and either $m_1 = 0$ or $\mathrm{row}_{\mathfrak{t}}(j-1) > \mathrm{row}_{\mathfrak{t}}(j)$ and $m_1 = (q-1)\{\mathfrak{t}\}$. This proves the lemma in the case where $i = j-1$. Assume, therefore, that $i < j-1$.

By induction

$$\{\mathfrak{t}s_{j-1}\}T_{j-1,i} = q^{k_2}\{\mathfrak{t}s_{j,i}\} + m_2$$

for some integer k_2, where m_2 is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ such that $\mathfrak{s}$ is row standard and $\mathrm{row}_{\mathfrak{s}}(i) > \mathrm{row}_{\mathfrak{t}s_{j-1,i}}(i) = \mathrm{row}_{\mathfrak{t}}(j-1)$. Furthermore, if $m_1 \ne 0$ then $m_1 = (q-1)\{\mathfrak{t}\}$ and induction shows that $\{\mathfrak{t}\}T_{j-1,i}$ is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ where $\mathfrak{s}$ is row standard and

$$\mathrm{row}_{\mathfrak{s}}(i) \ge \mathrm{row}_{\mathfrak{t}s_{j-1,i}}(i) = \mathrm{row}_{\mathfrak{t}}(j-1) > \mathrm{row}_{\mathfrak{t}}(j).$$

Therefore,

$$\{\mathfrak{t}\}T_{j,i} = \{\mathfrak{t}\}T_{j-1}T_{j-1,i} = q^k\{\mathfrak{t}s_{j,i}\} + m,$$

for some integer k, where m is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ where $\mathfrak{s}$ is row standard and $\mathrm{row}_{\mathfrak{s}}(i) > \mathrm{row}_{\mathfrak{t}s_{j,i}}(i)$. □

Corollary 3.17 *Assume that $\{\mathfrak{t}\}$ is a λ–bitabloid and that all the entries in $\mathfrak{t}$ are positive. Suppose that $0 \le a \le r$. Then for some integer k,*

$$\{\mathfrak{t}\}h_{a,r-a} = q^k\{\mathfrak{t}w_{a,r-a}\} + m,$$

where m is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ ($\mathfrak{s}$ row standard) such that for all i with $1 \le i \le r-a$,

$$\mathrm{row}_{\mathfrak{s}}(i) \ge \mathrm{row}_{\mathfrak{t}}(a+i)$$

with the inequality being strict for at least one i.

Proof: We claim that the following holds. For $0 \le j \le r-a$

$$\{\mathfrak{t}\}T_{a+1,1}T_{a+2,2}\cdots T_{a+j,j} = q^{k_1}\{\mathfrak{t}s_{a+1,1}\cdots s_{a+j,j}\} + m \qquad (3.18)$$

where k_1 is an integer, and m is a linear combination of λ–bitabloids $\{\mathfrak{s}\}$ where $\mathfrak{s}$ is row standard and, for $1 \le i \le j$, $\mathrm{row}_\mathfrak{s}(i) \ge \mathrm{row}_\mathfrak{t}(a+i)$, with strict inequality for some i with $1 \le i \le j$, and

$$\mathrm{row}_\mathfrak{s}(i) = \mathrm{row}_\mathfrak{t}(i) \quad \text{for} \quad i > a+j.$$

Equation (3.18) certainly holds if $j = 0$. Since $s_{a+j+1,j+1}$ permutes only numbers i with $j+1 \le i \le a+j+1$, Lemma 3.16 provides the induction step, so (3.18) holds.

Since $h_{a,r-a} = T_{a+1,1}T_{a+2,2}\cdots T_{r,r-a}$, by (2.2), the corollary is the special case of (3.18) when $j = r-a$. □

4 The (Q,q)-Schur algebra

The q–Schur algebra [8, 9] associated with the Hecke algebra $\mathcal{H}(\mathfrak{S}_r)$ is defined as the endomorphism ring

$$\mathrm{End}_{\mathcal{H}(\mathfrak{S}_r)}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} M^\lambda\Big)$$

where $\Lambda(n,r)$ is the set of compositions of r into n parts and M^λ is the $\mathcal{H}(\mathfrak{S}_r)$–module induced from the trivial module of the parabolic subalgebra $\mathcal{H}(\mathfrak{S}_\lambda)$. Because each partition in $\Lambda(n,r)$ corresponds to a parabolic subgroup of $\mathfrak{S}_r$, a natural generalization to a Schur algebra of type **B** is the algebra

$$\tilde{\mathcal{S}} = \mathrm{End}_{\mathcal{H}}\Big(\bigoplus_{J\subseteq\Delta_0} M^J\Big), \tag{4.1}$$

where $M^J = \mathrm{Ind}_J^{\Delta_0} x_J R$ and $x_J = \sum_{w\in W_J} T_w$. The algebra $\tilde{\mathcal{S}}$ (with Δ_0 replaced by Δ), has been investigated in [12, 15].

We wish to consider a larger algebra, which contains $\tilde{\mathcal{S}}$ as a subalgebra, in which we take endomorphisms of induced modules which correspond to arbitrary reflection subgroups of W_r, rather than just the parabolic subgroups.

Definition 4.2 *The* (Q,q)**-Schur algebra** *is the endomorphism ring*

$$\mathcal{S}_R(n,r) = \mathrm{End}_{\mathcal{H}}\Big(\bigoplus_{\lambda\in\Lambda_2(n,r)} M^\lambda\Big).$$

If $n \ge r$ then all bipartitions of r belong to $\Lambda_2(n,r)$, and we see from (3.2) that the next result holds.

Lemma 4.3 *If $n \ge r$ then the (Q,q)-Schur algebra is Morita equivalent to* $\mathrm{End}_{\mathcal{H}}\Big(\bigoplus_\lambda M^\lambda\Big)$ *where the direct sum is now over all bipartitions λ of r.*

If $n \le r$ the (Q,q)-Schur algebra $\mathcal{S}_R(n,r)$ is Morita equivalent to an algebra of the form $e\mathcal{S}_R(r,r)e$ for some idempotent e in $\mathcal{S}_R(r,r)$. Similarly, the algebra $\tilde{\mathcal{S}}$ defined in (4.1) is Morita equivalent to an algebra of the form $e\mathcal{S}_R(n,r)e$.

Henceforth, we fix n and r and let $\mathcal{S} = \mathcal{S}_R(n,r)$.

Definition 4.4 *For subsets K, L, M of Δ_0 we set*

$$\mathcal{D}^M_{K,L} = \mathcal{D}_K \cap \mathcal{D}_L^{-1} \cap W_M.$$

Thus, if $K \subseteq M$ and $L \subseteq M$ then $\mathcal{D}^M_{K,L}$ is the set of **distinguished double** (W_K, W_L)–**coset representatives** *in W_M. Similarly, we define $\mathcal{D}^M_K = \mathcal{D}_K \cap W_M$. If $M = \Delta_0$ we usually omit the superscript Δ_0.*

The next lemma summarizes the properties of the distinguished (double) coset representatives which we need; the proofs of these results can be found in [3, §2.7].

Lemma 4.5 *Suppose that $K, L \subseteq \Delta_0$ and let $d \in \mathcal{D}_{K,L}$.*

(i) *Every element of $W_K d W_L$ is uniquely expressible in the form xdw where $x^{-1} \in \mathcal{D}^K_{K\cap Ld^{-1}}$ and $w \in W_L$; moreover, $\ell(xdw) = \ell(x) + \ell(d) + \ell(w)$.*
(ii) *$W_{Kd\cap L} = d^{-1} W_K d \cap W_L$; consequently, $T_d^{-1}\mathcal{H}_K T_d \cap \mathcal{H}_L = \mathcal{H}_{Kd\cap L}$.*
(iii) *If $K \subseteq L$ then $\mathcal{D}_K = \mathcal{D}^L_K \mathcal{D}_L$; in particular, $\mathcal{D}_L \subseteq \mathcal{D}_K$.*

As in [7, 3.4] (compare [9, 1.4]) we have the following theorem.

Theorem 4.6 *The algebra $\tilde{\mathcal{S}}$ is free as an R–module with basis*

$$\{\psi^d_{I,J} \mid I, J \subseteq \Delta_0,\ d \in \mathcal{D}_{I,J}\},$$

where $\psi^d_{I,J} : x_J\mathcal{H} \to x_I\mathcal{H}$ is the homomorphism given by

$$\psi^d_{I,J}(x_J h) = \sum_{w \in W_I d W_J} T_w h, \qquad (h \in \mathcal{H}).$$

We now want to exhibit a generic basis of $\mathcal{S}$, that is a basis which is independent of the choice of the ring R and the values of the parameters Q and q. By construction it is enough to show that

$$\mathfrak{H}_{\lambda,\mu} = \operatorname{Hom}_{\mathcal{H}}(M^\lambda, M^\mu) \tag{4.7}$$

for $\lambda, \mu \in \Lambda_2(n,r)$ is free as an R–module and has a generic basis.

First, from the defining relations (2.1) for W_r we see that for any $w \in W_r$ the number of factors equal to s_0 is the same in every reduced expression of w. Thus the following definition makes sense.

Definition 4.8 *Suppose that $w \in W_r$ and $\ell(w) = a + b$ where a reduced expression for w has exactly b factors equal to s_0. We set*

$$\underline{q}^{\ell(w)} = q^a Q^b \quad \text{and} \quad \underline{q}^{-\ell(w)} = q^{-a} Q^{-b}.$$

Next we state Frobenius reciprocity and a result which shows that, as for Hecke algebras of type **A**, induction from parabolic subalgebras is not only a left adjoint functor to restriction, but a right adjoint functor as well.

Let $*$ be the antiautomorphism of $\mathcal{H}$ determined by $T_w^* = T_{w^{-1}}$ for all $w \in W_r$.

Theorem 4.9 *Let $J \subseteq \Delta_0$, and let M be an $\mathcal{H}_J$–module, N an $\mathcal{H}$–module. Then the following hold.*

(i) $\operatorname{Hom}_{\mathcal{H}}\Big(\operatorname{Ind}_J^{\Delta_0} M\,,\, N\Big) \cong \operatorname{Hom}_{\mathcal{H}_J}\Big(M\,,\, \operatorname{Res}_J^{\Delta_0} N\Big)$ *where an isomorphism is given by restricting a map in* $\operatorname{Hom}_{\mathcal{H}}\Big(\operatorname{Ind}_J^{\Delta_0} M\,,\, N\Big)$ *to the $\mathcal{H}_J$–subspace $M \otimes 1$ of* $\operatorname{Ind}_J^{\Delta_0} M = M \otimes_{\mathcal{H}_J} \mathcal{H}$.

(ii) $\operatorname{Hom}_{\mathcal{H}}\Big(N\,,\, \operatorname{Ind}_J^{\Delta_0} M\Big) \cong \operatorname{Hom}_{\mathcal{H}_J}\Big(\operatorname{Res}_J^{\Delta_0} N\,,\, M\Big)$ *where an isomorphism given by sending $\phi \in \operatorname{Hom}_{\mathcal{H}_J}\Big(\operatorname{Res}_J^{\Delta_0} N\,,\, M\Big)$ to the map $\hat\phi \in \operatorname{Hom}_{\mathcal{H}}\Big(N\,,\, \operatorname{Ind}_J^{\Delta_0} M\Big)$ where $\hat\phi$ is defined on $x \in N$ by*

$$\hat\phi(x) = \sum_{d \in \mathcal{D}_J} \underline{q}^{-\ell(d)} \phi(x T_d^*) \otimes T_d.$$

Proof: Part (i) is Frobenius reciprocity which holds for arbitrary rings and subrings. For part (ii) we refer to [7, 2.6]; the proof there can be adjusted easily. □

We remark that the inverse to the map $\phi \mapsto \hat\phi$ in part (ii) of the theorem is given by

$$\theta \mapsto \theta_1 \in \operatorname{Hom}_{\mathcal{H}_J}\Big(\operatorname{Res}_J^{\Delta_0} N\,,\, M\Big)$$

for $\theta \in \operatorname{Hom}_{\mathcal{H}}(N\,,\, \operatorname{Ind}_J^{\Delta_0} M)$, where θ_1 is defined as follows. For $x \in N$ we have $x_d \in M$ uniquely defined by

$$\theta(x) = \sum_{d \in \mathcal{D}_J} x_d \otimes T_d \in \operatorname{Ind}_J^{\Delta_0} M.$$

The map $\theta_d : N \to M : x \mapsto x_d$ is easily seen to be R–linear and, for $d = 1$, even $\mathcal{H}_J$–linear, (in general, θ_d is $T_d^{-1}\mathcal{H}_J T_d$–linear). Then θ_1 is the desired map.

We shall also use the Mackey Decomposition Theorem [7, 2.7].

Theorem 4.10 *Let I, J be subsets of Δ_0 and let M be an $\mathcal{H}_J$–module. Then*

$$\operatorname{Res}_I^{\Delta_0} \operatorname{Ind}_J^{\Delta_0} M = \bigoplus_{d \in \mathcal{D}_{J,I}} \operatorname{Ind}_{Jd \cap I}^{I} \operatorname{Res}_{Jd \cap I}^{Jd} M \otimes T_d,$$

considering $M \otimes T_d$ as an $\mathcal{H}_{Jd}$–module, where $\mathcal{H}_{Jd} = T_d^{-1}\mathcal{H}_J T_d$.

Now this result taken in conjunction with Theorem 4.9 produces the following intertwining number theorem (compare [7, 2.8]).

