Old and New Aspects of Semi-classical Spectral Estimates

1. Motivation: Counting Eigenvalues

Consider the one particle Schrödinger operator with the electrical potential $-V$

$$H(V;\hbar) = -\hbar^2\Delta - V(x) \quad \text{on} \quad L^2(\mathbb{R}^d)$$

Given local regularity and sufficient decay of of V at infinity, the spectrum of $H(V;\hbar)$ consists of two parts:

- $\sigma_{ess}(H(V;\hbar)) = [0,+\infty[$ and
- negative discrete spectrum $\sigma_d \ni -\lambda_n(V;\hbar)$

We are interested in the bound states $-\lambda_n(V;\hbar)$ trapped by $-V(x) \leq 0$.

$$S_{\sigma,d}(V;\hbar) = \operatorname{tr} H_-^\sigma(V;\hbar) = \sum_n \lambda_n^\sigma(V;\hbar)$$

$$S^{\text{cl}}_{\sigma,d}(V;\hbar) = \int\int_{h<0} (-h(\xi,x))^\sigma \frac{dx\,d\xi}{(2\pi\hbar)^d}, \quad h = |\xi|^2 - V(x), \quad \sigma \geq 0.$$

1.1 The semi-classical Limit

Physical Quantization Principle: Each bound state occupies a portion of the phase space corresponding to negative energy with the volume $(2\pi\hbar)^{-d}$

This principle is mathematically expressed in the following semi-classical formulae, which is proven initially for compactly supported and sufficiently smooth potentials V

$$S_{\sigma,d}(V;\hbar) = S^{\mathrm{cl}}_{\sigma,d}(V;\hbar)(1+o(1)), \quad \hbar \to 0. \tag{1}$$

The following questions arise:

- How does one efficiently count and sum up negative eigenvalues?
- Does (1) hold for all V with finite $S^{\mathrm{cl}}_{\sigma,d}(V;\hbar)$?
- Do the classical phase space averages deliver spectral estimates even in the non-asymptotical regime?

Such estimates are known as Cwikel-Lieb-Rosenblum [CLR] and the Lieb-Thirring [LTh] bounds.

2. Semi-classical Spectral Estimates

Cwikel-Lieb-Rosenblum and Lieb-Thirring bounds have the shape

$$S_{\sigma,d}(V;\hbar) \le R(\sigma,d) S^{\mathsf{cl}}_{\sigma,d}(V;\hbar)$$

Evaluating explicitly the phase space volume by integration in ξ, one finds

$$\begin{aligned} S^{\mathsf{cl}}_{\sigma,d}(V;\hbar) &= \int\int_{h<0} (-h(\xi,x))^\sigma \frac{dx\,d\xi}{(2\pi\hbar)^d} \\ &= \underbrace{\frac{\Gamma(\sigma+1)}{2^d \pi^{d/2} \Gamma\left(\sigma+\frac{d}{2}+1\right)}}_{L^{\mathsf{cl}}_{\sigma,d}} \hbar^{-d} \int V_+^{\sigma+\frac{d}{2}} dx. \end{aligned}$$

Hence these bounds turn into

2.1 The Cwikel-Lieb-Rosenblum and the Lieb-Thirring inequalities

$$\sum_n \lambda_n^\sigma(V;\hbar) \le L_{\sigma,d}\hbar^{-d}\int V_+^{\sigma+\frac{d}{2}}dx, \qquad L_{\sigma,d} = R(\sigma,d)L_{\sigma,d}^{\mathrm{cl}}. \qquad (2)$$

For $\sigma > 0$ they are known as Lieb-Thirring inequalities.

For $\sigma = 0$ one finds the Cwikel-Lieb-Rosenblum-inequality

$$N(V;\hbar) = S_{0,d}(V;\hbar) \le \hbar^{-d}L_{0,d}\int_{\mathbb{R}} V_+^{\frac{d}{2}}dx, \quad d \ge 3.$$

For which (σ,d), $\sigma \ge 0$, $d \in \mathbb{N}$, does (2) hold for *all* $\hbar > 0$ and *all* V with finite phase space volume $S_{\sigma,d}^{\mathrm{cl}}(V;\hbar)$, that means $V_+ \in L^{\sigma+\frac{d}{2}}(\mathbb{R}^d)$?

