Egorychev method and the evaluation of combinatorial sums: Part 3: complex variables

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The Egorychev method is from the book by G.P.Egorychev [Ego84]. We collect several examples, the focus being on computational methods to produce results. Those that are from posts to math.stackexchange.com have retained the question answer format from that site. The website for this document is at this hyperlink:
https://pnp.mathematik.uni-stuttgart.de/iadm/Riedel/egorychev.html.

The crux of the method is the use of integrals from the Cauchy Residue Theorem to represent binomial coefficients, exponentials, the Iverson bracket and Stirling numbers, Catalan numbers, Harmonic numbers, Eulerian numbers and Bernoulli numbers. There is a tutorial at the following article: [RM23].

We use these types of integrals:

- *First binomial coefficient integral (B_1)*

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^n}{z^{k+1}}\, dz = \operatorname*{res}_z \frac{(1+z)^n}{z^{k+1}}$$

  where $0 < \varepsilon < \infty$.

- *Second binomial coefficient integral (B_2)*

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{(1-z)^{k+1} z^{n-k+1}}\, dz = \operatorname*{res}_z \frac{1}{(1-z)^{k+1} z^{n-k+1}}$$

  where $0 < \varepsilon < 1$.

- *Exponentiation integral (E)*

$$n^k = \frac{k!}{2\pi i}\int_{|z|=\varepsilon} \frac{\exp(nz)}{z^{k+1}}\, dz = k! \operatorname*{res}_z \frac{\exp(nz)}{z^{k+1}}$$

  where $0 < \varepsilon < \infty$.

- *Iverson bracket (I)*

$$[[k \leq n]] = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{z^k}{z^{n+1}} \frac{1}{1-z} \; dz = \operatorname*{res}_z \frac{z^k}{z^{n+1}} \frac{1}{1-z}$$

  where $0 < \varepsilon < 1$.

- *Stirling numbers of the first kind*

$$\left[\begin{matrix} n \\ k \end{matrix}\right] = \frac{n!}{k!} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \left(\log \frac{1}{1-z} \right)^k \; dz = \frac{n!}{k!} \operatorname*{res}_z \frac{1}{z^{n+1}} \left(\log \frac{1}{1-z} \right)^k$$

  where $0 < \varepsilon < 1$.

- *Stirling numbers of the second kind*

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\} = \frac{n!}{k!} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \left(\exp(z) - 1\right)^k \; dz = \frac{n!}{k!} \operatorname*{res}_z \frac{1}{z^{n+1}} \left(\exp(z) - 1\right)^k$$

  where $0 < \varepsilon < \infty$.

The residue at infinity is coded R.

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List of identities in this document

section 1 B_1

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{k}{j}=(-1)^n\binom{n}{j}\binom{n+j}{j}.$$

section 2 B_1B_2

$$\sum_{k=0}^{r}\binom{r-k}{m}\binom{s+k}{n}=\binom{s+r+1}{n+m+1}.$$

section 3 B_1

$$\sum_{q=0}^{2m}(-1)^q\binom{p-1+q}{q}\binom{2m+2p+q-1}{2m-q}2^q=(-1)^m\binom{p-1+m}{m}.$$

section 4 B_1

$$\sum_{k=0}^{\lfloor m/2\rfloor}\binom{n}{k}(-1)^k\binom{m-2k+n-1}{n-1}=\binom{n}{m}.$$

section 5 B_2

$$\sum_{k=0}^{n}k\binom{2n}{n+k}=\frac{1}{2}n\binom{2n}{n}.$$

section 6 I_1

$$\sum_{k=0}^{n}2^{-k}\binom{n+k}{k}=2^n.$$

section 7 B_1

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{n+1}\frac{1}{a+bk+1}\binom{a+bk}{m}\binom{k-n-1}{k}=\binom{n}{m}.$$

section 8 B_1, R

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{m+k}{2n}=\binom{2m}{2n}.$$

section 9 B_1E

$$(-1)^p \sum_{q=r}^{p} \binom{p}{q}\binom{q}{r}(-1)^q q^{p-r} = \frac{p!}{r!}.$$

section 10 B_1

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor} = \binom{2n+1}{n}.$$

section 11 B_1

Verify that $f_1(n,k) = f_2(n,k)$ where

$$f_1(n,k) = \sum_{v=0}^{n} \frac{(2k+2v)!}{(k+v)! \times v! \times (2k+v)! \times (n-v)!} 2^{-v}$$

and

$$f_2(n,k) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{(k+m)! \times m! \times (n-2m)!} 2^{n-4m}.$$

section 12 B_1

If

$$T(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} T(n-k)$$

for $n \geq 2$ then

$$T(n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

section 13 B_1

$$\sum_{k=0}^{n}\sum_{l=0}^{n} (-1)^{k+l} \binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{k}\binom{n}{l} = (-1)^m \binom{2m}{m}.$$

section 14 B_1

$$\sum_{k=0}^{2m+1} \binom{n}{k} 2^k \binom{n-k}{\lfloor (2m+1-k)/2 \rfloor} = \binom{2n+1}{2m+1}.$$

section 15 B_1

$$\sum_{k=m}^{n} (-1)^{n+k} \frac{2k+1}{n+k+1} \binom{n}{k} \binom{n+k}{k}^{-1} \binom{k}{m} \binom{k+m}{m} = \delta_{mn}.$$

section 16 B_1

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=\lceil n/2 \rceil}^{n} \binom{n}{k}^2 \binom{2k}{n}.$$

section 17 B_1

$$\sum_{s} \binom{n+s}{k+l} \binom{k}{s} \binom{l}{s} = \binom{n}{k} \binom{n}{l}.$$

section 18 B_1

$$\sum_{k=-\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k \binom{2n}{n+3k} = 2 \times 3^{n-1}.$$

section ?? $B_1 B_2$

$$\sum_{j=0}^{b} \binom{b}{j}^2 \binom{n+j}{2b} = \binom{n}{b}^2.$$

section 19 B_1

$$\sum_{k=0}^{\rho} \binom{2x+1}{2k} \binom{x-k}{\rho-k} = \frac{2x+1}{2\rho+1} \binom{x+\rho}{2\rho} 2^{2\rho}.$$

section 20 $B_1 B_2$

$$\sum_{k=0}^{\min(a,b)} \binom{x+y+k}{k} \binom{x}{b-k} \binom{y}{a-k} = \binom{x+a}{b} \binom{y+b}{a}$$

section 21 $B_1 I$

$$\sum_{q=0}^{n} \binom{n}{2q} \binom{n-2q}{p-q} 2^{2q} = \binom{2n}{2p}.$$

section 22 $B_1 R$

$$\sum_{k=0}^{n-1} \left(\sum_{q=0}^{k} \binom{n}{q} \right) \left(\sum_{q=k+1}^{n} \binom{n}{q} \right) = \frac{1}{2} n \binom{2n}{n}.$$

section 23 B_1

$$(1-x)^{2k+1}\sum_{n\geq 0}\binom{n+k-1}{k}\binom{n+k}{k}x^n=\sum_{j\geq 0}\binom{k-1}{j-1}\binom{k+1}{j}x^j.$$

section 24 B_1

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k\binom{n+1}{k}\binom{2n-3k}{n}=\sum_{k=\lfloor n/2\rfloor}^{n}\binom{n+1}{k}\binom{k}{n-k}.$$

section 25 B_1

$$\sum_{k=0}^{\lfloor (m+n)/2\rfloor}\binom{n}{k}(-1)^k\binom{m+n-2k}{n-1}=\binom{n}{m+1}.$$

section 26 B_1

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}F_{k+1}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}F_{2k+1}.$$

section 27 B_2R

$$\sum_{p,q\geq 0}\binom{n-p}{q}\binom{n-q}{p}=F_{2n+2}.$$

section 28 B_1I

$$\sum_{r=0}^{n}\binom{r+n-1}{n-1}\binom{3n-r}{n}=\frac{1}{2}\left(\binom{4n}{2n}+\binom{2n}{n}^2\right).$$

section 29 B_1

$$[x^\mu y^\nu]\frac{1}{2}\left(1-x-y-\sqrt{1-2x-2y-2xy+x^2+y^2}\right)=\frac{1}{\mu+\nu-1}\binom{\mu+\nu-1}{\nu}\binom{\mu+\nu-1}{\mu}.$$

section 30 B_1

$$\sum_{r=1}^{n+1}\frac{1}{r+1}\binom{2r}{r}\binom{m+n-2r}{n+1-r}=\binom{m+n}{n}.$$

section 31 B_1

$$\sum_{k\geq 0}\binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}=\binom{n-1}{m-1}.$$

section 32 B_1

$$\sum_k \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{r}{tk+r} = \binom{tn+r+s}{n}$$

section 33 B_1

$$\sum_{q=0}^{n-2}\sum_{k=1}^{n}\binom{k+q}{k}\binom{2n-q-k-1}{n-k+1} = n \times \binom{2n}{n+2}$$

section 34 $B_1 I$

$$\sum_{q=0}^{l}\binom{q+k}{k}\binom{l-q}{k} = \binom{l+k+1}{2k+1},$$

section 35 $B_1 I$

$$\sum_{k=0}^{n} k\binom{m+k}{m+1} = \frac{nm+2n+1}{m+3}\binom{n+m+1}{m+2}.$$

section 36 $B_1 IR$

$$\sum_{k=1}^{n} 2^{n-k}\binom{k}{\lfloor k/2 \rfloor} = -2^{n+1} + (2n+2+(n \mod 2))\binom{n}{\lfloor n/2 \rfloor}.$$

section 37 IR

$$\sum_{q=0}^{m-1}\binom{n-1+q}{q}x^n(1-x)^q + \sum_{q=0}^{n-1}\binom{m-1+q}{q}x^q(1-x)^m = 1$$

where $n, m \geq 1$
as well as

$$\sum_{k=0}^{n}\binom{m+k}{k}2^{n-k} + \sum_{k=0}^{m}\binom{n+k}{k}2^{m-k} = 2^{n+m+1}.$$

section 38 B_1

$$\sum_{k=0}^{n}(-1)^k\binom{p+q+1}{k}\binom{p+n-k}{n-k}\binom{q+n-k}{n-k} = \binom{p}{n}\binom{q}{n}.$$

section 39 B_1

$$\sum_{k=0}^{n}\binom{n}{k}\binom{pn-n}{k}\binom{pn+k}{k} = \binom{pn}{n}^2.$$

section 40 B_1

$$\sum_{r=0}^{\min\{m,n,p\}} \binom{m}{r}\binom{n}{r}\binom{p+m+n-r}{m+n} = \binom{p+m}{m}\binom{p+n}{n}.$$

section 41 B_1

$$\sum_{p=0}^{l}\sum_{q=0}^{p}(-1)^q\binom{m-p}{m-l}\binom{n}{q}\binom{m-n}{p-q} = 2^l\binom{m-n}{l}$$

section 42 E

$$\sum_{q=0}^{n}(n-2q)^k\binom{n}{2q+1} = \sum_{q=0}^{k+1}\binom{n}{q}2^{n-q-1}\times q!\times\left\{{k+1 \atop q+1}\right\} - \frac{1}{2}\times n!\times\left\{{k+1 \atop n+1}\right\}.$$

section 43 B_1B_2R

$$\sum_{q=0}^{n} q\binom{2n}{n+q}\binom{m+q-1}{2m-1} = m\times 4^{n-m}\times\binom{n}{m}$$

where $n \geq m$.

section 44 B_1B_2R

$$\sum_{q=0}^{n} q\binom{2n}{n+q}\binom{m+q-1}{2m-1} = m\times 4^{n-m}\times\binom{n}{m}$$

where $n \geq m$.

(different proof).

section 45 B_1EIR

With

$$b_k^n = \sum_{l=1}^{k}(-1)^{k-l}l^n\binom{n+1}{k-l}$$

and we

show that $b_k^n = b_{n+1-k}^n$ where $0 \leq k \leq n+1$.

section 46 B_1B_2

Suppose we have a random variable X where

$$\mathrm{P}[X=k] = \binom{N}{2n+1}^{-1}\binom{N-k}{n}\binom{k-1}{n}$$

for $k = n+1, \ldots, N-n$ and zero otherwise.

We seek to show that these probabilities sum to one and compute the the mean and the variance.

section 47 B_1

Suppose we have the Narayana number

$$N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$$

and let

$$A(n,k,l) = \sum_{\substack{i_0+i_1+\cdots+i_k=n\\ j_0+j_1+\cdots+j_k=l}} \prod_{t=0}^{k} N(i_t, j_t+1)$$

where the compositions for n are regular and the ones for l are weak and we seek to verify that

$$A(n,k,l) = \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1}.$$

section 48 B_1

Same as previous, generalized.

section 49 B_1

$$(-1)^m \frac{(n+m)!}{(n-m)!}\left(\frac{d}{dz}\right)^{n-m}(1-z^2)^n = (1-z^2)^m \left(\frac{d}{dz}\right)^{n+m}(1-z^2)^n$$

section 50 B_1

$$r^k (r+n)! = \sum_{m=0}^{k} (r+n+m)!(-1)^{k+m} \sum_{p=0}^{k-m} \binom{k}{p} \left\{ {k+1-p \atop m+1} \right\} n^p.$$

section 51 B_1

$$\sum_{k=0}^{n}\sum_{q=0}^{k}(-1)^q \binom{k}{q}\binom{n-1-qm}{k-1} = [z^n]\frac{1}{1-w-w^2-\cdots-w^m}.$$

section 52

$$\left\{ {n \atop m} \right\} = \sum_{k=m}^{n}\binom{k}{m}\sum_{q=0}^{k}(-1)^n \left\{ {n+q-m \atop k} \right\} (-1)^k \left[{k \atop q} \right] \binom{n}{n+q-m}$$

section 53 B_1

$$\sum_{k=0}^{m} \frac{q}{pk+q}\binom{pk+q}{k}\binom{pm-pk}{m-k} = \binom{mp+q}{m}.$$

section 54 B_1B_2

$$\sum_{k=q}^{n-1} \frac{q}{k}\binom{2n-2k-2}{n-k-1}\binom{2k-q-1}{k-1} = \binom{2n-q-2}{n-1}.$$

section 55

$$\sum_{k=0}^{n} \frac{x}{x+kz}\binom{x+kz}{k}\frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}$$

section 56

$$P_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(y)$$

where

$$P_n(x) = x(x+an)^{n-1}$$

is an Abel polynomial.

section 57

$$\sum_{m=0}^{n}(-1)^m\binom{2n+2m}{n+m}\binom{n+m}{n-m} = (-1)^n 2^{2n}.$$

section 58

$$\sum_{\substack{k=0\\ k\text{ odd}}}^{m}\binom{2n}{2n-k}\binom{2m-2n}{m-k} = \frac{1}{2}\binom{2m}{m} + (-1)^{m+1}2^{2m-1}\binom{n-1/2}{m}$$

section 59

$$\sum_{j=0}^{n}(-1)^{n+j}\begin{bmatrix} n\\ j\end{bmatrix}\begin{Bmatrix} m+j\\ k\end{Bmatrix} = \frac{n!}{k!}\sum_{q=0}^{k}\binom{k}{q}\binom{q}{n}(-1)^{k-q}q^m.$$

section 60

$$[z^k]\frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n = \binom{n+2k}{k}$$

section 61.1

$$\sum_{k=1}^{m-1} \sin^{2q}(k\pi/m) = m\frac{1}{2^{2q}}\binom{2q}{q} + m\frac{1}{2^{2q-1}}\sum_{l=1}^{\lfloor q/m \rfloor}\binom{2q}{q-lm}(-1)^{lm}.$$

section 61.2 B_1

$$\sum_{j=-\lfloor n/p \rfloor}^{\lfloor n/p \rfloor}(-1)^j\binom{2n}{n-pj}$$
$$= [z^n]\left(\sum_{q=0}^{\lfloor p/2 \rfloor}\frac{p}{p-q}\binom{p-q}{q}(-1)^q z^q\right)^{-1}\sum_{q=0}^{\lfloor (p-1)/2 \rfloor}\binom{p-1-q}{q}(-1)^q z^q$$

section 61.3 B_1

$$\sum_{k=0}^{n}\binom{2k+1}{k}\binom{m-(2k+1)}{n-k} = \sum_{k=0}^{n}\binom{m+1}{k}.$$

section 61.4

$$\sum_{m\geq 0} m^{m+n}\frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}}\sum_{k=0}^{n}\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k$$

section 61.5

$$\sum_{n\geq 0}\left\{ {n+r \atop n} \right\} z^n = \frac{1}{(1-z)^{2r+1}}\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle z^k$$

section 61.5.1

$$\sum_{n\geq 0}\left[{n+r+1 \atop n+1} \right] z^n = \frac{1}{(1-z)^{2r+1}}\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^k$$

section 61.6

$$\binom{q-j+k}{k}+(-1)^k\binom{j}{k} = \sum_{\ell=0}^{\lfloor k/2 \rfloor}\binom{q/2+\ell}{2\ell}\left(\binom{q/2-j+k-\ell}{k-2\ell}+\binom{q/2-j+k-\ell-1}{k-2\ell}\right)$$

section 61.7

$$\sum_{j=0}^{k}\binom{2j}{j+q}\binom{2k-2j}{k-j} = 4^k - \sum_{j=k-q+1}^{k}\binom{2k+1}{j}$$

section 61.8

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \binom{m+k}{2n} = \binom{2m}{2n}$$

section 61.9

$$\sum_{q=0}^{n} \binom{q}{n-q} (-1)^{n-q} \binom{2q+1}{q+1} = 2^{n+1} - 1$$

section 61.10

$$\sum_{k=0}^{n} \left\{ {n+k \atop k} \right\} \binom{2n}{n+k} \frac{(-1)^k}{k+1} = B_n \binom{2n}{n} \frac{1}{n+1}$$

section 61.11

With $f(z)$ the OGF and $g(w)$ the EGF of a sequence we have

$$g(w) = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f(z)}{z} \exp(w/z)\, dz$$

section 61.12

$$\sum_{q=0}^{r} (-1)^{q+r} \left[{r \atop q} \right] \left\{ {n+q-1 \atop k} \right\} = \frac{(-1)^{k-r}}{(k-r)!} \sum_{p=0}^{k-r} \binom{k-r}{p} (-1)^p (p+r)^{n-1}$$

section 61.13

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k/2}{m} = \frac{n}{m} \binom{n-m-1}{m-1} 2^{n-2m}$$

section 61.14

$$[z^n] \log^2 \frac{2}{1+\sqrt{1-4z}} = \binom{2n}{n} (H_{2n-1} - H_n) \frac{1}{n}$$

section 61.15

$$\sum_{k=0}^{n} \left[{n+1 \atop k+1} \right] B_k = \frac{n!}{n+1}$$

section 61.16

$$\sum_{q=0}^{K} (-1)^q \binom{2n+1-q}{q} \binom{2n-2q}{K-q} = \frac{1}{2}(1 + (-1)^K)$$

1 Introductory example for the method (B_1)

Suppose we seek to evaluate

$$S_j(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k}{j}$$

which is claimed to be

$$(-1)^n \binom{n}{j} \binom{n+j}{j}.$$

Introduce

$$\binom{n+k}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+k}}{z^{k+1}} \; dz$$

and

$$\binom{k}{j} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^k}{w^{j+1}} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1+z)^k (1+w)^k}{z^k} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \left(1 - \frac{(1+w)(1+z)}{z}\right)^n \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \left(-1 - w - wz\right)^n \; dw \; dz$$

$$= \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \left(1 + w + wz\right)^n \; dw \; dz.$$

This is

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \sum_{q=0}^{n} \binom{n}{q} w^q (1+z)^q \; dw \; dz.$$

Extracting the residue at $w = 0$ we get

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \binom{n}{j} (1+z)^j \; dz$$

$$= \binom{n}{j} \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+j}}{z^{n+1}} \; dz$$

$$= (-1)^n \binom{n}{j} \binom{n+j}{n}.$$

thus proving the claim.

This is math.stackexchange.com problem 1331507.

2 Introductory example for the method, convergence about zero (B_1B_2)

Suppose we seek to evaluate

$$\sum_{k=0}^{r} \binom{r-k}{m} \binom{s+k}{n}$$

where $n \geq s$ and $m \leq r$.

Introduce

$$\binom{r-k}{m} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-k-m+1}} \frac{1}{(1-z)^{m+1}} \; dz.$$

Note that this is zero when $k > r-m$ so we may extend the sum in k to $k = \infty$.

Introduce furthermore

$$\binom{s+k}{n} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{s+k}}{w^{n+1}} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^s}{w^{n+1}} \sum_{k\geq 0} z^k (1+w)^k \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^s}{w^{n+1}} \frac{1}{1-(1+w)z} \; dw \; dz.$$

For the geometric series to converge we must have $|z(1+w)| < 1$, which also ensures that the inner pole is not inside the contour. Observe that $|z(1+w)| = \epsilon|1+w| \leq \epsilon(1+\gamma)$. So we need to choose $1+\gamma < 1/\epsilon$ with ϵ in a neighborhood of zero. The choice $\epsilon = 1/2$ and $\gamma = 1/2$ will work.

Continuing we find

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+2}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^s}{w^{n+1}} \frac{1}{1-wz/(1-z)} \; dw \; dz.$$

Extracting the inner residue we get

$$\sum_{q=0}^{n} \binom{s}{n-q} \frac{z^q}{(1-z)^q}.$$

Now
$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{r-m-q+1}}\frac{1}{(1-z)^{m+q+2}}\;dz = \binom{r+1}{m+q+1}$$
which yields for the sum
$$\sum_{q=0}^{n}\binom{s}{n-q}\binom{r+1}{m+q+1}.$$

Continue by re-indexing for
$$\sum_{q=0}^{s}\binom{s}{q}\binom{r+1}{m+n-q+1}$$

where we have lowered the upper limit to s since the first binomial coefficient is zero when $q > s$.

Using
$$\binom{r+1}{m+n-q+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{r+1}}{z^{m+n-q+2}}dz$$
we thus obtain for the sum
$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{r+1}}{z^{m+n+2}}\sum_{q=0}^{s}\binom{s}{q}z^q dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{r+s+1}}{z^{m+n+2}} = \binom{s+r+1}{n+m+1}.$$

Remark. This can be done using formal power series only.

We have for the sum
$$\sum_{k=0}^{r}\binom{r-k}{m}\binom{s+k}{n} = \sum_{k=0}^{r}[z^{r-k-m}]\frac{1}{(1-z)^{m+1}}[w^n](1+w)^{s+k}$$
$$= [z^{r-m}]\frac{1}{(1-z)^{m+1}}[w^n](1+w)^s\sum_{k=0}^{r}z^k(1+w)^k.$$

Now we may certainly extend the sum to infinity as there is no contribution to the coefficient extractor when $k > r-m$ (recall that $r \geq m$) getting
$$[z^{r-m}]\frac{1}{(1-z)^{m+1}}[w^n](1+w)^s\sum_{k\geq 0}z^k(1+w)^k$$
$$= [z^{r-m}]\frac{1}{(1-z)^{m+1}}[w^n](1+w)^s\frac{1}{1-z(1+w)}$$
$$= [z^{r-m}]\frac{1}{(1-z)^{m+1}}[w^n](1+w)^s\frac{1}{1-z-wz}$$

$$= [z^{r-m}]\frac{1}{(1-z)^{m+2}}[w^n](1+w)^s\frac{1}{1-wz/(1-z)}.$$

Now with $n \ge s$ we get for the inner coefficient

$$\sum_{q=0}^{s}\binom{s}{q}\frac{z^{n-q}}{(1-z)^{n-q}}.$$

Substitute into the outer coefficient extractor to get

$$[z^{r-m}]\frac{1}{(1-z)^{m+2}}\sum_{q=0}^{s}\binom{s}{q}\frac{z^{n-q}}{(1-z)^{n-q}} = [z^{r-m}]\sum_{q=0}^{s}\binom{s}{q}\frac{z^{n-q}}{(1-z)^{n+m+2-q}}$$

$$= \sum_{q=0}^{s}\binom{s}{q}[z^{r-m-n+q}]\frac{1}{(1-z)^{n+m+2-q}} = \sum_{q=0}^{s}\binom{s}{q}\binom{r+1}{n+m+1-q}$$

$$= \sum_{q=0}^{s}\binom{s}{q}[z^{n+m+1-q}](1+z)^{r+1} = [z^{n+m+1}](1+z)^{r+1}\sum_{q=0}^{s}\binom{s}{q}z^q$$

$$= [z^{n+m+1}](1+z)^{r+1}(1+z)^s = [z^{n+m+1}](1+z)^{r+s+1} = \binom{r+s+1}{n+m+1}.$$

This was math.stackexchange.com problem 928271.

3 Introductory example for the method, an interesting substitution (B_1)

Suppose we seek to verify that

$$\sum_{q=0}^{2m}(-1)^q\binom{p-1+q}{q}\binom{2m+2p+q-1}{2m-q}2^q = (-1)^m\binom{p-1+m}{m}.$$

Introduce

$$\binom{2m+2p+q-1}{2m-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-q+1}}(1+z)^{2m+2p+q-1}\;dz.$$

Observe that this controls the range being zero when $q > 2m$ so we may extend q to infinity to obtain for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m+1}}(1+z)^{2m+2p-1}\sum_{q\ge 0}\binom{p-1+q}{q}(-1)^q 2^q z^q(1+z)^q\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{2m+1}}(1+z)^{2m+2p-1}\frac{1}{(1+2z(z+1))^p}\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{2m+1}}(1+z)^{2m+2p-1}\frac{1}{((1+z)^2+z^2)^p}\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{2m+1}}(1+z)^{2m-1}\frac{1}{(1+z^2/(1+z)^2)^p}\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{2m}}(1+z)^{2m}\frac{1}{z(1+z)}\frac{1}{(1+z^2/(1+z)^2)^p}\; dz.$$

Now put

$$\frac{z}{1+z} = u \quad \text{so that} \quad z = \frac{u}{1-u} \quad \text{and} \quad dz = \frac{1}{(1-u)^2}du$$

to obtain for the integral

$$\frac{1}{2\pi i}\int_{|u|=\gamma} \frac{1}{u^{2m}}\frac{1}{u/(1-u)\times 1/(1-u)}\frac{1}{(1+u^2)^p}\frac{1}{(1-u)^2}\; du$$
$$= \frac{1}{2\pi i}\int_{|u|=\gamma} \frac{1}{u^{2m+1}}\frac{1}{(1+u^2)^p}\; du.$$

This is

$$[u^{2m}]\frac{1}{(1+u^2)^p} = [v^m]\frac{1}{(1+v)^p} = (-1)^m\binom{m+p-1}{m},$$

as claimed.

For the conditions on ϵ and γ we require convergence of the geometric series with $|2z(1+z)| < 1$ which holds for $\epsilon < (-1+\sqrt{3})/2$. Note that with $u = z + \cdots$ the image of $|z| = \epsilon$ makes one turn around zero. The closest it comes to the origin is at $\epsilon/(1+\epsilon)$ so we must choose $\gamma < \epsilon/(1+\epsilon)$ e.g. $\gamma = \epsilon^2/(1+\epsilon)$ for $|w| = \gamma$ to be entirely contained in the image of $|z| = \epsilon$. Taking $\epsilon = 1/5$ will work.

This was math.stackexchange.com problem 557982.

4 Introductory example for the method, another interesting substitution (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}(-1)^k\binom{m-2k+n-1}{n-1}$$

where $m \le n$ and introduce

$$\binom{m-2k+n-1}{n-1} = \binom{m-2k+n-1}{m-2k}$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m-2k+1}}(1+z)^{m-2k+n-1}\; dz$$

which has the property that it is zero when $2k > m$ so we may set the upper limit in the sum to n, getting

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+1}}(1+z)^{m+n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{z^{2k}}{(1+z)^{2k}}\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+1}}(1+z)^{m+n-1}\left(1-\frac{z^2}{(1+z)^2}\right)^n\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+1}}(1+z)^{m-n-1}(1+2z)^n\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^m}{z^m}\frac{1}{z(1+z)}\frac{(1+2z)^n}{(1+z)^n}\; dz.$$

Now put

$$\frac{1+2z}{1+z} = u \quad \text{so that} \quad z = -\frac{u-1}{u-2},\; 1+z = -\frac{1}{u-2},\; \frac{1+z}{z} = \frac{1}{u-1},$$
$$\frac{1}{z(1+z)} = \frac{(u-2)^2}{u-1} \quad \text{and} \quad dz = \frac{1}{(u-2)^2}\; du$$

to get for the integral

$$\frac{1}{2\pi i}\int_{|u-1|=\gamma} \frac{1}{(u-1)^m}\frac{(u-2)^2}{u-1}u^n\frac{1}{(u-2)^2}\; du$$
$$= \frac{1}{2\pi i}\int_{|u-1|=\gamma} \frac{1}{(u-1)^{m+1}}u^n\; du.$$

This is

$$[(u-1)^m]u^n = [(u-1)^m]\sum_{q=0}^{n}\binom{n}{q}(u-1)^q = \binom{n}{m}.$$

This solution is more complicated than the obvious one (which can be found at the stackexchange link) but it serves to illustrate the substitution aspect of the method.

Concerning the choice of ϵ and γ the closest that the image of $|z| = \epsilon$ which is $1 + \frac{z}{1+z}$, gets to one, is $\epsilon/(1+\epsilon)$ so that must be the upper bound for γ. Taking $\epsilon = 1/3$ and $\gamma = 1/5$ will work. Note also that $u = 1 + z + \cdots$ makes one turn around one.

This was math.stackexchange.com problem 1558659.

5 Introductory example for the method, yet another interesting substitution (B_2)

Suppose we seek to evaluate

$$\sum_{k=0}^{n} k \binom{2n}{n+k}.$$

Introduce

$$\binom{2n}{n+k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-k+1}} \frac{1}{(1-z)^{n+k+1}} \; dz.$$

Observe that this is zero when $k > n$ so we may extend k to infinity to obtain for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}} \sum_{k\ge 0} k \frac{z^k}{(1-z)^k} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}} \frac{z/(1-z)}{(1-z/(1-z))^2} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n}} \frac{1}{(1-z)^{n}} \frac{1}{(1-2z)^2} \; dz.$$

Now put $z(1-z) = w$ so that (observe that with $w = z + \cdots$ the image of $|z| = \epsilon$ with ϵ small is another closed circle-like contour which makes one turn and which we may certainly deform to obtain another circle $|w| = \gamma$)

$$z = \frac{1 - \sqrt{1-4w}}{2} \quad \text{and} \quad (1-2z)^2 = 1 - 4w$$

and furthermore

$$dz = -\frac{1}{2} \times \frac{1}{2} \times (-4) \times (1-4w)^{-1/2} \; dw = (1-4w)^{-1/2} \; dw$$

to get for the integral

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{1-4w} (1-4w)^{-1/2} \; dw = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{(1-4w)^{3/2}} \; dw.$$

This evaluates by inspection to

$$4^{n-1}\binom{n-1+1/2}{n-1} = 4^{n-1}\binom{n-1/2}{n-1} = \frac{4^{n-1}}{(n-1)!}\prod_{q=0}^{n-2}(n-1/2-q)$$

$$= \frac{2^{n-1}}{(n-1)!}\prod_{q=0}^{n-2}(2n-2q-1) = \frac{2^{n-1}}{(n-1)!}\frac{(2n-1)!}{2^{n-1}(n-1)!}$$

$$= \frac{n^2}{2n}\binom{2n}{n} = \frac{1}{2}n\binom{2n}{n}.$$

Here the mapping from $z = 0$ to $w = 0$ determines the choice of square root. For the conditions on ϵ and γ we have that for the series to converge we require $|z/(1-z)| < 1$ or $\epsilon/(1-\epsilon) < 1$ or $\epsilon < 1/2$. The closest that the image contour of $|z| = \epsilon$ comes to the origin is $\epsilon - \epsilon^2$ so we choose $\gamma < \epsilon - \epsilon^2$ for example $\gamma = \epsilon^2 - \epsilon^3$. This also ensures that $\gamma < 1/4$ so $|w| = \gamma$ does not intersect the branch cut $[1/4, \infty)$ (and is contained in the image of $|z| = \epsilon$). For example $\epsilon = 1/3$ and $\gamma = 2/27$ will work.

This was math.stackexchange.com problem 1585536.

Using formal power series

We may use the change of variables rule 1.8 (5) from the Egorychev text (page 16) on the integral

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^n}\frac{1}{(1-z)^n}\frac{1}{(1-2z)^2}\,dz = \operatorname*{res}_z \frac{1}{z^n}\frac{1}{(1-z)^n}\frac{1}{(1-2z)^2}$$

with $A(z) = \frac{z}{(1-2z)^2}$ and $f(z) = \frac{1}{1-z}$. We get $h(z) = z(1-z)$ and find

$$\operatorname*{res}_w \frac{1}{w^{n+1}}\left[\frac{A(z)}{f(z)h'(z)}\right]\Bigg|_{z=g(w).}$$

with g the inverse of h.

This becomes

$$\operatorname*{res}_w \frac{1}{w^{n+1}}\left[\frac{z/(1-2z)^2}{(1-2z)/(1-z)}\right]\Bigg|_{z=g(w)}$$

or alternatively

$$\operatorname*{res}_w \frac{1}{w^{n+1}}\left[\frac{z(1-z)}{(1-2z)^3}\right]\Bigg|_{z=g(w)} = \operatorname*{res}_w \frac{1}{w^n}\left[\frac{1}{(1-2z)^3}\right]\Bigg|_{z=g(w).}$$

Observe that $(1-2z)^2 = 1-4z+4z^2 = 1-4z(1-z) = 1-4w$ so this is

$$\operatorname*{res}_w \frac{1}{w^n}\frac{1}{(1-4w)^{3/2}}$$

and the rest of the computation continues as before.

This was math.stackexchange.com problem 4007052.

6 Introductory example for the method, using the Iverson bracket only (I_1)

Suppose we seek to verify that

$$S_n = \sum_{k=0}^{n} 2^{-k} \binom{n+k}{k} = 2^n.$$

We introduce the Iverson bracket

$$[[k \le n]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-k+1}} \frac{1}{1-z} \; dz$$

so we may extend k to infinity, getting

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1-z} \sum_{k\ge 0} 2^{-k} \binom{n+k}{n} z^k \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(1-z/2)^{n+1}} \; dz.$$

We evaluate this using the negative of the residues at $z=1, z=2$ and $z=\infty$. Here the contour does not include the other two finite poles which also ensures that the geometric series converges. We could choose $\epsilon = 1/2$. We get for the residue at $z=1$

$$-\frac{1}{(1/2)^{n+1}} = -2^{n+1}.$$

For the residue at $z=2$ we write

$$(-1)^{n+1} \mathrm{Res}_{z=2} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(z/2-1)^{n+1}}$$
$$= (-1)^{n+1} 2^{n+1} \mathrm{Res}_{z=2} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(z-2)^{n+1}}$$
$$= (-1)^n 2^{n+1} \mathrm{Res}_{z=2} \frac{1}{(2+(z-2))^{n+1}} \frac{1}{1+(z-2)} \frac{1}{(z-2)^{n+1}}$$
$$= (-1)^n \mathrm{Res}_{z=2} \frac{1}{(1+(z-2)/2)^{n+1}} \frac{1}{1+(z-2)} \frac{1}{(z-2)^{n+1}}.$$

This is

$$(-1)^n \sum_{q=0}^{n} (-1)^q \binom{n+q}{q} 2^{-q} (-1)^{n-q} = \sum_{q=0}^{n} \binom{n+q}{q} 2^{-q} = S_n.$$

Finally do the residue at $z = \infty$ getting (this also follows by inspection having degree zero in the numerator and degree $2n+3$ in the denominator)

$$\begin{aligned}
&\mathrm{Res}_{z=\infty} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(1-z/2)^{n+1}} \\
&= -\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n+1} \frac{1}{1-1/z} \frac{1}{(1-1/2/z)^{n+1}} \\
&= -\mathrm{Res}_{z=0} \frac{1}{z} z^{n+1} \frac{1}{z-1} \frac{z^{n+1}}{(z-1/2)^{n+1}} \\
&= -\mathrm{Res}_{z=0} z^{2n+1} \frac{1}{z-1} \frac{1}{(z-1/2)^{n+1}} = 0.
\end{aligned}$$

Using the fact that the residues sum to zero we thus obtain

$$S_n - 2^{n+1} + S_n = 0$$

which yields

$$\boxed{S_n = 2^n.}$$

This was math.stackexchange.com problem 389099.

7 Verifying that a certain sum vanishes (B_1)

Suppose we seek to evaluate

$$\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{n+1} \frac{1}{a+bk+1} \binom{a+bk}{m} \binom{k-n-1}{k}.$$

Now we have

$$\begin{aligned}
\binom{a+bk}{m} &= \sum_{q=0}^{m} (-1)^{m-q} \binom{a+bk+1}{q} \\
&= (-1)^m + \sum_{q=1}^{m} (-1)^{m-q} \binom{a+bk+1}{q}
\end{aligned}$$

and hence

$$\frac{1}{a+bk+1} \binom{a+bk}{m} = \frac{(-1)^m}{a+bk+1} + \sum_{q=1}^{m} \frac{1}{q} (-1)^{m-q} \binom{a+bk}{q-1}.$$

Now from the first component we get in the main sum

$$\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{n+1} \frac{(-1)^m}{a+bk+1} \binom{k-n-1}{k}$$
$$= \sum_{k=0}^{n+1} \frac{1}{a+bk+1} \binom{k-n-1}{k} \sum_{m=0}^{n} \binom{n}{m} (-1)^m = 0.$$

We are thus left with the following sum:

$$\sum_{k=0}^{n+1} \binom{k-n-1}{k} \sum_{m=0}^{n} \binom{n}{m} \sum_{q=1}^{m} \frac{1}{q} (-1)^{m-q} \binom{a+bk}{q-1}.$$

Working with the inner sum we obtain

$$\sum_{m=1}^{n} \binom{n}{m} \sum_{q=1}^{m} \frac{1}{q} (-1)^{m-q} \binom{a+bk}{q-1}$$
$$= \sum_{q=1}^{n} \frac{(-1)^q}{q} \binom{a+bk}{q-1} \sum_{m=q}^{n} \binom{n}{m} (-1)^m$$
$$= \sum_{q=1}^{n} \binom{n-1}{q-1} \frac{1}{q} \binom{a+bk}{q-1}$$
$$= \frac{1}{n} \sum_{q=1}^{n} \binom{n}{q} \binom{a+bk}{q-1}.$$

Now put

$$\binom{a+bk}{q-1} = \binom{a+bk}{a+bk-q+1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{a+bk-q+2}} (1+z)^{a+bk} \; dz$$

to get

$$\frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{a+bk+2}} (1+z)^{a+bk} \sum_{q=1}^{n} \binom{n}{q} z^q \; dz$$
$$= \frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{a+bk+2}} (1+z)^{a+bk} (-1 + (1+z)^n) \; dz$$

The inner constant term does not contribute and we are left with

$$\frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{a+bk+n}}{z^{a+bk+2}} \; dz = \frac{1}{n} \binom{a+bk+n}{a+bk+1} = \frac{1}{n} \binom{a+bk+n}{n-1}.$$

Returning to the main sum we thus have

$$\frac{1}{n}\sum_{k=0}^{n+1}\binom{k-n-1}{k}\binom{a+bk+n}{n-1}$$
$$=\frac{1}{n}\sum_{k=0}^{n+1}\binom{-k}{n+1-k}\binom{a+b(n+1)+n-bk}{n-1}.$$

Note that

$$\binom{-k}{n+1-k}=\frac{1}{(n+1-k)!}\prod_{q=0}^{n-k}(-k-q)=\frac{(-1)^{n-k+1}}{(n+1-k)!}\prod_{q=0}^{n-k}(k+q)$$
$$=\frac{(-1)^{n-k+1}}{(n+1-k)!}\frac{n!}{(k-1)!}=(-1)^{n-k+1}\binom{n}{k-1}.$$

This means for the main sum

$$\frac{(-1)^{n+1}}{n}\sum_{k=1}^{n+1}\binom{n}{k-1}(-1)^k\binom{a+b(n+1)+n-bk}{n-1}$$
$$=\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{a+bn+n-bk}{n-1}.$$

Introduce

$$\binom{a+bn+n-bk}{n-1}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}(1+z)^{a+bn+n-bk}\;dz$$

We get for the sum

$$\frac{(-1)^n}{n}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}(1+z)^{a+bn+n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(1+z)^{bk}}\;dz$$
$$=\frac{(-1)^n}{n}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}(1+z)^{a+bn+n}\left(1-\frac{1}{(1+z)^b}\right)^n\;dz$$
$$=\frac{(-1)^n}{n}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}(1+z)^{a+bn+n}\frac{((1+z)^b-1)^n}{(1+z)^{bn}}\;dz$$
$$=\frac{(-1)^n}{n}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}(1+z)^{a+n}((1+z)^b-1)^n\;dz.$$

This is

$$\frac{(-1)^n}{n}[z^{n-1}](1+z)^{a+n}((1+z)^b-1)^n.$$

Note however that

$$((1+z)^b - 1)^n = \left(\binom{b}{1}z + \binom{b}{2}z^2 + \cdots\right)^n = b^n z^n + \cdots$$

so there is no coefficient on $[z^{n-1}]$ because the powered term starts at z^n. Therefore the end result of the whole calculation is

$$0.$$

Remark. We have made several uses of

$$\binom{n}{m} = \sum_{q=0}^{m} (-1)^{m-q} \binom{n+1}{q}.$$

If this is not considered obvious we can prove it with the integral

$$\binom{n+1}{q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^{n+1} \; dz$$

to get

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z}(1+z)^{n+1} \sum_{q=0}^{m} (-1)^{m-q} \frac{1}{z^q} \; dz \\
&= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^m}{z} (1+z)^{n+1} \frac{1-(-1/z)^{m+1}}{1+1/z} \\
&= \frac{1}{2\pi i} \int_{|z|=\epsilon} (-1)^m (1+z)^{n+1} \frac{1-(-1/z)^{m+1}}{1+z} \; dz \\
&= \frac{1}{2\pi i} \int_{|z|=\epsilon} (-1)^m (1+z)^{n} (1-(-1/z)^{m+1}) \; dz \\
&= -(-1)^m \times (-1)^{m+1} \binom{n}{m} = \binom{n}{m}.
\end{aligned}$$

This was math.stackexchange.com problem 1789981.

8 A case of radical cancellation (B_1, R)

Suppose we seek to show that

$$\binom{2m}{2n} = \sum_{k=0}^{n} \binom{2n+1}{2k+1} \binom{m+k}{2n}.$$

where $m \geq n$. We introduce

$$\binom{2n+1}{2k+1} = \binom{2n+1}{2n-2k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n-2k+1}} (1+z)^{2n+1} \; dz.$$

Observe that this vanishes when $k > n$ so that we may use it to control the range and extend k to infinity. We also use

$$\binom{m+k}{2n} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{2n+1}}(1+w)^{m+k}\; dw.$$

We thus obtain

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \sum_{k\ge 0} z^{2k}(1+w)^k \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{1-(1+w)z^2} \; dw\; dz.$$

Evalute the inner integral using the negative of the residue at the pole at

$$w = \frac{1-z^2}{z^2}$$

(residues sum to zero) as in

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{1-z^2-wz^2} \; dw\; dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+3}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{w-(1-z^2)/z^2} \; dw\; dz.$$

The negative of the residue is

$$\frac{1}{z^{2m}} \frac{z^{4n+2}}{(1-z^2)^{2n+1}} = \frac{1}{z^{2m-4n-2}} \frac{1}{(1-z^2)^{2n+1}}$$

and we obtain from the outer integral

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+3}} \frac{1}{z^{2m-4n-2}} \frac{1}{(1-z^2)^{2n+1}} \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{2m-2n+1}} \frac{1}{(1-z)^{2n+1}} \; dz$$

$$= \binom{2m-2n+2n}{2n} = \binom{2m}{2n}.$$

This is the claim.

Remark. We also need to show that the contribution from the residue at infinity of the inner integral is zero. We get

$$\mathrm{Res}_{w=\infty} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{1-(1+w)z^2}$$

$$= -\mathrm{Res}_{w=0} \frac{1}{w^2}(1+1/w)^m w^{2n+1} \frac{1}{1-z^2-z^2/w}$$

$$= -\mathrm{Res}_{w=0}(1+w)^m w^{2n-m} \frac{1}{w(1-z^2)-z^2}.$$

No contribution when $2n \geq m$. Otherwise,

$$\frac{1}{z^2}\mathrm{Res}_{w=0}(1+w)^m \frac{1}{w^{m-2n}} \frac{1}{1-w(1-z^2)/z^2}$$

$$= \frac{1}{z^2} \sum_{q=0}^{m-2n-1} \binom{m}{m-2n-1-q} \frac{(1-z^2)^q}{z^{2q}}$$

$$= \frac{1}{z^2} \sum_{q=0}^{m-2n-1} \binom{m}{2n+1+q} \left(\frac{1}{z^2}-1\right)^q$$

Combining this with the integral in z yields

$$\sum_{q=0}^{m-2n-1} \binom{m}{2n+1+q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{z^2} \sum_{p=0}^{q} \binom{q}{p} (-1)^{q-p} \frac{1}{z^{2p}} \; dz.$$

The contribution from the residue is

$$[z^{2n+2+2p}](1+z)^{2n+1} = 0.$$

We can express this verbally by saying that the term from the integral is $[z^{2n}](1+z)^{2n+1} = 0$ and the sum only contributes negative powers of z with exponent starting at two.