Corollary 4.11 *Let I, J and M be as in Theorem 4.10 and let N be an $\mathcal{H}_I$–module. Let* $\mathfrak{H} = \operatorname{Hom}_{\mathcal{H}}\Big(\operatorname{Ind}_J^{\Delta_0} M \,,\, \operatorname{Ind}_I^{\Delta_0} N\Big)$. *Then*

$$
\begin{aligned}
\text{(i)}\quad \mathfrak{H} &\cong \bigoplus_{d\in\mathcal{D}_{I,J}} \operatorname{Hom}_{\mathcal{H}_J}\Big(M \,,\, \operatorname{Ind}_{Id\cap J}^{J} \operatorname{Res}_{Id\cap J}^{Id} N \otimes T_d\Big)\\
&\cong \bigoplus_{d\in\mathcal{D}_{I,J}} \operatorname{Hom}_{\mathcal{H}_{Id\cap J}}\Big(\operatorname{Res}_{Id\cap J}^{J} M \,,\, \operatorname{Res}_{Id\cap J}^{Id} N \otimes T_d\Big),\\
\text{(ii)}\quad \mathfrak{H} &\cong \bigoplus_{d\in\mathcal{D}_{J,I}} \operatorname{Hom}_{\mathcal{H}_I}\Big(\operatorname{Ind}_{Jd\cap I}^{I} \operatorname{Res}_{Jd\cap I}^{Jd} M \otimes T_d \,,\, N\Big)\\
&\cong \bigoplus_{d\in\mathcal{D}_{J,I}} \operatorname{Hom}_{\mathcal{H}_{Jd\cap I}}\Big(\operatorname{Res}_{Jd\cap I}^{Jd} M \otimes T_d \,,\, \operatorname{Res}_{Jd\cap I}^{I} N\Big).
\end{aligned}
$$

Remark 4.12 *We shall apply Corollary 4.11 in the special case where the modules M and N are the trivial modules $x_J\mathcal{H}_J$ and $x_I\mathcal{H}_I$ respectively. Note that the restriction of the trivial module to a parabolic subalgebra is again the trivial module. Moreover, the identity map generates the endomorphism ring of the trivial module. Corollary 4.11 provides ways to exhibit a basis of $\mathfrak{H}$, by choosing the identity maps in each summand. Using Theorem 4.9 the basis elements in parts (i) and (ii) which are labelled by $d \in \mathcal{D}_{I,J}$ and $d^{-1} \in \mathcal{D}_{J,I}$ differ by a factor of $\underline{q}^{-\underline{\ell}(d)}$.*

It is an immediate consequence of the way that W_r acts on V that the next result holds.

4.13 *Suppose that $w \in W_r$ and $J \subseteq \Delta_0$. Then the following are equivalent.*

(i) $\alpha_0 \in Jw$.
(ii) $\alpha_0 \in J$ *and* $\alpha_0 w = \alpha_0$.
(iii) $\alpha_0 \in J$ *and* $s_0 w = w s_0$ *and* $\ell(s_0 w) = \ell(w) + 1$.

From now on we fix the following notation.

Notation 4.14 *Let a and b be integers with $0 \le a, b \le r$ and fix an a–bicomposition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ and a b–bicomposition $\mu = (\mu^{(1)}, \mu^{(2)})$ with $\lambda, \mu \in \Lambda_2(n,r)$.*

(i) *Let J_1 be the subset of $\{\alpha_1, \dots, \alpha_{a-1}\}$ corresponding to $\lambda^{(1)}$ and J_2 be the subset of $\{\alpha_{a+1}, \dots, \alpha_{r-1}\}$ corresponding to $\lambda^{(2)}$. Similarly, define subsets I_1 and I_2 corresponding to $\mu^{(1)}$ and $\mu^{(2)}$ respectively. Let $J = J_1 \cup J_2$ and $I = I_1 \cup I_2$.*
(ii) *Let $\tilde{A} = \Delta_0 \setminus \{\alpha_a\}$ if $a \neq r$, and $\tilde{A} = \Delta_0$ if $a = r$. Similarly, $\tilde{B} = \Delta_0 \setminus \{\alpha_b\}$ is $b \neq r$, and $\tilde{B} = \Delta_0$ otherwise. Let $A = \tilde{A} \setminus \{\alpha_0\}$ and $B = \tilde{B} \setminus \{\alpha_0\}$.*

Note that
$$x_J = \sum_{w\in W_J} T_w = x_{J_1}x_{J_2} = x_{J_2}x_{J_1} = x_\lambda.$$
We also remark that $J \subseteq A \subseteq \tilde{A} \subseteq \Delta_0$ and that $I \subseteq B \subseteq \tilde{B} \subseteq \Delta_0$.

Note that we write $\mathcal{D}_{\tilde{B},\tilde{A}}$ instead of $\mathcal{D}^{\Delta_0}_{\tilde{B},\tilde{A}}$.

Lemma 4.15 *Let $d \in \mathcal{D}_{\tilde{B},\tilde{A}}$, and suppose that $\alpha_0 \in \tilde{B}d \cap \tilde{A}$. Then $a \geq 1$, $b \geq 1$ and $T_dT_0 = T_0T_d$. Moreover for $h \in \mathcal{H}_{\tilde{A}}$ we have*
$$u_a^+ x_J h T_0 = Q u_a^+ x_J h$$
and for $h \in \mathcal{H}_{\tilde{B}}T_d$ we have
$$u_b^+ x_I h T_0 = Q u_b^+ x_I h.$$

Proof: Since $\alpha_0 \in \tilde{B}d \cap \tilde{A}$, we have $\alpha_0 \in \tilde{B} \cap \tilde{A}$ and $s_0d = ds_0$ by (4.13). Therefore, $a \geq 1$ and $b \geq 1$ by (4.14). Further, since $d \in \mathcal{D}_{\tilde{B},\tilde{A}}$, we have $T_dT_0 = T_{ds_0} = T_{s_0d} = T_0T_d$.

Let $h \in \mathcal{H}_{\tilde{A}}$. Now $\tilde{A} = \{\alpha_0, \dots, \alpha_{a-1}\} \cup \{\alpha_{a+1}, \dots \alpha_{r-1}\}$ and hence we may write $h = h_1h_2 = h_2h_1$ with $h_1 \in \mathcal{H}_{\{\alpha_0,\dots,\alpha_{a-1}\}}$ and $h_2 \in \mathcal{H}_{\{\alpha_{a+1},\dots,\alpha_{r-1}\}}$. By (2.4),
$$u_a^+ x_J h T_0 = x_J u_a^+ h_1 h_2 T_0 = x_J h_1 u_a^+ T_0 h_2 = Q x_J h_1 u_a^+ h_2 = Q u_a^+ x_J h.$$
Similarly, $u_b^+ x_I h' T_0 = Q u_b^+ x_I h'$ for any $h' \in \mathcal{H}_{\tilde{B}}$. Since T_d commutes with T_0 we have $u_b^+ x_I h T_0 = Q u_b^+ x_I h$ for any $h \in \mathcal{H}_{\tilde{B}}T_d$. □

Definition 4.16 *We say that a triple (d, v, u) is* **admissible** *for (λ, μ) if $d \in \mathcal{D}_{\tilde{B},\tilde{A}}$ and $v \in \mathcal{D}^A_{J,Bd\cap A}$ and $u \in \mathcal{D}^B_{I,B\cap Jvd^{-1}}$.*

Our current aim is to show that $\mathfrak{H}_{\lambda,\mu}$ has a generic basis indexed by the set of admissible triples for (λ, μ). We first prove two technical lemmas about admissible triples; after that, the next three lemmas introduce in turn the elements d, v and u of an admissible triple and show how they determine a basis of the R–module $\mathfrak{H}_{\lambda,\mu}$.

Ultimately we are interested in bases which are indexed not by admissible triples but by elements of $\mathcal{D}_{I,J}$; this is the point of part (iv) below.

Lemma 4.17 *Suppose that (d, v, u) is an admissible triple. Then*

(i) $\ell(udv^{-1}) = \ell(u) + \ell(d) + \ell(v^{-1})$.
(ii) $ud \in \mathcal{D}_{I,A\cap Jv}$.
(iii) $dv^{-1} \in \mathcal{D}_{B,J}$.
(iv) $udv^{-1} \in \mathcal{D}_{I,J}$.

Proof: Note that $d \in \mathcal{D}_{\tilde{B},\tilde{A}} \subseteq \mathcal{D}_{B,A} \subseteq \mathcal{D}_{B,A\cap Jv}$ and that $u \in W_B$ and $v^{-1} \in \mathcal{D}^A_{Bd\cap A}$; hence (i) follows from Lemma 4.5(i).

To prove (ii), first note that $ud \in \mathcal{D}^B_I\mathcal{D}_B = \mathcal{D}_I$ by Lemma 4.5(iii) because $u \in \mathcal{D}^B_I$ and $d \in \mathcal{D}_{\tilde{B},\tilde{A}} \subseteq \mathcal{D}_{\tilde{B}} \subset \mathcal{D}_B$. On the other hand, $u \in (\mathcal{D}^B_{B\cap(A\cap Jv)d^{-1}})^{-1}$ and $d \in \mathcal{D}_{B,A\cap Jv}$ so, by taking $K = B$ and $L = A \cap Jv$ in Lemma 4.5(i),

$$\ell(udw) = \ell(u) + \ell(d) + \ell(w) = \ell(ud) + \ell(w)$$

for any $w \in W_{A\cap Jv}$. Therefore, $ud \in (\mathcal{D}_{A\cap Jv})^{-1}$ proving (ii). A similar argument proves (iii).

Finally, we prove (iv). By (iii), $dv^{-1} \in \mathcal{D}_B$ so $udv^{-1} \in \mathcal{D}^B_I\mathcal{D}_B = \mathcal{D}_I$ by Lemma 4.5(iii). Furthermore, if $w \in W_J$ then

$$\ell(udv^{-1}w) = \ell(u) + \ell(dv^{-1}) + \ell(w) = \ell(udv^{-1}) + \ell(w)$$

by Lemma 4.5(i), since $u^{-1} \in \mathcal{D}^B_{B\cap Jvd^{-1}}$ and $dv^{-1} \in \mathcal{D}_{B,J}$ by (iii). Therefore, $udv^{-1} \in \mathcal{D}_J^{-1}$ and the proof is complete. □

Lemma 4.18 *Let (d,v,u) be an admissible triple. Then*

(i) $B \cap Jvd^{-1} \subseteq Ad^{-1}$.
(ii) $I \cap Jvd^{-1}u^{-1} \subseteq Bu^{-1}$.

Proof: Let $\alpha \in B \cap Jvd^{-1}$. Then $d^{-1}s_\alpha d = v^{-1}s_\beta v$ for some $\beta \in J$. Now, $v^{-1}s_\beta v \in W_A$ so

$$\ell(d) + \ell(v^{-1}s_\beta v) = \ell(dv^{-1}s_\beta v) = \ell(s_\alpha d) = \ell(s_\alpha) + \ell(d) = 1 + \ell(d).$$

Therefore, $\ell(v^{-1}s_\beta v) = 1$ and so $\beta v \in A$; consequently, $\alpha = \beta vd^{-1} \in Ad^{-1}$ proving (i).

For (ii), suppose that $\alpha \in I \cap Jvd^{-1}u^{-1}$. Then $u^{-1}s_\alpha u = dv^{-1}s_\beta vd^{-1}$ for some $\beta \in J$. Since $u^{-1}s_\alpha u \in W_B$ and $dv^{-1} \in \mathcal{D}_{B,J}$ by Lemma 4.17(iii),

$$\ell(vd^{-1}) + \ell(u^{-1}s_\alpha u) = \ell(vd^{-1}u^{-1}s_\alpha u) = \ell(s_\beta vd^{-1}) = 1 + \ell(vd^{-1}).$$

Therefore, $\alpha u \in B$ and so the lemma is proved. □

We are now ready to investigate the Hom–space $\mathfrak{H}_{\lambda,\mu} = \operatorname{Hom}_{\mathcal{H}}(M^\lambda, M^\mu)$.

