Spectral Analysis of Matrices

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An Introduction for Engineers

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Plan of the Talk

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5. A practical problem:
   Torsion oscillations of a gear shaft

1. A practical problem: Stable rotation of a three-dimensional body

Consider a stiff body Ω with the density $\rho(\mathbf{x})$.

Chose the system of coordinates, which is fixed with the body at the centre of mass 0.

The description of the rotation of Ω invokes the matrix of the moments of inertia:

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \quad ,$$

$$t_{kl} = \int_\Omega \left[\delta_{kl} \sum_{j=1}^{3} x_j^2 - x_l x_k \right] \rho(\mathbf{x}) d^3\mathbf{x},$$

$$\delta_{kl} = \begin{cases} 1 & \text{if} \quad k = l \\ 0 & \text{if} \quad k \neq l \end{cases}, \qquad k, l = 1, 2, 3.$$

Let $\mathbf{w}$ be the angular velocity of Ω. This is a vector, directed along the axes of the rotation.

The energy E and the spin $\mathbf{s}$ are given by

$$E = \frac{1}{2} < T\mathbf{w}, \mathbf{w} >= \frac{1}{2} < \mathbf{s}, \mathbf{w} > \qquad \text{and} \qquad \mathbf{s} = T\mathbf{w}.$$

A body can rotate freely around the axes along $\mathbf{w}$, if and only if $\mathbf{s}$ and $\mathbf{w}$ are parallel:

$$\lambda\mathbf{w} = \mathbf{s} = T\mathbf{w}.$$

Mathematical problem: Find all λ and $\mathbf{w} \neq 0$, such that

$$T\mathbf{w} = \lambda\mathbf{w}. \tag{0}$$

2. Eigenvalues and eigenvectors of linear operators

Consider the finite-dimensional complex vector space $H = \mathbb{C}^n$. Let 1 denote the idendity on H.

Consider a linear operator T on H.

Definition 1. Assume that for some $\lambda \in \mathbb{C}$ and some $\mathbf{w} \in \mathbb{C}^n$, $\mathbf{w} \neq 0$, the identity

$$T\mathbf{w} = \lambda\mathbf{w} \tag{2}$$

is satisfied. Then λ is said to be an *eigenvalue* of T and $\mathbf{w}$ is the respective *eigenvector*.

Definition 2. Let λ be an eigernvalue of T. The *eigenspace* E_λ of T corresponding to λ is defined as the set of all solutions $\mathbf{w} \in \mathbb{C}^n$ of (2).

Remark. An eigenspace E_λ of T is a linear subspace of $H = \mathbb{C}^n$.

Indeed, for $\mathbf{v}, \mathbf{w} \in E_\lambda$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned} T(\alpha\mathbf{v} + \beta\mathbf{w}) &= \alpha T\mathbf{v} + \beta T\mathbf{w} = \alpha\lambda\mathbf{v} + \beta\lambda\mathbf{w} \\ &= \lambda(\alpha\mathbf{v} + \beta\mathbf{w}) \end{aligned}$$

and hence $\alpha\mathbf{v} + \beta\mathbf{w} \in E_\lambda$.

Definition 3. The set of all eigenvalues of T is called the *spectrum* $\sigma(T)$ of T.

Consider the operator function

$$R_T(\lambda) = (T - \lambda \cdot \mathbf{1})^{-1}.$$

It is well defined, iff the equation $(T - \lambda\mathbf{1})\mathbf{w} = 0$ does not have a non-trivial solution, that is iff $\lambda \notin \sigma(T)$.

Definition 4. The set $\rho(T) = \mathbb{C} \backslash \sigma(T)$ is called the *resolvent set*, and the operator function $R_T(\lambda)$ is called the *resolvent* of T.

3. The characteristic polynomial.

Definition 5. The function

$$\begin{aligned} d_T(\lambda) &:= \det(T - \lambda \cdot \mathbf{1}) \\ &= \det \begin{pmatrix} t_{11} - \lambda & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{12} - \lambda & \cdots & t_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{pmatrix} = 0. \end{aligned}$$

is called the *characteristic polynomial* of T.

Remark. The roots of $d_T(\lambda)$ are the eigenvalues of T.

Problem: Show that $d_T(\lambda)$ is a polynomial of degree n !

Remark. The spectrum $\sigma(T)$ consists of at least one and of not more than n points.