Remark, II. From the convergence we require that $|z^2(1+w)| < 1$ in the double integral and must choose our contours appropriately. We must also verify that $(1-z^2)/z^2$ is outside the contour $|w| = \gamma$. This is $1/z^2 - 1$ i.e. a circle of radius $1/\epsilon^2$ shifted by one to the left. Therefore when $\epsilon < 1/\sqrt{2}$ the pole is outside the contour.

This was math.stackexchange.com problem 1900578.

9 Basic usage of exponentiation integral (B_1E)

Suppose we seek to verify that

$$(-1)^p \sum_{q=r}^{p} \binom{p}{q} \binom{q}{r} (-1)^q q^{p-r} = \frac{p!}{r!}.$$

We use the integral representation

$$\binom{q}{r} = \binom{q}{q-r} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^q}{z^{q-r+1}} \; dz$$

which is zero when $q < r$ (pole vanishes) so we may extend q back to zero. We also use the integral

$$q^{p-r} = \frac{(p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{\exp(qw)}{w^{p-r+1}} \, dw.$$

We thus obtain for the sum

$$\frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} z^{r-1} \sum_{q=0}^{p} \binom{p}{q} (-1)^q \frac{(1+z)^q}{z^q} \exp(qw) \, dz \, dw$$
$$= \frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} z^{r-1} \left(1 - \frac{1+z}{z} \exp(w)\right)^p \, dz \, dw$$
$$= \frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p-r+1}} (-\exp(w) + z(1 - \exp(w)))^p \, dz \, dw$$
$$= \frac{(p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p-r+1}} (\exp(w) + z(\exp(w) - 1))^p \, dz \, dw.$$

We extract the residue on the inner integral to obtain

$$\frac{(p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}} \binom{p}{p-r} \exp(rw)(\exp(w) - 1)^{p-r} \, dw$$
$$= \frac{p!}{r!} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}} \exp(rw)(\exp(w) - 1)^{p-r} \, dw.$$

It remains to compute

$$[w^{p-r}] \exp(rw)(\exp(w) - 1)^{p-r}.$$

Observe that $\exp(w) - 1$ starts at w so $(\exp(w) - 1)^{p-r}$ starts at w^{p-r} and hence only the constant coefficient from $\exp(rw)$ contributes, the value being one, which finally yields

$$\frac{p!}{r!}.$$

This was math.stackexchange.com problem 1731648.

10 Introductory example for the method, eliminating odd-even dependence (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor} = \binom{2n+1}{n}.$$

This is

$$\sum_{q=0}^{n} \binom{n}{2q} 2^{n-2q} \binom{2q}{q} + \sum_{q=0}^{n} \binom{n}{2q+1} 2^{n-2q-1} \binom{2q+1}{q}.$$

We treat these in turn.
First sum. Observe that

$$\binom{n}{2q}\binom{2q}{q} = \binom{n}{q}\binom{n-q}{q}.$$

This yields for the sum

$$2^n \sum_{q=0}^{n} \binom{n}{q}\binom{n-q}{q} 2^{-2q}.$$

Introduce

$$\binom{n-q}{q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-q}}{z^{q+1}} \; dz$$

which yields for the sum

$$\frac{2^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \sum_{q=0}^{n} \binom{n}{q} 2^{-2q} \frac{1}{z^q (1+z)^q} \; dz$$

$$= \frac{2^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \left(1 + \frac{1}{4z(1+z)}\right)^n \; dz$$

$$= \frac{2^{-n}}{2\pi i} \int_{|z|=\epsilon} \frac{(1+2z)^{2n}}{z^{n+1}} \; dz = 2^{-n} \binom{2n}{n} 2^n = \binom{2n}{n}.$$

Second sum. Observe that

$$\binom{n}{2q+1}\binom{2q+1}{q} = \binom{n}{q}\binom{n-q}{q+1}.$$

This yields for the sum

$$2^{n-1}\sum_{q=0}^{n}\binom{n}{q}\binom{n-q}{q+1}2^{-2q}.$$

This time introduce

$$\binom{n-q}{q+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-q}}{z^{q+2}}\;dz$$

which yields for the sum

$$\frac{2^{n-1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^2}\sum_{q=0}^{n}\binom{n}{q}2^{-2q}\frac{1}{z^q(1+z)^q}\;dz$$

$$= \frac{2^{n-1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^2}\left(1+\frac{1}{4z(1+z)}\right)^n\;dz$$

$$= \frac{2^{-n-1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+2z)^{2n}}{z^{n+2}}\;dz = 2^{-n-1}\binom{2n}{n+1}2^{n+1} = \binom{2n}{n+1}.$$

Conclusion.
Collecting the two contributions we obtain

$$\binom{2n}{n}+\binom{2n}{n+1} = \binom{2n+1}{n}$$

as claimed.
This was math.stackexchange.com problem 1442436.

11 Introductory example for the method, proving equality of two double hypergeometrics (B_1)

Suppose we seek to verify that $f_1(n,k) = f_2(n,k)$ where

$$f_1(n,k) = \sum_{v=0}^{n}\frac{(2k+2v)!}{(k+v)!\times v!\times(2k+v)!\times(n-v)!}2^{-v}$$

and

$$f_2(n,k) = \sum_{m=0}^{\lfloor n/2\rfloor}\frac{1}{(k+m)!\times m!\times(n-2m)!}2^{n-4m}.$$

Multiplying by $(n+k)!$ we obtain

$$g_1(n,k) = \sum_{v=0}^{n} \binom{n+k}{n-v} \binom{2k+2v}{v} 2^{-v}$$

and

$$g_2(n,k) = 2^n \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+k}{m} \binom{n+k-m}{n-2m} 2^{-4m}.$$

We will work with the latter two. Re-write the first sum as follows:

$$2^{-n} \sum_{v=0}^{n} \binom{n+k}{v} \binom{2k+2n-2v}{n-v} 2^{v}$$

Introduce

$$\binom{2k+2n-2v}{n-v} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-v+1}} (1+z)^{2k+2n-2v} \; dz.$$

This integral is zero when $v > n$ so we may extend v to infinity.
We get for $g_1(n,k)$

$$2^{-n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2k+2n} \sum_{v \ge 0} \binom{n+k}{v} \frac{z^v}{(1+z)^{2v}} 2^v \; dz$$

$$= 2^{-n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2k+2n} \left(1 + 2 \frac{z}{(1+z)^2} \right)^{n+k} \; dz$$

$$= 2^{-n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \left(1 + 4z + z^2 \right)^{n+k} \; dz.$$

For the second sum introduce

$$\binom{n+k-m}{n-2m} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-2m+1}} (1+z)^{n+k-m} \; dz.$$

This is zero when $2m > n$ so we may extend m to infinity.
We get for $g_2(n,k)$

$$2^{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+k} \sum_{m \ge 0} \binom{n+k}{m} \frac{z^{2m}}{(1+z)^{m}} 2^{-4m} \; dz$$

$$= 2^{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+k} \left(1 + \frac{1}{16} \frac{z^2}{1+z} \right)^{n+k} \; dz$$

$$= 2^{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \left(1 + z + \frac{1}{16} z^2 \right)^{n+k} \; dz.$$

Finally put $z = 4w$ in this integral to get

$$2^n \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{4^{n+1} w^{n+1}} \left(1 + 4w + w^2\right)^{n+k} \; 4dw$$

$$= 2^{-n} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \left(1 + 4w + w^2\right)^{n+k} \; dw.$$

This concludes the argument.
This was math.stackexchange.com problem 924966.

12 A remarkable case of factorization (B_1)

We let $T(0) = 0$ and $T(1) = 1$ and prove that when

$$T(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} T(n-k)$$

for $n \geq 2$ then

$$T(n) = C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1} = \binom{2n-2}{n-1} - \binom{2n-2}{n}.$$

In fact the case of a zero argument to T is not reached as for $n \geq 2$ we also have $n - \lfloor n/2 \rfloor \geq 1$. Applying the induction hypothesis on the RHS we get two pieces, the first is

$$A = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k-1}$$

$$= \binom{2n-2}{n-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k-1}$$

and the second

$$B = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k}$$

$$= \binom{2n-2}{n} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k}.$$

As we subtract B from A we see that we only need to show that the contribution from the two sum terms call them A' and B' is zero.

For these two pieces we introduce the integral representation

$$\binom{n-k}{k} = \binom{n-k}{n-2k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-2k+1}}(1+z)^{n-k}\; dz.$$

This has the nice property that it vanishes when $k > \lfloor n/2 \rfloor$ so we may extend the upper limit of the sum to infinity. We also introduce for the first sum

$$\binom{2n-2k-2}{n-k-1} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n-k}}(1+w)^{2n-2k-2}\; dw.$$

We thus obtain

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^n}(1+w)^{2n-2}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^n \sum_{k\ge 0}(-1)^{k+1}\frac{z^{2k}w^k}{(1+z)^k(1+w)^{2k}}\; dz\; dw$$

$$= -\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^n}(1+w)^{2n-2}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^n \frac{1}{1+z^2w/(1+z)/(1+w)^2}\; dz\; dw$$

$$= -\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^n}(1+w)^{2n}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{n+1} \frac{1}{(1+z)(1+w)^2+z^2w}\; dz\; dw$$

$$= -\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(1+w)^{2n}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{n+1} \frac{1}{z+1+w}\frac{1}{z+(1+w)/w}\; dz\; dw.$$

We evaluate the inner integral by summing the residues at $z = -(1+w)$ and $z = -(1+w)/w$ and flipping the sign. (We will verify that the residue at infinity is zero.)

The residue at $z = -(1+w)$ yields

$$-\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(1+w)^{2n}$$

$$\times \frac{(-1)^{n+1}}{(1+w)^{n+1}}(-1)^{n+1}w^{n+1}\frac{1}{-(1+w)+(1+w)/w}\; dw$$

$$= -\frac{1}{2\pi i}\int_{|w|=\gamma} (1+w)^{n-1}\frac{w}{1-w^2}\; dw.$$

This is zero as the pole at zero has been canceled. Next for the residue at

$z = -(1+w)/w$ we get

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(1+w)^{2n}$$
$$\times\frac{(-1)^{n+1}w^{n+1}}{(1+w)^{n+1}}(-1)^{n+1}\frac{1}{w^{n+1}}\frac{1}{-(1+w)/w+1+w}\;dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(1+w)^{n-1}\frac{w}{1-w^2}dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n}}(1+w)^{n-2}\frac{1}{1-w}dw.$$

With $n \ge 2$ we can evaluate this as

$$\sum_{q=0}^{n-1}\binom{n-2}{q} = 2^{n-2}.$$

To wrap up the residue at infinity of the inner integral is

$$\mathrm{Res}_{z=\infty}\frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{z+1+w}\frac{1}{z+(1+w)/w}$$
$$= -\mathrm{Res}_{z=0}\frac{1}{z^2}z^{n+1}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{1}{1/z+1+w}\frac{1}{1/z+(1+w)/w}$$
$$= -\mathrm{Res}_{z=0}(1+z)^{n+1}\frac{1}{1+z(1+w)}\frac{1}{1+z(1+w)/w} = 0.$$

Collecting everything and flipping the sign we have shown that

$$A' = -2^{n-2}.$$

For piece B' we see that it only differs from A' in an extra $1/w$ factor on the extractor in w at the front. We thus obtain

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+2}}(1+w)^{2n}$$
$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{z+1+w}\frac{1}{z+(1+w)/w}\;dz\;dw.$$

The residue at $z = -(1+w)$ vanishes the same because there was an extra w to spare on the $w/(1-w^2)$ term:

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{n-1}\frac{1}{1-w^2}\;dw.$$

For the residue at $z = -(1+w)/w$ we are now extracting from

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(1+w)^{n-2}\frac{1}{1-w}dw.$$

to get

$$\sum_{q=0}^{n}\binom{n-2}{q} = 2^{n-2}$$

as before. The residue at infinity vanished in z and did not reach the front extractor in w, for another contribution of zero. This means that

$$B' = -2^{n-2}$$

and we may conclude the proof. The fact that the sum term from the geometric series factored as it did is the remarkable feature of this problem.

Addendum, four years later. In the present version with complex variables the proof requires the convergence of the geometric series. This is $|z^2w/(1+z)/(1+w)^2| < 1$ or $|z^2w| < |(1+z)(1+w)^2|$. Now we have $|(1+z)(1+w)^2| \ge (1-\epsilon)(1-\gamma)^2$ so $(1-\epsilon)(1-\gamma)^2 > \epsilon^2\gamma$ will do. Suppose we take $\epsilon = \gamma$. We obtain $(1-\gamma)^3 > \gamma^3$. Therefore e.g. $\epsilon = \gamma = 1/4$ ensures convergence of the series. This also ensures that the two poles at $-(1+w)$ and $-(1+w)/w$ are outside the contour $|z| = \epsilon$.

This was math.stackexchange.com problem 2113830

13 Evaluating a quadruple hypergeometric(B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{n}\sum_{l=0}^{n}(-1)^{k+l}\binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{k}\binom{n}{l}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{l=0}^{n}(-1)^l\binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{l}.$$

Evaluate the inner sum first and introduce

$$\binom{n+k-l}{n} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+k-l}}{z^{n+1}}\,dz.$$

and

$$\binom{k+l}{n} = \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{k+l}}{w^{n+1}}\,dw.$$

This yields for the inner sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+k}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^k}{w^{n+1}}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{(1+w)^l}{(1+z)^l}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+k}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^k}{w^{n+1}}\left(1-\frac{1+w}{1+z}\right)^n\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{k}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^k}{w^{n+1}}(z-w)^n\;dw\;dz.$$

Extracting the inner coefficient yields

$$\sum_{q=0}^{n}\binom{k}{q}\binom{n}{n-q}(-1)^{n-q}z^q.$$

The outer coefficient becomes

$$\sum_{q=0}^{n}\binom{k}{q}\binom{n}{n-q}(-1)^{n-q}\binom{k}{n-q}$$

$$=\sum_{q=0}^{n}\binom{k}{q}\binom{n}{q}(-1)^{n-q}\binom{k}{n-q}.$$

Call this S. By symmetry we have on re-indexing that

$$2S=\sum_{q=0}^{n}\binom{k}{q}\binom{n}{q}((-1)^q+(-1)^{n-q})\binom{k}{n-q}$$

$$=(1+(-1)^n)\sum_{q=0}^{n}\binom{k}{q}\binom{n}{q}(-1)^q\binom{k}{n-q}.$$

This is zero when n is odd so *the entire sum being evaluated vanishes when n is odd* and we may assume that $n=2m$ and get

$$\sum_{q=0}^{2m}\binom{k}{q}\binom{2m}{q}(-1)^q\binom{k}{2m-q}.$$

Substituting this into the outer sum yields

$$\sum_{q=0}^{2m}\binom{2m}{q}(-1)^q\sum_{k=0}^{2m}\binom{2m}{k}(-1)^k\binom{k}{q}\binom{k}{2m-q}.$$

We evaluate the inner sum with the integrals

$$\binom{k}{q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^k}{z^{q+1}} \; dz.$$

and

$$\binom{k}{2m-q} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^k}{w^{2m-q+1}} \; dw$$

to get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k (1+z)^k (1+w)^k \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} (z+w+wz)^{2m} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} (w(1+z)+z)^{2m} \; dw \; dz.$$

Extracting the coefficient we get for the inner term

$$\binom{2m}{2m-q} (1+z)^{2m-q} z^q$$

and for the outer integral

$$\binom{2m}{2m-q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z} (1+z)^{2m-q} \; dz = \binom{2m}{2m-q}.$$

We are now ready to conclude and return to the main sum which has been transformed into

$$\sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \binom{2m}{2m-q}$$

which is

$$[v^{2m}](1-v)^{2m}(1+v)^{2m} = [v^{2m}](1-v^2)^{2m} = [v^m](1-v)^{2m}$$

$$= (-1)^m \binom{2m}{m}.$$

This was math.stackexchange.com problem 1577907.

14 An integral representation of a binomial coefficient involving the floor function (B_1)

Suppose we seek to prove that

$$\sum_{k=0}^{2m+1} \binom{n}{k} 2^k \binom{n-k}{\lfloor (2m+1-k)/2 \rfloor} = \binom{2n+1}{2m+1}.$$

Observe that from first principles we have that

$$\binom{n}{\lfloor q/2 \rfloor} = \binom{n}{n - \lfloor q/2 \rfloor} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \left(1 + z + wz^2 + wz^3 + w^2 z^4 + w^2 z^5 + \cdots\right) \; dw \; dz.$$

This simplifies to

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \left(\frac{1}{1-wz^2} + z \frac{1}{1-wz^2} \right) \; dw \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \; dw \; dz.$$

This correctly enforces the range as the reader is invited to verify and we may extend k beyond $2m+1$, getting for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \sum_{k \geq 0} \binom{n}{k} 2^k z^k \frac{w^k}{(1+w)^k} \; dw \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \left(1 + \frac{2wz}{1+w}\right)^n \; dw \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w+2wz)^n}{w^{n+1}} \frac{1}{1-wz^2} \; dw \; dz.$$

Extracting the inner coefficient now yields

$$\sum_{q=0}^{n} \binom{n}{q} (1+2z)^q z^{2n-2q} = z^{2n} \sum_{q=0}^{n} \binom{n}{q} (1+2z)^q z^{-2q}$$
$$= z^{2n} \left(1 + \frac{1+2z}{z^2}\right)^n = (1+z)^{2n}.$$

We thus get from the outer coefficient

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+1}}{z^{2m+2}}\;dz$$

which is

$$\binom{2n+1}{2m+1}$$

as claimed. I do believe this is an instructive exercise.
This was math.stackexchange.com problem 2087559

15 Evaluating another quadruple hypergeometric(B_1)

Suppose we seek to verify that

$$\sum_{k=m}^{n}(-1)^{n+k}\frac{2k+1}{n+k+1}\binom{n}{k}\binom{n+k}{k}^{-1}\binom{k}{m}\binom{k+m}{m}=\delta_{mn}.$$

Here we may assume $n \ge m$, the equality holds trivially otherwise.
Now we have

$$\binom{n}{k}\binom{n+k}{k}^{-1}=\frac{n!}{k!(n-k)}\frac{k!n!}{(n+k)!}$$
$$=\frac{n!}{(n-k)}\frac{n!}{(n+k)!}=\binom{2n}{n+k}\binom{2n}{n}^{-1}.$$

We get for the sum

$$\sum_{k=m}^{n}(-1)^{n+k}\frac{2k+1}{n+k+1}\binom{2n}{n+k}\binom{k}{m}\binom{k+m}{m}=\delta_{mn}\times\binom{2n}{n}.$$

which is

$$\sum_{k=m}^{n}(-1)^{n+k}(2k+1)\binom{2n+1}{n+k+1}\binom{k}{m}\binom{k+m}{m}$$
$$=\delta_{mn}\times(2n+1)\times\binom{2n}{n}.$$

Introduce

$$\binom{2n+1}{n+k+1}=\binom{2n+1}{n-k}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-k+1}}(1+z)^{2n+1}\;dz.$$

Observe that this vanishes when $k > n$ so we may extend k upward to infinity.

Furthermore introduce

$$\binom{k}{m} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+1}}(1+w)^k \; dw.$$

Observe once again that the integral vanishes, this time when $0 \le k < m$ so we may extend k back to zero.

We thus get for the sum

$$(-1)^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{2n+1}$$
$$\times \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+1}} \sum_{k\ge 0}(-1)^k (2k+1)\binom{k+m}{m} z^k (1+w)^k \; dw \; dz.$$

The inner sum yields two pieces, the first is

$$\sum_{k\ge 0}(-1)^k \binom{k+m}{m} z^k (1+w)^k = \frac{1}{(1+z+wz)^{m+1}}$$
$$= \frac{1}{(1+z)^{m+1}} \frac{1}{(1+wz/(1+z))^{m+1}}.$$

On extracting the residue for the integral in w we obtain

$$(-1)^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{2n+1}$$
$$\times \frac{1}{(1+z)^{m+1}} \binom{2m}{m} (-1)^m \frac{z^m}{(1+z)^m} \; dz$$
$$= \binom{2m}{m}(-1)^{n+m} \int_{|z|=\epsilon} \frac{1}{z^{n-m+1}}(1+z)^{2n-2m} \; dz$$
$$= \binom{2m}{m}(-1)^{n+m}\binom{2n-2m}{n-m}.$$

The second piece from the sum is

$$2\sum_{k\ge 1}(-1)^k k \binom{k+m}{m} z^k (1+w)^k.$$

Write

$$k\binom{k+m}{m} = \frac{(k+m)!}{(k-1)!m!} = (m+1)\frac{(k+m)!}{(k-1)!(m+1)!}$$
$$= (m+1)\binom{k+m}{m+1}$$

to get for the sum

$$2(m+1)z(1+w)\sum_{k\ge 1}(-1)^k \binom{k+m}{m+1} z^{k-1}(1+w)^{k-1}$$

$$= -2(m+1)z(1+w)\frac{1}{(1+z+wz)^{m+2}}$$

$$= -2(m+1)z(1+w)\frac{1}{(1+z)^{m+2}}\frac{1}{(1+wz/(1+z))^{m+2}}.$$

Here we get two pieces, the first is

$$-2(m+1)(-1)^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z}{z^{n+1}}(1+z)^{2n+1}$$

$$\times \frac{1}{(1+z)^{m+2}}\binom{2m+1}{m}(-1)^m \frac{z^m}{(1+z)^m}\; dz$$

$$= -2(m+1)\binom{2m+1}{m}(-1)^{n+m}\int_{|z|=\epsilon}\frac{1}{z^{n-m}}(1+z)^{2n-2m-1}\; dz$$

We have two cases, we get zero when $n=m$ and when $n>m$ we have

$$-2(m+1)\binom{2m+1}{m}(-1)^{n+m}\binom{2n-2m-1}{n-m-1}.$$

The second piece is

$$-2(m+1)(-1)^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z}{z^{n+1}}(1+z)^{2n+1}$$

$$\times \frac{1}{(1+z)^{m+2}}\binom{2m}{m-1}(-1)^{m-1} \frac{z^{m-1}}{(1+z)^{m-1}}\; dz$$

$$= 2(m+1)\binom{2m}{m-1}(-1)^{n+m}\int_{|z|=\epsilon}\frac{1}{z^{n-m+1}}(1+z)^{2n-2m}\; dz$$

$$= 2(m+1)\binom{2m}{m-1}(-1)^{n+m}\binom{2n-2m}{n-m}.$$

Therefore when $n=m$ we get

$$\binom{2n-2m}{n-m}(-1)^{m+n}\left(2(m+1)\binom{2m}{m-1}+\binom{2m}{m}\right).$$

This simplifies to

$$(-1)^{2m}\left(2(m+1)\binom{2m}{m-1}+\binom{2m}{m}\right)$$

$$= 2m\binom{2m}{m}+\binom{2m}{m} = (2m+1)\binom{2m}{m}.$$

This is precisely the claim we were trying to prove. On the other hand when $n > m$ we obtain

$$\binom{2n-2m}{n-m}(-1)^{m+n}$$
$$\times \left(2(m+1)\binom{2m}{m-1} + \binom{2m}{m} - 2(m+1)\binom{2m+1}{m}\frac{n-m}{2n-2m}\right).$$

The factor is

$$(2m+1)\binom{2m}{m} - (m+1)\binom{2m+1}{m} = 0.$$

This concludes the argument.

Remark. For $n = m$ we could have evaluated the single term in the initial sum by expanding the four binomial coefficients and assumed $n > m$ thereafter.

This was math.stackexchange.com problem 1817122.

16 An identity by Strehl (B_1)

Suppose we seek to show that

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=\lceil n/2 \rceil}^{n} \binom{n}{k}^2 \binom{2k}{n}.$$

With

$$\binom{n}{k}\binom{2k}{n} = \frac{(2k)!}{k! \times (n-k)! \times (2k-n)!} = \binom{2k}{k}\binom{k}{n-k}$$

we find that the RHS is

$$\sum_{k=\lceil n/2 \rceil}^{n} \binom{n}{k}\binom{2k}{k}\binom{k}{n-k}.$$

Introduce

$$\binom{2k}{k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2k}}{z^{k+1}}\; dz$$

and (this integral is zero when $0 \le k < \lceil n/2 \rceil$)

$$\binom{k}{n-k} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{k}}{w^{n-k+1}}\; dw$$

to get for the RHS

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{w^k(1+w)^k(1+z)^{2k}}{z^k}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\left(1+\frac{w(1+w)(1+z)^2}{z}\right)^n\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(z+w(1+w)(1+z)^2)^n\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(z+w(z+1))^n(1+w(z+1))^n\;dw\;dz.$$

Extracting first the residue in w in next the residue in z we get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\sum_{q=0}^{n}\binom{n}{q}z^{n-q}(1+z)^q\binom{n}{n-q}(1+z)^{n-q}\;dz$$

$$=\sum_{q=0}^{n}\binom{n}{q}^2\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{q+1}}\;dz$$

$$=\sum_{q=0}^{n}\binom{n}{q}^3$$

QED.

Addendum May 27 2018. We compute this using formal power series as per request in comment. Start from

$$\binom{2k}{k}=[z^k](1+z)^{2k}$$

and

$$\binom{k}{n-k}=[w^{n-k}](1+w)^k.$$

Observe that this coefficient extractor is zero when $n-k>k$ or $k<\lceil n/2\rceil$ where $k\geq 0$. Hence we are justified in lowering k to zero when we substitute these into the sum and we find

$$\sum_{k=0}^{n}\binom{n}{k}[z^k](1+z)^{2k}[w^{n-k}](1+w)^k$$

$$=[z^0][w^n]\sum_{k=0}^{n}\binom{n}{k}\frac{1}{z^k}(1+z)^{2k}w^k(1+w)^k$$

$$= [z^0][w^n] \left(1 + \frac{(1+z)^2 w(1+w)}{z}\right)^n$$
$$= [z^n][w^n](z + (1+z)^2 w(1+w))^n$$
$$= [z^n][w^n](1 + w(1+z))^n (z + w(1+z))^n.$$

We extract the coefficient on $[w^n]$ then the one on $[z^n]$ and get

$$[z^n] \sum_{q=0}^{n} \binom{n}{q} (1+z)^q \binom{n}{n-q} (1+z)^{n-q} z^q$$
$$= \sum_{q=0}^{n} \binom{n}{q}^2 [z^{n-q}](1+z)^n = \sum_{q=0}^{n} \binom{n}{q}^2 \binom{n}{n-q} = \sum_{q=0}^{n} \binom{n}{q}^3.$$

The claim is proved.
This was math.stackexchange.com problem 586138.

17 Shifting the index variable and applying Leibniz' rule (B_1)

We seek to simplify

$$\sum_s \binom{n+s}{k+l} \binom{k}{s} \binom{l}{s}.$$

The substitution $s = t + k + l - n$ yields

$$\sum_t \binom{t+k+l}{k+l} \binom{k}{t+k+l-n} \binom{l}{t+k+l-n}.$$

Working with the assumption that the parameters are positive integers we find that from the first binomial coefficient we get that for it to be non-zero we must have $t \ge 0$ or $t < -(k+l)$. Note however that in the latter case the two remaining coefficients vanish, which leaves $t \ge 0$. Re-writing we find

$$\sum_{t \ge 0} \binom{t+k+l}{k+l} \binom{k}{n-l-t} \binom{l}{n-k-t}.$$

We introduce integral represenations for the two right coefficients that also enforce the fact that $t \le n - l$ and $t \le n - k$ so that we may then let t range to infinity. We use

$$\binom{k}{n-l-t} = \frac{1}{2\pi i} \int_{|z|=\epsilon_1} \frac{1}{z^{n-l-t+1}} (1+z)^k \, dz$$

as well as

$$\binom{l}{n-k-t} = \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k-t+1}}(1+v)^l \, dv.$$

We then get for the sum (no convergence issues here)

$$\frac{1}{2\pi i}\int_{|z|=\epsilon_1} \frac{1}{z^{n-l+1}}(1+z)^k \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k+1}}(1+v)^l \sum_{t\ge 0}\binom{k+l+t}{k+l} v^t z^t \, dv\, dz.$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon_1} \frac{1}{z^{n-l+1}}(1+z)^k \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k+1}}(1+v)^l \frac{1}{(1-vz)^{k+l+1}} \, dv\, dz.$$

We see that this vanishes when $n < k$ or $n < l$, which we label case A. Case B is that $n \ge k, l$. We evaluate the inner integral using the fact that residues sum to zero. With this in mind we write

$$\frac{(-1)^{k+l+1}}{2\pi i}\int_{|z|=\epsilon_1} \frac{1}{z^{n+k+2}}(1+z)^k \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k+1}}(1+v)^l \frac{1}{(v-1/z)^{k+l+1}} \, dv\, dz.$$

We thus require for the pole at $v = 1/z$

$$\frac{1}{(k+l)!}\left(\frac{1}{v^{n-k+1}}(1+v)^l\right)^{(k+l)}$$

which is (apply Leibniz)

$$\frac{1}{(k+l)!}\sum_{q=0}^{k+l}\binom{k+l}{q}(-1)^q\binom{n-k+q}{q}\frac{q!}{v^{n-k+1+q}}$$

$$\times\binom{l}{k+l-q}(k+l-q)!(1+v)^{l-(k+l-q)}$$

$$= \sum_{q=0}^{k+l}(-1)^q\binom{n-k+q}{q}\frac{1}{v^{n-k+1+q}}\binom{l}{k+l-q}(1+v)^{q-k}.$$

Evaluate at $v = 1/z$ to get

$$\sum_{q=0}^{k+l}(-1)^q\binom{n-k+q}{q}z^{n-k+1+q}\binom{l}{k+l-q}\frac{(1+z)^{q-k}}{z^{q-k}}.$$

Substituting this into the integral in z and flipping the sign yields

$$(-1)^{k+l}\sum_{q=0}^{k+l}(-1)^q\binom{n-k+q}{q}\binom{l}{k+l-q}\binom{q}{k}.$$

Now we have

$$\binom{q}{k}\binom{n-k+q}{q} = \frac{(n-k+q)!}{k! \times (q-k)! \times (n-k)!} = \binom{n}{k}\binom{n-k+q}{n}$$

and we obtain

$$(-1)^{k+l}\binom{n}{k}\sum_{q=0}^{k+l}(-1)^q\binom{l}{k+l-q}\binom{n-k+q}{n}$$

$$= \binom{n}{k}\sum_{q=0}^{k+l}(-1)^q\binom{l}{q}\binom{n+l-q}{n}$$

$$= \binom{n}{k}[w^n]\sum_{q=0}^{k+l}(-1)^q\binom{l}{q}(1+w)^{n+l-q}$$

$$= \binom{n}{k}[w^n](1+w)^{n+l}\sum_{q=0}^{k+l}(-1)^q\binom{l}{q}\frac{1}{(1+w)^q}$$

$$= \binom{n}{k}[w^n](1+w)^{n+l}\left(1-\frac{1}{1+w}\right)^l$$

$$= \binom{n}{k}[w^n]w^l(1+w)^n = \binom{n}{k}\binom{n}{n-l} = \binom{n}{k}\binom{n}{l}.$$

This is the claim, which we proved for case B.

Remark. To be perfectly rigorous we also need to show that the contribution from the residue at infinity is zero. We find

$$\mathrm{Res}_{v=\infty}\frac{1}{v^{n-k+1}}(1+v)^l\frac{1}{(1-vz)^{k+l+1}}$$

$$= -\mathrm{Res}_{v=0}\frac{1}{v^2}v^{n-k+1}\frac{(1+v)^l}{v^l}\frac{1}{(1-z/v)^{k+l+1}}$$

$$= -\mathrm{Res}_{v=0}\frac{1}{v^2}v^{n-k-l+1}(1+v)^l\frac{v^{k+l+1}}{(v-z)^{k+l+1}}$$

$$= -\mathrm{Res}_{v=0}v^n(1+v)^l\frac{1}{(v-z)^{k+l+1}} = 0$$

and the check goes through.

This was math.stackexchange.com problem 2381429.

18 Working with negative indices (B_1)

Suppose we seek to prove that

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} (-1)^k {2n\choose n+3k} = 2\times 3^{n-1}.$$

We start by introducing the integral

$${2n\choose n+3k} = {2n\choose n-3k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-3k+1}} (1+z)^{2n} \; dz.$$

Observe that this vanishes for $3k > n$ (pole canceled) and for $3k < -n$ (upper range of polynomial term exceeded) so we may extend the summation to $[-n, n]$ getting

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2n} \sum_{k=-n}^n (-1)^k z^{3k} \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2n} (-1)^n z^{-3n} \sum_{k=0}^{2n} (-1)^k z^{3k} \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{4n+1}} (1+z)^{2n} (-1)^n \frac{1-(-1)^{2n+1}z^{3(2n+1)}}{1+z^3} \; dz.$$

Only the first piece from the difference due to the geometric series contributes and we get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{4n+1}} (1+z)^{2n} (-1)^n \frac{1}{1+z^3} \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{4n+1}} (1+z)^{2n-1} (-1)^n \frac{1}{1-z+z^2} \; dz.$$

We have two poles other than zero and infinity at ρ and $1/\rho$ where

$$\rho = \frac{1+\sqrt{3}i}{2}$$

and using the fact that residues sum to zero we obtain

$$S + \frac{(-1)^n}{\rho(1+\rho)} \frac{1}{\rho-1/\rho} \left(\frac{(1+\rho)^2}{\rho^4}\right)^n + \frac{(-1)^n}{1/\rho(1+1/\rho)} \frac{1}{1/\rho-\rho} \left(\frac{(1+1/\rho)^2}{1/\rho^4}\right)^n$$

$$+\mathrm{Res}_{z=\infty} \frac{1}{z^{4n+1}} (1+z)^{2n-1} (-1)^n \frac{1}{1-z+z^2} = 0.$$

We get for the residue at infinity

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{4n+1} (1+1/z)^{2n-1} (-1)^n \frac{1}{1-1/z+1/z^2}$$

$$= -\mathrm{Res}_{z=0} z^{2n+2}(1+z)^{2n-1}(-1)^n \frac{1}{z^2-z+1} = 0.$$

Now if $z^2 = z - 1$ then $z^4 = z^2 - 2z + 1 = -z$ and thus

$$\frac{(1+1/\rho)^2}{1/\rho^4} = \frac{(1+\rho)^2}{\rho^4} = \frac{\rho - 1 + 2\rho + 1}{-\rho} = -3$$

and furthermore with $z(1+z)(z-1/z) = (1+z)(z^2-1)$ and $(1+z)(z-2) = z^2 - z - 2 = -3$ we finally get

$$S + (-1)^n \times \left(-\frac{1}{3}\right)(-3)^n + (-1)^n \times \left(-\frac{1}{3}\right)(-3)^n = 0$$

or

$$S = 2 \times 3^{n-1}.$$

This was math.stackexchange.com problem 2054777.

19 Two companion identities by Gould (B_1)

Suppose we seek to evaluate

$$Q(x,\rho) = \sum_{k=0}^{\rho} \binom{2x+1}{2k}\binom{x-k}{\rho-k}$$

where $x \ge \rho$.

Introduce

$$\binom{x-k}{\rho-k} = \binom{x-k}{x-\rho} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}}(1+z)^{x-k}\; dz.$$

Note that this controls the range being zero when $\rho < k \le x$ so we can extend the sum to x supposing that $x > \rho$. And when $x = \rho$ we may also set the upper limit to x.

We get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}}(1+z)^{x} \sum_{k=0}^{x}\binom{2x+1}{2k}\frac{1}{(1+z)^k}\; dz.$$

This is

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}}(1+z)^{x}\left(\left(1+\frac{1}{\sqrt{1+z}}\right)^{2x+1} + \left(1-\frac{1}{\sqrt{1+z}}\right)^{2x+1}\right)\; dz$$

$$= \frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}}\frac{1}{\sqrt{1+z}}\left((1+\sqrt{1+z})^{2x+1} + (1-\sqrt{1+z})^{2x+1}\right)\; dz.$$

Observe that the second term in the parenthesis (i.e. $1 - \sqrt{1+z}$) has no constant term and hence starts at z^{2x+1} making for a zero contribution. This leaves

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}} \frac{1}{\sqrt{1+z}}(1+\sqrt{1+z})^{2x+1}\; dz.$$

Now put $1+z = w^2$ so that $dz = 2w\; dw$ to get

$$\frac{1}{2\pi i}\int_{|w-1|=\epsilon} \frac{1}{(w^2-1)^{x-\rho+1}} \frac{1}{w}(1+w)^{2x+1}\; w\; dw$$

$$= \frac{1}{2\pi i}\int_{|w-1|=\epsilon} \frac{1}{(w-1)^{x-\rho+1}} \frac{1}{(w+1)^{x-\rho+1}}(1+w)^{2x+1}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w-1|=\epsilon} \frac{1}{(w-1)^{x-\rho+1}} (1+w)^{x+\rho}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w-1|=\epsilon} \frac{1}{(w-1)^{x-\rho+1}} \sum_{q=0}^{x+\rho} \binom{x+\rho}{q} 2^{x+\rho-q}(w-1)^q\; dw.$$

This is

$$[(w-1)^{x-\rho}] \sum_{q=0}^{x+\rho} \binom{x+\rho}{q} 2^{x+\rho-q}(w-1)^q$$

$$= \binom{x+\rho}{x-\rho} 2^{x+\rho-(x-\rho)} = \binom{x+\rho}{x-\rho} 2^{2\rho} = \binom{x+\rho}{2\rho} 2^{2\rho}.$$

We can also prove the companion identity from above. Suppose we seek to evaluate

$$Q(x,\rho) = \sum_{k=0}^{\rho} \binom{2x+1}{2k+1}\binom{x-k}{\rho-k}$$

where $x \geq \rho$.
Introduce

$$\binom{x-k}{\rho-k} = \binom{x-k}{x-\rho} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}}(1+z)^{x-k}\; dz.$$

Note that this controls the range being zero when $\rho < k \leq x$ so we can extend the sum to x supposing that $x > \rho$. And when $x = \rho$ we may also set the upper limit to x.

We get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}}(1+z)^x \sum_{k=0}^{x} \binom{2x+1}{2k+1} \frac{1}{(1+z)^k}\; dz.$$

This is

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^x}{z^{x-\rho+1}}\sqrt{1+z}\left(\left(1+\frac{1}{\sqrt{1+z}}\right)^{2x+1}-\left(1-\frac{1}{\sqrt{1+z}}\right)^{2x+1}\right)\,dz$$

$$=\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}\left((1+\sqrt{1+z})^{2x+1}-(1-\sqrt{1+z})^{2x+1}\right)\,dz.$$

Observe that the second term in the parenthesis (i.e. $1-\sqrt{1+z}$) has no constant term and hence starts at z^{2x+1} making for a zero contribution. This leaves

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}(1+\sqrt{1+z})^{2x+1}\,dz.$$

Now put $1+z=w^2$ so that $dz=2w\,dw$ to get

$$\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w^2-1)^{x-\rho+1}}(1+w)^{2x+1}\,w\,dw$$

$$=\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho+1}}\frac{1}{(w+1)^{x-\rho+1}}(1+w)^{2x+1}\,w\,dw$$

$$=\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho+1}}(1+w)^{x+\rho}\,w\,dw.$$

Writing $w=(w-1)+1$ this produces two pieces, the first is

$$\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho}}\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q\,dw.$$

This is

$$[(w-1)^{x-\rho-1}]\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q$$

$$=\binom{x+\rho}{x-\rho-1}2^{x+\rho-(x-\rho-1)}=\binom{x+\rho}{x-\rho-1}2^{2\rho+1}=\binom{x+\rho}{2\rho+1}2^{2\rho+1}.$$

The second piece is

$$[(w-1)^{x-\rho}]\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q$$

$$=\binom{x+\rho}{x-\rho}2^{x+\rho-(x-\rho)}=\binom{x+\rho}{x-\rho}2^{2\rho}=\binom{x+\rho}{2\rho}2^{2\rho}.$$

Joining the two pieces we finally obtain

$$\left(2 \times \frac{x-\rho}{2\rho+1} + 1\right) \times \binom{x+\rho}{2\rho} 2^{2\rho}$$
$$= \frac{2x+1}{2\rho+1} \binom{x+\rho}{2\rho} 2^{2\rho}.$$

This was math.stackexchange.com problem 1383343.

20 Exercise 1.3 from Stanley's Enumerative Combinatorics (B_2)

We will do this one using coefficient extractors as in the second half of this document. We seek to verify that

$$\sum_{k=0}^{\min a,b} \binom{x+y+k}{k} \binom{x}{b-k} \binom{y}{a-k} = \binom{x+a}{b} \binom{y+b}{a}.$$

where we take $y \geq a$ and $x \geq b$.

Now introduce

$$\binom{x}{b-k} = \binom{x}{x-b+k} = [z^{b-k}] \frac{1}{(1-z)^{x-b+k+1}}$$

and

$$\binom{y}{a-k} = \binom{y}{y-a+k} = [w^{a-k}] \frac{1}{(1-w)^{y-a+k+1}}.$$

We get for the sum

$$[z^b][w^a] \frac{1}{(1-z)^{x-b+1}} \frac{1}{(1-w)^{y-a+1}} \sum_{k=0}^{\min(a,b)} \binom{x+y+k}{k} \frac{z^k w^k}{(1-z)^k} \frac{1}{(1-w)^k}.$$

The coefficient extractors provide range control and we may continue with

$$[z^b][w^a] \frac{1}{(1-z)^{x-b+1}} \frac{1}{(1-w)^{y-a+1}} \sum_{k\geq 0} \binom{x+y+k}{k} \frac{z^k w^k}{(1-z)^k} \frac{1}{(1-w)^k}$$
$$= [z^b][w^a] \frac{1}{(1-z)^{x-b+1}} \frac{1}{(1-w)^{y-a+1}} \frac{1}{(1-zw/(1-z)/(1-w))^{x+y+1}}$$
$$= [z^b][w^a](1-z)^{y+b}(1-w)^{x+a} \frac{1}{(1-z-w)^{x+y+1}}$$

$$= [z^b]\frac{1}{(1-z)^{x-b+1}}[w^a](1-w)^{x+a}\frac{1}{(1-w/(1-z))^{x+y+1}}$$

$$= [z^b]\frac{1}{(1-z)^{x-b+1}}\sum_{k=0}^{a}\binom{x+a}{k}(-1)^k\binom{a-k+x+y}{x+y}\frac{1}{(1-z)^{a-k}}$$

$$= \sum_{k=0}^{a}\binom{x+a}{k}(-1)^k\binom{a-k+x+y}{x+y}[z^b]\frac{1}{(1-z)^{x+a-b-k+1}}$$

$$= \sum_{k=0}^{a}\binom{x+a}{k}(-1)^k\binom{a-k+x+y}{x+y}\binom{x+a-k}{b}.$$

Now

$$\binom{x+a}{k}\binom{x+a-k}{b} = \frac{(x+a)!}{k!\times b!\times(x+a-b-k)!} = \binom{x+a}{b}\binom{x+a-b}{k}$$

so we obtain

$$\binom{x+a}{b}\sum_{k=0}^{a}\binom{x+a-b}{k}(-1)^k\binom{a-k+x+y}{a-k}$$

$$= \binom{x+a}{b}[z^a](1+z)^{a+x+y}\sum_{k=0}^{a}\binom{x+a-b}{k}(-1)^k z^k\frac{1}{(1+z)^k}.$$

Here the coefficient extractor once more enforces the range and we get

$$\binom{x+a}{b}[z^a](1+z)^{a+x+y}\sum_{k\ge 0}\binom{x+a-b}{k}(-1)^k z^k\frac{1}{(1+z)^k}$$

$$= \binom{x+a}{b}[z^a](1+z)^{a+x+y}\left(1-\frac{z}{1+z}\right)^{x+a-b}$$

$$= \binom{x+a}{b}[z^a](1+z)^{y+b} = \binom{x+a}{b}\binom{y+b}{a}.$$

This is the claim.
This was math.stackexchange.com problem 1426447.

21 Counting m-subsets ($B_1 I$)

Permit me to contribute an algebraic proof.

