An integral theory of dominant dimension of Noetherian algebras

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Abstract

Dominant dimension is introduced into integral representation theory, extending the classical theory of dominant dimension of Artinian algebras to projective Noetherian algebras (that is, algebras which are finitely generated projective as modules over a commutative Noetherian ring). This new homological invariant is based on relative homological algebra introduced by Hochschild in the 1950s. Amongst the properties established here are a relative version of the Morita-Tachikawa correspondence and a relative version of Mueller's characterization of dominant dimension. The behaviour of relative dominant dimension of projective Noetherian algebras under change of ground ring is clarified and we explain how to use this property to determine the relative dominant dimension of projective Noetherian algebras. In particular, we determine the relative dominant dimension of Schur algebras and quantized Schur algebras.

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1 Introduction

An Artinian algebra is said to have dominant dimension at least n if the first n terms of an injective coresolution of the regular module are projective modules. In particular, it has positive dominant dimension if it admits a faithful projective-injective module. Dominant dimension has been proven to be a very useful tool to establish a connection between two algebras playing an important role in many correspondences in representation theory of Artinian algebras: for instance, the Morita-Tachikawa correspondence [Mue68, Theorem 2], Auslander's correspondence [Aus71], Iyama's Higher Auslander correspondence [Iya07]. Auslander's correspondence is a crucial result in representation theory of Artinian algebras providing a bijection between Artinian algebras of finite type and Artinian algebras with dominant dimension at least two and global dimension at most two. Many versions of Schur–Weyl duality and other double centralizer properties involve algebras having dominant dimension at least two ([KSX01], [Tac73, (7.1)]). Moreover, dominant dimension gives computation-free proofs of double centralizer properties in contrast to more traditional methods. As in [DDPW08, chapter 9], many double centralizer properties of interest also hold in the integral setup which is no longer an Artinian algebra. Unfortunately, so far, dominant dimension has not been used in these integral setups since the definition of dominant dimension for Artinian algebras does not carry over for Noetherian algebras. Indeed, projective-injective modules rarely exist for Noetherian algebras. Over the years, there were some approaches to extending dominant dimension by replacing the projective modules by flat (see [Hos89]) or even torsionless (see [Kat68]) modules. But, these notions do not provide much information in applications. In particular, they do not seem to be very useful to evaluate the connection between two Noetherian algebras. Our aim in this paper is to introduce a new generalization of dominant dimension for algebras which are finitely generated projective as modules over a Noetherian ring. This generalization is suitable for computations and it has the properties that a dominant dimension must have (see [Mue68]). For instance, it is left-right symmetric and there is a characterization of dominant dimension using homological algebra. In doing so, we dramatically increase the classical theory of dominant dimension to also include problems of integral representation theory. Moreover, this concept will help us, also in forthcoming work, reducing problems of integral representation theory to problems of finite dimensional algebras over algebraically closed fields and vice-versa.

This new relative dominant dimension of Noetherian algebras is based on relative injective modules instead of (absolute) injective modules (see Definition 3.1). The term relative means that we consider only the exact sequences over Noetherian algebras which split over the ground ring of the Noetherian algebra. This leads us to introduce other concepts like strongly faithful modules (see Definition 3.5) and also allows us to adapt the arguments used in the classical theory to work for Noetherian algebras. Here, strongly faithful modules replace the role that faithful modules have in classical theory of dominant dimension.

To simplify the language, by a projective Noetherian algebra we will mean an algebra which is finitely generated projective as module over a commutative Noetherian ring. At a first glance, we may think that it would be enough to simply replace the assumption of Artinian by Noetherian once this new definition of dominant dimension is in place. But this is not the case as we can see in the following relative version of the Morita-Tachikawa correspondence:

Theorem (see Theorem 4.1)**.** *Let R be a commutative Noetherian ring. There is a bijection:*

$$\left\{ (B,M)\colon \begin{array}{c} B \text{ a projective} \\ \text{Noetherian } R\text{-algebra,} \\ M \text{ a } B\text{-generator } (B,R)\text{-cogenerator,} \\ M \in R\text{-proj}, \\ DM \otimes_B M \in R\text{-proj} \end{array} \right\} \Big/ \sim_1 \longleftrightarrow \left\{ A\colon \begin{array}{c} A \text{ a projective Noetherian} \\ R\text{-algebra with} \\ \operatorname{domdim}(A,R) \geq 2 \end{array} \right\} \Big/ \sim_2 .$$

Here, a module being a generator means that its additive closure contains the regular module and D denotes the functor $\operatorname{Hom}_R(-,R)$. Further, we see that, in this relative setup, we are only interested in the generators whose additive closure contains also all relative injective modules and with this extra property $DM \otimes_B M \in R$-proj. This property ensures that the module M is strongly faithful over its endomorphism algebra. But, most importantly we will see in Theorem 6.14 that this property is equivalent to requiring a base change property on the endomorphism algebra of the generator. Integral Schur algebras, for example, possess this property. This extra condition reinforces the idea that using dominant dimension provides characteristic-free proofs for double centralizer properties also in the integral setup. For projective Noetherian algebras over a commutative Noetherian ring with Krull dimension at most one, this version of relative Morita-Tachikawa correspondence can be modified to not include the property $DM \otimes_B M \in R$-proj. In particular, the correspondence established by Auslander and Roggenkamp in [AR72] involving semisimple orders of finite lattice type is a special case of such a version of relative Morita-Tachikawa without $DM \otimes_B M \in R$-proj (see Theorem 7.3 and 4.3).

Another big difference with the classical case is visible in the relative version of a theorem by Mueller where we are forced to use Tor functors instead of Ext functors.

Theorem (see Theorem 5.2)**.** *Let A be a projective Noetherian R-algebra with positive relative dominant dimension and with V a projective (A,R)-injective-strongly faithful right A-module. Fix $C = \operatorname{End}_A(V)$. For any left A-module M being R-projective, the following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq n \geq 2$;

(ii) The map $\operatorname{Hom}_A(V, DM) \otimes_C V \to DM$*, given by* $f \otimes v \mapsto f(v)$*, is an isomorphism and for each* $1 \leq i \leq n-2$*,* $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0$.

A main property of relative dominant dimension of projective Noetherian algebras establishing a connection with the Artinian case is the following:

Theorem (see Theorem 6.13)**.** *Let A be a projective Noetherian R-algebra with positive relative dominant dimension. Let $M \in A\text{-mod} \cap R\text{-proj}$. Then*

$$\operatorname{domdim}_{(A,R)} M = \inf\{\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \colon \mathfrak{m} \textit{ maximal ideal in } R\}.$$

As an application of the theory developed here, we can extend the notions of Morita and gendo-symmetric algebras to Noetherian algebras. Further, we remark that there are many algebras with origins in invariant theory and Lie theory that can be covered by our approach that we present here. Not to mention many algebras arising in deformation theory. In particular, in this paper we compute the relative dominant dimension of Schur algebras $S_R(n, d)$ (when $n \geq d$) and q-Schur algebras in the integral setup, that is, when R is an arbitrary commutative Noetherian ring using the methods introduced in this paper.

The paper is structured as follows:

In section 2 we recall some notation and folklore results for Noetherian algebras. In subsection 2.3, we give a brief introduction to relative homological algebra with respect to (A, R)-exact sequences giving emphasis to relative injective modules. In section 3, we introduce the definition of relative dominant dimension for projective Noetherian algebras and the notion of strongly faithful module. We explain that for relative self-injective algebras the latter is exactly the notion of generator. In subsection 3.4, we present an alternative definition of relative dominant dimension based on the existence of a projective relative injective strongly faithful module. In subsection 3.5, we establish the equivalence of relative dominant dimension greater or equal than two with a stronger type of double centralizer property on a strongly faithful module, namely $DV \otimes_C V \simeq DA$. Along the way, we reprove many technical results which are well known for projective left ideals in the classical case. In section 4, we prove a relative version of Morita-Tachikawa correspondence which is valid for all projective Noetherian algebras. For the cases of Krull dimension one, a weaker version of the relative Morita-Tachikawa is also considered. In section 5, we give a generalization of Mueller's characterization of dominant dimension for the relative dominant dimension of modules that are projective over the ground ring. When the Krull dimension of the ground ring is one, these modules are known as lattices. We initiate here the study of the influence of the Krull dimension of the ground ring in the theory of relative dominant dimension of a projective Noetherian algebra. In subsection 5.1, we obtain more properties of relative dominant dimension including its right-left symmetry. In section 6, we explore the behaviour of relative dominant dimension under change of ground rings culminating in the proof of one of the main results of this paper, Theorem 6.13, clarifying the meaning behind the property $DM \otimes_B M \in R\text{-proj}$. In section 7, we aim to exhibit the usefulness of relative dominant dimension in practice. We observe that some old results like homological characterizations of lattices of finite type can be written in terms of relative dominant dimension. We can also see that both, properties of Artinian algebras and classes of such algebras, can be further extended to the realm of Noetherian algebras. In subsections 7.5 and 7.6, we conclude this paper computing the relative dominant dimension of Schur algebras and q-Schur algebras showing, in particular, how strongly faithful modules and the property $DM \otimes_B M \in R\text{-proj}$ appear in applications. A small appendix involving spectral sequences is attached for a better understanding of Theorem 6.13.

In forthcoming work based on the current paper, the technology introduced here will be used to deduce results on cover theory of Noetherian algebras.

2 Noetherian algebras and relative homological algebra

In this section, we will introduce the notation to be used throughout this paper and provide some elementary results involving standard duality with respect to a Noetherian ring to be used several times in the results ahead. The tensor product is shown to commute with extension of scalars (Proposition 2.3) and homomorphisms from a projective A-module to another module also commute with extension of scalars (Proposition 2.4). Afterwards, we will discuss the class of (A, R)-exact sequences in 2.3 together with the projective and injective objects with respect to this class of exact sequences. This constitutes the background for the concept of relative dominant dimension of Noetherian algebras.

2.1 Noetherian Algebras

Let R be a commutative Noetherian ring with identity. A is called a **Noetherian R-algebra** if A is an algebra over R such that A is finitely generated as an R-module. By a **projective Noetherian R-algebra** we mean a Noetherian R-algebra A so that A is a finitely generated projective R-module.

Throughout this paper, R will be a commutative Noetherian ring with identity and A will always be a projective Noetherian R-algebra, unless stated otherwise.

By a generator of A (or R) we mean a module whose additive closure contains the regular module. Observe that if A is a Noetherian R-algebra which is free over R, then A is a faithful over R. In such a case R is contained in the center of A. By A-mod we mean the category of finitely generated left A-modules and by A-proj the full subcategory of A-mod whose modules are the finitely generated projective A-modules. By an A-projective finitely generated module we mean a finitely generated projective A-module. We denote by $\operatorname{add}_A M$ (or just $\operatorname{add} M$ when A is fixed) the full subcategory of A-mod whose modules are direct summands of finite direct sums of $M \in A$-mod. We write A-proj to denote $\operatorname{add} A$. Similarly, mod-A and proj-A denote the previous subcategories but for right modules. By A-Mod we mean the category of left A-modules and by $\operatorname{Add}_A M$ the full subcategory of A-Mod whose modules are direct summands of direct sums of $M \in A$-Mod. For any $M \in A$-mod and $f, g \in \operatorname{End}_A(M)$ the multiplication fg is the composite $f \circ g$ of g and f. The opposite algebra of A will be denoted by A^{op}.

By D we mean the standard duality functor $\operatorname{Hom}_R(-, R)\colon A\text{-mod} \to A^{op}\text{-mod}$. For each prime ideal $\mathfrak{p}$ of R, we denote by $R_\mathfrak{p}$ the localization of R at $\mathfrak{p}$. For each $M \in A$-mod, $M_\mathfrak{p}$ is the localization of M at the prime ideal $\mathfrak{p}$. In particular, $M_\mathfrak{p} \simeq R_\mathfrak{p} \otimes_R M$. By $R(\mathfrak{p})$ we mean the residue field $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ associated to the prime ideal $\mathfrak{p}$ of R. For maximal ideals $\mathfrak{m}$ of R the residue field $R(\mathfrak{m})$ is also isomorphic to $R/\mathfrak{m}$. For more properties on localizations of rings, we refer to [Coh89, 11.3]

Given a finitely generated (A, B)-bimodule M, there is a **double centralizer property** on M between A and B provided that the multiplication maps on M induce isomorphisms $A \simeq \operatorname{End}_B(M)$ and $B \simeq \operatorname{End}_A(M)^{op}$.

The following results are quite elementary and folklore but they will be used several times throughout this paper.

Proposition 2.1. *Let A be a projective Noetherian R-algebra. Assume $M, N \in A\text{-mod} \cap R\text{-proj}$. Then, the map $\kappa_{M,N}\colon \operatorname{Hom}_A(M, N) \to D(DN \otimes_A M)$, given by $\kappa(g)(f \otimes m) = f(g(m))$, $g \in \operatorname{Hom}_A(M, N)$, $f \in DN$, $m \in M$, is an $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})$-bimodule isomorphism.*

Moreover, if $DN \otimes_A M \in R\text{-proj}$ the map $\iota_{M,N}\colon DN \otimes_A M \to D\operatorname{Hom}_A(M, N)$, given by $\iota(f \otimes m)(g) = f(g(m))$ for each $f \otimes m \in DN \otimes_A M$, $g \in \operatorname{Hom}_A(M, N)$, is an $(\operatorname{End}_A(N)^{op}, \operatorname{End}_A(M)^{op})$-bimodule isomorphism.

Proof. It follows by Tensor-Hom adjunction and D being a duality functor. □

Proposition 2.2. *Let A be a Noetherian R-algebra. Assume $M, N \in A\text{-mod} \cap R\text{-proj}$. Then, the map $\psi_{M,N}\colon \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{A^{op}}(DN, DM)$, given by, $\psi_{M,N}(g)(h) = h \circ g$, $g \in \operatorname{Hom}_A(M, N)$, $h \in DN$, is an $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})$-bimodule isomorphism, where D is the standard duality.*

Proof. Consider the map $e_M\colon M \to DDM$, given by $e_M(m)(g) = g(m)$. This is a $(A, \operatorname{End}_A(M)^{op})$-bimodule homomorphism. If M is a free R-module, then it is clear that e_M is an $(A, \operatorname{End}_A(M)^{op})$-bimodule isomorphism. Since $(e_M)_{M \in A\text{-mod}}$ is a natural transformation between the functors $\operatorname{Id}_{A\text{-mod}}$ and DD we obtain that e_M is also an e_M is $(A, \operatorname{End}_A(M)^{op})$-bimodule isomorphism for every $M \in A\text{-mod} \cap R\text{-proj}$.

Define the map $\delta\colon \operatorname{Hom}_A(DDM, DDN) \to \operatorname{Hom}_A(M, N)$, given by $\delta(h) = e_N^{-1} \circ h \circ e_M$, $h \in \operatorname{Hom}_A(DDM, DDN)$. This map is bijective since e_M and e_N^{-1} are. By simple computations, we deduce that $e_N \circ \delta \circ \psi_{DN,DM} \circ \psi_{M,N} = e_N \circ \operatorname{id}_{\operatorname{Hom}_A(M,N)}$. Hence, $\delta \circ \psi_{DN,DM} \circ \psi_{M,N} = \operatorname{id}_{\operatorname{Hom}_A(M,N)}$. As

δ is bijective, $\psi_{DN,DM}$ is surjective. By a symmetric argument, we obtain $\delta' \circ \psi_{DDN,DDM} \circ \psi_{DM,DN} = \mathrm{id}_{\mathrm{Hom}_A(DM,DN)}$. Hence, $\psi_{DM,DN}$ is also an injective map. It follows that $\psi_{M,N}$ is a bijective map. We leave to the reader to see that $\psi_{M,N}$ is an $(\mathrm{End}_A(M)^{op}, \mathrm{End}_A(N)^{op})$-bimodule homomorphism. $\square$

2.2 Change of ground rings

Changing the ground ring of a Noetherian algebra has many advantages. The most elementary advantage follows from the several versions of Nakayama's Lemma. For instance, a finitely generated R-module is zero if and only if $M(\mathfrak{m})$ is the zero module for all maximal ideals $\mathfrak{m}$ of R. Also, a finitely generated R-module is projective if and only if $\mathrm{Tor}_1^R(M, R(\mathfrak{m})) = 0$ for all maximal ideals $\mathfrak{m}$ of R (see for example [Rot09, Lemma 8.53] together with the exactness of localization). Here, we collect further elementary facts to be used later.

Proposition 2.3. *Let S be a commutative R-algebra and A a Noetherian R-algebra. Let $M \in \mathrm{mod}\text{-}A$, $N \in A\text{-}\mathrm{mod}$. Then, $S \otimes_R (M \otimes_A N) \simeq S \otimes_R M \otimes_{S\otimes_R A} S \otimes_R N$ as S-modules.*

Proof. Consider the map $\psi\colon S\times(M\otimes_A N) \to S\otimes_R M\otimes_{S\otimes_R A} S\otimes_R N$, given by $\psi(s, m\otimes n) = (s\otimes m\otimes 1_S \otimes n)$, $s \in S, m\otimes n \in M\otimes_A N$. ψ is linear in each term. Further, for every $r \in R$

$$\psi(rs, m\otimes n) = rs\otimes m\otimes 1_S\otimes n = s\otimes rm\otimes 1_S\otimes n = \psi(s, rm\otimes n).$$

So, ψ induces uniquely a map $\psi' \in \mathrm{Hom}(S\otimes_R M\otimes_A N, S\otimes_R M\otimes_{S\otimes_R A} S\otimes_R N)$ which maps $s\otimes m\otimes n$ to $s\otimes m\otimes 1_S\otimes n$. Such a map is an S-homomorphism since

$$\begin{aligned}\psi(ls\otimes(m\otimes n)) &= ls\otimes m\otimes 1_S\otimes n = sl\otimes m\otimes 1_S\otimes n = s\otimes m\cdot(l\otimes 1_A)\otimes 1_S\otimes n\\ &= s\otimes m\otimes(l\otimes 1_A)\cdot 1_S\otimes n = s\otimes m\otimes l\otimes n = l\psi(s\otimes m\otimes n), s,l\in S, m\in M, n\in N.\end{aligned}$$

Now, consider the map $\delta\colon S\otimes_R M\times S\otimes_R N \to S\otimes_R M\otimes_A N$, given by $\delta(s\otimes m, s'\otimes n) = ss'\otimes(m\otimes n)$, $m\in M$, $s, s'\in S, n\in N$. It is clear that this map is bilinear. Let $l\otimes a\in S\otimes_R A$. Then,

$$\begin{aligned}\delta(s\otimes m\cdot l\otimes a, s'\otimes n) &= \delta(sl\otimes ma, s'\otimes n) = (sl)s'\otimes(ma\otimes n) = s(ls')\otimes(m\otimes an) = \delta(s\otimes m, ls'\otimes an)\\ &= \delta(s\otimes m, (l\otimes a)\cdot(s'\otimes n)).\end{aligned}$$

So, δ induces uniquely a map $\delta' \in \mathrm{Hom}_S(S\otimes_R M\otimes_{S\otimes_R A} S\otimes_R N, S\otimes_R M\otimes_A N)$. The S-homomorphisms δ' and ψ' are inverse to each other, and thus the result follows. $\square$

Proposition 2.4. *Let S be a commutative R-algebra. Let A be a Noetherian R-algebra. Let $M \in A\text{-}\mathrm{proj}$ and $N \in A\text{-}\mathrm{mod}$. Then, $S\otimes_R \mathrm{Hom}_A(M,N) \simeq \mathrm{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R N)$ as S-modules.*

Proof. For each $M \in A\text{-}\mathrm{mod}$, consider the S-homomorphism

$$\psi_M\colon S\otimes_R \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R N),$$

given by $\psi_M(s\otimes f)(s'\otimes m) = ss'\otimes f(m)$, $s, s'\in S$, $m\in M$, $f\in \mathrm{Hom}_A(M,N)$. The homomorphism ψ_M is compatible with direct sums. This means that if M admits a decomposition $M = M_1 \oplus M2$, then there exists a commutative diagram

$$\begin{tikzcd} S\otimes_R \mathrm{Hom}_A(M_1\oplus M_2, N) \arrow[r, "\psi_{M_1\oplus M_2}"] \arrow[d, "\simeq"] & \mathrm{Hom}_{S\otimes_R A}(S\otimes_R (M_1\oplus M_2), S\otimes_R N) \arrow[d, "\simeq"] \\ S\otimes_R \mathrm{Hom}_A(X,N)\oplus S\otimes_R \mathrm{Hom}_A(Y,N) \arrow[r, "\psi_{M_1}\oplus\psi_{M_2}"] & \mathrm{Hom}_{S\otimes_R A}(S\otimes_R X, S\otimes_R N)\oplus \mathrm{Hom}_{S\otimes_R A}(S\otimes_R Y, S\otimes_R N) \end{tikzcd}.$$

Let $M = A$. Then, there exists a commutative diagram

$$\begin{array}{ccc} S\otimes_R \operatorname{Hom}_A(A,N) & \xrightarrow{\psi_A} & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R A, S\otimes_R N) \\ \simeq\Big\downarrow\psi_1 & & \simeq\Big\downarrow\psi_2 \\ S\otimes_R N & = & S\otimes_R N \end{array}.$$

In fact,

$$\psi_2\circ\psi_M(s\otimes f) = \psi_2(s\otimes f)(1_S\otimes 1_A) = s1_S\otimes f(1_A) = \psi_1(s\otimes f).$$

Therefore, ψ_A is bijective. Since ψ_M is compatible with direct sums it follows that ψ_M is an S-isomorphism whenever $M \in A$-proj. $\square$

The following result is [CPS90, Lemma 3.3.2].

Theorem 2.5. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $M \in A$-mod. Then, M is projective over A if and only if $M \in R$-proj and $M(\mathfrak{m})$ is $A(\mathfrak{m})$-projective for every maximal ideal $\mathfrak{m}$ in R.*

2.3 Relative homological algebra

In this subsection, we assume only that A is a Noetherian R-algebra. Hochschild introduced in the 1950's (see [Hoc56]) the concept of (A,R)-exact sequence. This concept did not get much attention at the time in representation theory of Noetherian algebras although it deserves more attention. Since this notion is not as commonly used in representation theory of Noetherian algebras as it might be we will recall with some detail the notions involved in this theory. An A-exact sequence $\cdots \to M_{i+1} \xrightarrow{t_{i+1}} M_i \xrightarrow{t_i} M_{i-1} \to \cdots$ is called (A,R)-exact if for each i there exists a map $h_i \in \operatorname{Hom}_R(M_i, M_{i+1})$ such that $h_{i-1}\circ t_i + t_{i+1}\circ h_i = \operatorname{id}_{M_i}$. It is a matter of bookkeeping to check that this last is equivalent to requiring that for each i, $\ker t_i = \operatorname{im} t_{i+1}$ is a summand of M_i as R-module. In this formulation, we can see that the (A,S)-short exact sequences are exactly the exact sequences of A-modules which are split as a sequence of S-modules. A homomorphism ϕ is called an (A,R)-**monomorphism** if $0 \to M \xrightarrow{\phi} N$ is (A,R)-exact. A homomorphism ϕ is called an (A,R)-**epimorphism** if $M \xrightarrow{\phi} N \to 0$ is (A,R)-exact.

An A-module Q is (A,R)**-projective** if every exact sequence of (A,R)-modules $0 \to M \to N \to Q \to 0$ splits as a sequence of R-modules. Analogously, we define (A,R)-injective modules. Due to [Hoc56, Lemma 1, Lemma 2], for each $M \in R$-Mod, $X \in \operatorname{add}(\operatorname{Hom}_R(A,M))$, $Y \in \operatorname{add}(A\otimes_R M)$, the functors $\operatorname{Hom}_A(Y,-)$ and $\operatorname{Hom}_A(-,X)$ are exact on (A,R)-exact sequences. These are exactly all the (A,R)-injective modules and (A,R)-projective modules, respectively, as we can see from the following elementary result.

Proposition 2.6. *Let $M \in A$-mod. The following assertions are equivalent.*

(a) M is (A,R)-injective, that is, every (A,R)-exact sequence $0 \to M \to V \to W \to 0$ is split over A;

(b) The natural homomorphism of A-modules $\varepsilon_M \colon M \xrightarrow{\simeq} \operatorname{Hom}_A(A,M) \to \operatorname{Hom}_R(A,M)$, $\varepsilon(m)(a) = am$, $\forall a\in A$, $m \in M$, splits over A;

(c) The functor $\operatorname{Hom}_A(-,M)$ is exact on (A,R)-exact sequences.

Proof. Assume (a). Notice that $\varepsilon'\colon \operatorname{Hom}_R(A,M) \to M$, given by $\varepsilon'(f) = f(1_A)$, $f \in \operatorname{Hom}_R(A,M)$, is an R-homomorphism since $\varepsilon'(rf) = rf(1_A) = f(1_A r) = r(f(1_A)) = r\varepsilon'(f)$, $\forall r \in R, f \in \operatorname{Hom}_R(A,M)$. Moreover, $\varepsilon'\circ\varepsilon_M = \operatorname{id}_M$. So, the exact sequence $0 \to M \xrightarrow{\varepsilon_M} \operatorname{Hom}_R(A,M) \to \operatorname{coker}\varepsilon_M \to 0$ is (A,R)-exact. By assumption, it splits over A. In particular, there exists $f \in \operatorname{Hom}_A(\operatorname{Hom}_R(A,M),M)$ satisfying $f\circ\varepsilon_M = \operatorname{id}_M$. So, (b) follows.

Assume now that (b) holds. By assumption, there exists $f \in \mathrm{Hom}_A(\mathrm{Hom}_R(A,M),M)$ such that $f \circ \varepsilon_M = \mathrm{id}_M$. Hence, $\varepsilon_M \circ f$ is an idempotent in $\mathrm{End}_A(\mathrm{Hom}_R(A,M))$. So, M is an A-summand of $\mathrm{Hom}_R(A,M)$. Thus, $\mathrm{Hom}_A(-,M)$ is exact on (A,R)-exact sequences.

Finally, assume that (c) holds. Since every (A,R)-exact sequence $0 \to M \to V \to W \to 0$ remains exact under $\mathrm{Hom}_A(-,M)$ they are split over A. $\square$

Remark 2.7. *It is immediate from Proposition 2.6 (b) that (A,R)-injective resolutions always exist. In fact, the following exact sequence*

$$\begin{tikzcd}
 & & & & & C_1 \arrow[dr, hook, "\varepsilon_{C_1}"] & & \\
0 \arrow[r] & M \arrow[r, "\varepsilon_M"] & \mathrm{Hom}_R(A,M) \arrow[rr] \arrow[dr, two heads] & & \mathrm{Hom}_R(A,C_0) \arrow[rr] \arrow[ur, two heads] & & \mathrm{Hom}_R(A,C_1) \arrow[r] & \cdots \\
 & & & C_0 \arrow[ur, hook, "\varepsilon_{C_0}"] & & & &
\end{tikzcd}$$

*is an (A,R)-injective resolution of M, where $C_i := \operatorname{coker} \varepsilon_{C_{i-1}}$ for $i \geq 1$ and $C_0 := \operatorname{coker} \varepsilon_M$. We call this resolution the **standard** (A,R)-injective resolution.*

Analogously, we have the same statement for (A,R)-projective modules. For absolute projective A-modules, we can say more.

Proposition 2.8. *Let M be a finitely generated projective left A-module. Denote $B = \mathrm{End}_A(M)^{op}$. Then, the functor $F = \mathrm{Hom}_A(M,-)$ sends (A,R)-exact sequences to (B,R)-exact sequences.*

Proof. Let $\cdots \to X_{i+1} \xrightarrow{t_{i+1}} X_i \xrightarrow{t_i} X_{i-1} \to \cdots$ be an (A,R)-exact sequence. In particular, $0 \to \ker t_i \xrightarrow{\nu_i} X_i \xrightarrow{\sigma_i} \ker t_{i-1} \to 0$ is (A,R)-exact satisfying $t_i = \nu_{i-1} \circ \sigma_i$ for all i. Applying F yields the B-exact sequence $0 \to \ker Ft_i \xrightarrow{F\nu_i} FX_i \xrightarrow{F\sigma_i} \ker Ft_{i-1} \to 0$, satisfying $Ft_i = F\nu_{i-1} \circ F\sigma_i$. So, it is enough to show that $\ker Ft_i$ is an R-summand of X_i with split monomorphism FK_i. So, it is enough to check that F sends (A,R)-monomorphisms to (B,R)-monomorphisms.

Let $0 \to Y \xrightarrow{\iota} X$ be an (A,R)-monomorphism. In particular, there exists a homomorphism $\pi \in \mathrm{Hom}_R(X,Y)$ satisfying $\pi \circ \iota = \mathrm{id}_Y$. Since $M \in A$-proj, there exists $n \in \mathbb{N}$ such that $A^n \simeq M \oplus K$. Denote $\pi_M \colon A^n \to M$ and $i_M \colon M \to A^n$ the canonical projection and inclusion, respectively. For each $i = 1, \ldots, n$, let $\pi_i \colon A^n \to A$ and $k_i \colon A \to A^n$ be the canonical projections and inclusions, respectively. Denote $\psi_X \colon \mathrm{Hom}_A(A^n, X) \to X^n$ and $\psi_Y^{-1} \colon Y^n \to \mathrm{Hom}_A(A^n, Y)$ the usual isomorphisms.

Consider $\psi := \mathrm{Hom}_A(k_M, Y) \circ \psi_Y^{-1} \circ (\pi, \cdots, \pi) \circ \psi_X \circ \mathrm{Hom}_A(\pi_M, X) \in \mathrm{Hom}_R(FX, FY)$. Let $g \in \mathrm{Hom}_A(M,Y)$ and $m \in M$. Then,

$$\begin{aligned}
\psi \circ \mathrm{Hom}_A(M,\iota)(g)(m) &= \psi(\iota \circ g)(m) = \mathrm{Hom}_A(k_M, Y) \circ \psi_Y^{-1} \circ (\pi, \cdots, \pi) \circ \psi_X \circ \mathrm{Hom}_A(\pi_M, X)(\iota \circ g)(m) \\
&= \psi_Y^{-1}((\pi, \cdots, \pi)(\psi_X(\iota \circ g \circ \pi_M)))(k_M(m)) \\
&= \psi_Y^{-1}((\pi, \cdots, \pi)(\iota \circ g \circ \pi_M \circ k_1(1_A), \cdots, \iota \circ g \circ \pi_M \circ k_n(1_A)))(k_M(m)) \\
&= \psi_Y^{-1}(g \circ \pi_M \circ k_1(1_A), \cdots, g \circ \pi_M \circ k_n(1_A))(k_M(m)) \\
&= \sum_{i=1}^{n} \pi(k_M(m)) g \circ \pi_M \circ k_i(1_A) = \sum_{i=1}^{n} g \circ \pi_M \circ k_i(\pi(k_M(m))) \\
&= g \circ \pi_M \circ k_M(m) = g(m).
\end{aligned}$$

Therefore, $\psi \circ \mathrm{Hom}_A(M,\iota) = \mathrm{id}_{FY}$. This concludes the proof. $\square$

Consequently, (A,R)-exact sequences and all the categorical notions in module categories involving (A,R)-exact sequences are Morita invariant properties. It will become clearer later on that if a functor $\mathrm{Hom}_A(N,-)$, with $N \in A$-mod, is exact on a certain (A,R)-exact sequence it will not necessarily map such (A,R)-exact sequence to relative exact sequence over the endomorphism ring.

Proposition 2.9. *Let V be a finitely generated projective right A-module. Denote $B = \operatorname{End}_A(V)$. Then, the functor $V \otimes_A -\colon A\text{-Mod} \to B\text{-Mod}$ sends (A,R)-exact sequences to (B,R)-exact sequences.*

Proof. Thanks to V being projective over A, the functors $V \otimes_A - \simeq \operatorname{Hom}_A(\operatorname{Hom}_A(V,A),A) \otimes_A - \simeq \operatorname{Hom}_A(\operatorname{Hom}_A(V,A),-)$ are equivalent. Since $\operatorname{Hom}_A(V,A) \in A\text{-proj}$, it follows by Proposition 2.8, that $V \otimes_A -$ sends (A,R)-exact sequences to (B,R)-exact sequences. $\square$

Thanks to the existence of the maps $h_i \in \operatorname{Hom}_R(M_i, M_{i+1})$ satisfying $h_{i-1} \circ t_i + t_{i+1} \circ h_i = \operatorname{id}_{M_i}$ for a given (A,R)-exact sequence t, the standard duality $D = \operatorname{Hom}_R(-,R)$ maps (A,R)-exact sequences to (A^{op},R)-exact sequences.

2.3.1 Forgetful functors

We say that we have a relative homological algebra if we choose an abelian category together with a class of exact sequences. A relative abelian category in the sense of Mac Lane [Mac95] consists of the following data: a pair of abelian categories $(\mathcal{A},\mathcal{B})$ together with a covariant additive, exact and faithful functor $F\colon \mathcal{A} \to \mathcal{B}$.

Consider the forgetful functor $F\colon A\text{-Mod} \to R\text{-Mod}$. Since it is a forgetful functor, it is faithful. This functor preserves biproducts, hence it is additive. Consider the functors $G, H\colon R\text{-Mod} \to A\text{-Mod}$, given by $GM = \operatorname{Hom}_R(A,M)$, $HM = A \otimes_R M$, and $Gf = \operatorname{Hom}_R(A,f)$, $Hf = A \otimes_R f$. It follows by tensor-hom adjunction that the functor G is a right adjoint of F and H is a left adjoint of F. The existence of left and right adjoint functors imply that F preserves all finite limits and all finite colimits. In particular, it preserves kernels and cokernels. Hence F is exact. In view of [Mac95, Chapter 9, 4], a short exact sequence of A-modules $0 \to X \to Y \to Z \to 0$ is said to be F-allowable if the exact sequence $0 \to FX \to FY \to FZ \to 0$ splits over R. These are exactly the (A,R)-exact sequences. We saw that the objects for which $\operatorname{Hom}_A(P,-)$ is exact on (A,R)-exact sequences are exactly the modules $P = A \otimes_R X$, $X \in R\text{-Mod}$. Conversely, using tensor-hom adjunction, we can see that the class of exact sequences which remains exact under $\operatorname{Hom}_A(A \otimes_R X, -)$ are exactly the (A,R)-exact sequences.

Nowadays, the most common approach to relative homological algebra is to first consider a class of objects $\mathcal{P}$ of an abelian category $\mathcal{A}$. Then, we can compute the class of exact sequences for which the class of objects $\mathcal{P}$ remain exact under $\operatorname{Hom}_{\mathcal{A}}(P,-)$ for every $P \in \mathcal{P}$. The class of (A,R)-exact sequences is closed in the sense of [EM65]. That is, these two approaches are equivalent for (A,R)-exact sequences. The literature of relative homological algebra, extending the classical homological algebra theory, is well developed in this point of view for Artinian algebras A. Hence the interested reader can obtain further properties on (A,R)-exact sequences like relative Ext and Tor functors by using the same arguments as the ones presented in [EJ11].

2.3.2 More details on relative injective modules

We will now shift our attention to modules that belong in $A\text{-mod} \cap R\text{-proj}$ once again assuming that A is a projective Noetherian R-algebra unless stated otherwise. Because of A being projective over R, the absolute projectives of A-mod are exactly the relative projectives of $A\text{-mod} \cap R\text{-proj}$. So, our interest will now be in the relative injective modules. Denote by (A,R)-inj the full subcategory of A-mod whose modules are (A,R)-injective.

Proposition 2.10. *Let $I \in A\text{-mod} \cap R\text{-proj}$. I is (A,R)-injective if and only if $\operatorname{Ext}_A^1(M,I) = 0$ for all $M \in A\text{-mod} \cap R\text{-proj}$. Moreover, if I is (A,R)-injective, then $\operatorname{Ext}_A^{i>0}(M,I) = 0$ for all $M \in A\text{-mod} \cap R\text{-proj}$.*

Proof. Suppose that I is (A,R)-injective. Any A-exact sequence $0 \to I \to X \to M \to 0$, with $M \in R\text{-proj}$, is (A,R)-exact and so it is split over A. Consider an A-projective resolution for $M \in A\text{-mod} \cap R\text{-proj}$, $\cdots \to P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$. In particular, there are (A,R)-exact sequences

$0 \to \operatorname{im}\alpha_j \to P_{j-1} \to \operatorname{im}\alpha_{j-1} \to 0$, thanks to the fact that $M \in R$-proj and consequently for every $j \geq 0$, $\operatorname{im}\alpha_j \in R$-proj. So, $\operatorname{Ext}_A^i(M, I) \simeq \operatorname{Ext}_A^1(\operatorname{im}\alpha_{i-1}, I) = 0$ for every $i > 0$.

Conversely, assume that $\operatorname{Ext}_A^1(M, I) = 0$ for all $M \in A\text{-mod} \cap R\text{-proj}$. Let $I \to \operatorname{Hom}_R(A, I)$ be the standard (A, R)-injective copresentation of I with cokernel X. Since A is projective over R, $\operatorname{Hom}_R(A, I)$ is R-projective making X an R-projective module. By assumption, the injective copresentation must split over A and therefore I is (A, R)-injective. $\square$

In [Rou08], the modules $I \in A\text{-mod} \cap R\text{-proj}$ satisfying the property $\operatorname{Ext}_A^1(M, I) = 0$ for all $M \in A\text{-mod} \cap R\text{-proj}$ are called relatively R-injective. Therefore, the relatively R-injective modules are exactly the (A, R)-injective modules which are R-projective. Furthermore, this characterization says that the (A, R)-injective modules which are R-projective are exactly the objects X of $\mathcal{A} = A\text{-mod} \cap R\text{-proj}$ which make $\operatorname{Hom}_{\mathcal{A}}(-, X)$ exact on $\mathcal{A}$.

Lemma 2.11. *Let $M \in A\text{-mod} \cap R\text{-proj}$. M is (A, R)-injective if and only if DM is A^{op}-projective.*

Proof. Let P be a projective (right) A-module. Then, DP is an A-summand of $\operatorname{Hom}_R(A^t, R) \simeq \operatorname{Hom}_R(A, R)^t$. Hence, DP is an (A, R)-injective left module.