The polynomial $d_T(\lambda)$ can be factorized as follows:

$$d_T(\lambda) = (\lambda - \lambda_1)^{\tau_1} \cdots (\lambda - \lambda_l)^{\tau_l},$$

where λ_l are the roots of order $\tau_l \in \mathbb{N}$. We have

$$\tau_1 + \cdots + \tau_l = n,$$

and hence $l \leq n$. We set

$$d_T(\lambda) = a_n \lambda^n + \cdots + a_1 \lambda + a_0.$$

Problem: Determine a_n, a_{n-1} and a_0 !

Definition 6. The order τ_k of the root λ_k of $d_T(\lambda)$ is called the *algebraic multiplicity* of the eigenvalue λ_k of T.

Definition 7. The number $\nu_k = \dim E_{\lambda_k}$ is called the *geometric multiplicity* of the eigenvalue λ_k of T.

Proposition 1. It holds

$$\nu_k \leq \tau_k \qquad \text{for all} \qquad k = 1, \ldots, l.$$

(Proof follows later).

4. Some examples and problems.

For $T = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ we have

$$\begin{aligned} d_T(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} &= (1-\lambda)(1-\lambda) + 4 \\ = \lambda^2 - 2\lambda + 5 &= (1 + 2i - \lambda)(1 - 2i - \lambda). \end{aligned}$$

Hence T has the eigenvalues

$$\lambda_1 = 1 - 2i, \qquad \lambda_2 = 1 + 2i$$

with the multiplicities $\nu_1 = \nu_2 = \tau_1 = \tau_2 = 1$. The eigenvectors $\mathbf{w}^{(1)}$ corresponding to λ_1 solve

$$\begin{pmatrix} 1-\lambda_1 & 2 \\ -2 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} w_1^{(1)} \\ w_2^{(1)} \end{pmatrix} = \begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix} \begin{pmatrix} w_1^{(1)} \\ w_2^{(1)} \end{pmatrix} = 0$$

We see that $\mathbf{w}^{(1)} = \beta \begin{pmatrix} i \\ 1 \end{pmatrix}$, $\beta \in \mathbb{C}$.

Determine the eigenspace E_{λ_2}!

In higher dimensions $n > 4$ the roots of the characteristic polynomial can usually not be found explicitly. Therefore other techniques are required to calculate or estimate the spectrum.

Let us consider the special case of diagonal matrices. Put

$$T = \begin{pmatrix} t_1 & 0 & 0 & \cdots & 0 \\ 0 & t_2 & 0 & \cdots & 0 \\ 0 & 0 & t_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t_n \end{pmatrix}.$$

$\boxed{\textbf{Determine } d_T(\lambda)\textbf{, } \sigma(T)\textbf{, } \rho(T) \textbf{ and } R_T(\lambda)\textbf{!}}$

Let T be some linear operator on $\mathbb{C}^n$ and let A be an invertible linear operator. Put $S = A^{-1}TA$.

$\boxed{\textbf{Show that } \sigma(S) = \sigma(T)\textbf{!}}$

5. A practical problem: Torsion oscillations of a gear shaft

Consider a gear shaft consisting of n wheels with the moments of inertia $m_1, \ldots, m_n > 0$.

Let $\mathbf{w}(t) = \begin{pmatrix} w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}$ be the vector of the torsion angles with respect to the normal position at the time t.

The torque acting on the l-th wheel is proportional to the relative torsion of the two near-by wheels

$$m_l \ddot{w}_l = (w_{l-1} - w_l)k_l - (w_l - w_{l+1})k_{l+1},$$

$$l = 1, \ldots, n, \qquad w_0 := w_1, \qquad w_{n+1} := w_n.$$

Let K be the matrix

$$K = \begin{pmatrix} -k_2 & k_2 & 0 & 0 & \cdots \\ k_2 & -k_2 - k_3 & k_3 & 0 & \cdots \\ 0 & k_3 & -k_3 - k_4 & k_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and let M be the matrix

$$M = \begin{pmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m_n \end{pmatrix}.$$

Then

$$M\ddot{\mathbf{w}}(t) = K\mathbf{w}(t).$$

We look for an oscillating solution of the type

$$\mathbf{w}(t) = e^{i\varkappa t}\mathbf{v}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

that is with $\lambda = -\varkappa^2$ for a solution of the

Mathematical problem: Find all λ and $\mathbf{v} \neq 0$, such that

$$\lambda M\mathbf{v} = K\mathbf{v} \Longleftrightarrow \lambda\mathbf{v} = (M^{-1}K)\mathbf{v} \tag{1}$$