Suppose we seek to verify that

$$\sum_{q=0}^{n}\binom{n}{2q}\binom{n-2q}{p-q}2^{2q} = \binom{2n}{2p}.$$

Observe that the sum is

$$\sum_{q=0}^{n} \binom{n}{p-q} \binom{n-p+q}{n-p-q} 4^q.$$

which is

$$\sum_{q=0}^{p} \binom{n}{p-q} \binom{n-p+q}{n-p-q} 4^q = 4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{n+q-2p} 4^{-q}.$$

Introduce the Iverson bracket

$$[[0 \le q \le p]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^q}{z^{p+1}} \frac{1}{1-z} \; dz.$$

This provides range control so we may extend q to n.
Introduce furthermore

$$\binom{n-q}{n+q-2p} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{n-q}}{w^{n+q-2p+1}} \; dw.$$

We thus get for the sum

$$\frac{4^p}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^n}{w^{n-2p+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} \sum_{q=0}^{n} \binom{n}{q} z^q \frac{1}{w^q(1+w)^q} 4^{-q} \; dz \; dw$$

$$= \frac{4^p}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^n}{w^{n-2p+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} \left(1 + z\frac{1}{4w(1+w)}\right)^n \; dz \; dw$$

$$= \frac{4^{p-n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} \left(4w(1+w)+z\right)^n \; dz \; dw.$$

We evaluate the inner integral using the negative of the residue of the pole at $z = 1$ which yields

$$\frac{4^{p-n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p+1}} (4w + 4w^2 + 1)^n \; dw$$

$$= \frac{4^{p-n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p+1}} (2w+1)^{2n} \; dw$$

$$= 4^{p-n} \binom{2n}{2n-2p} 2^{2n-2p} = \binom{2n}{2p}.$$

If we want to be rigorous we need to verify that the contribution from the residue at infinity of the last integral in z is zero when $n \ge p$. We get for the residue

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{p+1} \frac{1}{1-1/z} (4w(1+w)+1/z)^n$$

$$= -\mathrm{Res}_{z=0} z^p \frac{1}{z-1} (4w(1+w)+1/z)^n$$

$$= -\mathrm{Res}_{z=0} \frac{1}{z^{n-p}} \frac{1}{z-1} (4zw(1+w)+1)^n.$$

This is clearly zero when $n = p$. For $n > p$ we obtain

$$\sum_{q=0}^{n-p-1} \binom{n}{q} 4^q w^q (1+w)^q.$$

This polynomial has degree $2n-2p-2$ but the integral in w extracts the coefficient on $2n-2p$ for a zero contribution.

Addendum. We can use the same method to prove the companion identity

$$\sum_{q=0}^{n} \binom{n}{2q+1} \binom{n-2q-1}{p-q} 2^{2q+1} = \binom{2n}{2p+1}.$$

The sum is

$$\sum_{q=0}^{n} \binom{n}{p-q} \binom{n-p+q}{n-p-q-1} 2^{2q+1}$$

which is

$$\sum_{q=0}^{p} \binom{n}{p-q} \binom{n-p+q}{n-p-q-1} 2^{2q+1} = 2^{2p+1} \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{n+q-2p-1} 2^{-2q}.$$

Using exactly the same substitution as before we obtain the integral

$$\frac{2^{2p+1-2n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} (4w(1+w)+z)^n \; dz \; dw.$$

This time we get from the residue at the pole $z = 1$

$$2^{2p+1-2n} \binom{2n}{2n-2p-1} 2^{2n-2p-1} = \binom{2n}{2p+1}.$$

For the residue at infinity we are extracting the coefficient on $w^{2n-2p-1}$ but the inner term has degree $2n-2p-2$, again for a contribution of zero.

Addendum II. We can actually eliminate the Iverson bracket starting from

$$4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{n+q-2p} 4^{-q}.$$

and observing that this is

$$4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{2p-2q} 4^{-q}.$$

Now introduce

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p-2q+1}} (1+z)^{n-q} \; dz$$

This is zero when $q > p$ so it provides the range control, which we have now obtained without the Iverson bracket.

We get for the sum

$$4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} (1+z)^n \sum_{q\ge 0} \binom{n}{q} 4^{-q} \frac{z^{2q}}{(1+z)^q} \; dz$$

$$= 4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} (1+z)^n \left(1 + \frac{1}{4} \frac{z^2}{1+z}\right)^n \; dz$$

$$= 4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} \left(1 + z + \frac{1}{4} z^2\right)^n \; dz$$

Now put $z = 2w$ to get

$$4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{2^{2p+1} w^{2p+1}} \left(1 + 2w + w^2\right)^n \; 2dw$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{w^{2p+1}} (1+w)^{2n} \; dw.$$

This is

$$\binom{2n}{2p}$$

as claimed. This was math.stackexchange.com problem 1430202.

22 Method applied to an iterated sum (B_1R)

Suppose we seek to show that

$$\sum_{k=0}^{n-1} \left(\sum_{q=0}^{k} \binom{n}{q} \right) \left(\sum_{q=k+1}^{n} \binom{n}{q} \right) = \frac{1}{2} n \binom{2n}{n}.$$

Using the integral representation

$$\binom{n}{q} = \binom{n}{n-q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n-q+1}} \; dz$$

we get for the first factor

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\sum_{q=0}^{k} z^q\; dz = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1-z^{k+1}}{1-z}\; dz$$

$$= 2^n - \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{z^{k+1}}{1-z}\; dz$$

and for the second factor

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{z^{k+1}-z^{n+1}}{1-z}\; dz = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{z^{k+1}}{1-z}\; dz.$$

These add to 2^n as they obviously should.

Summing from $k=0$ to $n-1$ we get a positive and a negative piece. The positive piece is

$$2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n}\sum_{k=0}^{n-1}\frac{z^k}{1-z}\; dz$$
$$= 2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n}\frac{1-z^n}{(1-z)^2}\; dz$$
$$= 2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n}\frac{1}{(1-z)^2}\; dz.$$

The negative piece is

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{(1+z_1)^n}{z_1^n(1-z_1)}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{(1+z_2)^n}{z_2^n(1-z_2)}\sum_{k=0}^{n-1} z_1^k z_2^k\; dz_2\; dz_1$$

$$= \frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{(1+z_1)^n}{z_1^n(1-z_1)}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{(1+z_2)^n}{z_2^n(1-z_2)}\frac{1-z_1^n z_2^n}{1-z_1 z_2}\; dz_2\; dz_1$$
$$= \frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{(1+z_1)^n}{z_1^n(1-z_1)}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{(1+z_2)^n}{z_2^n(1-z_2)}\frac{1}{1-z_1 z_2}\; dz_2\; dz_1.$$

We evaluate the inner integral by taking the sum of the negatives of the residues of the poles at $z_2=1$ and $z_2=1/z_1$ instead of computing the residue of the pole at zero by using the fact that the residues sum to zero.

Re-write the integral as follows.

$$\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{(1+z_2)^n}{z_2^n(z_2-1)}\frac{1}{z_1 z_2-1}\; dz_2$$

$$= \frac{1}{z_1}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{(1+z_2)^n}{z_2^n(z_2-1)}\frac{1}{z_2-1/z_1}\; dz_2.$$

Now the negative of the residue at $z_2 = 1$ is

$$-\frac{1}{z_1} 2^n \frac{1}{1 - 1/z_1} = 2^n \frac{1}{1 - z_1}.$$

Substituting this into the outer integral we get

$$2^n \frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{(1 + z_1)^n}{z_1^n (1 - z_1)^2} \; dz_1.$$

We see that this piece precisely cancels the positive piece that we obtained first.

Continuing the negative of the residue at $z_2 = 1/z_1$ is

$$-\frac{1}{z_1} \frac{(1 + 1/z_1)^n}{1/z_1^n \times (1/z_1 - 1)} = -\frac{1}{z_1} \frac{(1 + z_1)^n}{(1/z_1 - 1)} = -\frac{(1 + z_1)^n}{(1 - z_1)}.$$

We now substitute this into the outer integral flipping the sign because this was the negative piece to get

$$\frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{(1 + z_1)^{2n}}{z_1^n (1 - z_1)^2} \; dz_1.$$

Extracting the residue at $z_1 = 0$ we get

$$\sum_{q=0}^{n-1} \binom{2n}{n - 1 - q} (q + 1) = \sum_{q=0}^{n-1} \binom{2n}{n + q + 1} (q + 1)$$

$$= -n \sum_{q=0}^{n-1} \binom{2n}{n + q + 1} + \sum_{q=0}^{n-1} \binom{2n}{n + q + 1} (n + q + 1)$$

$$= -n \left(\frac{1}{2} 2^{2n} - \frac{1}{2} \binom{2n}{n} \right) + 2n \sum_{q=0}^{n-1} \binom{2n - 1}{n + q}$$

$$= -n \left(\frac{1}{2} 2^{2n} - \frac{1}{2} \binom{2n}{n} \right) + 2n \frac{1}{2} 2^{2n-1}$$

$$= \frac{1}{2} n \binom{2n}{n}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity of the inner integral is zero. We use the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z} \right) \right]$$

which in the present case gives for the inner term in z_2

$$-\mathrm{Res}_{z_2=0}\frac{1}{z_2^2}\frac{(1+1/z_2)^n}{1/z_2^n\times(1-1/z_2)}\frac{1}{1-z_1/z_2}$$

$$=-\mathrm{Res}_{z_2=0}\frac{1}{z_2^2}\frac{(1+z_2)^n}{(1-1/z_2)}\frac{1}{1-z_1/z_2}$$

$$=-\mathrm{Res}_{z_2=0}\frac{(1+z_2)^n}{(z_2-1)}\frac{1}{z_2-z_1}$$

which is zero by inspection.

This was math.stackexchange.com problem 889892.

23 A pair of two double hypergeometrics (B_1)

We seek to show that

$$(1-x)^{2k+1}\sum_{n\ge 0}\binom{n+k-1}{k}\binom{n+k}{k}x^n=\sum_{j\ge 0}\binom{k-1}{j-1}\binom{k+1}{j}x^j.$$

Suppose we start by evalutating the two sums in turn, where the parameter $k\ge 1$. For the first we will be using the following integral representation:

$$\binom{n+k}{k}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+k}}{z^{k+1}}\;dz.$$

We seek

$$\sum_{n\ge 1}\binom{n-1+k}{k}\binom{n+k}{k}x^n.$$

Using the integral we find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\sum_{n\ge 1}\binom{n-1+k}{k}x^n\frac{(1+z)^{n+k}}{z^{k+1}}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^k}{z^{k+1}}\sum_{n\ge 1}\binom{n-1+k}{k}(1+z)^n x^n\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{x(1+z)^{k+1}}{z^{k+1}}\sum_{n\ge 1}\binom{n-1+k}{k}(1+z)^{n-1}x^{n-1}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{x(1+z)^{k+1}}{z^{k+1}}\frac{1}{(1-x(1+z))^{k+1}}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{x(1+z)^{k+1}}{z^{k+1}}\frac{1}{(1-x-xz))^{k+1}}\;dz$$

$$= \frac{1}{(1-x)^{k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{x(1+z)^{k+1}}{z^{k+1}} \frac{1}{(1-xz/(1-x)))^{k+1}} \; dz$$

$$= \frac{x}{(1-x)^{k+1}} \sum_{q=0}^{k} \binom{k+1}{k-q} \binom{q+k}{k} \left(\frac{x}{1-x}\right)^q$$

$$= \frac{x}{(1-x)^{k+1}} \sum_{q=0}^{k} \binom{k+1}{q+1} \binom{q+k}{k} \left(\frac{x}{1-x}\right)^q.$$

Applying the integral representation from the beginning a second time we obtain for this sum

$$\frac{x}{(1-x)^{k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{k} \binom{k+1}{q+1} \frac{(1+z)^{q+k}}{z^{k+1}} \left(\frac{x}{1-x}\right)^q \; dz$$

$$= \frac{x}{(1-x)^{k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^k}{z^{k+1}} \sum_{q=0}^{k} \binom{k+1}{q+1} (1+z)^q \left(\frac{x}{1-x}\right)^q \; dz$$

$$= \frac{1}{(1-x)^{k}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \sum_{q=0}^{k} \binom{k+1}{q+1} (1+z)^{q+1} \left(\frac{x}{1-x}\right)^{q+1} \; dz$$

$$= \frac{1}{(1-x)^{k}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(-1 + \left(1 + (1+z)\frac{x}{1-x}\right)^{k+1}\right) \; dz.$$

We have $k+1-(k-1)=2$, so the first component inside the parentheses drops out, leaving

$$\frac{1}{(1-x)^{k}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(1 + (1+z)\frac{x}{1-x}\right)^{k+1} \; dz$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(1-x+x(1+z)\right)^{k+1} \; dz$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(1+xz\right)^{k+1} \; dz.$$

We need one more simplification on this and put $z = 1/w$, getting

$$\frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+1/w)^{k-1}}{(1/w)^{k+1}} \left(1+x/w\right)^{k+1} \; \frac{1}{w^2} dw$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} w^2 (w+1)^{k-1} \left(\frac{w+x}{w}\right)^{k+1} \; \frac{1}{w^2} dw$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(w+1)^{k-1}}{w^{k+1}} (w+x)^{k+1} \; dw.$$

The reson this works is because we are essentially evaluating the residue at infinity and the residues sum to zero. This concludes the evaluation of the first sum. For the second we will be using the following integral representation:

$$\binom{k-1}{j-1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^j}\; dz.$$

We seek

$$\sum_{j\ge 1} \binom{k+1}{j}\binom{k-1}{j-1} x^j.$$

Using the integral we find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \sum_{j\ge 1}\binom{k+1}{j} x^j \frac{(1+z)^{k-1}}{z^j}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} (1+z)^{k-1} \sum_{j\ge 1}\binom{k+1}{j}\frac{x^j}{z^j}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} (1+z)^{k-1}\left(-1+(1+x/z)^{k+1}\right)\; dz.$$

The entire component drops out, leaving

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} (1+z)^{k-1}(1+x/z)^{k+1}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}}(z+x)^{k+1}\; dz.$$

This however is precisely the integral that we had for the first sum without the factor in front, done.

The only infinite sum appearing here is the first one with convergence when $|(1+z)x| < 1$. Therefore choosing $|x| < 1/Q$ and $|z| < 1/Q$ with $Q \ge 2$ we have $|(Q+1)/Q/Q| = |1/Q^2 + 1/Q| < 1$ and get convergence of the first LHS integral in a neighborhood of zero.

This is math.stackexchange.com problem 869982.

24 A two phase application of the method (B_1)

We seek to show that

$$\sum_{k=0}^{\lfloor n/3\rfloor} (-1)^k \binom{n+1}{k}\binom{2n-3k}{n} = \sum_{k=\lfloor n/2\rfloor}^{n} \binom{n+1}{k}\binom{k}{n-k}.$$

Note that the second binomial coefficient in both sums controls the range of the sum, so we can write our claim like this:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \binom{2n-3k}{n-3k} = \sum_{k=0}^{n+1} \binom{n+1}{k} \binom{k}{n-k}.$$

To evaluate the LHS introduce the integral representation

$$\binom{2n-3k}{n-3k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-3k}}{z^{n-3k+1}} \; dz.$$

We can check that this really is zero when $k > \lfloor n/3 \rfloor$.

This gives for the sum the representation

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \left(\frac{z^3}{(1+z)^3} \right)^k \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \left(1 - \frac{z^3}{(1+z)^3} \right)^{n+1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1+z)^{n+3}} \left(3z^2+3z+1\right)^{n+1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1+z)^{n+3}} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q z^q (1+z)^q \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q z^{q-n-1} (1+z)^{q-n-3} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q \frac{1}{z^{n+1-q}} \frac{1}{(1+z)^{n+3-q}} \; dz.$$

Computing the residue we find

$$\sum_{q=0}^{n+1} \binom{n+1}{q} 3^q (-1)^{n-q} \binom{n-q+n+2-q}{n+2-q}$$

$$= \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q (-1)^{n-q} \binom{2n-2q+2}{n-q+2}.$$

Now introduce the integral representation

$$\binom{2n-2q+2}{n-q+2} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-2q+2}}{z^{n-q+3}} \; dz$$

which gives for the sum the integral

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+3}}\sum_{q=0}^{n+1}\binom{n+1}{q}3^q(-1)^{n-q}\left(\frac{z}{(1+z)^2}\right)^q\;dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+3}}\left(\frac{3z}{(1+z)^2}-1\right)^{n+1}\;dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+3}}(-1+z-z^2)^{n+1}\;dz.$$

Put $w=-z$ which just rotates the small circle to get

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{(-w)^{n+3}}(-1-w-w^2)^{n+1}\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+3}}(1+w+w^2)^{n+1}\;dw.$$

We get for the final answer

$$[w^{n+2}](1+w+w^2)^{n+1}$$

but we have $2n+2-n-2=n$ and thus exploiting the symmetry of $1+w+w^2$ we get

$$[w^n](1+w+w^2)^{n+1}.$$

To evaluate the RHS introduce the integral representation

$$\binom{k}{n-k}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^k}{z^{n-k+1}}\;dz.$$

This gives for the sum the representation

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}((1+z)z)^k\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z(1+z))^{n+1}\;dz.$$

The answer is

$$[z^n](1+z+z^2)^{n+1},$$

the same as the LHS, and we are done.

This was math.stackexchange.com problem 664823.

25 An identity from Mathematical Reflections (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{\lfloor (m+n)/2 \rfloor} \binom{n}{k} (-1)^k \binom{m+n-2k}{n-1}.$$

Observe that in the second binomial coefficient we must have $m+n-2k \ge n-1$ in order to avoid hitting the zero value in the product in the numerator of the binomial coefficient, so the upper limit for the sum is in fact $m+1 \ge 2k$ with the sum being

$$\sum_{k=0}^{\lfloor (m+1)/2 \rfloor} \binom{n}{k} (-1)^k \binom{m+n-2k}{n-1}.$$

Introduce

$$\binom{m+n-2k}{n-1} = \binom{m+n-2k}{m+1-2k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n-2k}}{z^{m+2-2k}} \; dz.$$

This integral correctly encodes the range for k being zero when k is larger than $\lfloor (m+1)/2 \rfloor$. Therefore we may let k go to infinity in the sum and obtain for $n > m$

$$\begin{aligned} &\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n}}{z^{m+2}} \sum_{k\ge 0} \binom{n}{k} (-1)^k \frac{z^{2k}}{(1+z)^{2k}} \; dz \\ &= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n}}{z^{m+2}} \left(1 - \frac{z^2}{(1+z)^2}\right)^n \; dz \\ &= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{(1+z)^{n-m} z^{m+2}} (1+2z)^n \; dz. \end{aligned}$$

This produces the closed form

$$\sum_{q=0}^{m+1} \binom{n}{q} 2^q (-1)^{m+1-q} \binom{m+1-q+n-m-1}{n-m-1}$$

$$= (-1)^{m+1} \sum_{q=0}^{m+1} \binom{n}{q} (-1)^q 2^q \binom{n-q}{n-m-1}.$$

This is

$$(-1)^{m+1} \sum_{q=0}^{m+1} \binom{n}{q} (-1)^q 2^q \binom{n-q}{m+1-q}.$$

Introduce

$$\binom{n-q}{m+1-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-q}}{z^{m+2-q}}\; dz$$

which once more correctly encodes the range with the pole at $z=0$ disappearing when $q > m+1$. Therefore we may extend the range to n to get

$$\begin{aligned}
&\frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{m+2}} \sum_{q=0}^{n} \binom{n}{q} (-1)^q 2^q \frac{z^q}{(1+z)^q}\; dz \\
&= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{m+2}} \left(1 - 2\frac{z}{1+z}\right)^n\; dz \\
&= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{m+2}} \frac{(1-z)^n}{(1+z)^n}\; dz \\
&= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1-z)^n}{z^{m+2}}\; dz \\
&= (-1)^{m+1}\binom{n}{m+1}(-1)^{m+1} = \binom{n}{m+1}.
\end{aligned}$$

This was math.stackexchange.com problem 390321.

26 A triple Fibonacci-binomial coefficient convolution (B_1)

Here is a proof using complex variables. We seek to show that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} F_{k+1} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} (-1)^{n-k} F_{2k+1}.$$

Start from

$$\binom{n+k}{k} = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z^{k+1}}(1+z)^{n+k}\; dz.$$

This yields the following expression for the sum on the LHS

$$\frac{1}{2\pi i}\int_{|z|=1} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{z^{k+1}}(1+z)^{n+k} \frac{\varphi^{k+1} - (-1/\varphi)^{k+1}}{\sqrt{5}}\; dz$$

This simplifies to

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1} \frac{(1+z)^n}{z} \sum_{k=0}^{n} \binom{n}{k} \left(\varphi\left(\varphi\frac{1+z}{z}\right)^k + \frac{1}{\varphi}\left(-\frac{1}{\varphi}\frac{1+z}{z}\right)^k\right)\; dz$$

This finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z}\left(\varphi\left(1+\varphi\frac{1+z}{z}\right)^n+\frac{1}{\varphi}\left(1-\frac{1}{\varphi}\frac{1+z}{z}\right)^n\right)\,dz$$

or

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{n+1}}\left(\varphi\left(z+\varphi(1+z)\right)^n+\frac{1}{\varphi}\left(z-\frac{1}{\varphi}(1+z)\right)^n\right)\,dz$$

Continuing we have the following expression for the sum on the RHS

$$\frac{1}{2\pi i}\int_{|z|=1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{1}{z^{k+1}}(1+z)^{n+k}\frac{\varphi^{2k+1}-(-1/\varphi)^{2k+1}}{\sqrt{5}}\,dz$$

This simplifies to

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\varphi\left(\varphi^2\frac{1+z}{z}\right)^k+\frac{1}{\varphi}\left(\frac{1}{\varphi^2}\frac{1+z}{z}\right)^k\right)\,dz$$

This finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z}\left(\varphi\left(-1+\varphi^2\frac{1+z}{z}\right)^n+\frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}\frac{1+z}{z}\right)^n\right)\,dz$$

or

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{n+1}}\left(\varphi\left(-z+\varphi^2(1+z)\right)^n+\frac{1}{\varphi}\left(-z+\frac{1}{\varphi^2}(1+z)\right)^n\right)\,dz$$

Apply the substitution $z=1/w$ to this integral to obtain (the sign to correct the reverse orientation of the circle is canceled by the minus on the derivative)

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\left(1+\frac{1}{w}\right)^n w^{n+1}$$
$$\times\left(\varphi\left(-\frac{1}{w}+\varphi^2(1+\frac{1}{w})\right)^n+\frac{1}{\varphi}\left(-\frac{1}{w}+\frac{1}{\varphi^2}(1+\frac{1}{w})\right)^n\right)\frac{1}{w^2}\,dw$$

which is

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\left(1+\frac{1}{w}\right)^n\frac{1}{w}$$
$$\times\left(\varphi\left(-1+\varphi^2(w+1)\right)^n+\frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}(w+1)\right)^n\right)\,dw$$

which finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\frac{(1+w)^n}{w^{n+1}}$$
$$\times\left(\varphi\left(-1+\varphi^2(w+1)\right)^n+\frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}(w+1)\right)^n\right)\;dw$$

This shows that the LHS is the same as the RHS because

$$-1+\varphi^2(w+1) = -1+(1+\varphi)(w+1) = w+\varphi(w+1)$$

and

$$-1+\frac{1}{\varphi^2}(w+1) = -1+(1-\frac{1}{\varphi})(w+1)$$
$$= -1+(w+1)-\frac{1}{\varphi}(w+1) = w-\frac{1}{\varphi}(w+1).$$

This is math.stackexchange.com problem 53830.

27 Fibonacci numbers and the residue at infinity (B_2R)

Suppose we seek to evaluate in terms of Fibonacci numbers

$$\sum_{p,q\geq 0}\binom{n-p}{q}\binom{n-q}{p}.$$

We use the integrals

$$\binom{n-p}{q} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{(1-z)^{q+1}z^{n-p-q+1}}\;dz$$

and

$$\binom{n-q}{p} = \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{(1-w)^{p+1}w^{n-p-q+1}}\;dw.$$

These correctly control the range so we may let p and q go to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{(1-z)z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{(1-w)w^{n+1}}\sum_{p,q\geq 0}\frac{z^{p+q}w^{p+q}}{(1-w)^p(1-z)^q}\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{(1-z)z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{(1-w)w^{n+1}}$$
$$\times\frac{1}{1-zw/(1-w)}\frac{1}{1-zw/(1-z)}\;dw\;dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-w-zw} \frac{1}{1-z-zw} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+2}(1+z)} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{w-1/(1+z)} \frac{1}{w-(1-z)/z} \; dw \; dz.$$

We evaluate the inner integral using the fact that the residues of the function in w sum to zero. We have two simple poles. We get for the first pole at $w = (1-z)/z$

$$\frac{z^{n+1}}{(1-z)^{n+1}} \frac{1}{(1-z)/z - 1/(1+z)} = \frac{z^{n+1}}{(1-z)^{n+1}} \frac{z(1+z)}{(1-z)(1+z)-z}$$

$$= \frac{z^{n+2}}{(1-z)^{n+1}} \frac{1+z}{(1-z)(1+z)-z}.$$

Substituting this expression into the outer integral we see that the pole at $z = 0$ is canceled making for a contribution of zero.

For the second pole at $w = 1/(1+z)$ we get

$$(1+z)^{n+1} \frac{1}{1/(1+z) - (1-z)/z} = (1+z)^{n+1} \frac{z(1+z)}{z-(1-z)(1+z)}.$$

This yields the contribution (taking into account the sign flip from the sum of residues)

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+2}(1+z)} (1+z)^{n+1} \frac{z(1+z)}{1-z-z^2} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+1} \frac{1}{1-z-z^2} \; dz.$$

We evaluate this using again the fact that the residues sum to zero. There are simple poles at $z = -\varphi$ and $z = 1/\varphi$.

These yield

$$\left(\frac{1-\varphi}{-\varphi}\right)^{n+1} \frac{1}{-1+2\varphi} + \left(\frac{1+1/\varphi}{1/\varphi}\right)^{n+1} \frac{1}{-1-2/\varphi}$$

$$= \frac{1}{\sqrt{5}} \frac{1}{\varphi^{2n+2}} - \frac{1}{\sqrt{5}} \varphi^{2n+2}.$$

Taking into account the sign flip this is obviously Binet / de Moivre for

$$F_{2n+2}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity in both cases is zero. For example in the first application we use the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0}\left[-\frac{1}{z^2} h\left(\frac{1}{z}\right)\right]$$

which in the present case gives for the inner term in w

$$-\mathrm{Res}_{w=0} \frac{1}{w^2} w^{n+1} \frac{1}{1/w - 1/(1+z)} \frac{1}{1/w - (1-z)/z}$$

$$= -\mathrm{Res}_{w=0} w^{n+1} \frac{1}{1 - w/(1+z)} \frac{1}{1 - w(1-z)/z}$$

which is zero by inspection.

This was math.stackexchange.com problem 801730.

28 Permutations containing a given subsequence ($B_1 I$)

The WZ machinery is very powerful but it is also an incentive to evaluate these sums manually e.g. by using the Egorychev method which I hope will make for a rewarding read.

Suppose as before that we are trying to evaluate

$$S = \sum_{r=0}^{n} \binom{r+n-1}{n-1} \binom{3n-r}{n}$$

which is

$$S_2 - S_1 = \sum_{r=0}^{2n} \binom{r+n-1}{n-1} \binom{3n-r}{n} - \sum_{r=n+1}^{2n} \binom{r+n-1}{n-1} \binom{3n-r}{n}.$$

Start by evaluating S_2.

Put

$$\binom{3n-r}{n} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-r+1}} \frac{1}{(1-w)^{n+1}} \; dw.$$

and use the following Iverson bracket

$$[[0 \le r \le 2n]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^r}{z^{2n+1}} \frac{1}{1-z} \; dz.$$

This second integral controls the range so that we may extend the sum to infinity to get

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n+1}} \frac{1}{(1-w)^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n+1}} \frac{1}{1-z} \sum_{r=0}^{\infty} \binom{r+n-1}{n-1} z^r w^r \; dz \; dw.$$

This simplifies to

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{2n+1}}\frac{1}{(1-w)^{n+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n+1}}\frac{1}{1-z}\frac{1}{(1-wz)^n}\;dz\;dw.$$

We evaluate the inner integral using the fact that the residues at the three poles sum to zero. The residue at $z=0$ is the sum S_2 which we are trying to compute. The residue at $z=1$ yields

$$-\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{2n+1}}\frac{1}{(1-w)^{2n+1}}\;dw = -\binom{2n+2n}{2n} = -\binom{4n}{2n}.$$

For the residue at $z=1/w$ re-write the integral as follows:

$$\frac{(-1)^n}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{3n+1}}\frac{1}{(1-w)^{n+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n+1}}\frac{1}{1-z}\frac{1}{(z-1/w)^n}\;dz\;dw.$$

We require a derivative which we compute using Leibniz' rule:

$$\frac{1}{(n-1)!}\left(\frac{1}{z^{2n+1}}\frac{1}{1-z}\right)^{(n-1)}$$

$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^q\frac{(2n+q)!}{(2n)!\times z^{2n+1+q}}\frac{(n-1-q)!}{(1-z)^{1+n-1-q}}$$

$$= \sum_{q=0}^{n-1}\binom{2n+q}{2n}(-1)^q\frac{1}{z^{2n+1+q}}\frac{1}{(1-z)^{n-q}}.$$

Evaluate at $z=1/w$ to get

$$\sum_{q=0}^{n-1}\binom{2n+q}{2n}(-1)^q w^{2n+1+q}\frac{w^{n-q}}{(w-1)^{n-q}}.$$

Substitute this back into the integral in w to obtain

$$\sum_{q=0}^{n-1}\binom{2n+q}{2n}(-1)^q\frac{(-1)^n}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{3n+1}}\frac{1}{(1-w)^{n+1}}\frac{w^{3n+1}}{(w-1)^{n-q}}\;dw$$

$$= \sum_{q=0}^{n-1}\binom{2n+q}{2n}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{(1-w)^{2n-q+1}}\;dw = 0.$$

We have shown that

$$S_2 = \binom{4n}{2n}.$$

Continuing with S_1 we see that

$$S_1 = \sum_{r=0}^{n-1} \binom{r+2n}{n-1}\binom{2n-1-r}{n} = \sum_{r=0}^{n-1} \binom{r+2n}{n-1}\binom{2n-1-r}{n-1-r}.$$

For this sum no Iverson bracket is needed as the second binomial coefficient controls the range via the following integral:

$$\binom{2n-1-r}{n-1-r} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n-1-r}}{w^{n-r}}\; dw.$$

Furthermore introduce

$$\binom{r+2n}{n-1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{r+2n}}{z^{n}}\; dz.$$

This gives the integral

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n-1}}{w^{n}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n}} \sum_{r\ge 0} \frac{w^r(1+z)^r}{(1+w)^r}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n-1}}{w^{n}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n}} \frac{1}{1-w(1+z)/(1+w)}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n}}{w^{n}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n}} \frac{1}{1+w-w(1+z)}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n}}{w^{n}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n}} \frac{1}{1-wz}\; dz\; dw.$$

Extracting the inner residue we obtain

$$\sum_{q=0}^{n-1} \binom{2n}{n-1-q} w^q$$

which yields

$$\sum_{q=0}^{n-1} \binom{2n}{n-1-q} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n}}{w^{n-q}}\; dw$$

$$= \sum_{q=0}^{n-1} \binom{2n}{n-1-q}\binom{2n}{n-1-q}.$$

This is

$$\sum_{q=0}^{n-1} \binom{2n}{q}^2$$

which may be evaluated by inspection as in the first version and we are done.

Remark. To be fully rigorous we must also show that the residue at infinity of

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n+1}}\frac{1}{1-z}\frac{1}{(1-wz)^n}\;dz$$

is zero. Recall the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty}h(z)=\mathrm{Res}_{z=0}\left[-\frac{1}{z^2}h\left(\frac{1}{z}\right)\right]$$

which in this case yields

$$\begin{aligned}&-\mathrm{Res}_{z=0}\frac{1}{z^2}z^{2n+1}\frac{1}{1-1/z}\frac{1}{(1-w/z)^n}\\ &=-\mathrm{Res}_{z=0}z^{2n}\frac{1}{z-1}\frac{1}{(1-w/z)^n}\\ &=-\mathrm{Res}_{z=0}z^{3n}\frac{1}{z-1}\frac{1}{(z-w)^n}\end{aligned}$$

which is zero by inspection.

This is math.stackexchange.com problem 1255356.

29 An example of Lagrange inversion (B_1)

We seek to use Lagrange Inversion to show that

$$s(x,y)=\frac{1}{2}\left(1-x-y-\sqrt{1-2x-2y-2xy+x^2+y^2}\right)$$

has the series expansion

$$\sum_{p,q\geq 1}\frac{1}{p+q-1}\binom{p+q-1}{p}\binom{p+q-1}{q}x^py^q.$$

On squaring we obtain

$$\begin{aligned}4s(x,y)^2&=(1-x-y)^2+1-2x-2y-2xy+x^2+y^2\\&\quad-2(1-x-y)(1-x-y-2s(x,y))\\&=2(1-x-y)^2-4xy-2(1-x-y)(1-x-y-2s(x,y))\\&=-4xy+4(1-x-y)s(x,y).\end{aligned}$$

We finally get

$$s(x,y)^2=-xy+(1-x-y)s(x,y).$$

This implies

$$x = \frac{s(x,y)(1-y-s(x,y))}{y+s(x,y)}.$$

We get with $p \ge 1$

$$[x^p]s(x,y) = \frac{1}{p}[x^{p-1}]\frac{d}{dx}s(x,y) = \frac{1}{p}\frac{1}{2\pi i}\int_{|x|=\varepsilon} \frac{1}{x^p}\frac{d}{dx}s(x,y)\; dx.$$

Now put $s(x,y) = u$ so that $\frac{d}{dx}s(x,y)\; dx = du$ and $x = 0$ maps to $u = 0$ to get

$$\frac{1}{p}\frac{1}{2\pi i}\int_{|u|=\gamma} \frac{(y+u)^p}{u^p(1-y-u)^p}\; du$$
$$= \frac{1}{p}\frac{1}{(1-y)^p}\frac{1}{2\pi i}\int_{|u|=\gamma} \frac{(y+u)^p}{u^p(1-u/(1-y))^p}\; du.$$

This is

$$\frac{1}{p}\frac{1}{(1-y)^p}\sum_{r=0}^{p-1}\binom{p}{r}y^{p-r}\binom{2p-2-r}{p-1}\frac{1}{(1-y)^{p-1-r}}.$$

Extracting the coefficient on $[y^q]$ where we see that $q \ge 1$:

$$\frac{1}{p}\sum_{r=0}^{p-1}\binom{p}{r}[y^q]y^{p-r}\binom{2p-2-r}{p-1}\frac{1}{(1-y)^{2p-1-r}}$$
$$= \frac{1}{p}\sum_{r=0}^{p-1}\binom{p}{r}\binom{2p-2-r}{p-1}\binom{q+p-2}{2p-2-r}.$$

Next observe that

$$\binom{2p-2-r}{p-1}\binom{q+p-2}{2p-2-r} = \frac{(q+p-2)!}{(p-1)! \times (p-1-r)! \times (q+r-p)!}$$
$$= \binom{p+q-2}{p-1}\binom{q-1}{p-1-r}.$$

We get for our sum

$$\frac{1}{p}\binom{p+q-2}{p-1}\sum_{r=0}^{p-1}\binom{p}{r}\binom{q-1}{p-1-r}$$
$$= \frac{1}{p}\binom{p+q-2}{p-1}[z^{p-1}](1+z)^{q-1}\sum_{r=0}^{p-1}\binom{p}{r}z^r.$$

The term in $r = p$ does not pass the coefficient extractor and we may raise

r to p :

$$\frac{1}{p}\binom{p+q-2}{p-1}[z^{p-1}](1+z)^{q+p-1} = \frac{1}{p}\binom{p+q-2}{p-1}\binom{p+q-1}{p-1}.$$

This was Vandermonde. Some binomial coefficient algebra now yields

$$\boxed{\frac{1}{p+q-1}\binom{p+q-1}{p}\binom{p+q-1}{q}}$$

as claimed.

Remark. For the contour of the integral in u to make a single turn the coefficient on x must be non-zero. We differentiate the functional equation with respect to x to get

$$2s(x,y)s'(x,y) = -y - s(x,y) + (1-x-y)s'(x,y)$$

Together with the fact that we choose the branch with $s(0,y) = s(x,0) = 0$ this yields $s'(0,y) = y/(1-y)$ as required.

30 A binomial coefficient - Catalan number convolution (B_1)

Suppose we seek to show that

$$\sum_{r=1}^{n+1} \frac{1}{r+1}\binom{2r}{r}\binom{m+n-2r}{n+1-r} = \binom{m+n}{n}.$$

We will assume familiarity with the generating function of the Catalan numbers (which seems like a reasonable assumption). This generating function is given by

$$\sum_{r\ge 0} \frac{1}{r+1}\binom{2r}{r} z^r = \frac{1-\sqrt{1-4z}}{2z}$$

so that

$$\frac{1}{r+1}\binom{2r}{r} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{r+1}} \frac{1-\sqrt{1-4z}}{2z}\, dz.$$

Note in particular that this generating function is analytic in a neighborhood of the origin $|z| < 1/4$ with the branch cut $[1/4, \infty)$.

Furthermore introduce

$$\binom{m+n-2r}{n+1-r} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n-2r}}{w^{n+2-r}}\, dw.$$

Observe carefully that this last integral is zero when $r > n+1$, so we may extend the range of the sum to infinity.

This yields for the sum

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n}}{w^{n+2}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}\frac{1-\sqrt{1-4z}}{2z}\sum_{r\geq 1}\frac{w^r}{(1+w)^{2r}z^r}\;dz\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n}}{w^{n+2}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1-\sqrt{1-4z}}{2z^2}\frac{w/(1+w)^2/z}{1-w/(1+w)^2/z}\;dz\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n}}{w^{n+2}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1-\sqrt{1-4z}}{2z^2}\frac{1}{z(1+w)^2/w-1}\;dz\;dw.$$

Observe that with the principal branch of the logarithm

$$1-\sqrt{1-4z}=2z+2z^2+4z^3+\cdots$$

and

$$\frac{1}{z(1+w)^2/w-1}=-1-z\frac{(1+w)^2}{w}-z^2\frac{(1+w)^4}{w^2}-\cdots.$$

so that the contribution from the pole at $z=0$ is

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n}}{w^{n+2}}\frac{1}{2}\times(-2)\;dw=-\binom{m+n}{n+1}.$$

On the other hand the contribution from the simple pole at $z=w/(1+w)^2$ which is inside the contour is

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n}}{w^{n+2}}\frac{1-\sqrt{1-4w/(1+w)^2}}{2w^2/(1+w)^4}\frac{w}{(1+w)^2}\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n-2}}{w^{n+1}}\frac{(1+w)^4-(1+w)^3\sqrt{(1+w)^2-4w}}{2w^2}\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n-2}}{2w^{n+3}}((1+w)^4-(1-w)(1+w)^3)\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n-2}}{2w^{n+3}}(1+w)^3\times(2w)\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{m+n+1}}{w^{n+2}}\;dw.$$

which yields

$$\binom{m+n+1}{n+1}.$$

Collecting the two contributions we obtain

$$\binom{m+n+1}{n+1}-\binom{m+n}{n+1}=\left(\frac{m+n+1}{n+1}-\frac{m}{n+1}\right)\binom{m+n}{n}$$

$$= \binom{m+n}{n}$$

as claimed.

Addendum. In fact the above admits considerable simplification. Write

$$-\binom{m+n}{n+1} + \sum_{r=0}^{n+1} \frac{1}{r+1}\binom{2r}{r}\binom{m+n-2r}{n+1-r}$$

and use the same integral as before for the binomial coefficient to obtain

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \sum_{r\ge 0} \frac{1}{r+1}\binom{2r}{r}\frac{w^r}{(1+w)^{2r}}\; dw$$

which becomes

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1-\sqrt{1-4w/(1+w)^2}}{2w/(1+w)^2}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1+w-\sqrt{(1+w)^2-4w}}{2w/(1+w)}\; dw$$

$$= \frac{1}{2}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n+1}}{w^{n+3}} \left(1+w-\sqrt{(1-w)^2}\right)\; dw$$

$$= \frac{1}{2}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n+1}}{w^{n+3}} (2w)\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n+1}}{w^{n+2}}\; dw$$

$$= \binom{m+n+1}{n+1}.$$

We may then conclude as before.

Addendum, Feb 2021. For the first version to be complete we need the conditions on ε and γ for the geometric series to converge. This requires $|w/(1+w)^2| < |z|$. Note that if this holds then the pole at $z = w/(1+w)^2$ is guaranteed to be inside the contour in $|z|$ as claimed. We take $\gamma < 1$ positive somewhere close to zero and we then require $\gamma/(1-\gamma)^2 < \varepsilon < 1/4$ where the last term is from the Catalan GF. The values $\gamma = 1/7$ and $\varepsilon = 1/5$ will work. This also ensures convergence of the Catalan GF series in the compact version.

This was math.stackexchange.com problem 563307.

31 A new obstacle from Concrete Mathematics (Catalan numbers) (B_1)

Suppose we seek to evaluate

$$\sum_{k\ge 0} \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

where $m, n \ge 0$. In fact we may assume that $n \ge m$ because if $m > n$ when counting down from the non-negative value $n+k$ with $m+2k$ terms we invariably hit zero and the sum vanishes.

Furthermore observe that when $k = n-m+q$ with $q > 0$ we obtain $\binom{2n-m+q}{2n-m+2q}$ which is zero by the same argument.

This gives

$$\sum_{k=0}^{n-m} \binom{n+k}{n-m-k}\binom{2k}{k}\frac{(-1)^k}{k+1}.$$

Introduce

$$\binom{n+k}{n-m-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m-k+1}}(1+z)^{n+k}\; dz.$$

Observe that this is zero when $k > n-m$ so we may extend k to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+1}}(1+z)^n \sum_{k\ge 0}\binom{2k}{k}\frac{(-1)^k}{k+1} z^k (1+z)^k\; dz.$$

Here we recognize the generating function of the Catalan numbers

$$\sum_{k\ge 0}\binom{2k}{k}\frac{1}{k+1}w^k = \frac{1-\sqrt{1-4w}}{2w}$$

where the branch cut of the logarithm is on the negative real axis and hence the branch cut of the square root term is $[1/4, \infty)$ so we certainly have analyticity in a neighborhood of zero. We obtain

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+1}}(1+z)^n \frac{1-\sqrt{1+4z(1+z)}}{2z(1+z)}\; dz$$
$$= -\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+2}}(1+z)^{n-1}\left(1-\sqrt{(1+2z)^2}\right)\; dz.$$

Now with z in a neighborhood of zero the square root produces the positive root so we finally have

$$-\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-m+2}}(1+z)^{n-1}(-2z)\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-m+1}}(1+z)^{n-1}\;dz$$

which evaluates by inspection to $\binom{n-1}{n-m}$ which is

$$\binom{n-1}{m-1}.$$

Note that for the series to converge we need $|z(1+z)| < 1/4$ (radius of convergence is distance to the nearest singularity). Now $|z(1+z)| \le \epsilon(1+\epsilon)$. With $\epsilon \ll 1$ this is less than 2ϵ. Therefor $\epsilon = 1/8$ will work.

This problem has not yet appeared at math.stackexchange.com.

32 Abel-Aigner identity from Table 202 of Concrete Mathematics (B_1)

Seeking to prove that

$$\sum_k \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{r}{tk+r} = \binom{tn+r+s}{n}$$

we see that our identity is in fact

$$\sum_{k=0}^{n}\binom{tk+r}{k}\binom{tn-tk+s}{n-k} - \sum_{k=0}^{n}\binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{tk}{tk+r}$$

$$=\binom{tn+r+s}{n}.$$

With integers $t, r, s \ge 1$ and starting with the first sum we introduce

$$\binom{tk+r}{k} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{k+1}}(1+w)^{tk+r}\;dw$$

and

$$\binom{tn-tk+s}{n-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-k+1}}(1+z)^{tn-tk+s}\;dz.$$

This last integral vanishes when $k > n$ so we may extend the sum to infinity, getting

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{tn+s}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^r}{w}\sum_{k\ge 0} z^k(1+z)^{-tk}\frac{1}{w^k}(1+w)^{tk}\;dw\;dz.$$

Now with ϵ and γ small in a neighborhood of the origin we get that for this to converge we must have $\epsilon/(1-\epsilon)^t < \gamma/(1+\gamma)^t$. We shall see that we may solve this with an additional constraint, namely that $\gamma > \epsilon$. Doing the summation we find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{tn+s}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^r}{w}\frac{1}{1-z(1+w)^t/w/(1+z)^t}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{tn+s}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^r\frac{1}{w-z(1+w)^t/(1+z)^t}\;dw\;dz.$$

The pole at $w=0$ has been canceled. There is a pole at $w=z$ however and with the chosen parameters it is inside the contour. We get for the residue

$$(1+w)^r\frac{1}{1-tz(1+w)^{t-1}/(1+z)^t}\bigg|_{w=z} = (1+z)^r\frac{1}{1-tz/(1+z)}$$

The derivative would have vanished if the pole had not been simple. Substituting into the outer integral we get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{tn+r+s+1}}{z^{n+1}}\frac{1}{1-(t-1)z}\;dz.$$

Continuing with the second sum we obtain

$$\sum_{k=1}^{n}\binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{tk}{tk+r} = t\sum_{k=1}^{n}\binom{tk+r-1}{k-1}\binom{tn-tk+s}{n-k}$$

$$= t\sum_{k=0}^{n-1}\binom{tk+t+r-1}{k}\binom{t(n-1)-tk+s}{(n-1)-k}.$$

We can recycle the earlier computation and find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{t(n-1)+t+r-1+s+1}}{z^n}\frac{t}{1-(t-1)z}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{tn+r+s}}{z^{n+1}}\frac{tz}{1-(t-1)z}\;dz.$$

Subtracting the two the result is

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{tn+r+s}}{z^{n+1}}\frac{(1+z)-tz}{1-(t-1)z}\;dz = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{tn+r+s}}{z^{n+1}}\;dz.$$

This evaluates to

$$\binom{tn+r+s}{n}$$

by inspection and we have proved the theorem.