Let M be an (A, R)-injective and projective R-module. Then, M is an A-summand of $\operatorname{Hom}_R(A, M)$. Note that

$$
\begin{aligned}
D\operatorname{Hom}_R(A, M) &\simeq \operatorname{Hom}_R(\operatorname{Hom}_R(A, M), R) \simeq \operatorname{Hom}_R(\operatorname{Hom}_R(A, R) \otimes_R M, R) && (1)\\
&\simeq \operatorname{Hom}_R(M, \operatorname{Hom}_R(\operatorname{Hom}_R(A, R), R)) \simeq \operatorname{Hom}_R(M, A) \simeq \operatorname{Hom}_R(M, R) \otimes_R A && (2)\\
&= A \otimes_R DM. && (3)
\end{aligned}
$$

As $DM \in R$-proj, $D\operatorname{Hom}_R(A, M) \in A^{op}$-proj and consequently $DM \in A^{op}$-proj. $\square$

Using this Lemma, we can formulate the dual version of Theorem 2.5.

Corollary 2.12. *Let A be a projective Noetherian R-algebra. Let $P \in A\text{-mod} \cap R\text{-proj}$. Then, P is (A, R)-injective if and only if $P(\mathfrak{m})$ is an injective $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R.*

Proof. Assume that P is (A, R)-injective. Then, DP is (A^{op}, R)-projective. Since $P \in R$-proj, $DP \in A^{op}$-proj. Let $\mathfrak{m}$ be a maximal ideal in R. Then, $DP(\mathfrak{m}) = \operatorname{Hom}_R(P, R)(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(P(\mathfrak{m}), R(\mathfrak{m}))$ is a projective right $A(\mathfrak{m})$-module. Thus, $P(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(\operatorname{Hom}_{R(\mathfrak{m})}(P(\mathfrak{m}), R(\mathfrak{m})), R(\mathfrak{m}))$ is an injective left $A(\mathfrak{m})$-module.

Conversely, assume that $P(\mathfrak{m})$ is an injective left $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R. Then, for every maximal ideal $\mathfrak{m}$ of R, $\operatorname{Hom}_{R(\mathfrak{m})}(P(\mathfrak{m}), R(\mathfrak{m})) \simeq \operatorname{Hom}_R(P, R)(\mathfrak{m})$ is a projective right $A(\mathfrak{m})$-module. Thus, $DP = \operatorname{Hom}_R(P, R)$ is an projective right A-module since $DP \in R$-proj. Hence, $P \simeq DDP$ is (A, R)-injective. $\square$

Remark 2.13. *In this sense, relative injective modules can be viewed as a natural generalization of injective modules of finite dimensional algebras.*

Further evidence that (A, R)-monomorphisms behave like the inclusions between modules over finite dimensional algebras is the following version of Nakayama's Lemma for (A, R)-monomorphisms.

Lemma 2.14. *If $\phi\colon M \to N$ is (A, R)-monomorphism and $M \simeq N$ as finitely generated R-modules, then ϕ is an isomorphism.*

Proof. Since ϕ is (A, R)-mono, there exists $\varepsilon\colon N \to M$ such that $\varepsilon \circ \phi = \mathrm{id}_M$. Thus, ε is surjective. By Nakayama's Lemma, ε is an R-isomorphism. Therefore, $\phi = \varepsilon^{-1} \circ \varepsilon \circ \phi = \varepsilon^{-1}$ is bijective. $\square$

For Artinian rings, a module is a cogenerator if and only if contains all injective indecomposable modules. However, we are only interested in the relative injective modules which are projective over the ground ring. Thus, for our purposes, we can relax the notion of cogenerator. By an (A, R)-cogenerator we mean an A-module Q whose additive closure contains the module DA_A.

The following Lemma also holds for Noetherian R-algebras.

Lemma 2.15. *Let A be a projective Noetherian R-algebra. Let $M, N \in A^{op}\text{-mod} \cap R\text{-proj}$. Then, $\operatorname{Ext}^i_{A^{op}}(M,N) \simeq \operatorname{Ext}^i_A(DN, DM)$ for any $i \geq 0$.*

Proof. Let $M^\bullet$ be a left A^{op}-projective resolution $\cdots \to M_1 \xrightarrow{\alpha_1} M_0 \xrightarrow{\alpha_0} M \to 0$. Since $M \in R$-proj, all $\operatorname{im} \alpha_i \in R$-proj. Hence, $M^\bullet$ is an (A^{op}, R)-projective resolution. Moreover, applying D to $M^\bullet$ yields the exact sequence $0 \to DM \xrightarrow{D\alpha_0} DM_0 \to DM_1 \to \cdots$, since $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > 0$. Each DM_i is (A,R)-injective. Thus, $DM^\bullet$ is an (A,R)-injective resolution of DM. The following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{A^{op}}(M_0, N) \arrow[r, "\operatorname{Hom}_{A^{op}}(\alpha_1,\ N)"] \arrow[d, "\psi_{M_0,N}"] & \operatorname{Hom}_{A^{op}}(M_1, N) \arrow[r] \arrow[d, "\psi_{M_1,N}"] & \cdots \\
0 \arrow[r] & \operatorname{Hom}_A(DN, DM_0) \arrow[r, "\operatorname{Hom}_A(DN, D\alpha_1)"] & \operatorname{Hom}_A(DN, DM_1) \arrow[r] & \cdots
\end{tikzcd}.$$

Hence,

$$\operatorname{Ext}^i_{A^{op}}(M,N) = H^i(\operatorname{Hom}_{A^{op}}(M^\bullet, N)) = H^i(\operatorname{Hom}_A(DN, DM^\bullet)) = \operatorname{Ext}^i_{(A,R)}(DN, DM). \tag{4}$$

Here, $\operatorname{Ext}_{(A,R)}$ denotes the relative Ext functor. Due to every A-projective resolution for $DN \in R$-proj being (A,R)-exact, it follows that $\operatorname{Ext}^i_{(A,R)}(DN, DM) = \operatorname{Ext}^i_A(DN, DM)$ for every $i \geq 0$. ☐

3 Relative QF3 R-algebras

Now, we are ready to introduce the concept of relative dominant dimension of projective Noetherian algebras (Definition 3.1) and to explain what it means for a module over a Noetherian algebra to have positive relative dominant dimension (Proposition 3.4). This endeavour leads us to study modules which are simultaneously projective relative injective and strongly faithful. The latter concept, to be defined in 3.5, will become very natural to consider once we know the definition of relative dominant dimension. Further, we show here that for relative self-injective algebras, strongly faithful modules are exactly the generator objects in the module category (Theorem 3.12). We will extend some known results of Tachikawa [Tac73] for QF3 algebras to this integral setup and we end this section by developing the analogue of Mueller's characterization for smaller levels of relative dominant dimension, that is, for values of relative dominant dimension one or two (Lemma 3.21 and Proposition 3.23). This can be viewed as the preparations for the relative Morita-Tachikawa correspondence.

3.1 Definition of relative dominant dimension

Definition 3.1. Let $M \in A$-mod. We say that M has **relative dominant dimension at least** $t \in \mathbb{N}$ if there exists an (A,R)-exact sequence of finitely generated left A-modules

$$0 \to M \to I_1 \to \cdots \to I_t \tag{5}$$

with I_i both A-projective and (A,R)-injective. If M admits no such (A,R)-exact sequence, then we say that M has relative dominant dimension zero. Otherwise, the relative dominant dimension of M is the supremum of the set of all values t such that an (A,R)-exact sequence of the form 5 exists. We denote by $\operatorname{domdim}_{(A,R)} M$ the relative dominant dimension of M.

Analogously, we can define relative dominant dimension for right A-modules.

Note that if R is a field, A is a finite-dimensional algebra and $\operatorname{domdim}_{(A,R)} M$ is exactly the dominant dimension over A of M.

Proposition 3.2. *(A,R)-dominant dimension is invariant under Morita equivalence.*

Proof. Let B be an algebra which is Morita equivalent to A. Thus, B is a projective Noetherian R-algebra. Since (A,R)-exact sequences and (A,R)-injective modules are preserved under equivalence of module categories it follows that (A,R)-dominant dimension is invariant under Morita equivalence. $\square$

Observe that since the zero module is projective and relative injective, if a module admits a finite projective (A,R)-injective coresolution, then it has infinite relative dominant dimension. We can make more precise the case of infinite relative dominant dimension for a module in A-mod$\cap R$-proj with finite relative injective dimension. In view of Proposition 2.10, the **relative injective dimension** of $M \in A\text{-mod}\cap R\text{-proj}$ is the minimum number n (if it exists) such that $\operatorname{Ext}_A^{n+1}(N,M)=0$ for every $N \in A\text{-mod}\cap R\text{-proj}$. The relative injective dimension of $M \in A\text{-mod}\cap R\text{-proj}$ is infinite if no such a number n exists. Hence, the dual of the usual characterizations of projective dimension can be used for relative injective modules. We will denote by $\operatorname{injdim}_{(A,R)} M$ the relative injective dimension of $M \in A\text{-mod}\cap R\text{-proj}$.

Proposition 3.3. *Let $M \in A\text{-mod}\cap R\text{-proj}$ having $\operatorname{injdim}_{(A,R)} M < \infty$. The following assertions are equivalent.*

(a) $\operatorname{domdim}_{(A,R)} M = +\infty$*;*

(b) M is A-projective and (A,R)-injective.

Proof. Assume that (b) holds. Consider the (A,R)-exact sequence $0 \to M \to M \to 0$. By Definition 3.1, (a) holds.

Assume that (a) holds. In particular, $\operatorname{domdim}_{(A,R)} M \geq t = \operatorname{injdim}_{(A,R)} M$ so there exists an (A,R)-exact sequence $0 \to M \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_t} I_t$ with I_i both A-projective and (A,R)-injective (possibly with some of them being zero). Thanks to Proposition 2.10 together with $\operatorname{Ext}_A^{t+1}(L,M) \simeq \operatorname{Ext}^1_{(A,R)}(L, \operatorname{im}\alpha_t) = 0$ for every $L \in A\text{-mod}\cap R\text{-proj}$, we obtain that $\operatorname{im}\alpha_t$ is (A,R)-injective. So, it is an A-summand of I_t. Thus, it is also A-projective. Further, the exact sequence $0 \to M \to I_0 \to \cdots \to \operatorname{im}\alpha_t \to 0$ splits over A. Hence, $M \in \operatorname{add} I_0$. $\square$

The result can be generalized for modules in A-mod if one defines relative injective dimension in terms of relative Ext functors which is a theme that we will not pursue here.

3.2 Modules with relative dominant dimension at least one

The following result characterizes the modules with relative dominant dimension at least one.

Proposition 3.4. *Let $M \in A$-mod. Then,* $\operatorname{domdim}_{(A,R)} M > 0$ *if and only if M is an (A,R)-submodule of a (left) module that is both A-projective and (A,R)-injective. In particular,* $\operatorname{domdim}(A,R)_A A > 0$ *if and only if A is an (A,R)-submodule of an A-projective (A,R)-injective (left) module.*

Proof. Assume that M is not an (A,R)-submodule of a (left) module that is both A-projective and (A,R)-injective. Assume by contradiction that $\operatorname{domdim}_{(A,R)} M > 0$. Then, there exists by definition an (A,R)-monomorphism $M \to I_1$ with $I_1 \in A\text{-proj}\cap(A,R)\text{-inj}$. This contradicts our assumption. Then, $\operatorname{domdim}_{(A,R)} M = 0$. Conversely, assume that $\operatorname{domdim}_{(A,R)} M = 0$. By contradiction assume that M is an (A,R)-submodule of a (left) module that is both A-projective and (A,R)-injective, say N. Then, the monomorphism $M \to N$ is (A,R)-exact and by the definition we get $\operatorname{domdim}_{(A,R)} M > 0$. $\square$

As a consequence, we see that every module with positive relative dominant dimension is projective over the ground ring. Moreover, if A is an order, modules with positive relative dominant dimension are exactly the R-pure A-submodules of a projective relative injective module. (see [Rei03]). In the classical theory of dominant dimension the analogue of Proposition 3.4 motivates us to study faithful projective-injective modules. Indeed, faithful finitely generated modules M can be characterized by the existence of an A-monomorphism of the regular module A into a finite direct sum of copies of M. The natural choice is to consider a generator set of M, say $\{m_1, \dots, m_t\}$, together with the A-homomorphism $A \to M^t$, given by $a \mapsto (am_1, \dots, am_t)$. For faithful A-modules, this homomorphism is always injective. For Artinian algebras, such characterization is also valid for non-finitely generated modules since descending chains of intersection of kernels of homomorphisms are finite. The reason of interest in such a characterization of faithfulness comes from the fact this turns finitely generated faithful modules and faithful modules over Artinian algebras into a categorical concept. However, the bigger problem comes from the fact that even if M is a faithful finitely generated module, a homomorphism of A into a direct sum of copies of M is not necessarily split over R, whenever A is a Noetherian algebra. The reader may think for example of an order contained in an over-order.

For now, another thing to keep in mind that is different from the Artinian case is A-mod not being a Krull-Schmidt category, in general.

3.3 Strongly faithful modules

As discussed, in relative dominant dimension theory, faithful modules without further properties no longer play a key role in the study of relative dominant dimension of Noetherian algebras. Here they are replaced by the following concept.

Definition 3.5. We say that a (left) module M is (A, R)**-strongly faithful** if there is an (A, R)-monomorphism ${}_AA \hookrightarrow M^t$ for some $t > 0$. The definition for right modules is analogous.

If R is a field, then A becomes a finite dimensional algebra. Thus, if R is a field, then (A, R)-strongly faithful coincides with faithful.

Any generator of A-mod is (A, R)-strongly faithful. Because of M being a generator of A-mod there exists $t > 0$ such that $M^t \simeq A \oplus K$ as A-modules. In particular, the canonical monomorphism $A \hookrightarrow M^t$ splits over A, and thus is an (A, R)-monomorphism.

In terms of relative dominant dimension, Proposition 3.4 says that an algebra has relative dominant dimension greater or equal than one if and only if it has an (A, R)-strongly faithful, A-projective (A, R)-injective module. By an (A, R)**-injective-strongly faithful module** we mean a module that is simultaneously (A, R)-injective and (A, R)-strongly faithful.

Any (A, R)-strongly faithful contains as summand a minimal (A, R)-strongly faithful module in the following sense.

Proposition 3.6. *Let M be a finitely generated A-projective and (A, R)-injective-strongly faithful module. Then, there exists an (A, R)-strongly faithful module $N \in \operatorname{add}_A M$ which does not contain any proper (A, R)-strongly faithful module as A-summand.*

Proof. If M does not contain a proper (A, R)-strongly faithful module as A-summand, then we are done. Otherwise, we can write $M \simeq N_0 \bigoplus K_0$ where N_0 is an (A, R)-strongly faithful module. Then, we can apply the same reasoning to N_0. After a finite number of steps, we can construct a proper chain

$$0 \subsetneq K_0 \subsetneq K_1 \bigoplus K_0 \subsetneq \cdots \subsetneq K_n \bigoplus \cdots \bigoplus K_0. \tag{6}$$

Since M is a Noetherian module, this chain must stabilize. Hence this construction must stop after a finite number of steps, say t. The module N_{t-1} belongs to $\operatorname{add} M$ and does not contain any proper (A, R)-strongly faithful module as A-summand. ☐

Lemma 3.7. *Let M be a finitely generated A-projective and (A, R)-injective-strongly faithful module. Then, every A-projective (A, R)-injective module belongs to $\operatorname{add} M$. In particular, all endomorphism rings of modules N being finitely generated A-projective and (A, R)-injective-strongly faithful are Morita equivalent.*

Proof. Let N be a projective and (A,R)-injective A-module. Since $N \in A$-proj then there is an $n \in \mathbb{Z}_0^+$ and $L \in A$-mod such that $A^n \simeq N \bigoplus L$. Denote by k_N and π_N the canonical injection and projection, respectively. Since M is (A,R)-strongly faithful, there exists $i \in \operatorname{Hom}_A(A, M^t)$ and $\pi \in \operatorname{Hom}_R(M^t, A)$ such that $\pi \circ i = \operatorname{id}_A$. Define $f = (i, \cdots, i) \circ k_N \in \operatorname{Hom}_A(N, M^{tn})$. Then,

$$\pi_N \circ (\pi, \cdots, \pi) \circ f = \pi_N \circ (\pi, \cdots, \pi) \circ (i, \cdots, i) \circ k_N = \pi_N \circ \operatorname{id}_{A^n} \circ k_N = \operatorname{id}_N . \tag{7}$$

Thus, f is an (A,R)-monomorphism. Since N is (A,R)-injective f splits over A. In particular, $N \in \operatorname{add}_A M$.

If N is also (A,R)-strongly faithful, then by reversing the roles of M and N, we obtain $M \in \operatorname{add} N$. Thus, $\operatorname{add} N = \operatorname{add} M$. This concludes the proof. $\square$

For projective Noetherian algebras, it is easier to check the double centralizer property in the presence of (A,R)-strongly faithful modules. Using Nakayama's Lemma for (A,R)-monomorphisms 2.14, we can extend Lemma 2.1 of [KY14] to Noetherian algebras.

Proposition 3.8. *Let M be an (A,R)-strongly faithful and $B = \operatorname{End}_A(M)^{op}$. Then, the following assertions are equivalent.*

(i) (A,M) satisfies the double centralizer property, that is, the canonical map $A \to \operatorname{End}_B(M)$ is an R-isomorphism of algebras.

(ii) $A \simeq \operatorname{End}_B(M)$ as R-modules.

(iii) $A \simeq \operatorname{End}_B(M)$ as R-algebras.

Proof. $i) \Rightarrow iii) \Rightarrow ii)$ is clear. We shall prove $ii) \Rightarrow i)$. Denote by ρ the canonical map of R-algebras $A \to \operatorname{End}_B(M)$. Since M is (A,R)-strongly faithful, there are maps $i \in \operatorname{Hom}_A(A, M^t)$, $\varepsilon \in \operatorname{Hom}_R(M^t, A)$, for some t, satisfying $\varepsilon \circ i = \operatorname{id}_A$. Let π_j and k_j be the canonical projections and injections of the direct sum M^t, $j = 1, \ldots, t$, respectively. Consider $\psi \colon \operatorname{End}_B(M) \to A$, given by $\psi(f) = \sum_j \varepsilon \circ k_j \circ f(\pi_j \circ i(1_A))$ for each $f \in \operatorname{End}_B(M)$. This is an R-map and

$$\begin{aligned} \psi \circ \rho(a) &= \sum_j \varepsilon \circ k_j \circ \rho(a)(\pi_j \circ i(1_A)) = \sum_j \varepsilon \circ k_j(a\pi_j \circ i(1_A)) \\ &= \sum_j \varepsilon \circ k_j(\pi_j(i(a))) = \varepsilon \circ \sum_j k_k \circ \pi_j i(a) = \varepsilon \circ i(a) = a \end{aligned}$$

Hence, ρ is (A,R)-monomorphism. By Lemma 2.14, since $A \simeq \operatorname{End}_B(M)$ as finitely generated R-modules, it follows that ρ is an isomorphism. By definition, (A,M) satisfies the double centralizer property. $\square$

3.3.1 Relative self-injective algebras

(A,R)-strongly faithful modules play an important role for relative self-injective algebras in the same fashion that faithful modules play an important role for self-injective Artinian algebras.

Definition 3.9. An R-algebra B is called **relative (left) self-injective** if ${}_BB$ is (B,R)-injective.

For projective Noetherian R-algebras the notions of relative left and relative right self-injective R-algebra are equivalent.

Proposition 3.10. *Let B be a projective Noetherian R-algebra. Then, B is a relative left self-injective R-algebra if and only if B is a relative right self-injective R-algebra.*

Proof. Assume that B is a relative right self-injective R-algebra. Then, B is (B, R)-injective as a right module. By Corollary 2.12, $B(\mathfrak{m})$ is an injective right $B(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R. In particular, every right module being projective over $B(\mathfrak{m})$ is injective over $B(\mathfrak{m})$. It is well known that this implies that every finitely generated $B(\mathfrak{m})$-injective module is $B(\mathfrak{m})$-projective ([ARS95, IV. 3]). In particular, $\mathrm{Hom}_{R(\mathfrak{m})}(B(\mathfrak{m}), R(\mathfrak{m}))$ is $B(\mathfrak{m})$-projective as a right module. So, $B(\mathfrak{m})$ is $B(\mathfrak{m})$-injective as a left module for every maximal ideal $\mathfrak{m}$ in R. Again, by Theorem 2.12, B is left (B, R)-injective. Thus, B is a relative left self-injective R-algebra. □

Projective Noetherian R-algebras which are relative self-injective were considered several times during the 1960s. For example, the structure of these algebras that have global dimension at most one was determined in [End67].

Before we show their relation with strongly faithful modules, we shall see that these algebras are quite common. In fact, a class of examples of relative self-injective algebras are the group algebras over a commutative ring. This fact is folklore and its proof is essentially the same as for finite dimensional algebras (see [CR06, (62.1)]).

Proposition 3.11. *For every finite group G, the group algebra RG is a relative self-injective R-algebra for any commutative ring R.*

Proof. Consider the R-linear map $\pi\colon RG \to R$, given by $\pi(g) = \mathbb{1}_{\{e\}}(g)1_R$, $g \in G$, where e denotes the identity element of G. Define the RG-map $\phi\colon RG \to DRG$, given by $\phi(g)(h) = \pi(gh)$ for every $h \in RG$. Note that

$$\phi(hg)(x) = \pi((hg)x) = \pi(h(gx)) = \phi(h)(gx) = \phi(h)g(x), \forall g, h, x \in G. \tag{8}$$

Thus, ϕ is an RG-right homomorphism. We claim that ϕ is injective. In fact, let $x = \sum_{g\in G} x_g g \in \ker \phi$. Then, for all $h \in G$,

$$0 = \phi(x)(h) = \pi(xh) = \pi(\sum_{g\in G} x_g gh) = \sum_{g\in G} x_g \mathbb{1}_{\{e\}}(gh) = x_{h^{-1}}. \tag{9}$$

We shall now prove that ϕ is surjective. Observe that elements $g^* \in DRG$, given by $g(h) = \mathbb{1}_{\{g\}}(h)1_R$, $h \in G$, form an R-basis of DRG. Moreover, $g^*(\sum_{g\in G} h_g g) = h_g$. We claim that $\phi(g^{-1}) = g^*$ for every $g \in G$. In fact,

$$\phi(g^{-1})(x) = \pi(g^{-1} \sum_{h\in G} x_h h) = \sum_{h\in G} x_h \mathbb{1}_{\{e\}\}}(g^{-1}h) = \sum_{h\in G} x_h \mathbb{1}_{\{g\}}(h) = x_g = g^*(x), \quad \forall x \in RG. \tag{10}$$

Therefore, $RG \simeq D(RG)$ as right RG-modules. Consequently $RG \simeq DDRG \simeq D(RG)$ as left RG-modules, since $RG \in R\text{-proj}$. Hence RG is (RG, R)-injective. □

Here, $\mathbb{1}_A$ denotes the indicator function of a set A.

Theorem 3.12. *Let B be a relative (left and right) self-injective R-algebra. Let M be a (B, R)-strongly faithful module. Then, M is a generator (B, R)-cogenerator and it satisfies a double centralizer property: $A = \mathrm{End}_B(M)^{op}$ and $B = \mathrm{End}_A(M)$.*

Proof. Since M is (B, R)-strongly faithful, there exists a (B, R)-monomorphism $0 \to B \to M^t$. As B is (B, R)-injective, this monomorphism splits over B. Hence $B \in \mathrm{add}\, M$. In particular, M is a generator of B-mod. Since double centralizer properties hold on generators, it follows that $B \simeq \mathrm{End}_A(M)$ with $A = \mathrm{End}_B(M)$. Since B is right self-injective algebra then B_B belongs to $\mathrm{add}\, D_B B$. Consequently, DB_B belongs to $\mathrm{add}_B\, B \subset \mathrm{add}\, M$. So, M is a B-generator (B, R)-cogenerator. □

Note that every relative self-injective R-algebra has infinite relative dominant dimension. Indeed, we can consider the (A, R)-exact sequence $0 \to A \to A \to 0$. In parallel, we conjecture the following relative version of Nakayama conjecture:

Conjecture 3.13. *Given a projective Noetherian R-algebra,* $\operatorname{domdim}(A, R) = +\infty$ *if and only if A is a relative (left and right) self-injective R-algebra.*

As we will see afterwards in Theorem 6.17, this conjecture is equivalent to the Nakayama conjecture.

Theorem 3.12 motivates us to study endomorphism rings of generators-relative cogenerators. For finite dimensional algebras over a field, they can be characterized using dominant dimension. In order to obtain a relative version of this fact for Noetherian algebras, we need first to introduce another definition of relative dominant dimension.

3.4 Dominant dimension with respect to a projective relative injective module

We will now introduce an alternative definition of relative dominant dimension. This will be extremely useful for the arguments in the proof of relative Morita-Tachikawa correspondence.

Definition 3.14. Let P be an A-projective (A, R)-injective module. Let $X \in A\text{-mod} \cap R\text{-proj}$. If X is not an (A, R)-submodule of some module in the additive closure of P, then we say that the **relative dominant dimension of** X **with respect to** P is zero. Otherwise, the **relative dominant dimension of** X **with respect to** P, denoted by $P - \operatorname{domdim}_{(A,R)} X$, is the supremum of all $n \in \mathbb{N}$ such that there exists an (A, R)-exact sequence

$$0 \to X \to P_1 \to \cdots \to P_n \tag{11}$$

with all $P_i \in \operatorname{add}_A P$.

By convention, the empty direct sum is the zero module. So, the existence of a finite relative $\operatorname{add} P$-coresolutions implies that $P - \operatorname{domdim}_{(A,R)} X$ is infinite. In the same way, we can define the relative dominant dimension of a right module with respect a right projective relative injective module Q.

Definition 3.14 generalizes the concept of relative dominant dimension introduced in 3.1 as we can see in the following Proposition. Furthermore, this is a generalization of [Tac73, 7.3, 7.7].

Proposition 3.15. *Assume that A is a projective Noetherian R-algebra with A-projective* (A, R)*-injective-strongly faithful left A-module P. Then,*

$$P - \operatorname{domdim}_{(A,R)} X = \operatorname{domdim}_{(A,R)} X, \quad X \in A\text{-mod}. \tag{12}$$

Proof. Assume that there exists an (A, R)-exact sequence

$$0 \to X \to X_1 \to X_2 \to \cdots \to X_n \tag{13}$$

with X_i an A-projective (A, R)-injective left module for all $i \geq 1$. Since all X_i are projective there exists k_i such that $A^{k_i} \simeq X_i \oplus K_i$. Choose $k = \max\{k_1, \ldots, k_n\}$. So, each X_i can be embedded in A^k as A-summand. Denote by $f_i \colon X_i \to A^{k_i}$, $g_i \colon A^{k_i} \to A^k$ the canonical injections and denote by $f_i' \colon A^{k_i} \to X_i$, $g_i' \colon A^k \to A^{k_i}$ the canonical projections. Since P is (A, R)-strongly faithful there exists an (A, R)-monomorphism $l \colon A \to P^t$ for some $t > 0$. Hence, there exists $\pi \in \operatorname{Hom}_R(V^t, A)$ such that $\pi \circ l = \operatorname{id}_A$. Then, the composition $(\oplus_{j=1}^k l) \circ g_i \circ f_i \in \operatorname{Hom}_A(X_i, P^{tk})$ is an (A, R)-monomorphism. In fact, $f_i' \circ g_i' \circ (\oplus_{j=1}^k \pi) \in \operatorname{Hom}_R(V^{tk}, X_i)$ satisfies

$$f_i' \circ g_i' \circ (\oplus_{j=1}^k \pi) \circ (\oplus_{j=1}^k l) \circ g_i \circ f_i = \operatorname{id}_{X_i}.$$

As X_i is (A, R)-injective, then the map $(\oplus_{j=1}^k l) \circ g_i \circ f_i$ splits over A. Therefore, X_i is an A-summand of P^{tk}, hence $X_i \in \operatorname{add} P$.

If some $X_i = 0$, then $\operatorname{domdim}_{(A,R)} X = +\infty = P-\operatorname{domdim}_{(A,R)} X$. This shows that if $\operatorname{domdim}_{(A,R)} X \geq n$, then $P - \operatorname{domdim}_{(A,R)} X \geq n$. Hence $\operatorname{domdim}_{(A,R)} X \leq P - \operatorname{domdim}_{(A,R)} X$.

Now since each module in $\operatorname{add} P$ is projective (A, R)-injective, it follows that $P - \operatorname{domdim}_{(A,R)} X \leq \operatorname{domdim}_{(A,R)} X$. This concludes the proof. $\square$

Analogously, we have the right version,

Proposition 3.16. *Assume that A is a projective Noetherian R-algebra with a projective (A,R)-injective-strongly faithful right A-module V. Then,*

$$V - \operatorname{domdim}_{(A,R)} X = \operatorname{domdim}_{(A,R)} X, \quad X \in \text{mod-}A. \tag{14}$$

Proof. It is analogous to Proposition 3.15. □

3.5 Modules with relative dominant dimension at least two

For given $X \in A$-mod, $V \in$ mod-A, denote by C the endomorphism algebra $\operatorname{End}_A(V)$ and by α_X the map $X \to \operatorname{Hom}_C(V, V \otimes_A X)$ given by $\alpha_X(x)(v) = v \otimes x$, $v \in V$, $x \in X$. This is an $(A, \operatorname{End}_A(X)^{op})$-bimodule homomorphism. In fact,

$$\alpha_X(a \cdot x)(v) = v \otimes ax = va \otimes x = \alpha_X(x)(va) = (a \cdot \alpha_X(x))(v),\ a \in A,\ v \in V,\ x \in X$$
$$\alpha_X(x \cdot b)(v) = \alpha_X(b(x))(v) = v \otimes b(x) = v \otimes (x \cdot b) = (v \otimes x) \cdot b = (\alpha_X(x) \cdot b)(v),\ b \in \operatorname{End}_A(X)^{op},\ v \in V,\ x \in X.$$

In addition, α is a natural transformation between the functors $\operatorname{Id}_{A\text{-mod}}$ and $\operatorname{Hom}_C(V, -) \circ V \otimes_A -$.

The following two lemmas, although being very technical, are crucial to our purposes. We note also the following lemma involving the Schur functor which will be essential to relative dominant dimension.

Lemma 3.17. *Let $V \in$ proj-A. Let $C = \operatorname{End}_A(V)$ and the functors $F = V \otimes_A -\colon A$-mod $\to C$-mod $G = \operatorname{Hom}_C(V, -)\colon C$-mod $\to A$-mod. The composition of functors $F \circ G\colon C$-mod $\to C$-mod is an equivalence of categories. Moreover $\xi_M\colon V \otimes_A \operatorname{Hom}_C(V, M) \to M$, given by $\xi_M(v \otimes \phi) = \phi(v), v \in V, \phi \in \operatorname{Hom}_C(V, M)$ is a natural isomorphism for every $M \in C$-mod.*

Proof. Fix $f \in \operatorname{Hom}_C(M, N)$. We have the commutative diagram,

$$\begin{tikzcd} V \otimes_A \operatorname{Hom}_C(V, M) \arrow[r, "\xi_M"] \arrow[d, "V \otimes_A \operatorname{Hom}_C(V,f)"'] & M \arrow[d, "f"] \\ V \otimes_A \operatorname{Hom}_C(V, N) \arrow[r, "\xi_N"] & N \end{tikzcd} .$$

In fact, $\xi_N \circ V \otimes_A \operatorname{Hom}_C(V, f)(v \otimes \phi) = \xi_N(v \otimes f \circ \phi) = f \circ \phi(v)$. Whereas $f \circ \xi_M(v \otimes \phi) = f(\phi(v))$ for every $v \otimes \phi \in V \otimes_A \operatorname{Hom}_C(V, M)$.

Consider the diagram

$$\begin{tikzcd} V \otimes_A \operatorname{Hom}_C(V, M) \arrow[r, "w \otimes_A \operatorname{id}_{\operatorname{Hom}_C(V,M)}"'] \arrow[d, "\xi_M"] & \operatorname{Hom}_A(\operatorname{Hom}_A(V, A), A) \otimes_A \operatorname{Hom}_C(V, M) \arrow[d, "\psi_{\operatorname{Hom}_C(V,M)}"] \\ M \arrow[d, "\pi^{-1}"] & \operatorname{Hom}_A(\operatorname{Hom}_A(V, A), \operatorname{Hom}_C(V, M)) \arrow[d, "\rho"] \\ \operatorname{Hom}_C(C, M) \arrow[r, "\operatorname{Hom}_C(\psi_V, M)"] & \operatorname{Hom}_C(V \otimes_A \operatorname{Hom}_A(V, A), M) \end{tikzcd} .$$

Here some remarks about these maps are in order. The map $\psi_{\operatorname{Hom}_C(V,M)}$ is the canonical multiplication map which is an isomorphism since $\operatorname{Hom}_A(V, A) \in A$-proj. The map ρ is the map given by Tensor-hom adjunction, and hence it is an isomorphism. The map ψ_V is the multiplication map which is an isomorphism thanks to V being a projective right A-module, thus $\operatorname{Hom}_C(\psi_V, M)$ is an isomorphism. The map π is the canonical map given by evaluation at the identity, so an isomorphism as well. The map w is the natural transformation from the identity functor on V to its double dual. Since V is projective, then w is an isomorphism. We claim that this is a commutative diagram. In fact, for $v \otimes g \in V \otimes_A \mathrm{Hom}_C(V, M), v' \otimes g' \in V \otimes_A \mathrm{Hom}_A(V, A)$, we have

$$\begin{aligned}\mathrm{Hom}_C(\psi_V, M) \circ \pi^{-1} \circ \xi_M(v \otimes g)(v' \otimes g') &= \pi^{-1} \circ \xi_M(v \otimes g) \circ \psi_V(v' \otimes g') \\ &= \pi^{-1} \circ \xi_M(v \otimes g)(v'g'(-)) = \pi^{-1}(g(v))(v'g'(-)) \\ &= v'g'(-) \cdot g(v) = g(v'g'(-) \cdot v) = g(v'g'(v)).\end{aligned}$$

$$\begin{aligned}\rho \circ \psi_{\mathrm{Hom}_C(V,M)} \circ w \otimes \mathrm{id}_{\mathrm{Hom}_C(V,M)}(v \otimes g)(v' \otimes g') &= \rho \circ \psi_{\mathrm{Hom}_C(V,M)}(w(v) \otimes g)(v' \otimes g') \\ &= \rho(w(v)(-)g)(v' \otimes g') = w(v)() \cdot g(g')(v') \\ &= (w(v)(g') \cdot g)(v') = g'(v) \cdot g(v') = g(v'g'(v)).\end{aligned}$$

Now by the commutativity of this diagram, it follows that ξ_M is an isomorphism. □

The following can be seen as the relative version of Proposition 4.8 of [Tac73].

Lemma 3.18. *Let P be a projective (A, R)-injective left A-module and let V be a projective (A, R)-strongly faithful right A-module. Fix $C = \mathrm{End}_A(V)$, $B = \mathrm{End}_A(P)^{op}$. Then, the following assertions hold.*

(a) The canonical map $\alpha_P \colon P \to \mathrm{Hom}_C(V, V \otimes_A P)$, given by $\alpha_P(p)(v) = v \otimes p$, $v \in V$, $p \in P$, is an isomorphism of (A, B)-bimodules.

(b) The canonical map $\psi \colon B \to \mathrm{End}_C(V \otimes_A P)^{op}$, given by $\psi(f)(v \otimes p) = v \otimes f(p)$, $f \in B, v \in V, p \in P$, is an isomorphism as left B-modules and as R-algebras.

(c) $V \otimes_A P$ is (C, R)-injective as left C-module.

Proof. We will start by showing that α_P (which we will abbreviate to just α) is an (A, R)-monomorphism. Since P is A-projective there are maps $k_P \in \mathrm{Hom}_A(P, A^s)$, $\pi_P \in \mathrm{Hom}_A(A^s, P)$ satisfying $\pi_P \circ k_P = \mathrm{id}_P$. Since V is (A, R)-strongly faithful there exists $i \in \mathrm{Hom}_A(A, V^t)$ and $\varepsilon \in \mathrm{Hom}_R(V^t, A)$ such that $\varepsilon \circ i = \mathrm{id}_A$. In addition, consider the A-maps arising from the direct sum V^t: $\nu_j \in \mathrm{Hom}_A(V, V^t)$, $\lambda_j \in \mathrm{Hom}_A(V^t, V)$ satisfying $\lambda_j \circ \nu_j = \mathrm{id}_V$, the multiplication map $\mu \in \mathrm{Hom}_A(V \otimes_A A, V)$ and the canonical maps $\gamma_j \in \mathrm{Hom}_A(V^s, (V^t)^s)$, $\gamma_j(v_1, \ldots, v_s) = (\nu_j(v_1), \ldots, \nu_j(v_s))$ for $1 \leq j \leq t$.

Define τ the R-map $\mathrm{Hom}_C(V, V \otimes_A P) \to P$ given by

$$\tau(h) = \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j \circ \mu^s \circ \mathrm{id}_V \otimes_A k_P \circ h \circ \lambda_j \circ i(1_A), \; h \in \mathrm{Hom}_C(V, V \otimes_A P).$$

Hence,

$$\begin{aligned}\tau \circ \alpha(p) &= \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j \circ \mu^s \circ \mathrm{id}_V \otimes_A k_P \circ \alpha(p)(\lambda_j \circ i(1_A)) = \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j \circ \mu^s \circ \mathrm{id}_V \otimes_A k_P(\lambda_j \circ i(1_A) \otimes p) \\ &= \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j \circ \mu^s(\lambda_j \circ i(1_A) \otimes k_P(p)) = \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j \circ \mu^s(\lambda_j \circ i(1_A) \otimes (k_P(p)_1, \ldots, k_P(p)_s) \\ &= \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j(\lambda_j \circ i(1_A)k_P(p)_1, \ldots, \lambda_j \circ i(1_A)k_P(p)_s)) \\ &= \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j(\lambda_j \circ i(k_P(p)_1), \ldots, \lambda_j \circ i(k_P(p)_s)) \\ &= \sum_j \pi_P \circ \varepsilon^s \circ (\nu_j\lambda_j \circ i(k_P(p)_1), \ldots, \nu_j\lambda_j \circ i(k_P(p)_s)) = \pi_P \circ \varepsilon^s(i(k_P(p)_1), \ldots, i(k_P(p)_s)) \\ &= \pi_P(k_P(p)_1, \ldots, k_P(p)_s) = \pi_P(k_P(p)) = p, \quad p \in P. \qquad (15)\end{aligned}$$

Thus, $\tau \circ \alpha = \mathrm{id}_P$ and α is an (A,R)-monomorphism.

