2.4.1 Analytic continuation, Laurent series and classification of singularities

The use of Laurent expansions and residues is central to this thesis. The Laurent expansion of a function f generalizes the concept of a Taylor expansion. Unlike Taylor expansions, Laurent expansions apply not only to points where f is analytic, but also to points where it fails to be analytic, i.e. to *singularities.* In order to discuss the Laurent expansion, we must first define the terms *analytic continuation,* and *singularity.* These are important in their own right; the reader will recall from the introduction that the evaluation of harmonic sums by the Mellin transform requires that we obtain an analytic continuation of the Dirichlet series in the amplitudes to the left or right of its fundamental strip.

Definition 2.4.3 *An* **analytic function element** (f, D) *is an analytic function* $f(z)$ *along with its domain of definition* D. *A function element* (f_2, D_2) *is a* **direct analytic continuation** *of another element* (f_1, D_1) *if* $D = D_2 \cap D_1 \neq \emptyset$ *and* $f_1 \equiv f_2$ *in* D. *A* **complete analytic function** *is the collection of all possible analytic function elements* (f, D) *starting with a given element* (f_0, D_0) *such that a chain of direct continuations exists between* (f, D) *and* (f_0, D_0). *A* **singularity** *of a complete analytic function is a limit point of a domain of one or more elements that is not itself in the domain of any element.*

Note that a complete analytic function may be multi-valued; the standard example is the complex logarithm. We will briefly discuss this example, because it demonstrates one of the most common techniques for analytic continuation used in actual applications. This technique is analytic continuation by power series along curves; the domains D of the function elements are disks. Suppose $f(z)$ is analytic in a neighborhood U of z_0, and we wish to obtain its analytic continuation to a point z_1 not in U. Let γ be a curve that joins z_0 to z_1. If we can expand f into a power series of radius ρ that such that $\{z \mid |z - z_0| < \rho\}$ includes more of γ than U, this series defines an analytic continuation of f along γ. We can repeat this process with the goal of eventually reaching z_1. We will reach z_1 if the successive radii of convergence do not shrink to zero. There are theorems to decide when the resulting function will be single valued; if we continue a function f around a simple closed curve, we cannot return to the starting point with a different value unless there is a singularity of f inside the curve; alternatively, continuation along two different curves with the same start and end points leads to the same value of f at the end point, unless there is a singularity of f between the two curves.

Example. We study the complex logarithm. We will construct an instance where analytic continuation around a singularity results in a different value on return to the starting point. The initial function element (f_0, D_0) has $f_0(z) = \log z$ and $D_0 = \{z \mid |z-1| < 1\}$, where $f_0(z)$ is the branch of the logarithm defined by

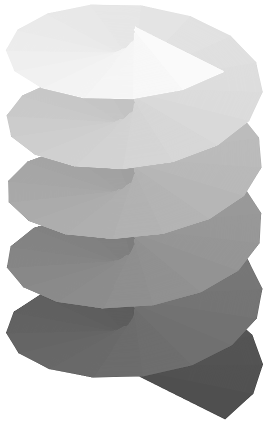
$$\sum_{k=0}^{\infty} \frac{(-1)^k (z-1)^{k+1}}{k}, \quad |z-1| < 1,$$

i.e. with $\log 1 = 0$. One way to continue $f_0(z)$ along $\gamma = \{z \mid |z| = 1\}$, the interior of which includes $z = 0$, is via the following nine function elements (here $0 \leq m \leq 8$).

$$f_m(z) = m\frac{i\pi}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k (z - e^{mi\pi/4})^{k+1}}{k e^{mi\pi/4}}, \quad D_m = \{z \mid |z - e^{mi\pi/4}| < 1\}.$$

The function element (f_m, D_m) is obtained from (f_{m-1}, D_{m-1}) by noting that the center z_m of the disk D_m lies inside D_{m-1}. Hence we can compute the Taylor expansion of f_{m-1} at z_m in terms of the expansion at z_{m-1}. This new expansion has radius of convergence $\rho = 1$ and defines the next function element in the analytic continuation of (f_{m-1}, D_{m-1}).

The function elements (f_0, D_0) and (f_8, D_8) clearly cover the same domain $\{z \mid |z-1| < 1\}$, but their values at the points of this domain differ by $2\pi i$. It is natural to ask whether there exists a paradigm to describe this behavior. This thesis is not concerned with multivalued analytic continuation. Nonetheless we remark in passing that such a paradigm does exist. It rests on a striking idea by Riemann, the so-called *Riemann surface* of the complete analytic function f. We construct this surface so that f is single-valued on it. We envision it as situated "over" the complex plane $\mathbb{C}$. In fact it is embedded in $\mathbb{C}^2$, i.e. a four-dimensional space, but there are cases when we can use the single dimension z ("height") of three-space to capture useful information about the shape of the surface in $\mathbb{C}^2$. The function elements (f, D) of the complete analytic function f are situated "over" D. They are patches of the surface. If two function elements are direct continuations of one another, their respective domains overlap; the corresponding patches on the surface overlap also. If two function elements have the same domain, but give different values for f, the corresponding two patches do not overlap. If the difference in values is one-dimensional, we can indicate it by situating the two patches at different heights over $\mathbb{C}$. If the image $f(z)$ of a point $z \in \mathbb{C}$ has cardinality $n = |f(z)| \geq 1$, we map z to n different points on the Riemann surface, one for each value in the image set. These points are said to lie on different *sheets* of the surface. The sheets of the Riemann surface are copies of $\mathbb{C}$ connected such that we can continuously pass from one sheet to another along suitably chosen curves and in this way obtain all the values of the complete analytic function $f(z)$.



Riemann surface of $\log z$; 5 sheets of the surface are shown.

The Riemann surface of $\log z$ is a surface that can be visualized in three-space. This is because the difference between the values "above" a specific z is an imaginary constant ($2\pi i$), and can therefore be mapped to the single three-space dimension "height".

Theorem 2.4.2 Laurent expansion. *Let $r_1, r_2 \in \mathbb{R}$, $r_1 \geq 0$, $r_2 > 0$ and $z_0 \in \mathbb{C}$. Define the annulus $A = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$. The combinations $r_1 = 0$ (deleted neighborhood) or $r_2 = \infty$ (open complement of a disk relative to $\mathbb{C}$) or both ($\mathbb{C} \setminus 0$) are permitted. Let f be analytic on A. There exist $b_n \in \mathbb{C}$ such that we may write*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$$

and this series converges absolutely on A and uniformly on any closed annulus $B \subset A$ of the form $B_{\rho_1,\rho_2} = \{z \in \mathbb{C} \mid \rho_1 \leq |z - z_0| \leq \rho_2\}$ where $r_1 < \rho_1 < \rho_2 < r_2$. For γ any circle $\{z_0 + re^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ around z_0 with radius r and $r_1 < r < r_2$ the coefficients are given by

$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

This expansion is the **Laurent expansion** *of f around z_0 in A and it is unique.*

A singularity of $f(z)$ in a region A can be classified according to the Laurent expansion about the singularity. We are concerned particularly with expansions in deleted neighborhoods, i.e. in annuli with $r_1 = 0$.

Definition 2.4.4 Classification of singularities. *Let f be analytic in a region A that contains a deleted neighborhood $N_\epsilon(z_0)\backslash\{z_0\}$, and let f fail to be analytic at z_0. We say that z_0 is an* **isolated singularity**. *Let $\{b_n\}$ be the coefficients of the Laurent expansion of f in the annulus $N_\epsilon(z_0) \setminus \{z_0\}$.*

- **Principal part.** *The expansion*
$$\sum_{n=-\infty}^{-1} b_n(z-z_0)^n$$
is the principal part of f at z_0.
- **Pole.** *An isolated singularity is a pole if the principal part has only a finite number of non-zero coefficients.*
- **Pole of order k.** *The point z_0 is a pole of order k if the principal part has the form*
$$\sum_{n=-k}^{-1} b_n(z-z_0)^n.$$
- **Simple pole.** *A simple pole is a pole of order* 1.
- **Essential singularity.** *If the number of zero coefficients in the principal part is finite, z_0 is an essential singularity.*
- **Residue.** *The coefficient b_{-1} is the residue of f at z_0. We write $b_{-1} = \mathrm{Res}\,(f(z); z = z_0)$.*
- **Removable singularity.** *The point z_0 is a removable singularity if all the coefficients of the principal part are zero.*
- **Meromorphic functions.** *A function is meromorphic in a region A if it is analytic in A with the exception of poles. A function f is meromorphic if it is meromorphic in $\mathbb{C}$.*

We use $\mathrm{Sing}\,(f(z))$ *to denote the set of finite singularities of f.*

It is important that we be able to compute the residues of a function f at its isolated singularities; e.g., the Mellin-Perron formula produces an integral that can be evaluated or estimated with the residue theorem.

Lemma 2.4.3 Computation of residues. *Let f have an isolated singularity at z_0 and let $k \in \mathbb{N}$ be the smallest integer such that $\lim_{s\to z_0}(z-z_0)^k f(z)$ exists. Then $f(z)$ has a pole of order k at z_0 and if we let $\phi(z) = (z-z_0)^k f(z)$, then ϕ can be defined uniquely at z_0 so that ϕ is analytic at z_0 and*
$$\mathrm{Res}\,(f(z); z = z_0) = \frac{\phi^{(k-1)}(z_0)}{(k-1)!}.$$

2.5 The Cauchy residue theorem

Definition 2.5.1 *Let γ be a closed curve in $\mathbb{C}$ and z_0 be a point not on γ. Then the* **index** *or* **winding number** *of γ with respect to z_0 is defined by*

$$I(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}.$$

The curve γ winds around z_0 $I(\gamma, z_0)$ times.

Theorem 2.5.1 (Cauchy residue theorem.) *Let A be a region and let $z_1, z_2 \ldots z_n$ be distinct points in A. Let f be analytic on $A \setminus \{z_1, z_2 \ldots z_n\}$. Let γ be a closed curve in A homotopic to, i.e. smoothly shrinkable to, a point in A. Assume none of $z_1, z_2 \ldots z_n$ lie on γ. We have*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{i=1}^{n} \operatorname{Res}\left(f(z); z = z_i\right) I(\gamma, z_i).$$

We will usually apply this theorem with $I(\gamma, z_i) = 1$.

2.6 Dirichlet series

The sequences considered in this section are of the form $\{a_n\}_{n\geq 1}$, $a_n \in \mathbb{C}$; i.e. $\{a_n\}_{n\geq 1}$ is an arithmetical function.

Definition 2.6.1 (Dirichlet Series.) *The* Dirichlet series *associated to $\{a_n\}_{n\geq 1}$, $a_n \in \mathbb{C}$ is*

$$A(s) = \sum_{n=1} \frac{a_n}{n^s}.$$

The function $A(s)$ is known as the Dirichlet generating function *of $\{a_n\}_{n\geq 1}$. The generalized Dirichlet series (A, Λ), with Λ as in Definition 2.0.3 and $A = \{a_n\}_{n\geq 1}$ is given by*

$$\sum_{n=1} a_n e^{-\lambda_n s}.$$

We will be concerned mostly with the first kind.

Theorem 2.6.1 (*Abscissae of convergence.*) *If $\sum |a_n n^{-s}|$ does not converge for all s or diverge for all s, then there exists $\sigma_a \in \mathbb{R}$ such that $A(s) = \sum a_n n^{-s}$ converges absolutely if $\sigma > \sigma_a$ but does not converge absolutely if $\sigma < \sigma_a$; σ_a is the* abscissa of absolute convergence *of $A(s)$.*

If $A(s) \sum a_n n^{-s}$ does not converge everywhere or diverge everywhere, then there exists $\sigma_c \in \mathbb{R}$ such that $A(s)$ converges if $\sigma > \sigma_c$ but does not converge if $\sigma < \sigma_c$; σ_c is the abscissa of convergence *of $A(s)$.*

The following remarkable theorem is due to S. Mandelbrojt.

Theorem 2.6.2 *Let Λ be of positive step h; let D be its upper density. Let $f(s) = (A, \Lambda)$ with σ_a finite. For $\alpha > 0$, $\beta \geq 0$ there exists a continuous function $A(\alpha, \beta)$ with $A(\alpha, 0) = 0$ such that for all $t_0 \in \mathbb{R}$ $f(s)$ has a singular point in the rectangle $\{\sigma + it \mid \sigma_a - A(h, D) \leq \sigma \leq \sigma_a, |t - t_0| \leq \pi D\}$. One such function $A(\alpha, \beta)$ is*

$$A(\alpha,\beta) = \begin{cases} \pi\beta - \left(3\log(\alpha\beta) - \frac{17}{2}\right)\beta & \textit{when } \beta > 0 \\ 0 & \textit{when } \beta = 0. \end{cases}$$

Example. For the Dirichlet series $\sum 2^{-ks}$, $\sigma_a = 0$, $h = \log 2$ and with $D = 1/\log 2$ the height of the rectangle, vertically centered at t_0, becomes $2\pi/\log 2$. Indeed $f(s) = -1 + 2^s/(2^s - 1)$ is meromorphic in all of $\mathbb{C}$ with poles at $2\pi ik/\log 2$, $k \in \mathbb{Z}$.

Theorem 2.6.2 has the following immediate corollary.

Corollary 2.6.1 (Theorem of Fabry-Pólya.) *If $h > 0$, $D = 0$ and σ_a is finite, then every point on the abscissa of absolute convergence is a singular point of f; i.e. σ_a is a* natural boundary *of f.*

Example. The Dirichlet series $A(s) = \sum 1/(k+1)!^s$ has $D = 0$ and $h = +\infty$, hence the line $\sigma = 0$ is a natural boundary of $f(s)$, and $f(s)$ has no analytic continuation into the left half-plane.

The following theorem lists analytic properties of Dirichlet series.

Theorem 2.6.3 *The Dirichlet series*

$$A(s) = \sum_{n=1} \frac{a_n}{n^s}$$

is analytic in its half-plane of convergence $\sigma > \sigma_c$. Its derivative $A'(s)$ is represented in this half-plane by the series

$$A'(s) = \sum_{n=1} \frac{a_n \log n}{n^s}.$$

$A'(s)$ has the same abscissa of convergence σ_c and abscissa of absolute convergence σ_a as $A(s)$.

2.6.1 The Riemann and Hurwitz ζ functions

Definition 2.6.2 *The* **Hurwitz zeta function** *is given by*

$$\zeta(s,a) = \sum_{n=0} \frac{1}{(n+a)^s}.$$

The **Riemann zeta function** *is the function*

$$\zeta(s) = \zeta(s,1) = \sum \frac{1}{n^s}.$$

Both series define analytic functions for $\sigma > 1$.

The Riemann ζ function is probably the most famous of all Dirichlet series. This is because it can be used to study the distribution of primes. The relation between the Riemann ζ function and the sequence of primes will be explained in the next section. The following theorems list various properties of the ζ function.

Theorem 2.6.4 *The equality*

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_0^\infty \left(x^{s/2-1} + x^{-s/2-1/2}\right)\omega(x)dx,$$

where $\omega(x) = \sum e^{-\pi n^2 x}$, holds for $\sigma > 1$.

The term

$$\frac{1}{s(s-1)} + \int_0^\infty \left(x^{s/2-1} + x^{-s/2-1/2}\right)\omega(x)dx$$

is meromorphic and provides the analytic continuation of $\zeta(s)$ to all of $\mathbb{C}$.