We shall first dwell more upon the case $\sigma = 0$.

3. The Birman-Schwinger-Principle

Put $\hbar = 1$, $\sigma = 0$. We consider the number $N_\lambda(V)$ of eigenvalues $-\lambda_n < -\lambda \leq 0$ of

$$H(V) = -\Delta - V(x) \quad \text{on} \quad L^2(\mathbb{R}^d).$$

Transform the problem $H(\alpha V)u = -\lambda u$ using $\Gamma_\lambda = (-\Delta + \lambda)^{-1}$ and $u = \Gamma_\lambda^{1/2} v$ into

$$\begin{aligned} -\Delta u - Vu = -\lambda u \quad &\Rightarrow \quad (-\Delta + \lambda)u = Vu \\ \Rightarrow \quad \Gamma_\lambda^{1/2}(-\Delta + \lambda)\Gamma_\lambda^{1/2} v = \Gamma_\lambda^{1/2} V \Gamma_\lambda^{1/2} v \quad &\Rightarrow \quad v = \Gamma_\lambda^{1/2} V \Gamma_\lambda^{1/2} v \end{aligned}$$

3.1 The Birman-Schwinger-Principle: Conclusion

$-\lambda < 0$ is an eigenvalue of $H(\alpha V)$ if and only if 1 is an eigenvalue of $\Gamma_\lambda^{1/2} V \Gamma_\lambda^{1/2}$

$$N_\lambda(V) = \#\left\{\text{e.v. of } \Gamma_\lambda^{1/2} V \Gamma_\lambda^{1/2} \text{ above } 1\right\} = \#\left\{\text{sing.v. of } V^{1/2}\Gamma_\lambda^{1/2} \text{above } 1\right\}$$

Estimating the counting function $N(\alpha V, \lambda)$ can be done by estimating the singular values of

$$V^{1/2}(x)\Gamma_\lambda^{1/2} = V^{1/2}(x) F^*(|\xi|^2 + \lambda)^{-1/2} F = a(x) b(i\nabla)$$

Below we shall study this class of operators more closely.

4. Operators of the Type $a(x)b(i\nabla)$

More precisely, let $(\mathcal{X}, dx), (\mathcal{Y}, dy)$ be separable spaces with σ-finite measures dx and dy. The integral operator

$$(Eu)(x) = \int \epsilon(x,y)u(y)dy$$

is said to be an *operator of the type* $\mathcal{E}(C_0, C_1)$, if the following two conditions are fulfilled:

$$E : L_1(\mathcal{Y}) \to L_\infty(\mathcal{X}) \text{ with the operator norm } C_0,$$
$$E : L_2(\mathcal{Y}) \to L_2(\mathcal{X}) \text{ with the operator norm } C_1.$$

Consider a function $q(x,y)$. We study the operators

$$(E_q u)(x) = \int q(x,y)\epsilon(x,y)u(y)dy\,.$$

In the special case $q(x,y) = a(x)b(y)$ this is the operator $E_{ab} = a(x)\ E\ b(y)$.

Note that the boundedness of $E : L_1(\mathcal{Y}) \to L_\infty(\mathcal{X})$ with operator norm C_0 is equivalent to

$$\|\epsilon\|_{L_\infty(\mathcal{X}\times\mathcal{Y})} = C_0.$$

4.1 Mapping properties of $\Pi : q \mapsto E_q$. First basic estimate.

Estimating singular values means the inclusion of E_q in suitable Neumann-Schatten ideals. For this we shall apply real interpolation to the mapping

$$\Pi : q \mapsto E_q \, .$$

First basic estimate: From $\epsilon \in L_\infty(\mathcal{X} \times \mathcal{Y})$ it follows, that for $q \in L_2(\mathcal{X} \times \mathcal{Y})$ the operator E_q is a Hilbert-Schmidt operator from $L_2(\mathcal{X})$ to $L_2(\mathcal{Y})$ and

$$\|E_q\|_{\mathfrak{S}_2} \leq \|\epsilon\|_{L_\infty(\mathcal{X}\times\mathcal{Y})} \|q\|_{L_2(\mathcal{X}\times\mathcal{Y})} = C_0 \|q\|_{L_2(\mathcal{X}\times\mathcal{Y})},$$

$$\Pi : L_2(\mathcal{X} \times \mathcal{Y}) \to \mathfrak{S}_2 \quad \text{with the operator norm} \quad C_0.$$