To show that the pole at $w = z$ is the only one inside the contour apply Rouche's theorem to

$$h(w) = w(1+z)^t - z(1+w)^t$$

with $f(w) = w(1+z)^t$ and $g(w) = z(1+w)^t$. We need $|g(w)| < |f(w)|$ on $|w| = \gamma$ and since $f(w)$ has only one root there so does $h(w)$, which must be $w = z$. We thus require

$$|g(w)| \le |z|(1+\gamma)^t < \gamma|1+z|^t = |f(w)|.$$

Now $\gamma/(1+\gamma)^t$ starts at zero and is increasing since $(1+\gamma-\gamma t)/(1+\gamma)^{t+1}$ is positive for $\gamma < 1/(t-1)$ with a local maximum there. Since $|z|/|1+z|^t \le \epsilon/(1-\epsilon)^t$ we may choose ϵ for this to take on a value from the range of $\gamma/(1+\gamma)^t$ on $[0, 1/(t-1)]$. Instantiating γ to the right of this point yields a value $> \epsilon$ that fulfils the requirements of the theorem. Here we have used that $\epsilon/(1+\epsilon)^t < \epsilon/(1-\epsilon)^t < \gamma/(1+\gamma)^t$ by construction. No need for Rouche when $t = 1$.

This was math.stackexchange.com problem 2814898.

33 Reducing the form of a double hypergeometric (B_1)

Suppose we seek to evaluate

$$S(n) = \sum_{q=0}^{n-2}\sum_{k=1}^{n}\binom{k+q}{k}\binom{2n-q-k-1}{n-k+1}.$$

which we re-write as

$$-\sum_{q=0}^{n-2}\binom{2n-q-1}{n+1} - \sum_{q=0}^{n-2}\binom{n+1+q}{n+1} + \sum_{q=0}^{n-2}\sum_{k=0}^{n+1}\binom{k+q}{k}\binom{2n-q-k-1}{n-k+1}.$$

Call these pieces up to sign from left to right S_1, S_2 and S_3.

The two pieces in front cancel the quantities introduced by extending k to include the values zero and $n+1$.

Evaluation of S_1.
Introduce

$$\binom{2n-q-1}{n+1} = \binom{2n-q-1}{n-q-2} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-q-1}}{z^{n-q-1}}\; dz.$$

This vanishes when $q > n-2$ so we may extend the sum to infinity to get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n-1}} \sum_{q\ge 0} \frac{z^q}{(1+z)^q}\; dz \\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n-1}} \frac{1}{1-z/(1+z)}\; dz \\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n-1}}\; dz \\
&= \binom{2n}{n-2}.
\end{aligned}$$

Evaluation of S_2.
Introduce

$$\binom{n+1+q}{n+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1+q}}{z^{n+2}}\; dz.$$

This yields for the sum

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+2}} \sum_{q=0}^{n-2} (1+z)^q\; dz \\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+2}} \frac{(1+z)^{n-1}-1}{1+z-1}\; dz \\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+3}} ((1+z)^{n-1}-1)\; dz \\
&= \binom{2n}{n+2}.
\end{aligned}$$

A more efficient evaluation is to notice that when we re-index q as $n-2-q$ in S_2 we obtain

$$\sum_{q=0}^{n-2} \binom{n+1+n-2-q}{n+1} = \sum_{q=0}^{n-2} \binom{2n-q-1}{n+1}$$

which is S_1.
Evaluation of S_3.

Introduce

$$\binom{2n-q-k-1}{n-k+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-q-k-1}}{z^{n-k+2}}\; dz.$$

This effectively controls the range so we can let k go to infinity to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+2}} \sum_{q=0}^{n-2}\sum_{k\ge 0}\binom{k+q}{q}\frac{z^k}{(1+z)^{q+k}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+2}} \sum_{q=0}^{n-2}\frac{1}{(1+z)^q}\frac{1}{(1-z/(1+z))^{q+1}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+2}} \sum_{q=0}^{n-2}\frac{1}{(1+z)^{q+1}}\frac{1}{(1-z/(1+z))^{q+1}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+2}} \times (n-1) \times\; dz$$

$$= (n-1)\times\binom{2n}{n+1}.$$

Finally collecting the three contributions we obtain

$$(n-1)\times\binom{2n}{n+1} - 2\binom{2n}{n+2} = (n+2)\binom{2n}{n+2} - 2\binom{2n}{n+2}$$

$$= n\times\binom{2n}{n+2}.$$

This is math.stackexchange.com problem 129913.

34 Basic usage of the Iverson bracket (B_1I)

Suppose we seek to evaluate

$$S(k,l) = \sum_{q=0}^{l}\binom{q+k}{k}\binom{l-q}{k}.$$

We start with the Iverson bracket valid for $q \ge 0$

$$[[0 \le q \le l]] = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z^q}{z^{l+1}}\frac{1}{1-z}\; dz$$

This gives for the sum

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^l}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\sum_{q\geq 0}\binom{q+k}{q}\frac{z^q}{(1+w)^q}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^l}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\frac{1}{(1-z/(1+w))^{k+1}}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{l+k+1}}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\frac{1}{(1+w-z)^{k+1}}\;dw\;dz.$$

We evaluate the inner integral by taking the negative of the sum of the residues at $z=1$ and at $z=1+w$ and $z=\infty$. With ϵ and γ small the second pole is not inside the contour.

The negative of the residue at $z=1$ is

$$\frac{1}{w^{k+1}}$$

which when substituted into the outer integral yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{l+k+1}}{w^{2k+2}}\;dw=\binom{l+k+1}{2k+1},$$

which is the formula we are trying to establish.

Next we prove that the residue at infinity is zero. This is given by

$$-\mathrm{Res}_{z=0}\frac{1}{z^2}z^{l+1}\frac{1}{1-1/z}\frac{1}{(1+w-1/z)^{k+1}}=-\mathrm{Res}_{z=0}z^l\frac{1}{z-1}\frac{z^{k+1}}{(z(1+w)-1)^{k+1}}$$

$$=-\frac{1}{(1+w)^{k+1}}\mathrm{Res}_{z=0}\frac{1}{z-1}\frac{z^{l+k+1}}{(z-1/(1+w))^{k+1}}.$$

This is zero by inspection, which leaves the residue at $z=1+w$. Write

$$\frac{(-1)^{k+1}}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{l+k+1}}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\frac{1}{(z-(1+w))^{k+1}}\;dw\;dz.$$

We require the derivative

$$\frac{1}{k!}\left(\frac{1}{z^{l+1}}\frac{1}{1-z}\right)^{(k)}=\frac{1}{k!}\sum_{q=0}^{k}\binom{k}{q}(-1)^q\frac{(l+q)!}{l!\times z^{l+1+q}}\frac{(k-q)!}{(1-z)^{1+k-q}}$$

$$=\sum_{q=0}^{k}\binom{l+q}{q}(-1)^q\frac{1}{z^{l+1+q}}\frac{1}{(1-z)^{1+k-q}}.$$

Evaluate this at $z = 1 + w$ to get

$$\sum_{q=0}^{k} \binom{l+q}{q} (-1)^q \frac{1}{(1+w)^{l+1+q}} \frac{1}{(-w)^{1+k-q}}$$

and substitute into the outer integral to obtain

$$\frac{(-1)^{k+1}}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \sum_{q=0}^{k} \binom{l+q}{q} (-1)^q \frac{1}{(1+w)^{l+1+q}} \frac{1}{(-w)^{1+k-q}} \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \sum_{q=0}^{k} \binom{l+q}{q} \frac{1}{(1+w)^{l+1+q}} \frac{1}{w^{1+k-q}} \; dw$$

$$= \sum_{q=0}^{k} \binom{l+q}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{k-q}}{w^{2k+2-q}} \; dw.$$

The inner term here is

$$[w^{2k+1-q}](1+w)^{k-q}.$$

But we have $2k+1-q \ge k+1$ while $k-q \le k$ so these terms are zero, thus concluding the proof.

Simplified solution. As observed elsewhere this can be done without the Iverson bracket.

Introduce

$$\binom{l-q}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-q-k+1}} \frac{1}{(1-z)^{k+1}} \; dz.$$

This controls the range becoming zero when $q > l-k$ so we may extend q to infinity.

We obtain for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{k+1}} \sum_{q\ge 0} \binom{q+k}{k} z^q \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{k+1}} \frac{1}{(1-z)^{k+1}} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{2k+2}} \; dz.$$

This evaluates by inspection to

$$\binom{l-k+2k+1}{2k+1} = \binom{l+k+1}{2k+1}.$$

This was math.stackexchange.com problem.

35 Basic usage of the Iverson bracket II ($B_1 I$)

Suppose we seek to compute

$$S(n, m) = \sum_{k=0}^{n} k \binom{m+k}{m+1}.$$

Introduce

$$\binom{m+k}{m+1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^{m+k} \; dz$$

as well as the Iverson bracket

$$[[0 \le k \le n]] = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^k}{w^{n+1}} \frac{1}{1-w} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^m \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1-w} \sum_{k \ge 0} k w^k (1+z)^k \; dw \; dz.$$

For this to converge we must have $|w(1+z)| < 1$. We get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^m \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1-w} \frac{w(1+z)}{(1-w(1+z))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^{m+1} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{1-w} \frac{1}{(1-w(1+z))^2} \; dw \; dz.$$

We evaluate the inner integral using the fact that the residues at the poles sum to zero. The residue at $w = 1$ produces

$$-\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^{m+1} \frac{1}{(-z)^2} \; dz = -\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+4}} (1+z)^{m+1} \; dz = 0.$$

For the residue at $w = 1/(1+z)$ we re-write the inner integral to get

$$\frac{1}{(1+z)^2} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{1-w} \frac{1}{(w - 1/(1+z))^2} \; dw.$$

We thus require

$$\left(\frac{1}{w^n}\frac{1}{1-w}\right)'\Bigg|_{w=1/(1+z)}$$

$$= \left(\frac{-n}{w^{n+1}}\frac{1}{1-w} + \frac{1}{w^n}\frac{1}{(1-w)^2}\right)\Bigg|_{w=1/(1+z)}$$

$$= -n(1+z)^{n+1}(1+z)/z + (1+z)^n(1+z)^2/z^2.$$

Substituting this into the outer integral and flipping signs we get two pieces which are

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{m+2}}(1+z)^{m-1}n(1+z)^{n+2}/z\;dz$$

$$= \frac{n}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{m+3}}(1+z)^{n+m+1}\;dz = n\times\binom{n+m+1}{m+2}.$$

The second piece is

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{m+2}}(1+z)^{m-1}(1+z)^{n+2}/z^2\;dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{m+4}}(1+z)^{n+m+1}\;dz = -\binom{n+m+1}{m+3}.$$

It follows that our answer is

$$\left(n - \frac{n-1}{m+3}\right)\binom{n+m+1}{m+2} = \frac{nm+2n+1}{m+3}\binom{n+m+1}{m+2}.$$

Remark. Being rigorous we also verify that the residue at infinity in the calculation of the inner integral is zero. We get

$$-\mathrm{Res}_{w=0}\frac{1}{w^2}w^n\frac{1}{1-1/w}\frac{1}{(1-(1+z)/w)^2}$$

$$= -\mathrm{Res}_{w=0}w^{n-2}\frac{w}{w-1}\frac{w^2}{(w-(1+z))^2} = -\mathrm{Res}_{w=0}\frac{w^{n+1}}{w-1}\frac{1}{(w-(1+z))^2}.$$

There is certainly no pole at zero here and the residue is zero as claimed (the term $1+z$ rotates in a circle around the point one on the real axis and with $\epsilon < 1$ it is never zero). This last result could also be obtained by comparing degrees of numerator and denominator.

This was math.stackexchange.com problem 1836190.

36 Use of a double Iverson bracket (B_1IR)

Suppose we seek to evaluate

$$Y(n) = \sum_{k=1}^{n} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor},$$

by considering

$$Y_1(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{2k}{k} \quad \text{and} \quad Y_2(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^{n-2k-1} \binom{2k+1}{k}.$$

We will use the following Iverson bracket:

$$[[0 \le k \le n]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^k}{z^{n+1}} \frac{1}{1-z} \; dz.$$

Evaluation of $Y_1(n)$. Introduce

$$\binom{2k}{k} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{k+1}} (1+w)^{2k} \; dw.$$

With the Iverson bracket controlling the range we can extend k to infinity to get for the sum

$$\frac{2^n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2 \rfloor + 1}} \frac{1}{1-z} \sum_{k \ge 0} 2^{-2k} z^k \frac{(1+w)^{2k}}{w^k} \; dz \; dw.$$

We can instantiate these contours to get convergence of the series. We thus obtain

$$\frac{2^n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2 \rfloor + 1}} \frac{1}{1-z} \frac{1}{1 - z(1+w)^2/w/4} \; dz \; dw$$

$$= \frac{2^{n+2}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{(1+w)^2} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2 \rfloor + 1}} \frac{1}{z-1} \frac{1}{z - 4w/(1+w)^2} \; dz \; dw.$$

We evaluate the inner piece by computing the negative of the sum of the residues at $z = 1$, $z = 4w/(1+w)^2$ and $z = \infty$. We get for $z = 1$

$$\frac{1}{1 - 4w/(1+w)^2} = \frac{(1+w)^2}{(1+w)^2 - 4w} = \frac{(1+w)^2}{(1-w)^2}$$

for a zero contribution.

We get for $z=\infty$

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{1}{1/z^{\lfloor n/2 \rfloor+1}} \frac{1}{1/z-1} \frac{1}{1/z-4w/(1+w)^2}$$

$$= -\mathrm{Res}_{z=0} z^{\lfloor n/2 \rfloor+1} \frac{1}{1-z} \frac{1}{1-4wz/(1+w)^2}$$

again for a zero contribution.

Finally for $z=4w/(1+w)^2$ we get

$$-\frac{(1+w)^{2\lfloor n/2 \rfloor+2}}{2^{2\lfloor n/2 \rfloor+2} \times w^{\lfloor n/2 \rfloor+1}} \frac{(1+w)^2}{(1-w)^2}.$$

Substitute into the outer integral to obtain

$$-\frac{2^{n \mod 2}}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{2\lfloor n/2 \rfloor+2}}{w^{\lfloor n/2 \rfloor+1}} \frac{1}{(1-w)^2} \; dw.$$

Extracting the negative of the residue we get the sum

$$2^{n \mod 2} \sum_{q=0}^{\lfloor n/2 \rfloor} \binom{2\lfloor n/2 \rfloor+2}{q} (\lfloor n/2 \rfloor - q + 1).$$

This yields

$$2^{n \mod 2} (\lfloor n/2 \rfloor+1) \frac{1}{2} \left(2^{2\lfloor n/2 \rfloor+2} - \binom{2\lfloor n/2 \rfloor+2}{\lfloor n/2 \rfloor+1} \right)$$

$$-2^{n \mod 2} (2\lfloor n/2 \rfloor+2) \sum_{q=1}^{\lfloor n/2 \rfloor} \binom{2\lfloor n/2 \rfloor+1}{q-1}$$

$$= 2^{n \mod 2} (\lfloor n/2 \rfloor+1) \frac{1}{2} \left(2^{2\lfloor n/2 \rfloor+2} - \binom{2\lfloor n/2 \rfloor+2}{\lfloor n/2 \rfloor+1} \right)$$

$$-2^{n \mod 2} (\lfloor n/2 \rfloor+1) \left(2^{2\lfloor n/2 \rfloor+1} - 2\binom{2\lfloor n/2 \rfloor+1}{\lfloor n/2 \rfloor} \right)$$

$$= 2^{n \mod 2} (\lfloor n/2 \rfloor+1) \left(2 - \frac{1}{2} \frac{2\lfloor n/2 \rfloor+2}{\lfloor n/2 \rfloor+1} \right) \binom{2\lfloor n/2 \rfloor+1}{\lfloor n/2 \rfloor}$$

$$= 2^{n \mod 2} (\lfloor n/2 \rfloor+1) \binom{2\lfloor n/2 \rfloor+1}{\lfloor n/2 \rfloor}.$$

Evaluation of $Y_2(n)$. This is obviously very similar to the first case. We get the integral

$$\frac{2^{n+1}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{1+w} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{\lfloor (n-1)/2 \rfloor+1}} \frac{1}{z-1} \frac{1}{z-4w/(1+w)^2} \; dz \; dw.$$

There is no contribution from $z=1$ and $z=\infty$ as before which leaves

$$-\frac{2^{(n+1) \mod 2}}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{2\lfloor (n-1)/2\rfloor+3}}{w^{\lfloor (n-1)/2\rfloor+1}}\frac{1}{(1-w)^2}\,dw.$$

Extracting the negative of the residue we obtain

$$2^{(n+1) \mod 2}\sum_{q=0}^{\lfloor (n-1)/2\rfloor}\binom{2\lfloor (n-1)/2\rfloor+3}{q}(\lfloor (n-1)/2\rfloor-q+1).$$

This yields

$$2^{(n+1) \mod 2}(\lfloor\frac{n-1}{2}\rfloor+1)\times\frac{1}{2}\left(2^{2\lfloor\frac{n-1}{2}\rfloor+3}-2\binom{2\lfloor\frac{n-1}{2}\rfloor+3}{\lfloor\frac{n-1}{2}\rfloor+1}\right)$$

$$-2^{(n+1) \mod 2}(2\lfloor\frac{n-1}{2}\rfloor+3)\sum_{q=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{2\lfloor\frac{n-1}{2}\rfloor+2}{q-1}$$

$$=2^{(n+1) \mod 2}(\lfloor\frac{n-1}{2}\rfloor+1)\times\frac{1}{2}\left(2^{2\lfloor\frac{n-1}{2}\rfloor+3}-2\binom{2\lfloor\frac{n-1}{2}\rfloor+3}{\lfloor\frac{n-1}{2}\rfloor+1}\right)$$

$$-2^{(n+1) \mod 2}(2\lfloor\frac{n-1}{2}\rfloor+3)$$

$$\times\frac{1}{2}\left(2^{2\lfloor\frac{n-1}{2}\rfloor+2}-2\binom{2\lfloor\frac{n-1}{2}\rfloor+2}{\lfloor\frac{n-1}{2}\rfloor}-\binom{2\lfloor\frac{n-1}{2}\rfloor+2}{\lfloor\frac{n-1}{2}\rfloor+1}\right).$$

Evaluation of $Y(n)$. Keeping in mind that $Y(n)$ does not include a term for $k=0$ we get for $n=2p$ the contributions

$$-2^{2p}+(p+1)\binom{2p+1}{p}+p\left(2^{2p+1}-2\binom{2p+1}{p}\right)$$

$$-(2p+1)\left(2^{2p}-2\binom{2p}{p-1}-\binom{2p}{p}\right)$$

$$=-2^{2p+1}+(4p+2)\binom{2p}{p}.$$

On the other hand for $n=2p+1$ we obtain

$$-2^{2p+1}+2(p+1)\binom{2p+1}{p}+\frac{1}{2}(p+1)\left(2^{2p+3}-2\binom{2p+3}{p+1}\right)$$

$$-\frac{1}{2}(2p+3)\left(2^{2p+2}-2\binom{2p+2}{p}-\binom{2p+2}{p+1}\right)$$

$$= -2^{2p+2} + (4p+5)\binom{2p+1}{p}.$$

Joining the two formulae we get the compact closed form

$$-2^{n+1} + (2n+2+(n \mod 2))\binom{n}{\lfloor n/2 \rfloor}.$$

I would conjecture that with the closed form being this simple now that it has been computed we can probably find a much simpler proof.

This was math.stackexchange.com problem 1219731.

37 Iverson bracket and an identity by Gosper, generalized (IR)

Suppose we seek to show that

$$\sum_{q=0}^{m-1} \binom{n-1+q}{q} x^n (1-x)^q + \sum_{q=0}^{n-1} \binom{m-1+q}{q} x^q (1-x)^m = 1$$

where $n, m \ge 1$.

We will evaluate the second term by a contour integral and show that is equal to one minus the first term which is the desired result.

Introduce the Iverson bracket

$$[[0 \le q \le n-1]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^q}{z^n} \frac{1}{1-z} \; dz.$$

With this bracket we may extend the sum in q to infinity to get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \sum_{q\ge 0} \binom{m-1+q}{q} z^q x^q (1-x)^m \; dz$$

$$= \frac{(1-x)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \sum_{q\ge 0} \binom{m-1+q}{q} z^q x^q \; dz$$

$$= \frac{(1-x)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \frac{1}{(1-xz)^m} \; dz.$$

Now we have three poles here at $z = 0$ and $z = 1$ and $z = 1/x$ and the residues at these poles sum to zero, so we can evaluate the residue at zero by computing the negative of the residues at $z = 1$ and $z = 1/x$.

Observe that the residue at infinity is zero as can be seen from the following computation:

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^n \frac{1}{1-1/z} \frac{1}{(1-x/z)^m}$$

$$-\mathrm{Res}_{z=0}\frac{1}{z^2}z^n\frac{z}{z-1}\frac{z^m}{(z-x)^m}$$

$$-\mathrm{Res}_{z=0}z^{n+m-1}\frac{1}{z-1}\frac{1}{(z-x)^m}=0.$$

Returning to the main thread the residue at $z=1$ as seen from

$$-\frac{(1-x)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{z-1}\frac{1}{(1-xz)^m}\;dz.$$

is

$$-(1-x)^m\frac{1}{(1-x)^m}=-1.$$

For the residue at $z=1/x$ we consider

$$\frac{(1-x)^m}{x^m\times 2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{1-z}\frac{1}{(1/x-z)^m}\;dz$$

$$=\frac{(-1)^m(1-x)^m}{x^m\times 2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{1-z}\frac{1}{(z-1/x)^m}\;dz.$$

and use the following derivative:

$$\frac{1}{(m-1)!}\left(\frac{1}{z^n}\frac{1}{1-z}\right)^{(m-1)}$$

$$=\frac{1}{(m-1)!}\sum_{q=0}^{m-1}\binom{m-1}{q}\frac{(-1)^q(n+q-1)!}{(n-1)!z^{n+q}}\frac{(m-1-q)!}{(1-z)^{m-q}}$$

$$=\sum_{q=0}^{m-1}\frac{1}{q!}\frac{(-1)^q(n+q-1)!}{(n-1)!z^{n+q}}\frac{1}{(1-z)^{m-q}}$$

$$=\sum_{q=0}^{m-1}\binom{n+q-1}{q}\frac{(-1)^q}{z^{n+q}}\frac{1}{(1-z)^{m-q}}.$$

Evaluate this at $z=1/x$ and multiply by the factor in front to get

$$\frac{(-1)^m(1-x)^m}{x^m}\times\sum_{q=0}^{m-1}\binom{n+q-1}{q}(-1)^q x^{n+q}\frac{1}{(1-1/x)^{m-q}}$$

$$=\frac{(-1)^m(1-x)^m}{x^m}\times\sum_{q=0}^{m-1}\binom{n+q-1}{q}(-1)^q x^{n+q}\frac{x^{m-q}}{(x-1)^{m-q}}$$

$$=(-1)^m(1-x)^m\times\sum_{q=0}^{m-1}\binom{n+q-1}{q}(-1)^q x^n(-1)^{m-q}\frac{1}{(1-x)^{m-q}}$$

$$= \sum_{q=0}^{m-1} {n+q-1 \choose q} x^n (1-x)^q.$$

This yields for the second sum term the value

$$1 - \sum_{q=0}^{m-1} {n+q-1 \choose q} x^n (1-x)^q$$

showing that when we add the first and the second sum by cancellation the end result is one, as claimed.

This was math.stackexchange.com problem 538309.

Special case by formal power series

Here we show the special case:

$$\sum_{k=0}^{n} {m+k \choose k} 2^{n-k} + \sum_{k=0}^{m} {n+k \choose k} 2^{m-k} = 2^{n+m+1}.$$

which is obtained from $x = 1/2$. We have by inspection i.e. same as before that

$$\sum_{k=0}^{n} {m+k \choose k} 2^{n-k} = 2^n [z^n] \frac{1}{1-z} \frac{1}{(1-z/2)^{m+1}}.$$

This is

$$2^n \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(1-z/2)^{m+1}}$$

$$= -2^n \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{2^{m+1}}{(2-z)^{m+1}}$$

$$= 2^{n+m+1} (-1)^m \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{1}{(z-2)^{m+1}}.$$

With

$$f(z) = 2^{n+m+1} (-1)^m \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{1}{(z-2)^{m+1}}$$

we will be using the fact that residues sum to zero i.e.

$$\mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=1} f(z) + \mathrm{Res}_{z=2} f(z) + \mathrm{Res}_{z=\infty} f(z) = 0.$$

The residue at infinity is zero since $\lim_{R\to\infty} 2\pi R/R^{n+1}/R/R^{m+1} = 0$.

The residue at one is

$$2^{n+m+1} (-1)^m \times (-1)^{m+1} = -2^{n+m+1}.$$

For the residue at two we use the Leibniz rule:

$$\frac{1}{m!}\left(\frac{1}{z^{n+1}}\frac{1}{z-1}\right)^{(m)}$$
$$= \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{(n+k)!}{n!}\frac{1}{z^{n+1+k}}(-1)^{m-k}\frac{(m-k)!}{(z-1)^{m-k+1}}$$
$$= (-1)^m\sum_{k=0}^{m}\binom{n+k}{k}\frac{1}{z^{n+1+k}}\frac{1}{(z-1)^{m-k+1}}.$$

Restore factor in front and evaluate at $z=2$:

$$2^{n+m+1}(-1)^m \times (-1)^m\sum_{k=0}^{m}\binom{n+k}{k}\frac{1}{2^{n+1+k}} = \sum_{k=0}^{m}\binom{n+k}{k}2^{m-k}.$$

Summing the residues we have shown that

$$\sum_{k=0}^{n}\binom{m+k}{k}2^{n-k} + \sum_{k=0}^{m}\binom{n+k}{k}2^{m-k} - 2^{n+m+1} = 0$$

which is the claim.
This was math.stackexchange.com problem 3024722.

38 Factoring a triple hypergeometric sum (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{n}(-1)^k\binom{p+q+1}{k}\binom{p+n-k}{n-k}\binom{q+n-k}{n-k}$$

which is claimed to be

$$\binom{p}{n}\binom{q}{n}.$$

Introduce

$$\binom{p+n-k}{n-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{p+n-k}}{z^{n-k+1}}\;dz$$

and

$$\binom{q+n-k}{n-k} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{q+n-k}}{w^{n-k+1}}\;dw.$$

Observe that these integrals vanish when $k > n$ and we may extend k to infinity.

We thus obtain for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{p+n}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{q+n}}{w^{n+1}}$$

$$\times\sum_{k\geq 0}\binom{p+q+1}{k}(-1)^k\frac{z^k w^k}{(1+z)^k(1+w)^k}\;dw\;dz.$$

Note that while there is no restriction on k the sum only contains a finite number of terms. Continuing,

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{p+n}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{q+n}}{w^{n+1}}$$

$$\times\left(1-\frac{zw}{(1+z)(1+w)}\right)^{p+q+1}\;dw\;dz$$

or

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-q-1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{n-p-1}}{w^{n+1}}(1+z+w)^{p+q+1}\;dw\;dz$$

Supposing that $p\geq n$ and $q\geq n$ and $\epsilon\ll 1$ and $\gamma\ll 1$ this may be re-written as

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}(1+z)^{q+1-n}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}(1+w)^{p+1-n}}$$

$$\times(1+z+w)^{p+q+1}\;dw\;dz$$

Put $w=(1+z)u$ so that $dw=(1+z)\;du$ to get with $\delta<\gamma/(1+\epsilon)$

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}(1+z)^{q+1-n}}\frac{1}{2\pi i}\int_{|u|=\delta}\frac{1}{(1+z)^{n+1}u^{n+1}(1+(1+z)u)^{p+1-n}}$$

$$\times(1+z)^{p+q+1}(1+u)^{p+q+1}\;(1+z)\;du\;dz$$

Note that the pole at $u=-1/(1+z)$ has norm $\delta/\gamma>\delta$ so it is not inside the contour in u. This yields

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^p}{z^{n+1}}\frac{1}{2\pi i}\int_{|u|=\delta}\frac{1}{u^{n+1}(1+u+zu)^{p+1-n}}$$

$$\times(1+u)^{p+q+1}\;du\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^p}{z^{n+1}}\frac{1}{2\pi i}\int_{|u|=\delta}\frac{(1+u)^{n+q}}{u^{n+1}(1+zu/(1+u))^{p+1-n}}\;du\;dz$$

Extracting the residue for z first we obtain

$$\sum_{k=0}^{n} \binom{p}{n-k} \frac{(1+u)^{n+q}}{u^{n+1}} \binom{k+p-n}{k} (-1)^k \frac{u^k}{(1+u)^k}.$$

The residue for u then yields

$$\sum_{k=0}^{n} (-1)^k \binom{p}{n-k} \binom{k+p-n}{k} \binom{n-k+q}{n-k}.$$

The sum term here is

$$\frac{p! \times (p+k-n)! \times (q+n-k)!}{(n-k)!(p+k-n)! \times k!(p-n)! \times (n-k)!q!}$$

which simplifies to

$$\frac{p! \times n! \times (q+n-k)!}{(n-k)! \times n! \times k!(p-n)! \times (n-k)!q!}$$

which is

$$\binom{n}{k} \binom{p}{n} \binom{q+n-k}{q}$$

so we have for the sum

$$\binom{p}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{q+n-k}{q}.$$

To evaluae the remaining sum we introduce

$$\binom{q+n-k}{q} = \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{q+n-k}}{v^{q+1}} \; dv$$

getting for the sum

$$\binom{p}{n} \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{q+n}}{v^{q+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(1+v)^k} \; dv$$

$$= \binom{p}{n} \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{q+n}}{v^{q+1}} \left(1 - \frac{1}{1+v}\right)^n \; dv$$

$$= \binom{p}{n} \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{q}}{v^{q-n+1}} \; dv = \binom{p}{n} \binom{q}{q-n}$$

which is

$$\binom{p}{n} \binom{q}{n}.$$

This concludes the argument.

Alternate proof

We have from first principles that the sum is (we will prove for $p, q \geq n$):

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{(1-z)^{p+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{(1-w)^{q+1}}$$
$$\times(1-wz)^{p+q+1}\;dw\;dz.$$

Now put $wz = v$ so that $z\;dw = dv$ to obtain

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+2}}\frac{1}{(1-z)^{p+1}}\frac{1}{2\pi i}\int_{|v|=\epsilon\gamma}\frac{z^{n+1}}{v^{n+1}}\frac{1}{(1-v/z)^{q+1}}$$
$$\times(1-v)^{p+q+1}\;dv\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{z^q}{(1-z)^{p+1}}\frac{1}{2\pi i}\int_{|v|=\epsilon\gamma}\frac{1}{v^{n+1}}\frac{1}{(z-v)^{q+1}}$$
$$\times(1-v)^{p+q+1}\;dv\;dz.$$

Now with $\epsilon \ll 1$ and $\gamma \ll 1$ the pole at $v = z$ is outside the contour in v, hence we may evaluate with minus the residue at that pole, minus the residue at infinity. We get for the inner term

$$-(-1)^{q+1}\frac{1}{(v-z)^{q+1}}\frac{1}{(z+v-z)^{n+1}}(1-z-(v-z))^{p+q+1}$$
$$=-\frac{(-1)^{q+1}}{z^{n+1}}\frac{1}{(v-z)^{q+1}}\frac{1}{(1+(v-z)/z)^{n+1}}(1-z-(v-z))^{p+q+1}$$

Computing the residue,

$$-\frac{(-1)^{q+1}}{z^{n+1}}\sum_{k=0}^{q}\binom{p+q+1}{k}(-1)^k(1-z)^{p+q+1-k}(-1)^{q-k}\binom{n+q-k}{q-k}\frac{1}{z^{q-k}}.$$

Substituting into the integral in z now yields

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-k+1}}\sum_{k=0}^{q}\binom{p+q+1}{k}(1-z)^{q-k}\binom{n+q-k}{q-k}\;dz$$
$$=(-1)^n\sum_{k=0}^{q}\binom{p+q+1}{k}(-1)^k\binom{q-k}{n-k}\binom{n+q-k}{q-k}.$$

Observe that

$$\binom{q-k}{n-k}\binom{n+q-k}{q-k} = \frac{(n+q-k)!}{(n-k)! \times (q-n)! \times n!} = \binom{q}{n}\binom{n+q-k}{q}.$$

so that we obtain

$$(-1)^n \binom{q}{n} \sum_{k=0}^{q} \binom{p+q+1}{k} (-1)^k \binom{n+q-k}{q}.$$

We must show that the remaining sum is $\binom{p}{n}$:

$$(-1)^{q-n} \sum_{k=0}^{q} \binom{p+q+1}{q-k} (-1)^k \binom{n+k}{q}$$
$$= (-1)^{q-n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^{p+q+1} \sum_{k \ge 0} z^k (-1)^k \binom{n+k}{q} \; dz.$$

Here we have extended to infinity due to the coefficient extractor in z. The sum is

$$\sum_{k \ge q-n} z^k (-1)^k \binom{n+k}{q} = (-1)^{q-n} z^{q-n} \sum_{k \ge 0} z^k (-1)^k \binom{k+q}{q}$$
$$= (-1)^{q-n} z^{q-n} \frac{1}{(1+z)^{q+1}}.$$

Substitute into the integral to obtain

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^p \; dz = \binom{p}{n}.$$

We have the claim. Now we just need to show that the residue at infinity in v makes a zero contribution. We obtain

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^q}{(1-z)^{p+1}} \frac{1}{2\pi i} \int_{|v|=\delta} \frac{1}{v^2} v^{n+1} \frac{1}{(z-1/v)^{q+1}}$$
$$\times (1-1/v)^{p+q+1} \; dv \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^q}{(1-z)^{p+1}} \frac{1}{2\pi i} \int_{|v|=\delta} \frac{1}{v^{p-n+1}} \frac{1}{(vz-1)^{q+1}}$$
$$\times (v-1)^{p+q+1} \; dv \; dz.$$

Now expanding $1/(vz-1)^{q+1}$ into a convergent power series about zero which converges in the chosen contour we have terms $\binom{k+q}{k} v^k z^k$, all of which make a zero contribution through the integral in z, there not being a pole at zero and the singularity at one not being inside the contour. Alternatively, switch the integrals and note that the poles of the integrand in z are at $z = 1$ and $z = 1/v$, outside the contour with δ being arbitrarily small due to its origin with the residue at infinity.

This is math.stackexchange.com problem 174054.

39 Factoring a triple hypergeometric sum II (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{pn-n}{k} \binom{pn+k}{k} = \binom{pn}{n}^2.$$

We use the integrals

$$\binom{pn-n}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z^{k+1}} \; dz$$

and

$$\binom{pn+k}{k} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pn+k}}{w^{k+1}} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pn}}{w} \sum_{k=0}^{n} \binom{n}{k} \frac{(1+w)^k}{z^k w^k} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pn}}{w} \left(1 + \frac{1+w}{zw}\right)^n \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pn}}{w^{n+1}} (1+w+zw)^n \; dw \; dz.$$

Expanding the binomial in the inner sum we get

$$\sum_{q=0}^{n} \binom{n}{q} w^q (1+z)^q$$

which yields

$$\sum_{q=0}^{n} \binom{n}{q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{pn-n+q}}{z^{n+1}} \binom{pn}{n-q} \; dz$$

$$= \sum_{q=0}^{n} \binom{n}{q} \binom{pn-n+q}{n} \binom{pn}{n-q}.$$

The inner term is

$$\binom{n}{q} \binom{pn-n+q}{n} \binom{pn}{pn-n+q}$$

$$= \frac{(pn)!}{q! \times (n-q)! \times (pn-2n+q)! \times (n-q)!}$$
$$= \binom{pn}{n} \frac{n! \times (pn-n)!}{q! \times (n-q)! \times (pn-2n+q)! \times (n-q)!}$$
$$= \binom{pn}{n} \binom{n}{q} \binom{pn-n}{n-q}.$$

Thus it remains to show that

$$\sum_{q=0}^{n} \binom{n}{q} \binom{pn-n}{n-q} = \binom{pn}{n}.$$

This can be done combinatorially or using the integral

$$\frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{pn-n}}{v^{n+1}} \sum_{q=0}^{n} \binom{n}{q} v^q \; dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{pn-n}}{v^{n+1}} (v+1)^n \; dv$$
$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{pn}}{v^{n+1}} = \binom{pn}{n}.$$

This was math.stackexchange.com problem 656116.

40 Factoring a triple hypergeometric sum III (B_1)

Suppose we seek to verify that

$$\sum_{r=0}^{\min\{m,n,p\}} \binom{m}{r} \binom{n}{r} \binom{p+m+n-r}{m+n} = \binom{p+m}{m} \binom{p+n}{n}.$$

Introduce

$$\binom{n}{r} = \binom{n}{n-r} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-r+1}} (1+z)^n \; dz$$

and

$$\binom{p+m+n-r}{m+n} = \binom{p+m+n-r}{p-r}$$
$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{p-r+1}} (1+w)^{p+m+n-r} \; dw.$$

Observe carefully that the first of these is zero when $r > n$ and the second one when $r > p$ so we may extend the range of r to infinity.

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+m+n}}{w^{p+1}}\sum_{r\ge 0}\binom{m}{r}z^r\frac{w^r}{(1+w)^r}\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+m+n}}{w^{p+1}}\left(1+\frac{zw}{1+w}\right)^m\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+n}}{w^{p+1}}(1+w+zw)^m\;dw\;dz.$$

The inner integral is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+n}}{w^{p+1}}\sum_{q=0}^{m}\binom{m}{q}(1+z)^q w^q\;dw$$

with residue

$$\sum_{q=0}^{\min(m,p)}\binom{m}{q}\binom{p+n}{p-q}(1+z)^q$$

which in combination with the outer integral yields

$$\sum_{q=0}^{\min(m,p)}\binom{m}{q}\binom{p+n}{n+q}\binom{n+q}{n}.$$

Now note that

$$\binom{p+n}{n+q}\binom{n+q}{n} = \frac{(p+n)!}{(p-q)!(n+q)!}\frac{(n+q)!}{q!n!}$$

$$= \frac{(p+n)!}{(p-q)!p!}\frac{p!}{q!n!} = \binom{p+n}{n}\binom{p}{q}.$$

Therefore we just need to verify that

$$\sum_{q=0}^{\min(m,p)}\binom{m}{q}\binom{p}{p-q} = \binom{p+m}{m}$$

which follows by inspection.

It can also be done with the integral

$$\binom{p}{p-q} = \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^p}{w^{p-q+1}}\;dw$$

which is zero when $q > p$ so we can extend q to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^p}{w^{p+1}}\sum_{q\ge 0}\binom{m}{q}w^q\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+m}}{w^{p+1}}\; dw$$

$$= \binom{p+m}{m}.$$

This was math.stackexchange.com problem 1460712.

41 A triple hypergeometric sum IV (B_1)

Suppose we seek to verify that

$$\sum_{p=0}^{l}\sum_{q=0}^{p}(-1)^q\binom{m-p}{m-l}\binom{n}{q}\binom{m-n}{p-q} = 2^l\binom{m-n}{l}$$

where $m \ge n$ and $m-n \ge l$.

This is

$$\sum_{p=0}^{l}\binom{m-p}{m-l}\sum_{q=0}^{p}(-1)^q\binom{n}{q}\binom{m-n}{p-q}.$$

Now introduce the integral

$$\binom{m-n}{p-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p-q+1}}(1+z)^{m-n}\; dz.$$

Note that this vanishes when $q > p$ so we may extend the range of q to infinity, getting for the sum

$$\sum_{p=0}^{l}\binom{m-p}{m-l}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p+1}}(1+z)^{m-n}\sum_{q\ge 0}(-1)^q\binom{n}{q}z^q\; dz$$

$$= \sum_{p=0}^{l}\binom{m-p}{l-p}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p+1}}(1+z)^{m-n}(1-z)^n\; dz.$$

Introduce furthermore

$$\binom{m-p}{l-p} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l-p+1}}(1+w)^{m-p}\; dw.$$

This too vanishes when $p > l$ so we may extend p to infinity, getting

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l+1}}(1+w)^m$$
$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}(1+z)^{m-n}(1-z)^n\sum_{p\ge 0}\frac{w^p}{z^p}\frac{1}{(1+w)^p}\;dz\;dw.$$

The geometric series converges when $|w/z/(1+w)|<1$. We get

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l+1}}(1+w)^m$$
$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}(1+z)^{m-n}(1-z)^n\frac{1}{1-w/z/(1+w)}\;dz\;dw$$
$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l+1}}(1+w)^m$$
$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}(1+z)^{m-n}(1-z)^n\frac{1}{z-w/(1+w)}\;dz\;dw.$$

Now from the convergence we have $|w/(1+w)|<|z|$ which means the pole at $z=w/(1+w)$ is inside the contour $|z|=\epsilon$. Extracting the residue yields (the pole at zero has disappeared)

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l+1}}(1+w)^m\left(1+\frac{w}{1+w}\right)^{m-n}\left(1-\frac{w}{1+w}\right)^n\;dw$$
$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l+1}}(1+2w)^{m-n}\;dw$$
$$=2^l\binom{m-n}{l}.$$

This was math.stackexchange.com problem 1767709.

42 Basic usage of exponentiation integral to obtain Stirling number formulae (E)

Suppose we seek to evaluate

$$\sum_{q=0}^{n}(n-2q)^k\binom{n}{2q+1}.$$

We observe that

$$(n-2q)^k=\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{k+1}}\exp((n-2q)z)\;dz.$$

This yields for the sum

$$\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{k+1}}\sum_{q=0}^{n}\binom{n}{2q+1}\exp((n-2q)z)\;dz$$

$$=\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp((n+1)z)}{z^{k+1}}\sum_{q=0}^{n}\binom{n}{2q+1}\exp((-2q-1)z)\;dz$$

which is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp((n+1)z)}{z^{k+1}}$$

$$\times\left(\sum_{q=0}^{n}\binom{n}{q}\exp(-qz)-\sum_{q=0}^{n}\binom{n}{q}(-1)^q\exp(-qz)\right)\;dz.$$

This yields two pieces, call them A_1 and A_2. Piece A_1 is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp((n+1)z)}{z^{k+1}}(1+\exp(-z))^n\;dz$$

$$=\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}(\exp(z)+1)^n\;dz$$

and piece A_2 is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp((n+1)z)}{z^{k+1}}(1-\exp(-z))^n\;dz$$

$$=\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}(\exp(z)-1)^n\;dz.$$

Recall the species equation for labelled set partitions:

$$\mathfrak{P}(\mathcal{U}\mathfrak{P}_{\geq 1}(\mathcal{Z}))$$

which yields the bivariate generating function of the Stirling numbers of the second kind

$$\exp(u(\exp(z)-1)).$$

This implies that

$$\sum_{n\geq q}\left\{ n \atop q \right\}\frac{z^n}{n!}=\frac{(\exp(z)-1)^q}{q!}$$

and

$$\sum_{n\geq q}\left\{ n \atop q \right\}\frac{z^{n-1}}{(n-1)!}=\frac{(\exp(z)-1)^{q-1}}{(q-1)!}\exp(z).$$

Now to evaluate A_1 proceed as follows:

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}(2+\exp(z)-1)^n\;dz$$

$$=\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}\sum_{q=0}^{n}{n\choose q}2^{n-q}(\exp(z)-1)^q\;dz$$

$$=\sum_{q=0}^{n}{n\choose q}2^{n-q}\times q!\times\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}\frac{(\exp(z)-1)^q}{q!}\;dz.$$

Recognizing the differentiated Stirling number generating function this becomes

$$\sum_{q=0}^{n}{n\choose q}2^{n-q-1}\times q!\times\left\{{k+1\atop q+1}\right\}.$$

Now observe that when $n>k+1$ the Stirling number for $k+1<q\le n$ is zero, so we may replace n by $k+1$. Similarly, when $n<k+1$ the binomial coefficient for $n<q\le k+1$ is zero so we may again replace n by $k+1$. This gives the following result for A_1 :

$$\sum_{q=0}^{k+1}{n\choose q}2^{n-q-1}\times q!\times\left\{{k+1\atop q+1}\right\}.$$

Moving on to A_2 we observe that when $k<n$ the contribution is zero because the series for $\exp(z)-1$ starts at z. This integral is simple and we have

$$\frac{1}{2}\frac{k!\times n!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}\frac{(\exp(z)-1)^n}{n!}\;dz.$$

Recognizing the Stirling number this yields

$$\frac{1}{2}\times n!\times\left\{{k+1\atop n+1}\right\}.$$

which correctly represents the fact that we have a zero contribution when $k<n$.