Lemma 4.19 *Suppose that λ and μ are bicompositions in $\Lambda_2(n,r)$. Then*

$$\mathfrak{H}_{\lambda,\mu} \cong \bigoplus_{d\in\mathcal{D}_{\tilde{B},\tilde{A}}} \operatorname{Hom}_{\mathcal{H}_{\tilde{B}d\cap\tilde{A}}}\Big(\operatorname{Res}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}} u_a^+ x_J \mathcal{H}_{\tilde{A}}\,,\ \operatorname{Res}^{\tilde{B}d}_{\tilde{B}d\cap\tilde{A}} u_b^+ x_I \mathcal{H}_{\tilde{B}} T_d\Big).$$

Proof: Note that $\{\alpha_0,\dots,\alpha_{a-1}\}\cup J_2\subseteq\tilde A$. Hence, by Lemma 3.7 and transitivity of induction, $M^\lambda\cong\operatorname{Ind}_{\tilde A}^{\Delta_0}u_a^+x_J\mathcal{H}_{\tilde A}$. Similarly, we have $M^\mu\cong\operatorname{Ind}_{\tilde B}^{\Delta_0}u_b^+x_I\mathcal{H}_{\tilde B}$. This is the key observation which allows us to apply Theorems 4.9 and 4.10 to $\mathfrak{H}_{\lambda,\mu}$.

$$\begin{aligned}\mathfrak{H}_{\lambda,\mu}&=\operatorname{Hom}_{\mathcal{H}}(M^\lambda,M^\mu)\\ &\cong\operatorname{Hom}_{\mathcal{H}}\Big(\operatorname{Ind}_{\tilde A}^{\Delta_0}u_a^+x_J\mathcal{H}_{\tilde A}\,,\,\operatorname{Ind}_{\tilde B}^{\Delta_0}u_b^+x_I\mathcal{H}_{\tilde B}\Big)\\ &\cong\operatorname{Hom}_{\mathcal{H}_{\tilde A}}\Big(u_a^+x_J\mathcal{H}_{\tilde A}\,,\,\operatorname{Res}_{\tilde A}^{\Delta_0}\operatorname{Ind}_{\tilde B}^{\Delta_0}u_b^+x_I\mathcal{H}_{\tilde B}\Big)\\ &\cong\bigoplus_{d\in\mathcal{D}_{\tilde B,\tilde A}}\operatorname{Hom}_{\mathcal{H}_{\tilde A}}\Big(u_a^+x_J\mathcal{H}_{\tilde A}\,,\,\operatorname{Ind}_{\tilde Bd\cap\tilde A}^{\tilde A}\operatorname{Res}_{\tilde Bd\cap\tilde A}^{\tilde Bd}u_b^+x_I\mathcal{H}_{\tilde B}T_d\Big)\\ &\cong\bigoplus_{d\in\mathcal{D}_{\tilde B,\tilde A}}\operatorname{Hom}_{\mathcal{H}_{\tilde Bd\cap\tilde A}}\Big(\operatorname{Res}_{\tilde Bd\cap\tilde A}^{\tilde A}u_a^+x_J\mathcal{H}_{\tilde A}\,,\,\operatorname{Res}_{\tilde Bd\cap\tilde A}^{\tilde Bd}u_b^+x_I\mathcal{H}_{\tilde B}T_d\Big)\end{aligned}$$

□

Recall that $\Delta=\Delta_0\setminus\{\alpha_0\}$.

Lemma 4.20 *Suppose that $d\in\mathcal{D}_{\tilde B,\tilde A}$ and let*

$$\mathfrak{H}^{(d)}_{\lambda,\mu}=\operatorname{Hom}_{\mathcal{H}_{\tilde Bd\cap\tilde A}}\Big(\operatorname{Res}_{\tilde Bd\cap\tilde A}^{\tilde A}u_a^+x_J\mathcal{H}_{\tilde A}\,,\,\operatorname{Res}_{\tilde Bd\cap\tilde A}^{\tilde Bd}u_b^+x_I\mathcal{H}_{\tilde B}T_d\Big).$$

Then

$$\mathfrak{H}^{(d)}_{\lambda,\mu}\cong\bigoplus_{v\in\mathcal{D}^A_{J,Bd\cap A}}\operatorname{Hom}_{\mathcal{H}_{S(v)}}\Big(\operatorname{Res}^{Jv}_{S(v)}x_JT_vR\,,\,\operatorname{Res}^{Bd}_{S(v)}(\operatorname{Ind}_I^B x_IR)T_d\Big),$$

where $S(v)=Bd\cap A\cap Jv\subseteq\Delta$.

Proof: If $\alpha_0\in\tilde Bd\cap\tilde A$ then T_0 acts on the modules $u_a^+x_J\mathcal{H}_{\tilde A}$ and $u_b^+x_I\mathcal{H}_{\tilde B}T_d$ as multiplication by Q, by Lemma 4.15; so, every $\mathcal{H}_{Bd\cap A}$–linear map between these modules is automatically $\mathcal{H}_{\tilde Bd\cap\tilde A}$–linear. If $\alpha_0\notin\tilde Bd\cap\tilde A$ then $\tilde Bd\cap\tilde A=Bd\cap A\subseteq\Delta$ by (4.13). Therefore, in either case,

$$\mathfrak{H}^{(d)}_{\lambda,\mu}=\operatorname{Hom}_{\mathcal{H}_{Bd\cap A}}\Big(\operatorname{Res}_{Bd\cap A}^{\tilde A}u_a^+x_J\mathcal{H}_{\tilde A}\,,\,\operatorname{Res}_{Bd\cap A}^{\tilde Bd}u_b^+x_I\mathcal{H}_{\tilde B}T_d\Big).$$

Now, by [10, 3.6], $u_a^+x_J\mathcal{H}_{\tilde A}\cong x_J\mathcal{H}_A$ as $\mathcal{H}_A$–modules and, similarly, $u_b^+x_I\mathcal{H}_{\tilde B}T_d\cong x_I\mathcal{H}_BT_d$ as $T_d^{-1}\mathcal{H}_BT_d$–modules. Therefore, by Theorems 4.10 and 4.9,

$$\begin{aligned}\mathfrak{H}^{(d)}_{\lambda,\mu}&\cong\operatorname{Hom}_{\mathcal{H}_{Bd\cap A}}\Big(\operatorname{Res}^A_{Bd\cap A}x_J\mathcal{H}_A\,,\,\operatorname{Res}^{Bd}_{Bd\cap A}x_I\mathcal{H}_BT_d\Big)\\ &=\operatorname{Hom}_{\mathcal{H}_{Bd\cap A}}\Big(\operatorname{Res}^A_{Bd\cap A}\operatorname{Ind}^A_Jx_JR\,,\,\operatorname{Res}^{Bd}_{Bd\cap A}x_I\mathcal{H}_BT_d\Big)\\ &\cong\bigoplus_{v\in\mathcal{D}^A_{J,Bd\cap A}}\operatorname{Hom}_{\mathcal{H}_{Bd\cap A}}\Big(\operatorname{Ind}^{Bd\cap A}_{S(v)}\operatorname{Res}^{Jv}_{S(v)}x_JT_vR\,,\,\operatorname{Res}^{Bd}_{Bd\cap A}x_I\mathcal{H}_BT_d\Big),\\ &\cong\bigoplus_{v\in\mathcal{D}^A_{J,Bd\cap A}}\operatorname{Hom}_{S(v)}\Big(\operatorname{Res}^{Jv}_{S(v)}x_JT_vR\,,\,\operatorname{Res}^{Bd\cap A}_{S(v)}\operatorname{Res}^{Bd}_{Bd\cap A}x_I\mathcal{H}_BT_d\Big).\end{aligned}$$

The lemma now follows from the transitivity of restriction and the observation that $\operatorname{Ind}_I^B x_I R \cong x_I \mathcal{H}_B$. □

Lemma 4.21 *Suppose that $d \in \mathcal{D}_{\tilde{B},\tilde{A}}$ and $v \in \mathcal{D}^A_{J,Bd\cap A}$ and let*

$$\mathfrak{H}^{(d,v)}_{\lambda,\mu} = \operatorname{Hom}_{\mathcal{H}_{S(v)}}\Big(\operatorname{Res}^{Jv}_{S(v)} x_J T_v R\,,\, \operatorname{Res}^{Bd}_{S(v)}(\operatorname{Ind}^B_I x_I R)T_d\Big).$$

Then

$$\mathfrak{H}^{(d,v)}_{\lambda,\mu} \cong \bigoplus_{u\in\mathcal{D}^B_{I,B\cap Jvd^{-1}}} \operatorname{Hom}_{\mathcal{H}_{S[u]}}\Big(\operatorname{Res}^{Jv}_{S[u]} x_J T_v R\,,\, \operatorname{Res}^{Iud}_{S[u]} x_I T_{ud} R\Big),$$

where $S[u] = S(v)\cap Iud$.

Proof: We observe that $S(v)d^{-1} = B\cap Ad^{-1}\cap Jvd^{-1} \subseteq \Delta$ determines the parabolic subgroup $W_B \cap d(W_A\cap v^{-1}W_J v)d^{-1}$ of W_r by Lemma 4.5(ii) since $d^{-1}\in\mathcal{D}_{\tilde{A},\tilde{B}} \subseteq \mathcal{D}_{A\cap Jv,B}$. Therefore,

$$\begin{aligned}
\mathfrak{H}^{(d,v)}_{\lambda,\mu} &= \operatorname{Hom}_{\mathcal{H}_{S(v)}}\Big(\operatorname{Res}^{Jv}_{S(v)} x_J T_v R\,,\, \operatorname{Res}^{Bd}_{S(v)}(\operatorname{Ind}^B_I x_I R)T_d\Big)\\
&\cong \operatorname{Hom}_{\mathcal{H}_{S(v)}}\Big(\operatorname{Res}^{Jv}_{S(v)} x_J T_v R\,,\, (\operatorname{Res}^{B}_{S(v)d^{-1}}\operatorname{Ind}^B_I x_I R)T_d\Big)\\
&\cong \bigoplus_u \operatorname{Hom}_{\mathcal{H}_{S(v)}}\Big(\operatorname{Res}^{Jv}_{S(v)} x_J T_v R\,,\, (\operatorname{Ind}^{S(v)d^{-1}}_{S[u]d^{-1}}\operatorname{Res}^{Iu}_{S[u]d^{-1}} x_I T_u R)T_d\Big),
\end{aligned}$$

observing that $S[u]d^{-1} = S(v)d^{-1}\cap Iu$, where the last isomorphism follows by the Mackey theorem 4.10, and u runs through $\mathcal{D}^B_{I,S(v)d^{-1}}$. Now

$$S(v)d^{-1} = B\cap Ad^{-1}\cap Jvd^{-1} = B\cap Jvd^{-1}$$

by Lemma 4.18(i) and $ud\in\mathcal{D}^B_{I,S(v)}$ by Lemma 4.17(ii). Therefore

$$d^{-1}W_{S(v)d^{-1}}d = W_{S(v)}.$$

Moreover $x_I T_u T_d R$ is a $T_{d^{-1}}T_{u^{-1}}\mathcal{H}_I T_u T_d$–module, and we have

$$(\operatorname{Ind}^{S(v)d^{-1}}_{S[u]d^{-1}}\operatorname{Res}^{Iu}_{S[u]d^{-1}} x_I T_u R)T_d = \operatorname{Ind}^{S(v)}_{S[u]}\operatorname{Res}^{Iud}_{S[u]}(x_I T_{ud} R),$$

since $T_u T_d = T_{ud}$ by Lemma 4.17(i). We have shown

$$\mathfrak{H}^{(d,v)}_{\lambda,\mu} \cong \bigoplus_{u\in\mathcal{D}^B_{I,B\cap Jvd^{-1}}} \operatorname{Hom}_{\mathcal{H}_{S(v)}}\Big(\operatorname{Res}^{Jv}_{S(v)} x_J T_v R\,,\, \operatorname{Ind}^{S(v)}_{S[u]}\operatorname{Res}^{Iud}_{S[u]} x_I T_{ud} R\Big).$$

So, Frobenius reciprocity and the transitivity of restriction complete the proof. □

Combining the last three lemmas we have shown that $\mathfrak{H}_{\lambda,\mu} \cong \bigoplus \mathfrak{H}_{\lambda,\mu}^{(d,v,u)}$ where the sum is over all admissible triples (d,v,u) and

$$\mathfrak{H}_{\lambda,\mu}^{(d,v,u)} = \mathrm{Hom}_{\mathcal{H}_{S[u]}}\Big(\mathrm{Res}^{Jv}_{S[u]}\, x_J T_v R\,,\ \mathrm{Res}^{Iud}_{S[u]}\, x_I T_{ud} R\Big).$$

Now, $\mathfrak{H}_{\lambda,\mu}^{(d,v,u)}$ is a free R–module of rank 1, since both modules involved are one dimensional. As a nonzero element in $\mathfrak{H}_{\lambda,\mu}^{(d,v,u)}$ we may choose

$$\phi_{I,J}^{(d,v,u)} : x_J T_v \mapsto x_I T_{ud}.$$

By Theorem 4.9, in the R–module $\mathfrak{H}_{\lambda,\mu}^{(d)}$ of Lemma 4.20, this map corresponds to the map (compare also [7, 3.4])

$$\hat{\phi}_{I,J}^{(d,v,u)} \in \mathrm{Hom}_{\mathcal{H}_{Bd\cap A}}\Big(\mathrm{Res}^{A}_{Bd\cap A}\, x_J \mathcal{H}_A\,,\ \mathrm{Res}^{Bd}_{Bd\cap A}\, x_I \mathcal{H}_B T_d\Big),$$

given by (noting that $ud \in \mathcal{D}^B_{I,S(v)}$),

$$\hat{\phi}_{I,J}^{(d,v,u)}(x_J T_v h) = \sum_{w\in W_I ud W_{S(v)}} T_w h \tag{4.22}$$

where $h \in \mathcal{H}_{Bd\cap A}$. The $\mathcal{H}_{Bd\cap A}$–direct summands of

$$\mathrm{Res}^{A}_{Bd\cap A}\, x_J \mathcal{H}_A = \bigoplus_{v'\in\mathcal{D}^A_{J,Bd\cap A}} \mathrm{Ind}^{Bd\cap A}_{S(v')}\, \mathrm{Res}^{Jv'}_{S(v')}\, x_J T_{v'} R \tag{4.23}$$

with $v' \neq v$ are taken to zero by $\hat{\phi}_{I,J}^{(d,v,u)}$.

We denote the image of $q^{\ell(v)}\hat{\phi}_{I,J}^{(d,v,u)}$ in $\mathfrak{H}_{\lambda,\mu}$ given by Lemmas 4.20 and 4.19 by $\varphi_{\mu,\lambda}^{(d,v,u)}$. (The power of q is inserted here because in Lemma 4.20 we used part (ii) of Corollary 4.11; see Remark 4.12.) We have proved the following.

