Egorychev method and the evaluation of combinatorial sums: Parts 1 and 2: formal power series and residue operators

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The Egorychev method is from the book by G.P.Egorychev [Ego84]. We collect several examples, the focus being on computational methods to produce results. Those that are from posts to math.stackexchange.com have retained the question answer format from that site. The website for this document is at this hyperlink:

https://pnp.mathematik.uni-stuttgart.de/iadm/Riedel/egorychev.html.

The crux of the method is the use of formal power series and residue operators to represent binomial coefficients, exponentials, the Iverson bracket and Stirling numbers, Catalan numbers, Harmonic numbers, Eulerian numbers and Bernoulli numbers. The residue operator algebra encapsulates the underlying complex integrals. There is a tutorial at the following article: [RM23].

We use these types of integrals:

- *First binomial coefficient integral (B_1)*

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^n}{z^{k+1}}\; dz = \operatorname*{res}_{z} \frac{(1+z)^n}{z^{k+1}}$$

  where $0 < \varepsilon < \infty$.

- *Second binomial coefficient integral (B_2)*

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{(1-z)^{k+1} z^{n-k+1}}\; dz = \operatorname*{res}_{z} \frac{1}{(1-z)^{k+1} z^{n-k+1}}$$

  where $0 < \varepsilon < 1$.

- *Exponentiation integral (E)*

$$n^k = \frac{k!}{2\pi i}\int_{|z|=\varepsilon} \frac{\exp(nz)}{z^{k+1}}\; dz = k! \operatorname*{res}_{z} \frac{\exp(nz)}{z^{k+1}}$$

  where $0 < \varepsilon < \infty$.

- *Iverson bracket (I)*

$$[[k \leq n]] = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{z^k}{z^{n+1}} \frac{1}{1-z} \; dz = \operatorname*{res}_z \frac{z^k}{z^{n+1}} \frac{1}{1-z}$$

  where $0 < \varepsilon < 1$.

- *Stirling numbers of the first kind*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \left(\log \frac{1}{1-z} \right)^k \; dz = \frac{n!}{k!} \operatorname*{res}_z \frac{1}{z^{n+1}} \left(\log \frac{1}{1-z} \right)^k$$

  where $0 < \varepsilon < 1$.

- *Stirling numbers of the second kind*

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{n!}{k!} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \left(\exp(z) - 1 \right)^k \; dz = \frac{n!}{k!} \operatorname*{res}_z \frac{1}{z^{n+1}} \left(\exp(z) - 1 \right)^k$$

  where $0 < \varepsilon < \infty$.

The residue at infinity is coded R.

Contents

List of identities in this document

section 1.1 B_1, B_2

$$\sum_{l=0}^{m}(-4)^l\binom{m}{l}\binom{2l}{l}^{-1}\sum_{k=0}^{n}\frac{(-4)^k}{2k+1}\binom{n}{k}\binom{2k}{k}^{-1}\binom{k+l}{l}=\frac{1}{2n+1-2m}.$$

section 1.2 B_1

$$\sum_{l=0}^{n}\binom{n}{l}^2(x+y)^{2l}(x-y)^{2n-2l}=\sum_{l=0}^{n}\binom{2l}{l}\binom{2n-2l}{n-l}x^{2l}y^{2n-2l}.$$

section 1.3 B_2

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+c}=\binom{n+c}{c}^{-1}\frac{(-1)^c}{c}\left(1-2^{n+1}\sum_{q=0}^{c-1}\binom{n+q}{q}(-1)^q\right).$$

section 1.4 B_1

$$\sum_{j=0}^{n-k}(-1)^j\binom{2k+2j}{j}\binom{n+k+j+1}{n-k-j}=[[(n-k)\text{ is even}]]=\frac{1+(-1)^{n-k}}{2}$$

section 1.5 B_2

$$\sum_{k=0}^{b-1}\binom{a+k-1}{a-1}p^a(1-p)^k=\sum_{k=a}^{a+b-1}\binom{a+b-1}{k}p^k(1-p)^{a+b-k-1}$$

section 1.6 B_1

$$(-1)^{n+k}\begin{bmatrix}n\\k\end{bmatrix}=\sum_{j=0}^{n-k}(-1)^j\binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j}\left\{\begin{matrix}n-k+j\\j\end{matrix}\right\}$$

section 1.7 B_1

$$\binom{m+n}{s+1}-\binom{n}{s+1}=\sum_{q=0}^{s}\frac{m}{q+1}\binom{m+1+2q}{q}\binom{n-2-2q}{s-q}$$

section 1.8 B_1

$$\sum_{k=q}^{2n}\binom{2n+k}{2k}\frac{(2k-1)!!}{(k-q)!}(-1)^k$$

is zero when q is odd, and

$$\frac{(-1)^{n+q/2}}{2^{2n}} \frac{(2n+q)!}{(n-q/2)! \times (n+q/2)!}$$

otherwise.

section 1.9

$$\sum_{j=n}^{2n} \sum_{k=j+1-n}^{j} (-1)^j 2^{j-k} \binom{2n}{j} \left\{ {j \atop k} \right\} \left[{k \atop j+1-n} \right] = 0.$$

section 1.10

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{n-j} \binom{n-j}{j} = \binom{n+k+m}{m+1}.$$

section 1.11

With Fibonacci numbers

$$F_{2n+2} = \sum_{p=0}^{n} \sum_{q=0}^{n} \binom{n-p}{q} \binom{n-q}{p}.$$

section 1.12

$$\sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^q b^K + a^K b^q}{(a+b)^{q+K}} = 1$$

section 1.13

$$\binom{r+2n-1}{n-1} - \binom{2n-1}{n-1} = \sum_{k=1}^{n-1} \binom{2k-1}{k} \binom{r+2(n-k)-1}{r+n-k}$$

section 1.14

$$\sum_{k=1}^{n} \left(-\frac{1}{4}\right)^k \binom{2k}{k}^2 \frac{1}{1-2k} \binom{n+k-2}{2k-2}$$

is zero when n is odd and

$$\left[\left(\frac{1}{4}\right)^m \binom{2m}{m} \frac{1}{1-2m} \right]^2$$

when $n = 2m$ is even.

section 1.15

$$\sum_{n=0}^{N}\sum_{k=0}^{N}\frac{(-1)^{n+k}}{n+k+1}\binom{N}{n}\binom{N}{k}\binom{N+n}{n}\binom{N+k}{k}=\frac{1}{2N+1}.$$

section 1.16

$$\sum_{k=3}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{k-2}\binom{j(n+1)+k-3}{n-2}=(-1)^{n-1}\left[\binom{n}{2}-\binom{2n+1}{n-2}\right]$$

section 1.17

$$n\sum_{k=0}^{n}\frac{(-1)^k}{2n-k}\binom{2n-k}{k}x^k y^{2n-2k}=\frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n}{2k}y^{2k}(y^2-4x)^{n-k}.$$

section 1.18

$$\sum_{k=0}^{n}(-1)^k 4^{n-k}\binom{2n-k}{k}=2n+1$$

section 1.19

$$\sum_{k=0}^{l}\binom{k}{m}\binom{k}{n}=\sum_{k=0}^{n}(-1)^k\binom{l+1}{m+k+1}\binom{l-k}{n-k}.$$

section 1.20

$$\sum_{j=0}^{k}\binom{2n}{2j}\binom{n-j}{k-j}=\frac{4^k n}{n+k}\binom{n+k}{n-k}.$$

section 1.21

$$\sum_{q=m}^{n-k}(-1)^{q-m}\binom{k-1+q}{k-1}\left\{{q \atop m}\right\}\left[{n \atop q+k}\right]=\binom{n-1}{m}\left[{n-m \atop k}\right].$$

section 1.22

$$\sum_{q=0}^{N}(-1)^q\binom{2q}{q}\binom{N+q}{N-q}\frac{q^2}{(q+1)^2}=(-1)^N+\frac{1}{N(N+1)}$$

section 1.23

$$\sum_{k=0}^{n}\binom{n}{k}^2\sum_{l=0}^{k}\binom{k}{l}\binom{n}{l}\binom{2n-l}{n}=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$$

section 1.24

$$\sum_{k=1}^{a}(-1)^{a-k}\binom{a}{k}\binom{b+k}{b+1} = \binom{b}{a-1}$$

section 1.25

$$\sum_{q=0}^{k}(-1)^{q-j}\binom{n+q}{q}\binom{n+k-q}{k-q}\binom{2n}{n+j-q} = \binom{2n}{n}$$

where $0 \le j \le k$.

section 1.26

$$S_{n,m} = \sum_{k=m}^{n}\binom{k+m}{2m}\binom{2n+1}{n+k+1} = \binom{n}{m}4^{n-m}.$$

section 1.27

$$\sum_{j=0}^{k}\binom{k}{j}\binom{j/2}{n}(-1)^{n-j} = \frac{k}{n}2^{k-2n}\binom{2n-k-1}{n-1}$$

section 1.28

$$\sum_{k=1}^{n}(-1)^{n-k}k^{n}\binom{n+1}{n-k} = 1$$

section 1.29

$$\sum_{q=a+1}^{n}\binom{q-1}{a}\binom{n-q}{k-a} = \binom{n}{k+1}$$

or alternatively

$$\sum_{q=0}^{n}\binom{q}{a}\binom{n-q}{b} = \binom{n+1}{a+b+1}.$$

section 1.30

$$\sum_{k\ge 0}\frac{(2k+1)^2}{(p+k+1)(q+k+1)}\binom{2p}{p-k}\binom{2q}{q-k} = \frac{1}{p+q+1}\binom{2p+2q}{p+q}$$

section 1.31

With

$$G_{n,j}=\sum_{k=1}^{n}\frac{k^j(-1)^{n-k}\binom{n}{k}}{\frac{1}{2}n(n+1)-k}$$

we have

$$G_{n,j}=\frac{(\frac{1}{2}n(n+1))^{j-1}n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)}-[[j>n]]n!\sum_{q=0}^{j-1-n}\left(\frac{1}{2}n(n+1)\right)^q\left\{{j-1-q \atop n}\right\}$$

section 1.32

$$\sum_{p=q}^{k}(-1)^p\binom{k}{p}(q-p)^k=\sum_{p=q}^{k}\left\langle{k \atop p}\right\rangle.$$

section 1.33

$$\sum_{k=0}^{n}(-1)^k\frac{2^{n-k}\binom{n}{k}}{(m+k+1)\binom{m+k}{k}}=\sum_{k=0}^{n}\frac{\binom{n}{k}}{m+k+1}$$

section 1.34

$$\sum_{k\geq 1}\left[\binom{\lfloor\frac{k}{2}\rfloor}{m}+\binom{\lceil\frac{k}{2}\rceil}{m}\right]\binom{n-1}{k-1}=2^{n-2m}\binom{n-m}{m-1}\frac{n+1}{m}$$

section 1.35

$$\sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle x^{n-k}=(1-x)^n\sum_{k=0}^{n}\left\{{n \atop k}\right\}k!\left(\frac{x}{1-x}\right)^k$$

section 1.36

$$\sum_{k=0}^{r}k^p\binom{m}{k}\binom{n}{r-k}=\sum_{j=0}^{p}m^{\underline{j}}\binom{m+n-j}{m+n-r}\left\{{p \atop j}\right\}$$

section 1.37

Li-Shanlan idenity:

$$\binom{m+k}{k}^2=\sum_{q=0}^{m}\binom{k}{m-q}^2\binom{2k+q}{q}$$

section 1.38

Two alternate representations of second order Eulerian numbers:

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{2n+1}{k-j}\left\{ n+j \atop j \right\} = \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k}(-1)^{j}\binom{2n+1}{j}\left[2n-k-j+1 \atop n-k-j+1 \right]$$

section 1.39

Two alternate representations of second order Eulerian numbers:

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}\left\{\!\!\left\{ n+j \atop j \right\}\!\!\right\} = \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k+1}(-1)^{n-k-j+1}\binom{n-j}{k-1}\left[\!\!\left[n+j \atop j \right]\!\!\right]$$

section 1.40

$$\left[n \atop n-k \right] - \left\{ n \atop n-k \right\} = \sum_{j=0}^{k}\left(\binom{n+j-1}{2k} - \binom{n+k-j}{2k}\right)\left\langle\!\!\left\langle k \atop j \right\rangle\!\!\right\rangle$$

section 1.41

$$\sum_{k=0}^{2n}(-1)^{k}\binom{n+k}{k}^{-1}\binom{2n}{k}\binom{2k}{k} = 1$$

section 1.42

$$\sum_{k=1}^{n}\binom{2n-2k}{n-k}\frac{H_{2k}-2H_{k}}{2n-2k-1}\binom{2k}{k} = \frac{1}{n}\left[4^{n} - 3\binom{2n-1}{n}\right]$$

section 1.43

$$\sum_{q=0}^{n}\binom{n}{q}q^{k} = \sum_{q=1}^{k}n^{\underline{q}}\left\{ k \atop q \right\}2^{n-q}$$

section 1.44

$$\sum_{r=0}^{n}r^{k} = (n+1)\sum_{q=1}^{k}n^{\underline{q}}\frac{1}{q+1}\left\{ k \atop q \right\}$$

section 1.45

$$\sum_{k=0}^{n}\binom{k}{m}\binom{n-k}{r-m} = \binom{n+1}{r+1}$$

section 1.46

$$\sum_{r=0}^{n} 2^{n-r}\binom{n+r}{r} = 4^n$$

section 1.47

With

$$\mathcal{K}_k(x;n) = \sum_{j=0}^{k}(-1)^j\binom{x}{j}\binom{n-x}{k-j}$$

we have

$$\sum_{\ell=0}^{n}\binom{n-\ell}{n-m}\mathcal{K}_\ell(x;n) = 2^m \times \binom{n-x}{m}.$$

section 1.48

$$B_n = \sum_{k=0}^{n}(-1)^k\frac{1}{k+1}H_{k+1}(k+2)!\left\{ {n+1 \atop k+1} \right\} + (-1)^{n+1}(n+1)$$

section 1.49

$$\sum_{q\geq k}\binom{m+1}{2q+1}\binom{q}{k} = \binom{m-k}{k}2^{m-2k}$$

section 1.50

$$\sum_{k=1}^{n-1} k\binom{n}{k}\frac{(2n-2k-1)!!}{(2n-1)!!} \sim \frac{1}{2}\sqrt{e}$$

section 1.51

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1+m} = \frac{(-1)^n \times m^{\underline{n}}}{(n+m+1)^{\underline{n+1}}}$$

section 1.52

$$\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n+k}{n-k}\binom{2k}{k} = \frac{1}{2n+1}$$

section 1.53

$$\left[{n \atop n-k} \right] = (-1)^k\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\sum_{q=0}^{j}(-1)^q\binom{j+1}{q+1}\binom{j+qn+q}{qn+q}^{-1}\left\{ {j+qn+q \atop qn+q} \right\}$$

section 1.54

$$\begin{bmatrix} n \\ n-k \end{bmatrix} = (-1)^k \binom{n-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \binom{jn+k}{k}^{-1} \left\{ \begin{matrix} jn+k \\ jn \end{matrix} \right\}$$

section 1.55

$$\begin{bmatrix} n \\ n-k \end{bmatrix} = \sum_{q=0}^{k} (-1)^{k-q} \binom{n+q-1}{n-k-1} \binom{n+k}{k-q} \left\{ \begin{matrix} k+q \\ q \end{matrix} \right\}$$

section 1.56

$$\begin{aligned} \sum_{k=0}^{n} k^p &= \sum_{j=1}^{p+1} n^j \sum_{k=j}^{p+1} \frac{1}{k} \left\{ \begin{matrix} p+1 \\ k \end{matrix} \right\} (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \\ &= \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \sum_{k=1}^{p-1} \binom{p}{k} \frac{B_{p+1-k}}{p+1-k} n^k \end{aligned}$$

section 1.57

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n+j} = \sum_{k=0}^{j} \binom{x-n}{k} (n+k)! \left\{ \begin{matrix} n+j \\ n+k \end{matrix} \right\}$$

section 1.58

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k} k^r = \sum_{k=0}^{r} (-1)^k \binom{x}{k} \binom{n-x}{n-k} k! \left\{ \begin{matrix} r \\ k \end{matrix} \right\}$$

section 1.59

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+j} = (-1)^n (n+j)! \sum_{k=0}^{j} \binom{j-n}{j-k} \binom{n}{k} \frac{k!}{(k+j)!} \left\{ \begin{matrix} k+j \\ k \end{matrix} \right\}$$

section 1.60

$$\sum_{k=0}^{n} (-1)^k \binom{n+x}{n-k} \frac{1}{k+1} = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} B_k(x)$$

section 1.61

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{k^r}{2^{2k}} = \frac{2n+1}{2^{2n}} \binom{2n}{n} \sum_{k=0}^{r} \binom{n}{k} \frac{1}{2k+1} k! \left\{ \begin{matrix} r \\ k \end{matrix} \right\}$$

section 1.62

$$x^n = (-1)^{m+n} \sum_{k=0}^{m+1} \binom{x+k-1}{m} \sum_{p=0}^{k} (-1)^p \binom{m+1}{p} (k-p)^n$$

section 1.63

$$\sum_{k=0}^{n} \binom{2k+1}{j} = \frac{(-1)^{j+1}}{2^{j+2}} \left\{ \sum_{k=0}^{j+1} (-1)^k \binom{2n+3}{k} 2^k + 1 \right\}$$

section 1.64

$$\sum_{k=0}^{n} (-1)^k \binom{j+k}{j} = \frac{(-1)^j}{2^{j+1}} \left\{ (-1)^n \sum_{k=0}^{j} (-1)^k \binom{n+j+1}{k} 2^k + (-1)^j \right\}$$

section 1.65

$$\sum_{k=0}^{n} \binom{x}{n-k} \binom{x}{n+k} = \frac{1}{2} \left\{ \binom{2x}{2n} + \binom{x}{n}^2 \right\}$$

section 1.66

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{x}{2k} \binom{x}{n-2k} = \frac{1}{2} \binom{2x}{n} + \frac{1}{2} (-1)^{n/2} \binom{x}{n/2} \frac{1+(-1)^n}{2}$$

section 1.67

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{x}{2k} \binom{2n-x}{n-2k} = \frac{1}{2} \left\{ \binom{2n}{n} + (-1)^n 2^{2n} \binom{\frac{x-1}{2}}{n} \right\}$$

section 1.68

$$\sum_{k=0}^{r} \binom{x}{k} \binom{-x}{n-k} = \frac{n-r}{n} \binom{x-1}{r} \binom{-x}{n-r}$$

section 1.69

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{x}{k} \binom{x-k}{n-2k} 2^{n-2k} = \binom{2x}{n}$$

section 1.70

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y+k}{n-k} 2^{2k} = \sum_{k=0}^{n} \binom{2x}{k} \binom{y}{n-k} 2^k = \sum_{k=0}^{n} \binom{2x}{k} \binom{2x+y-k}{n-k}$$

section 1.71

$$\sum_{k=0}^{n}\binom{2x}{2k}\binom{x-k}{n-k}=\frac{x}{x+n}\binom{x+n}{2n}2^{2n}=\frac{2^{2n}}{(2n)!}\prod_{k=0}^{n-1}(x^2-k^2)$$

section 1.72

$$\sum_{k=0}^{n}\binom{2x+1}{2k+1}\binom{x-k}{n-k}=\frac{2x+1}{2n+1}\binom{x+n}{2n}2^{2n}=\frac{2x+1}{(2n+1)!}\prod_{k=0}^{n-1}((2x+1)^2-(2k+1)^2)$$

section 1.73

$$\sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{x}{n+k}=\frac{1}{2}\left\{\binom{x}{n}+\binom{x}{n}^2\right\}$$

section 1.74

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2x-2n}{x-k}=\frac{1}{2}(-1)^n\left\{\binom{x}{n}+\binom{x}{n}^2\right\}\binom{2x}{x}\binom{2x}{2n}^{-1}$$

section 1.75

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2n-x}{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{x}{k}\binom{2n-2x}{n-2k}$$

$$=(-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n-k}{n-k}\binom{2n-x}{k}2^k$$

$$=\sum_{k=0}^{n}(-1)^k2^k\binom{x}{k}\binom{2n-k}{n}$$

$$=\frac{2^n}{n!}\prod_{k=0}^{n-1}(2k+1-x)=(-1)^n2^{2n}\binom{\frac{x-1}{2}}{n}$$

section 1.76

$$\sum_{k=0}^{n}\binom{n}{k}^2k^r=\sum_{k=0}^{r}\binom{n}{k}\binom{2n-k}{n}k!\left\{{r \atop k}\right\}$$

section 1.77

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^2=\frac{1}{2}(-1)^n\left\{\binom{2n}{n}+\binom{2n}{n}^2\right\}$$

section 1.78

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2n}{2k}^2=\frac{1}{4}\binom{4n}{2n}+\frac{1}{4}(-1)^n\binom{2n}{n}+\frac{1+(-1)^n}{4}\binom{2n}{n}^2$$

section 1.79

$$2^{2n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}=\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}5^k$$

$$2^{2n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}=\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{2k}{k}3^k$$

section 1.80

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{x}{x+k}=2^{2n}\binom{x+n}{n}^{-1}\binom{n+x-1/2}{n}$$

section 1.81

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{(2k-1)(2n-2k+1)}=\frac{2^{4n}}{2n(2n+1)}\binom{2n}{n}^{-1}$$

section 1.82

$$\sum_{k=0}^{n}\binom{4n-4k}{2n-2k}\binom{4k}{2k}=2^{4n-1}+2^{2n-1}\binom{2n}{n}$$

$$\sum_{k=0}^{n-1}\binom{4n-4k-2}{2n-2k-1}\binom{4k+2}{2k+1}=2^{4n-1}-2^{2n-1}\binom{2n}{n}$$

section 1.83

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}\binom{2k}{k}\frac{x}{x+k}=(-1)^n\binom{x+n}{n}^{-1}\binom{x-1}{n}$$

section 1.84

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+2r+k}{n+r}2^{n-k}=(-1)^{n/2}\frac{1+(-1)^n}{2}\binom{n+r}{n}^{-1}\binom{n+r}{n/2}\binom{n+2r}{r}$$

$$\sum_{k=0}^{n-r}(-1)^k\binom{n}{k+r}\binom{n+k+r}{k}2^{n-r-k}=(-1)^{(n-r)/2}\frac{1+(-1)^{n-r}}{2}\binom{n}{(n-r)/2}$$

section 1.85

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2k}{n+k}3^{2n-k}=\binom{2n}{n}$$

section 1.86

$$\sum_{k=0}^{n}\binom{4n+1}{2n-2k}\binom{k+n}{n}=2^{2n}\binom{3n}{n}$$

section 1.87

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n-k}\binom{2n+2k+1}{2k}=(-1)^n(n+1)2^{2n}$$

section 1.88

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-k}{n}\frac{k}{(2n-k)\times 2^k}=(-1)^n2^{2n}\binom{-1/4}{n}$$

section 1.89

$$\sum_{k=1}^{n}\binom{n}{k}^2H_k=\binom{2n}{n}(2H_n-H_{2n})$$

section 1.90

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}H_k=\frac{(-1)^n}{n}$$

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}H_{n+k-1}=\frac{(-1)^n}{n}$$

section 1.91

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{1}{k}=2H_n$$

section 1.92

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}$$

$$P_n(x) = \left[\frac{x-1}{2}\right]^n \sum_{k=0}^{n} \binom{n}{k}^2 \left[\frac{x+1}{x-1}\right]^k$$

$$P_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^k \left[\frac{x+1}{2}\right]^k$$

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left[\frac{x-1}{2}\right]^k$$

section 1.93

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} 2^{-k} \sqrt{x^2-1}^{\,k} \left[x - \sqrt{x^2-1}\right]^{n-k}$$

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} 2^{-2k} x^{n-2k} (x^2-1)^k$$

section 1.94

$$\begin{aligned} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} x^{2k} &= 2^{2n} x^n P_n((x+1/x)/2) \\ &= 2^{2n} \frac{2}{\pi} \int_0^{\pi/2} (x^2 \sin^2 t + \cos^2 t)^n \, dt \\ \sum_{k=0}^{n} \binom{-1/2}{k} \binom{-1/2}{n-k} x^{2k} &= (-1)^n x^n P_n((x+1/x)/2) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k-1/2}{n} x^{2k} \end{aligned}$$

section 1.95

$$\sum_{r=0}^{n} \frac{1}{4^r} \binom{2r}{r} = \frac{1}{2^n} \sum_{q=0}^{n} \binom{n+q}{n} \frac{1}{2^q} (n-q+1) = \binom{n+1/2}{n}$$

section 1.96

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}(x + \sqrt{x^2-1}\times\cos t)^n\ dt$$

$$P_n(x) = \frac{1}{m}\sum_{k=0}^{m-1}\left(x + \sqrt{x^2-1}\times\cos\frac{2\pi k}{m}\right)^n$$

$$P_n(x) = \frac{1}{2^n}\frac{1}{2\pi i}\int_{|t-x|=\varepsilon}\frac{(t^2-1)^n}{(t-x)^{n+1}}\ dt$$

section 1.97

$$\sum_{k=0}^{n}\binom{x+ky}{k}\binom{p-x-ky}{n-k} = \begin{cases} y^{p+1}(y-1)^{n-p-1}, & 0 \le p \le n-1 \\ \frac{y^{n+1}-1}{y-1}, & p = n \end{cases}$$

section 1.98

$$\sum_{k=0}^{n}\binom{x+kt}{k}\binom{y-kt}{n-k} = \sum_{k=0}^{n}\binom{x+y-k}{n-k}t^k$$

section 1.99

$$\sum_{k=1}^{n-1}\binom{kx}{k}\binom{nx-kx}{n-k}\frac{1}{kx(nx-kx)} = \frac{2}{nx}\binom{nx}{n}\sum_{k=1}^{n-1}\frac{1}{nx-n+k}$$

section 1.100

$$[z^n]\frac{1}{(1-z)^{\alpha+1}}\log\frac{1}{1-z} = \binom{n+\alpha}{n}(H_{n+\alpha} - H_\alpha)$$

section 1.101

$$\sum_{k=1}^{n}\frac{1}{k}\binom{kx-2}{k-1}\binom{nx-kx}{n-k} = \frac{1}{x}\binom{nx}{n}$$

section 1.102

$$C_{n-1} = \sum_{k=1}^{\lfloor n/2\rfloor}2^{n-2k}\binom{n-2}{n-2k}\frac{1}{k}\binom{2k-2}{k-1}$$

section 1.103

$$\sum_{k=0}^{n-1}\binom{2x}{2k+1}\binom{x-k-1}{n-k-1}=\frac{n}{x+n}2^{2n}\binom{x+n}{2n}$$
$$\sum_{k=0}^{n}\binom{2x}{2k+1}\binom{x-k-1}{n-k}=\frac{x+n}{2n+1}2^{2n+1}\binom{x+n-1}{2n}$$

section 1.104

$$\sum_{k=1}^{a-b}\frac{(a-b-k)!}{(a+1-k)!}=\frac{1}{b}\left[\frac{1}{b!}-\frac{(a-b)!}{a!}\right]$$

section 1.105

$$\sum_{k=a}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k}2^{2k}\frac{2n+1}{2k+1}=(-1)^n\binom{n+a}{2a}2^{2a}$$

section 1.106

$$\sum_{j=1}^{n+1}\binom{n+j}{2j-1}(-1)^{n+j}C_{n+j-1}=0$$

section 1.107

$$\sum_{k=0}^{n}\frac{2k+1}{n+k+1}\binom{x-k-1}{n-k}\binom{x+k}{n+k}=\binom{x}{n}^2$$

section 1.108

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\binom{x+k}{n}=\binom{2x}{n}$$
$$\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{n+1}{2k}\binom{x+k}{n}=\binom{2x+1}{n}$$

section 1.109

$$\sum_{k=0}^{n} \binom{x}{2k} \binom{x-2k}{n-k} 2^{2k} = \binom{2x}{2n}$$

$$\sum_{k=0}^{n} \binom{x+1}{2k+1} \binom{x-2k}{n-k} 2^{2k+1} = \binom{2x+2}{2n+1}$$

section 1.110

$$\sum_{k=0}^{n-p} \binom{2n+1}{2p+2k+1} \binom{p+k}{k} = \binom{2n-p}{p} 2^{2n-2p}$$

$$\sum_{k=0}^{n-p} \binom{2n}{2p+2k} \binom{p+k}{k} = \frac{n}{2n-p} \binom{2n-p}{p} 2^{2n-2p}$$

section 1.111

$$\sum_{k=r}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \binom{k}{r} 2^{n-2k} = (-1)^r \binom{n+1}{2r+1}$$

$$\sum_{k=r}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{k}{r} 2^{n-2k-1} = (-1)^r \binom{n}{2r}$$

section 1.112

$$\sum_{k=m}^{n} (-1)^k 2^{2k} \binom{k}{m} \frac{n}{n+k} \binom{n+k}{2k} = (-1)^n 2^{2m} \frac{n}{n+m} \binom{n+m}{2m}$$

section 1.113

$$\sum_{q=0}^{m} \frac{(-1)^{q-1}}{q+1} \binom{k+q}{q} \binom{k}{q} = \frac{(-1)^{m+1}}{k+1} \binom{k-1}{m} \binom{k+1+m}{k}$$

section 1.114

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{f(x-k)}{k} = H_n f(x) - f'(x)$$

section 1.115

$$f(x+y)=y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}$$

section 1.116

$$f(x+y)=(-1)^m\sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}f(j-k+y)$$

section 1.117

$$\sum_{k=0}^{n}\binom{m+k}{m}^{-1}=\frac{m}{m-1}\left[1-\binom{m+n}{m-1}^{-1}\right]$$

section 1.118

$$(-1)^n\sum_{g=0}^{m}\frac{B_{n+g+1}}{n+g+1}\binom{m}{g}+(-1)^m\sum_{g=0}^{n}\frac{B_{m+g+1}}{m+g+1}\binom{n}{g}=-\frac{1}{n+m+1}\binom{n+m}{m}^{-1}$$

section 1.119

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\frac{f(y+k^2)}{x^2-k^2}=(-1)^n\frac{f(x^2+y)}{2x(x-n)}\binom{x+n}{2n}^{-1}+\frac{1}{2}\binom{2n}{n}\frac{f(y)}{x^2}$$

section 1.120

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+r}{k}^{-1}f(y-k)=-\sum_{k=1}^{r}(-1)^k\binom{r}{k}\binom{k+n}{k}^{-1}f(y+k)$$

section 1.121

$$\sum_{q=0}^{\lfloor n/2\rfloor}(n-2q)^n\binom{n}{q}(-1)^q=2^{n-1}n!$$

section 1.122

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(j)=(-1)^n\binom{x-1}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{xf(k)}{x-k}$$

section 1.123

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}2^{2k}}{k}\binom{n}{k}\binom{2k}{k}^{-1}=2H_{2n}-H_n$$

section 1.124

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^{-1} = (1+(-1)^n)\frac{n+1}{n+2}$$

section 1.125

$$\alpha_n = \sum_{k=0}^{n}\binom{n+k}{n-k}\beta_k \Leftrightarrow \beta_n = \sum_{k=0}^{n}(-1)^{n-k}\frac{2k+1}{2n+1}\binom{2n+1}{n-k}\alpha_k$$

section 1.126

$$\sum_{k=0}^{n}x^k\binom{n}{k}^{-1} = (n+1)\left(\frac{x}{1+x}\right)^{n+1}\sum_{k=1}^{n+1}\frac{1}{k}\frac{1+x^k}{1+x}\left(\frac{1+x}{x}\right)^k$$

section 1.127

$$\sum_{k=1}^{2n-1}(-1)^{k-1}\binom{2n}{k}^{-1}H_k = \frac{1}{2}\frac{n}{(n+1)^2} + \frac{1}{2}\frac{1}{n+1}H_{2n}$$

section 1.128

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(x+1)^k$$

section 1.129

$$\begin{bmatrix}n\\m\end{bmatrix} = (-1)^{n-m}\sum_{k=0}^{n-m}(-1)^k\binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k}\left\{\begin{matrix}n-m+k\\k\end{matrix}\right\}, \quad \text{and}$$

$$\left\{\begin{matrix}n\\m\end{matrix}\right\} = (-1)^{n-m}\sum_{k=0}^{n-m}(-1)^k\binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k}\begin{bmatrix}n-m+k\\k\end{bmatrix}$$

section 1.130

$$B_n = \sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\left\{\begin{matrix}n+k\\k\end{matrix}\right\}\binom{n+k}{k}^{-1}$$

section 1.131

$$\binom{k}{h}\begin{bmatrix}n\\k\end{bmatrix} = \sum_{j=k-h}^{n-h}\binom{n}{j}\begin{bmatrix}n-j\\h\end{bmatrix}\begin{bmatrix}j\\k-h\end{bmatrix}, \quad \text{and}$$

$$\binom{k}{h}\left\{\begin{matrix}n\\k\end{matrix}\right\} = \sum_{j=k-h}^{n-h}\binom{n}{j}\left\{\begin{matrix}n-j\\h\end{matrix}\right\}\left\{\begin{matrix}j\\k-h\end{matrix}\right\}$$

section 1.132

$$\sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} \left(\frac{5}{9}\right)^k B_{2k} F_{n-2k} = \frac{n}{6} L_{n-1} + \frac{n}{3^n} L_{2n-2}$$

section 1.133

$$B_n = \frac{1}{m(1-m^n)} \sum_{k=0}^{n-1} m^k \binom{n}{k} B_k \sum_{j=1}^{m-1} j^{n-k}$$

section 1.134

$$B_n = \frac{1}{n+1} \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^j j^n \binom{n+1}{k-j} \binom{n}{k}^{-1}$$

section 1.135

$$B_{2n} = \sum_{j=2}^{2n+1} (-1)^{j-1} \binom{2n+1}{j} \frac{1}{j} \sum_{k=1}^{j-1} k^{2n}$$

section 1.136

$$B_{n+1} = \frac{n+1}{2(1-2^{n+1})} \sum_{k=0}^{n} \frac{(-1)^k}{2^k} k! \left\{ {n \atop k} \right\}, \quad \text{and}$$

$$B_{n+1} = \frac{n+1}{2^{n+1}(2^{n+1}-1)} \sum_{k=0}^{n} (-1)^{k+1} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n$$

section 1.137.1

$$\sum_{k=1}^{n} \binom{2n+1}{2k-1} \left(2^{n+1-k} - \frac{1}{2^{n+1-k}}\right)^2 B_{2n+2-2k} = (2n+1)\left(\frac{1}{4} + \frac{1}{2^{2n+1}}\right)$$

section 1.137.2

$$\sum_{k=1}^{n} \binom{2n}{2k} 2^{4k} B_{2k} = 4n - 1 - (2^{2n}-2) B_{2n}$$

section 1.137.3

$$\sum_{k=1}^{m} \frac{1}{k+1} (-1)^k k! \left\{ {m \atop k} \right\} = B_m \quad \text{and}$$

$$\sum_{k=1}^{m} \frac{1}{k(k+1)} (-1)^{k+1} k! \left\{ {m \atop k} \right\} = B_m$$

section 1.137.4

$$\sum_{k=1}^{m} \begin{bmatrix} 2m \\ 2k \end{bmatrix} B_{2k} = \frac{(2m-1)(2m-1)!}{2(2m+1)} \quad \text{and}$$

$$\sum_{k=1}^{m} \begin{bmatrix} 2m+1 \\ 2k \end{bmatrix} B_{2k} = \frac{m(2m)!}{2m+2}$$

section 1.137.5

$$B_{2m} = \frac{1}{2m(2m+1)} \sum_{k=0}^{m-1} (-1)^k (2m-1-2k) \binom{2m-1}{k}^{-1} \left\langle {2m \atop k} \right\rangle$$

section 1.137.6

$$\frac{1-2^{2m}}{2^{2m-1}} \frac{B_{2m}}{2m} = \binom{m-\frac{1}{2}}{2m} \left\langle {2m-1 \atop m-1} \right\rangle + 2 \sum_{k=1}^{m-1} \binom{k-\frac{1}{2}}{2m} \left\langle {2m-1 \atop k-1} \right\rangle$$

section 1.137.7

$$E_{2m} = \frac{1}{2m+1} \left\{ 4m+1 - \sum_{k=1}^{m} \binom{2m+1}{2k} B_{2k} 2^{4k} \right\}$$

section 1.137.8

$$B_{2m} = \frac{(2m)!}{2^{4m}(2^{2m}-1)} [z^{2m}] \prod_{q=0}^{2m-1} (\exp(\zeta_{2m}^q z) + \exp(-\zeta_{2m}^q z))$$

$$\text{where} \quad \zeta_{2m} = \exp(2\pi i/(2m))$$

section 1.138

$$x^n = \sum_{k=0}^{n-1} \binom{x+k}{n} \left\langle {n \atop k} \right\rangle$$

section 1.139

$$2^N = \sum_{m=0}^{N} \sum_{r=0}^{n} \sum_{s=0}^{m} (-1)^{n+m} (-2)^{r+s} \binom{n}{r} \binom{m}{s} \binom{N-r}{m} \binom{N-s}{n}$$

section 1.140

$$\sum_{k=0}^{m} k^{\ell} (-1)^k \binom{n}{k} = (-1)^m n \binom{n-1}{m} \sum_{k=1}^{\ell} \frac{1}{n-k} m^{\underline{k}} \left\{ {\ell \atop k} \right\}$$

section 1.141

$$\sum_{k=1}^{n} k^{\ell} \binom{n}{k}^2 = \sum_{k=1}^{\ell} \binom{2n-k}{n} n^{\underline{k}} \left\{ \begin{matrix} \ell \\ k \end{matrix} \right\}$$

section 1.142

$$\sum_{q=1}^{n} q^k = (n+1)n \sum_{q=1}^{k} \left\{ \begin{matrix} k \\ q \end{matrix} \right\} \frac{(n-1)^{\underline{q-1}}}{q+1}$$

section 1.143

$$\sum_{k=0}^{m} \binom{m}{k} \left\{ \begin{matrix} n+k+1 \\ k+1 \end{matrix} \right\} k! = \sum_{k=0}^{m} \binom{m}{k} (-k)^{m-k} (k+1)^{n+k}$$

section 1.144

$$\sum_{k=1}^{n} \frac{\prod_{1 \le r \le n, r \ne m} (x+k-r)}{\prod_{1 \le r \le n, r \ne k} (k-r)} = 1$$

section 1.145

$$\sum_{k=1}^{N} (-1)^k (\cos \frac{k\pi}{N})^{N-m} (\sin \frac{k\pi}{N})^m = \frac{1+(-1)^m}{2} (-1)^{m/2} \frac{N}{2^{N-1}}$$

section 1.146

$$1 = \sum_{p=0}^{n} (-1)^p \binom{n+q}{n-p-1} \binom{n+p}{n-q-1} \binom{p+q}{p}$$

$$1 = (-1)^q \sum_{p=0}^{n} (-1)^p \binom{n+q}{n-p-1} \binom{n+p}{n-q-1} \left[\begin{matrix} p+q \\ p \end{matrix} \right]$$

section 1.147.1

$$\left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} = (-1)^{n-m} \sum_{k=0}^{n} (-1)^k \binom{m-1+k}{m-n+k} \binom{2n-m}{n-m-k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

section 1.147.2

$$n^m = \sum_{k=0}^{n} (-1)^k \left[\begin{matrix} n \\ n-k \end{matrix} \right] \left\{ \begin{matrix} n-k+m \\ n \end{matrix} \right\}$$

section 1.147.3

$$\binom{n}{m}\binom{2n-m}{n} = (-1)^{n-m}\sum_{k=0}^{n-m}(-1)^k\binom{n+k}{n}^2\binom{2n-m}{n-m-k}$$

section 1.147.4

$$n = (-1)^{n+1}\sum_{k=0}^{n}(-1)^k\binom{m+k}{m}\binom{m+n}{n-k-1}\binom{m-1+k}{m-n+k}$$

section 1.147.5

$$\binom{n-2}{m-1} = \frac{(-1)^{m-1}}{n-1}\sum_{k=0}^{n}(-1)^k\binom{n+k}{n-m-1}\binom{n+1}{k+1}\binom{2k}{m-1}$$

section 1.147.6

$$\binom{n+m}{m-1}\binom{n-1}{m-1}\frac{1}{m} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{k}{m}\frac{1}{k+1}\binom{2k}{k}\binom{n+k}{n-k}$$

section 1.147.7

$$n+m = \sum_{k=0}^{n}(-1)^k\binom{n+m}{n-k-1}\binom{m-1+k}{k}\binom{n+m+k}{n}$$

section 1.147.8

$$\binom{n}{m}\binom{m+1}{n-m} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k}{m}\binom{k+1}{m+1}$$

section 1.147.9

$$\binom{n-1}{m} = (-1)^m\sum_{k=0}^{n}(-1)^k\binom{m+k}{m}\binom{2n}{n+k}\binom{n-1+k}{k}$$

section 1.147.10

$$n-m^2 = (-1)^m\sum_{k=0}^{n}(-1)^k\binom{n+m}{n-k-1}\binom{m-1+k}{k}\binom{m-1+k}{m}$$

section 1.147.11

$$\binom{n}{m}\binom{m}{p} = (-1)^m\sum_{k=0}^{n}(-1)^k\binom{m}{k}\binom{k}{p}\binom{k-p+n}{k}$$

section 1.147.12

$$\binom{2p-1}{p-1} = (-1)^p \sum_{k=0}^{n-m} (-1)^k \binom{2n-m}{n-m-k} \binom{n-1+k}{k} \binom{k-p+n}{p}$$

section 1.147.13

$$\binom{n-1}{p} = (-1)^p \sum_{k=0}^{n-m} (-1)^k \binom{p+k}{k} \binom{2n-m}{n-m-k} \binom{n-1+k}{k}$$

$$\binom{2n-m}{n-1} - \binom{n-1}{m-2} = (-1)^{n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n-m+1+k}{k} \binom{2n-m}{n-m-k} \binom{n-1+k}{k}$$

section 1.147.14

$$\binom{3n+1-m}{2n+1} = (-1)^{n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n-m+k}{k} \binom{2n-k}{n-m-k} \binom{2n+1}{k}$$

section 1.147.15

$$\binom{n}{m} = (-1)^{n+m} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n-m+k}{k} \binom{m-k+n}{m}$$

section 1.147.16

$$\binom{n-m-1}{m-1} = \sum_{k=0}^{n} (-1)^k \binom{n-1+k}{n-m+k} \binom{2m}{m+k} \binom{m-1+k}{k}$$

section 1.147.17

$$n^2 = \sum_{k=0}^{n} (-1)^k \binom{k-n}{n}^2 \binom{2n+1}{k}$$

section 1.147.18

$$(-1)^{(n+1)/2} \frac{n+1}{2} \times \frac{1+(-1)^{n+1}}{2} = \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} \binom{2k-n}{n} \binom{2k}{n-1}$$

section 1.147.19

$$\frac{(2n)!}{2^n n!} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \left[\begin{matrix} n+k \\ k \end{matrix}\right]$$

section 1.147.20

$$\binom{n}{m}n^m = (-1)^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{n-m+k}{k}\binom{kn+m}{m}$$

section 1.147.21

$$\binom{n}{m}2^m = (-1)^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{2k}{m}\binom{n-m+k}{k}$$

section 1.147.22

$$\binom{n}{m}^3 = \sum_{k=0}^{n} \binom{n}{k}\binom{k}{m}\binom{m}{k-m}\binom{n-m+k}{n}$$

section 1.147.23

$$\binom{n}{m}m! = (-1)^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{n-m+k}{n-m}k^m$$

section 1.147.24

$$\binom{2n-m-1}{n-1} = (-1)^n \sum_{k=0}^{n}(-1)^k \binom{2n}{n+k}\binom{m-1+k}{k-n+m}\binom{n-1+k}{n}$$

section 1.147.25

$$\binom{2n-m}{m} = (-1)^{n+m} \sum_{k=0}^{n} \binom{2n+1}{k}(-1)^k \binom{2m-k}{m}\binom{2n-k}{n}$$

section 1.147.26

$$\binom{n}{m} = (-1)^{n+m} \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{nm-k+m}{m}\binom{n-m+k}{k}$$

section 1.147.27

$$\binom{n}{m} = (-1)^{n+m} \sum_{k=0}^{n} \binom{n+m}{m+k}(-1)^k \binom{m-1+k}{k-n+m}\binom{n+m+k}{m}$$

section 1.147.28

$$\binom{n}{m}\binom{2n-m}{n}\binom{3n-2m}{n-m} = \sum_{k=0}^{n} \binom{2n-m}{n-m-k}^2 \binom{m+k}{k}\binom{n+k}{m+k}$$

section 1.147.29

$$1 = (-1)^m \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \binom{n-1+k}{n-m+k} \binom{n-1+k}{n}$$

section 1.147.30

$$\frac{1}{n-m+1} \binom{n+1}{m+1} \binom{n+1}{m} \left[(n+1) \binom{n+1}{m+1} - m \binom{n}{m+1} \right]$$
$$= \sum_{k=0}^{n} \binom{n+1}{k+1} \binom{n-k}{m-k} \binom{n+k+1}{k+1} \binom{n-m}{k}$$

section 1.147.31

$$\frac{(n+m)!}{2^m \times (n-m)! \times m!} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+m}{m+k} \begin{bmatrix} m+k \\ k \end{bmatrix} \binom{k-m}{n-m}$$

section 1.147.32

$$\frac{1}{n+1} \binom{n+m}{m} \binom{n-1}{m-1} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{k}{m} \binom{n+k}{2k} \frac{1}{k+1} \binom{2k}{k}$$

section 1.147.33

$$\binom{n}{m}^2 = \sum_{k=0}^{n} \binom{m+k}{m-k} \binom{2k}{k} \binom{n}{m+k}$$

section 1.147.34

$$2^{n-m} = \sum_{k=0}^{n} (-1)^k \binom{n+m-k}{k} \binom{2n-2k}{n-k}$$

section 1.148

$$\sum_{k=0}^{r} \binom{n}{2k} \binom{n-2k}{r-k} = \sum_{k=r}^{n} \binom{n}{k} \binom{2k}{2r} \left(\frac{3}{4}\right)^{n-k} \left(\frac{1}{2}\right)^{2k-2r}$$

section 1.149

$$\sum_{k=0}^{n} k^2 \binom{n+k}{k} = \frac{1}{2}(n+1)^2 \binom{2n+2}{n-1}$$

section 1.150

$$2^m \sum_{k=0}^{m} \sum_{j=0}^{p} (-1)^j \binom{k}{j} \binom{m-k}{p-j} \binom{m}{k} \binom{1/2(m+k-1)}{m} = (-1)^p \binom{m}{p}^2$$

section 1.151

$$\binom{n+c}{a} \binom{n+d}{b} = \sum_{q=0}^{a+b} \binom{a-c+d}{q-c} \binom{b-d+c}{q-d} \binom{n+q}{a+b}$$

section 1.152

$$[x^{2p}] P_{m,n}(x) = [x^{2p}] \sum_{k=0}^{m} \binom{2x+2k}{2k+1} \binom{n+m-k-x-1/2}{m-k} = 0$$

section 1.153

$$\sum_{k=0}^{n} (-1)^k \frac{1}{m-k} \binom{m-k}{k} \frac{1}{m+2n-2k} \binom{m+2n-2k}{n-k} = 0$$

section 1.154

$$E_n = 2^{2n-1} \sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell+1} \left\{ {n \atop \ell} \right\} \left(3 \left(\frac{1}{4} \right)^{\overline{\ell}} - \left(\frac{3}{4} \right)^{\overline{\ell}} \right)$$

$$E_{2n} = -4^{2n} \sum_{\ell=1}^{2n} \frac{(-1)^\ell}{\ell+1} \left\{ {2n \atop \ell} \right\} \left(\frac{3}{4} \right)^{\overline{\ell}}$$

section 1.155

$$\sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} = \frac{1}{2} \frac{n}{n-k} 2^{n-2k} \binom{n-k}{k}$$

section 1.156

$$K_k = \sum_{j=0}^{k} (-q)^j (q-1)^{k-j} \binom{n-j}{k-j} \binom{X}{j}$$

$$K_k = \sum_{j=0}^{k} (-1)^j q^{k-j} \binom{n-k+j}{j} \binom{n-X}{k-j}$$

section 1.157

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{\alpha^{2k+1}} \frac{B_{2k+2}}{(n-2k)!(2k+2)!}$$
$$= \frac{1}{2}\frac{n+2-2\alpha}{(n+2)!} + \frac{1}{\alpha^{n+1}(n+1)!}\sum_{j=0}^{\alpha-1} j^{n+1}.$$

section 1.158

$$\sum_{k=q}^{n} \binom{k-1}{q-1} p^q (1-p)^{k-q} = \sum_{k=q}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

section 1.159

$$a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k a_{n-k} \Longrightarrow a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}$$

section 1.160

$$\sum_{q=0}^{n} \binom{n}{q} \frac{q}{q+1} \binom{2n}{q+1}^{-1} = \frac{1}{n+1}$$

section 1.161

$$E_n = -\sqrt{2} \sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{k!}{\sqrt{2}^k} \cos(3\pi(k+1)/4) = 2\int_0^\infty \exp(-t)\cos(t) T_n(-t)\; dt$$

section 1.162

$$\sum_{k=j}^{\lfloor n/2 \rfloor} \frac{1}{4^k} \binom{n}{2k}\binom{k}{j}\binom{2k}{k} = \frac{1}{2^n}\binom{2n-2j}{n-j}\binom{n-j}{j}$$

section 1.163

$$\sum_{p=0}^{n-1}\sum_{q=0}^{n} |n-p-q| \binom{n+p-q}{p}\binom{n-p+q-1}{q} = n4^{n-1}$$

section 1.164

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+r}{k}\binom{2n}{2k}^{-1} = 4^n \binom{2n}{n}^{-1} \sum_{p=0}^{r-1} \binom{r-1}{p}\binom{2p}{p} 4^{-p} \binom{n+1/2}{n-p}$$

section 1.165.1

$$\binom{2n-p}{m-p} = (-1)^{n+m+p} \sum_{k=0}^{n} \binom{2n+1}{k} (-1)^k \binom{2n-k}{n} \binom{m-k}{m-p}$$

section 1.165.2

$$\binom{n-m-1}{p} = (-1)^{m+1} \sum_{k=0}^{n} (-1)^k \binom{m-1+k}{k}^2 \binom{n+m}{m+k} \binom{n-1+k}{p}$$

section 1.165.3

$$\binom{4p-2}{p} = (-1)^m \sum_{k=0}^{n} \binom{n+p}{k+p} (-1)^k \binom{k+p-1}{k+p-m} \binom{2k+1-p}{p}$$

section 1.165.4

$$\frac{(n+m)!}{2^m (n-m)! m!} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+m}{m+k} \binom{n-m+k}{k} \left\{ {m+k+1 \atop k+1} \right\}$$

section 1.165.5

$$\binom{n+m-1-p}{n-1} = (-1)^{n+p+1} \sum_{k=0}^{n} \binom{m-k}{p-k} (-1)^k \binom{2m}{k-n+m} \binom{-n+k}{m}$$

section 1.165.6

$$\binom{n-m-p}{m} = \sum_{k=0}^{n} (-1)^k \binom{p+m}{p+k} \binom{n-m+k}{m} \binom{p-1+k}{k}$$

section 1.165.7

$$(2n+1)^p = (-1)^{n+m} \sum_{k=0}^{n} \binom{2n}{k-1} \binom{2n-k}{n-m-k} (-1)^k k^p$$

section 1.165.8

$$\binom{2p}{p} = \sum_{k=0}^{n} (-1)^k \binom{2p}{k} \binom{m-k+p}{p} \binom{n+p-k}{p}$$

section 1.165.9

$$1 = (-1)^{n+m+p+1} \sum_{k=0}^{m-p} (-1)^k \binom{m-1+k}{n-1} \binom{m-1+k}{p-1} \binom{2m-p}{m-p-k}$$

section 1.165.10

$$\binom{n-1}{p-1}^2 = (-1)^m \sum_{k=0}^{n} \binom{p-1+k}{k} (-1)^k \binom{n-1+k}{m} \binom{2n}{n+k} \binom{k+n-p}{k}$$

section 1.165.11

$$\binom{2n-1}{p} = (-1)^{m+p} \sum_{k=0}^{n} \binom{n-1+k}{k} (-1)^k \binom{n-1+k}{m} \binom{2n}{n+k} \binom{2k+p}{p}$$

section 1.165.12

$$\binom{2n}{n}\binom{n}{m} = \sum_{k=0}^{n} \binom{n+1}{2k+1} \binom{n+k}{m+k} \binom{m+k}{m}$$

section 1.165.13

$$\binom{n+m}{2m}^2 = (-1)^n \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{2m} \binom{2k-2m}{k-m} (-1)^k$$

section 1.165.14

$$\binom{n+m}{m} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+k}{n} \binom{n+m}{m+k} \binom{p+k}{m}$$

section 1.165.15

$$\binom{2n}{n+m} = \sum_{k=0}^{n} (-1)^k \binom{n-m+k}{k} \binom{2n-k}{n-m-k} \binom{2n}{n+m-k}$$

section 1.165.16

$$\binom{n}{m}^2 = (-1)^m \sum_{k=0}^{n} (-1)^k \binom{k}{m} \binom{2n-k}{k} \binom{2n-2k}{n-k}$$

section 1.165.17

$$\binom{n+m}{2m} = (-1)^m \sum_{k=0}^{n} \binom{n+m}{m+k} (-1)^k \binom{p-k}{n-m} \binom{2m+k}{k}$$

section 1.165.18

$$\binom{n}{m} 2^m = (-1)^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{2k}{m} \binom{p-k}{n-m}$$

section 1.165.19

$$\frac{(2p)!}{p!} = (-1)^{p+m} \sum_{k=0}^{p+m} (n-k)^p (-1)^k \binom{2p}{p+m-k} \binom{r-k}{p}$$

section 1.165.20

$$2^{n-2m} \frac{(n+m)!}{(n-m)!m!} = (-1)^m \sum_{k=0}^{n} \binom{n+m}{m+k} (-1)^k \binom{p-2k}{n-m} \left\{ {m+k+1 \atop k+1} \right\}$$

section 1.165.21

$$2^p \binom{2p}{p} = (-1)^m \sum_{k=0}^{n} \binom{n-k+p}{p} (-1)^k \binom{2k}{p} \binom{2p}{m-k+p}$$

section 1.165.22

$$n^m = (-1)^m \sum_{k=0}^{n} \binom{n-1+k}{k} (-1)^k k^m \binom{2n}{n+k}$$

section 1.165.23

$$\binom{n}{m} \binom{n-1}{n-m} \frac{1}{n-m+1} = (-1)^{n+m} \sum_{k=0}^{n} \frac{1}{k+1} \binom{2k}{k} (-1)^k \binom{n-k}{m} \binom{n+k}{n-k}$$

section 1.165.24

$$\binom{y-x}{n} = (-1)^n \sum_{k=0}^{n} \binom{x-k}{n-k} \binom{x+1}{k} (-1)^k \binom{y+1-k}{n}$$

section 1.165.25

$$\binom{n+m}{n} \binom{n}{m} \frac{1}{m+1} = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{1}{k+1} \binom{2k}{k} \binom{k+1}{m+1} \binom{n+k}{n-k}$$

section 1.165.26

$$\binom{n}{m} m! q^{n-m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{p-qk}{n-m} (r-k)^m$$

section 1.165.27

$$p^{n-m} q^m \binom{n}{m} = (-1)^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{x-pk}{n-m} \binom{y+qk}{m}$$

section 1.165.28

$$\binom{n-1}{m-1} = (-1)^{m+1}\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^k\binom{m-1+k}{k}\binom{n-1+k}{k}$$

section 1.165.29

$$\binom{n+m}{2m}^2 = (-1)^n\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{m+k}(-1)^k\binom{k+m}{k-m}$$

section 1.165.30

$$\binom{2m-1}{p} = (-1)^{p+1}\sum_{k=0}^{n}\binom{n+m}{m+k}(-1)^k\binom{m-1+k}{m}\binom{k-m+p}{p}$$

section 1.165.31

$$\binom{n}{m} = (-1)^{m+p}\sum_{k=0}^{n-p}(-1)^k\binom{n}{p+k}\binom{x-k}{n-m}\binom{m+k}{k}$$

section 1.165.32

$$(n+1)^m = (-1)^{n+m}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{n}\binom{2n+1}{k}(n-k)^m$$

section 1.165.33

$$\binom{n}{m}\binom{n-1}{m} = (-1)^{n+1}\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^k\binom{n+k-m-1}{n-m-1}\binom{m-1+k}{m}\binom{n-1+k}{n-1}$$

section 1.165.34

$$\frac{(n+m)!}{n!} = (-1)^n\sum_{k=0}^{n}(r+k)^m\binom{n+k}{n}(-1)^k\binom{n+m}{m+k}$$

section 1.165.35

$$\frac{(n+p)!}{2^p(n-p)!p!}\binom{n-p}{m-p} = (-1)^m\sum_{k=0}^{m}\binom{n-k}{m-k}(-1)^k\left\{{p+k+1 \atop k+1}\right\}\binom{m+k}{p+k}\binom{p+n}{p+k}$$

section 1.165.36

$$\binom{n-1}{p-1}\binom{m}{p} = (-1)^p\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\binom{n+k}{p+k}\binom{k-m+p}{p}$$

section 1.165.37

$$m^m = (-1)^n \sum_{k=0}^{n} \binom{n+m}{m+k} (-1)^k k^m \binom{m-1+k}{m-n+k}$$

section 1.165.38

$$\binom{2p}{p} = (-1)^{m+p} \sum_{k=0}^{n} \binom{n-k+p}{p} \binom{m-k}{p} (-1)^k \binom{2p}{k-m+p}$$

1 Egorychev method in formal power series

1.1 MSE 2384932

We seek to evaluate

$$\sum_{\ell=0}^{m}(-4)^\ell {m\choose \ell}{2\ell\choose \ell}^{-1}\sum_{k=0}^{n}\frac{(-4)^k}{2k+1}{n\choose k}{2k\choose k}^{-1}{k+\ell\choose \ell}.$$

Recall the Beta function identity

$$\frac{1}{2k+1}{2k\choose k}^{-1} = \int_0^1 x^k(1-x)^k\; dx.$$

We substitute this into the inner sum to get

$$\begin{aligned}
&[z^\ell]\int_0^1 \sum_{k=0}^{n}(-4)^k{n\choose k}\frac{x^k(1-x)^k}{(1-z)^{k+1}}\; dx\\
&= [z^\ell]\frac{1}{1-z}\int_0^1\left[1-\frac{4x(1-x)}{1-z}\right]^n\; dx\\
&= [z^\ell]\frac{1}{(1-z)^{n+1}}\int_0^1\left[(2x-1)^2-z\right]^n\; dx\\
&= [z^\ell]\frac{1}{(1-z)^{n+1}}\int_0^1\sum_{q=0}^{n}{n\choose q}(-1)^q z^q(2x-1)^{2n-2q}\; dx\\
&= \sum_{q=0}^{\ell}{\ell-q+n\choose n}{n\choose q}\frac{(-1)^q}{2n-2q+1}.
\end{aligned}$$

Now for this sum introduce the function

$$f(z) = \frac{(-1)^n}{2n+1-2z}\prod_{p=1}^{n}(\ell+p-z)\prod_{r=0}^{n}\frac{1}{z-r}.$$

This has the property that where $0\le q\le n$

$$\begin{aligned}
\operatorname*{res}_{z=q} f(z) &= \frac{(-1)^n}{2n+1-2q}\prod_{p=1}^{n}(\ell+p-q)\prod_{r=0}^{q-1}\frac{1}{q-r}\prod_{r=q+1}^{n}\frac{1}{q-r}\\
&= \frac{(-1)^n}{2n+1-q}{\ell+n-q\choose n}n!\times\frac{1}{q!}\frac{(-1)^{n-q}}{(n-q)!}\\
&= \frac{(-1)^q}{2n+1-q}{\ell+n-q\choose n}{n\choose q}.
\end{aligned}$$

Here residues sum to zero and the residue at infinity is zero so we may evaluate using minus the residue at $z = (2n+1)/2$, getting

$$\frac{1}{2}(-1)^n \prod_{p=1}^{n} (\ell + p - (2n+1)/2) \prod_{r=0}^{n} \frac{1}{(2n+1)/2 - r}$$

$$= (-1)^n 2^n \prod_{p=1}^{n} (\ell + p - n - 1/2) \prod_{r=0}^{n} \frac{1}{2n+1-2r}$$

$$= (-1)^n 2^n \binom{\ell - 1/2}{n} n! \frac{2^n n!}{(2n+1)!} = (-1)^n 2^{2n} \binom{\ell - 1/2}{n} \frac{1}{2n+1} \binom{2n}{n}^{-1}.$$

This gives for our sum

$$(-1)^n 2^{2n} \frac{1}{2n+1} \binom{2n}{n}^{-1} \sum_{\ell=0}^{m} (-4)^\ell \binom{m}{\ell} \binom{2\ell}{\ell}^{-1} \binom{\ell - 1/2}{n}.$$

We work with the remaining inner sum

$$\sum_{\ell=0}^{m} (-4)^\ell \binom{m}{\ell} \binom{2\ell}{\ell}^{-1} \binom{\ell - 1/2}{n}$$

$$= \sum_{\ell=0}^{m} (-4)^\ell \binom{m}{\ell} \binom{2\ell}{\ell}^{-1} \frac{n+1}{\ell + 1/2} \binom{\ell + 1/2}{n+1}$$

$$= 2(n+1) \sum_{\ell=0}^{m} (-4)^\ell \binom{m}{\ell} \binom{2\ell}{\ell}^{-1} \frac{1}{2\ell+1} \binom{\ell + 1/2}{n+1}.$$

The beta function identity will now produce

$$2(n+1) \int_0^1 \sum_{\ell=0}^{m} (-4)^\ell \binom{m}{\ell} \binom{\ell + 1/2}{n+1} x^\ell (1-x)^\ell \, dx$$

$$= 2(n+1) [w^{n+1}] \sqrt{1+w} \int_0^1 \sum_{\ell=0}^{m} (-4)^\ell \binom{m}{\ell} (1+w)^\ell x^\ell (1-x)^\ell \, dx.$$

We continue with the core integral:

$$\int_0^1 (1 - 4x(1-x)(1+w))^m \, dx$$

$$= \int_0^1 ((2x-1)^2 + 4x(x-1)w)^m \, dx$$

$$= \int_0^1 \sum_{q=0}^{m} \binom{m}{q} (2x-1)^{2m-2q} ((2x-1)^2 - 1)^q w^q \, dx$$

$$= \int_0^1 \sum_{q=0}^m \binom{m}{q} (2x-1)^{2m} (1 - 1/(2x-1)^2)^q w^q \; dx$$

$$= \int_0^1 \sum_{q=0}^m \binom{m}{q} w^q \sum_{p=0}^q \binom{q}{p} (-1)^p (2x-1)^{2m-2p} \; dx$$

$$= \sum_{q=0}^m \binom{m}{q} w^q \sum_{p=0}^q \binom{q}{p} (-1)^p \frac{1}{2m+1-2p}.$$

Switching sums,

$$\sum_{p=0}^m \frac{(-1)^p}{2m+1-2p} \sum_{q=p}^m \binom{m}{q} \binom{q}{p} w^q.$$

Observe that

$$\binom{m}{q}\binom{q}{p} = \frac{m!}{(m-q)! \times p! \times (q-p)!} = \binom{m}{p}\binom{m-p}{m-q}.$$

We get for the sum

$$\sum_{p=0}^m \binom{m}{p} \frac{(-1)^p}{2m+1-2p} \sum_{q=p}^m \binom{m-p}{m-q} w^q$$

$$= \sum_{p=0}^m \binom{m}{p} \frac{(-1)^p}{2m+1-2p} w^p \sum_{q=0}^{m-p} \binom{m-p}{m-q} w^q$$

$$= \sum_{p=0}^m \binom{m}{p} \frac{(-1)^p}{2m+1-2p} w^p (1+w)^{m-p}.$$

Applying the extractor in w,

$$2(n+1) \sum_{p=0}^m \binom{m}{p} \frac{(-1)^p}{2m+1-2p} \binom{m-p+1/2}{n+1-p}.$$

Now working with the Gamma function representation of the binomial coefficient we get for the second coefficient (pole from the Gamma function when $m > n$)

$$\binom{m-p+1/2}{n+1-p} = \frac{\Gamma(m-p+3/2)}{\Gamma(m-n+1/2)\Gamma(n+2-p)}$$

$$= \frac{m-p+1/2}{m-n-1/2} \frac{\Gamma(m-p+1/2)}{\Gamma(m-n-1/2)\Gamma(n+2-p)} = \frac{2m-2p+1}{2m-2n-1} \binom{m-p-1/2}{n+1-p}.$$

We find for the remaining sum

$$\frac{2(n+1)}{2m-2n-1}\sum_{p=0}^{m}\binom{m}{p}(-1)^p\binom{m-p-1/2}{n+1-p}.$$

The sum is

$$[w^{n+1}](1+w)^{m-1/2}\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{w^p}{(1+w)^p} = [w^{n+1}](1+w)^{m-1/2}\left[1-\frac{w}{1+w}\right]^m$$

$$= [w^{n+1}]\frac{1}{\sqrt{1+w}} = \frac{1}{(-4)^{n+1}}[w^{n+1}]\frac{1}{\sqrt{1-4w}} = \frac{1}{(-4)^{n+1}}\binom{2n+2}{n+1}.$$

Collecting all the pieces we finally have

$$\frac{2(n+1)}{2m-2n-1}(-1)^{n+1}\frac{1}{2^{2n+1}}\binom{2n+1}{n+1}(-1)^n 2^{2n}\frac{1}{2n+1}\binom{2n}{n}^{-1}.$$

Multiplying everything there is cancellation and we are left with just

$$\frac{1}{2n+1-2m}.$$

This was math.stackexchange.com problem 2384932.

1.2 MSE 2472978

We seek to verify that

$$\sum_{l=0}^{n}\binom{n}{l}^2(x+y)^{2l}(x-y)^{2n-2l} = \sum_{l=0}^{n}\binom{2l}{l}\binom{2n-2l}{n-l}x^{2l}y^{2n-2l}.$$

Now we see on the LHS that the powers of x and y always add up to $2n$ and the exponent on x determines the one on y. Extracting the coefficient on $[x^q][y^{2n-q}]$ we obtain

$$\sum_{l=0}^{n}\binom{n}{l}^2\sum_{p=0}^{q}\binom{2l}{p}(-1)^{2n-2l-(q-p)}\binom{2n-2l}{q-p}$$

$$= \sum_{l=0}^{n}\binom{n}{l}^2\sum_{p=0}^{q}\binom{2l}{p}(-1)^{q-p}[z^{q-p}](1+z)^{2n-2l}$$

$$= [z^q](-1)^q\sum_{l=0}^{n}\binom{n}{l}^2(1+z)^{2n-2l}\sum_{p=0}^{q}\binom{2l}{p}(-1)^p z^p.$$

We may extend p to infinity because with $p > q$ there is no contribution to $[z^q]$, getting

$$[z^q](-1)^q \sum_{l=0}^{n} \binom{n}{l}^2 (1+z)^{2n-2l} \sum_{p\ge 0} \binom{2l}{p} (-1)^p z^p$$
$$= [z^q](-1)^q \sum_{l=0}^{n} \binom{n}{l}^2 (1+z)^{2n-2l} (1-z)^{2l}$$
$$= [z^q](-1)^q [w^n](1+w(1-z)^2)^n (1+w(1+z)^2)^n$$
$$= [z^q][w^n](1+w(1-z)^2)^n (1+w(1+z)^2)^n.$$

Re-write this as

$$[z^q][w^n]((w(1+z^2)+1)^2 - 4w^2z^2)^n$$
$$= [z^q][w^n] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} w^{2p} z^{2p} (w(1+z^2)+1)^{2n-2p}$$
$$= [z^q] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} z^{2p} [w^{n-2p}] (w(1+z^2)+1)^{2n-2p}$$
$$= [z^q] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} z^{2p} \binom{2n-2p}{n-2p} (1+z^2)^{n-2p}.$$

We observe at this point that we get zero here when q is odd, which agrees with the target formula. We are thus justified in putting $q = 2l$ to get

$$[z^l] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} z^p \binom{2n-2p}{n-2p} (1+z)^{n-2p}$$
$$= \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} \binom{2n-2p}{n-2p} \binom{n-2p}{l-p}.$$

Note that

$$\binom{n}{p}\binom{2n-2p}{n-2p}\binom{n-2p}{l-p} = \frac{(2n-2p)!}{p! \times (n-p)! \times (l-p)! \times (n-l-p)!}$$
$$= \binom{l}{p} \frac{(2n-2p)!}{(n-p)! \times l! \times (n-l-p)!} = \binom{l}{p}\binom{2n-2p}{n-p}\binom{n-p}{l}.$$

Re-indexing we get for the sum

$$(-1)^n 2^{2n} \sum_{p=0}^{n} \binom{l}{n-p}\binom{2p}{p}\binom{p}{l} (-1)^p 2^{-2p}$$

$$= (-1)^n 2^{2n} \sum_{p=0}^{n} \binom{2p}{p} (-1)^p 2^{-2p} [z^{n-p}](1+z)^l [w^l](1+w)^p$$

$$= (-1)^n 2^{2n} [z^n](1+z)^l [w^l] \sum_{p=0}^{n} \binom{2p}{p} (-1)^p 2^{-2p} z^p (1+w)^p.$$

We may once more extend p to infinity because there is no contribution from the sum term to the coefficient extractor $[z^n]$ when $p > n$, obtaining

$$(-1)^n 2^{2n} [z^n](1+z)^l [w^l] \sum_{p\ge 0} \binom{2p}{p} (-1)^p 2^{-2p} z^p (1+w)^p$$

$$= (-1)^n 2^{2n} [z^n](1+z)^l [w^l] \frac{1}{\sqrt{1+z(1+w)}}$$

$$= (-1)^n 2^{2n} [z^n](1+z)^l [w^l] \frac{1}{\sqrt{1+z+wz}}$$

$$= (-1)^n 2^{2n} [z^n](1+z)^{l-1/2} [w^l] \frac{1}{\sqrt{1+wz/(1+z)}}$$

$$= (-1)^n 2^{2n} [z^n](1+z)^{l-1/2} \binom{2l}{l} (-1)^l 2^{-2l} z^l \frac{1}{(1+z)^l}$$

$$= (-1)^{n-l} 2^{2n-2l} \binom{2l}{l} [z^{n-l}] \frac{1}{\sqrt{1+z}}$$

$$= (-1)^{n-l} 2^{2n-2l} \binom{2l}{l} \binom{2n-2l}{n-l} (-1)^{n-l} 2^{-(2n-2l)}$$

$$= \binom{2l}{l} \binom{2n-2l}{n-l}.$$

Alternate answer. We keep the preliminaries and start with the contribution from w being

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} (1+w(1-z)^2)^n (1+w(1+z)^2)^n.$$

Now put $v = w/(1+w(1-z)^2)$ so that $w = v/(1-v(1-z)^2)$ and $dw = 1/(1-v(1-z)^2)^2 \; dv$ to get

$$\operatorname*{res}_{v} \frac{1}{v^{n+1}} (1-v(1-z)^2) \frac{(1+4vz)^n}{(1-v(1-z)^2)^n} \frac{1}{(1-v(1-z)^2)^2}$$

$$= \operatorname*{res}_{v} \frac{1}{v^{n+1}} \frac{(1+4vz)^n}{(1-v(1-z)^2)^{n+1}}$$

$$= \sum_{p=0}^{n} \binom{n+p}{n} (1-z)^{2p} 4^{n-p} z^{n-p} \binom{n}{p}.$$

Observe that

$$\binom{n}{p}\binom{n+p}{n} = \frac{(n+p)!}{p! \times (n-p)! \times p!} = \binom{2p}{p}\binom{n+p}{n-p}.$$

We obtain

$$(-1)^n \sum_{p=0}^{n} \binom{2p}{p}\binom{-2p-1}{n-p}(-1)^p(1-z)^{2p}4^{n-p}z^{n-p}$$

$$= (-1)^n [w^n] \frac{1}{1+w} \sum_{p\ge 0} \binom{2p}{p}(-1)^p(1-z)^{2p}4^{n-p}z^{n-p}(1+w)^{-2p}w^p.$$

Here we have extended to infinity due to the coefficient extractor in w. Continuing,

$$(-1)^n 4^n z^n [w^n] \frac{1}{1+w} \sum_{p\ge 0} \binom{2p}{p}(-1)^p(1-z)^{2p}4^{-p}z^{-p}(1+w)^{-2p}w^p$$

$$= (-1)^n 4^n z^n [w^n] \frac{1}{1+w} \frac{1}{\sqrt{1+(1-z)^2 w/(1+w)^2/z}}$$

$$= (-1)^n 4^n [w^n] \frac{1}{1+wz} \frac{1}{\sqrt{1+(1-z)^2 w/(1+wz)^2}}$$

$$= (-1)^n 4^n [w^n] \frac{1}{\sqrt{(1+wz)^2+(1-z)^2 w}}$$

$$= (-1)^n 4^n [w^n] \frac{1}{\sqrt{1+w}} \frac{1}{\sqrt{1+wz^2}}.$$

Now here we see that there are no odd order coefficients in z and we may put $q = 2l$. Extracting the coefficient we get

$$[z^{2l}](-1)^n 4^n [w^n] \frac{1}{\sqrt{1+w}} \frac{1}{\sqrt{1+wz^2}}$$

$$= [z^l](-1)^n [w^n] \frac{1}{\sqrt{1+4w}} \frac{1}{\sqrt{1+4wz}}$$

$$= (-1)^n [w^n] \frac{1}{\sqrt{1+4w}} \binom{2l}{l}(-1)^l w^l$$

$$= (-1)^n \binom{2l}{l}(-1)^l \binom{2n-2l}{n-l}(-1)^{n-l}$$

$$= \binom{2l}{l}\binom{2n-2l}{n-l}.$$

This is the claim.

This was math.stackexchange.com problem 2472978.

1.3 MSE 2719320

The goal here was to investigate closed forms of

$$\binom{n}{k}\frac{1}{ak+b}.$$

We start by trying to prove the first closed form given to see if a pattern does emerge. We use with c a positive integer

$$\binom{n+c}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+c}$$

Now

$$\binom{n+c}{n}\binom{n}{k} = \frac{(n+c)!}{(c)!\times k!\times (n-k)!} = \binom{n+c}{k+c}\binom{k+c}{k}.$$

Hence we have for the sum

$$\sum_{k=0}^{n}\binom{n+c}{k+c}\binom{k+c}{k}\frac{1}{k+c} = \frac{1}{c}\sum_{k=0}^{n}\binom{n+c}{k+c}\binom{k+c-1}{c-1}.$$

This is

$$\frac{1}{c}\sum_{k=0}^{n}\binom{k+c-1}{c-1}[z^{n-k}]\frac{1}{(1-z)^{k+c+1}} = \frac{1}{c}\sum_{k=0}^{n}\binom{k+c-1}{c-1}[z^n]z^k\frac{1}{(1-z)^{k+c+1}}.$$

Here we get no contribution to $[z^n]$ when $k > n$ so we may continue with

$$\begin{aligned}
&\frac{1}{c}[z^n]\frac{1}{(1-z)^{c+1}}\sum_{k\ge 0}\binom{k+c-1}{c-1}z^k\frac{1}{(1-z)^k}\\
&= \frac{1}{c}[z^n]\frac{1}{(1-z)^{c+1}}\frac{1}{(1-z/(1-z))^c}\\
&= \frac{1}{c}[z^n]\frac{1}{1-z}\frac{1}{(1-2z)^c}.
\end{aligned}$$

This is

$$\begin{aligned}
&\frac{1}{c}\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}\frac{1}{1-z}\frac{1}{(1-2z)^c}\\
&= \frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}\frac{1}{z-1}\frac{1}{(z-1/2)^c}.
\end{aligned}$$

With residues summing to zero we can evaluate this using the residues at $z=1$, $z=1/2$ and $z=\infty$. We get for $z=1$ the residue

$$\frac{(-1)^{c+1}}{c}.$$

For the residue at infinity we find

$$\begin{aligned}
&-\frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}\frac{1}{z^2}\frac{1}{(1/z)^{n+1}}\frac{1}{1/z-1}\frac{1}{(1/z-1/2)^c}\\
&= -\frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}\frac{1}{z^2}z^{n+1}\frac{z}{1-z}\frac{z^c}{(1-z/2)^c}\\
&= -\frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}z^{n+c}\frac{1}{1-z}\frac{1}{(1-z/2)^c} = 0.
\end{aligned}$$

This also follows by inspection. The residue at $z=1/2$ requires the use of Leibniz' rule as in

$$\frac{1}{p!}\left(\frac{1}{z^{n+1}}\frac{1}{z-1}\right)^{(p)} = \frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^q(n+q)!}{n!z^{n+1+q}}(-1)^{p-q}\frac{(p-q)!}{(z-1)^{p-q+1}}$$

$$= (-1)^p\sum_{q=0}^{p}\binom{n+q}{q}\frac{1}{z^{n+1+q}}\frac{1}{(z-1)^{p-q+1}}.$$

We set $p=c-1$ and $z=1/2$ and restore the factor in front to get for the residue

$$\begin{aligned}
&\frac{(-1)^{c+1}}{c2^c}(-1)^{c-1}\sum_{q=0}^{c-1}\binom{n+q}{q}\frac{1}{(1/2)^{n+1+q}}\frac{(-1)^{c-q}}{(1/2)^{c-q}}\\
&= \frac{(-1)^c 2^{n+1}}{c}\sum_{q=0}^{c-1}\binom{n+q}{q}(-1)^q.
\end{aligned}$$

Collecting everything we thus obtain

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+c} = \binom{n+c}{c}^{-1}\frac{(-1)^c}{c}\left(1-2^{n+1}\sum_{q=0}^{c-1}\binom{n+q}{q}(-1)^q\right).$$

This is an improvement in the sense that if n is the variable and c is the constant then we have replaced the sum in n terms (variable) by a sum in c terms (fixed) of polynomials in n. We can make this more explicit by writing

$$\sum_{q=0}^{c-1} \binom{n+q}{q} (-1)^q = \sum_{q=0}^{c-1} \frac{(-1)^q}{q!} \sum_{p=0}^{q} n^p \left[{q+1 \atop p+1} \right]$$
$$= \sum_{p=0}^{c-1} n^p \sum_{q=p}^{c-1} \frac{(-1)^q}{q!} \left[{q+1 \atop p+1} \right].$$

We find

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+c} = \binom{n+c}{c}^{-1} \frac{(-1)^c}{c} \left(1 - 2^{n+1} \sum_{p=0}^{c-1} n^p \sum_{q=p}^{c-1} \frac{(-1)^q}{q!} \left[{q+1 \atop p+1} \right] \right).$$

With this last result we obtain closed forms for fixed c, e.g. for $c = 5$ it yields

$$\frac{-24 + 2^{n+1}(n^4 + 6n^3 + 23n^2 + 18n + 24)}{(n+5) \times \cdots \times (n+1)}.$$

Addendum. With the purpose of matching conjectures by OP we write

$$\sum_{q=0}^{c-1} \binom{n+q}{q} (-1)^q = \sum_{q=0}^{c-1} \binom{n+q}{q} (-1)^q [z^{c-1}] \frac{z^q}{1-z}$$
$$= [z^{c-1}] \frac{1}{1-z} \sum_{q \geq 0} \binom{n+q}{q} (-1)^q z^q = [z^{c-1}] \frac{1}{1-z} \frac{1}{(1+z)^{n+1}}$$
$$= (-1)^{c-1} [z^{c-1}] \frac{1}{1+z} \frac{1}{(1-z)^{n+1}} = (-1)^{c-1} [z^{c-1}] \frac{1}{1-z^2} \frac{1}{(1-z)^n}.$$

With $c = 2d + 1$ where $d \geq 0$ this becomes

$$[z^{2d}] \frac{1}{1-z^2} \frac{1}{(1-z)^n} = \sum_{q=0}^{d} \binom{2q+n-1}{2q}$$

and when $c = 2d$ where $d \geq 1$ it becomes

$$-[z^{2d-1}] \frac{1}{1-z^2} \frac{1}{(1-z)^n} = -\sum_{q=0}^{d-1} \binom{2q+n}{2q+1}.$$

We thus obtain in the first case the closed form

$$\binom{n+2d+1}{2d+1}^{-1} \frac{1}{2d+1} \left(-1 + 2^{n+1} \sum_{q=0}^{d} \binom{2q+n-1}{2q} \right)$$

and in the second case

$$\binom{n+2d}{2d}^{-1} \frac{1}{2d} \left(1 + 2^{n+1} \sum_{q=0}^{d-1} \binom{2q+n}{2q+1}\right).$$

These two confirm the conjectures by OP.