Theorem 2.6.5 *The Riemann ζ function is meromorphic with a single pole at $s = 1$. This pole is simple and* $\operatorname{Res}(\zeta(s); s = 1) = 1$. *Riemann's functional equation holds:*

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

We list some special values of $\zeta(s,a)$.

Theorem 2.6.6 *We have*

$$\zeta(0,a) = \frac{1}{2} - a \quad \textit{and} \quad \zeta'(0,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi)$$

where $\Gamma(a)$ is the Euler gamma function.

The Bernoulli polynomials, defined below, are used to compute the values of $\zeta(-m, a)$ where $m \in \mathbb{N}$.

Definition 2.6.3 *The set of* **Bernoulli polynomials** $\{B_n(x)\}$ *is defined by the following relation.*

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

The first four Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

Theorem 2.6.7 *Let* $m \in \mathbb{N}$.

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}$$

We recall some growth properties of $\zeta(s, a)$.

Theorem 2.6.8 (Whittaker-Watson.) *Let* $s = \sigma + it$ *and* $\delta \in (0, 1/2)$. *The following set of relations describes the growth of* $\zeta(s, a)$ *in* $\langle -\delta, \infty \rangle$.

$$\zeta(s, a) \in \begin{cases} \mathcal{O}\left(|t|^{1/2} \log |t|\right) & \text{if } s \in \langle -\delta, \delta \rangle \\ \mathcal{O}\left(|t|^{1/2}\right) & \text{if } s \in \langle \delta, 1 - \delta \rangle \\ \mathcal{O}\left(|t|^{1-\sigma} \log |t|\right) & \text{if } s \in \langle 1 - \delta, 1 + \delta \rangle \\ \mathcal{O}(1) & \text{if } s \in \langle 1 + \delta, \infty \rangle \end{cases}$$

2.7 The analytic version of the fundamental theorem of arithmetic

Definition 2.7.1 (Dirichlet product.) *Let*

$$c_n = \sum_{t|n} a_t b_{n/t};$$

$\{c_n\}_{n \geq 1}$ *is the Dirichlet product of* $\{a_n\}_{n \geq 1}$ *and* $\{b_n\}_{n \geq 1}$ *and is written*

$$c_n = a_n * b_n.$$

This product is also referred to as the Dirichlet convolution of $\{a_n\}_{n \geq 1}$ *and* $\{b_n\}_{n \geq 1}$.

Theorem 2.7.1 (Dirichlet products and Dirichlet series.) *Let*

$$A(s) = \sum \frac{a_n}{n^s}, \text{ and } B(s) = \sum \frac{b_n}{n^s}$$

with abscissae of absolute convergence a *and* b. *Let* $c_n = a_n * b_n$. *Then*

$$C(s) = \sum \frac{c_n}{n^s}$$

converges absolutely with abscissa $c = \max\{a, b\}$ *and*

$$C(s) = A(s)B(s).$$

The proof of this theorem uses the following computation.

$$\sum \frac{c_n}{n^s} = \sum \frac{1}{n^s} a_n * b_n = \sum \frac{1}{n^s} \sum_{t|n} a_t b_{n/t} = \sum \frac{1}{n^s} \sum_{kl=n} a_k b_l = \sum_{k \geq 1} \sum_{l \geq 1} \frac{a_k b_l}{(kl)^s}.$$

Definition 2.7.2 (Multiplicative and completely multiplicative arithmetical functions.) *An arithmetical function* $\{a_n\}_{n\geq 1}$ *is* multiplicative *if* $a_{n_1 n_2} = a_{n_1} a_{n_2}$ *when* $(n_1, n_2) = 1$; $\{a_n\}_{n \geq 1}$ *is* completely multiplicative *if* $a_{n_1 n_2} = a_{n_1} a_{n_2}$ *for all* n_1, n_2.

Note that this definition implies $a_1 = 1$ unless $\{a_n\}_{n\geq 1}$ vanishes everywhere; $\exists n : a_n \neq 0$ and hence $a_n = a_1 a_n$ or $a_1 = 1$.

Theorem 2.7.2 (Analytic version of the fundamental theorem of arithmetic.) *Let* $\{a_n\}_{n\geq 1}$ *be multiplicative; let* $A(s)$ *be its Dirichlet series with abscissa of absolute convergence* a. *We have*

$$\sum \frac{a_n}{n^s} = \prod_p \sum_{v=0} \frac{a_{p^v}}{p^{vs}}$$

when $\sigma > a$. *If* $\{a_n\}_{n\geq 1}$ *is completely multiplicative, this simplifies to*

$$\sum \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s}}.$$

The term on the right is known as the Euler product *of* $\{a_n\}_{n\geq 1}$.

Proof. Let

$$A_p(s) = \sum_{v=0}^{\infty} \frac{a_{p^v}}{p^{vs}}$$

and consider

$$\prod_{p \le p_r} A_p(s) = \sum_{v_1=0}^{\infty} \cdots \sum_{v_r=0}^{\infty} \frac{a_{p^{v_1}} \cdots a_{p^{v_r}}}{p^{v_1 s} \cdots p^{v_r s}} = \sum_r \frac{a_n}{n^s}$$

where the product ranges over the first r primes and the sum includes those n with prime factors $\le p_r$. We have

$$\lim_{r\to\infty} \left| \sum \frac{a_n}{n^s} - \sum_r \frac{a_n}{n^s} \right| = \lim_{r\to\infty} \left| \sum_{\exists \rho > r : p_\rho | n} \frac{a_n}{n^s} \right| < \lim_{r\to\infty} \left| \sum_{n=p_r+1} \frac{a_n}{n^s} \right| \le \lim_{r\to\infty} \sum_{n=p_r+1} \frac{|a_n|}{n^s} = 0.$$

We have used the absolute convergence of $A(s)$ in the last step. We conclude that

$$\lim_{r\to\infty} \sum_r \frac{a_n}{n^s} = \sum \frac{a_n}{n^s}.$$

It remains to verify that $\prod_{p \le p_r} A_p(s)$ converges. Recall that $1 + x < e^x = 1 + x + \sum_{n=2} \frac{x^n}{n!}$ and hence $\log(1+x) < x$ for $x \in \mathbb{R}, x > -1$. We use this inequality to obtain

$$\left| \log \prod_{p \le p_r} A_p(s) \right| \le \sum_{p \le p_r} |\log A_p(s)| = \sum_{p \le p_r} \left| \log \left(1 + \sum_{v=1}^{\infty} \frac{a_{p^v}}{p^{vs}} \right) \right| \le \sum_{p \le p_r} \sum_{v=1}^{\infty} \frac{|a_{p^v}|}{p^{vs}}.$$

All the partial sums are bounded and hence the series on the left and the product both converge. We have

$$\sum \frac{a_n}{n^s} = \prod_p A_p(s).$$

This is the desired result. ∎

We thus have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

This relation was already known to Euler.

2.7.1 Some useful Dirichlet generating functions

The examples in this section have been selected to illustrate Theorem 2.7.2, and demonstrate additional techniques for the evaluation of Dirichlet generating functions. We will use them later, when we evaluate digital sums.

Definition 2.7.3 *Let $\{\kappa(j)\}_{j\geq 0}$ be a sequence of positive integers such that $\kappa(0)=1$ and $\kappa(j)\mid\kappa(j+1)$, $\kappa(j)<\kappa(j+1)$, for $j\geq 0$. Then the function $v_\kappa:\mathbb{Z}^+\mapsto\mathbb{N}$ is defined as*

$$v_\kappa(n)=\max\{j \mid j\in\mathbb{N},\kappa(j)\mid n\}.$$

The special case $\kappa(j)=q^j$ where $q\geq 2$, is denoted by $v_q(n)$;

$$v_q(n)=\max\{v \mid v\in\mathbb{N},\ q^v\mid n\}.$$

The function $v_q(n)$ gives the highest power of q that divides n.

Definition 2.7.4 *The function $\kappa^{-1}(k)$ is defined to be an integer-valued inverse of $\kappa(j)$.*

$$\kappa^{-1}(k)=j \Leftrightarrow \kappa(j-1)<k\leq\kappa(j)$$

When $\kappa(j)=q^j$ where $q\geq 2$,

$$\kappa^{-1}(k)=\lceil\log_q k\rceil.$$

We evaluate several Dirichlet generating functions that contain v_κ and v_q.

Example. By definition of $v_q(n)$ the function $(-1)^{v_q(n)}$ is completely multiplicative when q is prime, and multiplicative when q is a prime power. We can evaluate the Dirichlet generating function of $(-1)^{v_q(n)}$ by Theorem 2.7.2 when q is prime. We have

$$\begin{aligned}\sum_{n\geq 1}\frac{(-1)^{v_q(n)}}{n^s} &= \prod_p\frac{1}{1-(-1)^{v_q(p)}p^{-s}}=\frac{1}{1+q^{-s}}\prod_{p\neq q}\frac{1}{1-p^{-s}}\\ &= \frac{1-q^{-s}}{1+q^{-s}}\prod_p\frac{1}{1-p^{-s}}=\zeta(s)\frac{q^s-1}{q^s+1}=\zeta(s)\left(1-2\frac{1}{q^s+1}\right).\end{aligned}$$

In fact this result holds for composite q as well.

$$\begin{aligned}\sum_{n\geq 1}\frac{(-1)^{v_q(n)}}{n^s} &= \sum_{k\geq 1}\frac{(-1)^{v_q(kq)}}{(kq)^s}+\sum_{r=1}^{q-1}\sum_{k\geq 0}\frac{(-1)^{v_q(kq+r)}}{(kq+r)^s}=\sum_{k\geq 1}\frac{1}{q^s}\frac{(-1)^{v_q(k)+1}}{k^s}+\sum_{r=1}^{q-1}\sum_{k\geq 0}\frac{1}{(kq+r)^s}\\ &= -\frac{1}{q^s}\sum_{k\geq 1}\frac{(-1)^{v_q(k)}}{k^s}+\sum_{k\geq 1}\frac{1}{k^s}-\sum_{k\geq 1}\frac{1}{(kq)^s}=-\frac{1}{q^s}\sum_{k\geq 1}\frac{(-1)^{v_q(k)}}{k^s}+\zeta(s)\left(1-\frac{1}{q^s}\right)\end{aligned}$$

Further manipulation yields

$$\begin{aligned}\left(1+\frac{1}{q^s}\right)\sum_{k\geq 1}\frac{(-1)^{v_q(k)}}{k^s} &= \zeta(s)\left(1-\frac{1}{q^s}\right)\\ \sum_{k\geq 1}\frac{(-1)^{v_q(k)}}{k^s} &= =\zeta(s)\frac{q^s-1}{q^s+1}=\zeta(s)\left(1-2\frac{1}{q^s+1}\right).\end{aligned}$$

Example. We consider the Dirichlet generating function of the following term:

$$v_q(n) \bmod 2.$$

This function can be evaluated in two ways. The first of these uses the relation

$$v_q(n) \bmod 2 = \frac{1}{2}\left(1-(-1)^{v_q(n)}\right).$$

It follows that

$$\sum_{v_q(n)\equiv 1(2)} \frac{1}{n^s} = \frac{1}{2}\zeta(s) - \frac{1}{2}\zeta(s)\left(1-2\frac{1}{q^s+1}\right) = \zeta(s)\frac{1}{q^s+1}.$$

The second approach is more general; it proceeds directly from the definition of $v_q(n)$ mod 2.

$$\begin{aligned}
\sum_{v_q(n)\equiv 1(2)} \frac{1}{n^s} &= \sum_{k=0} \sum_{v_q(n)=2k+1} \frac{1}{n^s} = \sum_{k=0}\sum_{q\nmid m} \frac{1}{(q^{2k+1}m)^s} \\
&= \frac{1}{q^s}\sum_{k=0}\frac{1}{q^{2ks}}\sum_{q\nmid m}\frac{1}{m^s} = \frac{1}{q^s}\left(\sum_{m}\frac{1}{m^s} - \sum_{q|m}\frac{1}{m^s}\right)\sum_{k=0}\frac{1}{q^{2ks}} \\
&= \frac{1}{q^s}\frac{1}{1-q^{-2s}}\zeta(s)\left(1-\frac{1}{q^s}\right) = \zeta(s)\frac{q^s-1}{(1-q^{-2s})q^{2s}} = \zeta(s)\frac{1}{q^s+1}.
\end{aligned}$$

The half-plane of convergence of the last two functions is obtained by a trivial comparison with $\zeta(s)$; hence the respective computations hold for $\sigma > 1$.

Lemma 2.7.1 *Let $\{\kappa(j)\}_{j\geq 0}$ be a sequence as in Definition 2.7.3. Let $m \geq 2$ and $0 \leq r < m$, $m, r \in \mathbb{Z}^+$. Then*

$$\sum_{v_\kappa(n)\equiv r(m)} \frac{1}{n^s} = \zeta(s)\sum_{k=0}\left(\frac{1}{\kappa(mk+r)^s} - \frac{1}{\kappa(mk+r+1)^s}\right)$$

with $\sigma > 1$. When $\kappa(j) = q^j$, this simplifies to

$$\zeta(s)q^{(m-r-1)s}\prod_{v=1}^{m-1}(q^s-\omega_m^v)^{-1},$$

where $\omega_m = e^{2\pi i/m}$ is the mth primitive root of unity.

Proof. The method used in the previous example ($m = 2$ and $r = 1$) can be used in the general case as well.

$$\sum_{v_\kappa(n)\equiv r(m)} \frac{1}{n^s} = \sum_{k=0}\sum_{v_\kappa(n)=mk+r}\frac{1}{n^s} = \sum_{k=0}\sum_{\frac{\kappa(mk+r+1)}{\kappa(mk+r)}\nmid l}\frac{1}{(\kappa(mk+r)l)^s}$$

$$= \sum_{k=0}\frac{1}{\kappa(mk+r)^s}\left(\sum - \sum_{\frac{\kappa(mk+r+1)}{\kappa(mk+r)}\mid l}\right)\frac{1}{l^s} = \zeta(s)\sum_{k=0}\frac{1}{\kappa(mk+r)^s}\left(1-\frac{\kappa(mk+r)^s}{\kappa(mk+r+1)^s}\right)$$

The special case $\kappa(j) = q^j$ gives $\kappa(mk+r) = (q^m)^k q^r$ and hence

$$\sum_{k=0} \frac{1}{\kappa(mk+r)^s} = \frac{1}{q^{rs}} \sum_{k=0} \frac{1}{(q^{ms})^k} = \frac{1}{q^{rs}} \frac{q^{ms}}{q^{ms}-1}.$$

With

$$q^{ms} - 1 = \prod_{v=0}^{m-1} (q^s - \omega_m^v) \quad \text{and} \quad \frac{1}{q^{rs}} - \frac{1}{q^{(r+1)s}} = \frac{q^s-1}{q^{(r+1)s}}$$

we have the result. ∎

Example. The third and last example in this series is the Dirichlet generating function of $v_q(n)$ itself. The computation is straightforward.

$$\sum \frac{v_q(n)}{n^s} = \sum_{q|n} \frac{v_q(n)}{n^s} = \sum \frac{v_q(qm)}{(qm)^s} = \frac{1}{q^s} \sum \frac{1+v_q(m)}{m^s} = \frac{1}{q^s} \left(\zeta(s) + \sum \frac{v_q(m)}{m^s} \right).$$

This gives

$$\sum \frac{v_q(n)}{n^s} = \left(1 - \frac{1}{q^s}\right)^{-1} \frac{\zeta(s)}{q^s} = \frac{\zeta(s)}{q^s-1}.$$

The previous example is a special case of the following lemma.