Theorem 4.24 *Let $\lambda,\mu \in \Lambda_2(n,r)$. Then $\mathfrak{H}_{\lambda,\mu}$ is a free R–module with basis*

$$\mathfrak{B}_{\lambda,\mu} = \{\varphi_{\mu,\lambda}^{(d,v,u)} \mid (d,v,u) \text{ an admissible triple for } (\lambda,\mu)\}.$$

Corollary 4.25 *The (Q,q)-Schur algebra $\mathcal{S} = \mathcal{S}_R(n,r)$ is a free R–module with basis*

$$\mathfrak{B} = \{\varphi_{\mu,\lambda}^{(d,v,u)} \mid \lambda,\mu \in \Lambda_2(n,r)\,,\ \text{and } (d,v,u) \text{ an admissible triple for } (\lambda,\mu)\}$$

which does not depend on the commutative ring R or the choice of the parameter values Q and q.

We next describe the μ–bitabloids which do not belong to M^μ_- (see Definition 3.13) and are in the support of the image of $\{\mathfrak{t}^\lambda\} = u_a^+ x_J$ under the homomorphism $\varphi^{(d,v,u)}_{\mu,\lambda}$ for an admissible triple (d,v,u) for (λ,μ).

Theorem 4.26 *Let (d,v,u) be an admissible triple for (λ,μ) and let $c = udv^{-1}$. Then $c \in \mathcal{D}_{I,J}$ and*

$$\varphi^{(d,v,u)}_{\mu,\lambda}(\{\mathfrak{t}^\lambda\}) \equiv u_b^+ \psi^c_{I,J}(x_J) \equiv \sum_{w\in W_I c W_J} u_b^+ T_w \pmod{M^\mu_-}.$$

Proof: We have already seen in Lemma 4.17 that $c = udv^{-1}$ is an element of $\mathcal{D}_{I,J}$.

In (4.22) we constructed the $\mathcal{H}_{Bd\cap A}$–linear map $\hat{\phi}^{(d,v,u)}_{I,J}$ in $\mathfrak{H}^{(d)}_{\lambda,\mu}$, and determined the image of $x_J T_v h$ for $h \in \mathcal{H}_{Bd\cap A}$ under this map.

We apply Lemma 4.15 (compare Lemma 4.20) to get an $\mathcal{H}_{\tilde{B}d\cap\tilde{A}}$–linear map $\hat{\varphi}^{(d,v,u)}_{I,J}$: $\operatorname{Res}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}} u_a^+ x_J \mathcal{H}_{\tilde{A}} \longrightarrow \operatorname{Res}^{\tilde{B}.d}_{\tilde{B}d\cap\tilde{A}} u_b^+ x_I \mathcal{H}_{\tilde{B}} T_d$ such that

$$\hat{\varphi}^{(d,v,u)}_{I,J}(u_a^+ x_J T_v h) = \sum_{w\in W_I u d W_{S(v)}} u_b^+ T_w h \tag{4.27}$$

for $h \in \mathcal{H}_{\tilde{B}d\cap\tilde{A}}$.

According to Lemma 4.19 we have to trace up this map using Theorem 4.9(ii), to get a map

$$\begin{aligned}\hat{\varphi}^{(d,v,u)}_{\mu,\lambda} &\in \operatorname{Hom}_{\mathcal{H}_{\tilde{A}}}\Big(u_a^+ x_J \mathcal{H}_{\tilde{A}}\,,\ \operatorname{Ind}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}} \operatorname{Res}^{\tilde{B}d}_{\tilde{B}d\cap\tilde{A}} u_b^+ x_I \mathcal{H}_{\tilde{B}} T_d\Big)\\ &\cong \operatorname{Hom}_{\mathcal{H}_{\tilde{B}d\cap\tilde{A}}}\Big(\operatorname{Res}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}} u_a^+ x_J \mathcal{H}_{\tilde{A}}\,,\ \operatorname{Res}^{\tilde{B}d}_{\tilde{B}d\cap\tilde{A}} u_b^+ x_I \mathcal{H}_{\tilde{B}} T_d\Big).\end{aligned}$$

Let $m \in u_a^+ x_J \mathcal{H}_{\tilde{A}}$. Then by Theorem 4.9 we have

$$\hat{\varphi}^{(d,v,u)}_{\mu,\lambda}(m) = \sum_{f\in\mathcal{D}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}}} \underline{q}^{-\underline{\ell}(f)} \hat{\varphi}^{(d,v,u)}_{I,J}(mT_f^*)T_f.$$

By (4.13), $\tilde{B}d \cap \tilde{A} \cap \Delta = Bd \cap A$; so $\mathcal{D}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}} \cap W_\Delta = \mathcal{D}^{A}_{Bd\cap A}$. Therefore, we may rewrite the last equation as

$$\hat{\varphi}^{(d,v,u)}_{\mu,\lambda}(m) = \sum_{f\in\mathcal{D}^{A}_{Bd\cap A}} q^{-\ell(f)} \hat{\varphi}^{(d,v,u)}_{I,J}(mT_f^*)T_f + \underline{z}. \tag{4.28}$$

where

$$\underline{z} = \sum_{\tilde{f}\in\mathcal{D}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}}\setminus W_\Delta} \underline{q}^{-\underline{\ell}(\tilde{f})} \hat{\varphi}^{(d,v,u)}_{I,J}(mT_{\tilde{f}}^*)T_{\tilde{f}}.$$

We claim that $\underline{z} \in M^\mu_-$. Observe that $\tilde{f} \in W_{\tilde{A}}$ and $d \in (\mathcal{D}_{\tilde{A}})^{-1}$, so $\ell(d\tilde{f}) = \ell(d) + \ell(\tilde{f})$. Moreover by (2.4), $u_b^+ x_I \mathcal{H}_{\tilde{B}} = u_b^+ x_I \mathcal{H}_B$; therefore

$$\underline{q}^{-\underline{\ell}(\tilde{f})} \hat{\varphi}^{(d,v,u)}_{I,J}(mT_{\tilde{f}}^*)T_{\tilde{f}} \in u_b^+ x_I \mathcal{H}_{\tilde{B}} T_{d\tilde{f}} = u_b^+ x_I \mathcal{H}_B T_{d\tilde{f}}.$$

Now $\tilde{f} \in \mathcal{D}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}}$ but $\tilde{f} \notin W_\Delta$; so, s_0 is involved in $\tilde{f}$, and in $d\tilde{f}$ as well, since $\ell(d\tilde{f}) = \ell(d) + \ell(\tilde{f})$. We apply [7, 1.4] and Lemmas 3.9 and 3.10 to conclude that $\underline{q}^{-\underline{\ell}(\tilde{f})}\hat{\varphi}^{(d,v,u)}_{I,J}(mT^*_{\tilde{f}})T_{\tilde{f}} \in M^\mu_-$. Consequently, $\underline{z} \in M^\mu_-$ and (4.28) becomes

$$\hat{\varphi}^{(d,v,u)}_{\mu,\lambda}(m) \equiv \sum_{f\in\mathcal{D}^A_{Bd\cap A}} q^{-\ell(f)}\hat{\varphi}^{(d,v,u)}_{I,J}(mT^*_f)T_f \pmod{M^\mu_-}. \tag{4.29}$$

We next investigate $\hat{\varphi}^{(d,v,u)}_{I,J}(mT^*_f)$ when $m = u^+_a x_J$. Fix $f \in \mathcal{D}^A_{Bd\cap A}$ and note that $f^{-1} \in \mathcal{D}^A_{\emptyset,Bd\cap A}$. Let v' be the distinguished double coset representative in $W_J f^{-1} W_{Bd\cap A}$. Then $v' \in \mathcal{D}^A_{J,Bd\cap A}$ and $f^{-1} = gv'$ for some $g \in \mathcal{D}^J_{(Bd\cap A)v'^{-1}\cap J}$. However, by (4.23), $\hat{\varphi}^{(d,v,u)}_{I,J}(u^+_a x_J T^*_f) = 0$ unless $v' = v$; therefore we assume that $f^{-1} = gv$ for some $g \in \mathcal{D}^J_{(Bd\cap A)v^{-1}\cap J}$. By Lemma 4.5(i), we have a disjoint union

$$W_J f^{-1} W_{Bd\cap A} = W_J v W_{Bd\cap A} = \bigcup_{g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}} gvW_{Bd\cap A},$$

recalling that $v \in \mathcal{D}^A_{J,Bd\cap A}$ and $S(v) = Bd\cap A\cap Jv$. Hence, modulo M^μ_-, we can rewrite (4.29) when $m = u^+_a x_J$ as

$$\begin{aligned}
\hat{\varphi}^{(d,v,u)}_{\mu,\lambda}(u^+_a x_J) &\equiv \sum_{g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}} q^{-\ell(gv)}\hat{\varphi}^{(d,v,u)}_{I,J}(u^+_a x_J T_{gv})T^*_{gv} \\
&\equiv \sum_{g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}} q^{-\ell(v)}\hat{\varphi}^{(d,v,u)}_{I,J}(u^+_a x_J T_v)T^*_{gv} \\
&\equiv \sum_{g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}} q^{-\ell(v)}u^+_b \sum_{w\in W_I udW_{S(v)}} T_w T^*_{gv}
\end{aligned}$$

where the second equality comes from the fact that $g \in W_J$ and the third equality from (4.27).

Using Theorem 4.9 we can lift the homomorphism $q^{\ell(v)}\hat{\varphi}^{(d,v,u)}_{\mu,\lambda}$ to the homomorphism $\varphi^{(d,v,u)}_{\mu,\lambda}$ in $\mathrm{Hom}_{\mathcal{H}}(M^\lambda, M^\mu)$ by extending it to the module $M^\lambda = \mathrm{Ind}^{\Delta_0}_{\tilde{A}}\, u^+_a x_J \mathcal{H}_{\tilde{A}}$. In particular the image of the generator $u^+_a x_J$ remains unchanged. Consequently, in order to complete the proof it suffices to prove the next lemma.

Lemma 4.30 *Suppose that (d, v, u) is an admissible triple and let $c = udv^{-1}$. Then*

$$\sum_{x\in W_I cW_J} T_x = \sum_{g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}} \sum_{y\in W_I udW_{S(v)}} T_y T^*_{gv}.$$

Proof: Since $c \in \mathcal{D}_{I,J}$, by Lemma 4.5(i)

$$\sum_{w\in W_I cW_J} T_w = \sum_{\substack{y\in W_J \\ x^{-1}\in\mathcal{D}^I_{I\cap Jc^{-1}}}} T_x T_c T_y = \sum_{x^{-1}\in\mathcal{D}^I_{I\cap Jc^{-1}}} T_x T_{ud} T^*_v \sum_{y\in W_J} T_y.$$

Recall that $S(v) = Bd \cap A \cap Jv$. Suppose that $f \in W_{S(v)v^{-1}} \subseteq W_J$, and let $e = v^{-1}fv$. Then $e \in W_{S(v)} \subseteq W_{Bd\cap A}$. Since $v \in \mathcal{D}_{J,Bd\cap A}$ we have $\ell(ev^{-1}) = \ell(e)+\ell(v^{-1})$ and $\ell(v^{-1}f) = \ell(v^{-1})+\ell(f)$; so,

$$T_eT_v^* = T_eT_{v^{-1}} = T_{v^{-1}}T_f = T_v^*T_f.$$

Therefore,

$$T_v^* \sum_{y\in W_J} T_y = \sum_{\substack{f\in W_{S(v)v^{-1}}\\ g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}}} T_v^*T_fT_g^* = \sum_{\substack{e\in W_{S(v)}\\ g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}}} T_eT_v^*T_g^*,$$

and consequently,

$$\begin{aligned}\sum_{w\in W_IcW_J} T_w &= \sum_{x^{-1}\in\mathcal{D}^I_{I\cap Jc^{-1}}} T_xT_{ud} \sum_{\substack{e\in W_{S(v)}\\ g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}}} T_eT_{gv}^*\\ &= \sum_{g^{-1}\in\mathcal{D}^J_{S(v)v^{-1}}} \Big(\sum_{\substack{e\in W_{S(v)}\\ x^{-1}\in\mathcal{D}^I_{I\cap Jc^{-1}}}} T_xT_{ud}T_e\Big)T_{gv}^*.\end{aligned}$$

Now $ud \in \mathcal{D}_{I,S(v)}$ by Lemma 4.17(ii) so by Lemma 4.5(i) it suffices to show that $I \cap S(v)d^{-1}u^{-1} = I\cap Jc^{-1}$. However,

$$\begin{aligned}I\cap S(v)d^{-1}u^{-1} &= I\cap(B\cap Ad^{-1}\cap Jvd^{-1})u^{-1}\\ &= I\cap(B\cap Jvd^{-1})u^{-1}\\ &= I\cap Jvd^{-1}u^{-1},\end{aligned}$$

by Lemma 4.18(i) and (ii) respectively. □

Remark 4.31 *In the group algebra case, that is specializing both Q and q to 1, one sees easily that*

$$u_a^+x_J = \sum_{w\in W_\lambda} w \in RW_r.$$

In this case $\varphi^{(d,v,u)}_{\mu,\lambda}$ takes the generator $u_a^+x_J$ of the module M^λ to

$$\varphi^{(d,v,u)}_{\mu,\lambda}(u_a^+x_J) = \sum_{w\in W_\lambda cW_\mu} w,$$

where $c = udv^{-1}$ and the subgroups W_λ and W_μ are defined in (2.9).