This was math.stackexchange.com problem 2719320.

1.4 MSE 2830860

Starting from (here evidently $n \ge k$ for it to be meaningful).

$$\sum_{j=0}^{n-k} (-1)^j \binom{2k+2j}{j} \binom{n+k+j+1}{n-k-j}$$
$$= (-1)^{n-k} \sum_{j=0}^{n-k} (-1)^j \binom{2n-2j}{n-k-j} \binom{2n-j+1}{j}$$
$$= (-1)^{n-k} \sum_{j=0}^{n-k} (-1)^j \binom{2n-2j}{n-k-j} \binom{2n+1-j}{2n+1-2j}.$$

we write

$$(-1)^{n-k} \sum_{j=0}^{n-k} (-1)^j \binom{2n+1-j}{2n+1-2j} [z^{n-k-j}](1+z)^{2n-2j}$$
$$= (-1)^{n-k} [z^{n-k}](1+z)^{2n} \sum_{j=0}^{n-k} (-1)^j \binom{2n+1-j}{2n+1-2j} z^j (1+z)^{-2j}$$

We get no contribution to the coefficient extractor when $j > n-k$ and hence may continue with

$$(-1)^{n-k} [z^{n-k}](1+z)^{2n} \sum_{j\ge 0} (-1)^j \binom{2n+1-j}{2n+1-2j} z^j (1+z)^{-2j}$$
$$= (-1)^{n-k} [z^{n-k}](1+z)^{2n} \sum_{j\ge 0} (-1)^j z^j (1+z)^{-2j} [w^{2n+1-2j}](1+w)^{2n+1-j}$$
$$= (-1)^{n-k} [z^{n-k}](1+z)^{2n} [w^{2n+1}](1+w)^{2n+1} \sum_{j\ge 0} (-1)^j z^j (1+z)^{-2j} w^{2j} (1+w)^{-j}$$
$$= (-1)^{n-k} [z^{n-k}](1+z)^{2n} [w^{2n+1}](1+w)^{2n+1} \frac{1}{1 + zw^2/(1+z)^2/(1+w)}$$
$$= (-1)^{n-k} [z^{n-k}](1+z)^{2n+2} [w^{2n+1}](1+w)^{2n+2} \frac{1}{(1+z)^2(1+w) + zw^2}$$

$$= (-1)^{n-k}[z^{n-k}](1+z)^{2n+2}[w^{2n+1}](1+w)^{2n+2}\frac{1}{(w+1+z)(wz+1+z)}$$

$$= (-1)^{n-k}[z^{n+1-k}](1+z)^{2n+2}[w^{2n+1}](1+w)^{2n+2}\frac{1}{(w+1+z)(w+(1+z)/z)}.$$

Now the inner term is

$$\mathrm{Res}_{w=0}\frac{1}{w^{2n+2}}(1+w)^{2n+2}\frac{1}{(w+1+z)(w+(1+z)/z)}.$$

Residues sum to zero and the residue at infinity is zero since $\lim_{R\to\infty} 2\pi R \times R^{2n+2}/R^{2n+2}/R^2 = 0$. Hence we may compute this from minus the sum of the residues at $-(1+z)$ and $-(1+z)/z$. The first one yields

$$-\frac{1}{(1+z)^{2n+2}}z^{2n+2}\frac{1}{-(1+z)+(1+z)/z}.$$

Replace this in the remaining coefficient extractor to get

$$(-1)^{n+1-k}[z^{n+1-k}]z^{2n+3}\frac{1}{1-z^2} = 0.$$

The second one yields

$$-\frac{z^{2n+2}}{(1+z)^{2n+2}}\frac{1}{z^{2n+2}}\frac{1}{-(1+z)/z+1+z}$$

Once more replace this in the remaining coefficient extractor to get

$$(-1)^{n+1-k}[z^{n+1-k}]\frac{1}{-(1+z)/z+1+z} = (-1)^{n+1-k}[z^{n+1-k}]\frac{z}{z^2-1}$$

$$= -[z^{n+1-k}]\frac{z}{z^2-1} = [z^{n-k}]\frac{1}{1-z^2}.$$

This is

$$[[(n-k) \text{ is even}]] = \frac{1+(-1)^{n-k}}{2}$$

as claimed.

This was math.stackexchange.com problem 2830860.

1.5 MSE 2904333

Starting from

$$\sum_{k=0}^{b-1}\binom{a+k-1}{a-1}p^a(1-p)^k = \sum_{k=a}^{a+b-1}\binom{a+b-1}{k}p^k(1-p)^{a+b-k-1}$$

we simplify to

$$\sum_{k=0}^{b-1} \binom{a+k-1}{a-1} p^a (1-p)^k = \sum_{k=0}^{b-1} \binom{a+b-1}{a+k} p^{a+k} (1-p)^{b-k-1}$$

or

$$\sum_{k=0}^{b-1} \binom{a+k-1}{a-1} (1-p)^k = \sum_{k=0}^{b-1} \binom{a+b-1}{a+k} p^{k} (1-p)^{b-k-1}.$$

We get for the LHS

$$\begin{aligned} &\sum_{k\ge 0} \binom{a+k-1}{a-1} (1-p)^k [[0 \le k \le b-1]] \\ &= \sum_{k\ge 0} \binom{a+k-1}{a-1} (1-p)^k [z^{b-1}] \frac{z^k}{1-z} \\ &= [z^{b-1}] \frac{1}{1-z} \sum_{k\ge 0} \binom{a+k-1}{a-1} (1-p)^k z^k \\ &= [z^{b-1}] \frac{1}{1-z} \frac{1}{(1-(1-p)z)^a}. \end{aligned}$$

The RHS is

$$\begin{aligned} &\sum_{k=0}^{b-1} p^k (1-p)^{b-k-1} [z^{b-1-k}] \frac{1}{(1-z)^{a+k+1}} \\ &= [z^{b-1}] \frac{1}{(1-z)^{a+1}} \sum_{k=0}^{b-1} p^k (1-p)^{b-k-1} \frac{z^k}{(1-z)^k}. \end{aligned}$$

There is no contribution to the coefficient extractor in front when $k > b-1$ and may extend k to infinity, getting

$$\begin{aligned} &(1-p)^{b-1} [z^{b-1}] \frac{1}{(1-z)^{a+1}} \sum_{k\ge 0} p^k (1-p)^{-k} \frac{z^k}{(1-z)^k} \\ &= (1-p)^{b-1} [z^{b-1}] \frac{1}{(1-z)^{a+1}} \frac{1}{1-pz/(1-p)/(1-z)} \\ &= (1-p)^{b-1} [z^{b-1}] \frac{1}{(1-z)^{a}} \frac{1}{1-z-pz/(1-p)} \\ &= [z^{b-1}] \frac{1}{(1-(1-p)z)^{a}} \frac{1}{1-(1-p)z-pz} \\ &= [z^{b-1}] \frac{1}{1-z} \frac{1}{(1-(1-p)z)^a}. \end{aligned}$$

The LHS and the RHS are seen to be the same and we may conclude.

Remark. The first one is the easy one and follows by inspection. The Iverson bracket may be of interest here as an example of the method.

This was math.stackexchange.com problem 2904333.

1.6 MSE 2950043

Starting from

$$(-1)^{n+k}\left[{n \atop k}\right] = \sum_{j=0}^{n-k}(-1)^j \binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j}\left\{{n-k+j \atop j}\right\}$$

we introduce the EGF for Stirling numbers of the second kind on the RHS, getting

$$\sum_{j=0}^{n-k}(-1)^j \binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j}(n-k+j)![z^{n-k+j}]\frac{(\exp(z)-1)^j}{j!}$$

$$= (n-k)![z^{n-k}]\sum_{j=0}^{n-k}(-1)^j \binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j}\binom{n-k+j}{j}\frac{(\exp(z)-1)^j}{z^j}.$$

Now

$$\binom{n-1+j}{n-k+j}\binom{n-k+j}{j} = \frac{(n-1+j)!}{(k-1)!\times j!\times(n-k)!} = \binom{n-1}{k-1}\binom{n-1+j}{n-1}$$

and we find

$$\frac{(n-1)!}{(k-1)!}[z^{n-k}]\sum_{j=0}^{n-k}(-1)^j\binom{n-1+j}{n-1}\binom{2n-k}{n-k-j}\frac{(\exp(z)-1)^j}{z^j}$$

$$= \frac{(n-1)!}{(k-1)!}[z^{n-k}]\sum_{j=0}^{n-k}(-1)^j\binom{n-1+j}{n-1}\frac{(\exp(z)-1)^j}{z^j}[w^{n-k-j}](1+w)^{2n-k}$$

$$= \frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}]\sum_{j=0}^{n-k}(-1)^j\binom{n-1+j}{n-1}\frac{(\exp(z)-1)^j}{z^j}w^j.$$

Note that there is no contribution to the coefficient extractor $[w^{n-k}]$ when $j > n-k$, so we may write

$$\frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}]\sum_{j\geq 0}(-1)^j\binom{n-1+j}{n-1}\frac{(\exp(z)-1)^j}{z^j}w^j$$

$$= \frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}]\frac{1}{(1+w(\exp(z)-1)/z)^n}$$
$$= \frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}]\frac{z^n/(\exp(z)-1)^n}{(w+z/(\exp(z)-1))^n}.$$

Working with

$$\mathrm{Res}_{w=0}\frac{1}{w^{n-k+1}}(1+w)^{2n-k}\frac{1}{(w-C)^n}$$

we compute the residues at C and at infinity in order to apply the fact that they must sum to zero. Starting with the first we require (Leibniz rule)

$$\frac{1}{(n-1)!}\left(\frac{1}{w^{n-k+1}}(1+w)^{2n-k}\right)^{(n-1)}$$
$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(n-k+q)!}{(n-k)!}(-1)^q\frac{1}{w^{n-k+1+q}}$$
$$\times\frac{(2n-k)!}{(2n-k-(n-1-q))!}(1+w)^{2n-k-(n-1-q)}$$
$$= \sum_{q=0}^{n-1}\binom{n-k+q}{q}(-1)^q\frac{1}{w^{n-k+1+q}}\binom{2n-k}{n-1-q}(1+w)^{n-k+1+q}$$
$$= \left(\frac{1+w}{w}\right)^{n-k+1}\sum_{q=0}^{n-1}\binom{n-k+q}{q}(-1)^q\binom{2n-k}{n-1-q}\left(\frac{1+w}{w}\right)^q.$$

We have two important observations, the first is that

$$\frac{z^n}{(\exp(z)-1)^n} = 1+\cdots$$

i.e. no pole at zero and that

$$\left.\frac{1+w}{w}\right|_{w=-z/(\exp(z)-1)} = \frac{1+z-\exp(z)}{z} = -\frac{1}{2}z+\cdots.$$

Hence on substituting into the coefficient extractor on $[z^{n-k}]$ we get for all sum terms

$$[z^{n-k}](1+\cdots)\left(-\frac{1}{2}z+\cdots\right)^{n-k+1}\times\left(-\frac{1}{2}z+\cdots\right)^q = 0,$$

i.e. due to the middle term there is zero contribution from the residue at $w=-z/(\exp(z)-1)$. Returning to the main computation we get for the residue at infinity

$$\begin{aligned}
&\mathrm{Res}_{w=\infty} \frac{1}{w^{n-k+1}} (1+w)^{2n-k} \frac{1}{(w-C)^n} \\
&= -\mathrm{Res}_{w=0} \frac{1}{w^2} w^{n-k+1} (1+1/w)^{2n-k} \frac{1}{(1/w-C)^n} \\
&= -\mathrm{Res}_{w=0} \frac{1}{w^2} w^{2n-k+1} \frac{(1+w)^{2n-k}}{w^{2n-k}} \frac{1}{(1-Cw)^n} \\
&= -\mathrm{Res}_{w=0} \frac{1}{w} (1+w)^{2n-k} \frac{1}{(1-Cw)^n} = -1.
\end{aligned}$$

On flipping the sign and substituting into the coefficient extractor on z we get

$$\begin{aligned}
&\frac{(n-1)!}{(k-1)!} [z^{n-k}] \frac{z^n}{(\exp(z)-1)^n} \\
&= \frac{(n-1)!}{(k-1)!} \mathrm{Res}_{z=0} \frac{1}{z^{n-k+1}} \frac{z^n}{(\exp(z)-1)^n}.
\end{aligned}$$

Summing we get for the OGF

$$\begin{aligned}
&\sum_{k=1}^{n} x^k \frac{(n-1)!}{(k-1)!} \mathrm{Res}_{z=0} \frac{z^{k-1}}{(\exp(z)-1)^n} \\
&= x(n-1)! \times \mathrm{Res}_{z=0} \frac{1}{(\exp(z)-1)^n} \sum_{k=1}^{n} \frac{x^{k-1} z^{k-1}}{(k-1)!} \\
&= x(n-1)! \times \mathrm{Res}_{z=0} \frac{\exp(xz)}{(\exp(z)-1)^n}.
\end{aligned}$$

Now we evaluate the residue for $1 \le x \le n$ an integer. We have

$$\begin{aligned}
\mathrm{Res}_{z=0} \frac{\exp(xz)}{(\exp(z)-1)^n} &= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(xz)}{(\exp(z)-1)^n} \; dz \\
&= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{\exp((x-1)z)}{(\exp(z)-1)^n} \exp(z) \; dz
\end{aligned}$$

and putting $\exp(z) = w$ so that $\exp(z) \; dz = dw$ we obtain

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{w^{x-1}}{(w-1)^n} \; dw \\
&= \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^n} \sum_{q=0}^{x-1} \binom{x-1}{q} (w-1)^q \; dw.
\end{aligned}$$

This is zero when $x-1 < n-1$ or $x < n$ and it is one when $x = n$. By construction the residue is a polynomial in x of degree $n-1$. We have the $n-1$

roots, they are at $x = 1, 2, \ldots, n-1$ so we know it is

$$Q(x-1)(x-2) \times \cdots \times (x-(n-1)).$$

But we also know that at $x = n$ it evaluates to one, so we must have

$$Q(n-1)(n-2) \times \cdots \times 1 = 1$$

or $Q = 1/(n-1)!$. Restoring the two terms in front we finally obtain

$$x(n-1)! \times \frac{1}{(n-1)!}(x-1)(x-2) \times \cdots \times (x-(n-1))$$
$$= x(x-1)(x-2) \times \cdots \times (x-(n-1)) = \sum_{k=1}^{n} (-1)^{n+k} \left[{n \atop k}\right] x^k$$

which is precisely the Stirling number OGF, first kind, and we are done. This was math.stackexchange.com problem 2950043.

1.7 MSE 3049572

Starting from the claim

$${m+n \choose s+1} - {n \choose s+1} = \sum_{q=0}^{s} \frac{m}{q+1} {m+1+2q \choose q} {n-2-2q \choose s-q}$$

we observe that

$${m+1+2q \choose q+1} - {m+1+2q \choose q}$$
$$= \frac{m+1+q}{q+1} {m+1+2q \choose q} - {m+1+2q \choose q}$$
$$= \frac{m}{q+1} {m+1+2q \choose q}.$$

Therefore we have two sums,

$$\sum_{q=0}^{s} {m+1+2q \choose q+1} {n-2-2q \choose s-q} - \sum_{q=0}^{s} {m+1+2q \choose q} {n-2-2q \choose s-q}.$$

For the first one we write

$$\sum_{q=0}^{s} [w^{q+1}](1+w)^{m+1+2q} [z^{s-q}](1+z)^{n-2-2q}$$

$$= \operatorname*{res}_w (1+w)^{m+1}[z^s](1+z)^{n-2}\sum_{q=0}^{s}\frac{1}{w^{q+2}}z^q(1+w)^{2q}(1+z)^{-2q}.$$

We may extend q beyond s because of the coefficient extractor $[z^s]$ in front, getting

$$\begin{aligned}
&\operatorname*{res}_w \frac{1}{w^2}(1+w)^{m+1}[z^s](1+z)^{n-2}\sum_{q\ge 0} z^q w^{-q}(1+w)^{2q}(1+z)^{-2q}\\
&= \operatorname*{res}_w (1+w)^{m+1}[z^s](1+z)^{n-2}\frac{1}{w^2}\frac{1}{1-z(1+w)^2/w/(1+z)^2}\\
&= \operatorname*{res}_w (1+w)^{m+1}[z^s](1+z)^{n}\frac{1}{w}\frac{1}{w(1+z)^2-z(1+w)^2}.
\end{aligned}$$

Repeat the calculation for the second one to get

$$\operatorname*{res}_w (1+w)^{m+1}[z^s](1+z)^{n}\frac{1}{w(1+z)^2-z(1+w)^2}.$$

Now we have

$$\begin{aligned}
\left(\frac{1}{w}-1\right)\frac{1}{w(1+z)^2-z(1+w)^2} &= \frac{1}{w-z}\frac{1}{w(1+w)} - \frac{1}{1-wz}\frac{1}{1+w}\\
&= \frac{1}{1-z/w}\frac{1}{w^2(1+w)} - \frac{1}{1-wz}\frac{1}{1+w}.
\end{aligned}$$

We thus obtain two components, the first is

$$\begin{aligned}
&\operatorname*{res}_w (1+w)^{m+1}[z^s](1+z)^{n}\frac{1}{1-z/w}\frac{1}{w^2(1+w)}\\
&= \operatorname*{res}_w \frac{1}{w^2}(1+w)^{m}[z^s](1+z)^{n}\frac{1}{1-z/w}\\
&= \operatorname*{res}_w \frac{1}{w^2}(1+w)^{m}\sum_{q=0}^{s}\binom{n}{q}\frac{1}{w^{s-q}} = \sum_{q=0}^{s}\binom{n}{q}\operatorname*{res}_w \frac{1}{w^{s-q+2}}(1+w)^m\\
&= \sum_{q=0}^{s}\binom{n}{q}[w^{s-q+1}](1+w)^m = [w^{s+1}](1+w)^m\sum_{q=0}^{s}\binom{n}{q}w^q\\
&= -\binom{n}{s+1} + [w^{s+1}](1+w)^m\sum_{q=0}^{s+1}\binom{n}{q}w^q.
\end{aligned}$$

We may extend q beyond $s+1$ due to the coefficient extractor in front, to get

$$-\binom{n}{s+1} + [w^{s+1}](1+w)^m \sum_{q\ge 0} \binom{n}{q} w^q = -\binom{n}{s+1} + [w^{s+1}](1+w)^{m+n}$$

This is

$$\binom{m+n}{s+1} - \binom{n}{s+1}.$$

We have the claim, so we just need to prove that the second component will produce zero. We obtain

$$\begin{aligned}
&\operatorname*{res}_w (1+w)^{m+1}[z^s](1+z)^n \frac{1}{1-wz}\frac{1}{1+w} \\
&= \operatorname*{res}_w (1+w)^m [z^s](1+z)^n \frac{1}{1-wz} \\
&= \operatorname*{res}_w (1+w)^m \sum_{q=0}^{s} \binom{n}{q} w^{s-q} = \sum_{q=0}^{s} \binom{n}{q} \operatorname*{res}_w w^{s-q}(1+w)^m = 0.
\end{aligned}$$

This concludes the argument.

This was math.stackexchange.com problem 3049572.

1.8 MSE 3051713

We seek to evaluate

$$\sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k-1)!!}{(k-q)!} (-1)^k.$$

or alternatively

$$\begin{aligned}
&\sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k-1)!}{(k-1)!\times 2^{k-1}} \frac{1}{(k-q)!} (-1)^k \\
&= \sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k)!}{k!\times 2^k} \frac{1}{(k-q)!} (-1)^k.
\end{aligned}$$

This is

$$\begin{aligned}
&q! \sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k)!}{k!\times k!\times 2^k} \frac{k!}{q!\times (k-q)!} \frac{(-1)^k}{2^k} \\
&= q! \sum_{k=q}^{2n} \binom{2n+k}{2k} \binom{2k}{k} \binom{k}{q} \frac{(-1)^k}{2^k}.
\end{aligned}$$

Observe that

$$\binom{2n+k}{2k}\binom{2k}{k} = \frac{(2n+k)!}{(2n-k)! \times k! \times k!} = \binom{2n+k}{2n}\binom{2n}{k}$$

and furthermore

$$\binom{2n}{k}\binom{k}{q} = \frac{(2n)!}{(2n-k)! \times q! \times (k-q)!} = \binom{2n}{q}\binom{2n-q}{k-q}.$$

We get for the sum

$$\binom{2n}{q} q! \sum_{k=q}^{2n} \binom{2n+k}{2n}\binom{2n-q}{k-q}\frac{(-1)^k}{2^k}$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} \sum_{k=0}^{2n-q} \binom{2n+q+k}{2n}\binom{2n-q}{k}\frac{(-1)^k}{2^k}.$$

This becomes

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} \sum_{k=0}^{2n-q} \binom{2n+q+k}{2n} [z^{2n-q-k}](1+z)^{2n-q}\frac{(-1)^k}{2^k}$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} \sum_{k=0}^{2n-q} \binom{2n+q+k}{2n}\frac{(-1)^k}{2^k} z^k.$$

Now we may extend k beyond $2n-q$ because of the coefficient extractor $[z^{2n-q}]$ (no contribution) and get

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} \sum_{k\ge 0} \binom{2n+q+k}{2n}\frac{(-1)^k}{2^k} z^k$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} [w^{2n}](1+w)^{2n+q} \sum_{k\ge 0} (1+w)^k \frac{(-1)^k}{2^k} z^k$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} [w^{2n}](1+w)^{2n+q} \frac{1}{1+z(1+w)/2}.$$

Re-write this as

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} [w^{2n}](1+w)^{2n+q} \operatorname*{res}_z \frac{1}{z^{2n-q+1}} (1+z)^{2n-q} \frac{1}{1+z(1+w)/2}.$$

Working with the residue we apply the substitution $z/(1+z) = v$ or $z = v/(1-v)$ to get

$$\operatorname*{res}_v \frac{1}{v^{2n-q}} \frac{1-v}{v} \frac{1}{1+(v/(1-v))(1+w)/2} \frac{1}{(1-v)^2}$$
$$= \operatorname*{res}_v \frac{1}{v^{2n-q+1}} \frac{1}{1-v+v(1+w)/2}$$
$$= \operatorname*{res}_v \frac{1}{v^{2n-q+1}} \frac{1}{1-v(1-w)/2} = \frac{1}{2^{2n-q}}(1-w)^{2n-q}.$$

Substitute into the remaining coefficient extractor to get

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} [w^{2n}](1+w)^{2n+q} \frac{1}{2^{2n-q}}(1-w)^{2n-q}$$
$$= \binom{2n}{q} q! \frac{(-1)^q}{2^{2n}} \sum_{p=0}^{2n-q} (-1)^p \binom{2n-q}{p} \binom{2n+q}{2n-p}.$$

Now

$$\binom{2n}{q}\binom{2n-q}{p} = \frac{(2n)!}{q! \times p! \times (2n-q-p)!} = \binom{2n}{p}\binom{2n-p}{q}$$

and

$$\binom{2n-p}{q}\binom{2n+q}{2n-p} = \frac{(2n+q)!}{q! \times (2n-p-q)! \times (p+q)!} = \binom{2n+q}{q}\binom{2n}{p+q}.$$

This yields

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} \sum_{p=0}^{2n-q} (-1)^p \binom{2n}{p}\binom{2n}{p+q}$$
$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} \sum_{p=0}^{2n-q} (-1)^p \binom{2n}{p} [z^{2n-p-q}](1+z)^{2n}$$
$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1+z)^{2n} \sum_{p=0}^{2n-q} (-1)^p \binom{2n}{p} z^p.$$

Now we may extend p beyond $2n-q$ because of the coeffcient extractor $[z^{2n-q}]$ in front. We find

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1+z)^{2n} \sum_{p\ge 0} (-1)^p \binom{2n}{p} z^p$$
$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1+z)^{2n}(1-z)^{2n}$$

$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1-z^2)^{2n}.$$

Concluding we immediately obtain zero when q is odd, and otherwise we find

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2(n-q/2)}](1-z^2)^{2n}$$
$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{n-q/2}](1-z)^{2n}.$$

This is

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} (-1)^{n-q/2} \binom{2n}{n-q/2}$$

or alternatively

$$\frac{(-1)^{n+q/2}}{2^{2n}} \frac{(2n+q)!}{(n-q/2)! \times (n+q/2)!}.$$

This was math.stackexchange.com problem 3051713.

1.9 MSE 3068381

We seek to show that

$$S_n = \sum_{j=n}^{2n} \sum_{k=j+1-n}^{j} (-1)^j 2^{j-k} \binom{2n}{j} \left\{ j \atop k \right\} \left[k \atop j+1-n \right] = 0.$$

With the usual EGFs we get

$$\sum_{j=n}^{2n} \sum_{k=j+1-n}^{j} (-1)^j 2^{j-k} \binom{2n}{j} j! [z^j] \frac{(\exp(z)-1)^k}{k!}$$
$$\times k! [w^k] \frac{1}{(j+1-n)!} \left(\log \frac{1}{1-w} \right)^{j+1-n}.$$

Now we have

$$\binom{2n}{j} j! \frac{1}{(j+1-n)!} = \frac{(2n)!}{(2n-j)! \times (j+1-n)!} = \frac{(2n)!}{(n+1)!} \binom{n+1}{j+1-n}.$$

This yields for the sum

$$\frac{(2n)!}{(n+1)!} \sum_{j=n}^{2n} \binom{n+1}{j+1-n} (-1)^j 2^j$$

$$\times [z^j] \sum_{k=j+1-n}^{j} 2^{-k} (\exp(z)-1)^k [w^k] \left(\log\frac{1}{1-w}\right)^{j+1-n}$$

$$= \frac{(2n)!}{(n+1)!} (-1)^n 2^n \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j 2^j$$

$$\times [z^{n+j}] \sum_{k=j+1}^{j+n} 2^{-k} (\exp(z)-1)^k [w^k] \left(\log\frac{1}{1-w}\right)^{j+1}.$$

Observe that $(\exp(z)-1)^k = z^k + \cdots$ and hence we may extend the inner sum beyond $j+n$ due to the coefficient extractor $[z^{n+j}]$. We find

$$\frac{(2n)!}{(n+1)!} (-1)^n 2^n \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j 2^j [z^{n+j}]$$

$$\times \sum_{k \ge j+1} 2^{-k} (\exp(z)-1)^k [w^k] \left(\log\frac{1}{1-w}\right)^{j+1}.$$

Furthermore note that $\left(\log\frac{1}{1-w}\right)^{j+1} = w^{j+1} + \cdots$ so that the coefficient extractor $[w^k]$ covers the entire series, producing

$$\frac{(2n)!}{(n+1)!} (-1)^n 2^n \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j 2^j [z^{n+j}] \left(\log\frac{1}{1-(\exp(z)-1)/2}\right)^{j+1}.$$

Working with formal power series we are justified in writing

$$[z^{n+j}] \left(\log\frac{1}{1-(\exp(z)-1)/2}\right)^{j+1} = [z^{n-1}] \frac{1}{z^{j+1}} \left(\log\frac{1}{1-(\exp(z)-1)/2}\right)^{j+1}$$

because the logarithmic term starts at $z^{j+1}/2^{j+1}$. To see this write

$$\frac{\exp(z)-1}{2} + \frac{1}{2}\frac{(\exp(z)-1)^2}{2^2} + \frac{1}{3}\frac{(\exp(z)-1)^3}{2^3} + \cdots$$

We continue

$$\frac{(2n)!}{(n+1)!} (-1)^{n-1} 2^{n-1}$$

$$\times [z^{n-1}] \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^{j+1} 2^{j+1} \frac{1}{z^{j+1}} \left(\log\frac{1}{1-(\exp(z)-1)/2}\right)^{j+1}$$

$$= \frac{(2n)!}{(n+1)!} (-1)^{n-1} 2^{n-1}$$

$$\times [z^{n-1}] \sum_{j=1}^{n+1} \binom{n+1}{j} (-1)^j 2^j \frac{1}{z^j} \left(\log \frac{1}{1-(\exp(z)-1)/2} \right)^j.$$

The term for $j = 0$ in the sum is one and hence only contributes to $n = 1$ so that we may write

$$-[[n=1]] + \frac{(2n)!}{(n+1)!} (-1)^{n-1} 2^{n-1}$$
$$\times [z^{n-1}] \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^j 2^j \frac{1}{z^j} \left(\log \frac{1}{1-(\exp(z)-1)/2} \right)^j$$
$$= -[[n=1]] + \frac{(2n)!}{(n+1)!} (-1)^{n-1} 2^{n-1}$$
$$\times [z^{n-1}] \left(1 - \frac{2}{z} \log \frac{1}{1-(\exp(z)-1)/2} \right)^{n+1}.$$

Finally observe that

$$\left(1 - \frac{2}{z} \log \frac{1}{1-(\exp(z)-1)/2} \right)^{n+1}$$
$$= \left(1 - \frac{2}{z} \left(\frac{\exp(z)-1}{2} + \frac{1}{2} \frac{(\exp(z)-1)^2}{2^2} + \frac{1}{3} \frac{(\exp(z)-1)^3}{2^3} + \cdots \right) \right)^{n+1}$$
$$= \left(-\frac{3}{4} z - \cdots \right)^{n+1}$$

and furthermore

$$[z^{n-1}] \left((-1)^{n+1} \frac{3^{n+1}}{4^{n+1}} z^{n+1} + \cdots \right) = 0$$

which is the claim.

This was math.stackexchange.com problem 3068381.

1.10 MSE 3138710

We seek to prove that with $n \ge m+2$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{n-j} \binom{n-j}{j} = \binom{n+k+m}{m+1}.$$

This is

$$\binom{m+k}{m+1} + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{n-j} \binom{n-j}{j} = \binom{n+k+m}{m+1}$$

or

$$\binom{m+k}{m+1} + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{j} \binom{n-j-1}{j-1} = \binom{n+k+m}{m+1}$$

Now observe that

$$\binom{n-j-1}{j} = \frac{n-2j}{j} \binom{n-j-1}{j-1} = \frac{n}{j} \binom{n-j-1}{j-1} - 2\binom{n-j-1}{j-1}.$$

We thus get two terms:

$$\binom{m+k}{m+1} + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{j} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{j}$$

and

$$2 \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{j-1}.$$

For the first one we have

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{n-2j-1}$$

$$= [z^{n-1}](1+z)^{n-1} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} (1+z)^{-j} z^{2j}.$$

We may extend j to infinity because of the coefficient extractor in front (note that the following representation in the variable w will produce a correct value of zero in the remaining binomial coefficient when $j > m+1$):

$$[z^{n-1}](1+z)^{n-1}[w^{m+1}](1+w)^{m+k} \sum_{j \geq 0} (1+z)^{-j} z^{2j} (1+w)^j w^j$$

$$= [z^{n-1}](1+z)^{n-1}[w^{m+1}](1+w)^{m+k} \frac{1}{1 - z^2 w(1+w)/(1+z)}$$

$$= [z^{n-1}](1+z)^{n}[w^{m+1}](1+w)^{m+k} \frac{1}{1 + z - z^2 w(1+w)}$$

$$= -[z^{n-1}](1+z)^{n}[w^{m+2}](1+w)^{m+k-1} \frac{1}{(z-1/w)(z+1/(1+w))}.$$

Extracting $[z^{n-1}]$ first we get

$$\mathrm{Res}_{z=0} \frac{1}{z^n}(1+z)^n \frac{1}{(z-1/w)(z+1/(1+w))}.$$

We see that the residue at infinity is zero. Residues sum to zero and we get for the residue at $z = 1/w$

$$w^n \frac{(1+w)^n}{w^n} \frac{1}{1/w+1/(1+w)} = w\frac{(1+w)^{n+1}}{1+2w}.$$

For the residue at $z = -1/(1+w)$ we find

$$-(-1)^n(1+w)^n \frac{w^n}{(1+w)^n} \frac{1}{1/(1+w)+1/w} = -(-1)^n w^{n+1}(1+w)\frac{1}{1+2w}.$$

Now the coefficient extractor is $[w^{m+2}]$ but we have $n \ge m+2$ so the contribution from this is zero.

It follows that the first sum is given by

$$[w^{m+1}]\frac{(1+w)^{n+k+m}}{1+2w}.$$

Continuing with the second sum we find

$$2\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1}\binom{n-j-1}{n-2j}$$

$$= 2[z^n](1+z)^{n-1} \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1}(1+z)^{-j} z^{2j}.$$

We may include $j=0$ here because

$$2[z^n](1+z)^{n-1}\binom{m+k}{m+1} = 0,$$

getting

$$2[z^n](1+z)^{n-1} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1}(1+z)^{-j} z^{2j}.$$

We skip forward to the residue computation since the intermediate steps are the same as before. We get for the residue at $z = 1/w$

$$2w^{n+1}\frac{(1+w)^n}{w^n}\frac{1}{1/w+1/(1+w)} = 2w^2\frac{(1+w)^{n+1}}{1+2w}.$$

For the residue at $z = -1/(1+w)$ we find

$$-(-1)^{n+1}(1+w)^{n+1}\frac{w^n}{(1+w)^n}\frac{1}{1/(1+w)+1/w} = (-1)^n w^{n+1}(1+w)^2\frac{1}{1+2w}.$$

We note once more that the coefficient extractor is $[w^{m+2}]$ but we have $n \ge m+2$ so the contribution from this is zero. It follows that the second sum is given by

$$[w^{m+1}]2w\frac{(1+w)^{n+k+m}}{1+2w}.$$

Adding the two sums we obtain at last

$$[w^{m+1}]\frac{(1+w)^{n+k+m}}{1+2w} + [w^{m+1}]2w\frac{(1+w)^{n+k+m}}{1+2w} = [w^{m+1}](1+w)^{n+k+m}.$$

or

$$\binom{n+k+m}{m+1}.$$

This was math.stackexchange.com problem 3138710.

1.11 MSE 3196998

As a preliminary, observe that the generating function of the Fibonacci numbers is

$$\frac{z}{1-z-z^2}.$$

so that we have $F_0 = 0$ and $F_1 = F_2 = 1$.

We seek to evaluate

$$\sum_{p=0}^{n}\sum_{q=0}^{n}\binom{n-p}{q}\binom{n-q}{p}$$
$$= \sum_{p=0}^{n}\sum_{q=0}^{n}\binom{n-p}{n-p-q}\binom{n-q}{n-p-q}.$$

Note that on the first line the binomial coefficient $\binom{n}{k} = n^{\underline{k}}/k!$ starts producing non-zero values when $p > n$ and $q > n$. This is not desired here, hence the upper limits. On the second line we use the convention that $\binom{n}{k} = 0$ when $k < 0$, which is also the behavior when residues are used. Continuing we find

$$\sum_{p=0}^{n}\sum_{q=0}^{n}[z^{n-p-q}](1+z)^{n-p}[w^{n-p-q}](1+w)^{n-q}$$

$$= [z^n](1+z)^n [w^n](1+w)^n \sum_{p=0}^n \sum_{q=0}^n z^{p+q}(1+z)^{-p} w^{p+q}(1+w)^{-q}$$

$$= [z^n](1+z)^n [w^n](1+w)^n \sum_{p=0}^n z^p w^p (1+z)^{-p} \sum_{q=0}^n z^q w^q (1+w)^{-q}.$$

Here the coefficient extractor controls the range and we may continue with

$$[z^n](1+z)^n [w^n](1+w)^n \sum_{p\ge 0} z^p w^p (1+z)^{-p} \sum_{q\ge 0} z^q w^q (1+w)^{-q}$$

$$= [z^n](1+z)^n [w^n](1+w)^n \frac{1}{1-zw/(1+z)} \frac{1}{1-zw/(1+w)}$$

$$= [z^n](1+z)^{n+1} [w^n](1+w)^{n+1} \frac{1}{1+z-zw} \frac{1}{1+w-zw}.$$

Now we have

$$\frac{1}{1+z-zw} \frac{1}{1+w-zw}$$
$$= \frac{1-w}{1+z-wz} \frac{1}{1+w-w^2} + \frac{w}{1+w-wz} \frac{1}{1+w-w^2}.$$

We get from the first piece treating z first

$$[z^n](1+z)^{n+1} \frac{1-w}{1+z-wz} = [z^n](1+z)^{n+1} \frac{1-w}{1-z(w-1)}$$

$$= (1-w) \sum_{p=0}^n \binom{n+1}{n-p} (w-1)^p = - \sum_{p=0}^n \binom{n+1}{p+1} (w-1)^{p+1}$$

$$= 1 - \sum_{p=-1}^n \binom{n+1}{p+1} (w-1)^{p+1} = 1 - w^{n+1}.$$

The contribution is

$$[w^n](1+w)^{n+1} \frac{1-w^{n+1}}{1+w-w^2} = [w^n](1+w)^{n+1} \frac{1}{1+w-w^2}.$$

The second piece yields

$$[z^n](1+z)^{n+1} \frac{w}{1+w-wz} = \frac{1}{1+w} [z^n](1+z)^{n+1} \frac{w}{1-wz/(1+w)}$$

$$= \frac{w}{1+w} \sum_{p=0}^n \binom{n+1}{n-p} \frac{w^p}{(1+w)^p} = \sum_{p=0}^n \binom{n+1}{p+1} \frac{w^{p+1}}{(1+w)^{p+1}}$$

$$= -1 + \sum_{p=-1}^{n} \binom{n+1}{p+1} \frac{w^{p+1}}{(1+w)^{p+1}} = -1 + \left(1 + \frac{w}{1+w}\right)^{n+1}$$

$$= -1 + \frac{(1+2w)^{n+1}}{(1+w)^{n+1}}.$$

The contribution is

$$[w^n](1+w)^{n+1} \left(-1 + \frac{(1+2w)^{n+1}}{(1+w)^{n+1}}\right) \frac{1}{1+w-w^2}.$$

Adding the first and the second contribution we find

$$[w^n](1+2w)^{n+1} \frac{1}{1+w-w^2}$$

$$= \operatorname*{res}_w \frac{1}{w^{n+1}} (1+2w)^{n+1} \frac{1}{1+w-w^2}.$$

Setting $w/(1+2w) = v$ or $w = v/(1-2v)$ so that $dw = 1/(1-2v)^2 \; dv$ we obtain

$$\operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{1 + v/(1-2v) - v^2/(1-2v)^2} \frac{1}{(1-2v)^2}$$

$$= \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{(1-2v)^2 + v(1-2v) - v^2}$$

$$= \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{1-3v+v^2}.$$

We have our answer:

$$[v^n] \frac{1}{1-3v+v^2} = F_{2n+2}.$$

It remains to prove that the coefficient extractor returns the Fibonacci number as claimed. The OGF of even-index Fibonacci numbers is

$$\sum_{n\ge 0} F_{2n} z^{2n} = \frac{1}{2} \frac{z}{1-z-z^2} + \frac{1}{2} \frac{(-z)}{1+z-z^2} = \frac{z^2}{1-3z^2+z^4}.$$

This implies that

$$\sum_{n\ge 0} F_{2n} z^n = \frac{z}{1-3z+z^2}.$$

Therefore

$$F_{2n+2} = [z^{n+1}] \frac{z}{1-3z+z^2} = [z^n] \frac{1}{1-3z+z^2}$$

as required.

This was math.stackexchange.com problem 3196998.

1.12 MSE 3245099

Starting from the claim that $S = 1$ where

$$S = \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^q b^K + a^K b^q}{(a+b)^{q+K}}$$

we get two pieces

$$\frac{b^K}{(a+b)^K} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^q}{(a+b)^q}$$
$$+ \frac{a^K}{(a+b)^K} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{b^q}{(a+b)^q}.$$

This is

$$\frac{b^K}{(a+b)^K} [z^{K-1}] \frac{1}{1-z} \frac{1}{(1-az/(a+b))^K}$$
$$+ \frac{a^K}{(a+b)^K} [z^{K-1}] \frac{1}{1-z} \frac{1}{(1-bz/(a+b))^K}.$$

Call these S_1 and S_2. The first sum is

$$\begin{aligned}
S_1 &= \frac{b^K}{(a+b)^K} \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{1-z} \frac{1}{(1-az/(a+b))^K} \\
&= b^K \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{1-z} \frac{1}{(a+b-az)^K} \\
&= \frac{b^K}{a^K} \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{1-z} \frac{1}{((a+b)/a-z)^K} \\
&= (-1)^{K+1} \frac{b^K}{a^K} \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{z-1} \frac{1}{(z-(a+b)/a)^K}.
\end{aligned}$$

Now residues sum to zero so we compute this from the residues at the poles at $z = 1$ and $z = (a+b)/a$. The residue at infinity is zero by inspection. The residue at $z = 1$ is

$$\begin{aligned}
(-1)^{K+1} \frac{b^K}{a^K} \frac{1}{(1-(a+b)/a)^K} &= (-1)^{K+1} b^K \frac{1}{(a-(a+b))^K} \\
&= (-1)^{K+1} b^K \frac{1}{(-b)^K} = -1.
\end{aligned}$$

For the residue at $z = (a+b)/a$ we require

$$\frac{1}{(K-1)!}\left(\frac{1}{z^K}\frac{1}{z-1}\right)^{(K-1)}$$

$$= \frac{1}{(K-1)!}\sum_{q=0}^{K-1}\binom{K-1}{q}(-1)^q\frac{(K-1+q)!}{(K-1)!}\frac{1}{z^{K+q}}(-1)^{K-1-q}\frac{(K-1-q)!}{(z-1)^{K-q}}$$

$$= (-1)^{K+1}\sum_{q=0}^{K-1}\binom{K-1+q}{K-1}\frac{1}{z^{K+q}}\frac{1}{(z-1)^{K-q}}.$$

Evaluating the residue we find

$$(-1)^{K+1}\frac{b^K}{a^K}(-1)^{K+1}\sum_{q=0}^{K-1}\binom{K-1+q}{K-1}\frac{1}{z^{K+q}}\frac{1}{(z-1)^{K-q}}\Bigg|_{z=(a+b)/a}$$

$$= \frac{b^K}{a^K}\sum_{q=0}^{K-1}\binom{K-1+q}{K-1}\frac{a^{K+q}}{(a+b)^{K+q}}\frac{1}{((a+b)/a-1)^{K-q}}$$

$$= \sum_{q=0}^{K-1}\binom{K-1+q}{K-1}\frac{a^{K+q}}{(a+b)^{K+q}}\frac{b^q}{a^q}\frac{b^{K-q}}{a^{K-q}}\frac{1}{((a+b)/a-1)^{K-q}}$$

$$= \sum_{q=0}^{K-1}\binom{K-1+q}{K-1}\frac{a^{K+q}}{(a+b)^{K+q}}\frac{b^q}{a^q}$$

$$= \frac{a^K}{(a+b)^K}\sum_{q=0}^{K-1}\binom{K-1+q}{K-1}\frac{b^q}{(a+b)^q} = S_2.$$

We recognise S_2 and hence we have shown that

$$S_1 - 1 + S_2 = 0$$

or

$$\sum_{q=0}^{K-1}\binom{K-1+q}{K-1}\frac{a^q b^K + a^K b^q}{(a+b)^{q+K}} = 1$$

as claimed.

This was math.stackexchange.com problem 3245099.

Remark. This is the formal power series version of the identity by Gosper in section **??**.

1.13 MSE 3260307

Starting from the claim (we treat the case r a positive integer)

$$\binom{r+2n-1}{n-1} - \binom{2n-1}{n-1} = S = \sum_{k=1}^{n-1} \binom{2k-1}{k} \binom{r+2(n-k)-1}{r+n-k} \frac{r}{n-k}$$

$$= \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{r+k} \frac{r}{k}$$

$$= \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{k-1} \frac{r}{k}$$

we use the fact that

$$\binom{r+2k-1}{k-1} \frac{r}{k} = \binom{r+2k-1}{k} - \binom{r+2k-1}{k-1}$$

to get two pieces, call them S_1 and S_2 where $S = S_1 - S_2$ and

$$S_1 = \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{k}$$

and

$$S_2 = \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{k-1}.$$

We find for S_1

$$\operatorname*{res}_{w} (1+w)^{r-1} \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \frac{(1+w)^{2k}}{w^{k+1}}$$

$$= \operatorname*{res}_{w} \frac{(1+w)^{r-1}}{w} [z^n](1+z)^{2n-1} \sum_{k=1}^{n-1} z^k (1+z)^{-2k} \frac{(1+w)^{2k}}{w^k}.$$

Including the term at $k = 0$ and compensating

$$-\binom{2n-1}{n-1} + \operatorname*{res}_{w} \frac{(1+w)^{r-1}}{w} [z^n](1+z)^{2n-1} \sum_{k=0}^{n-1} z^k (1+z)^{-2k} \frac{(1+w)^{2k}}{w^k}.$$

Including the term at $k = n$ and again compensating

$$-\binom{2n-1}{n-1} - \binom{r+2n-1}{n}$$

$$+\operatorname*{res}_{w} \frac{(1+w)^{r-1}}{w} [z^n](1+z)^{2n-1} \sum_{k=0}^{n} z^k (1+z)^{-2k} \frac{(1+w)^{2k}}{w^k}.$$

Now we may extend k beyond n owing to the coefficient extractor $[z^n]$ to get

$$-\binom{2n-1}{n-1} - \binom{r+2n-1}{n}$$

$$+ \operatorname*{res}_w \frac{(1+w)^{r-1}}{w}[z^n](1+z)^{2n-1}\frac{1}{1-z(1+w)^2/w/(1+z)^2}$$

$$= -\binom{2n-1}{n-1} - \binom{r+2n-1}{n}$$

$$+ \operatorname*{res}_w (1+w)^{r-1}[z^n](1+z)^{2n+1}\frac{1}{w(1+z)^2-z(1+w)^2}.$$

We get for S_2

$$\operatorname*{res}_w (1+w)^{r-1}[z^n](1+z)^{2n-1}\sum_{k=1}^{n-1} z^k(1+z)^{-2k}\frac{(1+w)^{2k}}{w^k}.$$

The term $k=0$ contributes zero. Compensating for $k=n$ we find

$$-\binom{r+2n-1}{n-1} + \operatorname*{res}_w (1+w)^{r-1}[z^n](1+z)^{2n-1}\sum_{k\ge 0} z^k(1+z)^{-2k}\frac{(1+w)^{2k}}{w^k}$$

$$= -\binom{r+2n-1}{n-1} + \operatorname*{res}_w w(1+w)^{r-1}[z^n](1+z)^{2n+1}\frac{1}{w(1+z)^2-z(1+w)^2}.$$

We therefore have

$$S = S_1 - S_2 = -\binom{2n-1}{n-1} - \binom{r+2n-1}{n} + \binom{r+2n-1}{n-1}$$

$$+\operatorname*{res}_w (1+w)^{r-1}[z^n](1+z)^{2n}\frac{(1-w)(1+z)}{w(1+z)^2-z(1+w)^2}.$$

Working with the remaining residue we note that

$$\frac{(1-w)(1+z)}{w(1+z)^2-z(1+w)^2} = \frac{1}{w}\frac{1}{1-z/w} - \frac{1}{1-zw}.$$

We see on substituting into the residue that we get no contribution from the second term. This leaves

$$\operatorname*{res}_w \frac{1}{w}(1+w)^{r-1}[z^n](1+z)^{2n}\frac{1}{1-z/w}$$

$$= \operatorname*{res}_w \frac{1}{w}(1+w)^{r-1}\sum_{q=0}^{n}\binom{2n}{n-q}w^{-q}$$

$$= \sum_{q=0}^{n} \binom{2n}{n-q} \binom{r-1}{q} = [z^n](1+z)^{2n} \sum_{q=0}^{n} \binom{r-1}{q} z^q.$$

The coefficient extractor once more enforces the range and we find

$$[z^n](1+z)^{2n} \sum_{q\ge 0} \binom{r-1}{q} z^q$$
$$= [z^n](1+z)^{2n}(1+z)^{r-1} = [z^n](1+z)^{r+2n-1} = \binom{r+2n-1}{n}.$$

Collecting all four pieces yields

$$S = S_1 - S_2 = -\binom{2n-1}{n-1} - \binom{r+2n-1}{n} + \binom{r+2n-1}{n-1} + \binom{r+2n-1}{n}$$
$$= \binom{r+2n-1}{n-1} - \binom{2n-1}{n-1}$$

which is the claim.

Remark. The next-to-last step may also be done as follows:

$$\operatorname*{res}_{w} \frac{1}{w} (1+w)^{r-1} [z^n](1+z)^{2n} \frac{1}{1-z/w}$$
$$= \operatorname*{res}_{w} \frac{1}{w} \sum_{q=0}^{r-1} \binom{r-1}{q} w^q [z^n](1+z)^{2n} \frac{1}{1-z/w}$$
$$= [z^n](1+z)^{2n} \sum_{q=0}^{r-1} \binom{r-1}{q} z^q = [z^n](1+z)^{2n}(1+z)^{r-1} = \binom{r+2n-1}{n}.$$

This was math.stackexchange.com problem 3260307.

1.14 MSE 3285142

Starting from (the contribution from $k=0$ is zero owing to the third binomial coefficient)

$$\sum_{k=1}^{n} \left(-\frac{1}{4}\right)^k \binom{2k}{k}^2 \frac{1}{1-2k} \binom{n+k-2}{2k-2}$$

we seek to show that this is zero when $n > 1$ is odd and

$$\left[\left(\frac{1}{4}\right)^m \binom{2m}{m} \frac{1}{1-2m}\right]^2$$

when $n = 2m$ is even.

We observe that with $k \ge 1$

$$\binom{2k}{k}\frac{1}{1-2k}\binom{n+k-2}{2k-2} = 2\binom{2k-1}{k-1}\frac{1}{1-2k}\binom{n+k-2}{2k-2}$$

$$= -2\binom{2k-2}{k-1}\frac{1}{k}\binom{n+k-2}{2k-2} = -\frac{2}{k}\frac{(n+k-2)!}{(k-1)!^2 \times (n-k)!}$$

$$= -\frac{2}{k}\binom{n+k-2}{k-1}\binom{n-1}{k-1} = -\frac{2}{n}\binom{n}{k}\binom{n+k-2}{k-1}.$$

We get for our sum

$$-\frac{2}{n}\sum_{k=1}^{n}\binom{n}{k}\left(-\frac{1}{4}\right)^k\binom{2k}{k}\binom{n+k-2}{k-1}$$

$$= -\frac{2}{n}\sum_{k=1}^{n}\binom{n}{k}\binom{-1/2}{k}\binom{n+k-2}{n-1}$$

$$= -\frac{2}{n}[z^{n-1}](1+z)^{n-2}\sum_{k=1}^{n}\binom{n}{k}\binom{-1/2}{k}(1+z)^k.$$

The value $k=0$ contributes zero:

$$-\frac{2}{n} \times \operatorname*{res}_{z} \frac{1}{w}(1+w)^{-1/2}[z^{n-1}](1+z)^{n-2}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{w^k}(1+z)^k$$

$$= -\frac{2}{n} \times \operatorname*{res}_{z} \frac{1}{w}(1+w)^{-1/2}[z^{n-1}](1+z)^{n-2}(1+(1+z)/w)^n$$

$$= -\frac{2}{n} \times \operatorname*{res}_{z} \frac{1}{w^{n+1}}(1+w)^{-1/2}[z^{n-1}](1+z)^{n-2}(1+w+z)^n$$

$$= -\frac{2}{n} \times \operatorname*{res}_{z} \frac{1}{w^{n+1}}(1+w)^{-1/2}[z^{n-1}](1+z)^{n-2}\sum_{q=0}^{n}\binom{n}{q}(1+w)^q z^{n-q}$$

$$= -\frac{2}{n} \times \sum_{q=1}^{n}\binom{n}{q}\binom{q-1/2}{n}\binom{n-2}{q-1}.$$

Now observe that with $q < n$ (third binomial coefficient is zero when $q = n$)

$$\binom{q-1/2}{n} = \frac{1}{n!}(q-1/2)^{\underline{n}} = \frac{1}{n!}\prod_{p=0}^{q-1}(q-1/2-p)\prod_{p=q}^{n-1}(q-1/2-p)$$

$$= \frac{1}{n! \times 2^n}\prod_{p=0}^{q-1}(2q-1-2p)\prod_{p=q}^{n-1}(2q-1-2p)$$

$$= \frac{1}{n! \times 2^n} \frac{(2q-1)!}{(q-1)! \times 2^{q-1}} \prod_{p=0}^{n-1-q} (-1-2p)$$

$$= \frac{(-1)^{n-q}}{n! \times 2^n} \frac{(2q-1)!}{(q-1)! \times 2^{q-1}} \frac{(2n-1-2q)!}{(n-1-q)! \times 2^{n-1-q}}$$

$$= \frac{(-1)^{n-q}}{2^{2n-2}} \binom{n}{q}^{-1} \binom{2q-1}{q-1} \binom{2n-1-2q}{n-q}.$$

We get for our sum

$$-\frac{1}{n \times 2^{2n-3}} \times \sum_{q=1}^{n-1} (-1)^{n-q} \binom{2q-1}{q-1} \binom{2n-1-2q}{n-q} \binom{n-2}{q-1}$$

$$= \frac{1}{n \times 2^{2n-3}} \times \sum_{q=0}^{n-2} \binom{n-2}{q} (-1)^{n-2-q} \binom{2q+1}{q} \binom{2n-3-2q}{n-q-1}.$$

This becomes

$$\frac{1}{n \times 2^{2n-3}} \times [z^{n-1}](1+z)^{2n-3} \sum_{q=0}^{n-2} \binom{n-2}{q} (-1)^{n-2-q} \binom{2q+1}{q} z^q (1+z)^{-2q}$$

$$= \frac{1}{n \times 2^{2n-3}} \operatorname*{res}_z \frac{1+w}{w} [z^{n-1}](1+z)^{2n-3}$$

$$\times \sum_{q=0}^{n-2} \binom{n-2}{q} (-1)^{n-2-q} \frac{1}{w^q} (1+w)^{2q} z^q (1+z)^{-2q}$$

$$= \frac{1}{n \times 2^{2n-3}} \operatorname*{res}_z \frac{1+w}{w} [z^{n-1}](1+z)^{2n-3} \left(\frac{z(1+w)^2}{w(1+z)^2} - 1 \right)^{n-2}$$

$$= \frac{1}{n \times 2^{2n-3}} \operatorname*{res}_z \frac{1+w}{w^{n-1}} [z^{n-1}](1+z) \left(z(1+w)^2 - w(1+z)^2 \right)^{n-2}$$

$$= \frac{1}{n \times 2^{2n-3}} \operatorname*{res}_z \frac{1+w}{w^{n-1}} [z^{n-1}](1+z)(z-w)^{n-2}(1-wz)^{n-2}.$$

The first piece in z is

$$[z^{n-1}](z-w)^{n-2}(1-wz)^{n-2}$$

$$= \sum_{q=1}^{n-2} \binom{n-2}{q} (-1)^{n-2-q} w^{n-2-q} \binom{n-2}{n-1-q} (-1)^{n-1-q} w^{n-1-q}$$

$$= -\sum_{q=1}^{n-2} \binom{n-2}{q} \binom{n-2}{q-1} w^{2n-3-2q}.$$

Here we require

$$([w^{n-2}] + [w^{n-3}])w^{2n-3-2q}$$

We get $q = (n-1)/2$ in the first case and $q = n/2$ in the second. As this is a pair of an integer and a fraction clearly only one of these extractors can return a non-zero value.

The second piece in z is

$$[z^{n-2}](z-w)^{n-2}(1-wz)^{n-2}$$
$$= \sum_{q=0}^{n-2} \binom{n-2}{q} (-1)^{n-2-q} w^{n-2-q} \binom{n-2}{n-2-q} (-1)^{n-2-q} w^{n-2-q}$$
$$= \sum_{q=0}^{n-2} \binom{n-2}{q} \binom{n-2}{q} w^{2n-4-2q}.$$

Solving for q again we require

$$([w^{n-2}] + [w^{n-3}])w^{2n-4-2q}$$

getting $q = n/2 - 1$ and $q = (n-1)/2$.

Supposing that n is odd i.e. $n = 2m+1$ we thus have

$$-\binom{2m-1}{m}\binom{2m-1}{m-1} + \binom{2m-1}{m}\binom{2m-1}{m} = 0,$$

and we have proved the second part of the claim. On the other hand with $n = 2m$ even we collect

$$-\binom{2m-2}{m}\binom{2m-2}{m-1} + \binom{2m-2}{m-1}\binom{2m-2}{m-1}$$
$$= \binom{2m-2}{m-1}^2 \left(1 - \frac{m-1}{m}\right) = \frac{m^2}{(2m-1)^2}\binom{2m-1}{m}^2 \frac{1}{m}$$
$$= \frac{m^2}{(2m-1)^2}\frac{m^2}{(2m)^2}\binom{2m}{m}^2 \frac{1}{m} = \frac{1}{4}\frac{m}{(2m-1)^2}\binom{2m}{m}^2.$$

Restoring the factor in front we obtain

$$\frac{1}{n \times 2^{2n-3}} \frac{1}{4} \frac{m}{(2m-1)^2}\binom{2m}{m}^2 = \frac{1}{2^{2n}}\frac{1}{(2m-1)^2}\binom{2m}{m}^2$$
$$= \frac{1}{2^{4m}}\frac{1}{(1-2m)^2}\binom{2m}{m}^2$$

This is

$$\left[\left(\frac{1}{4}\right)^m \binom{2m}{m}\frac{1}{1-2m}\right]^2$$

as was to be shown.

This was math.stackexchange.com problem 3285142.

1.15 MSE 3333597

We seek to verify that

$$\sum_{n=0}^N \sum_{k=0}^N \frac{(-1)^{n+k}}{n+k+1}\binom{N}{n}\binom{N}{k}\binom{N+n}{n}\binom{N+k}{k} = \frac{1}{2N+1}.$$

Now we have

$$\binom{N}{k}\binom{N+n}{n} = \frac{(N+n)!}{(N-k)!\times k!\times n!} = \binom{N+n}{n+k}\binom{n+k}{k}.$$

We get for the LHS

$$\sum_{n=0}^N \sum_{k=0}^N \frac{(-1)^{n+k}}{n+k+1}\binom{N+n}{n+k}\binom{N}{n}\binom{N+k}{k}\binom{n+k}{k}$$

$$= \sum_{n=0}^N \frac{1}{N+n+1}\sum_{k=0}^N (-1)^{n+k}\binom{N+n+1}{n+k+1}\binom{N}{n}\binom{N+k}{k}\binom{n+k}{k}$$

$$= \sum_{n=0}^N \frac{1}{N+n+1}\binom{N}{n}\sum_{k=0}^N (-1)^{n+k}\binom{N+n+1}{N-k}\binom{N+k}{k}\binom{n+k}{k}$$

$$= \sum_{n=0}^N \frac{1}{N+n+1}[z^N](1+z)^{N+n+1}\binom{N}{n}\sum_{k=0}^N (-1)^{n+k} z^k\binom{N+k}{N}\binom{n+k}{n}.$$

Now the coefficient extractor controls the range and we continue with

$$\sum_{n=0}^N \frac{1}{N+n+1}[z^N](1+z)^{N+n+1}\binom{N}{n}$$

$$\times \sum_{k\ge 0}(-1)^{n+k} z^k \binom{N+k}{N} \operatorname*{res}_w \frac{1}{w^{n+1}}\frac{1}{(1-w)^{k+1}}$$

$$= \sum_{n=0}^N \frac{1}{N+n+1}[z^N](1+z)^{N+n+1}\binom{N}{n}\operatorname*{res}_w \frac{1}{w^{n+1}}\frac{1}{1-w}$$

$$\times \sum_{k\ge 0}(-1)^{n+k} z^k \binom{N+k}{N}\frac{1}{(1-w)^k}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n}{N+n+1} [z^N](1+z)^{N+n+1} \binom{N}{n}$$

$$\times \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1}{1-w} \frac{1}{(1+z/(1-w))^{N+1}}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n}{N+n+1} [z^N](1+z)^{N+n+1} \binom{N}{n}$$

$$\times \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{(1-w)^N}{(1-w+z)^{N+1}}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n}{N+n+1} [z^N](1+z)^{n} \binom{N}{n}$$

$$\times \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{(1-w)^N}{(1-w/(1+z))^{N+1}}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n}{N+n+1} [z^N](1+z)^{n} \binom{N}{n}$$

$$\times \sum_{k=0}^{n} (-1)^k \binom{N}{k} \binom{n-k+N}{N} \frac{1}{(1+z)^{n-k}}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n}{N+n+1} \binom{N}{n}$$

$$\times \sum_{k=0}^{n} (-1)^k \binom{N}{k} \binom{n-k+N}{N} [z^N](1+z)^k.$$

Now for the coefficient extractor to be non-zero we must have $k \ge N$ which happens just once, namely when $n = N$ and $k = N$. We get

$$\frac{(-1)^N}{2N+1} \binom{N}{N} (-1)^N \binom{N}{N} \binom{N-N+N}{N}.$$

This expression does indeed simplify to

$$\frac{1}{2N+1}$$

as claimed.

This was math.stackexchange.com problem 3333597.

1.16 MSE 3342361

We seek to verify that

$$\sum_{k=3}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{k-2}\binom{j(n+1)+k-3}{n-2} = (-1)^{n-1}\left[\binom{n}{2}-\binom{2n+1}{n-2}\right].$$

where $n \geq 3$. Now for

$$\sum_{k=3}^{n}(-1)^k\binom{n}{k}\binom{k-3}{n-2}$$

to be non-zero we would need $k-3 \geq n-2$ or $k \geq n+1$, which is not in the range, so it is zero and we may work with

$$\sum_{k=3}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{k-2}\binom{j(n+1)+k-3}{n-2}$$

$$= \sum_{k=3}^{n}(-1)^k\binom{n}{k}\sum_{j\geq 0}\binom{j(n+1)+k-3}{n-2}[[0 \leq j \leq k-2]]$$

$$= \sum_{k=3}^{n}(-1)^k\binom{n}{k}\sum_{j\geq 0}\binom{j(n+1)+k-3}{n-2}\operatorname*{res}_{z}\frac{1}{z^{k-1}}\frac{z^j}{1-z}$$

$$= \operatorname*{res}_{z}\frac{z}{1-z}\sum_{k=3}^{n}(-1)^k\binom{n}{k}\frac{1}{z^k}\sum_{j\geq 0}\binom{j(n+1)+k-3}{n-2}z^j$$

$$= \operatorname*{res}_{z}\frac{z}{1-z}\sum_{k=3}^{n}(-1)^k\binom{n}{k}\frac{1}{z^k}\sum_{j\geq 0}\operatorname*{res}_{w}\frac{1}{w^{n-1}}(1+w)^{j(n+1)+k-3}z^j$$

$$= \operatorname*{res}_{z}\frac{z}{1-z}\operatorname*{res}_{w}\frac{1}{w^{n-1}}\sum_{k=3}^{n}(-1)^k\binom{n}{k}\frac{1}{z^k}(1+w)^{k-3}\sum_{j\geq 0}(1+w)^{j(n+1)}z^j$$

$$= \operatorname*{res}_{z}\frac{z}{1-z}\operatorname*{res}_{w}\frac{1}{w^{n-1}}\frac{1}{1-z(1+w)^{n+1}}\sum_{k=3}^{n}(-1)^k\binom{n}{k}\frac{1}{z^k}(1+w)^{k-3}$$

$$= \operatorname*{res}_{z}\frac{z}{1-z}\operatorname*{res}_{w}\frac{1}{w^{n-1}}\frac{1}{(1+w)^3}\frac{1}{1-z(1+w)^{n+1}}$$
$$\times\sum_{k=3}^{n}(-1)^k\binom{n}{k}\frac{1}{z^k}(1+w)^k.$$

We compute this by lowering the index to $k=0$ and subtracting the values for $k=0,1$ and $k=2$ from this completed sum. **First** (piece A), extending to $k=0$ we find

$$\operatorname*{res}_{z}\frac{z}{1-z}\operatorname*{res}_{w}\frac{1}{w^{n-1}}\frac{1}{(1+w)^3}\frac{1}{1-z(1+w)^{n+1}}\left(1-\frac{1+w}{z}\right)^n$$

$$= \operatorname*{res}_z \frac{1}{z^n}\frac{z}{1-z} \operatorname*{res}_w \frac{1}{w^{n-1}}\frac{1}{(1+w)^3}\frac{1}{1-z(1+w)^{n+1}}(z-1-w)^n.$$

We introduce $z/(1+w-z) = v$ so that $z = v(1+w)/(1+v)$ and $dz = (1+w)/(1+v)^2\, dv$ as well as $z/(1-z) = v(1+w)/(1-vw)$ to get

$$\operatorname*{res}_v \frac{(-1)^n}{v^n} \operatorname*{res}_w \frac{1}{w^{n-1}}\frac{1}{(1+w)^3}\frac{v(1+w)}{1-vw}\frac{1}{1-v(1+w)^{n+2}/(1+v)}\frac{1+w}{(1+v)^2}$$

$$= \operatorname*{res}_v \frac{(-1)^n}{v^{n-1}}\frac{1}{1+v} \operatorname*{res}_w \frac{1}{w^{n-1}}\frac{1}{1+w}\frac{1}{1-vw}\frac{1}{1-v((1+w)^{n+2}-1)}.$$

Observe that

$$\frac{1}{1+v}\frac{1}{1-vw} = \frac{1}{1+w}\frac{1}{1+v} + \frac{w}{1+w}\frac{1}{1-vw}.$$

We thus have piece A_1 :

$$\operatorname*{res}_v \frac{(-1)^n}{v^{n-1}}\frac{1}{1+v} \operatorname*{res}_w \frac{1}{w^{n-1}}\frac{1}{(1+w)^2}\frac{1}{1-v((1+w)^{n+2}-1)}$$

$$= \operatorname*{res}_w \frac{(-1)^n}{w^{n-1}}\frac{1}{(1+w)^2}\sum_{q=0}^{n-2}(-1)^{n-2-q}((1+w)^{n+2}-1)^q$$

$$= \operatorname*{res}_w \frac{1}{w^{n-1}}\frac{1}{(1+w)^2}\sum_{q=0}^{n-2}(1-(1+w)^{n+2})^q$$

$$= \operatorname*{res}_w \frac{1}{w^{n-1}}\frac{1}{(1+w)^2}\frac{1-(1-(1+w)^{n+2})^{n-1}}{(1+w)^{n+2}}$$

$$= [w^{n-2}]\frac{1-(-(n+2)w-\cdots-w^{n+2})^{n-1}}{(1+w)^{n+4}} = (-1)^{n-2}\binom{n-2+n+3}{n-2}$$

$$= (-1)^n\binom{2n+1}{n-2}.$$

We have one correct piece. Continuing with A_2 (which we conjecture to be zero) we find

$$\operatorname*{res}_v \frac{(-1)^n}{v^{n-1}} \operatorname*{res}_w \frac{1}{w^{n-2}}\frac{1}{(1+w)^2}\frac{1}{1-vw}\frac{1}{1-v((1+w)^{n+2}-1)}$$

$$= \operatorname*{res}_w \frac{(-1)^n}{w^{n-2}}\frac{1}{(1+w)^2}\sum_{q=0}^{n-2} w^{n-2-q}((1+w)^{n+2}-1)^q$$

$$= \operatorname*{res}_w \frac{(-1)^n}{w^{n-2}}\frac{1}{(1+w)^2}\sum_{q=0}^{n-2} w^{n-2-q}((n+2)w+\cdots+w^{n+2})^q$$

$$= \operatorname{res}_w \frac{(-1)^n}{w^{n-2}} \frac{1}{(1+w)^2} \sum_{q=0}^{n-2} ((n+2)^q w^{n-2} + \cdots + w^{(n+1)q+n-2})$$

$$= \operatorname{res}_w \frac{(-1)^n}{(1+w)^2} \sum_{q=0}^{n-2} ((n+2)^q + \cdots + w^{(n+1)q}) = 0.$$

Continuing with the **second** piece B which corresponds to $k = 0$

$$\operatorname{res}_z \frac{z}{1-z} \operatorname{res}_w \frac{1}{w^{n-1}} \frac{1}{(1+w)^3} \frac{1}{1-z(1+w)^{n+1}}.$$

This is zero by inspection because there is no pole at $z = 0$. More formally,

$$\operatorname{res}_w \frac{1}{w^{n-1}} \frac{1}{(1+w)^3}$$

$$\times \operatorname{res}_z z(1 + z + z^2 + \cdots)(1 + z(1+w)^{n+1} + z^2(1+w)^{2n+2} + \cdots) = 0.$$

For the **third** piece C which corresponds to $k = 1$ we get a factor of $-n(1+w)/z$ for

$$-n \operatorname{res}_w \frac{1}{w^{n-1}} \frac{1}{(1+w)^2}$$

$$\times \operatorname{res}_z (1 + z + z^2 + \cdots)(1 + z(1+w)^{n+1} + z^2(1+w)^{2n+2} + \cdots) = 0.$$

The factor for the **fourth** piece D is $\binom{n}{2}(1+w)^2/z^2$:

$$\binom{n}{2} \operatorname{res}_w \frac{1}{w^{n-1}} \frac{1}{1+w}$$

$$\times \operatorname{res}_z \frac{1}{z}(1 + z + z^2 + \cdots)(1 + z(1+w)^{n+1} + z^2(1+w)^{2n+2} + \cdots)$$

$$= \binom{n}{2} \operatorname{res}_w \frac{1}{w^{n-1}} \frac{1}{1+w} = (-1)^n \binom{n}{2}.$$

Subtracting B, C and D from A we finally obtain

$$(-1)^n \left[\binom{2n+1}{n-2} - \binom{n}{2} \right].$$

This was math.stackexchange.com problem 3342361.

1.17 MSE 3383557

We seek to show that

$$n \sum_{k=0}^{n} \frac{(-1)^k}{2n-k} \binom{2n-k}{k} x^k y^{2n-2k} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{2k} y^{2k} (y^2 - 4x)^{n-k}.$$

We compare the coefficient on $[x^q]$ of the LHS and the RHS where $0 \le q \le n$ and show that they are equal. We must therefore show that

$$n\frac{(-1)^q}{2n-q}\binom{2n-q}{q}y^{2n-2q} = [x^q]\frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n}{2k}y^{2k}(y^2-4x)^{n-k}.$$

The RHS is

$$[x^q]\frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n}{2n-2k}y^{2n-2k}(y^2-4x)^k$$

$$= \frac{1}{2^{2n}}\sum_{k=q}^{n}\binom{2n}{2n-2k}y^{2n-2k}[x^q](y^2-4x)^k$$

$$= \frac{1}{2^{2n}}\sum_{k=q}^{n}\binom{2n}{2k}y^{2n-2k}\binom{k}{q}(-4)^q y^{2k-2q}$$

$$= y^{2n-2q}\frac{1}{2^{2n}}\sum_{k=q}^{n}\binom{2n}{2k}\binom{k}{q}(-4)^q.$$

We have reduced the claim to

$$n\frac{(-1)^q}{2n-q}\binom{2n-q}{q} = \frac{1}{2^{2n}}\sum_{k=q}^{n}\binom{2n}{2k}\binom{k}{q}(-4)^q.$$

The RHS is

$$\frac{1}{2^{2n}}\sum_{k=q}^{n}\binom{k}{q}(-4)^q[z^{2n-2k}](1+z)^{2n}$$

$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n}](1+z)^{2n}\sum_{k=q}^{n}\binom{k}{q}z^{2k}.$$

Now when k exceeds n we get zero from the coefficient extractor, which enforces the range:

$$\frac{(-1)^q}{2^{2n-2q}}[z^{2n}](1+z)^{2n}\sum_{k\ge q}\binom{k}{q}z^{2k}$$

$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n}]z^{2q}(1+z)^{2n}\sum_{k\ge 0}\binom{k+q}{q}z^{2k}$$

$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n}]z^{2q}(1+z)^{2n}\frac{1}{(1-z^2)^{q+1}}$$

$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n-2q}](1+z)^{2n-q-1}\frac{1}{(1-z)^{q+1}}$$

$$= \frac{(-1)^q}{2^{2n-2q}} \sum_{p=0}^{2n-q-1} \binom{2n-q-1}{p} \binom{2n-2q-p+q}{q}$$

$$= \frac{(-1)^q}{2^{2n-2q}} \sum_{p=0}^{2n-q-1} \binom{2n-q-1}{2n-q-1-p} \binom{2n-q-p}{q}.$$

Then we have

$$\binom{2n-q-1}{2n-q-1-p} \binom{2n-q-p}{q} = \frac{(2n-q-1)!(2n-q-p)}{p! \times q! \times (2n-2q-p)!}$$

$$= \frac{1}{2n-q} \frac{(2n-q)!(2n-q-p)}{p! \times q! \times (2n-2q-p)!}$$

$$= \frac{1}{2n-q} \binom{2n-q}{q} \binom{2n-2q}{p} (2n-q-p).$$

Substituting we find (here we have included the value for $p = 2n - q$, which is zero):

$$\frac{(-1)^q}{2^{2n-2q}} \frac{1}{2n-q} \binom{2n-q}{q} \sum_{p=0}^{2n-q} \binom{2n-2q}{p} (2n-q-p).$$

Working with the remaining sum we note that $(2n-2q)^{\underline{p}} = 0$ when $p > 2n-2q$ and $2n-q \ge 2n-2q$ so we may continue with

$$\sum_{p=0}^{2n-2q} \binom{2n-2q}{p} (2n-q-p) = (2n-q)2^{2n-2q} - \sum_{p=1}^{2n-2q} \binom{2n-2q}{p} p$$

$$= (2n-q)2^{2n-2q} - (2n-2q) \sum_{p=1}^{2n-2q} \binom{2n-2q-1}{p-1}$$

$$= (2n-q)2^{2n-2q} - (2n-2q)2^{2n-2q-1} = (2n-q)2^{2n-2q} - (n-q)2^{2n-2q}$$
$$= n2^{2n-2q}.$$

Substituting we at last obtain

$$n \frac{(-1)^q}{2n-q} \binom{2n-q}{q}$$

which was to be shown.

This was math.stackexchange.com problem 3383557.