Lemma 2.7.2 *Let $\{\kappa(j)\}_{j \geq 0}$ be a sequence as in Definition 2.7.3 and consider a function $t : \mathbb{N} \mapsto \mathbb{C}$. Then*

$$\sum \frac{1}{n^s} \sum_{j=1}^{v_\kappa(n)} t(j) = \zeta(s) \sum_{j=1}^{\infty} \frac{t(j)}{\kappa(j)^s};$$

in particular,

$$\sum \frac{v_\kappa(n)}{n^s} = \zeta(s) \sum_{j=1}^{\infty} \frac{1}{\kappa(j)^s} \quad \text{and} \quad \sum \frac{v_\kappa(n)(v_\kappa(n)+1)}{2n^s} = \zeta(s) \sum_{j=1}^{\infty} \frac{j}{\kappa(j)^s}$$

with $\sigma > 1$.

Proof. It is an instructive exercise to adapt the technique employed in the evaluation of $\sum \frac{v_q(n)}{n^s}$ to the above lemma; indeed this yields a proof. We will use a restricted Dirichlet convolution to establish the result.

$$\sum \frac{1}{n^s} \sum_{j=1}^{v_\kappa(n)} t(j) = \sum \frac{1}{n^s} \sum_{\kappa(j)|n, j>0} t(j) = \sum \sum_{n=m\kappa(j), j>0} \frac{t(j)}{m^s \kappa(j)^s} = \sum_m \sum_{j=1}^{\infty} \frac{t(j)}{m^s \kappa(j)^s} = \zeta(s) \sum_{j=1}^{\infty} \frac{t(j)}{\kappa(j)^s}$$

The key step is the use of $\sum_{n=m\kappa(j), j>0}$. This is a Dirichlet convolution with the divisors restricted to $\{\kappa(j)\}_{j>0}$. The two particular instances are obtained from $v_\kappa(n) = \sum_{j=1}^{v_\kappa(n)} 1$ and $\frac{1}{2} v_\kappa(n)(v_\kappa(n)+1) = \sum_{j=1}^{v_\kappa(n)} j$. We need to verify that the latter two converge in $\sigma > 1$. By definition of $v_\kappa(n)$, $v_\kappa(n) \leq \log_2 n$. Hence $\sigma > 2$ and $\sigma > 3$ would suffice. In fact Theorem 2.6.3 shows that $\sigma > 1$ for both series. ∎

2.8 Integrals of the Hurwitz ζ-function

The integrals of the generalized ζ-function that we will encounter in the remainder of this thesis are all of a similar type. This section provides a *shifting lemma* that makes it possible to evaluate those integrals.

We will use the growth estimates of Theorem 2.6.8 for $\zeta(s,a)$ to prove the following lemma.

Lemma 2.8.1 (Shifting lemma.) *Let $\alpha \in (-1,0)$ and let $c > 1$. Let $\{T_j\}_{j\geq 1} \subset \mathbb{R}^+$ be an increasing sequence of real numbers such that $\lim_{j\to\infty} T_j = \infty$ and $T_j > T_0$, T_0 a fixed constant. Suppose $\Phi(s)$ is a function that is analytic on all $R_j = \langle \alpha, c\rangle \cap \{s = \sigma + it \mid |t| \leq T_j\}$ except for a finite number of singularities, and that there exists a constant $M \in \mathbb{R}^+$ such that $|\Phi(s)| < M$ independently of j on the boundary δR_j of R_j. Let $S' = \mathrm{Sing}\,(\Phi(s)\zeta(s,a)/(s(s+1))) \cap \langle \alpha, c\rangle$. Under these conditions*

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \Phi(s)\frac{\zeta(s,a)}{s(s+1)}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)\frac{\zeta(s,a)}{s(s+1)}ds - \sum_{\varsigma\in S'} \mathrm{Res}\left(\Phi(s)\frac{\zeta(s,a)}{s(s+1)}; s = \varsigma\right)$$

This is a shifting lemma because it tells us that the integral along a line parallel to the imaginary axis and situated at $c > 1$ maybe shifted to $\alpha \in (-1,0)$, taking into account the residues of the integrand.

Proof. We use the following rectangular contour, where $T = T_j > T_0$, and the contour is traversed counterclockwise.

$$\begin{array}{ll} \Gamma_1 = \{c + it \mid |t| \leq T\} & \Gamma_2 = \{\sigma + iT \mid \alpha \leq \sigma \leq c\} \\ \Gamma_3 = \{\alpha + it \mid |t| \leq T\} & \Gamma_4 = \{\sigma - iT \mid \alpha \leq \sigma \leq c\} \end{array}$$

We apply the Cauchy residue theorem:

$$\frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4} \frac{\Phi(s)\zeta(s,a)}{s(s+1)}ds = \sum_{\varsigma\in S'} \mathrm{Res}\left(\frac{\Phi(s)\zeta(s,a)}{s(s+1)}; s = \varsigma\right)$$

or

$$\begin{aligned} \frac{1}{2\pi i}\int_{-\Gamma_3} \frac{\Phi(s)\zeta(s,a)}{s(s+1)}ds \quad &= \quad \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\Phi(s)\zeta(s,a)}{s(s+1)}ds + \frac{1}{2\pi i}\int_{\Gamma_2\cup\Gamma_4} \frac{\Phi(s)\zeta(s,a)}{s(s+1)}ds \\ &\quad - \sum_{\varsigma\in S'} \mathrm{Res}\left(\frac{\Phi(s)\zeta(s,a)}{s(s+1)}; s = \varsigma\right). \end{aligned}$$

We have the result if we can show that (a) the integrals along Γ_1 and Γ_3 converge and (b) the integrals along Γ_2 and Γ_4 vanish as $T = T_j \to \infty$.

We verify (a) with a comparison test. Let $s = c + it$, $ds = idt$.

$$\begin{aligned}
\left|\int_{c-iT}^{c+iT} \frac{\Phi(s)\zeta(s,a)}{s(s+1)} ds\right| &\leq \left(\int_{-T}^{-T_0} + \int_{-T_0}^{T_0} + \int_{T_0}^{T}\right) \left|\frac{\Phi(c+it)\zeta(c+it)}{s(s+1)} i\right| dt \\
&< C + c_4 M \left(\int_{-T}^{-T_0} + \int_{T_0}^{T}\right) \frac{1}{\sqrt{c^2+t^2}\sqrt{(c+1)^2+t^2}} dt \\
&< C + 2c_4 M \int_{T_0}^{T} \frac{1}{t^2} dt = C + 2c_4 M \left[\frac{1}{t}\right]_T^{T_0} = C_1 - \frac{2c_4 M}{T}
\end{aligned}$$

The case for Γ_3 is similar. This time we use the first rather than the fourth case of Theorem 2.6.8. Let $s = \alpha + it$, $ds = idt$.

$$\begin{aligned}
\left|\int_{\alpha-iT}^{\alpha+iT} \frac{\Phi(s)\zeta(s,a)}{s(s+1)} ds\right| &\leq \left(\int_{-T}^{-T_0} + \int_{-T_0}^{T_0} + \int_{T_0}^{T}\right) \left|\frac{\Phi(\alpha+it)\zeta(\alpha+it)}{s(s+1)} i\right| dt \\
&< C + c_1 M \left(\int_{-T}^{-T_0} + \int_{T_0}^{T}\right) \frac{|t|^{1/2}\log|t|}{\sqrt{\alpha^2+t^2}\sqrt{(\alpha+1)^2+t^2}} dt \\
&< C + 2c_1 M \int_{T_0}^{T} \frac{t^{1/2}\log t}{t^2} dt
\end{aligned}$$

The last integral converges, as does, therefore, the integral along Γ_3.

In order to establish (b), we consider integrals along the four types of intervals listed in Theorem 2.6.8. Take $s = \sigma \pm iT$, $ds = d\sigma$, $\sigma \in [1-\delta, 1+\delta]$, $\delta \in (0, 1/2)$. (This is the only one of the four types we will treat; the other three are can be estimated in the same way.) We have

$$\begin{aligned}
\left|\int_{1-\delta\pm iT}^{1+\delta\pm iT} \frac{\Phi(s)\zeta(s,a)}{s(s+1)} ds\right| &= \left|\int_{1-\delta}^{1+\delta} \frac{\Phi(\sigma\pm iT)\zeta(\sigma\pm iT,a)}{s(s+1)} d\sigma\right| \\
&\leq \frac{M}{T^2}\int_{1-\delta}^{1+\delta} |\zeta(\sigma\pm iT)| d\sigma \leq \frac{c_3 M}{T^2}\int_{1-\delta}^{1+\delta} |\pm T|^{1-\sigma}\log|\pm T| d\sigma \\
&< \frac{c_3 M\log T}{T^2}\left[\frac{1}{\log T} T^{1-\sigma}\right]_{1+\delta}^{1-\delta} = \frac{c_3 M\left(T^{\delta} - T^{-\delta}\right)}{T^2}.
\end{aligned}$$

Hence the integrals along $\Gamma_{2,4} \cap \langle 1-\delta, 1+\delta\rangle$ vanish as $T \to \infty$; the same estimate works for the remaining three types. Clearly there always exists an appropriate decomposition of $[\alpha, c]$ into a sequence of intervals that can be treated as above, e.g. $[-1/4, 3/2] = [-1/4, 1/4]\,[1/4, 1-1/4]\,[1-1/4, 1+1/4]\,[1+1/4, 3/2]$. This shows that the integrals along the horizontal segments Γ_2 and Γ_4 vanish as claimed, and concludes the proof of the lemma. ∎

There is a straightforward generalization of this lemma.

Lemma 2.8.2 (Generalized shifting lemma.) *Let* $m \in \mathbb{Z}^+$ *and* $a_1, a_2 \ldots a_n \in (0,1]$. *Let* $\alpha \in (-1,0)$ *and let* $c > 1$. *Let* $\{T_j\}_{j\geq 1} \subset \mathbb{R}^+$ *be an increasing sequence of real numbers such that* $\lim_{j\to\infty} T_j = \infty$ *and* $T_j > T_0$, T_0 *a fixed constant. Suppose* $\Phi(s)$ *is a function that is analytic on all* $R_j = \langle \alpha, c\rangle \cap \{s = \sigma + it \mid |t| \leq T_j\}$ *except for a finite number of singularities, and that there exists a constant* $M \in \mathbb{R}^+$ *such that* $|\Phi(s)| < M$ *independently of* j *on the boundary* δR_j *of* R_j. *Let* $S' = \operatorname{Sing}\left(\Phi(s)\prod_{n=1}^m \zeta(s,a_n)/(s(s+1)\ldots(s+m))\right) \cap \langle\alpha, c\rangle$. *Under these conditions*

$$\begin{aligned}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Phi(s)\frac{\prod_{n=1}^m \zeta(s,a_n)}{s(s+1)\ldots(s+m)}ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Phi(s)\frac{\prod_{n=1}^m \zeta(s,a_n)}{s(s+1)\ldots(s+m)}ds \\ &- \sum_{\varsigma\in S'}\operatorname{Res}\left(\Phi(s)\frac{\prod_{n=1}^m \zeta(s,a)}{s(s+1)\ldots(s+m)}; s = \varsigma\right)\end{aligned}$$

Proof. The argument is the same as in the special case $m = 1$. We need to verify that the integrals along $\Gamma_{1,3}$ and $\Gamma_{2,4}$ converge and vanish, respectively. This is immediate when we consider that

- for Γ_1, we have the bound
$$C_1 - \frac{2c_4 M}{T^m},$$
- for Γ_3, the bound
$$C + 2c_1 M\int_{T_0}^{T}\frac{(\log t)^m}{t^{m/2+1}}dt,$$
which converges,
- and for $\Gamma_{2,4}$, the bound
$$\frac{c_3 M(\log T)^{m-1}\left(T^{m\delta} - T^{-m\delta}\right)}{T^{m+1}},$$
again for the type $[1-\delta, 1+\delta]$. ∎

2.9 The Mellin-Perron formula

This section contains the definitions and lemmata that are required to state the Mellin-Perron formula and define its domain of application. We will sketch the proof of the Mellin-Perron formula, as its use often requires a more than superficial appreciation of the method.

2.9.1 The Mellin transform

Definition 2.9.1 (Open strip.) *The* open strip *of complex numbers* $\langle\alpha,\beta\rangle$ *is the set* $\{s = \sigma + it \mid \alpha < \sigma < \beta\}$.

Definition 2.9.2 (Mellin transform.) *Let $f(x)$ be locally Lebesgue integrable over $(0, +\infty)$. The* Mellin transform *of $f(x)$ is defined by*

$$\mathfrak{M}\left[f(x); s\right] = f^*(s) = \int_0^{+\infty} f(x) x^{s-1} dx.$$

The fundamental strip *is the largest open strip where the integral converges.*

Lemma 2.9.1 *The conditions*

$$f(x)_{x \to 0+} \in \mathcal{O}\left(x^u\right), \qquad f(x)_{x \to +\infty} \in \mathcal{O}\left(x^v\right),$$

when $u > v$, guarantee that $f^*(x)$ exists in the strip $\langle -u, -v \rangle$.

We apply this lemma to a family of Heaviside-like step functions.

Definition 2.9.3 *Let*

$$H_0(x) = \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{if } x > 1 \end{cases}$$

be defined on $[0, +\infty)$ and let

$$H_m(x) = (1 - x)^m H_0(x) \qquad \text{when } m \in \mathbb{Z}^+.$$

Note that $H_0(x)$ has a discontinuity at $x = 1$; we have $\lim_{x \to 1-} H_0(x) = 1$ and $\lim_{x \to 1+} H_0(x) = 0$. Note also that $\lim_{x \to 1-} H_m(x) = \lim_{x \to 1+} H_m(x) = 0$ when $m \in \mathbb{Z}^+$; $H_m(x)$ is continuous at $x = 1$.

Lemma 2.9.2 *The Mellin transform $H_m^*(x)$ of $H_m(x)$, where $m \in \mathbb{N}$, exists in $\langle 0, +\infty \rangle$ and is given by*

$$H_m^*(x) = \frac{m!}{s(s+1)\ldots(s+m)}.$$

We have $H_m(x)_{x \to 0+} \in \mathcal{O}(1)$ and $H_m(x)_{x \to +\infty} \in \mathcal{O}\left(x^{-b}\right)$ for any $b > 0$ and for $m \in \mathbb{N}$, hence $H_m^*(x)$ exists in $\langle 0, +\infty \rangle$. Note that

$$H_0^*(x) = \int_0^1 x^{s-1} dx = \frac{1}{s}\left[x^s\right]_0^1 = \frac{1}{s}.$$

We also have

$$\begin{aligned} H_m^*(s) &= \int_0^1 H_m(x)x^{s-1}dx \\ &= \int_0^1 H_{m-1}(x)x^{s-1}dx - \int_0^1 H_{m-1}(x)x^s dx \\ &= H_{m-1}^*(x) - \int_0^1 \frac{(1-x)^m}{m} s x^{s-1}dx \\ &= H_{m-1}^*(x) - \frac{s}{m}H_m^*(x). \end{aligned}$$

This gives

$$H_m^*(x) = \frac{m}{s+m}H_{m-1}^*(x)$$

for $m \in \mathbb{Z}^+$ and the lemma follows. ▮

We will be concerned with the linarity and the rescaling property of the Mellin transform.