4.2 Mapping properties of $\Pi : q \mapsto E_q$. Second basic estimate.

Let $q(x, y) = a(x)b(y)$ and $a \in L_\infty(\mathcal{X})$, $b \in L_\infty(\mathcal{Y})$. Then $E_q = E_{ab} = aEb$ is bounded from $L_2(\mathcal{X})$ to $L_2(\mathcal{Y})$ and

$$\|E_{ab}\|_{\mathfrak{B}} \leq C_1 \|a\|_{L_\infty(\mathcal{X})} \|b\|_{L_\infty(\mathcal{Y})}.$$

Consider now $a_j \in L_\infty(\mathcal{X})$, $b_j \in L_\infty(\mathcal{Y})$. We denote by $L^\perp$ the set of all $q(x, y)$ of the type

$$q(x, y) = \sum_j a_j(x) b_j(y) \quad \text{with} \quad a_j a_k = b_j b_k = 0 \quad \text{for all} \quad j \neq k.$$

Then $(E_q)^* E_q = \oplus_j (E_{a_j b_j})^* E_{a_j b_j}$ and for $q \in L^\perp$

$$\|E_q\|_{\mathfrak{B}} = \sup_j \|E_{a_j b_j}\|_{\mathfrak{B}} \leq C_1 \sup_j \|a_j\|_{L_\infty(\mathcal{X})} \|b_j\|_{L_\infty(\mathcal{Y})} = C_1 \|q\|_{L_\infty(\mathcal{X} \times \mathcal{Y})}.$$

Let $L_\infty^\perp(\mathcal{X} \times \mathcal{Y})$ be the linear space of all $q(x, y) = \sum_l q^{(l)}(x, y)$ with $q^{(l)} \in L^\perp$, such that the norm

$$\|q\|_{L_\infty^\perp(\mathcal{X} \times \mathcal{Y})} = \inf_{q = \sum_l q^{(l)}, q^{(l)} \in L^\perp} \sum_l \|q^{(l)}\|_{L_\infty(\mathcal{X} \times \mathcal{Y})}$$

is finite.

The triangular inequality implies

$$\|E_q\| \leq C_1 \|q\|_{L_\infty^\perp(\mathcal{X}\times\mathcal{Y})} \quad \text{if} \quad q \in L_\infty^\perp(\mathcal{X}\times\mathcal{Y}).$$

That means the linear mapping $\Pi : q \mapsto E_q$ is bounded as follows

$$\Pi : L_\infty^\perp(\mathcal{X}\times\mathcal{Y}) \to \mathfrak{B} \quad \text{with the operator norm} \quad C_1.$$

4.3 Cwikels Theorem.

For any real K-interpolation functors Φ it holds

$$\|E_q\|_{\Phi(\mathfrak{S}_2,\mathfrak{B})} \le c_0(\Phi, C_0, C_1)\|q\|_{\Phi(L_2(\mathcal{X}\times\mathcal{Y}),L_\infty^\perp(\mathcal{X}\times\mathcal{Y}))}.$$

Problem: Give an analytic description of $\Phi(L_2(\mathcal{X}\times\mathcal{Y}), L_\infty^\perp(\mathcal{X}\times\mathcal{Y}))$.

For $q \in L^\perp$ and, in particular, for the special case $q(x,y) = a(x)b(y)$ the following two-sided bound for the K-function holds true

$$K(t,q;L_2,L_\infty) \le K(t,q;L_2,L_\infty^\perp) \le c_1 K(t,q;L_2,L_\infty), \quad q \in L_\infty^\perp.$$

This implies

$$\|E_q\|_{\Phi(\mathfrak{S}_2,\mathfrak{B})} \le c_2(\Phi, C_0, C_1)\|q\|_{\Phi(L_2(\mathcal{X}\times\mathcal{Y}),L_\infty(\mathcal{X}\times\mathcal{Y}))}, \quad q(x,y) = a(x)b(y).$$