We claim that α is an essential embedding, that is, $\mathrm{im}\,\alpha \cap A\beta \neq 0$ if $0 \neq \beta \in \mathrm{Hom}_C(V, V \otimes_A P)$.

Denote by $\pi_V\colon A^l \to V$, $k_V\colon V \to A^l$, $\pi_j \in \mathrm{Hom}_A(A^l, A), k_j \in \mathrm{Hom}_A(A, A^l)$ the canonical surjections and injections induced by the direct sum A^l, $1 \leq j \leq t$. For each j, define $e_{V,j} = \pi_V \circ k_j(1_A) \in V$ and for each $y \in V$, define $\phi_{y,j} \in \mathrm{End}_A(V) = C$ given by $\phi_{y,j}(x) = y \cdot \pi_j \circ k_V(x),\ x \in V$. Then,

$$\begin{aligned}\sum_j \phi_{e_{V,j},j} \cdot v &= \sum_j \phi_{e_{V,j},j}(v) = \sum_j e_{V,j} \cdot \pi_j \circ k_V(v) = \sum_j \pi_V \circ k_j(1_A) \cdot \pi_j \circ k_V(v) \\ &= \sum_j \pi_V \circ k_j(1_A \pi_j \circ k_V(v)) = \pi_V \circ k_V(v) = v. \end{aligned} \tag{16}$$

Let $0 \neq \beta \in \mathrm{Hom}_C(V, V \otimes_A P)$. Hence there exists $v \in V$ such that $\beta(v) \neq 0$. Moreover, for $y \in V$

$$\sum_j \pi_j \circ k_V(v) \cdot \beta(y) = \sum_j \beta(y\pi_j \circ k_V(v)) = \sum_j \beta(\phi_{y,j}(v)) = \sum_j \beta(\phi_{y,j} \cdot v) = \sum_j \phi_{y,j}\beta(v). \tag{17}$$

Assume that $\beta(v) = \sum_i x_i \otimes p_i \in V \otimes_A P$. Then,

$$\begin{aligned}\sum_j \phi_{y,j}\beta(v) &= \sum_{j,i} \phi_{y,j} x_i \otimes p_i = \sum_{i,j} (\phi_{y,j} \cdot x_i) \otimes p_i = \sum_{i,j} (y \cdot \pi_j \circ k_V(x_i)) \otimes p_i = \sum_{i,j} y \otimes \pi_j \circ k_V(x_i) p_i \\ &= \alpha(\sum_{i,j} \pi_j \circ k_V(x_i) p_i)(y) \implies \alpha(\sum_{i,j} \pi_j \circ k_V(x_i)p_i) = (\sum_j \pi_j \circ k_V(v)) \cdot \beta \in \mathrm{im}\,\alpha \cap A\beta.\end{aligned}$$

Since

$$\sum_j \pi_j \circ k_V(v)) \cdot \beta(e_{V,j}) = \sum_j \beta(e_{V,j}\pi_j \circ k_V(v))) = \sum_j \beta(\phi_{e_{V,j},j} v) = \beta(\sum_j \phi_{e_{V,j},j} v) = \beta(v) \neq 0,$$

it follows that α is an essential embedding.

Since P is (A,R)-injective and α is (A,R)-mono, there exists $h \in \mathrm{Hom}_A(\mathrm{Hom}_C(V, V \otimes_A P), P)$ such that $h \circ \alpha = \mathrm{id}_P$. Assume that there exists $0 \neq \beta \in \mathrm{im}(\mathrm{id}_{\mathrm{Hom}_C(V,V\otimes_A P)} - \alpha \circ h)$. As α is an essential embedding, $0 \neq \mathrm{im}\,\alpha \cap A\beta \subset \mathrm{im}\,\alpha \cap \mathrm{im}(\mathrm{id}_{\mathrm{Hom}_C(V,V\otimes_A P)} - \alpha \circ h) = 0$ contradicting the existence of β. Thus, $\alpha \circ h = \mathrm{id}_{\mathrm{Hom}_C(V,V\otimes_A P)}$. So, α is an isomorphism.

The map ψ given in (b) is an B-homomorphism since

$$\psi(g \circ f)(v \otimes p) = \psi(f \circ g)(v \otimes p) = v \otimes f \circ g(p) = v \otimes f(p \cdot g) = (\mathrm{id}_V \otimes f)(v \otimes p \cdot g) \tag{18}$$

$$= (g \cdot (\mathrm{id}_V \otimes f))(v \otimes p),\ v \otimes p \in V \otimes_A P,\ f, g \in B. \tag{19}$$

The map ψ is a homomorphism of R-algebras since

$$\psi(g \cdot f) = \mathrm{id}_V \otimes_A (f \circ g) = \mathrm{id}_V \otimes_A f \circ \mathrm{id}_V \otimes_A g = \mathrm{id}_V \otimes_A g \cdot \mathrm{id}_V \otimes_A f = \psi(g) \cdot \psi(f),\ f, g \in B \tag{20}$$

$$\psi(\mathrm{id}_P) = \mathrm{id}_{V \otimes_A P}\,. \tag{21}$$

We claim that ψ is bijective. Towards this goal, our procedure will be as follows. We will construct a commutative diagram

$$\begin{tikzcd} B \arrow[r, "k_B"] \arrow[d, "\psi"] & P^s \arrow[d, "\alpha_{P^s}"] \\ \mathrm{Hom}_C(V \otimes_A P, V \otimes_A P) \arrow[r, "H"] & \mathrm{Hom}_C(V, V \otimes_A P^s) \end{tikzcd}$$

where H will be a split mono and k_B is the natural injection.

Thanks to $(\alpha_X)_{X\in A\text{-mod}}$ being a natural transformation we obtain by (a) that α_{P^s} is an isomorphism.

We can see that, as right B-modules,

$$P^s \simeq \mathrm{Hom}_A(A^s,P) \simeq \mathrm{Hom}_A(P,P)\oplus \mathrm{Hom}_A(K,P) = B\oplus \mathrm{Hom}_A(K,P), \tag{22}$$

for some A-module K and π_K and k_K being the canonical maps making K an summand of A^s. We denote by k_B, k_X the canonical injections of this direct sum (22) and π_B and π_X the canonical surjections, where $X=\mathrm{Hom}_A(K,P)$. So, explicitly, $k_B(b)=b\circ\pi_P(1_A,\ldots,1_A)$. In order to define H, we first consider the following isomorphism τ given by the following commutative diagram:

$$\begin{array}{ccc}\mathrm{Hom}_C(V^s,V\otimes_A P) & \xrightarrow{\tau} & \mathrm{Hom}_C(V,V\otimes_A P^s)\\ \big\downarrow{\scriptstyle\simeq} & & \big\uparrow{\scriptstyle \mathrm{Hom}_C(V,\sigma)}\\ \mathrm{Hom}_C(V,V\otimes_A P)^s & \xrightarrow{\simeq} & \mathrm{Hom}_C(V,(V\otimes_A P)^s)\end{array}$$

where $\sigma(x_1\otimes p_1,\ldots,x_s\otimes p_s)=x_1\otimes(p_1,0,\ldots,0)+\ldots+x_s\otimes(0,\ldots,0,p_s)$.

Consider $H=\tau\circ\mathrm{Hom}_C(V\otimes_A\pi_P\circ\theta\circ(\mu^{-1})^s,V\otimes_A P)$, where θ is the isomorphism $(V\otimes_A A)^s\to V\otimes_A A^s$. Then,

$$\begin{aligned}
H\circ\psi(b)(v) &= \tau(\psi(b)\circ V\otimes_A\pi_P\circ\theta\circ(\mu^{-1})^s)(v)\\
&= \sigma(\psi(b)\circ V\otimes_A\pi_P\circ\theta\circ(\mu^{-1})^s(v,0,\ldots,0),\ldots,\psi(b)\circ V\otimes_A\pi_P\circ\theta\circ(\mu^{-1})^s(0,\ldots,0,v))\\
&= \sigma(\psi(b)\circ V\otimes_A\pi_P\theta(v\otimes 1_A,0,\ldots,0),\ldots,\psi(b)\circ V\otimes_A\pi_P\theta(0,\ldots,0,v\otimes_A 1_A)) && (23)\\
&= \sigma(\psi(b)\circ V\otimes_A\pi_P(v\otimes(1_A,0,\ldots,0)),\ldots,\psi(b)\circ V\otimes_A\pi_P(v\otimes(0,\ldots,0,1_A))) && (24)\\
&= \sigma(v\otimes b\pi_P(1_A,\ldots,0),\ldots,v\otimes b\pi_P(0,\ldots,1_A)) = v\otimes b\pi_P(1_A,\ldots,1_A) && (25)\\
\alpha_{P^s}\circ k_B(b)(v) &= \alpha_{P^s}(b\circ\pi_P(1_A,\ldots,1_A))(v) = v\otimes b\pi_P(1_A,\ldots,1_A),\ v\in V, b\in B. && (26)
\end{aligned}$$

Hence, $H\circ\psi$ is injective. In particular, ψ is injective. Since $V\otimes_A\pi_P\circ\theta\circ(\mu^{-1})^s\in\mathrm{Hom}_C(V^s,V\otimes_A P)$ is the surjection that gives $V\otimes_A P$ as C-summand of V^s the map $\mathrm{Hom}_C(V\otimes_A\pi_P\circ\theta\circ(\mu^{-1})^s,V\otimes_A P)$ is split monomorphism. So, H is a split monomorphism. Thus, there exists a map H' such that $H'\circ H=\mathrm{id}$. In particular, $\psi\circ\pi_B=H'\circ\alpha_{P^s}\circ k_B\circ\pi_B=H'\circ\alpha_{P^s}$ is surjective if $H'\circ\alpha_{P^s}\circ k_X\circ\pi_X=0$. So, it remains to show that $H'\circ\alpha_{P^s}\circ k_X\circ\pi_X=0$.

Observe that $H'=\mathrm{Hom}_C(\mu^s\circ\theta^{-1}\circ V\otimes_A k_P,V\otimes_A P)\circ\tau^{-1}$ and in the following $\pi_j^A\in\mathrm{Hom}_A(A^s,A)$, $k_j^A\in\mathrm{Hom}_A(A,A^s)$ will denote the surjections and injections of the direct sum A^s.

Thus,

$$\begin{aligned}
H'\alpha_{P^s}k_X\pi_X(p_1,\ldots,p_s)(v\otimes p) &= \tau^{-1}(\alpha_{P^s}k_X\pi_X(p_1,\ldots,p_s))(\mu^s\circ\theta^{-1}\circ V\otimes_A k_P(v\otimes p))\\
&= \tau^{-1}(\alpha_{P^s}k_X\pi_X(p_1,\ldots,p_s))(v\pi_1^A k_P(p),\ldots,v\pi_s^A k_P(p))\\
&= \sum_{i=1}^s v\pi_i^A k_P(p)\otimes\sum_j \pi_j^A k_K\pi_K k_i^A(1_A)p_j\\
&= v\otimes\sum_{i,j=1}^s \pi_j^A k_K\pi_K k_i^A\pi_i^A(k_P(p))p_j\\
&= v\otimes\sum_{j=1}^s p i_j^A k_K\pi_K k_P(p)p_j = 0,\quad p_i,p\in P,\ v\in V,\ 1\le i\le s.
\end{aligned}$$

The last equality follows since $\pi_K\circ k_P=0$. So, (b) follows.

Consider the canonical C-monomorphism $\varepsilon_{V\otimes_A P}\colon V\otimes_A P\to\mathrm{Hom}_R(C,V\otimes_A P)$. The following diagram is commutative

$$
\begin{array}{ccc}
\mathrm{Hom}_C(V, V\otimes_A P) & \xrightarrow{\mathrm{Hom}_C(V,\varepsilon)} & \mathrm{Hom}_C(V, \mathrm{Hom}_R(C, V\otimes_A P)) \\
{\scriptstyle \alpha_P}\uparrow & & \downarrow{\scriptstyle f} \\
P & \xrightarrow{\qquad\delta\qquad} & \mathrm{Hom}_R(V, V\otimes_A P)
\end{array}
$$

where $\delta\colon P \to \mathrm{Hom}_R(V, V\otimes_A P)$ is the morphism given by $\delta(p)(v) = v\otimes p$, and f is canonical map given by tensor-hom adjunction. We want to show that the map δ is an (A, R)-monomorphism. For that purpose, we need further notation. Define τ' the R-map $\mathrm{Hom}_R(V, V\otimes_A P) \to P$ given by

$$\tau(h) = \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j \circ \mu^s \circ \mathrm{id}_V \otimes_A k_P \circ h \circ \lambda_j \circ i(1_A),\ \ h \in \mathrm{Hom}_R(V, V\otimes_A P).$$

Using the same computations as in (15), it follows that $\tau'\circ\delta = \mathrm{id}_P$. Since P is (A, R)-injective, it follows that $P \in \mathrm{add}_A \mathrm{Hom}_R(V, V\otimes_A P)$. Therefore, $V\otimes_A P \in \mathrm{add}_C V\otimes_A \mathrm{Hom}_R(V, V\otimes_A P)$. By Tensor-Hom adjunction and Lemma 3.17,

$$V\otimes_A \mathrm{Hom}_R(V, V\otimes_A P) \simeq V\otimes_A \mathrm{Hom}_C(V, \mathrm{Hom}_R(C, V\otimes_A P)) \simeq \mathrm{Hom}_R(C, V\otimes_A P). \tag{27}$$

Thus, $V\otimes_A P \in \mathrm{add}_C \mathrm{Hom}_R(C, V\otimes_A P)$ and $V\otimes_A P$ is (C, R)-injective. □

Lemma 3.19. *Let P be a projective (A, R)-strongly faithful left A-module and let V be a projective (A, R)-injective right A-module. Denote $C = \mathrm{End}_A(V)$, $B = \mathrm{End}_A(P)^{op}$. Then, the following assertions hold.*

(a) The canonical map $\alpha_V\colon V \to \mathrm{Hom}_B(P, V\otimes_A P)$, given by $\alpha_V(v)(p) = v\otimes p$, $v\in V$, $p\in P$, is an isomorphism of (C, A)-bimodules.

(b) The canonical map $\psi_C\colon C \to \mathrm{End}_B(V\otimes_A P)$, given by $\psi_C(f)(v\otimes p) = f(v)\otimes p$, $f\in B, v\in V, p\in P$, is an isomorphism as left C-modules and as R-algebras.

(c) $V\otimes_A P$ is (B, R)-injective as right B-module.

Proof. It is the dual version of Theorem 3.18. □

At this point, it is not yet clear that the existence of a projective relative injective strongly faithful left module implies the existence of a projective relative injective strongly faithful right module. For this we will need change of rings techniques. We are aiming to obtain better tools to compute relative dominant dimension of modules for algebras with positive relative dominant dimension. Given that, we need to require for now the existence of both a projective relative injective strongly faithful left module and a a projective relative injective strongly faithful right module.

Definition 3.20. Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $P\in A$-mod and $V\in$ mod-A. We call a triple (A, P, V) a **relative QF3 R-algebra, or just RQF3 algebra** provided P is an A-projective (A, R)-injective-strongly faithful left A-module and V is an A-projective (A, R)-injective-strongly faithful right A-module.

It will become clear in Corollary 6.6 that RQF3 algebras are exactly the algebras having positive relative dominant dimension.

Given $X\in A$-mod, $V\in$ mod-A, denote by Φ_X the map $\mathrm{Hom}_A(V, DX)\otimes_C V \to DX$ defined by $\Phi_X(g\otimes v) = g(v)$, $v\in V$, $g\in \mathrm{Hom}_A(V, DX)$. This map is an $(\mathrm{End}_A(X)^{op}, A)$-bimodule homomorphism. In fact, if $b\in \mathrm{End}_A(X)^{op}$, $g\otimes v\in \mathrm{Hom}_A(V, DX)\otimes_C V$ and $a\in A$, then

$$\Phi_X(b\cdot(g\otimes v)) = \Phi_X(b\cdot g)\otimes v) = (b\cdot g)(v) = bg(v) = b\Phi_X(g\otimes v), \tag{28}$$

$$\Phi_X((g\otimes v)\cdot a) = \Phi_X(g\otimes v\cdot a) = g(v\cdot a) = g(v)a = \Phi_X(g\otimes v)\cdot a. \tag{29}$$

Dually, we can define the map $\delta_Y \colon P \otimes_B \operatorname{Hom}_A(P, DY) \to DY$, given by $\delta_Y(p \otimes h) = h(p)$, $p \in P$, $h \in \operatorname{Hom}_A(P, DY)$ for any $P \in A$-mod and $Y \in$ mod-A.

In the same manner, δ_Y is an $(A, \operatorname{End}_A(Y))$-bimodule homomorphism.

Lemma 3.21. *Let (A, P, V) be a RQF3 algebra. Denote $C = \operatorname{End}_A(V)$, $B = \operatorname{End}_A(P)^{op}$. Then, the following assertions hold.*

(a) $\operatorname{add}_A DV = \operatorname{add}_A P$. Furthermore, B is Morita equivalent to C.

(b) $V \otimes_A P$ satisfies a double centralizer property

$$\operatorname{End}_B(V \otimes_A P) \simeq C, \quad \operatorname{End}_C(V \otimes_A P)^{op} \simeq B$$

and $V \otimes_A P$ is a left (C, R)-injective-cogenerator and a right (B, R)-injective-cogenerator.

(c) $P \in$ mod-B is a B-generator (B, R)-cogenerator and R-projective;

(d) $V \in C$-mod is a C-generator (C, R)-cogenerator and R-projective.

(e) The canonical map $\Phi_X \colon \operatorname{Hom}_A(V, DX) \otimes_C V \to DX$, given by $\Phi_X(g \otimes v) = g(v)$, $v \in V$, $g \in \operatorname{Hom}_A(V, DX)$, is an A-isomorphism for any $X \in \operatorname{add}_A P$.

(f) The canonical map $\delta_Y \colon P \otimes_B \operatorname{Hom}_A(P, DY) \to DY$, given by $\delta_Y(p \otimes h) = h(p)$, $p \in P$, $h \in \operatorname{Hom}_A(P, DY)$, is an A-isomorphism for any $Y \in \operatorname{add}_A V$.

Proof. By Lemma 2.11, DP is an A-projective (A, R)-injective right module and DV is an A-projective (A, R)-injective left module. According to Lemma 3.7, $DP \in \operatorname{add} V$ and $DV \in \operatorname{add} P$. Hence, $P \in \operatorname{add} DV$ and $C \simeq \operatorname{End}_A(DV)^{op}$ is Morita equivalent to $B = \operatorname{End}_A(P)^{op}$. Thus, (a) follows.

Note that $D(V \otimes_A P) \simeq \operatorname{Hom}_A(P, DV)$. By (a), $P \in \operatorname{add}_A DV$. Hence,

$$ {}_B B = \operatorname{Hom}_A(P, P) \in \operatorname{add}_B \operatorname{Hom}_A(P, DV) = \operatorname{add}_B D(V \otimes_A P). \tag{30}$$

Hence $DB \in \operatorname{add}_B V \otimes_A P$. So, $V \otimes_A P$ is a right (B, R)-cogenerator. In the same fashion, by (a) $V \in \operatorname{add}_A DP$. Consequently, $C_C = \operatorname{Hom}_A(V, V) \in \operatorname{add}_C \operatorname{Hom}_A(V, DP) = \operatorname{add}_C D(V \otimes_A P)$. Then, $V \otimes_A P$ is a left (C, R)-cogenerator. Therefore, it holds the double centralizer property on $V \otimes_A P$ between C and B. By Lemma 3.19 (c) and Lemma 3.18 (c), (b) follows.

Since $P \in A$-proj there exists $s > 0$ such that $A^s \simeq P \oplus K$ as left A-modules. Thus, as right A-modules,

$$A^s \simeq \operatorname{Hom}_A(A, A_A)^s \simeq \operatorname{Hom}_A(A^s, A_A) \simeq \operatorname{Hom}_A(P \oplus K, A_A) \simeq \operatorname{Hom}_A(P, A_A) \oplus \operatorname{Hom}_A(K, A_A). \tag{31}$$

Therefore, as right B-modules

$$\begin{aligned} P^s \simeq A^s \otimes_A P &\simeq \operatorname{Hom}_A(P, A_A) \oplus \operatorname{Hom}_A(K, A_A) \otimes_A P \simeq \operatorname{Hom}_A(P, A_A) \otimes_A P \oplus \operatorname{Hom}_A(K, A_A) \otimes_A P \\ &\simeq \operatorname{Hom}_A(P, P) \oplus \operatorname{Hom}_A(K, A_A) \otimes_A P = B \oplus \operatorname{Hom}_A(K, A_A) \otimes_A P. \end{aligned} \tag{32}$$

Hence, P is a right B-generator. In the same fashion, V is a left C-generator.

Since V is projective as right A-module, there exists $t > 0$ such that $A^t \simeq V \oplus K'$ as right A-modules. So, as right B-modules,

$$P^t \simeq A^t \otimes_A P \simeq (V \oplus K') \otimes_A P \simeq V \otimes_A P \oplus K' \otimes_A P. \tag{33}$$

Hence $V \otimes_A P \in \operatorname{add}_B P$. In particular, by (b) P is also a right (B, R)-cogenerator. In the same way, V is a left (C, R)-cogenerator. This completes the proof for (c) and (d).

We claim that Φ_X and δ_X are compatible with direct sums. Let $X = X_1 \oplus X_2 \in A$-mod. Denote by k_i the canonical injections and π_i the canonical projections $i = 1, 2$. This follows from the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(V, D(X_1 \oplus X_2)) \otimes_C V \arrow[r, "\Phi_{X_1 \oplus X_2}"] \arrow[d, "(Dk_1\circ -, Dk_2 \circ -)\otimes_C \mathrm{id}_V"] & D(X_1 \oplus X_2) \arrow[d, "(Dk_1, Dk_2)"] \\
\operatorname{Hom}_A(V, DX_1) \otimes_C V \oplus \operatorname{Hom}_A(V, DX_2) \otimes_C V \arrow[r, "\Phi_{X_1} \oplus \Phi_{X_2}"] & DX_1 \oplus DX_2
\end{tikzcd}.$$

Since both columns are isomorphisms it follows our claim. The reasoning for δ_X is analogous.

Now since Φ_{DV} is the isomorphism $\operatorname{Hom}_A(V, DDV) \otimes_C V \simeq \operatorname{Hom}_A(V, V) \otimes_C V \simeq C \otimes_C V \simeq V \simeq DDV$ it follows that Φ_X is an isomorphism for any $X \in \operatorname{add} DV = \operatorname{add} P$. $\square$

We should remark that the statement of Theorem 3.21 is a generalization of (5.1) of [Tac73].

Remark 3.22. *The canonical map $\Phi\colon \operatorname{Hom}_A(V, Y) \otimes_C V \to Y$ is an A-isomorphism for any $Y \in \operatorname{Add}_A(V)$. This follows from the fact that the tensor product commutes with arbitrary coproducts and since V is a finitely generated A-projective module the Hom functor $\operatorname{Hom}_A(V, -)$ commutes with arbitrary coproducts (see [Zim14, Lemma 4.1.9]). Hence we can apply the same argument as in Lemma 3.21. The dual statement also holds for the canonical maps δ.*

The importance of these canonical maps Φ_X and α_X stems from the following theorem.

Proposition 3.23. *Let (A, P, V) be a RQF3 algebra. Denote $C = \operatorname{End}_A(V)$, $B = \operatorname{End}_A(P)^{op}$. Let $X \in A\text{-mod} \cap R\text{-proj}$ and let $Y \in \text{mod-}A \cap R\text{-proj}$, then:*

(a) $\operatorname{domdim}_{(A,R)} X \geq 1$ *if and only if the canonical map $\Phi_X\colon \operatorname{Hom}_A(V, DX) \otimes_C V \to DX$ is an epimorphism.*

(b) If $\operatorname{domdim}_{(A,R)} X \geq 1$*, then $\alpha_X\colon X \to \operatorname{Hom}_C(V, V \otimes_A X)$ is an (A, R)-monomorphism. Converse holds if* $\operatorname{Hom}_A(V, DX) \otimes_C V \in R\text{-proj}$.

(c) $\operatorname{domdim}_{(A,R)} Y \geq 1$ *if and only if the canonical map $\delta_Y\colon P \otimes_B \operatorname{Hom}_A(P, DY) \to DY$ is an epimorphism.*

(d) If $\operatorname{domdim}_{(A,R)} Y \geq 1$*, then $\alpha_Y\colon Y \to \operatorname{Hom}_B(P, Y \otimes_A P)$ is a right (A, R)-monomorphism. Converse holds if* $P \otimes_B \operatorname{Hom}_A(P, DY) \in R\text{-proj}$.

(e) The following assertions are equivalent:

(i) $\operatorname{domdim}_{(A,R)} X \geq 2$*;*

(ii) The canonical map $\Phi_X\colon \operatorname{Hom}_A(V, DX) \otimes_C V \to DX$ is a right A-isomorphism;

(iii) $\operatorname{Hom}_A(V, DX) \otimes_C V \in R\text{-proj}$ *and the canonical map $\alpha_X\colon X \to \operatorname{Hom}_C(V, V \otimes_A X)$ is a left A-isomorphism.*

(f) The following assertions are equivalent:

(i) $\operatorname{domdim}_{(A,R)} Y \geq 2$*;*

(ii) The canonical map $\delta_Y\colon P \otimes_B \operatorname{Hom}_A(P, DY) \to DY$ is a left A-isomorphism;

(iii) $\operatorname{Hom}_A(V, DX) \otimes_C V \in R\text{-proj}$ *and the canonical map $\alpha_Y\colon Y \to \operatorname{Hom}_B(P, Y \otimes_A P)$ is a right A-isomorphism.*

Proof. (a). Assume that $\operatorname{domdim}_{(A,R)} X \geq 1$. Then, there exists an (A, R)-monomorphism $f\colon X \to X_0$ with $X_0 \in \operatorname{add} DV = \operatorname{add} P$. In particular, Df is a surjective map. Applying $\operatorname{Hom}_A(V, D-) \otimes_C V$ yields the following diagram with exact rows

$$\begin{array}{ccccc}
\operatorname{Hom}_A(V, DX_0)\otimes_C V & \xrightarrow{\operatorname{Hom}_A(V,Df)\otimes_C \operatorname{id}_V} & \operatorname{Hom}_A(V, DX)\otimes_C V & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Phi_{X_0}} & & \downarrow{\scriptstyle \Phi_X} & & \\
DX_0 & \xrightarrow{\quad Df \quad} & DX & \longrightarrow & 0
\end{array}$$

Hence Φ_X is surjective because $Df \circ \Phi_{X_0}$ is. Conversely, assume that Φ_X is an epimorphism.

Observe that $\operatorname{Hom}_C(V, M)$ is an A-projective (A, R)-injective left module for any finitely generated left C-module M being (C, R)-injective and R-projective. In fact, $\operatorname{Hom}_C(V, DC) \simeq \operatorname{Hom}_R(C \otimes_C V, R) \simeq DV$ is an A-projective (A, R)-injective left module. Moreover, every (A, R)-injective R-projective module belongs to $\operatorname{add}_C DC$, so $\operatorname{Hom}_C(V, M) \in A\text{-proj} \cap \operatorname{add}_A DA$.

Consider a C-projective presentation $P_0 \xrightarrow{g} \operatorname{Hom}_A(V, DX) \to 0$. The functor $-\otimes_C V$ is right exact, so $g \otimes_C \operatorname{id}_V$ is surjective. So, $\Phi_X \circ g \otimes_C \operatorname{id}_V : P_0 \otimes_C V \to DX$ is surjective, by assumption. As $X \in R$-proj, $DX \in R$-proj and consequently, $\Phi_X \circ g \otimes_C \operatorname{id}_V$ is a right (A, R)-epimorphism. So, applying D yields an (A, R)-monomorphism $X \to D(P_0 \otimes_C V) \simeq \operatorname{Hom}_C(V, DP_0)$. Hence $\operatorname{domdim}_{(A,R)} X \geq 1$.

(b). We can relate the maps Φ_X and α_X using the following commutative diagram

$$\begin{array}{ccccc}
\operatorname{Hom}_C(V, V\otimes_A X) & \xrightarrow[\simeq]{\operatorname{Hom}_C(V, V\otimes_A w_X)} & \operatorname{Hom}_C(V, V\otimes_A DDX) & \xrightarrow[\simeq]{\operatorname{Hom}_C(V, \iota_{V,DX})} & \operatorname{Hom}_C(V, D\operatorname{Hom}_A(V, DX)) \\
 & & & & \simeq \downarrow {\scriptstyle \kappa_{V, D\operatorname{Hom}_A(V,DX)}} \\
\uparrow {\scriptstyle \alpha_X} & & & & D(DD\operatorname{Hom}_A(V, DX)\otimes_C V) \\
 & & & & \simeq \downarrow {\scriptstyle D(w_{\operatorname{Hom}_A(V,\ DX)}\otimes_C \operatorname{id}_V)} \\
X & \xrightarrow[\simeq]{w_X} & DDX & \xrightarrow{D\Phi_X} & D(\operatorname{Hom}_A(V, DX)\otimes_C V)
\end{array} \tag{34}$$

Here w_X denotes the natural transformation from the identity to the double dual functor. As $X \in R$-proj and $\operatorname{Hom}_A(V, DX) \in R$-proj w_X and $w_{\operatorname{Hom}_A(V,DX)}$ are isomorphisms. The isomorphism $\iota_{V,DX}$ and $\kappa_{V,\operatorname{Hom}_A(V,DX)}$ are according to Proposition 2.1.

The diagram 34 is commutative because

$$D\Phi_X \circ w_X(x)(f\otimes v) = w_X(x)\circ \Phi_X(f\otimes v) = w_X(x)(f(v)) \tag{35}$$

$$D(w_{\operatorname{Hom}_A(V,DX)}\otimes_C \operatorname{id}_V)\circ \kappa_{V,D\operatorname{Hom}_A(V,DX)} \circ \operatorname{Hom}_C(V, \iota_{V,DX}) \circ \operatorname{Hom}_C(V, V\otimes_A w_X)\circ \alpha_X(x)(f\otimes v) = \tag{36}$$

$$= \kappa_{V,D\operatorname{Hom}_A(V,DX)}(\iota_{V,DX}\circ V\otimes_A w_X \circ \alpha_X(x))\circ w_{\operatorname{Hom}_A(V,DX)}\otimes_C \operatorname{id}_V(f\otimes v) = \tag{37}$$

$$= w_{\operatorname{Hom}_A(V,DX)}(f)(\iota_{V,DX}\circ V\otimes_A w_X\circ \alpha_X(x)(v)) = w_{\operatorname{Hom}_A(V,DX)}(f)(\iota_{V,DX}(v\otimes w_X(x))) = \tag{38}$$

$$= \iota_{V,DX}(v\otimes w_X(x))(f) = w_X(x)(f(v)),\ x\in X, f\otimes v \in \operatorname{Hom}_A(V, DX)\otimes_C V. \tag{39}$$

Assume that $\operatorname{domdim}_{(A,R)} X \geq 1$. Then, by (a) Φ_X is an (A, R)-epimorphism. Thus, $D\Phi_X$ is an (A, R)-monomorphism. By diagram (34), α_X is an (A, R)-monomorphism. Assume now that α_X is an (A, R)-monomorphism and $\operatorname{Hom}_A(V, DX)\otimes_C V \in R$-proj. Then, $D\alpha_X$ is an (A, R)-epi. Applying D to (34), we deduce that $DD\Phi_X$ is surjective. Because of $\operatorname{Hom}_A(V, DX)\otimes_C V \in R$-proj $w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is an isomorphism. Thus, $w_{DX}\circ \Phi_X = DD\Phi_X \circ w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is surjective. Since $DX \in R$-proj, Φ_X is surjective. By (a), $\operatorname{domdim}_{(A,R)} X \geq 1$.

The assertions (c) and (d) are analogous to (a) and (b), respectively.

(e). Assume that (i) holds. By definition, there exists an (A, R)-exact sequence $0 \to X \xrightarrow{\varepsilon_0} P_0 \xrightarrow{\varepsilon_1} P_1$ with $P_0, P_1 \in \operatorname{add} P$. Applying D yields the exact sequence

$$DP_1 \xrightarrow{D\varepsilon_1} DP_0 \xrightarrow{D\varepsilon_0} X \to 0. \tag{40}$$

The functor $\operatorname{Hom}_A(V, -)\otimes_C V$ is right exact, hence applying $\operatorname{Hom}_A(V, -)\otimes_C V$ to (40) yields the following commutative diagram with exact rows

$$\begin{tikzcd}
DP_1 \arrow[r, "D\varepsilon_1"] & DP_0 \arrow[r, "D\varepsilon_0"] & DX \arrow[r] & 0 \\
\operatorname{Hom}_A(V, DP_1)\otimes_C V \arrow[u, "\Phi_{P_1}"] \arrow[r, "\operatorname{Hom}_A(V,D\varepsilon_1)\otimes_C V"] & \operatorname{Hom}_A(V, DP_0)\otimes_C V \arrow[u, "\Phi_{P_0}"] \arrow[r, "\operatorname{Hom}_A(V,D\varepsilon_0)\otimes_C V"] & \operatorname{Hom}_A(V, DX)\otimes_C V \arrow[u, "\Phi_X"] \arrow[r] & 0
\end{tikzcd}.$$

By Lemma 3.21, Φ_{P_0}, Φ_{P_1} are isomorphisms. By diagram chasing we deduce that Φ_X is an isomorphism. So, (ii) holds.

Assume that (ii) holds. Φ_X induces the isomorphism as R-modules $\operatorname{Hom}_A(V, DX)\otimes_C V \simeq DX \in R\text{-proj}$. In particular, $D\Phi_X$ is an isomorphism. Using diagram (34), we deduce that α_X is an isomorphism. Thus, (iii) follows. Now consider a C-projective resolution for $\operatorname{Hom}_A(V, DX)$, $P_1 \to P_0 \to \operatorname{Hom}_A(V, DX) \to 0$. Applying $-\otimes_C V$ we obtain the exact sequence

$$P_1 \otimes_C V \to P_0 \otimes_C V \to \operatorname{Hom}_A(V, DX)\otimes_C V \to 0. \tag{41}$$

Since Φ_X and $X \in R\text{-proj}$ is an isomorphism this yields an (A, R)-exact sequence

$$P_1 \otimes_C V \to P_0 \otimes_C V \to DX \to 0. \tag{42}$$

Finally, applying D yields an (A, R)-exact sequence $0 \to X \to D(P_0 \otimes_C V) \to D(P_1 \otimes_C V)$. As we have seen $D(P_i \otimes_C V) \in A\text{-proj} \cap \operatorname{add} DA$, $i = 1, 2$, therefore $\operatorname{domdim}_{(A,R)} X \geq 2$. So, (i) holds.

Assume that (iii) holds. By diagram (34), $D\Phi_X$ is an isomorphism. Since $\operatorname{Hom}_A(V, DX)\otimes_C V \in R\text{-proj}$ $w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is an isomorphism. So, $w_{DX} \circ \Phi_X = DD\Phi_X \circ w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is an isomorphism. Thus, (ii) follows.

The argument for (f) is analogous to (e). $\square$

Here we can see that for a commutative ring, a module having relative dominant dimension at least two is equivalent to a stronger type of the double centralizer property $DV \otimes_C V \simeq DA$, which over fields is exactly the double centralizer property $\operatorname{End}_C(V)^{op} \simeq A$.

This situation rises the question: in which situations can the R-module $DV \otimes_C V$ be at least R-projective? The following lemma answers this question for RQF3 algebras with left or right relative dominant dimension greater or equal than two.

The next result is a consequence of the following lemma.