Theorem 2.9.1 (Linearity and rescaling.) *Let $\mathcal{K} \subset \mathbb{Z}$ be a finite set of integers; let $\mu_k, \lambda_k \in \mathbb{R}^+$. Let the fundamental strip of $\mathfrak{M}[f(x); s]$ be $\langle \alpha, \beta \rangle$. We have*

$$\mathfrak{M}\left[\sum_k \lambda_k f(\mu_k x); s\right] = \left(\sum_k \frac{\lambda_k}{\mu_k^s}\right)\mathfrak{M}[f(x); s],$$

where $s \in \langle \alpha, \beta \rangle$.

Let $y = \mu_k x$ and $dy = \mu_k dx$. Note that

$$\begin{aligned} \int_0^\infty \left(\sum_k \lambda_k f(\mu_k x)\right) x^{s-1}dx &= \sum_k \lambda_k \int_0^\infty f(\mu_k x)x^{s-1}dx \\ &= \sum_k \lambda_k \int_0^\infty f(y)y^{s-1}\frac{dy}{\mu_k^s} = \left(\sum_k \frac{\lambda_k}{\mu_k^s}\right) f^*(s). \end{aligned}$$

We were able to exchange the integral with the summation because $\mathcal{K}$ is finite. It can be shown that this operation extends to infinite $\mathcal{K}$ as long as $\sum_k \lambda_k/\mu_k^s$ converges absolutely. The extended property holds in the intersection of the half-plane of convergence of $\sum_k \lambda_k/\mu_k^s$ and the fundamental strip $\langle \alpha, \beta \rangle$ of $f(x)$.

Definition 2.9.4 (Inverse Mellin transform.)

1. (Lebesgue integration.)

   Let $f(x)$ be integrable with fundamental strip $\langle \alpha, \beta \rangle$. If $c \in (\alpha, \beta)$ and $f^*(c+it)$ is integrable, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds = f(x)$$

   almost everywhere. If $f(x)$ is continuous, the equality holds everwhere on $(0, +\infty)$.

2. (Riemann integration.)

Let $f(x)$ be locally integrable with fundamental strip $\langle\alpha,\beta\rangle$ and be of bounded variation in a neighborhood of x_0. Then

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f^*(s)x^{-s}\,ds\bigg|_{x_0} = \frac{f(x_0^+)+f(x_0^-)}{2}$$

for $c\in(\alpha,\beta)$.

Of course if $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$ then

$$\frac{f(x_0^+)+f(x_0^-)}{2} = f(x_0).$$

2.9.2 The Mellin-Perron formula

Theorem 2.9.2 (Mellin-Perron formula.) *Let $c\in\mathbb{R}^+$ lie in the half-plane of absolute convergence of $\sum_k \lambda_k/k^s$. Then we have*

$$\frac{1}{m!}\sum_{1\le k<n}\lambda_k\left(1-\frac{k}{n}\right)^m = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\sum_{k\ge 1}\frac{\lambda_k}{k^s}\right)n^s\frac{ds}{s(s+1)\dots(s+m)}$$

for $m\in\mathbb{Z}^+$. We have

$$\sum_{1\le k<n}\lambda_k+\frac{\lambda_n}{2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\sum_{k\ge 1}\frac{\lambda_k}{k^s}\right)n^s\frac{ds}{s}$$

when $m=0$.

This theorem is a straightforward application of Mellin inversion.

Proof. Let $F(x)=\sum_k \lambda_k f(\mu_k x)$ and use the rescaling property to obtain

$$\mathfrak{M}\left[F(x);s\right] = F^*(s) = \left(\sum_k\frac{\lambda_k}{\mu_k^s}\right)f^*(s).$$

Consider Riemann-integrable $f(x)$ and apply the Mellin inversion formula.

$$\sum_k\lambda_k\frac{f(\mu_k x^+)+f(\mu_k x^-)}{2} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\sum_k\frac{\lambda_k}{\mu_k^s}\right)f^*(s)x^{-s}\,ds$$

Let $f(x) = H_m(x)$, $m \in \mathbb{N}$ and let $\mu_k = k$. Recall that the fundamental strip of $H_m(x)$ is $\langle 0, \infty \rangle$; let $x = 1/n$. This gives

$$\begin{aligned}
\sum_k \lambda_k \frac{f(\mu_k x^+) + f(\mu_k x^-)}{2} &= \sum_k \lambda_k \frac{H_m(\frac{k}{n^-}) + H_m(\frac{k}{n^+})}{2} \\
&= \sum_{1 \le k < n} \lambda_k \frac{\left(1 - \frac{k}{n^-}\right)^m + \left(1 - \frac{k}{n^+}\right)^m}{2} + \lambda_n \frac{H_m(1^+) + H_m(1^-)}{2} \\
&= \sum_{1 \le k < n} \lambda_k \left(1 - \frac{k}{n}\right)^m + \lambda_n \frac{H_m(1^+) + H_m(1^-)}{2}.
\end{aligned}$$

Note that

$$\lambda_n \frac{H_m(1^+) + H_m(1^-)}{2} = \begin{cases} \lambda_n/2 & \text{if } m = 0 \\ 0 & \text{if } m \in \mathbb{Z}^+. \end{cases}$$

Continuing the substitution, we have

$$\begin{aligned}
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\sum_k \frac{\lambda_k}{\mu_k^s} \right) f^*(s) x^{-s} ds &= \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\sum_k \frac{\lambda_k}{k^s} \right) \frac{m!}{s(s+1)\dots(s+m)} n^s ds \\
&= \frac{m!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_k \frac{\lambda_k}{k^s} \right) n^s \frac{ds}{s(s+1)\dots(s+m)}
\end{aligned}$$

This concludes the proof. Because the fundamental strip of $H_m(x)$ is $\langle 0, \infty \rangle$, the choice of $c > 0$ is determined by the half-plane of convergence of $\sum_k \lambda_k / k^s$ only. ∎

2.9.3 The use of the Mellin-Perron formula when $m = 1$

The lemma below summarizes the usage pattern of the Mellin-Perron formula when $m = 1$.

Lemma 2.9.3 *Let $\{a_n\}_{n \ge 1}$ be an arithmetical function; let $a_0 = 0$ and let $\{b_n\}_{n \ge 1} = \{\Delta \nabla a_n\}_{n \ge 1}$. If $B(s) = \sum b_n / n^s$ is the Dirichelet generating function of $\{b_n\}_{n \ge 1}$, we have*

$$\sum_{k=1}^{n-1} \sum_{l=1}^{k} b_l = a_n - n a_1 = \frac{n}{2\pi i} \int_{c-i\infty}^{c+i\infty} B(s) n^s \frac{ds}{s(s+1)}$$

where c is in the half-plane of convergence of $B(s)$.

To see this, note first that

$$\frac{1}{1!} \sum_{k=1}^{n-1} b_k \left(1 - \frac{k}{n}\right)^1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} B(s) n^s \frac{ds}{s(s+1)}$$

or

$$\sum_{k=1}^{n-1} b_k(n-k) = \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} B(s)n^s \frac{ds}{s(s+1)}.$$

We select $m=1$ because the iterated sum on the left cancels the operator $\triangle\nabla$.

$$\begin{aligned}\sum_{k=1}^{n-1} b_k(n-k) &= \sum_{k=1}^{n-1}\sum_{l=1}^{k} b_l = \sum_{k=1}^{n-1}\sum_{l=1}^{k}\triangle\nabla a_l = \sum_{k=1}^{n-1}\sum_{l=1}^{k}(\nabla a_{l+1}-\nabla a_l)\\ &= \sum_{k=1}^{n-1}(\nabla a_{k+1}-\nabla a_1) = \sum_{k=1}^{n-1}(a_{k+1}-a_k)-(n-1)\nabla a_1\\ &= a_n - a_1 - (n-1)a_1 = a_n - na_1\end{aligned}$$

2.10 Mellin-Perron formulae for the Hurwitz ζ-function

This section presents two Mellin-Perron formulae for the generalized ζ-function. They will be used in later chapters. We include them here because their respective derivations hardly differ from that of the standard Mellin-Perron formula.

We apply the Mellin inversion theorem to $F(x) = \sum_k \lambda_k f(\mu_k x)$ with $x = r/n$, $r, n \in \mathbb{Z}^+$, $\mu_k = k+a$, $\lambda_k = 1$, $a \in \mathbb{R}$, $a \in (0,1]$, $f(x) = H_1(x) = (1-x)H_0(x)$. As we require $\mu_k \in \mathbb{R}^+$ we take $k \in \mathbb{N}$. We have

$$F(x) = \sum_{k\in\mathbb{N}}\left(1-(k+a)\frac{r}{n}\right) H_0\left((k+a)\frac{r}{n}\right)$$

and

$$F^*(s) = \left(\sum_{k\in\mathbb{N}}\frac{1}{(k+a)^s}\right) f^*(s) = \frac{\zeta(s,a)}{s(s+1)}$$

where $\sigma > 1$. We need to evaluate $F(x)$. $H_0(x)$ vanishes outside of $[0,1)$, hence we require $0 \le (k+a)r/n < 1$ or $k < n/r - a$. Let $\mathbb{N}(u) = \{\nu < u \mid \nu \in \mathbb{N}\}$ where $u \in \mathbb{R}^+$. We have

$$F(x) = \sum_{k\in\mathbb{N}(n/r-a)}\left(1-(k+a)\frac{r}{n}\right).$$

With these settings the Mellin inversion formula yields the following theorem.

Theorem 2.10.1 *Let $c > 1$.*

$$\sum_{k\in\mathbb{N}(n/r-a)}\left(1-(k+a)\frac{r}{n}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds$$

This theorem has several useful corollaries. The first of these is obtained by setting $r=1$. Let $\alpha \in (-1,0)$.

Corollary 2.10.1 *Let $n \in \mathbb{N}$.*

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{s(s+1)}ds = 0$$

Let $c = 1$. The set of poles of $\zeta(s,a)n^s/(s(s+1))$ in $\langle \alpha, c\rangle$ is $\{1,0\}$. We apply the shifting lemma with $\Phi(s) = n^s$ and $T_j = j$. Because $|n^s| = n^\sigma$ we can take $M = n^c$.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{s(s+1)}ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s,a)\frac{n^s}{s(s+1)} \\
&\quad -\mathrm{Res}\left(\zeta(s,a)\frac{n^s}{s(s+1)}; s=1\right) - \mathrm{Res}\left(\zeta(s,a)\frac{n^s}{s(s+1)}; s=0\right) \\
&= \sum_{0\le k<n}\left(1-(k+a)\frac{1}{n}\right) - \frac{n}{2} - \zeta(0,a) \\
&= n - n\frac{a}{n} - \frac{1}{n}\frac{1}{2}(n-1)n - \frac{n}{2} - \zeta(0,a) = \frac{1}{2} - a - \zeta(0,a) = 0
\end{aligned}$$

The second corollary results from taking $r = 4$.

Corollary 2.10.2 *Let $n \in \mathbb{N}$. The value of*

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{4^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds$$

is given by the following table.

	$n = 4m$	$n = 4m+1$	$n = 4m+2$	$n = 4m+3$
$0 < a \le \frac{1}{4}$	0	$-\frac{1}{n}\left(3a+\frac{3}{8}\right)$	$-\frac{1}{n}\left(2a-\frac{1}{2}\right)$	$-\frac{1}{n}\left(a-\frac{3}{8}\right)$
$\frac{1}{4} < a \le \frac{1}{2}$	0	$\frac{1}{n}\left(a-\frac{5}{8}\right)$	$-\frac{1}{n}\left(2a-\frac{1}{2}\right)$	$-\frac{1}{n}\left(a-\frac{3}{8}\right)$
$\frac{1}{2} < a \le \frac{3}{4}$	0	$\frac{1}{n}\left(a-\frac{5}{8}\right)$	$\frac{1}{n}\left(2a-\frac{3}{2}\right)$	$-\frac{1}{n}\left(a-\frac{3}{8}\right)$
$\frac{3}{4} < a \le 1$	0	$\frac{1}{n}\left(a-\frac{5}{8}\right)$	$\frac{1}{n}\left(2a-\frac{3}{2}\right)$	$\frac{1}{n}\left(3a-\frac{21}{8}\right)$

We let $c = 1$ as before and consider the poles of $\zeta(s,a)n^s/(4^s s(s+1))$ in $\langle\alpha, c\rangle$, which are at 1 and 0. We apply the shifting lemma with $\Phi(s) = (n/4)^s$, $T_j = j$ and take $M = (n/4)^c$.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{4^s s(s+1)}ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s,a)\frac{n^s}{4^s s(s+1)} \\
&\quad -\mathrm{Res}\left(\zeta(s,a)\frac{n^s}{4^s s(s+1)}; s=1\right) - \mathrm{Res}\left(\zeta(s,a)\frac{n^s}{4^s s(s+1)}; s=0\right) \\
&= \sum_{k\in\mathbb{N}(n/4-a)}\left(1-(k+a)\frac{4}{n}\right) - \frac{n}{8} - \zeta(0,a) = \epsilon(n,a) - \frac{n}{8} - \zeta(0,a)
\end{aligned}$$

Suppose $n = 4m + m_1$ where $m_1 \in \{0, 1, 2, 3\}$. We have $n/4 - a = \lfloor n/4 \rfloor + m_1/4 - a$. If $m_1/4 < a$, the sum over $\mathbb{N}(n/4 - a)$ ranges from 0 to $\lfloor n/4 \rfloor - 1$. If $m_1/4 \geq a$ the sum includes $\lfloor n/4 \rfloor$. We have two cases.

$$\epsilon(n,a) = \begin{cases} \left\lfloor \frac{n}{4} \right\rfloor - a\frac{4}{n}\left\lfloor \frac{n}{4} \right\rfloor - \frac{2}{n}\left(\left\lfloor \frac{n}{4} \right\rfloor - 1\right)\left\lfloor \frac{n}{4} \right\rfloor & \text{if } \frac{m_1}{4} < a \\ \left\lfloor \frac{n}{4} \right\rfloor + 1 - a\frac{4}{n}\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right) - \frac{2}{n}\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right)\left\lfloor \frac{n}{4} \right\rfloor & \text{if } \frac{m_1}{4} \geq a \end{cases}$$

We note that $\lfloor n/4 \rfloor = (n - m_1)/4$ and $\lfloor n/4 \rfloor 4/n = 1 - m_1/n$. Hence the two terms evaluate to

$$\frac{1}{8}n + \frac{1}{2} - a + \frac{1}{n}\left(am_1 - \frac{1}{2}m_1 - \frac{1}{8}m_1^2\right)$$

and

$$\frac{1}{8}n + \frac{1}{2} - a + \frac{1}{n}\left(a(m_1 - 4) + \frac{1}{2}m_1 - \frac{1}{8}m_1^2\right).$$

We conclude that

$$\frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \zeta(s,a)\frac{n^s}{4^s s(s+1)}ds = \epsilon(n,a) - \frac{1}{8}n - \frac{1}{2} + a.$$

This gives the tabled values when $\epsilon(n,a)$ is evaluated according to m_1 and a.