This general bound contains Cwikels inequality as a particular case: Because of

$$\|q\|_{L_p^w(\mathcal{X}\times\mathcal{Y})} \le \|a\|_{L_p(\mathcal{X})}\|b\|_{L_p^w(\mathcal{Y})}, \quad q(x,y) = a(x)b(y),$$

the general bound turns for a special choice of the interpolation functor into

$$\|E_{ab}\|_{\mathfrak{S}_p^w} \le C(p, C_0, C_1)\|q\|_{L_p^w(\mathcal{X}\times\mathcal{Y})} \le C(p, C_0, C_1)\|a\|_{L_p(\mathcal{X})}\|b\|_{L_p^w(\mathcal{Y})},$$

where $2 < p < \infty$.

5. Cwikels Theorem without interpolation spaces

For the measurable function $q(x, y)$ on the measure space $(\mathcal{X} \times \mathcal{Y}, dx \cdot dy)$ consider its distribution function m_q and the non-increasing rearrangement q^*

$$\begin{aligned} m_q(s) &= \int_{(x,y):|q(x,y)|>s} dx \cdot dy, \qquad s > 0, \\ q^*(t) &= \inf\{s > 0 : m_q(s) \leq t\}, \qquad t > 0, \end{aligned}$$

For example, the weak spaces $L_p^w(\mathcal{X} \times \mathcal{Y})$ are equipped with the (quasi)-norm

$$\|q\|_{L_p^w} = \sup_{\tau > 0} \tau^{1/p} q^*(\tau), \qquad p > 0.$$

5.1 The key inequality

If $q \in L_2 + L_\infty$, we introduce the non-increasing average

$$\langle q \rangle (t) = \left(t^{-1} \int_0^t (q^*(\tau))^2 \, d\tau \right)^{1/2}, \qquad t > 0.$$

Assume that $q(x,y) = a(x)b(y) \in (L_2 + L_\infty)(\mathcal{X} \times \mathcal{Y})$. Then the estimate

$$s_n(E_q) \le (1 + 4C_1) \langle q \rangle (C_0^{-2} n), \qquad n \in \mathbb{N},$$

holds true.

In the special case $q(x, i\nabla) = a(x)b(i\nabla)$ on $L^2(\mathbb{R}^d)$ this turns into

$$s_n\left(a(x)b(i\nabla)\right) \leq 5\,\langle q\rangle\left((2\pi)^d n\right), \qquad n \in \mathbb{N}\,.$$

By the Birman-Schwinger principle the counting function $N_0(V)$ for $-\Delta - V$ equals the number of singular values of $V^{1/2}(-\Delta)^{-1/2}$ above 1 and Cwikels bound looks as follows:

$$1 \leq 5\,\langle q\rangle\left((2\pi)^d N_0(V)\right), \qquad q(x,\xi) = V^{1/2}(x)|\xi|^{-1}.$$

5.2 Applications. The two-dimensional Schrödinger operator.

Cwikels inequality in the dimension $d = 2$

$$N_0(\alpha V) \leq L_{0,2} \int \alpha V \, dx$$

does *not* hold.

- Weak coupling bound state for $\alpha \to 0$.
- There are compactly supported potentials $V \in L^1$ with a non-Weyl behavior of the counting functions $N_\lambda(\alpha V)$ for $\alpha \to \infty$.

Nevertheless one can apply the undressed inequality above for $\lambda = 1$ and finds:

For $\beta \in (0,1)$ define $V_\beta(x) = V(x)$ for all x where $V(x) \geq \beta$ and $V_\beta(x) = 0$ elsewhere.

There exist finite constants $C(\beta)$, such that

$$N_1(V) \leq C(\beta) \left\{ \| V_\beta \ln_+ (V_\beta \|V_\beta\|_{L_1}^{-1}) \|_{L_1} + \|V_\beta\|_{L_1} (1 + \ln_+ \|V_\beta\|_{L_1}) \right\}.$$

5.3 Applications. The pseudo-relativistic two-body pair Hamiltonian.

Consider non-relativistic two particles with the masses $m_\pm$ in the absence of external fields:

$$-\frac{1}{2m_+}\Delta^+ - \frac{1}{2m_-}\Delta^- - V(x^+ - x^-) \quad \text{on} \quad L^2(\mathbb{R}^{2d}),\ x^+, x^- \in \mathbb{R}^d,$$

where $-V$ stands for the interaction between the particles.