Lemma 3.24. *Let D be a projective Noetherian R-algebra. Let X be a left D-progenerator and $E = \operatorname{End}_D(X)^{op}$. Consider the equivalence functors $F = \operatorname{Hom}_D(X, -)\colon D\text{-mod} \to E\text{-mod}$ and $G = \operatorname{Hom}_D(\operatorname{Hom}_D(X, D), -)\colon \text{mod-}D \to \text{mod-}E$. Then, for any $M \in \text{mod-}D$, $N \in D\text{-mod}$, $\operatorname{add}_R(M \otimes_D N) = \operatorname{add}_R(GM \otimes_E FN)$.*

Proof. By Morita theory,

$$\begin{aligned}
GM \otimes_E FN &\simeq \operatorname{Hom}_D(\operatorname{Hom}_D(X, D), M) \otimes_E \operatorname{Hom}_D(X, N) \simeq M \otimes_D \operatorname{Hom}_D(\operatorname{Hom}_D(X, M), D) \otimes_E \operatorname{Hom}_D(X, D) \otimes_D N \\
&\simeq M \otimes_D X \otimes_E \operatorname{Hom}_D(X, D) \otimes_D N \simeq M \otimes_D X \otimes_E \operatorname{Hom}_E(X, E) \otimes_D N \qquad (43)\\
&\simeq M \otimes_D \operatorname{Hom}_E(X, X) \otimes_D N \simeq M \otimes_D D \otimes_D N \simeq M \otimes_D N.
\end{aligned}$$
$\square$

Lemma 3.25. *Let (A, P, V) be a RQF3 algebra. Denote $C = \operatorname{End}_A(V)$, $B = \operatorname{End}_A(P)^{op}$. If $\operatorname{domdim}_{(A,R)} {}_AA \geq 2$ or $\operatorname{domdim}_{(A,R)} A_A \geq 2$, then $DV \otimes_C V \in R\text{-proj}$ and $P \otimes_B DP \in R\text{-proj}$.*

Proof. By Lemma 3.21(b), $C \simeq \operatorname{End}_B(D(V \otimes_A P))$ with $D(V \otimes_A P)$ a left B-progenerator. Thus, $F = \operatorname{Hom}_B(D(V \otimes_A P), -)$ and $G = \operatorname{Hom}_B(\operatorname{Hom}_B(D(V \otimes_A P), B), -)$. Note that by Lemma 3.19(a),

$$FDP = \operatorname{Hom}_B(D(V \otimes_A P), DP) \simeq \operatorname{Hom}_B(P, V \otimes_A P) \simeq V, \tag{44}$$

$$\begin{aligned}
GP &\simeq \operatorname{Hom}_B(\operatorname{Hom}_B(D(V \otimes_A P), B), DDP) \simeq \operatorname{Hom}_R(\operatorname{Hom}_B(D(V \otimes_A P), B) \otimes_B DP, R) \qquad (45)\\
&\simeq D\operatorname{Hom}_B(D(V \otimes_A P), DP) \simeq D\operatorname{Hom}_B(P, V \otimes_A P) \simeq DV \qquad (46)
\end{aligned}$$

The last isomorphism follows from Lemma 3.19. Consequently,

$$\operatorname{add}_R(P \otimes_B DP) = \operatorname{add}_R(GP \otimes_C FDP) = \operatorname{add}_R(DV \otimes_C V).$$

If $\operatorname{domdim}(A,R)_A A \geq 2$, then according to Proposition 3.23(e),

$$DV \otimes_C V \simeq \operatorname{Hom}_A(V, DA) \otimes_C V \simeq DA \in R\text{-proj}\,. \tag{47}$$

If $\operatorname{domdim}(A,R) A_A \geq 2$, then according to Proposition 3.23(f),

$$P \otimes_B DP \simeq P \otimes_B \operatorname{Hom}_A(P, DA) \simeq DA \in R\text{-proj}\,. \qquad \square$$

4 Relative Morita-Tachikawa correspondence

For finite dimensional algebras the Morita-Tachikawa correspondence states that every finite dimensional algebra with dominant dimension greater or equal to two is the endomorphism algebra of a generator-cogenerator. In this integral situation, there are two situations worth distinguishing. The case where the ground ring is a regular Noetherian ring with Krull dimension one (Theorem 4.3) and the general case where we do not look at the Krull dimension of the ground ring. We will present in the following the relative version of this statement now for projective Noetherian R-algebras where R is a commutative Noetherian ring (not necessarily regular).

Theorem 4.1 (General case)**.** *Let R be a commutative Noetherian ring. There is a bijection:*

$$\left\{ (B,M)\colon \begin{array}{c} B \text{ a projective} \\ \text{Noetherian } R\text{-algebra,} \\ M \text{ a } B\text{-generator } (B,R)\text{-cogenerator,} \\ M \in R\text{-proj,} \\ DM \otimes_B M \in R\text{-proj} \end{array} \right\} \Big/ \sim_1 \longleftrightarrow \left\{ A\colon \begin{array}{c} A \text{ a projective Noetherian} \\ R\text{-algebra with} \\ \operatorname{domdim}_{(A,R)} {}_A A \geq 2, \\ \operatorname{domdim}_{(A,R)} A_A \geq 2 \end{array} \right\} \Big/ \sim_2$$

In this notation, $A \sim_2 A'$ if and only if A and A' are isomorphic, whereas, $(B,M) \sim_1 (B',M')$ if and only if there is an equivalence of categories $F\colon B\text{-mod} \to B'\text{-mod}$ such that $M' = FM$.

$$\begin{aligned} (B,M) &\mapsto A = \operatorname{End}_B(M)^{op} \\ (\operatorname{End}_A(N), N) &\leftarrow\!\shortmid A \end{aligned}$$

where N is an A-projective (A,R)-injective-strongly faithful right module.

Proof. It is immediate that $\sim_1$ is an equivalence relation. Let A be a projective Noetherian R-algebra with right and left relative dominant dimension greater or equal than two. Hence, by definition, there exists $P \in A\text{-mod} \cap R\text{-proj}$ and $V \in \text{mod-}A \cap R\text{-proj}$ such that (A,P,V) is a RQF3 algebra. Let $B = \operatorname{End}_A(V)$. Since V is an A-projective right module B is a projective Noetherian R-algebra. Since R is Noetherian, it follows that B is Noetherian. By Lemma 3.21(d), V is a left B-generator (B,R)-cogenerator and R-projective. By Lemma 3.25, $DV \otimes_B V \in R\text{-proj}$. Furthermore, by Proposition 3.23, there holds the double centralizer property $A \simeq \operatorname{End}_B(V)^{op}$. If there exists another pair (P',V') such that (A,P',V') is RQF3, then we deduce by Lemma 3.7 that $\operatorname{add}_A V = \operatorname{add}_A V'$. So, $(\operatorname{End}_A(V'),V') \sim_1 (B,V)$.

Conversely, let (B,M) be a pair such that B is a projective Noetherian R-algebra and M is a B-generator (B,R)-cogenerator satisfying $M, DM \otimes_B M \in R\text{-proj}$. Define $A = \operatorname{End}_B(M)^{op}$. Since $DM \otimes_R M$, it follows that $A = \operatorname{Hom}_B(M,M) \simeq D(DM \otimes_B M) \in R\text{-proj}$. Thus, A is a projective Noetherian R-algebra. As M is a B-generator $M^t \simeq B \oplus K$. In particular, there exists a surjective B-homomorphism $\phi\colon M^t \twoheadrightarrow B$ for some $t > 0$. Let $\pi_j \in \operatorname{Hom}_B(M^t, M)$ and $k_j \in \operatorname{Hom}_B(M, M^t)$, $1 \leq j \leq t$, be the canonical surjections and injections, respectively. In particular, $1_B = \sum_j \phi \circ k_j(m_j)$ for some $m_j \in M$, $1 \leq j \leq t$.

For any $x \in M$, define $h_x \in \operatorname{Hom}_B(B, M)$ satisfying $h_x(1_B) = x$. Then, $t_x \circ \phi \circ k_j \in \operatorname{Hom}_B(M, M) = A$, $1 \leq j \leq t$. Then, for any $x \in M$,

$$x = t_x(1_B) = t_x(\sum_j \phi \circ k_j(m_j)) = \sum_j t_x \circ \phi \circ k_j(m_j) = \sum_j m_j \cdot t_x \circ \phi \circ k_j. \tag{48}$$

This shows that M is finitely generated as right A-module.

As a result of M being a B-generator, we can write

$$\begin{aligned} A^t &\simeq \operatorname{Hom}_B(M, M_A)^t \simeq \operatorname{Hom}_B(M^t, M_A) \simeq \operatorname{Hom}_B(B \oplus K, M_A) \simeq \operatorname{Hom}_B(B, M_A) \oplus \operatorname{Hom}_B(K, M_A) \\ &\simeq M \oplus \operatorname{Hom}_B(K, M_A). \end{aligned} \tag{49}$$

Hence, M is projective as right A-module. On the other hand, as M is a (B, R)-cogenerator, we can write

$$\begin{aligned} A^s &\simeq \operatorname{Hom}_B(M_A, M)^s \simeq \operatorname{Hom}_B(M_A, M^s) \simeq \operatorname{Hom}_B(M_A, DB \oplus K') \simeq \operatorname{Hom}_B(M_A, DB) \oplus \operatorname{Hom}_B(M, K') \\ &\simeq \operatorname{Hom}_B(B, DM) \oplus \operatorname{Hom}_B(M, K') \simeq DM \oplus \operatorname{Hom}_B(M, K') \end{aligned} \tag{50}$$

for some $s > 0$ and $K' \in B$-mod. Therefore, DM is an A-projective left module, and consequently, M is an (A, R)-injective right module. Hence, M is an A-projective (A, R)-injective right module. Consider a left B-projective resolution for M, $P_1 \to P_0 \to M \to 0$. Due to $DM \otimes_B M \in R$-proj applying $DM \otimes_B -$ yields the (A, R)-exact sequence

$$DM \otimes_B P_1 \to DM \otimes_B P_0 \to DM \otimes_B M \to 0. \tag{51}$$

Now applying D yields the right (A, R)-exact sequence

$$0 \to A \to D(DM \otimes_B P_0) \to D(DM \otimes_A P_1). \tag{52}$$

Observe that $D(DM \otimes_B P_i) \simeq \operatorname{Hom}_B(P_i, M) \in \operatorname{add} M$, $i = 1, 2$. Hence the (A, R)-monomorphism $A \to D(DM \otimes_B P_0)$ makes M an (A, R)-strongly faithful module and (52) implies $\operatorname{domdim}_{(A,R)} A_A \geq 2$. Consider now a right B-projective resolution for DM, $Q_1 \to Q_0 \to DM \to 0$. Applying $- \otimes_B M$ yields the (A, R)-exact sequence

$$Q_1 \otimes_B M \to Q_0 \otimes_B M \to DM \otimes_B M \to 0. \tag{53}$$

Applying D we obtain the (A, R)-exact sequence

$$0 \to A \to D(Q_0 \otimes_B M) \to D(Q_1 \otimes_B M). \tag{54}$$

Here $D(Q_i \otimes_B M) \simeq \operatorname{Hom}_B(Q_i, DM) \in \operatorname{add} DM$. Therefore, (54) yields that $\operatorname{domdim}_{(A,R)\ A} A \geq 2$ and DM is an (A, R)-strongly faithful module.

As generators satisfy the double centralizer property we have that $B \simeq \operatorname{End}_A(M)$. If $(B, M) \simeq_1 (B', M')$, then by Morita theory, $A = \operatorname{End}_B(M)^{op} \simeq \operatorname{End}_{B'}(M')$. This concludes the proof. $\square$

We should emphasize the importance of R being a commutative Noetherian ring in the proof of the relative Morita-Tachikawa correspondence. Furthermore, we remark that using finitely generated modules in Definition 3.1 of relative dominant dimension instead of general modules is no mistake. One of the reasons is that the Hom functors do not preserve in general arbitrary direct sums. Consequently, the techniques employed in relative Morita-Tachikawa correspondence would not hold in such a general setting.

Moreover, the following result is a consequence of equation (48). This result goes back to [Mor58].

Corollary 4.2. *Let B be a projective Noetherian R-algebra. Let M be a generator in B-*Mod. *Then, M is finitely generated as $\operatorname{End}_B(M)^{op}$-module.*

Therefore, it is not expected that a version of Morita-Tachikawa correspondence can hold in general for arbitrary commutative non-Noetherian rings. Nonetheless, if such version happens to exist it should involve at very least compact modules in order to solve the problems of Hom regarding direct sums.

The surprise in this relative version is that we are only interested in the generators relative cogenerators that satisfy $DM \otimes_B M \in R$-proj. Modules are faithful over its endomorphism algebras. The importance of the property $DM \otimes_B M \in R$-proj lies on the fact that this is a sufficient condition for a given B-module M to be strongly faithful over its endomorphism algebra. Later, we will see a characterization of this property and what it means for the endomorphism algebra $\operatorname{End}_B(M)$ in terms of base change properties.

4.1 Relative Morita-Tachikawa correspondence in case of Krull dimension one

For regular commutative Noetherian rings with Krull dimension less or equal to one, we can drop the condition $DM \otimes_B M \in R$-proj in the relative Morita-Tachikawa correspondence and we can reformulate the relative Morita-Tachikawa correspondence in the following way.

Theorem 4.3. *Let R be a commutative regular Noetherian ring with Krull dimension less than or equal to one. There is a bijection between*

$$\left\{(B,M)\colon \begin{array}{c} B \textit{ a projective Noetherian} \\ R\textit{-algebra, } M \in R\text{-proj} \\ M \textit{ a } B\textit{-generator } (B,R)\textit{-cogenerator}\end{array}\right\}\Big/\sim_1 \longleftrightarrow \left\{A\colon \begin{array}{c} A \textit{ a projective} \\ \textit{Noetherian } R\textit{-algebra with} \\ \operatorname{domdim}_{(A,R)}\, {}_AA \geq 1, \\ \operatorname{domdim}_{(A,R)}\, A_A \geq 1, \\ \textit{all } (A,R)\textit{-injective-strongly faithful} \\ \textit{projective modules} \\ \textit{satisfy the} \\ \textit{double centralizer property}\end{array}\right\}\Big/\sim_2$$

In this notation, $A \sim_2 A'$ if and only if A and A' are isomorphic, whereas, $(B,M) \sim_1 (B',M')$ if and only if there is an equivalence of categories $F\colon B\text{-mod} \to B'\text{-mod}$ such that $M' = FM$.

$$\begin{aligned} (B,M) &\mapsto A = \operatorname{End}_B(M)^{op} \\ (\operatorname{End}_A(N), N) &\leftarrow\!\shortmid A \end{aligned}$$

where N is an A-projective (A,R)-injective-strongly faithful right module.

Proof. Let A be a projective Noetherian R-algebra with $\operatorname{domdim}_{(A,R)} A_A \geq 1$, $\operatorname{domdim}_{(A,R)}\, {}_AA \geq 1$ and all projective (A,R)-injective-strongly faithful modules satisfy the double centralizer property. Hence, there exists $P \in A$-mod and $V \in$ mod-A such that (A,P,V) is a RQF3 algebra. Define $B = \operatorname{End}_A(V)$. As V is an A-projective right module, B is a projective Noetherian R-algebra. By Lemma 3.21, V is a left B-generator (B,R)-cogenerator. By assumption, V satisfies the double centralizer property, thus $A \simeq \operatorname{End}_B(V)^{op}$. By the same argument as in relative Morita-Tachikawa correspondence, the mapping $\leftarrow\!\shortmid$ is well defined.

Conversely, let (B,M) with $M \in B\text{-mod} \cap R\text{-proj}$ a B-generator (B,R)-cogenerator. Define $A = \operatorname{End}_B(M)^{op}$. Note that $A = \operatorname{Hom}_B(M,M) \subset \operatorname{Hom}_R(M,M) \in \operatorname{add}_R M$. Since R has Krull dimension less or equal than one, and A is an R-submodule of a projective then A is projective as R-module. Thus, A is a projective Noetherian$R$-algebra. As in the proof of Theorem 4.1, M is an A-projective (A,R)-injective finitely generated module that satisfies the double centralizer property. Consider a projective resolution for M, $P_1 \to P_0 \to M \to 0$. Applying $DM \otimes_B -$ we get the exact sequence

$$DM \otimes_B P_1 \to DM \otimes_B P_0 \to DM \otimes_B M \to 0. \tag{55}$$

Now, applying D yields the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & D(DM\otimes_B P_0) & \longrightarrow & D(DM\otimes_B P_1) & & \\
 & & \| & & \| & & \uparrow & & . \\
0 & \longrightarrow & A & \longrightarrow & D(DM\otimes_B P_0) & \longrightarrow & \operatorname{coker} & \longrightarrow & 0
\end{array}$$

By Snake Lemma, the map $\operatorname{coker} \to D(DM\otimes_B P_1) \simeq \operatorname{Hom}_B(P_1, M)$ is a monomorphism and $\operatorname{Hom}_B(P_1, M) \in \operatorname{add} M$. As $\dim R \leq 1$, $\operatorname{coker} \in R\text{-proj}$. Thus, the monomorphism $A \to D(DM \otimes_B P_0)$ is an (A, R)-monomorphism. It follows that $\operatorname{domdim}(A, R)A_A \geq 1$. Using a projective resolution for DM and applying $D \circ - \otimes_B M$ we deduce that $\operatorname{domdim}(A, R)_A A \geq 1$. In particular, (A, DM, M) is a RQF3 algebra and there exists an A-exact sequence $0 \to A \to X_0 \to X_1$, with $X_0, X_1 \in \operatorname{add} DM$. Now assume that T is another right projective (A, R)-injective-strongly faithful module. Then, (A, DM, T) is a RQF3 algebra. By Lemma 3.21(a), $\operatorname{add}_A M = \operatorname{add}_A T$. Denote by C the endomorphism algebra $\operatorname{End}_A(T)$. By Morita theory, $(C, T) \sim_1 (B, M)$. Hence, $A \simeq \operatorname{End}_B(M)^{op} \simeq \operatorname{End}_C(T)^{op}$. So, all T satisfies the double centralizer property between C and A. $\square$

As we will see in Corollary 6.6, in the right hand side of Theorem 4.3 it is enough to consider only the dominant dimension of the regular left module or only the dominant dimension of the regular right module.

4.2 Splitting map between endomorphism algebras

In general, we know very little about the splitness over R of the natural inclusion

$$\operatorname{End}_C(V) \to \operatorname{End}_R(V) \tag{56}$$

even in the case where V is a left C-generator. A relation between this property and relative dominant dimension can be found in the next proposition.

Proposition 4.4. *Let (A, P, V) be a RQF3 algebra. Fix $C = \operatorname{End}_A(V)$. The following assertions hold.*

(a) If $\operatorname{domdim}(A, R) \geq 2$, then the canonical inclusion

$$i\colon \operatorname{End}_C(V) \hookrightarrow \operatorname{End}_R(V) \tag{57}$$

splits over R.

(b) Assume also that the splitting map $\tau\colon \operatorname{End}_R(V) \to \operatorname{End}_C(V)$ satisfies the following two properties:

$$\tau(h \circ g) = h \circ \tau(g), \qquad \tau(g \circ h) = \tau(g) \circ h,\ g \in \operatorname{End}_R(V), h \in \operatorname{End}_C(V). \tag{58}$$

Let $\delta\colon M_{i+1} \to M_i \to M_{i-1}$ be a (C, R)-exact sequence. If $\operatorname{Hom}_C(V, M_{i+1}) \to \operatorname{Hom}_C(V, M_i) \to \operatorname{Hom}_C(V, M_{i-1})$ is exact and $M_i \in R\text{-proj}$, then the sequence $\operatorname{Hom}_C(V, \delta)$ is (A, R)-exact.

Proof. By Proposition 3.23, $\Phi_A\colon DV \otimes_C V \to DA$ is an isomorphism. In particular, $DV \otimes_C V \in R\text{-proj}$. Consider the canonical R-epimorphism $\varepsilon\colon DV \otimes_R V \to DV \otimes_C V$, given by $f \otimes v \mapsto f \otimes v, f \in DV, v \in V$. So, ε splits over R. Using the commutativity of the diagram with bijective columns

$$\begin{array}{ccccc}
0 & \longrightarrow & D(DV\otimes_C V) & \xrightarrow{D\varepsilon} & D(DV\otimes_R V) \\
 & & \simeq\uparrow & & \simeq\uparrow \\
0 & \longrightarrow & \operatorname{Hom}_C(V, DDV) & \longrightarrow & \operatorname{Hom}_R(V, DDV) \\
 & & \simeq\uparrow & & \simeq\uparrow \\
0 & \longrightarrow & \operatorname{Hom}_C(V, V) & \xrightarrow{i} & \operatorname{Hom}_R(V, V)
\end{array}$$

we obtain that the natural inclusion i splits over R.

Assume that the splitting map $\tau\colon \operatorname{End}_R(V) \to \operatorname{End}_C(V)$ satisfies the following two properties:

$$\tau(h \circ g) = h \circ \tau(g), \qquad \tau(g \circ h) = \tau(g) \circ h, \; g \in \operatorname{End}_R(V), h \in \operatorname{End}_C(V). \tag{59}$$

Let $M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1}$ be a (C,R)-exact sequence. Hence, there are maps $h_j \in \operatorname{Hom}_R(M_j, M_{j+1})$ satisfying $f_{i+1} \circ h_i + h_{i-1} \circ f_i = \operatorname{id}_{M_i}$, $j = i, i-1$.

Since V is C-generator there exists a surjective $\pi^{(i)}\colon V^{t_i} \to M_i$. As $M_i \in R$-proj, there exists $k^{(i)} \in \operatorname{Hom}_R(M_i, V^{t_i})$ such that $\pi^{(i)} \circ k^{(i)} = \operatorname{id}_{M_i}$. Let $\pi_j^{(i)}$ and $k_j^{(i)}$ be the canonical surjections and inclusions of the direct sum V^{t_i}. Since V is a (C,R)-cogenerator, M_i can be embedded in V^s through a map $l^{(i)}$. Denote by ϕ_z and ν_z the canonical projections and injections of the direct sum V^s. Define the map $H_i\colon \operatorname{Hom}_C(V, M_i) \to \operatorname{Hom}_C(V, M_{i+1})$, given by $H_i(g) = \sum_j \pi^{(i+1)} k_j^{(i+1)} \tau(\pi_j^{(i+1)} k^{(i+1)} h_i g)$ for each $g \in \operatorname{Hom}_C(V, M_i)$. For any $g \in \operatorname{Hom}_C(V, M_i)$,

$$l^{(i)}(\operatorname{Hom}_C(V, f_{i+1} \circ H_i + H_{i-1} \circ \operatorname{Hom}_C(V, f_i)))(g) = l^{(i)}(f_{i+1} \circ H_i(g) + H_{i-1}(f_i \circ g)) \tag{60}$$

$$= \sum_{z,j} \nu_z(\phi_z l^{(i)} f_{i+1} \pi^{(i+1)} k_j^{(i+1)} \tau(\pi_j^{(i+1)} k^{(i+1)} h_i g) + \phi_z l^{(i)} \pi^{(i)} k_j^{(i)} \tau(\pi_j^{(i)} k^{(i)} h_{i-1} f_i g)) \tag{61}$$

$$= \sum_z \nu_z(\tau(\phi_z l^{(i)} f_{i+1} \pi^{(i+1)} \sum_j k_j^{(i+1)} \pi_j^{(i+1)} h_i g) + \tau(\phi_z l^{(i)} \pi^{(i)} \sum_j k_j^{(i)} \pi_j^{(i)} k^{(i)} h_{i-1} f_i g)) \tag{62}$$

$$= \sum_z \nu_z \tau(\phi_z l^{(i)} f_{i+1} h_i g + \phi_z l^{(i)} h_{i-1} f_i g) = \sum_z \nu_z \tau(\phi_z l^{(i)} g) = \sum_z \nu_z \phi_z l^{(i)} g = l^{(i)} g. \tag{63}$$

Therefore, $\operatorname{Hom}_C(V, f_{i+1} \circ H_i + H_{i-1} \circ \operatorname{Hom}_C(V, f_i) = \operatorname{id}_{\operatorname{Hom}_C(V, M_i)}$. Analogously, we can see the same statement holds for the functor $\operatorname{Hom}_C(-, V)$. $\square$

The existence of a map τ in the conditions of Proposition 4.4(b) may not exist in general, otherwise, every module should satisfy the property $\operatorname{Hom}_A(V, DM) \otimes_C V \in R$-proj. However, such a map τ with the given properties exists for relative separable algebras (see for example [Hat63, 2.2]).

5 Mueller's characterization of relative dominant dimension

We will now study how to compute the relative dominant dimension of a module in terms of the homology over the endomorphism algebra of a projective relative injective strongly faithful module (Theorem 5.2). This will be the analogue of Mueller's characterization of dominant dimension. It turns out that in the integral setup, vanishing of cohomology is weaker than vanishing of homology. Actually, we will see that it is the global dimension of the ground ring which causes obstructions suggesting the use of Tor functors instead of Ext functors to study relative dominant dimension (Theorem 5.2, Proposition 5.5 and Theorem 5.6). In general, without further assumptions, the larger the global dimension of the ground ring, the less vanishing of Ext groups tell us about the value of relative dominant dimension of a module. Similarly to the classical case, thanks to the Mueller's characterization of relative dominant dimension we can deduce several additional properties of relative dominant dimension. For instance, we can establish the left and right symmetry of relative dominant dimension of relative QF-3 algebras (Corollary 5.9) and how relative dominant dimension behaves on short (A,R)-exact sequences (Lemma 5.12).

The following technical lemma will be useful for the relative Mueller theorem.

Lemma 5.1. *Consider the following commutative diagram with one exact row*

$$\begin{tikzcd} X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[rr, "\alpha_1"] \arrow[dr, "\varepsilon"'] & & X_2 \arrow[r, "\alpha_2"] & X_3 \\ & & Y \arrow[ur, "t"'] & & \end{tikzcd}.$$

The following assertions hold.

(i) If ε is surjective and $\varepsilon \circ \alpha_0 = 0$, then t is mono.

(ii) If t is mono and $\alpha_2 \circ t = 0$, then ε is surjective.

Proof. (i). Let $y \in \ker t$. Since ε is surjective, we can write $y = \varepsilon(x)$ for some $x \in X_1$. Thus, $\alpha_1(x) = t\varepsilon(x) = t(y) = 0$. So, $x \in \operatorname{im} \alpha_0 = \ker \alpha_1$. Hence, $y = \varepsilon(\alpha_0(z)) = 0$ for some $z \in X_0$. Hence, t is injective.

(ii). Let $y \in Y$. Then, $t(y) \in \ker \alpha_2 = \operatorname{im} \alpha_1$. So, we can write $t(y) = \alpha_1(x) = t\varepsilon(x)$ for some $x \in X_1$. As t is injective, $y = \varepsilon(x)$. $\square$

Let $X \in A$-mod. Denote by $\Omega^i(X, P^\bullet)$ the i-th syzygy of X with respect to an A-projective resolution $P^\bullet$. Naturally, $\Omega^0(X, P^\bullet) \simeq X$ for any $P^\bullet$ and $\Omega^i(X, P^\bullet) \in R$-proj whenever $X \in R$-proj.

Theorem 5.2. *Let (A, P, V) be a RQF3 algebra. Fix $C = \operatorname{End}_A(V)$. For any R-projective left A-module M, the following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq n \geq 2$;

(ii) $\Phi_M \colon \operatorname{Hom}_A(V, DM) \otimes_C V \to DM$ *is an isomorphism and* $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0$, $1 \leq i \leq n-2$;

(iii) $\alpha_M \colon M \to \operatorname{Hom}_C(V, V \otimes_A M)$ *is an isomorphism,* $\Omega^i(\operatorname{Hom}_A(V, DM), P^\bullet) \otimes_C V \in R$-proj, $0 \leq i \leq n-2$ *for every C-projective resolution $P^\bullet$ of* $\operatorname{Hom}_A(V, DM)$ *and* $\operatorname{Ext}_C^i(V, V \otimes_A M) = 0$, $1 \leq i \leq n-2$.

Proof. $(i) \implies (ii)$. By Proposition 3.23, Φ_M is an isomorphism. By definition, there exists an (A, R)-exact sequence

$$0 \to M \xrightarrow{\varepsilon} X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \to X_{n-1}, \tag{64}$$

with X_i an A-projective (A, R)-injective module. The functor $\operatorname{Hom}_A(V, -)$ is exact, and since D preserves (A, R)-exact sequences, applying $\operatorname{Hom}_A(V, D-)$ yields the exact sequence

$$\operatorname{Hom}_A(V, DX_{n-1}) \xrightarrow{\operatorname{Hom}_A(V, Df_{n-1})} \operatorname{Hom}_A(V, DX_{n-2}) \to \cdots \to \operatorname{Hom}_A(V, DX_0) \xrightarrow{\operatorname{Hom}_A(V, D\varepsilon)} \operatorname{Hom}_A(V, DM) \to 0. \tag{65}$$

As $\operatorname{Hom}_A(V, DX_i) \in \operatorname{add} \operatorname{Hom}_A(V, V) = C$-proj, we can extend (65) to a C-projective resolution of $\operatorname{Hom}_A(V, DM)$, $P^\bullet$ where $P_i = \operatorname{Hom}_A(V, DX_i)$, $0 \leq i \leq n-1$. Applying $- \otimes_C V$ we get the following commutative diagram with the top row exact.

$$\begin{tikzcd}
DX_{n-1} \arrow[r, "Df_{n-1}"] & DX_{n-2} \arrow[r] & \cdots \arrow[r] & DX_0 \arrow[r, two heads, "D\varepsilon"] & DM \\
\operatorname{Hom}_A(V, DX_{n-1}) \otimes_C V \arrow[u, "\Phi_{X_{n-1}}"] \arrow[r, "\operatorname{Hom}_A(V, Df_{n-1}) \otimes_C V"] & \operatorname{Hom}_A(V, DX_{n-2}) \otimes_C V \arrow[u, "\Phi_{X_{n-2}}"] \arrow[r] & \cdots \arrow[r] & \operatorname{Hom}_A(V, DX_0) \otimes_C V \arrow[u, "\Phi_{X_0}"] \arrow[r, two heads, "\operatorname{Hom}_A(V, D\varepsilon) \otimes_C V"] & \operatorname{Hom}_A(V, DM) \otimes_C V \arrow[u, "\Phi_M"]
\end{tikzcd}.$$

According to Lemma 3.21, the maps Φ_M and Φ_{X_i}, $i = 1, \ldots, n-1$ are isomorphisms. Thus, the bottom row is exact. Thus,

$$\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = \ker \operatorname{Hom}_A(V, Df_i) \otimes_C V / \operatorname{im} \operatorname{Hom}_A(V, Df_{i+1}) \otimes_C V = 0, \quad 1 \leq i \leq n-2.$$

$(ii) \implies (iii)$. By Proposition 3.23, $\operatorname{Hom}_A(V, DM) \otimes_C V \simeq D(V \otimes_A M) \otimes_C V \in R$-proj and α_M is an isomorphism. Let

$$\cdots \to P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} D(V \otimes_A M) \to 0. \tag{66}$$

be an arbitrary C-projective resolution of $D(V \otimes_A M)$. In particular, for every $1 \leq i \leq n-2$, we have the following exact sequence

$$0 \to \Omega^i(\mathrm{Hom}_A(V, DM), P^\bullet) \xrightarrow{k_i} P_{i-1} \xrightarrow{p_{i-1}} P_{i-2} \to \cdots \to P_0 \to D(V \otimes_A M) \to 0, \tag{67}$$

where $P^\bullet$ is the deleted projective resolution of (66). It follows from $\mathrm{Tor}_i^C(\mathrm{Hom}_A(V, DM), V) = 0$, $1 \leq i \leq n-2$ the existence of the following exact sequence and factorization of $p_i \otimes_C V$

$$P_{n-1} \otimes_C V \xrightarrow{p_{n-1} \otimes_C V} P_{n-2} \otimes_C V \to \cdots \to P_0 \otimes_C V \to D(V \otimes_A M) \otimes_C V \to 0, \tag{68}$$

$$\begin{tikzcd}[column sep=small]
P_{i+1} \otimes_C V \arrow[r, "p_{i+1} \otimes_C V"] & P_i \otimes_C V \arrow[rr, "p_i \otimes_C V"] \arrow[dr, two heads, "\varepsilon_i \otimes_C V"'] & & P_{i-1} \otimes_C V \arrow[r, "p_{i-1} \otimes_C V"] & P_{i-2} \otimes_C V \\
& & \Omega^i(\mathrm{Hom}_A(V, DM), P^\bullet) \otimes_C V \arrow[ur, "k_i \otimes_C V"'] & &
\end{tikzcd},$$

where ε_i is the map given in the factorization (epi, mono) $k_i \varepsilon_i = p_i$. For the case $i = 1$, we can take $P_{-1} = D(V \otimes_A M)$. Observe that $0 = p_i p_{i+1} = k_i \varepsilon_i p_{i+1}$. Hence, $\varepsilon_i p_{i+1} = 0$ because k_i is a mono. Consequently, $\varepsilon_i \otimes_C V p_{i+1} \otimes_C V = 0$. By Lemma 5.1, $k_i \otimes_C V$ is a monomorphism and thus

$$\Omega^i(\mathrm{Hom}_A(V, DM), P^\bullet) \otimes_C V \simeq \mathrm{im}(p_i \otimes_C V) = \ker(p_{i-1} \otimes_C V) \in R\text{-proj} \tag{69}$$

since $D(V \otimes_A M) \otimes_C V \in R$-proj and every $P_i \in R$-proj. By Tensor-Hom adjunction there exists the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D(D(V \otimes_A M) \otimes_C V) \arrow[r] \arrow[d, "\simeq"] & D(P_0 \otimes_C V) \arrow[r] \arrow[d, "\simeq"] & \cdots \arrow[r] & D(P_{n-1} \otimes_C V) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Hom}_C(V, V \otimes_A M) \arrow[r] & \mathrm{Hom}_C(V, DP_0) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_C(V, DP_{n-1})
\end{tikzcd}, \tag{70}$$

such that every column is an isomorphism. The upper row is just the exact sequence obtained by applying D to the (A, R)-exact sequence (68), and therefore it is exact. Now, the commutativity of diagram (70) yields that the bottom row of (70) is exact. Taking into account that $0 \to V \otimes_A M \to DP_0 \to DP_1 \to \cdots$ is a (C, R)-injective resolution and $V \otimes_A M \in R$-proj, the exactness of the bottom row of (70) means that $\mathrm{Ext}^i_{(C,R)}(V, V \otimes_A M) = 0$, $1 \leq i \leq n-2$. Again, since $V \otimes_A M \in R$-proj and $V \in R$-proj the standard (C, R)-projective resolution of V is a C-projective resolution of V. Therefore,

$$\mathrm{Ext}^i_C(V, V \otimes_A M) = \mathrm{Ext}^i_{(C,R)}(V, V \otimes_A M) = 0, \ 1 \leq i \leq n-2.$$

(iii) $\implies$ (i). We shall proceed by induction on k to show that if $\alpha_M \colon M \to \mathrm{Hom}_C(V, V \otimes_A M)$ is an isomorphism, $\Omega^i(\mathrm{Hom}_A(V, DM), P^\bullet) \otimes_C V \in R$-proj, $0 \leq i \leq k-2$ for every C-projective resolution $P^\bullet$ of $\mathrm{Hom}_A(V, DM)$ and $\mathrm{Ext}^i_C(V, V \otimes_A M) = 0$ for $1 \leq i \leq k-2$, then $\mathrm{domdim}_{(A,R)} M \geq k \geq 2$. If $k = 2$, then the result holds by Proposition 3.23. Assume that the result holds for a given k satisfying $n > k > 2$. Assume, in addition, that $\alpha_M \colon M \to \mathrm{Hom}_C(V, V \otimes_A M)$ is an isomorphism, $\Omega^i(\mathrm{Hom}_A(V, DM), P^\bullet) \otimes_C V \in R$-proj, $0 \leq i \leq k-1$ for every C-projective resolution $P^\bullet$ of $\mathrm{Hom}_A(V, DM)$ and $\mathrm{Ext}^i_C(V, V \otimes_A M) = 0$, $1 \leq i \leq k-1$. By induction, $\mathrm{domdim}_{(A,R)} M \geq k$. So, there exists a (A, R)-exact sequence

$$0 \to M \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} X_1 \to \cdots \to X_{k-1}, \tag{71}$$

with all $X_i \in \mathrm{add}\, DV$. Applying $V \otimes_A -$ yields the (C, R)-exact sequence

$$0 \to V \otimes_A M \to V \otimes_A X_0 \to V \otimes_A X_1 \to \cdots \to V \otimes_A X_{k-1}. \tag{72}$$

Now, observe that, $D(V \otimes_A X_i) \simeq \mathrm{Hom}_A(V, DX_i) \in \mathrm{add}\, \mathrm{Hom}_A(V, DDV) = C$-proj. So, we can extend (72) to a (C, R)-injective resolution of $V \otimes_A M$, $I^\bullet$. Furthermore, we have the (epi, mono) factorization

$$\begin{tikzcd}[column sep=small]
V \otimes_A X_{k-2} \arrow[rr, "V\otimes_A \alpha_{k-1}"] \arrow[dr, two heads, "\varepsilon"'] & & V \otimes_A X_{k-1} \\
& D\Omega^{k-1}(D(V \otimes_A M), D((V \otimes_A X)^\bullet) \arrow[ur, hook, "t"'] &
\end{tikzcd},$$

where $(V \otimes_A X)^\bullet$ denotes the deleted (C,R)-injective resolution obtained by $I^\bullet$. Denote by Ω the module $D\Omega^{k-1}(D(V \otimes_A M), D((V \otimes_A X)^\bullet)$. Since $\mathrm{Ext}^i_C(V, V \otimes_A M) = \mathrm{Ext}^i_{(C,R)}(V, V \otimes_A M) = 0$, $i \leq k-1$ applying $\mathrm{Hom}_C(V,-)$ to the (C,R)-injective $I^\bullet$ we obtain the exact sequence

$$\begin{tikzcd}[column sep=small]
\mathrm{Hom}_C(V, V\otimes_A M) \arrow[r, hook] & \mathrm{Hom}_C(V, V\otimes_A X_0) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_C(V, V\otimes_A X_{k-2}) \arrow[r, "\mathrm{Hom}_C(V, V\otimes_A \alpha_{k-1})"] \arrow[d, "\mathrm{Hom}_C(V,\varepsilon)"'] & \mathrm{Hom}_C(V, V\otimes_A X_{k-1}) \arrow[r, "\mathrm{Hom}_C(V, i_k)"] & \mathrm{Hom}_C(V, I_k) \\
& & & \mathrm{Hom}_C(V,\Omega) \arrow[ur, hook, "\mathrm{Hom}_C(V,t)"'] & &
\end{tikzcd} \tag{73}$$

where $\mathrm{Hom}_C(V,t)$ is injective and $\ker i_k = \mathrm{im}\, V \otimes_A \alpha_{k-1}$. Note that $0 = i_k \circ V \otimes_A \alpha_{k-1} = i_k t \varepsilon$. Thus, $i_k t = 0$ since ε is surjective. Now, as $\mathrm{Hom}_C(V, i_k) \circ \mathrm{Hom}_C(V,t) = \mathrm{Hom}_C(V, i_k t) = 0$, it follows by Lemma 5.1(ii) that $\mathrm{Hom}_C(V, \varepsilon)$ is surjective. On the other hand,

$$\mathrm{Hom}_C(V,\Omega) \simeq D(\Omega^{k-1}(D(V \otimes_A M), D((V \otimes_A X)^\bullet) \otimes_C V) \in R\text{-proj}\,. \tag{74}$$

Hence, the exact sequence

$$0 \to \mathrm{Hom}_C(V, V\otimes_A M) \to \mathrm{Hom}_C(V, V \otimes_A X_0) \to \cdots \to \mathrm{Hom}_C(V, V\otimes_A X_{k-2}) \to \mathrm{Hom}_C(V,\Omega) \to 0 \tag{75}$$

is (A,R)-exact. As $M \simeq \mathrm{Hom}_C(V, V \otimes_A M)$ and each $\mathrm{Hom}_C(V, V\otimes_A X_i) \simeq X_i \in \mathrm{add}\, DV$ it is enough to show that $\mathrm{Hom}_C(V,\Omega)$ has relative dominant dimension greater or equal than two. In such a case, there exists $Y_0, Y_1 \in \mathrm{add}\, DV$ and an (A,R)-exact sequence $0 \to \mathrm{Hom}_C(V,\Omega) \to Y_0 \to Y_1$. Combining this (A,R)-exact sequence with (75) we obtain an (A,R)-exact sequence

$$0 \to M \to \mathrm{Hom}_C(V, V\otimes_A X_0) \to \cdots \to \mathrm{Hom}_C(V, V \otimes_A X_{k-2}) \to Y_0 \to Y_1.$$

This would imply that $\mathrm{domdim}_{(A,R)} M \geq k+1$.