Example. We can use this corollary to verify the following relation.

$$\frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} (\zeta(s, 7/16) + \zeta(s, 15/16))\frac{n^s}{4^s s(s+1)}ds = \begin{cases} 0 & \text{if } n \equiv 0 \mod 2 \\ \frac{1}{8}n & \text{if } n \equiv 1 \mod 2 \end{cases}$$

We have $7/16 \in (1/4, 1/2]$ and $15/16 \in (3/4, 1]$. Hence we need only add the second and fourth rows of the table, with a set to $7/16$ and $15/16$ respectively. This result will be useful in a later chapter.

2.11 Notes

I consulted [Cla82] as an introduction to elementary real analysis; the preliminaries of this chapter are from [Cla82, p. 167].

There are many texts on basic complex analysis. The definitions pertaining to point sets, complex limits, and analyticity are from [Det84, p. 13-17, 27-39]; analytic continuation is discussed on [Det84, p. 152-162] and [Mar87, p. 397-411]. Convergent series of analytic functions are discussed on [Mar87, p. 206-213]. The material on Laurent series is from [Mar87, p. 246-252, 266-272] and [Det84, p. 163-170]. The winding number is discussed on [Mar87, p. 165]; the residue theorem is given on [Mar87, p. 280].

The material on Dirichlet series will be found in any good text on number theory. I have used the presentation in [Apo86, p. 224-248]. This text also includes a detailed technical proof of the Mellin-Perron formula for $m = 0$. I also consulted [Kra81, p. 86-87] for the proof of the analytic version of the fundamental theorem of algebra. Theorem 2.6.2 is from [Man72, p. 1, 2, 7, 102-104].

An introduction to the Riemann ζ function is found on [Lan93, p. 415-421]. This includes a presentation of the functional equation; the reader may also wish to consult [Kar92, p. 9-11] or [WW15, p. 262-265]. The formula for $\zeta(m, a)$ with m a negative integer is proved on [BMP55c, p. 24-27, 35-37] or [WW15, p. 260-262], for example. The growth estimates for $\zeta(s, a)$ are from [WW15, p. 269-270].

[FGK$^+$94] note and use the shifting lemma. A statement of Corollary 2.10.1 for the case $a = 1$ can be found on [FGK$^+$94, p. 297].

The presentation of the Mellin-Perron formula is from [FGK$^+$94, p. 295-297]. The note on the use of the Mellin transform when $m = 1$ is based on observations found on [FG94, p. 680-681]. The background material on Mellin transforms is from [FGD95, p. 9-14].

Chapter 4

Digital Sums

The main topic of this chapter is the average order of digital sums in various bases. The results of [FGK+94] show how to obtain a Fourier series expansion of the sum-of-digits function with constant or exponential weights. We begin with basic definitions; then we demonstrate the method of [FGK+94] by considering alternating digital sums in base q, a special case of a kind of base known as a *Cantor representation.* We extend this method to periodic weights by giving a new proof of a result concerning alternating digital sums from [KPT85]. Thereafter we return to the general problem, and show that some classes of Cantor digital sums can also be dealt with by this method. We use the Mellin-Perron formula to obtain an asymptotic result that is similar to a theorem in [KPT85].

4.1 Definitions

Definition 4.1.1 *Let $q \in \mathbb{Z}^+$ and $q \geq 2$. For $n \in \mathbb{N}$,*

$$(d_r d_{r-1} \dots d_1 d_0)_q$$

denotes the unique q-ary representation of n, i.e.

$$n = \sum_{j=0}^{r} d_j q^j$$

where $0 \leq d_j < q$.

Example. We have $m = q^{v_q(m)} m'$ with $q \nmid m'$, hence $(m)_q = (m')_q \underbrace{000 \ \dots \ 000}_{v_q(m) \text{ 0 digits}}$. (Side-by-side placement of two base-q expansions means concatenation, not product.)

Cantor representations of integers generalize the concept of base-q representations.

Definition 4.1.2 *Let $\{q(j)\}_{j \geq 0} \subseteq \mathbb{Z}^+$ be a sequence of positive integers such that $q(0) = 1$ and $q(j) > 1$ when $j \geq 1$. Let $\kappa(j) = \prod_{k=0}^{j} q(k)$ for $j \geq 0$. For $n \in \mathbb{N}$,*

$$(d_r d_{r-1} \dots d_1 d_0)_\kappa$$

denotes the unique base-κ or Cantor representation of n with respect to κ, i.e.

$$n = \sum_{j=0}^{r} d_j \kappa(j)$$

where $0 \leq d_j < q(j)$.

It should be pointed out that $\{\kappa(j)\}$ fulfills the requirements of Definition 2.7.3.

Lemma 4.1.1 *The number of trailing zeros in the base-κ representation of n is given by $v_\kappa(n)$.*

To see this, note that $\sum_{j=0}^{r}(q(j)-1)\kappa(j) < \kappa(r+1)$. This holds for $r=0$ and with

$$\kappa(r+1) + (q(r+1)-1)\kappa(r+1) = \kappa(r+1)q(r+1) < \kappa(r+2)$$

for all $r \in \mathbb{Z}^+$.
Now $\kappa(l) \mid n$ implies $n \equiv \sum_{j=0}^{r} d_j\kappa(j) \equiv \sum_{j=0}^{l-1} d_j\kappa(j) \equiv 0 \quad (\kappa(l))$. Because $0 \leq \sum_{j=0}^{l-1} d_j\kappa(j) < \kappa(l)$, this requires $d_0 = d_1 = \ldots d_{l-1} = 0$.

Example. The $(2,3)$-number system; $\{q(j)\} = \{1,2,3,2,3,2,3\ldots\}$ and $\{\kappa(j)\} = \{1,2,6,12,36,\ldots\}$.

$(n)_{10}$	$(n)_\kappa$	$v_\kappa(n)$	$(n)_{10}$	$(n)_\kappa$	$v_\kappa(n)$	$(n)_{10}$	$(n)_\kappa$	$v_\kappa(n)$	$(n)_{10}$	$(n)_\kappa$	$v_\kappa(n)$
0	0	-	4	20	1	8	110	1	12	1000	3
1	1	0	5	21	0	9	111	0	13	1001	0
2	10	1	6	100	2	10	120	1	14	1010	1
3	11	0	7	101	0	11	121	0	15	1011	0

Definition 4.1.3 *A weight function $w(j)$ is a function $w : \mathbb{N} \mapsto \mathbb{C}$. The weighted base-$\kappa$ digital sum $v(n)$ of $n = \sum_{j=0}^{r} d_j\kappa(j)$ is*

$$v(n) = \sum_{j=0}^{r} w(j) d_j.$$

An alternating digital sum *uses the weight function $w(j) = (-1)^j$.*

Example. Let $m = q^r - 1$, $r \in \mathbb{Z}^+$. Hence $(m)_q = \underbrace{(q-1)(q-1)\ \ldots\ (q-1)}_{r \text{ digits}}$. The corresponding alternating digital sum is

$$v(m) = \begin{cases} 0 & \text{if } r \text{ is even} \\ q-1 & \text{if } r \text{ is odd.} \end{cases}$$

4.2 Alternating digital sums

In the remainder of this section $v(n)$ will always refer to an alternating digital sum, i.e. with weight function $w(j) = (-1)^j$. We have the following theorem.

Theorem 4.2.1 [KPT85] *The average order*

$$\frac{1}{n}\sum_{k=1}^{n-1} v(k)$$

of the alternating digital sum $v(n)$ is given by $F(\log_q n)$ where $F(u)$ is a Fourier series

$$F(u) = f_0 + \sum_{k\in\mathbb{Z}} f_k e^{(2k+1)\pi i u}$$

with coefficients

$$f_0 = \frac{q-1}{4} \text{ and } f_k = \frac{q+1}{(2k+1)\pi i}\,\zeta\left(\frac{(2k+1)\pi i}{\log q}\right)\left(1+\frac{(2k+1)\pi i}{\log q}\right)^{-1}.$$

The proof in [KPT85] uses elementary methods and builds on an earlier result by Delange. The remainder of this section will present a self-contained proof that uses the Mellin-Perron formula for $m = 1$. The method is that of [FGK+94].

4.2.1 Application of the Mellin-Perron formula

We wish to evaluate $\sum_{k=1}^{n-1} v(k)$. Consider $\nabla v(n) = v(n) - v(n-1)$, i.e. the change in $v(n)$ from $n-1$ to n. The following diagram shows how to evaluate $\nabla v(n)$. We assume that $(n)_q$ contains a prefix of unspecified length, which ends in the digit $d+1$, followed by $v_q(n)$ zeros.

$$\begin{array}{rcc|ccccc}
w(j) = & \ldots & \mp 1 & \pm 1 & \mp 1 & \ldots & -1 & 1 \\
(n-1)_q = & \ldots & d & (q-1) & (q-1) & \ldots & (q-1) & (q-1) \\
(n)_q = & \ldots & (d+1) & 0 & 0 & \ldots & 0 & 0
\end{array}$$

There are $v_q(n)$ columns to the right of the vertical line. Note that $d < q-1$ by definition of $v_q(n)$. Using the diagram, we have

$$\nabla v(n) = -d(-1)^{v_q(n)} + (d+1)(-1)^{v_q(n)} - \begin{cases} 0 & \text{if } v_q(n) \text{ is even} \\ q-1 & \text{if } v_q(n) \text{ is odd} \end{cases}$$

or

$$\nabla v(n) = (-1)^{v_q(n)} - (q-1)\ (v_q(n) \bmod 2)\,.$$

Note that $v(n) = v(n) - v(0) = \sum_{l=1}^{n} \nabla v(l)$ and hence $\sum_{k=1}^{n-1} v(k) = \sum_{k=1}^{n-1}\sum_{l=1}^{k} \nabla v(l)$. We apply Lemma 2.9.3, i.e. the Mellin-Perron formula with $m = 1$ and $b_l = \nabla v(l)$ to obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=1}^{k} \nabla v(l) = \frac{1}{n}\frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} V(s) n^s \frac{ds}{s(s+1)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} V(s) n^s \frac{ds}{s(s+1)}$$

where $V(s) = \sum \frac{\nabla v(n)}{n^s}$ is the Dirichlet generating function of $\nabla v(n)$.

The next step is to determine $V(s)$ and hence c. Evidently

$$V(s) = \sum \frac{(-1)^{v_q(n)} - (q-1)\ (v_q(n) \bmod 2)}{n^s}.$$

These terms were evaluated earlier, when we discussed the analytic version of the fundamental theorem of arithmetic (Theorem 2.7.2). We have

$$\sum \frac{(-1)^{v_q(n)}}{n^s} = \zeta(s)\left(1 - 2\frac{1}{q^s+1}\right) \text{ and } \sum \frac{v_q(n) \bmod 2}{n^s} = \zeta(s)\frac{1}{q^s+1}$$

with $\sigma > 1$ and hence

$$V(s) = \zeta(s)\left(1 - 2\frac{1}{q^s+1}\right) - (q-1)\zeta(s)\frac{1}{q^s+1} = \zeta(s)\left(1 - \frac{q+1}{q^s+1}\right),$$

also with $\sigma > 1$. Any $c > 1$ will suffice.

4.2.2 Evaluating the integral

We evaluate this integral by means of the shifting lemma (Lemma 2.8.1), taking

$$\Phi(s) = \left(1 - \frac{q+1}{q^s+1}\right) n^s = \frac{q^s - q}{q^s+1} n^s \text{ and } T_j = \frac{2j\pi}{\log q}$$

with $j > 0$. Along the vertical segments situated at α and c (recall that the contour is rectangular, with $c > 1$ being the right vertical boundary, and $\alpha \in (-1, 0)$ the left one)

$$|\Phi(s)| = \left|\frac{q^s - q}{q^s+1}\right| n^\alpha \le \frac{q^\alpha + q}{1 - q^\alpha} n^\alpha = M_\alpha \text{ and } |\Phi(s)| = \left|\frac{q^s - q}{q^s+1}\right| n^c \le \frac{q^c + q}{q^c - 1} = M_c$$

respectively. Along the two horizontal segments $\langle \alpha, c\rangle \cap \{s \mid s = \sigma \pm iT_j\}$ we have

$$|\Phi(s)| = \left|\frac{q^\sigma - q}{q^\sigma+1}\right| n^\sigma \le \frac{q^c - q}{q^\alpha+1} n^c = M_T\,.$$

Therefore the constant M required by the lemma is $M = \max\{M_\alpha, M_c, M_T\}$.

The poles of $\Phi(s)\frac{\zeta(s)}{s(s+1)}$ in $\langle \alpha, c\rangle$ are at $s=0$ and $s = \rho + \chi_k$ where $\rho = \frac{i\pi}{\log q}$ and $\chi_k = \frac{2\pi i k}{\log q}$, $k \in \mathbb{Z}$. (There is no pole at $s=1$ because the pole of $\zeta(s)$ at $s=1$ is simple and $\Phi(1) = 0$; the zero of $\Phi(s)$ cancels the pole, as in $(1/(s-1)+\ldots)(\phi_1(s-1)+\ldots) = \phi_1 + \ldots$) Hence the lemma gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Phi(s)\frac{\zeta(s)}{s(s+1)}ds &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Phi(s)\frac{\zeta(s)}{s(s+1)}ds \\ &+\mathrm{Res}\left(\Phi(s)\frac{\zeta(s)}{s(s+1)}; s=0\right) + \sum_{k=0}\mathrm{Res}\left(\Phi(s)\frac{\zeta(s)}{s(s+1)}; s=\rho+\chi_k\right).\end{aligned}$$

Note that

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Phi(s)\frac{\zeta(s)}{s(s+1)}ds = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s)\frac{n^s}{s(s+1)}ds - (q+1)\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{q^s+1}\zeta(s)\frac{n^s}{s(s+1)}ds.$$

Note also that the expansion $\frac{1}{1+q^s} = 1 - q^s + q^{2s} - q^{3s} + \ldots$ is convergent since $|q^s| = q^\alpha \in (-1, 0)$. Corollary 2.10.1 applies. We conclude that $\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Phi(s)\frac{\zeta(s)}{s(s+1)}ds = 0$; both the constant term and the terms of the series expansion vanish.