1.18 MSE 3441855

We seek to evaluate the LHS of the first equation below and start as follows:

$$\sum_{k=0}^{n}(-1)^k 4^{n-k}\binom{2n-k}{k} = \sum_{k=0}^{n}(-1)^k 4^{n-k}\binom{2n-k}{2n-2k}$$
$$= \sum_{k=0}^{n}(-1)^k 4^{n-k}[z^{2n-2k}](1+z)^{2n-k}$$
$$= [z^{2n}](1+z)^{2n}\sum_{k=0}^{n}(-1)^k 4^{n-k} z^{2k}(1+z)^{-k}.$$

Now when $k > n$ we get zero contribution due to the coefficient extractor $[z^{2n}]$ and the factor z^{2k}, so this enforces the range of the sum and we may continue with

$$[z^{2n}](1+z)^{2n}\sum_{k\ge 0}(-1)^k 4^{n-k} z^{2k}(1+z)^{-k}$$
$$= 4^n[z^{2n}](1+z)^{2n}\frac{1}{1+z^2/(1+z)/4}$$
$$= 4^{n+1}[z^{2n}](1+z)^{2n+1}\frac{1}{4+4z+z^2} = 4^{n+1}[z^{2n}](1+z)^{2n+1}\frac{1}{(z+2)^2}.$$

This is

$$4^{n+1}\operatorname*{res}_{z}\frac{1}{z^{2n+1}}(1+z)^{2n+1}\frac{1}{(z+2)^2}.$$

We introduce $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2\, dw$, to obtain

$$4^{n+1}\operatorname*{res}_{w}\frac{1}{w^{2n+1}}\frac{1}{(w/(1-w)+2)^2}\frac{1}{(1-w)^2}$$
$$= 4^{n+1}\operatorname*{res}_{w}\frac{1}{w^{2n+1}}\frac{1}{(2-w)^2}$$
$$= 4^{n+1}[w^{2n}]\frac{1}{(2-w)^2} = 4^n[w^{2n}]\frac{1}{(1-w/2)^2} = 4^n(2n+1)\frac{1}{2^{2n}}$$
$$= 2n+1.$$

Remark. This can also be done using the fact that residues sum to zero, which starting from the residue in z we see that the residue at infinity is zero, so our sum is

$$-4^{n+1}\mathrm{Res}_{z=-2}\frac{1}{z^{2n+1}}(1+z)^{2n+1}\frac{1}{(z+2)^2}$$
$$= -4^{n+1}\left.\left(\frac{1}{z^{2n+1}}(1+z)^{2n+1}\right)'\right|_{z=-2}$$

$$= -4^{n+1}\left(-\frac{2n+1}{z^{2n+2}}(1+z)^{2n+1} + \frac{(2n+1)}{z^{2n+1}}(1+z)^{2n}\right)\Bigg|_{z=-2}$$

$$= (2n+1)\times 4^{n+1}\left(\frac{(-1)^{2n+1}}{(-2)^{2n+2}} - \frac{(-1)^{2n}}{(-2)^{2n+1}}\right)$$

$$= (2n+1)\times 2^{2n+2}\left(-\frac{1}{2^{2n+2}} + \frac{1}{2^{2n+1}}\right) = 2n+1.$$

This was math.stackexchange.com problem 3441855.

1.19 MSE 3577193

We seek to show that

$$\sum_{k=0}^{l}\binom{k}{m}\binom{k}{n} = \sum_{k=0}^{n}(-1)^k\binom{l+1}{m+k+1}\binom{l-k}{n-k}.$$

The RHS is

$$[z^n]\sum_{k=0}^{n}(-1)^k\binom{l+1}{m+k+1}z^k(1+z)^{l-k}.$$

The coefficient extractor enforces the range:

$$[z^n]\sum_{k\ge 0}(-1)^k\binom{l+1}{l-m-k}z^k(1+z)^{l-k}$$

$$= [z^n](1+z)^l[w^{l-m}](1+w)^{l+1}\sum_{k\ge 0}(-1)^k w^k z^k(1+z)^{-k}$$

$$= [z^n](1+z)^l[w^{l-m}](1+w)^{l+1}\frac{1}{1+wz/(1+z)}$$

$$= [z^n](1+z)^{l+1}[w^{l-m}](1+w)^{l+1}\frac{1}{1+z+wz}$$

$$= [z^n](1+z)^{l+1}[w^{l-m}](1+w)^{l+1}\frac{1}{1+z(1+w)}$$

$$= [z^n](1+z)^{l+1}[w^{l-m}]\sum_{k\ge 0}(-1)^k z^k(1+w)^{k+l+1}$$

$$= [z^n](1+z)^{l+1}\sum_{k\ge 0}(-1)^k z^k\binom{k+l+1}{l-m}.$$

This is

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{l+1}{n-k}\binom{k+l+1}{l-m}}.$$

The LHS is

$$\sum_{k\ge 0}[[0 \le k \le l]][z^m](1+z)^k[w^n](1+w)^k$$

$$= [z^m][w^n]\sum_{k\ge 0}(1+z)^k(1+w)^k[v^l]\frac{v^k}{1-v}$$

$$= [z^m][w^n][v^l]\frac{1}{1-v}\sum_{k\ge 0}(1+z)^k(1+w)^k v^k$$

$$= [z^m][w^n][v^l]\frac{1}{1-v}\frac{1}{1-(1+z)(1+w)v}$$

$$= [z^m][w^n][v^l]\frac{1}{v-1}\frac{1/(1+z)/(1+w)}{v-1/(1+z)/(1+w)}.$$

The inner term is

$$\mathrm{Res}_{v=0}\frac{1}{v^{l+1}}\frac{1}{v-1}\frac{1/(1+z)/(1+w)}{v-1/(1+z)/(1+w)}.$$

Residues sum to zero and the residue at infinity in v is zero. The contribution from minus the residue at $v = 1/(1+z)/(1+w)$ is

$$-[z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/(1+z)/(1+w)}{1/(1+z)/(1+w)-1}$$

$$= -[z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/(1+z)}{1/(1+z)-(1+w)}$$

$$= [z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/(1+z)}{w+z/(1+z)}$$

$$= [z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/z}{w(1+z)/z+1}.$$

Now with l, m, n positive integers we must have $l \ge n, m$ or else there is no contribution to $k^{\underline{m}}k^{\underline{n}}$. This means we continue with

$$[z^m](1+z)^{l+1}\sum_{k=0}^{n}\binom{l+1}{k}\frac{1}{z}(-1)^{n-k}\frac{(1+z)^{n-k}}{z^{n-k}}$$

$$= \sum_{k=0}^{n}(-1)^{n-k}\binom{l+1}{k}\binom{l+1+n-k}{m+1+n-k}.$$

This is

$$\boxed{\sum_{k=0}^{n}(-1)^{n-k}\binom{l+1}{k}\binom{l+1+n-k}{l-m}.}$$

We have the same closed form for LHS and RHS, thus proving the claim.

For a full proof we also need to show that the contribution from $v = 1$ is zero. We get

$$[z^m][w^n]\frac{1/(1+z)/(1+w)}{1-1/(1+z)/(1+w)} = [z^m][w^n]\frac{1}{(1+z)(1+w)-1}$$
$$= [z^m][w^n]\frac{1}{z+w+zw} = [z^{m+1}][w^n]\frac{1}{1+w(1+z)/z}$$
$$= [z^{m+1}](-1)^n\frac{(1+z)^n}{z^n} = (-1)^n\binom{n}{n+m+1} = 0.$$

This was math.stackexchange.com problem 3577193.

1.20 MSE 3583191

Goal here is

$$\sum_{j=0}^{k}\binom{2n}{2j}\binom{n-j}{k-j} = \frac{4^k n}{n+k}\binom{n+k}{n-k}.$$

Start as follows:

$$\sum_{j=0}^{k}\binom{2n}{2j}\binom{n-j}{k-j} = \sum_{j=0}^{k}\binom{2n}{2k-2j}\binom{n-k+j}{j}$$
$$= [z^{2k}](1+z)^{2n}\sum_{j=0}^{k} z^{2j}\binom{n-k+j}{j}.$$

Here the coefficient extractor enforces the range:

$$[z^{2k}](1+z)^{2n}\sum_{j\ge 0} z^{2j}\binom{n-k+j}{j}$$
$$= [z^{2k}](1+z)^{2n}\frac{1}{(1-z^2)^{n-k+1}} = [z^{2k}](1+z)^{n+k-1}\frac{1}{(1-z)^{n-k+1}}.$$

This is

$$\mathrm{Res}_{z=0}\frac{1}{z^{2k+1}}(1+z)^{n+k-1}\frac{1}{(1-z)^{n-k+1}}$$
$$= (-1)^{n-k+1}\mathrm{Res}_{z=0}\frac{1}{z^{2k+1}}(1+z)^{n+k-1}\frac{1}{(z-1)^{n-k+1}}.$$

Now the residue at infinity is zero so this is minus the residue at one:

$$(-1)^{n-k}\mathrm{Res}_{z=1}\frac{1}{(1+(z-1))^{2k+1}}(2+(z-1))^{n+k-1}\frac{1}{(z-1)^{n-k+1}}$$

$$= (-1)^{n-k}\sum_{j=0}^{n-k}\binom{n+k-1}{j}2^{n+k-1-j}(-1)^{n-k-j}\binom{n-k-j+2k}{2k}$$

$$= 2^{n+k-1}\sum_{j=0}^{n-k}\binom{n+k-1}{j}2^{-j}(-1)^j\binom{n+k-j}{n-k-j}.$$

Coefficient extractor enforces range:

$$2^{n+k-1}[z^{n-k}](1+z)^{n+k}\sum_{j\ge 0}\binom{n+k-1}{j}2^{-j}(-1)^j\frac{z^j}{(1+z)^j}$$

$$= 2^{n+k-1}[z^{n-k}](1+z)^{n+k}\left(1-\frac{z}{2(1+z)}\right)^{n+k-1}$$

$$= [z^{n-k}](1+z)(2+z)^{n+k-1}$$

$$= [z^{n-k}](2+z)^{n+k-1} + [z^{n-k-1}](2+z)^{n+k-1}$$

$$= \binom{n+k-1}{n-k}2^{n+k-1-(n-k)} + \binom{n+k-1}{n-k-1}2^{n+k-1-(n-k-1)}$$

$$= \frac{1}{2}4^k\frac{2k}{n+k}\binom{n+k}{n-k} + \frac{n-k}{n+k}4^k\binom{n+k}{n-k}$$

$$= \frac{4^k n}{n+k}\binom{n+k}{n-k}.$$

This was math.stackexchange.com problem 3583191.

1.21 MSE 3592240

We seek to verify that

$$\sum_{q=m}^{n-k}(-1)^{q-m}\binom{k-1+q}{k-1}\left\{{q \atop m}\right\}\left[{n \atop q+k}\right] = \binom{n-1}{m}\left[{n-m \atop k}\right].$$

Using the standard EGFs the LHS becomes

$$\sum_{q=m}^{n-k}(-1)^{q-m}\binom{k-1+q}{k-1}q![z^q]\frac{(\exp(z)-1)^m}{m!}n![w^n]\frac{1}{(q+k)!}\left(\log\frac{1}{1-w}\right)^{q+k}$$

$$= \frac{n!}{(k-1)! \times m!} [w^n] \sum_{q=m}^{n-k} (-1)^{q-m} [z^q] (\exp(z) - 1)^m \frac{1}{q+k} \left(\log \frac{1}{1-w} \right)^{q+k}$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \sum_{q=m}^{n-k} (-1)^{q-m} [z^q] (\exp(z) - 1)^m \left(\log \frac{1}{1-w} \right)^{q+k-1} \frac{1}{1-w}$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times \sum_{q=m}^{n-k} (-1)^{q-m} [z^{q+k-1}] z^{k-1} (\exp(z) - 1)^m \left(\log \frac{1}{1-w} \right)^{q+k-1}$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times \sum_{q=m+k-1}^{n-1} (-1)^{q-(k-1)-m} [z^q] z^{k-1} (\exp(z) - 1)^m \left(\log \frac{1}{1-w} \right)^{q}.$$

Now as $\log \frac{1}{1-w} = w + \cdots$ when $q > n-1$ there is no contribution from the logarithmic power term due to the coefficient extractor $[w^{n-1}]$ so we find

$$(-1)^{m+(k-1)} \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times \sum_{q \ge m+k-1} (-1)^q \left(\log \frac{1}{1-w} \right)^q [z^q] z^{k-1} (\exp(z) - 1)^m.$$

Note that $z^{k-1}(\exp(z)-1)^m = z^{m+k-1} + \cdots$ which means that the remaining sum / coefficient etractor pair covers the entire series and we get

$$(-1)^{m+(k-1)} \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times (-1)^{k-1} \left(\log \frac{1}{1-w} \right)^{k-1} \left(\exp \left(-\log \frac{1}{1-w} \right) - 1 \right)^m$$

$$= (-1)^{m+(k-1)} \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times (-1)^{k-1} \left(\log \frac{1}{1-w} \right)^{k-1} (-w)^m$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1-m}] \frac{1}{1-w} \left(\log \frac{1}{1-w} \right)^{k-1}$$

$$= \frac{(n-1)!}{m!} [w^{n-1-m}] \frac{1}{1-w} \frac{1}{(k-1)!} \left(\log \frac{1}{1-w} \right)^{k-1}$$

$$= \frac{(n-1)!}{m!}(n-m)[w^{n-m}]\frac{1}{k!}\left(\log\frac{1}{1-w}\right)^k$$

$$= \frac{(n-1)!}{m! \times (n-1-m)!}(n-m)![w^{n-m}]\frac{1}{k!}\left(\log\frac{1}{1-w}\right)^k$$

$$= \binom{n-1}{m}\left[{n-m \atop k}\right].$$

This is the claim.

This was math.stackexchange.com problem 3592240.

1.22 MSE 3604802

We seek to evaluate

$$S(N) = \sum_{q=0}^{N} (-1)^q \binom{2q}{q}\binom{N+q}{N-q}\frac{q^2}{(q+1)^2}$$

or alternatively

$$S(N) = \sum_{q=0}^{N} (-1)^q \frac{(N+q)!}{(N-q)!(q-1)!^2}\frac{1}{(q+1)^2}.$$

This is

$$S(N) = \sum_{q=0}^{N} q^2(-1)^q \frac{(N+q)!}{(N-q)!(q+1)!^2}$$

$$= \sum_{q=0}^{N} q^2(-1)^q \binom{N+1}{q+1}\frac{(N+q)!}{(N+1)!(q+1)!}$$

$$= \frac{1}{N(N+1)}\sum_{q=0}^{N} q^2(-1)^q \binom{N+1}{q+1}\frac{(N+q)!}{(N-1)!(q+1)!}$$

$$= \frac{1}{N(N+1)}\sum_{q=0}^{N} q^2(-1)^q \binom{N+1}{q+1}\binom{N+q}{q+1}.$$

We continue with

$$\frac{1}{N(N+1)}\sum_{q=0}^{N} q^2(-1)^q \binom{N+1}{N-q}\binom{N+q}{q+1}$$

$$= \frac{1}{N(N+1)}[z^N](1+z)^{N+1}\sum_{q=0}^{N} q^2(-1)^q z^q \binom{N+q}{q+1}.$$

Here the coefficient extractor enforces the upper limit of the sum:

$$\begin{aligned}
&\frac{1}{N(N+1)}[z^N](1+z)^{N+1}\sum_{q\ge 0} q^2(-1)^q z^q \binom{N+q}{N-1} \\
&= \frac{1}{N(N+1)}[z^N](1+z)^{N+1}[w^{N-1}](1+w)^N \sum_{q\ge 0} q^2(-1)^q z^q (1+w)^q \\
&= \frac{1}{N(N+1)}[z^N](1+z)^{N+1}[w^{N-1}](1+w)^N \frac{-z(1+w)(1-z(1+w))}{(1+z(1+w))^3} \\
&= -\frac{1}{N(N+1)}[z^{N-1}](1+z)^{N+1}[w^{N-1}](1+w)^{N+1} \frac{1-z(1+w)}{(1+z(1+w))^3}.
\end{aligned}$$

We have two pieces here, the first one is

$$\begin{aligned}
&-\frac{1}{N(N+1)}[z^{N-1}](1+z)^{N+1}[w^{N-1}](1+w)^{N+1} \frac{1}{(1+z(1+w))^3} \\
&= -\frac{1}{N(N+1)}[z^{N-1}](1+z)^{N-2}[w^{N-1}](1+w)^{N+1} \frac{1}{(1+zw/(1+z))^3}.
\end{aligned}$$

The inner term is

$$\sum_{q=0}^{N-1} \binom{N+1}{N-1-q}(-1)^q \binom{q+2}{2} \frac{z^q}{(1+z)^q}.$$

Now

$$\binom{N+1}{N-1-q}\binom{q+2}{2} = \frac{(N+1)!}{(N-1-q)! \times q! \times 2!} = \binom{N+1}{2}\binom{N-1}{q}$$

and we find for the inner term

$$\begin{aligned}
\binom{N+1}{2}\sum_{q=0}^{N-1}\binom{N-1}{q}(-1)^q \frac{z^q}{(1+z)^q} &= \binom{N+1}{2}\left(1-\frac{z}{1+z}\right)^{N-1} \\
&= \binom{N+1}{2}\frac{1}{(1+z)^{N-1}}.
\end{aligned}$$

Substitute into the outer term to get

$$\begin{aligned}
&-\frac{1}{N(N+1)}[z^{N-1}](1+z)^{N-2}\binom{N+1}{2}\frac{1}{(1+z)^{N-1}} \\
&= -\frac{1}{2}[z^{N-1}]\frac{1}{1+z} = \frac{1}{2}(-1)^N.
\end{aligned}$$

The second piece is

$$\frac{1}{N(N+1)}[z^{N-2}](1+z)^{N-2}[w^{N-1}](1+w)^{N+2}\frac{1}{(1+zw/(1+z))^3}.$$

For this piece we obtain

$$\frac{1}{N(N+1)}[z^{N-2}](1+z)^{N-2}\sum_{q=0}^{N-1}\binom{N+2}{N-1-q}(-1)^q\binom{q+2}{2}\frac{z^q}{(1+z)^q}.$$

The remaining coefficient extractor cancels the term for $q = N-1$:

$$\frac{1}{N(N+1)}[z^{N-2}](1+z)^{N-2}\sum_{q=0}^{N-2}\binom{N+2}{N-1-q}(-1)^q\binom{q+2}{2}\frac{z^q}{(1+z)^q}$$

$$= \frac{1}{N(N+1)}\sum_{q=0}^{N-2}\binom{N+2}{N-1-q}(-1)^q\binom{q+2}{2}$$

$$= -\frac{1}{N(N+1)}(-1)^{N-1}\binom{N+1}{2} + \frac{1}{N(N+1)}\sum_{q=0}^{N-1}\binom{N+2}{N-1-q}(-1)^q\binom{q+2}{2}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)}\sum_{q=0}^{N-1}\binom{N+2}{N-1-q}(-1)^q\binom{q+2}{2}.$$

Continuing, with the coefficient extractor enforcing the range,

$$\frac{1}{2}(-1)^N + \frac{1}{N(N+1)}[z^{N-1}](1+z)^{N+2}\sum_{q\ge 0} z^q(-1)^q\binom{q+2}{2}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)}[z^{N-1}](1+z)^{N+2}\frac{1}{(1+z)^3}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)}[z^{N-1}](1+z)^{N-1}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)}.$$

Collecting the contributions from the two pieces we obtain at last

$$\boxed{(-1)^N + \frac{1}{N(N+1)}.}$$

This was math.stackexchange.com problem 3604802.

1.23 MSE 3619182

We seek to verify that

$$\sum_{k=0}^n {n\choose k}^2 \sum_{l=0}^k {k\choose l}{n\choose l}{2n-l\choose n} = \sum_{k=0}^n {n\choose k}^2 {n+k\choose k}^2.$$

Starting with the inner term on the LHS we have

$$\sum_{l=0}^k {k\choose k-l}{n\choose l}{2n-l\choose n}$$
$$= [z^k](1+z)^k \sum_{l=0}^k z^l {n\choose l}{2n-l\choose n}$$
$$= [z^k](1+z)^k [w^n](1+w)^{2n} \sum_{l=0}^k z^l {n\choose l}(1+w)^{-l}.$$

The coefficient extractor $[z^k]$ enforces the upper limit of the sum and we find

$$[z^k](1+z)^k [w^n](1+w)^{2n} \sum_{l\ge 0} z^l {n\choose l}(1+w)^{-l}$$
$$= [z^k](1+z)^k [w^n](1+w)^{2n} \left(1+\frac{z}{1+w}\right)^n$$
$$= [z^k](1+z)^k [w^n](1+w)^n (1+w+z)^n.$$

We get from the outer sum

$$\sum_{k=0}^n {n\choose k}^2 [z^k](1+z)^k [w^n](1+w)^n(1+w+z)^n$$
$$= \sum_{k=0}^n {n\choose k}^2 [z^n] z^k (1+z)^{n-k} [w^n](1+w)^n(1+w+z)^n$$
$$= [z^n](1+z)^n [w^n](1+w)^n(1+w+z)^n \sum_{k=0}^n {n\choose k}^2 z^k (1+z)^{-k}$$
$$= [z^n](1+z)^n [w^n](1+w)^n(1+w+z)^n [v^n](1+v)^n \sum_{k=0}^n {n\choose k} v^k z^k (1+z)^{-k}$$
$$= [z^n](1+z)^n [w^n](1+w)^n(1+w+z)^n [v^n](1+v)^n \left(1+\frac{vz}{1+z}\right)^n$$
$$= [z^n][w^n](1+w)^n(1+w+z)^n [v^n](1+v)^n (1+z+vz)^n.$$

Extracting the coefficient on $[z^n]$ we obtain

$$\sum_{k=0}^{n}([z^{n-k}][w^n](1+w)^n(1+w+z)^n)([z^k][v^n](1+v)^n(1+z(1+v))^n)$$

$$= \sum_{k=0}^{n}\left(\binom{n}{n-k}[w^n](1+w)^{n+k}\right)\left(\binom{n}{k}[v^n](1+v)^{n+k}\right)$$

$$= \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}^2.$$

This is the claim.
This was math.stackexchange.com problem 3619182.

1.24 MSE 3638162

Suppose we seek to verify that

$$\sum_{k=1}^{a}(-1)^{a-k}\binom{a}{k}\binom{b+k}{b+1} = \binom{b}{a-1}.$$

We get

$$\sum_{k=1}^{a}[z^{a-k}](-1)^{a-k}\frac{1}{(1-z)^{k+1}}\binom{b+k}{b+1}$$

$$= [z^a]\frac{1}{1-z}\sum_{k=1}^{a}z^k(-1)^{a-k}\frac{1}{(1-z)^k}\binom{b+k}{b+1}.$$

Here the coefficient extractor enforces the upper limit of the sum and we find

$$[z^a]\frac{1}{1-z}\sum_{k\ge 1}z^k(-1)^{a-k}\frac{1}{(1-z)^k}\binom{b+k}{b+1}$$

$$= [z^a]\frac{1}{1-z}(-1)^{a-1}\frac{z}{1-z}\sum_{k\ge 0}z^k(-1)^k\frac{1}{(1-z)^k}\binom{b+1+k}{b+1}$$

$$= [z^{a-1}]\frac{(-1)^{a-1}}{(1-z)^2}\frac{1}{(1+z/(1-z))^{b+2}} = [z^{a-1}]\frac{(-1)^{a-1}}{(1-z)^2}(1-z)^{b+2}$$

$$= [z^{a-1}](-1)^{a-1}(1-z)^b = [z^{a-1}](1+z)^b = \binom{b}{a-1}.$$

This is the claim.
This was math.stackexchange.com problem 3638162.

1.25 MSE 3661349

We seek to show that

$$\sum_{q=0}^{k}(-1)^{q-j}\binom{n+q}{q}\binom{n+k-q}{k-q}\binom{2n}{n+j-q} = \binom{2n}{n}$$

where $0 \le j \le k$.

The LHS is

$$(-1)^j[w^{n+j}](1+w)^{2n}\sum_{q=0}^{k}(-1)^q\binom{n+q}{q}w^q\binom{n+k-q}{k-q}$$

$$= (-1)^j[w^{n+j}](1+w)^{2n}[z^k]\frac{1}{(1+wz)^{n+1}}\frac{1}{(1-z)^{n+1}}.$$

The inner term is

$$\mathrm{Res}_{z=0}\frac{1}{z^{k+1}}\frac{1}{(1+wz)^{n+1}}\frac{1}{(1-z)^{n+1}}.$$

Residues sum to zero and the residue at infinity is zero by inspection. We get for the residue at $z=1$

$$(-1)^{n+1}\mathrm{Res}_{z=1}\frac{1}{z^{k+1}}\frac{1}{(1+wz)^{n+1}}\frac{1}{(z-1)^{n+1}}$$

$$= (-1)^{n+1}\mathrm{Res}_{z=1}\frac{1}{(1+(z-1))^{k+1}}\frac{1}{(1+w+w(z-1))^{n+1}}\frac{1}{(z-1)^{n+1}}$$

$$= \frac{(-1)^{n+1}}{(1+w)^{n+1}}\mathrm{Res}_{z=1}\frac{1}{(1+(z-1))^{k+1}}\frac{1}{(1+w(z-1)/(1+w))^{n+1}}\frac{1}{(z-1)^{n+1}}$$

$$= \frac{(-1)^{n+1}}{(1+w)^{n+1}}\sum_{q=0}^{n}\binom{n+q}{q}(-1)^q\frac{w^q}{(1+w)^q}(-1)^{n-q}\binom{k+n-q}{k}$$

$$= -\sum_{q=0}^{n}\binom{n+q}{q}\frac{w^q}{(1+w)^{n+1+q}}\binom{k+n-q}{k}.$$

Substitute into the coefficient extractor in w to get

$$-(-1)^j\sum_{q=0}^{n}\binom{n+q}{q}\binom{k+n-q}{k}[w^{n+j-q}](1+w)^{n-1-q}.$$

Now with $0 \le q \le n-1$ and $j \ge 0$ we have $[w^{n+j-q}](1+w)^{n-1-q} = 0$. This leaves $q = n$ which yields

$$-(-1)^j\binom{2n}{n}\binom{k}{k}[w^j]\frac{1}{1+w} = -\binom{2n}{n}.$$

This is the claim. We have the result if we can show that the residue at $z = -1/w$ makes for a zero contribution. We get

$$\frac{1}{w^{n+1}} \mathrm{Res}_{z=-1/w} \frac{1}{z^{k+1}} \frac{1}{(z+1/w)^{n+1}} \frac{1}{(1-z)^{n+1}}.$$

This requires

$$\frac{1}{n!}\left(\frac{1}{z^{k+1}}\frac{1}{(1-z)^{n+1}}\right)^{(n)} = \frac{1}{n!}\sum_{q=0}^{n}\binom{n}{q}\frac{(-1)^q(k+q)!}{z^{k+1+q}\times k!}\frac{(n+n-q)!}{(1-z)^{n+1+n-q}\times n!}$$

$$= \sum_{q=0}^{n}\binom{k+q}{k}(-1)^q\frac{1}{z^{k+1+q}}\binom{2n-q}{n}\frac{1}{(1-z)^{2n+1-q}}.$$

Evaluate at $z = -1/w$ and restore the factor in front:

$$\frac{1}{w^{n+1}}\sum_{q=0}^{n}\binom{k+q}{k}(-1)^{k+1}w^{k+1+q}\binom{2n-q}{n}\frac{1}{(1+1/w)^{2n+1-q}}.$$

Applying the coefficient extractor in w we get

$$(-1)^j[w^{n+j}](1+w)^{2n}\frac{1}{w^{n+1}}w^{k+1+q}\frac{w^{2n+1-q}}{(1+w)^{2n+1-q}}$$
$$= (-1)^j[w^{n+j}](1+w)^{q-1}w^{n+k+1} = (-1)^j[w^j](1+w)^{q-1}w^{k+1} = 0$$

because $j \le k$. This concludes the argument.
This was math.stackexchange.com problem 3661349.

1.26 MSE 3706767

We seek to verify that

$$S_{n,m} = \sum_{k=m}^{n}\binom{k+m}{2m}\binom{2n+1}{n+k+1} = \binom{n}{m}4^{n-m}.$$

The LHS is

$$\sum_{k=0}^{n-m}\binom{k+2m}{2m}\binom{2n+1}{n+m+k+1}$$

$$= \sum_{k=0}^{n-m}\binom{k+2m}{2m}[z^{n-m-k}]\frac{1}{(1-z)^{n+m+k+2}}$$

$$= [z^{n-m}]\frac{1}{(1-z)^{n+m+2}}\sum_{k=0}^{n-m}\binom{k+2m}{2m}\frac{z^k}{(1-z)^k}.$$

Now when $k > n - m$ there is no contribution to the coefficient extractor and we may continue with

$$[z^{n-m}]\frac{1}{(1-z)^{n+m+2}}\sum_{k\ge 0}\binom{k+2m}{2m}\frac{z^k}{(1-z)^k}$$
$$= [z^{n-m}]\frac{1}{(1-z)^{n+m+2}}\frac{1}{(1-z/(1-z))^{2m+1}}$$
$$= [z^{n-m}]\frac{1}{(1-z)^{n-m+1}}\frac{1}{(1-2z)^{2m+1}}.$$

This yields

$$S_{n,m} = \mathrm{Res}_{z=0}\frac{1}{z^{n-m+1}}\frac{1}{(1-z)^{n-m+1}}\frac{1}{(1-2z)^{2m+1}}.$$

Residues sum to zero and the residue at infinity is zero by inspection. We get for the residue at $z = 1$

$$\mathrm{Res}_{z=1}\frac{1}{z^{n-m+1}}\frac{1}{(1-z)^{n-m+1}}\frac{1}{(1-2z)^{2m+1}}.$$

Setting $z = 1 - u$ we get

$$-\mathrm{Res}_{u=0}\frac{1}{(1-u)^{n-m+1}}\frac{1}{u^{n-m+1}}\frac{1}{(1-2(1-u))^{2m+1}}$$
$$= -\mathrm{Res}_{u=0}\frac{1}{(1-u)^{n-m+1}}\frac{1}{u^{n-m+1}}\frac{1}{(2u-1)^{2m+1}}$$
$$= \mathrm{Res}_{u=0}\frac{1}{(1-u)^{n-m+1}}\frac{1}{u^{n-m+1}}\frac{1}{(1-2u)^{2m+1}} = S_{n,m}.$$

Here the contour in z is given by the circle $|z - 1| = \varepsilon$ where $\varepsilon < 1/2$ so the image contour is $|-u| = \varepsilon$, now multiplication by -1 is a rotation by π radians so this is $|u| = \varepsilon$, at the end use $dz = -du$.

Continuing with the residue at $z = 1/2$ we find

$$-\frac{1}{2^{2m+1}}\mathrm{Res}_{z=1/2}\frac{1}{z^{n-m+1}}\frac{1}{(1-z)^{n-m+1}}\frac{1}{(z-1/2)^{2m+1}}$$
$$= -\frac{1}{2^{2m+1}}\mathrm{Res}_{z=1/2}\frac{1}{(1/2+(z-1/2))^{n-m+1}}\frac{1}{(1/2-(z-1/2))^{n-m+1}}$$
$$\times\frac{1}{(z-1/2)^{2m+1}}$$
$$= -\frac{1}{2^{2m+1}}\mathrm{Res}_{z=1/2}\frac{1}{(1/4-(z-1/2)^2)^{n-m+1}}\frac{1}{(z-1/2)^{2m+1}}$$
$$= -\frac{1}{2^{2m+1}}\mathrm{Res}_{z=1/2}\frac{4^{n-m+1}}{(1-4(z-1/2)^2)^{n-m+1}}\frac{1}{(z-1/2)^{2m+1}}$$

$$= -\frac{2^{2n-2m+2}}{2^{2m+1}}[(z-1/2)^{2m}]\frac{1}{(1-4(z-1/2)^2)^{n-m+1}}$$

$$= -\frac{2^{2n-2m+2}}{2^{2m+1}}[(z-1/2)^{m}]\frac{1}{(1-4(z-1/2))^{n-m+1}}$$

$$= -\frac{2^{2n-2m+2}}{2^{2m+1}}\binom{m+n-m}{n-m}2^{2m}.$$

We have shown that

$$S_{n,m} + S_{n,m} - 2^{2n-2m+1}\binom{n}{m} = 0$$

which is at last

$$\boxed{S_{n,m} = \binom{n}{m}4^{n-m}.}$$

This was math.stackexchange.com problem 3706767.

1.27 MSE 3737197

We seek to show that

$$\sum_{j=0}^{k}\binom{k}{j}\binom{j/2}{n}(-1)^{n-j} = \frac{k}{n}2^{k-2n}\binom{2n-k-1}{n-1}$$

where $n \geq k \geq 0$. We get for the even component

$$\sum_{p=0}^{\lfloor k/2 \rfloor}\binom{k}{2p}\binom{p}{n}(-1)^n = 0$$

because $n > p$ and $p \geq 0$. This leaves the odd component

$$-(-1)^n\sum_{p=0}^{\lfloor (k-1)/2 \rfloor}\binom{k}{2p+1}\binom{p+1/2}{n}.$$

Now we have

$$\binom{p+1/2}{n} = \frac{1}{n!}\prod_{q=0}^{n-1}(p+1/2-q) = \frac{1}{2^n n!}\prod_{q=0}^{n-1}(2p+1-2q)$$

$$= \frac{1}{2^n n!}\prod_{q=0}^{p}(2p+1-2q)\prod_{q=p+1}^{n-1}(2p+1-2q)$$

$$\begin{aligned}
&= \frac{1}{2^n n!} \frac{(2p+2)!}{2^{p+1}(p+1)!} (-1)^{n-p-1} \prod_{q=p+1}^{n-1} (2q-2p-1)\\
&= \frac{1}{2^n n!} \frac{(2p+2)!}{2^{p+1}(p+1)!} (-1)^{n-p-1} \frac{(2n-2p-2)!}{2^{n-p-1}(n-p-1)!}\\
&= \frac{(-1)^{n-p-1}(2n)!}{2^{2n} n!^2} {2n \choose 2p+2}^{-1} {n \choose p+1}\\
&= \frac{(-1)^{n-p-1}}{2^{2n}} {2n \choose n} {2n \choose 2p+2}^{-1} {n \choose p+1}.
\end{aligned}$$

where $p < n$. It will be helpful to re-write this as

$$\begin{aligned}
&\frac{p+1}{n} \frac{(-1)^{n-p-1}}{2^{2n}} {2n \choose n} {2n-1 \choose 2p+1}^{-1} {n \choose p+1}\\
&= \frac{(-1)^{n-p-1}}{2^{2n}} {2n \choose n} {2n-1 \choose 2p+1}^{-1} {n-1 \choose p}.
\end{aligned}$$

We thus get for our sum

$$\frac{1}{2^{2n}} {2n \choose n} \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} (-1)^p {k \choose 2p+1} {2n-1 \choose 2p+1}^{-1} {n-1 \choose p}.$$

Now observe that

$$\begin{aligned}
{k \choose 2p+1} {2n-1 \choose 2p+1}^{-1} &= \frac{k!}{(k-2p-1)!} \frac{(2n-2p-2)!}{(2n-1)!}\\
&= {2n-1 \choose k}^{-1} {2n-2p-2 \choose k-2p-1}.
\end{aligned}$$

This yields for the sum

$$\frac{1}{2^{2n}} {2n \choose n} {2n-1 \choose k}^{-1} \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} (-1)^p {2n-2p-2 \choose k-2p-1} {n-1 \choose p}.$$

Now to treat the remaining sum we have

$$[z^k](1+z)^{2n-2} \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} (-1)^p z^{2p+1} (1+z)^{-2p} {n-1 \choose p}.$$

The coefficient extractor enforces the upper limit $\lfloor (k-1)/2 \rfloor \geq p$ so we may continue with

$$[z^k](1+z)^{2n-2} \sum_{p \geq 0} (-1)^p z^{2p+1} (1+z)^{-2p} {n-1 \choose p}$$

$$= [z^k](1+z)^{2n-2} z \left(1 - \frac{z^2}{(1+z)^2}\right)^{n-1}$$
$$= [z^k] z(1+2z)^{n-1}.$$

This means for $k = 0$ the sum is zero. For $k \ge 1$ we get including the factor in front

$$\boxed{\frac{1}{2^{2n}} \binom{2n}{n} \binom{2n-1}{k}^{-1} \binom{n-1}{k-1} 2^{k-1}.}$$

To simplify this we expand the binomial coefficients

$$\frac{1}{2^{2n-k+1}} \frac{(2n)! \times k! \times (2n-1-k)! \times (n-1)!}{n! \times n! \times (2n-1)! \times (k-1)! \times (n-k)!}$$
$$= \frac{1}{2^{2n-k+1}} \frac{(2n) \times k \times (2n-1-k)!}{n \times n! \times (n-k)!}$$
$$= \frac{1}{2^{2n-k}} \frac{k \times (2n-1-k)!}{n! \times (n-k)!}.$$

This yields at last

$$\boxed{\frac{1}{2^{2n-k}} \frac{k}{n} \binom{2n-1-k}{n-1}.}$$

This was math.stackexchange.com problem 3737197.

1.28 MSE 3825092

We seek to show that

$$S(n) = \sum_{k=1}^{n} (-1)^{n-k} k^n \binom{n+1}{n-k} = 1.$$

This is

$$\sum_{k=0}^{n-1} (-1)^k (n-k)^n \binom{n+1}{k} = 1 + \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k (n-k)^n$$
$$= 1 + n![z^n] \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \exp((n-k)z)$$
$$= 1 + n![z^n] \exp(nz) \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \exp(-kz)$$
$$= 1 + n![z^n] \exp(nz)(1 - \exp(-z))^{n+1}$$

$$= 1 + n![z^n] \exp(-z)(\exp(z) - 1)^{n+1} = 1$$

because $\exp(z) - 1 = z + \cdots$ and hence $(\exp(z) - 1)^{n+1} = z^{n+1} + \cdots$ which is the claim.

This was math.stackexchange.com problem 3825092.

1.29 MSE 3845061

We seek to show that

$$\sum_{q=a+1}^{n} \binom{q-1}{a} \binom{n-q}{k-a} = \binom{n}{k+1}$$

or alternatively

$$\sum_{q=0}^{n} \binom{q}{a} \binom{n-q}{b} = \binom{n+1}{a+b+1}.$$

where $k \ge a$ for the binomial coefficient to be defined, and $n \ge a+1$ or alternatively

$$\sum_{q=0}^{n-a-1} \binom{q+a}{a} \binom{n-a-1-q}{k-a} = \binom{n}{k+1}.$$

The LHS is

$$[z^{k-a}](1+z)^{n-a-1} \sum_{q \ge 0} \binom{q+a}{a} (1+z)^{-q} [[q \le n-a-1]]$$

$$= [z^{k-a}](1+z)^{n-a-1} \sum_{q \ge 0} \binom{q+a}{a} (1+z)^{-q} [w^{n-a-1}] \frac{w^q}{1-w}$$

$$= [z^{k-a}](1+z)^{n-a-1} [w^{n-a-1}] \frac{1}{1-w} \sum_{q \ge 0} \binom{q+a}{a} (1+z)^{-q} w^q$$

$$= [z^{k-a}](1+z)^{n-a-1} [w^{n-a-1}] \frac{1}{1-w} \frac{1}{(1-w/(1+z))^{a+1}}$$

$$= [z^{k-a}](1+z)^{n} [w^{n-a-1}] \frac{1}{1-w} \frac{1}{(1+z-w)^{a+1}}.$$

This is

$$[z^{k-a}](1+z)^n (-1)^a \mathrm{Res}_{w=0} \frac{1}{w^{n-a}} \frac{1}{w-1} \frac{1}{(w-(1+z))^{a+1}}.$$

Now the residue at infinity for w is zero by inspection, residues sum to zero and the residue at $w = 1$ yields

$$[z^{k-a}](1+z)^n(-1)^a \frac{1}{(-1)^{a+1}z^{a+1}} = -\binom{n}{k+1}.$$

This is the claim if we can show that the contribution from the pole at $w = 1+z$ is zero. We get (Leibniz rule)

$$\frac{1}{a!}\left(\frac{1}{w^{n-a}}\frac{1}{w-1}\right)^{(a)} = \frac{1}{a!}\sum_{q=0}^{a}\binom{a}{q}\frac{(-1)^q(n-1-a+q)!}{(n-1-a)!\times w^{n-a+q}}\frac{(-1)^{a-q}(a-q)!}{(w-1)^{a+1-q}}$$

$$= (-1)^a \sum_{q=0}^{a}\binom{n-1-a+q}{q}\frac{1}{w^{n-a+q}}\frac{1}{(w-1)^{a+1-q}}.$$

We thus obtain for the contribution

$$[z^{k-a}](1+z)^n \sum_{q=0}^{a}\binom{n-1-a+q}{q}\frac{1}{(1+z)^{n-a+q}}\frac{1}{z^{a+1-q}}$$

$$= \sum_{q=0}^{a}\binom{n-1-a+q}{q}[z^{k+1-q}](1+z)^{a-q} = 0$$

because $a \ge q$ and $k+1 > a$. This concludes the argument.
This was math.stackexchange.com problem 3845061.

1.30 MSE 3885278

Introduction

The identity

$$\sum_{k\ge 0}\frac{(2k+1)^2}{(p+k+1)(q+k+1)}\binom{2p}{p-k}\binom{2q}{q-k} = \frac{1}{p+q+1}\binom{2p+2q}{p+q}$$

is identical to

$$\sum_{k=0}^{\min(p,q)}(2k+1)^2\binom{2p+1}{p+k+1}\binom{2q+1}{q+k+1} = \frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}$$

or

$$\sum_{k=0}^{\min(p,q)}(2k+1)^2\binom{2p+1}{p-k}\binom{2q+1}{q-k} = \frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}.$$

The LHS is

$$S = [z^p](1+z)^{2p+1}[w^q](1+w)^{2q+1} \sum_{k=0}^{\min(p,q)} (2k+1)^2 z^k w^k.$$

The two coefficient extractors enforce the upper limit of the sum:

$$\begin{aligned}
&[z^p](1+z)^{2p+1}[w^q](1+w)^{2q+1} \sum_{k\ge 0} (2k+1)^2 z^k w^k \\
&= [z^p](1+z)^{2p+1}[w^q](1+w)^{2q+1} \frac{z^2w^2+6zw+1}{(1-zw)^3} \\
&= -[z^p]\frac{1}{z^3}(1+z)^{2p+1}[w^q](1+w)^{2q+1} \frac{z^2w^2+6zw+1}{(w-1/z)^3} \\
&= -[z^{p+3}](1+z)^{2p+1}[w^q](1+w)^{2q+1} \frac{z^2w^2+6zw+1}{(w-1/z)^3}.
\end{aligned}$$

The coefficient extractor in w is

$$\mathrm{Res}_{w=0} \frac{1}{w^{q+1}}(1+w)^{2q+1} \frac{z^2w^2+6zw+1}{(w-1/z)^3}.$$

Residue at infinity

Now residues sum to zero and the residue at infinity is given by

$$\begin{aligned}
&-\mathrm{Res}_{w=0} \frac{1}{w^2} w^{q+1} \frac{(1+w)^{2q+1}}{w^{2q+1}} \frac{z^2/w^2+6z/w+1}{(1/w-1/z)^3} \\
&= -\mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+2}} \frac{z^2w+6zw^2+w^3}{(1-w/z)^3} \\
&= -\mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \frac{z^2+6zw+w^2}{(1-w/z)^3}.
\end{aligned}$$

Next applying the coefficient extractor in z we find

$$\begin{aligned}
&\mathrm{Res}_{z=0} \frac{(1+z)^{2p+1}}{z^{p+4}} \mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \frac{z^2+6zw+w^2}{(1-w/z)^3} \\
&= \mathrm{Res}_{z=0} \frac{(1+z)^{2p+1}}{z^{p+2}} \mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \frac{1+6w/z+w^2/z^2}{(1-w/z)^3} \\
&= \mathrm{Res}_{z=0} \frac{(1+z)^{2p+1}}{z^{p+2}} \mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \sum_{k\ge 0} (2k+1)^2 \frac{w^k}{z^k} \\
&= \sum_{k\ge 0} (2k+1)^2 \binom{2p+1}{p+k+1}\binom{2q+1}{q-k} = S.
\end{aligned}$$

This means that S is minus half the residue at $w = 1/z$, substituted into the coefficient extractor in z.

Residue at $w = 1/z$

The residue at $w = 1/z$ is

$$\mathrm{Res}_{w=1/z} \frac{1}{w^{q+1}} (1+w)^{2q+1} \frac{z^2w^2 + 6zw + 1}{(w-1/z)^3}$$

$$= \mathrm{Res}_{w=1/z} \frac{1}{w^{q+1}} (1+w)^{2q+1} \left(\frac{8}{(w-1/z)^3} + \frac{8z}{(w-1/z)^2} + \frac{z^2}{w-1/z} \right).$$

Evaluating the three pieces in turn we start with

$$8\frac{1}{2} \left(\frac{(1+w)^{2q+1}}{w^{q+1}} \right)'' = 4(q+1)(q+2) \frac{(1+w)^{2q+1}}{w^{q+3}}$$
$$-8(q+1)(2q+1) \frac{(1+w)^{2q}}{w^{q+2}} + 4(2q+1)(2q) \frac{(1+w)^{2q-1}}{w^{q+1}}.$$

Evaluate at $w = 1/z$ to get

$$4(q+1)(q+2) \frac{(1+z)^{2q+1}}{z^{q-2}}$$
$$-8(q+1)(2q+1) \frac{(1+z)^{2q}}{z^{q-2}} + 4(2q+1)(2q) \frac{(1+z)^{2q-1}}{z^{q-2}}.$$

Substituting into the coefficient extractor in z we find

$$-4(q+1)(q+2) \binom{2p+2q+2}{p+q+1}$$
$$+8(q+1)(2q+1) \binom{2p+2q+1}{p+q+1} - 4(2q+1)(2q) \binom{2p+2q}{p+q+1}.$$

Continuing with the middle piece we have

$$8z \left(\frac{(1+w)^{2q+1}}{w^{q+1}} \right)' = -8z(q+1) \frac{(1+w)^{2q+1}}{w^{q+2}} + 8z(2q+1) \frac{(1+w)^{2q}}{w^{q+1}}.$$

Evaluate at $w = 1/z$ to get

$$-8(q+1) \frac{(1+z)^{2q+1}}{z^{q-2}} + 8(2q+1) \frac{(1+z)^{2q}}{z^{q-2}}.$$

The coefficient extractor now yields

$$8(q+1) \binom{2p+2q+2}{p+q+1} - 8(2q+1) \binom{2p+2q+1}{p+q+1}.$$

The third and last piece produces

$$\frac{(1+z)^{2q+1}}{z^{q-2}}$$

which when substituted into the coefficient extractor yields

$$-\binom{2p+2q+2}{p+q+1}.$$

Collecting the three pieces

We get

$$-(2q+1)^2\binom{2p+2q+2}{p+q+1} + 8q(2q+1)\binom{2p+2q+1}{p+q+1} - 8q(2q+1)\binom{2p+2q}{p+q+1}$$

$$= -(2q+1)^2\binom{2p+2q+2}{p+q+1} + 8q(2q+1)\binom{2p+2q}{p+q}$$

$$= -2(2q+1)^2\binom{2p+2q+1}{p+q} + 8q(2q+1)\binom{2p+2q}{p+q}$$

$$= -2(2q+1)^2\frac{2p+2q+1}{p+q+1}\binom{2p+2q}{p+q} + 8q(2q+1)\binom{2p+2q}{p+q}$$

$$= -2\frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}.$$

Halve this value and flip the sign to obtain the coveted

$$\boxed{\frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}.}$$

This was math.stackexchange.com problem 3885278.

1.31 MSE 3559223

We seek to evaluate

$$G_{n,j} = \sum_{k=1}^{n} \frac{k^j(-1)^{n-k}\binom{n}{k}}{\frac{1}{2}n(n+1)-k}.$$

With this in mind we introduce the function

$$F_n(z) = n!\frac{z^{j-1}}{\frac{1}{2}n(n+1)-z}\prod_{q=1}^{n}\frac{1}{z-q}.$$

This has the property that the residue at $z = k$ where $1 \le k \le n$ is the desired sum term. We find

$$\begin{aligned}
\mathrm{Res}_{z=k} F_n(z) &= n! \frac{k^{j-1}}{\frac{1}{2}n(n+1) - k} \prod_{q=1}^{k-1} \frac{1}{k-q} \prod_{q=k+1}^{n} \frac{1}{k-q} \\
&= n! \frac{k^j}{\frac{1}{2}n(n+1) - k} \frac{1}{k} \frac{1}{(k-1)!} \frac{(-1)^{n-k}}{(n-k)!} \\
&= \frac{k^j}{\frac{1}{2}n(n+1) - k} (-1)^{n-k} \binom{n}{k}.
\end{aligned}$$

We will evaluate this using the fact that residues sum to zero and if $(n+1) - (j-1) \ge 2$ or $n \ge j$ the residue at infinity is zero, so we have in this case

$$G_{n,j} = -\mathrm{Res}_{z=\frac{1}{2}n(n+1)} F_n(z) = n! \frac{(\frac{1}{2}n(n+1))^{j-1}}{\prod_{q=1}^n (\frac{1}{2}n(n+1) - q)}.$$

We thus have

$$\boxed{G_{n,1} = \frac{n!}{\prod_{q=1}^n (\frac{1}{2}n(n+1) - q)}}$$

and

$$\boxed{G_{n,n} = \frac{(\frac{1}{2}n(n+1))^{n-1} n!}{\prod_{q=1}^n (\frac{1}{2}n(n+1) - q)}.}$$

When $j > n$ we must use the formula

$$G_{n,j} = -\mathrm{Res}_{z=\frac{1}{2}n(n+1)} F_n(z) - \mathrm{Res}_{z=\infty} F_n(z).$$

We have

$$\begin{aligned}
-\mathrm{Res}_{z=\infty} F_n(z) &= \mathrm{Res}_{z=0} \frac{1}{z^2} F_n(1/z) \\
&= n! \times \mathrm{Res}_{z=0} \frac{1}{z^2} \frac{1}{z^{j-1}} \frac{1}{\frac{1}{2}n(n+1) - 1/z} \prod_{q=1}^{n} \frac{1}{1/z - q} \\
&= n! \times \mathrm{Res}_{z=0} \frac{1}{z^{j+1}} \frac{z}{\frac{1}{2}n(n+1)z - 1} \prod_{q=1}^{n} \frac{z}{1 - qz} \\
&= n! \times \mathrm{Res}_{z=0} \frac{1}{z^{j-n}} \frac{1}{\frac{1}{2}n(n+1)z - 1} \prod_{q=1}^{n} \frac{1}{1 - qz}.
\end{aligned}$$

In particular when $j = n + 1$ we just need the constant term and find

$$n!\frac{1}{\frac{1}{2}n(n+1)\times 0-1}\prod_{q=1}^{n}\frac{1}{1-q\times 0} = -n!$$

we thus have

$$\boxed{G_{n,n+1} = \frac{(\frac{1}{2}n(n+1))^n n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)} - n!.}$$

The general case for $j > n$ is

$$n! \times \mathrm{Res}_{z=0}\frac{1}{z^j}\frac{1}{\frac{1}{2}n(n+1)z-1}\prod_{q=1}^{n}\frac{z}{1-qz}$$

which yields

$$-n!\sum_{q=0}^{j-1}\left(\frac{1}{2}n(n+1)\right)^q \left\{ j-1-q \atop n \right\}$$

so that the closed form is (here we must have $j-1-q \ge n$)

$$\boxed{G_{n,j} = \frac{(\frac{1}{2}n(n+1))^{j-1}n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)} - [[j>n]]n!\sum_{q=0}^{j-1-n}\left(\frac{1}{2}n(n+1)\right)^q \left\{ j-1-q \atop n \right\}.}$$

This was math.stackexchange.com problem 3559223.

1.32 MSE 3926409

Suppose we seek an alternate representation of

$$\sum_{p=q}^{k}(-1)^p\binom{k}{p}(q-p)^k.$$

This is

$$\sum_{p=0}^{k}(-1)^p\binom{k}{p}(q-p)^k - \sum_{p=0}^{q-1}(-1)^p\binom{k}{p}(q-p)^k.$$

We get for the first piece

$$k![z^k]\sum_{p=0}^{k}(-1)^p\binom{k}{p}\exp((q-p)z)$$

$$= k![z^k] \exp(qz) \sum_{p=0}^{k} (-1)^p \binom{k}{p} \exp(-pz)$$

$$= k![z^k] \exp(qz)(1 - \exp(-z))^k.$$

Now $(1 - \exp(-z))^k = z^k + \cdots$ so this evaluates to $k!$. We thus have

$$k! - \sum_{p=0}^{q-1} (-1)^p \binom{k}{p} (q-p)^k.$$

Using an Iverson bracket we get for the sum component

$$[w^{q-1}] \frac{1}{1-w} \sum_{p \geq 0} (-1)^p \binom{k}{p} (q-p)^k w^p$$

$$= k![z^k][w^{q-1}] \frac{1}{1-w} \exp(qz)(1 - w\exp(-z))^k$$

$$= k! \operatorname*{res}_z \frac{1}{z^{k+1}} \operatorname*{res}_w \frac{1}{w^q} \frac{1}{1-w} \exp(qz)(1 - w\exp(-z))^k.$$

We now apply Jacobi's Residue Formula. We put $w = v\exp((1-v)u)$ and $z = (1-v)u$. The scalar to obtain a non-zero constant term in u and v for z and w is u for z and v for w. Using the determinant of the Jacobian we obtain

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^{-1} \begin{vmatrix} 1-v & -u \\ v(1-v)\exp((1-v)u) & \exp((1-v)u) - uv\exp((1-v)u) \end{vmatrix}$$

$$= \exp((1-v)u) \begin{vmatrix} 1-v & -u \\ v(1-v) & 1-uv \end{vmatrix}$$

$$= \exp((1-v)u)(1 - uv - v + uv^2 + uv - uv^2)$$

$$= \exp((1-v)u)(1-v).$$

Doing the substitution we find

$$k! \operatorname*{res}_u \frac{1}{u^{k+1}} \frac{1}{(1-v)^{k+1}} \operatorname*{res}_v \frac{1}{v^q} \frac{1}{\exp(q(1-v)u)}$$

$$\times \frac{1}{1 - v\exp((1-v)u)} \exp(q(1-v)u)(1 - v\exp((1-v)u)\exp(-(1-v)u))^k$$

$$\times \exp((1-v)u)(1-v)$$

$$= k! \operatorname*{res}_u \frac{1}{u^{k+1}} \frac{1}{(1-v)^{k+1}} \operatorname*{res}_v \frac{1}{v^q} \frac{1}{1 - v\exp((1-v)u)} (1-v)^k$$

$$\times \exp((1-v)u)(1-v)$$

$$= k! \operatorname*{res}_{u} \frac{1}{u^{k+1}} \operatorname*{res}_{v} \frac{1}{v^q} \frac{1}{1 - v\exp((1-v)u)} \exp((1-v)u)$$

$$= k! \operatorname*{res}_{u} \frac{1}{u^{k+1}} \operatorname*{res}_{v} \frac{1}{v^q} \frac{1}{\exp((v-1)u) - v}.$$

Consider on the other hand the quantity

$$\sum_{p=0}^{q-1} \left\langle {k \atop p} \right\rangle.$$

This is

$$k![z^k] \sum_{p=0}^{q-1} [w^p] \frac{1-w}{\exp((w-1)z) - w}$$

$$= k![z^k][w^{q-1}] \frac{1}{1-w} \frac{1-w}{\exp((w-1)z) - w}$$

$$= k![z^k][w^{q-1}] \frac{1}{\exp((w-1)z) - w}$$

$$= k! \operatorname*{res}_{z} \frac{1}{z^{k+1}} \operatorname*{res}_{w} \frac{1}{w^q} \frac{1}{\exp((w-1)z) - w}.$$

This is the same as the sum term and we conclude the argument having shown that

$$\sum_{p=q}^{k} (-1)^p \binom{k}{p} (q-p)^k = k! - \sum_{p=0}^{q-1} \left\langle {k \atop p} \right\rangle$$

which is

$$\boxed{\sum_{p=q}^{k} (-1)^p \binom{k}{p} (q-p)^k = \sum_{p=q}^{k} \left\langle {k \atop p} \right\rangle.}$$

The reference for Jacobi's Residue Formula is Theorem 3 in [Ges87].

This was math.stackexchange.com problem 3926409.

1.33 MSE 3942039

We seek to verify that

$$\sum_{k=0}^{n} (-1)^k \frac{2^{n-k}\binom{n}{k}}{(m+k+1)\binom{m+k}{k}} = \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k+1}.$$

We can re-write this as

$$\frac{m!n!}{(n+m+1)!} 2^n \sum_{k=0}^{n} (-1)^k 2^{-k} \binom{n+m+1}{n-k} = \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k+1}$$

or

$$2^n \sum_{k=0}^{n} (-1)^k 2^{-k} \binom{n+m+1}{n-k} = (m+1) \binom{n+m+1}{n} \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k+1}.$$

We get for the LHS

$$2^n \sum_{k=0}^{n} (-1)^k 2^{-k} \binom{n+m+1}{m+k+1} = 2^n \sum_{k=0}^{n} (-1)^k 2^{-k} [z^{n-k}] \frac{1}{(1-z)^{m+k+2}}$$

$$= 2^n [z^n] \frac{1}{(1-z)^{m+2}} \sum_{k=0}^{n} (-1)^k 2^{-k} z^k \frac{1}{(1-z)^k}.$$

Here the coefficient extractor enforces the range and we find

$$2^n [z^n] \frac{1}{(1-z)^{m+2}} \sum_{k\ge 0} (-1)^k 2^{-k} z^k \frac{1}{(1-z)^k} = 2^n [z^n] \frac{1}{(1-z)^{m+2}} \frac{1}{1+z/(1-z)/2}$$

$$= [z^n] \frac{1}{(1-2z)^{m+2}} \frac{1}{1+z/(1-2z)} = [z^n] \frac{1}{(1-2z)^{m+1}} \frac{1}{1-z}.$$

On the other hand we have

$$\binom{n+m+1}{n} \binom{n}{k} = \frac{(n+m+1)!}{(m+1)! \times k! \times (n-k)!} = \binom{n+m+1}{n-k} \binom{m+k+1}{m+1}$$

which gives for the RHS

$$\sum_{k=0}^{n} \binom{n+m+1}{n-k} \frac{m+1}{m+k+1} \binom{m+k+1}{m+1} = \sum_{k=0}^{n} \binom{n+m+1}{m+k+1} \binom{m+k}{m}$$

$$= \sum_{k=0}^{n} \binom{m+k}{m} [z^{n-k}] \frac{1}{(1-z)^{m+k+2}} = [z^n] \sum_{k=0}^{n} \binom{m+k}{m} \frac{1}{(1-z)^{m+k+2}} z^k.$$

We once more have the coefficient extractor enforcing the range and we get

$$[z^n] \frac{1}{(1-z)^{m+2}} \sum_{k\ge 0} \binom{m+k}{m} \frac{1}{(1-z)^k} z^k$$

$$= [z^n]\frac{1}{(1-z)^{m+2}}\frac{1}{(1-z/(1-z))^{m+1}} = [z^n]\frac{1}{1-z}\frac{1}{(1-2z)^{m+1}}.$$

The LHS is the same as the RHS which concludes the argument. The coefficient extractor evaluates to

$$\boxed{\sum_{k=0}^{n} \binom{k+m}{m} 2^k.}$$

This was math.stackexchange.com problem 3942039.

1.34 MSE 3956698

The sum in the problem statement here is

$$\sum_{k\ge 1}\left[\binom{\lfloor \frac{k}{2}\rfloor}{m} + \binom{\lceil \frac{k}{2}\rceil}{m}\right]\binom{n-1}{k-1}$$

$$= 2\sum_{k\ge 0}\binom{k}{m}\binom{n-1}{2k-1} + \sum_{k\ge 0}\binom{k}{m}\binom{n-1}{2k} + \sum_{k\ge 0}\binom{k+1}{m}\binom{n-1}{2k}$$

which we seek to prove is equal to

$$2^{n-2m}\binom{n-m}{m-1}\frac{n+1}{m}$$

where we will take $m \ge 1$. We get for the first term

$$2\sum_{k\ge 0}\binom{k}{m}\binom{n-1}{n-2k} = 2[z^n](1+z)^{n-1}\sum_{k\ge 0}\binom{k}{m}z^{2k}$$

$$= 2[z^n](1+z)^{n-1}\sum_{k\ge m}\binom{k}{m}z^{2k} = 2[z^{n-2m}](1+z)^{n-1}\sum_{k\ge 0}\binom{k+m}{m}z^{2k}$$

$$= 2[z^{n-2m}](1+z)^{n-1}\frac{1}{(1-z^2)^{m+1}}.$$

The second term is

$$\sum_{k\ge 0}\binom{k}{m}\binom{n-1}{n-1-2k} = [z^{n-1}](1+z)^{n-1}\sum_{k\ge 0}\binom{k}{m}z^{2k}$$

$$= [z^{n-2m-1}](1+z)^{n-1}\frac{1}{(1-z^2)^{m+1}}.$$

The third term is

$$\sum_{k\ge 0}\binom{k+1}{m}\binom{n-1}{n-1-2k} = [z^{n-1}](1+z)^{n-1}\sum_{k\ge 0}\binom{k+1}{m}z^{2k}$$

$$= [z^{n-1}](1+z)^{n-1} \sum_{k \ge m-1} \binom{k+1}{m} z^{2k} = [z^{n-2m+1}](1+z)^{n-1} \sum_{k \ge 0} \binom{k+m}{m} z^{2k}$$
$$= [z^{n-2m+1}](1+z)^{n-1} \frac{1}{(1-z^2)^{m+1}}.$$

Adding these together we get

$$[z^{n-2m+1}](1+z^2+2z)(1+z)^{n-1} \frac{1}{(1-z^2)^{m+1}} = [z^{n-2m+1}](1+z)^{n+1} \frac{1}{(1-z^2)^{m+1}}$$
$$= [z^{n-2m+1}](1+z)^{n-m} \frac{1}{(1-z)^{m+1}}.$$

The coefficient extractor now yields

$$\sum_{q=0}^{n+1-2m} \binom{n-m}{q} \binom{n+1-2m-q+m}{m} = \sum_{q=0}^{n+1-2m} \binom{n-m}{q} \binom{n+1-m-q}{m}$$
$$= \sum_{q=0}^{n+1-2m} \binom{n-m}{q} \frac{n+1-m-q}{m} \binom{n-m-q}{m-1}.$$

Now

$$\binom{n-m}{q} \binom{n-m-q}{m-1} = \frac{(n-m)!}{q! \times (m-1)! \times (n+1-2m-q)!}$$
$$= \binom{n-m}{m-1} \binom{n+1-2m}{q}.$$

We get for the sum

$$\frac{1}{m} \binom{n-m}{m-1} \sum_{q=0}^{n+1-2m} (n+1-m-q) \binom{n+1-2m}{q}$$
$$= \frac{1}{m} \binom{n-m}{m-1} \sum_{q=0}^{n+1-2m} (n+1-2m-q) \binom{n+1-2m}{q}$$
$$+ \binom{n-m}{m-1} \sum_{q=0}^{n+1-2m} \binom{n+1-2m}{q}$$
$$= \frac{1}{m} \binom{n-m}{m-1} \sum_{q=0}^{n+1-2m} q \binom{n+1-2m}{q} + \binom{n-m}{m-1} 2^{n+1-2m}$$

$$= \frac{n+1-2m}{m}\binom{n-m}{m-1}\sum_{q=1}^{n+1-2m}\binom{n-2m}{q-1} + \binom{n-m}{m-1}2^{n+1-2m}$$

$$= \frac{n+1-2m}{m}\binom{n-m}{m-1}2^{n-2m} + \binom{n-m}{m-1}2^{n+1-2m}.$$

This simplifies to

$$\boxed{\frac{n+1}{m}\binom{n-m}{m-1}2^{n-2m}.}$$

Addendum. Following the hint by OP in view of the intermediate closed form we see that we can simplify the three terms first. We get

$$2\sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k-1} + \sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k} + \sum_{k\ge m-1}\binom{k+1}{m}\binom{n-1}{2k}$$

$$= 2\sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k-1} + \sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k} + \sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k-2}$$

$$= \sum_{k\ge m}\binom{k}{m}\binom{n}{2k} + \sum_{k\ge m}\binom{k}{m}\binom{n}{2k-1} = \sum_{k\ge m}\binom{k}{m}\binom{n+1}{2k}.$$

We then find

$$\sum_{k\ge m}\binom{k}{m}\binom{n+1}{n+1-2k} = [z^{n+1}](1+z)^{n+1}\sum_{k\ge m}\binom{k}{m}z^{2k}$$

$$= [z^{n+1-2m}](1+z)^{n+1}\sum_{k\ge 0}\binom{k+m}{m}z^{2k} = [z^{n+1-2m}](1+z)^{n+1}\frac{1}{(1-z^2)^{m+1}}.$$

From this point on the computation continues as before.

This was math.stackexchange.com problem 3956698.

1.35 MSE 3993530

We seek to verify that (with $n \ge 1$, $n = 0$ holds by inspection)

$$\sum_{k=0}^{n}\left\langle {n \atop k} \right\rangle x^{n-k} = (1-x)^n \sum_{k=0}^{n}\left\{ {n \atop k} \right\} k! \left(\frac{x}{1-x}\right)^k.$$

We get using standard EGFs for the RHS

$$n![z^n](1-x)^n\sum_{k=0}^{n}\frac{(\exp(z)-1)^k}{k!}k!\left(\frac{x}{1-x}\right)^k$$

$$= n![z^n](1-x)^n \sum_{k=0}^{n} (\exp(z)-1)^k \left(\frac{x}{1-x}\right)^k.$$

Now because $\exp(z) - 1 = z + \cdots$ we have $(\exp(z)-1)^k = z^k + \cdots$ so when $k > n$ there is no contribution to the coefficient extractor and we get

$$n![z^n](1-x)^n \sum_{k\ge 0} (\exp(z)-1)^k \left(\frac{x}{1-x}\right)^k$$

$$= n![z^n](1-x)^n \frac{1}{1-(\exp(z)-1)x/(1-x)}$$

$$= n![z^n](1-x)^n \frac{1-x}{1-x-(\exp(z)-1)x}$$

$$= n![z^n](1-x)^n \frac{1-x}{1-x\exp(z)}$$

$$= n![z^n] \frac{1-x}{1-x\exp(z(1-x))}.$$

On the other hand we have for the LHS by the mixed GF of the Eulerian numbers

$$n![z^n] \sum_{k=0}^{n} x^{n-k} [w^k] \frac{w-1}{w-\exp((w-1)z)}$$

Now we have $\left\langle {n \atop k} \right\rangle = 0$ when $k \ge n$ so this is

$$n![z^n] x^n \sum_{k\ge 0} x^{-k} [w^k] \frac{w-1}{w-\exp((w-1)z)}$$

$$= n![z^n] x^n \frac{1/x-1}{1/x-\exp((1/x-1)z)}$$

$$= n![z^n] x^n \frac{1-x}{1-x\exp((1/x-1)z)}$$

$$= n![z^n] \frac{1-x}{1-x\exp((1/x-1)zx)}$$

$$= n![z^n] \frac{1-x}{1-x\exp((1-x)z)}.$$

The LHS is the same as the RHS and we have the claim.

Addendum. We have

$$n![z^n][w^k] \frac{w-1}{w-\exp((w-1)z)}$$

$$= n![z^n][w^{k+1}] \frac{w-1}{1-\exp((w-1)z)/w}$$

$$= n![z^n][w^{k+1}](w-1)\sum_{q\ge 0}\frac{1}{w^q}\exp(q(w-1)z)$$

$$= [w^{k+1}]\sum_{q\ge 0}\frac{1}{w^q}q^n(w-1)^{n+1} = \sum_{q\ge 0}[w^{k+1+q}]q^n(w-1)^{n+1}$$

$$= (-1)^{n-k}\sum_{q=1}^{n-k}(-1)^q q^n\binom{n+1}{k+1+q}.$$

This justifies that $\left\langle {n \atop k} \right\rangle = 0$ when $k \ge n$ and hence the two coefficient extractors combined return zero in that case as claimed.

This was math.stackexchange.com problem 3993530.

1.36 MSE 4008277

We seek to show that

$$\sum_{k=0}^{r} k^p \binom{m}{k}\binom{n}{r-k} = \sum_{j=0}^{p} m^{\underline{j}}\binom{m+n-j}{m+n-r}\left\{ {p \atop j} \right\}.$$

The LHS is

$$p![w^p]\sum_{k=0}^{r}\exp(kw)\binom{m}{k}\binom{n}{r-k}$$

$$= p![w^p][z^r](1+z)^n\sum_{k=0}^{r}\exp(kw)\binom{m}{k}z^k.$$

Now the coefficient extractor enforces the upper limit of the range and we may continue with

$$p![w^p][z^r](1+z)^n\sum_{k\ge 0}\exp(kw)\binom{m}{k}z^k$$

$$= p![w^p][z^r](1+z)^n(1+z\exp(w))^m$$

$$= p![w^p][z^r](1+z)^n(1+z+z(\exp(w)-1))^m$$

$$= p![w^p][z^r](1+z)^n\sum_{j=0}^{m}\binom{m}{j}(1+z)^{m-j}z^j(\exp(w)-1)^j$$

$$= [z^r]\sum_{j=0}^{m}\binom{m}{j}(1+z)^{m+n-j}z^j j!\left\{ {p \atop j} \right\}$$

$$= \sum_{j=0}^{m}\binom{m}{j}\binom{m+n-j}{r-j}j!\left\{ {p \atop j} \right\}.$$

Note that if $m > p$ the values with $m \ge j > p$ produce a zero Stirling number so we may lower m to p. If $m < p$ the values with $p \ge j > m$ produce a zero binomial coefficient and we may raise m to p. We thus obtain

$$\boxed{\sum_{j=0}^{p} \binom{m}{j} \binom{m+n-j}{m+n-r} j! \left\{ p \atop j \right\}.}$$

a sum with p non-zero terms except for $p = 0$, when it has one term. (We could also use $\min(m, p)$ as the upper limit but we want to emphasize the dependence on p.) Note that in the initial sum for it to be non-zero with non-negative k we must have $m \ge k$ and $n \ge r - k$ or $k \ge r - n$ so that $m \ge k \ge r - n$ and for the range not to be empty we must have $m \ge r - n$ or $m + n - r \ge 0$ which ensures that the middle binomial coefficient in the boxed form is well defined. Observe that with $p = 0$ we obtain $\binom{m+n}{m+n-r} = \binom{m+n}{r}$ which is Vandermonde. A slight variation is

$$\boxed{\sum_{j=0}^{p} m^{\underline{j}} \binom{m+n-j}{m+n-r} \left\{ p \atop j \right\}.}$$

Remark. We may keep the $\binom{m+n-j}{r-j}$ if we remember that it originates with $[z^r](1+z)^{m+n-j} z^j$ and hence is zero when $j > r$.

This was math.stackexchange.com problem 4008277.

1.37 MSE 4031272

We seek

$$\binom{m+k}{k}^2 = \sum_{q=0}^{m} \binom{k}{m-q}^2 \binom{2k+q}{q}$$

Starting with the RHS we find

$$\sum_{q=0}^{m} \binom{k}{q}^2 \binom{2k+m-q}{m-q}$$

$$= \sum_{q=0}^{m} \binom{k}{q} [z^k] z^q (1+z)^k [w^m] w^q (1+w)^{2k+m-q}.$$

Now we may extend q to infinity because the coefficient extractor $[w^m]$ enforces the upper limit. We get

$$[z^k](1+z)^k [w^m](1+w)^{2k+m} \sum_{q \ge 0} \binom{k}{q} z^q w^q (1+w)^{-q}$$

$$= [z^k](1+z)^k [w^m](1+w)^{2k+m} (1 + zw/(1+w))^k$$

$$= [z^k](1+z)^k [w^m](1+w)^{k+m}(1+w+zw)^k$$

Re-expanding we find

$$[z^k](1+z)^k [w^m](1+w)^{k+m} \sum_{q=0}^{k} \binom{k}{q} w^q (1+z)^q.$$

We may set the upper limit of the sum to m. (If $k < m$ the values $k < q \le m$ produce zero from the binomial coefficient and we may raise q to m. If $k > m$ the values $m < q \le k$ produce zero by the coefficient extractor $[w^m]$ and we may lower q to m.) We get

$$[z^k](1+z)^k [w^m](1+w)^{k+m} \sum_{q=0}^{m} \binom{k}{q} w^q (1+z)^q$$
$$= \sum_{q=0}^{m} \binom{k}{q} \binom{k+q}{k} \binom{k+m}{m-q}.$$

Now observe that

$$\binom{k+q}{k}\binom{k+m}{m-q} = \frac{(k+m)!}{k! \times q! \times (m-q)!} = \binom{m+k}{k}\binom{m}{m-q}.$$

This yields for our sum

$$\binom{m+k}{k} \sum_{q=0}^{m} \binom{k}{q}\binom{m}{m-q}.$$

Using Vandermonde we obtain at last

$$\boxed{\binom{m+k}{k}^2.}$$

This was math.stackexchange.com problem 4031272 and this identity is the Li Shanlan identity.

1.38 MSE 4034224

We seek to show that with $0 \le k \le n$ the following identity holds: (two alternate representations of second order Eulerian numbers)

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{2n+1}{k-j} \left\{ {n+j \atop j} \right\} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k} (-1)^j \binom{2n+1}{j} \left[{2n-k-j+1 \atop n-k-j+1} \right].$$

We will start with the LHS. The chapter 6.2 on Eulerian Numbers of *Concrete Mathematics* by Knuth et al. [GKP89] proposes the formula

$$\left\{ n \atop m \right\} = (-1)^{n-m+1} \frac{n!}{(m-1)!} \sigma_{n-m}(-m)$$

where $\sigma_n(x)$ is a Stirling polynomial and we have the identity

$$\left(\frac{1}{z} \log \frac{1}{1-z} \right)^x = x \sum_{n \ge 0} \sigma_n(x+n) z^n.$$

We get

$$[z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^x = x\sigma_{n-m}(x+n-m)$$

and hence

$$[z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n} = -n\sigma_{n-m}(-m)$$

which implies that for $n \ge m \ge 1$

$$\boxed{\left\{ n \atop m \right\} = (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n}.}$$

This gives for the LHS

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{2n+1}{k-j} (-1)^n \frac{(n+j-1)!}{(j-1)!} [z^n] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n-j}$$

$$= (-1)^{n-k+1} n! [z^n] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n-1} [w^{k-1}](1+w)^{2n+1}$$

$$\times \sum_{j=1}^{k} \binom{n+j-1}{n} (-1)^{j-1} w^{j-1} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-j+1}.$$

Now the coefficient extractor in w enforces the upper limit of the sum and we may extend j to infinity, getting

$$(-1)^{n-k+1} n! [z^n] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n-1} [w^{k-1}](1+w)^{2n+1} \frac{1}{(1 + w/(\frac{1}{z} \log \frac{1}{1-z}))^{n+1}}$$

$$= (-1)^{n-k+1} n! [z^n] [w^{k-1}](1+w)^{2n+1} \frac{1}{(w + \frac{1}{z} \log \frac{1}{1-z})^{n+1}}.$$

Continuing,

$$(-1)^{n-k+1} n! [z^n] [w^{n+k}](1+w)^{2n+1} \frac{1}{(1 + \frac{1}{w} \frac{1}{z} \log \frac{1}{1-z})^{n+1}}$$

$$= (-1)^{n-k+1} n! [z^n][w^{n+k}](1+w)^{2n+1} \sum_{q\ge 0} \binom{n+q}{n} (-1)^q \frac{1}{w^q} \left(\frac{1}{z}\log\frac{1}{1-z}\right)^q$$

$$= (-1)^{n-k+1} n! [z^n] \sum_{j=n+k}^{2n+1} \binom{2n+1}{j} \binom{n+j-(n+k)}{n} (-1)^{j-(n+k)}$$

$$\times \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{j-(n+k)}$$

$$= (-1)^{n-k+1} n! [z^n] \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} \binom{n+j}{n} (-1)^j \left(\frac{1}{z}\log\frac{1}{1-z}\right)^j$$

$$= (-1)^{n-k+1} n! \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} \binom{n+j}{n} (-1)^j [z^{n+j}] \left(\log\frac{1}{1-z}\right)^j$$

$$= (-1)^{n-k+1} n! \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} \binom{n+j}{n} (-1)^j$$

$$\times \frac{j!}{(n+j)!} \times (n+j)! [z^{n+j}] \frac{1}{j!} \left(\log\frac{1}{1-z}\right)^j$$

$$= (-1)^{n-k+1} \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} (-1)^j \begin{bmatrix} n+j \\ j \end{bmatrix}$$

$$= (-1)^{n-k+1} \sum_{j=0}^{n-k+1} \binom{2n+1}{2n-j+1} (-1)^{n-k-j+1} \begin{bmatrix} 2n-k-j+1 \\ n-k-j+1 \end{bmatrix}$$

$$= \sum_{j=0}^{n-k+1} \binom{2n+1}{j} (-1)^j \begin{bmatrix} 2n-k-j+1 \\ n-k-j+1 \end{bmatrix}.$$

The Stirling number is zero for $j = n-k+1$ and we get at last

$$\sum_{j=0}^{n-k} \binom{2n+1}{j} (-1)^j \begin{bmatrix} 2n-k-j+1 \\ n-k-j+1 \end{bmatrix}.$$

This is the RHS and we have the claim.