This finally yields the closed form formula

$$\sum_{q=0}^{k+1}{n\choose q}2^{n-q-1}\times q!\times\left\{{k+1\atop q+1}\right\}-\frac{1}{2}\times n!\times\left\{{k+1\atop n+1}\right\}.$$

confirming the previous results.

This was math.stackexchange.com problem 1353963

43 Three phase application including Leibniz' rule (B_1B_2R)

Suppose we seek to verify that

$$\sum_{q=0}^{n} q \binom{2n}{n+q} \binom{m+q-1}{2m-1} = m \times 4^{n-m} \times \binom{n}{m}$$

where $n \geq m$.

We use the integrals

$$\binom{2n}{n+q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-q+1}} \frac{1}{(1-z)^{n+q+1}} \; dz.$$

and

$$\binom{m+q-1}{2m-1} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m+q-1}}{w^{2m}} \; dw.$$

Observe that the first integral is zero when $q > n$ so we may extend q to infinity.

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \sum_{q \geq 0} q \frac{z^q (1+w)^q}{(1-z)^q} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \frac{z(1+w)/(1-z)}{(1-z(1+w)/(1-z))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \frac{z(1+w)(1-z)}{(1-z-z(1+w))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n}} \frac{1}{(1-z)^{n}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m}}{w^{2m}} \frac{1}{(1-2z-zw)^2} \; dw \; dz.$$

We evaluate the inner integral using the negative of the residue at the pole at $w = (1-2z)/z$, starting from

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+2}} \frac{1}{(1-z)^{n}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m}}{w^{2m}} \frac{1}{(w-(1-2z)/z)^2} \; dw \; dz.$$

Differentiating we have

$$m\frac{(1+w)^{m-1}}{w^{2m}} - 2m\frac{(1+w)^m}{w^{2m+1}} = (w - 2(1+w))\, m\frac{(1+w)^{m-1}}{w^{2m+1}}$$

$$= (-w-2)m\frac{(1+w)^{m-1}}{w^{2m+1}}.$$

The negative of this evaluated at $w = (1-2z)/z$ is

$$\frac{1}{z} \times m \times \frac{(1-z)^{m-1}}{z^{m-1}} \times \frac{z^{2m+1}}{(1-2z)^{2m+1}}$$

which finally yields

$$\frac{m}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \frac{1}{(1-2z)^{2m+1}}\; dz.$$

We have that the residues at zero, one and one half sum to zero with the first one being the sum we are trying to compute. Therefore we evaluate these in turn. We will restore the front factor of m at the end.

For the residue at zero we have using the Cauchy product that

$$\sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{n-m-q} \binom{2m+n-m-q}{n-m-q}$$

$$= \sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{n-m-q} \binom{m+n-q}{2m}.$$

For the residue at one we have that

$$\frac{(-1)^{n-m+1}}{(n-m)!} \left(\frac{1}{z^{n-m+1}} \frac{1}{(1-2z)^{2m+1}}\right)^{(n-m)}$$

$$= \frac{(-1)^{n-m+1}}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times 2^{n-m-q} \frac{(2m+n-m-q)!}{(2m)! \times (1-2z)^{2m+1+n-m-q}}$$

$$= \frac{(-1)^{n-m+1} 2^{n-m}}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times 2^{-q} \frac{(m+n-q)!}{(2m)! \times (1-2z)^{m+1+n-q}}.$$

Evaluate this at one to get

$$2^{n-m} \sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{-q} \binom{m+n-q}{2m}.$$

The residue at one evaluates to the sum we seek just like the residue at zero. This leaves the residue at one half, where we find

$$\frac{(-1)^{2m+1}}{(2m)! \times 2^{2m+1}} \left(\frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \right)^{(2m)}$$

$$= \frac{(-1)^{2m+1}}{(2m)! \times 2^{2m+1}} \sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times \frac{(n-m+2m-q)!}{(n-m)! \times (1-z)^{n-m+1+2m-q}}$$

$$= \frac{(-1)^{2m+1}}{(2m)! \times 2^{2m+1}} \sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times \frac{(n+m-q)!}{(n-m)! \times (1-z)^{n+m+1-q}}.$$

Evaluate this at one half to get

$$-\frac{1}{2^{2m+1}} \sum_{q=0}^{2m} \binom{n-m+q}{q} (-1)^q 2^{n-m+1+q} \binom{n+m-q}{2m-q} 2^{n+m+1-q}$$

$$= -2^{2n-2m+1} \sum_{q=0}^{2m} \binom{n-m+q}{q} (-1)^q \binom{n+m-q}{2m-q}.$$

For this last sum use the integral

$$\binom{n+m-q}{2m-q} = \binom{n+m-q}{n-m} = \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m-q+1}} \frac{1}{(1-v)^{n-m+1}} \; dv.$$

This controls the range so we can let q go to infinity in the sum to get

$$\frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m+1}} \frac{1}{(1-v)^{n-m+1}} \sum_{q\ge 0} \binom{n-m+q}{q} (-1)^q v^q \; dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m+1}} \frac{1}{(1-v)^{n-m+1}} \frac{1}{(1+v)^{n-m+1}} \; dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m+1}} \frac{1}{(1-v^2)^{n-m+1}} \; dv = \binom{n-m+m}{m} = \binom{n}{m}.$$

We have shown that

$$2S - m \times 2 \times 2^{2n-2m} \times \binom{n}{m} = 0$$

and hence may conclude that

$$S = m \times 4^{n-m} \times \binom{n}{m}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity of the integral in w is zero. Recall the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z}\right) \right]$$

In the present case this becomes

$$\begin{aligned} &-\mathrm{Res}_{w=0} \frac{1}{w^2} \frac{(1+1/w)^m}{1/w^{2m}} \frac{1}{(1-2z-z/w)^2} \\ &= -\mathrm{Res}_{w=0} \frac{(1+1/w)^m}{1/w^{2m}} \frac{1}{(w(1-2z)-z)^2} \\ &= -\mathrm{Res}_{w=0} (1+w)^m w^m \frac{1}{(w(1-2z)-z)^2} \end{aligned}$$

which is zero by inspection.

The same procedure applied to the main integral yields

$$\begin{aligned} &-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n-m+1} \frac{1}{(1-1/z)^{n-m+1}} \frac{1}{(1-2/z)^{2m+1}} \\ &= -\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n-m+1} \frac{z^{n-m+1}}{(z-1)^{n-m+1}} \frac{z^{2m+1}}{(z-2)^{2m+1}} \\ &= -\mathrm{Res}_{z=0} z^{2n+1} \frac{1}{(z-1)^{n-m+1}} \frac{1}{(z-2)^{2m+1}} \end{aligned}$$

which is zero as well.

This was math.stackexchange.com problem 1247818.

44 Same problem, streamlined proof (B_1B_2R)

Suppose we seek to verify that

$$S = \sum_{q=0}^{n} q \binom{2n}{n+q} \binom{m+q-1}{2m-1} = m \times 4^{n-m} \times \binom{n}{m}$$

where $n \ge m$.

This is

$$\sum_{q=0}^{n} (n-q) \binom{2n}{q} \binom{m+n-q-1}{2m-1}$$

which has two pieces. We use the integral

$$\binom{m+n-q-1}{2m-1} = \binom{m+n-q-1}{n-m-q} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m-q+1}} (1+w)^{m+n-q-1} \; dw.$$

Observe that this integral vanishes when $q > n-m$ and we may extend q to $2n$. We get for the first piece

$$\frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} (1+w)^{m+n-1} \sum_{q=0}^{2n} \binom{2n}{q} \frac{w^q}{(1+w)^q} \; dw$$

$$= \frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n} \; dw.$$

The second piece is the negative of

$$\sum_{q=0}^{n} q \binom{2n}{q} \binom{m+n-q-1}{2m-1} = \sum_{q=1}^{n} q \binom{2n}{q} \binom{m+n-q-1}{2m-1}$$

$$= 2n \sum_{q=1}^{n} \binom{2n-1}{q-1} \binom{m+n-q-1}{2m-1} = 2n \sum_{q=0}^{n-1} \binom{2n-1}{q} \binom{m+n-q-2}{2m-1}$$

$$= 2n \sum_{q=0}^{n-1} \binom{2n-1}{q} \binom{m+n-q-2}{n-m-q-1}.$$

This vanishes through its integral representation when $q > n-m-1$ and we obtain

$$\frac{2n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \; dw.$$

Joining the two pieces we arrive at the single integral

$$\frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \; dw.$$

We know the residues at zero, minus one and infinity sum to zero, where the first represents the queried sum. For the residue at minus one it is given by

$$\frac{n}{2\pi i} \int_{|w+1|=\gamma} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \; dw$$

$$= \frac{n}{2\pi i} \int_{|v|=\gamma} \frac{1}{(v-1)^{n-m+1}} \frac{1}{v^{n+1-m}} (2v-1)^{2n-1} \; dv$$

$$= -\frac{n}{2\pi i} \int_{|v|=\gamma} \frac{1}{(-v-1)^{n-m+1}} \frac{1}{(-v)^{n+1-m}} (-1-2v)^{2n-1} \; dv$$

$$= \frac{n}{2\pi i} \int_{|v|=\gamma} \frac{1}{(1+v)^{n-m+1}} \frac{1}{v^{n+1-m}} (1+2v)^{2n-1} \; dv.$$

We see that this residue also represents the queried sum. This leaves the residue at infinity which is

$$\mathrm{Res}_{w=\infty} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1}$$

$$= -\mathrm{Res}_{w=0} \frac{1}{w^2} w^{n-m+1} \frac{1}{(1+1/w)^{n+1-m}} (1+2/w)^{2n-1}$$

$$= -\mathrm{Res}_{w=0} w^{n-m-1} \frac{w^{n+1-m}}{(1+w)^{n+1-m}} \frac{(2+w)^{2n-1}}{w^{2n-1}}$$

$$= -\mathrm{Res}_{w=0} \frac{1}{w^{2m-1}} \frac{(2+w)^{2n-1}}{(1+w)^{n+1-m}}.$$

Extracting coefficients we find

$$-n \sum_{q=0}^{2m-2} \binom{2n-1}{2m-2-q} 2^{2n-2m+1+q} (-1)^q \binom{n-m+q}{q}.$$

Introduce (this vanishes when $q > 2m-2$)

$$\binom{2n-1}{2m-2-q} = \binom{2n-1}{2n+1-2m+q}$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-1-q}} \frac{1}{(1-z)^{2n-2m+2+q}} \; dz$$

to get for the sum

$$-\frac{n2^{2n-2m+1}}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-1}} \frac{1}{(1-z)^{2n-2m+2}} \sum_{q\ge 0} \binom{n-m+q}{q} 2^q (-1)^q \frac{z^q}{(1-z)^q} \; dz$$

$$= -\frac{n2^{2n-2m+1}}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-1}} \frac{1}{(1-z)^{2n-2m+2}} \frac{1}{(1+2z/(1-z))^{n-m+1}} \; dz$$

$$= -\frac{n2^{2n-2m+1}}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-1}} \frac{1}{(1-z)^{n-m+1}} \frac{1}{(1+z)^{n-m+1}} \; dz$$

$$= -\frac{n2^{2n-2m+1}}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-1}} \frac{1}{(1-z^2)^{n-m+1}} \; dz$$

$$= -n2^{2n-2m+1}[z^{2m-2}]\frac{1}{(1-z^2)^{n-m+1}} = -n2^{2n-2m+1}[z^{m-1}]\frac{1}{(1-z)^{n-m+1}}$$

$$= -n2^{2n-2m+1}\binom{n-m+m-1}{m-1}.$$

It follows that

$$2S - n2^{2n-2m+1}\binom{n-1}{m-1} = 0 \quad \text{or} \quad S = n4^{n-m}\frac{m}{n}\binom{n}{m}$$

which yields

$$S = m \times 4^{n-m} \times \binom{n}{m}$$

as claimed.

45 Symmetry of the Euler-Frobenius coefficient (B_1EIR)

Suppose we have the coefficient of the Euler-Frobenius polynomial

$$b_k^n = \sum_{l=1}^{k}(-1)^{k-l}l^n\binom{n+1}{k-l}$$

and we seek to show that $b_k^n = b_{n+1-k}^n$ where $0 \le k \le n+1$.

First re-write this as

$$\sum_{l=0}^{k}(-1)^l(k-l)^n\binom{n+1}{l}.$$

Introduce the Iverson bracket

$$[[0 \le l \le k]] = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{z^l}{z^{k+1}}\frac{1}{1-z}\;dz$$

and the exponentiation integral

$$(k-l)^n = \frac{n!}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+1}}\exp((k-l)w)\;dw.$$

to get for the sum (extend the summation to $n+1$ since the Iverson bracket controls the range)

$$\frac{n!}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+1}}\exp(kw)\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{k+1}}\frac{1}{1-z}\sum_{l=0}^{n+1}\binom{n+1}{l}(-1)^l z^l\exp(-lw)\;dz\;dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw) \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+1}} \frac{1}{1-z} (1 - z\exp(-w))^{n+1} \; dz \; dw.$$

Evaluate this using the residues at the poles at $z = 1$ and at infinity. We obtain for $z = 1$

$$-\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw)(1 - \exp(-w))^{n+1} \; dw,$$

note however that $1 - \exp(-w)$ starts at w so the power starts at w^{n+1} making for a zero contribution.

We get for the residue at infinity

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{k+1} \frac{1}{1-1/z} (1 - \exp(-w)/z)^{n+1}$$

$$= -\mathrm{Res}_{z=0} z^k \frac{1}{z-1} (1 - \exp(-w)/z)^{n+1}$$

$$= \mathrm{Res}_{z=0} \frac{z^k}{z^{n+1}} \frac{1}{1-z} (z - \exp(-w))^{n+1}.$$

We need to flip the sign on this one more time since we are exploiting the fact that the residues at the three poles sum to zero. Actually extracting the coefficient we get

$$-\sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n+1-q} \exp(-(n+1-q)w).$$

Substitute this into the integral in w to get

$$-\sum_{q=0}^{n-k} \binom{n+1}{q} \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw)(-1)^{n+1-q} \exp(-(n+1-q)w) \; dw$$

$$= -\sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n+1-q} (-1)^n (n+1-k-q)^n$$

$$= \sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^q (n+1-k-q)^n.$$

Using the fact that $n+1-k-q$ is zero at $q = n+1-k$ we finally obtain

$$\sum_{q=0}^{n+1-k} \binom{n+1}{q} (-1)^q (n+1-k-q)^n$$

which is precisely b^n_{n+1-k} by definition, QED.

Addendum. An alternate proof (variation on the theme from above) starts from the unmodified definition and introduces

$$\binom{n+1}{k-l} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-l+1}} (1+z)^{n+1} \; dz.$$

This controls the range so we may extend l to infinity. Introduce furthermore

$$l^n = \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(lw) \; dw.$$

These two yield for the sum

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \sum_{l\ge 0} (-1)^l z^l \exp(lw) \; dz \; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \frac{1}{1+z\exp(w)} \; dz \; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(-w)}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \frac{1}{z+\exp(-w)} \; dz \; dw.$$

We evaluate this using the negatives of the residues at $z = -\exp(-w)$ and at infinity. We get for $z = -\exp(-w)$

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(-w)}{w^{n+1}} \frac{(-1)^k}{(-1)^{k+1}\exp(-(k+1)w)} (1-\exp(-w))^{n+1} \; dw$$

$$= -\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(kw)}{w^{n+1}} (1-\exp(-w))^{n+1} \; dw.$$

As before the exponentiated term starts at w^{n+1} so there is no coefficient on w^n for a contribution of zero.

We get for the residue at infinity (starting from the next-to-last version of the integral)

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} (-1)^k z^{k+1} \frac{(1+z)^{n+1}}{z^{n+1}} \frac{1}{1+\exp(w)/z}$$

$$= -\mathrm{Res}_{z=0} \frac{1}{z^2} (-1)^k z^{k+1} \frac{(1+z)^{n+1}}{z^{n+1}} \frac{z/\exp(w)}{1+z/\exp(w)}$$

$$= -\mathrm{Res}_{z=0} (-1)^k z^k \frac{(1+z)^{n+1}}{z^{n+1}} \frac{\exp(-w)}{1+z/\exp(w)}.$$

Doing the sign flip and simplifying we obtain

$$\exp(-w)(-1)^k \times \mathrm{Res}_{z=0} \frac{(1+z)^{n+1}}{z^{n-k+1}} \frac{1}{1+z/\exp(w)}.$$

Extract the residue to get

$$\exp(-w)(-1)^k \sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n-k-q} \exp(-(n-k-q)w)$$

Substitute into the integral in w to obtain

$$\sum_{q=0}^{n-k} \binom{n+1}{q} \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} (-1)^{n-q} \exp(-(n+1-k-q)w)\; dw$$

$$= \sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n-q} (-1)^n (n+1-k-q)^n$$

$$= \sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^q (n+1-k-q)^n.$$

We have obtained b^n_{n+1-k} as before.
This was math.stackexchange.com problem 1435648.

46 A probability distribution with two parameters ($B_1 B_2$)

A sum of binomial coefficients CLXVII

Suppose we have a random variable X where

$$\mathrm{P}[X = k] = \binom{N}{2n+1}^{-1} \binom{N-k}{n} \binom{k-1}{n}$$

for $k = n+1, \ldots, N-n$ and zero otherwise.

We seek to show that these probabilities sum to one and compute the the mean and the variance.

Sum of probabilities. This is given by

$$\binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} \binom{N-k}{n} \binom{k-1}{n}.$$

Introduce

$$\binom{N-k}{n} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{N-n-k+1}} \frac{1}{(1-z)^{n+1}}\; dz$$

and

$$\binom{k-1}{n} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{k-1}}{w^{n+1}}\; dw.$$

Observe carefully that the first integral is zero when $k > N - n$ and the second one when $1 \le k \le n$ so we may extend the range of the sum to $1 \le k$.

This gives for the sum (without the scalar)

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \sum_{k\ge 1} z^{k-1}(1+w)^{k-1} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-z(1+w)} \; dw \; dz.$$

The integral in w is

$$\frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-wz/(1-z)} \; dw$$

which yields for the integral in z

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{z^n}{(1-z)^{n+1}} \; dz$$

which is

$$\binom{N-2n-1+2n+1}{2n+1} = \binom{N}{2n+1}.$$

This confirms that the probabilities sum to one.

Expectation. This is given by

$$\mathrm{E}[X] = \binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} k \binom{N-k}{n}\binom{k-1}{n}.$$

Introduce

$$k\binom{k-1}{n} = \frac{k!}{n! \times (k-1-n)!} = (n+1)\binom{k}{n+1}$$
$$= (n+1)\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^k}{w^{n+2}} \; dw.$$

The range control from this integral produces zero when $0 \le k \le n$ so we may extend the sum to zero, getting

$$(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \sum_{k\ge 0} z^k (1+w)^k \; dw \; dz.$$

The integral in w is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \frac{1}{1-z(1+w)} \; dw$$

$$= \frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+2}}\frac{1}{1-wz/(1-z)}\;dw$$

which yields for the integral in z including the factor in front

$$(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{N-n+1}}\frac{1}{(1-z)^{n+1}}\frac{z^{n+1}}{(1-z)^{n+2}}\;dz$$

which is

$$(n+1)\binom{N-2n-1+2n+2}{2n+2} = (n+1)\binom{N+1}{2n+2}.$$

We will scale this at the end, same as the variance.
Variance. Start by computing

$$\mathrm{E}[(X+1)X] = \binom{N}{2n+1}^{-1}\sum_{k=n+1}^{N-n}(k+1)k\binom{N-k}{n}\binom{k-1}{n}.$$

Introduce

$$(k+1)k\binom{k-1}{n} = \frac{(k+1)!}{n!\times(k-1-n)!}$$
$$= (n+2)(n+1)\binom{k+1}{n+2} = (n+2)(n+1)\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{k+1}}{w^{n+3}}\;dw.$$

The range control from this integral produces zero when $0 \le k \le n$ as before so we may extend the sum to zero, getting

$$(n+2)(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{N-n+1}}\frac{1}{(1-z)^{n+1}}$$
$$\times\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1+w}{w^{n+3}}\sum_{k\ge 0}z^k(1+w)^k\;dw\;dz.$$

The integral in w is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1+w}{w^{n+3}}\frac{1}{1-z(1+w)}\;dw$$
$$= \frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1+w}{w^{n+3}}\frac{1}{1-wz/(1-z)}\;dw$$

which yields for the integral in z including the factor in front

$$(n+2)(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{N-n+1}}\frac{1}{(1-z)^{n+1}}\left(\frac{z^{n+2}}{(1-z)^{n+3}}+\frac{z^{n+1}}{(1-z)^{n+2}}\right)\;dz$$

which is

$$(n+2)(n+1)\left(\binom{N-2n-2+2n+3}{2n+3}+\binom{N-2n-1+2n+2}{2n+2}\right)$$

$$= (n+2)(n+1)\left(\binom{N+1}{2n+3}+\binom{N+1}{2n+2}\right).$$

Simplification for ease of interpretation.
We get for the expectation

$$\mathrm{E}[X] = (n+1)\frac{(N+1)!}{(N-2n-1)!(2n+2)!}\frac{(N-2n-1)!(2n+1)!}{N!}$$

$$= \frac{1}{2}(N+1).$$

We obtain furthermore

$$\mathrm{E}[(X+1)X] = (n+2)(n+1)$$

$$\times\left(\frac{(N+1)!}{(N-2n-2)!(2n+3)!}+\frac{(N+1)!}{(N-2n-1)!(2n+2)!}\right)\frac{(N-2n-1)!(2n+1)!}{N!}$$

$$= \frac{1}{2}(N+1)(n+2)\left(\frac{N-2n-1}{2n+3}+1\right)$$

$$= \frac{1}{2}(N+2)(N+1)\frac{n+2}{2n+3}.$$

This yields for the variance

$$\mathrm{Var}[X] = \mathrm{E}[X^2]-\mathrm{E}[X]^2$$

$$= \frac{1}{2}(N+2)(N+1)\frac{n+2}{2n+3}-\frac{1}{2}(N+1)-\frac{1}{4}(N+1)^2.$$

which simplifies to

$$\mathrm{Var}[X] = \frac{1}{4}(N+1)\frac{N-2n-1}{2n+3}.$$

This was math.stackexchange.com problem 1257644.

47 An identity involving Narayana numbers (B_1)

Suppose we have the Narayana number

$$N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$$

and let

$$A(n,k,l) = \sum_{\substack{i_0+i_1+\cdots+i_k=n\\ j_0+j_1+\cdots+j_k=l}} \prod_{t=0}^{k} N(i_t, j_t+1)$$

where the compositions for n are regular and the ones for l are weak and we seek to verify that

$$A(n,k,l) = \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1}.$$

Introducing

$$\begin{aligned} G(z,u) &= \sum_{p\ge 1} z^p \sum_{q\ge 0} u^q \frac{1}{p}\binom{p}{q+1}\binom{p}{q} \\ &= \sum_{p\ge 1} \frac{1}{p} z^p \sum_{q\ge 0} u^q \binom{p}{q+1}\binom{p}{q} \end{aligned}$$

we have by inspection that

$$A(n,k,l) = [z^n][u^l]G(z,u)^{k+1}.$$

To evaluate this introduce for the inner sum term

$$\binom{p}{q+1} = \binom{p}{p-q-1} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p-q}}(1+w)^p \; dw.$$

We get for the inner sum

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p}}(1+w)^p \sum_{q\ge 0}\binom{p}{q} u^q w^q \; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p}}(1+w)^p (1+uw)^p \; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p}}(1+w(1+u+uw)))^p \; dw.$$

Extracting the coefficient from this we get

$$[w^{p-1}]\sum_{q=0}^{p}\binom{p}{q} w^q (1+u+uw)^q$$

$$= \sum_{q=0}^{p-1}\binom{p}{q} [w^{p-1-q}](1+u+uw)^q$$

$$= \sum_{q=0}^{p-1} \binom{p}{q} \binom{q}{p-1-q} u^{p-1-q} (1+u)^{2q+1-p}.$$

This is

$$\sum_{q=0}^{p-1} \binom{p}{p-1-q} \binom{p-1-q}{q} u^q (1+u)^{p-1-2q}$$

$$= \sum_{q=0}^{p-1} \binom{p}{q+1} \binom{p-1-q}{q} u^q (1+u)^{p-1-2q}.$$

Now observe that

$$\frac{1}{p} \binom{p}{q+1} \binom{p-1-q}{q} = \frac{1}{q+1} \binom{p-1}{q} \binom{p-1-q}{q}$$

$$= \frac{1}{q+1} \binom{p-1}{p-1-q} \binom{p-1-q}{q} = \frac{1}{q+1} \binom{p-1}{2q} \binom{2q}{q}.$$

where

$$C_q = \frac{1}{q+1} \binom{2q}{q}$$

is a Catalan number.

We thus get for the sum

$$\sum_{p \geq 1} z^p \sum_{q=0}^{p-1} \binom{p-1}{2q} C_q u^q (1+u)^{p-1-2q}$$

$$= z \sum_{p \geq 0} z^p \sum_{q=0}^{p} \binom{p}{2q} C_q u^q (1+u)^{p-2q}$$

$$= z \sum_{q \geq 0} C_q u^q (1+u)^{-2q} \sum_{p \geq q} \binom{p}{2q} z^p (1+u)^p$$

$$= z \sum_{q \geq 0} C_q u^q (1+u)^{-2q} \sum_{p \geq 2q} \binom{p}{2q} z^p (1+u)^p$$

$$= z \sum_{q \geq 0} C_q u^q (1+u)^{-2q} (1+u)^{2q} z^{2q} \sum_{p \geq 0} \binom{p+2q}{2q} z^p (1+u)^p$$

$$= z \sum_{q \geq 0} C_q u^q z^{2q} \frac{1}{(1-z(1+u))^{2q+1}}.$$

Using the generating function of the Catalan numbers

$$Q(w) = \sum_{q \geq 0} C_q w^q = \frac{1 - \sqrt{1-4w}}{2w}$$

which has functional equation

$$Q(w) = 1 + wQ(w)^2$$

we obtain

$$Q\left(\frac{uz^2}{(1-z(1+u))^2}\right) = 1 + \frac{uz^2}{(1-z(1+u))^2} Q\left(\frac{uz^2}{(1-z(1+u))^2}\right)^2$$

which is

$$G(z,u)\frac{1-z(1+u)}{z} = 1 + uG(z,u)^2.$$

Extract the coefficient in z first. We get from the functional equation

$$z = \frac{G(z,u)}{uG(z,u)^2 + (1+u)G(z,u) + 1}.$$

The coefficient extractor integral is

$$[z^n]G(z,u) = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} G(z,u)^{k+1}\, dz.$$

which becomes with $G(z,u) = v$

$$\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(uv^2+(1+u)v+1)^{n+1}}{v^{n+1}}$$

$$\times v^{k+1}\left(\frac{1}{uv^2+(1+u)v+1} - \frac{v}{(uv^2+(1+u)v+1)^2}(2uv+(1+u))\right)\, dv$$

$$= \frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(uv^2+(1+u)v+1)^{n-1}}{v^{n-k}}(1-uv^2)\, dv.$$

This is

$$\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{n-1}(1+uv)^{n-1}}{v^{n-k}}(1-uv^2)\, dv.$$

Extracting the coefficient on $[u^l]$ we get two pieces which are, first piece A

$$\binom{n-1}{l}\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{n-1}v^l}{v^{n-k}}\, dv = \binom{n-1}{l}\binom{n-1}{n-k-l-1}$$

which is

$$\binom{n-1}{l}\binom{n-1}{k+l} = \binom{n-1}{l}\frac{k+l+1}{n}\binom{n}{k+l+1}$$

$$= (n-l)\frac{k+l+1}{n^2}\binom{n}{l}\binom{n}{k+l+1}.$$

and piece B which is

$$-\binom{n-1}{l-1}\frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{(1+v)^{n-1}v^{l-1}}{v^{n-k}}\;v^2 dv = -\binom{n-1}{l-1}\binom{n-1}{n-k-l-2}$$

which is

$$-\binom{n-1}{l-1}\binom{n-1}{k+l+1} = -\binom{n-1}{l-1}\frac{n-k-l-1}{n}\binom{n}{k+l+1}$$

$$= -l\frac{n-k-l-1}{n^2}\binom{n}{l}\binom{n}{k+l+1}.$$

Collecting the two pieces we finally obtain

$$\left((n-l)\frac{k+l+1}{n^2} + l\frac{-n+k+l+1}{n^2}\right)\binom{n}{l}\binom{n}{k+l+1}$$

$$= \left(n\frac{k+l+1}{n^2} + l\frac{-n}{n^2}\right)\binom{n}{l}\binom{n}{k+l+1}$$

$$= \frac{k+1}{n}\binom{n}{l}\binom{n}{k+l+1}$$

as claimed, QED.

Remark. The closed form of $G(z, u)$ can be computed as follows:

$$\frac{z}{1-z(1+u)}\frac{1-\sqrt{1-4uz^2/(1-z(1+u))^2}}{2uz^2/(1-z(1+u))^2}$$

$$= \frac{z}{(1-z(1+u))^2}\frac{1-z(1+u)-\sqrt{1-2z(1+u)+z^2(1+u)^2-4uz^2}}{2uz^2/(1-z(1+u))^2}$$

$$= \frac{1-z(1+u)-\sqrt{1-2z(1+u)+z^2(1+u)^2-4uz^2}}{2uz}.$$

The above material incorporates data from OEIS A055151 and from OEIS A001263 on Narayana numbers.

This was math.stackechange.com problem 1498014.

48 Convolution of Narayana polynomials (B_1)

This is basically a re-write of the previous entry with a more general conclusion.

Suppose we define

$$C_0^{(1)}(t) = 1 \quad \text{and} \quad C_n^{(1)}(t) = \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n+1}{k+1}\frac{1}{n+1}t^k$$

and let for $m \ge 2$

$$C_n^{(m)}(t) = \sum_{q=0}^{n} C_q^{(m-1)}(t) C_{n-q}^{(1)}(t).$$

This definition is equivalent to introducing

$$G(w) = \sum_{n\ge 1} C_n^{(1)}(t) w^n$$

and letting

$$C_n^{(m)}(t) = [w^n](1+G(w))^m = [w^n]\sum_{p=0}^{m}\binom{m}{p}G(w)^p.$$

We seek to show that

$$C_0^{(m)}(t) = 1 \quad \text{and} \quad C_n^{(m)}(t) = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+m}{k+m}\frac{m}{n+m}t^k.$$

The proof will consist in doing a coefficient extraction operation from the powers of $G(w)$ and showing that these match the proposed formula. We require an alternate representation of $C_n^{(1)}(t)$ and observe that

$$\begin{aligned}\binom{n-1}{k}\binom{n+1}{k+1}\frac{1}{n+1} &= \binom{n}{k+1}\frac{k+1}{n}\binom{n}{k}\frac{n+1}{k+1}\frac{1}{n+1} \\ &= \frac{1}{n}\binom{n}{k+1}\binom{n}{k}.\end{aligned}$$

and introduce

$$\binom{n}{k+1} = \binom{n}{n-k-1} = \frac{1}{2\pi i}\int_{|z|=1}\frac{1}{z^{n-k}}(1+z)^n \; dz$$

which conveniently vanishes for $k \ge n$. We get for $n \ge 1$

$$C_n^{(1)}(t) = \frac{1}{n}\frac{1}{2\pi i}\int_{|z|=1}\frac{1}{z^n}(1+z)^n\sum_{k\ge 0}\binom{n}{k}z^k t^k \; dz$$

$$= \frac{1}{n} \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z^n} (1+z)^n (1+tz)^n \; dz.$$

We re-write this as

$$\frac{1}{n} \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z^n} (1+z(1+t+tz))^n \; dz.$$

Extracting the coefficient yields

$$\frac{1}{n} \sum_{q=0}^{n-1} \binom{n}{q} [z^{n-1-q}](1+t+tz)^q$$

$$= \frac{1}{n} \sum_{q=0}^{n-1} \binom{n}{q} \binom{q}{n-1-q} (1+t)^{q-(n-1-q)} t^{n-1-q}$$

$$= \frac{1}{n} \sum_{q=0}^{n-1} \binom{n}{q} \binom{q}{n-1-q} (1+t)^{2q+1-n} t^{n-1-q}$$

$$= \frac{1}{n} \sum_{q=0}^{n-1} \binom{n}{n-1-q} \binom{n-1-q}{q} (1+t)^{n-1-2q} t^q.$$

Now we have

$$\frac{1}{n} \binom{n}{n-1-q} \binom{n-1-q}{q} = \frac{1}{n} \frac{n!}{(q+1)!q!(n-1-2q)!}$$
$$= \frac{1}{n} \binom{2q+1}{q} \binom{n}{2q+1} = \frac{1}{2q+1} \binom{2q+1}{q} \binom{n-1}{2q}$$
$$= \frac{1}{2q+1} \binom{2q+1}{q+1} \binom{n-1}{2q} = \frac{1}{q+1} \binom{2q}{q} \binom{n-1}{2q}$$

where

$$C_q = \frac{1}{q+1} \binom{2q}{q}$$

is a Catalan number. We thus obtain for $G(w)$

$$\sum_{n\ge 1} w^n \sum_{q=0}^{n-1} C_q \binom{n-1}{2q} (1+t)^{n-1-2q} t^q$$

$$= w \sum_{n\ge 0} w^n \sum_{q=0}^{n} C_q \binom{n}{2q} (1+t)^{n-2q} t^q$$

$$= w \sum_{q\ge 0} C_q (1+t)^{-2q} t^q \sum_{n\ge q} \binom{n}{2q} w^n (1+t)^n$$

$$= w\sum_{q\geq 0} C_q(1+t)^{-2q}t^q \sum_{n\geq 2q}\binom{n}{2q}w^n(1+t)^n$$

$$= w\sum_{q\geq 0} C_q(1+t)^{-2q}t^q w^{2q}(1+t)^{2q}\sum_{n\geq 0}\binom{n+2q}{2q}w^n(1+t)^n$$

$$= w\sum_{q\geq 0} C_q t^q w^{2q}\frac{1}{(1-w(1+t))^{2q+1}}$$

$$= \frac{w}{(1-w(1+t))}\sum_{q\geq 0} C_q t^q w^{2q}\frac{1}{(1-w(1+t))^{2q}}.$$

Now the classic generating function of the Catalan numbers is

$$Q(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

We have computed the closed form of $G(w)$ which is

$$G(w) = \frac{w}{(1-w(1+t))}\frac{1-\sqrt{1-4tw^2/(1-w(1+t))^2}}{2tw^2/(1-w(1+t))^2}$$

$$= w\frac{1-\sqrt{1-4tw^2/(1-w(1+t))^2}}{2tw^2/(1-w(1+t))}$$

$$= w\frac{1-w(1+t)-\sqrt{(1-w(1+t))^2-4tw^2}}{2tw^2}$$

$$= \frac{1-w(1+t)-\sqrt{(1-w(1+t))^2-4tw^2}}{2tw}.$$

Recall the functional equation of the Catalan number generating function which is

$$Q(x) = 1 + xQ(x)^2.$$

We thus obtain

$$Q\left(\frac{tw^2}{(1-w(1+t))^2}\right) = 1 + \frac{tw^2}{(1-w(1+t))^2}Q\left(\frac{tw^2}{(1-w(1+t))^2}\right)^2$$

or

$$\frac{1-w(1+t)}{w}G(w) = 1 + tG(w)^2.$$

Solving for w we get

$$G(w) - w(1+t)G(w) = w(1+tG(w)^2)$$

or

$$G(w) = w(1 + (1+t)G(w) + tG(w)^2)$$

which is

$$w = \frac{G(w)}{1 + (1+t)G(w) + tG(w)^2}.$$

Recall that we seek

$$[t^k] \sum_{p=0}^{m} \binom{m}{p} [w^n] G(w)^p.$$

We establish the coefficient extraction integral (Lagrange inversion)

$$[w^n] G(w)^p = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} G(w)^p \; dw.$$

Setting $v = G(w)$ and observing that $w = 0$ is mapped to $v = 0$ we get

$$\frac{1}{2\pi i} \int_{|v|=\gamma} \frac{(1 + (1+t)v + tv^2)^{n+1}}{v^{n+1}}$$
$$\times v^p \times \left(\frac{1}{1 + (1+t)v + tv^2} - \frac{v(1 + t + 2tv)}{(1 + (1+t)v + tv^2)^2} \right) \; dv.$$

This is

$$\frac{1}{2\pi i} \int_{|v|=\gamma} \frac{(1 + (1+t)v + tv^2)^{n-1}}{v^{n-p+1}} (1 + (1+t)v + tv^2 - v(1 + t + 2tv)) \; dv$$
$$= \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{(1+v)^{n-1}(1+tv)^{n-1}}{v^{n-p+1}} (1 - tv^2) \; dv.$$

Now substituting this into the target formula yields

$$C_n^{(m)}(t) = \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{(1+v)^{n-1}(1+tv)^{n-1}}{v^{n+1}} (1 - tv^2) \sum_{p=0}^{m} \binom{m}{p} v^p \; dv$$
$$= \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{(1+v)^{n+m-1}(1+tv)^{n-1}}{v^{n+1}} (1 - tv^2) \; dv.$$

To conclude the proof we must treat the case of $n = 0$ which is different from the case $n \geq 1$ and extract coefficients on $[t^k]$ in the latter case. With $n = 0$ we get

$$\frac{1}{2\pi i} \int_{|v|=\gamma} (1+v)^{m-1} \frac{1}{v} \frac{1}{1+tv} (1 - tv^2) \; dv.$$

This is the constant coefficient and is equal to

$$(1+v)^{m-1}\frac{1}{1+tv}(1-tv^2)\Bigg|_{v=0} = 1,$$

as required. For the case of $n \ge 1$ we have two subcases, $k = 0$ and $k \ge 1$. For $k = 0$ the second term in $1 - tv^2$ does not contribute and we have just

$$\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n+m-1}}{v^{n+1}}\, dv$$
$$= \binom{n+m-1}{n} = \binom{n+m-1}{m-1} = \binom{n+m}{m}\frac{m}{n+m}.$$

This is again the required value. Finally when $n \ge 1$ and $k \ge 1$ we get two pieces, namely

$$\binom{n-1}{k}\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n+m-1}}{v^{n-k+1}}\, dv = \binom{n-1}{k}\binom{n+m-1}{n-k}$$

and

$$-\binom{n-1}{k-1}\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n+m-1}}{v^{n-k}}\, dv = -\binom{n-1}{k-1}\binom{n+m-1}{n-k-1}.$$

Note that when $k = n$ the first of these integrals never appears in the first place because $[t^k](1+tv)^{n-1} = 0$ and the second vanishes due to the residue. When $k > n$ we have $[t^k](1+tv)^{n-1}(1-tv^2) = 0$ and everything vanishes. This is the required behavior. We get for the non-zero cases

$$\binom{n-1}{k}\binom{n+m-1}{k+m-1} - \binom{n-1}{k-1}\binom{n+m-1}{k+m}$$
$$= \binom{n-1}{k}\binom{n+m}{k+m}\frac{k+m}{n+m} - \binom{n-1}{k}\frac{k}{n-k}\binom{n+m}{k+m}\frac{n-k}{n+m}$$
$$= \binom{n-1}{k}\binom{n+m}{k+m}\frac{m}{n+m}.$$

We have the required value as was to be shown and may end the computation. This was math.stackexchange.com problem 1997791.

49 A property of Legendre polynomials (B_1)

Suppose we seek to determine the constant Q in the equality

$$Q_{n,m}\left(\frac{d}{dz}\right)^{n-m}(1-z^2)^n = (1-z^2)^m\left(\frac{d}{dz}\right)^{n+m}(1-z^2)^n$$

where $n \geq m$. We will compute the coefficients on $[z^q]$ on the LHS and the RHS. Writing $1 - z^2 = (1+z)(1-z)$ we get for the LHS

$$\sum_{p=0}^{n-m} \binom{n-m}{p} \binom{n}{p} p! (1+z)^{n-p}$$
$$\times \binom{n}{n-m-p} (n-m-p)! (-1)^{n-m-p} (1-z)^{m+p}$$
$$= (n-m)! (-1)^{n-m} \sum_{p=0}^{n-m} \binom{n}{p} \binom{n}{n-m-p} (1+z)^{n-p} (-1)^p (1-z)^{m+p}.$$

Extracting the coefficient we get

$$(n-m)! (-1)^{n-m} \sum_{p=0}^{n-m} \binom{n}{p} \binom{n}{n-m-p} (-1)^p$$
$$\times \sum_{k=0}^{n-p} \binom{n-p}{k} (-1)^{q-k} \binom{m+p}{q-k}.$$

We use the same procedure on the RHS and merge in the $(1-z^2)^m$ term to get

$$(n+m)! (-1)^{n+m} \sum_{p=0}^{n+m} \binom{n}{p} \binom{n}{n+m-p} (-1)^p$$
$$\times \sum_{k=0}^{n+m-p} \binom{n+m-p}{k} (-1)^{q-k} \binom{p}{q-k}.$$

Working in parallel with LHS and RHS we treat the inner sum of the LHS first, putting

$$\binom{m+p}{q-k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q-k+1}} (1+z)^{m+p} \; dz$$

to get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^{m+p} \sum_{k=0}^{n-p} \binom{n-p}{k} (-1)^{q-k} z^k \; dz$$
$$= \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^{m+p} (1-z)^{n-p} \; dz.$$

Adapt and repeat to obtain for the inner sum of the RHS

$$\frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^p (1-z)^{n+m-p} \; dz.$$

Moving on to the two outer sums we introduce

$$\binom{n}{n-m-p} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m-p+1}} (1+w)^n \; dw$$

to obtain for the LHS

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^m (1-z)^n \sum_{p=0}^{n-m} \binom{n}{p} (-1)^p w^p \frac{(1+z)^p}{(1-z)^p} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^m (1-z)^n \left(1 - w\frac{1+z}{1-z}\right)^n \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^m (1-z-w-wz)^n \; dz \; dw.$$

Repeat for the RHS to get

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1-z)^m (1-z-w-wz)^n \; dz \; dw.$$

Extracting coefficients from the first integral (LHS) we write

$$(1-z-w-wz)^n = (2-(1+z)(1+w))^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+z)^k (1+w)^k 2^{n-k}$$

and the inner integral yields

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{m+k}{q} (1+w)^k 2^{n-k}$$

followed by the outer one which gives

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{m+k}{q} \binom{n+k}{n-m} 2^{n-k}.$$

For the second integral (RHS) we write

$$(1 - z - w - wz)^n = ((1 - z)(1 + w) - 2w)^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} (1 - z)^k (1 + w)^k (-1)^{n-k} 2^{n-k} w^{n-k}$$

and the inner integral yields

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{q} (-1)^q (1 + w)^k (-1)^{n-k} 2^{n-k} w^{n-k}$$

followed by the outer one which produces

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{q} \binom{n+k}{k+m} (-1)^{n-k} 2^{n-k}.$$

The two sums are equal up to a sign and the RHS for the coefficient on $[z^q]$ is obtained from the LHS by multiplying by

$$\frac{(n+m)!}{(n-m)!} (-1)^{n-q}.$$

Observe that powers of z that are present in the LHS and the RHS always have the same parity, the coefficients being zero otherwise (either all even powers or all odd). Therefore $(-1)^{n-q}$ is in fact a constant not dependent on q, the question is which. The leading term has degree $2n - (n - m) = n + m = (2n - (n + m)) + 2m$ on both sides and the sign on the LHS is $(-1)^n$ and on the RHS it is $(-1)^{n+m}$. The conclusion is that the queried factor is given by

$$Q_{n,m} = (-1)^m \frac{(n+m)!}{(n-m)!}.$$

This was math.stackexchange.com problem 2066340.

50 A sum of factorials, OGF and EGF of the Stirling numbers of the second kind (B_1)

We are given that

$$r^k (r + n)! = \sum_{m=0}^{k} \lambda_m (r + n + m)!$$

and seek to determine the λ_m independent of r. We claim and prove that

$$\lambda_m = (-1)^{k+m} \sum_{p=0}^{k-m} \binom{k}{p} \left\{ {k+1-p \atop m+1} \right\} n^p.$$

With this in mind we re-write the initial condition as

$$r^k = \sum_{m=0}^{k} \lambda_m m! \binom{r+n+m}{m}.$$

We evaluate the RHS starting with λ_m using the EGF of the Stirling numbers of the second kind which in the present case says that

$$\left\{ {k+1-p \atop m+1} \right\} = \frac{(k+1-p)!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2-p}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \; dz.$$

We obtain for λ_m

$$(-1)^{k+m} \sum_{p=0}^{k-m} n^p \binom{k}{p} \frac{(k+1-p)!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2-p}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \; dz.$$

The inner term vanishes when $p \ge k+2$ but in fact even better it also vanishes when $p > k-m$ which implies $m+1 > k+1-p$ because $(\exp(z)-1)^{m+1}$ starts at $[z^{m+1}]$ and we are extracting the term on $[z^{k+1-p}]$.