It remains to compute the residues. Let $\log_q n = u$, and $W(s) = \Phi(s)\frac{\zeta(s)}{s(s+1)}$.

$$\begin{aligned}\mathrm{Res}\,(W(s); s=0) &= \lim_{s\to 0}\Phi(s)\frac{\zeta(s)}{s+1} = \left(1 - \frac{q+1}{2}\right)\zeta(0) = \frac{q+1}{4} - \frac{1}{2} = \frac{q-1}{4} \\ \mathrm{Res}\,(W(s); s=\rho+\chi_k) &= \lim_{s\to\rho+\chi_k}(s-\rho-\chi_k)\left(1 - \frac{q+1}{q^s+1}\right)n^s\frac{\zeta(s)}{s+1} \\ &= n^{\rho+\chi_k}\frac{\zeta(\rho+\chi_k)}{(\rho+\chi_k)(\rho+\chi_k+1)}\lim_{s\to\rho+\chi_k}\left(s-\rho-\chi_k-(q+1)\frac{s-\rho-\chi_k}{q^s+1}\right) \\ &= e^{u(2k+1)\pi i}\zeta\left(\frac{(2k+1)\pi i}{\log q}\right)\frac{(q+1)\log q}{(2k+1)\pi i}\left(1+\frac{(2k+1)\pi i}{\log q}\right)^{-1}\frac{-1}{-\log q} \\ &= e^{u(2k+1)\pi i}\frac{q+1}{(2k+1)\pi i}\zeta\left(\frac{(2k+1)\pi i}{\log q}\right)\left(1+\frac{(2k+1)\pi i}{\log q}\right)^{-1}\end{aligned}$$

This concludes the proof of Theorem 4.2.1. ∎

4.3 Periodic weights in general

The proof of Theorem 4.2.1 serves to illustrate the general method of treating digital sums with periodic weights. We will sketch the case

$$w(j) = \begin{cases} 2 & \text{if } j \equiv 0(3) \\ 5 & \text{if } j \equiv 1(3) \\ 11 & \text{if } j \equiv 2(3).\end{cases}$$

We consider the function $\nabla v(n)$, which is computed in a manner analogous to the case of an alternating digital sum ($v(n)$ uses the new $w(j)$). We have

$$\nabla v(n) = -dw(v_q(n)) + (d+1)w(v_q(n)) - 18(q-1)\left\lfloor \frac{v_q(n)}{3} \right\rfloor - \begin{cases} 0 & \text{if } v_q(n) \equiv 0(3) \\ 2(q-1) & \text{if } v_q(n) \equiv 1(3) \\ 7(q-1) & \text{if } v_q(n) \equiv 2(3). \end{cases}$$

With $\lfloor n/3 \rfloor = n/3 - (n \bmod 3)/3$ this simplifies to

$$\nabla v(n) = -6(q-1)v_q(n) + \begin{cases} 2 & \text{if } v_q(n) \equiv 0(3) \\ 5 + 4(q-1) & \text{if } v_q(n) \equiv 1(3) \\ 11 + 5(q-1) & \text{if } v_q(n) \equiv 2(3). \end{cases}$$

This is the general form of $\nabla v(n)$, i.e. $\nabla v(n)$ is a multiple of $v_q(n)$ and a term linear in q, plus a second term linear in q, one for each of the residues of $v_q(n)$ modulo the period length. We recall that

$$\sum \frac{v_q(n)}{n^s} = \frac{\zeta(s)}{q^s - 1} \text{ and } \sum_{v_q(n) \equiv r(m)} \frac{1}{n^s} = \zeta(s) q^{(m-r-1)s} \prod_{v=1}^{m-1} (q^s - \omega_m^v)^{-1}.$$

The Dirichlet generating function $V(s)$ of $\nabla v(n)$ is a linear combination of these two kinds of terms, with poles corresponding to $q^s - \omega_m^v = 0$ or

$$s = \frac{2\pi i v/m + 2\pi i k}{\log q}$$

for $0 \le v < m$ and $k \in \mathbb{Z}$. (The generating function of $v_q(n)$ contributes $v = 0$ and the residues r modulo m the rest. Note that there may be some cancellation of poles, such as that of $s = 1$ in the $v_q(n)$ term.) Continuing the example, we have

$$\begin{aligned} V(s) &= -6(q-1)\frac{\zeta(s)}{q^s - 1} + \zeta(s)\left(2q^{2s} + 5q^s + 4q^s(q-1) + 11 + 5(q-1)\right)\frac{1}{(q^s - \omega_3)(q^s - \omega_3^2)} \\ &= \zeta(s)\left(\frac{2q^{2s} + 4q^{s+1} + q^s + 5q + 6}{(q^s - \omega_3)(q^s - \omega_3^2)} - 6\frac{1}{(q^s - 1)/(q-1)}\right) \end{aligned}$$

At this point it is a matter of routine computation to obtain the Fourier series expansion of the average order of $v(n)$. Two questions must be considered.

- Does the shifting lemma apply to $V(s)n^s/(s(s+1))$, i.e. how do we choose M and T_j for $\Phi(s) = V(s)n^s/\zeta(s)$?

Note that all the poles of $\Phi(s)$ are staggered along the imaginary axis. Furthermore, there is only a finite number of singularities in the interval $2\pi i/\log q[k, k+1]$. Hence we can choose T_j such that $\Phi(s)$ is analytic on a closed rectangular band that includes δR_j. Because $\Phi(s)$ is analytic there, $|\Phi(s)|$ is bounded. We need to verify that this bound is independent of j. But $\Phi(s)$ contains only terms in n^s and q^s, with $|n^s| = n^\sigma$ and $|q^s| = q^\sigma$. This observation and the analyticity of $\Phi(s)$ yield the claim.

- Does $\int_{\alpha-i\infty}^{\alpha+i\infty} V(s)n^s/(s(s+1))ds$ vanish? (We need this in order to ensure that there are no terms other than the Fourier series; compare the proof of Theorem 5.1.2, where the corresponding term does not vanish.)

  This is a question of expanding $(q^s - \omega_m^v)^{-1}$ with $0 \leq v < m$. (We use the partial fraction decomposition of $\prod_{v=1}^{m-1}(q^s - \omega_m^v)^{-1}$.) Note that

$$\frac{1}{q^s - \omega_m^v} = -\frac{1}{\omega_m^v}\frac{1}{1 - q^s/\omega_m^v} = -\frac{1}{\omega_m^v}\left(1 + \frac{q^s}{\omega_m^v} + \left(\frac{q^s}{\omega_m^v}\right)^2 + \right)\dots$$

  converges since $|q^s/\omega_m^v| = q^\alpha$ and $\alpha \in (-1, 0)$. Hence we may apply Corollary 2.10.1 to $(qn)^{ks}$ ($qn \in \mathbb{N}$ and $1/\omega_m^{kv}$ is a constant factor with respect to s). The integral vanishes as claimed.

The above observations lead to the following statement. *Suppose $w(j)$ is a periodic weight function and $v(n)$ the associated digital sum. Then the average order of $v(n)$ can be expanded into a sum of Fourier series with terms corresponding to $s = \frac{2\pi iv/m + 2\pi ik}{\log q}$.*

4.4 Intermezzo: digital sum paradigms

The preceding discussion should suffice to demonstrate that the problem of computing the average order of a general digital sum by Mellin-transform methods requires that the two following conditions hold.

- There must exist a closed form of $\nabla v(n)$ in terms of a polynomial of the "number of trailing zeros"-function in $\mathbb{Z}$ or $\mathbb{Z}_q$, or in exponentials of this function.

  Recall that the "trailing zeros"-function is given by v_κ (Lemma 4.1.1.) It follows that the types of $\nabla v(n)$ generated by periodic, constant or exponential weights all fit this condition.

- The corresponding Dirichlet generating function $V(s)$ must have a closed form and $V(s)n^s/\zeta(s)$ must satisfy the requirements of the shifting lemma.

The Lemmata 2.7.1 and 2.7.2 show that the behavior of $\sum_{j=1}\kappa(j)^{-s}$ and $\sum_{k=0}\kappa(mk+r)^{-s}$, $m\geq 2$, $0\leq r<m$ determines that of $V(s)$. E.g. if $\sum_{j=1}\kappa(j)^{-s}$ does not represent a meromorphic function in $\langle\alpha,c\rangle$, $V(s)$ fails the shifting lemma.

The above criteria constitute an informal quick test for the computability of a Fourier expansion by the Mellin-Perron formula for a given a digital sum problem. The significant part of the test is the investigationn of the properties (read: analyticity and location of poles) of $\sum_{j=1}\kappa(j)^{-s}$ and $\sum_{k=0}\kappa(mk+r)^{-s}$. We will present two additional examples in the remainder of this chapter. The first of these exhibits a $V(s)$ that is well-behaved; the second shows how $V(s)$ may fail the second condition.

4.5 Digital sums relative to κ when $\kappa(j+1)/\kappa(j)=q(j+1)$ is periodic

We treat the case

$$\{q(j)\}=\{1,2,\ldots a+1,2,\ldots a+1\ldots\}$$

and

$$\{\kappa(j)\}=\{1,2!,\ldots(a+1)!,2(a+1)!\ldots(a+1)!^2\ldots\},$$

where $a>1$. We select this case because it is one of a series of κ that have the factorial number system as their limit, see section 4.6.

Step 1. What is $\nabla v(n)$? Every complete sequence of a zeros corresponds to digits $a,a-1,\ldots 1$ lost from $(n-1)_\kappa$; the remainder corresponds to digits $r,r-1\ldots 1$, where $r=v_\kappa(n)\bmod a$. These digits are replaced by zeros; we gain a 1 in the first non-zero digit of $(n-1)_\kappa$. Hence

$$\nabla v(n)=1-\frac{1}{2}a(a+1)\left\lfloor\frac{v_\kappa(n)}{a}\right\rfloor-\frac{1}{2}(v_\kappa(n)\bmod a)(v_\kappa(n)\bmod a+1).$$

Step 2. What is $V(s)$? Using $\lfloor v_\kappa(n)/a\rfloor=v_\kappa(n)/a-(v_\kappa(n)\bmod a)/a$, the problem reduces to finding the Dirichlet generating functions of $v_\kappa(n)\bmod a$, $\frac{1}{2}(v_\kappa(n)\bmod a)(v_\kappa(n)\bmod a+1)$ and $v_\kappa(n)$. We apply Lemma 2.7.2 to obtain the Dirichlet generating function of $v_\kappa(n)$. Note that $\kappa(ak+r)=(a+1)!^k(r+1)!$ and hence

$$\sum_{j=1}\frac{1}{\kappa(j)^s}=\sum_{r=1}^{a}\sum_{k=0}\frac{1}{\kappa(ak+r)^s}=\frac{(a+1)!^s}{(a+1)!^s-1}\sum_{r=1}^{a}\frac{1}{(r+1)!^s}.$$

This function contributes poles at $(a+1)!^s-1=0$. We use Lemma 2.7.1 to evaluate the remaining two types, i.e.

$$\sum\left\{\begin{array}{c}v_\kappa(n)\bmod a\\ \frac{1}{2}(v_\kappa(n)\bmod a)(v_\kappa(n)\bmod a+1)\end{array}\right\}\frac{1}{n^s}=\sum_{r=1}^{a-1}\left\{\begin{array}{c}r\\ \frac{1}{2}r(r+1)\end{array}\right\}\sum_{v_\kappa(n)\equiv r(a)}\frac{1}{n^s}$$

and the relevant terms of the respective generating functions are

$$\frac{(a+1)!^s}{(a+1)!^s-1}\sum_{r=1}^{a-1} r\left(\frac{1}{(r+1)!^s}-\frac{1}{(r+2)!^s}\right) \quad\text{and}\quad \frac{(a+1)!^s}{(a+1)!^s-1}\sum_{r=1}^{a-1}\frac{1}{2}r(r+1)\left(\frac{1}{(r+1)!^s}-\frac{1}{(r+2)!^s}\right).$$

Step 3. Does the shifting lemma apply? We note that $V(s)$ is the product of $\zeta(s)$ and finite sum of terms, which are in turn the product of a meromorphic function with poles at $s = 2\pi ik/\log(a+1)!$ and a finite sum of entire functions (the $1/(r+1)!^s$ terms). Therefore the shifting lemma applies with $T_j = \pi i(2j+1)/\log(a+1)!$; the term $(a+1)!^s/((a+1)!^s-1)$ was evaluated in the base-q problem and the norm of the terms from $1/(r+1)!^s$ is bounded on the horizontal and constant on the vertical segments.

4.6 Digital sums in the factorial number system

The factorial number system has $q(j) = j+1$ and $\kappa(j) = (j+1)!$.

Step 1. What is $\nabla v(n)$? We have $v_\kappa(n)$ zeros, which correspond to the digits $v_\kappa(n), \ldots 2, 1$, hence $\nabla v(n) = 1 - 1/2 v_\kappa(n)(v_\kappa(n)+1)$.

Step 2. What is $V(s)$? We use Lemma 2.7.2 to obtain

$$V(s) = \zeta(s)\left(1-\sum_{j=1}^{\infty}\frac{j}{\kappa(j)^s}\right) = \zeta(s)\left(1-\sum_{j=1}^{\infty}\frac{j}{(j+1)!^s}\right).$$

(Compare this with the corresponding function in the previous section. The contribution from the period length a vanishes and the finite sums in $r/(r+1)!^s$ become an infinite series.)

Step 3. Does the shifting lemma apply? Recall that $\sigma = 0$ is a natural boundary of $\sum_{j=1}^{\infty}\frac{j}{(j+1)!^s}$ by Corollary 2.6.1. Hence we cannot continue this sum into the left half-plane. The shifting lemma does not apply.

We can however extract some information from the Mellin-Perron formula. Evidently the problem requires the evaluation of

$$\int_{c-i\infty}^{c+i\infty}\zeta(s)\left(1-\sum_{j=1}^{\infty}\frac{j}{(j+1)!^s}\right)\frac{n^s}{s(s+1)}ds.$$

With Corollary 2.10.1, this simplifies to

$$\frac{1}{2}(n-1) - \int_{c-i\infty}^{c+i\infty}\zeta(s)\left(\sum_{j=1}^{\infty}\frac{j}{(j+1)!^s}\right)\frac{n^s}{s(s+1)}ds.$$

(We could also have evaluated the first term directly, i.e. at Step 1.) Note that Theorem 2.10.1 applies to

$$j\int_{c-i\infty}^{c+i\infty}\zeta(s)\frac{1}{(j+1)!^s}\frac{n^s}{s(s+1)}ds$$

with $r=(j+1)!$, $a=1$. Hence

$$j\int_{c-i\infty}^{c+i\infty}\zeta(s)\frac{1}{(j+1)!^s}\frac{n^s}{s(s+1)}ds = j\sum_{k\in\mathbb{N}(n/(j+1)!-1)}\left(1-(k+1)\frac{(j+1)!}{n}\right).$$

Note that $\mathbb{N}(n/(j+1)!-1)=\emptyset$ when $(j+1)!\geq n$. Therefore only a finite number of terms actually contribute to the integral in $\sum_{j=1}^{\infty} j/(j+1)!^s$, which justifies writing

$$\int_{c-i\infty}^{c+i\infty}\zeta(s)\left(\sum_{j=1}^{\infty}\frac{j}{(j+1)!^s}\right)\frac{n^s}{s(s+1)}ds = \sum_{j\geq 1}^{(j+1)!<n} j\sum_{k\in\mathbb{N}(n/(j+1)!-1)}\left(1-(k+1)\frac{(j+1)!}{n}\right).$$

By definition of $v_\kappa(n)$,

$$\sum_{\mathbb{N}(n/(j+1)!-1)} = \begin{cases}\sum_0^{n/(j+1)!-2} & \text{if } j\leq v_\kappa(n)\\ \sum_0^{\lfloor n/(j+1)!\rfloor-1} & \text{if } j> v_\kappa(n).\end{cases}$$

Hence the integral splits into two sums,

$$\begin{aligned}\sum_{j=1}^{v_\kappa(n)} j\left(\frac{n}{(j+1)!}-1\right)\left(1-\frac{(j+1)!}{2n}\frac{n}{(j+1)!}\right) &= -\frac{1}{4}v_\kappa(n)(v_\kappa(n)+1)+\frac{1}{2}n\sum_{j=1}^{v_\kappa(n)}\frac{j}{(j+1)!}\\ &= -\frac{1}{4}v_\kappa(j)(v_\kappa(j)+1)+\frac{1}{2}n\left(1-\frac{1}{(v_\kappa(n)+1)!}\right)\end{aligned}$$

and

$$\sum_{j=v_\kappa(n)+1}^{(j+1)!<n} j\left\lfloor\frac{n}{(j+1)!}\right\rfloor\left(1-\frac{(j+1)!}{2n}\left(\left\lfloor\frac{n}{(j+1)!}\right\rfloor+1\right)\right).$$

This formula has some utility as we shall see below. Nonetheless it must be pointed out that it can equally well be derived by an elementary counting argument (the reader is urged to supply this proof). There is no *qualitative* gain with respect to elementary methods because the function $V(s)$ is not meromorphic and the terms of the series merely transcribe the problem definition.