Remark. The Stirling number identity from [GKP89] may be derived from first principles. Start using the combinatorial EGF of set partitions

$$\begin{Bmatrix} n \\ m \end{Bmatrix} = \frac{n!}{m!} [z^n] (\exp(z) - 1)^m$$

which is for $n \ge 1$, $n \ge m \ge 1$

$$\frac{(n-1)!}{(m-1)!}[z^{n-1}]\exp(z)(\exp(z)-1)^{m-1}.$$

The corresponding integral is by the Cauchy Coefficient Formula with $\varepsilon \ll 1$

$$\frac{(n-1)!}{(m-1)!}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^n}\exp(z)(\exp(z)-1)^{m-1}\;dz.$$

Now put $\exp(z)-1=w$ so that in a neighborhood of zero $z=\log(1+w)$ (branch cut is $[-1,\infty)$) and $\exp(z)\;dz=dw$. With $w=z+z^2/2+z^3/6+\cdots$ the image of $|z|=\varepsilon$ makes one turn around zero. We obtain

$$\frac{(n-1)!}{(m-1)!}\frac{1}{2\pi i}\int_{|w|=\gamma}(\log(1+w))^{-n}w^{m-1}\;dw.$$

As for the choice of γ the image of $|z|=\varepsilon$ is contained in the annulus defined by two circles centered at the origin of radius $1-\exp(-\varepsilon)$ and $\exp(\varepsilon)-1$. Hence we may take $\gamma=\varepsilon-\varepsilon^2/2$ (the branch point is at $w=-1$). Continuing we find

$$\begin{aligned}&\frac{(n-1)!}{(m-1)!}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^n}\left(\frac{1}{w}\log(1+w)\right)^{-n}w^{m-1}\;dw\\ &=\frac{(n-1)!}{(m-1)!}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n-m+1}}\left(\frac{1}{w}\log(1+w)\right)^{-n}\;dw.\end{aligned}$$

We may recover a formal power series result from this which is

$$\begin{aligned}&\frac{(n-1)!}{(m-1)!}[w^{n-m}]\left(\frac{1}{w}\log(1+w)\right)^{-n}\\ &=\frac{(n-1)!}{(m-1)!}[w^{n-m}](-1)^n\left(\frac{1}{w}\log\frac{1}{1+w}\right)^{-n}\\ &=\frac{(n-1)!}{(m-1)!}(-1)^{n-m}[w^{n-m}](-1)^n\left(-\frac{1}{w}\log\frac{1}{1-w}\right)^{-n}\\ &=\frac{(n-1)!}{(m-1)!}(-1)^{n-m}[w^{n-m}]\left(\frac{1}{w}\log\frac{1}{1-w}\right)^{-n}.\end{aligned}$$

This is the cited result. The powered term is in fact a formal power series as the logarithmic term being zero cancels the $1/w$ factor.

This was math.stackexchange.com problem 4034224.

1.39 MSE 4037172

We seek to show that with $0 \le k \le n$ the following identity holds: (two alternate representations of second order Eulerian numbers)

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}\left\{\!\!\left\{ {n+j \atop j} \right\}\!\!\right\} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k+1}(-1)^{n-k-j+1}\binom{n-j}{k-1}\left[\!\!\left[{n+j \atop j} \right]\!\!\right]$$

where we have associated Stirling numbers of the first and second kind.

Now from the combinatorial meaning of these numbers (cancel fixed points resp. singleton sets) we have that

$$\left[\!\!\left[{n \atop k} \right]\!\!\right] = \sum_{q=0}^{k}(-1)^q\binom{n}{q}\left[{n-q \atop k-q} \right]$$

and

$$\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\} = \sum_{q=0}^{k}(-1)^q\binom{n}{q}\left\{ {n-q \atop k-q} \right\}.$$

Consult OEIS A008306 and OEIS A008299 for more information. We will only use the second of these but we show the pair to illustrate the similarity in their construction (PIE). The combinatorial classes for these are $\textsc{SET}(\mathcal{U}\times \textsc{CYC}_{\ge 2}(\mathcal{Z}))$ and $\textsc{SET}(\mathcal{U}\times \textsc{SET}_{\ge 2}(\mathcal{Z}))$.

We start with the LHS and obtain

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}\sum_{q=0}^{j}(-1)^q\binom{n+j}{q}\left\{ {n+j-q \atop j-q} \right\}.$$

With $n \ge 1$ this is

$$(-1)^k\sum_{j=1}^{k}\binom{n-j}{k-j}\sum_{q=1}^{j}(-1)^q\binom{n+j}{j-q}\left\{ {n+q \atop q} \right\}.$$

Recall e.g. from *Concrete Mathematics* chapter 6.2. [GKP89] that

$$\boxed{\left\{ {n \atop m} \right\} = (-1)^{n-m}\frac{(n-1)!}{(m-1)!}[z^{n-m}]\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-n}.}$$

We find for the LHS

$$(-1)^k\sum_{j=1}^{k}\binom{n-j}{k-j}\sum_{q=1}^{j}(-1)^q\binom{n+j}{j-q}(-1)^n\frac{(n+q-1)!}{(q-1)!}[z^n]\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-n-q}$$

$$= (-1)^{n-k+1}n![z^n]\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-n-1}\sum_{j=1}^{k}\binom{n-j}{k-j}$$

$$\times \sum_{q=1}^{j} (-1)^{q-1} \binom{n+j}{j-q} \binom{n+q-1}{q-1} \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-q+1}$$

$$= (-1)^{n-k+1} n! [z^n] \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-n-1} \sum_{j=1}^{k} \binom{n-j}{k-j}$$

$$\times [w^{j-1}](1+w)^{n+j} \sum_{q=1}^{j} (-1)^{q-1} w^{q-1} \binom{n+q-1}{q-1} \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-q+1}.$$

Now the coefficient extractor enforces the upper limit of the inner sum and we may extend q to infinity, getting

$$(-1)^{n-k+1} n! [z^n] \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-n-1} \sum_{j=1}^{k} \binom{n-j}{k-j}$$

$$\times [w^{j-1}](1+w)^{n+j} \frac{1}{(1 + w/(\frac{1}{z} \log \frac{1}{1-z}))^{n+1}}$$

$$= (-1)^{n-k+1} n! [z^n] \sum_{j=1}^{k} \binom{n-j}{k-j} [w^{j-1}](1+w)^{n+j} \frac{1}{(\frac{1}{z} \log \frac{1}{1-z} + w)^{n+1}}.$$

The inner term is

$$[w^{j-1}](1+w)^{n+j} \frac{1}{(\frac{1}{z}(\log \frac{1}{1-z} - z) + 1 + w)^{n+1}}$$

$$= [w^{j-1}](1+w)^{j-1} \frac{1}{(1 + \frac{1}{1+w} \frac{1}{z}(\log \frac{1}{1-z} - z))^{n+1}}.$$

Re-expanding the series,

$$(-1)^{n-k+1} n! [z^n] \sum_{j=1}^{k} \binom{n-j}{k-j} [w^{j-1}](1+w)^{j-1}$$

$$\times \sum_{q=j}^{n} \binom{n+q}{n} (-1)^q \frac{1}{(1+w)^q} \left(\frac{1}{z}(\log \frac{1}{1-z} - z)\right)^q.$$

The upper limit on the inner sum results from $[z^n]$ because $\frac{1}{z}(\log \frac{1}{1-z} - z) = \frac{1}{2} z + \cdots$ and the lower one from the fact that $[w^{j-1}](1+w)^{j-1-q} = 0$ when $1 \le q \le j-1$; $q = 0$ produces a constant. Continuing,

$$(-1)^{n-k+1} n! [z^n] \sum_{j=1}^{k} \binom{n-j}{k-j} [w^{j-1}]$$

$$\times \sum_{q=j}^{n} \binom{n+q}{n} (-1)^q \frac{1}{(1+w)^{q-(j-1)}} \left(\frac{1}{z}(\log\frac{1}{1-z} - z)\right)^q$$

$$= (-1)^{n-k} n! [z^n] \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} \binom{n+q}{n} (-1)^{q-j} \binom{q-1}{q-j} \left(\frac{1}{z}(\log\frac{1}{1-z} - z)\right)^q$$

$$= (-1)^{n-k} n! \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} \binom{n+q}{n} (-1)^{q-j} \binom{q-1}{q-j} [z^{n+q}] \left(\log\frac{1}{1-z} - z\right)^q$$

$$= (-1)^{n-k} n! \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} \binom{n+q}{n} (-1)^{q-j} \binom{q-1}{q-j}$$

$$\times \frac{q!}{(n+q)!} \times (n+q)! [z^{n+q}] \frac{1}{q!} \left(\log\frac{1}{1-z} - z\right)^q$$

$$= (-1)^{n-k} \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} (-1)^{q-j} \binom{q-1}{q-j} \left[\!\left[n+q \atop q \right]\!\right].$$

It remains to simplify the binomial coefficients:

$$(-1)^{n-k} \sum_{j=1}^{k} [u^k] u^j (1+u)^{n-j} \sum_{q=j}^{n} (-1)^{q-j} \binom{q-1}{q-j} \left[\!\left[n+q \atop q \right]\!\right].$$

We see that we may raise j to n owing to $[u^k]$:

$$(-1)^{n-k} \sum_{j=1}^{n} [u^k] u^j (1+u)^{n-j} \sum_{q=j}^{n} (-1)^{q-j} \binom{q-1}{q-j} \left[\!\left[n+q \atop q \right]\!\right]$$

$$= (-1)^{n-k} \sum_{q=1}^{n} \left[\!\left[n+q \atop q \right]\!\right] [u^k] (1+u)^n \sum_{j=1}^{q} \binom{q-1}{q-j} (-1)^{q-j} u^j (1+u)^{-j}.$$

The inner term is

$$[u^k](1+u)^n \frac{u^q}{(1+u)^q} \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j \frac{(1+u)^j}{u^j}$$

$$= [u^k](1+u)^{n-q} u^q \left(1 - \frac{1+u}{u}\right)^{q-1}$$

$$= [u^k](1+u)^{n-q} u (-1)^{q-1} = (-1)^{q-1} [u^{k-1}] (1+u)^{n-q}.$$

This yields

$$\sum_{q=0}^{n-k+1} (-1)^{n-k-q+1} \binom{n-q}{k-1} \left[\!\left[n+q \atop q \right]\!\right]$$

which is the claim. (Here we must have $n-q \ge k-1$ or $n-k+1 \ge q$ else the binomial coefficient vanishes and we may lower the upper limit from n to $n-k+1$.)

This was math.stackexchange.com problem 4037172.

1.40 MSE 4037946

We seek to show that with $0 \le k \le n$ the following identity holds:

$$\left[{n \atop n-k}\right] - \left\{{n \atop n-k}\right\} = \sum_{j=0}^{k} \left({n+j-1 \choose 2k} - {n+k-j \choose 2k} \right) \left\langle\!\!\left\langle {k \atop j} \right\rangle\!\!\right\rangle.$$

Recall from the previous example the identity

$$\sum_{j=0}^{k} (-1)^{k-j} {n-j \choose k-j} \left\{\!\!\left\{ {n+j \atop j} \right\}\!\!\right\} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k+1} (-1)^{n-k-j+1} {n-j \choose k-1} \left[\!\!\left[{n+j \atop j} \right]\!\!\right]$$

We get for the first piece

$$\sum_{j=1}^{k} {n+j-1 \choose 2k} \sum_{p=0}^{k-j+1} (-1)^{k-j-p+1} {k-p \choose j-1} \left[\!\!\left[{k+p \atop p} \right]\!\!\right]$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] (-1)^{k-p+1} \sum_{j=1}^{k+1-p} (-1)^{j} {n+j-1 \choose 2k} {k-p \choose j-1}$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] (-1)^{k-p} [z^{2k}] (1+z)^n \sum_{j=1}^{k+1-p} (-1)^{j-1} (1+z)^{j-1} {k-p \choose j-1}$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] (-1)^{k-p} [z^{2k}] (1+z)^n (1-(1+z))^{k-p}$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] [z^{k+p}] (1+z)^n = \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] {n \choose k+p}.$$

Now this last piece evaluates combinatorially to $\left[{n \atop n-k}\right]$ when written as $\left[\!\left[{k+p \atop p} \right]\!\right] {n \choose n-k-p}$ namely we choose $n-k-p$ fixed points and split the remaining $k+p$ elements into p cycles of size at least two for a total of $n-k$ cycles. Here we must have $k+p \ge 2p$ or $p \le k$. (We have classified by the number of fixed points).

We get for the second piece

$$\sum_{j=1}^{k} \binom{n+k-j}{2k} \sum_{p=0}^{j} (-1)^{j-p} \binom{k-p}{j-p} \left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\} (-1)^p \sum_{j=p}^{k} (-1)^j \binom{n+k-j}{2k} \binom{k-p}{j-p}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\} \sum_{j=0}^{k-p} (-1)^j \binom{n+k-j-p}{2k} \binom{k-p}{j}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\} [z^{2k}](1+z)^{n+k-p} \sum_{j=0}^{k-p} (-1)^j (1+z)^{-j} \binom{k-p}{j}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\} [z^{2k}](1+z)^{n+k-p} \left(1 - \frac{1}{1+z}\right)^{k-p}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\} [z^{2k}](1+z)^n z^{k-p} = \sum_{p=0}^{k} \left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\} \binom{n}{k+p}.$$

With this piece we get exactly the same reasoning as with the first one, namely it evaluates to $\left\{ n \atop n-k \right\}$. We write it as $\left\{\!\!\left\{ k+p \atop p \right\}\!\!\right\} \binom{n}{n-k-p}$ in choosing the number of singletons, of which there are $n-k-p$. The remaining $k+p$ elements are distributed into p disjoint sets of at least two elements for a total of $n-k$ sets. We once more have the condition that $k+p \ge 2p$ or $p \le k$. (We have classified by the number of singleton sets.)

This was math.stackexchange.com problem 4037946.

1.41 MSE 4055292

In trying to verify the identity

$$\sum_{k=0}^{2n} (-1)^k \binom{n+k}{k}^{-1} \binom{2n}{k} \binom{2k}{k} = 1$$

we see that

$$\binom{n+k}{k}^{-1} \binom{2n}{k} = \frac{(2n)!/(2n-k)!}{(n+k)!/n!} = \binom{3n}{n}^{-1} \binom{3n}{2n-k}$$

so that we seek to prove

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{2n-k} \binom{2k}{k} = \binom{3n}{n}.$$

The LHS is

$$\sum_{k=0}^{2n}(-1)^k\binom{3n}{k}\binom{4n-2k}{2n-k} = [z^{2n}](1+z)^{4n}\sum_{k=0}^{2n}(-1)^k\binom{3n}{k}\frac{z^k}{(1+z)^{2k}}$$

Here the coefficient extractor enforces the range of the sum and we find

$$[z^{2n}](1+z)^{4n}\sum_{k\ge 0}(-1)^k\binom{3n}{k}\frac{z^k}{(1+z)^{2k}} = [z^{2n}](1+z)^{4n}\left(1-\frac{z}{(1+z)^2}\right)^{3n}$$

$$= [z^{2n}]\frac{1}{(1+z)^{2n}}(1+z+z^2)^{3n}.$$

Expanding the second powered term

$$[z^{2n}]\frac{1}{(1+z)^{2n}}\sum_{q=0}^{3n}\binom{3n}{q}(1+z)^{3n-q}z^{2q}$$

The coefficient extractor sets the upper limit of the sum to n and we get (note that the powers of $1+z$ do not have a pole at zero hence the expansion about zero starts with z^{2q} and there is no contribution to $[z^{2n}]$ when $q>n$):

$$[z^{2n}]\sum_{q=0}^{n}\binom{3n}{q}(1+z)^{n-q}z^{2q} = \sum_{q=0}^{n}\binom{3n}{q}\binom{n-q}{2n-2q} = \binom{3n}{n}.$$

Observe that the the power $n-q$ to which $1+z$ is raised is a non-negative integer and hence we are justified in writing $[z^{2n}]z^{2q}(1+z)^{n-q} = [z^{2n-2q}](1+z)^{n-q} = \binom{n-q}{2n-2q}$. The only q in the range $0 \le q \le n$ where this binomial coefficient is not zero is $q=n$, producing a contribution of $\binom{3n}{n}$ and we have the claim.

This was math.stackexchange.com problem 4055292.

1.42 MSE 4054024

We seek to verify the identity

$$\sum_{k=1}^{n}\binom{2n-2k}{n-k}\frac{H_{2k}-2H_k}{2n-2k-1}\binom{2k}{k} = \frac{1}{n}\left[4^n - 3\binom{2n-1}{n}\right].$$

Preliminary. We get for the first piece in H_{2k} call it A that

$$\sum_{k=1}^{n}\binom{2n-2k}{n-k}\frac{1}{2n-2k-1}\binom{2k}{k}[z^{2k}]\frac{1}{1-z}\log\frac{1}{1-z}$$

$$= \sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{2k-1} \binom{2n-2k}{n-k} [z^{2n-2k}] \frac{1}{1-z} \log \frac{1}{1-z}$$

We may raise k to n because the function in z has no constant term:

$$[z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} \binom{2n-2k}{n-k} z^{2k}$$

Now the coefficient extractor enforces the upper limit of the sum and we get (in fact expansions start at z^{2k+1} which cancels $k = n$ already)

$$[z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \sum_{k \ge 0} \binom{2k}{k} \frac{1}{2k-1} \binom{2n-2k}{n-k} z^{2k}$$

$$= -[z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} [w^n] \sqrt{1-4wz^2} \frac{1}{\sqrt{1-4w}}.$$

The same method yields for the second piece in H_k call it B

$$-[z^n] \frac{1}{1-z} \log \frac{1}{1-z} [w^n] \sqrt{1-4wz} \frac{1}{\sqrt{1-4w}}.$$

First part. Continuing with piece B

$$[w^n] \sqrt{1 + \frac{4w(1-z)}{1-4w}} = -[w^n] \sum_{k \ge 0} \binom{2k}{k} \frac{1}{2k-1} (-1)^k \frac{w^k (1-z)^k}{(1-4w)^k}$$

$$= -\sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k [w^{n-k}] \frac{(1-z)^k}{(1-4w)^k}$$

$$= -4^n \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k (1-z)^k 4^{-k} \binom{n-1}{k-1}$$

and extracting the coefficient in $[z^n]$

$$4^n \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k 4^{-k} \binom{n-1}{k-1} [z^n] (1-z)^{k-1} \log \frac{1}{1-z}$$

$$= 4^n \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k 4^{-k} \binom{n-1}{k-1} \sum_{q=0}^{k-1} (-1)^q \binom{k-1}{q} \frac{1}{n-q}.$$

Now

$$\binom{n-1}{k-1} \binom{k-1}{q} = \frac{(n-1)!}{(n-k)! \times q! \times (k-1-q)!} = \binom{n-1}{q} \binom{n-1-q}{k-1-q}$$

Switching the order of the summation,

$$4^n \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{(-1)^q}{n-q} \sum_{k=q+1}^{n} \binom{n-1-q}{k-1-q} \binom{2k}{k} \frac{1}{2k-1} (-1)^k 4^{-k}$$
$$= \frac{4^n}{n} \sum_{q=0}^{n-1} \binom{n}{q} (-1)^q \sum_{k=q+1}^{n} \binom{n-1-q}{k-1-q} \binom{2k}{k} \frac{1}{2k-1} (-1)^k 4^{-k}$$
$$= -\frac{4^n}{n} \sum_{q=0}^{n-1} \binom{n}{q} (-1)^q \sum_{k=q+1}^{n} \binom{n-1-q}{k-1-q} [z^k]\sqrt{1+z}.$$

The inner sum is

$$\sum_{k=0}^{n-1-q} \binom{n-1-q}{k} [z^{k+q+1}]\sqrt{1+z} = \sum_{k=0}^{n-1-q} \binom{n-1-q}{k} [z^{n-k}]\sqrt{1+z}$$
$$= [z^n]\sqrt{1+z} \sum_{k=0}^{n-1-q} \binom{n-1-q}{k} z^k = [z^n]\sqrt{1+z}(1+z)^{n-1-q}.$$

Substitute into the outer sum to get

$$-\frac{4^n}{n}[z^n]\sqrt{1+z} \sum_{q=0}^{n-1} \binom{n}{q} (-1)^q (1+z)^{n-1-q} = -\frac{4^n}{n}[z^n]\frac{1}{\sqrt{1+z}}(-(-1)^n + z^n)$$
$$= -\frac{4^n}{n}\left(-4^{-n}\binom{2n}{n} + 1\right) = -\frac{4^n}{n} + \binom{2n}{n}\frac{1}{n}.$$

Second part. Here we may recycle the first segment from the easy piece B and obtain for piece A

$$4^n \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k 4^{-k} \binom{n-1}{k-1} [z^{2n}](1-z^2)^{k-1}(1+z) \log \frac{1}{1-z}.$$

The coefficient extractor in z has two parts, the first of which is

$$\sum_{q=0}^{k-1} (-1)^q \binom{k-1}{q} \frac{1}{2n-2q}$$

which contributes half the value of the piece B. The second is

$$\sum_{q=0}^{k-1} (-1)^q \binom{k-1}{q} \frac{1}{2n-1-2q}.$$

This yields

$$-4^n[z^n]\sqrt{1+z}\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(-1)^q}{2n-1-2q}(1+z)^{n-1-q}$$

$$= -4^n[z^n]\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(-1)^q}{2n-1-2q}(1+z)^{n-1/2-q}$$

$$= -4^n\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(-1)^q}{2n-1-2q}(n-1/2-q)^{\underline{n}}/n!.$$

We have for the falling factorial

$$\prod_{p=0}^{n-1}(n-1/2-q-p) = \frac{1}{2^n}\prod_{p=0}^{n-1}(2n-1-2q-2p)$$

$$= \frac{1}{2^n}\prod_{p=-(n-1)}^{0}(1-2q-2p) = \frac{(-1)^n}{2^n}\prod_{p=q-(n-1)}^{q}(2p-1)$$

$$= \frac{(-1)^{n+1}}{2^n}\frac{(2q-1)!}{2^{q-1}(q-1)!}\prod_{p=q-(n-1)}^{-1}(2p-1).$$

With $2q-2(n-1)-1 = 2q-2n+1$ this finally becomes

$$\frac{(-1)^q}{2^n}\frac{(2q-1)!}{2^{q-1}(q-1)!}\frac{(2n-1-2q)!}{2^{n-1-q}(n-1-q)!}$$

$$= \frac{(-1)^q}{2^{2n-1}}\frac{(2q)!}{q!}\frac{(2n-1-2q)!}{(n-1-q)!}.$$

This was for $1 \le q \le n-1$. We get for $q=0$

$$\frac{1}{2^n}\prod_{p=-(n-1)}^{0}(1-2p) = \frac{1}{2^n}\frac{(2n-1)!}{2^{n-1}(n-1)!}$$

and we see that the generic term in four factorials represents this case correctly as well.

Returning to the sum we obtain

$$-\frac{2}{n}\sum_{q=0}^{n-1}\binom{2q}{q}\binom{2n-2-2q}{n-1-q}$$

$$= -\frac{2}{n}[z^{n-1}]\frac{1}{\sqrt{1-4z}}\frac{1}{\sqrt{1-4z}} = -\frac{2}{n}[z^{n-1}]\frac{1}{1-4z} = -\frac{2}{n}4^{n-1} = -\frac{1}{2}\frac{4^n}{n}.$$

Conclusion. We now collect the three pieces with A first then B :

$$-\frac{1}{2}\frac{4^n}{n} - \frac{1}{2}\frac{4^n}{n} + \frac{1}{2}\frac{1}{n}\binom{2n}{n}$$

$$+2\frac{4^n}{n} - 2\frac{1}{n}\binom{2n}{n} = \frac{4^n}{n} - \frac{3}{2}\frac{1}{n}\binom{2n}{n} = \frac{4^n}{n} - 3\frac{1}{n}\binom{2n-1}{n-1}.$$

This is indeed

$$\boxed{\frac{1}{n}\left[4^n - 3\binom{2n-1}{n}\right].}$$

This was math.stackexchange.com problem 4054024.

1.43 MSE 4084763

We seek to evaluate

$$\sum_{q=0}^{n} \binom{n}{q} q^k,$$

k a positive integer. We get

$$k![z^k] \sum_{q=0}^{n} \binom{n}{q} \exp(qz) = k![z^k](\exp(z)+1)^n$$

$$= k![z^k] \sum_{q=0}^{n} \binom{n}{q} (\exp(z)-1)^q 2^{n-q} = \sum_{q=0}^{n} \binom{n}{q} q! \left\{ {k \atop q} \right\} 2^{n-q}$$

$$= \sum_{q=0}^{n} n^{\underline{q}} \left\{ {k \atop q} \right\} 2^{n-q}.$$

Now we may set the upper limit to k. If $n > k$ we may lower to k because the extra range $k < q \le n$ produces zero from the Stirling number. If $n < k$ we may raise to k because the extra range $n < q \le k$ produces zero from the falling factorial. We get

$$\boxed{\sum_{q=1}^{k} n^{\underline{q}} \left\{ {k \atop q} \right\} 2^{n-q}.}$$

In this way we obtain e.g. for $k = 4$

$$2^{n-1} n^{\underline{1}} \left\{ {4 \atop 1} \right\} + 2^{n-2} n^{\underline{2}} \left\{ {4 \atop 2} \right\} + 2^{n-3} n^{\underline{3}} \left\{ {4 \atop 3} \right\} + 2^{n-4} n^{\underline{4}} \left\{ {4 \atop 4} \right\}.$$

Now the Stirling numbers can be evaluated by inspection:

$$2^{n-1}n^{\underline{1}} \times 1 + 2^{n-2}n^{\underline{2}} \times \left(\frac{1}{2}\binom{4}{2} + \binom{4}{1}\right) + 2^{n-3}n^{\underline{3}} \times \binom{4}{2} + 2^{n-4}n^{\underline{4}} \times 1.$$

We find at last

$$2^{n-1}n^{\underline{1}} + 7 \times 2^{n-2}n^{\underline{2}} + 6 \times 2^{n-3}n^{\underline{3}} + 2^{n-4}n^{\underline{4}}.$$

We may expand the falling factorial if desired:

$$2^{n-1} \times n + 7 \times 2^{n-2} \times n(n-1)$$
$$+6 \times 2^{n-3} \times n(n-1)(n-2) + 2^{n-4} \times n(n-1)(n-2)(n-3).$$

This was math.stackexchange.com problem 4084763.

1.44 MSE 4095795

We seek to evaluate

$$\sum_{r=0}^{n} r^k.$$

We may also express this in terms of Stirling numbers of the second kind and falling factorials. We start with

$$\sum_{r=0}^{n} r^k = k![z^k] \sum_{r=0}^{n} \exp(rz) = k![z^k] \frac{\exp((n+1)z) - 1}{\exp(z) - 1}$$
$$= k![z^k] \frac{1}{\exp(z) - 1} \sum_{q=1}^{n+1} \binom{n+1}{q} (\exp(z) - 1)^q$$
$$= k![z^k] \sum_{q=1}^{n+1} \binom{n+1}{q} (\exp(z) - 1)^{q-1}$$
$$= k![z^k] \sum_{q=1}^{n+1} (n+1)^{\underline{q}} \frac{1}{q} \frac{(\exp(z) - 1)^{q-1}}{(q-1)!}$$
$$= \sum_{q=1}^{n+1} (n+1)^{\underline{q}} \frac{1}{q} \left\{ {k \atop q-1} \right\}.$$

Note that we may set the upper limit of the sum to $k+1$. If $n+1 > k+1$ we may lower to $k+1$ because the removed terms from the range $k+2 \le q \le n+1$ produce zero by the Stirling number. If $k+1 > n+1$ we may raise to $k+1$ because the extra terms from the range $n+2 \le q \le k+1$ produce zero through the falling factorial.

We get

$$\sum_{q=2}^{k+1} (n+1)^{\underline{q}} \frac{1}{q} \left\{ {k \atop q-1} \right\} = (n+1) \sum_{q=2}^{k+1} n^{\underline{q-1}} \frac{1}{q} \left\{ {k \atop q-1} \right\}$$

or alternatively

$$\boxed{\sum_{r=0}^{n} r^k = (n+1) \sum_{q=1}^{k} n^{\underline{q}} \frac{1}{q+1} \left\{ {k \atop q} \right\}.}$$

In this way we get e.g. with $k = 4$

$$(n+1) \times \left[n^{\underline{1}} \frac{1}{2} \left\{ {4 \atop 1} \right\} + n^{\underline{2}} \frac{1}{3} \left\{ {4 \atop 2} \right\} + n^{\underline{3}} \frac{1}{4} \left\{ {4 \atop 3} \right\} + n^{\underline{4}} \frac{1}{5} \left\{ {4 \atop 4} \right\} \right]$$

The Stirling numbers may be evaluated by inspectiona as before and we find

$$\sum_{r=0}^{n} r^4 = (n+1) \times \left[\frac{1}{2} n^{\underline{1}} + \frac{7}{3} n^{\underline{2}} + \frac{3}{2} n^{\underline{3}} + \frac{1}{5} n^{\underline{4}} \right].$$

This was math.stackexchange.com problem 4095795.

1.45 MSE 4098492

We seek to verify that

$$\sum_{k=0}^{n} \binom{k}{m} \binom{n-k}{r-m} = \binom{n+1}{r+1}$$

where $n \geq 0$ and $0 \leq m \leq n$ and $m \leq r \leq n$.
We get for the LHS

$$\begin{aligned} &[z^m][w^{r-m}] \sum_{k=0}^{n} (1+z)^k (1+w)^{n-k} \\ &= [z^m][w^{r-m}](1+w)^n \sum_{k=0}^{n} (1+z)^k (1+w)^{-k} \\ &= [z^m][w^{r-m}](1+w)^n \frac{(1+z)^{n+1}/(1+w)^{n+1} - 1}{(1+z)/(1+w) - 1} \\ &= [z^m][w^{r-m}](1+w)^n \frac{(1+z)^{n+1}/(1+w)^n - (1+w)}{z-w} \\ &= [z^m][w^{r-m}] \frac{(1+z)^{n+1} - (1+w)^{n+1}}{z-w}. \end{aligned}$$

Now we have

$$\frac{(1+z)^{n+1}-(1+w)^{n+1}}{z-w} = \sum_{q=0}^{n} \binom{n+1}{q+1} \sum_{p=0}^{q} z^p w^{q-p}.$$

This is because the RHS is

$$\sum_{q=0}^{n} \binom{n+1}{q+1} w^q \sum_{p=0}^{q} z^p/w^p = \sum_{q=0}^{n} \binom{n+1}{q+1} w^q \frac{z^{q+1}/w^{q+1}-1}{z/w-1}$$
$$= \sum_{q=0}^{n} \binom{n+1}{q+1} w^q \frac{z^{q+1}/w^q - w}{z-w} = \sum_{q=0}^{n} \binom{n+1}{q+1} \frac{z^{q+1}-w^{q+1}}{z-w}$$
$$= \frac{1}{z-w}\left[\sum_{q=0}^{n} \binom{n+1}{q+1} z^{q+1} - \sum_{q=0}^{n} \binom{n+1}{q+1} w^{q+1}\right]$$
$$= \frac{1}{z-w}\left[\sum_{q=-1}^{n} \binom{n+1}{q+1} z^{q+1} - \sum_{q=-1}^{n} \binom{n+1}{q+1} w^{q+1}\right]$$
$$= \frac{1}{z-w}\left[(1+z)^{n+1}-(1+w)^{n+1}\right].$$

Returning to the main sum we now see that it is given by

$$[z^m][w^{r-m}] \sum_{q=0}^{n} \binom{n+1}{q+1} \sum_{p=0}^{q} z^p w^{q-p} = [z^m][w^{r-m}] \sum_{p=0}^{n} \frac{z^p}{w^p} \sum_{q=p}^{n} \binom{n+1}{q+1} w^q.$$

With $m \le n$ we obtain

$$[w^{r-m}] \frac{1}{w^m} \sum_{q=m}^{n} \binom{n+1}{q+1} w^q = [w^r] \sum_{q=m}^{n} \binom{n+1}{q+1} w^q.$$

With $m \le r \le n$ this becomes at last

$$\boxed{\binom{n+1}{r+1}.}$$

The concluding step also follows by inspection seeing that $p=m$ and $q=r$ are the only combination $z^p w^{q-p}$ that can possibly contribute to $[z^m][w^{r-m}]$.

This was math.stackexchange.com problem 4098492.

1.46 MSE 4127695

In seeking to evaluate

$$S_n = \sum_{r=0}^{n} 2^{n-r} \binom{n+r}{r}$$

we find that it is

$$[z^n] \frac{1}{1-2z} \frac{1}{(1-z)^{n+1}} = \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{1-2z} \frac{1}{(1-z)^{n+1}}.$$

We will use the fact that residues sum to zero, which requires the residue at $z = 1/2$ and the residue at $z = 1$ as well as the residue at infinity. The latter is zero by inspection, however . We get for the residue at $z = 1/2$

$$-\frac{1}{2} \mathrm{Res}_{z=1/2} \frac{1}{z^{n+1}} \frac{1}{z-1/2} \frac{1}{(1-z)^{n+1}}$$

We obtain

$$-\frac{1}{2} 2^{n+1} 2^{n+1} = -2 \times 4^n.$$

We also have for the residue at $z = 1$

$$\mathrm{Res}_{z=1} \frac{1}{z^{n+1}} \frac{1}{1-2z} \frac{1}{(1-z)^{n+1}}$$

$$= \mathrm{Res}_{z=1} \frac{1}{(1+(z-1))^{n+1}} \frac{1}{-1-2(z-1)} \frac{1}{(1-z)^{n+1}}$$

$$= (-1)^n \mathrm{Res}_{z=1} \frac{1}{(1+(z-1))^{n+1}} \frac{1}{1+2(z-1)} \frac{1}{(z-1)^{n+1}}.$$

This is

$$(-1)^n \sum_{r=0}^{n} (-1)^r \binom{n+r}{r} (-1)^{n-r} 2^{n-r} = \sum_{r=0}^{n} 2^{n-r} \binom{n+r}{r} = S_n.$$

We have shown that $S_n - 2 \times 4^n + S_n = 0$ or

$$\boxed{S_n = 4^n.}$$

For the residue at infinity we get

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n+1} \frac{1}{1-2/z} \frac{1}{(1-1/z)^{n+1}} = -\mathrm{Res}_{z=0} z^n \frac{1}{z-2} \frac{z^{n+1}}{(z-1)^{n+1}}$$

$$= -\mathrm{Res}_{z=0} z^{2n+1} \frac{1}{z-2} \frac{1}{(z-1)^{n+1}} = 0.$$

This was math.stackexchange.com problem 4127695.

Additional answers appeared at math.stackexchange.com problem 1874816.

1.47 MSE 4131219

Defining the Kravchuck polynomial as (the definition in its full generality is at Wikipedia)

$$\mathcal{K}_k(x;n) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j}$$

we seek to show that

$$\sum_{\ell=0}^{n} \binom{n-\ell}{n-m} \mathcal{K}_\ell(x;n) = 2^m \times \binom{n-x}{m}.$$

We prove this for $x = p$ an integer and then it holds for all x because $\mathcal{K}_k(x;n)$ is a polynomial in x.

We have

$$\mathcal{K}_k(p;n) = [z^k](1+z)^{n-p} \sum_{j=0}^{k} (-1)^j \binom{p}{j} z^j.$$

Here the coefficient extractor enforces the upper limit of the sum and we get

$$\mathcal{K}_k(p;n) = [z^k](1+z)^{n-p} \sum_{j\ge 0} (-1)^j \binom{p}{j} z^j = [z^k](1+z)^{n-p}(1-z)^p.$$

We also get for the coveted identity that it is

$$\begin{aligned}\sum_{\ell=0}^{n} \binom{\ell}{n-m} \mathcal{K}_{n-\ell}(p;n) &= \sum_{\ell=0}^{n} \binom{\ell}{n-m} [z^{n-\ell}](1+z)^{n-p}(1-z)^p \\ &= [z^n](1+z)^{n-p}(1-z)^p \sum_{\ell=0}^{n} \binom{\ell}{n-m} z^\ell \\ &= [z^n](1+z)^{n-p}(1-z)^p \sum_{\ell=n-m}^{n} \binom{\ell}{n-m} z^\ell \\ &= [z^m](1+z)^{n-p}(1-z)^p \sum_{\ell=0}^{m} \binom{\ell+n-m}{n-m} z^\ell.\end{aligned}$$

Now here we have another coefficient extractor enforcing the upper range of the sum and we get

$$[z^m](1+z)^{n-p}(1-z)^p \sum_{\ell\ge 0} \binom{\ell+n-m}{n-m} z^\ell$$

$$= [z^m](1+z)^{n-p}(1-z)^p \frac{1}{(1-z)^{n-m+1}}.$$

This is

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{m+1}}(1+z)^{n-p}(1-z)^{m-1}\frac{1}{(1-z)^{n-p}}\; dz.$$

Now put $(1+z)/(1-z)=w$ so that $z=(w-1)/(1+w)$ and $dz = 2/(1+w)^2\; dw$ to obtain (observe that due to the fact that $w = 1+2z+\cdots$ the image of a small circle $|z|=\varepsilon$ can be deformed to another small circle $|w-1|=\gamma$ because when z makes one turn around zero so does w around one)

$$\frac{1}{2\pi i}\int_{|w-1|=\gamma} \frac{(1+w)^{m+1}}{(w-1)^{m+1}} w^{n-p} \frac{2^{m-1}}{(1+w)^{m-1}} \frac{2}{(1+w)^2}\; dw$$

$$= \frac{2^m}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{m+1}} w^{n-p}\; dw$$

$$= \frac{2^m}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{m+1}} \sum_{r\ge 0}\binom{n-p}{r}(w-1)^r\; dw.$$

There were no poles other than $w=1$ inside the image contour and the series in $w-1$ converges including for $n-p<0$ because $\gamma \ll 1$.

This yields

$$\boxed{2^m \times \binom{n-p}{m}}$$

as claimed.

This was math.stackexchange.com problem 4131219.

1.48 MSE 4139722

We seek to prove the identity

$$B_n = \sum_{k=0}^n (-1)^k \frac{1}{k+1} H_{k+1} (k+2)! \left\{ {n+1 \atop k+1} \right\} + (-1)^{n+1}(n+1).$$

The sum is

$$(n+1)![z^{n+1}] \sum_{k=0}^n (-1)^k \left(1+\frac{1}{k+1}\right) H_{k+1} (\exp(z)-1)^{k+1}.$$

With $\exp(z) - 1 = z + \cdots$ the coefficient extractor enforces the upper limit of the sum and we get

$$(n+1)![z^{n+1}] \sum_{k\ge 0} (-1)^k \left(1 + \frac{1}{k+1}\right) (\exp(z)-1)^{k+1} [w^{k+1}] \frac{1}{1-w} \log \frac{1}{1-w}.$$

Note that the term in w starts at w. We get for the first piece

$$\begin{aligned} & -(n+1)![z^{n+1}] \exp(-z) \log \exp(-z) \\ & = (n+1)![z^n] \exp(-z) = (-1)^n (n+1). \end{aligned}$$

We see that this cancels the extra term from the initial closed form. Therefore the remaining term must give the Bernoulli numbers:

$$(n+1)![z^{n+1}] \sum_{k\ge 0} (-1)^k \frac{1}{k+1} (\exp(z)-1)^{k+1} [w^{k+1}] \frac{1}{1-w} \log \frac{1}{1-w}.$$

Differentiate to get

$$\begin{aligned} & n![z^n] \exp(z) \sum_{k\ge 0} (-1)^k (\exp(z)-1)^k [w^{k+1}] \frac{1}{1-w} \log \frac{1}{1-w} \\ & = n![z^n] \exp(z) \sum_{k\ge 0} (-1)^k (\exp(z)-1)^k [w^k] \frac{1}{w} \frac{1}{1-w} \log \frac{1}{1-w}. \end{aligned}$$

This is

$$\begin{aligned} & n![z^n] \exp(z) \frac{1}{1-\exp(z)} \exp(-z) \log \exp(-z) \\ & = n![z^n] \frac{z}{\exp(z)-1} = B_n \end{aligned}$$

as claimed.

This was math.stackexchange.com problem 4139722.

1.49 MSE 4192271

We seek to show that

$$\sum_{q\ge k} \binom{m+1}{2q+1} \binom{q}{k} = \binom{m-k}{k} 2^{m-2k}.$$

For the initial analysis note that the first binomial coefficient requires $m \ge 2q$ so that when $k > m/2$ which would imply $2q > m$ the LHS evaluates to zero, even though the RHS is nonzero when $k > m$. We will therefore restrict to $k \le m/2$. We get for the LHS

$$\sum_{q\ge 0}\binom{m+1}{2q+2k+1}\binom{q+k}{k} = \sum_{q\ge 0}\binom{m+1}{m-2k-2q}\binom{q+k}{k}$$
$$= [z^{m-2k}](1+z)^{m+1}\sum_{q\ge 0} z^{2q}\binom{q+k}{k}.$$

Observe that this coefficient extractor produces a finite sum with no contribution from $2q > m-2k$. Continuing,

$$[z^{m-2k}](1+z)^{m+1}\frac{1}{(1-z^2)^{k+1}} = [z^{m-2k}](1+z)^{m-k}\frac{1}{(1-z)^{k+1}}.$$

This is

$$\operatorname*{res}_z \frac{1}{z^{m-2k+1}}(1+z)^{m-k}\frac{1}{(1-z)^{k+1}} = \operatorname*{res}_z \frac{z^{k-1}}{z^{m-k}}(1+z)^{m-k}\frac{1}{(1-z)^{k+1}}$$

See how the residue vanishes when $2k > m$. Now put $z/(1+z) = w$ so that $z = w/(1-w)$ and $1/(1-z) = (1-w)/(1-2w)$ and $dz = 1/(1-w)^2\; dw$ to obtain

$$\operatorname*{res}_w \frac{1}{w^{m-k}}\frac{w^{k-1}}{(1-w)^{k-1}}\frac{(1-w)^{k+1}}{(1-2w)^{k+1}}\frac{1}{(1-w)^2}$$
$$= \operatorname*{res}_w \frac{1}{w^{m-2k+1}}\frac{1}{(1-2w)^{k+1}}.$$

This is

$$\boxed{2^{m-2k}\binom{m-k}{k}}$$

as claimed.

This was math.stackexchange.com problem 4192271.

1.50 MSE 4212878

We are interested in the asymptotics of

$$g(n) = \sum_{k=1}^{n-1} k\binom{n}{k}\frac{(2n-2k-1)!!}{(2n-1)!!} = n\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(2n-2k-1)!!}{(2n-1)!!}.$$

Now we have

$$(2n-1)!! = \frac{(2n-1)!}{2^{n-1} \times (n-1)!}$$

so we get for our sum

$$\frac{n! \times 2^{n-1}}{(2n-1)!} \sum_{k=1}^{n-1} \binom{n-1}{k-1} \frac{(2n-2k-1)!}{2^{n-k-1} \times (n-k-1)!}$$

$$= \frac{n! \times 2^{n-1}}{(2n-1)!} \sum_{k=0}^{n-2} \binom{n-1}{k} \frac{(2n-2k-3)!}{2^{n-k-2} \times (n-k-2)!}$$

$$= \frac{n! \times 2^{n-1}}{(2n-1)!} \sum_{k=0}^{n-2} \binom{n-1}{n-2-k} \frac{(2k+1)!}{2^k \times k!}$$

$$= 2^{n-1} \binom{2n-1}{n}^{-1} \sum_{k=0}^{n-2} \frac{1}{(n-2-k)!} \frac{1}{2^k} \binom{2k+1}{k}$$

$$= 2^{n-1} \binom{2n-1}{n}^{-1} [z^{n-2}] \exp(z) \sum_{k=0}^{n-2} z^k \frac{1}{2^k} \binom{2k+1}{k}.$$

Here the coefficient extractor enforces the upper limit of the sum and we obtain

$$2^{2n-2} \binom{2n}{n}^{-1} [z^{n-2}] \exp(z/2) \sum_{k \ge 0} z^k \frac{1}{2^{2k}} \binom{2k+1}{k}$$

The sum is

$$-\frac{2}{z} + \frac{2}{z} \frac{1}{\sqrt{1-z}}.$$

We get from the first piece

$$-2^{2n-1} \binom{2n}{n}^{-1} [z^{n-1}] \exp(z/2) = -2^n \binom{2n}{n}^{-1} \frac{1}{(n-1)!}$$

Now from the asymptotic $\binom{2n}{n}^{-1} \sim \sqrt{\pi n}/2^{2n}$ we get for the modulus $\sqrt{\pi n}/2^n/(n-1)!$ so this vanishes quite rapidly. Continuing with the second piece we obtain

$$2^{2n-1} \binom{2n}{n}^{-1} [z^{n-1}] \frac{\exp(z/2)}{\sqrt{1-z}}.$$

We apply the Darboux method here as documented on page 180 section 5.3 of Wilf's *generatingfunctionology* [Wil94] where we expand $\exp(z/2)$ about 1 and take the first term, extracting the corresponding factor from the singular term. This yields

$$\exp(1/2) \times 2^{2n-1} \binom{2n}{n}^{-1} [z^{n-1}] \frac{1}{\sqrt{1-z}}$$
$$= \exp(1/2) \times 2^{2n-1} \binom{2n}{n}^{-1} \binom{n-3/2}{n-1}$$
$$= \exp(1/2) \times 2^{2n-1} \binom{2n}{n}^{-1} \frac{n}{n-1/2} \binom{n-1/2}{n}.$$

Using the Gamma function approximation of the second binomial coefficient from the Wilf text we get

$$\exp(1/2) \times 2^{2n-1} \binom{2n}{n}^{-1} \frac{n}{n-1/2} \frac{1}{\sqrt{n} \times \Gamma(1/2)}$$
$$\sim \exp(1/2) \times 2^{2n-1} \times \frac{\sqrt{\pi n}}{2^{2n}} \frac{n}{n-1/2} \frac{1}{\sqrt{\pi n}} \sim \frac{1}{2} \exp(1/2)$$

We have obtained

$$\frac{\sqrt{e}}{2}$$

the same as in the contributions that were first to appear.

This was math.stackexchange.com problem 4212878.

1.51 A different obstacle

We seek to evaluate for $n, m \ge 0$ the sum

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1+m} \binom{n+k}{2k} \binom{2k}{k}.$$

First note that

$$\binom{n+k}{2k} \binom{2k}{k} = \frac{(n+k)!}{(n-k)! \times k! \times k!} = \binom{n+k}{k} \binom{n}{k}$$

We get for our sum

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1+m} \binom{n+k}{n}.$$

Now introduce

$$f(z) = \frac{(-1)^n}{z+1+m} \prod_{q=0}^{n} \frac{1}{z-q} \prod_{p=0}^{n-1} (n+z-p).$$

Note that $n+z-p \neq 0$ for $z=q$ with $0 \le q \le n$ so the simple poles from the first product are preserved.

This function $f(z)$ has the property that with $0 \le k \le n$

$$\begin{aligned}\mathrm{Res}_{z=k}\; f(z) &= \frac{(-1)^n}{k+1+m} \prod_{q=0}^{k-1} \frac{1}{k-q} \prod_{q=k+1}^{n} \frac{1}{k-q} \prod_{p=0}^{n-1} (n+k-p) \\ &= \binom{n+k}{n} \frac{n! \times (-1)^n}{k+1+m} \frac{1}{k!} \frac{(-1)^{n-k}}{(n-k)!}.\end{aligned}$$

Upon simplifying we find that our sum is given by

$$\sum_{k=0}^{n} \mathrm{Res}_{z=k}\; f(z).$$

Now using the fact that residues sum to zero and that the residue at infinity of $f(z)$ is zero by inspection (compare degree of denominator and numerator which are $n+2$ and n resp.) we have that the sum must be

$$-\mathrm{Res}_{z=-m-1}\; f(z).$$

Compute this to get

$$\begin{aligned}-(-1)^n \prod_{q=0}^{n} \frac{1}{-1-m-q} \prod_{p=0}^{n-1} (n-1-m-p) \\ = \prod_{q=0}^{n} \frac{1}{q+m+1} \prod_{p=0}^{n-1} (p-m).\end{aligned}$$

Here we get a zero value when $0 \le m \le n-1$ or $n > m$. Otherwise the terms in the second product are all negative and we get

$$\begin{aligned}(-1)^n \frac{m!}{(m+n+1)!} \prod_{p=0}^{n-1} (m-p) &= (-1)^n \frac{m!}{(m+n+1)!} \frac{m!}{(m-n)!} \\ &= (-1)^n \frac{m! \times n!}{(m+n+1)!} \binom{m}{n}.\end{aligned}$$

Here the last binomial coefficient produces zero when $n > m$ as required.

This is a simplified version of an earlier answer prompted by an observation by Markus Scheuer at math.stackexchange.com problem 4504576.

This problem has not yet appeared at math.stackexchange.com. The source is problem 8 “A different obstacle” from section 5.2 of *Concrete Mathematics* by Graham, Knuth and Patashnik [GKP89].

1.52 MO 291738

We seek a closed form of

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n+k}{n-k} \binom{2k}{k}.$$

This follows the template from the previous section very closely with only the type of the auxiliary residue being different. First note that

$$\binom{n+k}{n-k}\binom{2k}{k} = \frac{(n+k)!}{(n-k)! \times k! \times k!} = \binom{n+k}{k}\binom{n}{k}$$

We get for our sum

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1} \binom{n+k}{n}.$$

Now introduce

$$f(z) = \frac{(-1)^n}{2z+1} \prod_{q=0}^{n} \frac{1}{z-q} \prod_{p=0}^{n-1} (n+z-p).$$

Note that $n+z-p \neq 0$ for $z=q$ with $0 \leq q \leq n$ so the simple poles from the first product are preserved.

This function $f(z)$ has the property that with $0 \leq k \leq n$

$$\begin{aligned} \mathrm{Res}_{z=k}\; f(z) &= \frac{(-1)^n}{2k+1} \prod_{q=0}^{k-1} \frac{1}{k-q} \prod_{q=k+1}^{n} \frac{1}{k-q} \prod_{p=0}^{n-1} (n+k-p) \\ &= \binom{n+k}{n} \frac{n! \times (-1)^n}{2k+1} \frac{1}{k!} \frac{(-1)^{n-k}}{(n-k)!}. \end{aligned}$$

Upon simplifying we find that our sum is given by

$$\sum_{k=0}^{n} \mathrm{Res}_{z=k}\; f(z).$$

Now using the fact that residues sum to zero and that the residue at infinity of $f(z)$ is zero by inspection (compare degree of denominator and numerator which are $n+2$ and n resp.) we have that the sum must be

$$-\mathrm{Res}_{z=-1/2}\; f(z).$$

Compute this to get

$$-\frac{(-1)^n}{2} \prod_{q=0}^{n} \frac{1}{-1/2-q} \prod_{p=0}^{n-1} (n-1/2-p)$$

$$= \frac{1}{2}\prod_{q=0}^{n}\frac{1}{1/2+q}\prod_{p=0}^{n-1}(n-1/2-p)$$

$$= \prod_{q=0}^{n}\frac{1}{1+2q}\prod_{p=0}^{n-1}(2n-1-2p) = \frac{1}{2n+1}.$$

This is a simplified version of an earlier answer prompted by an observation by Markus Scheuer at math.stackexchange.com problem 4504576.

This was mathoverflow.net problem 291738.

1.53 Stirling number identity by Gould

We seek to show that

$$\left[{n \atop n-k} \right] = (-1)^k \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} \sum_{q=0}^{j} (-1)^q \binom{j+1}{q+1} \binom{j+qn+q}{qn+q}^{-1} \left\{ {j+qn+q \atop qn+q} \right\}.$$

Using the standard EGF on the RHS we find for the inner sum

$$j! \sum_{q=0}^{j} (-1)^q \binom{j+1}{q+1} [z^{j+qn+q}](\exp(z)-1)^{qn+q}$$

$$= j! \sum_{q=0}^{j} (-1)^{j-q} \binom{j+1}{q} [z^{j+jn+j-qn-q}](\exp(z)-1)^{jn+j-qn-q}$$

$$= j![z^{j(n+2)}](\exp(z)-1)^{jn+j} \sum_{q=0}^{j} (-1)^{j-q} \binom{j+1}{q} z^{qn+q} (\exp(z)-1)^{-qn-q}$$

$$= -j![z^{j(n+2)}](\exp(z)-1)^{jn+j} \sum_{q=0}^{j} (-1)^{j+1-q} \binom{j+1}{q} \left(\frac{z}{\exp(z)-1}\right)^{qn+q}.$$

Observe that when we raise q to $j+1$ we obtain for the sum

$$-j![z^{j(n+2)}](\exp(z)-1)^{j(n+1)} \left[\left(\frac{z}{\exp(z)-1}\right)^{n+1} - 1 \right]^{j+1}$$

but note that $(\exp(z)-1)^{j(n+1)} = z^{j(n+1)} + \cdots$ and

$$\left[\left(\frac{z}{\exp(z)-1}\right)^{n+1} - 1 \right]^{j+1} = (-1)^{j+1}((n+1)/2)^{j+1} z^{j+1} + \cdots$$

We have however that $[z^{j(n+2)}]((-1)^{j+1}((n+1)/2)^{j+1} z^{j(n+2)+1} + \cdots) = 0$. Hence the sum is minus the value at $q = j+1$ and we get

$$
\begin{aligned}
&j![z^{j(n+2)}](\exp(z)-1)^{j(n+1)}\left(\frac{z}{\exp(z)-1}\right)^{(j+1)(n+1)}\\
&= j![z^j]\frac{(\exp(z)-1)^{j(n+1)}}{z^{j(n+1)}}\left(\frac{z}{\exp(z)-1}\right)^{(j+1)(n+1)}\\
&= j![z^j]\left(\frac{z}{\exp(z)-1}\right)^{n+1}.
\end{aligned}
$$

We obtain for the outer sum in j

$$
k!\sum_{j=0}^{k}\frac{1}{(k-j)!}[z^j]\left(\frac{z}{\exp(z)-1}\right)^{n+1}
$$

$$
= k![z^k]\left(\frac{z}{\exp(z)-1}\right)^{n+1}\sum_{j=0}^{k}\frac{1}{j!}z^j.
$$

We may raise the upper limit beyond k because there is no contribution to the coeffcient extractor in front and find

$$
k![z^k]\exp(z)\left(\frac{z}{\exp(z)-1}\right)^{n+1}.
$$

This is

$$
\frac{k!}{2\pi i}\int_{|z|=\varepsilon}\exp(z)\frac{z^{n-k}}{(\exp(z)-1)^{n+1}}\;dz.
$$

Note that with the arithmetic we have preserved the pole at $z=0$. Now put $\exp(z)-1=w$ so that $\exp(z)\;dz=dw$ and $z=\log(1+w)$. (Branch cut of the logarithm is $(-\infty,-1]$.) This yields

$$
\frac{k!}{2\pi i}\int_{|w|=\gamma}\frac{(\log(1+w))^{n-k}}{w^{n+1}}\;dw.
$$

Putting it all together we have

$$
(-1)^k\binom{n}{k}k![w^n](\log(1+w))^{n-k} = (-1)^{n-k}\binom{n}{k}k![w^n](\log(1-w))^{n-k}
$$

$$
= \binom{n}{k}k![w^n]\left(\log\frac{1}{1-w}\right)^{n-k} = n![w^n]\frac{1}{(n-k)!}\left(\log\frac{1}{1-w}\right)^{n-k} = \left[\begin{matrix} n \\ n-k \end{matrix}\right]
$$

as claimed. Concerning the choice for ε and γ we have for the image of $|z|=\varepsilon$ using $|\exp(\varepsilon\exp(i\theta))| = \exp(\varepsilon\cos(\theta))$ that $1-\exp(-\varepsilon)\le|\exp(z)-1|\le\exp(\varepsilon)-1$. The image is contained in two circles of radius $\varepsilon-\frac{1}{2}\varepsilon^2$ and $\varepsilon/(1-\varepsilon)$ and we may take $\gamma=\varepsilon-\frac{1}{2}\varepsilon^2$.

This problem has not appeared at math.stackexchange.com. It is from page 179 eqn. 13.10 of H.W.Gould's *Combinatorial Identities for Stirling Numbers* [Gou16].

1.54 Stirling number identity by Gould II

The claim here is

$$\left[{n \atop n-k} \right] = (-1)^k \binom{n-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \binom{jn+k}{k}^{-1} \left\{ {jn+k \atop jn} \right\}.$$

Using the standard EGF this becomes

$$(-1)^k \binom{n-1}{k} k! \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} [z^{jn+k}](\exp(z)-1)^{jn}$$

$$= \binom{n-1}{k} k! [z^{k(n+1)}] \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} z^{jn} (\exp(z)-1)^{kn-jn}.$$

Raising the index to $k+1$ we obtain for the sum

$$(\exp(z)-1)^{kn} \left[1 - \left(\frac{z}{\exp(z)-1} \right)^n \right]^{k+1}.$$

Note that $(\exp(z)-1)^{kn} = z^{kn} + \cdots$ and

$$\left[1 - \left(\frac{z}{\exp(z)-1} \right)^n \right]^{k+1} = (n/2)^{k+1} z^{k+1} + \cdots$$

We have however that $[z^{k(n+1)}]((n/2)^{k+1} z^{k(n+1)+1} + \cdots) = 0$. Hence the sum is minus the value at $j = k+1$ and we get

$$\binom{n-1}{k} k! [z^{k(n+1)}] (-1)^k z^{(k+1)n} \frac{1}{(\exp(z)-1)^n}$$

$$= (-1)^k \binom{n-1}{k} k! [z^k] \left(\frac{z}{\exp(z)-1} \right)^n$$

$$= \binom{n-1}{k} k! [z^k] \left(\frac{z}{1-\exp(-z)} \right)^n$$

$$= \binom{n-1}{k} k! [z^k] \left(\frac{z\exp(z)}{\exp(z)-1} \right)^n.$$

Here we recognize the generating function of the Stirling polynomials (consult e.g. *Concrete Mathematics* [GKP89] section 6.2) and we obtain at last

$$\binom{n-1}{k} k! \times n\sigma_k(n) = \frac{n!}{(n-1-k)!} \left[n \atop n-k \right] \prod_{q=0}^{k} \frac{1}{n-q} = \left[n \atop n-k \right]$$

as claimed.

This problem has not appeared at math.stackexchange.com. It is from page 183 eqn. 13.28 of H.W.Gould's *Combinatorial Identities for Stirling Numbers* [Gou16].

1.55 Schläfli's identity for Stirling numbers

Gould [Gou16] presents the following version of Schläfli's formula linking the two kinds of Stirling numbers: where $n \ge 1$ and $n > k$ (the first binomial coefficient vanishes when $n = k$)

$$\left[n \atop n-k \right] = \sum_{q=0}^{k} (-1)^{k-q} \binom{n+q-1}{n-k-1} \binom{n+k}{k-q} \left\{ k+q \atop q \right\}.$$

The RHS is

$$\sum_{q=0}^{k} (-1)^{q} \binom{n+k-q-1}{n-k-1} \binom{n+k}{q} \left\{ 2k-q \atop k-q \right\}.$$

Using the standard EGF this becomes

$$\frac{(n-1)!}{(n-k-1)!} \sum_{q=0}^{k} (-1)^{q} \binom{n+k-q-1}{k-q} \binom{n+k}{q} [z^{2k-q}](\exp(z)-1)^{k-q}$$

$$= \frac{(n-1)!}{(n-k-1)!} [z^{2k}](\exp(z)-1)^{k} [w^k](1+w)^{n+k-1}$$
$$\times \sum_{q\ge 0} (-1)^{q} \frac{w^q}{(1+w)^q} \binom{n+k}{q} z^q (\exp(z)-1)^{-q}.$$

Here we have extended q to infinity because of the coefficient extractor in w. Continuing,

$$\frac{(n-1)!}{(n-k-1)!} [z^{2k}](\exp(z)-1)^{k} [w^k](1+w)^{n+k-1}$$
$$\times \left[1 - \frac{wz}{(1+w)(\exp(z)-1)} \right]^{n+k}$$
$$= \frac{(n-1)!}{(n-k-1)!} [z^{2k}](\exp(z)-1)^{-n} [w^k] \frac{1}{1+w}$$

$$\times \left[(1+w)(\exp(z)-1) - wz\right]^{n+k}.$$

Now we have for the inner powered term

$$[w(\exp(z)-1-z) + (\exp(z)-1)]^{n+k}$$
$$= \sum_{q=0}^{n+k} \binom{n+k}{q} w^q (\exp(z)-1-z)^q (\exp(z)-1)^{n+k-q}.$$

Extracting the coefficient on $[w^k]$ (note the upper range)

$$\frac{(n-1)!}{(n-k-1)!}[z^{2k}](\exp(z)-1)^{-n}$$
$$\times \sum_{q=0}^{k} \binom{n+k}{q} (-1)^{k-q} (\exp(z)-1-z)^q (\exp(z)-1)^{n+k-q}.$$

Observe that $(\exp(z)-1-z)^q = z^{2q}/2^{2q} + \cdots$ so that the sum terms start at z to the power $2q+n+k-q-n = k+q$ so we may raise q to $n+k$ once more due to the extractor in z (the outer exponential has a pole of order n which gets canceled however, yielding a FPS). We get

$$\frac{(n-1)!}{(n-k-1)!}[z^{2k}](\exp(z)-1)^{-n}(-1)^k z^{n+k}.$$

Revealing the formal power series we finally have

$$\frac{(n-1)!}{(n-k-1)!}(-1)^k [z^k] \left[\frac{z}{\exp(z)-1}\right]^n.$$

The core term is

$$\operatorname*{res}_z \frac{1}{z^{k+1}} \frac{z^n}{(\exp(z)-1)^n}.$$

Now put $\exp(z)-1 = w$ so that $z = \log(1+w)$ and $dz = 1/(1+w)\; dw$ to get

$$\operatorname*{res}_w \frac{1}{w^n} (\log(1+w))^{n-k-1} \frac{1}{1+w}$$
$$= [w^{n-1}](\log(1+w))^{n-k-1}\frac{1}{1+w} = \frac{n}{n-k}[w^n](\log(1+w))^{n-k}.$$

Collecting everything we have

$$\frac{n!}{(n-k)!}(-1)^k [w^n](\log(1+w))^{n-k} = \frac{n!}{(n-k)!}(-1)^{n+k}[w^n](\log(1-w))^{n-k}$$

$$= n![w^n]\frac{1}{(n-k)!}\left(\log\frac{1}{1-w}\right)^{n-k} = \left[{n \atop n-k}\right].$$

Consult also for a generalization 1.129.

This problem has not appeared at math.stackexchange.com. It is from page 183 eqn. 13.32 of H.W.Gould's *Combinatorial Identities for Stirling Numbers* [Gou16].

1.56 Stirling numbers and Faulhaber's formula

Suppose we seek to prove that with $p \ge 1$ (polynomial representation of the power sum)

$$\sum_{k=0}^{n} k^p = \sum_{j=1}^{p+1} n^j \sum_{k=j}^{p+1} \frac{1}{k}\left\{{p+1 \atop k}\right\}(-1)^{k-j}\left[{k \atop j}\right].$$

With the usual EGFs we obtain for the inner sum

$$(p+1)![z^{p+1}]\sum_{k=j}^{p+1}\frac{1}{k}(\exp(z)-1)^k(-1)^{k-j}[w^k]\frac{1}{j!}\left(\log\frac{1}{1-w}\right)^j$$

$$= (-1)^j p![z^p]\exp(z)\sum_{k=j}^{p+1}(\exp(z)-1)^{k-1}(-1)^k[w^k]\frac{1}{j!}\left(\log\frac{1}{1-w}\right)^j.$$

Now with $\exp(z)-1 = z+\cdots$ the coefficient extractor in z enforces the upper limit of the sum and we get

$$(-1)^j p![z^p]\frac{\exp(z)}{\exp(z)-1}\sum_{k\ge j}(\exp(z)-1)^k(-1)^k[w^k]\frac{1}{j!}\left(\log\frac{1}{1-w}\right)^j.$$

Since $\log\frac{1}{1-z} = z+\cdots$ the coefficient extractor in w covers the whole of the powered logarithmic term and we find

$$p![z^p]\frac{\exp(z)}{\exp(z)-1}\frac{1}{j!}z^j.$$

Substitute into the outer sum to obtain (here the exponential terms yield two formal power series):

$$p![z^p]\frac{\exp(z)}{\exp(z)-1}\sum_{j=1}^{p+1} n^j\frac{z^j}{j!}$$

$$= p![z^{p+1}]\exp(z)\frac{z}{\exp(z)-1}\sum_{j=1}^{p+1} n^j\frac{z^j}{j!}.$$

The coefficient extractor once more enforces the upper limit of the sum and we have

$$p![z^{p+1}]\exp(z)\frac{z}{\exp(z)-1}(\exp(nz)-1) = p![z^{p+1}]z\exp(z)\sum_{k=0}^{n-1}\exp(kz)$$

$$= p![z^p]\sum_{k=0}^{n-1}\exp((k+1)z) = \sum_{k=0}^{n-1}(k+1)^p = \sum_{k=1}^{n} k^p.$$

This is the claim. With $p \geq 1$ we may restore the $k = 0$ value with no change. Note that this also yields

$$\frac{1}{p+1}(p+1)![z^{p+1}]\frac{z}{1-\exp(-z)}(\exp(nz)-1)$$

$$= \frac{(-1)^{p+1}}{p+1}(p+1)![z^{p+1}]\frac{z}{\exp(z)-1}(\exp(-nz)-1)$$

$$= \frac{(-1)^{p+1}}{p+1}(p+1)!\sum_{k=1}^{p+1}\frac{B_{p+1-k}}{(p+1-k)!}(-1)^k n^k \frac{1}{k!}$$

$$= \frac{1}{p+1}\sum_{k=1}^{p+1}\binom{p+1}{k}B_{p+1-k}(-1)^{p+1-k}n^k$$

$$= \frac{1}{p+1}n^{p+1} + \sum_{k=1}^{p}\binom{p}{k}\frac{(-1)^{p+1-k}B_{p+1-k}}{p+1-k}n^k$$

$$= \frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + \sum_{k=1}^{p-1}\binom{p}{k}\frac{(-1)^{p+1-k}B_{p+1-k}}{p+1-k}n^k.$$

Now for $q \geq 2$ we have that B_q is non-zero only if q is even so we may write

$$\frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + \sum_{k=1}^{p-1}\binom{p}{k}\frac{B_{p+1-k}}{p+1-k}n^k.$$

We have derived Faulhaber's formula.

This problem has not appeared at math.stackexchange.com. It is from page 214 eqn. 15.32 of H.W.Gould's *Combinatorial Identities for Stirling Numbers* [Gou16].

1.57 Stirling number and binomial coefficient

Suppose we seek to prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^{n+j} = \sum_{k=0}^{j}\binom{x-n}{k}(n+k)!\left\{ {n+j \atop n+k} \right\}.$$

Starting with the LHS we obtain

$$(n+j)![z^{n+j}]\sum_{k=0}^{n}(-1)^k\binom{n}{k}\exp((x-k)z)$$

$$= (n+j)![z^{n+j}]\exp(xz)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\exp(-kz)$$

$$= (n+j)![z^{n+j}]\exp(xz)(1-\exp(-z))^n$$

$$= (n+j)![z^{n+j}]\exp((x-n)z)(\exp(z)-1)^n.$$

Now observing that $\exp(z)-1 = z+\cdots$ we find

$$(n+j)!\sum_{k=n}^{n+j}[z^{n+j-k}]\exp((x-n)z)[z^k](\exp(z)-1)^n$$

$$= (n+j)!\sum_{k=0}^{j}[z^{j-k}]\exp((x-n)z)[z^{n+k}](\exp(z)-1)^n$$

$$= (n+j)!\sum_{k=0}^{j}\frac{(x-n)^{j-k}}{(j-k)!}\frac{n!}{(n+k)!}\left\{ {n+k \atop n} \right\}$$

$$= (n+j)!\sum_{k=0}^{j}\frac{(x-n)^k}{k!}\frac{n!}{(n+j-k)!}\left\{ {n+j-k \atop n} \right\}$$

$$= n!\sum_{k=0}^{j}\binom{n+j}{k}(x-n)^k\left\{ {n+j-k \atop n} \right\}.$$

Expanding the powered term in x yields

$$n!\sum_{k=0}^{j}\binom{n+j}{k}\left\{ {n+j-k \atop n} \right\}\sum_{p=0}^{k}\left\{ {k \atop p} \right\}(x-n)^{\underline{p}}$$

$$= n!\sum_{p=0}^{j}\binom{x-n}{p}p!\sum_{k=p}^{j}\binom{n+j}{k}\left\{ {n+j-k \atop n} \right\}\left\{ {k \atop p} \right\}.$$

It remains to simplify

$$n!p!\sum_{k=p}^{j}\binom{n+j}{k}\left\{ {n+j-k \atop n} \right\}\left\{ {k \atop p} \right\}$$

$$= (n+j)! \sum_{k=p}^{j} [z^{n+j-k}](\exp(z)-1)^n [w^k](\exp(w)-1)^p$$

$$= (n+j)![z^{n+j}](\exp(z)-1)^n \sum_{k=p}^{j} z^k [w^k](\exp(w)-1)^p.$$

Now with $\exp(w) - 1 = w + \cdots$ the coefficient extractor in w starts at the first non-zero coefficient on $[w^p]$. We may extend k beyond j to infinity owing to the powered exponential in n because $k > j$ is $n + k > n + j$ and there is no contribution due to the coefficient extractor in z. We obtain at last

$$(n+j)![z^{n+j}](\exp(z)-1)^n \sum_{k\ge p} z^k [w^k](\exp(w)-1)^p$$

$$= (n+j)![z^{n+j}](\exp(z)-1)^{n+p} = (n+p)! \left\{ {n+j \atop n+p} \right\}.$$

This is the claim because we have the coefficient on the falling factorial in x and we may conclude.

This problem has not appeared at math.stackexchange.com. It is from page 2 eqn. 1.16 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.58 Stirling number and double binomial coefficient

We seek to show that

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k} k^r = \sum_{k=0}^{r} (-1)^k \binom{x}{k} \binom{n-x}{n-k} k! \left\{ {r \atop k} \right\}.$$

We will prove this for x a positive integer, it then follows for all x including complex because LHS and RHS are polynomials in x. (Use e.g. $\binom{x}{k} = x^{\underline{k}}/k!$.) Starting with the RHS we find

$$\sum_{k=0}^{r} (-1)^k [z^k](1+z)^x [w^{n-k}](1+w)^{n-x} r! [v^r](\exp(v)-1)^k$$

$$= r![v^r][w^n](1+w)^{n-x} \sum_{k=0}^{r} (-1)^k w^k (\exp(v)-1)^k [z^k](1+z)^x$$

Now with $\exp(v) - 1 = v + \cdots$ we may raise the upper limit of the sum to infinity because the additional values do not pass the coefficient extractor in v:

$$r![v^r][w^n](1+w)^{n-x} \sum_{k\ge 0} (-1)^k w^k (\exp(v)-1)^k [z^k](1+z)^x$$

$$= r![v^r][w^n](1+w)^{n-x}(1 - w(\exp(v)-1))^x$$

$$= r![v^r][w^n](1+w)^{n-x}(1 + w - w\exp(v))^x$$

$$= r![v^r][w^n](1+w)^{n-x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k w^k \exp(kv)(1+w)^{x-k}.$$

The coefficient extractor in w is $[w^{n-k}](1+w)^{n-k}$ which is one when $n \ge k$ and zero otherwise (residue definition). Hence if $x > n$ we may lower the upper limit of the sum to n because the range $x \ge k > n$ does not contribute. On the other hand when $x < n$ we may raise the limit to n because we get zero from $\binom{x}{k}$ for the range $x < k \le n$. This leaves

$$r![v^r] \sum_{k=0}^{n} \binom{x}{k} (-1)^k \exp(kv) = \sum_{k=0}^{n} (-1)^k \binom{x}{k} k^r$$

as claimed.

This problem has not appeared at math.stackexchange.com. It is from page 1 eqn. 1.6 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.59 Stirling number and double binomial coefficient II

We seek to show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+j} = (-1)^n (n+j)! \sum_{k=0}^{j} \binom{j-n}{j-k} \binom{n}{k} \frac{k!}{(k+j)!} \left\{ {k+j \atop k} \right\}.$$

Start with the LHS to get

$$(n+j)![z^{n+j}] \sum_{k=0}^{n} (-1)^k \binom{n}{k} \exp(kz) = (n+j)![z^{n+j}](1-\exp(z))^n$$

$$= (-1)^n (n+j)![z^{n+j}](\exp(z)-1)^n = (-1)^n n! \left\{ {n+j \atop n} \right\}.$$

Proof for $j = 0$

This follows by substituting $j = 0$ into LHS and RHS and observing that they produce the same value.

Proof for $n > j$ with $n, j \ge 1$

Re-write the sum without the scalar in front as

$$\sum_{k=0}^{j} (-1)^{j-k} \binom{n-k-1}{j-k} \binom{n}{k} \frac{k!}{(k+j)!} \left\{ {k+j \atop k} \right\}.$$

Recall the following result from [GKP89] that we used in section 1.38:

$$\left\{ n \atop m \right\} = (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n}.$$

We apply this to the RHS. Because we assumed $j \ge 1$ we have that $\left\{ k+j \atop k \right\} = 0$ when $k = 0$ and we may start the sum at $k = 1$. We obtain

$$\begin{aligned} &[w^j](1+w)^{n-1} \sum_{k=1}^{j} \binom{n}{k} (-1)^{j-k} \frac{w^k}{(1+w)^k} \\ &\times \frac{k!}{(k+j)!} (-1)^j \frac{(k+j-1)!}{(k-1)!} [z^j] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-k-j} \\ &= n[w^j](1+w)^{n-1} \sum_{k=1}^{j} \binom{n-1}{k-1} (-1)^k \frac{w^k}{(1+w)^k} \\ &\times \frac{1}{k+j} [z^j] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-k-j}. \end{aligned}$$

We may raise k to n due to the coefficient extractor in w:

$$\begin{aligned} &-n \times \operatorname*{res}_v \frac{1}{v^{j+2}} \log \frac{1}{1-v} [w^{j-1}](1+w)^{n-2} \\ &\times \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{k-1} \frac{1}{v^{k-1}} \frac{w^{k-1}}{(1+w)^{k-1}} [z^j] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-k-j} \\ &= -n \times \operatorname*{res}_v \frac{1}{v^{j+2}} \log \frac{1}{1-v} [w^{j-1}](1+w)^{n-2} [z^j] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-j-1} \\ &\times \left(1 - \frac{1}{v} \frac{w}{1+w} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-1} \right)^{n-1} \\ &= -n \times \operatorname*{res}_v \frac{1}{v^{j+2}} \log \frac{1}{1-v} [w^{j-1}] \frac{1}{1+w} [z^j] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-j-1} \\ &\times \left(1 + w \left(1 - \frac{1}{v} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-1} \right) \right)^{n-1}. \end{aligned}$$

Re-expand the powered term in $n-1$ being extracted by $[w^{j-1}]$:

$$\begin{aligned} &-n \times \operatorname*{res}_v \frac{1}{v^{j+2}} \log \frac{1}{1-v} [z^j] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-j-1} \\ &\times \sum_{q=0}^{j-1} (-1)^{j-1-q} \binom{n-1}{q} \left(1 - \frac{1}{v} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-1} \right)^{q} \end{aligned}$$

$$= -n \times \operatorname*{res}_v \frac{1}{v^{j+2}} \log \frac{1}{1-v} [z^j] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-j-1}$$

$$\times \sum_{q=0}^{j-1} (-1)^{j-1-q} \binom{n-1}{q} \sum_{p=0}^{q} \binom{q}{p} (-1)^p \frac{1}{v^p} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-p}$$

$$= -n \sum_{q=0}^{j-1} (-1)^{j-1-q} \binom{n-1}{q}$$

$$\times \sum_{p=0}^{q} \binom{q}{p} (-1)^p \frac{1}{j+p+1} \left\{ {j+p+1 \atop p+1} \right\} (-1)^j \frac{p!}{(j+p)!}$$

$$= n \sum_{q=0}^{j-1} (-1)^q \binom{n-1}{q}$$

$$\times \sum_{p=0}^{q} \binom{q}{p} (-1)^p \frac{1}{p+1} \frac{(p+1)!}{(j+p+1)!} \left\{ {j+p+1 \atop p+1} \right\}.$$

There follows some simple binomial coefficient manipulation:

$$n \sum_{q=0}^{j-1} (-1)^q \binom{n-1}{q} \frac{1}{q+1}$$

$$\times \sum_{p=0}^{q} \binom{q+1}{p+1} (-1)^p \frac{(p+1)!}{(j+p+1)!} \left\{ {j+p+1 \atop p+1} \right\}$$

$$= \sum_{q=0}^{j-1} (-1)^q \binom{n}{q+1}$$

$$\times \sum_{p=0}^{q} \binom{q+1}{p+1} (-1)^p \frac{(p+1)!}{(j+p+1)!} \left\{ {j+p+1 \atop p+1} \right\}.$$

Continue with the standard Stirling number EGF:

$$\operatorname*{res}_z \frac{1}{z^{j+1}} \sum_{q=0}^{j-1} (-1)^{q+1} \binom{n}{q+1}$$

$$\times \sum_{p=0}^{q} \binom{q+1}{p+1} (-1)^{p+1} \frac{1}{z^{p+1}} (\exp(z) - 1)^{p+1}.$$

With $j \ge 1$ we may include $p = -1$ as it makes no contribution and obtain

$$\operatorname*{res}_z \frac{1}{z^{j+1}} \sum_{q=0}^{j-1} (-1)^{q+1} \binom{n}{q+1} (1 + (-1) \times (\exp(z) - 1)/z)^{q+1}$$

$$= [z^j] \sum_{q=0}^{j-1} (-1)^{q+1} \binom{n}{q+1} \frac{1}{z^{q+1}} (1+z-\exp(z))^{q+1}.$$

Now we may extend q to $n-1$ because $(1+z-\exp(z))^{q+1}/z^{q+1} = (-1)^{q+1}/2^{q+1} \times z^{q+1} + \cdots$ and hence when $q+1 > j$ or $q > j-1$ there is no contribution to the coefficient extractor in z. We may also include $q = -1$ because $j \ge 1$ and the sum term is zero in this case:

$$[z^j] \sum_{q=-1}^{n-1} (-1)^{q+1} \binom{n}{q+1} \frac{1}{z^{q+1}} (1+z-\exp(z))^{q+1}$$

$$= [z^j](1 + (-1) \times (1+z-\exp(z))/z)^n = [z^j] \frac{1}{z^n} (-1+\exp(z))^n.$$

It remains to restore the scalar in front:

$$(-1)^n (n+j)! [z^{n+j}] (\exp(z)-1)^n = (-1)^n n! \left\{ {n+j \atop n} \right\}$$

as claimed.