By (4.14), $\mathcal{D}_{\tilde B,\tilde A}\cap W_\Delta = \mathcal{D}^\Delta_{B,A}$. Therefore, $\mathfrak{H}_{\lambda,\mu}$ in (4.19) splits into two R–modules $\mathfrak{H}^+_{\lambda,\mu}$ and $\mathfrak{H}^-_{\lambda,\mu}$ where

$$\mathfrak{H}^+_{\lambda,\mu} = \bigoplus_{d\in\mathcal{D}^\Delta_{B,A}} \mathrm{Hom}_{\mathcal{H}_{\tilde Bd\cap\tilde A}}\Big(\mathrm{Res}^{\tilde A}_{\tilde Bd\cap\tilde A}\, u_a^+x_J\mathcal{H}_{\tilde A}\,,\, \mathrm{Res}^{\tilde Bd}_{\tilde Bd\cap\tilde A}\, u_b^+x_I\mathcal{H}_{\tilde B}T_d\Big)$$

and

$$\mathfrak{H}^-_{\lambda,\mu} = \bigoplus_{d\in\mathcal{D}_{B,A}\setminus W_\Delta} \operatorname{Hom}_{\mathcal{H}_{\tilde{B}d\cap\tilde{A}}}\Big(\operatorname{Res}^{\tilde{A}}_{\tilde{B}d\cap\tilde{A}} u_a^+ x_J\mathcal{H}_{\tilde{A}}\,,\ \operatorname{Res}^{\tilde{B}d}_{\tilde{B}d\cap\tilde{A}} u_b^+ x_I\mathcal{H}_{\tilde{B}}T_d\Big).$$

We apply Lemma 4.15 and Theorem 4.9 again to get

$$\begin{aligned}\mathfrak{H}^+_{\lambda,\mu} &\cong \bigoplus_{d\in\mathcal{D}^\Delta_{B,A}} \operatorname{Hom}_{\mathcal{H}_{Bd\cap A}}\Big(\operatorname{Res}^{A}_{Bd\cap A} u_a^+ x_J\mathcal{H}_A\,,\ \operatorname{Res}^{Bd}_{Bd\cap A} u_b^+ x_I\mathcal{H}_B T_d\Big)\\ &\cong \bigoplus_{d\in\mathcal{D}^\Delta_{B,A}} \operatorname{Hom}_{\mathcal{H}_A}\Big(u_a^+ x_J\mathcal{H}_A\,,\ \operatorname{Ind}^A_{Bd\cap A}\operatorname{Res}^{Bd}_{Bd\cap A} u_b^+ x_I\mathcal{H}_B T_d\Big)\\ &\cong \operatorname{Hom}_{\mathcal{H}_\Delta}\Big(x_J\mathcal{H}_\Delta\,,\ x_I\mathcal{H}_\Delta\Big).\end{aligned}$$

The last Hom–set has basis $\mathfrak{B}_\Delta = \{\tilde\varphi^c_{I,J} \mid c\in\mathcal{D}^\Delta_{I,J}\}$, where the map $\tilde\varphi^c_{I,J}$ is defined by

$$\tilde\varphi^c_{I,J}(x_J h) = \sum_{w\in W_I cW_J} T_w h \qquad (h\in\mathcal{H}_\Delta).$$

We observe that the map $\tilde\varphi^c_{I,J}$ involves the same double coset W_IcW_J as the map $\psi^c_{I,J}$ in Theorem 4.26. Moreover, by Lemma 4.19 and Theorem 4.24,

$$\mathfrak{B}^+_{\lambda,\mu} = \{\varphi^{(d,v,u)}_{\mu,\lambda}\in\mathfrak{B}_{\lambda,\mu} \mid d\in\mathcal{D}^\Delta_{B,A} \text{ and } (d,v,u) \text{ admissible for } (\lambda,\mu)\}$$

is also a basis of $\mathfrak{H}^+_{\lambda,\mu}$; so $\mathfrak{B}^+_{\lambda,\mu}$ and $\mathfrak{B}_\Delta$ have the same cardinality. This proves the next lemma (it is also easily checked directly).

Lemma 4.32 *Let $\lambda,\mu\in\Lambda_2(n,r)$ and let I and J be as in* (4.14). *Then*

$$\mathcal{D}^\Delta_{I,J} = \{udv^{-1} \mid (d,v,u) \text{ an admissible triple for } (\lambda,\mu) \text{ and } d\in\mathcal{D}^\Delta_{B,A}\}$$

In view of the last result we make the following definition.

Definition 4.33 *Given $c\in\mathcal{D}^\Delta_{I,J}$ let $\varphi^c_{\mu,\lambda}\in\mathfrak{B}_{\lambda,\mu}$ be the map $\varphi^{(d,v,u)}_{\mu,\lambda}$ where (d,v,u) is the admissible triple such that $c=udv^{-1}$.*

Observe that Theorem 4.26 implies that for an admissible triple (d,v,u) and $c=udv^{-1}$

(4.34) $$\varphi^{(d,v,u)}_{\mu,\lambda}(\{\mathfrak{t}^\lambda\}) \equiv \sum_{b\in\mathcal{D}^J_{I^c\cap J}} \{\mathfrak{t}^\mu\}T_{cb} \pmod{M^\mu_-}.$$

From this the first part of the next result follows easily. The second part is a special case of Theorem 4.26.

Corollary 4.35 *Let (d,v,u) be admissible, and let $c=udv^{-1}$.*

(i) *If $c\notin\mathcal{D}^\Delta_{I,J}$ then $\varphi^{(d,v,u)}_{\mu,\lambda}(\{\mathfrak{t}^\lambda\})\equiv 0 \pmod{M^\mu_-}$.*
(ii) *If $c\in\mathcal{D}^\Delta_{I,J}$, then*

$$\varphi^c_{\mu,\lambda}(\{\mathfrak{t}^\lambda\}) \equiv u_b^+\sum_{w\in W_IcW_J} T_w \pmod{M^\mu_-}.$$

5 (Q,q)–tensor space and (Q,q)–Weyl modules

In view of Lemma 4.3 we assume hereafter that $n \geq r$. We denote the bicomposition $((-),(1^r))$ by ω. Thus $\omega \in \Lambda_2(n,r)$, since $n \geq r$.

Definition 5.1 *(Q,q)–**tensor space** is the R–module*

$$\mathcal{E} = \mathcal{E}_R(n,r) = \bigoplus_{\lambda\in\Lambda_2(n,r)} M^\lambda.$$

Definition 5.2 *Let λ be an a–bicomposition in $\Lambda_2(n,r)$.*

(i) *The homomorphism $\varphi_{\lambda,\lambda}$ is the identity map on M^λ and maps M^μ to zero when $\lambda \neq \mu$.*
(ii) *The homomorphism $\varphi_{\lambda,\omega}$ is premultiplication of $\mathcal{H} = M^\omega$ by $u_a^+ x_\lambda$ and maps M^μ to zero when $\mu \neq \omega$.*

Thus $\varphi_{\lambda,\lambda}$ is the projection of $\mathcal{E}$ onto M^λ and $\varphi_{\lambda,\omega}$ is the canonical epimorphism of $\mathcal{H}$ onto the cyclic $\mathcal{H}$–module M^λ. In particular $\varphi_{\omega,\omega}$ is the identity on $\mathcal{H}$ and maps M^λ to zero for $\lambda \neq \omega$. From the proof of Theorem 4.26, using in particular (4.28), one sees easily that the following holds.

Lemma 5.3 *Let λ be a bicomposition.*

(i) *The restriction of $\varphi_{\lambda,\lambda}$ to M^λ equals $\varphi^{(1,1,1)}_{\lambda,\lambda}$.*
(ii) *The restriction of $\varphi_{\lambda,\omega}$ to $\mathcal{H}$ equals $\varphi^{(1,1,1)}_{\lambda,\omega}$.*

By construction $\{\varphi_{\lambda,\lambda} \mid \lambda \in \Lambda_2(n,r)\}$ is a set of orthogonal idempotents of $\mathcal{S}$ whose sum is the identity element of $\mathcal{S}$. Observe that

$$\varphi_{\lambda,\lambda}\varphi_{\lambda,\omega} = \varphi_{\lambda,\omega} = \varphi_{\lambda,\omega}\varphi_{\omega,\omega}, \tag{5.4}$$

hence, as in [9, 2.5], we see that postmultiplication by $\varphi_{\lambda,\omega}$ embeds the left ideal $\mathcal{S}\varphi_{\lambda,\lambda}$ of $\mathcal{S}$ in the left ideal $\mathcal{S}\varphi_{\omega,\omega}$. Hence, $\mathcal{S}$ acts faithfully on $\mathcal{S}\varphi_{\omega,\omega}$ and, when R is a field, every irreducible left $\mathcal{S}$–module occurs as a composition factor of the left ideal $\mathcal{S}\varphi_{\omega,\omega}$ of $\mathcal{S}$. We have the following (compare [9, 2.10 and 2.6]).

Lemma 5.5 *The Hecke algebra $\mathcal{H}$ is canonically isomorphic to $\varphi_{\omega,\omega}\mathcal{S}\varphi_{\omega,\omega}$ and acts on $\mathcal{S}\varphi_{\omega,\omega}$ as a set of $\mathcal{S}$–linear maps. The $\mathcal{H}$–submodule $\varphi_{\lambda,\omega}\mathcal{H}$ of $\mathcal{S}\varphi_{\omega,\omega}$ is isomorphic to M^λ and (Q,q)–tensor space $\mathcal{E}$ is isomorphic to the left ideal $\mathcal{S}\varphi_{\omega,\omega}$ of $\mathcal{S}$ as an $(\mathcal{S},\mathcal{H})$–bimodule.*

Proof: We have canonical isomorphisms

$$\mathcal{H} \cong \mathrm{End}_{\mathcal{H}}(\mathcal{H}) \cong \mathrm{Hom}_{\mathcal{H}}(x_\omega\mathcal{H}\,,\, x_\omega\mathcal{H}) \cong \varphi_{\omega,\omega}\mathcal{S}\varphi_{\omega,\omega}.$$

Identifying $\mathcal{H}$ and $\varphi_{\omega,\omega}\mathcal{S}\varphi_{\omega,\omega}$ we see that $\mathcal{H}$ acts on $\mathcal{S}\varphi_{\omega,\omega}$ on the right as a set of $\mathcal{S}$–linear maps. Also, $\varphi_{\lambda,\omega}h \mapsto u_a^+ x_\lambda h$ for $h \in \mathcal{H}$ gives an $\mathcal{H}$–isomorphism between $\varphi_{\lambda,\omega}\mathcal{H}$ and M^λ.

By premultiplying $\mathcal{S}\varphi_{\omega,\omega}$ by $\sum_{\lambda\in\Lambda_2(n,r)} \varphi_{\lambda,\lambda}$, which is the identity of $\mathcal{S}$, we obtain

$$\mathcal{S}\varphi_{\omega,\omega} = \bigoplus_{\lambda\in\Lambda_2(n,r)} M^\lambda = \mathcal{E}. \qquad \square$$

Definition 5.6 *Let U be a left $\mathcal{S}$–module, and let $\lambda \in \Lambda_2(n,r)$. Then the R–submodule $U_\lambda = \varphi_{\lambda,\lambda}U$ of U is the* **weight space** *of U of* **weight** *λ.*

Note that $\varphi_{\lambda,\lambda}U$ is free as an R–module. We have

$$U = \bigoplus_{\lambda \in \Lambda_2(n,r)} \varphi_{\lambda,\lambda}U, \tag{5.7}$$

since $\{\varphi_{\lambda,\lambda} \mid \lambda \in \Lambda_2(n,r)\}$ is a set of orthogonal idempotents whose sum is the identity of $\mathcal{S}$. This decomposition of U is called the **weight space decomposition** of U, (compare [9, 2.13]).

Lemma 5.8 *The weight space decomposition of (Q,q)–tensor space $\mathcal{E}$ is given as*

$$\mathcal{E} = \bigoplus_{\lambda \in \Lambda_2(n,r)} M^\lambda.$$

More generally, if U is any R–submodule of $\mathcal{E}$ then its weight space decomposition is

$$U = \bigoplus_{\lambda \in \Lambda_2(n,r)} U \cap M^\lambda.$$

Let $\lambda \in \Lambda_2(n,r)$ and recall the definition of z_λ from (2.16). With the above identifications we have

$$z_\lambda = \varphi_{\lambda,\omega} h_{a,r-a} T_{\hat{\pi}_\lambda} u^-_{r-a} \hat{y}_\lambda. \tag{5.9}$$

Contrast the next definition with our definition of the Specht module $S^\lambda = z_\lambda \mathcal{H}$.

Definition 5.10 *Suppose that λ is an a–bicomposition of r.*

(i) *Let $W^\lambda = \mathcal{S} z_\lambda$.*
(ii) *Let $L^\lambda = \mathcal{S} u^-_{r-a} \hat{y}_\lambda$.*

*We call W^λ a (Q,q)***-Weyl module**.

As in [9, 3.9] we have the following.

5.11 *If λ and μ are associated bicompositions then $W^\lambda \cong W^\mu$ and $L^\lambda \cong L^\mu$.*

We next define a bilinear form $\langle\ ,\ \rangle_\lambda$ on M^λ by specifying its values at pairs of basis elements, which are given in (3.8) by λ–bitabloids. For such a pair $\{\mathfrak{t}\}, \{\mathfrak{t}'\}$, where $\mathfrak{t} = \mathfrak{t}^\lambda w$ is row standard and $w \in W_r$, we set

$$\langle \{\mathfrak{t}\}, \{\mathfrak{t}'\} \rangle_\lambda = \begin{cases} \underline{q}^{\ell(w)} & \text{if } \{\mathfrak{t}\} = \{\mathfrak{t}'\} \\ 0 & \text{if } \{\mathfrak{t}\} \neq \{\mathfrak{t}'\}. \end{cases} \tag{5.12}$$

Extend these bilinear forms to a non–degenerate bilinear form $\langle\ ,\ \rangle$ on (Q,q)–tensor space $\mathcal{E}=\bigoplus M^\lambda$ by specifying that M^λ and M^μ are orthogonal when $\lambda\neq\mu$.