We remark in passing that $n=(r+1)!$ gives $v_\kappa(n)=v_\kappa((r+1)!)=r$, in which case the second term of the above sum drops out and $1/n\sum_{m=1}^{n-1}v(m)$ is given by

$$\frac{1}{2}(n-1)-\left(-\frac{1}{4}v_\kappa(n)(v_\kappa(n)+1)+\frac{1}{2}n\left(1-\frac{1}{(v_\kappa(n)+1)!}\right)\right)=\frac{1}{4}r(r+1).$$

The average order of the sum of factorial digits of the first $n = (r+1)!$ non-negative integers is quadratic in the factorial inverse of n.

The work of this section suggests the following question. Considering the fact that

$$\frac{1}{n}\sum_{m=1}^{n-1} v(m) \sim \frac{1}{2}\log_q n \sim \frac{1}{2}\sum_{j=0}^{\kappa(j)<n} (q(j+1)-1)$$

for q-ary digital sums, $\kappa(j) = q^j$, $q \geq 2$ and all n, and

$$\frac{1}{n}\sum_{m=1}^{n-1} v(m) = \frac{1}{2}\left(\frac{1}{2}r(r+1)\right) \sim \frac{1}{2}\sum_{j=0}^{\kappa(j)<n} (q(j+1)-1)$$

for digital sums in the factorial number system, $\kappa(j) = (j+1)!$, $n = (r+1)!$, what are the conditions such that

$$\frac{1}{2}\sum_{j=0}^{\kappa(j)<n} w(j)\,(q(j+1)-1)$$

is the asymptotically dominant term of the general sum-of-digits function? This question will be answered in the next section.

We will need the following observation. Suppose $q(j) \to \infty$ as $j \to \infty$. Then $\kappa(j+1) < n$ implies

$$\frac{\kappa(j)}{n} \in o(1) \qquad \text{as} \qquad n \to \infty.$$

To see this, note that

$$\frac{\kappa(j)}{n} = \frac{\kappa(\kappa^{-1}(n)-1)}{n}\prod_{k=j+1}^{\kappa^{-1}(n)-1}\frac{1}{q(k)} < \prod_{k=j+1}^{\kappa^{-1}(n)-1}\frac{1}{q(k)} \to 0 \qquad \text{as} \qquad n \to \infty.$$

4.7 The general digital sum problem

We wish to examine the role of

$$\frac{1}{2}\sum_{j=0}^{\kappa(j)<n} w(j)\,(q(j+1)-1)$$

in the behavior of

$$\frac{1}{n}\sum_{m=1}^{n-1} v(m)$$

for an arbitrary weight function and an arbitrary Cantor system κ. We require an expression of $\nabla v(n)$. Define $s : \mathbb{N} \mapsto \mathbb{C}$ by

$$\begin{aligned} s(v) &= w(v) - \sum_{j=0}^{v-1} w(j)\,(q(j+1)-1) \\ &= w(0) + w(v) - w(0) - \sum_{j=1}^{v} w(j-1)\,(q(j)-1) = w(0) + \sum_{j=1}^{v} (w(j) - w(j-1)q(j))\,. \end{aligned}$$

Evidently

$$\nabla v(n) = s(v_\kappa(n)).$$

By Lemma 2.7.2, the Dirichlet generating function $V(s)$ is

$$\zeta(s)\left(w(0) + \sum \frac{1}{\kappa(j)^s}\left(w(j) - w(j-1)q(j)\right)\right).$$

We proceed by the same method that was used to obtain a formula for the average order of digital sums in the factorial number system in the previous section. The $w(0)$ term corresponds to

$$w(0)\frac{n-1}{2}$$

by Corollary 2.10.1. Theorem 2.10.1 is used to evaluate the series. We have

$$\int_{c-i\infty}^{c+i\infty} \zeta(s)\frac{1}{\kappa(j)^s}\frac{n^s}{s(s+1)}ds = \sum_{k\in\mathbb{N}(n/\kappa(j)-1)} \left(1-(k+1)\frac{\kappa(j)}{n}\right).$$

The sum is zero when $\kappa(j) \geq n$. There are two cases when $\kappa(j) < n$.

Case 1. $\kappa(j) \mid n$

$$\sum_{k=0}^{n/\kappa(j)-2}\left(1-(k+1)\frac{\kappa(j)}{n}\right) = \left(\frac{n}{\kappa(j)}-1\right)\left(1-\frac{1}{2}\frac{\kappa(j)}{n}\frac{n}{\kappa(j)}\right) = \frac{1}{2}\left(\frac{n}{\kappa(j)}-1\right)$$

Case 2. $\kappa(j) \nmid n$

$$\begin{aligned}
&\sum_{k=0}^{\lfloor n/\kappa(j)\rfloor-1}\left(1-(k+1)\frac{\kappa(j)}{n}\right) = \left\lfloor\frac{n}{\kappa(j)}\right\rfloor\left(1-\frac{1}{2}\frac{\kappa(j)}{n}\left(\left\lfloor\frac{n}{\kappa(j)}\right\rfloor+1\right)\right)\\
=\;& \left(\frac{n}{\kappa(j)}-\frac{n \bmod \kappa(j)}{\kappa(j)}\right)\left(1-\frac{1}{2}\frac{\kappa(j)}{n}\left(\frac{n}{\kappa(j)}-\frac{n \bmod \kappa(j)}{\kappa(j)}+1\right)\right)\\
=\;& \left(\frac{n}{\kappa(j)}-\frac{n \bmod \kappa(j)}{\kappa(j)}\right)\left(\frac{1}{2}-\frac{1}{2}\frac{\kappa(j)}{n}+\frac{1}{2}\frac{n \bmod \kappa(j)}{n}\right)\\
=\;& \frac{1}{2}\left(\frac{n}{\kappa(j)}-1\right)+\frac{1}{2}\frac{n \bmod \kappa(j)}{\kappa(j)}-\frac{1}{2}\left(\frac{n \bmod \kappa(j)}{\kappa(j)}-\frac{n \bmod \kappa(j)}{n}+\frac{(n \bmod \kappa(j))^2}{n\kappa(j)}\right)\\
=\;& \frac{1}{2}\left(\frac{n}{\kappa(j)}-1\right)+\frac{1}{2}\frac{n \bmod \kappa(j)}{n}\left(1-\frac{n \bmod \kappa(j)}{\kappa(j)}\right)
\end{aligned}$$

Evaluating the contribution from the

$$\frac{1}{2}\left(\frac{n}{\kappa(j)}-1\right)$$

terms, we have

$$
\begin{aligned}
&\sum_{j=1}^{\kappa(j)<n} \frac{1}{2}\left(\frac{n}{\kappa(j)}-1\right)(w(j)-w(j-1)q(j)) \\
= & \frac{1}{2}n\sum_{j=1}^{\kappa(j)<n}\left(\frac{w(j)}{\kappa(j)}-\frac{w(j-1)}{\kappa(j-1)}\right)-\frac{1}{2}\sum_{j=1}^{\kappa(j)<n} w(j)+\frac{1}{2}\sum_{j=1}^{\kappa(j)<n} w(j-1)q(j) \\
= & \frac{1}{2}n\left(\frac{w(\kappa^{-1}(n)-1)}{\kappa(\kappa^{-1}(n)-1)}-w(0)\right)+\frac{1}{2}w(0)-\frac{1}{2}\sum_{j=0}^{\kappa(j)<n} w(j) \\
- & \frac{1}{2}w(\kappa^{-1}(n)-1)q(\kappa^{-1}(n))+\frac{1}{2}\sum_{j=0}^{\kappa(j)<n} w(j)q(j+1) \\
= & -w(0)\frac{n-1}{2}+\frac{1}{2}\sum_{j=0}^{\kappa(j)<n} w(j)\left(q(j+1)-1\right)+\frac{1}{2}w(\kappa^{-1}(n)-1)\left(\frac{n}{\kappa(\kappa^{-1}(n)-1)}-q(\kappa^{-1}(n))\right)
\end{aligned}
$$

It remains to evaluate the contribution from the second term of the case 2 sum. With

$$\mu(n,j)=(n \bmod \kappa(j))(\kappa(j)-n \bmod \kappa(j)),$$

$$\mu(n,\kappa^{-1}(n))=n(\kappa(\kappa^{-1}(n))-n)$$

and

$$\frac{1}{2}\frac{n \bmod \kappa(j)}{n}\left(1-\frac{n \bmod \kappa(j)}{\kappa(j)}\right)=\frac{1}{2n\kappa(j)}\mu(n,j),$$

this becomes

$$
\begin{aligned}
&\frac{1}{2n}\sum_{j=v_\kappa(n)+1}^{\kappa(j)<n}\mu(n,j)\left(\frac{w(j)}{\kappa(j)}-\frac{w(j-1)}{\kappa(j-1)}\right) \\
= & \frac{1}{2n}\left(\sum_{j=v_\kappa(n)}^{\kappa(j)<n}\frac{w(j)}{\kappa(j)}\left(\mu(n,j)-\mu(n,j+1)\right)-\mu(n,v_\kappa(n))\frac{w(v_\kappa(n))}{\kappa(v_\kappa(n))}+\mu(n,\kappa^{-1}(n))\frac{w(\kappa^{-1}(n)-1)}{\kappa(\kappa^{-1}(n)-1)}\right) \\
= & -\frac{1}{2n}\sum_{j=v_\kappa(n)}^{\kappa(j)<n}\frac{w(j)}{\kappa(j)}\left(\mu(n,j+1)-\mu(n,j)\right)+\frac{1}{2}w(\kappa^{-1}(n)-1)\left(q(\kappa^{-1}(n))-\frac{n}{\kappa(\kappa^{-1}(n)-1)}\right)
\end{aligned}
$$

We combine these results to obtain the following theorem.

Theorem 4.7.1 *Let $w(j)$ be an arbitrary weight function, κ any Cantor system, and define*

$$\mu(n,j)=(n \bmod \kappa(j))(\kappa(j)-n \bmod \kappa(j)).$$

The sum-of-digits function for w and κ is given by

$$\frac{1}{n}\sum_{m=1}^{n-1}v(m)=\frac{1}{2}\sum_{j=0}^{\kappa(j)<n}w(j)\left(q(j+1)-1\right)-\frac{1}{2n}\sum_{j=v_\kappa(n)}^{\kappa(j)<n}\frac{w(j)}{\kappa(j)}\left(\mu(n,j+1)-\mu(n,j)\right).$$

We can use this theorem to answer the question posed in the previous section. In the following, we will assume that the $w(j)$ are positive and that $q(j) \to \infty$ as $j \to \infty$. The term

$$\frac{1}{2} \sum_{j=0}^{\kappa(j)<n} w(j)\,(q(j+1)-1)$$

will dominate asymptotically if it dominates the second term. It is not difficult to see that

$$\mu(n,j) \le \frac{1}{4}\kappa(j)^2$$

when $\kappa(j) < n$. This gives the following estimate for the $-\mu(n,j)$ part of the second term.

$$\sum_{j=v_\kappa(n)}^{\kappa(j)<n} \frac{w(j)}{n\kappa(j)}\mu(n,j) \le \sum_{j=v_\kappa(n)}^{\kappa(j)<n} w(j)q(j)\frac{\kappa(j-1)}{4n} \le \sum_{j=v_\kappa(n)}^{\kappa(j)<n} w(j)(q(j+1)-1)\frac{\kappa(j-1)}{4n}$$

Using our earlier observation and a term-by-term comparison we see that the first term dominates the $-\mu(n,j)$ part asymptotically.

We split the $\mu(n,j+1)$ part into $\sum_{j=v_\kappa(n)}^{\kappa(j+1)<n}$ and $j = \kappa^{-1}(n)-1$. For the first part we again have

$$\begin{aligned}
\sum_{j=v_\kappa(n)}^{\kappa(j+1)<n} \frac{w(j)}{n\kappa(j)}\mu(n,j+1) &\le \sum_{j=v_\kappa(n)}^{\kappa(j+1)<n} w(j)(q(j+1)-1)\frac{\kappa(j)}{4n}\frac{q(j+1)^2}{q(j+1)-1} \\
&\le \sum_{j=v_\kappa(n)}^{\kappa(j+1)<n} w(j)(q(j+1)-1)\frac{\kappa(j)}{4n}\left(q(j+1)+1+\frac{1}{q(j+1)-1}\right).
\end{aligned}$$

With

$$1 + \frac{1}{q(j+1)-1} \le 2$$

this is less than or equal to

$$\sum_{j=v_\kappa(n)}^{\kappa(j+1)<n} w(j)(q(j+1)-1)\left(\frac{\kappa(j+1)}{4n} + \frac{\kappa(j)}{2n}\right).$$

This part is also dominated by the first term, except for the first half of the sum when $j = \kappa^{-1}(n)-2$, which is

$$w(\kappa^{-1}(n)-2)(q(\kappa^{-1}(n)-1)-1)\frac{\kappa(\kappa^{-1}(n)-1)}{4n}.$$

Suppose we have

$$w(v-1)(q(v)-1) \in o\left(\frac{1}{q(v)}\sum_{j=0}^{v-1} w(j)\,(q(j+1)-1)\right)$$

for v sufficiently large. Then certainly

$$w(v-1)(q(v)-1)q(v)\frac{\kappa(v-1)}{4n} \in o\left(w(v-1)(q(v)-1)q(v)\right) \subset o\left(\sum_{j=0}^{v} w(j)\left(q(j+1)-1\right)\right)$$

for $\kappa(v) < n$. Taking $v = \kappa^{-1}(n) - 1$, we see that we have a sufficient condition for the first term to dominate. It remains to test $j = \kappa^{-1}(n) - 1$, in which case $w(j)/(nk(j))\mu(n, j+1)$ becomes

$$\begin{aligned}\frac{w(\kappa^{-1}(n)-1)}{\kappa(\kappa^{-1}(n)-1)}(\kappa(\kappa^{-1}(n))-n) &< \frac{w(\kappa^{-1}(n)-1)}{\kappa(\kappa^{-1}(n)-1)}(\kappa(\kappa^{-1}(n))-\kappa(\kappa^{-1}(n)-1)) \\ &= w(\kappa^{-1}(n)-1)(q(\kappa^{-1}(n))-1)\end{aligned}$$

assuming $\kappa(\kappa^{-1}(n)) \nmid n$ (if $\kappa(\kappa^{-1}(n)) \mid n$ the term is zero and we are done). We need only take $v = \kappa^{-1}(n)$ and point out that $q(v) \to \infty$ as $v \to \infty$; hence

$$w(v-1)(q(v)-1) \in o\left(\frac{1}{q(v)}\sum_{j=0}^{v-1} w(j)\left(q(j+1)-1\right)\right) \subset o\left(\sum_{j=0}^{v-1} w(j)\left(q(j+1)-1\right)\right)$$

and we have proved the following theorem.