Due to translational invariance, this operator is unitary equivalent to the direct integral $\int_{\mathbb{R}^d}^{\oplus} h(P)dP$, where

$$h(P) = -\frac{M}{2m_+m_-}\Delta_y - V(y) + \frac{p^2}{2M}, \quad p = |P|,$$

acts on $L^2(\mathbb{R}^d)$. The parameter $M = m_+ + m_-$ is the total mass of the system and $P \in \mathbb{R}^d$ is the total momentum.

Notice that $h(P)$ depends on P only by a shift of $\frac{p^2}{2M}$, and the spectra of all $h(P)$ coincide modulo the respective shift.

In other words, the fundamental properties of the pair operator do not depend on the choice of the inertial system of coordinates.

On the other hand, if we consider the pseudo-relativistic Hamiltonian

$$\sqrt{-\Delta^+ + m_+^2} + \sqrt{-\Delta^- + m_-^2} - V(x^+ - x^-),$$

the decomposition into a direct integral $\int_{\mathbb{R}^d}^{\oplus} h_{rel}(P)dP$ gives rise to the pair operators

$$h_{rel}(P) = \sqrt{|\mu_+ P - i\nabla_y|^2 + \mu_+^2 M^2} + \sqrt{|\mu_- P + i\nabla_y|^2 + \mu_-^2 M^2} - V(y),$$

where $\mu_\pm = m_\pm M^{-1}$.

Obviously these operators show a much more involved dependence on the total momentum $P \in \mathbb{R}^d$.

In particular, the distribution of the negative eigenvalues of

$$q_{rel}(P) = h_{rel}(P) - \sqrt{p^2 + M^2}, \quad p = |P|,$$

depends on P.

Even if the attractive force $-V$ is too weak to induce negative bound states for small p, eigenvalues will appear as p grows and their total number tends to infinity as $p \to \infty$.

Already the volume Ξ_p of the classical phase space for negative energies $q_{rel}(P)$ behaves somehow unexpectedly.

Put $V \geq 0$. Then Ξ_p is finite if and only if $V \in L^{\frac{d}{2}}(\mathbb{R}^d) \cap L^d(\mathbb{R}^d)$. But for these potentials Ξ_p shows various asymptotics as $p \to \infty$.

- On the one hand one has

$$\Xi_p(V) = \frac{\omega_d p^{\frac{d+1}{2}}(1+o(1))}{2^{\frac{3d+1}{2}}\pi^d} \int V^{\frac{d-1}{2}} dy \quad \text{if} \quad V \in L^{\frac{d-1}{2}} \cap L^d, \quad p \to \infty.$$

- On the other hand, consider the model potentials $V_\theta(y) = \min\{1, v|y|^{-d/\theta}\}$.

If $\frac{d-1}{2} < \theta < d$ then $V_\theta \in L^\theta_w \cap L^d \subset (L^{\frac{d}{2}} \cap L^d) \setminus L^{\frac{d-1}{2}}$ and it holds

$$\Xi_p(V_\theta) = c_1(d,\theta,\mu_\pm) p^{\theta+1} v^\theta M^{d-1-2\theta}(1+o(1)), \quad \frac{d-1}{2} < \theta < d, p \to \infty.$$

5.4 Estimates on the Counting Function

The inhomogeneous symbol defies standard versions of Cwikel inequality. Instead we apply the undressed version, where the estimate follows the phase space distribution as close as possible.

In particular, we show that for $p \geq M > 0$

$$N_p(V) \leq c\left(p^2\left(1+\ln pM^{-1}\right)\|V\|_{L^1} + \|V\|_{L^3}^3\right), \quad d=3, \tag{3}$$

$$N_p(V) \leq c\left(p^{\frac{d+1}{2}}\|V\|_{L^{\frac{d-1}{2}}}^{\frac{d-1}{2}} + \|V\|_{L^d}^d\right), \quad d \geq 4, \tag{4}$$

$$N_p(V) \leq c\left(p^{1+\theta}M^{d-1-2\theta}\|V\|_{L_\theta^w}^\theta + \|V\|_{L^d}^d\right), \quad d \geq 3, \tag{5}$$

where $\frac{d-1}{2} < \theta < d$ in (5), whenever the respective r.h.s. is finite. The leading terms in the bounds (4) and (5) reduplicate the correct asymptotic order as $p \to \infty$.