Proof for $n \le j$ with $n, j \ge 1$

We start with the RHS and obtain for the sum without the scalar

$$[z^j](1+z)^{j-n} \sum_{k=0}^{j} \binom{n}{k} z^k [w^{k+j}] (\exp(w)-1)^k$$

$$= [z^j](1+z)^{j-n} [w^j] \sum_{k=0}^{j} \binom{n}{k} z^k \frac{(\exp(w)-1)^k}{w^k}$$

Now when $k > j$ there is no contribution to the coefficient extractor in z and we may write

$$[z^j](1+z)^{j-n} [w^j] \sum_{k \ge 0} \binom{n}{k} z^k \frac{(\exp(w)-1)^k}{w^k}$$

$$= [z^j](1+z)^{j-n} [w^j] \left(1 + z \frac{\exp(w)-1}{w} \right)^n$$

$$= [z^j](1+z)^{j-n} [w^{n+j}] \left(w - z + z \exp(w) \right)^n .$$

Expanding the powered term

$$[z^j](1+z)^{j-n} [w^{n+j}] \sum_{k=0}^{n} \binom{n}{k} z^k \exp(kw) \sum_{p=0}^{n-k} \binom{n-k}{p} w^p (-1)^{n-k-p} z^{n-k-p}$$

$$= [z^j](1+z)^{j-n} \sum_{k=0}^{n} \binom{n}{k} \sum_{p=0}^{n-k} \binom{n-k}{p} \frac{k^{n+j-p}}{(n+j-p)!} (-1)^{n-k-p} z^{n-p}.$$

The coefficient extractor in z is $[z^{j-n+p}](1+z)^{j-n}$. Now when $j-n \geq 0$ or $j \geq n$ the only contribution originates with $p=0$ and we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k^{n+j}}{(n+j)!} (-1)^{n-k}.$$

Multiply by the scalar from the start to get

$$\sum_{k=0}^{n} \binom{n}{k} k^{n+j} (-1)^k$$

which is the claim.

This problem has not appeared at math.stackexchange.com. It is from page 3 eqn. 1.17 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.60 Stirling number and Bernoulli polynomials

We seek to show that

$$\sum_{k=0}^{n} (-1)^k \binom{n+x}{n-k} \frac{1}{k+1} = \frac{1}{n!} \sum_{k=0}^{n} \left[{n+1 \atop k+1}\right] B_k(x)$$

where $B_k(x)$ is a Bernoulli polynomial. The EGF of these polynomials is

$$\frac{t\exp(xt)}{\exp(t)-1}$$

so we get for the RHS

$$\frac{1}{n!} \sum_{k=0}^{n} (n+1)! [z^{n+1}] \frac{1}{(k+1)!} \left(\log \frac{1}{1-z} \right)^{k+1} k! [t^k] \frac{t\exp(xt)}{\exp(t)-1}$$
$$= \frac{1}{n!} \sum_{k=0}^{n} n! [z^n] \frac{1}{k!} \frac{1}{1-z} \left(\log \frac{1}{1-z} \right)^{k} k! [t^k] \frac{t\exp(xt)}{\exp(t)-1}.$$

Now because $\log \frac{1}{1-z} = z + \cdots$ there is no contribution to the coefficient extractor in $[z^n]$ when $k > n$ and we may extend k to infinity, obtaining (there is no pole at $t=0$)

$$[z^n] \frac{1}{1-z} \sum_{k \geq 0} \left(\log \frac{1}{1-z} \right)^{k} [t^k] \frac{t\exp(xt)}{\exp(t)-1}$$

$$= [z^n] \frac{1}{1-z} \frac{\left(\log \frac{1}{1-z}\right) \frac{1}{(1-z)^x}}{\frac{1}{1-z} - 1} = [z^{n+1}] \frac{1}{(1-z)^x} \log \frac{1}{1-z}.$$

This is

$$\operatorname*{res}_{z} \frac{1}{z^{n+2}}(1-z)^{n+2}\frac{1}{(1-z)^{x+n+2}}\log\frac{1}{1-z}.$$

Now put $z/(1-z) = w$ so that $z = w/(1+w)$ and $dz = 1/(1+w)^2\ dw$ which yields

$$\begin{aligned}&\operatorname*{res}_{w} \frac{1}{w^{n+2}}(1+w)^{x+n+2}\log(1+w)\frac{1}{(1+w)^2}\\ &= -\operatorname*{res}_{w} \frac{1}{w^{n+2}}(1+w)^{x+n}\log\frac{1}{1+w}.\end{aligned}$$

Extract the coefficient to get

$$-\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}\binom{x+n}{n+1-(k+1)} = \sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\binom{n+x}{n-k}.$$

This is the claim.

This problem has not appeared at math.stackexchange.com. It is from page 13 eqn. 1.102 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.61 Central binomial coefficient and Stirling numbers

We seek to show that

$$\sum_{k=0}^{n}\binom{2k}{k}\frac{k^r}{2^{2k}} = \frac{2n+1}{2^{2n}}\binom{2n}{n}\sum_{k=0}^{r}\binom{n}{k}\frac{1}{2k+1}k!\left\{ {r \atop k} \right\}.$$

First part

We get for the sum on the RHS without the scalar in front using the combinatorial EGF of the Stirling numbers of the second kind:

$$r![z^r]\sum_{k=0}^{r}\binom{n}{k}\frac{1}{2k+1}(\exp(z)-1)^k.$$

Now we may extend k to infinity because with $(\exp(z)-1)^k = z^k + \cdots$ the coefficient extractor cancels contributions from $k > r$:

$$r![z^r]\sum_{k\ge 0}\binom{n}{k}\frac{1}{2k+1}(\exp(z)-1)^k.$$

With n non-negative we may now set the upper limit to n because the binomial coefficient is zero when $k > n$ (we also reverse the index on the sum):

$$r![z^r]\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2n-2k+1}(\exp(z)-1)^{n-k}$$

$$= r![z^r](\exp(z)-1)^n[w^{2n+1}]\log\frac{1}{1-w}\sum_{k=0}^{n}\binom{n}{k}w^{2k}(\exp(z)-1)^{-k}$$

$$= r![z^r](\exp(z)-1)^n[w^{2n+1}]\log\frac{1}{1-w}\left(1+\frac{w^2}{\exp(z)-1}\right)^n$$

$$= r![z^r][w^{2n+1}]\log\frac{1}{1-w}(\exp(z)-1+w^2)^n.$$

Expanding the powered term we find

$$r![z^r][w^{2n+1}]\log\frac{1}{1-w}\sum_{k=0}^{n}\binom{n}{k}\exp(kz)(w^2-1)^{n-k}$$

$$= [w^{2n+1}]\log\frac{1}{1-w}\sum_{k=0}^{n}\binom{n}{k}k^r(w^2-1)^{n-k}$$

$$= \sum_{q=0}^{n}[w^{2q+1}]\log\frac{1}{1-w}\sum_{k=0}^{n}\binom{n}{k}k^r[w^{2n-2q}](w^2-1)^{n-k}$$

$$= \sum_{q=0}^{n}\frac{1}{2q+1}\sum_{k=0}^{n}\binom{n}{k}k^r[w^{n-q}](w-1)^{n-k}$$

$$= \sum_{q=0}^{n}\frac{1}{2q+1}\sum_{k=0}^{n}\binom{n}{k}k^r\binom{n-k}{n-q}(-1)^{q-k}.$$

Switching sums we find

$$\sum_{k=0}^{n}\binom{n}{k}k^r\sum_{q=0}^{n}\frac{1}{2q+1}\binom{n-k}{n-q}(-1)^{q-k}.$$

We have the claim if we can show that

$$\frac{1}{2^{2k}}\binom{2k}{k} = \frac{2n+1}{2^{2n}}\binom{2n}{n}\binom{n}{k}\sum_{q=0}^{n}\frac{1}{2q+1}\binom{n-k}{n-q}(-1)^{q-k}.$$

Second part

Working with the inner sum we obtain

$$(-1)^{n-k}\sum_{q=0}^{n}\frac{1}{2n-2q+1}\binom{n-k}{q}(-1)^q.$$

Now with $n \ge k \ge 0$ the binomial coefficient is zero when $q > n-k$ so we may discard the upper range to obtain

$$(-1)^{n-k} \sum_{q=0}^{n-k} \frac{1}{2n-2q+1} \binom{n-k}{q} (-1)^q.$$

Next introduce

$$f(z) = \frac{(n-k)!}{2n+1-2z} \prod_{r=0}^{n-k} \frac{1}{z-r}$$

This has the property that with $0 \le q \le n-k$

$$\mathrm{Res}_{z=q} f(z) = \frac{(n-k)!}{2n+1-2q} \prod_{r=0}^{q-1} \frac{1}{q-r} \prod_{r=q+1}^{n-k} \frac{1}{q-r}$$

$$= \frac{(n-k)!}{2n+1-2q} \frac{1}{q!} \frac{(-1)^{n-k-q}}{(n-k-q)!} = (-1)^{n-k} \frac{1}{2n+1-2q} \binom{n-k}{q} (-1)^q.$$

With the residue at infinity of $f(z)$ being zero by inspection and residues adding up to zero we get for the sum that it is

$$-\mathrm{Res}_{z=(2n+1)/2} f(z) = \frac{1}{2}(n-k)! \times \mathrm{Res}_{z=(2n+1)/2} \frac{1}{z-(2n+1)/2} \prod_{r=0}^{n-k} \frac{1}{z-r}$$

$$= \frac{1}{2}(n-k)! \prod_{r=0}^{n-k} \frac{1}{(2n+1)/2-r} = 2^{n-k}(n-k)! \prod_{r=0}^{n-k} \frac{1}{2n+1-2r}.$$

When $k=0$ the product works out to

$$\frac{n! \times 2^n}{(2n+1)!}.$$

When $k \ge 1$ we find

$$\prod_{r=0}^{n} \frac{1}{2n+1-2r} \prod_{r=n-k+1}^{n} (2n+1-2r)$$

$$= \frac{n! \times 2^n}{(2n+1)!} \prod_{r=0}^{k-1} (2k-1-2r) = \frac{n! \times 2^n}{(2n+1)!} \frac{(2k-1)!}{(k-1)! \times 2^{k-1}}$$

$$= \frac{n! \times 2^n}{(2n+1)!} \frac{(2k)!}{k! \times 2^k}.$$

We observe that this formula correctly represents the case $k=0$. To put it all together restore the two factors in front to obtain

$$\frac{2n+1}{2^{2n}} \binom{2n}{n} \binom{n}{k} 2^{n-k}(n-k)! \frac{n! \times 2^n}{(2n+1)!} \frac{(2k)!}{k! \times 2^k}$$

$$= \frac{2^{2n-2k}}{2^{2n}} \frac{(2n)!}{n! \times k! \times (n-k)!} (n-k)! \frac{n!}{(2n)!} \frac{(2k)!}{k!}$$
$$= \frac{1}{2^{2k}} \binom{2k}{k}.$$

This concludes the argument.

This problem has not appeared at math.stackexchange.com. It is from page 14 eqn. 1.108 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.62 Single variable monomial and two binomial coefficients

We seek to show that with $m \ge n$

$$x^n = (-1)^{m+n} \sum_{k=0}^{m+1} \binom{x+k-1}{m} \sum_{p=0}^{k} (-1)^p \binom{m+1}{p} (k-p)^n.$$

We will prove this for x a positive integer. It then holds for arbitrary x since both LHS and RHS are polynomials in x.

First part

We get for the inner sum

$$\sum_{p=0}^{k} (-1)^{k-p} \binom{m+1}{k-p} p^n = n![w^n][z^k](1+z)^{m+1} \sum_{p=0}^{k} (-1)^{k-p} z^p \exp(pw).$$

Here the coefficient extractor in z enforces the upper limit of the sum and we may raise p to infinity to get

$$n![w^n][z^k](1+z)^{m+1}(-1)^k \frac{1}{1+z\exp(w)}$$
$$= n![w^n][z^k](1-z)^{m+1} \frac{1}{1-z\exp(w)}.$$

We get from the outer sum (reverse index)

$$(-1)^{n+m} n![w^n][z^{m+1}](1-z)^{m+1} \frac{1}{1-z\exp(w)} \sum_{k=0}^{m+1} \binom{x+m-k}{m} z^k.$$

Applying the coefficient extractor to limit the sum we find

$$(-1)^{n+m} n![w^n](-1)^{m+n}[z^{m+1}](1-z)^{m+1}[v^m](1+v)^{x+m} \frac{1}{1-z\exp(w)} \frac{1}{1-z/(1+v)}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{m+2}}(1-z)^{m+1}\frac{1}{1-z\exp(w)}\frac{1}{1-z/(1+v)}.$$

Now put $z/(1-z)=u$ so that $z=u/(1+u)$ and $dz = 1/(1+u)^2\, du$ to get

$$\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1+u}{u}\frac{1}{1-u\exp(w)/(1+u)}\frac{1}{1-u/(1+v)/(1+u)}\frac{1}{(1+u)^2}$$

$$= \operatorname*{res}_u \frac{1}{u^{m+2}}(1+u)\frac{1}{1+u-u\exp(w)}\frac{1}{1+u-u/(1+v)}$$

$$= \operatorname*{res}_u \frac{1}{u^{m+2}}(1+u)\frac{1}{1-u(\exp(w)-1)}\frac{1+v}{1+v(1+u)}.$$

Extract the coefficient on $[w^n]$ to get

$$\operatorname*{res}_u \frac{1}{u^{m+2}}(1+u)\frac{1+v}{1+v(1+u)}\sum_{q=0}^{n} u^q q! {n \brace q}.$$

Next do the coefficient on v to find

$$\operatorname*{res}_u \frac{1}{u^{m+2}}(1+u)\sum_{p=0}^{m}\binom{x+m+1}{p}(-1)^{m-p}(1+u)^{m-p}\sum_{q=0}^{n} u^q q! {n \brace q}.$$

Resolve the residue in u and obtain

$$(-1)^n\sum_{p=0}^{m}\binom{x+m+1}{p}(-1)^p\sum_{q=0}^{n} q! {n \brace q}\binom{m+1-p}{m+1-q}.$$

Second part

The binomial coefficient in q is fine because $m\ge n$ and $q\le n$ so that $q\le m$. Switch sums to get

$$(-1)^n\sum_{q=0}^{n} q! {n \brace q}\sum_{p=0}^{m}\binom{x+m+1}{p}(-1)^p\binom{m+1-p}{m+1-q}.$$

We have for the inner sum where we take $q\ge 1$:

$$\sum_{p=0}^{m}\binom{x+m+1}{m-p}(-1)^{m-p}\binom{p+1}{m+1-q}$$

$$= [z^m](1+z)^{x+m+1}[w^{m+1-q}](1+w)\sum_{p=0}^{m}(-1)^{m-p}z^p(1+w)^p.$$

The coefficient extractor in z once more enforces the upper limit of the sum and we may extend to infinity:

$$(-1)^m [z^m](1+z)^{x+m+1}[w^{m+1-q}](1+w)\frac{1}{1+z+zw}$$
$$= (-1)^m [z^m](1+z)^{x+m}[w^{m+1-q}](1+w)\frac{1}{1+zw/(1+z)}$$
$$= (-1)^m [z^m](1+z)^{x+m}\left[(-1)^{m+1-q}\frac{z^{m+1-q}}{(1+z)^{m+1-q}} + (-1)^{m-q}\frac{z^{m-q}}{(1+z)^{m-q}}\right]$$
$$= (-1)^{q-1}\binom{x+q-1}{q-1} + (-1)^q\binom{x+q}{q} = (-1)^{q-1}\binom{x+q-1}{q-1}\left[1-\frac{x+q}{q}\right]$$
$$= (-1)^q\binom{x+q-1}{q-1}\frac{x}{q} = (-1)^q\binom{x+q-1}{q}.$$

Note that when $q=0$ only $p=0$ contributes for a contribution of one, so this is covered by the previous formula as well.

To conclude the argument substitute this into the remaining outer sum to find

$$(-1)^n n![w^n]\sum_{q=0}^{n}(\exp(w)-1)^q(-1)^q\binom{x+q-1}{q}.$$

Here the coefficient extractor enforces the range one last time because $\exp(w)-1 = w+\cdots$ and we have

$$(-1)^n n![w^n]\frac{1}{(1+(\exp(w)-1))^x} = (-1)^n n![w^n]\exp(-xw) = x^n$$

which is the claim. QED.

This problem has not appeared at math.stackexchange.com. It is from page 16 eqn. 1.128 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.63 Use of an Iverson bracket

We seek to show that

$$\sum_{k=0}^{n}\binom{2k+1}{j} = \frac{(-1)^{j+1}}{2^{j+2}}\left\{\sum_{k=0}^{j+1}(-1)^k\binom{2n+3}{k}2^k+1\right\}.$$

We get for the LHS using an Iverson bracket

$$\sum_{k\ge 0}\binom{2k+1}{j}[z^n]\frac{z^k}{1-z}$$

$$= [w^j](1+w)[z^n]\frac{1}{1-z}\sum_{k\ge 0}(1+w)^{2k}z^k$$

$$= [w^j](1+w)[z^n]\frac{1}{1-z}\frac{1}{1-z(1+w)^2}$$

$$= [w^j]\frac{1}{1+w}[z^n]\frac{1}{z-1}\frac{1}{z-1/(1+w)^2}.$$

The contribution from z is

$$\mathrm{Res}_{z=0}\ \frac{1}{z^{n+1}}\frac{1}{z-1}\frac{1}{z-1/(1+w)^2}.$$

Residues sum to zero and the residue at infinity is zero by inspection. Hence we may use minus the residues at $z = 1$ and $z = 1/(1 + w)^2$. We get from the first one

$$-[w^j]\frac{1}{1+w}\frac{1}{1-1/(1+w)^2} = -[w^j](1+w)\frac{1}{w(w+2)}$$

$$= -[w^{j+1}](1+w)\frac{1}{2}\frac{1}{1+w/2} = -\frac{1}{2}\left[\frac{(-1)^{j+1}}{2^{j+1}} + \frac{(-1)^j}{2^j}\right]$$

$$= \frac{(-1)^{j+1}}{2^{j+2}}\,[-1+2] = \frac{(-1)^{j+1}}{2^{j+2}}.$$

This looks good, we have recovered one of the target terms from the RHS. The next residue is $1/(1 + w)^2$:

$$-[w^j]\frac{1}{1+w}(1+w)^{2n+2}\frac{1}{1/(1+w)^2-1} = [w^j](1+w)^{2n+3}\frac{1}{w(w+2)}$$

$$= \frac{1}{2}[w^{j+1}](1+w)^{2n+3}\frac{1}{1+w/2} = \frac{1}{2}\sum_{k=0}^{j+1}\binom{2n+3}{k}\frac{(-1)^{j+1-k}}{2^{j+1-k}}$$

$$= \frac{(-1)^{j+1}}{2^{j+2}}\sum_{k=0}^{j+1}\binom{2n+3}{k}(-1)^k 2^k.$$

This is the second target term and we may conclude.

This problem has not appeared at math.stackexchange.com. It is from page 17 eqn. 1.129 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.64 Use of an Iverson bracket II

We seek to show that

$$\sum_{k=0}^{n}(-1)^k\binom{j+k}{j} = \frac{(-1)^j}{2^{j+1}}\left\{(-1)^n\sum_{k=0}^{j}(-1)^k\binom{n+j+1}{k}2^k + (-1)^j\right\}.$$

We get for the LHS using an Iverson bracket

$$\sum_{k\ge 0}(-1)^k\binom{j+k}{j}[z^n]\frac{z^k}{1-z}$$
$$= [w^j](1+w)^j[z^n]\frac{1}{1-z}\sum_{k\ge 0}(-1)^k z^k(1+w)^k$$
$$= [w^j](1+w)^j[z^n]\frac{1}{1-z}\frac{1}{1+z(1+w)}$$
$$= -[w^j](1+w)^{j-1}[z^n]\frac{1}{z-1}\frac{1}{z+1/(1+w)}.$$

The contribution from z is

$$\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}\frac{1}{z-1}\frac{1}{z+1/(1+w)}.$$

Residues sum to zero and the residue at infinity is zero by inspection. Hence we may use minus the residues at $z=1$ and $z=-1/(1+w)$. We get from the first one

$$[w^j](1+w)^{j-1}\frac{1}{1+1/(w+1)} = [w^j](1+w)^j\frac{1}{2}\frac{1}{1+w/2}$$
$$= \frac{1}{2}\sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}\frac{1}{2^{j-k}} = \frac{1}{2}(1-1/2)^j = \frac{(-1)^{2j}}{2^{j+1}}.$$

Nice, we have recovered one of the terms. Continuing with minus the residue at $z=-1/(1+w)$ we get

$$[w^j](1+w)^{j-1}(-1)^{n+1}(1+w)^{n+1}\frac{1}{-1/(w+1)-1}$$
$$= [w^j](1+w)^{n+j+1}(-1)^n\frac{1}{2}\frac{1}{1+w/2}$$
$$= \frac{1}{2}(-1)^n\sum_{k=0}^{j}\binom{n+j+1}{k}(-1)^{j-k}\frac{1}{2^{j-k}}$$
$$= \frac{(-1)^j}{2^{j+1}}(-1)^n\sum_{k=0}^{j}(-1)^k\binom{n+j+1}{k}2^k.$$

We have recovered the second term and may conclude.

This problem has not appeared at math.stackexchange.com. It is from page 17 eqn. 1.130 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.65 Use of an Iverson bracket III

We seek to show that

$$S_n(x) = \sum_{k=0}^{n} \binom{x}{n-k}\binom{x}{n+k} = \frac{1}{2}\left\{\binom{2x}{2n} + \binom{x}{n}^2\right\}.$$

We will prove this for x a non-negative integer and it then holds for all x because both sides are polynomials in x. It also holds by inspection when $n = 0$ and we may assume that $n \geq 1$. We have

$$\sum_{k=0}^{n} \binom{x}{k}\binom{x}{2n-k} = \binom{x}{n}^2 + [w^{2n}](1+w)^x \sum_{k=0}^{n-1} \binom{x}{k} w^k$$

$$= \binom{x}{n}^2 + [w^{2n}](1+w)^x \sum_{k\geq 0} \binom{x}{k} w^k [z^{n-1}] \frac{z^k}{1-z}.$$

We momentarily omit the term in front:

$$[w^{2n}](1+w)^x [z^{n-1}] \frac{1}{1-z} \sum_{k\geq 0} \binom{x}{k} w^k z^k$$

$$= [w^{2n}](1+w)^x [z^{n-1}] \frac{1}{1-z} (1+wz)^x.$$

Examination of this last expression with respect to w reveals a value of zero when $2x < 2n$ or $x < n$, which agrees with the proposed closed form. Henceforth we shall assume that $x \geq n$. The contribution from z is

$$\mathrm{Res}_{z=0} \frac{1}{z^n} \frac{1}{1-z} (1+wz)^x.$$

Residues sum to zero and thus this term contributes through minus the residue at $z = 1$ and $z = \infty$. We get for the first one

$$[w^{2n}](1+w)^x(1+w)^x = \binom{2x}{2n}.$$

The negative of the residue at infinity is

$$\mathrm{Res}_{z=0} \frac{1}{z^2} z^n \frac{1}{1-1/z} (1+w/z)^x = -\mathrm{Res}_{z=0} \frac{1}{z^{x-n+1}} \frac{1}{1-z} (w+z)^x.$$

Expanding the powered term and substituting yields

$$-[w^{2n}](1+w)^x \sum_{k=0}^{x-n} \binom{x}{k} w^{x-k} = -\sum_{k=0}^{x-n} \binom{x}{k} \binom{x}{2n-x+k}.$$

Put $k = x - q$ to get

$$-\sum_{q=n}^{x} \binom{x}{x-q} \binom{x}{2n-q} = -\sum_{q=n}^{x} \binom{x}{q} \binom{x}{2n-q}$$

$$= -\sum_{p=0}^{x-n} \binom{x}{n+p} \binom{x}{n-p}.$$

Now when $x - n > n$ we have in the range $x - n \geq p > n$ that the second binomial coefficient is zero (residue definition) and we may lower the upper limit to n. On the other hand when $n > x - n$ we have in the added range $n \geq p > x - n$ the first binomial coefficient is zero and we may raise the upper limit to n, getting at last

$$-\sum_{p=0}^{n} \binom{x}{n+p} \binom{x}{n-p} = -S_n(x).$$

We have shown that

$$S_n(x) = \binom{x}{n}^2 + \binom{2x}{2n} - S_n(x).$$

Solve for $S_n(x)$ to obtain the claim, which we have now verified for x a non-negative integer and hence for complex x with both sides being polynomials in x. QED.

This problem has not appeared at math.stackexchange.com. It is from page 22 eqn. 3.6 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.66 Basic example

We seek to show that

$$S_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{x}{2k} \binom{x}{n-2k} = \frac{1}{2}\binom{2x}{n} + \frac{1}{2}(-1)^{n/2}\binom{x}{n/2}\frac{1+(-1)^n}{2}.$$

We will prove this for x a non-negative integer and it then holds for all x because both sides are polynomials in x. We start with the LHS to get

$$[z^x](1+z)^x [w^n](1+w)^x \sum_{k=0}^{\lfloor n/2 \rfloor} z^{2k} w^{2k}.$$

Here the coefficient extractor in w enforces the upper limit of the sum and we have

$$[z^x](1+z)^x[w^n](1+w)^x \sum_{k\ge 0} z^{2k}w^{2k}$$
$$= [z^x](1+z)^x[w^n](1+w)^x \frac{1}{1-z^2w^2}.$$

With

$$\frac{1}{1-z^2w^2} = \frac{1}{2}\frac{1}{1+wz} + \frac{1}{2}\frac{1}{1-wz}$$

we get two pieces. The first one is

$$\frac{1}{2}[w^{n+1}](1+w)^x \mathrm{Res}_{z=0} \frac{1}{z^{x+1}}(1+z)^x \frac{1}{z+1/w}.$$

Here the residue at infinity in z is zero so we may take minus the residue at $z=-1/w$ to obtain

$$-\frac{1}{2}[w^{n+1}](1+w)^x(-1)^{x+1}w^{x+1}\left(1-\frac{1}{w}\right)^x$$
$$= \frac{1}{2}[w^n](1-w^2)^x.$$

We get for n even

$$\frac{1}{2}[w^{n/2}](1-w)^x = \frac{1}{2}(-1)^{n/2}\binom{x}{n/2}.$$

Continuing with the second piece we find

$$-\frac{1}{2}[w^{n+1}](1+w)^x \mathrm{Res}_{z=0} \frac{1}{z^{x+1}}(1+z)^x \frac{1}{z-1/w}.$$

We once more have a residue of zero at infinity and hence we may evaluate at minus the residue at $z=1/w$ to get

$$\frac{1}{2}[w^{n+1}](1+w)^x w^{x+1}\left(1+\frac{1}{w}\right)^x = \frac{1}{2}[w^n](1+w)^{2x} = \frac{1}{2}\binom{2x}{n}.$$

Joining the two pieces we have the claim.

This problem has not appeared at math.stackexchange.com. It is from page 22 eqn. 3.8 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.67 Basic example continued

We seek to show that

$$\sum_{k=0}^{\lfloor n/2\rfloor} {x\choose 2k} {2n-x\choose n-2k} = \frac{1}{2} \left\{ {2n\choose n} + (-1)^n 2^{2n} {\frac{x-1}{2}\choose n} \right\}.$$

We will prove this for x a non-negative integer and it then holds for all x because both sides are polynomials in x. We start with the LHS to get

$$[z^x](1+z)^x [w^n](1+w)^{2n-x} \sum_{k=0}^{\lfloor n/2\rfloor} z^{2k} w^{2k}.$$

Here the coefficient extractor in w enforces the upper limit of the sum and we obtain

$$[z^x](1+z)^x [w^n](1+w)^{2n-x} \sum_{k\ge 0} z^{2k} w^{2k}$$
$$= [z^x](1+z)^x [w^n](1+w)^{2n-x} \frac{1}{1-z^2w^2}.$$

With

$$\frac{1}{1-z^2w^2} = \frac{1}{2}\frac{1}{1+wz} + \frac{1}{2}\frac{1}{1-wz}$$

we get two pieces. The first one is

$$\frac{1}{2}[w^{n+1}](1+w)^{2n-x} \mathrm{Res}_{z=0} \frac{1}{z^{x+1}} (1+z)^x \frac{1}{z+1/w}.$$

Here the residue at infinity in z is zero so we may take minus the residue at $z=-1/w$ to obtain

$$-\frac{1}{2}[w^{n+1}](1+w)^{2n-x} (-1)^{x+1} w^{x+1} \left(1-\frac{1}{w}\right)^x$$
$$= \frac{1}{2}[w^n](1+w)^{2n-x}(1-w)^x$$
$$= \frac{1}{2} \operatorname*{res}_w \frac{1}{w^{n+1}} (1+w)^{n+1} (1+w)^{n-1-x} (1-w)^x.$$

Now we put $w/(1+w)=v$ so that $w=v/(1-v)$ and $dw = 1/(1-v)^2\; dv$ and get

$$\frac{1}{2} \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{(1-v)^{n-1-x}} \frac{(1-2v)^x}{(1-v)^x} \frac{1}{(1-v)^2}$$
$$= \frac{1}{2} \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{(1-v)^{n+1}} (1-2v)^x.$$

Next put $v(1-v)=u$ so that $v=(1-\sqrt{1-4u})/2$ and $dv = 1/\sqrt{1-4u}\; du$. We get

$$\frac{1}{2}\operatorname*{res}_{u}\frac{1}{u^{n+1}}\sqrt{1-4u}^{x}\frac{1}{\sqrt{1-4u}}$$

$$=\frac{1}{2}\operatorname*{res}_{u}\frac{1}{u^{n+1}}(1-4u)^{(x-1)/2}=\frac{1}{2}(-1)^{n}2^{2n}\binom{\frac{x-1}{2}}{n}.$$

This concludes the computation of the first piece which we recognize from the proposed closed form. Continuing with the second piece we obtain

$$-\frac{1}{2}[w^{n+1}](1+w)^{2n-x}\mathrm{Res}_{z=0}\frac{1}{z^{x+1}}(1+z)^{x}\frac{1}{z-1/w}.$$

We once more have a residue of zero at infinity and hence we may evaluate at minus the residue at $z=1/w$ to get

$$\frac{1}{2}[w^{n+1}](1+w)^{2n-x}w^{x+1}\left(1+\frac{1}{w}\right)^{x}=\frac{1}{2}[w^{n}](1+w)^{2n}=\frac{1}{2}\binom{2n}{n}.$$

We also recognize this piece as the second one from the closed form. Joining the two pieces we have the claim.

This problem has not appeared at math.stackexchange.com. It is from page 23 eqn. 3.12 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.68 An identity by Erik Sparre Andersen

We seek to show that with $n\geq 1$ and $0\leq r\leq n$

$$S_{n,r}(x)=\sum_{k=0}^{r}\binom{x}{k}\binom{-x}{n-k}=\frac{n-r}{n}\binom{x-1}{r}\binom{-x}{n-r}.$$

We will prove this for x a positive integer and it then holds for all x because both sides are polynomials in x. We start with the LHS to get using an Iverson bracket

$$\sum_{k\geq 0}\binom{x}{k}\binom{-x}{n-k}[z^{r}]\frac{z^{k}}{1-z}$$

$$=[z^{r}]\frac{1}{1-z}[w^{n}](1+w)^{-x}\sum_{k\geq 0}\binom{x}{k}z^{k}w^{k}$$

$$=[z^{r}]\frac{1}{1-z}[w^{n}](1+w)^{-x}(1+wz)^{x}.$$

The contribution from w is

$$\mathrm{Res}_{w=0}\frac{1}{w^{n+1}}\frac{1}{(1+w)^{x}}(1+wz)^{x}.$$

With $n \ge 1$ we have that the residue at infinity is zero by inspection and we may evaluate through minus the residue at $w = -1$ because residues sum to zero. We write

$$-(-1)^{n+1}\mathrm{Res}_{w=0} \frac{1}{(1-(w+1))^{n+1}} \frac{1}{(1+w)^x} (1 - z + (1+w)z)^x$$

$$= (-1)^n (1-z)^x \mathrm{Res}_{w=0} \frac{1}{(1-(w+1))^{n+1}} \frac{1}{(1+w)^x} (1 + (1+w)z/(1-z))^x.$$

This yields

$$(-1)^n (1-z)^x \sum_{q=0}^{x-1} \binom{x}{q} \frac{z^q}{(1-z)^q} \binom{x-1-q+n}{n}.$$

Restore z to find

$$(-1)^n \sum_{q=0}^{x-1} \binom{x}{q} (-1)^{r-q} \binom{x-1-q}{r-q} \binom{x-1-q+n}{n}.$$

Observe that the second binomial coefficient is zero when $r > x-1$ which agrees with the proposed RHS. Thus we may henceforth assume that $r \le x-1$. We may lower the upper limit to r because the range $r < q \le x-1$ produces zero from that same binomial coefficient by construction from the coefficient extractor in z. We thus have

$$(-1)^n \sum_{q=0}^{r} \binom{x}{q} (-1)^{r-q} \binom{x-1-q}{r-q} \binom{x-1-q+n}{n}.$$

Next observe that

$$\binom{x-1-q}{r-q} \binom{x-1-q+n}{n} = \frac{(x-1-q+n)!}{(r-q)! \times (x-1-r)! \times n!}$$
$$= \binom{x-1-r+n}{n} \binom{x-1-q+n}{r-q}.$$

We thus have for our sum

$$(-1)^n \binom{x-1-r+n}{n} \sum_{q=0}^{r} \binom{x}{q} (-1)^{r-q} \binom{x-1-q+n}{r-q}.$$

Working with the remaining sum,

$$[z^r](1+z)^{x-1+n} \sum_{q\ge 0} \binom{x}{q} (-1)^{r-q} \frac{z^q}{(1+z)^q}$$

$$= (-1)^r [z^r](1+z)^{x-1+n} \left(1 - \frac{z}{1+z}\right)^x$$

$$= (-1)^r [z^r](1+z)^{n-1} = (-1)^r \binom{n-1}{r}.$$

We have obtained the preliminary closed form

$$(-1)^{n+r} \binom{x-1-r+n}{n} \binom{n-1}{r}.$$

which produces zero when $n - 1 < r$ so we may suppose that $n - 1 \geq r$, a refinement of the initial $r \leq n$. This is

$$(-1)^{n+r} \binom{x-1-r+n}{n} \frac{n-r}{n} \binom{n}{r}$$

$$= \frac{n-r}{n} (-1)^{n+r} (x-1-r+n)^{\underline{n}} \frac{1}{r! \times (n-r)!}.$$

With (this also goes through for $r = 0$)

$$(x-1-r+n)^{\underline{n}} = \prod_{p=0}^{n-1} (x-r+p) = \prod_{p=0}^{r-1} (x-r+p) \prod_{p=r}^{n-1} (x-r+p)$$

$$= (x-1)^{\underline{r}} \prod_{p=0}^{n-1-r} (x+p) = (x-1)^{\underline{r}} (-1)^{n+r} \prod_{p=0}^{n-1-r} (-x-p)$$

$$= (x-1)^{\underline{r}} (-1)^{n+r} (-x)^{\underline{n-r}}$$

we at last have the claim. QED.

Remark

Apparently we also have

$$S_{n,r} = - \sum_{k=r+1}^{n} \binom{x}{k} \binom{-x}{n-k}.$$

This entails showing

$$S_{n,n} = \sum_{k=0}^{n} \binom{x}{k} \binom{-x}{n-k} = 0$$

This is the case of $r = n$ which was shown to be zero in the previous section.

This problem has not appeared at math.stackexchange.com. It is from page 23 eqn. 3.14 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.69 Very basic example

We seek to show that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} {x \choose k} {x-k \choose n-2k} 2^{n-2k} = {2x \choose n}.$$

We will prove this for x a positive integer and it then holds for all x because both sides are polynomials in x. We start with the LHS to get

$$[z^n](1+z)^x \sum_{k=0}^{\lfloor n/2 \rfloor} {x \choose k} 2^{n-2k} \frac{z^{2k}}{(1+z)^k}.$$

Here the coefficient extractor enforces the upper limit of the sum and we get

$$[z^n](1+z)^x \sum_{k \ge 0} {x \choose k} 2^{n-2k} \frac{z^{2k}}{(1+z)^k}$$

$$= 2^n [z^n](1+z)^x \left(1 + \frac{z^2}{4(1+z)}\right)^x$$

$$= 2^n [z^n](1+z+z^2/4)^x = 2^n [z^n](1+z/2)^{2x} = {2x \choose n}.$$

This problem has not appeared at math.stackexchange.com. It is from page 24 eqn. 3.22 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.70 An identity by Karl Goldberg

We seek to show that

$$\sum_{k=0}^{n} {x \choose k} {y+k \choose n-k} 2^{2k} = \sum_{k=0}^{n} {2x \choose k} {y \choose n-k} 2^k = \sum_{k=0}^{n} {2x \choose k} {2x+y-k \choose n-k}.$$

We will prove this for x, y positive integers and it then holds for all x and y because both sides are polynomials in x and y.

We start with the first sum to get

$$[z^n](1+z)^y \sum_{k=0}^{n} {x \choose k} (1+z)^k z^k 2^{2k}.$$

Here the coefficient extractor enforces the upper limit of the sum and we find

$$[z^n](1+z)^y \sum_{k \ge 0} {x \choose k} (1+z)^k z^k 2^{2k}$$

$$= [z^n](1+z)^y(1+4z(1+z))^x = [z^n](1+z)^y(1+2z)^{2x}.$$

We extract the coefficient to obtain

$$\sum_{k=0}^{n} \binom{2x}{k} 2^k \binom{y}{n-k}.$$

This is the second sum. Observe carefully that the coefficient extractor returns zero when $n > 2x+y$. We may henceforth assume that $n \le 2x+y$. This yields

$$\operatorname*{res}_{z} \frac{1}{z^{n+1}} (1+z)^y (1+2z)^{2x}$$
$$= \operatorname*{res}_{z} \frac{1}{z^{n+1}} (1+z)^{n+1} (1+z)^{y-1-n} (1+2z)^{2x}.$$

Next put $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2 \; dw$ to obtain

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1}{(1-w)^{y-1-n}} \frac{(1+w)^{2x}}{(1-w)^{2x}} \frac{1}{(1-w)^2}$$
$$= \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1}{(1-w)^{2x+y+1-n}} (1+w)^{2x}.$$

This is using $n \le 2x+y$

$$\sum_{k=0}^{n} \binom{2x}{k} \binom{2x+y-n+n-k}{n-k} = \sum_{k=0}^{n} \binom{2x}{k} \binom{2x+y-k}{n-k}.$$

We have found the third sum and may conclude. Note that there is another substitution we can make by writing

$$\operatorname*{res}_{z} \frac{1}{z^{n+1}} (1+2z)^{n+1} (1+z)^y (1+2z)^{2x-n-1}.$$

We put $z/(1+2z) = w$ so that $z = w/(1-2w)$ and $dz = 1/(1-2w)^2 \; dw$ to get

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{(1-w)^y}{(1-2w)^y} \frac{1}{(1-2w)^{2x-n-1}} \frac{1}{(1-2w)^2}$$
$$= \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1}{(1-2w)^{2x+y+1-n}} (1-w)^y.$$

This yields a fourth sum:

$$\sum_{k=0}^{n} \binom{y}{k} (-1)^k 2^{n-k} \binom{2x+y-n+n-k}{n-k}$$

$$= \sum_{k=0}^{n} \binom{y}{k} (-1)^k 2^{n-k} \binom{2x+y-k}{n-k}.$$

This problem has not appeared at math.stackexchange.com. It is from page 24 eqn. 3.21 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.71 Sum producing a square root

We seek to show that

$$\sum_{k=0}^{n} \binom{2x}{2k} \binom{x-k}{n-k} = \frac{x}{x+n} \binom{x+n}{2n} 2^{2n} = \frac{2^{2n}}{(2n)!} \prod_{k=0}^{n-1} (x^2 - k^2).$$

We will prove this for x a positive integer and it then holds for all x because both sides are polynomials in x. We need some preliminary observations about the definition of the binomial coefficients that we are using. We have

$$\binom{x-k}{n-k} = \mathrm{Res}_{z=0} \frac{1}{z^{n-k+1}} (1+z)^{x-k}.$$

This is zero when $k > n$ or $n - k > x - k$ i.e. $n > x$ and $x \ge k$. Otherwise we may evaluate through minus the residue at infinity to get

$$\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n-k+1} (1 + 1/z)^{x-k} = \mathrm{Res}_{z=0} \frac{1}{z^{x-n+1}} (1+z)^{x-k} = \binom{x-k}{x-n}.$$

This residue vanishes when $x < n$ or when $x - n > x - k$ i.e. $k > n$ and $x \ge k$. As the closed form is also zero when $x < n$ we will henceforth assume that $x \ge n$.

We start with the LHS to get

$$\mathop{\mathrm{res}}_{z} \frac{1}{z^{x-n+1}} (1+z)^x \sum_{k=0}^{n} \binom{2x}{2k} \frac{1}{(1+z)^k}$$

Here we may raise the upper limit to x because with $x \ge n$ for the range $x \ge k > n$ the residue is zero:

$$\mathop{\mathrm{res}}_{z} \frac{1}{z^{x-n+1}} (1+z)^x \sum_{k=0}^{x} \binom{2x}{2k} \frac{1}{(1+z)^k}$$

$$= \mathop{\mathrm{res}}_{z} \frac{1}{z^{x-n+1}} (1+z)^x \sum_{k=0}^{x} \binom{2x}{2k} \frac{1}{\sqrt{1+z}^{2k}}$$

$$= \mathop{\mathrm{res}}_{z} \frac{1}{z^{x-n+1}} (1+z)^x \sum_{k=0}^{2x} \binom{2x}{k} \frac{1}{\sqrt{1+z}^{k}} \frac{1+(-1)^k}{2}$$

$$= \frac{1}{2}\operatorname*{res}_{z} \frac{1}{z^{x-n+1}}(1+z)^x \left[\left(1+\frac{1}{\sqrt{1+z}}\right)^{2x} + \left(1-\frac{1}{\sqrt{1+z}}\right)^{2x}\right].$$

Next we put $1 - 1/\sqrt{1+z} = w$ so that $z = w(2-w)/(1-w)^2$ and $dz = 2/(1-w)^3\, dw$ to get

$$\frac{1}{2}\operatorname*{res}_{w} \frac{(1-w)^{2x-2n+2}}{w^{x-n+1}(2-w)^{x-n+1}} \frac{1}{(1-w)^{2x}} \left[(2-w)^{2x} + w^{2x}\right] \frac{2}{(1-w)^3}$$

$$= \operatorname*{res}_{w} \frac{1}{w^{x-n+1}(2-w)^{x-n+1}} \frac{1}{(1-w)^{2n+1}} \left[(2-w)^{2x} + w^{2x}\right]$$

The term w^{2x} does not contribute and we are left with

$$\operatorname*{res}_{w} \frac{1}{w^{x-n+1}}(2-w)^{x+n-1}\frac{1}{(1-w)^{2n+1}}.$$

Extracting the coefficient yields (recall that $x \ge n$)

$$\sum_{k=0}^{x-n} \binom{x+n-1}{k} (-1)^k 2^{x+n-1-k} \binom{x+n-k}{2n}$$

$$= 2^{x+n-1} \sum_{k=0}^{x-n} \binom{x+n-1}{k} (-1)^k 2^{-k} \binom{x+n-k}{x-n-k}$$

$$= 2^{x+n-1} [z^{x-n}] (1+z)^{x+n} \sum_{k=0}^{x-n} \binom{x+n-1}{k} (-1)^k 2^{-k} \frac{z^k}{(1+z)^k}.$$

The coefficient extractor enforces the upper limit of the sum and we have

$$2^{x+n-1} [z^{x-n}] (1+z)^{x+n} \sum_{k\ge 0} \binom{x+n-1}{k} (-1)^k 2^{-k} \frac{z^k}{(1+z)^k}$$

$$= 2^{x+n-1} [z^{x-n}] (1+z)^{x+n} \left(1 - \frac{1}{2}\frac{z}{1+z}\right)^{x+n-1}$$

$$= 2^{x+n-1} [z^{x-n}] (1+z)(1+z/2)^{x+n-1}$$

This gives

$$2^{x+n-1}\binom{x+n-1}{x-n} 2^{-(x-n)} + 2^{x+n-1}\binom{x+n-1}{x-n-1} 2^{-(x-n-1)}$$

$$= 2^{2n-1}\binom{x+n-1}{2n-1} + 2^{2n}\binom{x+n-1}{2n}.$$

We thus have for our answer

$$2^{2n-1}\frac{2n}{x+n}\binom{x+n}{2n} + 2^{2n}\frac{x-n}{x+n}\binom{x+n}{2n} = \frac{x}{x+n}\binom{x+n}{2n}2^{2n}$$

which is the claim. As for the alternate form we get without the multiplier $2^{2n}/(2n)!$ in front

$$\frac{x}{x+n}\prod_{k=0}^{2n-1}(x+n-k) = x\prod_{k=1}^{2n-1}(x+n-k)$$

$$= x\prod_{k=1}^{n}(x+n-k)\prod_{k=n+1}^{2n-1}(x+n-k) = x\prod_{k=0}^{n-1}(x+k)\prod_{k=1}^{n-1}(x-k).$$

We obtain at last

$$\frac{2^{2n}}{(2n)!}\prod_{k=0}^{n-1}(x^2-k^2).$$

This problem has not appeared at math.stackexchange.com. It is from page 25 eqn. 3.26 of H.W.Gould's *Combinatorial Identities* [Gou72a]. For additional information the reader is asked to consult math.stackexchange.com problem 1098257.

1.72 Sum producing a square root II

We seek to show that

$$\sum_{k=0}^{n}\binom{2x+1}{2k+1}\binom{x-k}{n-k} = \frac{2x+1}{2n+1}\binom{x+n}{2n}2^{2n} = \frac{2x+1}{(2n+1)!}\prod_{k=0}^{n-1}((2x+1)^2-(2k+1)^2).$$

We will prove this for x a positive integer and it then holds for all x because both sides are polynomials in x. The assumptions here are the same as in the previous section.

We start with the LHS to get

$$\operatorname*{res}_{z}\frac{1}{z^{x-n+1}}(1+z)^x\sum_{k=0}^{n}\binom{2x+1}{2k+1}\frac{1}{(1+z)^k}$$

Here we may raise the upper limit to x because with $x \geq n$ for the range $x \geq k > n$ the residue is zero:

$$\operatorname*{res}_{z}\frac{1}{z^{x-n+1}}(1+z)^x\sum_{k=0}^{x}\binom{2x+1}{2k+1}\frac{1}{(1+z)^k}$$

$$= \operatorname*{res}_{z}\frac{1}{z^{x-n+1}}(1+z)^{x+1/2}\sum_{k=0}^{x}\binom{2x+1}{2k+1}\frac{1}{\sqrt{1+z}^{2k+1}}$$

$$= \operatorname*{res}_z \frac{1}{z^{x-n+1}} (1+z)^{x+1/2} \sum_{k=0}^{2x+1} \binom{2x+1}{k} \frac{1}{\sqrt{1+z}^k} \frac{1-(-1)^k}{2}$$

$$= \frac{1}{2} \operatorname*{res}_z \frac{1}{z^{x-n+1}} (1+z)^{x+1/2} \left[\left(1 + \frac{1}{\sqrt{1+z}}\right)^{2x+1} - \left(1 - \frac{1}{\sqrt{1+z}}\right)^{2x+1} \right].$$

Next we put $1 - 1/\sqrt{1+z} = w$ so that $z = w(2-w)/(1-w)^2$ and $dz = 2/(1-w)^3 \; dw$ to get

$$\frac{1}{2} \operatorname*{res}_w \frac{(1-w)^{2x-2n+2}}{w^{x-n+1}(2-w)^{x-n+1}} \frac{1}{(1-w)^{2x+1}} \left[(2-w)^{2x+1} - w^{2x+1}\right] \frac{2}{(1-w)^3}$$

$$= \operatorname*{res}_w \frac{1}{w^{x-n+1}(2-w)^{x-n+1}} \frac{1}{(1-w)^{2n+2}} \left[(2-w)^{2x+1} - w^{2x+1}\right]$$

The term w^{2x+1} does not contribute and we are left with

$$\operatorname*{res}_w \frac{1}{w^{x-n+1}} (2-w)^{x+n} \frac{1}{(1-w)^{2n+2}}.$$

Extracting the coefficient yields (recall that $x \ge n$)

$$\sum_{k=0}^{x-n} \binom{x+n}{k} (-1)^k 2^{x+n-k} \binom{x+n+1-k}{2n+1}$$

$$= 2^{x+n} \sum_{k=0}^{x-n} \binom{x+n}{k} (-1)^k 2^{-k} \binom{x+n+1-k}{x-n-k}$$

$$= 2^{x+n} [z^{x-n}] (1+z)^{x+n+1} \sum_{k=0}^{x-n} \binom{x+n}{k} (-1)^k 2^{-k} \frac{z^k}{(1+z)^k}.$$

The coefficient extractor enforces the upper limit of the sum and we have

$$2^{x+n} [z^{x-n}] (1+z)^{x+n+1} \sum_{k\ge 0} \binom{x+n}{k} (-1)^k 2^{-k} \frac{z^k}{(1+z)^k}$$

$$= 2^{x+n} [z^{x-n}] (1+z)^{x+n+1} \left(1 - \frac{1}{2} \frac{z}{1+z}\right)^{x+n}$$

$$= 2^{x+n} [z^{x-n}] (1+z)(1+z/2)^{x+n}$$

This gives

$$2^{x+n} \binom{x+n}{x-n} 2^{-(x-n)} + 2^{x+n} \binom{x+n}{x-n-1} 2^{-(x-n-1)}$$

$$= 2^{2n} \binom{x+n}{2n} + 2^{2n+1} \binom{x+n}{2n+1}.$$

We thus have for our answer

$$2^{2n}\binom{x+n}{2n} + 2^{2n+1}\frac{x-n}{2n+1}\binom{x+n}{2n} = \frac{2x+1}{2n+1}\binom{x+n}{2n}2^{2n}$$

which is the claim. As for the alternate form we get without the multiplier $\frac{2x+1}{(2n+1)!}$ in front

$$\prod_{k=0}^{n-1}((2x+1)^2-(2k+1)^2)$$

$$= \prod_{k=0}^{n-1}(2x+2k+2)(2x-2k) = 2^{2n}\prod_{k=0}^{n-1}(x+k+1)(x-k)$$

$$= 2^{2n}(x+n)^{\underline{n}}x^{\underline{n}} = 2^{2n}(x+n)^{\underline{2n}}.$$

Restore the multiplier to obtain at last

$$\frac{2x+1}{(2n+1)!}2^{2n}(x+n)^{\underline{2n}} = \frac{2x+1}{2n+1}2^{2n}\binom{x+n}{2n}$$

as desired.

This problem has not appeared at math.stackexchange.com. It is from page 25 eqn. 3.27 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.73 Use of an Iverson bracket IV

We seek to show that

$$S_n(x) = \sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{x}{n+k} = \frac{1}{2}\left\{\binom{x}{n} + \binom{x}{n}^2\right\}.$$

We will prove this for x a non-negative integer and it then holds for all x because both sides are polynomials in x. It also holds by inspection when $n = 0$ and we may assume that $n \geq 1$. We have

$$(-1)^n\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{2n-k} = \binom{x}{n}^2 + (-1)^n[w^{2n}](1+w)^x\sum_{k=0}^{n-1}\binom{x}{k}(-1)^k w^k$$

$$= \binom{x}{n}^2 + (-1)^n[w^{2n}](1+w)^x\sum_{k\geq 0}\binom{x}{k}(-1)^k w^k[z^{n-1}]\frac{z^k}{1-z}.$$

We momentarily omit the term in front:

$$(-1)^n[w^{2n}](1+w)^x[z^{n-1}]\frac{1}{1-z}\sum_{k\geq 0}\binom{x}{k}(-1)^k w^k z^k$$

$$= (-1)^n [w^{2n}](1+w)^x [z^{n-1}] \frac{1}{1-z}(1-wz)^x.$$

Examination of this last expression with respect to w reveals a value of zero when $2x < 2n$ or $x < n$, which agrees with the proposed closed form. Henceforth we shall assume that $x \ge n$. The contribution from z is

$$\mathrm{Res}_{z=0} \frac{1}{z^n} \frac{1}{1-z}(1-wz)^x.$$

Residues sum to zero and thus this term contributes through minus the residue at $z = 1$ and $z = \infty$. We get for the first one

$$(-1)^n [w^{2n}](1+w)^x (1-w)^x = (-1)^n [w^{2n}](1-w^2)^x$$
$$= (-1)^n [w^n](1-w)^x = \binom{x}{n}.$$

The negative of the residue at infinity is

$$\mathrm{Res}_{z=0} \frac{1}{z^2} z^n \frac{1}{1-1/z}(1-w/z)^x = -\mathrm{Res}_{z=0} \frac{1}{z^{x-n+1}} \frac{1}{1-z}(z-w)^x.$$

Expanding the powered term and substituting yields

$$-(-1)^n [w^{2n}](1+w)^x \sum_{k=0}^{x-n} \binom{x}{k} (-1)^{x-k} w^{x-k}$$
$$= -(-1)^{n+x} \sum_{k=0}^{x-n} \binom{x}{k} (-1)^k \binom{x}{2n-x+k}.$$

The term being summed is zero by construction when $2n < x - k$ or $2n - x + k < 0$. Put $k = x - q$ to get

$$-(-1)^n \sum_{q=n}^{x} \binom{x}{x-q} (-1)^q \binom{x}{2n-q} = -(-1)^n \sum_{q=n}^{x} \binom{x}{q} (-1)^q \binom{x}{2n-q}$$
$$= -\sum_{p=0}^{x-n} \binom{x}{n+p} (-1)^p \binom{x}{n-p}.$$

Now when $x - n > n$ we have in the range $x - n \ge p > n$ that the second binomial coefficient is zero (residue definition) and we may lower the upper limit to n. On the other hand when $n > x - n$ we have in the added range $n \ge p > x - n$ the first binomial coefficient is zero and we may raise the upper limit to n, getting at last

$$-\sum_{p=0}^{n}(-1)^p \binom{x}{n+p}\binom{x}{n-p} = -S_n(x).$$

We have shown that

$$S_n(x) = \binom{x}{n}^2 + \binom{x}{n} - S_n(x).$$

Solve for $S_n(x)$ to obtain the claim, which we have now verified for x a non-negative integer and hence for complex x with both sides being polynomials in x. QED.

This problem has not appeared at math.stackexchange.com. It is from page 26 eqn. 3.35 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.74 Binomial coefficient manipulation

We seek to show that

$$\sum_{k=0}^{n}(-1)^k \binom{2n}{k}\binom{2x-2n}{x-k} = \frac{1}{2}(-1)^n \left\{ \binom{x}{n} + \binom{x}{n}^2 \right\} \binom{2x}{x}\binom{2x}{2n}^{-1}.$$

We will prove this for $n \ge 0$ a non-negative integer and $x \ge n$ a non-negative integer. With

$$\binom{2n}{k}\binom{2x}{2n} = \frac{(2x)!}{k! \times (2n-k)! \times (2x-2n)!} = \binom{2x}{k}\binom{2x-k}{2x-2n}$$

this is equivalent to

$$\sum_{k=0}^{n}(-1)^k \binom{2x}{k}\binom{2x-k}{2x-2n}\binom{2x-2n}{x-k} = \frac{1}{2}(-1)^n \left\{ \binom{x}{n} + \binom{x}{n}^2 \right\} \binom{2x}{x}.$$

We also have (the second binomial coefficient vanishes when $x+k-2n < 0$ or $x < 2n-k$ in accordance with the residue definition and agrees with the factorials otherwise)

$$\binom{2x-k}{2x-2n}\binom{2x-2n}{x-k} = \frac{(2x-k)!}{(2n-k)! \times (x-k)! \times (x+k-2n)!} = \binom{2x-k}{x-k}\binom{x}{2n-k}.$$

so the LHS becomes

$$\sum_{k=0}^{n}(-1)^k \binom{2x}{k}\binom{2x-k}{x-k}\binom{x}{2n-k}.$$

Next observe that

$$\binom{2x}{k}\binom{2x-k}{x-k} = \frac{(2x)!}{k! \times x! \times (x-k)!} = \binom{2x}{x}\binom{x}{k}$$

and we may divide by $\binom{2x}{x}$ to get as the goal

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{x}{2n-k} = \frac{1}{2}(-1)^n \left\{ \binom{x}{n} + \binom{x}{n}^2 \right\}.$$

This is the identity from the previous section and we are done.

This problem has not appeared at math.stackexchange.com. It is from page 29 eqn. 3.60 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.75 Four binomial sums

We seek to show that

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2n-x}{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{x}{k}\binom{2n-2x}{n-2k}$$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n-k}{n-k}\binom{2n-x}{k} 2^k$$

$$= \frac{2^n}{n!} \prod_{k=0}^{n-1} (2k+1-x) = (-1)^n 2^{2n} \binom{\frac{x-1}{2}}{n}.$$

There is a fourth sum which will appear during the computation. We will prove this for x a positive integer and then it holds for all i.e. complex x because the expressions involved are all polynomials in x. We start with the first formula and obtain

$$[z^n](1+z)^{2n-x} \sum_{k=0}^{n} (-1)^k \binom{x}{k} z^k.$$

Here the coefficient extractor enforces the upper limit of the sum and we get

$$[z^n](1+z)^{2n-x} \sum_{k\ge 0} (-1)^k \binom{x}{k} z^k$$

$$= [z^n](1+z)^{2n-x}(1-z)^x.$$

We can re-write this as

$$[z^n](1+z)^{2n-2x}(1-z^2)^x.$$

Extract the coefficient to obtain

$$\sum_{k=0}^{\lfloor n/2 \rfloor} [z^{2k}](1-z^2)^x [z^{n-2k}](1+z)^{2n-2x} = \sum_{k=0}^{\lfloor n/2 \rfloor} [z^k](1-z)^x \binom{2n-2x}{n-2k}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{x}{k} \binom{2n-2x}{n-2k}.$$

This is the second formula. Continuing with the initial closed form we write

$$\operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^{n+1} (1+z)^{n-x-1} (1-z)^x.$$

We put $z/(1+z) = v$ so that $z = v/(1-v)$ and $dz = \frac{1}{(1-v)^2}\, dv$ to get

$$\operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{(1-v)^{n-x-1}} \frac{(1-2v)^x}{(1-v)^x} \frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{(1-v)^{n+1}} (1-2v)^x.$$

This is

$$\sum_{k=0}^{n} (-1)^k 2^k \binom{x}{k} \binom{2n-k}{n}$$

which was not listed in the Gould text. We put $v(1-v) = w$ so that $v = (1-\sqrt{1-4w})/2$ and $dv = 1/\sqrt{1-4w}\, dw$ to find

$$\operatorname*{res}_w \frac{1}{w^{n+1}} \sqrt{1-4w}^{\,x} \frac{1}{\sqrt{1-4w}} = \operatorname*{res}_w \frac{1}{w^{n+1}} (1-4w)^{(x-1)/2}$$

$$= (-1)^n 2^{2n} \binom{\frac{x-1}{2}}{n}.$$

This is the fifth and last formula. We get for the fourth formula

$$(-1)^n 2^{2n} \frac{1}{n!} \prod_{p=0}^{n-1} ((x-1)/2 - p) = (-1)^n 2^n \frac{1}{n!} \prod_{p=0}^{n-1} (x-1-2p)$$

$$= 2^n \frac{1}{n!} \prod_{p=0}^{n-1} (2p+1-x).$$

It remains to show the third formula. We start with the inital closed form and write

$$\operatorname*{res}_z \frac{1}{z^{n+1}} (1-z)^{n+1} (1+z)^{2n-x} (1-z)^{x-n-1}.$$

We now put $z/(1-z) = v$ so that $z = v/(1+v)$ and $dz = 1/(1+v)^2\, dv$ to obtain

$$\operatorname*{res}_{v} \frac{1}{v^{n+1}} \frac{(1+2v)^{2n-x}}{(1+v)^{2n-x}} \frac{1}{(1+v)^{x-n-1}} \frac{1}{(1+v)^2}$$
$$= \operatorname*{res}_{v} \frac{1}{v^{n+1}} (1+2v)^{2n-x} \frac{1}{(1+v)^{n+1}}.$$

Extracting the coefficient we find

$$\sum_{k=0}^{n} \binom{2n-x}{k} 2^k (-1)^{n-k} \binom{2n-k}{n}.$$

This was the missing formula and we may conclude.

This problem has not appeared at math.stackexchange.com. It is from page 27 eqn. 3.42 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.76 Power term and two binomial coefficients

We seek to show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 k^r = \sum_{k=0}^{r} \binom{n}{k} \binom{2n-k}{n} k! {r \brace k}.$$

We find for the LHS

$$r![w^r] \sum_{k=0}^{n} \binom{n}{k}^2 \exp(kw)$$
$$= r![w^r][z^n](1+z)^n \sum_{k=0}^{n} \binom{n}{k} z^k \exp(kw)$$
$$= r![w^r][z^n](1+z)^n (1+z\exp(w))^n.$$

Continuing, we obtain

$$r![w^r][z^n](1+z)^n(1+z+z(\exp(w)-1))^n$$
$$= r![w^r][z^n](1+z)^n \sum_{k=0}^{n} \binom{n}{k} (1+z)^{n-k} (\exp(w)-1)^k$$
$$= r![w^r] \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} k! \frac{(\exp(w)-1)^k}{k!}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} k! {r \brace k}.$$

Now observe that we may set the upper range to r because if $r > n$ the first binomial coefficient is zero in the added range $r \ge k > n$ and if $n > r$ the

Stirling number is zero in the removed range $n \ge k > r$. Hence we obtain

$$\sum_{k=0}^{r} \binom{n}{k} \binom{2n-k}{n} k! \left\{ {r \atop k} \right\}.$$

This problem has not appeared at math.stackexchange.com. It is from page 31 eqn. 3.77 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.77 Use of an Iverson bracket V

We seek to show that

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{2n}{k}^2 = \frac{1}{2}(-1)^n \left\{ \binom{2n}{n} + \binom{2n}{n}^2 \right\}.$$

We start by writing for the LHS

$$(-1)^n \binom{2n}{n}^2 + \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k}^2$$

and introduce an Iverson bracket for the sum

$$[z^{n-1}] \frac{1}{1-z} \sum_{k \ge 0} (-1)^k \binom{2n}{k}^2 z^k$$

$$= [w^{2n}](1+w)^{2n} [z^{n-1}] \frac{1}{1-z} \sum_{k \ge 0} (-1)^k \binom{2n}{k} w^k z^k$$

$$= [w^{2n}](1+w)^{2n} [z^{n-1}] \frac{1}{1-z} (1-wz)^{2n}.$$

The contribution from z is

$$\mathrm{Res}_{z=0} \frac{1}{z^n} \frac{1}{1-z} (1-wz)^{2n}.$$

Residues sum to zero hence this term is given by minus the sum of the residues at $z=1$ and $z=\infty$. We get for the first one

$$[w^{2n}](1+w)^{2n}(1-w)^{2n} = [w^{2n}](1-w^2)^{2n} = [w^n](1-w)^{2n}$$
$$= (-1)^n \binom{2n}{n}.$$

For minus the residue at infinity we find

$$\mathrm{Res}_{z=0} \frac{1}{z^2} z^n \frac{1}{1-1/z} (1-w/z)^{2n} = \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} (z-w)^{2n}.$$

Restoring the coefficient extractor in w we obtain

$$-[w^{2n}](1+w)^{2n} \sum_{k=0}^{n} [z^{n-k}] \frac{1}{1-z} [z^k](z-w)^{2n}$$

$$= -[w^{2n}](1+w)^{2n} \sum_{k=0}^{n} \binom{2n}{k} (-1)^{2n-k} w^{2n-k}$$

$$= -\sum_{k=0}^{n} \binom{2n}{k} (-1)^k \binom{2n}{k} = -S_n.$$

We have shown that $S_n = (-1)^n \binom{2n}{n}^2 + (-1)^n \binom{2n}{n} - S_n$ which is the claim.

This problem has not appeared at math.stackexchange.com. It is from page 31 eqn. 3.82 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.78 Use of an Iverson bracket VI

We seek to show that

$$S_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n}{2k}^2 = \frac{1}{4} \binom{4n}{2n} + \frac{1}{4} (-1)^n \binom{2n}{n} + \frac{1+(-1)^n}{4} \binom{2n}{n}^2.$$

Start by observing that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n}{2k}^2 = \frac{1+(-1)^n}{2} \binom{2n}{n}^2 + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n}{2k}^2.$$

We introduce an Iverson bracket to treat the remaining sum:

$$[z^{n-1}] \frac{1}{1-z} \sum_{k \ge 0} \binom{2n}{2k}^2 z^{2k}$$

$$= [z^{n-1}] \frac{1}{1-z} \sum_{k \ge 0} \binom{2n}{k}^2 z^k \frac{1+(-1)^k}{2}$$

$$= [w^{2n}](1+w)^{2n} [z^{n-1}] \frac{1}{1-z} \sum_{k \ge 0} \binom{2n}{k} w^k z^k \frac{1+(-1)^k}{2}$$

$$= \frac{1}{2} [w^{2n}](1+w)^{2n} [z^{n-1}] \frac{1}{1-z} ((1+wz)^{2n} + (1-wz)^{2n}).$$

The contribution from z is

$$\frac{1}{2} \mathrm{Res}_{z=0} \frac{1}{z^n} \frac{1}{1-z} ((1+wz)^{2n} + (1-wz)^{2n}).$$

Residues sum to zero so this is minus the sum of the residues at $z = 1$ and $z = \infty$. We get for the first one

$$\frac{1}{2}[w^{2n}](1+w)^{2n}((1+w)^{2n} + (1-w)^{2n}) = \frac{1}{2}[w^{2n}]((1+w)^{4n} + (1-w^2)^{2n})$$
$$= \frac{1}{2}\binom{4n}{2n} + \frac{1}{2}[w^n](1-w)^{2n} = \frac{1}{2}\binom{4n}{2n} + \frac{1}{2}(-1)^n\binom{2n}{n}.$$

There remains minus the residue at infinity:

$$\frac{1}{2}\mathrm{Res}_{z=0}\frac{1}{z^2}z^n\frac{1}{1-1/z}((1+w/z)^{2n} + (1-w/z)^{2n})$$
$$= -\frac{1}{2}\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}\frac{1}{1-z}((z+w)^{2n} + (z-w)^{2n})$$
$$= -\frac{1}{2}[w^{2n}](1+w)^{2n}\sum_{k=0}^{n}\binom{2n}{k}w^{2n-k}(1+(-1)^{2n-k})$$
$$= -\sum_{k=0}^{n}\binom{2n}{k}^2\frac{1+(-1)^k}{2} = -\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2n}{2k}^2 = -S_n.$$

We have shown that $S_n = \frac{1+(-1)^n}{2}\binom{2n}{n}^2 + \frac{1}{2}\binom{4n}{2n} + \frac{1}{2}(-1)^n\binom{2n}{n} - S_n$ which is the claim.

This problem has not appeared at math.stackexchange.com. It is from page 30 eqn. 3.72 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.79 Appearance of constants three and five

We seek to verify the two related sum identities

$$2^{2n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k} = \sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}5^k$$

and

$$2^{2n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k} = \sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{2k}{k}3^k.$$

Observe that for $k \ge 1$

$$\binom{-1/2}{k} = \frac{1}{k!}\prod_{q=0}^{k-1}(-1/2-q) = \frac{1}{k!}\frac{(-1)^k}{2^k}\prod_{q=0}^{k-1}(2q+1)$$
$$= \frac{1}{k!}\frac{(-1)^k}{2^k}\frac{(2k-1)!}{(k-1)! \times 2^{k-1}}.$$

This is

$$\frac{1}{k!}\frac{(-1)^k}{2^k}\frac{(2k)!}{k! \times 2^k}$$

which also holds for $k = 0$. We get

$$\binom{-1/2}{k} = \frac{(-1)^k}{2^{2k}}\binom{2k}{k}.$$

or alternatively

$$\binom{2k}{k} = (-1)^k 2^{2k} [z^k]\frac{1}{\sqrt{1+z}} = [z^k]\frac{1}{\sqrt{1-4z}}.$$

First identity

We start with the LHS to get

$$2^{2n}\sum_{k=0}^{n}\binom{n}{k}\binom{2n-2k}{n-k} = 2^{2n}[z^n]\frac{1}{\sqrt{1-4z}}\sum_{k=0}^{n}\binom{n}{k}z^k$$
$$= 2^{2n}[z^n]\frac{1}{\sqrt{1-4z}}(1+z)^n.$$

This is

$$2^{2n}\operatorname*{res}_z \frac{1}{z^{n+1}}(1+z)^n\frac{1}{\sqrt{1-4z}}.$$

Now put $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2\ dw$ to obtain

$$2^{2n}\operatorname*{res}_w \frac{1}{w^{n+1}}(1-w)\frac{1}{\sqrt{1-4w/(1-w)}}\frac{1}{(1-w)^2}$$
$$= 2^{2n}\operatorname*{res}_w \frac{1}{w^{n+1}}\frac{1}{\sqrt{1-5w}}\frac{1}{\sqrt{1-w}}$$
$$= 2^{2n}[w^n]\frac{1}{\sqrt{1-5w}}\frac{1}{\sqrt{1-w}}$$
$$= [w^n]\frac{1}{\sqrt{1-20w}}\frac{1}{\sqrt{1-4w}} = \sum_{k=0}^{n}\binom{2k}{k}5^k\binom{2n-2k}{n-k}.$$

This is the claim.

Second identity

This is very similar to the first. We obtain

$$2^{2n}(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n-k} = 2^{2n}(-1)^n [z^n] \frac{1}{\sqrt{1-4z}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k z^k$$

$$= 2^{2n}(-1)^n [z^n] \frac{1}{\sqrt{1-4z}} (1-z)^n.$$

This is

$$2^{2n}(-1)^n \operatorname*{res}_{z} \frac{1}{z^{n+1}} (1-z)^n \frac{1}{\sqrt{1-4z}}.$$

Now put $z/(1-z) = w$ so that $z = w/(1+w)$ and $dz = 1/(1+w)^2 \; dw$ to obtain

$$2^{2n}(-1)^n \operatorname*{res}_{w} \frac{1}{w^{n+1}} (1+w) \frac{1}{\sqrt{1-4w/(1+w)}} \frac{1}{(1+w)^2}$$

$$= 2^{2n}(-1)^n \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1}{\sqrt{1-3w}} \frac{1}{\sqrt{1+w}}$$

$$= 2^{2n}(-1)^n [w^n] \frac{1}{\sqrt{1-3w}} \frac{1}{\sqrt{1+w}}$$

$$= [w^n] \frac{1}{\sqrt{1+12w}} \frac{1}{\sqrt{1-4w}} = \sum_{k=0}^{n} \binom{2k}{k} (-1)^k 3^k \binom{2n-2k}{n-k}.$$

Once more we have the claim.

This problem has not appeared at math.stackexchange.com. It is from page 32 eqns. 3.88 and 3.87 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.80 Generating function of a binomial term

We seek to show that

$$\sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k} \frac{x}{x+k} = 2^{2n} \binom{x+n}{n}^{-1} \binom{n+x-1/2}{n}.$$

Note that we get a polynomial in x on the LHS and the RHS on multiplication by $\binom{x+n}{n}$ so we just need to prove it for x a positive integer and it will hold for all i.e. complex x. We start with the following claim where $q \ge 1$ is a positive integer:

$$\binom{2k}{k} \frac{1}{k+q} = [z^{k+q}] \frac{1 - \sqrt{1-4z} \sum_{p=0}^{q-1} \binom{2p}{p} z^p}{q \times \binom{2q}{q}}.$$

We also claim that the first non-zero coefficient of this OGF is on z^q. The constant coefficient is zero by inspection (compare Catalan number GF). We have for the coefficient on $[z^m]$ where $1 \le m \le q-1$ without the scalar in q and the sign

$$\sum_{p=0}^{m} \binom{2p}{p} [z^{m-p}]\sqrt{1-4z} = \sum_{p=0}^{m} [z^p] \frac{1}{\sqrt{1-4z}} [z^{m-p}]\sqrt{1-4z} = [z^m]1 = 0.$$

On the other hand for $m = q$ we get

$$-\binom{2q}{q} + \sum_{p=0}^{q} \binom{2p}{p} [z^{q-p}]\sqrt{1-4z} = -\binom{2q}{q}$$

so the coefficient on q is $\binom{2q}{q}/q/\binom{2q}{q} = 1/q$.

This leaves $m \ge q$ which corresponds to $k \ge 0$. On differentiating the OGF we must obtain

$$\frac{z^{q-1}}{\sqrt{1-4z}}.$$

Doing the differentiation of the functional term we find

$$-\frac{2}{\sqrt{1-4z}} \sum_{p=0}^{q-1} \binom{2p}{p} z^p + \frac{1-4z}{\sqrt{1-4z}} \sum_{p=1}^{q-1} p \binom{2p}{p} z^{p-1}.$$

Without the square root we have

$$-2\sum_{p=0}^{q-1} \binom{2p}{p} z^p + \sum_{p=0}^{q-2} (p+1) \binom{2p+2}{p+1} z^p - 4\sum_{p=0}^{q-1} p \binom{2p}{p} z^p.$$

The contribution from $p \le q-2$ is

$$-2\binom{2p}{p} + (p+1)\binom{2p+2}{p+1} - 4p\binom{2p}{p} = 0.$$

Restoring the scalar and the sign we get for $p = q-1$

$$-z^{q-1}\frac{1}{q}\binom{2q}{q}^{-1}\left[-2\binom{2q-2}{q-1} - 4(q-1)\binom{2q-2}{q-1}\right] = z^{q-1}$$

as desired. Using the newly established closed form for the OGF of $\binom{2k}{k}\frac{1}{k+q}$ we have by convolution of formal power series (in fact two functions that are analytic in a neighborhood of the origin) that the LHS of the proposed identity is

$$[z^n] q \frac{1 - \sqrt{1-4z}\sum_{p=0}^{q-1}\binom{2p}{p} z^p}{q \times \binom{2q}{q} \times z^q} \frac{1}{\sqrt{1-4z}}$$

$$= [z^{q+n}]\binom{2q}{q}^{-1}\frac{1}{\sqrt{1-4z}} - [z^{q+n}]\binom{2q}{q}^{-1}\sum_{p=0}^{q-1}\binom{2p}{p}z^p.$$

The second term does not contribute to the coefficient eqtractor and we get

$$[z^{n+q}]\binom{2q}{q}^{-1}\frac{1}{\sqrt{1-4z}} = \binom{2q}{q}^{-1}\binom{2q+2n}{q+n}.$$

We simplify to the required form:

$$\frac{q!\times q!}{(2q)!}\frac{(2q+2n)!}{(q+n)!\times(q+n)!}$$

$$= \binom{q+n}{n}^{-1}\frac{q!\times(2q+2n)!}{n!\times(2q)!\times(q+n)!}$$

$$= \binom{q+n}{n}^{-1}\frac{q!}{n!\times(2q)!}2^{q+n}\prod_{p=0}^{n-1}(2q+2n-1-2p)\frac{(2q)!}{q!\times 2^q}$$

$$= \binom{q+n}{n}^{-1}2^{2n}\binom{q+n-1/2}{n}.$$

With the definition $\binom{n}{k} = n^{\underline{k}}/k!$ this extends to

$$\binom{x+n}{n}^{-1}2^{2n}\binom{x+n-1/2}{n}$$

which is the claim. Note that it can be re-written for $n \ge 1$ as

$$2^n\prod_{p=1}^{n}\frac{1}{x+p}\prod_{p=0}^{n-1}(2x+2n-1-2p) = 2^n\prod_{p=1}^{n}\frac{1}{x+p}\prod_{p=0}^{n-1}(2x+2p+1)$$

which shows the singularities and zeros.

This problem is from page 33 eqn. 3.95 of H.W.Gould's *Combinatorial Identities* [Gou72a]. This has also appeared at math.stackexchange.com problem 4461543. Markus Scheuer has written a detailed explanation of the above which can be found at math.stackexchange.com problem 4537379.