Theorem 4.7.2 *Let w be a weight function such that $w(j) > 0$ and let $q(j) \to \infty$ as $j \to \infty$. If*

$$w(v-1)(q(v)-1) \in o\left(\frac{1}{q(v)}\sum_{j=0}^{v-1} w(j)\left(q(j+1)-1\right)\right)$$

then

$$\frac{1}{n}\sum_{m=1}^{n-1} v(m) \sim \frac{1}{2}\sum_{j=0}^{\kappa(j)<n} w(j)\left(q(j+1)-1\right), \quad n \to \infty.$$

For example, the combination $w(j) = (j+1)^{-\alpha}$, $\alpha \leq -1$ and $q(j) = j+1$ fulfills the conditions of the theorem.

4.8 Notes

The survey [KPT85] defines the general digital sum problem for Cantor representations of integers and includes an extensive bibliography; the introduction to this chapter is modelled on [KPT85, p. 55-56]; I also consulted [KT84]. Theorem 4.2.1 can be found on [KPT85, p. 63-64]. Theorem 4.7.1 is similar to a result on [KPT85, p. 56], which is obtained with real-variable, Delange-type methods and contains a different form of the error term. A different proof of Theorem 4.7.2 is given on [KPT85, p. 58].

A more up-to-date introduction can be found on [FGK$^+$94, p. 292-295]; the proof of Delange's theorem concerning binary digital sums is on [FGK$^+$94, p. 297-299]; digital sums with exponential weights are treated on [FGK$^+$94, p. 303-304].

Chapter 5

Counting sums of three squares

This chapter presents a new proof of a result due to Osbaldestin and Shiu concerning integers representable as sums of three squares. Their papers ([Shi88], [OS89]) use real-variable methods of the Delange type; we will use the Mellin-Perron formula for $m = 1$.

5.1 Preliminaries

Definition 5.1.1 *Let Q be the set of integers $n \in \mathbb{Z}^+$ representable as sums of three squares including 0.*

Lemma 5.1.1 *If $n = 4^l(8k+7)$, where $l, k \in \mathbb{Z}^+$, then n is not representable as a sum of three squares; i.e. $n \in \bar{Q}$.*

Proof. Note that $0, 1, 4$ are the only quadratic residues modulo 8. Hence $x_1^2 + x_2^2 + x_3^2 \not\equiv 7(8)$. Suppose $4^l(8k+7)$ cannot be represented as a sum of three squares and $4^{l+1}(8k+7)$ can, i.e. $4^{l+1}(8k+7) = x_1^2 + x_2^2 + x_3^2$. This implies $4^l(8k+7) = (x_1/2)^2 + (x_2/2)^2 + (x_3/2)^2$, a contradiction. (Note that $x_1^2 + x_2^2 + x_3^2 \equiv 0 \mod 4$ implies $x_{1,2,3} \equiv 0 \mod 2$.) ∎

In fact all n not of the form $4^l(8k+7)$ are representable as a sum of three squares. The following theorem is due to Gauss.

Theorem 5.1.1 *A positive integer n is representable as the sum of three squares if and only if there do not exist $k, l \in \mathbb{Z}^+$ such that $n = 4^l(8k+7)$.*

We let $k(n)$ be the characteristic function of $\bar{Q}$, i.e.

$$k(n) = \begin{cases} 1 & \text{if } n \in \bar{Q}, \\ 0 & \text{if } n \in Q \end{cases}$$

or equivalently

$$k(n) = \begin{cases} 1 & \text{if } n = 4^l(8k+7), \text{ where } l, k \in \mathbb{Z}^+ \\ 0 & \text{otherwise.} \end{cases}$$

Note that $8k+7 = 4(2k+1)+3$. This shows that $k(n)$ is the characteristic function of those integers whose base-four representation ends in a 1 or a 3, followed by a 3, followed by a possibly empty string of zeros. Let

$$Q(N) = \sum_{n \in Q, 0<n\le N} 1 = N - \sum_{0<n\le N} k(n).$$

Define $\triangle(N)$ as follows:

$$Q(N) = \frac{5}{6}N + \triangle(N),$$

i.e.

$$\triangle(N) = \frac{1}{6}N - \sum_{0<n\le N} k(n)$$

and let $\triangle(0) = 0$. Osbaldestin and Shiu consider the average order of $\triangle(N)$, which is given by

$$\begin{aligned} \frac{1}{N}\sum_{0\le n<N} \triangle(n) &= \frac{1}{N}\sum_{n=1}^{N-1}\left(\frac{1}{6}n - \sum_{0<l\le n} k(l)\right) \\ &= \frac{1}{N}\frac{1}{6}\frac{1}{2}(N-1)N - \frac{1}{N}\sum_{n=1}^{N-1}\sum_{l=1}^{n} k(l) = \frac{1}{12}N - \frac{1}{12} - \frac{1}{N}\sum_{n=1}^{N-1}\sum_{l=1}^{n} k(l). \end{aligned}$$

They prove the following theorem.

Theorem 5.1.2 [OS89] *There exists a periodic function $F(u)$ with period 1 such that for $N \ge 1$,*

$$\frac{1}{N}\sum_{0\le n<N} \triangle(n) = \frac{3}{8}L + F(L) + \frac{\delta(N)}{N}$$

where

$$L = \frac{\log N}{\log 4} \quad \text{and} \quad \delta(N) = \begin{cases} \frac{1}{8} & N \text{ odd,} \\ 0 & N \text{ even.} \end{cases}$$

The function $F(u)$ is a Fourier series

$$\sum_{k\in\mathbb{Z}} c_k e^{2\pi i k u}$$

with coefficients

$$c_0 = -\frac{31}{48} - \frac{3}{8\log 4} - \frac{1}{\log 4}(\zeta'(0, 7/16) + \zeta'(0, 15/16))$$

and

$$c_k = -\frac{1}{2\pi i k}\left(1 + \frac{2\pi i k}{\log 4}\right)^{-1}\left(\zeta\left(\frac{2\pi i k}{\log 4}, \frac{7}{16}\right) + \zeta\left(\frac{2\pi i k}{\log 4}, \frac{15}{16}\right)\right), k \ne 0.$$

We present a new proof of this theorem in the remainder of this chapter.

5.2 Application of the Mellin-Perron formula

The closed form of the Dirichlet generating function $K(s) = \sum k(n)/n^s$ of $k(n)$ is obtained as follows.

$$\begin{aligned} K(s) &= \sum \frac{k(n)}{n^s} = \sum \frac{k(4n)}{(4n)^s} + \sum_{n\geq 0} \frac{k(4n+3)}{(4n+3)^s} \\ &= \frac{1}{4^s} \sum \frac{k(n)}{n^s} + \sum_{n\geq 0} \frac{k(16n+(13)_4)}{(16n+(13)_4)^s} + \sum_{n\geq 0} \frac{k(16n+(33)_4)}{(16n+(33)_4)^s} \\ &= \frac{1}{4^s} K(s) + \frac{1}{16^s} \sum_{n\geq 0} \frac{1}{(n+7/16)^s} + \frac{1}{16^s} \sum_{n\geq 0} \frac{1}{(n+15/16)^s} \end{aligned}$$

We conclude that

$$K(s) = \frac{4^s}{4^s-1} \frac{1}{16^s}(\zeta(s,7/16)+\zeta(s,15/16)) = \frac{1}{4^s-1}\frac{1}{4^s}(\zeta(s,7/16)+\zeta(s,15/16)).$$

Let

$$L(s) = K(s)\frac{N^s}{s(s+1)}.$$

The Mellin-Perron formula (Lemma 2.9.3) tells us that

$$\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l=1}^{n} k(l) = \frac{1}{N}\frac{N}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} L(s)ds = \frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} L(s)ds.$$

We evaluate this integral by means of the shifting lemma (Lemma 2.8.1), taking

$$\Phi(s) = \frac{1}{4^s-1}\frac{N^s}{4^s}$$

and

$$T_j = \frac{(2j+1)\pi}{\log 4}.$$

Note that

$$|\Phi(s)| = \left(\frac{N}{4}\right)^{\sigma} \frac{1}{|4^s-1|}.$$

Along the vertical segments situated at α and c

$$|\Phi(s)| < \left(\frac{N}{4}\right)^{\alpha} \frac{1}{1-4^{\alpha}} = M_{\alpha} \quad \text{and} \quad |\Phi(s)| < \left(\frac{N}{4}\right)^{c} \frac{1}{4^c-1} = M_c$$

respectively. Along the horizontal segments at $\pm iT_j$

$$|\Phi(s)| < \left(\frac{N}{4}\right)^{c} \frac{1}{|-4^{\sigma}-1|} < \left(\frac{N}{4}\right)^{c} \frac{1}{1+4^{\alpha}} = M_T.$$

Therefore the bound independent of σ, t on $\Phi(s)$ that we require in order to apply the lemma is $M = \max\{M_c, M_\alpha, M_T\}$. We have

$$\begin{aligned}\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} L(s)ds &= \frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} L(s)ds \\ &+\mathrm{Res}\left(L(s); s=1\right) + \mathrm{Res}\left(L(s); s=0\right) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathrm{Res}\left(L(s); s = \frac{2\pi ik}{\log 4}\right).\end{aligned}$$

Let $\zeta_0 = \zeta(0, 7/16) + \zeta(0, 15/16) = 1/2 - 7/16 + 1/2 - 15/16 = -3/8$ and let $\zeta_1 = \zeta'(0, 7/16) + \zeta'(0, 15/16)$. Finally, let $\chi_k = 2\pi ik/\log 4$ when $k \neq 0$.

$$\begin{aligned}
\mathrm{Res}(L(s); s=0) &= \lim_{s\to 0}\left(K(s)\frac{N^s}{s(s+1)}s^2\right)' = \lim_{s\to 0}\left(sK(s)\frac{N^s}{s+1}\right)' \\
&= \lim_{s\to 0} sK(s)\left(\log N\ N^s(s+1)^{-1} - N^s(s+1)^{-2}\right) + \lim_{s\to 0}\left(sK(s)\right)'\frac{N^s}{s+1} \\
&= (\log N - 1)\lim_{s\to 0}\frac{s}{4^s-1}\frac{1}{4^s}(\zeta(s,7/16)+\zeta(s,15/16)) + \\
&\quad \lim_{s\to 0}\frac{s}{4^s-1}\frac{1}{4^s}(\zeta'(s,7/16)+\zeta'(s,15/16)) - \\
&\quad \lim_{s\to 0}\frac{s}{4^s-1}\frac{\log 4}{4^s}(\zeta(s,7/16)+\zeta(s,15/16)) + \\
&\quad \lim_{s\to 0}\frac{4^s-1-\log 4\ s4^s}{(4^s-1)^2}\frac{1}{4^s}(\zeta(s,7/16)+\zeta(s,15/16)) \\
&= (\log n - 1)\zeta_0 \lim_{s\to 0}\frac{1}{\log 4\ 4^s} + \\
&\quad \zeta_1\lim_{s\to 0}\frac{1}{\log 4\ 4^s} - \log 4\zeta_0\lim_{s\to 0}\frac{1}{\log 4\ 4^s} - \zeta_0\lim_{s\to 0}\frac{\log 4\log 4s4^s}{2(4^s-1)\log 4\ 4^s} \\
&= (\log_4 N - 1/\log 4 - 1)\zeta_0 + \frac{1}{\log 4}\zeta_1 - \zeta_0\lim_{s\to 0}\frac{\log 4s}{2(4^s-1)} \\
&= (\log_4 N - 1/\log 4 - 1)\zeta_0 + \frac{1}{\log 4}\zeta_1 - \zeta_0\lim_{s\to 0}\frac{1}{2\ 4^s} \\
&= -\frac{3}{8}(\log_4 N - 1/\log 4 - 3/2) + \frac{1}{\log 4}\zeta_1 \\
\mathrm{Res}(L(s); s=1) &= \lim_{s\to 1}(s-1)K(s)\frac{N^s}{s(s+1)} = \frac{N}{2}\lim_{s\to 1}\frac{s}{4^s-1}\frac{1}{4^s}(s-1)(\zeta(s,7/16)+\zeta(s,15/16)) \\
&= \frac{N}{2}\frac{1}{12}2 = \frac{N}{12} \\
\mathrm{Res}\left(L(s); s=\frac{2\pi ik}{\log 4}\right) &= \lim_{s\to\chi_k}(s-\chi_k)K(s)\frac{N^s}{s(s+1)} \\
&= \frac{e^{2\pi ik\log_4 N}}{\chi_k(\chi_k+1)}\lim_{s\to\chi_k}\frac{s-\chi_k}{4^s-1}\frac{1}{4^s}((\zeta(s,7/16)+\zeta(s,15/16)) \\
&= \frac{e^{2\pi ik\log_4 N}}{\chi_k(\chi_k+1)}((\zeta(\chi_k,7/16)+\zeta(\chi_k,15/16))\lim_{s\to\chi_k}\frac{1}{\log 4\ 4^s} \\
&= \frac{1}{\log 4}\frac{e^{2\pi ik\log_4 N}}{\chi_k(\chi_k+1)}((\zeta(\chi_k,7/16)+\zeta(\chi_k,15/16))
\end{aligned}$$

We note that

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} \operatorname{Res}\left(L(s); s = \frac{2\pi i k}{\log 4}\right) = -(F(\log_4 N) - c_0).$$

In order to evaluate

$$\frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} L(s)ds = \frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} \frac{1}{4^s-1}\frac{1}{4^s}(\zeta(s,7/16)+\zeta(s,15/16))\frac{N^s}{s(s+1)}ds$$

we note that we may use the expansion

$$\frac{1}{4^s-1}\frac{1}{4^s} = -\sum_{k=0}\frac{4^{ks}}{4^s} = -\frac{1}{4^s} - \sum_{k=0} 4^{ks}$$

since $\sigma = -1/4$. By Corollary 2.10.1 all integrals of the form

$$\frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} (\zeta(s,7/16)+\zeta(s,15/16))\frac{(4^k n)^s}{s(s+1)}ds$$

are zero. (The integer $4^k N$ takes the place of the integer n.) We apply Corollary 2.10.2 in the manner shown in the associated example and obtain

$$\frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} L(s)ds = -\frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} \frac{1}{4^s}(\zeta(s,7/16)+\zeta(s,15/16))\frac{N^s}{s(s+1)}ds = -\frac{\delta(N)}{N}.$$

We are now able to compute the average order of $\triangle(n)$ and complete the proof of the Osbaldestin-Shiu result.

$$\begin{aligned}
\frac{1}{N}\sum_{0\le n<N}\triangle(n) &= \frac{1}{12}N - \frac{1}{12} - \frac{1}{N}\sum_{n=1}^{N-1}\sum_{l=1}^{n} k(l) = \frac{1}{12}N - \frac{1}{12} - \frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} L(s)ds \\
&= \frac{1}{12}N - \frac{1}{12} \\
&\quad - \left(-\frac{\delta(N)}{N} - \frac{3}{8}(\log_4 N - 1/\log 4 - 3/2) + \frac{1}{\log 4}\zeta_1 + \frac{N}{12} - (F(\log_4 N) - c_0)\right) \\
&= \frac{3}{8}\log_4 N + F(\log_4 N) + \frac{\delta(N)}{N} - c_0 - \frac{1}{12} - \frac{27}{48} - \frac{3}{8\log 4} - \frac{1}{\log 4}\zeta_1 \\
&= \frac{3}{8}\log_4 N + F(\log_4 N) + \frac{\delta(N)}{N}
\end{aligned}$$

This concludes the proof. ∎

5.3 Notes

The introduction is modelled on [Kra81, p. 162-163]. The Fourier series expansion is developed on [OS89, p. 373-374].

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