6. Back to CLR and LTh

We study inequalities of the shape

$$S_{\sigma,d}(V;\hbar) \le R(\sigma,d) S^{\mathrm{cl}}_{\sigma,d}(V;\hbar) \qquad (6)$$

For which pairs (σ, d), $\sigma \ge 0$, $d \in \mathbb{N}$ does this bound hold true for all $V_+ \in L^{\sigma+\frac{d}{2}}(\mathbb{R}^d)$?

- For $d = 1$ if and only if $\sigma \ge 1/2$.
- For $d = 2$ if and only if $\sigma > 0$.
- For $d \ge 3$ if and only if $\sigma \ge 0$.

Using (6) one extends *in these cases* the semi-classical asymptotic formula to *all* $0 \le V \in L^{\sigma+\frac{d}{2}}$:

$$S_{\sigma,d}(V;\hbar) = S^{\mathrm{cl}}_{\sigma,d}(V;\hbar)(1+o(1)), \quad \hbar \to 0. \qquad (7)$$

negative spectrum for $d = 3$ and all $p > 0$.

What are the sharp values and the meaning of $R(\sigma, d)$? Because of

$$S_{\sigma,d}(V;\hbar) = S^{\mathrm{cl}}_{\sigma,d}(V;\hbar)(1+o(1)), \quad \hbar \to 0. \tag{8}$$

it holds

$$R(\sigma, d) \geq 1 \qquad \text{and} \qquad L_{\sigma,d} \geq L^{\mathrm{cl}}_{\sigma,d}.$$

For which pairs (σ, d) does one have $R(\sigma, d) = 1$?

6.1 Sharp values of Lieb-Thirring constants

The Riesz mean

$$S_{\sigma,d}(V;\hbar) = \operatorname{tr} H_-^\sigma(V;\hbar) = \sum_n \lambda_n^\sigma(V;\hbar)$$

becomes more regular with increasing values of σ. In particular, the constants $R(\sigma, d)$ are by Lieb-Aizenmann monotone decreasing in σ, that means

$$1 \leq R(\sigma, d) \leq R(\sigma', d) \quad \text{for} \quad \sigma \geq \sigma' \ .$$

The case with minimal σ is difficult, because the inequality is most difficult to prove. The cases with high σ are difficult, because we want really good bounds on the constants $R(\sigma, d)$.

6.2 Lieb-Thirring constants for $d = 1$.

In the dimension $d = 1$ the following is known about the constants $R(\sigma, 1)$:

- $R(1/2, 1) = 2$.
- $R(\sigma, 1) \leq 2$ for $\frac{1}{2} < \sigma < \frac{3}{2}$ by monotony.
- $R(3/2, 1) = 1$ and by monotony $R(\sigma, 1) = 1$ for all $\sigma \geq \frac{3}{2}$.

The sharp values for $R(\sigma, 1)$ for $\frac{1}{2} < \sigma < \frac{3}{2}$ are unknown. It is expected, that they are reached on potentials with one single negative bound state.

This is the case for $R(1/2, 1) = 2$. Here the δ-Potential is the maximizer. This constant corresponds also to the weak coupling limit.

The constant $R(3/2, 1) = 1$ is achieved on all reflection-less potentials.