Recall that the anti–automorphism * on $\mathcal{H}$ is defined by $T_w^*=T_{w^{-1}}$ for $w\in W_r$. As in [9, 4.1] for all $h\in\mathcal{H}$, $x,y\in\mathcal{E}$

$$\langle xh,y\rangle=\langle x,yh^*\rangle, \tag{5.13}$$

and consequently $\mathcal{E}$ is self–dual as an $\mathcal{H}$–module. Here $M^*=\operatorname{Hom}_{\mathcal{H}}(M,R)$ is the dual of an $\mathcal{H}$–module M, the right action of $\mathcal{H}$ on M^* being given by $fh(x)=f(xh^*)$, for $f\in M^*$, $h\in\mathcal{H}$ and $x\in M$. The discussion in [9, section 1] shows that $*$ extends to an anti–isomorphism of $\mathcal{S}$, which we also denote by $*$. This allows us to define the dual of a left $\mathcal{S}$–module. We then have the following.

Lemma 5.14 *Let $x,y\in\mathcal{E}$, $h\in\mathcal{H}$ and $s\in\mathcal{S}$. Then*

(i) $$\langle sx,y\rangle=\langle x,s^*y\rangle$$

(ii) $$\langle xh,y\rangle=\langle x,yh^*\rangle$$

In particular, (Q,q)–tensor space $\mathcal{E}$ is self–dual as a left $\mathcal{S}$–module.

Since $(u_{r-a}^-\hat{y}_\lambda)^*=u_{r-a}^-\hat{y}_\lambda$ we can contract the bilinear form which is given by restricting $\langle\ ,\ \rangle$ to L^λ to obtain a bilinear form $\ll\ ,\ \gg$ on L^λ such that

$$\begin{aligned}\ll s_1u_{r-a}^-\hat{y}_\lambda\,,\,s_2u_{r-a}^-\hat{y}_\lambda\gg&=\ll s_1\varphi_{\omega,\omega}u_{r-a}^-\hat{y}_\lambda\,,\,s_2\varphi_{\omega,\omega}u_{r-a}^-\hat{y}_\lambda\gg\\&=\langle s_1\varphi_{\omega,\omega}\,,\,s_2u_{r-a}^-\hat{y}_\lambda\rangle,\end{aligned}$$

for all $s_1,s_2\in\mathcal{S}$, (compare [14, section 5.5] and [9, 4.2]). The proof of [9, 4.4] now gives the following theorem.

Theorem 5.15 (The Submodule Theorem) *Suppose that R is a field and $\lambda\in\Lambda_2(n,r)$. Let U be an $\mathcal{S}$–submodule of L^λ. Then $W^\lambda\subseteq U$ or $U\subseteq W^{\lambda\perp}$.*

Corollary 5.16 *Suppose that R is a field and $\lambda\in\Lambda_2(n,r)$. Then $W^\lambda\cap W^{\lambda\perp}$ is the unique maximal submodule of W^λ, and the quotient $W^\lambda/(W^\lambda\cap W^{\lambda\perp})$ is an absolutely irreducible self–dual $\mathcal{S}$–module.*

Proof: The corollary will be an immediate consequence of the Submodule Theorem once we prove that the generator z_λ of W^λ is anisotropic. We have using (5.9) and (3.8)

$$\begin{aligned}\ll z_\lambda,z_\lambda\gg&=\langle\varphi_{\lambda,\omega}h_{a,r-a}T_{\hat{\pi}_\lambda}\,,\,\varphi_{\lambda,\omega}h_{a,r-a}T_{\hat{\pi}_\lambda}u_{r-a}^-\hat{y}_\lambda\rangle\\&=\langle\{\mathsf{t}\}\,,\,\{\mathsf{t}\}u_{r-a}^-\hat{y}_\lambda\rangle,\end{aligned}$$

where $\mathsf{t}=\hat{\mathsf{t}}_\lambda=\mathsf{t}^\lambda w_{a,r-a}\hat{\pi}_\lambda$ (see Definitions 2.8 and 2.14). Hence , by Corollary 3.15,

$$\ll z_\lambda,z_\lambda\gg=Q^{r-a}q^{(r-a)(r-a-1)/2}\langle\{\mathsf{t}\},\{\mathsf{t}\}\hat{y}_\lambda\rangle.$$

But $\{\mathtt{t}\}\hat{y}_\lambda = \{\mathtt{t}\} + v$, where v is a linear combination of λ–bitabloids distinct from $\{\mathtt{t}\}$ (cf. [7, 4.1]). Therefore

$$\ll z_\lambda, z_\lambda \gg = Q^{r-a} q^{(r-a)(r-a-1)/2} \langle \{\mathtt{t}\}, \{\mathtt{t}\} \rangle \neq 0.$$

□

Definition 5.17 *Suppose that R is a field and $\lambda \in \Lambda_2(n,r)$. Let F^λ be the irreducible $\mathcal{S}_R(n,r)$–module $W^\lambda/(W^\lambda \cap W^{\lambda\perp})$.*

Theorem 5.18 *Suppose that R is a field and that $\lambda, \mu \in \Lambda_2(n,r)$. We have $F^\lambda \cong F^\mu$ if and only if λ and μ are associated bicompositions. Thus*

$$\{F^\lambda \mid \lambda \textit{ is a bipartition of } r\}$$

is a set of non–isomorphic absolutely irreducible self–dual $\mathcal{S}_R(n,r)$–modules.

If λ and μ are bipartitions of r then let $d_{\lambda\mu}$ be the multiplicity of F^μ as a composition factor of W^λ. If the bipartitions of r are ordered lexicographically then the matrix $(d_{\lambda\mu})$ is upper unitriangular.

Proof: This theorem is proved in exactly the same way as [9, 4.11 and 4.13]. □

6 The semistandard basis theorem

Hereafter, we assume that $\lambda, \mu \in \Lambda_2(n,r)$, with λ being an a–bipartition.

Definition 6.1 (i) *A* **λ–bitableau of type** *μ is an array $\mathtt{T} = (\mathtt{T}^{(1)}, \mathtt{T}^{(2)})$ of integers obtained from the diagram $[\lambda]$ by replacing each cross by a non–zero integer according to the following restrictions. For $1 \le i \le n$, the number of entries j with $j = i$ is equal to the ith part of $\mu^{(1)}$; and the number of entries j with $|j| = n + i$ is equal to the ith part of $\mu^{(2)}$. For $k = 1, 2$, $\mathtt{T}^{(k)}$ denotes the array of integers replacing the crosses in $\lambda^{(k)}$. We denote the set of λ–bitableaux of type μ by $\mathfrak{T}(\lambda,\mu)$. Note that all entries less than n in $\mathtt{T} \in \mathfrak{T}(\lambda,\mu)$ are positive.*

(ii) *An element $\mathtt{T} = (\mathtt{T}^{(1)}, \mathtt{T}^{(2)})$ of $\mathfrak{T}(\lambda,\mu)$ is* **positive** *if all the entries in $\mathtt{T}^{(1)}$ are greater than 0 and all the entries in $\mathtt{T}^{(2)}$ are greater than n. We denote the set of positive elements of $\mathfrak{T}(\lambda,\mu)$ by $\mathfrak{T}^+(\lambda,\mu)$.*

(iii) *A* **semistandard λ–bitableau of type** *μ is an element of $\mathfrak{T}^+(\lambda,\mu)$ in which the entries are weakly increasing along each row and strictly increasing down each column. We denote the set of semistandard λ–bitableaux of type μ by $\mathfrak{T}_\circ(\lambda,\mu)$.*

Example 6.2 *If $r = n = 7$ and $\lambda = ((3,2),(1,1))$ and $\mu = ((3,1),(2,1))$ then*

$$\mathtt{T}_1 = \begin{pmatrix} 1 & 8 & 9 & , & 1 \\ 2 & 1 & & & -8 \end{pmatrix} \in \mathfrak{T}(\lambda,\mu),$$

$$\text{and} \qquad \mathtt{T}_2 = \begin{pmatrix} 1 & 1 & 1 & , & 8 \\ 2 & 8 & & & 9 \end{pmatrix} \in \mathfrak{T}_\circ(\lambda,\mu).$$

Note that there are no positive λ–bitableaux of type μ if λ is an a–bipartition and μ is a b–bicomposition of r with $a < b$.

Definition 6.3 *Given $\mathtt{T} \in \mathfrak{T}^+(\lambda,\mu)$, let $\mathtt{T}(i)$ equal the integer which appears in $\mathtt{T}$ in the place which is occupied by i in $\hat{\mathfrak{t}}_\lambda$. Choose a total order $<$ on $\mathfrak{T}^+(\lambda,\mu)$ such that if $\mathtt{A}$ and $\mathtt{B}$ are elements of $\mathfrak{T}^+(\lambda,\mu)$ then $\mathtt{A} < \mathtt{B}$ if one of the following holds.*

(i) *For all j,k.*

$$\#\{i \,|\, \mathtt{A}(i) = j \text{ and } i \text{ belongs to column } k \text{ of } \hat{\mathfrak{t}}_\lambda\} \\ = \#\{i \,|\, \mathtt{B}(i) = j \text{ and } i \text{ belongs to column } k \text{ of } \hat{\mathfrak{t}}_\lambda\},$$

and $\mathtt{A}(i) < \mathtt{B}(i)$ for the smallest integer i such that $\mathtt{A}(i) \neq \mathtt{B}(i)$.

(ii) $\sum_{i=1}^{r-a} \mathtt{A}(i) < \sum_{i=1}^{r-a} \mathtt{B}(i)$.

(iii) $\sum_{i=1}^{r-a} \mathtt{A}(i) = \sum_{i=1}^{r-a} \mathtt{B}(i)$, *and for all j,k*

$$\#\{i \,|\, \mathtt{A}(i) \leq j \text{ and } i \text{ belongs to the first } k \text{ columns of } \hat{\mathfrak{t}}_\lambda\} \\ \geq \#\{i \,|\, \mathtt{B}(i) \leq j \text{ and } i \text{ belongs to the first } k \text{ columns of } \hat{\mathfrak{t}}_\lambda\}.$$

Let $V(\lambda,\mu)$ be the free R–module spanned by all $\mathtt{T} \in \mathfrak{T}(\lambda,\mu)$. Given $\mathtt{T} \in \mathfrak{T}^+(\lambda,\mu)$, define $E_\mathtt{T}$ to be the element of $V(\lambda,\mu)$ obtained by summing over those $\mathtt{T}'$ which are row equivalent to $\mathtt{T}$, and then applying the signed column symmetrizer in a fashion similar to that in [17, 8.1.11].

Example 6.4 *If*

$$\mathtt{T} = \begin{pmatrix} 1 & 1 & , & 5 \\ 2 & & & 6 \end{pmatrix},$$

then

$$E_\mathtt{T} = \begin{pmatrix} 1 & 1 & , & 5 \\ 2 & & & 6 \end{pmatrix} - \begin{pmatrix} 2 & 1 & , & 5 \\ 1 & & & 6 \end{pmatrix} - \begin{pmatrix} 1 & 1 & , & 6 \\ 2 & & & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & , & 6 \\ 1 & & & 5 \end{pmatrix}$$

We shall need the following result, which follows easily from the corresponding theorem [17, 8.1.16] for Weyl modules for general linear groups.

Theorem 6.5 *The R–submodule of $V(\lambda,\mu)$ spanned by $\{E_{\mathtt{T}} \mid \mathtt{T} \in \mathfrak{T}^+(\lambda,\mu)\}$ is a free R–module with basis $\{E_{\mathtt{T}} \mid \mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu)\}$.*

Definition 6.6 (i) *If $\mathtt{T} = (\mathtt{T}^{(1)}, \mathtt{T}^{(2)}) \in \mathfrak{T}(\lambda,\mu)$ then let $\varepsilon(\mathtt{T}) = 1$ if an even number of entries in $\mathtt{T}^{(2)}$ are negative , and $\varepsilon(\mathtt{T}) = -1$, otherwise.*

(ii) *Let σ be the linear map on $V(\lambda,\mu)$ which sends each $\mathtt{T} \in \mathfrak{T}(\lambda,\mu)$ to*

$$\sigma(\mathtt{T}) = \sum \varepsilon(\mathtt{T}')\mathtt{T}',$$

where the sum ranges over all those $\mathtt{T}' \in \mathfrak{T}(\lambda,\mu)$ which are obtained from $\mathtt{T}$ by changing the sign of some of the entries in $\mathtt{T}$ which are greater than n.

Example 6.7 *Suppose that $r = 5$ and $n = 3$. If*

$$\mathtt{T} = \begin{pmatrix} 1 & 1 & , & 5 & 6 \\ 2 & & & & \end{pmatrix},$$

then

$$\sigma(\mathtt{T}) = \begin{pmatrix} 1 & 1 & , & 5 & 6 \\ 2 & & & & \end{pmatrix} - \begin{pmatrix} 1 & 1 & , & -5 & 6 \\ 2 & & & & \end{pmatrix} - \begin{pmatrix} 1 & 1 & , & 5 & -6 \\ 2 & & & & \end{pmatrix} + \begin{pmatrix} 1 & 1 & , & -5 & -6 \\ 2 & & & & \end{pmatrix}.$$

Theorem 6.5 immediately gives the following corollary.