Hence we may extend p to infinity without picking up any extra contributions to get

$$(-1)^{k+m} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \sum_{p\ge 0} (k+1-p) \frac{n^p z^p}{p!} \; dz.$$

This is

$$(-1)^{k+m} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} ((k+1)-nz) \exp(nz) \; dz.$$

Substitute this into the outer sum to get

$$(-1)^k \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1)-nz) \exp(nz)$$
$$\times \sum_{m=0}^{k} \binom{r+n+m}{m} (-1)^m \frac{(\exp(z)-1)^{m+1}}{m+1} \; dz.$$

We have

$$\binom{r+n+m}{m} \frac{1}{m+1} = \binom{r+n+m}{m+1} \frac{1}{r+n}$$

and hence obtain

$$\frac{(-1)^k}{r+n}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{k+2}}((k+1)-nz)\exp(nz)$$

$$\times\sum_{m=0}^{k}\binom{r+n+m}{m+1}(-1)^m(\exp(z)-1)^{m+1}\,dz.$$

We may extend m to $m>k$ in the remaining sum because the term $(\exp(z)-1)^{m+1}$ as before starts at $[z^{m+1}]$ which would then be $>k+1$ but we are extracting the coefficient on $[z^{k+1}]$, which makes for a zero contribution.

Continuing we find

$$-\sum_{m\geq 0}\binom{r+n+m}{r+n-1}(-1)^{m+1}(\exp(z)-1)^{m+1}$$

$$=1-\frac{1}{(1-(1-\exp(z)))^{r+n}}=1-\exp(-(r+n)z).$$

We get two pieces on substituting this back into the main integral, the first is

$$\frac{(-1)^k}{r+n}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{k+2}}((k+1)-nz)\exp(nz)\,dz$$

$$=\frac{(-1)^k}{r+n}(k+1)!\frac{n^{k+1}}{(k+1)!}-\frac{(-1)^k}{r+n}k!n\frac{n^k}{k!}=0.$$

and the second is

$$\frac{(-1)^{k+1}}{r+n}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{k+2}}((k+1)-nz)\exp(nz)\exp(-(r+n)z)\,dz$$

$$=\frac{(-1)^{k+1}}{r+n}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{k+2}}((k+1)-nz)\exp(-rz)\,dz$$

$$=\frac{(-1)^{k+1}}{r+n}(k+1)!\frac{(-r)^{k+1}}{(k+1)!}-\frac{(-1)^{k+1}}{r+n}k!n\frac{(-r)^k}{k!}$$

$$=\frac{1}{r+n}(k+1)!\frac{r^{k+1}}{(k+1)!}+\frac{1}{r+n}k!n\frac{r^k}{k!}$$

$$=\frac{1}{r+n}r^{k+1}+\frac{1}{r+n}nr^k=r^k.$$

This concludes the argument.

Addendum Nov 27 2016. Markus Scheuer proposes the identity

$$\lambda_m = (-1)^{m+k} \sum_{p=m}^{k} \left\{ p \atop m \right\} \binom{k}{p} (n+1)^{k-p}.$$

To see that this is the same as what I presented we extract the coefficient on $[n^q]$ to get

$$(-1)^{m+k} \sum_{p=m}^{k} \left\{ p \atop m \right\} \binom{k}{p} \binom{k-p}{q}.$$

Now we have

$$\binom{k}{p}\binom{k-p}{q} = \frac{k!}{p!q!(k-p-q)!} = \binom{k}{q}\binom{k-q}{p}.$$

We get

$$(-1)^{m+k} \binom{k}{q} \sum_{p=m}^{k} \left\{ p \atop m \right\} \binom{k-q}{p}.$$

We now introduce

$$\binom{k-q}{p} = \binom{k-q}{k-q-p} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-q-p+1}} (1+z)^{k-q} \; dz.$$

This certainly vanishes when $p > k-q$ so we may extend p to infinity, getting for the sum

$$(-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-q+1}} (1+z)^{k-q} \sum_{p \ge m} \left\{ p \atop m \right\} z^p \; dz.$$

Using the OGF of the Stirling numbers of the second kind this becomes

$$(-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-q+1}} (1+z)^{k-q} \prod_{l=1}^{m} \frac{z}{1-lz} \; dz.$$

Now put $z/(1+z) = w$ to get $z = w/(1-w)$ and $dz = 1/(1-w)^2 \; dw$ to get

$$\begin{aligned}
&(-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k-q}} \frac{1-w}{w} \frac{1}{(1-w)^2} \prod_{l=1}^{m} \frac{w/(1-w)}{1-lw/(1-w)} \; dw \\
&= (-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k-q+1}} \frac{1}{1-w} \prod_{l=1}^{m} \frac{w}{1-w-lw} \; dw \\
&= (-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k-q+1}} \frac{1}{1-w} \prod_{l=1}^{m} \frac{w}{1-(l+1)w} \; dw
\end{aligned}$$

$$= (-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k-q+2}} \frac{w}{1-w} \prod_{l=2}^{m+1} \frac{w}{1-lw} \; dw$$

$$= (-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k-q+2}} \prod_{l=1}^{m+1} \frac{w}{1-lw} \; dw$$

$$= (-1)^{m+k} \binom{k}{q} \left\{ {k-q+1 \atop m+1} \right\}.$$

This is the claim and we are done.
This was math.stackexchange.com problem 2028293.

51 Fibonacci, Tribonacci, Tetranacci (B_1)

Suppose we seek to evaluate the following sum (with a condition on the binomial coefficient)

$$G(n,m) = \sum_{k=0}^{n} \sum_{q=0}^{k} (-1)^q \binom{k}{q} \binom{n-1-qm}{k-1}.$$

Now when $n-1-qm < 0$ we usually get a non-zero value for the binomial coefficient but this is not wanted here. Therefore we have

$$G(n,m) = \sum_{k=0}^{n} \sum_{q=0}^{\lfloor (n-k)/m \rfloor} (-1)^q \binom{k}{q} \binom{n-1-qm}{k-1}.$$

If we have lost any values for q above $\lfloor (n-k)/m \rfloor$ these would render the second binomial coefficient zero. If we have added in any values for q above k the first binomial coefficient is zero there.

Now with the integral

$$\binom{n-1-qm}{k-1} = \binom{n-1-qm}{n-k-qm} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1-qm}}{z^{n-k-qm+1}} \; dz$$

we get range control because the pole vanishes when $q > (n-k)/m$ and we may extend q to infinity. We thus obtain for the inner sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n-k+1}} \sum_{q \ge 0} (-1)^q \binom{k}{q} \frac{z^{qm}}{(1+z)^{qm}} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n-k+1}} \left(1 - \frac{z^m}{(1+z)^m}\right)^k \; dz$$

This yields for the outer sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^{n+1}}\left(1-z\left(1-\frac{z^m}{(1+z)^m}\right)\right)^{-1}$$
$$\times\left(1-z^{n+1}\left(1-\frac{z^m}{(1+z)^m}\right)^{n+1}\right)\;dz$$

which is

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+m-1}}{z^{n+1}}\left((1-z)(1+z)^m+z^{m+1}\right)^{-1}$$
$$\times\left(1-z^{n+1}\left(1-\frac{z^m}{(1+z)^m}\right)^{n+1}\right)\;dz$$

Extracting the second component from the difference we get

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon}(1+z)^{n+m-1}\left((1-z)(1+z)^m+z^{m+1}\right)^{-1}\left(1-\frac{z^m}{(1+z)^m}\right)^{n+1}\;dz$$

The pole at zero has vanished. We now have non-zero poles at $z=-1$ and from the inverted term. These depend on m and we can certainly choose ϵ small enough so that none of them are inside the contour. Therefore this term does not contribute, leaving only

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+m-1}}{z^{n+1}}\frac{1}{(1-z)(1+z)^m+z^{m+1}}\;dz.$$

The generating function $f(w)$ of these numbers is thus given by

$$f(w)=\sum_{n\ge 0}w^n\sum_{q=0}^{n}\binom{n+m-1}{n-q}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}.$$

This is

$$\sum_{q\ge 0}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}\sum_{n\ge q}w^n\binom{n+m-1}{n-q}$$
$$=\sum_{q\ge 0}w^q[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}\sum_{n\ge 0}w^n\binom{n+m-1+q}{n}$$
$$=\frac{1}{(1-w)^m}\sum_{q\ge 0}\frac{w^q}{(1-w)^q}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}.$$

What we have here is an *annihilated coefficient extractor* that simplifies to

$$f(w) = \frac{1}{(1-w)^m} \frac{1}{(1-w/(1-w))(1+w/(1-w))^m + (w/(1-w))^{m+1}}$$

$$= \frac{1}{(1-w)^m} \frac{1}{(1-2w)/(1-w)/(1-w)^m + w^{m+1}/(1-w)^{m+1}}$$

$$= \frac{1-w}{1-2w+w^{m+1}}.$$

Now observe that

$$1 - 2w + w^{m+1} = (1-w)(1 - w - w^2 - \cdots - w^{m-1} - w^m)$$

so we finally have

$$f(w) = \left(1 - \sum_{q=1}^{m} w^q\right)^{-1} = \frac{1}{1 - w - w^2 - \cdots - w^m}.$$

We see that by the basic theory of linear recurrences what we have here is a Fibonacci, Tribonacci, Tetranacci etc. recurrence. The question is what are the initial values.

Observe however that $[w^0]f(w) = 1$ and for $1 \le q \le m$ we have

$$[w^q]\frac{1-w}{1-2w+w^{m+1}} = [w^q]\frac{1}{1-2w+w^{m+1}} - [w^{q-1}]\frac{1}{1-2w+w^{m+1}}.$$

But

$$\frac{1}{1-2w+w^{m+1}} = \frac{1}{1-2w(1-w^m/2)} = \sum_{n\ge 0} 2^n w^n (1-w^m/2)^n$$

With the condition on q and $n \ge 1$ only the constant term from the term $(1-w^m/2)^n$ contributes because the degree would be more than m otherwise. This produces just one matching term with coefficient 2^q.

This yields for $f(w)$

$$[w^q]f(w) = 2^q - 2^{q-1} = 2^{q-1}.$$

Therefore we get for the intial terms starting at $q = 0$

$$1, 1, 2, 4, 8, 16, \ldots, 2^{m-1} \quad \text{with recurrence} \quad f_n = \sum_{q=1}^{m} f_{n-q}.$$

This recurrence also shows (by subtraction) that the sequence may be produced starting from $m-1$ zero terms followed by one.

The OEIS has the Fibonacci numbers, OEIS A000045

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

and the Tribonacci numbers, OEIS A000073

$$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, \ldots$$

and the Tetranacci numbers, OEIS A000078

$$1, 2, 4, 8, 15, 29, 56, 108, 208, 401, \ldots$$

and more.
This was math.stackexcange.com problem 1626949.

52 Stirling numbers of two kinds, binomial coefficients

Suppose we seek to verify that

$$\left\{ {n \atop m} \right\} = (-1)^n \sum_{k=m}^{n} \binom{k}{m} (-1)^k \sum_{q=0}^{k} \left\{ {n+q-m \atop k} \right\} \left[{k \atop q} \right] \binom{n}{m-q}$$

where presumably $n \ge m$. We need for the second binomial coefficient that $m \ge q$ so this is

$$\left\{ {n \atop m} \right\} = (-1)^n \sum_{k=m}^{n} \binom{k}{m} (-1)^k \sum_{q=0}^{m} \left\{ {n+q-m \atop k} \right\} \left[{k \atop q} \right] \binom{n}{m-q}.$$

Observe that the Stirling number of the second kind vanishes when $k > n$ so we may extend the summation to infinity, getting

$$\left\{ {n \atop m} \right\} = (-1)^n \sum_{k \ge m} \binom{k}{m} (-1)^k \sum_{q=0}^{m} \left\{ {n+q-m \atop k} \right\} \left[{k \atop q} \right] \binom{n}{m-q}.$$

Recall that

$$\left[{k \atop q} \right] = [w^q] k! \times \binom{w+k-1}{k}.$$

Starting with the inner sum we obtain

$$n! \sum_{q=0}^{m} \frac{1}{(m-q)!} [z^{n+q-m}] (\exp(z) - 1)^k [w^q] \binom{w+k-1}{k}$$

$$= n! \sum_{q=0}^{m} \frac{1}{q!} [z^{n-q}] (\exp(z)-1)^k [w^{m-q}] \binom{w+k-1}{k}.$$

Now when $q > m$ the coefficient extractor in w yields zero, hence we may extend the sum in q to infinity:

$$n! \sum_{q\ge 0} \frac{1}{q!} [z^{n-q}] (\exp(z)-1)^k [w^{m-q}] \binom{w+k-1}{k}.$$

We thus obtain

$$\frac{n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^k \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+k-1}{k} \sum_{q\ge 0} \frac{1}{q!} z^q w^q \; dw \; dz$$

$$= \frac{n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^k \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+k-1}{k} \exp(zw) \; dw \; dz.$$

Preparing the outer sum we obtain

$$\sum_{k\ge m} \binom{k}{m} (-1)^k (\exp(z)-1)^k \binom{w+k-1}{k}$$

$$= \sum_{k\ge m} \binom{k}{m} (-1)^k (\exp(z)-1)^k [v^k] \frac{1}{(1-v)^w}.$$

Note that for a formal power series $Q(v)$ we have

$$\sum_{k\ge m} \binom{k}{m} (-1)^{k-m} u^{k-m} [v^k] Q(v) = \frac{1}{m!} \left. (Q(v))^{(m)} \right|_{v=-u}.$$

We get for the derivative in v

$$\left(\frac{1}{(1-v)^w} \right)^{(m)} = m! \binom{w+m-1}{m} \frac{1}{(1-v)^{w+m}}.$$

Substituting $u = \exp(z) - 1$ yields

$$m! \binom{w+m-1}{m} \exp(-(w+m)z).$$

Returning to the double integral we find

$$\frac{(-1)^n \times n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^m (-1)^m$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \exp(zw) \binom{w+m-1}{m} \exp(-(w+m)z) \; dw \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z) - 1)^m (-1)^m \exp(-mz)$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+m-1}{m} \; dw \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z) - 1)^m (-1)^m \exp(-mz) \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1 - \exp(-z))^m (-1)^m \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(-z) - 1)^m \; dz.$$

Finally put $z = -v$ to get

$$-\frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|v|=\epsilon} \frac{(-1)^{n+1}}{v^{n+1}} (\exp(v) - 1)^m \; dv$$

$$= \frac{n!}{2\pi i \times m!} \int_{|v|=\epsilon} \frac{1}{v^{n+1}} (\exp(v) - 1)^m \; dv.$$

This is

$$n![v^n] \frac{(\exp(v) - 1)^m}{m!} = \left\{ n \atop m \right\}$$

and we have the claim.
This was math.stackexchange.com problem 1926107.

53 An identity involving two binomial coefficients and a fractional term (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{m} \frac{q}{pk+q} \binom{pk+q}{k} \binom{pm-pk}{m-k} = \binom{mp+q}{m}.$$

Observe that

$$\binom{pk+q}{k} = \frac{pk+q}{k} \binom{pk+q-1}{k-1}$$

so that

$$\binom{pk+q}{k} - p\binom{pk+q-1}{k-1} = \frac{q}{k} \binom{pk+q-1}{k-1} = \frac{q}{pk+q} \binom{pk+q}{k}.$$

This yields two pieces for the sum, call them S_1

$$\sum_{k=0}^{m} \binom{pk+q}{k} \binom{pm-pk}{m-k}$$

and S_2

$$-p \sum_{k=0}^{m} \binom{pk+q-1}{k-1} \binom{pm-pk}{m-k}.$$

For S_1 introduce the integrals

$$\binom{pk+q}{k} = \frac{1}{2\pi i} \int_{|z|=\gamma} \frac{(1+z)^{pk+q}}{z^{k+1}} \; dz$$

and

$$\binom{pm-pk}{m-k} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm-pk}}{w^{m-k+1}} \; dw.$$

The second one controls the range of the sum because the pole at zero vanishes when $k > m$ so we may extend k to infinity, getting for the sum

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm}}{w^{m+1}} \frac{1}{2\pi i} \int_{|z|=\gamma} \frac{(1+z)^q}{z} \sum_{k \geq 0} \frac{w^k}{z^k} \frac{(1+z)^{pk}}{(1+w)^{pk}} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm}}{w^{m+1}} \frac{1}{2\pi i} \int_{|z|=\gamma} \frac{(1+z)^q}{z} \frac{1}{1 - w(1+z)^p/z/(1+w)^p} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm+p}}{w^{m+1}} \frac{1}{2\pi i} \int_{|z|=\gamma} (1+z)^q \frac{1}{z(1+w)^p - w(1+z)^p} \; dz \; dw.$$

Suppose $|\epsilon| < |\gamma|$ which makes $\left| \frac{w(1+z)^p}{z(1+w)^p} \right| < 1$ so that we have convergence of the geometric series and suppose we can prove that $z = w$ is the only pole inside the contour and it is simple. We have

$$((1+w)^p z - w(1+z)^p)' = (1+w)^p - pw(1+z)^{p-1}$$
$$= (1+w)^{p-1}(1+w-wp).$$

We can choose $|\epsilon|$ small enough such that $|1+w-wp| > 0$ so the pole is order one which yields

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm+p}}{w^{m+1}} (1+w)^q \frac{1}{(1+w)^{p-1}} \frac{1}{1+w-pw} \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm+q+1}}{w^{m+1}} \frac{1}{1+w-pw} \; dw.$$

Following exactly the same procedure we obtain for S_2

$$-p\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm+q}}{w^m}\frac{1}{1+w-pw}\,dw.$$

Adding these two pieces now yields

$$\begin{aligned}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm+q}}{w^m}\left(\frac{1+w}{w}-p\right)\frac{1}{1+w-pw}\,dw\\ =\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm+q}}{w^{m+1}}\,dw\\ =\binom{pm+q}{m}.\end{aligned}$$

Remark Mon Jan 25 2016.

An alternate proof which is completely rigorous and does not depend on assumptions about the poles of a bivariate complex function proceeds from the integral

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm}}{w^{m+1}}\sum_{k\ge 0}\frac{w^k}{(1+w)^{pk}}\frac{1}{2\pi i}\int_{|z|=\gamma}\frac{(1+z)^q}{z^{k+1}}(1+z)^{pk}\,dz\,dw$$

Now put

$$u=\frac{z}{(1+z)^p}\quad\text{and introduce}\quad g(u)=z.$$

We then have

$$du=\left(\frac{1}{(1+z)^p}-p\frac{z}{(1+z)^{p+1}}\right)\,dz=\left(\frac{u}{g(u)}-\frac{pu}{1+g(u)}\right)\,dz$$

and

$$dz=\frac{1}{u}\frac{g(u)(1+g(u))}{1+g(u)-pg(u)}\,du.$$

This yields

$$\begin{aligned}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm}}{w^{m+1}}\sum_{k\ge 0}\frac{w^k}{(1+w)^{pk}}\\ \times\frac{1}{2\pi i}\int_{|u|=\gamma}\frac{1}{g(u)u^k}(1+g(u))^q\frac{1}{u}\frac{g(u)(1+g(u))}{1+g(u)-pg(u)}\,du\,dw\end{aligned}$$

or

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm}}{w^{m+1}}\,(1+g(u))^q\frac{1+g(u)}{1+g(u)-pg(u)}\bigg|_{u=w/(1+w)^p}\,dw.$$

Now observe that $g(w/(1+w)^p) = w$ by definition so we get

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm}}{w^{m+1}}(1+w)^q \frac{1+w}{1+w-pw}\; dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm+q+1}}{w^{m+1}} \frac{1}{1+w-pw}\; dw.$$

This is exactly the same as before and the rest of the proof continues unchanged.

This was math.stackexchange.com problem 1620083.

54 Double chain of a total of three integrals ($B_1 B_2$)

Suppose we seek to verify that

$$\sum_{k=q}^{n-1} \frac{q}{k} \binom{2n-2k-2}{n-k-1} \binom{2k-q-1}{k-1} = \binom{2n-q-2}{n-1}.$$

This is the same as

$$\sum_{k=q}^{n} \frac{q}{k} \binom{2n-2k}{n-k} \binom{2k-q-1}{k-1} = \binom{2n-q}{n}.$$

which is equivalent to

$$\sum_{k=q}^{n} \frac{q-k}{k} \binom{2n-2k}{n-k} \binom{2k-q-1}{k-1} + \sum_{k=q}^{n} \binom{2n-2k}{n-k} \binom{2k-q-1}{k-1}$$
$$= \binom{2n-q}{n}.$$

Now

$$\frac{q-k}{k}\binom{2k-q-1}{k-1} = \frac{q-k}{k}\frac{(2k-q-1)!}{(k-1)!(k-q)!}$$
$$= -\frac{(2k-q-1)!}{k!(k-q-1)!} = -\binom{2k-q-1}{k}.$$

It follows that what we have is in fact

$$\sum_{k=q}^{n} \binom{2n-2k}{n-k} \left(\binom{2k-q-1}{k-1} - \binom{2k-q-1}{k} \right) = \binom{2n-q}{n}$$

or alternatively

$$\sum_{k=q}^{n} \binom{2n-2k}{n-k} \left(\binom{2k-q-1}{k-q} - \binom{2k-q-1}{k-q-1} \right) = \binom{2n-q}{n}.$$

There are two pieces here, call them A and B. We use the integral representation

$$\binom{2n-2k}{n-k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-2k}}{z^{n-k+1}} \; dz$$

which is zero when $k > n$ (pole vanishes) so we may extend k to infinity. We also use the integral

$$\binom{2k-q-1}{k-q} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{2k-q-1}}{w^{k-q+1}} \; dw$$

which is zero when $k < q$ so we may extend k back to zero. We obtain for piece A

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \sum_{k\ge 0} \frac{z^k}{(1+z)^{2k}} \frac{(1+w)^{2k}}{w^k} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \frac{1}{1 - z(1+w)^2/w/(1+z)^2} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1}{w(1+z)^2 - z(1+w)^2} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1}{(z-w)(z-1/w)} \; dz \; dw.$$

The derivation for piece B is the same and yields

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1}{(z-w)(z-1/w)} \; dz \; dw.$$

The difference of these two is

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1-w}{(z-w)(z-1/w)} \; dz \; dw.$$

Using partial fractions by residues we get

$$\frac{1-w}{(z-w)(z-1/w)} = \frac{1-w}{w-1/w} \frac{1}{z-w} + \frac{1-w}{1/w-w} \frac{1}{z-1/w}$$

$$= \frac{w(1-w)}{w^2-1}\frac{1}{z-w} + \frac{w(1-w)}{1-w^2}\frac{1}{z-1/w} = -\frac{w}{1+w}\frac{1}{z-w} + \frac{w}{1+w}\frac{1}{z-1/w}$$
$$= \frac{1}{1+w}\frac{1}{1-z/w} - \frac{w^2}{1+w}\frac{1}{1-wz}.$$

At this point we can see that there will be no contribution from the second term but this needs to be verified. We get for the residue in z

$$-\frac{w^2}{1+w}\sum_{p=0}^{n}\binom{2n+2}{p}w^{n-p}$$

There is no pole at zero in the outer integral for a contribution of zero.

Continuing with the first term we get

$$\frac{1}{1+w}\sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{w^{n-p}}$$

which yields

$$\sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q-1}}{(1+w)^{q+2}}\frac{1}{w^{n-p}}\;dw$$
$$= \sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{(1+w)^{q+2}}\frac{1}{w^{n-q-p+1}}\;dw$$
$$= \sum_{p=0}^{n}\binom{2n+2}{p}(-1)^{n-q-p}\binom{n-p+1}{q+1}.$$

This is

$$\sum_{p=0}^{n}\binom{2n+2}{p}(-1)^{n-q-p}\binom{n-p+1}{n-p-q}.$$

The last integral we will be using is

$$\binom{n-p+1}{n-p-q} = \frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(1+v)^{n-p+1}}{v^{n-p-q+1}}\;dv.$$

Observe that this is zero when $p \ge n$ so we may extend p to infinity, getting

$$\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(1+v)^{n+1}}{v^{n-q+1}}\sum_{p\ge 0}\binom{2n+2}{p}(-1)^{n-q-p}\frac{v^p}{(1+v)^p}\;dv$$
$$= (-1)^{n-q}\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(1+v)^{n+1}}{v^{n-q+1}}\left(1-\frac{v}{1+v}\right)^{2n+2}\;dv$$

$$= (-1)^{n-q}\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{1}{v^{n-q+1}}\frac{1}{(1+v)^{n+1}}\;dv$$

$$= (-1)^{n-q}(-1)^{n-q}\binom{n-q+n}{n} = \binom{2n-q}{n}.$$

This is the claim. QED.
This was math.stackexchange.com problem 1708435.

55 Rothe-Hagen identity

The claim we set out to prove is the Rothe-Hagen identity

$$\sum_{k=0}^{n}\frac{x}{x+kz}\binom{x+kz}{k}\frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}.$$

We prove it for x, y, z positive integers and since the LHS and the RHS are in fact polynomials in x, y, z (the fractional terms cancel with the corresponding binomial coefficients e.g. $\frac{x}{x+kz}\binom{x+kz}{k} = \frac{x}{k!}(x+kz-1)^{\underline{k-1}}$ as long as $x+kz \neq 0$ (consult problem statement)) we then have it for arbitrary values (we also get polynomials when $k=0$ or $k=n$.)

Consider the generating function $C(v)$ that satisfies the functional equation again with z a positive integer

$$C(v) = 1 + vC(v)^z.$$

We ask about again with x a positive integer

$$[v^k]C(v)^x = \frac{1}{k}[v^{k-1}]xC(v)^{x-1}C'(v).$$

This is by the Cauchy Coefficient Formula

$$\frac{x}{k\times 2\pi i}\int_{|v|=\epsilon}\frac{1}{v^k}C(v)^{x-1}C'(v)\;dv.$$

Now we put $C(v) = w$ and we have from the functional equation

$$v = \frac{w-1}{w^z}$$

which yields

$$\frac{x}{k\times 2\pi i}\int_{|w-1|=\gamma}\frac{w^{zk}}{(w-1)^k}w^{x-1}\;dw$$

$$= \frac{x}{k\times 2\pi i}\int_{|w-1|=\gamma}\frac{1}{(w-1)^k}\sum_{p=0}^{kz+x-1}\binom{kz+x-1}{p}(w-1)^p\;dw$$

$$= \frac{x}{k}\binom{kz+x-1}{k-1} = \frac{x}{x+kz}\binom{x+kz}{k}.$$

Note that this yields the correct value including for $k=0$.

Now starting from the left of the desired identity we find

$$\sum_{k=0}^{n} [v^k] C_z(v)^x [v^{n-k}] C_z(v)^y = [v^n] C_z(v)^x C_z(v)^y = [v^n] C_z(v)^{x+y}.$$

This is the claim.

The same result may be obtained using Lagrange inversion.

For the LIF computation we put $D(v) = C(v) - 1$ so that we get the functional equation

$$D(v) = v(D(v)+1)^z.$$

Using the notation from Wikipedia on LIF we have $\phi(w) = (w+1)^z$ and $H(v) = (v+1)^x$ and obtain

$$\frac{1}{k}[w^{k-1}](x(w+1)^{x-1}((w+1)^z)^k) = \frac{x}{k}[w^{k-1}](1+w)^{kz+x-1} = \frac{x}{k}\binom{kz+x-1}{k-1}.$$

This matches the first result.

This was math.stackexchange.com problem 3573304.

56 Abel polynomials are of binomial type

We seek to prove that

$$P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y)$$

where

$$P_n(x) = x(x+an)^{n-1}$$

is an Abel polynomial. Introduce $T(z)$ with functional equation

$$T(z) = z\exp(aT(z))$$

Viewing this as an EGF we seek the coefficient

$$n![z^n]\exp(xT(z)) = x(n-1)![z^{n-1}]\exp(xT(z))T'(z).$$

Note that $[z^0]\exp(xT(z)) = 1$. With the Cauchy Coefficient Formula we find for $n \ge 1$

$$\frac{x(n-1)!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\exp(xT(z))T'(z)\;dz.$$

Now we put $T(z) = w$ to get $z = w/\exp(aw)$ and

$$\frac{x(n-1)!}{2\pi i}\int_{|w|=\gamma}\frac{\exp(anw)\exp(xw)}{w^n}\;dw$$

$$=\frac{x(n-1)!}{2\pi i}\int_{|w|=\gamma}\frac{\exp((x+an)w)}{w^n}\;dw$$

$$= x(x+an)^{n-1}.$$

This means that

$$\exp(xT(z)) = 1 + \sum_{n\ge 1} x(x+an)^{n-1}\frac{z^n}{n!}$$

$$= \sum_{n\ge 0} x(x+an)^{n-1}\frac{z^n}{n!} = \sum_{n\ge 0} P_n(x)\frac{z^n}{n!}.$$

By convolution of EGFs we thus have

$$P_n(x+y) = n![z^n]\exp((x+y)T(z)) = n![z^n]\exp(xT(z))\exp(yT(z))$$

$$= n!\sum_{k=0}^{n}[z^k]\exp(xT(z))[z^{n-k}]\exp(yT(z))$$

$$= n!\sum_{k=0}^{n}\frac{P_k(x)}{k!}\frac{P_{n-k}(y)}{(n-k)!} = \sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(y).$$

The CCF can also be done by Lagrange Inversion, which goes as follows. Using the notation from Wikipedia on Lagrange-Buermann we have $\phi(w) = \exp(aw)$ and $H(w) = \exp(xw)$ and we find

$$n![z^n]\exp(xT(z)) = n!\frac{1}{n}[w^{n-1}]x\exp(xw)\exp(anw)$$

$$= (n-1)!x[w^{n-1}]\exp((x+an)w) = x(x+an)^{n-1}.$$

This was math.stackexchange.com problem 3704156.

57 A summation identity with four poles (B_2)

We seek to show that

$$\sum_{m=0}^{n}(-1)^m\binom{2n+2m}{n+m}\binom{n+m}{n-m} = (-1)^n 2^{2n}.$$

The LHS is

$$[z^n](1+z)^n\sum_{m=0}^{n}(-1)^m\binom{2n+2m}{n+m}(1+z)^m z^m.$$

The coefficient extractor enforces the upper limit of the sum and we may continue with

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\sum_{m\ge 0}(-1)^m\binom{2n+2m}{n+m}(1+z)^m z^m \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{(1-w)^{n+1}}$$

$$\times\sum_{m\ge 0}(-1)^m\frac{1}{w^m}\frac{1}{(1-w)^m}(1+z)^m z^m \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{(1-w)^{n+1}}\frac{1}{1+z(1+z)/w/(1-w)} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n}}\frac{1}{(1-w)^{n}}\frac{1}{w(1-w)+z(1+z)} \; dw \; dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n}}\frac{1}{(1-w)^{n}}\frac{1}{(w+z)(w-(1+z))} \; dw \; dz.$$

The contribution from the pole at $w=-z$ is

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{(-1)^n}{z^n}\frac{1}{(1+z)^n}\frac{1}{1+2z} \; dz$$

$$= \frac{(-1)^n}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n+1}}\frac{1}{1+2z} \; dz = (-1)^n[z^{2n}]\frac{1}{1+2z} = (-1)^n(-1)^{2n}2^{2n}$$

$$= \boxed{(-1)^n 2^{2n}.}$$

This is the claim. We will document a choice of γ and ϵ so that $w=0$ and $w=-z$ are the only poles inside the contour (pole at $w=1$ not included, nor the pole at $w=1+z$.)

Now we have for the pole at $w=0$

$$-\frac{1}{(w+z)(w-(1+z))} = \frac{1}{1+2z}\frac{1}{w+z} - \frac{1}{1+2z}\frac{1}{w-(1+z)}$$

$$= \frac{1}{z}\frac{1}{1+2z}\frac{1}{1+w/z} + \frac{1}{1+z}\frac{1}{1+2z}\frac{1}{1-w/(1+z)}.$$

We get from the first piece

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+2}}\frac{1}{1+2z}\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\frac{1}{z^{n-1-q}}\;dz$$

$$=-\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{2n+1-q}}\frac{1}{1+2z}\;dz$$

$$=-\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\sum_{p=0}^{n}\binom{n}{p}(-1)^{2n-q-p}2^{2n-q-p}$$

$$=\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}\sum_{p=0}^{n}\binom{n}{p}(-1)^{n-p}2^{n-p}$$

$$=(-1)^n\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}.$$

The second piece yields

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^{n+1}}\frac{1}{1+2z}\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\frac{1}{(1+z)^{n-1-q}}\;dz$$

$$=-\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^q}{z^{n+1}}\frac{1}{1+2z}\;dz$$

$$=-\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\sum_{p=0}^{q}\binom{q}{p}(-1)^{n-p}2^{n-p}$$

$$=-\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-q}2^{n-q}\sum_{p=0}^{q}\binom{q}{p}(-1)^{q-p}2^{q-p}$$

$$=-(-1)^n\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}.$$

We see that the two pieces from $w=0$ cancel so that the contribution is zero. This almost completes the proof, we only need to choose the contour so that $w=1$ and $w=1+z$ are not included. For the initial geometric series to converge we need $|1+z|\epsilon<|1-w|\gamma$. With ϵ and γ in a neigborhood of zero we have $|1+z|\epsilon\le(1+\epsilon)\epsilon$ and $(1-\gamma)\gamma\le|1-w|\gamma$. The series converges if $(1+\epsilon)\epsilon<(1-\gamma)\gamma$. Therefore a good choice is $\epsilon=1/10$ and $\gamma=1/5$. The contour in γ clearly includes $w=0$ and $w=-z$ and definitely does not include

$w=1$ and $w=1+z$ with leftmost value $9/10$. This concludes the proof.

We are not required to simplify the sum that appears in $w=0$, but we may do so. We get

$$S_n = \sum_{q=0}^{n-1} \binom{q+n-1}{n-1} 2^{n-q} = 2^n [z^{n-1}] \frac{1}{1-z} \frac{1}{(1-z/2)^n}$$
$$= (-1)^{n+1} 2^{2n} \mathrm{Res}_{z=0} \frac{1}{z^n} \frac{1}{z-1} \frac{1}{(z-2)^n}.$$

Residues sum to zero and the residue at infinity is zero by inspection. The residue at $z=1$ contributes -2^{2n}. The residue at $z=2$ requires

$$\frac{1}{(2+(z-2))^n} \frac{1}{1+(z-2)} = \frac{1}{2^n} \frac{1}{(1+(z-2)/2)^n} \frac{1}{1+(z-2)}.$$

and we get the contribution

$$(-1)^{n+1} 2^n \sum_{q=0}^{n-1} \binom{q+n-1}{n-1} (-1)^q 2^{-q} (-1)^{n-1-q} = S_n.$$

This shows that $2S_n - 2^{2n} = 0$ or $S_n = 2^{2n-1}$.

This was math.stackexchange.com problem 3729998.

58 A summation identity over odd indices with a branch cut (B_2)

In trying to evaluate

$$\sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \binom{2n}{2n-k} \binom{2m-2n}{m-k}$$

we require

$$\sum_{k=0}^{m} \binom{2n}{2n-k} \binom{2m-2n}{m-k} \quad \text{and} \quad \sum_{k=0}^{m} \binom{2n}{2n-k} (-1)^k \binom{2m-2n}{m-k}.$$

For the first one we find

$$\sum_{k=0}^{m} \binom{2n}{k} \binom{2m-2n}{m-k} = [z^m](1+z)^{2m-2n} \sum_{k=0}^{m} \binom{2n}{k} z^k.$$

Here the coefficient extractor enforces the range and we get

$$[z^m](1+z)^{2m-2n}\sum_{k\ge 0}\binom{2n}{k}z^k = [z^m](1+z)^{2m-2n}(1+z)^{2n}$$

$$= [z^m](1+z)^{2m} = \binom{2m}{m}.$$

This also follows from Chu-Vandermonde.

Continuing with the second piece we obtain

$$\sum_{k=0}^{m}\binom{2n}{k}(-1)^k\binom{2m-2n}{m-k} = (-1)^m\sum_{k=0}^{m}\binom{2n}{m-k}(-1)^k\binom{2m-2n}{k}$$

$$= (-1)^m\sum_{k=0}^{m}(-1)^k\binom{2m-2n}{k}[z^{2n+k-m}]\frac{1}{(1-z)^{m-k+1}}.$$

Now when $k > m$ we have $[z^{2n+k-m}](1-z)^{k-m-1} = 0$ so the coefficient extractor again enforces the range and we find

$$\frac{(-1)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n-m+1}}\frac{1}{(1-z)^{m+1}}\sum_{k\ge 0}(-1)^k\binom{2m-2n}{k}\frac{(1-z)^k}{z^k}\;dz$$

$$= \frac{(-1)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n-m+1}}\frac{1}{(1-z)^{m+1}}\left(1-\frac{1-z}{z}\right)^{2m-2n}\;dz$$

$$= \frac{(-1)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{m+1}}\frac{(1-2z)^{2m-2n}}{(1-z)^{m+1}}\;dz$$

$$= \frac{(-1)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{m+1}}\frac{(1-2z)^{2(m+1)}}{(1-z)^{m+1}}\frac{1}{(1-2z)^{2n+2}}\;dz.$$

Now put $z(1-z)/(1-2z)^2 = w$ so that

$$z = \frac{1}{2}\pm\frac{1}{2}\frac{1}{\sqrt{1+4w}}.$$

We have that $w = z + 3z^2 + 8z^3 + \cdots$ so $z = 0$ should be mapped to $w = 0$ and in fact we work with

$$z = \frac{1}{2}-\frac{1}{2}\frac{1}{\sqrt{1+4w}}.$$

We also see from the series expansion that the small circle around the origin $|z| = \epsilon$ is mapped to a contour that encircles $w = 0$ once and may in turn be deformed to a small circle $|w| = \gamma$. We choose the branch cut on $(-\infty, -1/4]$ so that we get analyticity in a neighborhood of the origin. We also have

$$dz = \frac{1}{(1+4w)^{3/2}} \; dw.$$

At last making the substitution we obtain

$$\frac{(-1)^m}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{1}{(1/\sqrt{1+4w})^{2n+2}} \frac{1}{(1+4w)^{3/2}} \; dw$$

$$= \frac{(-1)^m}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} (1+4w)^{n-1/2} \; dw = (-1)^m 4^m \binom{n-1/2}{m}.$$

Collecting the two pieces we find

$$\boxed{\frac{1}{2}\binom{2m}{m} + (-1)^{m+1} 2^{2m-1} \binom{n-1/2}{m}.}$$

This was math.stackexchange.com problem 3782050.

59 A stirling number identity

We seek to evaluate (note that this is zero by inspection when $k > n+m$):

$$\sum_{j=0}^{n} (-1)^{n+j} \left[n \atop j \right] \left\{ m+j \atop k \right\}$$

where $k \le n$. It is claimed that it is zero for $k < n$ and n^m for $k = n$. Using standard EGFs this becomes

$$n![z^n] \sum_{j=0}^{n} (-1)^{n+j} \frac{1}{j!} \left(\log \frac{1}{1-z} \right)^j (m+j)! [w^{m+j}] \frac{(\exp(w)-1)^k}{k!}$$

$$= (-1)^n n! m! [z^n] \sum_{j=0}^{n} (-1)^j \binom{m+j}{j} \left(\log \frac{1}{1-z} \right)^j$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+j+1}} \frac{(\exp(w)-1)^k}{k!} \; dw$$

$$= (-1)^n n! m! [z^n] \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!}$$

$$\times \sum_{j=0}^{n} (-1)^j \binom{m+j}{j} \left(\log \frac{1}{1-z} \right)^j \frac{1}{w^j} \; dw.$$

Now $\left(\log \frac{1}{1-z} \right)^j = z^j + \cdots$ so the coefficient extractor $[z^n]$ enforces the upper limit of the sum:

$$(-1)^n n!m![z^n]\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{m+1}}\frac{(\exp(w)-1)^k}{k!}$$

$$\times\sum_{j\ge 0}(-1)^j\binom{m+j}{j}\left(\log\frac{1}{1-z}\right)^j\frac{1}{w^j}\;dw$$

$$=(-1)^n n!m!\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{m+1}}\frac{(\exp(w)-1)^k}{k!}$$

$$\times\sum_{j\ge 0}(-1)^j\binom{m+j}{j}\left(\log\frac{1}{1-z}\right)^j\frac{1}{w^j}\;dw\;dz$$

$$=(-1)^n n!m!\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}$$

$$\times\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{m+1}}\frac{(\exp(w)-1)^k}{k!}\frac{1}{(1+\frac{1}{w}\log\frac{1}{1-z})^{m+1}}\;dw\;dz$$

$$=(-1)^n n!m!\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}$$

$$\times\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(\exp(w)-1)^k}{k!}\frac{1}{(w+\log\frac{1}{1-z})^{m+1}}\;dw\;dz.$$

Now observe that for the geometric series in j to converge we must have $|\log\frac{1}{1-z}|<|w|$. Note that with $\log\frac{1}{1-z}=z+\cdots$ the image of $|z|=\epsilon$ makes one turn around the origin, a circle of radius ϵ plus additional lower order fluctuations. We therefore choose ϵ to shrink this pseudo-circle to be entirely contained in $|w|=\gamma$. With this choice the pole at $-\log\frac{1}{1-z}$ is inside the contour in w. We thus require

$$\frac{1}{k!\times m!}\left(\sum_{q=0}^{k}\binom{k}{q}(-1)^{k-q}\exp(qw)\right)^{(m)}=\frac{1}{k!\times m!}\sum_{q=0}^{k}\binom{k}{q}(-1)^{k-q}q^m\exp(qw).$$

Evaluating the integral in w we find

$$(-1)^n\frac{n!}{k!}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\sum_{q=0}^{k}\binom{k}{q}(-1)^{k-q}q^m(1-z)^q\;dz$$

which is

$$\boxed{\frac{n!}{k!}\sum_{q=0}^{k}\binom{k}{q}\binom{q}{n}(-1)^{k-q}q^m.}$$

Now when $k < n$ we have $\binom{q}{n} = 0$ so the entire sum vanishes as claimed. We get just one term when $k = n$ namely

$$\frac{n!}{n!}\binom{n}{n}\binom{n}{n}(-1)^{n-n}n^m = n^m$$

also as claimed. This concludes the argument.

This was math.stackexchange.com problem 3852633.

60 A Catalan-Central Binomial Coefficient Convolution

We seek to show that with

$$Q(z) = \frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n$$

we have

$$[z^k]Q(z) = \binom{n+2k}{k}.$$

Now with the branch cut on $[1/4, \infty)$ for $\sqrt{1-4z}$ we have analyticity of $Q(z)$ in a neighborhood of the origin (note that the exponentiated term does not in fact have a pole at $z = 0$) and the Cauchy Coefficient Formula applies. We obtain

$$[z^k]Q(z) = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{k+1}}\frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n \; dz.$$

We put $\sqrt{1-4z} = w$ so that $\frac{1}{\sqrt{1-4z}}\; dz = -\frac{1}{2}\; dw$ and $z = (1-w^2)/4$. With $w = 1 - 2z - \cdots$ we get as the image of $|z| = \varepsilon$ a contour that winds around $w = 1$ counterclockwise once and may be deformed to a circle, so that we obtain

$$[z^k]Q(z) = -\frac{1}{2}\frac{1}{2\pi i}\int_{|w-1|=\gamma} \frac{4^{k+1}}{(1-w^2)^{k+1}}(1-w)^n\frac{1}{2^n}\frac{4^n}{(1-w^2)^n}\; dw$$

$$= \frac{(-1)^k \times 2^{n+2k+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{(w-1)^n}{(w^2-1)^{n+k+1}}\; dw$$

$$= \frac{(-1)^k \times 2^{n+2k+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{k+1}}\frac{1}{(w+1)^{n+k+1}}\; dw$$

$$= \frac{(-1)^k \times 2^k}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{k+1}}\frac{1}{(1+(w-1)/2)^{n+k+1}}\; dw$$

Apply the Cauchy Residue Theorem to get

$$(-1)^k \times 2^k \times (-1)^k \frac{1}{2^k} \binom{n+2k}{n+k} = \boxed{\binom{n+2k}{k}}$$

as claimed.
This was math.stackexchange.com problem 4025969.

61 Post Scriptum additions

61.1 A trigonometric sum

Suppose we seek to evaluate

$$S = \sum_{k=1}^{m-1} \sin^{2q}(k\pi/m) = \sum_{k=0}^{m-1} \sin^{2q}(2\pi k/2/m).$$

Introducing $\zeta_k = \exp(2\pi i k/2/m)$ (root of unity) we get

$$S = \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}.$$

We also have

$$\sum_{k=m}^{2m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k \exp(2\pi i m/2/m) - 1/\zeta_k/\exp(2\pi i m/2/m))^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (-\zeta_k + 1/\zeta_k)^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q} = S.$$

We conclude that

$$S = \frac{1}{2} \sum_{k=0}^{2m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}.$$

Introducing

$$f(z) = \frac{(-1)^q}{2^{2q+1}} \left(z - \frac{1}{z}\right)^{2q} \frac{2mz^{2m-1}}{z^{2m}-1}$$

$$= \frac{(-1)^q}{2^{2q+1}} \frac{(z^2-1)^{2q}}{z^{2q}} \frac{2mz^{2m-1}}{z^{2m}-1}$$

we then have

$$S = \sum_{k=0}^{2m-1} \mathrm{Res}_{z=\zeta_k} f(z).$$

Observe that the term $(z^2-1)^{2q}$ cancels the poles at ± 1 produced by $z^{2m}-1$ which however is perfectly acceptable as they correspond to $\zeta_0 = 1$ and $\zeta_m = -1$ where $\zeta_k - 1/\zeta_k$ is zero as well.