1.81 Double square root

We seek to show that

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{(2k-1)(2n-2k+1)} = \frac{2^{4n}}{2n(2n+1)}\binom{2n}{n}^{-1}.$$

Re-write the LHS to obtain

$$(-1)^n \sum_{k=0}^{n} \binom{-n+k-1}{n-k} \binom{2k}{k} (-1)^k \frac{1}{(2k-1)(2n-2k+1)}.$$

This is

$$(-1)^n [z^{2n+1}] \log \frac{1}{1-z} [w^n] (1+w)^{-n-1} \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k w^k (1+w)^k z^{2k}.$$

Here the coefficient extractor in w enforces the upper limit of the sum and we may extend to infinity:

$$(-1)^{n+1} [z^{2n+1}] \log \frac{1}{1-z} [w^n] \frac{1}{(1+w)^{n+1}} \sqrt{1+4z^2 w(1+w)}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1}{(1+w)^{n+1}} \sqrt{1+4z^2 w(1+w)}.$$

Now put $w(1+w) = v$ so that $w = (-1+\sqrt{1+4v})/2$ and $dw = 1/\sqrt{1+4v}\; dv$:

$$\operatorname*{res}_{v} \frac{1}{v^{n+1}} \sqrt{1+4z^2 v} \frac{1}{\sqrt{1+4v}}.$$

This could have been obtained by inspection. Continuing,

$$\operatorname*{res}_{v} \frac{1}{v^{n+1}} \sqrt{1+4\frac{(z^2-1)v}{1+4v}}$$

$$= -[v^n] \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k (z^2-1)^k \frac{v^k}{(1+4v)^k}$$

$$= -\sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k (z^2-1)^k \binom{n-1}{k-1} (-1)^{n-k} 4^{n-k}$$

$$= (-1)^{n+1} \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} \binom{n-1}{k-1} 4^{n-k} \sum_{q=0}^{k} \binom{k}{q} (-1)^{k-q} z^{2q}.$$

Activating the coefficient extractor in z will produce

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} \binom{n-1}{k-1} 4^{n-k} \sum_{q=0}^{k} \binom{k}{q} \frac{(-1)^{k-q}}{2n-2q+1}.$$

For the inner sum we introduce

$$f(z) = k! \frac{1}{2n+1-2z} \prod_{p=0}^{k} \frac{1}{z-p}$$

which has the property that with $0 \le q \le k$

$$\mathrm{Res}_{z=q} f(z) = k! \frac{1}{2n+1-2q} \prod_{p=0}^{q-1} \frac{1}{q-p} \prod_{p=q+1}^{k} \frac{1}{q-p}$$

$$= k! \frac{1}{2n+1-2q} \frac{1}{q!} \frac{(-1)^{k-q}}{(k-q)!} = \binom{k}{q} \frac{(-1)^{k-q}}{2n+1-2q}.$$

With residues summing to zero and the residue at infinity being zero by inspection the sum is minus the residue at $z = (2n+1)/2$:

$$\frac{1}{2} k! \prod_{p=0}^{k} \frac{1}{(2n+1)/2 - p} = 2^k k! \prod_{p=0}^{k} \frac{1}{2n+1-2p}$$

$$= 2^k k! \frac{1}{(2n+1)!!} (2n-2k-1)!! = 2^k k! \frac{2^n \times n!}{(2n+1)!} \frac{(2n-2k-1)!}{2^{n-k-1} \times (n-k-1)!}.$$

Merging in the case $k = n$ yields

$$2^k k! \frac{2^n \times n!}{(2n+1)!} \frac{(2n-2k)!}{2^{n-k} \times (n-k)!} = 2^{2k} k! \frac{n!}{(2n+1)!} \frac{(2n-2k)!}{(n-k)!}.$$

We find for our sum

$$4^n \frac{n!}{(2n+1)!} \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} \binom{n-1}{k-1} k! \frac{(2n-2k)!}{(n-k)!}.$$

This is

$$4^n \frac{n! \times (n-1)!}{(2n+1)!} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{k}{2k-1}.$$

With

$$\frac{k}{2k-1} = \frac{1}{2} + \frac{1}{2} \frac{1}{2k-1}$$

we get two pieces.
We can evaluate the first one by inspection and find

$$2^{2n} \frac{1}{n(2n+1)} \binom{2n}{n}^{-1} \frac{1}{2} [z^n] \frac{1}{1-4z} = 2^{4n} \frac{1}{2n(2n+1)} \binom{2n}{n}^{-1}.$$

This is precisely the claim. It remains to show that the other piece is zero.
We get

$$-\frac{1}{2}[z^n]\sqrt{1-4z}\frac{1}{\sqrt{1-4z}} = -[z^n]1 = 0$$

when $n \ge 1$. This completes the proof.
Remark. In the above we have used the following coefficient extractors:

$$[z^n]\frac{1}{\sqrt{1-4z}} = \operatorname*{res}_z \frac{1}{z^{n+1}}\frac{1}{\sqrt{1-4z}}.$$

With $w = \frac{1-\sqrt{1-4z}}{2}$ we get $z = w(1-w)$ and $dz = (1-2w)\,dw$ and we have

$$\operatorname*{res}_w \frac{1}{w^{n+1}(1-w)^{n+1}}\frac{1}{1-2w}(1-2w) = \binom{2n}{n}.$$

We also use

$$-[z^n]\sqrt{1-4z} = -\operatorname*{res}_z \frac{1}{z^{n+1}}\sqrt{1-4z}$$

$$= -\operatorname*{res}_w \frac{1}{w^{n+1}(1-w)^{n+1}}(1-2w)(1-2w)$$

$$= -\binom{2n}{n} + 4\binom{2n-1}{n} - 4\binom{2n-2}{n}$$

$$= \binom{2n}{n} \times \left[-1 + 4\frac{n}{2n} - 4\frac{n(n-1)}{2n(2n-1)}\right]$$

$$= \binom{2n}{n}\frac{1}{2n-1}.$$

This problem has not appeared at math.stackexchange.com. It is from page 33 eqn. 3.94 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.82 Central Delannoy Numbers

We seek to show that

$$\sum_{k=0}^{n} \binom{4n-4k}{2n-2k}\binom{4k}{2k} = 2^{4n-1} + 2^{2n-1}\binom{2n}{n}$$

and

$$\sum_{k=0}^{n-1} \binom{4n-4k-2}{2n-2k-1}\binom{4k+2}{2k+1} = 2^{4n-1} - 2^{2n-1}\binom{2n}{n}.$$

First sum

We get for the first sum

$$\sum_{k=0}^{2n} \binom{4n-2k}{2n-k} \binom{2k}{k} \frac{1+(-1)^k}{2}.$$

The first piece is

$$\frac{1}{2} \sum_{k=0}^{2n} \binom{4n-2k}{2n-k} \binom{2k}{k}$$
$$= \frac{1}{2} [z^{2n}] (1+z)^{4n} \sum_{k=0}^{2n} z^k (1+z)^{-2k} \binom{2k}{k}.$$

Here the coefficient extractor enforces the upper limit of the sum and we may extend to infinity:

$$\frac{1}{2} [z^{2n}] (1+z)^{4n} \frac{1}{\sqrt{1-4z/(1+z)^2}} = \frac{1}{2} [z^{2n}] (1+z)^{4n+1} \frac{1}{1-z}$$
$$= \frac{1}{2} \sum_{q=0}^{2n} \binom{4n+1}{q} = \frac{1}{4} 2^{4n+1} = 2^{4n-1}.$$

Good, we have obtained the first term of the closed form. The second piece is

$$\frac{1}{2} [z^{2n}] (1+z)^{4n} \frac{1}{\sqrt{1+4z/(1+z)^2}}$$
$$= \frac{1}{2} [z^{2n}] (1+z)^{4n+1} \frac{1}{\sqrt{1+6z+z^2}}.$$

Now observe that

$$[z^q] \frac{1}{\sqrt{1+6z+z^2}} = [z^q] \frac{1}{\sqrt{1+4z(3/2+z/4)}}$$
$$= [z^q] \sum_{p=0}^{q} \binom{2p}{p} (-1)^p z^p (3/2+z/4)^p$$
$$= \sum_{p=0}^{q} \binom{2p}{p} (-1)^p \binom{p}{q-p} \frac{1}{4^{q-p}} \frac{3^{2p-q}}{2^{2p-q}}$$
$$= \frac{1}{2^q} \sum_{p=0}^{q} \binom{2p}{p} (-1)^p \binom{p}{q-p} 3^{2p-q}.$$

Next use

$$\binom{2p}{p} \binom{p}{q-p} = \frac{(2p)!}{p! \times (q-p)! \times (2p-q)!} = \binom{2p}{q} \binom{q}{p}.$$

Note that the residue definition of the binomial coefficient $\operatorname{res}_z \frac{1}{z^{q-p+1}}(1+z)^p$ has that we get zero when $p \ge 0$ and $q-p > p$ or $q > 2p$ so there is no singularity when $2p < q$. This is enforced by the first binomial coefficient of the re-written form. We obtain

$$\frac{1}{2^q}[z^q]\sum_{p=0}^{q}\binom{q}{p}(-1)^p(1+z)^{2p}3^{2p-q}$$

$$= \frac{1}{6^q}[z^q](1-9(1+z)^2)^q = \frac{1}{6^q}[z^q](4+3z)^q(-2-3z)^q$$

$$= \frac{(-1)^q}{6^q}[z^q](4+3z)^q(2+3z)^q$$

$$= \frac{(-1)^q}{6^q}\sum_{p=0}^{q}\binom{q}{p}3^p2^{2q-2p}\binom{q}{q-p}3^{q-p}2^p$$

$$= (-1)^q\sum_{p=0}^{q}\binom{q}{p}^2 2^{q-p} = (-1)^q\sum_{p=0}^{q}\binom{q}{p}^2 2^p = (-1)^q[w^q](1+2w)^q(1+w)^q.$$

We recognize that we have a signed version of the central Delannoy numbers here. Substitute into the coefficient extractor for the sum to get

$$\frac{1}{2}\sum_{q=0}^{2n}\binom{4n+1}{q}(-1)^q[w^{2n-q}](1+2w)^{2n-q}(1+w)^{2n-q}$$

$$= \frac{1}{2}[w^{2n}](1+2w)^{2n}(1+w)^{2n}\sum_{q=0}^{2n}\binom{4n+1}{q}\frac{(-1)^q w^q}{(1+2w)^q(1+w)^q}.$$

Here the coefficient extractor again enforces the upper limit and we may extend to infinity:

$$\frac{1}{2}[w^{2n}](1+2w)^{2n}(1+w)^{2n}\left[1-\frac{w}{(1+2w)(1+w)}\right]^{4n+1}$$

$$= \frac{1}{2}[w^{2n}]\frac{1}{(1+2w)^{2n+1}}\frac{1}{(1+w)^{2n+1}}\left[2w(1+w)+1\right]^{4n+1}$$

$$= \frac{1}{2}\operatorname{res}_w \frac{1}{w^{2n+1}}\frac{1}{(1+2w)^{2n+1}}\frac{1}{(1+w)^{2n+1}}\left[2w(1+w)+1\right]^{4n+1}.$$

Now put $w(1+w) = v$ so that $w = (-1+\sqrt{1+4v})/2$ and $dw = 1/\sqrt{1+4v}\; dv$ to obtain

$$\frac{1}{2}\operatorname{res}_v \frac{1}{v^{2n+1}}\frac{1}{\sqrt{1+4v}^{2n+1}}(2v+1)^{4n+1}\frac{1}{\sqrt{1+4v}}$$

$$= \frac{1}{2}[v^{2n}]\frac{(1+2v)^{4n+1}}{(1+4v)^{n+1}} = \frac{1}{2}\sum_{q=0}^{2n}\binom{4n+1}{2n-q}2^{2n-q}(-1)^q 4^q \binom{q+n}{q}$$

$$= 2^{2n-1}\sum_{q=0}^{2n}\binom{-2n-q-2}{2n-q}2^q\binom{q+n}{q}$$

$$= 2^{2n-1}[w^{2n}]\frac{1}{(1+w)^{2n+2}}\sum_{q=0}^{2n}\frac{w^q}{(1+w)^q}2^q\binom{q+n}{q}.$$

Using the coefficient extractor to enforce the range,

$$2^{2n-1}[w^{2n}]\frac{1}{(1+w)^{2n+2}}\frac{1}{(1-2w/(1+w))^{n+1}}$$

$$= 2^{2n-1}[w^{2n}]\frac{1}{(1+w)^{n+1}}\frac{1}{(1-w)^{n+1}} = 2^{2n-1}[w^{2n}]\frac{1}{(1-w^2)^{n+1}}$$

$$= 2^{2n-1}[w^n]\frac{1}{(1-w)^{n+1}} = 2^{2n-1}\binom{2n}{n}.$$

This is the second term from the closed form and concludes the proof.

Second sum

We get for the second sum

$$\sum_{k=0}^{2n-1}\binom{4n-2k}{2n-k}\binom{2k}{k}\frac{1-(-1)^k}{2}.$$

Now we just recombine the pieces from the previous calculation to obtain the result.

This problem has not appeared at math.stackexchange.com. It is from page 33 eqn. 3.97 and eqn. 3.98 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.83 A case of factorization

We seek to show that

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}\binom{2k}{k}\frac{x}{x+k} = (-1)^n\binom{x+n}{n}^{-1}\binom{x-1}{n}.$$

Note that we get a polynomial in x on the LHS and the RHS on multiplication by $\binom{x+n}{n}$ so we just need to prove it for x a positive integer and it will hold for all i.e. complex x. Recall from section 1.80 that

$$\binom{2k}{k}\frac{1}{k+q} = [z^{k+q}]\frac{1-\sqrt{1-4z}\sum_{p=0}^{q-1}\binom{2p}{p}z^p}{q\times\binom{2q}{q}}.$$

Note also that the first non-zero coefficient of this OGF is on z^q. We get for the LHS

$$\sum_{k=0}^n (-1)^k \binom{n+k}{n-k} \binom{2k}{k} \frac{x}{x+k}$$

$$= [z^n](1+z)^n \sum_{k=0}^n (-1)^k z^k (1+z)^k \binom{2k}{k} \frac{x}{x+k}$$

$$= x(-1)^x [z^n](1+z)^n \frac{1-\sqrt{1+4z(1+z)}\sum_{p=0}^{x-1}\binom{2p}{p}(-1)^p z^p (1+z)^p}{x \times \binom{2x}{x} \times z^x \times (1+z)^x}.$$

Here the coefficient extractor has enforced the range of the sum. Continuing,

$$(-1)^x \binom{2x}{x}^{-1} \operatorname*{res}_z \frac{1}{z^{x+n+1}} (1+z)^{n-x} \left[1-(1+2z)\sum_{p=0}^{x-1}\binom{2p}{p}(-1)^p z^p (1+z)^p\right].$$

We get three pieces here without the scalar in front, the first is

$$\binom{n-x}{n+x}.$$

The second is (write $1+2z = z+(1+z)$)

$$-\sum_{p=0}^{x-1} \binom{2p}{p} (-1)^p \binom{n-x+p}{n+x-p-1}.$$

The third is

$$-\sum_{p=0}^{x-1} \binom{2p}{p} (-1)^p \binom{n-x+p+1}{n+x-p}.$$

Continuing with the second,

$$(-1)^x \sum_{p=0}^{x-1} \binom{2x-2-2p}{x-1-p} (-1)^p \binom{n-1-p}{n+p}$$

$$= (-1)^{x+n} \sum_{p=0}^{x-1} \binom{2x-2-2p}{x-1-p} \binom{2p}{n+p}$$

$$= (-1)^{x+n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^x} (1+z)^{2x-2}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \sum_{p=0}^{x-1} \frac{z^p}{(1+z)^{2p}} \frac{(1+w)^{2p}}{w^p} \; dw \; dz.$$

The residue in z enforces the range and we may extend the sum to infinity. For this to converge we need $|z/(1+z)^2| < |w/(1+w)^2|$. We have $|z/(1+z)^2| \leq \varepsilon/(1-\varepsilon)^2$ and $\gamma/(1+\gamma)^2 \leq |w/(1+w)^2|$. Note that we have $\varepsilon/(1-\varepsilon)^2 < 2\varepsilon$ when $1/2 < (1-\varepsilon)^2$ or $\varepsilon < 1 - 1/\sqrt{2}$. This will be our choice of ε. We also have $\gamma/2 < \gamma/(1+\gamma)^2$ when $(1+\gamma)^2 < 2$ or $\gamma < \sqrt{2} - 1$. This will be our choice of γ. Now we just need to impose with these two conditions a third, which is $2\varepsilon < \gamma/2$ or $\varepsilon < \gamma/4$. A possible pair that works is $\gamma = 1/5$ and $\varepsilon = 1/21$. Continuing,

$$\begin{aligned}
&(-1)^{x+n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^x} (1+z)^{2x-2} \\
&\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1 - z(1+w)^2/w/(1+z)^2} \; dw \; dz \\
&= (-1)^{x+n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^x} (1+z)^{2x} \\
&\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{w(1+z)^2 - z(1+w)^2} \; dw \; dz \\
&= (-1)^{x+n+1} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x+1}} (1+z)^{2x} \\
&\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{(w-z)(w-1/z)} \; dw \; dz.
\end{aligned}$$

Note here that with $\varepsilon < \gamma/4$ the pole at $w = z$ is now inside the contour in addition to the pole at zero. The pole at $w = 1/z$ has norm $1/\varepsilon > 1$ and is definitely not inside the contour. Since residues sum to zero and the residue at infinity is zero by inspection our integral in w is minus the residue at $w = 1/z$, which yields

$$\begin{aligned}
&(-1)^{x+n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x+1}} (1+z)^{2x} z^n \frac{1}{1/z - z} \; dz \\
&= (-1)^{x+n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-n}} (1+z)^{2x} \frac{1}{1-z^2} \; dz \\
&= (-1)^{x+n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-n}} (1+z)^{2x-1} \frac{1}{1-z} \; dz.
\end{aligned}$$

We discover here that we require $x \geq n+1$. We still have agreement at an infinite number of values for our polynomials, so the initial equality is not questioned. The remaining integral yields

$$(-1)^{x+n} \sum_{p=0}^{x-n-1} \binom{2x-1}{p}.$$

Next observe that the third piece is just the second with n replaced by $n+1$

so we get

$$-(-1)^{x+n}\sum_{p=0}^{x-n-2}\binom{2x-1}{p}.$$

Adding these two pieces yields

$$(-1)^{x+n}\binom{2x-1}{x-n-1} = (-1)^{x+n}\binom{2x-1}{n+x} = \binom{n-x}{n+x}.$$

We have established the closed form

$$2\times(-1)^x\binom{2x}{x}^{-1}\binom{n-x}{n+x}.$$

Now to morph this into the RHS of the proposed identity:

$$2\times(-1)^x\frac{x!\times x!}{(2x)!}\times\frac{1}{(n+x)!}\prod_{p=0}^{n+x-1}(n-x-p)$$

$$= \binom{x+n}{n}^{-1}\times 2\times(-1)^x\frac{x!}{(2x)!}\times\frac{1}{n!}\prod_{p=0}^{n-1}(n-x-p)\prod_{p=n}^{n+x-1}(n-x-p)$$

$$= \binom{x+n}{n}^{-1}\times 2\times(-1)^x\frac{x!}{(2x)!}\times\frac{(-1)^n}{n!}\prod_{p=0}^{n-1}(x+p-n)\prod_{p=0}^{x-1}(-x-p)$$

$$= (-1)^n\binom{x+n}{n}^{-1}\binom{x-1}{n}\times 2\times\frac{x!}{(2x)!}\times\prod_{p=0}^{x-1}(x+p)$$

$$= (-1)^n\binom{x+n}{n}^{-1}\binom{x-1}{n}\times 2\times\frac{x!}{(2x)!}\times\frac{(2x-1)!}{(x-1)!}$$

$$= (-1)^n\binom{x+n}{n}^{-1}\binom{x-1}{n}.$$

This is the claim and we may conclude. We have a quotient of two polynomials that factor very nicely.

This problem has not appeared at math.stackexchange.com. It is from page 34 eqn. 3.100 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.84 Two identities due to Grosswald

We seek to show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+2r+k}{n+r}2^{n-k} = (-1)^{n/2}\frac{1+(-1)^n}{2}\binom{n+r}{n}^{-1}\binom{n+r}{n/2}\binom{n+2r}{r}$$

as well as

$$\sum_{k=0}^{n-r}(-1)^k \binom{n}{k+r}\binom{n+k+r}{k} 2^{n-r-k} = (-1)^{(n-r)/2}\frac{1+(-1)^{n-r}}{2}\binom{n}{(n-r)/2}.$$

First identity

We get for the LHS

$$\begin{aligned}
&2^n [z^{n+r}](1+z)^{n+2r} \sum_{k=0}^{n}(-1)^k \binom{n}{k} 2^{-k}(1+z)^k \\
&= 2^n [z^{n+r}](1+z)^{n+2r}(1-(1+z)/2)^n \\
&= [z^{n+r}](1+z)^{n+2r}(1-z)^n \\
&= \operatorname*{res}_{z} \frac{1}{z^{n+r+1}}(1+z)^{n+2r}(1-z)^n.
\end{aligned}$$

Now put $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2\; dw$ to get

$$\begin{aligned}
&\operatorname*{res}_{w} \frac{1}{w^{n+r+1}} \frac{1}{(1-w)^{r-1}} \frac{(1-2w)^n}{(1-w)^n} \frac{1}{(1-w)^2} \\
&= \operatorname*{res}_{w} \frac{1}{w^{n+r+1}} \frac{1}{(1-w)^{n+r+1}} (1-2w)^n.
\end{aligned}$$

Next put $w(1-w) = v$ so that $w = (1-\sqrt{1-4v})/2$ and $dw = 1/\sqrt{1-4v}\; dv$:

$$\operatorname*{res}_{v} \frac{1}{v^{n+r+1}} \sqrt{1-4v}^n \frac{1}{\sqrt{1-4v}} = [v^{n+r}](1-4v)^{(n-1)/2}.$$

Now if n is positive and odd we have $n > (n-1)/2$ and the powered term is a finite series so we obtain zero as per the factor in the RHS. If n is even we get

$$(-1)^{n+r} 2^{2n+2r} \binom{(n-1)/2}{n+r}.$$

This is

$$\begin{aligned}
&\binom{n+r}{n}^{-1} \times (-1)^{n+r} 2^{2n+2r} \frac{1}{n!}\frac{1}{r!} \prod_{p=0}^{n+r-1} (n/2 - 1/2 - p) \\
&= \binom{n+r}{n}^{-1} \times (-1)^{r} 2^{n+r} \frac{1}{n!}\frac{1}{r!} \prod_{p=0}^{n+r-1} (n - 1 - 2p)
\end{aligned}$$

$$= \binom{n+r}{n}^{-1} \times (-1)^r 2^{n+r} \frac{1}{n!} \frac{1}{r!} \prod_{p=0}^{n/2-1} (n-1-2p) \prod_{p=n/2}^{n+r-1} (n-1-2p)$$

$$= \binom{n+r}{n}^{-1} \times (-1)^r 2^{n+r} \frac{1}{n!} \frac{1}{r!} (n-1)!! \prod_{p=0}^{n/2+r-1} (-1-2p)$$

$$= \binom{n+r}{n}^{-1} \times (-1)^{n/2} 2^{n+r} \frac{1}{n!} \frac{1}{r!} \frac{(n-1)!}{2^{n/2-1} \times (n/2-1)!} \prod_{p=0}^{n/2+r-1} (2p+1)$$

$$= \binom{n+r}{n}^{-1} \times (-1)^{n/2} 2^{n+r} \frac{1}{n!} \frac{1}{r!} \frac{n!}{2^{n/2} \times (n/2)!} (n+2r-1)!!$$

$$= \binom{n+r}{n}^{-1} \times (-1)^{n/2} 2^{n+r} \frac{1}{n!} \frac{1}{r!} \frac{n!}{2^{n/2} \times (n/2)!} \frac{(n+2r-1)!}{2^{n/2+r-1} \times (n/2+r-1)!}$$

$$= \binom{n+r}{n}^{-1} \times (-1)^{n/2} 2^{n+r} \frac{1}{n!} \frac{1}{r!} \frac{n!}{2^{n/2} \times (n/2)!} \frac{(n+2r)!}{2^{n/2+r} \times (n/2+r)!}$$

$$= \binom{n+r}{n}^{-1} \times (-1)^{n/2} \frac{(n+2r)!}{r!(n+r)!} \frac{1}{(n/2)!} \frac{(n+r)!}{(n/2+r)!}$$

$$= (-1)^{n/2} \binom{n+r}{n}^{-1} \binom{n+2r}{r} \binom{n+r}{n/2}.$$

Second identity

We get for the LHS where $n \ge r$

$$\sum_{k=0}^{n-r} (-1)^k \binom{n}{n-r-k} \binom{n+k+r}{n+r} 2^{n-r-k}$$

$$= 2^{n-r} [z^{n-r}] (1+z)^n [w^{n+r}] (1+w)^{n+r} \sum_{k=0}^{n-r} (-1)^k z^k (1+w)^k 2^{-k}.$$

The coefficient extractor in z enforces the upper limit of the sum:

$$2^{n-r} [z^{n-r}] (1+z)^n [w^{n+r}] (1+w)^{n+r} \frac{1}{1+z(1+w)/2}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+r+1}} (1+w)^{n+r} \frac{1}{1+z(1+w)/2}$$

Now put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2 \, dv$ to get

$$\begin{aligned}
&\operatorname*{res}_v \frac{1}{v^{n+r+1}}(1-v)\frac{1}{1+z/2/(1-v)}\frac{1}{(1-v)^2}\\
&= \operatorname*{res}_v \frac{1}{v^{n+r+1}}\frac{1}{1-v+z/2} = \frac{1}{1+z/2}\operatorname*{res}_v \frac{1}{v^{n+r+1}}\frac{1}{1-v/(1+z/2)}.
\end{aligned}$$

Substituting into the coefficient extractor in z we obtain

$$2^{n-r}\operatorname*{res}_z \frac{1}{z^{n-r+1}}(1+z)^n\frac{1}{(1+z/2)^{n+r+1}}.$$

Here the residue at infinity is zero and residues sum to zero so we may evaluate through minus the residue at $z=-2$. We write

$$\begin{aligned}
&-2^{2n+1}\mathrm{Res}_{z=-2}\frac{1}{((z+2)-2)^{n-r+1}}((z+2)-1)^n\frac{1}{(z+2)^{n+r+1}}\\
&= (-1)^r 2^{n+r}\mathrm{Res}_{z=-2}\frac{1}{(1-(z+2)/2)^{n-r+1}}(1-(z+2))^n\frac{1}{(z+2)^{n+r+1}}.
\end{aligned}$$

This is

$$\begin{aligned}
&(-1)^r 2^{n+r}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n-r}\frac{1}{2^{n+r-k}}\\
&= (-1)^r\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n-r}2^k\\
&= (-1)^r[z^{n-r}](1+z)^{2n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^k}{(1+z)^k}\\
&= (-1)^r[z^{n-r}](1+z)^{2n}\left[1-\frac{2}{1+z}\right]^n\\
&= (-1)^r[z^{n-r}](1+z)^n(-1+z)^n = (-1)^{n-r}[z^{n-r}](1-z^2)^n.
\end{aligned}$$

This is zero if n and r do not have the same parity, precisely as in the proposed RHS. If they do have the same parity we obtain

$$(-1)^{(n-r)/2}\binom{n}{(n-r)/2}$$

as claimed.

This problem has not appeared at math.stackexchange.com. It is from page 34 eqns. 3.103 and 3.104 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.85 Appearance of the constant three

We seek to show that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2k}{n+k}3^{2n-k} = \binom{2n}{n}.$$

We have for the LHS

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{2n-k}\binom{2n+2k}{n+k}3^{2n-k}$$

$$= 3^{2n} \operatorname*{res}_w \frac{(1+w)^{2n}}{w^{n+1}}[z^{2n}](1+z)^{2n}\sum_{k=0}^{2n}(-1)^k z^k \frac{(1+w)^{2k}}{w^k}3^{-k}.$$

The coefficient extractor in z enforces the upper range of the sum and we may extend to infinity to obtain

$$3^{2n} \operatorname*{res}_w \frac{(1+w)^{2n}}{w^{n+1}}[z^{2n}](1+z)^{2n}\frac{1}{1+z(1+w)^2/w/3}$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{2n+1}}(1+z)^{2n}\frac{1}{1+z(1+w)^2/w/3}$$

Now put $z/(1+z) = v$ so that $z = v/(1-v)$ and $dz = 1/(1-v)^2\ dv$ to get

$$\operatorname*{res}_v \frac{1}{v^{2n+1}}(1-v)\frac{1}{1+v(1+w)^2/w/3/(1-v)}\frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_v \frac{1}{v^{2n+1}}\frac{1}{1-v(1-(1+w)^2/w/3)}.$$

Substitute into the residue in w to find

$$3^{2n} \operatorname*{res}_w \frac{(1+w)^{2n}}{w^{n+1}}(1-(1+w)^2/w/3)^{2n}$$

$$= \operatorname*{res}_w \frac{(1+w)^{2n}}{w^{3n+1}}(3w-(1+w)^2)^{2n}$$

$$= \operatorname*{res}_w \frac{(1+w)^{2n}}{w^{3n+1}}(-1+w-w^2)^{2n}$$

$$= \operatorname*{res}_w \frac{(1+w)^{2n}}{w^{3n+1}}(1-w+w^2)^{2n}$$

$$= \operatorname*{res}_w \frac{(1+w^3)^{2n}}{w^{3n+1}} = [w^{3n}](1+w^3)^{2n} = [w^n](1+w)^{2n} = \binom{2n}{n}.$$

This is the claim.

This problem has not appeared at math.stackexchange.com. It is from page 34 eqn. 3.106 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.86 Very basic example

We seek to show that

$$\sum_{k=0}^{n} \binom{4n+1}{2n-2k} \binom{k+n}{n} = 2^{2n} \binom{3n}{n}.$$

The LHS is

$$[z^{2n}](1+z)^{4n+1} \sum_{k=0}^{n} z^{2k} \binom{k+n}{n}.$$

Here the coefficient extractor enforces the upper limit of the sum and we may extend to infinity:

$$[z^{2n}](1+z)^{4n+1} \frac{1}{(1-z^2)^{n+1}} = [z^{2n}](1+z)^{3n} \frac{1}{(1-z)^{n+1}}.$$

This is

$$\operatorname*{res}_{z} \frac{1}{z^{2n+1}} (1+z)^{3n} \frac{1}{(1-z)^{n+1}}.$$

Now put $z/(1+z) = v$ so that $z = v/(1-v)$ and $dz = 1/(1-v)^2 \; dv$ to get

$$\operatorname*{res}_{v} \frac{1}{v^{2n+1}} \frac{1}{(1-v)^{n-1}} \frac{(1-v)^{n+1}}{(1-2v)^{n+1}} \frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_{v} \frac{1}{v^{2n+1}} \frac{1}{(1-2v)^{n+1}} = [v^{2n}] \frac{1}{(1-2v)^{n+1}} = 2^{2n} \binom{3n}{2n} = 2^{2n} \binom{3n}{n}$$

as claimed.

This problem has not appeared at math.stackexchange.com. It is from page 35 eqn. 3.115 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.87 Very basic example II

We seek to show that

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n-k} \binom{2n+2k+1}{2k} = (-1)^n (n+1) 2^{2n}.$$

The LHS is

$$[z^n](1+z)^{2n} \sum_{k=0}^{n} (-1)^k z^k \binom{2n+2k+1}{2n+1}$$

$$= [w^{2n+1}](1+w)^{2n+1} [z^n](1+z)^{2n} \sum_{k=0}^{n} (-1)^k z^k (1+w)^{2k}.$$

Here the coefficient extractor in z enforces the upper limit of the sum and we may extend to infinity:

$$[w^{2n+1}](1+w)^{2n+1}[z^n](1+z)^{2n}\frac{1}{1+z(1+w)^2}.$$

The contribution from z is

$$\begin{aligned}&\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}(1+z)^{2n}\frac{1}{1+z(1+w)^2}\\ &=\frac{1}{(1+w)^2}\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}(1+z)^{2n}\frac{1}{z+1/(1+w)^2}.\end{aligned}$$

With residues summing to zero this is minus the residue at $z=-1/(1+w)^2$ plus minus the residue at infinity. We get for the first one

$$\begin{aligned}&-[w^{2n+1}](1+w)^{2n+1}\frac{1}{(1+w)^2}(-1)^{n+1}(1+w)^{2n+2}\left[1-\frac{1}{(1+w)^2}\right]^{2n}\\ &=(-1)^n[w^{2n+1}](1+w)(2w+w^2)^{2n}=(-1)^n[w^1](1+w)(2+w)^{2n}\\ &=(-1)^n\binom{2n}{1}2^{2n-1}+(-1)^n\binom{2n}{0}2^{2n}=(-1)^n(n+1)2^{2n}\end{aligned}$$

as claimed. Now we just need to verify that the contribution from the residue at infinity is zero. We obtain

$$\begin{aligned}&[w^{2n+1}](1+w)^{2n+1}\ \mathrm{Res}_{z=0}\frac{1}{z^2}z^{n+1}\frac{(1+z)^{2n}}{z^{2n}}\frac{1}{1+(1+w)^2/z}\\ &=[w^{2n+1}](1+w)^{2n-1}\ \mathrm{Res}_{z=0}\frac{1}{z}z^{n+1}\frac{(1+z)^{2n}}{z^{2n}}\frac{1}{1+z/(1+w)^2}\\ &=[w^{2n+1}](1+w)^{2n-1}\ \mathrm{Res}_{z=0}\frac{(1+z)^{2n}}{z^n}\frac{1}{1+z/(1+w)^2}.\end{aligned}$$

Computing the residue,

$$\begin{aligned}&[w^{2n+1}](1+w)^{2n-1}\ \sum_{q=0}^{n-1}\binom{2n}{n-1-q}(-1)^q\frac{1}{(1+w)^{2q}}\\ &=\sum_{q=0}^{n-1}\binom{2n}{n-1-q}(-1)^q[w^{2n+1}](1+w)^{2n-1-2q}=0\end{aligned}$$

as desired. This went through with the maximum upper range of the sum in q, it does not work with $q=n$.

This problem has not appeared at math.stackexchange.com. It is from page 35 eqn. 3.114 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.88 Nested square root

We seek to show that

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-k}{n} \frac{k}{(2n-k) \times 2^k} = (-1)^n 2^{2n} \binom{-1/4}{n}.$$

The LHS is

$$\begin{aligned} &\frac{1}{n} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-k-1}{n-1} \frac{k}{2^k} \\ &= \frac{1}{n} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-k-1}{n-k} \frac{k}{2^k} \\ &= \frac{1}{n} [z^n] (1+z)^{2n-1} \sum_{k=0}^{n} \binom{2k}{k} \frac{k}{2^k} \frac{z^k}{(1+z)^k}. \end{aligned}$$

Here we may extend to infinity because of the coefficient extractor in z:

$$\frac{1}{n} \operatorname*{res}_{z} \frac{1}{z^{n+1}} (1+z)^{2n-1} \sum_{k \geq 0} \binom{2k}{k} \frac{k}{2^k} \frac{z^k}{(1+z)^k}.$$

Now put $z/(1+z) = v$ so that $z = v/(1-v)$ and $dz = 1/(1-v)^2 \; dv$ to obtain

$$\begin{aligned} &\frac{1}{n} \operatorname*{res}_{v} \frac{1}{v^{n+1}} \frac{1}{(1-v)^{n-2}} \frac{1}{(1-v)^2} \sum_{k \geq 0} \binom{2k}{k} \frac{k}{2^k} v^k \\ &= \frac{1}{n} \operatorname*{res}_{v} \frac{1}{v^n} \frac{1}{(1-v)^n} \frac{1}{\sqrt{1-2v}^3}. \end{aligned}$$

Next put $v(1-v) = w$ so that $v = (1-\sqrt{1-4w})/2$ and $dv = 1/\sqrt{1-4w} \; dw$ to get

$$\begin{aligned} &\frac{1}{n} \operatorname*{res}_{w} \frac{1}{w^n} \frac{1}{\sqrt{\sqrt{1-4w}}^3} \frac{1}{\sqrt{1-4w}} \\ &= \frac{1}{n} \operatorname*{res}_{w} \frac{1}{w^n} \frac{1}{(1-4w)^{5/4}} \\ &= \frac{1}{n} (-1)^{n-1} 2^{2n-2} \binom{-5/4}{n-1} = (-1)^{n-1} 2^{2n-2} \binom{-1/4}{n} (-4) \\ &= (-1)^n 2^{2n} \binom{-1/4}{n}. \end{aligned}$$

This is the claim. Here we have made use of the fact that

$$[z^n]\frac{2z}{\sqrt{1-4z}^3} = 2[z^{n-1}]\frac{1}{\sqrt{1-4z}^3} = 2(-1)^{n-1}2^{2n-2}\binom{-3/2}{n-1}$$
$$= (-1)^{n-1}2^{2n-1}\binom{-1/2}{n}n(-2) = (-1)^n 2^{2n} n\binom{-1/2}{n} = n\binom{2n}{n}.$$

This problem has not appeared at math.stackexchange.com. It is from page 35 eqn. 3.110 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.89 Harmonic numbers and a squared binomial coefficient

We seek to show that

$$\sum_{k=1}^{n}\binom{n}{k}^2 H_k = \binom{2n}{n}(2H_n - H_{2n}).$$

The LHS is

$$\sum_{k=0}^{n-1}\binom{n}{k}^2 H_{n-k} = [z^n]\frac{1}{1-z}\log\frac{1}{1-z}\sum_{k=0}^{n-1}\binom{n}{k}^2 z^k.$$

Here the contribution from $k = n$ is zero and we may include this value in our sum:

$$[z^n]\frac{1}{1-z}\log\frac{1}{1-z}\sum_{k=0}^{n}\binom{n}{k}^2 z^k$$
$$= [z^n]\frac{1}{1-z}\log\frac{1}{1-z}[w^n](1+w)^n\sum_{k=0}^{n}\binom{n}{k}w^k z^k$$
$$= [z^n]\frac{1}{1-z}\log\frac{1}{1-z}[w^n](1+w)^n(1+wz)^n.$$

The contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^{n+1}}(1+w)^n(1+wz)^n.$$

Now put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2\ dv$ to get

$$\operatorname*{res}_{v}\frac{1}{v^{n+1}}(1-v)(1+zv/(1-v))^n\ \frac{1}{(1-v)^2}$$
$$= \operatorname*{res}_{v}\frac{1}{v^{n+1}(1-v)^{n+1}}(1-(1-z)v)^n.$$

A binomial identity

Introduce with $q \ge 1$

$$f(z) = n!(-1)^n \frac{1}{z+q} \prod_{p=0}^{n} \frac{1}{z-p}.$$

This has the property that for $0 \le r \le n$

$$\begin{aligned}\mathrm{Res}_{z=r} f(z) &= n!(-1)^n \frac{1}{r+q} \prod_{p=0}^{r-1} \frac{1}{r-p} \prod_{p=r+1}^{n} \frac{1}{r-p} \\ &= n!(-1)^n \frac{1}{r+q} \frac{1}{r!} \frac{(-1)^{n-r}}{(n-r)!} = \binom{n}{r} \frac{(-1)^r}{r+q}.\end{aligned}$$

With the residue at infinity being zero by inspection we obtain

$$\sum_{r=0}^{n} \binom{n}{r} \frac{(-1)^r}{r+q} = -\mathrm{Res}_{z=-q} f(z)$$

$$= -n!(-1)^n \prod_{p=0}^{n} \frac{1}{-q-p} = n! \prod_{p=0}^{n} \frac{1}{q+p} = n! \frac{(q-1)!}{(q+n)!} = \frac{1}{q} \binom{n+q}{q}^{-1}.$$

Therefore with $1 \le k \le n$

$$\begin{aligned}\frac{1}{k} \binom{n}{k}^{-1} &= \sum_{r=0}^{n-k} \binom{n-k}{r} \frac{(-1)^r}{r+k} = (-1)^{n-k} \sum_{r=0}^{n-k} \binom{n-k}{r} \frac{(-1)^r}{n-r} \\ &= [z^n] \log \frac{1}{1-z} (-1)^{n-k} \sum_{r=0}^{n-k} \binom{n-k}{r} (-1)^r z^r \\ &= [z^n] \log \frac{1}{1-z} (-1)^{n-k} (1-z)^{n-k}.\end{aligned}$$

Processing the first and second piece

Returning to the residue in v we find

$$\operatorname*{res}_{v} \frac{1}{v^{n+1}(1-v)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (1-z)^{n-k} v^{n-k}.$$

Evaluating at $k=n$ and substituting into the coefficient extractor in z yields

$$\binom{2n}{n} H_n$$

which is our first piece. Restoring the coefficient extractor in z will produce

$$[z^n] \log \frac{1}{1-z} \operatorname*{res}_v \frac{1}{v^{n+1}(1-v)^{n+1}} \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} (1-z)^{n-k-1} v^{n-k}$$

$$= -[z^n] \log \frac{1}{1-z} \operatorname*{res}_v \frac{1}{v^{n+1}(1-v)^{n+1}} \sum_{k=1}^{n} \binom{n}{k-1} (-1)^{n-k} (1-z)^{n-k} v^{n+1-k}$$

$$= - \operatorname*{res}_v \frac{1}{v^{n+1}(1-v)^{n+1}} \sum_{k=1}^{n} \binom{n}{k-1} \frac{1}{k} \binom{n}{k}^{-1} v^{n+1-k}$$

$$= - \sum_{k=1}^{n} \binom{n}{k-1} \frac{1}{k} \binom{n}{k}^{-1} \binom{n+k-1}{n}.$$

Observe that

$$\binom{n}{k-1} \frac{1}{k} \binom{n}{k}^{-1} = \frac{n! \times k! \times (n-k)!}{(n+1-k)! \times (k-1)! \times k \times n!}$$

$$= \frac{1}{n+1-k}$$

so that our second piece becomes

$$- \sum_{k=1}^{n} \frac{1}{n+1-k} \binom{n+k-1}{n} = - \sum_{k=1}^{n} \frac{1}{k} \binom{2n-k}{n}$$

$$= - \sum_{k=1}^{n} \frac{1}{k} \binom{2n-k}{n-k} = -[z^n](1+z)^{2n} \sum_{k=1}^{n} \frac{1}{k} \frac{z^k}{(1+z)^k}.$$

Here we may extend to infinity owing to the coefficient extractor:

$$-[z^n](1+z)^{2n} \log \frac{1}{1-z/(1+z)} = [z^n](1+z)^{2n} \log \frac{1}{1+z}$$

$$= \sum_{q=0}^{n-1} \binom{2n}{q} \frac{(-1)^{n-q}}{n-q}$$

$$= [z^{2n}] \log \frac{1}{1-z} \sum_{q=0}^{n-1} \binom{2n}{q} \binom{2n-q}{n-q} (-1)^q z^q (1-z)^n.$$

Next observe that

$$\binom{2n}{q} \binom{2n-q}{n-q} = \frac{(2n)!}{q! \times (n-q)! \times n!} = \binom{2n}{n} \binom{n}{q}$$

so we obtain

$$\binom{2n}{n}[z^{2n}]\log\frac{1}{1-z}(1-z)^n\sum_{q=0}^{n-1}\binom{n}{q}(-1)^q z^q$$

Two parts of the second piece

The piece now splits into two subpieces, which are (without the central binomial coefficient)

$$\operatorname*{res}_z (-1)^{n+1}\frac{1}{z^{n+1}}\log\frac{1}{1-z}(1-z)^n$$

and

$$\operatorname*{res}_z \frac{1}{z^{2n+1}}\log\frac{1}{1-z}(1-z)^{2n}.$$

We put $z/(1-z)=w$ so that $z=w/(1+w)$ and $dz=1/(1+w)^2\,dw$ to get

$$\operatorname*{res}_w (-1)^{n+1}\frac{1}{w^{n+1}}(1+w)\log\frac{1}{1-w/(1+w)}\frac{1}{(1+w)^2}$$
$$= \operatorname*{res}_w (-1)^n\frac{1}{w^{n+1}}\frac{1}{1+w}\log\frac{1}{1+w} = [w^n]\frac{1}{1-w}\log\frac{1}{1-w} = H_n$$

and

$$-\operatorname*{res}_w (-1)^{2n}\frac{1}{w^{2n+1}}\frac{1}{1+w}\log\frac{1}{1+w} = -[w^{2n}]\frac{1}{1-w}\log\frac{1}{1-w} = -H_{2n}.$$

Collecting all three components we find

$$\binom{2n}{n}H_n+\binom{2n}{n}H_n-\binom{2n}{n}H_{2n}$$

as claimed and we may conclude. Regarding this computation consult 1.100 for a generalization.

This problem has not appeared at math.stackexchange.com. It is from page 36 eqn. 3.125 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.90 Harmonic numbers and a double binomial coefficient

We seek to show that

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}H_k = \frac{(-1)^n}{n}$$

as well as

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k-1}{k} H_{n+k-1} = \frac{(-1)^n}{n}.$$

First identity

The LHS is

$$\sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} \binom{2n-1-k}{n-k} H_{n-k}$$
$$= [z^n] \frac{1}{1-z} \log \frac{1}{1-z} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} \binom{2n-1-k}{n-k} z^k.$$

Here the contribution from $k = n$ is zero and we may include this value in our sum:

$$[z^n] \frac{1}{1-z} \log \frac{1}{1-z} [w^n](1+w)^{2n-1} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{w^k}{(1+w)^k} z^k$$
$$= [z^n] \frac{1}{1-z} \log \frac{1}{1-z} [w^n](1+w)^{2n-1} \left[-1 + \frac{wz}{1+w}\right]^n$$
$$= [z^n] \frac{1}{1-z} \log \frac{1}{1-z} [w^n](1+w)^{n-1} [-1-w+wz]^n.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} (1+w)^{n-1} (-1-w+wz)^n.$$

Now put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2 \; dv$ to get

$$\operatorname*{res}_{v} \frac{1}{v^{n+1}} (1-v)^2 (-1 - v/(1-v) + vz/(1-v))^n \; \frac{1}{(1-v)^2}$$
$$= \operatorname*{res}_{v} \frac{1}{v^{n+1}(1-v)^n} (-1+vz)^n$$
$$= \operatorname*{res}_{v} \frac{1}{v^{n+1}(1-v)^n} \sum_{k=0}^{n} \binom{n}{k} (v-1)^k v^{n-k} (z-1)^{n-k}$$
$$= \operatorname*{res}_{v} \frac{1}{v^{n+1}(1-v)^n} \sum_{k=0}^{n} \binom{n}{k} (v-1)^k v^{n-k} (-1)^{n-k} (1-z)^{n-k}.$$

Note that this makes for a zero contribution when $k = n$. Recall from the previous section that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [z^n] \log\frac{1}{1-z}(-1)^{n-k}(1-z)^{n-k}.$$

We get for the remaining sum on dividing by $1-z$ from the coefficient extractor in z

$$\sum_{k=0}^{n-1}\binom{n}{k}(v-1)^k v^{n-k}(-1)^{n-k}(1-z)^{n-1-k}$$
$$= -\sum_{k=1}^{n}\binom{n}{k-1}(v-1)^{k-1}v^{n+1-k}(-1)^{n-k}(1-z)^{n-k}.$$

Applying the coefficient extractor yields

$$-\operatorname*{res}_v \frac{1}{v^{n+1}(1-v)^n}\sum_{k=1}^{n}\binom{n}{k-1}(v-1)^{k-1}v^{n+1-k}\frac{1}{k}\binom{n}{k}^{-1}.$$

Observe that

$$\binom{n}{k-1}\frac{1}{k}\binom{n}{k}^{-1} = \frac{k!\times(n-k)!}{(k-1)!\times(n+1-k)!\times k}$$
$$= \frac{1}{n+1-k}.$$

We obtain

$$\sum_{k=1}^{n}\frac{1}{n+1-k}(-1)^k[v^{k-1}]\frac{1}{(1-v)^{n+1-k}}$$
$$= \sum_{k=1}^{n}\frac{1}{n+1-k}(-1)^k\binom{n-1}{k-1}$$
$$= -[w^n]\log\frac{1}{1-w}\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}w^{k-1} = -[w^n]\log\frac{1}{1-w}(1-w)^{n-1}$$
$$= -\operatorname*{res}_w \frac{1}{w^{n+1}}\log\frac{1}{1-w}(1-w)^{n-1}.$$

Now put $w/(1-w) = v$ so that $w = v/(1+v)$ and $dw = 1/(1+v)^2\ dv$ to get

$$-\operatorname*{res}_v \frac{1}{v^{n+1}}(1+v)^2\log\frac{1}{1-v/(1+v)}\frac{1}{(1+v)^2}$$
$$= -[v^n]\log(1+v) = [v^n]\log\frac{1}{1+v} = \frac{(-1)^n}{n}.$$

This is the claim.

Second identity

Recapitulating the work from the first identity we find

$$[z^{2n-1}]\frac{1}{1-z}\log\frac{1}{1-z}[w^n](1+w)^{n-1}[-1-w+wz]^n.$$

Start with the contribution from w. It is given by

$$\operatorname*{res}_w \frac{1}{w^{n+1}}(1+w)^{n-1}[-1-w+wz]^n.$$

We put $w/(-1-w+wz) = v$ so that $w = v/(v(z-1)-1)$ and $dw = -1/(v(z-1)-1)^2\, dv$ to get

$$-\operatorname*{res}_v \frac{1}{v^{n+1}}(v(z-1)-1)\frac{(vz-1)^{n-1}}{(v(z-1)-1)^{n-1}}\frac{1}{(v(z-1)-1)^2}$$
$$= (-1)^{n+1}\operatorname*{res}_v \frac{1}{v^{n+1}}\frac{(vz-1)^{n-1}}{(1-v(z-1))^n}.$$

This is

$$(-1)^{n+1}\sum_{q=0}^{n}\binom{n-1}{n-q}z^{n-q}(-1)^{q-1}\binom{n-1+q}{q}(z-1)^q$$

Here $q=0$ makes no contribution by the first binomial coefficient and we have

$$(-1)^n\sum_{q=1}^{n}\binom{n-1}{n-q}(-1)^{q-1}\binom{n-1+q}{q}[z^{n+q-1}]\log\frac{1}{1-z}(z-1)^{q-1}.$$

Now we apply the quoted identity for inverse binomial coefficients replacing n by $n+q-1$ and k by n to find

$$(-1)^n\frac{1}{n}\sum_{q=1}^{n}\binom{n-1}{n-q}(-1)^{q-1}\binom{n-1+q}{q}\binom{n+q-1}{n}^{-1}.$$

Observe that

$$\binom{n-1+q}{q}\binom{n+q-1}{n}^{-1} = \frac{(n-1+q)!\times n!\times(q-1)!}{q!\times(n-1)!\times(n+q-1)!} = \frac{n}{q}$$

so that we get for our sum

$$(-1)^n\frac{1}{n}\sum_{q=1}^{n}\binom{n-1}{q-1}(-1)^{q-1}\frac{n}{q} = (-1)^{n+1}\frac{1}{n}\sum_{q=1}^{n}\binom{n}{q}(-1)^q$$

$$= (-1)^n \frac{1}{n} + \sum_{q=0}^{n} \binom{n}{q} (-1)^q = (-1)^n \frac{1}{n} + (-1+1)^n = (-1)^n \frac{1}{n}.$$

This is the claim. The identity is attributed to R. R. Goldberg.

This problem has not appeared at math.stackexchange.com. It is from pages 36 eqns. 3.123 and 3.124 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.91 Two instances of a harmonic number

We seek to show that

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \binom{n+k}{k} \frac{1}{k} = 2H_n.$$

The LHS is

$$\sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n}{k} \binom{2n-k}{n-k} \frac{1}{n-k}$$

$$= [z^n] \log \frac{1}{1-z} \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n}{k} \binom{2n-k}{n-k} z^k$$

$$= [z^n] \log \frac{1}{1-z} [w^n] (1+w)^{2n} \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n}{k} \frac{w^k}{(1+w)^k} z^k.$$

Note that we get a zero contribution from $k = n$ hence we may include it in our sum to obtain

$$-[z^n] \log \frac{1}{1-z} [w^n] (1+w)^{2n} \left[\frac{wz}{1+w} - 1 \right]^n$$

$$= -[z^n] \log \frac{1}{1-z} [w^n] (1+w)^n [w(z-1) - 1]^n.$$

Expanding the last powered term yields

$$-[z^n] \log \frac{1}{1-z} [w^n] (1+w)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k w^{n-k} (z-1)^{n-k}.$$

Recall from the previous section that with $1 \le k \le n$

$$\frac{1}{k} \binom{n}{k}^{-1} = [z^n] \log \frac{1}{1-z} (-1)^{n-k} (1-z)^{n-k}.$$

In order to apply this formula we first need the term for $k = 0$ from the sum where it does not apply. We get

$$- \operatorname*{res}_{z} \frac{1}{z^{n+1}} \log \frac{1}{1-z} (-1)^n (1-z)^n.$$

We put $z/(1-z) = v$ so that $z = v/(1+v)$ and $dz = 1/(1+v)^2\; dv$ to obtain

$$-(-1)^n \operatorname*{res}_v \frac{1}{v^{n+1}} \log \frac{1}{1-v/(1+v)} (1+v) \frac{1}{(1+v)^2}$$
$$= (-1)^n \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{1+v} \log \frac{1}{1+v}$$
$$= (-1)^n [v^n] \frac{1}{1+v} \log \frac{1}{1+v} = [v^n] \frac{1}{1-v} \log \frac{1}{1-v} = H_n.$$

We have produced one instance of the harmonic number and thus the remaining sum must produce the second one. We find

$$-[w^n](1+w)^n \sum_{k=1}^{n} \binom{n}{k} (-1)^k w^{n-k} \frac{1}{k} \binom{n}{k}^{-1} = -\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{1}{k}$$

$$= -[z^n] \log \frac{1}{1-z} \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} z^k.$$

There is no contribution from $k = n$ and we get

$$-[z^n] \log \frac{1}{1-z} (z-1)^n = H_n.$$

This is the term that we evaluated for the first instance and hence we get a second instance of H_n which proves the claim.

This problem has not appeared at math.stackexchange.com. It is from page 36 eqn. 3.122 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.92 Legendre Polynomials

We seek to prove the following identities for Legendre polyomials where we first show that they are all equivalent and then connect them to the generating function

$$\sum_{n \ge 0} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}}.$$

The four identities are

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}.$$

and

$$P_n(x) = \left[\frac{x-1}{2}\right]^n \sum_{k=0}^{n} \binom{n}{k}^2 \left[\frac{x+1}{x-1}\right]^k$$

as well as

$$P_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^k \left[\frac{x+1}{2}\right]^k.$$

and

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left[\frac{x-1}{2}\right]^k.$$

We get for the first identity

$$\frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n-2k} x^{n-2k}$$

$$= \frac{x^n}{2^n} [z^n] (1+z)^{2n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \frac{z^{2k}}{(1+z)^{2k}} x^{-2k}.$$

Here the coefficient extractor enforces the upper limit of the sum and we may extend to infinity, getting

$$\frac{x^n}{2^n} [z^n] (1+z)^{2n} \left[1 - \frac{z^2}{x^2(1+z)^2}\right]^n$$

$$= \frac{1}{2^n \times x^n} [z^n] (x^2(1+z)^2 - z^2)^n$$

$$= \frac{1}{2^n \times x^n} [z^n] (x + z(x+1))^n (x + z(x-1))^n$$

$$= \frac{1}{2^n \times x^n} \sum_{k=0}^{n} \binom{n}{k} (x+1)^k x^{n-k} \binom{n}{n-k} (x-1)^{n-k} x^k$$

$$= \left[\frac{x-1}{2}\right]^n \sum_{k=0}^{n} \binom{n}{k}^2 \left[\frac{x+1}{x-1}\right]^k.$$

This is the second identity. Continuing we have

$$\frac{1}{2^n \times x^n} \operatorname*{res}_z \frac{1}{z^{n+1}} (x + z(x+1))^n (x + z(x-1))^n$$

Now we put $z/(x + z(x+1)) = w$ so that $z = wx/(1 - w(x+1))$ and $dz = x/(1 - w(x+1))^2 \, dw$ to obtain

$$\frac{1}{2^n \times x^n} \operatorname*{res}_w \frac{1}{w^{n+1}} \frac{1 - w(x+1)}{x} \frac{x^n (1-2w)^n}{(1 - w(x+1))^n} \frac{x}{(1 - w(x+1))^2}$$

$$= \frac{1}{2^n} \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{(1-2w)^n}{(1-w(x+1))^{n+1}}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+k}{k} (x+1)^k \binom{n}{n-k} (-1)^{n-k} 2^{n-k}$$

$$= (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^k \left[\frac{x+1}{2}\right]^k.$$

This is the third identity. With an alternate substitution we put $z/(x+z(x-1)) = w$ so that $z = wx/(1-w(x-1))$ and $dz = x/(1-w(x-1))^2\, dw$ to obtain

$$\frac{1}{2^n \times x^n} \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1-w(x-1)}{x} \frac{x^n(1+2w)^n}{(1-w(x-1))^n} \frac{x}{(1-w(x-1))^2}$$

$$= \frac{1}{2^n} \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{(1+2w)^n}{(1-w(x-1))^n}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+k}{k} (x-1)^k \binom{n}{n-k} 2^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left[\frac{x-1}{2}\right]^k.$$

This is the fourth identity.

Connection to generating function

For the remainder we compute

$$[t^n] \frac{1}{\sqrt{1-2xt+t^2}} = [t^n] \frac{1}{\sqrt{1-4t(x/2-t/4)}}$$

$$= [t^n] \sum_{k=0}^{n} \binom{2k}{k} t^k (x/2-t/4)^k = \sum_{k=0}^{n} \binom{2k}{k} [t^{n-k}] (x/2-t/4)^k$$

$$= \sum_{k=0}^{n} \binom{2k}{k} \binom{k}{n-k} (-1)^{n-k} 4^{k-n} 2^{n-2k} x^{2k-n}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{2k}{k} \binom{k}{n-k} (-1)^{n-k} x^{2k-n}.$$

Observe that the second binomial coefficient is zero unless $2k \geq n$. With this condition we have

$$\binom{2k}{k}\binom{k}{n-k} = \frac{(2k)!}{k! \times (n-k)! \times (2k-n)!} = \binom{2k}{n}\binom{n}{k}.$$

The first binomial coefficient enforces the range condition on k and there is no singular factorial. At last we have

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{n}(-1)^{n-k}x^{2k-n}$$
$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2n-2k}{n}(-1)^{k}x^{n-2k}$$

which links us to the first identity, where with n not negative the condition $2n-2k \ge n$ or $n \ge 2k$ enforces the upper range of the sum.

This problem has not appeared at math.stackexchange.com. It is from page 38 eqns. 3.133, 3.134 and 3.135 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.93 Legendre Polynomials and a square root

We seek to prove the following identity

$$P_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}2^{-k}\sqrt{x^2-1}^{k}\left[x-\sqrt{x^2-1}\right]^{n-k}$$

as well as

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{2k}\binom{2k}{k}2^{-2k}x^{n-2k}(x^2-1)^k.$$

Expanding the powered term we find for the RHS

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}2^{-k}\sqrt{x^2-1}^{k}\sum_{q=0}^{n-k}\binom{n-k}{q}(-1)^q\sqrt{x^2-1}^{q}x^{n-k-q}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}2^{-k}\sum_{q=0}^{n-k}\binom{n-k}{q}(-1)^q\sqrt{x^2-1}^{q+k}x^{n-k-q}$$
$$= \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}2^{-k}\sum_{q=k}^{n}\binom{n-k}{q-k}(-1)^{q-k}\sqrt{x^2-1}^{q}x^{n-q}.$$

Switching summations we obtain

$$\sum_{q=0}^n x^{n-q}\sqrt{x^2-1}^q \sum_{k=0}^q {n\choose k}{2k\choose k} 2^{-k} {n-k\choose q-k} (-1)^{q-k}.$$

Note that

$${n\choose k}{n-k\choose q-k} = \frac{n!}{k!\times(n-q)!\times(q-k)!} = {n\choose q}{q\choose k}$$

so we have

$$\sum_{q=0}^n {n\choose q} x^{n-q}\sqrt{x^2-1}^q \sum_{k=0}^q {q\choose k}{2k\choose k} 2^{-k} (-1)^{q-k}.$$

Working with the inner sum we find

$$\sum_{k=0}^q {q\choose k}{2q-2k\choose q-k} 2^{k-q}(-1)^k$$

$$= [z^q](1+z)^{2q} \sum_{k=0}^q {q\choose k} \frac{z^k}{(1+z)^{2k}} 2^{k-q}(-1)^k$$

$$= 2^{-q}[z^q](1+z)^{2q}\left(1-\frac{2z}{(1+z)^2}\right)^q$$

$$= 2^{-q}[z^q](1+z^2)^q.$$

This is zero when q is odd and $2^{-2p}[z^{2p}](1+z^2)^{2p} = 2^{-2p}[z^p](1+z)^{2p} = 2^{-2p}{2p\choose p}$ when q is even i.e. $q=2p$. We get for our sum (the second identity appears)

$$\sum_{p=0}^{\lfloor n/2\rfloor} {n\choose 2p} x^{n-2p}(x^2-1)^p 2^{-2p}{2p\choose p}.$$

This is

$$x^n[z^n](1+z)^n \sum_{p=0}^{\lfloor n/2\rfloor} z^{2p}x^{-2p}(x^2-1)^p 2^{-2p}{2p\choose p}.$$

Here the coefficient extractor enforces the upper range of the sum and we may extend to infinity, getting

$$x^n[z^n](1+z)^n \frac{1}{\sqrt{1-z^2(x^2-1)/x^2}}$$

$$= [z^n](1+xz)^n \frac{1}{\sqrt{1-z^2(x^2-1)}}$$

$$= \operatorname*{res}_z \frac{1}{z^{n+1}} (1+xz)^n \frac{1}{\sqrt{1-z^2(x^2-1)}}.$$

Now we put $z/(1+xz) = w$ so that $z = w/(1-xw)$ and $dz = 1/(1-xw)^2 \; dw$ to get

$$\operatorname*{res}_w \frac{1}{w^{n+1}} (1-xw) \frac{1}{\sqrt{1-w^2(x^2-1)/(1-xw)^2}} \frac{1}{(1-xw)^2}$$
$$= \operatorname*{res}_w \frac{1}{w^{n+1}} \frac{1}{\sqrt{(1-xw)^2 - w^2(x^2-1)}}$$
$$= \operatorname*{res}_w \frac{1}{w^{n+1}} \frac{1}{1-2xw+w^2}.$$

This is precisely the OGF of the Legendre polynomials which concludes the proof.

This problem has not appeared at math.stackexchange.com. It is from page 39 eqns. 3.136 and 3.137 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.94 Legendre Polynomials and a double square root

We seek to prove the following four identities which form two pairs:

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} x^{2k} = 2^{2n} x^n P_n((x+1/x)/2)$$
$$= 2^{2n} \frac{2}{\pi} \int_0^{\pi/2} (x^2 \sin^2 t + \cos^2 t)^n \; dt.$$

and

$$\sum_{k=0}^n \binom{-1/2}{k} \binom{-1/2}{n-k} x^{2k} = (-1)^n x^n P_n((x+1/x)/2)$$
$$= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k-1/2}{n} x^{2k}.$$

First identity

We have by inspection for the LHS of the first identity

$$[t^n] \frac{1}{\sqrt{1-4x^2t}} \frac{1}{\sqrt{1-4t}}.$$

The RHS is by the OGF of the Legendre polynomials

$$2^{2n} x^n [t^n] \frac{1}{\sqrt{1-2\times 1/2(x+1/x)t+t^2}}$$

$$= 2^{2n}[t^n]\frac{1}{\sqrt{1-(x^2+1)t+x^2t^2}}$$

$$= [t^n]\frac{1}{\sqrt{1-4(x^2+1)t+16x^2t^2}}.$$

Now observe that

$$(1-4x^2t)(1-4t) = 1-4(x^2+1)t+16x^2t^2$$

to conclude. For the trigonometric integral we get

$$2^{2n}\frac{1}{2\pi}\int_0^{2\pi}(x^2\sin^2 t+\cos^2 t)^n \; dt$$

$$= 2^{2n}\frac{1}{2\pi}\int_0^{2\pi}((x^2-1)\sin^2 t+1)^n \; dt$$

We put $z = \exp(it)$ so that $dz = iz\; dt$ to obtain

$$2^{2n}\frac{1}{2\pi}\int_{|z|=1}((x^2-1)((z-1/z)/2/i)^2+1)^n \; \frac{dz}{iz}$$

$$= 2^{2n}\frac{1}{2\pi i}\int_{|z|=1}\frac{1}{z^{2n+1}}((1-x^2)(z^2-1)^2/4+z^2)^n \; dz.$$

Evaluating the residue yields a coefficient extractor:

$$2^{2n}[z^{2n}]((1-x^2)(z^2-1)^2/4+z^2)^n$$

$$= 2^{2n}[z^n]((1-x^2)(z-1)^2/4+z)^n$$

$$= 2^{2n}[z^n]\sum_{q=0}^{n}\binom{n}{q}(1-x^2)^{n-q}(z-1)^{2n-2q}/4^{n-q}z^q$$

$$= 2^{2n}\sum_{q=0}^{n}\binom{n}{q}(1-x^2)^{n-q}\binom{2n-2q}{n-q}(-1)^{n-q}/4^{n-q}$$

$$= 2^{2n}[z^n]\frac{1}{\sqrt{1+(1-x^2)z}}\sum_{q=0}^{n}\binom{n}{q}z^q$$

$$= 2^{2n}\operatorname*{res}_z\frac{1}{z^{n+1}}\frac{1}{\sqrt{1+(1-x^2)z}}(1+z)^n.$$

Next we put $z/(1+z) = t$ so that $z = t/(1-t)$ and $dz = 1/(1-t)^2 \; dt$ to find

$$2^{2n}\operatorname*{res}_t\frac{1}{t^{n+1}}(1-t)\frac{1}{\sqrt{1+(1-x^2)t/(1-t)}}\frac{1}{(1-t)^2}$$

$$= 2^{2n} \operatorname*{res}_t \frac{1}{t^{n+1}} \frac{1}{\sqrt{(1-t)^2 + (1-x^2)t(1-t)}}$$

Now $1 - 2t + t^2 + (1-x^2)t - (1-x^2)t^2 = 1 - (x^2+1)t + x^2t^2$ and we have at last

$$2^{2n}[t^n]\frac{1}{\sqrt{1-(x^2-1)t + x^2t^2}}$$
$$= [t^n]\frac{1}{\sqrt{1-4(x^2-1)t + 16x^2t^2}}$$

and we may conclude. Gould references R. P. Kelisky for this identity.

Second identity

The first part of the second identity is very similar to the first and we get for the LHS

$$[t^n]\frac{1}{\sqrt{1+x^2t}}\frac{1}{\sqrt{1+t}}.$$

We get for the RHS

$$(-1)^n x^n [t^n]\frac{1}{\sqrt{1-2\times 1/2(x+1/x)t + t^2}}$$
$$= (-1)^n [t^n]\sqrt{1-(x^2+1)t + x^2t^2}$$
$$= [t^n]\sqrt{1+(x^2+1)t + x^2t^2}.$$

Now observe that

$$(1+x^2t)(1+t) = 1 + (x^2+1)t + x^2t^2$$

to conclude. For the second part of the second identity we get

$$[z^n]\frac{1}{\sqrt{1+z}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+z)^k x^{2k}$$
$$= [z^n]\frac{1}{\sqrt{1+z}}(1-x^2(1+z))^n$$
$$= \operatorname*{res}_z \frac{1}{z^{n+1}}\frac{1}{\sqrt{1+z}}(1-x^2(1+z))^n.$$

Next we put $z/(1-x^2(1+z)) = t$ to get $z = t(1-x^2)/(1+x^2t)$ and $dz = (1-x^2)/(1+x^2t)^2 \; dt$ for

$$\operatorname*{res}_t \frac{1}{t^{n+1}}\frac{1+x^2t}{1-x^2}\frac{1}{\sqrt{1+t(1-x^2)/(1+x^2t)}}\frac{1-x^2}{(1+x^2t)^2}$$

$$= \operatorname*{res}_t \frac{1}{t^{n+1}} \frac{1}{\sqrt{(1+x^2t)^2 + t(1-x^2)(1+x^2t)}}.$$

The last step is to note that

$$\begin{aligned}
&(1+x^2t)^2 + t(1-x^2)(1+x^2t) \\
&= 1 + 2x^2t + x^4t^2 + t - x^2t + x^2t^2 - x^4t^2 \\
&= 1 + (x^2+1)t + x^2t^2.
\end{aligned}$$

We have a match of the generating function from the first part and may conclude.

These identities are from page 39 eqns. 3.138 and 3.139 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.95 MSE 4304623

We have from first principles that

$$\boxed{S_n = \sum_{r=0}^n \frac{1}{4^r} \binom{2r}{r} = \operatorname*{res}_z \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{\sqrt{1-z}}.}$$

and seek to use this to find a closed form of S_n.

Now put $1 - \sqrt{1-z} = w$ so that $z = w(2-w)$ and $dz = 2(1-w)\, dw$ to get

$$\begin{aligned}
&\operatorname*{res}_w \frac{1}{w^{n+1}(2-w)^{n+1}} \frac{1}{(1-w)^2} \frac{1}{1-w} 2(1-w) \\
&= 2(-1)^{n+1} \operatorname*{res}_w \frac{1}{w^{n+1}(w-2)^{n+1}} \frac{1}{(w-1)^2}.
\end{aligned}$$

The residue at infinity is zero by inspection so we need the residues at $w = 1$ and $w = 2$. For the former we get without the scalar in front

$$\begin{aligned}
&\left. \left(\frac{1}{w^{n+1}} \frac{1}{(w-2)^{n+1}} \right)' \right|_{w=1} \\
&= \left. \left(-(n+1) \frac{1}{w^{n+2}} \frac{1}{(w-2)^{n+1}} - \frac{1}{w^{n+1}} (n+1) \frac{1}{(w-2)^{n+2}} \right) \right|_{w=1} \\
&= -(n+1)(-1)^{n+1} - (n+1)(-1)^{n+2} = 0.
\end{aligned}$$

With this our sum is minus the residue at $w = 2$. We write

$$\begin{aligned}
&2(-1)^n \operatorname{Res}_{w=2} \frac{1}{((w-2)+2)^{n+1}} \frac{1}{(w-2)^{n+1}} \frac{1}{((w-2)+1)^2} \\
&= \frac{(-1)^n}{2^n} \operatorname{Res}_{w=2} \frac{1}{(1+(w-2)/2)^{n+1}} \frac{1}{(w-2)^{n+1}} \frac{1}{(1+(w-2))^2}.
\end{aligned}$$

This will produce

$$\boxed{S_n = \frac{1}{2^n}\sum_{q=0}^{n} \binom{n+q}{n} \frac{1}{2^q}(n-q+1).}$$

First piece

Now we get two pieces here, where $S_n = A_n + B_n$, the first is

$$\begin{aligned} A_n &= \frac{n+1}{2^n} \operatorname{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(1-z/2)^{n+1}} \\ &= (-1)^n 2(n+1) \operatorname{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{1}{(z-2)^{n+1}}. \end{aligned}$$

We evaluate this using the residues at $z=1$ and $z=2$. We get for the former the value $-2(n+1)$. We write for the latter

$$\begin{aligned} &(-1)^n 2(n+1) \operatorname{Res}_{z=2} \frac{1}{((z-2)+2)^{n+1}} \frac{1}{(z-2)+1} \frac{1}{(z-2)^{n+1}} \\ &= (-1)^n \frac{n+1}{2^n} \operatorname{Res}_{z=2} \frac{1}{((z-2)/2+1)^{n+1}} \frac{1}{(z-2)+1} \frac{1}{(z-2)^{n+1}} \end{aligned}$$

This yields

$$(-1)^n \frac{n+1}{2^n} \sum_{q=0}^{n} \binom{n+q}{q} (-1)^q \frac{1}{2^q} (-1)^{n-q}.$$

Simplify to obtain A_n. With residues adding to zero, we have established that for the first piece A_n it evaluates to $A_n = n+1$.

Second piece

For the second piece we find

$$\begin{aligned} B_n &= -\frac{n+1}{2^n} \sum_{q=1}^{n} \binom{n+q}{n+1} \frac{1}{2^q} = -\frac{n+1}{2^{n+1}} \sum_{q=0}^{n-1} \binom{n+1+q}{n+1} \frac{1}{2^q} \\ &= -\frac{n+1}{2^{n+1}} \operatorname{Res}_{z=0} \frac{1}{z^n} \frac{1}{1-z} \frac{1}{(1-z/2)^{n+2}} \\ &= (-1)^n 2(n+1) \operatorname{Res}_{z=0} \frac{1}{z^n} \frac{1}{z-1} \frac{1}{(z-2)^{n+2}}. \end{aligned}$$

Again evaluate using residues at $z=1$ and $z=2$. We get for the former the value $2(n+1)$. For the latter we write

$$(-1)^n \frac{n+1}{2^{n-1}} \operatorname{Res}_{z=2} \frac{1}{((z-2)/2+1)^n} \frac{1}{(z-2)+1} \frac{1}{(z-2)^{n+2}}$$

This yields

$$(-1)^n \frac{n+1}{2^{n-1}} \sum_{q=0}^{n+1} \binom{n-1+q}{q} (-1)^q \frac{1}{2^q} (-1)^{n+1-q}$$

$$= -\frac{n+1}{2^{n-1}} \sum_{q=0}^{n-1} \binom{n-1+q}{q} \frac{1}{2^q} - \frac{n+1}{2^{2n-1}} \binom{2n-1}{n} - \frac{n+1}{2^{2n}} \binom{2n}{n+1}.$$

The sum is

$$-\frac{n+1}{2^{n-1}} A_{n-1} \frac{2^{n-1}}{n} = -(n+1)$$

hence piece B_n evaluates as

$$\frac{n+1}{2^{2n-1}} \binom{2n-1}{n} + \frac{n+1}{2^{2n}} \binom{2n}{n+1} - (n+1).$$

Conclusion

Adding the two pieces we have shown that

$$\boxed{S_n = \frac{2n+1}{2^{2n}} \binom{2n}{n} = \binom{n+1/2}{n}.}$$

as claimed. This may be seen from (evaluate LHS)

$$(2n+1)[z^n] \frac{1}{\sqrt{1-z}} = (2n+1) \binom{-1/2}{n} (-1)^n$$

$$= (2n+1) \binom{n-1/2}{n} = \binom{n+1/2}{n}.$$

This was math.stackexchange.com problem 4304623.

1.96 Legendre Polynomials, trigonometric terms and a contour integral

We seek to prove the following three identities:

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2-1} \times \cos t)^n \, dt$$

and for positive integer $m > n$

$$P_n(x) = \frac{1}{m} \sum_{k=0}^{m-1} \left(x + \sqrt{x^2-1} \times \cos \frac{2\pi k}{m} \right)^n.$$

as well as

$$P_n(x) = \frac{1}{2^n} \frac{1}{2\pi i} \int_{|t-x|=\varepsilon} \frac{(t^2-1)^n}{(t-x)^{n+1}} \; dt.$$

First identity

We get for the first one by symmetry

$$\frac{1}{2\pi} \int_0^{2\pi} (x + \sqrt{x^2-1} \times \cos t)^n \; dt.$$

Now we put $z = \exp(it)$ so that $dt = \frac{dz}{iz}$ to obtain

$$\frac{1}{2\pi} \int_{|z|=1} (x + \sqrt{x^2-1} \times (z+1/z)/2)^n \; \frac{dz}{iz}$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z^{n+1}} (xz + \sqrt{x^2-1} \times (z^2+1)/2)^n \; dz.$$

Computing the residue we find

$$[z^n] \sum_{q=0}^{n} \binom{n}{q} x^{n-q} z^{n-q} \sqrt{x^2-1}^q (z^2+1)^q / 2^q$$

$$= \sum_{q=0}^{n} \binom{n}{q} x^{n-q} \sqrt{x^2-1}^q [z^q] (z^2+1)^q / 2^q.$$

The only contribution comes from even $q = 2p$ and we get

$$\sum_{p=0}^{\lfloor n/2 \rfloor} \binom{n}{2p} x^{n-2p} (x^2-1)^p [z^{2p}] (z^2+1)^{2p} / 2^{2p}$$

$$= \sum_{p=0}^{\lfloor n/2 \rfloor} \binom{n}{2p} x^{n-2p} (x^2-1)^p [z^p] (z+1)^{2p} / 2^{2p}$$

$$= \sum_{p=0}^{\lfloor n/2 \rfloor} \binom{n}{p} x^{n-2p} (x^2-1)^p \binom{2p}{p} \frac{1}{2^{2p}}.$$

We have recovered the form from section 1.93 and may conclude.

Second identity

Starting with the second identity we have

$$\frac{1}{m} \sum_{k=0}^{m-1} \sum_{q=0}^{n} \binom{n}{q} x^{n-q} \sqrt{x^2-1}^q \cos^q \frac{2\pi k}{m}$$

$$= \frac{1}{m} \sum_{q=0}^{n} \binom{n}{q} x^{n-q} \sqrt{x^2-1}^q \sum_{k=0}^{m-1} \cos^q \frac{2\pi k}{m}.$$

For the inner sum we introduce $\rho = \exp(2\pi i/m)$ so that it becomes

$$\frac{1}{m} \sum_{k=0}^{m-1} (\rho^k + \rho^{-k})^q \frac{1}{2^q} = \frac{1}{2^q} \frac{1}{m} \sum_{k=0}^{m-1} \mathrm{Res}_{z=\rho^k} \left(z + \frac{1}{z}\right)^q \frac{mz^{m-1}}{z^m - 1}$$

$$= \frac{1}{2^q} \frac{1}{m} \sum_{k=0}^{m-1} \mathrm{Res}_{z=\rho^k} \left(z + \frac{1}{z}\right)^q \frac{m/z}{z^m - 1} = \frac{1}{2^q} \mathrm{Res}_{z=0} \frac{1}{z^{q+1}} (z^2+1)^q \frac{1}{1-z^m}.$$

This is

$$\frac{1}{2^q} [z^q] (z^2+1)^q \frac{1}{1-z^m}.$$

Note however that $q \le n < m$ as given in the statement of the identity so that only the constant term from $\frac{1}{1-z^m}$ contributes, which yields

$$\frac{1}{2^q} [z^q] (z^2+1)^q.$$

This requires $q = 2p$ and we get

$$\frac{1}{2^{2p}} [z^{2p}] (1+z^2)^{2p} = \frac{1}{2^{2p}} [z^p] (1+z)^{2p} = \frac{1}{2^{2p}} \binom{2p}{p}.$$

Substituting this into the sum we obtain the same closed form as in the first identity and may conclude. This is credited to I.J. Good.