Corollary 6.8 *The R–module spanned by $\{\sigma(E_{\mathtt{T}}) \mid \mathtt{T} \in \mathfrak{T}^+(\lambda,\mu)\}$ is free with basis*

$$\{\sigma(E_{\mathtt{T}}) \mid \mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu)\}.$$

To facilitate our calculations, we introduce for fixed bicompositions $\lambda,\mu \in \Lambda_2(n,r)$ three linear maps, α, β, and γ from M^μ into $V(\lambda,\mu)$.

Definition 6.9 *Let α,β,γ be the linear transformations from M^μ into $V(\lambda,\mu)$ which are given as follows. Suppose that $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ is a μ–bitableau. The λ–bitableau $\alpha\{\mathfrak{t}\}$ of type μ is obtained from $\mathfrak{t}^\lambda$ and $\mathfrak{t}$ as follows. For $1 \le i \le r$*

(i) *replace the entry i in $\mathfrak{t}^\lambda$ by j if i or $-i$ occurs in row j of $\mathfrak{t}^{(1)}$, and*

(ii) *replace the entry i in $\mathfrak{t}^\lambda$ by $n+j$ (respectively $-n-j$) if i (respectively $-i$) occurs in row j of $\mathfrak{t}^{(2)}$.*

The definitions of $\beta\{\mathfrak{t}\}$ and $\gamma\{\mathfrak{t}\}$ are obtained in a similar way, replacing $\mathfrak{t}^\lambda$ by $\hat{\mathfrak{t}}^\lambda$ and $\hat{\mathfrak{t}}_\lambda$ respectively.

Observe that these maps are independent of the choice of tableau $\mathfrak{t}$ in $\{\mathfrak{t}\}$ and so are well defined.

Example 6.10 *Assume that $r=n=6$, and let $\lambda=((2^2),(2))$, $\mu=((3,1),(1^2))$. Then*

$$\mathfrak{t}^\lambda=\begin{pmatrix}1&2&,&5&6\\3&4&&&\end{pmatrix}\quad \hat{\mathfrak{t}}^\lambda=\begin{pmatrix}3&4&,&1&2\\5&6&&&\end{pmatrix}\quad \hat{\mathfrak{t}}_\lambda=\begin{pmatrix}3&5&,&1&2\\4&6&&&\end{pmatrix}.$$

If

$$\mathfrak{t}=\begin{pmatrix}3&4&6&,&-2\\1&&&&5\end{pmatrix}$$

then

$$\alpha\{\mathfrak{t}\}=\begin{pmatrix}2&-7&,&8&1\\1&1&&&\end{pmatrix}\quad,\quad \beta\{\mathfrak{t}\}=\begin{pmatrix}1&1&,&2&-7\\8&1&&&\end{pmatrix}$$

and

$$\gamma\{\mathfrak{t}\}=\begin{pmatrix}1&8&,&2&-7\\1&1&&&\end{pmatrix}.$$

Note that all three maps defined above have inverses. Given $\mathtt{T}\in\mathfrak{T}(\lambda,\mu)$ we define the μ–bitabloid $\{\mathfrak{t}\}=\alpha^{-1}\mathtt{T}$ as follows. If the place occupied by i in $\mathfrak{t}^\lambda$ is occupied by j (respectively by $-j$) in $\mathtt{T}$ put i (respectively $-i$) in row j of $\mathfrak{t}$, counting the rows of $\mathfrak{t}^{(1)}$ as row $1,2,\dots$ and the rows of $\mathfrak{t}^{(2)}$ as row $n+1,n+2,\dots$, and then take the bitabloid $\{\mathfrak{t}\}$ containing the bitableau $\mathfrak{t}$. For the maps β and γ use $\hat{\mathfrak{t}}^\lambda$ and $\hat{\mathfrak{t}}_\lambda$ respectively instead of $\mathfrak{t}^\lambda$.

Lemma 6.11 *The maps α, β and γ induce bijections between the set of μ–bitabloids and $\mathfrak{T}(\lambda,\mu)$.*

The maps defined in (6.9) can be used to define an action of W_r on the set $\mathfrak{T}(\lambda,\mu)$ by taking the preimage under one of these maps, acting on the resulting bitabloid and taking the image under the same map again. The action of $\mathfrak{S}_r$ on $\mathfrak{T}(\lambda,\mu)$ is given by place permutations, where the numbering of the places is determined by $\mathfrak{t}^\lambda$ if we use α, by $\hat{\mathfrak{t}}^\lambda$ if we use β, and by $\hat{\mathfrak{t}}_\lambda$ if we use γ.

Our main aim in this section is to prove that the Weyl module W^λ has a basis which is indexed by the semistandard λ–bitableaux of various types. Indeed, we shall show that the weight space $W^\lambda\cap M^\mu$ (see Definition 5.6) is free as an R–module with basis indexed by $\mathfrak{T}_\circ(\lambda,\mu)$. We begin with a special case.

Theorem 6.12 *Let R be a field of characteristic zero and $Q=q=1$. Then*

$$\dim(W^\lambda\cap M^\mu)=|\mathfrak{T}_\circ(\lambda,\mu)|.$$

Proof: In this case $\mathcal{H}$ is isomorphic to the group algebra RW_r so $h_{a,r-a}=w_{a,r-a}$, and

$$u_a^+=\prod_{i=1}^{a}(1+(i,-i))\quad\text{and}\quad u_{r-a}^-=\prod_{i=1}^{r-a}(1-(i,-i)).$$

Since R is a field of characteristic zero, $W^\lambda \cap M^\mu$ is spanned by the elements of the form $\{\mathfrak{t}\}z_\lambda$ where $\{\mathfrak{t}\}$ varies over the μ–bitabloids. This follows immediately from Definition 5.6 and Lemma 5.8.

Let $\mathfrak{t}$ be a row standard μ–bitableau. For $i \in \boldsymbol{r}^+$, we have

$$\{\mathfrak{t}\}(i,-i) = \begin{cases} \{\mathfrak{t}\}, & \text{if } i \text{ is an entry of } \mathfrak{t}^{(1)}, \\ \{\mathfrak{t}(i,-i)\}, & \text{if } i \text{ or } -i \text{ is an entry of } \mathfrak{t}^{(2)}. \end{cases}$$

Let $X = \{i \in \boldsymbol{r}^+ \mid i \text{ or } -i \in \mathfrak{t}^{(2)}\}$ and $X_a = X \cap \{1, 2, \dots, a\}$. Then

$$\{\mathfrak{t}\}u_a^+ = 2^{a-|X_a|} \sum \{\{\mathfrak{t}_1\} \mid \mathfrak{t}_1 \in A_\mathfrak{t}\}, \tag{6.13}$$

where $A_\mathfrak{t}$ is the set of row standard μ–bitableaux which agree with $\mathfrak{t}$, except that the integers in X_a are allowed to have either sign.

Next,

$$\tilde{u}_{r-a}^- := w_{a,r-a} u_{r-a}^- w_{a,r-a}^{-1} = \prod_{i=a+1}^{r} (1 - (i,-i)).$$

Therefore

$$\{\mathfrak{t}\}u_a^+ \tilde{u}_{r-a}^- = \begin{cases} 2^{a-|X_a|} \sum \{\pm\{\mathfrak{t}_1\} \mid \mathfrak{t}_1 \in B_\mathfrak{t}\}, & \text{if } \{a+1, \dots, r\} \subseteq X, \\ 0, & \text{otherwise.} \end{cases} \tag{6.14}$$

where the sum is over the set $B_\mathfrak{t}$ of the $2^{|X|}$ row standard μ–bitableaux $\mathfrak{t}_1 = (\mathfrak{t}_1^{(1)}, \mathfrak{t}_1^{(2)})$ which are the same as $\mathfrak{t}$ except that the integers in $\mathfrak{t}_1^{(2)}$ are allowed to have either sign. (The coefficient of $\{\mathfrak{t}_1\}$ is $+1$ if and only if $|\{i \in \mathfrak{t}^{(2)} \mid a+1 \le i \le r\}| - |\{i \in \mathfrak{t}_1^{(2)} \mid a+1 \le i \le r\}|$ is even.)

Assume now that $\{a+1, \dots, r\} \subseteq X$. Then

$$\{\mathfrak{t}\}u_a^+ w_{a,r-a} u_{r-a}^- = 2^{a-|X_a|} \sum \{\pm\{\mathfrak{t}_1\} \mid \mathfrak{t}_1 \in C_\mathfrak{t}\},$$

where the sum is over the set $C_\mathfrak{t}$ of row standard μ–bitableaux $\mathfrak{t}_1 = (\mathfrak{t}_1^{(1)}, \mathfrak{t}_1^{(2)})$ which are row equivalent to $\mathfrak{t}w_{a,r-a}$, except that the integers in $\mathfrak{t}_1^{(2)}$ can have either sign. Let $\{\mathfrak{t}^+\}$ be the μ–bitabloid obtained from $\mathfrak{t}w_{a,r-a}$ by changing the signs of all the negative entries and taking the row equivalence class. Since $\{a+1, \dots, r\} \subseteq X$, we have

$$\{a+1, \dots, r\}w_{a,r-a} = \{1, 2, \dots, r-a\} \subseteq \mathfrak{t}^{+(2)}.$$

Hence $\beta(\{\mathfrak{t}^+\}) \in \mathfrak{T}^+(\lambda, \mu)$ and $2^{|X_a|-a}\beta(\{\mathfrak{t}\}u_a^+ w_{a,r-a} u_{r-a}^-) = \pm\sigma(\beta\{\mathfrak{t}^+\})$. Therefore, by (2.17),

$$2^{|X_a|-a}\beta(z_\lambda) = \pm\sigma(\beta(\{\mathfrak{t}^+\})\hat{x}_\lambda T_{\hat{\pi}_\lambda} \hat{y}_\lambda).$$

If $\beta(\{\mathtt{t}^+\}) = \mathtt{T}$, then $\mathtt{T}\hat{x}_\lambda$ is a multiple of the sum over those $\mathtt{T}'$ which are row equivalent to $\mathtt{T}$ (since $\hat{x}_\lambda$ is the sum over the row symmetrizer of $\mathtt{T}$). Hence $\beta(z_\lambda)$ is a non–zero multiple of $\sigma(E_{\mathtt{T}})$. The theorem now follows from Corollary 6.8. □

We shall generalise Theorem 6.12, making use of the special case in the course of the proof. First we need to reformulate some results from section 4 in the language of λ–tableaux using the map α introduced in Definition 6.9. Define $\tilde{\mu}$ to be the composition of r into $2n$ parts $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_{2n})$ as follows. Suppose that $\mu^{(1)} = (\mu_1^{(1)}, \dots, \mu_{n_1}^{(1)})$ and $\mu^{(2)} = (\mu_1^{(2)}, \dots, \mu_{n_2}^{(2)})$. Thus $n_1 + n_2 = n$. We define

$$\tilde{\mu}_i = \begin{cases} \mu_i^{(1)}, & \text{for } 1 \le i \le n_1, \\ \mu_i^{(2)}, & \text{for } n+1 \le i \le n+n_2, \\ 0, & \text{otherwise.} \end{cases} \tag{6.15}$$

Define $\tilde{\lambda}$ similarly.

Denote the set of $\tilde{\lambda}$–tableaux of type $\tilde{\mu}$ by $\mathfrak{T}(\tilde{\lambda}, \tilde{\mu})$. We can turn $\mathtt{T} \in \mathfrak{T}^+(\lambda, \mu)$ into an element $\tilde{\mathtt{T}}$ of $\mathfrak{T}(\tilde{\lambda}, \tilde{\mu})$ by combining the two components of $\mathtt{T}$.

Example 6.16 *If $r = n = 7$ and $\lambda, \mu \in \Lambda_2(n, r)$ as in Example 6.2. Then*

$$\begin{aligned} \tilde{\lambda} &= (3, 2, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0) \qquad \textit{and} \\ \tilde{\mu} &= (3, 1, 0, 0, 0, 0, 0, 2, 1, 0, 0, 0, 0, 0). \end{aligned}$$

Moreover if $\mathtt{T} = \begin{pmatrix} 1\ 1\ 1 & & 8 \\ 2\ 8 & , & 9 \end{pmatrix} \in \mathfrak{T}^+(\lambda, \mu)$, *then*

$$\tilde{\mathtt{T}} \quad = \quad \begin{array}{lll} 1 & 1 & 1 \\ 2 & 8 & \\ - & & \\ - & & \\ - & & \\ - & & \\ - & & \\ 8 & & \\ 9 & & \\ - & & \\ - & & \\ - & & \\ - & & \\ - & & \end{array}$$

in $\mathfrak{T}(\tilde{\lambda}, \tilde{\mu})$.

As in (4.14) we let I and J be the subsets of Δ such that $x_\lambda = x_J$ and $x_\mu = x_I$.

We have the following lemma. The first part is trivial; the second part follows from part one and the corresponding result for type **A** [7, 1.7(i)].

Lemma 6.17 (i) *The map* $\mathtt{T} \mapsto \tilde{\mathtt{T}}$ *defines a bijection between* $\mathfrak{T}^+(\lambda,\mu)$ *and* $\mathfrak{T}(\tilde{\lambda},\tilde{\mu})$.