Residues sum to zero so we obtain

$$S + \mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=\infty} f(z) = 0.$$

Now for the residue at zero we see that when $2q-1 < 2m-1$ or $q < m$ the residue is zero. Otherwise we get

$$\frac{(-1)^q}{2^{2q+1}} [z^{2q-2m}](z^2-1)^{2q} \frac{2m}{z^{2m}-1}$$

$$= \frac{(-1)^q}{2^{2q+1}} [z^{2q}](z^2-1)^{2q} \frac{2mz^{2m}}{z^{2m}-1}$$

$$= -2m \frac{(-1)^q}{2^{2q+1}} \sum_{p=0}^{q} \binom{2q}{p} (-1)^{2q-p} [z^{2q-2p}] \frac{z^{2m}}{1-z^{2m}}.$$

We must have $p = q - lm$ where $l \geq 1$. This yields

$$-2m \frac{1}{2^{2q+1}} \sum_{l=1}^{\lfloor q/m \rfloor} \binom{2q}{q-lm} (-1)^{lm}.$$

This is correct even when $q < m$.

Continuing with the residue at infinity we find

$$\mathrm{Res}_{z=\infty} f(z) = -\mathrm{Res}_{z=0} \frac{1}{z^2} f(1/z)$$

$$= -\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{(-1)^q}{2^{2q+1}} \frac{(1/z^2-1)^{2q}}{1/z^{2q}} \frac{2m/z^{2m-1}}{1/z^{2m}-1}$$

$$= -\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{(-1)^q}{2^{2q+1}} \frac{(1-z^2)^{2q}}{z^{2q}} \frac{2mz}{1-z^{2m}}$$

$$= -\mathrm{Res}_{z=0} \frac{(-1)^q}{2^{2q+1}} \frac{(z^2-1)^{2q}}{z^{2q+1}} \frac{2m}{1-z^{2m}}.$$

This is the same as the first residue at zero except now l starts at $l = 0$ and we obtain

$$-2m\frac{1}{2^{2q+1}}\sum_{l=0}^{\lfloor q/m\rfloor}\binom{2q}{q-lm}(-1)^{lm}.$$

Joining the two pieces we finally have

$$m\frac{1}{2^{2q}}\binom{2q}{q}+m\frac{1}{2^{2q-1}}\sum_{l=1}^{\lfloor q/m\rfloor}\binom{2q}{q-lm}(-1)^{lm}.$$

This was math.stackexchange.com problem 2051454.

61.2 A class of polynomials similar to Fibonacci and Lucas Polynomials (B_1)

Suppose we seek to collect information concerning

$$\sum_{j=-\lfloor n/p\rfloor}^{\lfloor n/p\rfloor}(-1)^j\binom{2n}{n-pj}.$$

We will construct a generating function in n with $p\ge 1$ fixed. We introduce

$$\binom{2n}{n-pj}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-pj+1}}(1+z)^{2n}\;dz.$$

Now as we examine this integral we see immediately that it vanishes if $j>\lfloor n/p\rfloor$ (pole at zero disappears). Moreover when $j<-\lfloor n/p\rfloor$ we have that $[z^{n-pj}](1+z)^{2n}=0$ so this vanishes as well. Hence with this integral in place we may let j range from $-n$ to infinity and get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2n}\sum_{j=-n}^{\infty}(-1)^j z^{pj}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2n}\sum_{j=0}^{\infty}(-1)^{j-n}z^{pj-pn}\;dz$$

$$=\frac{(-1)^n}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{(p+1)n+1}}(1+z)^{2n}\frac{1}{1+z^p}\;dz.$$

We get zero for the residue at infinity, as can be seen from

$$\mathrm{Res}_{z=\infty}\frac{1}{z^{(p+1)n+1}}(1+z)^{2n}\frac{1}{1+z^p}$$

$$=-\mathrm{Res}_{z=0}\frac{1}{z^2}z^{(p+1)n+1}\frac{(1+z)^{2n}}{z^{2n}}\frac{z^p}{1+z^p}$$

$$=-\mathrm{Res}_{z=0}z^{(p-1)(n+1)}(1+z)^{2n}\frac{1}{1+z^p}=0.$$

With residues adding to zero and introducing $\rho_k = \exp(\pi i/p + 2\pi i k/p)$ we thus obtain

$$-\sum_{k=0}^{p-1} (-1)^n \frac{1}{\rho_k^{(p+1)n+1}} (1+\rho_k)^{2n} \frac{1}{p\rho_k^{p-1}} = \frac{1}{p} \sum_{k=0}^{p-1} (-1)^n \frac{1}{\rho_k^{pn+n}} (1+\rho_k)^{2n}$$

$$= \frac{1}{p} \sum_{k=0}^{p-1} \left(\frac{1}{\rho_k} + 2 + \rho_k \right)^n .$$

At this point we can compute a generating function using the fact that

$$\sum_{q \ge 0} \rho^q z^q = \frac{1}{1-\rho z} = -\frac{1}{\rho} \frac{1}{z - 1/\rho}$$

and we obtain as a first attempt

$$G_p(z) = \frac{1}{p} \sum_{k=0}^{p-1} \frac{1}{1 - 2(1 + \cos(\pi/p + 2\pi k/p))z}.$$

Observe that this correctly represents the cancelation of the pole at $z = -1$ when p is odd, contributing zero when $n \ge 1$ and $1/p$ otherwise. Furthermore note that with $\rho_k = \exp((2k+1)\pi i/p)$ we have

$$\frac{1}{\rho_{p-1-k}} = \exp(-(2(p-1-k)+1)\pi i/p) = \exp((2(k+1-p)-1)\pi i/p)$$

$$= \exp((2(k+1)-1)\pi i/p - 2\pi i) = \exp((2k+1)\pi i/p) = \rho_k$$

so the poles come in pairs with no pole at -1 when p is odd. Therefore the set of poles generated by this sum corresponds to the first $(p-1)/2$ poles when p is odd and the first $p/2$ when p is even. Joining these two we get the degree of the denominator once the sum is computed being $\lfloor p/2 \rfloor$.

This first formula enables us to compute a few of these, like for $p = 8$ we get (no complex number algebra required, basic trigonometry only)

$$G_8(z) = \frac{1 - 6z + 10z^2 - 4z^3}{1 - 8z + 20z^2 - 16z^3 + 2z^4}.$$

Looking up the coefficients we find for the denominator OEIS A034807 and for the numerator OEIS A011973 which point us to three types of polynomials, Fibonacci polynomials, Dickson polynomials and Lucas polynomials. With these data we are able to state a *conjecture* for the closed form of the generating function, which is

$$G_p(z) = \left(\sum_{q=0}^{\lfloor p/2 \rfloor} \frac{p}{p-q} \binom{p-q}{q} (-1)^q z^q\right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q} (-1)^q z^q.$$

To verify this we must show that the poles are at

$$\left(\frac{1}{\rho_k} + 2 + \rho_k\right)^{-1} \quad \text{with residue} \quad -\frac{2}{p}\left(\frac{1}{\rho_k} + 2 + \rho_k\right)^{-1}$$

where the factor two appears because the poles have been paired.

We therefore require the generating functions of the polynomials that appear in $G_p(z)$. Call the numerator $A_p(z)$ and the denominator $B_p(z)$. We first compute the auxiliary generating function

$$Q_1(t,z) = \sum_{p\ge 0} t^p \sum_{q=0}^{\lfloor p/2 \rfloor} \binom{p-q}{q} (-1)^q z^q = \sum_{q\ge 0} (-1)^q z^q \sum_{p\ge 2q} \binom{p-q}{q} t^p$$

$$= \sum_{q\ge 0} (-1)^q z^q t^{2q} \sum_{p\ge 0} \binom{p+q}{q} t^p = \sum_{q\ge 0} (-1)^q z^q t^{2q} \frac{1}{(1-t)^{q+1}}$$

$$= \frac{1}{1-t} \frac{1}{1+zt^2/(1-t)} = \frac{1}{1-t+zt^2}.$$

We then have $A(t,z) = tQ_1(t,z)$. With $p/(p-q) = 1 + q/(p-q)$ we get two pieces for $B(t,z)$, the first is $Q_1(t,z)$ and the second is

$$Q_2(t,z) = \sum_{p\ge 0} t^p \sum_{q=1}^{\lfloor p/2 \rfloor} \binom{p-1-q}{q-1} (-1)^q z^q = \sum_{q\ge 1} (-1)^q z^q \sum_{p\ge 2q} \binom{p-1-q}{q-1} t^p$$

$$= \sum_{q\ge 1} (-1)^q z^q t^{2q} \sum_{p\ge 0} \binom{p+q-1}{q-1} t^p = \sum_{q\ge 1} (-1)^q z^q t^{2q} \frac{1}{(1-t)^q}$$

$$= -\frac{zt^2/(1-t)}{1+zt^2/(1-t)} = -\frac{zt^2}{1-t+zt^2}$$

and hence we have $B(t,z) = Q_1(t,z) + Q_2(t,z)$. This yields the closed form

$$G_p(z) = \frac{[t^p]\frac{t}{1-t+zt^2}}{[t^p]\frac{1-zt^2}{1-t+zt^2}}.$$

Now introducing (we meet a shifted generating function of the Catalan numbers)

$$\alpha(z) = \frac{1+\sqrt{1-4z}}{2} \quad \text{and} \quad \beta(z) = \frac{1-\sqrt{1-4z}}{2}$$

we have a relationship that is analogous to that between Fibonacci and Lucas polynomials, namely,

$$A_p(z) = \frac{1}{\alpha(z)-\beta(z)}(\alpha(z)^p - \beta(z)^p) \quad \text{and} \quad B_p(z) = \alpha(z)^p + \beta(z)^p.$$

We now verify that $B_p(z) = 0$ for z a value from the claimed poles. Using $1/(1/\rho_k + 2 + \rho_k) = \rho_k/(1+\rho_k)^2$ ($\rho_k = -1$ is not included here) we find

$$\alpha(z) = \frac{1+\sqrt{1-4\rho_k/(1+\rho_k)^2}}{2} = \frac{1+(1-\rho_k)/(1+\rho_k)}{2} = \frac{1}{1+\rho_k}$$

and similarly

$$\beta(z) = \frac{\rho_k}{1+\rho_k}.$$

Raising to the power p we find

$$\alpha(z)^p + \beta(z)^p = \frac{1^p + \rho_k^p}{(1+\rho_k)^p} = \frac{1-1}{(1+\rho_k)^p} = 0.$$

We have located $\lfloor p/2 \rfloor$ distinct zeros here which means given the degree of $B_p(z)$ the poles are all simple. This means we may evaluate the residue by setting $z = \rho_k/(1+\rho_k)^2$ in (differentiate the denominator)

$$\frac{1}{p}\left(\sum_{q=0}^{\lfloor p/2 \rfloor} \frac{1}{p-q}\binom{p-q}{q}(-1)^q q z^{q-1}\right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q}(-1)^q z^q$$

which is

$$\frac{z}{p}\left(\sum_{q=1}^{\lfloor p/2 \rfloor} \binom{p-1-q}{q-1}(-1)^q z^q\right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q}(-1)^q z^q$$

The numerator is $A_p(z)$ and we get

$$\frac{1+\rho_k}{1-\rho_k}\frac{2}{(1+\rho_k)^p} = \frac{2}{(1-\rho_k)(1+\rho_k)^{p-1}}.$$

The denominator is $[t^p]Q_2(t,z)$ which is

$$[t^p]\frac{-zt^2}{1-t+zt^2} = [t^p]\frac{1-zt^2}{1-t+zt^2} - [t^p]\frac{1}{1-t+zt^2}$$

$$= [t^p]\frac{1-zt^2}{1-t+zt^2} - [t^{p+1}]\frac{t}{1-t+zt^2} = B_p(z) - A_{p+1}(z) = -A_{p+1}(z).$$

We get

$$-\frac{1+\rho_k}{1-\rho_k}\frac{1^{p+1}-\rho_k^{p+1}}{(1+\rho_k)^{p+1}} = -\frac{(1+\rho_k)^2}{(1-\rho_k)(1+\rho_k)^{p+1}} = -\frac{1}{(1-\rho_k)(1+\rho_k)^{p-1}}.$$

Joining numerator and denominator and multiplying by z/p finally produces

$$\frac{1}{p}\left(\frac{1}{\rho_k}+2+\rho_k\right)^{-1}\frac{2/(1-\rho_k)/(1+\rho_k)^{p-1}}{-1/(1-\rho_k)/(1+\rho_k)^{p-1}} = -\frac{2}{p}\left(\frac{1}{\rho_k}+2+\rho_k\right)^{-1}$$

as claimed. We have proved that the formula from the Egorychev method matches the conjectured form in terms of a certain class of polynomials that are related to Fibonacci and Lucas polynomials as well as Catalan numbers.

This was math.stackexchange.com problem 2237745.

61.3 Partial row sums of Pascal's triangle (B_1)

Here we seek to prove that

$$\sum_{k=0}^{n}\binom{2k+1}{k}\binom{m-(2k+1)}{n-k} = \sum_{k=0}^{n}\binom{m+1}{k}.$$

This is

$$[z^n]\sum_{k=0}^{n}\binom{2k+1}{k}z^k(1+z)^{m-(2k+1)}$$
$$= [z^n](1+z)^{m-1}\sum_{k=0}^{n}\binom{2k+1}{k}z^k(1+z)^{-2k}.$$

Here $[z^n]$ enforces the range of the sum and we find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m-1}}{z^{n+1}}\sum_{k\ge 0}\binom{2k+1}{k}z^k(1+z)^{-2k}\;dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m-1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w}\sum_{k\ge 0}\frac{(1+w)^{2k}}{w^k}z^k(1+z)^{-2k}\;dw\;dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m-1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w}\frac{1}{1-z(1+w)^2/w/(1+z)^2}\;dw\;dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m+1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w(1+z)^2-z(1+w)^2}\;dw\;dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+1}}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1+w}{(1-wz)(w-z)} \; dw \; dz.$$

There is no pole at $w = 0$ here. Note however that for the geometric series to converge we must have $|z(1+w)^2| < |w(1+z)^2|$. We can achieve this by taking $\gamma = 2\epsilon$ so that

$$|z(1+w)^2| \le \epsilon(1+2\epsilon)^2 = 4\epsilon^3 + 4\epsilon^2 + \epsilon\Big|_{\epsilon=1/20} = \frac{242}{4000}$$

and

$$|w(1+z)^2| \ge 2\epsilon(1-\epsilon)^2 = 2\epsilon^3 - 4\epsilon^2 + 2\epsilon\Big|_{\epsilon=1/20} = \frac{361}{4000}.$$

With these values the pole at $w = z$ is inside the contour and we get as the residue

$$\frac{1+z}{1-z^2} = \frac{1}{1-z}.$$

This yields on substitution into the outer integral

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+1}}{z^{n+1}} \frac{1}{1-z} \; dz = [z^n] \frac{(1+z)^{m+1}}{1-z}$$

$$= \sum_{k=0}^{n} [z^k](1+z)^{m+1} [z^{n-k}] \frac{1}{1-z} = \sum_{k=0}^{n} \binom{m+1}{k}.$$

This is the claim.

Remark. For the pole at $w = 1/z$ to be inside the contour we would need $1/\epsilon < 2\epsilon$ or $1 < 2\epsilon^2$ which does not hold here so this pole does not contribute.

This was math.stackexchange.com problem 3640984.

61.4 The Tree function and Eulerian numbers of the second order

We seek to show that the following identity holds:

$$2^{n+1} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \frac{1}{2^k} = n! [x^n] \frac{1}{1 + W(-\exp((x-1)/2)/2)}.$$

We will be using data from Wikipedia on Lambert W and work with the combinatorial branch which is $W_0(z)$.

Recall that

$$W'(z) \frac{z}{W(z)} = \frac{1}{1+W(z)}.$$

We obtain

$$[z^m]\frac{1}{1+W(z)} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^m}\frac{1}{W(z)} W'(z)\; dz.$$

Putting $W(z) = v$ we find

$$\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v^m \exp(mv)}\frac{1}{v}\; dv = \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v^{m+1}} \exp(-mv)\; dv = \frac{(-1)^m m^m}{m!}.$$

so that

$$\frac{1}{1+W(z)} = \sum_{m\ge 0} (-1)^m m^m \frac{z^m}{m!}.$$

We get for the original RHS

$$n![x^n]\sum_{m\ge 0}\frac{m^m}{m!}\exp(m(x-1)/2)\frac{1}{2^m}$$

$$= n![x^n]\sum_{m\ge 0}\frac{m^m}{m!}\frac{\exp(-m/2)}{2^m}\exp(mx/2)$$

$$= \sum_{m\ge 0}\frac{m^{m+n}}{m!}\frac{\exp(-m/2)}{2^{m+n}}.$$

First part. Introduce the tree function $T(z)$ from combinatorics where $T(z) = z\exp T(z)$ and $T(z) = -W_0(-z)$. Note that we have by Cayley's theorem that $T(z) = \sum_{m\ge 1} m^{m-1}\frac{z^m}{m!}$. We claim that with $n \ge 1$

$$Q_n(z) = \sum_{m\ge 0} m^{m+n}\frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}}\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k.$$

This means the RHS is $\frac{1}{2^n}Q_n(\exp(-1/2)/2)$. To verify this last identity note that $Q_{n+1}(z) = z\frac{d}{dz}Q_n(z)$ so we may prove it by induction.

We get for the RHS of the series identity on differentiating and multiplying by z

$$\frac{(2n+1)zT'(z)}{(1-T(z))^{2n+2}}\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k + \frac{z}{(1-T(z))^{2n+1}}\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle kT(z)^{k-1}T'(z)$$

Extracting the term $zT'(z)/(1-T(z))^{2n+2}$ in front leaves us with

$$(2n+1)\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k + (1-T(z))\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle kT(z)^{k-1}$$

$$= (2n+1)\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k + \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n \atop k+1} \right\rangle\!\!\right\rangle (k+1)T(z)^k - \sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle kT(z)^k$$

$$= \sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle (2n+2-(k+1))T(z)^k + \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n \atop k+1} \right\rangle\!\!\right\rangle (k+1)T(z)^k.$$

We may include $k = 0$ in the first sum and $k = n$ in the second. Now the Eulerian number recurrence (second order) according to OEIS A349556 is

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle k + \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle (2n-k)$$

We have shown that

$$Q_{n+1}(z) = \frac{zT'(z)}{(1-T(z))^{2n+2}} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n+1 \atop k+1} \right\rangle\!\!\right\rangle T(z)^k$$

$$= \frac{zT'(z)}{T(z)(1-T(z))^{2n+2}} \sum_{k=1}^{n+1} \left\langle\!\!\left\langle {n+1 \atop k} \right\rangle\!\!\right\rangle T(z)^k.$$

Now we just have to verify that

$$\frac{zT'(z)}{T(z)(1-T(z))^{2n+2}} = \frac{1}{(1-T(z))^{2n+3}} \quad \text{or} \quad zT'(z)(1-T(z)) = T(z).$$

The functional equation tells us that $T'(z) = \exp T(z) + z \exp T(z) T'(z)$ so that $T'(z)(1-T(z)) = \exp T(z) = T(z)/z$ which is just what we need. It remains to verify the base case so the induction starts properly. We seek

$$Q_1(z) = \sum_{m\ge 0} m^{m+1} \frac{z^m}{m!} = \frac{T(z)}{(1-T(z))^3}.$$

We verify this by coefficient extraction. We get

$$m![z^m]Q_1(z) = \frac{m!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{m+1}} \frac{T(z)}{(1-T(z))^3} \; dz.$$

With $T(z) = z + \cdots$ this integral will produce the correct value zero for $m = 0$. For $m \ge 1$, we put $T(z) = w$ so that $z = w \exp(-w)$ and $dz = \exp(-w)(1-w)\; dw$ and obtain

$$\frac{m!}{2\pi i} \int_{|w|=\gamma} \frac{\exp((m+1)w)}{w^{m+1}} \frac{w}{(1-w)^3} \exp(-w)(1-w)\; dw$$

$$= \frac{m!}{2\pi i} \int_{|w|=\gamma} \frac{\exp(mw)}{w^m} \frac{1}{(1-w)^2} \; dw.$$

This is

$$m! \sum_{q=0}^{m-1} \frac{m^q}{q!}(m-q) = m! \sum_{q=0}^{m-1} \frac{m^{q+1}}{q!} - m! \sum_{q=1}^{m-1} \frac{m^q}{(q-1)!}$$

$$= m! \sum_{q=0}^{m-1} \frac{m^{q+1}}{q!} - m! \sum_{q=0}^{m-2} \frac{m^{q+1}}{q!} = m! \frac{m^m}{(m-1)!} = m^{m+1}$$

as desired.

Sequel. Note that in the identity for $Q_n(z)$ we have by the definition of the Eulerian numbers that $\left\langle\!\left\langle {n \atop 0} \right\rangle\!\right\rangle$ is zero when $n \ge 1$. Therefore we may extend k to include zero (with $n \ge 1$ for the moment) which yields

$$\boxed{Q_n(z) = \sum_{m\ge 0} m^{m+n} \frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k.}$$

Now observe that this will produce $Q_0(z) = \sum_{m\ge 0} m^m \frac{z^m}{m!} = \frac{1}{1-T(z)}$ due to $\left\langle\!\left\langle {0 \atop 0} \right\rangle\!\right\rangle = 1$ which is in fact correct because unlike $Q_n(z)$ with $n \ge 1$, $Q_0(z)$ has a constant term, which is one (this is because $m^{m+n} = 0$ for $m = 0$ and $n \ge 1$ and $m^{m+n} = 1$ for $m = 0$ and $n = 0$). Therefore

$$Q_0(z) = 1 + zT'(z) = 1 + \frac{T(z)}{1-T(z)} = \frac{1}{1-T(z)}$$

as obtained from the boxed version of the main identity, which is seen to hold for all $n \ge 0$.

Conclusion. We are now ready to answer the original question. We have shown that the RHS is $\frac{1}{2^n} Q_n(\exp(-1/2)/2)$. By our formula for $Q_n(z)$ in terms of the tree function we obtain with $T(\exp(-1/2)/2) = \frac{1}{2}$ at last the closed form

$$\frac{1}{2^n} \frac{1}{(1-1/2)^{2n+1}} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \frac{1}{2^k} = 2^{n+1} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \frac{1}{2^k}$$

which is the LHS and hence the claim.

This was math.stackexchange.com problem 4040942.

61.5 A Stirling set number generating function and Eulerian numbers of the second order

Supposing that $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind giving the count of partitions of a set of n distinguishable objects into k non-empty subsets we seek to show that

$$\left\{ {n+r \atop n} \right\}$$

is a polynomial of degree $2r$ in n. We start with the following claim for $r \ge 0$:

$$\boxed{Q_r(z) = \sum_{n\ge 0} {n+r \brace n} z^n = \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle z^k.}$$

We will prove this by induction. Note that depending on whether ball $n+r+1$ joins an existing set or becomes a singleton we have

$${n+r+1 \brace n} = n{n+r \brace n} + {n+r \brace n-1}.$$

Multiply by z^n and sum over $n \ge 0$ to get

$$Q_{r+1}(z) = zQ_r'(z) + \sum_{n\ge 1} {n+r \brace n-1} z^n = zQ_r'(z) + z\sum_{n\ge 0} {n+r+1 \brace n} z^n$$

$$= zQ_r'(z) + zQ_{r+1}(z).$$

This means we have

$$Q_{r+1}(z) = \frac{z}{1-z} Q_r'(z).$$

Now to prove the claim it certainly holds for $r = 0$ by inspection. It also holds for $r = 1$ since

$$\sum_{n\ge 1} \binom{n+1}{2} z^n = z\sum_{n\ge 1} \binom{n+1}{2} z^{n-1} = z\sum_{n\ge 0} \binom{n+2}{2} z^n = \frac{z}{(1-z)^3}.$$

For the induction step supposing it holds for $r \ge 1$ we differentiate and multiply by $z/(1-z)$ to get for $Q_{r+1}(z)$

$$\frac{z}{1-z} \frac{2r+1}{(1-z)^{2r+2}} \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle z^k + \frac{z}{1-z} \frac{1}{(1-z)^{2r+1}} \sum_{k=1}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^{k-1}.$$

Factoring out $1/(1-z)^{2r+3}$ for the moment we are left with

$$(2r+1)\sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle z^{k+1} + (1-z)\sum_{k=1}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^k$$

$$= (2r+1)\sum_{k=1}^{r+1} \left\langle\!\!\left\langle r \atop k-1 \right\rangle\!\!\right\rangle z^k + \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^k - \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^{k+1}$$

$$= (2r+1)\sum_{k=1}^{r+1} \left\langle\!\!\left\langle r \atop k-1 \right\rangle\!\!\right\rangle z^k + \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^k - \sum_{k=1}^{r+1} \left\langle\!\!\left\langle r \atop k-1 \right\rangle\!\!\right\rangle (k-1)z^k$$

Now with $r \geq 1$ we may extend the first and the third sum to include $k = 0$ and the second to include $k = r + 1$ to obtain

$$\sum_{k=0}^{r+1} \left[(2r + 2 - k) \left\langle\!\!\left\langle {r \atop k-1} \right\rangle\!\!\right\rangle + k \left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle \right] z^k.$$

The Eulerian number recurrence (second order) according to OEIS A349556 is

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = k \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n - k) \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle$$

so this is with the factor in front

$$\frac{1}{(1-z)^{2r+3}} \sum_{k=0}^{r+1} \left\langle\!\!\left\langle {r+1 \atop k} \right\rangle\!\!\right\rangle z^k$$

and the induction goes through.

Now to see that $\left\{ {n+r \atop n} \right\}$ is a polynomial in n of degree $2r$ we extract the coefficient on $[z^n]$ of $Q_r(z)$ to get

$$\sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle \binom{2r+n-k}{2r} = \frac{1}{(2r)!} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle (n + 2r - k)^{\underline{2r}}.$$

The sum terms are products of $2r$ linear terms in n times a coefficient that does not depend on n (Eulerian number) and neither does the range of the sum (finite, $r + 1$ terms) and we have the claim. The coefficient on n^{2r} is

$$\frac{1}{(2r)!} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle = \frac{1}{(2r)!}(2r-1)!! = \frac{1}{(2r)!} \frac{(2r)!}{2^r r!} = \frac{1}{2^r r!} \neq 0.$$

This was math.stackexchange.com problem 4121168.

61.5.1 A Stirling cycle number generating function and Eulerian numbers of the second order (II)

We start with the following claim

$$\boxed{Q_r(z) = \sum_{n \geq 0} \left[{n+r+1 \atop n+1} \right] z^n = \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^k.}$$

We will prove this by induction. Introduce $P_r(z) = z^r Q_r(z)$. Note that depending on whether ball $n + r + 2$ joins an existing cycle or turns into a fixed point we have

$$\left[{n+r+2 \atop n+2} \right] = (n + r + 1) \left[{n+r+1 \atop n+2} \right] + \left[{n+r+1 \atop n+1} \right].$$

Multiply by z^{n+r} and sum over $n \ge 0$ to get

$$\sum_{n\ge 0} \left[{n+r+2 \atop n+2} \right] z^{n+r} = \sum_{n\ge 0} (n+r+1) \left[{n+r+1 \atop n+2} \right] z^{n+r} + P_r(z).$$

The first term is

$$\frac{1}{z}(P_r(z) - r! z^r)$$

and the second one

$$\left(z \sum_{n\ge 0} \left[{n+2+(r-1) \atop n+2} \right] z^{n+1+r-1} \right)'$$

$$= (-(r-1)! z^r + z P_{r-1}(z))' = -r! z^{r-1} + P_{r-1}(z) + z P'_{r-1}(z).$$

This gives the recurrence

$$P_r(z) - r! z^r = -r! z^r + z P_{r-1}(z) + z^2 P'_{r-1}(z) + z P_r(z).$$

We obtain

$$P_r(z) = \frac{z}{1-z}(P_{r-1}(z) + z P'_{r-1}(z)) = \frac{z}{1-z}(z P_{r-1}(z))'.$$

We now prove by induction that

$$P_r(z) = \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+k}.$$

It certainly holds for $r = 0$ where the infinite series gives $1/(1-z)$ and it also holds at $r = 1$ as well where the sum gives

$$\sum_{n\ge 1} \binom{n+1}{2} z^n = z \sum_{n\ge 0} \binom{n+2}{2} z^n = \frac{z}{(1-z)^3}$$

and the Eulerian numbers produce

$$\frac{1}{(1-z)^3} \left[\left\langle\!\!\left\langle {1 \atop 1} \right\rangle\!\!\right\rangle z + \left\langle\!\!\left\langle {1 \atop 0} \right\rangle\!\!\right\rangle z^2 \right] = \frac{z}{(1-z)^3}.$$

Now supposing it holds with $r \ge 1$ we must show that it holds for $r+1$. Doing the differentiation and multiplication we obtain

$$\frac{z}{1-z} \frac{2r+1}{(1-z)^{2r+2}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+1+k}$$

$$+ \frac{z}{1-z} \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k) z^{r+k}.$$

Factoring out $1/(1-z)^{2r+3}$ for the moment this becomes

$$z(2r+1)\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+1+k} + (z-z^2)\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+k}.$$

or

$$(2r+1)\sum_{k=-1}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+2+k} + \sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+1+k}$$
$$-\sum_{k=-1}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+2+k}.$$

Here we have included three zero terms, one in every sum. Continuing,

$$(2r+1)\sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r+1-k} \right\rangle\!\!\right\rangle z^{r+1+k} + \sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+1+k}$$
$$-\sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r+1-k} \right\rangle\!\!\right\rangle (r+k)z^{r+1+k}.$$

We obtain

$$\sum_{k=0}^{r+1}\left[(r+1-k)\left\langle\!\!\left\langle {r \atop r+1-k} \right\rangle\!\!\right\rangle + (r+1+k)\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle\right] z^{r+1+k}.$$

The Eulerian number recurrence (second order) according to OEIS A349556 is

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = k\left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n-k)\left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle$$

Putting $n := r+1$ and $k := r+1-k$ and restoring the factor in front now yields

$$\frac{1}{(1-z)^{2r+3}}\sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r+1 \atop r+1-k} \right\rangle\!\!\right\rangle z^{r+1+k}$$

thus concluding the induction.

Addendum. The reader might well wonder how the conjecture from the beginning was obtained i.e. how we find the closed form for small r for lookup in the OEIS, which then points us to Eulerian numbers, enabling the whole computation.

Recall e.g. from *Concrete Mathematics* chapter 6.2. [GKP89] that

$$\boxed{\left[n \atop m \right] = \frac{(n-1)!}{(m-1)!} [w^{n-m}] \left(\frac{w \exp(w)}{\exp(w) - 1} \right)^n.}$$

We get for our series

$$Q_r(z) = [w^r] \sum_{n\ge 0} z^n \frac{(n+r)!}{n!} \left(\frac{w}{1-\exp(-w)} \right)^{n+r+1}$$

$$= r![w^r] \left(\frac{w}{1-\exp(-w)} \right)^{r+1} \sum_{n\ge 0} z^n \binom{n+r}{r} \left(\frac{w}{1-\exp(-w)} \right)^n$$

$$= r![w^r] \left(\frac{w}{1-\exp(-w)} \right)^{r+1} \frac{1}{(1 - zw/(1-\exp(-w)))^{r+1}}$$

$$= r![w^r] \frac{w^{r+1}}{(1-\exp(-w) - zw)^{r+1}}.$$

Note that the fraction is a formal power series in w with no pole at zero. Continuing,

$$r! \operatorname*{res}_{w} \frac{1}{(1-\exp(-w) - zw)^{r+1}}.$$

A CAS like Maple for example can recognize the pole of order $r+1$ at zero which has now appeared and quickly compute the residue by differentiation. This will produce e.g.

$$Q_5(z) = \frac{z^4 + 52\,z^3 + 328\,z^2 + 444\,z + 120}{(1-z)^{11}}$$

which is enough to spot the pattern.

This was math.stackexchange.com problem 4480877.

61.6 Another case of factorization

In seeking to evaluate

$$\sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{q/2+\ell}{2\ell} \left(\binom{q/2-j+k-\ell}{k-2\ell} + \binom{q/2-j+k-\ell-1}{k-2\ell} \right)$$

We get for the first piece of the sum

$$\sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{q/2+\ell}{2\ell} \binom{q/2-j+k-\ell}{k-2\ell}$$

$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{k+1}} (1+z)^{q/2-j+k} \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{q/2+\ell}{2\ell} \frac{z^{2\ell}}{(1+z)^\ell} \; dz.$$

Now here the residue vanishes when $2\ell > k$ so it enforces the upper limit of the sum and we obtain

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k}}{z^{k+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{q/2}}{w} \sum_{\ell \ge 0} \frac{z^{2\ell}}{(1+z)^\ell} \frac{(1+w)^\ell}{w^{2\ell}} \; dw \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k}}{z^{k+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{q/2}}{w} \frac{1}{1 - z^2(1+w)/(1+z)/w^2} \; dw \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k+1}}{z^{k+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} (1+w)^{q/2} \frac{w}{(w-z)(w(1+z)+z)} \; dw \; dz.$$

The pole at $w = 0$ has been canceled. Now observe that for the geometric series to converge we must have

$$|z^2(1+w)/w^2/(1+z)| < 1.$$

We will choose a contour that includes both simple poles. The first pole is at $-z/(1+z)$. We thus require $|z/(1+z)| < \gamma$. With $|z/(1+z)| \le \varepsilon/(1-\varepsilon)$ we get $\varepsilon/(1-\varepsilon) < \gamma$ and we furthermore need $|z^2/(1+z)| < |w^2/(1+w)|$. The latter holds if $\varepsilon^2/(1-\varepsilon) < \gamma^2/(1+\gamma)$. Both hold if $\varepsilon\gamma < \gamma^2/(1+\gamma)$ or $\varepsilon < \gamma/(1+\gamma)$. So $\varepsilon = \gamma^2/(1+\gamma)$ will work. Observe that this contour also includes the pole at $w = z$.

First pole. Now to extract the residue at $w = -z/(1+z)$ we write

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k}}{z^{k+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} (1+w)^{q/2} \frac{w}{(w-z)(w+z/(1+z))} \; dw \; dz$$

and obtain

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k}}{z^{k+1}} (1+z)^{-q/2} \frac{-z/(1+z)}{-z/(1+z)-z} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{k-j}}{z^{k+1}} \frac{1}{z+2} \; dz.$$

Repeating for the second sum we get

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{k-j-1}}{z^{k+1}} \frac{1}{z+2} \; dz.$$

Adding the two we find

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{k-j-1}(1+(1+z))}{z^{k+1}} \frac{1}{z+2} \; dz = \binom{k-j-1}{k}.$$

Second pole. For the residue at $w = z$ we obtain for the first sum

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k+1}}{z^{k+1}} (1+z)^{q/2} \frac{z}{(z(1+z)+z)} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q-j+k+1}}{z^{k+1}} \frac{1}{z+2} \; dz.$$

Repeating for the second sum we get

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q-j+k}}{z^{k+1}} \frac{1}{z+2} \; dz.$$

Adding the two we find

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q-j+k}(1+(1+z))}{z^{k+1}} \frac{1}{z+2} \; dz = \binom{q-j+k}{k}.$$

Conclusion. Collecting everything we obtain

$$\binom{q-j+k}{k} + \binom{k-j-1}{k}.$$

The second term is $(k-j-1)^{\underline{k}}/k!$. Now if $0 \le j < k$ this is indeed zero because the falling factorial hits the zero value. If $j \ge k$ all k terms are negative and we get $(-j)^{\overline{k}}/k!$.

We have at last

$$\boxed{\binom{q-j+k}{k} + (-1)^k \binom{j}{k}.}$$

as claimed.

Remark. The potential square roots that appeared in the above all use the principal branch of the logarithm with branch cut $(-\infty, -1]$ which means everything is analytic in a neighborhood of zero as required.

This was math.stackexchange.com problem 4155443.

61.7 An additional case of factorization

Supposing we seek to simplify

$$\sum_{j=0}^{k} \binom{2j}{j+q}\binom{2k-2j}{k-j}.$$

where $0 \le q \le k$. This is

$$[z^k](1+z)^{2k} \sum_{j=0}^{k} \binom{2j}{j+q} \frac{z^j}{(1+z)^{2j}}.$$

Here the coefficient extractor enforces the upper limit of the sum and we find

$$[z^k](1+z)^{2k} \sum_{j\ge 0} \binom{2j}{j+q} \frac{z^j}{(1+z)^{2j}}.$$

At this point we see that we will require residues and complex integration and continue with

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}} \sum_{j\ge 0} \frac{(1+w)^{2j}}{w^j} \frac{z^j}{(1+z)^{2j}} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}} \frac{1}{1-z(1+w)^2/w/(1+z)^2} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k+2}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q}} \frac{1}{w(1+z)^2-z(1+w)^2} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k+2}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q}} \frac{1}{(w-z)(1-wz)} \; dw \; dz.$$

For the geometric series to converge we must have $|z(1+w)^2/w/(1+z)^2| < 1$ or $|z/(1+z)^2| < |w/(1+w)^2|$. This requires $\varepsilon/(1-\varepsilon)^2 < \gamma/(1+\gamma)^2$. We will also require $w = z$ to be inside the contour for w so we need $\varepsilon < \gamma$. With $\varepsilon \ll 1$ and $\gamma \ll 1$ we may take $\varepsilon = \gamma^2$ for the latter inquality. We then get for the inquality from the geometric series $\gamma^2/(1-\gamma^2)^2 < \gamma/(1+\gamma)^2$ or $\gamma < (1-\gamma^2)^2/(1+\gamma)^2$ or $\gamma < (1-\gamma)^2$. This holds for $\gamma < 1 - 1/\varphi$ with φ the golden mean.

Now we have the pole at zero and the one at $w = z$ inside the contour in w. This means we can evaluate the integral by using the fact that residues sum to zero, taking minus the residue at $w = 1/z$ and minus the residue at infinity, which is zero by inspection, however. (The pole at $w = 1/z$ has modulus $1/\varepsilon$ and is outside the contour.) Computing minus the residue at $w = 1/z$ we write

$$-\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+2}}{z^{k+2}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^q}\frac{1}{(w-z)(w-1/z)}\;dw\;dz.$$

With the sign change we obtain

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+2}}{z^{k+2}}z^q\frac{1}{1/z-z}\;dz = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+2}}{z^{k-q+1}}\frac{1}{1-z^2}\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+1}}{z^{k-q+1}}\frac{1}{1-z}\;dz.$$

This is zero when $q > k$ and otherwise

$$\sum_{j=0}^{k-q}\binom{2k+1}{j} = \sum_{j=0}^{k}\binom{2k+1}{j} - \sum_{j=k-q+1}^{k}\binom{2k+1}{j}$$

or alternatively

$$\boxed{4^k - \sum_{j=k-q+1}^{k}\binom{2k+1}{j}}$$

which is a closed form term plus a sum of q terms. E.g. with $q=0$ we obtain 4^k and with $q=1$, $4^k - \binom{2k+1}{k}$. For $q=2$ we have $4^k - \binom{2k+1}{k-1} - \binom{2k+1}{k}$ and so on.

This was math.stackexchange.com problem 4174584.

61.8 Contours and a binomial square root

Suppose we seek to prove that

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{m+k}{2n} = \binom{2m}{2n}.$$

Introduce the integral representation

$$\binom{m+k}{2n} = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{2n+1}}(1+z)^{m+k}\;dz.$$

This gives the following integral

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\sum_{k=0}^{n}\binom{2n+1}{2k+1}\frac{1}{z^{2n+1}}(1+z)^{m+k}\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^m}{z^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (1+z)^k \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{m-1/2}}{z^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{2k+1} \sqrt{1+z}^{2k+1} \; dz.$$

The sum is

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} \sqrt{1+z}^{k} \frac{1}{2}(1-(-1)^k)$$
$$= \frac{1}{2}((1+\sqrt{1+z})^{2n+1} - (1-\sqrt{1+z})^{2n+1})$$

and we get for the integral

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{m-1/2}}{2z^{2n+1}} \left((1+\sqrt{1+z})^{2n+1} - (1-\sqrt{1+z})^{2n+1}\right) \; dz.$$

By way of ensuring analyticity we observe that we must have $\varepsilon < 1$ owing to the branch cut $(-\infty, -1]$ of the square root. Now put $1+z = w^2$ so that $dz = 2w\; dw$ and the integral becomes

$$\frac{1}{2\pi i}\int_{|w-1|=\gamma} \frac{w^{2m-1}}{(w^2-1)^{2n+1}} \left((1+w)^{2n+1} - (1-w)^{2n+1}\right) \; w \; dw.$$

This is

$$\frac{1}{2\pi i}\int_{|w-1|=\gamma} w^{2m} \left(\frac{1}{(w-1)^{2n+1}} + \frac{1}{(w+1)^{2n+1}}\right) \; dw.$$

Treat the two terms in the parentheses in turn. The first contributes

$$[(w-1)^{2n}]w^{2m} = [(w-1)^{2n}] \sum_{q=0}^{2m} \binom{2m}{q} (w-1)^q = \binom{2m}{2n}.$$

The second term is analytic on and inside the circle that w traces round the value 1 with no poles (pole is at $w=-1$) and hence does not contribute anything. This concludes the argument.

Remark. We must document the choice of γ so that $|w-1| = \gamma$ is entirely contained in the image of $|z| = \varepsilon$, which since $w = 1 + \frac{1}{2}z + \cdots$ makes one turn around $w = 1$ and may then be continuously deformed to the circle $|w - 1| = \gamma$. We need a bound on where this image comes closest to one. We have $w = 1 + \frac{1}{2}z + \sum_{q\ge 2} (-1)^{q+1} \frac{1}{4^q} \frac{1}{2q-1} \binom{2q}{q} z^q$. The modulus of the series term is bounded by $\sum_{q\ge 2} \frac{1}{4^q} \frac{1}{2q-1} \binom{2q}{q} |z|^q = 1 - \frac{1}{2}|z| - \sqrt{1-|z|}$. Therefore choosing

$\gamma = \frac{1}{2}\varepsilon - 1 + \frac{1}{2}\varepsilon + \sqrt{1-\epsilon} = \sqrt{1-\varepsilon} + \varepsilon - 1$ will fit the bill. For example with $\varepsilon = 1/2$ we get $\gamma = (\sqrt{2}-1)/2$. It is a matter of arithmetic to verify that with the formula we have $\gamma < 1$.

This was math.stackexchange.com problem 601940.

61.9 Careful examination of a contour

We seek to show that

$$\sum_{q=0}^{n} \binom{q}{n-q} (-1)^{n-q} \binom{2q+1}{q+1} = 2^{n+1} - 1.$$

The LHS is

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \sum_{q=0}^{n} (-1)^q z^q (1+z)^q \frac{(1+w)^{2q+1}}{w^{q+2}} \; dw \; dz.$$

There is no contribution when $q > n$ and we may extend q to infinity:

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1+w}{w^2} \frac{1}{1+z(1+z)(1+w)^2/w} \; dw \; dz$$

$$= \frac{(-1)^n}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1+w}{w} \frac{1}{(1+z+wz)(z+(1+z)w)} \; dw \; dz.$$

Now we determine ε and γ so that the geometric series converges and the pole at $w = -z/(1+z)$ is inside $|w| = \gamma$ while the pole at $w = -(1+z)/z$ is not. For the series we require $|z(1+z)(1+w)^2/w| < 1$. With $|z(1+z)| \le \varepsilon(1+\varepsilon)$ and $|w/(1+w)^2| \ge \gamma/(1+\gamma)^2$ we need $\varepsilon(1+\varepsilon) < \gamma/(1+\gamma)^2$. Observe that on $[0,1]$ we have $\gamma/(1+\gamma)^2 \ge \gamma/4$ since $4 \ge (1+\gamma)^2$. For $\gamma/4 > \varepsilon(1+\varepsilon)$ we choose $\gamma = 8\varepsilon$ with $\varepsilon \ll 1$ and we have our pair. Now for the pole at $-z/(1+z)$ we need for the maximum norm $\varepsilon/(1-\varepsilon) < \gamma = 8\varepsilon$ which holds with $\varepsilon < 7/8$ which we will enforce. The second pole under consideration is $-(1+z)/z = -1 - 1/z$. The closest this comes to the origin is $-1 + \varepsilon = -1 + \gamma/8$. To see that this is outside $|w| = \gamma$ we need $-1 + \gamma/8 < -\gamma$ or $\gamma < 8/9$. This means we instantiate ε to $\varepsilon < 1/9$, which completes the discussion of the contour.