In fact, in the limiting case $\sigma = 1/2$ for $d = 1$ a two-sided bound holds true

$$\frac{\alpha}{4}\int_{\mathbb{R}} V\,dx \le \sum_n \sqrt{\lambda_n(\alpha V)} \le \frac{\alpha}{2}\int_{\mathbb{R}} V\,dx, \quad V \ge 0, \alpha > 0.$$

6.3 The Lieb-Thirring Hypothesis

Lieb and Thirring stated in 1972 the following hypothesis: In all dimensions d exists a threshold $\sigma(d)$, such that

$$S_{\sigma,d}(V) \leq S^{\text{cl}}_{\sigma,d}(V) \quad \text{for all} \quad \sigma \geq \sigma(d).$$

We prove this hypothesis in A. Laptev, T. Weidl: "Sharp Lieb-Thirring Inequalities in High Dimensions", Acta Mathematica 184 (2000) 87-111

$$\sigma(d) \leq \frac{3}{2} \quad \text{for all} \quad d \in \mathbb{N}.$$

The question whether $\sigma(3) = 1$ holds true, remains unresolved. We show in D. Hundertmark, A. Laptev, T. Weidl: "New bounds on the Lieb-Thirring constants", Inv math 140 3 (2000) 693-704

$$S_{\sigma,d}(V) \leq 2S^{\mathrm{cl}}_{\sigma,d}(V) \quad \text{for} \quad 1 \leq \sigma < \frac{3}{2}, \quad d \in \mathbb{N}.$$

6.4 The Key Inequality

The key to sharp bounds in higher dimensions is an induction procedure in the dimension based upon a one-dimensional sharp inequality for matrix potentials. Consider

$$H = -\frac{d^2}{dx^2} \otimes 1_G - W(x) \quad \text{on} \quad L_2(\mathbb{R}, G)$$

where G is an auxiliary Hilbert space and $W : \mathbb{R} \to \mathfrak{S}_\infty(G)$ an operator valued non-negative potential.

Let $-\lambda_n(W)$ be the negative eigenvalues of H. The following bound holds true:

$$\sum_n \lambda_n^{3/2} \le L^{\mathsf{cl}}_{3/2,1} \int \operatorname{tr}_G W^2(x)\,dx, \qquad L^{\mathsf{cl}}_{3/2,1} = \frac{3}{16}.$$

7. The Dirichlet Laplacian.

Let $\Omega \subset \mathbb{R}^d$ be an open domain. We consider $-\Delta_D^\Omega$ with Dirichlet boundary conditions on $L^2(\Omega)$. If Ω is bounded, the spectrum of the Dirichlet Laplacian is discrete. We denote it by $0 < \lambda_1^D(\Omega) \le \lambda_2^D(\Omega) \le \cdots$

We study the spectral quantities

$$S_{\sigma,d}(\Omega;\Lambda) := \sum_n (\Lambda - \lambda_n^D)_+^{3/2}, \quad \Lambda \ge 0,$$

in comparison with the classical analog

$$S_{\sigma,d}^{\mathsf{cl}}(\Omega;\Lambda) := (2\pi)^{-d} \int_{x\in\Omega} dx \int_{\xi\in\mathbb{R}^d} (\Lambda - |\xi|^2)_+^\sigma d\xi = L_{\sigma,d}^{\mathsf{cl}} \Lambda^{\sigma+d/2} \mathsf{vol}(\Omega)$$

7.1 Weyls law and Berezin-Li-Yau bounds

Let us point out the following relations, which hold for general domains Ω:

$$\begin{aligned} S_{\sigma,d}(\Omega;\Lambda) &= (1+o(1))S^{\mathrm{cl}}_{\sigma,d}(\Omega;\Lambda) \quad \text{for} \quad \Lambda\to+\infty,\\ S_{\sigma,d}(\Omega;\Lambda) &\le r(\sigma,d)S^{\mathrm{cl}}_{\sigma,d}(\Omega;\Lambda) \quad \text{for all } d\in\mathbb{N} \quad \text{and } \sigma\ge 0,\\ S_{\sigma,d}(\Omega;\Lambda) &\le S^{\mathrm{cl}}_{\sigma,d}(\Omega;\Lambda), \quad \text{for all } d\in\mathbb{N} \quad \text{and } \sigma\ge 1. \end{aligned}$$

In some sense the later bound is optimal. Can one improve an optimal bound? Yes - we do so for $d=2$ and $\sigma=3/2$.

$$\Omega(x_0)=\{y\in\mathbb{R}|(x_0,y)\in\Omega\},\quad L(x_0)=\mathrm{meas}(\Omega(x_0)),$$

$$L_\Lambda:=\pi\Lambda^{-1/2},\quad I_\Lambda=\{x\in\mathbb{R}|L(x)>L_\Lambda\},\quad d_\Lambda=\mathrm{meas}(I_\Lambda).$$