Third identity

We require the derivative

$$\frac{1}{n!} \left((t-1)^n (t+1)^n\right)^{(n)}$$

which by the Leibniz rule is

$$\frac{1}{n!} \sum_{q=0}^{n} \binom{n}{q} \frac{n!}{(n-q)!} (t-1)^{n-q} \frac{n!}{q!} (t+1)^q$$

$$= \sum_{q=0}^{n} \binom{n}{q}^2 (t-1)^{n-q} (t+1)^q.$$

Using the derivative to evaluate the contour integral by the Cauchy Residue Theorem we obtain

$$P_n(x) = \frac{1}{2^n} \sum_{q=0}^{n} \binom{n}{q}^2 (x-1)^{n-q} (x+1)^q$$

$$= \left[\frac{x-1}{2}\right]^n \sum_{q=0}^{n} \binom{n}{q}^2 \left[\frac{x+1}{x-1}\right]^q.$$

This is one of the entries in the list from section 1.92 and we may conclude. This is credited to L. Schläfli.

These identities are from page 39 eqns. 3.139 and 3.140 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.97 Sum independent of a variable

We seek to show that

$$\sum_{k=0}^{n} \binom{x+ky}{k}\binom{p-x-ky}{n-k} = \begin{cases} y^{p+1}(y-1)^{n-p-1}, & 0 \le p \le n-1 \\ \frac{y^{n+1}-1}{y-1}, & p = n. \end{cases}$$

As both sides are polynomials in x and y we may prove it for positive integer values for x and y and it then holds for all i.e. complex x and y. We have for the LHS

$$[z^n](1+z)^{p-x} \sum_{k\ge 0} \binom{x+ky}{k} z^k \frac{1}{(1+z)^{ky}}$$

Here we have extended to infinity because the coefficient extractor enforces the upper range of the sum. Now note that

$$\binom{x+ky}{k} = \operatorname*{res}_{w} (1+w)^x \frac{(1+w)^{ky}}{w^{k+1}}.$$

Next introduce $w/(1+w)^y = v$ and let the inverse be $w = f(v)$ so that $f(v)/(1+f(v))^y = v$ and the binomial coefficient becomes

$$\operatorname*{res}_{v} \frac{1}{v^{k+1}} (1+f(v))^{x-y} f'(v)$$

Substitute into our sum to get

$$[z^n](1+z)^{p-x} \sum_{k\ge 0} z^k \frac{1}{(1+z)^{ky}} [v^k](1+f(v))^{x-y} f'(v)$$
$$= [z^n](1+z)^{p-y} f'(z/(1+z)^y).$$

We also have

$$1 = f'(v)/(1+f(v))^y - yf(v)/(1+f(v))^{y+1} f'(v)$$

or

$$f'(v) = (1+f(v))^{y+1}/(1-f(v)(y-1)).$$

This gives for the sum

$$[z^n](1+z)^{p+1}\frac{1}{1-z(y-1)}.$$

With $0 \le p \le n-1$ this is

$$\sum_{q=0}^{p+1}\binom{p+1}{q}(y-1)^{n-q}$$
$$= (y-1)^n\left(1+\frac{1}{y-1}\right)^{p+1} = (y-1)^{n-p-1}y^{p+1}$$

as claimed. For $p \ge n$ we get

$$\sum_{q=0}^{n}\binom{p+1}{q}(y-1)^{n-q}$$

which for $p = n$ works out to

$$-\frac{1}{y-1} + (y-1)^n\left(1+\frac{1}{y-1}\right)^{n+1} = -\frac{1}{y-1} + \frac{y^{n+1}}{y-1} = \frac{y^{n+1}-1}{y-1}$$

also as claimed. We can simplify the general case some more, getting

$$\sum_{q=0}^{n}\binom{p+1}{q}\sum_{r=0}^{n-q}\binom{n-q}{r}y^r(-1)^{n-q-r}$$
$$= \sum_{r=0}^{n}y^r(-1)^{n-r}\sum_{q=0}^{n-r}\binom{p+1}{q}\binom{n-q}{r}(-1)^q$$
$$= \sum_{r=0}^{n}y^r(-1)^{n-r}[z^{n-r}](1+z)^n\sum_{q\ge 0}\binom{p+1}{q}(-1)^q\frac{z^q}{(1+z)^q}$$
$$= \sum_{r=0}^{n}y^r(-1)^{n-r}[z^{n-r}](1+z)^n\left(1-\frac{z}{1+z}\right)^{p+1}$$
$$= \sum_{r=0}^{n}y^r(-1)^{n-r}[z^{n-r}]\frac{1}{(1+z)^{p+1-n}} = \sum_{r=0}^{n}y^r\binom{p-r}{p-n}.$$

This last formula covers all cases if we use a certain type of binomial coefficient for negative lower index. We get the case $p = n$ by inspection. With $p < n$ we write

$$\sum_{r=0}^{n}y^r\binom{p-r}{n-r} = [z^n](1+z)^p\sum_{r\ge 0}y^r\frac{z^r}{(1+z)^r}$$

$$= [z^n](1+z)^p \frac{1}{1-yz/(1+z)} = [z^n](1+z)^{p+1}\frac{1}{1-(y-1)z}.$$

This is the closed form we obtained earlier.

This problem is from page 41 eqn. 3.145 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.98 Polynomial in three variables

We seek to show that

$$\sum_{k=0}^{n} \binom{x+kt}{k}\binom{y-kt}{n-k} = \sum_{k=0}^{n} \binom{x+y-k}{n-k} t^k.$$

As both sides are polynomials in x, y and t we may prove it for positive integer values for x, y and t and it then holds for all i.e. complex x, y and t. We have for the LHS

$$[z^n](1+z)^y \sum_{k\ge 0} \binom{x+kt}{k} \frac{z^k}{(1+z)^{kt}}$$

Here we have extended to infinity because the coefficient extractor enforces the upper range of the sum. Now note that

$$\binom{x+kt}{k} = \operatorname*{res}_{w} (1+w)^x \frac{(1+w)^{kt}}{w^{k+1}}.$$

Next introduce $w/(1+w)^t = v$ and let the inverse be $w = f(v)$ so that $f(v)/(1+f(v))^t = v$ and the binomial coefficient becomes

$$\operatorname*{res}_{v} \frac{1}{v^{k+1}} (1+f(v))^{x-t} f'(v)$$

Substitute into our sum to get

$$[z^n](1+z)^y \sum_{k\ge 0} \frac{z^k}{(1+z)^{kt}} [v^k](1+f(v))^{x-t} f'(v)$$

$$= [z^n](1+z)^{x+y-t} f'(z/(1+z)^t).$$

We also have

$$1 = f'(v)/(1+f(v))^t - tf(v)/(1+f(v))^{t+1} f'(v)$$

or

$$f'(v) = (1+f(v))^{t+1}/(1-f(v)(t-1)).$$

This gives for the sum

$$[z^n](1+z)^{x+y+1}\frac{1}{1-z(t-1)}.$$

This duplicates the calculation from the previous section. We may also write

$$[z^n](1+z)^{x+y}\frac{1}{1-tz/(1+z)} = [z^n](1+z)^{x+y}\sum_{k=0}^{n} t^k \frac{z^k}{(1+z)^k}$$

$$= \sum_{k=0}^{n} \binom{x+y-k}{n-k} t^k$$

which is the claim.

This problem is from page 41 eqn. 3.144 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.99 An identity by Van der Corput

We seek to show that

$$\sum_{k=1}^{n-1} \binom{kx}{k}\binom{nx-kx}{n-k}\frac{1}{kx(nx-kx)} = \frac{2}{nx}\binom{nx}{n}\sum_{k=1}^{n-1}\frac{1}{nx-n+k}.$$

As we get a polynomial in x on multiplication by x^2 we may prove it for x a positive integer and it then holds for all i.e. complex x. Observe how the binomial coefficient cancels the rational terms.

First phase

A first simplification we can make to the LHS is

$$\frac{2}{nx^2}\sum_{k=1}^{n-1}\binom{kx}{k}\binom{nx-kx}{n-k}\frac{1}{k}$$

$$= -\frac{2}{n^2x^2}\binom{nx}{n} + \frac{2}{nx}\sum_{k=1}^{n}\frac{1}{k}\binom{kx-1}{k-1}\binom{nx-kx}{n-k}$$

$$= -\frac{2}{n^2x^2}\binom{nx}{n} + \frac{2}{nx}[z^n](1+z)^{nx}\sum_{k\ge 1}\frac{1}{k}\binom{kx-1}{k-1}\frac{z^k}{(1+z)^{kx}}.$$

Here we have extended the sum to infinity because the coefficient extractor enforces the upper range. Observe that

$$\binom{kx-1}{k-1} = \operatorname*{res}_{w}\frac{1}{w^k}(1+w)^{kx-1}.$$

Next introduce $w/(1+w)^x = v$ and let the inverse be $w = f(v)$ so that $f(v)/(1+f(v))^x = v$ and the binomial coefficient becomes

$$\operatorname*{res}_v \frac{1}{v^k}\frac{1}{1+f(v)}f'(v).$$

Substitute into our sum to get

$$\frac{2}{nx}[z^n](1+z)^{nx}\sum_{k\ge 1}\frac{z^k}{(1+z)^{kx}}\frac{1}{k}[v^{k-1}]\frac{1}{1+f(v)}f'(v)$$

Integrating the functional term in v yields

$$-\log\frac{1}{1+f(v)}.$$

Note that owing to $f(0)=0$ we did not pick up a constant. Substitute once more to obtain

$$-\frac{2}{nx}[z^n](1+z)^{nx}\log\frac{1}{1+z}.$$

We get without the scalar and using that x is a positive integer

$$\sum_{q=0}^{n-1}\binom{nx}{q}(-1)^{n-q}\frac{1}{n-q}.$$

Second phase

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [z^n]\log\frac{1}{1-z}(-1)^{n-k}(1-z)^{n-k}.$$

We let n be $nx-q$ and k be $n-q$ and obtain

$$[z^{nx}]\log\frac{1}{1-z}\sum_{q=0}^{n-1}\binom{nx}{q}(-1)^{nx-q}\binom{nx-q}{n-q}(1-z)^{nx-n}z^q.$$

Next observe that

$$\binom{nx}{q}\binom{nx-q}{n-q} = \frac{(nx)!}{q!\times(n-q)!\times(nx-n)!} = \binom{nx}{n}\binom{n}{q}$$

Working with the remainder of the sum,

$$\sum_{q=0}^{n-1}\binom{n}{q}(-1)^{nx-q}z^q = -(-1)^{nx-n}z^n + (-1)^{nx}(1-z)^n.$$

Just to recapitulate, we are now left with two pieces, the first being

$$\frac{2}{nx}\binom{nx}{n}[z^{nx-n}](-1)^{nx-n}\log\frac{1}{1-z}(1-z)^{nx-n}$$

and

$$-\frac{2}{nx}\binom{nx}{n}[z^{nx}](-1)^{nx}\log\frac{1}{1-z}(1-z)^{nx}.$$

We are therefore tasked with

$$\operatorname*{res}_{z}\frac{1}{z^{m+1}}\log\frac{1}{1+z}(1+z)^{m}.$$

We use the standard substitution $z/(1+z)=w$ so that $z=w/(1-w)$ and $dz=1/(1-w)^2\ dw$ to get

$$\operatorname*{res}_{w}\frac{1}{w^{m+1}}\log\frac{1}{1+w/(1-w)}(1-w)\frac{1}{(1-w)^2}$$
$$=\operatorname*{res}_{w}\frac{1}{w^{m+1}}\frac{1}{1-w}\log(1-w)=-H_m.$$

Collecting everything we have so far we get

$$-\frac{2}{n^2x^2}\binom{nx}{n}+\frac{2}{nx}\binom{nx}{n}(H_{nx}-H_{nx-n}).$$

The top term from the first harmonic number is canceled and we have at last

$$\frac{2}{nx}\binom{nx}{n}(H_{nx-1}-H_{nx-n}).$$

We want to manipulate this to obtain a rational function in x with the variable x not in the summation limits so we write for the harmonic numbers

$$\sum_{k=nx-n+1}^{nx-1}\frac{1}{k}=\sum_{k=-n+1}^{-1}\frac{1}{nx+k}$$
$$=\sum_{k=1}^{n-1}\frac{1}{nx-k}=\sum_{k=1}^{n-1}\frac{1}{nx-n+k}$$

and we get

$$\frac{2}{nx}\binom{nx}{n}\sum_{k=1}^{n-1}\frac{1}{nx-n+k}$$

as claimed.

This problem is from page 41 eqn. 3.147 of H.W.Gould's *Combinatorial Identities* [Gou72a]. It was credited to Van der Corput.

1.100 MSE 4316307: Logarithm, binomial coefficient and harmonic numbers

We seek to show that

$$[z^n]\frac{1}{(1-z)^{\alpha+1}}\log\frac{1}{1-z} = \binom{n+\alpha}{n}(H_{n+\alpha}-H_\alpha).$$

with α a non-negative integer.

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1}[z^n]\log\frac{1}{1-z}(-1)^{n-k}(1-z)^{n-k}.$$

We get for the LHS from first principles that it is (apply identity setting n to $n+\alpha$)

$$\sum_{q=1}^{n}\binom{n-q+\alpha}{n-q}\frac{1}{q}$$

$$= [z^{n+\alpha}]\log\frac{1}{1-z}\sum_{q=1}^{n}\binom{n+\alpha}{q}\binom{n-q+\alpha}{\alpha}(-1)^{n+\alpha-q}(1-z)^{n+\alpha-q}.$$

Note that for $q=0$ we get

$$\binom{n+\alpha}{\alpha}[z^{n+\alpha}]\log\frac{1}{1-z}(-1)^{n+\alpha}(1-z)^{n+\alpha}.$$

This will be our first piece. We include it in our sum at this time. Next observe that

$$\binom{n+\alpha}{q}\binom{n-q+\alpha}{\alpha} = \frac{(n+\alpha)!}{q!\times\alpha!\times(n-q)!} = \binom{n+\alpha}{\alpha}\binom{n}{q}.$$

We have for the augmented sum without the binomial scalar in front

$$[z^{n+\alpha}]\log\frac{1}{1-z}\sum_{q=0}^{n}\binom{n}{q}(z-1)^{n+\alpha-q}$$

$$= [z^{n+\alpha}]\log\frac{1}{1-z}(z-1)^{n+\alpha}\left[1+\frac{1}{z-1}\right]^n$$

$$= [z^{n+\alpha}]\log\frac{1}{1-z}(z-1)^{\alpha}z^n = [z^\alpha]\log\frac{1}{1-z}(z-1)^\alpha.$$

This is the second piece. Now to evaluate these two pieces we evidently require

$$\operatorname*{res}_z \frac{1}{z^{m+1}}\log\frac{1}{1-z}(-1)^m(1-z)^m.$$

This was evaluated e.g. in section 1.99 and found to be $-H_m$. Hence our first piece is $-\binom{n+\alpha}{\alpha} H_{n+\alpha}$ while the second is $-\binom{n+\alpha}{\alpha} H_\alpha$. Subtract the first from the second to obtain our claim,

$$\binom{n+\alpha}{n}(H_{n+\alpha} - H_\alpha).$$

This was math.stackexchange.com problem 4316307.

1.101 An identity credited to Chung

We seek to show that

$$\sum_{k=1}^{n} \frac{1}{k}\binom{kx-2}{k-1}\binom{nx-kx}{n-k} = \frac{1}{x}\binom{nx}{n}.$$

As we get a polynomial in x on both sides we may prove it for x a positive integer and it then holds for all i.e. complex x. Observe how the binomial coefficient cancels the rational term.

We have from first principles for the LHS

$$[z^n](1+z)^{nx}\sum_{k\ge 1}\frac{1}{k}\binom{kx-2}{k-1}\frac{z^k}{(1+z)^{kx}}.$$

Here we have extended the sum to infinity because the coefficient extractor enforces the upper range. Observe that

$$\binom{kx-2}{k-1} = \operatorname*{res}_{w} \frac{1}{w^k}(1+w)^{kx-2}.$$

Next introduce $w/(1+w)^x = v$ and let the inverse be $w = f(v)$ so that $f(v)/(1+f(v))^x = v$ and the binomial coefficient becomes

$$\operatorname*{res}_{v} \frac{1}{v^k}\frac{1}{(1+f(v))^2} f'(v).$$

Substitute into our sum to get

$$[z^n](1+z)^{nx}\sum_{k\ge 1}\frac{z^k}{(1+z)^{kx}}\frac{1}{k}[v^{k-1}]\frac{1}{(1+f(v))^2} f'(v)$$

Integrating the functional term in v yields

$$1 - \frac{1}{1+f(v)}.$$

We have picked up a constant minus one which we have canceled. Continuing,

$$[z^n](1+z)^{nx}\left(1-\frac{1}{1+z}\right) = [z^{n-1}](1+z)^{nx-1} = \binom{nx-1}{n-1} = \frac{1}{x}\binom{nx}{n}.$$

This problem is from page 41 eqn. 3.148 of H.W.Gould's *Combinatorial Identities* [Gou72a]. It was credited to Chung.

1.102 A Catalan number convolution

In seeking to evaluate where the sum is zero when $n < 2$

$$\sum_{k=1}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n-2}{n-2k} \frac{1}{k} \binom{2k-2}{k-1}$$

we recognize the Catalan number and obtain

$$\sum_{k=1}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n-2}{n-2k} [z^k] \frac{1-\sqrt{1-4z}}{2}$$

$$= [w^n](1+w)^{n-2} \sum_{k=1}^{\lfloor n/2 \rfloor} 2^{n-2k} w^{2k} [z^k] \frac{1-\sqrt{1-4z}}{2}.$$

We get a zero contribution when $k = 0$ from the coefficient extractor in z as well as when $2k > n$ from the coefficient extractor in w so we may extend the sum to

$$2^n [w^n](1+w)^{n-2} \sum_{k \ge 0} 2^{-2k} w^{2k} [z^k] \frac{1-\sqrt{1-4z}}{2}$$

$$= 2^n [w^n](1+w)^{n-2} \frac{1-\sqrt{1-w^2}}{2}.$$

This is

$$2^n \operatorname*{res}_w \frac{1}{w^{n+1}} (1+w)^{n-2} \frac{1-\sqrt{1-w^2}}{2}.$$

Now we put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2 \; dv$ to get

$$2^n \operatorname*{res}_v \frac{1}{v^{n+1}} (1-v)^3 \frac{1-\sqrt{1-v^2/(1-v)^2}}{2} \frac{1}{(1-v)^2}$$

$$= 2^n [v^n] \frac{1-v-\sqrt{1-2v}}{2} = [v^n] \frac{1-2v-\sqrt{1-4v}}{2}$$

$$= [v^{n-1}] \frac{1-2v-\sqrt{1-4v}}{2v} = [v^{n-1}](-1 + \frac{1-\sqrt{1-4v}}{2v}).$$

This is the Catalan number C_{n-1} when $n \ge 2$ and when $n = 1$ we get $-1 + C_0 = 0$.

This was math.stackexchange.com problem 4317353.

1.103 Odd index binomial coefficients

We seek to show that

$$\sum_{k=0}^{n-1} \binom{2x}{2k+1} \binom{x-k-1}{n-k-1} = \frac{n}{x+n} 2^{2n} \binom{x+n}{2n}$$

and

$$\sum_{k=0}^{n} \binom{2x}{2k+1} \binom{x-k-1}{n-k} = \frac{x+n}{2n+1} 2^{2n+1} \binom{x+n-1}{2n}.$$

As these are polynomials in x on the LHS and the RHS we may prove them for positive $x \ge n$ and we then have the results for all i.e. complex x.

First identity

We get for the LHS

$$[z^{x-n}] \frac{1}{(1-z)^n} \sum_{k \ge 0} \binom{2x}{2k+1} (1-z)^k.$$

Here we have extended to infinity because the remaining binomial coefficient enforces $x \ge k + 1/2$ so that with $k \ge n$ the term $[z^{x-n}](1-z)^{k-n}$ is zero. Continuing,

$$[z^{x-n}] \frac{1}{(1-z)^n} \frac{1}{\sqrt{1-z}} \sum_{k \ge 0} \binom{2x}{2k+1} (1-z)^{k+1/2}$$

$$= [z^{x-n}] \frac{1}{(1-z)^n} \frac{1}{\sqrt{1-z}} \sum_{k \ge 0} \binom{2x}{k} (1-z)^{k/2} \frac{1-(-1)^k}{2}$$

$$= \frac{1}{2} \operatorname*{res}_{z} \frac{1}{z^{x-n+1}} \frac{1}{(1-z)^n} \frac{1}{\sqrt{1-z}} \left[\left(1+\sqrt{1-z}\right)^{2x} - \left(1-\sqrt{1-z}\right)^{2x} \right].$$

Now put $1 - \sqrt{1-z} = u$ so that $z = u(2-u)$ and $dz = 2(1-u)\, du$ to get

$$\operatorname*{res}_{u} \frac{1}{u^{x-n+1}} \frac{1}{(2-u)^{x-n+1}} \frac{1}{(1-u)^{2n+1}} [(2-u)^{2x} - u^{2x}](1-u).$$

The second term in the square bracket cancels the pole at zero and hence we are left with

$$\operatorname*{res}_{u} \frac{1}{u^{x-n+1}} \frac{1}{(1-u)^{2n}} (2-u)^{x+n-1}.$$

This is

$$\sum_{k=0}^{x-n} \binom{2n+k-1}{2n-1} \binom{x+n-1}{x-n-k} (-1)^{x-n-k} 2^{2n-1+k}$$

Observe that

$$\begin{aligned}\binom{2n+k-1}{2n-1} \binom{x+n-1}{x-n-k} &= \frac{(x+n-1)!}{(2n-1)! \times k! \times (x-n-k)!} \\ &= \binom{x+n-1}{2n-1} \binom{x-n}{k}\end{aligned}$$

Substitute into the sum to get

$$\begin{aligned}(-1)^{x-n} 2^{2n-1} \binom{x+n-1}{2n-1} \sum_{k=0}^{x-n} \binom{x-n}{k} (-1)^k 2^k = 2^{2n-1} \binom{x+n-1}{2n-1} \\ = 2^{2n-1} \frac{2n}{x+n} \binom{x+n}{2n} = 2^{2n} \frac{n}{x+n} \binom{x+n}{2n}.\end{aligned}$$

This is the claim.

Second identity

This is obviously very similar to the first. Here we prove it for $x \ge n+1$. We find

$$[z^{x-n-1}] \frac{1}{(1-z)^{n+1}} \sum_{k \ge 0} \binom{2x}{2k+1} (1-z)^k.$$

The binomial coefficient once more enforces $x \ge k+1/2$ so that with $k \ge n+1$ the term $[z^{x-n-1}](1+z)^{k-n-1}$ is zero. Repeating the previous computation we obtain

$$\operatorname*{res}_u \frac{1}{u^{x-n}} \frac{1}{(1-u)^{2n+2}} (2-u)^{x+n}.$$

This is

$$\sum_{k=0}^{x-n-1} \binom{2n+k+1}{2n+1} \binom{x+n}{x-n-1-k} (-1)^{x-n-1-k} 2^{2n+1+k}.$$

We once more observe that

$$\binom{2n+k+1}{2n+1} \binom{x+n}{x-n-1-k} = \frac{(x+n)!}{(2n+1)! \times k! \times (x-n-1-k)!}$$

$$= \binom{x+n}{2n+1}\binom{x-n-1}{k}.$$

Substitute into the sum to get

$$(-1)^{x-n-1}2^{2n+1}\binom{x+n}{2n+1}\sum_{k=0}^{x-n-1}\binom{x-n-1}{k}(-1)^k 2^k$$
$$= 2^{2n+1}\binom{x+n}{2n+1} = 2^{2n+1}\frac{x+n}{2n+1}\binom{x+n-1}{2n}.$$

We have the claim.

This problem is from page 42 eqns. 3.157 and 3.158 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.104 A sum of inverse binomial coefficients

We seek to show that

$$\sum_{k=1}^{a-b}\frac{(a-b-k)!}{(a+1-k)!} = \frac{1}{b}\left[\frac{1}{b!} - \frac{(a-b)!}{a!}\right]$$

Recall from section 1.89 the following identity which was proved there: with $1 \le k \le n$

$$\binom{n}{k}^{-1} = k[z^n]\log\frac{1}{1-z}(-1)^{n-k}(1-z)^{n-k}.$$

We thus have with positive integers a, b where $a-b \ge 1$ that

$$\sum_{k=1}^{a-b}\frac{(a-b-k)!}{(a+1-k)!} = \sum_{k=0}^{a-b-1}\frac{k!}{(b+1+k)!} = \frac{1}{(b+1)!}\sum_{k=0}^{a-b-1}\binom{b+1+k}{k}^{-1}$$

$$= \frac{1}{(b+1)!} + \frac{1}{(b+1)!}\sum_{k=1}^{a-b-1}\binom{b+1+k}{k}^{-1}$$

$$= \frac{1}{(b+1)!} + \frac{1}{(b+1)!}\sum_{k=1}^{a-b-1}k[z^{b+1+k}]\log\frac{1}{1-z}(-1)^{b+1}(1-z)^{b+1}.$$

We may lower k to zero because there is zero contribution and get for the sum term

$$\sum_{k=0}^{a-b-1}k[z^{b+1+k}]\log\frac{1}{1-z}(-1)^{b+1}(1-z)^{b+1}$$

$$= \sum_{k=b+1}^{a} (k-(b+1))[z^k] \log \frac{1}{1-z} (-1)^{b+1} (1-z)^{b+1}.$$

Two pieces

We thus require two pieces, the first is

$$[w^m] \frac{1}{1-w} \sum_{k \geq 0} w^k k [z^k] \log \frac{1}{1-z} (-1)^{b+1} (1-z)^{b+1}.$$

This is

$$[w^{m-1}] \frac{1}{1-w} \left(\log \frac{1}{1-z} (z-1)^{b+1} \right)' \Bigg|_{z=w}$$

$$= [w^{m-1}] \frac{1}{1-w} \left(-(z-1)^b + (b+1) \log \frac{1}{1-z} (z-1)^b \right) \Bigg|_{z=w}$$

$$= [w^{m-1}] \left(((w-1)^{b-1} - (b+1) \log \frac{1}{1-w} (w-1)^{b-1} \right).$$

The second main piece is

$$-(b+1)[w^m] \frac{1}{1-w} \sum_{k \geq 0} w^k [z^k] \log \frac{1}{1-z} (-1)^{b+1} (1-z)^{b+1}$$

$$= (b+1)[w^m] \log \frac{1}{1-w} (w-1)^b.$$

Evaluating the pieces at $m = a$ and $m = b$

Evaluating at $m = a$ and $m = b$ we get for the first one

$$-\frac{b+1}{a-b} \binom{a-1}{a-b}^{-1}$$

and the second one

$$1 - (b+1)[w^{b-1}] \log \frac{1}{1-w} (w-1)^{b-1}.$$

Evaluate the second piece again at $m = a$ and $m = b$ we find

$$\frac{b+1}{a-b} \binom{a}{a-b}^{-1}$$

and

$$(b+1)[w^b] \log \frac{1}{1-w} (w-1)^b.$$

We evidently require

$$(-1)^b \operatorname*{res}_w \frac{1}{w^{b+1}}(1-w)^b \log \frac{1}{1-w}$$

This was also evaluated at the cited section and found to be $-H_b$.

Collecting everything

We obtain at last for the sum component

$$-\frac{b+1}{a-b}\binom{a-1}{a-b}^{-1} - 1 - (b+1)H_{b-1} + \frac{b+1}{a-b}\binom{a}{a-b}^{-1} + (b+1)H_b$$

$$= \frac{1}{b} - \frac{b+1}{a-b}\binom{a}{b}^{-1}\frac{a}{b} + \frac{b+1}{a-b}\binom{a}{b}^{-1} = \frac{1}{b} + \frac{b+1}{a-b}\binom{a}{b}^{-1}\frac{b-a}{b}.$$

We get for the complete sum

$$\frac{1}{(b+1)!} + \frac{1}{b \times (b+1)!} - \frac{(a-b)!}{b \times a!},$$

which is the claim.

This was math.stackexchange.com problem 4325592.

1.105 Inverted sum index

We seek to show that

$$\sum_{k=a}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} \frac{2n+1}{2k+1} = (-1)^n \binom{n+a}{2a} 2^{2a}.$$

We will henceforth assume $n \ge a$. First observe that

$$\binom{n+k}{2k}\frac{2n+1}{2k+1} = 2\binom{n+k+1}{2k+1} - \binom{n+k}{2k}.$$

We thus have two pieces, the first is

$$2\sum_{k=a}^{n} (-1)^k \binom{k}{a} 2^{2k} \binom{n+k+1}{2k+1}$$

$$= 2\sum_{k=a}^{n} (-1)^k \binom{k}{a} 2^{2k} \binom{n+k+1}{n-k}$$

$$= (-1)^a 2^{2a+1} \sum_{k=0}^{n-a} (-1)^k \binom{k+a}{a} 2^{2k} \binom{n+a+k+1}{n-a-k}$$

Here we may extend to infinity because of the coefficient extractor for the second binomial coefficient:

$$(-1)^a 2^{2a+1} [z^{n-a}] (1+z)^{n+a+1} \sum_{k\ge 0} (-1)^k \binom{k+a}{a} 2^{2k} z^k (1+z)^k$$
$$= (-1)^a 2^{2a+1} [z^{n-a}] (1+z)^{n+a+1} \frac{1}{(1+4z(1+z))^{a+1}}$$
$$= (-1)^a 2^{2a+1} [z^{n-a}] (1+z)^{n+a+1} \frac{1}{(1+2z)^{2a+2}}.$$

We get for the second piece

$$\sum_{k=a}^{n} (-1)^k \binom{k}{a} 2^{2k} \binom{n+k}{n-k}$$
$$= (-1)^a 2^{2a} \sum_{k=0}^{n-a} (-1)^k \binom{k+a}{a} 2^{2k} \binom{n+a+k}{n-a-k}$$

We extend to infinity same as before

$$(-1)^a 2^{2a} [z^{n-a}] (1+z)^{n+a} \sum_{k\ge 0} (-1)^k \binom{k+a}{a} 2^{2k} z^k (1+z)^k$$
$$= (-1)^a 2^{2a} [z^{n-a}] (1+z)^{n+a} \frac{1}{(1+2z)^{2a+2}}.$$

Subtract the second from the first to get

$$\boxed{(-1)^a 2^{2a} [z^{n-a}] (1+z)^{n+a} \frac{1}{(1+2z)^{2a+1}}.}$$

This is

$$(-1)^a 2^{2a} \sum_{k=0}^{n-a} \binom{n+a}{k} (-1)^{n-a-k} 2^{n-a-k} \binom{n+a-k}{2a}.$$

Note that

$$\binom{n+a}{k}\binom{n+a-k}{2a} = \frac{(n+a)!}{k! \times (2a)! \times (n-a-k)!} = \binom{n+a}{2a}\binom{n-a}{k}$$

which gives for the sum

$$(-1)^n 2^{n+a} \binom{n+a}{2a} \sum_{k=0}^{n-a} \binom{n-a}{k} (-1)^{-k} 2^{-k}$$

$$= (-1)^n 2^{n+a} \binom{n+a}{2a} \frac{1}{2^{n-a}} = (-1)^n 2^{2a} \binom{n+a}{2a}.$$

This is the claim.

This problem is from page 43 eqn. 3.161 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.106 MSE 4351714: A Catalan number recurrence

We seek to show that with regular Catalan numbers

$$\sum_{j=1}^{n+1} \binom{n+j}{2j-1} (-1)^{n+j} C_{n+j-1} = 0.$$

The LHS is setting j to $n+1-j$

$$\sum_{j=0}^{n} \binom{2n+1-j}{2n-2j+1} (-1)^{j+1} C_{2n-j}.$$

This is (discarding the sign because we are trying to verify that the sum is zero):

$$[z^{2n}] \frac{1-\sqrt{1-4z}}{2z} [w^{2n+1}](1+w)^{2n+1} \sum_{j\ge 0} (-1)^j \frac{w^{2j}}{(1+w)^j} z^j.$$

Here we have extended the sum to infinity because of the coefficient extractor in w and obtain

$$[z^{2n}] \frac{1-\sqrt{1-4z}}{2z} [w^{2n+1}](1+w)^{2n+1} \frac{1}{1+w^2 z/(1+w)}$$
$$= [z^{2n}] \frac{1-\sqrt{1-4z}}{2z} [w^{2n+1}](1+w)^{2n+2} \frac{1}{1+w+w^2 z}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{2n+1}} \frac{1-\sqrt{1-4z}}{2z} \frac{1}{1+w+w^2 z}$$

Now put $1-\sqrt{1-4z} = v$ so that $z = v(2-v)/4$ and $dz = (1-v)/2\; dv$ to get

$$\operatorname*{res}_v \frac{4^{2n+1}}{v^{2n+1}(2-v)^{2n+1}} \frac{v}{v(2-v)/2} \frac{(1-v)/2}{1+w+w^2 v(2-v)/4}$$
$$= \operatorname*{res}_v \frac{4^{2n+1}}{v^{2n+1}(2-v)^{2n+2}} \frac{1-v}{1+w+w^2 v(2-v)/4}$$
$$= 2^{2n} \operatorname*{res}_v \frac{1}{v^{2n+1}(1-v/2)^{2n+2}} \frac{1-v}{1+w+w^2 v(2-v)/4}.$$

Observe that

$$\frac{1-v}{1+w+w^2v(2-v)/4} = -\frac{v}{2+vw} + \frac{2-v}{2(1+w)-vw}.$$

The contribution from the first term is

$$2^{2n}\operatorname*{res}_v \frac{1}{v^{2n+1}(1-v/2)^{2n+2}}\left(-\frac{1}{2}v\right)[w^{2n+1}](1+w)^{2n+2}\sum_{q=0}^{2n+1}(-1)^q\frac{1}{2^q}v^q w^q$$

$$= -2^{2n-1}\operatorname*{res}_v \frac{1}{v^{2n}(1-v/2)^{2n+2}}\sum_{q=0}^{2n+1}\binom{2n+2}{2n+1-q}(-1)^q\frac{1}{2^q}v^q$$

$$= -2^{2n-1}\sum_{q=0}^{2n}\binom{2n+2}{q+1}(-1)^q\frac{1}{2^q}\binom{2n-1-q+2n+1}{2n+1}\frac{1}{2^{2n-1-q}}$$

$$= -\sum_{q=0}^{2n}\binom{2n+2}{q+1}(-1)^q\binom{4n-q}{2n+1}.$$

The contribution from the second term is

$$2^{2n}\operatorname*{res}_v \frac{1}{v^{2n+1}(1-v/2)^{2n+2}}\left(1-\frac{1}{2}v\right)[w^{2n+1}](1+w)^{2n+1}\sum_{q=0}^{2n+1}\frac{1}{2^q}v^q\frac{w^q}{(1+w)^q}$$

$$= 2^{2n}\operatorname*{res}_v \frac{1}{v^{2n+1}(1-v/2)^{2n+1}}\sum_{q=0}^{2n+1}\binom{2n+1-q}{2n+1-q}\frac{1}{2^q}v^q$$

$$= 2^{2n}\sum_{q=0}^{2n}\frac{1}{2^q}\binom{2n-q+2n}{2n}\frac{1}{2^{2n-q}} = \sum_{q=0}^{2n}\binom{4n-q}{2n}.$$

We thus have to show that

$$\sum_{q=0}^{m}\binom{m+2}{q+1}(-1)^q\binom{2m-q}{m+1} = \sum_{q=0}^{m}\binom{2m-q}{m}.$$

The LHS is

$$\sum_{q=1}^{m+1}\binom{m+2}{q}(-1)^{q-1}\binom{2m+1-q}{m+1}$$

$$= \binom{2m+1}{m} - [z^{m+1}](1+z)^{2m+1}\sum_{q=0}^{m+2}\binom{m+2}{q}\frac{(-1)^q}{(1+z)^q}$$

$$= \binom{2m+1}{m} - [z^{m+1}](1+z)^{2m+1}\left(1 - \frac{1}{1+z}\right)^{m+2}$$

$$= \binom{2m+1}{m} - [z^{m+1}](1+z)^{m-1}z^{m+2} = \binom{2m+1}{m}.$$

The RHS is

$$\sum_{q=0}^{m} \binom{2m-q}{m-q} = [z^m](1+z)^{2m} \sum_{q\ge 0} \frac{z^q}{(1+z)^q}$$

$$= [z^m](1+z)^{2m} \frac{1}{1-z/(1+z)} = [z^m](1+z)^{2m+1} = \binom{2m+1}{m}.$$

This concludes the argument.

This was math.stackexchange.com problem 4351714.

1.107 An identity by Graham and Riordan

We seek to show that

$$\sum_{k=0}^{n} \frac{2k+1}{n+k+1} \binom{x-k-1}{n-k} \binom{x+k}{n+k} = \binom{x}{n}^2.$$

As both sides are polynomials of degree $2n$ in x we prove it for x a positive integer such that $x > n$ and we then have it for all i.e. complex x. This sum is the difference of

$$\sum_{k=0}^{n} \binom{x-k-1}{n-k} \binom{x+k}{n+k}$$

and

$$\sum_{k=0}^{n} \frac{n-k}{n+k+1} \binom{x-k-1}{n-k} \binom{x+k}{n+k}.$$

First piece

We get

$$\sum_{k=0}^{n} \binom{x-k-1}{n-k} \binom{x+k}{n+k}$$

$$= [z^n](1+z)^{x-1} \sum_{k=0}^{n} \frac{z^k}{(1+z)^k} \binom{x+k}{x-n}.$$

Here we may extend the sum to infinity because the coefficient extractor enforces the upper limit. Continuing,

$$\begin{aligned}
&[z^n](1+z)^{x-1}[w^{x-n}](1+w)^x \sum_{k\ge 0} \frac{z^k}{(1+z)^k}(1+w)^k \\
&= [z^n](1+z)^{x-1}[w^{x-n}](1+w)^x \frac{1}{1-z(1+w)/(1+z)} \\
&= [z^n](1+z)^{x}[w^{x-n}](1+w)^x \frac{1}{1-wz}.
\end{aligned}$$

Second piece

This is very similar to the first. We find

$$\begin{aligned}
&\sum_{k=0}^{n-1} (x-n)\binom{x-k-1}{n-k-1}\binom{x+k}{n+k+1}\frac{1}{x-n} \\
&= \sum_{k=0}^{n-1} \binom{x-k-1}{n-k-1}\binom{x+k}{n+k+1} \\
&= [z^{n-1}](1+z)^{x-1}\sum_{k=0}^{n-1} \frac{z^k}{(1+z)^k}\binom{x+k}{x-n-1}.
\end{aligned}$$

Once more we extend the sum to infinity because the coefficient extractor enforces the upper limit. Continuing,

$$\begin{aligned}
&[z^{n-1}](1+z)^{x-1}[w^{x-n-1}](1+w)^x \sum_{k\ge 0} \frac{z^k}{(1+z)^k}(1+w)^k \\
&= [z^{n-1}](1+z)^{x-1}[w^{x-n-1}](1+w)^x \frac{1}{1-z(1+w)/(1+z)} \\
&= [z^{n-1}](1+z)^{x}[w^{x-n-1}](1+w)^x \frac{1}{1-wz}.
\end{aligned}$$

Collecting everything

Observe that the second piece may be written as

$$[z^n](1+z)^x[w^{x-n}](1+w)^x \frac{wz}{1-wz}.$$

Subtract the second piece from the first to get

$$[z^n](1+z)^x[w^{x-n}](1+w)^x = \binom{x}{n}\binom{x}{x-n} = \binom{x}{n}^2$$

as claimed.

This problem is from page 44 eqn. 3.168 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.108 Square root term

We seek to prove that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} {n+1 \choose 2k+1} {x+k \choose n} = {2x \choose n}$$

and

$$\sum_{k=0}^{\lfloor (n+1)/2 \rfloor} {n+1 \choose 2k} {x+k \choose n} = {2x+1 \choose n}.$$

As these are both polynomials in x of degree n we prove it for $x \ge n$ a positive integer and then have it for all i.e. complex x.

First formula

We get for the LHS

$$[z^n](1+z)^x \sum_{k=0}^{n+1} {n+1 \choose k} \frac{1-(-1)^k}{2} (1+z)^{(k-1)/2}$$
$$= [z^n](1+z)^x \frac{1}{\sqrt{1+z}} \sum_{k=0}^{n+1} {n+1 \choose k} \frac{1-(-1)^k}{2} (1+z)^{k/2}.$$

The first piece here is

$$\frac{1}{2}[z^n](1+z)^x \frac{1}{\sqrt{1+z}} (1+\sqrt{1+z})^{n+1}$$
$$= \frac{1}{2} \operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^x \frac{1}{\sqrt{1+z}} \frac{(-1)^{n+1} z^{n+1}}{(1-\sqrt{1+z})^{n+1}}.$$

Now we put $1-\sqrt{1+z} = w$ so that $z = w(w-2)$ and $dz = 2(w-1)\; dw$ to obtain

$$\frac{1}{2} \operatorname*{res}_w (1-w)^{2x} \frac{1}{1-w} (-1)^{n+1} \frac{1}{w^{n+1}} 2(w-1)$$
$$= \operatorname*{res}_w \frac{1}{w^{n+1}} (-1)^n (1-w)^{2x} = {2x \choose n}.$$

It remains to show that the second piece is zero and we get

$$\frac{1}{2} \operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^x \frac{1}{\sqrt{1+z}} (1-\sqrt{1+z})^{n+1}.$$

With the same substitution as before we have

$$\frac{1}{2} \operatorname*{res}_{w} \frac{1}{w^{n+1}(w-2)^{n+1}} (1-w)^{2x} \frac{1}{1-w} w^{n+1} 2(w-1)$$
$$= - \operatorname*{res}_{w} \frac{1}{(w-2)^{n+1}} (1-w)^{2x} = 0.$$

Second formula

This is very similar to the first. We get

$$[z^n](1+z)^x \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1+(-1)^k}{2} (1+z)^{k/2}.$$

The first piece is

$$\frac{1}{2} \operatorname*{res}_{z} \frac{1}{z^{n+1}} (1+z)^x (1+\sqrt{1+z})^{n+1}$$
$$= \frac{1}{2} \operatorname*{res}_{z} \frac{1}{z^{n+1}} (1+z)^x \frac{(-1)^{n+1} z^{n+1}}{(1-\sqrt{1+z})^{n+1}}.$$

Repeat the substitution to get

$$\frac{1}{2} \operatorname*{res}_{w} (1-w)^{2x} (-1)^{n+1} \frac{1}{w^{n+1}} 2(w-1)$$
$$= \operatorname*{res}_{w} \frac{1}{w^{n+1}} (-1)^n (1-w)^{2x+1} = \binom{2x+1}{n}.$$

To conclude show that the second piece is zero as in

$$\frac{1}{2} \operatorname*{res}_{z} \frac{1}{z^{n+1}} (1+z)^x (1-\sqrt{1+z})^{n+1}$$

which becomes

$$\frac{1}{2} \operatorname*{res}_{w} \frac{1}{w^{n+1}(w-2)^{n+1}} (1-w)^{2x} w^{n+1} 2(w-1)$$
$$= - \operatorname*{res}_{w} \frac{1}{(w-2)^{n+1}} (1-w)^{2x+1} = 0.$$

This problem is from page 44 eqns. 3.169 and 3.170 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.109 Identity by Machover and Gould

We seek to prove that

$$\sum_{k=0}^{n} \binom{x}{2k} \binom{x-2k}{n-k} 2^{2k} = \binom{2x}{2n}$$

and

$$\sum_{k=0}^{n} \binom{x+1}{2k+1} \binom{x-2k}{n-k} 2^{2k+1} = \binom{2x+2}{2n+1}.$$

As these are both polynomials in x of degree $2n$ and $2n+1$ we prove it for $x \geq 2n$ a positive integer and then have it for all i.e. complex x.

First formula

We get for the LHS

$$[z^n](1+z)^x \sum_{k\geq 0} \binom{x}{2k} \frac{z^k}{(1+z)^{2k}} 2^{2k}.$$

Here we have extended the sum to infinity because the coefficient extractor enforces the upper limit. Continuing,

$$[z^n](1+z)^x \sum_{k\geq 0} \binom{x}{k} \frac{2^k \sqrt{z}^k}{(1+z)^k} \frac{1+(-1)^k}{2}.$$

The first piece is

$$\frac{1}{2}[z^n](1+z)^x \left(1 + \frac{2\sqrt{z}}{1+z}\right)^x = \frac{1}{2}[z^n]\left(1+2\sqrt{z}+z\right)^x$$
$$= \frac{1}{2}[z^n](1+\sqrt{z})^{2x}.$$

Incorporating the second piece we have

$$\frac{1}{2}[z^n](1+\sqrt{z})^{2x} + \frac{1}{2}[z^n](1-\sqrt{z})^{2x}$$
$$= \frac{1}{2}\binom{2x}{2n} + \frac{1}{2}\binom{2x}{2n}(-1)^{2n} = \binom{2x}{2n}.$$

This is the claim.

Second formula

We get for the LHS

$$[z^n](1+z)^x \sum_{k\geq 0} \binom{x+1}{2k+1} \frac{z^k}{(1+z)^{2k}} 2^{2k+1}.$$

This was the same extension to infinity of the sum. Continuing,

$$[z^n](1+z)^x \sum_{k\geq 0} \binom{x+1}{k} \frac{2^k \sqrt{z}^{k-1}}{(1+z)^{k-1}} \frac{1-(-1)^k}{2}.$$

The first piece is

$$\frac{1}{2}[z^n](1+z)^{x+1}\frac{1}{\sqrt{z}}\left(1+\frac{2\sqrt{z}}{1+z}\right)^{x+1} = \frac{1}{2}[z^n]\frac{1}{\sqrt{z}}\left(1+2\sqrt{z}+z\right)^{x+1}$$

$$= \frac{1}{2}[z^n]\frac{1}{\sqrt{z}}(1+\sqrt{z})^{2x+2}.$$

Incorporate the second piece to get

$$\frac{1}{2}[z^n]\frac{1}{\sqrt{z}}(1+\sqrt{z})^{2x+2} - \frac{1}{2}[z^n]\frac{1}{\sqrt{z}}(1-\sqrt{z})^{2x+2}$$

$$= \frac{1}{2}\binom{2x+2}{2n+1} - \frac{1}{2}\binom{2x+2}{2n+1}(-1)^{2n+1} = \binom{2x+2}{2n+1}.$$

Once more we have the claim.

This problem is from page 44 eqns. 3.175 and 3.176 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.110 Moriarty identity by H.T.Davis et al.

We seek to prove that with $n \ge p$

$$\sum_{k=0}^{n-p}\binom{2n+1}{2p+2k+1}\binom{p+k}{k} = \binom{2n-p}{p}2^{2n-2p}$$

and

$$\sum_{k=0}^{n-p}\binom{2n}{2p+2k}\binom{p+k}{k} = \frac{n}{2n-p}\binom{2n-p}{p}2^{2n-2p}.$$

First formula

We get for the LHS

$$\sum_{k=0}^{n-p}\binom{2n+1}{2n-2p-2k}\binom{p+k}{k}$$

$$= [z^{2n-2p}](1+z)^{2n+1}\sum_{k\ge 0}\binom{p+k}{k}z^{2k}.$$

Here we have extended the sum to infinity because the coefficient extractor enforces the range. Continuing,

$$[z^{2n-2p}](1+z)^{2n+1}\frac{1}{(1-z^2)^{p+1}} = [z^{2n-2p}](1+z)^{2n-p}\frac{1}{(1-z)^{p+1}}.$$

This is

$$\operatorname*{res}_z \frac{1}{z^{2n-2p+1}} (1+z)^{2n-p} \frac{1}{(1-z)^{p+1}}.$$

Now put $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2\ dw$ to get

$$\begin{aligned}
&\operatorname*{res}_w \frac{1}{w^{2n-2p+1}} \frac{1}{(1-w)^{p-1}} \frac{(1-w)^{p+1}}{(1-2w)^{p+1}} \frac{1}{(1-w)^2} \\
&= \operatorname*{res}_w \frac{1}{w^{2n-2p+1}} \frac{1}{(1-2w)^{p+1}} = \binom{2n-p}{p} 2^{2n-2p}.
\end{aligned}$$

This is the claim.

Second formula

Re-capitulating the previous computation we get

$$\operatorname*{res}_z \frac{1}{z^{2n-2p+1}} (1+z)^{2n-p-1} \frac{1}{(1-z)^{p+1}}$$

which leads to

$$\begin{aligned}
\operatorname*{res}_w \frac{1}{w^{2n-2p+1}} \frac{1-w}{(1-2w)^{p+1}} &= \binom{2n-p}{p} 2^{2n-2p} - \binom{2n-p-1}{p} 2^{2n-2p-1} \\
&= \binom{2n-p}{p} 2^{2n-2p} \left[1 - \frac{1}{2} \frac{2n-2p}{2n-p}\right] = \binom{2n-p}{p} 2^{2n-2p} \frac{n}{2n-p}
\end{aligned}$$

again as claimed.

This problem is from page 44 eqns. 3.177 and 3.178 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.111 Inverse Moriarty identity by Marcia Ascher

We seek to prove that with $n \ge 2r$

$$\sum_{k=r}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \binom{k}{r} 2^{n-2k} = (-1)^r \binom{n+1}{2r+1}$$

and

$$\sum_{k=r}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{k}{r} 2^{n-2k-1} = (-1)^r \binom{n}{2r}.$$

First formula

We have for the LHS

$$\sum_{k=r}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{n-2k} \binom{k}{r} 2^{n-2k}$$
$$= [z^n](1+z)^n \sum_{k\ge r} (-1)^k \frac{z^{2k}}{(1+z)^k} \binom{k}{r} 2^{n-2k}.$$

Here we have extended the sum to infinity because the coefficient extractor enforces the upper range. Continuing,

$$(-1)^r 2^{n-2r} [z^{n-2r}](1+z)^{n-r} \sum_{k\ge 0} (-1)^k \frac{z^{2k}}{(1+z)^k} \binom{k+r}{r} 2^{-2k}$$
$$= (-1)^r 2^{n-2r} [z^{n-2r}](1+z)^{n-r} \frac{1}{(1+z^2/4/(1+z))^{r+1}}$$
$$= (-1)^r 2^{n+2} [z^{n-2r}](1+z)^{n+1} \frac{1}{(4+4z+z^2)^{r+1}}.$$

This is

$$(-1)^r 2^{n+2} \operatorname*{res}_z \frac{1}{z^{n+1-2r}} (1+z)^{n+1} \frac{1}{(2+z)^{2r+2}}.$$

Now we put $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2 \; dw$ to obtain

$$(-1)^r 2^{n+2} \operatorname*{res}_w \frac{1}{w^{n+1-2r}} \frac{1}{(1-w)^{2r}} \frac{1}{(2+w/(1-w))^{2r+2}} \frac{1}{(1-w)^2}$$
$$= (-1)^r 2^{n+2} \operatorname*{res}_w \frac{1}{w^{n+1-2r}} \frac{1}{(2-w)^{2r+2}}$$
$$= (-1)^r 2^{n-2r} \operatorname*{res}_w \frac{1}{w^{n+1-2r}} \frac{1}{(1-w/2)^{2r+2}}$$
$$= (-1)^r 2^{n-2r} \binom{n-2r+2r+1}{2r+1} \frac{1}{2^{n-2r}} = (-1)^r \binom{n+1}{2r+1}.$$

This is the claim.

Second formula

The LHS is the sum of

$$\sum_{k=r}^{\lfloor n/2\rfloor} (-1)^k {n-k\choose k} {k\choose r} 2^{n-2k-1}$$

and

$$\sum_{k=r}^{\lfloor n/2\rfloor} (-1)^k \frac{k}{n-k} {n-k\choose k} {k\choose r} 2^{n-2k-1}.$$

The first one was evaluated in the previous section and yields

$$\frac{1}{2}(-1)^r {n+1\choose 2r+1}.$$

For the second one we get

$$\sum_{k=r}^{\lfloor n/2\rfloor} (-1)^k {n-k-1\choose k-1} {k\choose r} 2^{n-2k-1} = \sum_{k=r}^{\lfloor n/2\rfloor} (-1)^k {n-k-1\choose n-2k} {k\choose r} 2^{n-2k-1}.$$

Recapitulating the previous computation this becomes

$$(-1)^r 2^{n+1} \operatorname*{res}_z \frac{1}{z^{n+1-2r}} (1+z)^n \frac{1}{(2+z)^{2r+2}}.$$

Continuing,

$$\begin{aligned}
&(-1)^r 2^{n-1-2r} \operatorname*{res}_w \frac{1}{w^{n+1-2r}} \frac{1-w}{(1-w/2)^{2r+2}} \\
&= (-1)^r 2^{n-1-2r} {n-2r+2r+1\choose 2r+1} \frac{1}{2^{n-2r}} \\
&-(-1)^r 2^{n-1-2r} {n-2r-1+2r+1\choose 2r+1} \frac{1}{2^{n-2r-1}} \\
&= \frac{1}{2}(-1)^r {n+1\choose 2r+1} - (-1)^r {n\choose 2r+1}.
\end{aligned}$$

Collecting the three binomial coefficients now yields

$$(-1)^r {n+1\choose 2r+1} - (-1)^r {n\choose 2r+1} = (-1)^r {n\choose 2r}.$$

Once more we have the claim.

This problem is from page 45 eqns. 3.179 and 3.180 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.112 Moriarty identity by Egorychev

Suppose we seek to evaluate

$$S_{n,m} = \sum_{k=m}^{n} (-1)^k 2^{2k} \binom{k}{m} \frac{n}{n+k} \binom{n+k}{2k}.$$

We have

$$\frac{n}{n+k}\binom{n+k}{2k} = \binom{n+k}{2k} - \frac{1}{2}\binom{n+k-1}{2k-1}.$$

Therefore we get for the first component

$$\sum_{k=m}^{n} (-1)^k 2^{2k} \binom{k}{m} \binom{n+k}{n-k}$$
$$= [z^n](1+z)^n \sum_{k\ge m} (-1)^k 2^{2k} \binom{k}{m} z^k (1+z)^k.$$

Here we have extended the sum to infinity because the coefficient extractor enforces the upper limit of the range. Continuing,

$$[z^{n-m}](1+z)^{n+m}(-1)^m 2^{2m} \sum_{k\ge 0} (-1)^k 2^{2k} \binom{k+m}{m} z^k (1+z)^k$$
$$= (-1)^m 2^{2m} [z^{n-m}](1+z)^{n+m} \frac{1}{(1+4z(1+z))^{m+1}}$$
$$= (-1)^m 2^{2m} \operatorname*{res}_{z} \frac{1}{z^{n-m+1}} (1+z)^{n+m} \frac{1}{(1+2z)^{2m+2}}.$$

Now we put $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2 \, dw$ to get

$$(-1)^m 2^{2m} \operatorname*{res}_{w} \frac{1}{w^{n-m+1}} \frac{1}{(1-w)^{2m-1}} \frac{(1-w)^{2m+2}}{(1+w)^{2m+2}} \frac{1}{(1-w)^2}$$
$$= (-1)^m 2^{2m} \operatorname*{res}_{w} \frac{1}{w^{n-m+1}} \frac{1-w}{(1+w)^{2m+2}}.$$

Repeating the above calculation we get for the second component

$$-\frac{1}{2}(-1)^m 2^{2m} \operatorname*{res}_{w} \frac{1}{w^{n-m+1}} \frac{(1-w)^2}{(1+w)^{2m+2}}.$$

Observing

$$(1-w) - \frac{1}{2}(1-w)^2 = (1+w) - \frac{1}{2}(1+w)^2$$

we thus obtain

$$(-1)^n 2^{2m}\left[\binom{n+m}{n-m} - \frac{1}{2}\binom{n+m-1}{n-m}\right] = (-1)^n 2^{2m}\binom{n+m}{n-m}\left[1 - \frac{1}{2}\frac{2m}{n+m}\right]$$
$$= (-1)^n 2^{2m}\frac{n}{n+m}\binom{n+m}{2m}.$$

This was page 11 from [Ego84].

1.113 MSE 4462359: Two binomial coefficients

Suppose for we seek to verify that

$$\sum_{q=0}^{m} \frac{(-1)^{q-1}}{q+1}\binom{k+q}{q}\binom{k}{q} = \frac{(-1)^{m+1}}{k+1}\binom{k-1}{m}\binom{k+1+m}{k}$$

where $1 \le m < k$. This is (Iverson bracket)

$$[z^m]\frac{1}{1-z}\sum_{q\ge 0} z^q \frac{(-1)^{q-1}}{q+1}\binom{k+q}{q}\binom{k}{q}$$
$$= \frac{1}{k+1}[z^m]\frac{1}{1-z}\sum_{q\ge 0} z^q (-1)^{q-1}\binom{k+q}{q}\binom{k+1}{q+1}$$
$$= \frac{1}{k+1}[z^m]\frac{1}{1-z}[w^k](1+w)^{k+1}\sum_{q\ge 0} z^q (-1)^{q-1}\binom{k+q}{q} w^q$$
$$= \frac{1}{k+1}[z^m]\frac{1}{z-1}[w^k](1+w)^{k+1}\frac{1}{(1+wz)^{k+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{k+1}}(1+w)^{k+1}\frac{1}{(1+wz)^{k+1}}.$$

We put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2\ dv$ to get

$$\operatorname*{res}_{v} \frac{1}{v^{k+1}}\frac{1}{(1+zv/(1-v))^{k+1}}\frac{1}{(1-v)^2}$$
$$= \operatorname*{res}_{v} \frac{1}{v^{k+1}}\frac{(1-v)^{k-1}}{(1-v(1-z))^{k+1}}.$$

This is

$$\sum_{q=0}^{k-1} (-1)^q \binom{k-1}{q}\binom{2k-q}{k-q}(1-z)^{k-q}.$$

Applying the coefficient extractor in z we find

$$\frac{(-1)^{m-1}}{k+1}\sum_{q=0}^{k-1}(-1)^q \binom{k-1}{q}\binom{2k-q}{k-q}\binom{k-1-q}{m}.$$

Observe that

$$\binom{k-1}{q}\binom{k-1-q}{m} = \frac{(k-1)!}{q! \times m! \times (k-1-q-m)!} = \binom{k-1}{m}\binom{k-1-m}{q}.$$

This will correctly evaluate to zero when $k-1-m < q$. Continuing we find

$$\frac{(-1)^{m-1}}{k+1}\binom{k-1}{m}\sum_{q=0}^{k-1}(-1)^q \binom{2k-q}{k-q}\binom{k-1-m}{q}.$$

Working with the sum we see that we may lower to $q = k-1-m$ due to the third binomial coefficient and the condition $1 \le m < k$. We thus obtain

$$\sum_{q=0}^{k-1-m}(-1)^q \binom{2k-q}{k-q}\binom{k-1-m}{q}$$

$$= [z^k](1+z)^{2k}\sum_{q=0}^{k-1-m}(-1)^q \frac{z^q}{(1+z)^q}\binom{k-1-m}{q}$$

$$= [z^k](1+z)^{2k}\left[1-\frac{z}{1+z}\right]^{k-1-m} = [z^k](1+z)^{k+1+m} = \binom{k+1+m}{k}.$$

Collecting everything we finally have

$$\boxed{\frac{(-1)^{m+1}}{k+1}\binom{k-1}{m}\binom{k+1+m}{k}.}$$

This was math.stackexchange.com problem 4462359.

1.114 Polynomial identity

We seek to prove that with $f(x) = \sum_{q=0}^{n} a_q x^q$ a polynomial of degree at most n we have

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{f(x-k)}{k} = H_n f(x) - f'(x).$$

Substituting into the LHS we find

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}\sum_{q=0}^{n} a_q (x-k)^q$$

$$= \sum_{q=0}^n a_q \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k} (x-k)^q$$

$$= \sum_{q=0}^n a_q \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n}{k} \frac{1}{n-k} (x-n+k)^q$$

$$= [z^n] \log \frac{1}{1-z} \sum_{q=0}^n a_q \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n}{k} z^k (x-n+k)^q.$$

We may raise k to n due to the coefficient extractor and the fact that $\log \frac{1}{1-z} = z + \cdots$. Continuing with the sum term,

$$\sum_{q=0}^n a_q q! [w^q] \exp((x-n)w) \sum_{k=0}^n (-1)^{n-k-1} \binom{n}{k} z^k \exp(kw)$$

$$= - \sum_{q=0}^n a_q q! [w^q] \exp((x-n)w)(z\exp(w)-1)^n$$

$$= - \sum_{q=0}^n a_q q! [w^q] \exp(xw)(z-\exp(-w))^n$$

$$= - \sum_{q=0}^n a_q q! [w^q] \exp(xw) \sum_{p=0}^n \binom{n}{p} (1-\exp(-w))^{n-p} (z-1)^p.$$

First piece

Now for $p = n$ we get

$$-[z^n] \log \frac{1}{1-z} (z-1)^n \sum_{q=0}^n a_q x^q = -f(x)[z^n] \log \frac{1}{1-z} (z-1)^n.$$

The contribution from z is

$$- \operatorname{res}_z \frac{1}{z^{n+1}} \log \frac{1}{1-z} (z-1)^n.$$

Now put $z/(z-1) = v$ so that $z = v/(v-1)$ and $dz = -1/(v-1)^2 \; dv$ to get

$$\operatorname{res}_v \frac{1}{v^{n+1}} \log \frac{1}{1-v/(v-1)} (v-1) \frac{1}{(v-1)^2}$$

$$= \operatorname{res}_v \frac{1}{v^{n+1}} \log(1-v) \frac{1}{v-1} = \operatorname{res}_v \frac{1}{v^{n+1}} \frac{1}{1-v} \log \frac{1}{1-v} = H_n.$$

We have recovered the first term $f(x)H_n$.

Second piece

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [z^n] \log \frac{1}{1-z}(z-1)^{n-k}.$$

Here we put $k := n - p$ to obtain including the logarithm in front

$$\begin{aligned}
&- \sum_{q=0}^{n} a_q q! [w^q] \exp(xw) \sum_{p=0}^{n-1} \frac{(1-\exp(-w))^{n-p}}{n-p} \\
&= - \sum_{q=0}^{n} a_q q! [w^q] \exp(xw) \sum_{p=1}^{n} \frac{(1-\exp(-w))^{p}}{p}.
\end{aligned}$$

Now we can certainly extend p to infinity because $(1-\exp(-w))^p = w^p + \cdots$ so there is no contribution when $p > n$. We get

$$\begin{aligned}
&- \sum_{q=0}^{n} a_q q! [w^q] \exp(xw) \log \frac{1}{1-(1-\exp(-w))} \\
&= - \sum_{q=0}^{n} a_q q! [w^q] \exp(xw) w = - \sum_{q=1}^{n} a_q q! [w^{q-1}] \exp(xw) \\
&= - \sum_{q=1}^{n} a_q q! x^{q-1} \frac{1}{(q-1)!} = - \sum_{q=1}^{n} a_q q x^{q-1} = -f'(x).
\end{aligned}$$

We have recovered the second term $-f'(x)$. This formula will produce e.g. the identity

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} \binom{x-k}{n} = \binom{x}{n} \left\{ H_n - \sum_{k=0}^{n-1} \frac{1}{x-k} \right\}.$$

Note that here both sides are polynomials in x.

This problem is from page 82 eqn. Z.7 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.115 Polynomial identity II

We seek to prove that with $f(x) = \sum_{q=0}^{n} a_q x^q$ a polynomial of degree at most n we have

$$f(x+y) = y\binom{y+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x-k)}{y+k}.$$

Starting with the sum term, proving it for y a positive integer (we have the result because the LHS and RHS are polynomials in x and y)

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{f(x-n+k)}{y+n-k}$$

$$= [z^{n+y}]\log\frac{1}{1-z}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}z^k\sum_{q=0}^{n}a_q(x-n+k)^q$$

$$= [z^{n+y}]\log\frac{1}{1-z}\sum_{q=0}^{n}a_q q![w^q]\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}z^k\exp((x-n+k)w)$$

$$= [z^{n+y}]\log\frac{1}{1-z}\sum_{q=0}^{n}a_q q![w^q]\exp((x-n)w)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}z^k\exp(kw)$$

$$= [z^{n+y}]\log\frac{1}{1-z}\sum_{q=0}^{n}a_q q![w^q]\exp((x-n)w)(z\exp(w)-1)^n$$

$$= [z^{n+y}]\log\frac{1}{1-z}\sum_{q=0}^{n}a_q q![w^q]\exp(xw)(z-\exp(-w))^n$$

$$= [z^{n+y}]\log\frac{1}{1-z}\sum_{q=0}^{n}a_q q![w^q]\exp(xw)\sum_{p=0}^{n}\binom{n}{p}(z-1)^p(1-\exp(-w))^{n-p}.$$

Now the contribution from z is

$$[z^{n+y}]\log\frac{1}{1-z}(z-1)^p.$$

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [z^n]\log\frac{1}{1-z}(z-1)^{n-k}.$$

Here we put $n := n+y$ and $k := n+y-p$ to get (we have $k \ge 1$ because we chose $y \ge 1$)

$$\sum_{q=0}^{n}a_q q![w^q]\exp(xw)\sum_{p=0}^{n}\binom{n}{p}\binom{n+y}{p}^{-1}\frac{(1-\exp(-w))^{n-p}}{n+y-p}.$$

Next observe that

$$\binom{n}{p}\binom{n+y}{p}^{-1} = \frac{n!\times(n+y-p)!}{(n-p)!\times(n+y)!} = \binom{n+y}{n}^{-1}\binom{n+y-p}{n-p}.$$

Restoring the binomial factor in front from the beginning, which now disappears,

$$y\sum_{q=0}^{n} a_q q![w^q]\exp(xw)\sum_{p=0}^{n}\binom{n+y-p}{n-p}\frac{(1-\exp(-w))^{n-p}}{n+y-p}$$
$$= y\sum_{q=0}^{n} a_q q![w^q]\exp(xw)\sum_{p=0}^{n}\binom{y+p}{y}\frac{(1-\exp(-w))^{p}}{y+p}$$
$$= \sum_{q=0}^{n} a_q q![w^q]\exp(xw)\sum_{p=0}^{n}\binom{y-1+p}{y-1}(1-\exp(-w))^{p}.$$

Finally observe that $(1-\exp(-w))^p = w^p + \cdots$ so we may extend p beyond n with no contribution due to the coefficient extractor in w to get

$$\sum_{q=0}^{n} a_q q![w^q]\exp(xw)\frac{1}{(1-(1-\exp(-w)))^y}$$
$$= \sum_{q=0}^{n} a_q q![w^q]\exp(xw)\exp(yw) = \sum_{q=0}^{n} a_q q![w^q]\exp((x+y)w) = f(x+y)$$

and we have the claim.

This problem is from page 82 eqn. Z.5 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.116 Worpitzky-Nielsen series

We seek to prove that with $f(x) = \sum_{q=0}^{n} a_q x^q$ a polynomial of degree at most n we have for $m \ge n$

$$f(x+y) = (-1)^m \sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}f(j-k+y).$$

With both sides polynomials in x and y it will suffice to prove this for x and y positive integers. We get for the RHS

$$(-1)^m \sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\sum_{q=0}^{n} a_q (j-k+y)^q$$
$$= (-1)^m \sum_{q=0}^{n} a_q q![w^q]\sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\exp((j-k+y)w)$$
$$= (-1)^m \sum_{q=0}^{n} a_q q![w^q]\exp(yw)\sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^{k-j}\binom{m+1}{k-j}\exp(-jw).$$

Continuing with the innermost sum,

$$[z^k](1+z)^{m+1}\sum_{j=0}^{k}(-1)^{k-j}z^j\exp(-jw) = (-1)^k[z^k]\frac{(1+z)^{m+1}}{1+z\exp(-w)}.$$

Here we have extended j to infinity due to the coefficient extractor in front. With the second binomial coefficient (which comes before the first) we find

$$\sum_{k=0}^{m+1}\binom{x+m-k}{m}(-1)^{m+1-k}[z^{m+1-k}]\frac{(1+z)^{m+1}}{1+z\exp(-w)}$$
$$= (-1)^{m+1}[z^{m+1}]\frac{(1+z)^{m+1}}{1+z\exp(-w)}[v^m](1+v)^{x+m}\sum_{k\ge 0}\frac{(-1)^k}{(1+v)^k}z^k.$$

Here the coefficient extractor in z is applied a second time to enforce the upper limit of the sum so we may raise to infinity to get (we will restore the factor $(-1)^{m+1}$ in the next phase)

$$[z^{m+1}]\frac{(1+z)^{m+1}}{1+z\exp(-w)}[v^m](1+v)^{x+m}\frac{1}{1+z/(1+v)}$$
$$= [z^{m+1}]\frac{(1+z)^{m+1}}{1+z\exp(-w)}[v^m](1+v)^{x+m+1}\frac{1}{1+v+z}$$
$$= [z^{m+1}]\frac{(1+z)^{m}}{1+z\exp(-w)}[v^m](1+v)^{x+m+1}\frac{1}{1+v/(1+z)}$$
$$= [z^{m+1}]\frac{(1+z)^{m}}{1+z\exp(-w)}\sum_{j=0}^{m}\frac{(-1)^j}{(1+z)^j}\binom{x+m+1}{m-j}$$
$$= \sum_{j=0}^{m}\binom{x+m+1}{m-j}(-1)^j\sum_{k=0}^{m+1}\binom{m-j}{k}(-1)^{m+1-k}\exp(-(m+1-k)w).$$

We merge in the factor $(-1)^{m+1}$ and note that with $m-j\ge 0$ and $k\ge 0$ we have $\binom{m-j}{k}=0$ when $k>m-j$ which yields

$$\exp(-(m+1)w)\sum_{j=0}^{m}\binom{x+m+1}{m-j}(-1)^j\sum_{k=0}^{m-j}\binom{m-j}{k}(-1)^k\exp(kw)$$
$$= \exp(-(m+1)w)\sum_{j=0}^{m}\binom{x+m+1}{m-j}(-1)^j(1-\exp(w))^{m-j}.$$

We at last compute the coefficient on a_q which is given by

$$q![w^q]\exp(yw)\exp(-(m+1)w)\sum_{j=0}^{m}\binom{x+m+1}{j}(-1)^j(1-\exp(w))^j.$$

Now with $q \le n \le m$ we may extend j to $x+m+1$ beyond m because there is no contribution due to $(\exp(w)-1)^j = w^j + \cdots$, getting

$$\sum_{q=0}^{n} a_q q![w^q]\exp(yw)\exp(-(m+1)w)\exp((x+m+1)w)$$
$$= \sum_{q=0}^{n} a_q q![w^q]\exp((x+y)w) = \sum_{q=0}^{n} a_q (x+y)^q = f(x+y)$$

and we have the claim. Note that the formula will prove e.g.

$$\binom{x+y}{n} = (-1)^m \sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{j-k+y}{n}.$$

This problem is from page 82 eqn. Z.4 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.117 MSE 4517120: A sum of inverse binomial coefficients

We seek to show that for $m > 1$

$$S_{m,n} = \sum_{k=0}^{n}\binom{m+k}{m}^{-1} = \frac{m}{m-1}\left[1-\binom{m+n}{m-1}^{-1}\right].$$

We have for the LHS using an Iverson bracket:

$$[w^n]\frac{1}{1-w}\sum_{k\ge 0}\binom{m+k}{m}^{-1} w^k.$$

Recall the following identity from 1.89: with $1 \le k \le n$

$$\binom{n}{k}^{-1} = k[z^n]\log\frac{1}{1-z}(z-1)^{n-k}.$$

We get with $m \ge 1$ as per requirement on k

$$m \operatorname*{res}_z \frac{1}{z^{m+1}}\log\frac{1}{1-z}[w^n]\frac{1}{1-w}\sum_{k\ge 0} w^k z^{-k}(z-1)^k$$

$$= m \operatorname*{res}_z \frac{1}{z^{m+1}} \log \frac{1}{1-z} [w^n] \frac{1}{1-w} \frac{1}{1-w(z-1)/z}$$
$$= m \operatorname*{res}_z \frac{1}{z^m} \log \frac{1}{1-z} \operatorname*{res}_w \frac{1}{w^{n+1}} \frac{1}{1-w} \frac{1}{z-w(z-1)}.$$

Now residues sum to zero and the residue at infinity in w is zero by inspection, so we may evaluate by taking minus the residue at $w = 1$ and minus the residue at $w = z/(z-1)$. For $w = 1$ start by writing

$$-m \operatorname*{res}_z \frac{1}{z^m} \log \frac{1}{1-z} \operatorname*{res}_w \frac{1}{w^{n+1}} \frac{1}{w-1} \frac{1}{z-w(z-1)}.$$

The residue then leaves

$$-m \operatorname*{res}_z \frac{1}{z^m} \log \frac{1}{1-z} = -m \frac{1}{m-1}.$$

On flipping the sign we get $m/(m-1)$ which is the first term so we are on the right track. Note that when $m = 1$ this term will produce zero. For the residue at $w = z/(1-z)$ we write

$$-m \operatorname*{res}_z \frac{1}{z^m} \frac{1}{z-1} \log \frac{1}{1-z} \operatorname*{res}_w \frac{1}{w^{n+1}} \frac{1}{1-w} \frac{1}{w-z/(z-1)}.$$

Doing the evaluation of the residue yields

$$-m \operatorname*{res}_z \frac{1}{z^m} \frac{1}{z-1} \log \frac{1}{1-z} \frac{(z-1)^{n+1}}{z^{n+1}} \frac{1}{1-z/(z-1)}$$
$$= m \operatorname*{res}_z \frac{1}{z^{m+n+1}} \log \frac{1}{1-z} (z-1)^{n+1}$$
$$= m[z^{m+n}] \log \frac{1}{1-z} (z-1)^{n+1}.$$

Using the cited formula a second time we put $n := m + n$ and $k := m - 1$ to get

$$m \frac{1}{m-1} \binom{m+n}{m-1}^{-1}.$$

On flipping the sign we get the second term as required and we have the claim.

Remark. In the above we have $m > 1$. We get for $m = 1$

$$[z^{n+1}] \log \frac{1}{1-z} (z-1)^{n+1} = \operatorname*{res}_z \frac{1}{z^{n+2}} \log \frac{1}{1-z} (z-1)^{n+1}.$$

Now we put $z/(z-1) = v$ so that $z = v/(v-1)$ and $dz = -1/(v-1)^2 \, dv$ to get

$$-\operatorname*{res}_v \frac{1}{v^{n+2}} \log \frac{1}{1-v/(v-1)} (v-1) \frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_{v} \frac{1}{v^{n+2}} \frac{1}{1-v} \log(1-v).$$

On flipping the sign we obtain

$$\operatorname*{res}_{v} \frac{1}{v^{n+2}} \frac{1}{1-v} \log \frac{1}{1-v} = H_{n+1},$$

again as claimed. This particular value follows by inspection, of course. This was math.stackexchange.com problem 4517120.