We can see that by Lemma 3.17 and by assumption on the R-projectivity of the $k-1$ syzygy that

$$\mathrm{Hom}_A(V, D\,\mathrm{Hom}_C(V,\Omega)) \otimes_C V \simeq D(V \otimes_A \mathrm{Hom}_C(V,\Omega)) \otimes_C V) \simeq D(\Omega) \otimes_C V \tag{76}$$

$$\simeq \Omega^{k-1}(D(V \otimes_A M), D((V\otimes_A X)^\bullet) \otimes_C V \in R\text{-proj}\,. \tag{77}$$

By Lemma 3.17 the map ξ_Ω is an isomorphism. Moreover,

$$\mathrm{Hom}_C(V, \xi_\Omega) \circ \alpha_{\mathrm{Hom}_C(V,\Omega)}(f)(v) = \xi_\Omega(v \otimes f) = f(v), \ \ f \in \mathrm{Hom}_C(V,\Omega), v \in V. \tag{78}$$

Thus, $\mathrm{Hom}_C(V,\xi_\Omega) \circ \alpha_{\mathrm{Hom}_C(V,\Omega)} = \mathrm{id}_{\mathrm{Hom}_C(V,\Omega)}$. It follows that $\alpha_{\mathrm{Hom}_C(V,\Omega)}$ is an isomorphism. By Proposition 3.23, $\mathrm{domdim}_{(A,R)} \mathrm{Hom}_C(V,\Omega) \geq 2$. $\square$

Theorem 5.3. *Let (A,P,V) be a RQF3 algebra. Denote $B = \mathrm{End}_A(P)^{op}$. For any R-projective right A-module M, the following assertions are equivalent.*

(a) $\mathrm{domdim}_{(A,R)} M \geq n \geq 2$;

(b) $\delta_M \colon P \otimes_B \mathrm{Hom}_A(P, DM) \to DM$ *is an isomorphism and* $\mathrm{Tor}_i^B(P, \mathrm{Hom}_A(P, DM)) = 0$, $1 \leq i \leq n-2$;

(c) $\alpha_M \colon M \to \mathrm{Hom}_B(P, M \otimes_A P)$ *is an isomorphism,* $P \otimes_B \Omega^i(\mathrm{Hom}_A(P,DM), Q^\bullet) \in R\text{-proj}$, $0 \leq i \leq n-2$ *for every left B-projective resolution $Q^\bullet$ of* $\mathrm{Hom}_A(P,DM)$ *and* $\mathrm{Ext}^i_B(P, M \otimes_A P) = 0$, $1 \leq i \leq n-2$.

Proof. The proof is analogous to Theorem 5.2. ☐

Remark 5.4. *By Observation 3.22 we can deduce as in Theorem 5.2 that the existence of an (A, R)-exact sequence*

$$Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y \to 0, \tag{79}$$

where $Y_i \in \operatorname{Add}_A V$, $1 \leq i \leq n$, for a given $Y \in \text{Mod-}A$, is equivalent to requiring $\Phi\colon \operatorname{Hom}_A(V, Y) \otimes_C V \to Y$ to be an isomorphism and $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, Y), V) = 0$, $1 \leq i \leq n-2$.

Comparing this version with the Mueller theorem for Artinian algebras, we can see that the functors Tor take a more important role than Ext. Furthermore, condition (c) does not seem very practical to use in applications since we have to test every syzygy of a projective resolution of $\operatorname{Hom}_A(V, DM)$. However, using Ext can still be useful if we know the Krull dimension of the ground ring. Recall that for **commutative Noetherian regular rings** (by definition the localization at every prime ideal is a regular local ring) the Krull dimension coincides with the global dimension (see for example [Rot09, Theorem 8.62, Proposition 8.60]).

Proposition 5.5. *Let R be a commutative Noetherian regular ring. Let (A, P, V) be a RQF3 algebra. Fix $C = \operatorname{End}_A(V)$ and $B = \operatorname{End}_A(P)^{op}$. Let $n \geq 2, M \in A\text{-mod} \cap R\text{-proj}$, and $N \in \text{mod-}A \cap R\text{-proj}$. The following assertions hold.*

(i) If $\alpha_M\colon M \to \operatorname{Hom}_C(V, V \otimes_A M)$ is an isomorphism and $\operatorname{Ext}_C^i(V, V \otimes_A M) = 0$ for every $1 \leq i \leq n-2$, then $\operatorname{domdim}_{(A,R)} M \geq n - \dim R$.

(ii) If $\alpha_N\colon N \to \operatorname{Hom}_B(P, N \otimes_A P)$ is an isomorphism and $\operatorname{Ext}_B^i(P, N \otimes_A P) = 0$ for every $1 \leq i \leq n-2$, then $\operatorname{domdim}_{(A,R)} N \geq n - \dim R$.

Proof. If $\dim R \geq n$, then there is nothing to prove. Assume that $n > \dim R$. Let $j = n - \dim R$. Let

$$0 \to V \otimes_A M \xrightarrow{\alpha_0} Y_0 \xrightarrow{\alpha_1} Y_1 \to \cdots \tag{80}$$

be a (C, R)-injective resolution of $V \otimes_A M$. The modules Y_i can be chosen to be R-projective as well. Since $\operatorname{Ext}_C^i(V, V \otimes_A M) = 0$, $1 \leq i \leq n-2$, applying $\operatorname{Hom}_C(V, -)$ yields the exact sequence

$$0 \to M \simeq \operatorname{Hom}_C(V, V \otimes_A M) \xrightarrow{\operatorname{Hom}_C(V, \alpha_0)} \operatorname{Hom}_C(V, Y_0) \xrightarrow{\operatorname{Hom}_C(V, \alpha_1)} \cdots \to \operatorname{Hom}_C(V, Y_{n-1}). \tag{81}$$

Note that $\operatorname{Hom}_C(V, Y_i) \in \operatorname{add} \operatorname{Hom}_C(V, DC) = \operatorname{add} DV = \operatorname{add} P$. Let $C_i = \operatorname{im} \operatorname{Hom}_C(V, \alpha_i), \forall i$. The exact sequence (81) induces the exact sequence

$$0 \to C_j \to \operatorname{Hom}_C(V, Y_j) \to \cdots \to \operatorname{Hom}_C(V, Y_{n-2}) \to C_{n-1} \to 0. \tag{82}$$

Note that this sequence has length $\dim R + 1$. Furthermore, since $\operatorname{pdim}_R C_{n-1} \leq \dim R$, we must have that C_j is R-projective. This implies that the exact sequence

$$0 \to M \to \operatorname{Hom}_C(V, Y_0) \to \cdots \to \operatorname{Hom}_C(V, Y_{j-1}) \tag{83}$$

is (A, R)-exact. Therefore, it follows that $\operatorname{domdim}_{(A,R)} M \geq j = n - \dim R$. (ii) is analogous to (i). ☐

When the Krull dimension is at most one, we can formulate the Mueller theorem in the following way.

Theorem 5.6. *Let R be a commutative Noetherian regular ring with Krull dimension at most one. Let (A, P, V) be a RQF3 algebra. Denote $C = \operatorname{End}_A(V)$. Let $M \in A\text{-mod} \cap R\text{-proj}$ and $n \geq 2$. The following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq n-1$ *where the* (A,R)*-exact sequence*

$$0 \to M \to X_1 \to \cdots \to X_{n-1}, \tag{84}$$

X_i *an* (A,R)*-injective* A*-projective module, can be continued to an exact sequence*

$$0 \to M \to X_1 \to \cdots \to X_{n-1} \to Y \tag{85}$$

where Y *is* (A,R)*-injective* A*-projective.*

(ii) α_M *is an isomorphism and* $\operatorname{Ext}^i_C(V, V\otimes_A M) = 0$, $1 \leq i \leq n-2$.

Proof. Assume that (ii) holds. Using Proposition 5.5, we see that $\operatorname{domdim}_{(A,R)} M \geq n-1$. Moreover, using the (A,R)-exact constructed there we have

$$0 \to M \to \operatorname{Hom}_C(V, Y_0) \to \cdots \to \operatorname{Hom}_C(V, Y_{n-2}) \to C_{n-1} \to 0. \tag{86}$$

Since C_{n-1} can be embedded into $\operatorname{Hom}_C(V, Y_{n-1})$ (i) follows.

Conversely, assume that (i) holds. Since $n \geq 2$, there exists an exact sequence $0 \to M \to X_1 \to X_2$ where $X_i \in \operatorname{add} DV$. The functor $\operatorname{Hom}_C(V, V\otimes_A -)$ is left exact, so it yields the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\alpha_M"] & X_1 \arrow[r] \arrow[d, "\alpha_{X_1}"] & X_2 \arrow[d, "\alpha_{X_2}"] \\
0 \arrow[r] & \operatorname{Hom}_C(V, V\otimes_A M) \arrow[r] & \operatorname{Hom}_C(V, V\otimes_A X_1) \arrow[r] & \operatorname{Hom}_C(V, V\otimes_A X_2)
\end{tikzcd} \tag{87}$$

By diagram chasing, it follows that α_M is an isomorphism. Applying $V\otimes_A -$ to (85) we obtain the exact sequence

$$0 \to V\otimes_A M \to V\otimes_A X_1 \to \cdots \to V\otimes_A X_{n-1} \to V\otimes_A Y. \tag{88}$$

Note that by deleting $V\otimes_A Y$ we obtain a (C,R)-exact sequence. We can continue such (C,R)-exact to a (C,R)-injective resolution of $V\otimes_A M$. Now consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\alpha_M", "\simeq"'] & \cdots \arrow[r] & X_{n-1} \arrow[r] \arrow[d, "\alpha_{X_{n-1}}", "\simeq"'] & Y \arrow[d, "\alpha_Y", "\simeq"'] \\
0 \arrow[r] & \operatorname{Hom}_C(V, V\otimes_A M) \arrow[r] & \cdots \arrow[r] & \operatorname{Hom}_C(V, V\otimes_A X_{n-1}) \arrow[r] & \operatorname{Hom}_C(V, V\otimes_A Y)
\end{tikzcd} \tag{89}$$

It follows that the bottom row is exact. In particular, $\operatorname{Ext}_{(C,R)}(V, V\otimes_A M) = 0$, $1 \leq i \leq n-3$. Notice that by continuing the (C,R)-injective resolution we have the following commutative diagram

$$\begin{tikzcd}
V\otimes_A X_{n-1} \arrow[rr, "\lambda_n"] \arrow[dr, two heads, "\varepsilon"] & & V\otimes_A Y & \\
& V\otimes_A \operatorname{coker} \arrow[ur, hook, "t"] \arrow[rr, hook, "\nu"] & & \tilde{X_n}
\end{tikzcd} \quad . \tag{90}$$

Since $\operatorname{Hom}_C(V, -)$ is left exact,

$$\ker \operatorname{Hom}_C(V, \nu\circ\varepsilon) = \ker \operatorname{Hom}_C(V, \varepsilon) = \ker \operatorname{Hom}_C(V, t\circ\varepsilon) = \operatorname{im} \operatorname{Hom}_C(V, \lambda_{n-1}). \tag{91}$$

This last equality follows from the exactness of (89). This means that

$$0 \to \operatorname{Hom}_C(V, V\otimes_A M) \to \cdots \to \operatorname{Hom}_C(V, V\otimes_A X_{n-1}) \to \operatorname{Hom}_C(V, \tilde{X_n}) \tag{92}$$

is exact. So, (ii) holds. $\square$

This method gives a hint why for Krull dimension one we can say that by continuing an (A, R)-exact sequence of projective relative injectives to a non-(A, R)-exact sequence of projective relative injectives is still enough to recover information about Ext. The method here used requires that at each step to compute the exact sequence we might have to replace the projective (A, R)-injective. This happens in general because we do not have a standard choice here unless the algebra is semiperfect. In such a case, the projective covers can take that role.

Proposition 5.7. *Let A be a semi-perfect R-algebra and a projective Noetherian R-algebra. Let $M \in A\text{-mod} \cap R\text{-proj}$. Let*

$$\cdots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} DM \to 0 \tag{93}$$

be a minimal right A-projective resolution. Then, $\operatorname{domdim}_{(A,R)} M \geq n$ *if and only if for each $i = 0, \ldots, n-1$, P_i is (A, R)-injective right module.*

Proof. One of the implications is clear. Assume that $\operatorname{domdim}_{(A,R)} M \geq n$. Then, there exists an (A, R)-exact sequence $0 \to M \xrightarrow{\alpha_0} I_0 \to \cdots \xrightarrow{\alpha_{n-1}} I_{n-1}$, with all I_i being A-projective (A, R)-injective. Hence applying D we obtain an exact sequence $DI_{n-1} \xrightarrow{D\alpha_{n-1}} \cdots \to DI_0 \xrightarrow{D\alpha_0} DM \to 0$. Since P_0 and DI_0 are A-projective there are maps $f_0 \in \operatorname{Hom}_A(P_0, DI_0)$, $g_0 \in \operatorname{Hom}_A(DI_0, P_0)$ satisfying $p_0 \circ g_0 = D\alpha_0$ and $D\alpha_0 \circ f_0 = p_0$. Hence, $p_0 \circ g_0 \circ f_0 = p_0$. Since (P_0, p_0) is the projective cover of DM, it follows that $g_0 \circ f_0 \in \operatorname{End}_A(P_0)$ is an isomorphism. Consequently, g_0 is surjective and thus, P_0 is an A-summand of DI_0. In particular, P_0 is (A, R)-injective. Observe that $p_0 \circ g_0 \circ D\alpha_1 = D\alpha_0 \circ D\alpha_1 = 0$. Hence, $\operatorname{im} g_0 \circ D\alpha_1 \subset \ker p_0$. Let $x \in \ker p_0 =$. Then, by the surjectivity of g_0, there exists $y \in DI_0$ such that $g_0(y) = x$. Therefore, $D\alpha_0(y) = p_0(x) = 0$. Thus, $y \in \ker D\alpha_0 = \operatorname{im} D\alpha_1$. So, $x \in \operatorname{im} g_0 \circ D\alpha_1$. We deduced that the sequence $DI_{n-1} \to \cdots \to DI_1 \xrightarrow{g \circ D\alpha_1} P_0 \xrightarrow{p_0} DM \to 0$ is exact. Now we can proceed by induction, where in the next step $\ker p_0$ takes the place of DM, to obtain that each P_i is an A-summand of DI_i. □

To clarify, if A is not semi-perfect there is no canonical (A, R)-injective resolution to pick for a module to compute its relative dominant dimension. For example, looking at the standard (A, R)-injective resolution gives us no information here because if $M \in A\text{-mod} \cap R\text{-proj}$ has positive relative dominant dimension and if it is free as R-module, then $DA \in \operatorname{add} \operatorname{Hom}_R(A, M) = \operatorname{add} DA \otimes_R M$. Hence, the first term of the standard (A, R)-injective resolution of M $I_0 := \operatorname{Hom}_R(A, M)$ cannot be projective over A.

5.1 Further consequences

We shall now see some properties of relative dominant dimension that follow from the relative Mueller theorem. In particular, the relative Mueller characterization applied to A takes the following form. This result is the relative analogue of [Mue68, Lemma 3] and [Tac73, 7.5].

Theorem 5.8. *Let A be an (A, P, V) RQF3 algebra with* $\operatorname{domdim}_{(A,R)} {}_AA \geq 2$ *and* $\operatorname{domdim}_{(A,R)} A_A \geq 2$. *For $n \geq 3$, the following are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} {}_AA \geq n$;

(ii) $\operatorname{Tor}_i^C(DV, V) = 0, \ i = 1, \ldots, n-2$;

(iii) $\operatorname{Ext}_C^i(V, V) = 0, \ i = 1, \ldots, n-2$ *and* $\Omega^j(DV, Q^\bullet) \otimes_C V \in R\text{-proj}$, $0 \leq j \leq n-2$, *for every C-projective resolution $Q^\bullet$ of DV;*

(iv) $\operatorname{Tor}_i^B(P, DP) = 0 \ i = 1, \ldots, n-2$;

(v) $\mathrm{Ext}^i_C(P,P) = 0$, $i = 1, \dots, n-2$ *and* $P \otimes_B \Omega^j(DP, Q^\bullet) \in R\text{-proj}$, $0 \leq j \leq n-2$, *for every B-projective resolution $Q^\bullet$ of DP;*

(vi) $\mathrm{domdim}_{(A,R)} A_A \geq n$.

Proof. The implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ follow from Theorem 5.2 and Theorem 5.3, respectively. We will, therefore, focus on the implication $(ii) \Leftrightarrow (iv)$.

Consider a left B-projective resolution

$$\cdots \to P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} DP \to 0. \tag{94}$$

Applying the exact functor $\mathrm{Hom}_B(D(V \otimes_A P), -)$ we get the exact sequence

$$\cdots \to \mathrm{Hom}_B(D(V \otimes_A P), P_{n-1}) \to \cdots \to \mathrm{Hom}_B(D(V \otimes_A P), P_0) \to \mathrm{Hom}_B(D(V \otimes_A P), DP) \to 0. \tag{95}$$

Since $D(V \otimes_A P)$ is a B-generator, each $P_i \in \mathrm{add}\, D(V \otimes_A P)$, therefore $\mathrm{Hom}_B(D(V \otimes_A P), P_i) \in C\text{-proj}$. Also, $\mathrm{Hom}_B(D(V \otimes_A P), DP) \simeq \mathrm{Hom}_B(P, V \otimes_A P) \simeq V$ as left C-modules. Thus, (95) is a C-projective resolution for V.

We recall that in Lemma 3.25, we saw that for

$$F = \mathrm{Hom}_B(D(V \otimes_A P), -) \text{ and } G = \mathrm{Hom}_B(\mathrm{Hom}_B(D(V \otimes_A P), B), -)$$

there was an isomorphism $GM \otimes_C FN \simeq M \otimes_B N$ for every $M \in \text{mod-}B$ and $N \in B\text{-mod}$. Since all the isomorphisms involved are functorial, it follows that there exists a natural isomorphism of bifunctors $\theta\colon G(-) \otimes_C F(-) \to \mathrm{id}(-) \otimes_B \mathrm{id}(-)$. In particular, the following diagram is commutative

$$\begin{tikzcd} P \otimes_B P_{i-1} \arrow[r, "\mathrm{id}_P \otimes_B f_i"] & P \otimes_B P_{i-2} \arrow[r] & \cdots \arrow[r] & P \otimes_B P_0 \arrow[r, "\mathrm{id}_P \otimes_B f_0"] & P \otimes_B DP \arrow[r] & 0 \\ GP \otimes_B FP_{i-1} \arrow[u, "\theta_{P,P_{i-1}}", "\simeq"'] \arrow[r, "G\,\mathrm{id}_P \otimes_B Ff_i"] & GP \otimes_B FP_{i-2} \arrow[u, "\theta_{P,P_{i-2}}", "\simeq"'] \arrow[r] & \cdots \arrow[r] & GP \otimes_C FP_0 \arrow[u, "\theta_{P,P_0}", "\simeq"'] \arrow[r, "G\,\mathrm{id}_P \otimes_C Ff_0"] & GP \otimes_C FDP \arrow[u, "\theta_{P,DP}", "\simeq"'] \arrow[r] & 0 \end{tikzcd}. \tag{96}$$

So, the upper row is exact if and only if the bottom row is exact. Furthermore, the bottom row is exactly the complex obtained by applying $DV \otimes_C -$ to the exact sequence (95). It follows that $\mathrm{Tor}_i^C(DV, V) = 0$ if and only if $\mathrm{Tor}_i^B(P, DP) = 0$. $\square$

Corollary 5.9. *Let (A, P, V) be a RQF3 algebra. Then,* $\mathrm{domdim}_{(A,R)}\, {}_A A = \mathrm{domdim}_{(A,R)} A_A$.

Proof. Assume that $\mathrm{domdim}_{(A,R)}\, {}_A A \geq 2$. By Lemma 3.21, V is a left C-generator (C,R)-cogenerator. In view of Lemma 3.25, $DV \otimes_C V \in R\text{-proj}$. By Theorem 5.2, V satisfies the double centralizer property. By Morita-Tachikawa correspondence $\mathrm{End}_C(V) \simeq A$ has left and right relative dominant dimension greater or equal than two. By Theorem 5.8, we have $\mathrm{domdim}_{(A,R)} A_A \geq \mathrm{domdim}_{(A,R)}\, {}_A A$. Symmetrically, $\mathrm{domdim}_{(A,R)}\, {}_A A \geq \mathrm{domdim}_{(A,R)} A_A$. $\square$

Another consequence of Theorem 5.8 is that we can characterize every endomorphism algebra of a generator relative cogenerator such that the generator remains R-projective under tensor product over its dual. In fact, Let B be the endomorphism algebra over A of a generator (A,R)-cogenerator such that $DM \otimes_A M \in R\text{-proj}$. By relative Morita-Tachikawa, B has left and right relative dominant dimension greater or equal than two. Now Theorem 5.8 gives that $\mathrm{domdim}(B,R) \geq n+2$ if and only if $\mathrm{Tor}_i^A(DM, M) = 0$, $1 \leq i \leq n$.

Corollary 5.10. *Let (A, P, V) be a RQF3 algebra. Let $M_i \in A\text{-mod} \cap R\text{-proj}$, $i \in I$, for some finite set I. Then,*

$$\mathrm{domdim}_{(A,R)} \left(\bigoplus_{i \in I} M_i \right) = \inf\{\mathrm{domdim}_{(A,R)} M_i \colon i \in I\}. \tag{97}$$

Proof. Since the maps Φ_X are compatible with direct sums, we get that Φ_{M_i} is surjective/bijective for every $i \in I$ if and only if $\Phi_{\bigoplus_{i\in I}}$ is surjective/bijective. Thus, $\operatorname{domdim}_{(A,R)} \bigoplus_{i\in I} M_i \geq 1$ (resp. 2) if and only if $\operatorname{domdim}_{(A,R)} M_i \geq 1$ (resp. 2) for every $i \in I$. Now since for every n

$$\operatorname{Tor}_n^C(\operatorname{Hom}_A(V, D(\bigoplus_{i\in I} M_i)), V) \simeq \operatorname{Tor}_n^C(\operatorname{Hom}_A(V, \bigoplus_{i\in I} DM_i), V) \simeq \bigoplus_{i\in I} \operatorname{Tor}_n^C(\operatorname{Hom}_A(V, DM_i), V), \quad (98)$$

the result follows by Theorem 5.2. □

Remark 5.11. *It follows that the value of the relative dominant dimension is independent of the direct sum decomposition of the module.*

The following Lemma is another consequence of relative Mueller characterization. In the field case, this proof is quicker using the relations between dominant dimension and the socle of the regular module and it was first stated in [FK11b, Proposition 3.6].

Lemma 5.12. *Let (A, P, V) be a RQF3 algebra. Let $M \in R$-proj and consider the following (A, R)-exact*

$$0 \to M_1 \to M \to M_2 \to 0. \quad (99)$$

Let $n = \operatorname{domdim}_{(A,R)} M$ and $n_i = \operatorname{domdim}_{(A,R)} M_i$. Then, the following holds.

(a) $n \geq \min\{n_1, n_2\}$.

(b) *If $n_1 < n$, then $n_2 = n_1 - 1$.*

(c) *(i)* $n_1 = n \implies n_2 \geq n - 1$.
  (ii) $n_1 = n + 1 \implies n_2 \geq n$.
  (iii) $n_1 \geq n + 2 \implies n_2 = n$.

(d) $n < n_2 \implies n_1 = n$.

(e) *(i)* $n = n_2 \implies n_1 \geq n_2$.
  (ii) $n = n_2 + 1 \implies n_1 \geq n_2 + 1$.
  (iii) $n \geq n_2 + 2 \implies n_1 = n_2 + 1$.

Proof. Applying D and $\operatorname{Hom}_A(V, D-) \otimes_C V$ we get the commutative diagram with exact rows

$$\begin{tikzcd} \operatorname{Tor}_1^C(\operatorname{Hom}_A(V, DM_1), V) \arrow[r] & \operatorname{Hom}_A(V, DM_2) \otimes_C V \arrow[r] \arrow[d, "\Phi_{M_2}"] & \operatorname{Hom}_A(V, DM) \otimes_C V \arrow[r, two heads] \arrow[d, "\Phi_M"] & \operatorname{Hom}_A(V, DM_1) \otimes_C V \arrow[d, "\Phi_{M_1}"] \\ 0 \arrow[r] & DM_2 \arrow[r] & DM \arrow[r, two heads] & DM_1 \end{tikzcd} . \quad (100)$$

By Snake Lemma, Φ_M is an epimorphism/isomorphism if Φ_{M_1} and Φ_{M_2} are epimorphisms/isomorphisms. Thus, $\min\{n_1, n_2\} \geq k$, $k \leq 2$, implies that $n \geq k$. Consider the long exact sequence

$$\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM_1), V) \to \operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) \to \operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM_2), V) \quad (101)$$

we obtain that if $n_1, n_2 \geq k \geq 2$, $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM_1), V) = \operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM_2), V) = 0$ for $i = 1, \ldots, k-2$, then $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0$. Thus, $n \geq \min\{n_1, n_2\}$. By Theorem 5.2, (a) follows.

(b). If $n_1 = 0$, then Φ_{M_1} is not surjective. By diagram chasing, if Φ_M is surjective, then Φ_{M_1} is surjective. Thus, $n > 0$ implies that $n_1 > 0$. Assume $n_1 = 1$ and $n > n_1$. Thus, Φ_M is bijective and Φ_{M_1} is surjective. If Φ_{M_2} is surjective, then by Snake Lemma, Φ_{M_1} is also injective. This would imply that $n_1 \geq 2$. So, $n_2 = 0$. Assume now $n_1 \geq 2$. By Snake Lemma, Φ_{M_2} is surjective. So, $n_2 \geq 1$.

If $n_2 \geq 2$, then, in particular, Φ_{M_2} is surjective. The exactness of the bottom row of (100) makes $\operatorname{Hom}_A(V, DM_2) \otimes_C V \to \operatorname{Hom}_A(V, DM) \otimes_C V$ injective. Since $\operatorname{Tor}_1^C(\operatorname{Hom}_A(V, DM), V) = 0$, the long exact sequence induces that $\operatorname{Tor}_1^C(\operatorname{Hom}_A(V, DM_1), V) = 0$. This contradicts $n_1 = 2$. Thus, $n_2 = 1$. Now assume that $n_1 \geq 3$. Thus, (100) becomes

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(V, DM_2) \otimes_C V \arrow[r] \arrow[d, "\Phi_{M_2}"] & \operatorname{Hom}_A(V, DM) \otimes_C V \arrow[r, two heads] \arrow[d, "\Phi_M"] & \operatorname{Hom}_A(V, DM_1) \otimes_C V \arrow[d, "\Phi_{M_1}"] \\
0 \arrow[r] & DM_2 \arrow[r] & DM \arrow[r, two heads] & DM_1
\end{tikzcd} . \tag{102}$$

Thus, by Snake Lemma Φ_{M_2} is bijective. Furthermore, using the long exact sequences and as $n > n_1$ we deduce that

$$\operatorname{Tor}_{i+1}^C(\operatorname{Hom}_A(V, DM_1), V) \simeq \operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM_2), V), \ 1 \leq i \leq n_1 - 2. \tag{103}$$

Thus, $n_2 = n_1 - 1$.

Analogously, (c), (d), (e) hold. $\square$

6 Relative dominant dimension under change of rings

In this section, we will see that relative dominant dimension is stable under change of rings. Furthermore, in practice, computations of relative dominant dimension of projective Noetherian algebras can be reduced to computations of dominant dimension of finite dimensional algebras over algebraically closed fields (Proposition 6.10 and Theorem 6.13). This is the interpretation of the main result of the current paper (Theorem 6.13). We can see using change of rings that projective relative injective strongly faithful modules are sharp generalizations of projective-injective faithful modules according to Proposition 6.4. Using this, we can conclude the left right symmetry of the relative dominant dimension of projective Noetherian algebras (Corollary 6.6). At this point, we also obtain a better picture of the relative Morita-Tachikawa correspondence. The extra condition appearing on relative Morita-Tachikawa correspondence is a manifestation of the fact that the algebras on both sides of the correspondence must satisfy a base change property (Proposition 6.14). We will see using the main result that the Nakayama conjecture is equivalent to the relative Nakayama conjecture for projective Noetherian algebras (Theorem 6.17).

6.1 Strongly faithful modules revisited

The proofs of the following two lemmas are technical however they are very useful to characterize strongly faithful modules.

Lemma 6.1. *Let A be a projective Noetherian R-algebra. Let $V \in \operatorname{mod-}A \cap R\text{-proj}$. Consider the A-map $\delta_V \colon \bigoplus_{g \in \operatorname{Hom}_A(DV, DA)} DV \to DA$, given by $\delta_V(f_g) = g(f)$, where $M_g := DV$ for $g \in \operatorname{Hom}_A(DV, DA)$. For each $f \in DV$ and $g \in \operatorname{Hom}_A(DV, DA)$, f_g denotes the function from $\operatorname{Hom}_A(DV, DA)$ to the disjoint union of all modules M_h, $h \in \operatorname{Hom}_A(DV, DA)$, so that $f_g(h) = 0$ if $g \neq h$ and $f_g(g) = f$. Then, δ_V is surjective if and only if V is (A, R)-strongly faithful.*

Proof. First, we need to check that δ_V is well defined. Let $g \in \operatorname{Hom}_A(DV, DA)$. Let $\theta_g \colon DV \to DA$ be the map given by $\theta_g(f) = g(f)$, $f \in DV$. This is clearly an A-map since $g \in \operatorname{Hom}_A(DV, DA)$. Taking the direct sum of maps θ_g over $g \in \operatorname{Hom}_A(DV, DA)$ yields the map δ_V. Thus, δ_V is well defined.

Assume that δ_V is surjective. Let $\{f_1, \ldots, f_t\}$ be an R-generator set for DA. By assumption, there exists for each $1 \leq i \leq t$ a natural number $s_i > 0$ and elements $w_{i,j} \in DV$, $g_{i,j} \in \operatorname{Hom}_A(DV, DA)$ with $j = 1, \ldots, s_i$ such that

$$f_i = \delta_V(\sum_{j=1}^{s_i} (w_{i,j})_{g_{i,j}}). \tag{104}$$

Let $h \in DA$. Then,

$$h = \sum_{i=1}^t \alpha_i f_i = \sum_{i=1}^t \alpha_i \delta_V (\sum_{j=1}^{s_i} (w_{i,j})_{g_{i,j}}) = \delta_V (\sum_{i=1}^t \sum_{j=1}^{s_i} \alpha_i (w_{i,j})_{g_{i,j}}), \quad \alpha_i \in R. \tag{105}$$

Therefore, the restriction of δ_V to the summands indexed by $g_{i,j}$ $1 \leq i \leq t, 1 \leq j \leq s_i$ is surjective. Denote by o the number of such indexes. Then, we found a surjective A-map $(DV)^o \twoheadrightarrow DA$. As $DA \in R$-proj, this map is an (A, R)-epimorphism. Thus, applying D yields an (A, R)-monomorphism $A \to V^o$. So, V is (A, R)-strongly faithful.

Conversely, assume that V is (A, R)-strongly faithful. Hence there is an (A, R)-monomorphism $A \to V^t$ for some $t > 0$. Applying D we obtain a surjective map $DV^t \to DA$. Denote this map by ε. Let $k_j \in \operatorname{Hom}_A(DV, DV^t)$ and $\pi_j \in \operatorname{Hom}_A(DV^t, DV)$ be the canonical injections and projections, respectively. Define $g_j = \varepsilon \circ k_j \in \operatorname{Hom}_A(DV, DA)$. For every $h \in DA$, there exists $y \in DV^t$ such that $\varepsilon(y) = h$. Therefore,

$$h = \sum_{j=1}^t \varepsilon \circ k_j \circ \pi_j(y) = \delta_V (\sum_{j=1}^t \pi_j(y)_{g_j}). \tag{106}$$

So, δ_V is surjective. □

Lemma 6.2. *Let A be a projective Noetherian R-algebra. For every commutative R-algebra S, and $X, Y \in A$-mod there exists a map*

$$\theta_S \colon S \bigotimes_R \bigoplus_{g \in \operatorname{Hom}_A(X,Y)} X \longrightarrow \bigoplus_{h \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R X, S \otimes_R Y)} S \otimes_R X,$$

given by $\theta_S(s \otimes x_g) = (s \otimes x)_{1_S \otimes g}$.

Moreover, if $X \in A$-proj, then $\theta_{R(\mathfrak{m})}$ is surjective for every maximal ideal $\mathfrak{m}$ in R.

Proof. Consider the map

$$\theta \colon S \times \bigoplus_{g \in \operatorname{Hom}_A(X,Y)} X \to \bigoplus_{h \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R X, S \otimes_R Y)} S \otimes_R X,$$

given by $\theta(s, x_g) = (s \otimes x)_{1_S \otimes g}$ for $s \in S$, $x \in X$, $g \in \operatorname{Hom}_A(X, Y)$. By definition, this map is linear in each term. Let $r \in R$. Then,

$$\theta(rs, x_g) = (rs \otimes x)_{1_S \otimes g} = (s \otimes rx)_{1_S \otimes g} = \theta(s, (rx)_g). \tag{107}$$

So, θ induces uniquely the S-map θ_S. Assume that $X \in A$-proj. Let $\mathfrak{m}$ be a maximal ideal in R. Then, $\operatorname{Hom}_{A(\mathfrak{m})}(X(\mathfrak{m}), Y(\mathfrak{m})) \simeq \operatorname{Hom}_A(X, Y)(\mathfrak{m})$. Thus, every element in $\operatorname{Hom}_{A(\mathfrak{m})}(X(\mathfrak{m}), Y(\mathfrak{m}))$ can be written in the form $h \otimes (r + m) = (rh) \otimes 1_{R(\mathfrak{m})}$ for $rh \in \operatorname{Hom}_A(X, Y)$. Moreover, every element in $\bigoplus_{h \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R X, S \otimes_R Y)} S \otimes_R X$ is the sum of elements $(1_{R(\mathfrak{m})} \otimes x)_{1_{R(\mathfrak{m})} \otimes h} = \theta_{R(\mathfrak{m})}(1_{R(\mathfrak{m})} \otimes x_h)$, $h \in \operatorname{Hom}_A(X, Y)$. This implies that $\theta_{R(\mathfrak{m})}$ is surjective. □

Proposition 6.3. *Let A be a projective Noetherian R-algebra. Let $V \in$ mod-$A \cap R$-proj. Then, the following assertions are equivalent.*

(a) V is an A-projective (A, R)-injective-strongly faithful right module.

(b) $S \otimes_R V$ is an $S \otimes_R A$-projective $(S \otimes_R A, S)$-injective-strongly faithful right module for every commutative R-algebra S.

(c) $V_{\mathfrak{m}}$ is an $A_{\mathfrak{m}}$-projective $(A_{\mathfrak{m}}, R_{\mathfrak{m}})$-injective-strongly faithful right module for every maximal ideal $\mathfrak{m}$ in R.

(d) $V(\mathfrak{m})$ is right $A(\mathfrak{m})$-projective-injective faithful for every maximal ideal $\mathfrak{m}$ in R.

Proof. $(i) \implies (ii)$. Let S be a commutative R-algebra. The module V is a right A-summand of A^t for some $t > 0$. Hence $S \otimes_R V$ is a right $S \otimes_R A$-summand of $S \otimes_R A^t \simeq (S \otimes_R A)^t$. Thus, $S \otimes_R V$ is a right $S \otimes_R A$-projective module. As V is (A, R)-injective, V is an A-summand of $\operatorname{Hom}_R(A, V)$. So, $S \otimes_R V$ is an $S \otimes_R A$-summand of $S \otimes_R \operatorname{Hom}_R(A, V) \simeq \operatorname{Hom}_S(S \otimes_R A, S \otimes_R V)$ since $A \in R$-proj. Hence, $S \otimes_R V$ is a projective $(S \otimes_R A, S)$-injective. By Lemma 6.1, the map $\delta_V \in \operatorname{Hom}_A(\bigoplus_{g \in \operatorname{Hom}_A(DV, DA)} DV, DA)$ is surjective. Applying the functor $S \otimes_R -$ we have the following commutative diagram

$$\begin{tikzcd}
S \bigotimes_R \bigoplus_{g \in \operatorname{Hom}_A(DV, DA)} DV \arrow[rr, "S \otimes_R \delta_V"] \arrow[d, "\theta_S"] & & S \otimes_R DA \arrow[dd, "l_S"] \\
\bigoplus_{g \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R DV, S \otimes_R DA)} S \otimes_R DV \arrow[d, "\kappa_S"] & & \\
\bigoplus_{h \in \operatorname{Hom}_{S \otimes_R A}(\operatorname{Hom}_S(S \otimes_R V, S), \operatorname{Hom}_S(S \otimes_R A, S))} \operatorname{Hom}_S(S \otimes_R V, S) \arrow[rr, "\delta_{S \otimes_R V}"] & & \operatorname{Hom}_S(S \otimes_R A, S)
\end{tikzcd}, \tag{108}$$

where l_S and κ_l are the canonical isomorphisms (as $V, A \in R$-proj). This diagram is commutative since:

$$\begin{aligned}
\delta_{S \otimes_R V} \circ \kappa_S \circ \theta_S(s \otimes x_g) = \delta_{S \otimes_R V} \circ \kappa_S(s \otimes x)_{1_S \otimes g} = \delta_{S \otimes_R V}((s \otimes x)_{1_S \otimes g}) = 1_S \otimes g(s \otimes x) = s \otimes g(x) \\
l_S \circ S \otimes_R \delta_V(s \otimes x_g) = l(s \otimes g(x)) = s \otimes g(x), \; s \in S, x \in DV, g \in \operatorname{Hom}_A(DV, DA).
\end{aligned}$$

The right exactness of $S \otimes_R -$implies that $S \otimes_R \delta_V$ is surjective. Using the commutativity of the diagram $\delta_{S \otimes_R V} \circ \kappa_S \circ \theta_S$ is surjective. Hence, $\delta_{S \otimes_R V}$ is surjective. By Lemma 6.1, (ii) follows.