7.2 Induction in the dimension

A simple variational argument implies

$$-\Delta_D^\Omega - \Lambda = -\frac{\partial^2}{\partial x^2} + \left(-\frac{\partial^2}{\partial y^2} - \Lambda\right) \geq -\frac{d^2}{dx^2} - W_-(x) \text{ on } L_2(\mathbb{R}, L_2(\mathbb{R}))$$

where $W(x) = \left(-\frac{d^2}{dy^2}\right)_D^{\Omega(x)} - \Lambda$ is the shifted second derivative in y-direction on the section $\Omega(x)$ with Dirichlet boundary conditions.

The k-th eigenvalue of $W(x)$ satisfies the bound

$$\mu_k(x) \geq \pi^2 k^2 L^{-2}(x) - \Lambda, \quad k \in \mathbb{N},$$

which turns into an equality if $\Omega(x)$ consists of only one segment. The Lieb-Thirring inequality for operator valued potentials implies now

$$
\begin{aligned}
S_{3/2,2}(\Omega,\Lambda) &\le \operatorname{tr}\left(-\frac{d^2}{dx^2} - W_-(x)\right)_-^{3/2} \\
&\le \frac{3}{16}\int_{\mathbb{R}} \operatorname{tr} W_-^2(x)dx \le \frac{3}{16}\int_{\mathbb{R}} \sum_k (\Lambda - \pi^2 k^2 L^{-2}(x))_+^2 dx \\
&\le \frac{3}{16}\int_{\mathbb{R}} \frac{\pi^4}{L^4(x)} \sum_{k=1}^{[A(x)]} (A(x) - k^2)^2 dx, \qquad A(x) = \frac{L(x)}{L_\Lambda}.
\end{aligned}
$$

The integrand in non-zero only where $A(x) > 1$, that means $L(x) > L_\Lambda$ or equivalently $x \in I_\Lambda$. For $A > 1$ the following bound holds true

$$
\sum_{k=1}^{[A]} (A - k^2)^2 dx \le \frac{8}{15}A^5 - \frac{1}{3}A^4 .
$$

7.3 Improved Berezin-Li-Yau bounds

This gives finally

$$\begin{aligned} S_{3/2,2}(\Omega,\Lambda) &\leq \frac{3}{16}\int_{I_\Lambda}\frac{\pi^4}{L^4(x)}\left(\frac{8}{15}\left(\frac{L(x)\Lambda^{1/2}}{\pi}\right)^5-\frac{1}{3}\left(\frac{L(x)\Lambda^{1/2}}{\pi}\right)^4\right)dx \\ &= \frac{3}{16}\cdot\frac{8}{15}\cdot\frac{1}{\pi}\cdot\Lambda^{5/2}\int_{I_\Lambda}L(x)dx-\frac{1}{16}\Lambda^2\int_{I_\Lambda}dx \\ &= L^{\mathsf{cl}}_{3/2,2}\Lambda^{5/2}\mathsf{vol}(\Omega_\Lambda)-\frac{d_\Lambda}{16}\Lambda^2 = S^{\mathsf{cl}}_{3/2,2}(\Omega_\Lambda,\Lambda)-\frac{d_\Lambda}{16}\Lambda^2\,. \end{aligned}$$

Here we use the notation

$$\Omega_\Lambda = \{(x,y)\in\Omega | x\in I_\Lambda\}\subset\Omega\,.$$

The bound

$$S_{3/2,2}(\Omega,\Lambda) \le S^{\mathsf{cl}}_{3/2,2}(\Omega_\Lambda,\Lambda) - \frac{d_\Lambda}{16}\Lambda^2$$

improves the Berezin-Li-Yau bound for $\sigma = 3/2$

$$S_{3/2,2}(\Omega,\Lambda) \le S^{\mathsf{cl}}_{3/2,2}(\Omega,\Lambda)$$

in several ways:

- It contains a negative remainder term $-\frac{d_\Lambda}{16}\Lambda^2$.
- It contains the volume of a suitable subset of Ω only.
- It can deal with sets Ω of infinite volume, as long as the volume of Ω_Λ is finite.