Now residues sum to zero and the residue at infinity in w is zero by inspection which means that the inner integral is minus the residue at $w = -(1+z)/z$, as it is equal to the sum of the residues at zero and at $w = -z/(1+z)$. We write

$$-\frac{1}{z} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1+w}{w} \frac{1}{((1+z)/z + w)(z + (1+z)w)} \; dw.$$

We get from this being a simple pole the contribution (here $(1+w)/w = 1/(1+z) $)

$$-\frac{1}{z}\frac{1}{1+z}\frac{1}{z-(1+z)^2/z} = \frac{1}{1+z}\frac{1}{1+2z}$$

which combined with the integral in z gives

$$(-1)^n [z^n] \frac{1}{1+z}\frac{1}{1+2z} = (-1)^n \sum_{q=0}^n (-1)^q 2^q (-1)^{n-q} = \sum_{q=0}^n 2^q.$$

This is indeed

$$\boxed{2^{n+1}-1}$$

as claimed.
This was math.stackexchange.com problem 4196412.

61.10 Stirling numbers, Bernoulli numbers and Catalan numbers

Suppose we seek to prove that

$$\sum_{k=0}^n \left\{ {n+k \atop k} \right\} {2n \choose n+k} \frac{(-1)^k}{k+1} = B_n {2n \choose n} \frac{1}{n+1}$$

a unique identity that connects three types of significant combinatorial numbers. We get for the LHS

$$\sum_{k=0}^n \left\{ {n+k \atop k} \right\} {2n \choose n-k} \frac{(-1)^k}{k+1} = [z^n](1+z)^{2n} \sum_{k=0}^n \left\{ {n+k \atop k} \right\} \frac{(-1)^k z^k}{k+1}.$$

The coefficient extractor $[z^n]$ combined with the factor z^k enforces the upper limit of the sum so we may let k range to infinity:

$$[z^n](1+z)^{2n} \sum_{k\ge 0} (n+k)! [w^{n+k}] \frac{(\exp(w)-1)^k}{k!} \frac{(-1)^k z^k}{k+1}.$$

Here we see that we will require complex methods and switch to

$$\frac{(n-1)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{2n}}{z^{n+1}}$$
$$\times \sum_{k\ge 0} {n+k \choose n-1} (-1)^k z^k \frac{(\exp(w)-1)^k}{w^k} \; dz \; dw.$$

Computing the sum we find

$$-\frac{1}{z}\frac{w}{\exp(w)-1}\sum_{k\ge 1}\binom{n-1+k}{n-1}(-1)^k z^k \frac{(\exp(w)-1)^k}{w^k}$$
$$= \frac{1}{z}\frac{w}{\exp(w)-1} - \frac{1}{z}\frac{w}{\exp(w)-1}\frac{1}{(1+z(\exp(w)-1)/w)^n}.$$

The first component yields by inspection

$$(n-1)! \times \binom{2n}{n+1} \times B_n \frac{1}{n!} = B_n \frac{1}{n}\binom{2n}{n+1} = B_n \binom{2n}{n}\frac{1}{n+1}.$$

We have the claim if we can show the second component yields zero. We get

$$-\frac{(n-1)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{(\exp(w)-1)^{n+1}}$$
$$\times \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2n}}{z^{n+2}}\frac{1}{(z+w/(\exp(w)-1))^n}\; dz\; dw.$$

At this time we must instantiate our contours. We need for the binomial series to converge that $|z(\exp(w)-1)/w| < 1$ or $|z| < |w/(\exp(w)-1)|$. Observe that this means the pole at $z = -w/(\exp(w)-1)$ is outside the circle $|z| = \varepsilon$. To get a lower bound on the norm of the image of $|w| = \gamma$ we first take $\gamma \ll 1$ and observe that by expanding the series and bounding $|\sum_{m\ge 1} w^{m-1}/m!|$ by $\sum_{m\ge 1}\gamma^{m-1}/m!$ we have $|(\exp(w)-1)/w| \le (\exp(\gamma)-1)/\gamma$. Since the term in w is non-zero on and inside $|w| = \gamma$ (there is no pole at zero and the value there is one and the nearest zero is at $\pm 2\pi i$) we may invert to get $|w/(\exp(w)-1)| \ge \gamma/(\exp(\gamma)-1)$. Now we also have $\gamma/(\exp(\gamma)-1) > 1 - \frac{1}{2}\gamma$ as can be seen by comparing $\sum_{m\ge 1}\frac{1}{2^{m-1}}\gamma^m$ to $\sum_{m\ge 1}\frac{1}{m!}\gamma^m$, certainly both convergent for $\gamma \ll 1$. Hence $\varepsilon = 1 - \frac{1}{2}\gamma$ is an admissible choice and we have determined the contour. The pair $\gamma = 1/3$ and $\varepsilon = 5/6$ will work.

We thus must verify that the pole at $z = -w/(\exp(w)-1)$ makes a zero contribution (residues sum to zero and the residue at infinity is zero by inspection). This requires (Leibniz rule)

$$\frac{1}{(n-1)!}\left(\frac{1}{z^{n+2}}(1+z)^{2n}\right)^{(n-1)}$$
$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^q\frac{(n+1+q)!}{(n+1)!}\frac{1}{z^{n+2+q}}$$
$$\times \frac{(2n)!}{(2n-(n-1-q))!}(1+z)^{2n-(n-1-q)}$$
$$= \sum_{q=0}^{n-1}(-1)^q\binom{n+1+q}{q}\frac{1}{z^{n+2+q}}\binom{2n}{n+1+q}(1+z)^{n+1+q}.$$

Observe that

$$\binom{n+1+q}{q}\binom{2n}{n+1+q} = \frac{(2n)!}{q! \times (n+1)! \times (n-1-q)!} = \binom{2n}{n+1}\binom{n-1}{q}$$

so the sum becomes

$$\binom{2n}{n+1}\frac{(1+z)^{n+1}}{z^{n+2}}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^q\frac{(1+z)^q}{z^q}$$
$$= \binom{2n}{n+1}\frac{(1+z)^{n+1}}{z^{n+2}}\left(1-\frac{1+z}{z}\right)^{n-1}$$
$$= \binom{2n}{n+1}(-1)^{n-1}\frac{(1+z)^{n+1}}{z^{2n+1}}.$$

Making the substitution we are left with the integral

$$\binom{2n}{n+1}(-1)^n\frac{(n-1)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{(\exp(w)-1)^{n+1}}$$
$$\times\frac{(1-w/(\exp(w)-1))^{n+1}}{(-w/(\exp(w)-1))^{2n+1}}\;dw.$$

The inner term is

$$-(\exp(w)-1)^n\frac{1}{w^{2n+1}}(1-w/(\exp(w)-1))^{n+1}$$
$$= -\frac{1}{w^{2n+1}}\frac{1}{\exp(w)-1}(\exp(w)-1-w)^{n+1}.$$

We get for the integral

$$\binom{2n}{n+1}(-1)^{n+1}\frac{(n-1)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{2n+2}}\frac{w}{\exp(w)-1}(\exp(w)-1-w)^{n+1}\;dw.$$

Now this is

$$[w^{2n+1}]\frac{w}{\exp(w)-1}(\exp(w)-1-w)^{n+1} = 0$$

because $(\exp(w)-1-w)^{n+1} = \frac{1}{2^{n+1}}w^{2n+2}+\cdots$ which concludes the argument. (The poles at $\pm 2\pi ik, k \geq 1$ are not inside the contour.)

This problem has not yet appeared at math.stackexchange.com. The source is exercise 6.74 from *Concrete Mathematics* by Graham, Knuth and Patashnik, [GKP89] credited to B.F.Logan.

61.11 Computing an EGF from an OGF

We seek to compute the EGF of a sequence from its OGF. There may be some cases where complex variables, the residue theorem and the residue at infinity are helpful. Suppose your OGF is $f(z)$ and the desired EGF is $g(w)$. Then we have

$$g(w) = \sum_{n\ge 0} \frac{w^n}{n!} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} f(z)\, dz.$$

This will simplify together with some conditions on convergence to give

$$g(w) = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f(z)}{z} \sum_{n\ge 0} \frac{1}{n!} \frac{w^n}{z^n}\, dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f(z)}{z} \exp(w/z)\, dz.$$

Example I. Suppose

$$f(z) = \frac{1}{1-z},$$

which yields

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{1-z} \frac{1}{z} \exp(w/z)\, dz.$$

Now for $z = R\exp(i\theta)$ with R going to infinity we have

$$2\pi R \times \frac{1}{R^2} \times \exp(|w|/R) \to 0$$

as $R \to \infty$ so this integral is

$$-\mathrm{Res}_{z=1} \frac{1}{1-z} \frac{1}{z} \exp(w/z)$$

and we get

$$g(w) = \exp(w)$$

which is the correct answer.

Example II. This time suppose that

$$f(z) = \frac{z}{(1-z)^2}$$

so that we should get

$$g(w) = \sum_{n\ge 1} n \frac{w^n}{n!} = w \sum_{n\ge 1} \frac{w^{n-1}}{(n-1)!} = w\exp(w).$$

The integral formula yields

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z}{(1-z)^2}\frac{1}{z}\exp(w/z)\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)^2}\exp(w/z)\; dz.$$

The residue at infinity is zero as before and we have

$$\exp(w/z) = \sum_{n\ge 0} \left.(\exp(w/z))^{(n)}\right|_{z=1} \frac{(z-1)^n}{n!}$$

The coefficient on $(z-1)$ is

$$\left.-\frac{1}{z^2} w\exp(w/z)\right|_{z=1} = -w\exp(w)$$

which is the correct answer taking into account the sign flip due to $z = 1$ not being inside the contour.

Remark. Good news. The sum in the integral converges everywhere.

Addendum: somewhat more involved example. The OGF of Stirling numbers of the second kind for set partitions into k non-empty sets is

$$\sum_{n\ge 0} \left\{ n \atop k \right\} z^n = \prod_{q=1}^{k} \frac{z}{1-qz}.$$

We thus have that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}\exp(w/z)\prod_{q=1}^{k}\frac{z}{1-qz}\; dz$$

$$= \frac{(-1)^k}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}\exp(w/z)\prod_{q=1}^{k}\frac{z}{qz-1}\; dz$$

$$= \frac{(-1)^k}{k!\times 2\pi i}\int_{|z|=\epsilon} \frac{1}{z}\exp(w/z)\prod_{q=1}^{k}\frac{z}{z-1/q}\; dz.$$

Computing the sum of the residues at the finite poles not including zero we get

$$\frac{(-1)^k}{k!}\sum_{q=1}^{k} q\exp(qw)\times\frac{1}{q}\prod_{m=1}^{q-1}\frac{1/q}{1/q-1/m}\prod_{m=q+1}^{k}\frac{1/q}{1/q-1/m}$$

$$= \frac{(-1)^k}{k!}\sum_{q=1}^{k}\exp(qw)\prod_{m=1}^{q-1}\frac{m}{m-q}\prod_{m=q+1}^{k}\frac{m}{m-q}$$

$$= \frac{(-1)^k}{k!} \sum_{q=1}^{k} \exp(qw) \frac{k!}{q} \prod_{m=1}^{q-1} \frac{1}{m-q} \prod_{m=q+1}^{k} \frac{1}{m-q}$$

$$= \frac{(-1)^k}{k!} \sum_{q=1}^{k} \exp(qw) \frac{k!}{q} \frac{(-1)^{q-1}}{(q-1)!} \frac{1}{(k-q)!}$$

$$= -\frac{1}{k!} \sum_{q=1}^{k} \exp(qw)(-1)^{k-q} \binom{k}{q}$$

$$= -\left(\frac{(\exp(w)-1)^k}{k!} - \frac{(-1)^k}{k!} \right).$$

This is a case where the residue at infinity is not zero. We have the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z}\right) \right]$$

This yields for the present case

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z \exp(wz) \prod_{q=1}^{k} \frac{1/z}{1-q/z} = -\mathrm{Res}_{z=0} \frac{1}{z} \exp(wz) \prod_{q=1}^{k} \frac{1}{z-q}$$

$$= -\frac{1}{k!} \mathrm{Res}_{z=0} \frac{1}{z} \exp(wz) \prod_{q=1}^{k} \frac{1}{z/q-1}$$

$$= -\frac{(-1)^k}{k!} \mathrm{Res}_{z=0} \frac{1}{z} \exp(wz) \prod_{q=1}^{k} \frac{1}{1-z/q} = -\frac{(-1)^k}{k!}.$$

Adding the residue at infinity to the residues from the poles at $z = 1/q$ we finally obtain

$$-\left(\frac{(\exp(w)-1)^k}{k!} - \frac{(-1)^k}{k!} \right) - \frac{(-1)^k}{k!} = -\frac{(\exp(w)-1)^k}{k!}.$$

Taking into account the sign flip we have indeed computed the EGF of the Stirling numbers of the second kind

$$\sum_{n \geq 0} \left\{ {n \atop k} \right\} \frac{z^n}{n!}$$

as can be seen from the combinatorial class equation

$$\mathrm{SET}(\mathcal{U} \times \mathrm{SET}_{\geq 1}(\mathcal{Z}))$$

which gives the bivariate generating function

$$G(z,u) = \exp(u(\exp(z)-1)).$$

This was math.stackexchange.com problem 1289377.

61.12 Stirling numbers of the first and second kind

We seek an alternate closed form of

$$\sum_{q=0}^{r} (-1)^{q+r} \begin{bmatrix} r \\ q \end{bmatrix} \left\{ \begin{matrix} n+q-1 \\ k \end{matrix} \right\}.$$

With the usual EGFs this becomes

$$\sum_{q=0}^{r} (-1)^{q+r} \frac{r!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}} \frac{1}{q!} \left(\log\frac{1}{1-z}\right)^q$$
$$\times \frac{(n+q-1)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+q}} \frac{1}{k!} (\exp(w)-1)^k \; dw \; dz.$$

Now we may extend q beyond r because $\left(\log\frac{1}{1-z}\right)^q = z^q + \cdots$ and hence $q > r$ produces no pole in a neighborhood of zero (the branch cut of the logarithmic term is $[1,\infty)$). We find

$$\frac{(-1)^r \times r! \times (n-1)!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{k!} (\exp(w)-1)^k$$
$$\times \sum_{q\ge 0} \binom{n+q-1}{n-1} \frac{(-1)^q}{w^q} \left(\log\frac{1}{1-z}\right)^q \; dw \; dz.$$

Next we will sum the binomial series which requires $|\log\frac{1}{1-z}| < |w|$. Observe that for the image of $|z|=\varepsilon$ we have $|\log\frac{1}{1-z}| < \frac{\varepsilon}{1-\varepsilon}$. Therefore choosing γ so that $\frac{\varepsilon}{1-\varepsilon} \le \gamma$ will work e.g. for $\varepsilon = 1/Q$ we take $\gamma = 1/(Q-1)$. This yields

$$\frac{(-1)^r \times r! \times (n-1)!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{k!} (\exp(w)-1)^k \frac{1}{(1+\frac{1}{w}\log\frac{1}{1-z})^n} \; dw \; dz$$
$$= \frac{(-1)^r \times r! \times (n-1)!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{k!} (\exp(w)-1)^k \frac{1}{(w+\log\frac{1}{1-z})^n} \; dw \; dz.$$

The pole at zero for w has been canceled but the pole at $w = -\log \frac{1}{1-z}$ now lies inside the contour. Therefore we require

$$\frac{1}{(n-1)!} \left(\frac{1}{k!} (\exp(w) - 1)^k \right)^{(n-1)}$$

$$= \frac{1}{(n-1)! \times k!} \left(\sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} \exp(pw) \right)^{(n-1)}$$

$$= \frac{1}{(n-1)! \times k!} \sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} p^{n-1} \exp(pw)$$

Evaluate at $w = -\log \frac{1}{1-z}$ and substitute into the integral in z to obtain

$$\frac{(-1)^r \times r!}{k! \times 2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}} \sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} p^{n-1} (1-z)^p \; dz$$

$$= (-1)^r \frac{r!}{k!} \sum_{p=r}^{k} \binom{k}{p} (-1)^{k-p} p^{n-1} (-1)^r \binom{p}{r}.$$

We have established that the sum vanishes when $k < r$. Note that

$$\binom{k}{p} \binom{p}{r} = \frac{k!}{(k-p)! \times r! \times (p-r)!} = \binom{k}{r} \binom{k-r}{p-r}$$

so this simplifies to

$$\frac{1}{(k-r)!} \sum_{p=r}^{k} \binom{k-r}{p-r} (-1)^{k-p} p^{n-1}.$$

We have proved that the alternate closed form is

$$\boxed{\frac{(-1)^{k-r}}{(k-r)!} \sum_{p=0}^{k-r} \binom{k-r}{p} (-1)^p (p+r)^{n-1}.}$$

An interesting special case is that this evaluates to r^{n-1} when $k = r$.

This problem has not appeared at math.stackexchange.com. It is from page 172 eqn. 12.22 of H.W.Gould's *Combinatorial Identities for Stirling Numbers* [Gou16] where it is attributed to Frank Olson.

61.13 An identity by Carlitz

We seek to show that where $m \geq 1$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k/2}{m} = \frac{n}{m} \binom{n-m-1}{m-1} 2^{n-2m}.$$

We get for the LHS

$$[z^m] \sum_{k=0}^{n} \binom{n}{k} \sqrt{1+z}^k = [z^m](1+\sqrt{1+z})^n.$$

This is

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{m+1}} (1+\sqrt{1+z})^n \; dz.$$

Now put $1+\sqrt{1+z} = w$ so that $z = w(w-2)$ and $dz = 2(w-1)\; dw$ to get

$$\frac{1}{2\pi i} \int_{|w-2|=\gamma} \frac{1}{w^{m+1}(w-2)^{m+1}} w^n 2(w-1)\; dw.$$

Now we have (series need not be finite)

$$\begin{aligned} w^{n-m} &= (2+(w-2))^{n-m} = 2^{n-m}(1+(w-2)/2)^{n-m} \\ &= 2^{n-m} \sum_{q \geq 0} \binom{n-m}{q} (w-2)^q/2^q \end{aligned}$$

so we get for the integral

$$\begin{aligned} &2^{n-m+1} \binom{n-m}{m} 2^{-m} - 2^{n-m} \binom{n-m-1}{m} 2^{-m} \\ &= 2^{n-2m} \binom{n-m-1}{m-1} \left[2\frac{n-m}{m} - \frac{n-2m}{m} \right] \\ &= 2^{n-2m} \binom{n-m-1}{m-1} \frac{n}{m}. \end{aligned}$$

This is the claim.

Remark. We need to document the choice of γ in terms of $\varepsilon \ll 1$. (The square root has the branch cut on $(-\infty, -1]$.) The image of $|z| = \varepsilon$ is contained in an annulus centered at two of radius $\sqrt{1+\varepsilon} - 1$ and $1 - \sqrt{1-\varepsilon}$. We may deform the image to a circle $|w-2| = \gamma$ where $\gamma = \epsilon/2$. This means the pole at $w = 0$ is definitely not inside the contour.

Using the residue operator

We get

$$\operatorname*{res}_{z} \frac{1}{z^{m+1}} (1+\sqrt{1+z})^n$$

$$= \operatorname*{res}_z \frac{1}{z^{m+1}} (-1)^n z^n \frac{1}{(1-\sqrt{1+z})^n}.$$

Now we put $1-\sqrt{1+z} = w$ so that $z = w(w-2)$ and $dz = 2(w-1)\, dw$ so that we obtain

$$\operatorname*{res}_w \frac{1}{w^{m+1}(w-2)^{m+1}} (-1)^n w^n (w-2)^n \frac{1}{w^n} 2(w-1)$$

$$= \operatorname*{res}_w \frac{1}{w^{m+1}(w-2)^{m-n+1}} (-1)^n 2(w-1)$$

$$= 2^{n-m} \operatorname*{res}_w \frac{1}{w^{m+1}(w/2-1)^{m-n+1}} (-1)^n (w-1)$$

$$= 2^{n-m} (-1)^{m+1} \operatorname*{res}_w \frac{1}{w^{m+1}(1-w/2)^{m-n+1}} (w-1).$$

Extracting the residue yields

$$2^{n-m}(-1)^{m+1} \left(\binom{2m-n-1}{m-1} \frac{1}{2^{m-1}} - \binom{2m-n}{m} \frac{1}{2^m} \right)$$

$$= 2^{n-m}(-1)^{m+1} \left((-1)^{m-1} \binom{n-m-1}{m-1} \frac{1}{2^{m-1}} - (-1)^m \binom{n-m-1}{m} \frac{1}{2^m} \right)$$

$$= 2^{n-2m} \left(2\binom{n-m-1}{m-1} + \binom{n-m-1}{m} \right)$$

$$= 2^{n-2m} \left(2\binom{n-m-1}{m-1} + \frac{n-2m}{m} \binom{n-m-1}{m-1} \right).$$

Merge the two binomial coefficients to obtain the same answer as before.

This problem is from page 43 eqn. 3.163 of H.W.Gould's *Combinatorial Identities* [Gou72].

61.14 Logarithm squared of the Catalan number OGF

Suppose we seek to find

$$[z^n] \log \left(\frac{2}{1+\sqrt{1-4z}} \right).$$

This is given by

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \log \left(\frac{2}{1+\sqrt{1-4z}} \right) \, dz.$$

Now put $1-4z = w^2$ so that $z = 1/4(1-w^2)$ and $-2\, dz = w\, dw$ to get

$$\frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{4^{n+1}}{(1-w^2)^{n+1}} \log \left(\frac{2}{1+w} \right) \left(-\frac{1}{2} \right) w\, dw.$$

This is

$$-\frac{1}{2}\frac{4^{n+1}}{2\pi i}\int_{|w-1|=\gamma}\frac{1}{(1-w)^{n+1}}\frac{1}{(1+w)^{n+1}}\log\left(\frac{1}{1+(w-1)/2}\right)\times w\, dw$$

or

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma}\frac{1}{(w-1)^{n+1}}\frac{1}{(1+w)^{n+1}}\log\left(\frac{1}{1+(w-1)/2}\right)\times w\, dw.$$

This has two parts, part A_1 is

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma}\frac{1}{(w-1)^{n}}\frac{1}{(1+w)^{n+1}}\log\left(\frac{1}{1+(w-1)/2}\right)\, dw$$

and part A_2 is

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma}\frac{1}{(w-1)^{n+1}}\frac{1}{(1+w)^{n+1}}\log\left(\frac{1}{1+(w-1)/2}\right)\, dw$$

Part A_1 is

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma}\frac{1}{(w-1)^{n}}\frac{1}{(2+(w-1))^{n+1}}\log\left(\frac{1}{1+(w-1)/2}\right)\, dw$$

$$=\frac{(-1)^n\times 2^{n}}{2\pi i}\int_{|w-1|=\gamma}\frac{1}{(w-1)^{n}}\frac{1}{(1+(w-1)/2)^{n+1}}\log\left(\frac{1}{1+(w-1)/2}\right)\, dw.$$

Extracting coefficients we get

$$(-1)^n 2^n\sum_{q=0}^{n-2}\binom{q+n}{n}\frac{(-1)^q}{2^q}\frac{(-1)^{n-1-q}}{2^{n-1-q}\times(n-1-q)}$$

which is

$$-2\sum_{q=0}^{n-2}\binom{q+n}{n}\frac{1}{n-1-q}.$$

Part A_2 is

$$(-1)^n 2^n\sum_{q=0}^{n-1}\binom{q+n}{n}\frac{(-1)^q}{2^q}\frac{(-1)^{n-q}}{2^{n-q}\times(n-q)}$$

which is

$$\sum_{q=0}^{n-1}\binom{q+n}{n}\frac{1}{n-q}.$$

Re-index A_1 to match A_2, getting

$$-2\sum_{q=1}^{n-1}\binom{q-1+n}{n}\frac{1}{n-q}.$$

Collecting the two contributions we obtain

$$\frac{1}{n}+\sum_{q=1}^{n-1}\left(\binom{q+n}{n}-2\binom{q-1+n}{n}\right)\frac{1}{n-q}$$

which is

$$\frac{1}{n}+\sum_{q=1}^{n-1}\left(\frac{q+n}{q}\binom{q-1+n}{n}-2\binom{q-1+n}{n}\right)\frac{1}{n-q}$$

$$=\frac{1}{n}+\sum_{q=1}^{n-1}\frac{n-q}{q}\binom{q-1+n}{n}\frac{1}{n-q}$$

$$=\frac{1}{n}+\sum_{q=1}^{n-1}\frac{1}{q}\binom{q-1+n}{n}$$

$$=\frac{1}{n}+\sum_{q=1}^{n-1}\frac{(q-1+n)!}{q!\times n!}$$

$$=\frac{1}{n}+\frac{1}{n}\sum_{q=1}^{n-1}\frac{(q-1+n)!}{q!\times (n-1)!}$$

$$=\frac{1}{n}+\frac{1}{n}\sum_{q=1}^{n-1}\binom{q-1+n}{n-1}=\frac{1}{n}\sum_{q=0}^{n-1}\binom{q-1+n}{n-1}.$$

To evaluate this last sum we use the integral

$$\binom{n-1+q}{n-1}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1+q}}{z^n}\;dz$$

which gives for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^n}\sum_{q=0}^{n-1}(1+z)^q\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^n}\frac{(1+z)^n-1}{1+z-1}\;dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n+1}} ((1+z)^n - 1) \; dz.$$

This also has two components, the second is zero and given by

$$-\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n+1}} \; dz$$

leaving

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+1}} \; dz$$

which evaluates to

$$\binom{2n-1}{n}.$$

We have shown that

$$[z^n] \log\left(\frac{2}{1+\sqrt{1-4z}}\right) = \frac{1}{n}\binom{2n-1}{n}.$$

Addendum Feb 27 2022. It appears from the comments that OP wanted to prove

$$[z^n] \log^2 \frac{2}{1+\sqrt{1-4z}} = \binom{2n}{n}(H_{2n-1} - H_n)\frac{1}{n}.$$

Using the result from the previous section the LHS becomes

$$\sum_{k=1}^{n-1} \frac{1}{k}\binom{2k-1}{k} \frac{1}{n-k}\binom{2n-2k-1}{n-k}.$$

Using

$$\frac{1}{k}\frac{1}{n-k} = \frac{1}{n}\frac{1}{k} + \frac{1}{n}\frac{1}{n-k}$$

this becomes

$$\frac{2}{n}\sum_{k=1}^{n-1} \frac{1}{n-k}\binom{2k-1}{k}\binom{2n-2k-1}{n-k}$$

$$= \frac{1}{2n}\sum_{k=1}^{n-1} \frac{1}{n-k}\binom{2k}{k}\binom{2n-2k}{n-k}$$

$$= -\frac{1}{2n^2}\binom{2n}{n} + \frac{1}{2n}[w^n] \log\frac{1}{1-w} \sum_{k\ge 0} w^k \binom{2k}{k}\binom{2n-2k}{n-k}.$$

Here we have extended to infinity due to the coefficient extractor in w (note that $\log \frac{1}{1-w} = w + \cdots$) and canceled the value for $k=0$ that was included in the sum. Continuing with the inner sum term

$$\begin{aligned}
&[z^n]\frac{1}{\sqrt{1-4wz}}\frac{1}{\sqrt{1-4z}}\\
&= [z^n]\frac{1}{\sqrt{(1-4z)^2-4z(1-4z)(w-1)}}\\
&= [z^n]\frac{1}{1-4z}\frac{1}{\sqrt{1-4z(w-1)/(1-4z)}}\\
&= [z^n]\sum_{k=0}^{n}\binom{2k}{k}z^k(w-1)^k\frac{1}{(1-4z)^{k+1}}.
\end{aligned}$$

This is

$$\frac{1}{2n}[w^n]\log\frac{1}{1-w}\sum_{k=0}^{n}\binom{2k}{k}(w-1)^k\binom{n}{k}4^{n-k}.$$

Recall from section **??** that with $1 \le k \le n$

$$\frac{1}{k} = \binom{n}{k}[w^n]\log\frac{1}{1-w}(w-1)^{n-k}.$$

Hence we get two pieces, the first is

$$\frac{1}{2n}\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{n-k}4^{n-k}.$$

and

$$\frac{1}{2n}[w^n]\log\frac{1}{1-w}\binom{2n}{n}(w-1)^n.$$

We get for the second

$$\binom{2n}{n}\frac{1}{2n}\operatorname*{res}_{w}\frac{1}{w^{n+1}}\log\frac{1}{1-w}(-1)^n(1-w)^n.$$

We put $w/(1-w) = v$ so that $w = v/(1+v)$ and $dw = 1/(1+v)^2\; dv$ to get (without the scalar in front)

$$\begin{aligned}
&\operatorname*{res}_{v}\frac{1}{v^{n+1}}(1+v)\log\frac{1}{1-v/(1+v)}(-1)^n\frac{1}{(1+v)^2}\\
&= \operatorname*{res}_{v}\frac{1}{v^{n+1}}(-1)^n\frac{1}{1+v}\log(1+v) = -(-1)^n[v^n]\frac{1}{1+v}\log\frac{1}{1+v}\\
&= -[v^n]\frac{1}{1-v}\log\frac{1}{1-v}.
\end{aligned}$$

With the scalar we get

$$-\binom{2n}{n}\frac{1}{2n}H_n.$$

We have the result if we can show that the first piece is

$$\binom{2n}{n}\left(H_{2n-1}+\frac{1}{2n}-\frac{1}{2}H_n\right)\frac{1}{n}=\binom{2n}{n}\left(H_{2n}-\frac{1}{2}H_n\right)\frac{1}{n}$$

i.e.

$$F_n=\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{n-k}4^{n-k}=\binom{2n}{n}(2H_{2n}-H_n).$$

We have for the LHS

$$4^n[w^n]\log\frac{1}{1-w}\sum_{k=0}^{n-1}\binom{2k}{k}w^k 4^{-k}.$$

The coefficient extractor enforces the upper limit, we may extend to infinity and we find

$$4^n[w^n]\log\frac{1}{1-w}\frac{1}{\sqrt{1-w}}=[w^n]\log\frac{1}{1-4w}\frac{1}{\sqrt{1-4w}}.$$

Call the OGF $F(w)$. We get

$$F'(w)=\frac{4}{\sqrt{1-4w}^3}+\frac{2}{1-4w}F(w).$$

Extracting the coefficient on $[w^n]$ we get

$$(n+1)F_{n+1}=4^{n+1}(-1)^n\binom{-3/2}{n}+2\sum_{q=0}^{n}F_q 4^{n-q}$$

$$=4^{n+1}(-1)^n\frac{n+1}{(-1/2)}\binom{-1/2}{n+1}+2\sum_{q=0}^{n}F_q 4^{n-q}$$

$$=2(n+1)\binom{2n+2}{n+1}+2\sum_{q=0}^{n}F_q 4^{n-q}$$

which also yields

$$\frac{1}{4}(n+2)F_{n+2}=\frac{1}{2}(n+2)\binom{2n+4}{n+2}+2\sum_{q=0}^{n+1}F_q 4^{n-q}.$$

Subtract to get

$$\frac{1}{4}(n+2)F_{n+2}$$

$$= (n+1)F_{n+1} + \frac{1}{2}(n+2)\binom{2n+4}{n+2} - 2(n+1)\binom{2n+2}{n+1} + \frac{1}{2}F_{n+1}.$$

Introducing $G_n = F_n\binom{2n}{n}^{-1}$ and dividing by $\binom{2n+2}{n+1}$ we get

$$\frac{1}{2}(2n+3)G_{n+2} = (n+3/2)G_{n+1} + 1 \quad \text{or} \quad G_n = G_{n-1} + \frac{1}{n-1/2}.$$

so that

$$G_n = \sum_{q=1}^{n} \frac{1}{q-1/2} = 2\sum_{q=1}^{n} \frac{1}{2q-1} = 2H_{2n-1} - H_{n-1} = 2H_{2n} - H_n.$$

This is the claim (we have $F_0 = G_0 = 0$ from the generating function) and it completes the entire argument.

This was math.stackexchange.com problem 1148203.

61.15 Bernoulli / Stirling number identity

We seek to show that

$$\sum_{k=0}^{n} \left[{n+1 \atop k+1}\right] B_k = \frac{n!}{n+1}.$$

The LHS is

$$\sum_{k=0}^{n} \left[{n+1 \atop n-k+1}\right] B_{n-k}$$

$$= \sum_{k=0}^{n} \left[{n+1 \atop n-k+1}\right] (n-k)![z^{n-k}]\frac{z}{\exp(z)-1}$$

$$= [z^n]\frac{z}{\exp(z)-1}\sum_{k=0}^{n}(n-k)!z^k(n+1)![w^{n+1}]\frac{1}{(n-k+1)!}\left(\log\frac{1}{1-w}\right)^{n-k+1}$$

$$= (n+1)![z^n][w^{n+1}]\frac{z}{\exp(z)-1}\sum_{k=0}^{n}\frac{z^k}{n-k+1}\left(\log\frac{1}{1-w}\right)^{n-k+1}$$

$$= n![z^n][w^n]\frac{z}{\exp(z)-1}\frac{1}{1-w}\sum_{k=0}^{n} z^k\left(\log\frac{1}{1-w}\right)^{n-k}.$$

We can certainly extend the sum to infinity because of the coefficient extractor in z and we find

$$n![z^n][w^n]\frac{z}{\exp(z)-1}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^n\frac{1}{1-z/\log\frac{1}{1-w}}.$$

At this point we recognize that we need complex variables and write

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^n$$
$$\times\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}\frac{z}{\exp(z)-1}\frac{1}{1-z/\log\frac{1}{1-w}}\,dz\,dw.$$

Now when we summed the geometric series we introduced the condition $|z| < \left|\log\frac{1}{1-w}\right|$ for convergence. Continuing,

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1}$$
$$\times\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}\frac{z}{\exp(z)-1}\frac{1}{z-\log\frac{1}{1-w}}\,dz\,dw.$$

Now we have with $\gamma \ll 1$ that

$$\left|\log\frac{1}{1-w}\right| > \gamma-\sum_{q\ge 2}\frac{\gamma^q}{q} > \gamma-\sum_{q\ge 2}\gamma^q = \gamma-\frac{\gamma^2}{1-\gamma}.$$

This is because $\log\frac{1}{1-w} = w + w^2/2 + w^3/3 + \cdots$ and the first term has modulus γ so that we minimize the whole if we subtract the maximum modulus of all remaining terms. If we choose this last term for ε we have convergence. As an example $\gamma = 1/10$ and $\varepsilon = 4/45$ will work.

Summarizing we have that all the poles in z at $\rho_k = 2\pi ik$ where $|k| \ge 1$ are outside the contour in z, as is the pole at $\log\frac{1}{1-w}$. Note also that with the principal branch of the logarithm $|\arg(\log\frac{1}{1-w})| \le \pi$ so the pole at $\log\frac{1}{1-w}$ does not coincide with any of the ρ_k.

We evaluate the inner integral using the fact that residues sum to zero. Hence we require minus the contribution from the logarithm pole and minus the contribution from the ρ_k. The former yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1}\left(\log\frac{1}{1-w}\right)^{-n}\frac{1}{1/(1-w)-1}\,dw$$
$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+2}}\log\frac{1}{1-w}\,dw = \frac{1}{n+1}.$$

Multiply by $n!$ to get $n!/(n+1)$ which is the claim. It remains to show that the ρ_k contribute zero. These are all simple. We get

$$\lim_{z\to\rho_k} \frac{z-\rho_k}{\exp(z)-1} = \lim_{z\to\rho_k} \frac{1}{\exp(z)} = 1.$$

This yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1} \frac{1}{\rho_k^n}\frac{1}{\rho_k - \log\frac{1}{1-w}}\; dw.$$

Note that

$$\left|\log\frac{1}{1-w}\right| = \left|\sum_{q\ge 1}\frac{w^q}{q}\right| \le \sum_{q\ge 1}\frac{\gamma^q}{q} < \frac{\gamma}{1-\gamma}.$$

With the choice of contour above we have a bound of 1/9 on this term. Hence we may expand into a convergent geometric series, of which only the initial segment can possibly contribute:

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1} \frac{1}{\rho_k^{n+1}}\frac{1}{1-\frac{1}{\rho_k}\log\frac{1}{1-w}}\; dw$$

$$= [w^n]\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1}\frac{1}{\rho_k^n}\sum_{q=0}^{n}\frac{1}{\rho_k^q}\left(\log\frac{1}{1-w}\right)^q.$$

Note however that we are extracting a coefficient on $[w^n]$ from a term that starts at w^{n+1+q}, and we have a zero contribution, which concludes the argument.

This was DLMF [DLMF, Eq. 24.15.8].

61.16 Formal power series vs contour integration

We seek to show that

$$\sum_{q=0}^{K}(-1)^q\binom{2n+1-q}{q}\binom{2n-2q}{K-q} = \frac{1}{2}(1+(-1)^K).$$

We do it two ways, using formal power series and residue operators as well as contour integration.

Formal power series

The LHS is

$$\sum_{q=0}^{K}(-1)^q\binom{2n+1-q}{2n+1-2q}\binom{2n-2q}{K-q}$$

$$= [z^K](1+z)^{2n} \sum_{q=0}^{K} (-1)^q \binom{2n+1-q}{2n+1-2q} \frac{z^q}{(1+z)^{2q}}.$$

Here we may extend to infinity because of the coefficient extractor in z. We find

$$[z^K](1+z)^{2n}[w^{2n+1}](1+w)^{2n+1} \sum_{q\ge 0} (-1)^q \frac{z^q}{(1+z)^{2q}} \frac{w^{2q}}{(1+w)^q}$$

$$= [z^K](1+z)^{2n}[w^{2n+1}](1+w)^{2n+1} \frac{1}{1+zw^2/(1+z)^2/(1+w)}$$

$$= [z^K](1+z)^{2n+2}[w^{2n+1}](1+w)^{2n+2} \frac{1}{(1+z)^2(1+w)+zw^2}.$$

Important note: what we have here is that $\binom{2n+1-q}{2n+1-2q}$ is zero when $2n+1-2q$ goes negative. This is not always what CAS systems might use. Maple for example uses that $\binom{a}{b}$ for $b<0$ is $\binom{a}{a-b}$ if $a\ge b$. This is a generalization that we will use in the second answer. It applies here because we replace $\binom{2n+1-q}{q}$ by $\binom{2n+1-q}{2n+1-2q}$ which is zero when $2q > 2n+1$ under the first rule.

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{2n+2}}(1+w)^{2n+2} \frac{1}{(1+z)^2(1+w)+zw^2}.$$

Now we put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = \frac{1}{(1-v)^2}\, dv$ to get

$$\operatorname*{res}_{v} \frac{1}{v^{2n+2}} \frac{1}{(1+z)^2(1+v/(1-v)) + zv^2/(1-v)^2} \frac{1}{(1-v)^2}.$$

Restoring the coefficient extractor in z we have

$$[z^K](1+z)^{2n+2}[v^{2n+1}] \frac{1}{(1+z)^2(1-v)+zv^2}$$

$$= [z^K](1+z)[v^{2n+1}] \frac{1}{(1+z)^2 - (1+z)^3 v + zv^2(1+z)^2}$$

$$= [z^K] \frac{1}{1+z} [v^{2n+1}] \frac{1}{1-(1+z)v+zv^2}$$

$$= [z^K] \frac{1}{1+z} [v^{2n+1}] \frac{1}{(1-v)(1-vz)}$$

$$= [z^K] \frac{1}{1+z} \sum_{q=0}^{2n+1} z^q = [z^K] \frac{1-z^{2n+2}}{1-z^2}.$$

This is clearly an even function hence zero when K is odd. When K is even and $K < 2n+2$ we get a value of one, and when K is even and $K \ge 2n+2$ the two contributions cancel, for a value of zero. Thus we have

$$\frac{1}{2}(1+(-1)^K)[[K<2n+2]].$$

Contour integration

We again seek to show that

$$\sum_{q=0}^{K}(-1)^q\binom{2n+1-q}{q}\binom{2n-2q}{K-q}=\frac{1}{2}(1+(-1)^K).$$

This time we will not flip the lower index of the first binomial coefficient and we get an answer that agrees with the second rule, which is used by CAS.

The LHS is

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\sum_{q=0}^{K}(-1)^q\frac{1}{z^{q+1}}(1+z)^{2n+1-q}\frac{1}{w^{K-q+1}}(1+w)^{2n-2q}\;dz\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1+w)^{2n}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}$$

$$\times\sum_{q=0}^{K}(-1)^q\frac{1}{z^q}(1+z)^{-q}w^q(1+w)^{-2q}\;dz\;dw.$$

Here we may extend q beyond K to infinity because the pole at zero in w is canceled for the extra values. We obtain

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1+w)^{2n}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}$$

$$\times\frac{1}{1+w/(1+w)^2/z/(1+z)}\;dz\;dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1+w)^{2n+2}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}$$

$$\times\frac{1}{(1+z(1+w))(w+z(1+w))}\;dz\;dw.$$

The pole at zero in z is gone but a new pole has appeared inside the contour. Note that when we summed the geometric series we required $|w/(1+w)^2|<|z(1+z)|$. We have with $\gamma\ll 1$ and $\varepsilon\ll 1$ that $|w/(1+w)^2|\le\gamma/(1-\gamma)^2<2\gamma$ and $|z(1+z)|\ge\varepsilon(1-\varepsilon)>\frac{1}{2}\varepsilon$. Therefore taking $\varepsilon=4\gamma$ will work e.g. $\gamma=1/11$ and $\varepsilon=4/11$.

We have for the first simple pole at $z_0=-1/(1+w)$ that $|-1/(1+w)|>1/(1+\gamma)>4\gamma=\varepsilon$. This pole is not inside the contour. The second pole is at $z_1=-w/(1+w)$ and we have $|-w/(1+w)|<\gamma/(1-\gamma)<4\gamma=\varepsilon$. This pole is inside the contour. We thus write

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1+w)^{2n+1}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}$$
$$\times\frac{1}{(1+z(1+w))(w/(1+w)+z)}\;dz\;dw.$$

Evaluating the residue from the simple pole at z_1 we find

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1+w)^{2n+1}(1-w/(1+w))^{2n+2}\frac{1}{1-(1+w)w/(1+w)}\;dw$$
$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{1-w^2}\;dw.$$

This is

$$[w^K]\frac{1}{1-w^2}=\frac{1}{2}(1+(-1)^K)$$

as claimed.

Alternate evaluation

Returning to the start we seek

$$\sum_{q=0}^{K}(-1)^q\binom{2n+1-q}{q}\binom{2n-2q}{K-q}=\frac{1}{2}(1+(-1)^K).$$

With

$$\binom{2n-2q}{K-q}=\binom{2n-2q}{2n-K-q}=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K-q+1}}\frac{1}{(1-w)^{2n-K-q+1}}\;dw$$

the LHS becomes

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\sum_{q=0}^{K}(-1)^q\frac{1}{z^{q+1}}(1+z)^{2n+1-q}\frac{1}{w^{K-q+1}}\frac{1}{(1-w)^{2n-K-q+1}}\;dz\;dw.$$

Here we may extend the sum to infinity due to the pole at $w=0$ vanishing when $q>K$. This yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}$$
$$\times\sum_{q\ge 0}(-1)^q\frac{1}{z^q}(1+z)^{-q}w^q(1-w)^q\;dz\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}$$
$$\times\frac{1}{1+w(1-w)/z/(1+z)}\;dz\;dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}$$
$$\times\frac{1}{z(1+z)+w(1-w)}\;dz\;dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}$$
$$\times\frac{1}{(z+w)(z-(w-1))}\;dz\;dw.$$

Once more the pole at $z=0$ is gone but a new one has appeared (two of them, in fact). To see this note that in the summation of the infinite series we require for convergence that $|w(1-w)|<|z(1+z)|$. We have for $\varepsilon\ll 1$ and $\gamma\ll 1$ that $|z(1+z)|\ge\varepsilon(1-\varepsilon)\ge\frac{1}{2}\varepsilon$ and $|w(1-w)|\le\gamma(1+\gamma)\le\frac{3}{2}\gamma$. Hence $3\gamma<\varepsilon$ will work e.g. take $\gamma=\varepsilon/4$ as in $\gamma=1/16$ and $\varepsilon=1/4$. In particular $|w|<|z|$ so the pole at $z_0=-w$ is inside the contour. On the other hand the closest that the pole at $z_1=w-1$ which is on a circle at $z=-1$, rotating with radius γ, gets to the origin is $1-\gamma=1-\varepsilon/4>\varepsilon$ as long as $\varepsilon<4/5$, so this is definitely not inside the contour. Hence the contribution from $z_0=-w$ is the only one and it yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}(1-w)^{2n+2}\frac{1}{1-2w}\;dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1-w)^{K+1}\frac{1}{1-2w}\;dw$$
$$= \sum_{q=0}^{K}\binom{K+1}{q}(-1)^q 2^{K-q} = -(-1)^{K+1}\frac{1}{2}+\frac{1}{2}\sum_{q=0}^{K+1}\binom{K+1}{q}(-1)^q 2^{K+1-q}$$
$$= -(-1)^{K+1}\frac{1}{2}+\frac{1}{2}1^{K+1} = \frac{1}{2}(1+(-1)^K).$$

We once more have the claim.

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