$(ii) \implies (iii)$. For every maximal ideal $\mathfrak{m}$ in R, consider $S = R_{\mathfrak{m}}$.

$(iii) \implies (iv)$. Let $\mathfrak{m}$ be a maximal ideal in R. Recall that

$$X_{\mathfrak{m}}(\mathfrak{m}) = X_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} = X \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} = X \otimes_R R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} = X(\mathfrak{m}). \tag{109}$$

Hence, using the same argument as discussed in $(i) \implies (ii)$ now with $S = R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ yields that $V(\mathfrak{m})$ is $A(\mathfrak{m})$-projective $(A(\mathfrak{m}), R(\mathfrak{m}))$-injective-strongly faithful. Since $R(\mathfrak{m})$ is a field, every $(A(\mathfrak{m}), R(\mathfrak{m}))$-injective is $A(\mathfrak{m})$-injective and strongly faithful coincides with faithful. So, (iv) follows.

$(iv) \implies (i)$. Since $V(\mathfrak{m})$ is an $A(\mathfrak{m})$-projective right module for every maximal ideal $\mathfrak{m}$ in R and $V \in R$-proj, we deduce that V is an A-projective right module. By Theorem 2.12, V is (A, R)-injective. By Lemma 6.1, $\delta_{V(\mathfrak{m})}$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By Lemma 6.2, $\theta_{R(\mathfrak{m})}$ is surjective. By the commutative diagram (108) with $S = R(\mathfrak{m})$ we get that $l_{R(\mathfrak{m})} \circ R(\mathfrak{m}) \otimes_R \delta_V$ is surjective. Since $l_{R(\mathfrak{m})}$ is bijective, it follows that $R(\mathfrak{m}) \otimes_R \delta_V$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, δ_V is surjective. So, V is also (A, R)-strongly faithful. ☐

By symmetry one obtains:

Proposition 6.4. *Let A be a projective Noetherian R-algebra. Let $P \in A$-mod$\cap R$-proj. Then, the following assertions are equivalent.*

(a) P is an A-projective (A, R)-injective-strongly faithful left module.

(b) $S \otimes_R P$ *is an* $S \otimes_R A$*-projective* $(S \otimes_R A, S)$*-injective-strongly faithful left module for every commutative* R*-algebra* S*.*

(c) $P_\mathfrak{m}$ *is an* $A_\mathfrak{m}$*-projective* $(A_\mathfrak{m}, R_\mathfrak{m})$*-injective-strongly faithful left module for every maximal ideal* $\mathfrak{m}$ *in* R*.*

(d) $P(\mathfrak{m})$ *is left* $A(\mathfrak{m})$*-projective-injective faithful for every maximal ideal* $\mathfrak{m}$ *in* R*.*

6.2 Left-right symmetry of relative dominant dimension

For finite dimensional algebras there exists a left faithful projective-injective if and only if there exists a right faithful projective-injective [Tac63, Theorem 2]. Although, we do not have an argument for (A, R)-strongly faithfulness being preserved under standard duality, we can recover the following statement.

Lemma 6.5. *Let* A *be a projective Noetherian* R*-algebra. Then,* $\operatorname{domdim}_{(A,R)} A_A \geq 1$ *if and only if* $\operatorname{domdim}_{(A,R)}\, {}_AA \geq 1$*. In particular, if* $\operatorname{domdim}_{(A,R)}\, {}_AA \geq 1$ *or* $\operatorname{domdim}_{(A,R)} A_A \geq 1$*, then there exist* P *and* V *such that* (A, P, V) *is a RQF3 algebra.*

Proof. Assume that $\operatorname{domdim}_{(A,R)} A_A \geq 1$. Then, there exists a right A-module V which is A-projective (A, R)-injective-strongly faithful. Since $A \in R\text{-proj}$, it follows that $V \in R\text{-proj}$. By Proposition 6.3, $V(\mathfrak{m})$ is an $A(\mathfrak{m})$-projective-injective faithful right module for every maximal ideal $\mathfrak{m}$ in R. Then, $\operatorname{Hom}_{R(\mathfrak{m})}(V(\mathfrak{m}), R(\mathfrak{m}))$ is an $A(\mathfrak{m})$-projective-injective left module for every maximal ideal $\mathfrak{m}$ in R.

Observe that in general if a finitely generated module X over a finite dimensional algebra B over a field K is faithful then $\operatorname{Hom}_K(X, K)$ is faithful as left B-module. In fact, let $b \in B$ and assume that $b \cdot f = 0$ for every $f \in \operatorname{Hom}_K(X, K)$. Then, for each $x \in X$,

$$0 = bf(x) = f(xb), \forall f \in \operatorname{Hom}_K(X, K).$$

Since X is finitely generated, we deduce that $xb = 0$. Now using that X is faithful over B yields $b = 0$.

Therefore, $DV(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(V(\mathfrak{m}), R(\mathfrak{m}))$ is an $A(\mathfrak{m})$-projective-injective faithful left module for every maximal ideal $\mathfrak{m}$ in R. By Proposition 6.4, DV is an A-projective (A, R)-injective-strongly faithful left module. Thus, $\operatorname{domdim}_{(A,R)}\, {}_AA \geq 1$. The converse implication is analogous. We also showed that (A, DV, V) is a RQF3 algebra. ☐

Corollary 6.6. *Let* A *be a projective Noetherian* R*-algebra. Then,* $\operatorname{domdim}_{(A,R)} A_A = \operatorname{domdim}_{(A,R)}\, {}_AA$*.*

Proof. Assume that $\operatorname{domdim}_{(A,R)} A_A \geq n$ for some $n \geq 1$. By Lemma 6.5, $\operatorname{domdim}_{(A,R)}\, {}_AA \geq 1$. By Corollary 5.9, $\operatorname{domdim}_{(A,R)}\, {}_AA \geq n$. Hence $\operatorname{domdim}_{(A,R)}\, {}_AA \geq \operatorname{domdim}_{(A,R)} A_A$.

Similarly, $\operatorname{domdim}_{(A,R)} A_A \geq \operatorname{domdim}_{(A,R)}\, {}_AA$. ☐

Thus, we will write $\operatorname{domdim}(A, R)$ avoiding the left and right notation to denote the relative dominant dimension of the regular module.

6.3 Relative dominant dimension on closed points

Proposition 6.7. *Let* (A, P, V) *be a RQF3 algebra. Let* $M \in A\text{-mod} \cap R\text{-proj}$*. Then, the following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq 1$.

(ii) $\operatorname{domdim}_{(S\otimes_R A,S)} (S \otimes_R M) \geq 1$ *for every commutative* R*-algebra* S*.*

(iii) $\operatorname{domdim}_{(A_\mathfrak{m},R_\mathfrak{m})} M_\mathfrak{m} \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *in* R*.*

(iv) $\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *in* R*.*

Proof. Let $C = \mathrm{End}_A(V)$. Denote by D_S the standard duality with respect to S, $\mathrm{Hom}_S(-, S)$. Consider the map $\Phi_M\colon \mathrm{Hom}_A(V, DM) \otimes_C V \to DM$. Applying the functor $S \otimes_R -$ we get the commutative diagram

$$\begin{tikzcd}
S \otimes_R \mathrm{Hom}_A(V, DM) \otimes_C V \arrow[r, "S\otimes_R \Phi_M"] \arrow[d, "\theta_{S,M}", "\simeq"'] & S \otimes_R DM \arrow[dd, "l_{S,M}", "\simeq"'] \\
S \otimes_R \mathrm{Hom}_A(V, DM) \otimes_{S\otimes_R C} S \otimes_R V \arrow[d, "\kappa_{S,M}", "\simeq"'] & \\
\mathrm{Hom}_{S\otimes_R A}(S \otimes_R V, D_S(S \otimes_R M)) \otimes_{S\otimes_R C} S \otimes_R V \arrow[r, "\Phi_{S\otimes_R M}"] & D_S(S \otimes_R M)
\end{tikzcd}, \tag{110}$$

where the $\theta_{S,M}$, $\kappa_{S,M}$ and $l_{S,M}$ are the natural maps. These are isomorphisms since $V \in \text{proj-}A$ and $M \in R\text{-proj}$.

$(i) \implies (ii)$. Since Φ_M is an epimorphism, it follows by diagram 110 that $\Phi_{S\otimes_R M}$ is an epimorphism. As $(S \otimes_R A, S \otimes_R P, S \otimes_R V)$ is a RQF3 S-algebra, (ii) follows by Theorem 5.2.

The implication $(ii) \implies (iii)$ follows by using (ii) with $S = R_{\mathfrak{m}}$. The implication $(iii) \implies (iv)$ follows by using the same argument as in the implication $(i) \implies (ii)$ with $S = R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$.

$(iv) \implies (i)$. By the diagram (110), it follows that $R(\mathfrak{m}) \otimes_R \Phi_M$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, Φ_M is surjective. Finally (i) follows by Theorem 5.2. $\square$

This last Proposition is not surprising since $S\otimes_R-$ is right exact and modules having relative dominant dimension at least one can be determined using surjective maps. By the same reason, flat extensions are compatible with relative dominant dimension of a module.

Proposition 6.8. *Let (A, P, V) be a RQF3 R-algebra. Let $M \in A\text{-mod} \cap R\text{-proj}$. The following assertions are equivalent. Let $n \in \mathbb{N}$.*

(i) $\mathrm{domdim}_{(A,R)} M \geq n \geq 1$.

(ii) $\mathrm{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq n \geq 1$ *for every flat commutative R-algebra.*

(iii) $\mathrm{domdim}_{(A_{\mathfrak{m}},R_{\mathfrak{m}})} M_{\mathfrak{m}} \geq n \geq 1$ *for every maximal ideal $\mathfrak{m}$ in R.*

Proof. By Proposition 6.3, $(S \otimes_R A, S \otimes_R P, S \otimes_R V)$ is a RQF3 S-algebra. Note that $S \otimes_R C \simeq S \otimes_R \mathrm{End}_A(V) \simeq \mathrm{End}_{S\otimes_R A}(S \otimes_R V)$. By Proposition 6.7, $\mathrm{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq 1$. Assume that $n \geq 2$. Hence, Φ_M is an isomorphism. By the diagram (110), $\Phi_{S\otimes_R M}$ is an isomorphism. So, $\mathrm{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq 2$. Now assume that $n \geq 3$. Then,

$$\begin{aligned}
0 = S \otimes_R \mathrm{Tor}_i^C(\mathrm{Hom}_A(V, DM), V) &= \mathrm{Tor}_i^{S\otimes_R C}(S \otimes_R \mathrm{Hom}_A(V, DM), S \otimes_R V) \\
&= \mathrm{Tor}_i^{S\otimes_R C}(\mathrm{Hom}_{S\otimes_R A}(S \otimes_R V, D_S(S \otimes_R M)), S \otimes_R V),\ 1 \leq i \leq n-2.
\end{aligned}$$

Now, (ii) follows by Theorem 5.2.

The implication $(ii) \implies (iii)$ follows by applying $S = R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ in R.

$(iii) \implies (i)$. If $n \geq 1$, then by Proposition 6.7 $\mathrm{domdim}_{(A,R)} M \geq 1$. If $n \geq 2$, then $\Phi_{M_{\mathfrak{m}}}$ is isomorphism for every maximal ideal $\mathfrak{m}$ in R. By the diagram (110), $R_{\mathfrak{m}} \otimes_R \Phi_M$ is isomorphism for every maximal ideal $\mathfrak{m}$ in R. Hence, Φ_M is an isomorphism. Moreover,

$$\mathrm{Tor}_i^C(\mathrm{Hom}_A(V, DM), V)_{\mathfrak{m}} = \mathrm{Tor}_i^{C_{\mathfrak{m}}}(\mathrm{Hom}_{A_{\mathfrak{m}}}(V_{\mathfrak{m}}, D_{\mathfrak{m}} M_{\mathfrak{m}}), V_{\mathfrak{m}}) = 0,\ 1 \leq i \leq n-2.$$

By Theorem 5.2,$\mathrm{domdim}_{(A,R)} M \geq n \geq 1$. $\square$

Proposition 6.9. *Let (A, P, V) be a RQF3 R-algebra. Let $M \in A\text{-mod} \cap R\text{-proj}$. If S is a Noetherian faithfully flat R-algebra, then*

$$\mathrm{domdim}_{(S\otimes_R A,S)} S \otimes_R M = \mathrm{domdim}_{(A,R)} M. \tag{111}$$

Proof. By Proposition 6.8, $\operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M \geq \operatorname{domdim}_{(A,R)} M$. The map $\Phi_{S\otimes_R M}$ is epi (resp. iso) if and only the map $S\otimes_R \Phi_M$ is epi (resp. iso). Recall that since S is faithfully flat an R-module is zero if and only if it is the zero module under the functor $S\otimes_R -$. In particular, the map $\Phi_{S\otimes_R M}$ is epi (resp. iso) if and only if the map Φ_M is epi (resp. iso). By flatness of S,

$$\operatorname{Tor}_i^{S\otimes_R C}(\operatorname{Hom}_{S\otimes_R A}(S\otimes_R V, \operatorname{Hom}_S(S\otimes_R M, S), S\otimes_R V) \simeq S\otimes_R \operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V), \quad \forall i>0.$$

Therefore, $\operatorname{Tor}_i^{S\otimes_R C}(\operatorname{Hom}_{S\otimes_R A}(S\otimes_R V, \operatorname{Hom}_S(S\otimes_R M, S), S\otimes_R V)$ is zero if and only if $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V)$ is zero. The result follows Theorem 5.2 and Proposition 3.23. $\square$

An immediate application of Proposition 6.9 is for polynomial rings $R[X_1,\dots,X_n]$. Further, $R[X_1,\dots,X_n]$ is free of infinite rank over R, and so it is faithfully flat.

An example of the importance of changing the ground ring to compute dominant dimension is that for finite dimensional algebras the computation of dominant dimension can be reduced to the computation of dominant dimension over algebraically closed fields. This is a known fact, and it can be found in [Mue68, Lemma 5].

Proposition 6.10. *Let A be a finite dimensional algebra over a field K. Assume that A is QF3 algebra. Then,* $\operatorname{domdim} A = \operatorname{domdim} \overline{K}\otimes_K A$.

Proof. Let $\overline{K}$ be the algebraic closure of K. In particular, $\overline{K}$ can be regarded as K-vector space, hence it is K-free. Furthermore, $\overline{K}$ is faithfully flat over K. By Proposition 6.9, the claim follows. $\square$

The idea here used can be generalized to the next Proposition. For the second part of its proof, we will require the following lemma.

Lemma 6.11. *Let $f\colon R\to S$ be a surjective R-algebra map. Let A be a projective Noetherian R-algebra. Then, for every $Y\in S\otimes_R A$-mod, $S\otimes_R Y\simeq Y$ as $S\otimes_R A$-modules.*

Proof. Let $Y\in S\otimes_R A$-mod. Y can be regarded as an A-module with action $a\cdot y = (f(1_R)\otimes_R a)\cdot y = (1_S\otimes a)\cdot y$. Consider the multiplication map $\mu\colon S\otimes_R Y\to Y$. μ is an $S\otimes_R A$-homomorphism. The map $\nu\colon Y\to S\otimes_R Y$, given by $\nu(y) = 1_S\otimes y$, is an $S\otimes_R A$-homomorphism. Further, ν and μ are inverse to each other. It follows that μ is an $S\otimes_R A$-isomorphism. $\square$

Proposition 6.12. *Let S be a commutative algebra over a commutative Noetherian ring R. Let A be projective Noetherian R-algebra. Let $M\in A$-mod$\cap R$-proj.*

Then, $\operatorname{domdim}_{(A,R)} M \leq \operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M$. *Assume, additionally the following*

- *(A,P,V) is a RQF3 R-algebra;*
- *there is a surjective map of R-algebras $R\to S$ making S an R-projective module.*

Then, $\operatorname{domdim}_{(A,R)} M = \operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M$.

Proof. Let $\operatorname{domdim}_{(A,R)} M\geq n$. Then, there exists an (A,R)-exact sequence

$$0\to M\to X_1\to\cdots\to X_n \tag{112}$$

such that each X_i is A-projective (A,R)-injective. Applying D yields the (A,R)-exact sequence

$$DX_n\to DX_{n-1}\to\cdots\to DX_1\to DM\to 0. \tag{113}$$

The functor $S\otimes_R -$ is exact on (A,R)-exact sequences, so we have the $S\otimes_R A$-exact sequence

$$S\otimes_R DX_n\to S\otimes_R DX_{n-1}\to\cdots\to S\otimes_R DX_1\to S\otimes_R DM\to 0. \tag{114}$$

Observe that $S \otimes_R DM = S \otimes_R \operatorname{Hom}_R(M, R) \simeq \operatorname{Hom}_{S\otimes_R R}(S \otimes_R M, S \otimes_R R) = D_S(S \otimes_R M)$ and each $S \otimes_R DX_i$ is a $S \otimes_R A$-projective $(S \otimes_R A, S)$-injective right module. As $S \otimes_R M \in S$-proj, (114) is $(S \otimes_R A, S)$-exact. Applying D_S yields that $\operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq n$. This shows that, $\operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq \operatorname{domdim}_{(A,R)} M$.

Now assume that there is a surjective map of R-algebras $R \to S$. In particular, S can be regarded as an R-module by restriction of scalars. Assume that this map makes S an R-projective module. Let $\operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq n$ for some integer $n \geq 0$. Then, there exists an $(S \otimes_R A, S)$-exact sequence

$$0 \to S \otimes_R M \to Y_1 \to \cdots \to Y_n, \tag{115}$$

where Y_i, $1 \leq i \leq n$, is $(S\otimes_R A)$-projective $(S\otimes_R A, S)$-injective. Applying D_S we obtain the $(S\otimes_R A, S)$-exact sequence

$$D_S Y_n \to \cdots \to D_S Y_1 \to D_S(S \otimes_R M) \to 0. \tag{116}$$

Observe that $(S \otimes_R A, S \otimes_R P, S \otimes_R V)$ is a RQF3 S-algebra. Thus, each $D_S Y_i \in \operatorname{add}_{S\otimes_R A} S \otimes_R V$. As S is projective over R, S is an R-summand of $\oplus_I R$ for some set I. Hence, $D_S Y_i$ is an A-summand of $S \otimes_R V^t$ which is an A-summand of $\oplus_I V^t$. Therefore, $D_S Y_i \in \operatorname{Add}_A V$. By Observation 5.4, the canonical map $\Phi\colon \operatorname{Hom}_A(V, D_S(S \otimes_R M)) \otimes_C V \to D_S(S \otimes_R M)$ is an isomorphism and for every $1 \leq i \leq n-2$ $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, D_S(S \otimes_R M)), V) = 0$. Now, note that

$$D_S(S \otimes_R M) \simeq \operatorname{Hom}_S(S \otimes_R M, S) \simeq S \otimes_R \operatorname{Hom}_R(M, R) = S \otimes_R DM$$

is an A-summand of $\oplus_I DM$. In particular, Φ_M is an isomorphism and $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0$, $1 \leq i \leq n-2$. So, $\operatorname{domdim}_{(A,R)} M \geq n$. This shows that $\operatorname{domdim}_{(A,R)} M \geq \operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M$. $\square$

In the following, we will see that we can reduce the computation of relative dominant dimension to computing dominant dimension over fields.

Theorem 6.13. *Let (A, P, V) be a RQF3 algebra over a Noetherian ring R. Let $M \in A$-mod$\cap R$-proj. Then,*

$$\operatorname{domdim}_{(A,R)} M = \inf\{\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m})\colon \mathfrak{m} \textit{ maximal ideal in } R\}.$$

Proof. Let $\mathfrak{m}$ be a maximal ideal in R. By Proposition 6.12, $\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq \operatorname{domdim}_{(A,R)} M$.

Assume that $\inf\{\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m})\colon \mathfrak{m} \text{ maximal ideal in } R\} \geq n$. We want to show that $\operatorname{domdim}_{(A,R)} M \geq n$. By Proposition 6.3, $(A(\mathfrak{m}), P(\mathfrak{m}), V(\mathfrak{m}))$ is a QF3 algebra for every maximal ideal $\mathfrak{m}$ in R. Denote by $D_{(\mathfrak{m})}$ the standard duality with respect to $R(\mathfrak{m})$ and denote $C = \operatorname{End}_A(V)$.

If $n = 0$ there is nothing to show. Assume that $n = 1$. Consider the following commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})) \otimes_{C(\mathfrak{m})} V(\mathfrak{m}) & \xrightarrow{\Phi_{M(\mathfrak{m})}} & D_{(\mathfrak{m})}M(\mathfrak{m}) \\ \simeq\uparrow & & \simeq\uparrow \\ R(\mathfrak{m}) \otimes_R \operatorname{Hom}_A(V, DM) \otimes_C V & \xrightarrow{\Phi_M(\mathfrak{m})} & DM(\mathfrak{m}) \end{array} . \tag{117}$$

By assumption, $\Phi_{M(\mathfrak{m})}$ is an epimorphism. Thus, $\Phi_M(\mathfrak{m})$ is an epimorphism for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, Φ_X is an epimorphism. By Proposition 3.23, $\operatorname{domdim}_{(A,R)} M \geq 1$.

Assume that $n = 2$. By the commutative diagram (117) $\Phi_M(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R. Since Φ_M is epi and $M \in R$-proj, Φ_M splits over R. That is, there is a map $t \in \operatorname{Hom}_R(DM, \operatorname{Hom}_A(V, DM) \otimes_C V)$ such that $\Phi_M \circ t = \operatorname{id}_{DM}$. In particular, t is a monomorphism. Applying $R(\mathfrak{m}) \otimes_R -$, we get $\operatorname{id}_{DM(\mathfrak{m})} = \Phi_M \circ t(\mathfrak{m}) = \Phi_M(\mathfrak{m}) \circ t(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Since $\Phi_M(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R it follows that $t(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, t is surjective. So, t is an R-isomorphism. It follows that Φ_M is bijective. By Proposition 3.23, $\operatorname{domdim}_{(A,R)} M \geq 2$.

Assume now that $n \geq 3$. In particular, $\operatorname{domdim}_{(A,R)} M \geq 2$. Hence $\operatorname{Hom}_A(V, DM) \otimes_C V \simeq DM \in R\text{-proj}$. By Theorem 5.2, $\operatorname{Tor}_i^{C(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m}), V(\mathfrak{m})) = 0$, $1 \leq i \leq n-2$ for every maximal ideal $\mathfrak{m}$ in R. Let

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to V \to 0 \tag{118}$$

be a C-projective resolution of V. Since $V \in R$-proj, this resolution is (C, R)-exact. Thus,

$$\cdots \to Q_2(\mathfrak{m}) \to Q_1(\mathfrak{m}) \to Q_0(\mathfrak{m}) \to V(\mathfrak{m}) \to 0 \tag{119}$$

is a $C(\mathfrak{m})$-projective resolution of V. Consider the chain complex $P^\bullet = \operatorname{Hom}_A(V, DM) \otimes_C Q^\bullet$, where $Q^\bullet$ denotes the deleted projective resolution (118). Each object $\operatorname{Hom}_A(V, DM) \otimes_C Q_i \in \operatorname{add}_R \operatorname{Hom}_A(V, DM) \subset R$-proj, since $\operatorname{Hom}_A(V, DM) \in R$-proj. By Lemma A.5, we obtain the Künneth Spectral sequence

$$E^2_{i,j} = \operatorname{Tor}_i^R(H_j(\operatorname{Hom}_A(V, DM) \otimes_C Q^\bullet), R(\mathfrak{m})) \implies H_{i+j}(\operatorname{Hom}_A(V, DM) \otimes_C Q^\bullet(\mathfrak{m})). \tag{120}$$

We have that

$$\operatorname{Hom}_A(V, DM) \otimes_C Q^\bullet(\mathfrak{m}) \simeq \operatorname{Hom}_A(V, DM)(\mathfrak{m}) \otimes_{C(\mathfrak{m})} Q(\mathfrak{m})^\bullet \simeq \operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})) \otimes_{C(\mathfrak{m})} Q(\mathfrak{m})^\bullet, \tag{121}$$

where $Q(\mathfrak{m})^\bullet$ is a $C(\mathfrak{m})$-projective resolution of $V(\mathfrak{m})$. Hence,

$$H_{i+j}(\operatorname{Hom}_A(V, DM) \otimes_C Q^\bullet(\mathfrak{m})) = \operatorname{Tor}_{i+j}^{C(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})) \tag{122}$$

and

$$H_j(\operatorname{Hom}_A(V, DM) \otimes_C Q^\bullet) = \operatorname{Tor}_j^C(\operatorname{Hom}_A(V, DM), C). \tag{123}$$

Thus, for every maximal ideal $\mathfrak{m}$ in R,

$$E^2_{i,j} = \operatorname{Tor}_i^R(\operatorname{Tor}_j^C(\operatorname{Hom}_A(V, DM), V), R(\mathfrak{m})) \implies \operatorname{Tor}_{i+j}^{C(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})). \tag{124}$$

We shall prove by induction on $1 \leq i \leq n-2$ that $\operatorname{Tor}_j^C(\operatorname{Hom}_A(V, DM), V) = 0$. By Lemma A.3 there is an exact sequence

$$E^2_{2,0} \to E^2_{0,1} \to \operatorname{Tor}_1^{C(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})) = 0. \tag{125}$$

As $\operatorname{Hom}_A(V, DM) \otimes_C V \in R$-proj, $E^2_{2,0} = \operatorname{Tor}_2^R(\operatorname{Hom}_A(V, DM) \otimes_C V, R(\mathfrak{m})) = 0$. Thus, for every maximal ideal $\mathfrak{m}$ in R, $0 = E^2_{0,1} = \operatorname{Tor}_1^C(\operatorname{Hom}_A(V, DM), V) \otimes_R R(\mathfrak{m})$. Therefore, $\operatorname{Tor}_1^C(\operatorname{Hom}_A(V, DM), V) = 0$.

Assume now that $\operatorname{Tor}_l^C(\operatorname{Hom}_A(V, DM), V) = 0$ for some $1 \leq l < n-2$. Then,

$$E^2_{i,j} = \operatorname{Tor}_i^R(\operatorname{Tor}_j^C(\operatorname{Hom}_A(V, DM), V), R(\mathfrak{m})) = \operatorname{Tor}_i^R(0, R(\mathfrak{m})) = 0, \ 1 \leq j \leq l, \ i \geq 0. \tag{126}$$

By Lemma A.4, there exists an exact sequence

$$E^2_{l+2,0} \to E^2_{0,l+1} \to \operatorname{Tor}_{l+1}^{C(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})) = 0, \tag{127}$$

where $E^2_{l+2,0} = \operatorname{Tor}_{l+2}^R(\operatorname{Hom}_A(V, DM) \otimes_C V, R(\mathfrak{m})) = 0$. Therefore, $E^2_{0,l+1} = \operatorname{Tor}_{l+1}^C(\operatorname{Hom}_A(V, DM), V)(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ in R. Therefore, $\operatorname{Tor}_{l+1}^C(\operatorname{Hom}_A(V, DM), V) = 0$. Hence, we obtain

$$\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0, \ 1 \leq i \leq n-2. \tag{128}$$

By Theorem 5.2, $\operatorname{domdim}_{(A,R)} M \geq n$. $\square$

Combining this theorem with Proposition 6.10, we deduce that the computation of relative dominant dimension of a projective Noetherian R-algebra can be reduced to computing the dominant dimension of finite dimensional algebras over algebraically closed fields. This shows that the dominant dimension is more static under change of ring than other homological invariants. For example, the global dimension of an algebra can heavily depend on the ground field of the algebra, and even worse it can depend on the Krull dimension of the ground ring in case of Noetherian algebras.

This reduction theorem also explains the meaning behind the generators relative cogenerators which arise in the relative Morita-Tachikawa correspondence. These are the ones which make its endomorphism algebra to admit a base change property like the Schur algebra.

6.4 Base change property

Proposition 6.14. *Let B be a projective Noetherian R-algebra. Let $M \in B\text{-mod} \cap R\text{-proj}$ be a B-generator (B,R)-cogenerator. The following assertions are equivalent.*

(i) $DM \otimes_B M \in R\text{-proj}$.

(ii) For every commutative R-algebra S, $S \otimes_R \operatorname{End}_B(M)^{op} \simeq \operatorname{End}_{S\otimes_R B}(S \otimes_R M)^{op}$ as S-algebras.

Proof. Assume that $DM \otimes_B M \in R\text{-proj}$ holds. Let S be a commutative R-algebra. Denote by D_S the standard duality over S. As $S \otimes_R -$ preserves coproducts,

$$D_S(S \otimes_R M) \otimes_{S\otimes_R B} S \otimes_R M = \operatorname{Hom}_S(S \otimes_R M, S) \otimes_{S\otimes_R B} S \otimes_R M \simeq S \otimes_R \operatorname{Hom}_R(M,R) \otimes_B M \in S\text{-proj}\,. \tag{129}$$

Denote by μ the canonical map $S \otimes_R \operatorname{Hom}_B(M,M) \to \operatorname{Hom}_{S\otimes_R B}(S \otimes_R M, S \otimes_R M)$. By Proposition 2.3, the canonical map $S \otimes_R DM \otimes_B M \to D_S(S \otimes_R M) \otimes_{S\otimes_R B} S \otimes_R M$ is an isomorphism. Consider the following commutative diagram

$$\begin{tikzcd}
D_S \operatorname{Hom}_{S\otimes_R B}(S \otimes_R M, S \otimes_R M) \arrow[r, "D_S\mu"] \arrow[d, "\simeq"] & D_S(S \otimes_R \operatorname{Hom}_B(M,M)) \arrow[d, "\simeq"] \\
D_S(S \otimes_R M) \otimes_{S\otimes_R B} S \otimes_R M & S \otimes_R DM \otimes_B M \arrow[l, "\simeq"']
\end{tikzcd}, \tag{130}$$

where the columns are isomorphisms by Proposition 2.1 since

$$DM \otimes_B M \in R\text{-proj}, \quad D_S(S \otimes_R M) \otimes_{S\otimes_R B} S \otimes_R M \in S\text{-proj}\,. \tag{131}$$

Consequently, $D_S\mu$ is an isomorphism. Again, since $D_S(S \otimes_R M) \otimes_{S\otimes_R B} S \otimes_R M \in S\text{-proj}$ it follows that μ is bijective.

Conversely, assume that (ii) holds. In particular, for every maximal ideal $\mathfrak{m}$ in R, $\operatorname{End}_{B(\mathfrak{m})}(M(\mathfrak{m})) \simeq \operatorname{End}_B(M)(\mathfrak{m})$. Since $R(\mathfrak{m}) \otimes_R -$ preserves direct sums, we get that $M(\mathfrak{m})$ is a generator-cogenerator over $B(\mathfrak{m})$. Hence by Morita-Tachikawa correspondence, $\operatorname{domdim} \operatorname{End}_{B(\mathfrak{m})}(M(\mathfrak{m}))^{op} \geq 2$. Now, for each maximal ideal $\mathfrak{m}$ in R, (ii) yields $\operatorname{domdim} \operatorname{End}_B(M)^{op}(\mathfrak{m}) \geq 2$. By Proposition 6.3, M is an $\operatorname{End}_B(M)^{op}$-projective $(\operatorname{End}_B(M)^{op}, R)$-injective-strongly faithful module. By Proposition 6.13, $\operatorname{domdim}(\operatorname{End}_B(M)^{op}, R) \geq 2$. By relative Morita-Tachikawa correspondence $DM \otimes_B M \in R\text{-proj}$. ☐

As usual, we can compare this situation with what happens to regular rings with Krull dimension at most one.

Lemma 6.15. *Let R be a commutative Noetherian regular ring with Krull dimension at most one. Let A be a projective Noetherian R-algebra. Then, the canonical map $S \otimes_R \operatorname{Hom}_A(M,X) \to \operatorname{Hom}_{S\otimes_R A}(S \otimes_R M, S \otimes_R X)$ is a monomorphism for every $M, X \in A\text{-mod}$ and every commutative R-algebra S.*

Proof. Let $M, X \in A$-mod and let S be a commutative R-algebra. Consider an A-projective presentation

$$P_1 \to P_0 \to M \to 0. \tag{132}$$

The functor $\operatorname{Hom}_{S\otimes_R A}(-, S\otimes_R X)\circ S\otimes_R -\colon A\text{-mod} \to S\otimes_R A\text{-mod}$ is contravariant left exact. So, the induced sequence

$$0 \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R X) \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_0, S\otimes_R X) \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_1, S\otimes_R X). \tag{133}$$

THe functor $\operatorname{Hom}_A(-, X)$ is left exact, thus we have the exact sequence

$$0 \to \operatorname{Hom}_A(M, X) \to \operatorname{Hom}_A(P_0, X) \to \operatorname{Hom}_A(P_1, X). \tag{134}$$

Denote by f the map $\operatorname{Hom}_A(M, X) \to \operatorname{Hom}_A(P_0, X)$. By exactness of (134), the cokernel of f is a submodule of $\operatorname{Hom}_A(P_1, X)$. Since $\dim R \leq 1$, the cokernel of f is R-projective. In particular, f is a split R-mono and so it remains a monomorphism under $S\otimes_R -$. Using the commutative diagram

$$\begin{tikzcd}
S\otimes_R \operatorname{Hom}_A(M, X) \arrow[r, "S\otimes_R f"] \arrow[d] & S\otimes_R \operatorname{Hom}_A(P_0, X) \arrow[d, "\simeq"] \\
\operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R X) \arrow[r] & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_0, S\otimes_R X)
\end{tikzcd}, \tag{135}$$

we conclude that the canonical map $S\otimes_R \operatorname{Hom}_A(M, X) \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R X)$ is a monomorphism. $\square$

6.5 Relative Nakayama conjecture

As in the field case, the relative dominant dimension is bounded by the global dimension.

Proposition 6.16. *Let A be a projective Noetherian R-algebra. If* $\operatorname{domdim}(A, R) < \infty$*, then*

$$\operatorname{domdim}(A, R) \leq \operatorname{injdim}_{(A,R)}\, {}_A A, \quad \operatorname{domdim}(A, R) \leq \operatorname{gldim} A.$$

Proof. Assume that $\operatorname{domdim}(A, R) = n < +\infty$. So, there exists an (A, R)-exact sequence

$$0 \to A \to X_0 \to X_1 \to \cdots \to X_{n-1}, \tag{136}$$

with all X_i being (A, R)-injective A-projective. Applying D we obtain the right A-exact sequence

$$DX_{n-1} \to \cdots \to DX_1 \to DX_0 \to DA \to 0. \tag{137}$$

In particular, there exists an exact sequence

$$0 \to K_{n-2} \to DX_{n-2} \to \cdots \to DX_1 \to DX_0 \to DA \to 0. \tag{138}$$

By contradiction, assume that $n > \operatorname{pdim}_A DA$. Since all DX_i are A-projective, it follows that K_{n-2} must be A-projective. Hence DK_{n-2} is (A, R)-injective and R-projective. Moreover, we have a factorization

$$\begin{tikzcd}
X_{n-2} \arrow[rr] \arrow[dr, two heads] & & X_{n-1} \\
& DK_{n-2} \arrow[ur, hook] &
\end{tikzcd}, \tag{139}$$

and the monomorphism is an (A, R)-monomorphism since this factorization is given by (136). So, it must split over A, and therefore DK_{n-2} is also A-projective. Applying D to (138), it follows that $\operatorname{domdim}(A, R)$ is infinite. Therefore, we must have

$$\operatorname{injdim}_{(A,R)}\, {}_A A = \operatorname{pdim}_A DA \geq n = \operatorname{domdim}(A, R)$$
$$\operatorname{gldim} A \geq \operatorname{pdim}_A DA \geq n = \operatorname{domdim}(A, R). \qquad \square$$

Theorem 6.17. *If the Nakayama conjecture holds for finite dimensional algebras over a field, then the relative Nakayama Conjecture holds for any projective Noetherian R-algebra.*

Proof. Assume that $\operatorname{domdim}(A, R) = +\infty$. By Theorem 6.13, $\operatorname{domdim} A(\mathfrak{m}) = +\infty$ for every maximal ideal $\mathfrak{m}$ in R. If the Nakayama conjecture holds for finite dimensional algebras over fields, then $A(\mathfrak{m})$ is $A(\mathfrak{m})$-injective for every maximal ideal $\mathfrak{m}$ in R. As A is projective when regarded as R-module, it follows that the (left) regular module A is (A, R)-injective by Theorem 2.12. In the same way, the right regular module A is (A, R)-injective. Thus, A is a relative self-injective R-algebra. $\square$

6.5.1 Center of a Noetherian algebra as ground ring

At this point we can ask why we never considered computing relative dominant dimension of algebra A over its center $Z(A)$ to obtain information of relative dominant dimension of A as R-algebra for some commutative Noetherian ring R. The main problem lies in the fact that Noetherian algebras in many instances are not projective as modules over their center which is a crucial assumption made throughout this paper. So, nice properties like base change properties might not hold in such scenario. For example, let k an algebraically closed field and let A be the following quiver k-algebra

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \ , \quad \alpha\beta = 0.$$

Note that we read the arrows in a path like morphisms, that is, from right to left. Denote by e_i the idempotent of A associated with the vertex i, $i = 1, 2$. It is not difficult to see that A has dominant dimension two as finite-dimensional k-algebra. The center of A is the subring of A generated by the elements $\beta\alpha$ and $e_1 + e_2 = 1_A$, that is, $Z(A) \simeq k[x]/(x^2)$. Since $Z(A)$ is a principal ideal domain, A cannot be projective over $Z(A)$ since it has dimension 5 over k. We can see that all the $(A, Z(A))$-projective modules are in the additive closure of either $A \otimes_{Z(A)} Z(A) \simeq A$ or $A \otimes_{Z(A)} k$. The latter is the direct sum of Ae_2 with the injective A-module associated with the vertex 2. Therefore, $\operatorname{Hom}_k(A_A, k)$ is a left $(A, Z(A))$-projective module. Furthermore,

$$A_A \simeq \operatorname{Hom}_k(\operatorname{Hom}_k(A_A, k), k) \in \operatorname{add}_A \operatorname{Hom}_k({}_A A \otimes_{Z(A)} M, k) = \operatorname{add}_A \operatorname{Hom}_{Z(A)}({}_A A, \operatorname{Hom}_k(M, k)).$$

Since $A \simeq A^{op}$, it follows that A is a relative self-injective algebra over its center $Z(A)$.