(ii) *There is a bijection between* $\mathcal{D}^{\Delta}_{I,J}$ *and the set of positive row standard* λ*–bitableaux of type* μ *given by* $c \mapsto \mathtt{T}_c$ *for* $c \in \mathcal{D}^{\Delta}_{I,J}$, *where* $\mathtt{T}_c = \alpha\{\mathfrak{t}^\mu c\}$.

Definition 6.18 *Given* $\mathtt{T}_1, \mathtt{T}_2 \in \mathfrak{T}(\lambda,\mu)$ *we write* $\mathtt{T}_1 \sim \mathtt{T}_2$ *if* $\mathtt{T}_1$ *and* $\mathtt{T}_2$ *are row equivalent.*

Corollary 4.35 (see also 4.34), taken in conjunction with [7, 1.7, 3.4] now gives the following theorem (note that the map $A \mapsto 1_A$ in [7, 1.7] is the analogue of the map α from Definition 6.9).

Theorem 6.19 *Let* $c \in \mathcal{D}^{\Delta}_{I,J}$. *Then*

$$\varphi^c_{\mu,\lambda}(\{\mathfrak{t}^\lambda\}) \equiv \sum_{\alpha\{\mathfrak{t}_1\}\sim \mathtt{T}_c} \{\mathfrak{t}_1\} \pmod{M^\mu_-}$$

where $\mathtt{T}_c = \alpha\{\mathfrak{t}^\mu c\}$.

We are now prepared to embark on the generalisation of Theorem 6.12.

Let $\mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu)$ and let $\mathfrak{t}$ be the row standard μ–bitableau such that $\alpha\{\mathfrak{t}\} = \mathtt{T}$, which is given by Lemma 6.11.

The semistandard tableau $\mathtt{T}$ corresponds to an element $c \in \mathcal{D}^{\Delta}_{I,J}$ by Lemma 6.17. By Theorem 6.19 the $\mathcal{H}$-homomorphism $\varphi^c_{\mu,\lambda}$ maps the generator $\{\mathfrak{t}^\lambda\}$ of M^λ to an element v of M^μ which satisfies

$$v \equiv \sum_{\alpha\{\mathfrak{t}_1\}\sim \mathtt{T}} \{\mathfrak{t}_1\} \pmod{M^\mu_-}.$$

Since $h_{a,r-a} \in \mathcal{H}(\mathfrak{S}_r)$,

$$vh_{a,r-a} \equiv \sum_{\alpha\{\mathfrak{t}_1\}\sim \mathtt{T}} \{\mathfrak{t}_1\} h_{a,r-a} \pmod{M^\mu_-}$$

by (3.14). Next, Corollary 3.17 gives

$$vh_{a,r-a} \equiv \sum_{\alpha\{\mathfrak{t}_1\}\sim \mathtt{T}} r_{\mathfrak{t}_1}\{\mathfrak{t}_1 w_{a,r-a}\} + v_1 \pmod{M^\mu_-},$$

where each $r_{\mathfrak{t}_1}$ is a unit in R and v_1 is a linear combination of μ–bitabloids $\{\mathfrak{t}_1'\}$ satisfying $\beta\{\mathfrak{t}_1'\} > \beta\{\mathfrak{t} w_{a,r-a}\}$ (see part (ii) of Definition 6.3).

Now, $\alpha\{\mathfrak{t}_1\} = \beta\{\mathfrak{t}_1 w_{a,r-a}\}$, so

$$vh_{a,r-a} \equiv \sum_{\beta\{\mathfrak{t}_2\}\sim \mathtt{T}} r_{\mathfrak{t}_2}\{\mathfrak{t}_2\} + v_1 \pmod{M^\mu_-}, \tag{6.20}$$

where each $r_{\mathfrak{t}_2}$ is a unit.

Note that the numbers $1,2,\dots,r-a$ belong to $\hat{\mathfrak{t}}^{\lambda(2)}$. On the other hand, since $\mathtt{T}$ is semistandard by assumption, all the numbers in $\mathtt{T}^{(2)}$ are greater than n. Thus for all $\beta\{\mathfrak{t}_2\}$ which are row equivalent to $\mathtt{T}$, all of the entries in the second component of $\beta\{\mathfrak{t}_2\}$ are greater than n. By Definition 6.9 we conclude that all the μ–bitabloids $\{\mathfrak{t}_2\}$ which appear in (6.20) have $1,2,\dots,r-a$ in $\mathfrak{t}^{(2)}$. A similar result applies to the μ–bitabloids which occur in v_1. Therefore, by Corollary 3.15,

$$vh_{a,r-a}u^-_{r-a} \equiv \sum_{\beta\{\mathfrak{t}_2\}\sim\mathtt{T}} r'_{\mathfrak{t}_2}\{\mathfrak{t}_2\} + v_2 \pmod{M^\mu_-}. \tag{6.21}$$

where each $r'_{\mathfrak{t}_2}$ is a unit and v_2 is a linear combination of μ–bitabloids $\{\mathfrak{t}'_1\}$ which are precisely the μ–bitabloids which are involved with nonzero coefficients in v_1. (Indeed Corollary 3.15 implies that the coefficients of $\{\mathfrak{t}'_1\}$ in v_1 and v_2 differ only by multiplication by a unit.)

Next, let $\{\mathfrak{t}^*\}$ be the μ–bitabloid such that $\gamma\{\mathfrak{t}^*\} = \mathtt{T}$. Note that the row standard μ–bitableau $\mathfrak{t}^*$ in $\{\mathfrak{t}^*\}$ is $\mathfrak{t}\hat{\pi}_\lambda$.

We have that

$$vh_{a,r-a}u^-_{r-a}T_{\hat{\pi}_\lambda} \equiv r\{\mathfrak{t}^*\} + v_3 \pmod{M^\mu_-}$$

where r is a unit and v_3 is a linear combination of μ–bitabloids $\{\mathfrak{t}_3\}$ such that $\gamma\{\mathfrak{t}_3\} > \mathtt{T}$. To see this, note part (iii) of Definition 6.3 and compare with [9, 7.26]. The matrix $\chi(\mathfrak{t},\mathfrak{t}^\mu w_\mu)$ which appears in [9, 7.26] is defined in such a way that its (j,k)th entry is equal to the number of entries less than or equal to j in the first k columns of the μ–tableau of type λ obtained by replacing each entry i of $\mathfrak{t}^\mu w_\mu$ by $\mathrm{row}_{\mathfrak{t}}(i)$.

We arrive at the following element of $W^\lambda \cap M^\mu$.

$$\varphi^c_{\mu,\lambda}(z_\lambda) = vh_{a,r-a}u^-_{r-a}T_{\hat{\pi}_\lambda}\hat{y}_\lambda \equiv r\{\mathfrak{t}^*\} + v_4 \pmod{M^\mu_-},$$

where v_4 is a linear combination of μ–bitabloids $\{\mathfrak{t}_4\}$ such that

$$\gamma\{\mathfrak{t}_4\} > \mathtt{T},$$

by part (i) of Definition 6.3. This is justified as follows. All of the μ–bitabloids $\{\mathfrak{t}_3\}$ involved in v_3 when acted upon by the terms T_w, appearing in $\hat{y}_\lambda$, are linear combinations of μ–bitabloids $\{\mathfrak{t}_4\}$ which are obtained from $\{\mathfrak{t}_3\}$ by permuting the entries of the columns in $\mathfrak{t}_3$. Since $\gamma\{\mathfrak{t}_3\} > \mathtt{T}$, Definition 6.3 shows that $\gamma\{\mathfrak{t}_4\} > \mathtt{T}$. Also, $\{\mathfrak{t}^*\}\mathtt{T}_w$ is a linear combination of terms of the form $\{\mathfrak{t}^*w'\}$ where w' is an element in the column stabilizer of $\hat{\mathfrak{t}}_\lambda$. From part (i) of Definition 6.3 we conclude that $\gamma\{\mathfrak{t}^*w'\} > \mathtt{T}$ for $w \neq 1$ since $\mathtt{T}$ is semi–standard by assumption. We have shown the following.

Lemma 6.22 *Let* $\mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu)$ *and let*

$$v_{\lambda,\mu}(\mathtt{T}) = \varphi^c_{\mu,\lambda}(\{\mathfrak{t}\})h_{a,r-a}u^-_{r-a}T_{\hat{\pi}_\lambda}\hat{y}_\lambda,$$

where $\mathfrak{t}$ *is the unique row standard* μ*–bitableau such that* $\alpha\{\mathfrak{t}\} = \mathtt{T}$*. Let* $\mathfrak{t}^*$ *be the row standard* μ*–bitableau such that* $\gamma\{\mathfrak{t}^*\} = \mathtt{T}$*. Then* $v_{\lambda,\mu}(\mathtt{T}) \in W^\lambda \cap M^\mu$ *and* $v_{\lambda,\mu}(\mathtt{T})$ *is congruent modulo* M^μ_- *to a linear combination of* μ*–bitabloids* $\{\mathfrak{t}_4\}$ *such that*

$$\gamma\{\mathfrak{t}_4\} \geq \mathtt{T}$$

and the coefficient of $\{\mathfrak{t}^*\}$ *in* $v_{\lambda,\mu}(\mathtt{T})$ *is invertible.*

Recall that by Lemma 5.8 $W^\lambda \cap M^\mu$ is a weight space of the (Q, q)–Weyl module W^λ.

Corollary 6.23 *Let* $\lambda, \mu \in \Lambda_2(n, r)$*. Then* $\{v_{\lambda,\mu}(\mathtt{T}) \mid \mathtt{T} \in \mathfrak{T}_\circ(\lambda, \mu)\}$ *is a linearly independent subset of the weight space* $W^\lambda \cap M^\mu$ *of* W^λ*.*

We now prove our main result.

Theorem 6.24 (The Semistandard Basis Theorem)
Let $\lambda, \mu \in \Lambda_2(n, r)$ *and let* Q, q *be invertible elements of* R*. Then the weight space* $W^\lambda \cap M^\mu$ *of the Weyl module* W^λ *is free as an* R*–module with basis* $\{v_{\lambda,\mu}(\mathtt{T}) \mid \mathtt{T} \in \mathfrak{T}_\circ(\lambda, \mu)\}$*. Consequently,* W^λ *is free as an* R*–module with basis*

$$\{v_{\lambda,\mu}(\mathtt{T}) \mid \mu \in \Lambda_2(n, r), \mathtt{T} \in \mathfrak{T}_\circ(\lambda, \mu)\}.$$

Proof: Suppose, for the moment, that $R = \mathbb{Q}(q, Q)$ where Q and q are independent transcendentals. Then $\mathcal{H}_{R,q,Q}(W_r)$ is isomorphic to RW_r by [2]. Hence, in this case, $\dim(W^\lambda \cap M^\mu) = |\mathfrak{T}_\circ(\lambda, \mu)|$ by Theorem 6.12, so $\{v_{\lambda,\mu}(\mathtt{T}) \mid \mathtt{T} \in \mathfrak{T}_\circ(\lambda, \mu)\}$ is a basis of $W^\lambda \cap M^\mu$.

Assume that m is a nonzero element of $W^\lambda \cap M^\mu$ and that the coefficient of every μ–bitabloid which is involved in m belongs to $\mathbb{Z}[q, q^{-1}, Q, Q^{-1}]$. We write

$$m = \sum_{\mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu)} r_{\mathtt{T}} v_{\lambda,\mu}(\mathtt{T})$$

with coefficients $r_{\mathtt{T}}$ in $\mathbb{Q}(q, Q)$. We claim that $r_{\mathtt{T}} \in \mathbb{Z}[q, q^{-1}, Q, Q^{-1}]$. In the total order of Definition 6.3 let $\mathtt{T}_1$ be the first element of $\mathfrak{T}_\circ(\lambda, \mu)$ such that $r_{\mathtt{T}_1} \neq 0$, and let $\mathfrak{t}_1$ be the row standard μ–bitableau with $\gamma\{\mathfrak{t}_1\} = \mathtt{T}_1$. The coefficient of $\{\mathfrak{t}_1\}$ in m belongs to $\mathbb{Z}[q, q^{-1}, Q, Q^{-1}]$, so by Lemma 6.22

$$r_{\mathtt{T}_1} \in \mathbb{Z}[q, q^{-1}, Q, Q^{-1}].$$

Using Lemma 6.22, we see that

$$m - r_{\mathtt{T}_1} v_{\lambda,\mu}(\mathtt{T}_1) = \sum_{\substack{\mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu) \\ \mathtt{T} \neq \mathtt{T}_1}} r_{\mathtt{T}} v_{\lambda,\mu}(\mathtt{T})$$

has the property that the first element $\mathtt{T}_2$ of $\mathfrak{T}_\circ(\lambda, \mu)$ such that $r_{\mathtt{T}_2} \neq 0$ satisfies $\mathtt{T}_2 > \mathtt{T}_1$. Hence, by induction, every $r_{\mathtt{T}}$ belongs to $\mathbb{Z}[q, q^{-1}, Q, Q^{-1}]$.

Now assume that R is an arbitrary commutative ring and q, Q are invertible elements of R. The result of the last paragraph shows that every non–zero element of $W^\lambda \cap M^\mu$ can be written as a linear combination of $\{v_{\lambda,\mu}(\mathtt{T}) \mid \mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu)\}$. Taken in conjunction with (6.23), this proves that

$$\{v_{\lambda,\mu}(\mathtt{T}) \mid \mathtt{T} \in \mathfrak{T}_\circ(\lambda,\mu)\}$$

is a basis of $W^\lambda \cap M^\mu$ and concludes the proof of the theorem. □

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