So, choosing the center as the ground ring might not give much additional information about the algebra A. In particular, such an approach might not give any information on the relative dominant dimension of a projective Noetherian R-algebra for some commutative Noetherian ring distinct from the center of the algebra.

7 Applications and some examples

We will now give some applications of the theory developed here. First, we will start by observing that Roggenkamp and Auslander's correspondence (see [AR72]) for orders of finite type can be formulated in terms of relative dominant dimension (Theorem 7.3). Secondly, we see that reflexive modules of projective Noetherian algebras can be determined using relative dominant dimension (Theorem 7.5). In addition, we are now able to extend the concepts of Morita algebras (see Theorem 7.6) and gendo-symmetric algebras (see Theorem 7.8) to the integral setup. We finish this section computing the relative dominant dimension of Schur algebras (see Theorem 7.12) and quantized Schur algebras (see Theorem 7.20) for the parameters $n \geq d$. Along the way, we give an alternative description of a basis of quantized Schur algebras (Proposition 7.16).

7.1 Orders of finite lattice type

When the ground ring R is a Dedekind domain, projective Noetherian R-algebras A are known in the literature as **R-orders**. For a more detailed exposure of representation theory of R-orders, we refer to [Rei70]. The modules belonging to A-mod$\cap R$-proj are known as **A-lattices**. Let F be the quotient field of R, then $F \otimes_R A$ is a finite dimensional algebra over F. We can identify A with $1 \otimes_R A$, so A is embedded in the finite dimensional algebra $F \otimes_R A$. The same idea holds for the A-lattices. Every A-lattice M can be embedded in the vector space $F \otimes_R M$. The (A, R)-monomorphisms also receive special attention in order theory. Given two A-lattices M, N, M is said to be R-pure A-sublattice of N if there exists an (A, R)-monomorphism $M \to N$. Moreover, the (A, R)-monomorphisms arise as inclusions of $F \otimes_R A$-modules.

Theorem 7.1. *[Zas38] Let R be a Dedekind domain and let A be an R-order. Let F be the quotient field of R. Given any A-lattice N, there is a bijection between A-submodules W of $F \otimes_R N$ and R-pure A-sublattices M of N. The correspondence is given by*

$$M = N \cap W, \quad W = F \otimes_R M.$$

Moreover, each $V \in F \otimes_R A\text{-mod}$ is of the form $F \otimes_R N$ for some A-lattice N in V.

We can deduce in this section that the characterization of orders of Finite Lattice-Type by Auslander and Roggenkamp [AR72] is a particular case of relative Morita-Tachikawa correspondence (Theorem 4.3). We say that an R-order A has **finite lattice-type** if A has a finite number of indecomposable A-lattices. Otherwise, we say that A is of **infinite lattice-type**.

By [Fad65, Proposition 25.1], if $F \otimes_R A$ is not semi-simple, then A is of infinite lattice type. We remark that semi-simple algebras over algebraic number fields are separable. In [AR72], R is assumed to be a complete discrete valuation ring such that its quotient field is a completion of an algebraic number field. This is due to the following fact:

Theorem 7.2. *[Kne66, Jon63] Let R be a Dedekind domain such that its quotient field is an algebraic number field. Let G be a finite group and RG the group algebra of G over R. Then, RG is of finite lattice type if and only if $\hat{RG}_m$ is of finite lattice type for every maximal ideal m in R.*

This reduction technique is useful because for every projective Noetherian algebra over a Noetherian local complete ring, A, A-mod is a Krull-Schmidt category. In particular, this allowed Jones, Heller and Reiner to completely determine all group algebras of finite type.

Theorem 7.3. *Let R be a local complete discrete valuation ring such that its quotient field K is a completion of an algebraic number field. There is a bijection between*

$$\left\{ A \colon \begin{array}{c} A \text{ an } R\text{-order in a} \\ \text{semi-simple } K\text{-algebra} \\ \text{of finite type} \end{array} \right\} \Big/ \sim \;\longleftrightarrow\; \left\{ B \colon \begin{array}{c} B \text{ an } R\text{-order in a semi-simple } K\text{-algebra with} \\ \operatorname{domdim}(B,R) \geq 1, \operatorname{gldim} B \leq 2, \text{ and} \\ \text{every minimal } (B,R)\text{-injective-strongly faithful} \\ \text{projective module satisfies} \\ \text{the double centralizer property} \end{array} \right\} \Big/ \text{iso}$$

In this notation, $A \sim A'$ if and only if A and A' are Morita equivalent. This correspondence is given by:

$$\begin{aligned} A \mapsto & B = \operatorname{End}_A(G)^{op} \\ (\operatorname{End}_B(N)) \leftarrow\!\shortmid & B \end{aligned}$$

where N is an B-projective (B, R)-injective-strongly faithful right module and G is an additive generator of A-mod$\cap R$-proj.

Proof. Let A be an R-order such that $K \otimes_R A$ is a semi-simple algebra and A is of finite type. Consider $G = \oplus_{i \in I} M_i$, where M_i are all non-isomorphic indecomposable A-lattices for some finite set I. In particular, every module of A-mod belongs to $\operatorname{add} G$. Thus, G is an additive generator of A-mod. So, G is a generator (A, R)-cogenerator. As $A \in R$-proj, it follows by Theorem 4.3 that $B = \operatorname{End}_A(G)^{op}$ has relative dominant dimension $\operatorname{domdim}(B, R)$ greater or equal than one and all minimal projective (B, R)injective-strongly faithful modules satisfy the double centralizer property between A and B. Since K is flat as R-module B is an R-order in the semi-simple K-algebra

$$K \otimes_R B = K \otimes_R \operatorname{End}_A(G) \simeq \operatorname{End}_{K \otimes_R A}(K \otimes_R G). \tag{140}$$

In fact, $K \otimes_R G$ is a semi-simple module over $K \otimes_R A$ and consequently, its endomorphism algebra is semi-simple by the Wedderburn Theorem. It remains to show that $\operatorname{gldim} B \leq 2$.

Let $X \in B$-mod. Let $P_1 \xrightarrow{h} P_0 \to X \to 0$ be the beginning of a B-projective resolution of X. By projectivization, the functor $\operatorname{Hom}_A(G, -)\colon A\text{-mod} \to B\text{-mod}$ induces an equivalence between $A\text{-mod} \cap R\text{-proj} = \operatorname{add} G$ and B-proj. Hence, there exists modules $M_0, M_1 \in A\text{-mod} \cap R\text{-proj}$ such that $P_i \simeq \operatorname{Hom}_A(G, M_i)$, $i = 0, 1$. Further, there exists a map $f \in \operatorname{Hom}_A(M_1, M_0)$ satisfying $h = \operatorname{Hom}_A(G, f)$. Applying $\operatorname{Hom}_A(G, -)$ to $0 \to \ker f \to M_1 \xrightarrow{f} M_0$ yields the exact sequence

$$0 \to \operatorname{Hom}_A(G, \ker f) \to P_1 \xrightarrow{h} P_0 \to X \to 0. \tag{141}$$

R has Krull dimension one, therefore $\ker f$ is an A-lattice. By assumption, $\ker f \in \operatorname{add} G$. This shows that $\operatorname{Hom}_A(G, \ker f) \in \operatorname{add} \operatorname{Hom}_A(G, G) = B$-proj. Hence, $\operatorname{pdim}_B X \leq 2$.

Conversely, assume that B is an R-order in a semi-simple K-algebra $K \otimes_R B$ with $\operatorname{domdim}(B, R) \geq 1$, $\operatorname{gldim} B \leq 2$ and all minimal (B, R)-injective-strongly faithful projective modules M satisfy a double centralizer property between B and $\operatorname{End}_B(M)$. Let M be a B-lattice such that (B, DM, M) is a RQF3 algebra. By Theorem 4.3, $A = \operatorname{End}_B(M) \in R$-proj and M is an A-generator (A, R)-cogenerator such that $B \simeq \operatorname{End}_A(M)^{op}$ as R-algebras. So, A is an R-order in the semi-simple K-algebra

$$K \otimes_R A \simeq K \otimes_R \operatorname{End}_B(M) \simeq \operatorname{End}_{K \otimes_R B}(K \otimes_R M). \tag{142}$$

Since A-mod is a Krull-Schmidt category, the number of indecomposable A-lattices summands of M is finite and unique up to isomorphism. Therefore, it is enough to prove that $\operatorname{add}_A M = A\text{-mod} \cap R\text{-proj}$.

Let $X \in A\text{-mod} \cap R\text{-proj}$. Let $0 \to X \to I_0 \to I_1$ be the standard (A, R)-injective resolution of X. Applying the functor $\operatorname{Hom}_A(M, -)$ yields the B-exact sequence

$$0 \to \operatorname{Hom}_A(M, X) \to \operatorname{Hom}_A(M, I_0) \to \operatorname{Hom}_A(M, I_1) \to Y \to 0 \tag{143}$$

for some $Y \in B$-mod. Now, the fact that M is an (A, R)-cogenerator implies that $\operatorname{Hom}_A(M, I_i) \in \operatorname{add} \operatorname{Hom}_A(M, M)$. The projective dimension of Y is at most two, and consequently, $\operatorname{Hom}_A(M, X)$ is B-projective. By projectivization, there exists $M_0 \in \operatorname{add}_A M$ satisfying $\operatorname{Hom}_A(M, X) \simeq \operatorname{Hom}_A(M, M_0)$. Now, thanks to the exactness of $M \otimes_B -$ and the standard (A, R)-injective resolution of X, $M_0 \simeq M \otimes_B \operatorname{Hom}_A(M, X)$ is isomorphic to X. ∎

7.2 Relative torsionless and reflexive modules

Given $M \in A$-mod, we say that M is (A, R)-**torsionless** if there exists a projective $P \in A$-proj and an (A, R)-monomorphism $M \to P$.

In [FOY18] Fang, Kerner and Yamagata showed that the theory of dominant dimension over finite dimensional algebras over a field was related to the exactness of left adjoint of the double dual functor

$$(-)^{**}\colon A\text{-Mod} \to A\text{-Mod},\ M \mapsto \operatorname{Hom}_{A^{op}}(\operatorname{Hom}_A(M, A), A). \tag{144}$$

For relative dominant dimension, the relevant functor to consider is the following functor

$$\mathcal{D}\colon A\text{-Mod} \to \text{Mod-}A, \quad M \mapsto \operatorname{Hom}_A(M, A) \otimes_A DA. \tag{145}$$

Proposition 7.4. *Let (A, P, V) be a RQF3 algebra with* $\operatorname{domdim}(A, R) \geq 2$.

Define the natural transformation $\gamma\colon \mathcal{D} \to D$ with morphisms $\gamma_X\colon \operatorname{Hom}_A(X, A) \otimes_A DA \to DX$, given by $\gamma_X(f \otimes g)(x) = g(f(x))$, $f \otimes g \in \operatorname{Hom}_A(X, A) \otimes_A DA$, $x \in X$.

There exists a natural equivalence $\Sigma\colon \operatorname{Hom}_A(V, D-) \otimes_C V \to \mathcal{D}$ making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Hom}_A(V, DX) \otimes_C V \arrow[r, "\simeq"] \arrow[d, "\Phi_X"] & \mathcal{D}X \arrow[d, "\gamma_X"] \\ DX \arrow[r, equal] & DX \end{tikzcd}, \quad \forall X \in A\text{-Mod}. \tag{146}$$

Proof. Let $X \in A$-mod. By assumption $\Phi_A\colon \operatorname{Hom}_A(V, DA) \otimes_C V \to DA$ is an isomorphism. Consider the C-isomorphism
$\kappa_X\colon \operatorname{Hom}_A(V, DX) \to \operatorname{Hom}_R(V \otimes_A X, R) \to \operatorname{Hom}_A(X, DV)$ given by $\kappa_X(g)(x)(v) = g(v)(x)$, $g \in \operatorname{Hom}_A(V, DX)$, $x \in X$, $v \in V$. By Tensor-Hom adjunction the following composition of C-maps is a C-isomorphism

$$\begin{tikzcd} \operatorname{Hom}_A(V, DX) \arrow[r, "\kappa_X"] & \operatorname{Hom}_A(X, DV) \arrow[r, "\operatorname{Hom}_A(X, w_{DV})"] & \operatorname{Hom}_A(X, \operatorname{Hom}_A(\operatorname{Hom}_A(DV, A), A)) \arrow[d, "\rho_{X,\, \operatorname{Hom}_A(DV, A)}"] \\ & \operatorname{Hom}_A(\operatorname{Hom}_A(DV, A), \operatorname{Hom}_A(X, A)) & \operatorname{Hom}_A(\operatorname{Hom}_A(DV, A) \otimes_A X, A) \arrow[l, "\sigma_{\operatorname{Hom}_A(DV, A), X}"'] \end{tikzcd}.$$

Denote this isomorphism by $\Sigma_X^{(1)}$. By Tensor-Hom adjunction and since $DV \in A^{op}$-proj the following map is an C-isomorphism

$$\operatorname{Hom}_A(X, A) \otimes_A DV \xrightarrow{\operatorname{Hom}_A(X, A) \otimes_A w_{DV}} \operatorname{Hom}_A(X, A) \otimes_A (DV)^{**} \xrightarrow{\psi_{\operatorname{Hom}_A(DV, A)}} \operatorname{Hom}_A(\operatorname{Hom}_A(DV, A), \operatorname{Hom}_A(X, A)).$$

Denote this isomorphism by $\Sigma_X^{(2)}$, where $*$ denotes the dual functor $\operatorname{Hom}_A(-, A)$. Taking into account that ${\Sigma_X^{(1)}}^{-1} = \kappa_X^{-1} \circ \operatorname{Hom}_A(X, w_{DV})^{-1} \circ \sigma_{X, \operatorname{Hom}_A(DV, A)} \circ \rho_{\operatorname{Hom}_A(DV, A), X}$ the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Hom}_A(X, A) \otimes_A DV \otimes_C V \arrow[rr, "\operatorname{Hom}_A(X, A) \otimes_A \Phi_A"] \arrow[d, "\Sigma_X^{(2)} \otimes_C \mathrm{id}_V"'] & & \operatorname{Hom}_A(X, A) \otimes_A DA \arrow[d, "\gamma_X"] \\ \operatorname{Hom}_A(\operatorname{Hom}_A(DV, A), \operatorname{Hom}_A(X, A)) \otimes_C V & \operatorname{Hom}_A(V, DX) \otimes_C V \arrow[l, "\Sigma_X^{(1)} \otimes_C \mathrm{id}_V"'] \arrow[r, "\Phi_X"] & DX \end{tikzcd}. \tag{147}$$

Let Σ_X be the composition $(\operatorname{Hom}_A(X, A) \otimes_A \Phi_A)^{-1} \circ (\Sigma_X^{(2)} \otimes_C \mathrm{id}_V)^{-1} \circ \Sigma_X^{(1)} \otimes_C \mathrm{id}_V$. Since all these maps are functorial then Σ is a natural equivalence between the functors $\operatorname{Hom}_A(V, D-) \otimes_C V$ and $\mathcal{D}$ which satisfies $\gamma_X \circ \Sigma_X = \Phi_X$ for all $X \in A$-mod. $\square$

Recall that $M \in A$-mod is called **reflexive** if the canonical A-map $M \to \operatorname{Hom}_{A^{op}}(\operatorname{Hom}_A(M, A), A)$ is an isomorphism.

Theorem 7.5. *Let (A, P, V) be a RQF3 algebra with* $\operatorname{domdim}(A, R) \geq 2$. *Let $M \in A$-mod$\cap R$-proj. The following assertions are equivalent.*

(i) *M is (A, R)-torsionless.*

(ii) $\operatorname{domdim}_{(A,R)} M \geq 1$.

(iii) *The map $\Phi_M\colon \operatorname{Hom}_A(V, DM) \otimes_C V \to DM$ is surjective.*

(iv) *The map $\gamma_M\colon \operatorname{Hom}_A(M, A) \otimes_A DA \to DM$ is surjective.*

The following assertions are equivalent.

(a) M *is reflexive over* A *and* $\mathrm{Hom}_A(M,A)\otimes_A DA \in R\text{-proj}$.

(b) $\mathrm{domdim}_{(A,R)} M \geq 2$.

(c) *The map* $\Phi_M\colon \mathrm{Hom}_A(V,DM)\otimes_C V \to DM$ *is bijective.*

(d) *The map* $\gamma_M\colon \mathrm{Hom}_A(M,A)\otimes_A DA \to DM$ *is bijective.*

Proof. By Proposition 7.4, the implications $(iii) \Leftrightarrow (iv)$ and $(c) \Leftrightarrow (d)$ hold. By relative Mueller characterization 3.23, $(ii) \Leftrightarrow (iii)$ and $(b) \Leftrightarrow (c)$ follow. Assume that (i) holds. Since $\mathrm{domdim}(A,R) \geq 1$ there exists a projective (A,R)-injective module X such that $A \to X$ is an (A,R)-monomorphism. Using the (A,R)-monomorphism $M \to P \to A^t \to X^t$ (ii) follows. Assume that (ii) holds. Then, there exists an (A,R)-monomorphism of M into an A-projective (A,R)-injective module. In particular, M is (A,R)-torsionless.

It remains to show that (a) is equivalent to (d).

The diagram

$$\begin{tikzcd} DDM \arrow[r, "D\gamma_M"] & D(\mathrm{Hom}_A(M,A)\otimes_A DA) \\ M \arrow[u, "w_M"] \arrow[r, "\tau_M"] & \mathrm{Hom}_A(\mathrm{Hom}_A(M,A),A) \arrow[u, "\kappa", "\simeq"'] \end{tikzcd} \tag{148}$$

is commutative. In fact, for $m\in M, f \in \mathrm{Hom}_A(M,A),\ g\in DA$

$$\kappa\tau_M(m)(f\otimes g) = g(\tau_M(m)(f)) = g(f(m))$$
$$D\gamma_M \circ w_M(m)(f\otimes g) = \mathrm{Hom}_R(\gamma_M, R)w_M(m)(f\otimes g) = w_M(m)\circ\gamma_M(f\otimes g) = \gamma_M(f\otimes g)(m) = g(f(m)).$$

Assume that (a) holds. Then, τ_M is an isomorphism. So, by the diagram (148) $D\gamma_M$ is an isomorphism. Since $\mathrm{Hom}_A(M,A)\otimes_A DA \in R\text{-proj}$, γ_M is an isomorphism. Assume now that (d) holds. As $DM \in R\text{-proj}$, it follows that $\mathrm{Hom}_A(M,A)\otimes_A DA \in R\text{-proj}$. Also, $D\gamma_M$ is an isomorphism. By the diagram (148), τ_M is an isomorphism. So, M is reflexive over A. $\square$

7.3 Relative Morita algebras

We shall now introduce a generalization of Morita algebras introduced in [KY13] to algebras over Noetherian rings. This also generalizes [Cru21, Theorem 11] and [FHK21, Proposition 2.9].

Theorem 7.6. *Let* A *be a projective Noetherian algebra over a commutative Noetherian ring* R. *The following assertions are equivalent.*

(a) (A,P,DP) *is a RQF3 algebra so that* $\mathrm{domdim}\,(A,R) \geq 2$ *and the restriction of the Nakayama functor* $DA\otimes_A -\colon \mathrm{add}\,P \to \mathrm{add}\,P$ *is well defined;*

(b) (A,P,DP) *is a RQF3 algebra so that* $\mathrm{domdim}\,(A,R)\geq 2$ *and* $\mathrm{add}_A DA\otimes_A P = \mathrm{add}_A P$.

(c) A *is the endomorphism algebra of a generator* $M\in B\text{-mod}\cap R\text{-proj}$ *satisfying* $DM\otimes_B M \in R\text{-proj}$ *over a relative self-injective* R*-algebra* B, *where* $B\in R\text{-proj}$.

(a') (A,P,DP) *is a RQF3 algebra so that* $\mathrm{domdim}\,(A,R) \geq 2$ *and the restriction of the Nakayama functor* $-\otimes_A DA\colon \mathrm{add}\,DP \to \mathrm{add}\,DP$ *is well defined;*

(b') (A,P,DP) *is a RQF3 algebra so that* $\mathrm{domdim}\,(A,R)\geq 2$ *and* $\mathrm{add}_A DP\otimes_A DA = \mathrm{add}_A DP$.

Proof. The argument is essentially the same as presented in [Cru21, Theorem 11] once we replace dominant dimension by relative dominant dimension. It is enough to prove $(a) \implies (c) \implies (b)$ since $(b) \implies (a)$ is clear and the implications $(b') \implies (a') \implies (c) \implies (b')$ are analogous.

Assume that (a) holds. By relative Morita-Tachikawa correspondence (see Theorem 4.1) $P \otimes_B DP \in R\text{-proj}$, $B = \operatorname{End}_A(P)^{op} = \operatorname{End}_A(DP)$ and $A \simeq \operatorname{End}_B(P) \simeq \operatorname{End}_B(DP)^{op}$. It remains to show that B is relative self-injective. But this follows immediately from observing that

$$B = \operatorname{Hom}_A(P, P) \simeq \operatorname{Hom}_A(P, A) \otimes_A P \simeq D(DA \otimes_A P) \otimes_A P \in \operatorname{add} DP \otimes_A P = \operatorname{add} DB. \tag{149}$$

Hence, B is (B, R)-injective.

Assume that (c) holds. By the relative Morita-Tachikawa correspondence, $\operatorname{domdim}(A, R) \geq 2$ so that (A, DM, M) is RQF3 and $A = \operatorname{End}_B(M)^{op}$. Moreover,

$$DA \otimes_A DM \simeq DM \otimes_B M \otimes_A DM \simeq DM \otimes_B DB. \tag{150}$$

Since B is a relative self-injective algebra DB is a B-progenerator. Hence, $\operatorname{add}_A DM \otimes_B DB = \operatorname{add}_A DM$. This completes the proof. ☐

The pair (A, P) (or (A, DP) if one prefers to work with right modules) is called a **relative Morita R-algebra** if it satisfies one of the conditions of Theorem 7.6.

Using Theorem 7.6(c), we see that relative Morita algebras generalize relative self-injective algebras.

7.4 Relative Gendo-symmetric algebras

Definition 7.7. Let B be a projective Noetherian algebra over a commutative Noetherian ring R. B is said to be **relative symmetric R-algebra** if there exists a (B, B)-bimodule isomorphism $DB \simeq B$.

Using the proof of Proposition 3.11, we see that group algebras RG are relative symmetric R-algebras for any commutative Noetherian ring R and finite groups G. We refer to [Yam96] for the study of symmetric finite dimensional algebras.

Theorem 7.8. *Let A be a projective Noetherian algebra over a commutative Noetherian ring R. The following assertions are equivalent.*

- *(a)* $\operatorname{domdim}(A, R) \geq 2$ *and* $V \simeq V \otimes_A DA$ *as* $(\operatorname{End}_A(V), A)$*-bimodules where* V *is a projective* (A, R)*-injective-strongly faithful right module.*
- *(b)* $\operatorname{domdim}(A, R) \geq 2$ *and* $P \simeq DA \otimes_A P$ *as* $(A, \operatorname{End}_A(P)^{op})$*-bimodules where* P *is a projective* (A, R)*-injective-strongly faithful left module.*
- *(c)* *A is the endomorphism algebra of a generator* $M \in B\text{-mod} \cap R\text{-proj}$ *satisfying* $DM \otimes_B M \in R\text{-proj}$ *over a relative symmetric R-algebra B.*

Proof. Assume that (a) holds. Let $B = \operatorname{End}_A(V)$. By relative Morita-Tachikawa correspondence 4.1, V is a left B-generator satisfying $DV \otimes_B V \in R\text{-proj}$ and $A = \operatorname{End}_B(V)^{op}$. In particular $DA \simeq DV \otimes_B V$ as (A, A)-bimodules. Furthermore, $DV \simeq D(V \otimes_A DA) \simeq \operatorname{Hom}_A(V, A)$ as (A, B)-bimodules. Thus, as (B, B)-bimodules

$$DB \simeq V \otimes_A DV \simeq V \otimes_A \operatorname{Hom}_A(V, A) \simeq \operatorname{Hom}_A(V, V) \simeq B. \tag{151}$$

Hence B is a relative symmetric R-algebra. So, (c) follows.

Conversely, assume that (c) holds. Every generator over a relative symmetric algebra is a generator relative cogenerator. By relative Morita-Tachikawa correspondence 4.1, $A = \operatorname{End}_B(M)^{op}$ has $\operatorname{domdim}(A, R) \geq 2$ and M is a projective (A, R)-injective-strongly faithful right module. In particular, $DA \simeq DM \otimes_B M$ as (A, A)-bimodules. Moreover, as (B, A)-bimodules

$$M \otimes_A DA \simeq M \otimes_A DM \otimes_B M \simeq DB \otimes_B M \simeq B \otimes_B M \simeq M.$$ ☐

Analogously, one can show the equivalence between (b) and (c)

By a **relative gendo-symmetric** R**-algebra** we mean a pair (A, V) satisfying (a) and (c) of Theorem 7.8 or a pair (A, P) satisfying (b) and (c) of Theorem 7.8.

Proposition 7.9. *Let (A, V) be a relative gendo-symmetric R-algebra. Then,*

(i) $DA \otimes_A DA \simeq DA$ *as (A, A)-bimodules.*

(ii) $DV \simeq DA \otimes_A DV$ *as $(A, \mathrm{End}_A(V))$-bimodules.*

Proof. Let $B = \mathrm{End}_A(V)$. We can identify as (A, A)-bimodules

$$DA \otimes_A DA \simeq DV \otimes_B V \otimes_A DV \otimes_B V \simeq DV \otimes_B DB \otimes_B V \simeq DV \otimes_B B \otimes_B V \simeq DV \otimes_B V \simeq DA.$$

So, (i) follows. By assumption, $V \simeq V \otimes_A DA$ as (B, A)-bimodules. Hence, as (A, B)-bimodules

$$DV \simeq D(V \otimes_A DA) \simeq \mathrm{Hom}_A(V, DDA) \simeq \mathrm{Hom}_A(V, A). \tag{152}$$

In particular, there exists an (A, B)-bimodule isomorphism

$$DA \otimes_A DV \simeq DA \otimes_A \mathrm{Hom}_A(V, A) \simeq \mathrm{Hom}_A(V, DA) \simeq \mathrm{Hom}_R(V \otimes_A A, R) \simeq DV. \qquad \square$$

Over fields, these class of algebras were introduced by Fang and Koenig in [FK11a] to give a homological characterization of a class of algebras that generalize Schur algebras and the blocks of the category $\mathcal{O}$.

Proposition 7.9 allows us to construct a comultiplication on A in the same fashion as in [FK16]. The advantage here is of course that the ground ring is any commutative Noetherian ring instead of a field.

A question that arises in this setup is whether the condition (i) in Proposition 7.9 is enough to deduce that there exists $V \in \text{proj-}A$ such that (A, V) is a relative gendo-symmetric R-algebra. The difficulty lies in fact in the construction of V. It is also unclear for the author if an algebra being symmetric can be characterized in terms of closed points.

7.5 Classical Schur algebras

A classical reference for the study of Schur algebras (over infinite fields) is [Gre07].

Let R be a commutative ring with identity. Fix natural numbers n, d. The symmetric group on d letters S_d acts by place permutation on the d-fold tensor product $(R^n)^{\otimes d}$, that is,

$$(v_1 \otimes \cdots \otimes v_d)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \ \sigma \in S_d, \ v_i \in R^n.$$

We will write $V_R^{\otimes d}$ instead of $(R^n)^{\otimes d}$ or simply $V^{\otimes d}$ when the ground ring is well understood. In particular, $V^{\otimes d}$ is a right module over the group algebra RS_d.

Definition 7.10. [Gre07] The subalgebra $\mathrm{End}_{RS_d}\left(V^{\otimes^d}\right)$ of the endomorphism algebra $\mathrm{End}_R\left(V^{\otimes d}\right)$ is called the **Schur algebra**. We will denote it by $S_R(n, d)$.

We recall some facts about these algebras.

Let $I(n, d)$ be the set of maps $i\colon \{1, \ldots, d\} \to \{1, \ldots, n\}$. We write $i(a) = i_a$. We can associate to $I(n, d)$ a right S_d-action by place permutation. In the same way, S_d acts on $I(n, d) \times I(n, d)$, by setting:

$$(i, j)\sigma = (i\sigma, j\sigma), \quad \forall i, j \in I(n, d), \forall \sigma \in S_d. \tag{153}$$

We will write $(i, j) \sim (f, g)$ if (i, j) and (f, g) belong to the same S_d-orbit. Then, $S_R(n, d)$ has a basis over R $\{\xi_{i,j} \mid (i, j) \in I(n, d) \times I(n, d)\}$ satisfying

$$\xi_{i,j}(e_{s_1} \otimes \cdots \otimes e_{s_d}) = \sum_{\substack{l \in I(n,d) \\ (l,s) \sim (i,j)}} e_{l_1} \otimes \cdots \otimes e_{l_d}, \tag{154}$$

for a given basis $\{e_{s_1} \otimes \cdots \otimes e_{s_d} \colon 1 \leq s_1, \dots, s_d \leq n\}$ of $V^{\otimes d}$. In particular, $\xi_{i,j} = \xi_{f,g}$ if and only if $(i,j) \sim (f,g)$.

An immediate consequence of the existence of an R-basis for $S_R(n,d)$ satisfying (154) is the existence of a base change property

$$R \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n,d) \simeq S_R(n,d). \tag{155}$$

It also follows that $R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^{\otimes d} \simeq V_R^{\otimes d}$ as $S_R(n,d)$-modules.

We will now focus on the case $n \geq d$. In this case,

$$V^{\otimes d} \simeq S_R(n,d)\xi_{(1,\dots,d),(1,\dots,d)}, \quad DV^{\otimes d} \simeq \xi_{(1,\dots,d),(1,\dots,d)} S_R(n,d). \tag{156}$$

Hence, $V^{\otimes d}$ is an $S_R(n,d)$-projective $(S_R(n,d), R)$-injective module.

Our aim is to compute the relative dominant dimension of $S_R(n,d)$ extending the results of Fang and Koenig [FK11b] contained in the following Theorem.

Theorem 7.11. *[FK11b, Theorem 5.1] Let K be a field.*

$$\operatorname{domdim} S_K(n,d) = \begin{cases} 2(\operatorname{char} K - 1) & \text{if } d \geq \operatorname{char} K > 0 \\ +\infty, & \text{otherwise.} \end{cases} \tag{157}$$

The dominant dimension of the Schur algebra $S_K(n,d)$ is always even because the Schur algebra $S_K(n,d)$ admits an involution fixing a complete set of primitive orthogonal idempotents.

In the following, we will show that we can compute the dominant dimension of $S_R(n,d)$ by knowing the invertible elements of R, denoted by $U(R)$.

Theorem 7.12. *Let R be a commutative Noetherian ring. If $n \geq d$, then $(S_R(n,d), V^{\otimes d})$ is a relative gendo-symmetric R-algebra and*

$$\operatorname{domdim}(S_R(n,d), R) = \inf\{2k \in \mathbb{N} \mid (k+1) \cdot 1_R \notin U(R),\ k < d\} \geq 2. \tag{158}$$

Proof. $V_K^{\otimes d}$ is a projective-injective faithful $S_K(n,d)$ module for every field. By Proposition 6.4, $(S_R(n,d), V^{\otimes d}, DV^{\otimes d})$ is a RQF3 algebra. Denote by $\operatorname{MaxSpec}(R)$ the set of maximal ideals of $\mathfrak{m}$. By Theorem 6.13,

$$\begin{aligned} \operatorname{domdim}(S_R(n,d), R) &= \inf\{\operatorname{domdim} S_R(n,d) \otimes_R R(\mathfrak{m}) | \mathfrak{m} \in \operatorname{MaxSpec}(R)\} && (159)\\ &= \inf\{\operatorname{domdim} S_{R(\mathfrak{m})}(n,d) | \mathfrak{m} \in \operatorname{MaxSpec}(R)\} \geq 2. && (160) \end{aligned}$$

By relative Morita-Tachikawa correspondence, $V^{\otimes d}$ is a generator of RS_d satisfying $V^{\otimes d} \otimes_{RS_d} DV^{\otimes d} \in R\text{-proj}$. Therefore, $(S_R(n,d), V^{\otimes d})$ is a relative gendo-symmetric R-algebra because RS_d is a relative symmetric R-algebra.

Let $k \in \mathbb{N}$ such that $(k+1)1_R \notin U(R)$ and $k < d$. Then, $(k+1)1_R \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. In particular, $\operatorname{char} R(\mathfrak{m})$ is positive and it is less or equal to $k+1 \leq d$. Hence, $\operatorname{domdim} S_{R(\mathfrak{m})}(n,d) \leq 2k$, for some maximal ideal $\mathfrak{m}$ of R. This shows that

$$\operatorname{domdim}(S_R(n,d), R) \leq \inf\{2k \in \mathbb{N} \mid (k+1) \cdot 1_R \notin U(R),\ k < d\}. \tag{161}$$

In particular, if $\operatorname{domdim}(S_R(n,d), R) = +\infty$ there is nothing more to prove.

Assume now that $\operatorname{domdim}(S_R(n,d), R) = l \geq 2$. So, there exists $\mathfrak{m} \in \operatorname{MaxSpec}(R)$ such that

$$2(\operatorname{char} R(\mathfrak{m}) - 1) = l, \quad \text{and} \quad \operatorname{char} R(\mathfrak{m}) \leq d. \tag{162}$$

In particular, the image of char $R(\mathfrak{m})1_R$ in $R(\mathfrak{m})$ is zero and so char $R(\mathfrak{m})1_R \in \mathfrak{m}$. Hence, char $R(\mathfrak{m})1_R \notin U(R)$. Therefore,

$$l \in \{2k \in \mathbb{N} | (k+1)1_R \notin U(R),\ k < d\}. \tag{163}$$

This finishes the proof. $\square$

Once again, we see that the invertible elements of the ground ring determine the quality of a double centralizer property. In [Cru19], a ring having sufficiently many invertible elements under some mild assumptions was a sufficient condition for Schur–Weyl duality to hold.

In Theorem 7.12, we saw that $V^{\otimes d}$ is an $(S_R(n,d), R)$-strongly faithful module. In general for Noetherian algebras, it is difficult to prove directly that a module is strongly faithful and whenever possible we always prefer to show this property using change of rings techniques. However, it is not difficult to show directly that $V^{\otimes d}$ is strongly faithful. This is the aim of the next example.

Example 7.13. Let $\{e_{s_1} \otimes \cdots \otimes e_{s_d} \colon 1 \leq s_1, \ldots, s_d \leq n\}$ be an R-basis of $V^{\otimes d}$. Choose Λ to be a set of representatives of S_d-orbits on $I(n,d) \times I(n,d)$. Define the R-map $\upsilon \in \operatorname{Hom}_R(S_R(n,d), V^{\otimes d^t})$, satisfying

$$\upsilon(\varphi) = \sum_{(i,j)\in\Lambda} \kappa_{i,j}(\varphi(e_{j_1} \otimes \cdots \otimes e_{j_d})), \quad \varphi \in S_R(n,d), \tag{164}$$

with $\kappa_{i,j}$ and $\pi_{i,j}$, $(i,j) \in \Lambda$, being the inclusion and projection mappings of $V^{\otimes}$ into the direct sum $(V^{\otimes d})^t$ as $S_R(n,d)$-modules, respectively, where $t = \binom{n^2+d-1}{d}$. Observe that

$$\upsilon(\eta\varphi) = \sum_{(i,j)\in\Lambda} \kappa_{i,j}(\eta\varphi(e_{j_1} \otimes \cdots \otimes e_{j_d})) = \sum_{(i,j)\in\Lambda} \eta\kappa_{i,j}\varphi(e_{j_1} \otimes \cdots \otimes e_{j_d}) = \eta\upsilon(\varphi),\ \varphi, \eta \in S_R(n,d). \tag{165}$$

Thus, $\upsilon \in \operatorname{Hom}_{S_R(n,d)}(S_R(n,d), V^{\otimes d^t}))$. For each $(i,j) \in \Lambda$, define $f_{i,j} \in \operatorname{Hom}_R(V^{\otimes d}, S_R(n,d))$ satisfying

$$f_{i,j}(e_{s_1} \otimes \cdots \otimes e_{s_d}) = \begin{cases} \xi_{i,j} & \text{if } (s_1, \ldots, s_d) = i \\ 0, & \text{otherwise.} \end{cases} \tag{166}$$

Finally, denote by ϵ the R-map $\sum_{(i,j)\in\Lambda} f_{i,j} \circ \pi_{i,j} \in \operatorname{Hom}_R((V^{\otimes d})^t, S_R(n,d))$. Then, the following holds,

$$\epsilon \circ \upsilon(\xi_{f,g}) = \epsilon\left(\sum_{(i,j)\in\Lambda} \kappa_{i,j}\xi_{f,g}(e_{j_1} \otimes \cdots \otimes e_{j_d})\right) = \sum_{(t,u)\in\Lambda}\sum_{(i,j)\in\Lambda} f_{t,u}\pi_{t,u}\kappa_{i,j}\xi_{f,g}(e_{j_1} \otimes \cdots \otimes e_{j_d}) \tag{167}$$

$$= \sum_{(i,j)\in\Lambda} f_{i,j}\xi_{f,g}(e_{j_1} \otimes \cdots \otimes e_{j_d}) = \sum_{(i,j)\in\Lambda} f_{i,j}\left(\sum_{\substack{l\in I(n,d)\\(l,j)\sim(f,g)}} e_{l_1} \otimes \cdots \otimes e_{l_d}\right) \tag{168}$$

$$= \sum_{(i,j)\in\Lambda}\sum_{\substack{l\in I(n,d)\\(l,j)\sim(f,g)}} \mathbb{1}_{\{i\}}(l)\xi_{i,j} = \sum_{\substack{(i,j)\in\Lambda\\(i,j)\sim(f,g)}} \xi_{i,j} = \xi_{f,g}. \tag{169}$$

Therefore, υ is an $(S_R(n,d), R)$-monomorphism. $\triangle$

7.6 q-Schur algebras

The Hecke algebra of the symmetric group (usually called the Iwahori-Hecke algebra) is obtained by a small perturbation q on the group algebra of symmetric group. By a small perturbation q we mean replacing the identity of the group algebra in some of its defining relations by a non-trivial root of unity. Although, one usually is more general and defines it for an invertible element q. Usually, the name quantum is referred to q being a small perturbation.

Let R be a commutative ring with identity. Fix natural numbers n, d. Let u be an invertible element of R and put $q = u^{-2}$. The **Iwahori-Hecke algebra** $H_{R,q}(d)$ is the R-algebra with basis $\{T_\sigma : \sigma \in S_d\}$ satisfying the relations

$$T_\sigma T_s = \begin{cases} T_{\sigma s}, & \text{if } l(\sigma s) = l(\sigma) + 1 \\ (u - u^{-1})T_\sigma + T_{\sigma s}, & \text{if } l(\sigma s) = l(\sigma) - 1, \end{cases} \tag{170}$$

where $s \in S := \{(1, 2), (2, 3), \cdots, (d - 1, d)\}$ is a set of transpositions and l is the length function, that is, $l(\sigma)$, $\sigma \in S_d$, is the minimum number of simple transpositions belonging to S needed to write σ.

There are many ways to define Hecke algebras. Here, we are following the definition of Hecke algebras according to Parshall-Wang [PW91] (but we use u instead of q and q instead of h). In [Mat99], they use a different basis for $H_{R,q}(d)$ which is the same as Definition (11.3a) of [PW91]. We would also like to point out that $\mathcal{H}_{R,q}$ in Definition 4.4.1 of [DD91] is exactly $H_{R,q}(d)$ in our notation.

Due to the relations (170), T_s, $s \in S$, generates as algebra $H_{R,q}(d)$.

The Iwahori-Hecke algebra $H_{R,q}(d)$ admits a base change property.

$$H_{R,q}(d) \simeq R \otimes_{\mathbb{Z}[u,u^{-1}]} H_{\mathbb{Z}[u,u^{-1}],u^{-2}}(d) \tag{171}$$

Under this isomorphism of R-algebras $1_R \otimes_{\mathbb{Z}[u,u^{-1}]} T_\sigma$ is mapped to $T_\sigma \in H_{R,q}(d)$.

We can regard $V^{\otimes d}$ as right $H_{R,q}(d)$-module by imposing to an R-basis $\{e_{i_1} \otimes \cdots \otimes e_{i_d} \mid i \in I(n, d)\}$ of $V^{\otimes d}$,

$$e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot T_s = \begin{cases} e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot s & \text{if } i_t < i_{t+1} \\ ue_{i_1} \otimes \cdots \otimes e_{i_d} & \text{if } i_t = i_{t+1}\ , \\ (u - u^{-1})e_{i_1} \otimes \cdots \otimes e_{i_d} + e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot s & \text{if } i_t > i_{t+1} \end{cases} \quad s = (t, t + 1) \in S, \quad 1 \leq t < d. \tag{172}$$

By considering $q = 1$, we recover the action on $V^{\otimes d}$ of the symmetric group by place permutation.

Definition 7.14. The subalgebra $\operatorname{End}_{H_{R,q}(d)}\left(V^{\otimes^d}\right)$ of the endomorphism algebra $\operatorname{End}_R\left(V^{\otimes d}\right)$ is called the **q-Schur algebra**. We will denote it by $S_{R,q}(n, d)$.

The q-Schur algebras were introduced by Dipper and James [DJ91, DJ89].

By [Du92, 2.d] (see also [DD91, Lemma 4.4.3]) $S_{R,q}(n, d) = S_{R,u^{-2}}(n, d)$ is isomorphic to the q-Schur algebra of Dipper and James [DJ91].

It is now a good opportunity to exhibit an R-basis of $V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$. By dualizing such R-basis we will obtain an R-basis for $S_{R,q}(n, d)$. Note, once more, that in general if $\operatorname{End}_B(M)$ has an R-basis nothing can be said about $DM \otimes_B M$, $M \in B$-mod.

Lemma 7.15. *Let $\{e^*_{i_1} \otimes \cdots \otimes e^*_{i_d} \mid i \in I(n, d)\}$ be an R-basis of $DV^{\otimes d}$. $DV^{\otimes d}$ is a left $H_{R,q}(d)$-module with action*

$$T_s \cdot e^*_{i_1} \otimes \cdots \otimes e^*_{i_d} = \begin{cases} s \cdot e^*_{i_1} \otimes \cdots \otimes e^*_{i_d} & \textit{if } i_t < i_{t+1} \\ ue^*_{i_1} \otimes \cdots \otimes e^*_{i_d} & \textit{if } i_t = i_{t+1}\ , \\ (u - u^{-1})e^*_{i_1} \otimes \cdots \otimes e^*_{i_d} + s \cdot e^*_{i_1} \otimes \cdots \otimes e^*_{i_d} & \textit{if } i_t > i_{t+1} \end{cases} \quad s = (t, t + 1) \in S, \quad 1 \leq t < d.$$

We can associate to $I(n, d) \times I(n, d)$ the lexicographical order. Each S_d-orbit of $I(n, d) \times I(n, d)$ has a representative (i, j) satisfying $(i_1, j_1) \leq \cdots \leq (i_d, j_d)$.

Proposition 7.16. *$V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$ is a free R-module with basis*

$$\{e_{i_1} \otimes \cdots \otimes e_{i_d} \otimes_{H_{R,q}(d)} e^*_{j_1} \otimes \cdots \otimes e^*_{j_d} \colon i, j \in I(n,d),\ (i_1, j_1) \leq \cdots \leq (i_d, j_d)\}. \tag{173}$$

Proof. Since $\{e_{i_1} \otimes \cdots \otimes e_{i_d} \mid i \in I(n,d)\}$ is an R-basis of $V^{\otimes d}$ and $\{e^*_{j_1} \otimes \cdots \otimes e^*_{j_d} \mid j \in I(n,d)\}$ is an R-basis of $DV^{\otimes d}$ the set $\{e_{i_1} \otimes \cdots \otimes e_{i_d} \otimes_{H_{R,q}(d)} e^*_{j_1} \otimes \cdots \otimes e^*_{j_d} | i, j \in I(n,d)\}$ generate (over R) $V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$.

Denote by Λ the set

$$\Lambda := \{(i,j) \in I(n,d) \times I(n,d) \colon (i_1, j_1) \leq \cdots \leq (i_d, j_d)\}. \tag{174}$$

Let $(l,s) \in I(n,d) \times I(n,d)$. Assume that $(l,s) \notin \Lambda$. Then, there exists $1 \leq k < d$ such that $(l_k, s_k) \not\leq (l_{k+1}, s_{k+1})$. Hence, either $l_k > l_{k+1}$ or $l_k = l_{k+1}$ and $s_k > s_{k+1}$. Assume that $l_k > l_{k+1}$. Take $i = l \cdot (k, k+1)$ and $\omega = (k, k+1)$. Then, $i_k < i_{k+1}$ and

$$e_{l_1} \otimes \cdots \otimes e_{l_d} = (e_{i_1} \otimes \cdots \otimes e_{i_d}) \cdot (k, k+1) = e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot T_\omega. \tag{175}$$

Hence,

$$\begin{aligned} e_{l_1} \otimes \cdots \otimes e_{l_d} \otimes_{H_{R,q}(d)} e^*_{s_1} \otimes \cdots \otimes e^*_{s_d} &= e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot T_\omega \otimes_{H_{R,q}(d)} e^*_{s_1} \otimes \cdots \otimes e^*_{s_d} && (176)\\ &= e_{i_1} \otimes \cdots \otimes e_{i_d} \otimes_{H_{R,q}(d)} T_\omega e^*_{s_1} \otimes \cdots \otimes e^*_{s_d} && (177) \end{aligned}$$

Therefore, we can write $e_{l_1} \otimes \cdots \otimes e_{l_d} \otimes_{H_{R,q}(d)} e^*_{s_1} \otimes \cdots \otimes e^*_{s_d}$ as a linear combination of elements $e_i \otimes_{H_{R,q}(d)} e^*_f$ where $i_1 \leq \ldots i_k \leq i_{k+1}$, $i, f \in I(n,d)$. Now, assume that $l_k = l_{k+1}$ and $s_k > s_{k+1}$ for some k. Put $j = s \cdot \omega$, $\omega = (k, k+1)$. Then, $j_k < j_{k+1}$ and

$$\begin{aligned} e_{l_1} \otimes \cdots \otimes e_{l_d} \otimes_{H_{R,q}(d)} e^*_{s_1} \otimes \cdots \otimes e^*_{s_d} &= e_{l_1} \otimes \cdots \otimes e_{l_d} \otimes_{H_{R,q}(d)} \omega e^*_{j_1} \otimes \cdots \otimes e^*_{j_d} && (178)\\ &= e_{l_1} \otimes \cdots \otimes e_{l_d} \otimes_{H_{R,q}(d)} T_\omega e^*_{j_1} \otimes \cdots \otimes e^*_{j_d} && (179)\\ &= e_{l_1} \otimes \cdots \otimes e_{l_d} T_\omega \otimes_{H_{R,q}(d)} e^*_{j_1} \otimes \cdots \otimes e^*_{j_d} && (180)\\ &= u e_{l_1} \otimes \cdots \otimes e_{l_d} \otimes_{H_{R,q}(d)} e^*_{j_1} \otimes \cdots \otimes e^*_{j_d}. && (181) \end{aligned}$$

So, we can order the elements (for example using Bubble sort) $(l,s) \in I(n,d) \times I(n,d)$ into $(i,j) \in I(n,d) \times I(n,d)$ with $(i,j) \in \Lambda$ and we obtain that each element $e_{l_1} \otimes \cdots \otimes e_{l_d} \otimes_{H_{R,q}(d)} e^*_{s_1} \otimes \cdots \otimes e^*_{s_d}$, $s, l \in I(n,d)$ can be written as a linear combination of elements $e_{i_1} \otimes \cdots \otimes e_{i_d} \otimes_{H_{R,q}(d)} e^*_{j_1} \otimes \cdots \otimes e^*_{j_d}$, $i, j \in \Lambda$. Moreover, the coefficients appearing in this linear combination belong to the image of $\mathbb{Z}[u, u^{-1}] \to R$. Denote these coefficients by $p^{l,s}_{i,j}(u)$. We claim that our desired set is linearly independent. For each $(i,j) \in \Lambda$, we define the map $\psi_{i,j} \colon V^{\otimes d} \times DV^{\otimes d} \to R$ satisfying

$$\psi_{i,j} = \sum_{l,s \in I(n,d)} p^{l,s}_{i,j}(u)(e_l, e^*_s)^*, \tag{182}$$

where $(e_l, e^*_s)^*$ is the dual element of (e_l, e^*_s). So, this map is R-bilinear. By construction, the coefficients $p^{l,s}_{i,j}(u)$ satisfy the following relations: For each $\omega = (k, k+1)$, we have

$$\begin{cases} p^{l,s\omega}_{i,j}(u) = u p^{l\omega,s}_{i,j}(u) & \text{if } l_t = l_{t+1},\ s_t < s_{t+1} \\ p^{l\omega,s}_{i,j}(u) = p^{l,s\omega}_{i,j}(u) & \text{if } l_t < l_{t+1},\ s_t < s_{t+1} \\ p^{l\omega,s}_{i,j}(u) = u p^{l,s}_{i,j}(u) & \text{if } l_t < l_{t+1},\ s_t = s_{t+1} \\ p^{l\omega,s}_{i,j}(u) = (u - u^{-1}) p^{l,s}_{i,j}(u) + p^{l,s\omega}_{i,j}(u) & \text{if } l_t < l_{t+1},\ s_t > s_{t+1} \end{cases}. \tag{183}$$

We are now ready to check that $\psi_{i,j}$ satisfies the relation $\psi_{i,j}(e_f T_\omega, e_g^*) = \psi_{i,j}(e_f, T_\omega e_g^*)$ for all $f, g \in I(n,d)$. For $f, g \in I(n,d)$ and $\omega = (t, t+1)$,

$$\psi(e_f T_\omega, e_g^*) = \begin{cases} \sum_{l,s\in I(n,d)} p_{i,j}^{l,s}(u)\mathbb{1}_{\{(f\omega,g)\}}(l,s) & \text{if } f_t < f_{t+1} \\ \sum_{l,s\in I(n,d)} p_{i,j}^{l,s}(u)\mathbb{1}_{\{(f,g)\}}(l,s)u & \text{if } f_t = f_{t+1} \\ \sum_{l,s\in I(n,d)} p_{i,j}^{l,s}(u)\mathbb{1}_{\{(f,g)\}}(l,s)(u-u^{-1}) + p_{i,j}^{l,s}(u)\mathbb{1}_{\{(f\omega,g)\}}(l,s) & \text{if } f_t > f_{t+1} \end{cases} \tag{184}$$

$$= \begin{cases} p_{i,j}^{f\omega,g}(u) & \text{if } f_t < f_{t+1} \\ up_{i,j}^{f,g}(u) & \text{if } f_t = f_{t+1} \\ (u-u^{-1})p_{i,j}^{f,g}(u) + p_{i,j}^{f\omega,g}(u) & \text{if } f_t > f_{t+1} \end{cases}. \tag{185}$$

On the other hand,

$$\psi(e_f, T_\omega e_g^*) = \begin{cases} p_{i,j}^{f,g\omega}(u) & \text{if } g_t < g_{t+1} \\ up_{i,j}^{f,g}(u) & \text{if } g_t = g_{t+1} \\ (u-u^{-1})p_{i,j}^{f,g}(u) + p_{i,j}^{f,g\omega}(u) & \text{if } g_t > g_{t+1} \end{cases}. \tag{186}$$

Using the relations (183) we obtain our claim. Hence, $\psi_{i,j}$ induces a unique map $\psi'_{i,j}\colon V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d} \to R$, satisfying

$$\psi'_{i,j}(e_f \otimes_{H_{R,q}(d)} e_g^*) = p_{i,j}^{f,g}(u), \quad f, g \in I(n,d). \tag{187}$$

In particular $\psi'_{i,j}(e_i \otimes_{H_{R,q}(d)} e_j^*) = 1$ and $\psi'_{i,j}(e_f \otimes_{H_{R,q}(d)} e_g^*) = 0$ for all $(f,g) \in \Lambda$ distinct from (i,j). This shows that (173) is an R-basis of $V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$. $\square$

The dual elements of $e_i \otimes_{H_{R,q}(d)} e_j^*$, $(i,j) \in \Lambda$, denoted by $\xi_{j,i} \in D(V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}) \simeq S_{R,q}(n,d)$, form an R-basis of the q-Schur algebra. Moreover, (by a tensor-hom adjunction argument)

$$e_g^*(\xi_{j,i}(e_f)) = \psi'_{i,j}(e_f \otimes_{H_{R,q}(d)} e_g^*) = p_{i,j}^{f,g}(u), \quad f, g \in I(n,d). \tag{188}$$

Thus, we can write

$$\xi_{j,i}(e_f) = \sum_{f\in I(n,d)} p_{i,j}^{f,g}(u)e_g, \quad \forall f \in I(n,d). \tag{189}$$

Using our approach to a basis of the q-Schur algebra it is clear that the q-Schur algebra admits a base change property.

Lemma 7.17. *Let R be a commutative ring with an invertible element u. Fix $q = u^{-2}$. For any commutative R-algebra S,*

$$S_{R,q}(n,d) \simeq R \otimes_{\mathbb{Z}[u,u^{-1}]} S_{\mathbb{Z}[u,u^{-1}],u^{-2}}(n,d), \tag{190}$$

$$S_{S,q1_S}(n,d) \simeq S \otimes_R S_{R,q}(n,d). \tag{191}$$

Proof. Since $V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$ is a free R-module and $H_{R,q}(d)$ admit a base change property the q-Schur algebra $S_{R,q}(n,d)$ has also a base change property:

$$S_{S,q1_S}(n,d) \simeq S \otimes_R S_{R,q}(n,d). \tag{192}$$

The first equation follows by fixing $R = \mathbb{Z}[u, u^{-1}]$. $\square$

We will now focus on the case $n \geq d$. There are isomorphisms,

$$V^{\otimes d} \simeq S_{R,q}(n,d)\xi_{(1,\dots,d),(1,\dots,d)}, \quad DV^{\otimes d} \simeq \xi_{(1,\dots,d),(1,\dots,d)}S_{R,q}(n,d). \tag{193}$$

Hence, $V^{\otimes d}$ is an $S_{R,q}(n,d)$-projective $(S_{R,q}(n,d),R)$-injective. Note that these facts follow by extending the results of Donkin (see [Don98]) to commutative rings. In particular, the arguments of the results [Don98, Section 2.1 (5), (6),(7)] can easily be extended to commutative rings. Alternatively, we can see these facts as applications of Proposition 6.3 and Nakayama's Lemma.

For the Schur algebra, the dominant dimension is directly related with the characteristics of the residue fields of the ground ring. So, it is natural to consider a quantum version of the characteristic of the ring. This is done by replacing the identity by q on the definition of characteristic of a ring.

Definition 7.18. The *q-characteristic of R*, denoted by $q\,\mathrm{char}\,R$, is the smallest positive number s such that $1+q+\cdots+q^{s-1}=0$ if such s exists, and zero otherwise.

We shall refer to $q\,\mathrm{char}\,R$ as the quantum characteristic of R when there is no misunderstanding about q. Note that $(1-q)(1+q+\cdots+q^{s-1})=1-q^s$, for all $s>0$. So, for integral domains the quantum characteristic is zero if and only if either q is not a root of unity or $q=1$ and $\mathrm{char}\,R=0$. We refer to [LQ13] for a more detailed discussion of quantum characteristic.

The computation of dominant dimension for quantised Schur algebras over fields is due to Fang and Miyachi.

Theorem 7.19. *[FM19, Theorem 3.13] Let K be a field. Assume that $q=u^{-2}$ for some non-zero element $u\in K$.*

$$\operatorname{domdim} S_{K,q}(n,d) = \begin{cases} 2(q\,\mathrm{char}\,K-1) & \text{if } d \geq q\,\mathrm{char}\,K > 0 \\ +\infty, & \text{otherwise.}\end{cases} \tag{194}$$

We will now extend this computation for all q-Schur algebras satisfying $n \geq d$. Further, we can determine the relative dominant dimension of the q-Schur algebra by knowing the invertible elements of R.

Theorem 7.20. *Let R be a commutative ring with invertible element $u\in R$. Put $q=u^{-2}$. If $n\geq d$, then $(S_{R,q}(n,d),V^{\otimes d})$ is a relative gendo-symmetric R-algebra and*

$$\operatorname{domdim}(S_{R,q}(n,d),R) = \inf\{2s\in\mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\}. \tag{195}$$

Proof. By Proposition 6.3, $V^{\otimes d}$ is a projective $(S_{R,q}(n,d))$-injective-strongly faithful module. Hence, $(S_{R,q}(n,d),V^{\otimes d},DV^{\otimes d})$ is a RQF3 algebra. Let $\mathrm{MaxSpec}(R)$ be the set of maximal ideals of R.

By Theorem 6.13,

$$\operatorname{domdim}(S_{R,q}(n,d),R) = \inf\{\operatorname{domdim} S_{R,q}(n,d)\otimes_R R(\mathfrak{m}) | \mathfrak{m}\in\mathrm{MaxSpec}(R)\} \tag{196}$$

$$= \inf\{\operatorname{domdim} S_{R(\mathfrak{m}),q_{\mathfrak{m}}}(n,d) | \mathfrak{m}\in\mathrm{MaxSpec}(R)\} \geq 2, \tag{197}$$

where $q_{\mathfrak{m}}$ is the image of q in $R(\mathfrak{m})$. In particular, $V^{\otimes d}$ is a generator-cogenerator of $H_{R,q}(d)$. Similarly to Proposition 3.11, we can define an R-linear map $\pi\colon H_{R,q}(d)\to R$, given by

$$\pi(T_\sigma) = \begin{cases} 1_R, & \text{if } \sigma = e \\ 0, & \text{otherwise}\end{cases}, \quad \sigma\in S_d.$$

Afterwards, we can define the $H_{R,q}(d)$-isomorphism $\phi\colon H_{R,q}(d)\to DH_{R,q}(d)$, given by $\phi(T_\sigma)(T_\omega)=\pi(T_\sigma T_\omega)$ for every $\sigma,\omega\in S_d$. This yields that the Hecke algebra $H_{R,q}(d)$ is a relative symmetric R-algebra. By Theorem 7.8, $(S_{R,q}(n,d),V^{\otimes d})$ is a relative gendo-symmetric R-algebra. First, we will show that

$$\operatorname{domdim}(S_{R,q}(n,d),R) \leq \inf\{2s\in\mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\}. \tag{198}$$

If the right hand side is infinite, then there is nothing to prove. Assume that there exists $s < d$ such that $1 + q + \cdots + q^s \notin U(R)$. Then, $1 + q + \cdots + q^s$ belongs to some maximal ideal of R, say $\mathfrak{m}$. Therefore, $1 + q_\mathfrak{m} + \ldots + q_\mathfrak{m}^s$ is zero in $R(\mathfrak{m})$. Assume that $q_\mathfrak{m} = 1$ in $R(\mathfrak{m})$. Then, $0 \neq q_\mathfrak{m} \operatorname{char} R(\mathfrak{m}) = \operatorname{char} R(\mathfrak{m}) \leq s + 1 \leq d - 1 + 1 = d$, so $\operatorname{domdim} S_{R(\mathfrak{m}),q_\mathfrak{m}}(n, d) \leq 2s$. Now, assume that $q_\mathfrak{m} \neq 1$. Then,

$$0 < q_\mathfrak{m} \operatorname{char} R(\mathfrak{m}) \leq s + 1 \leq d - 1 + 1 = d. \tag{199}$$

Hence,

$$\operatorname{domdim}(S_{R(\mathfrak{m}),q_\mathfrak{m}}(n, d), R) = 2(q_\mathfrak{m} \operatorname{char} - 1) \leq 2s. \tag{200}$$

So, our claim follows. If $\operatorname{domdim}(S_{R,q}(n, d), R)$ is infinite, then, of course, that the equality (195) holds. Suppose that $\operatorname{domdim}(S_{R,q}(n, d), R) = l > 0$. So, there exists a maximal ideal $\mathfrak{m}$ of R such that

$$l = \operatorname{domdim} S_{R(\mathfrak{m}),q\mathfrak{m}}(n, d) = 2(q_\mathfrak{m} \operatorname{char} R(\mathfrak{m}) - 1), \tag{201}$$

and $0 < q_\mathfrak{m} \operatorname{char} R(\mathfrak{m}) \leq d$. By definition of quantum characteristic, the image of $1+q+\cdots+q^{q_\mathfrak{m} \operatorname{char} R(\mathfrak{m})-1}$ in $R(\mathfrak{m})$ is zero. So, $1 + q + \cdots + q^{q_\mathfrak{m} \operatorname{char} R(\mathfrak{m})-1}$ belongs to $\mathfrak{m}$. Since $q_\mathfrak{m} \operatorname{char} R(\mathfrak{m}) - 1 \leq d - 1 < d$ then $l \in \{2s \in \mathbb{N} \mid 1 + q + \cdots + q^s \notin U(R),\ s < d\}$. This finishes the proof. ☐

We can now compute $\operatorname{domdim}(S_{\mathbb{Z}[u,u^{-1}],u^{-2}}(n, d), \mathbb{Z}[u, u^{-1}])$. The invertible elements of $\mathbb{Z}[u, u^{-1}]$ are the powers of u and the constants 1 and -1. Hence, $1 + q = 1 + u^{-2}$ is not invertible. So,

$$\operatorname{domdim}(S_{\mathbb{Z}[u,u^{-1}],u^{-2}}(n, d), \mathbb{Z}[u, u^{-1}]) = 2, \quad d \geq 2. \tag{202}$$

A Appendix On Spectral sequences

In most cases, the computation of Ext and Tor groups is not done directly by the definition since it is not practical. Instead, spectral sequences provide useful ways to compute homology and cohomology of complexes. For a more detailed approach, we refer to ([Wei03], [Rot09]).

Definition A.1. A **(homology) spectral sequence** (starting with E^a) in an abelian category $\mathcal{A}$ consists of the following data:

- For $r \geq a$, the r-page is a collection of objects of $\mathcal{A}$ $\{E^r_{i,j}\},\ i, j \in \mathbb{Z}$.
- Maps $d^r_{i,j} \colon E^r_{i,j} \to E^r_{i-r,j+r-1}$ satisfying $d^r_{i,j} \circ d^r_{i+r,j-r+1} = 0$ and $E^{r+1}_{i,j} = \ker d^r_{i,j} / \operatorname{im} d^r_{i+r,j-r+1}$.

If $E^r_{i,j} = 0$ unless $i \geq 0$ and $j \geq 0$, then we say that $\{E^r_{i,j}\}$ is a **first quadrant homology spectral sequence.**

Hence the $(r+1)$-page contains the homology of the differential of the r-page. If the value at (i, j)-spot stabilizes from some page on, then we denote this value by $E^\infty_{i,j}$.

Definition A.2. We say that a first quadrant (homology) spectral sequence converges to H_*, written as

$$E^a_{i,j} \implies H_{i+j}$$

if we are given a collection of objects H_n of $\mathcal{A}$, each having a finite filtration

$$0 = H^{-1}_n \subset H^0_n \subset H^1_n \subset \cdots \subset H^n_n = H_n$$

such that $E^\infty_{i,n-i} \simeq H^i_n / H^{i-1}_n$ for $0 \leq i \leq n$.

Lemma A.3. *Assume that $E^2_{i,j} \implies H_{i+j}$ is a first quadrant spectral sequence. Then, there is an exact sequence*

$$H_2 \to E^2_{2,0} \to E^2_{0,1} \to H_1 \to E^2_{1,0} \to 0. \tag{203}$$

Proof. By convergence, we have the filtration

$$0 = H_1^{-1} \subset H_1^0 \subset H_1^1 = H_1. \tag{204}$$

with $E^\infty_{1,0} \simeq H_1^1/H_1^0$ and $E^\infty_{0,1} \simeq H_1^0/H_1^{-1} = H_1^0$. In particular, there is an exact sequence

$$0 \to E^\infty_{0,1} \to H_1 \to E^\infty_{1,0} \to 0. \tag{205}$$

Let $n \geq 2$. Then,

$$\begin{aligned} E^{n+1}_{1,0} &= \ker\left(d^n_{1,0} \colon E^n_{1,0} \to E^n_{1-n,n-1}\right)/\operatorname{im}\left(d^n_{1+n,1-n} \colon E^n_{1+n,-n+1} \to E^n_{1,0}\right) && (206)\\ &= E^n_{1,0}. && (207) \end{aligned}$$

By induction, $E^n_{1,0} = E^2_{1,0}$ for $n \geq 2$. By definition, $E^\infty_{1,0} = E^2_{1,0}$. We will now compute $E^\infty_{0,1}$. For $n \geq 3$,

$$E^{n+1}_{0,1} = \ker d^n_{0,1}/\operatorname{im} d^n_{n,2-n} = \ker d^n_{0,1} = \ker(E^n_{0,1} \to E^n_{-n,n}) = E^n_{0,1}. \tag{208}$$

By induction, it follows that

$$E^\infty_{0,1} = E^3_{0,1} = \ker d^2_{0,1}/\operatorname{im} d^2_{2,0} = E^2_{0,1}/\operatorname{im}(E^2_{2,0} \to E^2_{0,1}) = \operatorname{coker}(E^2_{2,0} \to E^2_{0,1}). \tag{209}$$

Now, $E^\infty_{2,0} = H_2^2/H_2^1 = H_2/H_2^1$. For $n \geq 2$,

$$E^{n+1}_{2,0} = \ker d^n_{2,0}/\operatorname{im} d^n_{2+n,1-n} = \ker(E^n_{2,0} \to E^n_{2-n,n-1}). \tag{210}$$

Therefore, $E^\infty_{2,0} = \ker(E^2_{2,0} \to E^2_{0,1})$. We constructed an exact sequence

$$\begin{tikzcd} H_2 \arrow[rr] \arrow[dr, two heads] & & E^2_{2,0} \arrow[r] & E^2_{0,1} \arrow[rr] \arrow[dr, two heads] & & H_1 \arrow[r] & E^2_{1,0} \arrow[r] & 0 \\ & E^\infty_{2,0} \arrow[ur, hook] & & & E^\infty_{0,1} \arrow[ur, hook] & & & \end{tikzcd}.$$

∎

Lemma A.4. *Let $q \geq 1$. Assume that $E^2_{i,j} \implies H_{i+j}$ is a first quadrant spectral sequence and $E^2_{i,j} = 0$ for $1 \leq j \leq q$. Then, $E^2_{i,0} \simeq H_i$, $1 \leq i \leq q$ and there is an exact sequence*

$$H_{q+2} \to E^2_{q+2,0} \to E^2_{0,q+1} \to H_{q+1} \to E^2_{q+1,0} \to 0. \tag{211}$$

Proof. We will start by showing by induction that $E^s_{i,j} = 0$ for every $s \geq 2$, $1 \leq j \leq q$ and every $i \geq 0$. Let $1 \leq j \leq q$ and $i \geq 0$. The case $s = 2$ follows by assumption. Assume that $E_l^{i,j} = 0$ for some $s \geq 2$ and $l \leq s$. Then,

$$E^{s+1}_{i,j} = \ker d^s_{i,j}/\operatorname{im} d^s_{i+s,j-s+1} = 0, \tag{212}$$

since by induction $\ker d^s_{i,j} \subset E^s_{i,j} = 0$ and thus $\ker d^s_{i,j} = 0$.

Therefore, $E^s_{i,j} = 0$ for every $s \geq 2$, $1 \leq j \leq q$ and every $i \geq 0$. In particular,

$$E^\infty_{i,j} = 0, \ 1 \leq j \leq q, \ i \geq 0. \tag{213}$$

Since $1 - s$ is a negative value, $E^s_{i+s,1-s} = 0$ and thus $\operatorname{im} d^s_{i+s,-s+1} = 0$. If $s \leq q+1$ or $i + 1 \leq s$, then $E_s^{i-s,s-1} = 0$. For $s \leq q+1$ or $i+1 \leq s$, we have

$$E^{s+1}_{i,0} = \ker\left(d^s_{i,0} \colon E^s_{i,0} \to E^s_{i-s,s-1}\right)/\operatorname{im} d^s_{i+s,-s+1} = E^s_{i,0}. \tag{214}$$

In particular, by an induction argument

$$E^{q+2}_{q+2,0} = E^{q+1}_{q+2,0} = E^2_{q+2,0} \tag{215}$$

$$E^{s+1}_{i,0} = E^s_{i,0} = E^2_{i,0}, \forall s \geq 2,\ 1 \leq i \leq q+1. \tag{216}$$

Thus,

$$E^\infty_{i,0} = E^2_{i,0},\ 1 \leq i \leq q+1. \tag{217}$$

For $s \geq q+3$, we have $E_s^{q+2-s,s-1} = 0$ and thus $\ker d^s_{q+2,0} = E^s_{q+2,0}$.

Therefore, we have

$$E^{s+1}_{q+2,0} = \ker d^s_{q+2,0}/\operatorname{im} d^s_{q+2+s,-s+1} = E^s_{q+2,0}, s \geq q+3 \text{ and} \tag{218}$$

$$E^\infty_{q+2,0} = E^{q+3}_{q+2,0} = \ker d^{q+2}_{q+2,0}/\operatorname{im} d^{q+2}_{q+2+(q+2),-(q+2)+1} = \ker\left(d^{q+2}_{q+2,0} \colon E^{q+2}_{q+2,0} \to E^{q+2}_{0,q+1}\right) \tag{219}$$

$$\underset{(215)}{=} \ker\left(E^2_{q+2,0} \to E^{q+2}_{0,q+1}\right). \tag{220}$$

Now we are ready to establish $E^2_{n,0} \simeq H_n$, $1 \leq n \leq q$.

Let $1 \leq n \leq q$ and $1 \leq i \leq n-1$. Then, $1 \leq n-i \leq q$. Hence by convergence and (213)

$$0 = E^\infty_{i,n-i} \simeq H^i_n/H^{i-1}_n, \text{ and} \tag{221}$$

$$H^{n-1}_n = H^{n-2}_n = H^0_n \simeq E^\infty_{0,n} \underset{(213)}{=} 0 \tag{222}$$

$$H_n = H^n_n = H^n_n\ H^{n-1}_n \simeq E^\infty_{n,0} \underset{(217)}{=} E^2_{n,0}. \tag{223}$$

Now we shall proceed to construct the desired exact sequence. By the filtration given by convergence, we have for any $n \geq 0$, $E^\infty_{n,0} \simeq H^n_n/H^{n-1}_n = H_n/H^{n-1}_n$. Thus, we have a canonical epimorphism $H_n \twoheadrightarrow E^\infty_{n,0}$ with kernel H^{n-1}_n for any $n \geq 0$. In particular, we have the exact sequence and the epimorphism

$$0 \to H^q_{q+1} \to H_{q+1} \to E^\infty_{q+1,0} \underset{(217)}{=} E^2_{q+1,0} \to 0, \quad H_{q+2} \twoheadrightarrow E^\infty_{q+2,0} \tag{224}$$

For $2 \leq s \leq q+1$, $1 \leq q+2-s \leq q$. Hence $E^s_{s,q+2-s} = 0$, for $2 \leq s \leq q+1$. Consequently, $\operatorname{im} d^s_{s,q+2-s} = 0$, for $2 \leq s \leq q+1$. Therefore, for $2 \leq s \leq q+1$,

$$E^{s+1}_{0,q+1} = \ker\left(E^s_{0,q+1} \to E^s_{-s,q+2-s}\right) = E^s_{0,q+1}. \tag{225}$$

We conclude that

$$E^{q+2}_{0,q+1} = E^{q+1}_{0,q+1} = E^2_{0,q+1}. \tag{226}$$

In view of (220),

$$E^\infty_{q+2,0} = \ker\left(E^2_{q+2,0} \to E^2_{0,q+1}\right) \tag{227}$$

For $s \geq q+3, \operatorname{im} d^s_{s,q+2-s} = 0$, and thus

$$E^{s+1}_{0,q+1} = \ker d^s_{0,q+1}/\operatorname{im} d^s_{s,q+2-s} = \ker\left(d^s_{0,q+1} \colon E^s_{0,q+1} \to E^s_{-s,q+s}\right) = E^s_{0,q+1}. \tag{228}$$

Therefore, $E^\infty_{0,q+1} = E^{q+3}_{0,q+1}$.

By (213) and using the filtration given by convergence for $1 \leq i \leq q$

$$0 = E^\infty_{i,q+1-i} = H^i_{q+1}/H^{i-1}_{q+1}. \tag{229}$$

This gives us

$$\begin{aligned} H^q_{q+1} &= H^{q-1}_{q+1} = H^0_{q+1} = E^\infty_{0,q+1} = E^{q+3}_{0,q+1} = \ker d^{q+2}_{0,q+1}/\operatorname{im} d^{q+2}_{q+2,0} && (230)\\ &= E^{q+2}_{0,q+1}/\operatorname{im}\left(E^{q+2}_{q+2,0} \to E^{q+2}_{0,q+1}\right) \underset{(226,215)}{=} E^2_{0,q+1}/\operatorname{im}\left(E^2_{q+2,0} \to E^2_{0,q+1}\right). && (231)\end{aligned}$$

Combining (231), (224) and (227) we obtain the exact sequence

$$\begin{tikzcd} H_{q+2} \arrow[rr] \arrow[dr, two heads] & & E^2_{q+2,0} \arrow[r] & E^2_{0,q+1} \arrow[rr] \arrow[dr, two heads] & & H_{q+1} \arrow[r] & E^2_{q+1,0} \arrow[r] & 0 \\ & E^\infty_{q+2,0} \arrow[ur, hook] & & & H^q_{q+1} \arrow[ur, hook] & & & \end{tikzcd}.$$ ☐

Lemma A.5. *(Künneth spectral sequence for chain complexes) Let P be a flat chain complex of R-modules $\cdots \to P_1 \to P_0 \to 0$. Let M be an R-module. Then,*

$$E^2_{i,j} = \operatorname{Tor}^R_i(H_j(P), M) \implies H_{i+j}(P \otimes_R M), \quad i, j \geq 0.$$

Proof. See for example [Wei03, Theorem 5.6.4]. ☐

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