1.118 MSE 4520057: Symmetric Bernoulli number identity

We are trying to prove the statement about Bernoulli numbers

$$(-1)^n \sum_{g=0}^{m} \frac{B_{n+g+1}}{n+g+1} \binom{m}{g} + (-1)^m \sum_{g=0}^{n} \frac{B_{m+g+1}}{m+g+1} \binom{n}{g} = -\frac{1}{n+m+1} \binom{n+m}{m}^{-1}$$

We prove this for $n \ge m$, it then follows by symmetry for $m \ge n$. Using

$$B_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_k$$

we get for the first piece

$$\sum_{g=0}^{m} \frac{1}{n+g+1} \binom{m}{g} (-1)^{g+1} \sum_{k=0}^{n+g+1} \binom{n+g+1}{k} B_k.$$

Extracting $k=0$ we get

$$\sum_{g=0}^{m} \frac{1}{n+g+1} \binom{m}{g} (-1)^{g+1} = \sum_{g=0}^{m} \frac{1}{n+m-g+1} \binom{m}{g} (-1)^{m-g+1}$$

$$= [z^{n+m+1}] \log \frac{1}{1-z} \sum_{g=0}^{m} z^g \binom{m}{g} (-1)^{m-g+1}$$

$$= (-1)^{m+1} [z^{n+m+1}] \log \frac{1}{1-z} (1-z)^m = -[z^{n+m+1}] \log \frac{1}{1-z} (z-1)^m.$$

Recall from 1.89 that with $1 \le k \le n$

$$\binom{n}{k}^{-1} = k[z^n] \log \frac{1}{1-z} (z-1)^{n-k}.$$

We put $n := n+m+1$ and $k := n+1$ to get

$$-\frac{1}{n+1}\binom{n+m+1}{n+1}^{-1} = -\frac{1}{n+m+1}\binom{n+m}{n}^{-1}.$$

This also could have been obtained by summing residues of a suitable function. Good, we have the RHS. Now we get for the remainder

$$\sum_{g=0}^{m} \frac{1}{n+g+1}\binom{m}{g}(-1)^{g+1}\sum_{k=1}^{n+g+1}\binom{n+g+1}{k}B_k$$

$$= \sum_{g=0}^{m} \binom{m}{g}(-1)^{g+1}\sum_{k=1}^{n+g+1}\binom{n+g}{k-1}\frac{B_k}{k}$$

$$= \sum_{g=0}^{m} \binom{m}{g}(-1)^{g+1}\sum_{k=0}^{n+g}\binom{n+g}{k}\frac{B_{k+1}}{k+1}.$$

We get two components, the first is,

$$\sum_{g=0}^{m} \binom{m}{g}(-1)^{g+1}\sum_{k=0}^{m-1}\binom{n+g}{k}\frac{B_{k+1}}{k+1}$$

$$= \sum_{k=0}^{m-1} \frac{B_{k+1}}{k+1}\sum_{g=0}^{m}\binom{m}{g}(-1)^{g+1}\binom{n+g}{k}.$$

The inner sum is

$$[z^k](1+z)^n\sum_{g=0}^{m}\binom{m}{g}(-1)^{g+1}(1+z)^g = -[z^k](1+z)^n(-1)^m z^m = 0$$

since $k < m$. That leaves

$$\sum_{g=0}^{m} \binom{m}{g}(-1)^{g+1}\sum_{k=m}^{n+g}\binom{n+g}{k}\frac{B_{k+1}}{k+1}$$

$$= \sum_{g=0}^{m} \binom{m}{g}(-1)^{g+1}\sum_{k=0}^{n-m+g}\binom{n+g}{m+k}\frac{B_{m+k+1}}{m+k+1}$$

$$= \sum_{k=0}^{n} \frac{B_{m+k+1}}{m+k+1}\sum_{g=k+m-n}^{m}\binom{m}{g}(-1)^{g+1}\binom{n+g}{m+k}.$$

Now when $n+g < m+k$ or $g < m-n+k$ the second binomial coefficient is zero, so we may lower g to zero (observe that the first binomial coefficient is zero when $g < 0$ so we also may raise to zero when $k+m-n < 0$):

$$\sum_{k=0}^{n} \frac{B_{m+k+1}}{m+k+1} \sum_{g=0}^{m} \binom{m}{g} (-1)^{g+1} \binom{n+g}{m+k}.$$

The inner sum is

$$[z^{m+k}](1+z)^n \sum_{g=0}^{m} \binom{m}{g} (-1)^{g+1} (1+z)^g = -[z^{m+k}](1+z)^n (-1)^m z^m$$

$$= -(-1)^m [z^k](1+z)^n = -(-1)^m \binom{n}{k}.$$

We have obtained

$$-(-1)^m \sum_{k=0}^{n} \frac{B_{m+k+1}}{m+k+1} \binom{n}{k},$$

which is minus the second piece and concludes the argument.

Addendum. Obviously what we have proved here is with $b_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} a_k$ then

$$(-1)^n \sum_{g=0}^{m} \frac{b_{n+g+1}}{n+g+1} \binom{m}{g} + (-1)^m \sum_{g=0}^{n} \frac{a_{m+g+1}}{m+g+1} \binom{n}{g} = -\frac{a_0}{n+m+1} \binom{n+m}{m}^{-1}.$$

We get for an ordinary binomial transform $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$ the relation

$$\sum_{g=0}^{m} (-1)^{g+1} \frac{b_{n+g+1}}{n+g+1} \binom{m}{g} + (-1)^m \sum_{g=0}^{n} \frac{a_{m+g+1}}{m+g+1} \binom{n}{g} = -\frac{a_0}{n+m+1} \binom{n+m}{m}^{-1}.$$

This was math.stackexchange.com problem 4520057.

1.119 Polynomial identity III

We seek to prove that with $f(x) = \sum_{q=0}^{n} a_q x^q$ a polynomial of degree at most n we have

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \frac{f(y+k^2)}{x^2-k^2} = (-1)^n \frac{f(x^2+y)}{2x(x-n)} \binom{x+n}{2n}^{-1} + \frac{1}{2} \binom{2n}{n} \frac{f(y)}{x^2}.$$

Consider

$$g(z) = (-1)^n (2n)! f(z^2+y) \frac{1}{z-x} \prod_{q=-n}^{n} \frac{1}{z-q}.$$

The residues of $g(z)$ sum to zero and the residue at infinity is zero because we have degree $2n$ in the numerator and degree $2n+2$ in the denominator. Here we have that x is an integer with $|x| > n$ which is necessary for the original LHS and RHS to be defined. We have polynomials in x and y upon multiplication by

$$x(x-n)\binom{x+n}{2n}$$

so that the identity then holds for all x, y including complex. We get for the residue at $z = x$

$$(-1)^n \frac{f(x^2+y)}{x-n} \binom{x+n}{2n}^{-1}.$$

The residue at $z = 0$ yields

$$-(-1)^n (2n)! f(y) \frac{1}{x} \prod_{q=-n}^{-1} \frac{1}{-q} \prod_{q=1}^{n} \frac{1}{-q} = -\binom{2n}{n} \frac{f(y)}{x}.$$

The residues at $|z| = k$ with $1 \le k \le n$ will produce

$$(-1)^n (2n)! f(y+k^2) \frac{1}{k-x} \prod_{q=-n}^{k-1} \frac{1}{k-q} \prod_{q=k+1}^{n} \frac{1}{k-q}$$

$$= (-1)^n (2n)! f(y+k^2) \frac{1}{k-x} \frac{1}{(n+k)!} \frac{(-1)^{n-k}}{(n-k)!}$$

$$= (-1)^k \binom{2n}{n-k} \frac{f(y+k^2)}{k-x}.$$

Adding these last we get

$$\sum_{k=1}^{n} (-1)^k \binom{2n}{n-k} \frac{f(y+k^2)}{k-x} + \sum_{k=-n}^{-1} (-1)^k \binom{2n}{n-k} \frac{f(y+k^2)}{k-x}$$

$$= \sum_{k=1}^{n} (-1)^k \binom{2n}{n-k} \frac{f(y+k^2)}{k-x} + \sum_{k=1}^{n} (-1)^k \binom{2n}{n+k} \frac{f(y+k^2)}{-k-x}$$

$$= \sum_{k=1}^{n} (-1)^k \binom{2n}{n-k} \frac{2x f(y+k^2)}{k^2-x^2}$$

$$= \binom{2n}{n} \frac{2x f(y)}{x^2} + \sum_{k=0}^{n} (-1)^k \binom{2n}{n-k} \frac{2x f(y+k^2)}{k^2-x^2}.$$

Collecting everything and dividing by $2x$ we have

$$0 = \binom{2n}{n}\frac{f(y)}{x^2} + \sum_{k=0}^{n}(-1)^k\binom{2n}{n-k}\frac{f(y+k^2)}{k^2-x^2}$$

$$+(-1)^n\frac{f(x^2+y)}{2x(x-n)}\binom{x+n}{2n}^{-1} - \frac{1}{2}\binom{2n}{n}\frac{f(y)}{x^2}.$$

Upon re-arrangeing we have the claim. Note that this will produce e.g. the identity

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\binom{y+k^2}{n}\frac{1}{x^2-k^2}$$

$$= (-1)^n\binom{x^2+y}{n}\frac{1}{2x(x-n)}\binom{x+n}{2n}^{-1} + \frac{1}{2}\binom{2n}{n}\binom{y}{n}\frac{1}{x^2}.$$

This problem is from page 83 eqn. Z.10 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.120 Polynomial identity IV

We seek to prove that with $f(x) = \sum_{q=0}^{n+r-1} a_q x^q$ a polynomial of degree at most $n+r-1$ we have where $n, r \ge 1$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+r}{k}^{-1} f(y-k) = -\sum_{k=1}^{r}(-1)^k\binom{r}{k}\binom{k+n}{k}^{-1} f(y+k).$$

Note that

$$\binom{x+r}{r}^{-1} = r!\prod_{m=1}^{r}\frac{1}{x+m} = r!\sum_{m=1}^{r}\frac{1}{x+m}\mathrm{Res}_{x=-m}\prod_{\ell=1}^{r}\frac{1}{x+\ell}$$

$$= r!\sum_{m=1}^{r}\frac{1}{x+m}\prod_{\ell=1}^{m-1}\frac{1}{-m+\ell}\prod_{\ell=m+1}^{r}\frac{1}{-m+\ell}$$

$$= r!\sum_{m=1}^{r}\frac{1}{x+m}\frac{(-1)^{m-1}}{(m-1)!}\frac{1}{(r-m)!} = r\sum_{m=1}^{r}\frac{1}{x+m}(-1)^{m-1}\binom{r-1}{m-1}.$$

We get for the LHS

$$r\sum_{m=1}^{r}(-1)^{m-1}\binom{r-1}{m-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+m}f(y-k).$$

We have for the inner sum

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{1}{n-k+m}f(y-n+k)$$

$$= [z^{n+m}]\log\frac{1}{1-z}\sum_{k=0}^{n}(-1)^{n-k}z^k\binom{n}{k}f(y-n+k)$$

$$= \sum_{q=0}^{n+r-1} a_q q![w^q]\exp((y-n)w)[z^{n+m}]\log\frac{1}{1-z}\sum_{k=0}^{n}(-1)^{n-k}z^k\binom{n}{k}\exp(kw)$$

$$= \sum_{q=0}^{n+r-1} a_q q![w^q]\exp((y-n)w)[z^{n+m}]\log\frac{1}{1-z}(z\exp(w)-1)^n.$$

We get for the coefficient extractor in z

$$[z^{n+m}]\log\frac{1}{1-z}(z\exp(w)-1)^n$$

$$= [z^{n+m}]\log\frac{1}{1-z}\sum_{\ell=0}^{n}\binom{n}{\ell}z^\ell(\exp(w)-1)^\ell(z-1)^{n-\ell}.$$

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [z^n]\log\frac{1}{1-z}(z-1)^{n-k}$$

so this becomes

$$\sum_{\ell=0}^{n}\binom{n}{\ell}(\exp(w)-1)^\ell\frac{1}{m}\binom{n+m-\ell}{m}^{-1}.$$

Let us recapitulate what we have so far:

$$\sum_{m=1}^{r}(-1)^{m-1}\binom{r}{m}\sum_{q=0}^{n+r-1}a_q q![w^q]\exp((y-n)w)$$
$$\times\sum_{\ell=0}^{n}\binom{n}{\ell}(\exp(w)-1)^\ell\binom{n+m-\ell}{m}^{-1}.$$

Next observe that

$$\binom{n}{\ell}\binom{n+m-\ell}{m}^{-1} = \frac{n!\times m!}{\ell!\times(n+m-\ell)!} = \binom{n+m}{\ell}\binom{n+m}{m}^{-1}.$$

We get

$$\sum_{m=1}^{r}(-1)^{m-1}\binom{r}{m}\binom{n+m}{m}^{-1}\sum_{q=0}^{n+r-1}a_q q![w^q]\exp((y-n)w)$$
$$\times\sum_{\ell=0}^{n}\binom{n+m}{\ell}(\exp(w)-1)^{\ell}.$$

Extending ℓ to $n+m$ we obtain $\exp((n+m)w)$ for the innermost sum which will produce

$$\sum_{m=1}^{r}(-1)^{m-1}\binom{r}{m}\binom{n+m}{m}^{-1}f(y+m)$$

which is the claim. It remains to show that there is a zero contribution from

$$\sum_{\ell=n+1}^{n+m}\binom{n+m}{\ell}(\exp(w)-1)^{\ell}=\sum_{\ell=1}^{m}\binom{n+m}{n+\ell}(\exp(w)-1)^{n+\ell}.$$

We get

$$\sum_{m=1}^{r}(-1)^{m-1}\binom{r}{m}\binom{n+m}{m}^{-1}\sum_{\ell=1}^{m}\binom{n+m}{n+\ell}(\exp(w)-1)^{n+\ell}.$$

This is

$$-(\exp(w)-1)^{n}+\sum_{m=1}^{r}(-1)^{m-1}\binom{r}{m}\binom{n+m}{m}^{-1}\sum_{\ell=0}^{m}\binom{n+m}{n+\ell}(\exp(w)-1)^{n+\ell}.$$

We also have

$$\binom{r}{m}\binom{n+m}{m}^{-1}=\frac{r!\times n!}{(r-m)!\times(n+m)!}=\binom{r+n}{n}^{-1}\binom{r+n}{r-m}.$$

Therefore we get

$$-(\exp(w)-1)^{n}+\binom{r+n}{n}^{-1}\sum_{m=1}^{r}(-1)^{m-1}\binom{r+n}{r-m}\sum_{\ell=0}^{m}\binom{n+m}{m-\ell}(\exp(w)-1)^{n+\ell}.$$

Including $m=0$ we have

$$\binom{r+n}{n}^{-1}\sum_{m=0}^{r}(-1)^{m-1}\binom{r+n}{r-m}\sum_{\ell=0}^{m}\binom{n+m}{m-\ell}(\exp(w)-1)^{n+\ell}$$

$$= \binom{r+n}{n}^{-1} (\exp(w)-1)^n \sum_{m=0}^{r} (-1)^{m-1} \binom{r+n}{r-m} \sum_{\ell=0}^{m} \binom{n+m}{m-\ell} (\exp(w)-1)^{\ell}.$$

Working with the sum we find

$$\sum_{m=0}^{r} (-1)^{m-1} \binom{r+n}{r-m} [z^m](1+z)^{n+m} \sum_{\ell \ge 0} z^{\ell} (\exp(w)-1)^{\ell}$$

$$= \sum_{m=0}^{r} (-1)^{m-1} \binom{r+n}{r-m} \operatorname*{res}_{z} \frac{1}{z^{m+1}} (1+z)^{n+m} \frac{1}{1-z(\exp(w)-1)}.$$

Now put $z/(1+z) = v$ so that $z = v/(1-v)$ and $dz = 1/(1-v)^2 \; dv$ to get for the residue

$$\operatorname*{res}_{v} \frac{1}{v^{m+1}} \frac{1}{(1-v)^{n-1}} \frac{1}{1-v(\exp(w)-1)/(1-v)} \frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_{v} \frac{1}{v^{m+1}} \frac{1}{(1-v)^{n}} \frac{1}{1-v\exp(w)}.$$

Substituting into the sum we have

$$-[u^r](1+u)^{r+n} \sum_{m \ge 0} (-1)^m u^m [v^m] \frac{1}{(1-v)^n} \frac{1}{1-v\exp(w)}$$

$$= -[u^r](1+u)^r \frac{1}{1+u\exp(w)} = -\sum_{p=0}^{r} \binom{r}{p} (-1)^{r-p} \exp((r-p)w)$$

$$= -(1-\exp(w))^r = (-1)^{r+1}(\exp(w)-1)^r.$$

We have computed for the remainder term that it is

$$(-1)^{r+1} \binom{r+n}{n}^{-1} (\exp(w)-1)^{n+r}.$$

Note however that $(\exp(w)-1)^{n+r} = w^{n+r} + \cdots$ yet the coefficient extractor $[w^q]$ has $q \le n+r-1$ as per the initial conditions. Hence it returns zero and the remainder term vanishes as claimed, concluding the proof.

Observe that with this identity we can prove special cases like

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+r}{k}^{-1} \binom{y-k}{n} = -\sum_{k=1}^{r} (-1)^k \binom{r}{k} \binom{k+n}{k}^{-1} \binom{y+k}{n}.$$

This problem is from page 85 eqn. Z.16 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.121 MSE 4540192: Symmetry in a simple proof

We seek to show that

$$\sum_{q=0}^{\lfloor n/2 \rfloor} (n-2q)^n \binom{n}{q} (-1)^q = 2^{n-1} n!.$$

Observe that

$$(n-2q)^n \binom{n}{q} (-1)^q = \frac{1}{2} \left[(n-2q)^n \binom{n}{q} (-1)^q + (2q-n)^n \binom{n}{n-q} (-1)^{n-q} \right].$$

Hence we get for our sum

$$\frac{1}{2} \sum_{q=0}^{\lfloor n/2 \rfloor} (n-2q)^n \binom{n}{q} (-1)^q + \frac{1}{2} \sum_{q=0}^{\lfloor n/2 \rfloor} (2q-n)^n \binom{n}{n-q} (-1)^{n-q}$$

$$= \frac{1}{2} \sum_{q=0}^{\lfloor n/2 \rfloor} (n-2q)^n \binom{n}{q} (-1)^q + \frac{1}{2} \sum_{q=n-\lfloor n/2 \rfloor}^{n} (n-2q)^n \binom{n}{q} (-1)^q$$

$$= \frac{1}{2} \sum_{q=0}^{n} (n-2q)^n \binom{n}{q} (-1)^q.$$

Introducing a coefficient extractor,

$$\frac{1}{2} \sum_{q=0}^{n} \binom{n}{q} (-1)^q n! [z^n] \exp((n-2q)z)$$

$$= \frac{1}{2} n! [z^n] \exp(nz) \sum_{q=0}^{n} \binom{n}{q} (-1)^q \exp((-2q)z)$$

$$= \frac{1}{2} n! [z^n] \exp(nz) (1-\exp(-2z))^n.$$

Note however that $(1-\exp(-2z))^n = (2z - 2z^2 \pm \cdots)^n$ so the only contribution to the coefficient extractor $[z^n]$ is from the first term of the series so that $[z^n] \exp(nz)(2z - 2z^2 \pm \cdots)^n = 2^n$ and we finally have

$$2^{n-1} n!$$

as claimed.

Alternative computation

We might try to use an Iverson bracket $[[2q \leq n]]$ in attempting to evaluate

$$S_n = \sum_{q=0}^{\lfloor n/2 \rfloor} (n-2q)^n \binom{n}{q} (-1)^q.$$

We obtain

$$[v^n] \frac{1}{1-v} \sum_{q\ge 0} v^{2q} (n-2q)^n \binom{n}{q} (-1)^q$$

Using a coefficient extractor,

$$n![z^n] \exp(nz) \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{1-v} \sum_{q\ge 0} v^{2q} \exp(-2qz) \binom{n}{q} (-1)^q$$

$$= n![z^n] \exp(nz) \operatorname*{res}_v \frac{1}{v^{n+1}} \frac{1}{1-v} (1 - v^2 \exp(-2z))^n.$$

Now residues sum to zero and the residue at one yields

$$-n![z^n] \exp(nz)(1-\exp(-2z))^n.$$

We have that since $(1-\exp(-2z))^n = (2z - 2z^2 \pm \cdots)^n = 2^n z^n + \cdots$ so this evaluates to $-2^n n!$. We find for the residue at infinity

$$-n![z^n] \exp(nz) \operatorname*{res}_v \frac{1}{v^2} v^{n+1} \frac{1}{1-1/v} (1 - \exp(-2z)/v^2)^n$$

$$= n![z^n] \exp(nz) \operatorname*{res}_v \frac{1}{v^n} \frac{1}{1-v} (v^2 - \exp(-2z))^n$$

$$= n![z^n] \exp(nz) \operatorname*{res}_v \frac{1}{v^n} \frac{1}{1-v} \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} \exp(-2(n-q)z) v^{2q}$$

$$= \operatorname*{res}_v \frac{1}{v^n} \frac{1}{1-v} \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} (2q-n)^n v^{2q}$$

$$= \sum_{q=0}^{n} \binom{n}{q} (-1)^q (n-2q)^n [[2q \le n-1]].$$

Now when n is odd this gives the upper limit $\lfloor n/2 \rfloor$ and when n is even $\lfloor n/2 \rfloor - 1$ however in the latter case we may raise to $\lfloor n/2 \rfloor$ because the added term is zero in the sum per $(n-2q)^n = 0$. We have obtained

$$\sum_{q=0}^{\lfloor n/2 \rfloor} \binom{n}{q} (-1)^q (n-2q)^n = S_n.$$

Collecting everything we have shown that $S_n - 2^n n! + S_n = 0$ or $S_n = 2^{n-1} n!$. This was math.stackexchange.com problem 4540192.

1.122 Free functional term

We seek to verify that

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(j) = (-1)^n\binom{x-1}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{xf(k)}{x-k}.$$

As both sides are polynomials in x it will suffice to prove it for $x > n$ a positive integer. We get for the LHS

$$\sum_{j=0}^{n}(-1)^j f(j)\sum_{k=j}^{n}(-1)^k\binom{x}{k}\binom{k}{j}.$$

We thus have for the coefficient on $f(j)$

$$(-1)^j\sum_{k=j}^{n}(-1)^k\binom{x}{n+1}\frac{(n+1)!}{k!}\binom{x-k}{n+1-k}^{-1}\frac{1}{(n+1-k)!}\binom{k}{j}$$

$$= (-1)^j\binom{x}{n+1}\sum_{k=j}^{n}(-1)^k\binom{n+1}{k}\binom{x-k}{n+1-k}^{-1}\binom{k}{j}$$

$$= (-1)^j[w^j]\binom{x}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{x-k}{n+1-k}^{-1}(1+w)^k.$$

Here we have lowered the lower range because the coefficient extractor returns zero when $k < j$.

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [z^n]\log\frac{1}{1-z}(z-1)^{n-k}.$$

This yields for our sum

$$\binom{x}{n+1}(-1)^j$$

$$\times[w^j][z^x]\log\frac{1}{1-z}(z-1)^{x-n-1}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}(n+1-k)z^k(1+w)^k$$

$$= (n+1)\binom{x}{n+1}(-1)^j$$

$$\times[w^j][z^x]\log\frac{1}{1-z}(z-1)^{x-n-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}z^k(1+w)^k$$

$$= (n+1)\binom{x}{n+1}(-1)^j [w^j][z^x] \log\frac{1}{1-z}(z-1)^{x-n-1}(1-z-wz)^n$$

$$= (-1)^n(n+1)\binom{x}{n+1}(-1)^j [w^j][z^x] \log\frac{1}{1-z}(z-1)^{x-n-1}(wz+z-1)^n.$$

Continuing with the coefficient extractor in w,

$$\binom{n}{j}[z^x]\log\frac{1}{1-z}(z-1)^{x-n-1}z^j(z-1)^{n-j}$$

$$= \binom{n}{j}[z^{x-j}]\log\frac{1}{1-z}(z-1)^{x-j-1} = \binom{n}{j}\binom{x-j}{1}^{-1}.$$

Restoring the scalar we find

$$(-1)^n(n+1)\binom{x}{n+1}(-1)^j\binom{n}{j}\frac{1}{x-j}.$$

With some algebra we arrive at

$$(-1)^{n+j}\binom{x-1}{n}\binom{n}{j}\frac{x}{x-j}$$

which is the claim.
Note that with this identity we can prove e.g.

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{j+m}{m} = (-1)^n\binom{x-1}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x}{x-k}\binom{k+m}{m}.$$

This problem is from page 87 eqn. Z.24 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.123 MSE 4547110: Inverse central binomial coefficient

We seek to verify that

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}2^{2k}}{k}\binom{n}{k}\binom{2k}{k}^{-1} = 2H_{2n} - H_n.$$

Recall from 1.89 the following identity which was proved there: with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [z^n]\log\frac{1}{1-z}(z-1)^{n-k}.$$

We get for our sum

$$\sum_{k=1}^{n}(-1)^{k+1}2^{2k}\binom{n}{k}[z^{2k}]\log\frac{1}{1-z}(z-1)^k$$
$$= (-1)^n 2^{2n}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{k+1}2^{-2k}[z^{2n-2k}]\log\frac{1}{1-z}(z-1)^{n-k}$$
$$= (-1)^{n+1}2^{2n}[z^{2n}]\log\frac{1}{1-z}(z-1)^n\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k 2^{-2k}z^{2k}(z-1)^{-k}.$$

We see that we may raise k to n because this is a zero contribution owing to the fact that $\log\frac{1}{1-z}$ does not have a constant term, getting

$$(-1)^{n+1}2^{2n}[z^{2n}]\log\frac{1}{1-z}(z-1)^n\left[1-\frac{z^2}{4(z-1)}\right]^n$$
$$= (-1)^{n+1}[z^{2n}]\log\frac{1}{1-z}\left[4z-4-z^2\right]^n$$
$$= -[z^{2n}]\log\frac{1}{1-z}(z-2)^{2n}.$$

This is

$$-\operatorname*{res}_z \frac{1}{z^{2n+1}}\log\frac{1}{1-z}(z-2)^{2n}.$$

Now put $z/(z-2)=v$ so that $z=2v/(v-1)$ and $dz=-2/(v-1)^2\,dv$ to obtain

$$\operatorname*{res}_v \frac{1}{v^{2n+1}}\log\frac{1}{1-2v/(v-1)}\frac{v-1}{2}2\frac{1}{(v-1)^2}$$
$$= -\operatorname*{res}_v \frac{1}{v^{2n+1}}\frac{1}{1-v}\log\frac{v-1}{-v-1}$$
$$= -\operatorname*{res}_v \frac{1}{v^{2n+1}}\frac{1}{1-v}\log\frac{1-v}{1+v}.$$

We get two pieces, the first is

$$\operatorname*{res}_v \frac{1}{v^{2n+1}}\frac{1}{1-v}\log\frac{1}{1-v} = H_{2n}.$$

The second is

$$-\operatorname*{res}_v \frac{1}{v^{2n+1}}\frac{1}{1-v}\log\frac{1}{1+v} = -\sum_{q=1}^{2n}\frac{(-1)^q}{q} = -\left[\sum_{p=1}^{n}\frac{1}{2p}-\sum_{p=0}^{n-1}\frac{1}{2p+1}\right]$$

$$= -\frac{1}{2}H_n + H_{2n} - \sum_{q=1}^n \frac{1}{2q} = H_{2n} - H_n.$$

Collecting the two pieces we obtain

$$\boxed{2H_{2n} - H_n}$$

as claimed.

This was math.stackexchange.com problem 4547110.

1.124 MSE 1402886: Inverse binomial coefficient

We seek to find a closed form of

$$\sum_{r=0}^n (-1)^r {n\choose r}^{-1}.$$

Recall from 1.89 the following identity which was proved there: with $1 \le k \le n$

$$\frac{1}{k}{n\choose k}^{-1} = [z^n] \log\frac{1}{1-z}(z-1)^{n-k}.$$

We get for our sum

$$1 + \sum_{r=0}^n (-1)^r r [z^n] \log\frac{1}{1-z}(z-1)^{n-r}$$
$$= 1 + [z^n]\log\frac{1}{1-z}(z-1)^n [v^n]\frac{1}{1-v}\sum_{r\ge 0}(-1)^r r (z-1)^{-r} v^r$$
$$= 1 + [z^n]\log\frac{1}{1-z}(z-1)^n [v^n]\frac{1}{v-1}\frac{v/(z-1)}{(1+v/(z-1))^2}$$
$$= 1 + [z^n]\log\frac{1}{1-z}(z-1)^{n+1} [v^{n-1}]\frac{1}{v-1}\frac{1}{(z-1+v)^2}.$$

The contribution from v is

$$\operatorname*{res}_v \frac{1}{v^n}\frac{1}{v-1}\frac{1}{(z-1+v)^2}.$$

The residue at infinity is zero so we may compute this by taking minus the residue at one and minus the residue at $v = 1 - z$. (Residues sum to zero.) We get for the former

$$-[z^{n+2}]\log\frac{1}{1-z}(z-1)^{n+1} = -{n+2\choose 1}^{-1} = -\frac{1}{n+2}$$

and the latter

$$-\left(\frac{1}{v^n}\frac{1}{v-1}\right)'\Bigg|_{v=1-z} = -\left(-\frac{n}{v^{n+1}}\frac{1}{v-1} - \frac{1}{v^n}\frac{1}{(v-1)^2}\right)\Bigg|_{v=1-z}$$

$$= -\frac{n}{z(1-z)^{n+1}} + \frac{1}{(1-z)^n z^2}.$$

We get for the first term

$$-(-1)^{n+1} n [z^{n+1}] \log\frac{1}{1-z} = (-1)^n \frac{n}{n+1}.$$

and for the second

$$(-1)^n [z^{n+2}] \log\frac{1}{1-z}(z-1) = (-1)^n \frac{1}{n+1} - (-1)^n \frac{1}{n+2}.$$

Collecting everything we find

$$1 + (-1)^n - \frac{1}{n+2}(1+(-1)^n).$$

This will produce zero when n is odd. For n even we find

$$\boxed{2\frac{n+1}{n+2}.}$$

This was math.stackexchange.com problem 1402886.

1.125 MSE 4552694: A pair of binomial transforms

We seek to show that

$$\alpha_n = \sum_{k=0}^{n} \binom{n+k}{n-k} \beta_k \Leftrightarrow \beta_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} \alpha_k$$

By substituting the left into the right and vice versa and interchanging summations we obtain multiples of the inner sequence times a sum, which must be shown to be an indicator variable / Iverson bracket $[[n=j]]$.

First indicator

We seek to evaluate where $n \ge j$

$$\sum_{k=j}^{n} (-1)^{n-k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} \binom{k+j}{k-j}.$$

This is

$$\sum_{k=0}^{n-j}(-1)^k\frac{2n-2k+1}{2n+1}\binom{2n+1}{k}\binom{n-k+j}{n-k-j}$$

$$= [z^{n-j}](1+z)^{n+j}\sum_{k\ge 0}(-1)^k\frac{2n-2k+1}{2n+1}\binom{2n+1}{k}z^k(1+z)^{-k}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing we get two pieces, the first is

$$[z^{n-j}](1+z)^{n+j}\sum_{k\ge 0}(-1)^k\binom{2n+1}{k}z^k(1+z)^{-k}$$

$$= [z^{n-j}](1+z)^{n+j}\left[1-\frac{z}{1+z}\right]^{2n+1}$$

$$= [z^{n-j}]\frac{1}{(1+z)^{n+1-j}} = (-1)^{n-j}\binom{2n-2j}{n-j}.$$

The second is

$$-[z^{n-j}](1+z)^{n+j}\sum_{k\ge 0}(-1)^k\frac{2k}{2n+1}\binom{2n+1}{k}z^k(1+z)^{-k}$$

$$= -\frac{2}{2n+1}[z^{n-j}](1+z)^{n+j}\sum_{k\ge 1}(-1)^k k\binom{2n+1}{k}z^k(1+z)^{-k}$$

$$= -2[z^{n-j}](1+z)^{n+j}\sum_{k\ge 1}(-1)^k\binom{2n}{k-1}z^k(1+z)^{-k}$$

$$= 2[z^{n-j}]z(1+z)^{n+j-1}\sum_{k\ge 0}(-1)^k\binom{2n}{k}z^k(1+z)^{-k}.$$

This will produce zero when $n=j$. Continuing with $n>j$,

$$2[z^{n-j-1}](1+z)^{n+j-1}\left[1-\frac{z}{1+z}\right]^{2n}$$

$$= 2[z^{n-j-1}]\frac{1}{(1+z)^{n+1-j}} = 2(-1)^{n-j-1}\binom{2n-2j-1}{n-j}.$$

Collecting the contributions we find for $n=j$

$$(-1)^{n-n}\binom{2n-2n}{n-n} = 1$$

and for $n>j$

$$(-1)^{n-j}\binom{2n-2j}{n-j}\left[1-2\frac{n-j}{2n-2j}\right]=0.$$

This is the claim.

Second indicator

We seek to evaluate where $n \ge j$

$$\sum_{k=j}^{n}\binom{n+k}{n-k}(-1)^{k-j}\frac{2j+1}{2k+1}\binom{2k+1}{k-j}.$$

Expanding the sum term we find

$$(2j+1)\frac{(n+k)!}{(n-k)!}(-1)^{k-j}\frac{1}{(k-j)!\times(k+j+1)!}$$
$$=(2j+1)\binom{n-j}{k-j}(-1)^{k-j}\binom{n+k}{n-j}\frac{1}{k+j+1}.$$

We get for the sum

$$(2j+1)\sum_{k=0}^{n-j}\binom{n-j}{n-k-j}(-1)^{n-k-j}\binom{2n-k}{n-j}\frac{1}{n+j-k+1}$$
$$=(2j+1)[z^{n-j}](1+z)^{2n}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^{n-k-j}\frac{1}{(1+z)^k}\frac{1}{n+j-k+1}$$
$$=(2j+1)[z^{n-j}](1+z)^{2n}[w^{n+j+1}]\log\frac{1}{1-w}\left[\frac{w}{1+z}-1\right]^{n-j}$$
$$=(2j+1)[z^{n-j}](1+z)^{n+j}[w^{n+j+1}]\log\frac{1}{1-w}\left[w-1-z\right]^{n-j}.$$

For $n=j$ this will produce

$$(2n+1)[z^0](1+z)^{2n}\frac{1}{2n+1}=1$$

as required. Continuing with $n>j$ and the extractor in w,

$$[w^{n+j+1}]\log\frac{1}{1-w}\sum_{q=0}^{n-j}\binom{n-j}{q}(w-1)^q(-1)^{n-j-q}z^{n-j-q}.$$

Recall from 1.89 the following identity which was proved there: with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1}=[w^n]\log\frac{1}{1-w}(w-1)^{n-k}.$$

so this becomes

$$(2j+1)[z^{n-j}](1+z)^{n+j}\sum_{q=0}^{n-j}\binom{n-j}{q}\frac{1}{n+j+1-q}\binom{n+j+1}{q}^{-1}(-1)^{n-j-q}z^{n-j-q}$$

$$= (2j+1)\sum_{q=0}^{n-j}\binom{n-j}{q}\frac{1}{n+j+1-q}\binom{n+j+1}{q}^{-1}(-1)^{n-j-q}\binom{n+j}{q}.$$

We have

$$\binom{n+j}{q}\binom{n+j+1}{q}^{-1} = \frac{(n+j)! \times (n+j+1-q)!}{(n+j-q)! \times (n+j+1)!}$$

and we obtain at last

$$\frac{2j+1}{n+j+1}\sum_{q=0}^{n-j}\binom{n-j}{q}(-1)^{n-j-q} = [[n=j]].$$

Once more we have the claim. It appears that the special case $n = j$ was subsumed by the above computation.

This was math.stackexchange.com problem 4552694.

1.126 Polynomial with inverse binomial coefficients

We seek to verify that

$$\sum_{k=0}^{n} x^k \binom{n}{k}^{-1} = (n+1)\left(\frac{x}{1+x}\right)^{n+1}\sum_{k=1}^{n+1}\frac{1}{k}\frac{1+x^k}{1+x}\left(\frac{1+x}{x}\right)^k.$$

First term

We get two terms for the sum on the RHS, the first is

$$\sum_{k=1}^{n+1}\frac{1}{k}\left(\frac{1+x}{x}\right)^k = [v^{n+1}]\frac{1}{1-v}\sum_{k\ge 1}\frac{1}{k}\left(\frac{1+x}{x}\right)^k v^k$$

$$= [v^{n+1}]\frac{1}{1-v}\log\frac{1}{1-v(1+x)/x}.$$

On multiplying by the term in front of the sum this becomes

$$\frac{n+1}{1+x}[v^{n+1}]\frac{1}{1-vx/(1+x)}\log\frac{1}{1-v}$$

$$= (n+1)[v^{n+1}]\frac{1}{1-x(v-1)}\log\frac{1}{1-v}.$$

Extracting the coefficient on x^k where $0 \le k \le n$ we find

$$(n+1)[v^{n+1}] \log \frac{1}{1-v} (v-1)^k.$$

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k} {n \choose k}^{-1} = [v^n] \log \frac{1}{1-v} (v-1)^{n-k}.$$

Making the subsitution yields

$$(n+1) \frac{1}{n+1-k} {n+1 \choose n+1-k}^{-1} = {n \choose n-k}^{-1} = {n \choose k}^{-1}.$$

This is the required result so the second term on the RHS must cancel the powers x^k and coefficients thereon where $k > n$.

Second term

We get

$$\sum_{k=1}^{n+1} \frac{1}{k} (1+x)^k = [v^{n+1}] \frac{1}{1-v} \log \frac{1}{1-v(1+x)}.$$

Multiplying by the term in front this becomes

$$\frac{n+1}{1+x} [v^{n+1}] \frac{1+x}{1-x(v-1)} \log \frac{1}{1-xv}$$

$$= (n+1)[v^{n+1}] \frac{1}{1-x(v-1)} \log \frac{1}{1-xv}.$$

Extracting the coefficient on k yields

$$(n+1)[v^{n+1}] \sum_{q=1}^{k} \frac{v^q}{q} (v-1)^{k-q}$$

This is zero when $k \le n$ because the degree of the sum term in v is k. Good, so it does not affect the inverse binomial coefficients in that range. On the other hand for $k > n$ we find

$$(n+1)[v^{n+1}](v-1)^k [u^k] \frac{1}{1-u} \sum_{q \ge 1} \frac{v^q}{q} (v-1)^{-q} u^q$$

$$= (n+1)[v^{n+1}](v-1)^k [u^k] \frac{1}{1-u} \log \frac{1}{1-uv/(v-1)}.$$

Here we get two pieces, the first is

$$
\begin{aligned}
&-(n+1)[v^{n+1}](v-1)^k[u^k]\frac{1}{1-u}\log\frac{1}{1-v}\\
&= -(n+1)[v^{n+1}]\log\frac{1}{1-v}(v-1)^k.
\end{aligned}
$$

This precisely cancels the upper range from the first term as required. The second piece must make a zero contribution:

$$
\begin{aligned}
&(n+1)[v^{n+1}](v-1)^k[u^k]\frac{1}{1-u}\log\frac{1}{1-v(1-u)}\\
&= (n+1)[u^k]\frac{1}{1-u}[v^{n+1}](v-1)^k\log\frac{1}{1-v(1-u)}\\
&= (n+1)[u^k]\sum_{q=1}^{n+1}\frac{(1-u)^{q-1}}{q}\binom{k}{n+1-q}(-1)^{k-(n+1-q)}\\
&= (n+1)\sum_{q=1}^{n+1}\frac{1}{q}(-1)^{n+1-q}\binom{q-1}{k}\binom{k}{n+1-q}.
\end{aligned}
$$

Note that with $k > n$ the binomial coefficient $\binom{q-1}{k}$ is zero. This concludes the argument.

Addendum

Observe that with the above polynomial being called $S_n(x)$ we have the following functional equation:

$$\left(1+\frac{1}{x}\right)S_{n+1}(x) = \frac{n+2}{n+1}S_n(x) + x^{n+1} + \frac{1}{x}.$$

This is because we get starting on the LHS

$$
\begin{aligned}
&(n+2)\left(\frac{x}{1+x}\right)^{n+1}\sum_{k=1}^{n+2}\frac{1}{k}\frac{1+x^k}{1+x}\left(\frac{1+x}{x}\right)^k\\
&= \frac{n+2}{n+1}S_n(x) + (n+2)\left(\frac{x}{1+x}\right)^{n+1}\frac{1}{n+2}\frac{1+x^{n+2}}{1+x}\left(\frac{1+x}{x}\right)^{n+2}\\
&= \frac{n+2}{n+1}S_n(x) + \frac{1+x^{n+2}}{1+x}\frac{1+x}{x}\\
&= \frac{n+2}{n+1}S_n(x) + x^{n+1} + \frac{1}{x}.
\end{aligned}
$$

This problem is from page 18 eqns. 2.4 and 2.5 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.127 Harmonic numbers with inverse binomial coefficients

We seek to verify that with harmonic numbers

$$\sum_{k=1}^{2n-1} (-1)^{k-1} \binom{2n}{k}^{-1} H_k = \frac{1}{2}\frac{n}{(n+1)^2} + \frac{1}{2}\frac{1}{n+1} H_{2n}.$$

We start with the LHS

$$\sum_{k=1}^{2n-1} (-1)^{k-1} \binom{2n}{k}^{-1} [z^k] \frac{1}{1-z} \log \frac{1}{1-z}$$
$$= \sum_{k=1}^{2n-1} (-1)^{k-1} \binom{2n}{k}^{-1} [z^{2n-k}] \frac{1}{1-z} \log \frac{1}{1-z}$$
$$= [z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \sum_{k=1}^{2n-1} (-1)^{k-1} \binom{2n}{k}^{-1} z^k.$$

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [w^n] \log \frac{1}{1-w} (w-1)^{n-k}.$$

Making the substitution yields

$$[w^{2n}] \log \frac{1}{1-w} (w-1)^{2n} [z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \sum_{k=1}^{2n-1} (-1)^{k-1} k (w-1)^{-k} z^k.$$

Given that $\log \frac{1}{1-z} = z + \cdots$ we may extend k to infinity, getting

$$[w^{2n}] \log \frac{1}{1-w} (w-1)^{2n} [z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \frac{z/(w-1)}{(1+z/(w-1))^2}$$
$$= [w^{2n}] \log \frac{1}{1-w} (w-1)^{2n} [z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \frac{z(w-1)}{(w-1+z)^2}$$
$$= [w^{2n}] \log \frac{1}{1-w} (w-1)^{2n+1} [z^{2n-1}] \frac{1}{1-z} \log \frac{1}{1-z} \frac{1}{(1-w-z)^2}.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{2n+1}} \log \frac{1}{1-w} (w-1)^{2n+1} \frac{1}{(1-z-w)^2}.$$

Now put $w/(w-1) = v$ so that $w = v/(v-1)$ and $dw = -1/(v-1)^2\, dv$ to get

$$
\begin{aligned}
&- \operatorname*{res}_v \frac{1}{v^{2n+1}} \log \frac{1}{1-v/(v-1)} \frac{1}{(1-z-v/(v-1))^2} \frac{1}{(v-1)^2} \\
&= - \operatorname*{res}_v \frac{1}{v^{2n+1}} \log(1-v) \frac{1}{(1-z+vz)^2} \\
&= [v^{2n}] \log \frac{1}{1-v} \frac{1}{(1-z+vz)^2} \\
&= \frac{1}{(1-z)^2} [v^{2n}] \log \frac{1}{1-v} \frac{1}{(1+vz/(1-z))^2} \\
&= \frac{1}{(1-z)^2} \sum_{q=1}^{2n} \frac{1}{q} (2n-q+1)(-1)^q \frac{z^{2n-q}}{(1-z)^{2n-q}}.
\end{aligned}
$$

We get from z

$$
[z^{2n-1}] \log \frac{1}{1-z} \frac{z^{2n-q}}{(1-z)^{2n+3-q}} = [z^{q-1}] \log \frac{1}{1-z} \frac{1}{(1-z)^{2n+3-q}}.
$$

This is zero when $q = 1$ so we finally have

$$
\begin{aligned}
&\sum_{q=2}^{2n} \frac{1}{q} (2n-q+1)(-1)^q \sum_{p=1}^{q-1} \frac{1}{p} \binom{2n+1-p}{q-1-p} \\
&= \sum_{p=1}^{2n-1} \frac{1}{p} \sum_{q=p+1}^{2n} \frac{1}{q} (2n-q+1)(-1)^q \binom{2n+1-p}{q-1-p} \\
&= \sum_{p=1}^{2n-1} \frac{1}{p} \sum_{q=0}^{2n-1-p} \frac{1}{q+p+1} (2n-q-p)(-1)^{q+p+1} \binom{2n+1-p}{q}.
\end{aligned}
$$

Now we can certainly raise q to $2n-p$ as the inner sum term is zero there. We will raise to $2n+1-p$ to get

$$
\frac{1}{2n+2} H_{2n-1} + \sum_{p=1}^{2n-1} \frac{1}{p} \sum_{q=0}^{2n+1-p} \frac{1}{q+p+1} (2n-q-p)(-1)^{q+p+1} \binom{2n+1-p}{q}.
$$

We omit the zero contribution to get (note that $\sum_{q=0}^{2n+1-p} (-1)^q \binom{2n+1-p}{q} = 0$)

$$
\frac{1}{2n+2} H_{2n-1} + (2n+1) \sum_{p=1}^{2n-1} \frac{(-1)^{p+1}}{p} \sum_{q=0}^{2n+1-p} \frac{1}{q+p+1} (-1)^q \binom{2n+1-p}{q}.
$$

Now introduce the function

$$f(z) = (2n+1-p)!(-1)^{p+1}\frac{1}{z+p+1}\prod_{r=0}^{2n+1-p}\frac{1}{z-r}.$$

We have that with $0 \le q \le 2n+1-p$

$$\mathrm{Res}(f(z); z=q) = (2n+1-p)!(-1)^{p+1}\frac{1}{q+p+1}\prod_{r=0}^{q-1}\frac{1}{q-r}\prod_{r=q+1}^{2n+1-p}\frac{1}{q-r}$$

$$= (2n+1-p)!(-1)^{p+1}\frac{1}{q+p+1}\frac{1}{q!}\frac{(-1)^{2n+1-p-q}}{(2n+1-p-q)!}$$

$$= \frac{1}{q+p+1}(-1)^q\binom{2n+1-p}{q}.$$

With residues summing to zero and the residue at infinity being zero we can thus evaluate the inner sum as minus the residue at $z=-p-1$ to get

$$-(2n+1-p)!(-1)^{p+1}\prod_{r=0}^{2n+1-p}\frac{1}{-p-1-r} = (2n+1-p)!\prod_{r=0}^{2n+1-p}\frac{1}{p+1+r}$$

$$= (2n+1-p)!\prod_{r=p+1}^{2n+2}\frac{1}{r} = (2n+1-p)!\frac{p!}{(2n+2)!}.$$

Substituting into the outer sum we have

$$\frac{1}{2n+2}H_{2n-1} + \frac{1}{2n+2}\sum_{p=1}^{2n-1}(-1)^{p+1}\binom{2n}{p-1}^{-1}$$

$$= \frac{1}{2n+2}H_{2n-1} + \frac{1}{2n+2}\sum_{p=0}^{2n-2}(-1)^{p}\binom{2n}{p}^{-1}$$

$$= \frac{1}{2n+2}H_{2n-1} + \frac{1}{2n+2}\frac{1}{2n} - \frac{1}{2n+2} + \frac{1}{2n+2}\sum_{p=0}^{2n}(-1)^{p}\binom{2n}{p}^{-1}$$

$$= \frac{1}{2n+2}H_{2n} - \frac{1}{2n+2} + \frac{1}{2n+2}\sum_{p=0}^{2n}(-1)^{p}\binom{2n}{p}^{-1}.$$

Recall from 1.124 that

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^{-1} = (1+(-1)^n)\frac{n+1}{n+2}$$

so we finally have

$$\frac{1}{2n+2} H_{2n} + \frac{1}{2n+2}\left[-1 + 2\frac{2n+1}{2n+2}\right]$$
$$= \frac{1}{2}\frac{1}{n+1} H_{2n} + \frac{1}{2}\frac{n}{(n+1)^2}.$$

This is the claim.

This problem is from page 20 eqn. 2.18 of H.W.Gould's *Combinatorial Identities* [Gou72a].

1.128 Simon's identity

We seek to verify that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} (-1)^{n-k}(x+1)^k.$$

We get for the RHS

$$[z^n](1+z)^n \sum_{k=0}^{n} \binom{n}{k} (1+z)^k (-1)^{n-k} (x+1)^k$$
$$= [z^n](1+z)^n((1+z)(x+1)-1)^n$$
$$= [z^n](1+z)^n(z+x+xz)^n = [z^n](1+z)^n(z+x(1+z))^n$$
$$= [z^n](1+z)^n \sum_{k=0}^{n} \binom{n}{k} x^k (1+z)^k z^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k [z^k](1+z)^{n+k}$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k.$$

This is the claim.

1.129 Identity from Abramowitz and Stegun

We seek to verify the two identities

$$\left[{n \atop m}\right] = (-1)^{n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k} \left\{{n-m+k \atop k}\right\}, \quad \text{and}$$

$$\left\{{n \atop m}\right\} = (-1)^{n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k} \left[{n-m+k \atop k}\right].$$

These two are close variations on Schläfli's formula: 1.55.

First identity

We get for the RHS

$$\sum_{k=0}^{n-m} (-1)^k \binom{2n-1-m-k}{2n-2m-k} \binom{2n-m}{k} \left\{ 2n-2m-k \atop n-m-k \right\}$$

$$= \sum_{k=0}^{n-m} (-1)^k \binom{2n-1-m-k}{2n-2m-k} \binom{2n-m}{k}$$

$$\times \frac{(2n-2m-k)!}{(n-m-k)!} [w^{2n-2m-k}] (\exp(w)-1)^{n-m-k}$$

$$= \frac{(n-1)!}{(m-1)!} [w^{2n-2m}] (\exp(w)-1)^{n-m}$$

$$\times \sum_{k=0}^{n-m} (-1)^k \binom{2n-1-m-k}{n-1} \binom{2n-m}{k} w^k (\exp(w)-1)^{-k}.$$

This is

$$\frac{(n-1)!}{(m-1)!} [w^{2n-2m}] (\exp(w)-1)^{n-m}$$

$$\times \sum_{k=0}^{n-m} (-1)^k \binom{2n-1-m-k}{n-m-k} \binom{2n-m}{k} w^k (\exp(w)-1)^{-k}$$

$$= \frac{(n-1)!}{(m-1)!} [w^{2n-2m}] (\exp(w)-1)^{n-m} [z^{n-m}] (1+z)^{2n-1-m}$$

$$\times \sum_{k=0}^{2n-m} (-1)^k \binom{2n-m}{k} w^k (\exp(w)-1)^{-k} \frac{z^k}{(1+z)^k}.$$

Here we have extended to $2n-m$ due to the coefficient extractor in z. Continuing,

$$\frac{(n-1)!}{(m-1)!} [w^{2n-2m}] (\exp(w)-1)^{n-m} [z^{n-m}] (1+z)^{2n-1-m}$$

$$\times \left[1 - \frac{wz}{(\exp(w)-1)(1+z)} \right]^{2n-m}$$

$$= \frac{(n-1)!}{(m-1)!} [w^{2n-2m}] \frac{1}{(\exp(w)-1)^n} [z^{n-m}] \frac{1}{1+z}$$

$$\times [(\exp(w)-1)(1+z) - wz]^{2n-m}$$

$$= \frac{(n-1)!}{(m-1)!} [w^{2n-2m}] \frac{1}{(\exp(w)-1)^n} [z^{n-m}] \frac{1}{1+z}$$

$$\times \sum_{q=0}^{2n-m} \binom{2n-m}{q} z^q (\exp(w)-1-w)^q (\exp(w)-1)^{2n-m-q}$$

$$= (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [w^{2n-2m}] \frac{1}{(\exp(w)-1)^n}$$

$$\times \sum_{q=0}^{2n-m} \binom{2n-m}{q} (-1)^q (\exp(w)-1-w)^q (\exp(w)-1)^{2n-m-q}$$

$$= (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [w^{2n-2m}] \frac{1}{(\exp(w)-1)^n}$$

$$\times [\exp(w)-1-\exp(w)+1+w]^{2n-m}$$

$$= (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [w^{n-m}] \frac{w^n}{(\exp(w)-1)^n}.$$

This is

$$(-1)^{n-m} \frac{(n-1)!}{(m-1)!} \operatorname*{res}_w \frac{w^{m-1}}{(\exp(w)-1)^n}.$$

Now put $\exp(w)-1 = v$ so that $w = \log(1+v)$ and $dw = \frac{1}{1+v}\, dv$ to get

$$(-1)^{n-m} \frac{(n-1)!}{(m-1)!} \operatorname*{res}_v (\log(1+v))^{m-1} \frac{1}{v^n} \frac{1}{1+v}$$

$$= (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [v^{n-1}] (\log(1+v))^{m-1} \frac{1}{1+v}$$

$$= (-1)^{n-m} \frac{n!}{m!} [v^n] (\log(1+v))^m = (-1)^m \frac{n!}{m!} [v^n] \log(1-v)^m$$

$$= \frac{n!}{m!} [v^n] \left(\log \frac{1}{1-v}\right)^m = \left[n \atop m \right].$$

This is the claim.

Second identity

Replacing $\exp(w)-1$ by $\log \frac{1}{1-w}$ and re-capitulating the above computation we arrive at

$$(-1)^{n-m} \frac{(n-1)!}{(m-1)!} [w^{n-m}] w^n \left(\log \frac{1}{1-w}\right)^{-n}.$$

This is

$$(-1)^{n-m} \frac{(n-1)!}{(m-1)!} \operatorname*{res}_w w^{m-1} \left(\log \frac{1}{1-w}\right)^{-n}.$$

Now put $\log \frac{1}{1-w} = v$ so that $w = 1 - \exp(-v)$ and $dw = \exp(-v)\, dv$ to get

$$\begin{aligned}
&(-1)^{n-m} \frac{(n-1)!}{(m-1)!} \operatorname*{res}_v (1-\exp(-v))^{m-1} \frac{1}{v^n} \exp(-v) \\
&= (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [v^{n-1}](1-\exp(-v))^{m-1} \exp(-v) \\
&= (-1)^{n-m} \frac{n!}{m!} [v^n](1-\exp(-v))^m = (-1)^m \frac{n!}{m!} [v^n](1-\exp(v))^m \\
&= \frac{n!}{m!} [v^n](\exp(v)-1)^m = \left\{ {n \atop m} \right\}.
\end{aligned}$$

This is from page 824 numbers 24.1.3 and 24.1.4 of Abramowitz and Stegun, *Handbook of Mathematical Functions*, [AS72].

1.130 Bernoulli number / Stirling number identity

We seek to show that

$$B_n = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \left\{ {n+k \atop k} \right\} \binom{n+k}{k}^{-1}.$$

We get for the RHS

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} \left\{ {2n-k \atop n-k} \right\} \binom{2n-k}{n-k}^{-1} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} \binom{2n-k}{n-k}^{-1} (2n-k)! [z^{2n-k}] \frac{1}{(n-k)!} (\exp(z)-1)^{n-k} \\
&= n! [z^{2n}] (\exp(z)-1)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} z^k (\exp(z)-1)^{-k}.
\end{aligned}$$

We add in the $k = n+1$ term and cancel the extra contribution to get

$$\begin{aligned}
&n! [z^{2n}] \frac{z^{n+1}}{\exp(z)-1} - n! [z^{2n}] (\exp(z)-1)^n \left[\frac{z}{\exp(z)-1} - 1 \right]^{n+1} \\
&= n! [z^n] \frac{z}{\exp(z)-1} - n! [z^{2n}] \frac{1}{\exp(z)-1} [1 + z - \exp(z)]^{n+1}.
\end{aligned}$$

Now since $\exp(z) = 1 + z + z^2/2 + \cdots$ the exponentiated zerm in z starts at z^{2n+2} and the denominator at z, for a total of a start at z^{2n+1} and hence the second term is zero. That leaves just the first term which is B_n by the canonical EGF.

This was DLMF [DLMF, Eq. 24.15.7].

1.131 Recurrence relation from DLMF

We seek to show that with $n \ge k \ge h$

$$\binom{k}{h}\left[{n \atop k}\right] = \sum_{j=k-h}^{n-h} \binom{n}{j} \left[{n-j \atop h}\right]\left[{j \atop k-h}\right]$$

and

$$\binom{k}{h}\left\{{n \atop k}\right\} = \sum_{j=k-h}^{n-h} \binom{n}{j} \left\{{n-j \atop h}\right\}\left\{{j \atop k-h}\right\}.$$

First identity

We get from the basic EGF for Stirling cycle numbers that the RHS of the identity is

$$\sum_{j=k-h}^{n-h} \binom{n}{j} (n-j)![z^{n-j}]\frac{1}{h!}\left(\log\frac{1}{1-z}\right)^h j![w^j]\frac{1}{(k-h)!}\left(\log\frac{1}{1-w}\right)^{k-h}$$

$$= \binom{k}{h}\frac{n!}{k!}\sum_{j=k-h}^{n-h}[z^n]z^j\left(\log\frac{1}{1-z}\right)^h [w^j]\left(\log\frac{1}{1-w}\right)^{k-h}$$

$$= \binom{k}{h}\frac{n!}{k!}[z^n]\left(\log\frac{1}{1-z}\right)^h \sum_{j=k-h}^{n-h} z^j[w^j]\left(\log\frac{1}{1-w}\right)^{k-h}.$$

We claim we can extend the sum to infinity. This is because in fact we have

$$\binom{k}{h}\frac{n!}{k!}[z^n]z^h\left(\frac{1}{z}\log\frac{1}{1-z}\right)^h \sum_{j=k-h}^{n-h} z^j[w^j]\left(\log\frac{1}{1-w}\right)^{k-h}$$

$$= \binom{k}{h}\frac{n!}{k!}[z^{n-h}]\left(\frac{1}{z}\log\frac{1}{1-z}\right)^h \sum_{j=k-h}^{n-h} z^j[w^j]\left(\log\frac{1}{1-w}\right)^{k-h}.$$

Here we see immediately that the coefficient extractor in z enforces the upper limit of the sum through the z^j term. We obtain

$$\binom{k}{h}\frac{n!}{k!}[z^{n-h}]\left(\frac{1}{z}\log\frac{1}{1-z}\right)^h \sum_{j\ge k-h} z^j[w^j]\left(\log\frac{1}{1-w}\right)^{k-h}.$$

But now we have covered the entire range of the logarithmic term in w so we find

$$\binom{k}{h}\frac{n!}{k!}[z^{n-h}]\left(\frac{1}{z}\log\frac{1}{1-z}\right)^h\left(\log\frac{1}{1-z}\right)^{k-h}$$
$$=\binom{k}{h}[z^n]\frac{n!}{k!}\left(\log\frac{1}{1-z}\right)^k=\binom{k}{h}\left[{n \atop k}\right].$$

This is the claim.

Second identity

This identity now follows by inspection using the EGF

$$\left\{{n \atop k}\right\}=n![z^n]\frac{1}{k!}(\exp(z)-1)^k.$$

We have that $\exp(z)-1$ is a formal power series that starts at z, just like $\log\frac{1}{1-z}$. Hence the computation from the Stirling set numbers goes through just as with the Stirling cycle numbers, where we use the fact that both $\frac{1}{z}\log\frac{1}{1-z}$ and $\frac{1}{z}(\exp(z)-1)$ are formal power series. There is no pole at zero here.

This was DLMF [DLMF, Eq. 26.8.19] and [DLMF, Eq. 26.8.23].

1.132 Bernoulli, Fibonacci and Lucas numbers

We seek to show that

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\left(\frac{5}{9}\right)^k B_{2k}F_{n-2k}=\frac{n}{6}L_{n-1}+\frac{n}{3^n}L_{2n-2}.$$

Using the fact that odd-index Bernoulli numbers are zero except for $k=1$ this is equivalent to

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}}{3}\right)^k B_kF_{n-k}=\frac{n}{6}L_{n-1}+\frac{n}{3^n}L_{2n-2}-\frac{\sqrt{5}n}{6}F_{n-1}.$$

Now observe that

$$\left(\frac{\sqrt{5}}{3}\right)^k B_k=k![z^k]\frac{z\sqrt{5}/3}{\exp(z\sqrt{5}/3)-1}$$

and

$$F_k=k![z^k]\frac{1}{\sqrt{5}}(\exp(\varphi z)-\exp(\psi(z)).$$

We thus obtain for the LHS by convolution of EGFs the following closed form:

$$\boxed{\frac{1}{3} n! [z^n] \frac{z}{\exp(z\sqrt{5}/3) - 1} (\exp(\varphi z) - \exp(\psi z)).}$$

Expanding we find

$$\frac{1}{3} n! [z^n] \frac{z}{\exp(z\sqrt{5}/3) - 1}$$

$$\times (\exp(\varphi z) - \exp(z(3+\sqrt{5})/6) + \exp(z(3+\sqrt{5})/6) - \exp(\psi z)).$$

Here we get for the first piece

$$\begin{aligned}\frac{1}{3} n! [z^n] z \exp(z(3+\sqrt{5})/6) &= \frac{1}{3} n! [z^{n-1}] \exp(z(3+\sqrt{5})/6) \\ &= \frac{1}{3} n ((3+\sqrt{5})/6)^{n-1}.\end{aligned}$$

We get for the second piece

$$\frac{1}{3} n! [z^n] \frac{z}{\exp(z\sqrt{5}/3) - 1}$$

$$\times (\exp(z(3+\sqrt{5})/6) - \exp(z(3-\sqrt{5})/6) + \exp(z(3-\sqrt{5})/6) - \exp(\psi z)).$$

This has the component

$$\frac{1}{3} n! [z^n] z \exp(\psi z) = \frac{1}{3} n \psi^{n-1}.$$

The last remaining component is

$$\begin{aligned}&\frac{1}{3} n! [z^n] \frac{z}{\exp(z\sqrt{5}/3) - 1} \exp(z/2)(\exp(z\sqrt{5}/6) - \exp(-z\sqrt{5}/6)) \\ &= \frac{1}{3} n! [z^n] z \exp(z/2) \exp(-z\sqrt{5}/6) = \frac{1}{3} n ((3-\sqrt{5})/6)^{n-1}.\end{aligned}$$

If we factor out the $n/3$ we have to show that the following holds:

$$\boxed{((3+\sqrt{5})/6)^{n-1} + ((3-\sqrt{5})/6)^{n-1} + \psi^{n-1} = \frac{1}{2} L_{n-1} + \frac{1}{3^{n-1}} L_{2n-2} - \frac{\sqrt{5}}{2} F_{n-1}.}$$

Now we have e.g. by induction

$$\psi^{n-1} = \frac{1}{2} L_{n-1} - \frac{\sqrt{5}}{2} F_{n-1}.$$

This leaves us with

$$((3+\sqrt{5})/6)^{n-1} + ((3-\sqrt{5})/6)^{n-1} = \frac{1}{3^{n-1}} L_{2n-2}.$$

Here the RHS is

$$\frac{1}{3^{n-1}}(\varphi^{2n-2} + \psi^{2n-2})$$

but with $\varphi^2/3 = (3+\sqrt{5})/6$ and $\psi^2/3 = (3-\sqrt{5})/6$ this follows immediately, which concludes the argument.

This was DLMF [DLMF, Eq. 24.15.11].

1.133 Bernoulli / exponential convolution

We seek to show that with $n \ge 1$ and $m \ge 2$

$$B_n = \frac{1}{m(1-m^n)} \sum_{k=0}^{n-1} m^k \binom{n}{k} B_k \sum_{j=1}^{m-1} j^{n-k}.$$

Extending k to n we find

$$B_n + \frac{1}{m(1-m^n)} m^n B_n(m-1) = \frac{1}{m(1-m^n)} \sum_{k=0}^{n} m^k \binom{n}{k} B_k \sum_{j=1}^{m-1} j^{n-k}.$$

Multiply by $m(1-m^n)$ to get

$$B_n m(1-m^n) + m^n B_n(m-1) = \sum_{k=0}^{n} m^k \binom{n}{k} B_k \sum_{j=1}^{m-1} j^{n-k}.$$

or

$$B_n(m-m^n) = \sum_{k=0}^{n} m^k \binom{n}{k} B_k \sum_{j=1}^{m-1} j^{n-k}.$$

Observe that we have

$$m^k B_k = k![z^k] \frac{mz}{\exp(mz)-1}$$

and

$$\sum_{j=1}^{m-1} j^k = k![z^k] \sum_{j=1}^{m-1} \exp(jz) = k![z^k] \exp(z) \sum_{j=0}^{m-2} \exp(jz)$$
$$= k![z^k] \exp(z) \frac{1-\exp((m-1)z)}{1-\exp(z)}.$$

We therefore have by convolution of EGFs the EGF of the RHS:

$$n![z^n] \frac{mz}{\exp(mz)-1} \frac{\exp(z)-\exp(mz)}{1-\exp(z)}$$

$$
\begin{aligned}
&= n![z^n]\frac{mz}{\exp(mz)-1}\frac{\exp(z)-1+1-\exp(mz)}{1-\exp(z)} \\
&= n![z^n]\frac{mz}{\exp(mz)-1}\left[-1+\frac{1-\exp(mz)}{1-\exp(z)}\right] = -m^n B_n + n![z^n]\frac{mz}{\exp(z)-1} \\
&= -m^n B_n + mB_n = (m-m^n)B_n.
\end{aligned}
$$

This is the claim.

This was WFS [WFS, Eq. 04.13.17.0001].

1.134 Bernoulli identity by Munch

We seek to show that

$$B_n = \frac{1}{n+1}\sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^j j^n \binom{n+1}{k-j}\binom{n}{k}^{-1}.$$

where $n \geq 1$. The RHS is

$$
\begin{aligned}
&\frac{1}{n+1}\sum_{k=1}^{n}\binom{n}{k}^{-1}\sum_{j=1}^{k}(-1)^j j^n\binom{n+1}{k-j} \\
&= n![z^n]\frac{1}{n+1}\sum_{k=1}^{n}\binom{n}{k}^{-1}\sum_{j=0}^{k}(-1)^j\exp(jz)\binom{n+1}{k-j}.
\end{aligned}
$$

Here we have included $j=0$ because it makes no contribution to the coefficient extractor in z. Continuing,

$$n![z^n]\frac{1}{n+1}\sum_{k=1}^{n}\binom{n}{k}^{-1}[w^k](1+w)^{n+1}\sum_{j\geq 0}(-1)^j\exp(jz)w^j.$$

We have extended j to infinity because of the coefficient extractor in w. Continuing,

$$
\begin{aligned}
&n![z^n]\frac{1}{n+1}\sum_{k=1}^{n}\binom{n}{k}^{-1}[w^k](1+w)^{n+1}\frac{1}{1+w\exp(z)} \\
&= n![z^n]\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n}{k}^{-1}[w^{n-k}](1+w)^{n+1}\frac{1}{1+w\exp(z)} \\
&= n![z^n]\frac{1}{n+1}[w^n](1+w)^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}^{-1}w^k\frac{1}{1+w\exp(z)}.
\end{aligned}
$$

Introducing the Beta function,

$$n![z^n][w^n](1+w)^{n+1}\sum_{k=0}^{n-1}\mathrm{B}(k+1,n-k+1)w^k\frac{1}{1+w\exp(z)}$$

$$= n![z^n][w^n](1+w)^{n+1}\frac{1}{1+w\exp(z)}\int_0^1 u^n\sum_{k=0}^{n-1}w^k\frac{(1-u)^k}{u^k}\,du$$

$$= n![z^n][w^n]\frac{(1+w)^{n+1}}{1+w\exp(z)}\int_0^1 u^n\frac{1-w^n(1-u)^n/u^n}{1-w(1-u)/u}\,du.$$

Now the second term in the fraction in the integral makes no contribution because the constant term respective w does not contain a term in z and we get

$$n![z^n][w^n]\frac{(1+w)^{n+1}}{1+w\exp(z)}\int_0^1 \frac{u^{n+1}}{u-w+wu}\,du.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{n+1}}\frac{(1+w)^{n+1}}{1+w\exp(z)}\int_0^1 \frac{u^{n+1}}{u-w+wu}\,du.$$

Now put $w/(1+w)=v$ so that $w=v/(1-v)$ and $dw=1/(1-v)^2\,dv$ to get

$$\operatorname*{res}_v \frac{1}{v^{n+1}}\frac{1}{1+v\exp(z)/(1-v)}\frac{1}{(1-v)^2}\int_0^1 u^{n+1}\frac{1-v}{u-v}\,du$$

$$= \operatorname*{res}_v \frac{1}{v^{n+1}}\frac{1}{1-v+v\exp(z)}\int_0^1 u^{n+1}\frac{1}{u-v}\,du$$

$$= \operatorname*{res}_v \frac{1}{v^{n+1}}\frac{1}{1-v(1-\exp(z))}\int_0^1 u^{n}\frac{1}{1-v/u}\,du$$

$$= \sum_{q=0}^{n}(1-\exp(z))^q\int_0^1 u^n\frac{1}{u^{n-q}}\,du = \sum_{q=0}^{n}\frac{(1-\exp(z))^q}{q+1}.$$

Returning to the coefficient extractor in z and noting that $1-\exp(z)$ has no constant term we finally obtain

$$n![z^n]\frac{1}{1-\exp(z)}\sum_{q\ge 0}\frac{(1-\exp(z))^{q+1}}{q+1}$$

$$= n![z^n]\frac{1}{1-\exp(z)}\log\frac{1}{1-(1-\exp(z))} = n![z^n]\frac{1}{1-\exp(z)}(-z)$$

$$= n![z^n]\frac{z}{\exp(z)-1} = B_n$$

and we have the claim.

This was from H.W. Goulds *Explicit formulas for Bernoulli numbers*, page

45, [Gou72b, Eq. 3].

1.135 Bernoulli identity by Kronecker

We seek to show that

$$B_{2n} = \sum_{j=2}^{2n+1} (-1)^{j-1} \binom{2n+1}{j} \frac{1}{j} \sum_{k=1}^{j-1} k^{2n}.$$

We get for the RHS

$$(2n)![z^{2n}] \sum_{j=2}^{2n+1} (-1)^{j-1} \binom{2n+1}{j} \frac{1}{j} \sum_{k=0}^{j-1} \exp(kz).$$

Here we have lowered k to zero because there is no contribution to the coefficient extractor in that case. Continuing,

$$(2n)![z^{2n}] \sum_{j=0}^{2n-1} (-1)^{j} \binom{2n+1}{j} \frac{1}{2n+1-j} \sum_{k=0}^{2n-j} \exp(kz)$$

$$= (2n)![z^{2n}][w^{2n+1}] \log \frac{1}{1-w} \sum_{j=0}^{2n-1} (-1)^{j} \binom{2n+1}{j} w^j \frac{\exp((2n-j+1)z)-1}{\exp(z)-1}.$$

Evaluating for $j = 2n$ we get

$$(2n)![z^{2n}][w^{2n+1}] \log \frac{1}{1-w} (2n+1) w^{2n} = 0$$

owing to the coefficient extractor in z. We also get zero when $j = 2n+1$ since $\log \frac{1}{1-w}$ starts at w. Hence we are justified in raising j to $2n+1$. We find

$$(2n)![z^{2n}][w^{2n+1}] \log \frac{1}{1-w} \sum_{j=0}^{2n+1} (-1)^{j} \binom{2n+1}{j} w^j \frac{\exp((2n-j+1)z)-1}{\exp(z)-1}.$$

The first piece here is

$$-(2n)![z^{2n+1}] \frac{z}{\exp(z)-1} [w^{2n+1}] \log \frac{1}{1-w} \sum_{j=0}^{2n+1} (-1)^{j} \binom{2n+1}{j} w^j$$

Note that

$$[z^{2n+1}] \frac{z}{\exp(z)-1} = \frac{B_{2n+1}}{(2n+1)!} = 0.$$

This leaves

$$(2n)![z^{2n}][w^{2n+1}]\log\frac{1}{1-w}\sum_{j=0}^{2n-1}(-1)^j\binom{2n+1}{j}w^j\frac{\exp((2n-j+1)z)}{\exp(z)-1}$$

$$= (2n)![z^{2n}]\frac{\exp((2n+1)z)}{\exp(z)-1}[w^{2n+1}]\log\frac{1}{1-w}\sum_{j=0}^{2n-1}(-1)^j\binom{2n+1}{j}w^j\exp(-jz)$$

$$= (2n)![z^{2n}]\frac{\exp((2n+1)z)}{\exp(z)-1}[w^{2n+1}]\log\frac{1}{1-w}(1-w\exp(-z))^{2n+1}$$

$$= (2n)![z^{2n+1}]\frac{z}{\exp(z)-1}[w^{2n+1}]\log\frac{1}{1-w}(\exp(z)-w)^{2n+1}$$

$$= (2n)![z^{2n+1}]\frac{z}{\exp(z)-1}$$

$$\times\sum_{q=0}^{2n+1}\binom{2n+1}{q}(\exp(z)-1)^q[w^{2n+1}]\log\frac{1}{1-w}(1-w)^{2n+1-q}.$$

Splitting the sum into the initial value for $q=0$ and the rest we get for the latter with the term in w and $1 \le q \le 2n+1$

$$[w^{2n+1}]\log\frac{1}{1-w}(1-w)^{2n+1-q}.$$

Recall from section 1.89 that with $1 \le k \le n$

$$\frac{1}{k}\binom{n}{k}^{-1} = [w^n]\log\frac{1}{1-w}(-1)^{n-k}(1-w)^{n-k}.$$

We thus have

$$(2n)![z^{2n+1}]\frac{z}{\exp(z)-1}$$

$$\times\sum_{q=1}^{2n+1}\binom{2n+1}{q}(\exp(z)-1)^q(-1)^{2n+1-q}\frac{1}{q}\binom{2n+1}{q}^{-1}$$

$$= -(2n)![z^{2n+1}]\frac{z}{\exp(z)-1}\sum_{q=1}^{2n+1}\frac{(1-\exp(z))^q}{q}.$$

Due to the coefficient extractor we may raise q to infinity to get

$$-(2n)![z^{2n+1}]\frac{z}{\exp(z)-1}\sum_{q\ge 1}\frac{(1-\exp(z))^q}{q}$$

$$= -(2n)![z^{2n+1}]\frac{z}{\exp(z)-1}\log\frac{1}{1-(1-\exp(z))}$$

$$= -(2n)![z^{2n+1}]\frac{z}{\exp(z)-1}(-z) = (2n)![z^{2n}]\frac{z}{\exp(z)-1} = B_{2n}.$$

This is the claim. It remains to show that the contribution from $q = 0$ is zero. This follows by inspection as we are again extracting an odd index Bernoulli number.

This was from H.W. Goulds *Explicit formulas for Bernoulli numbers*, page 46, [Gou72b, Eq. 4].

1.136 Computing Bernoulli numbers

We seek to show that

$$B_{n+1} = \frac{n+1}{2(1-2^{n+1})}\sum_{k=0}^{n}\frac{(-1)^k}{2^k}k!\left\{ n \atop k \right\}$$

as well as

$$B_{n+1} = \frac{n+1}{2^{n+1}(2^{n+1}-1)}\sum_{k=0}^{n}(-1)^{k+1}\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n.$$

First identity

We have for the sum component using the Stirling set number EGF

$$n![z^n]\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\exp(z)-1)^k$$

We may extend to infinity due to the coefficient extractor and the fact that $\exp(z) - 1 = z + \cdots$ to get

$$n![z^n]\frac{1}{1+(\exp(z)-1)/2} = n![z^n]\frac{2}{1+\exp(z)}.$$

On the other hand we have the following EGF

$$\frac{B_{n+1}}{n+1} = n![z^n]\left(-\frac{1}{z}+\frac{1}{\exp(z)-1}\right).$$

The sum component must therefore have EGF

$$\left(-\frac{2}{z}+\frac{2}{\exp(z)-1}\right) - \left(-\frac{4}{2z}+\frac{4}{\exp(2z)-1}\right).$$

This is

$$\frac{2(\exp(z)+1)}{\exp(2z)-1} - \frac{4}{\exp(2z)-1} = \frac{2(\exp(z)-1)}{\exp(2z)-1} = \frac{2}{\exp(z)+1}.$$

We have the claim.

Second identity

We get for the double sum

$$\begin{aligned}
&n![z^n]\sum_{k=0}^{n}(-1)^{k+1}\exp(kz)\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}\exp(-jz)\\
&= n![z^n]\sum_{k=0}^{n}(-1)^{k+1}\exp(kz)[w^k]\frac{1}{1-w}\sum_{j\ge 0}(-1)^j\binom{n+1}{j}w^j\exp(-jz)\\
&= n![z^n]\sum_{k=0}^{n}(-1)^{k+1}\exp(kz)[w^k]\frac{1}{1-w}(1-w\exp(-z))^{n+1}\\
&= n![z^n]\sum_{k=0}^{n}(-1)^{k+1}[w^k]\frac{1}{1-w\exp(z)}(1-w)^{n+1}\\
&= n![z^n]\sum_{k=0}^{n}(-1)^{n-k+1}[w^{n-k}]\frac{1}{1-w\exp(z)}(1-w)^{n+1}\\
&= n![z^n][w^n]\sum_{k=0}^{n}(-1)^{n-k+1}w^k\frac{1}{1-w\exp(z)}(1-w)^{n+1}\\
&= n![z^n][w^n]\sum_{k\ge 0}(-1)^k w^k\frac{1}{1-w\exp(z)}(w-1)^{n+1}.
\end{aligned}$$

Here we have extended k to infinity due to the coefficient extractor in w. We finally have

$$n![z^n][w^n]\frac{1}{1+w}\frac{1}{1-w\exp(z)}(w-1)^{n+1}.$$

Here the contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^{n+1}}\frac{1}{1+w}\frac{1}{1-w\exp(z)}(w-1)^{n+1}.$$

Now we put $w/(w-1)=v$ so that $w=v/(v-1)$ and $dw=-1/(v-1)^2\,dv$ to get

$$\begin{aligned}
&-\operatorname*{res}_{v}\frac{1}{v^{n+1}}\frac{1}{1+v/(v-1)}\frac{1}{1-v\exp(z)/(v-1)}\frac{1}{(v-1)^2}\\
&= \operatorname*{res}_{v}\frac{1}{v^{n+1}}\frac{1}{1-2v}\frac{1}{v-1-v\exp(z)}\\
&= -\operatorname*{res}_{v}\frac{1}{v^{n+1}}\frac{1}{1-2v}\frac{1}{1-v(1-\exp(z))}.
\end{aligned}$$

Restoring the coefficient extractor in z we have

$$-n![z^n]\sum_{q=0}^{n} 2^{n-q}(1-\exp(z))^q = -n!2^n[z^n]\sum_{q\geq 0} 2^{-q}(1-\exp(z))^q.$$

Here we have extended q to infinity due to the coefficient extractor and the fact that $1-\exp(z) = -z - \cdots$. Continuing,

$$-n!2^n[z^n]\frac{1}{1-(1-\exp(z))/2} = -n!2^n[z^n]\frac{2}{1+\exp(z)}$$
$$= -n![z^n]\frac{2}{1+\exp(2z)}.$$

Repeating the construction of the EGF from the first identity we must replicate this through

$$\left(-\frac{4}{4z}+\frac{4}{\exp(4z)-1}\right)-\left(-\frac{2}{2z}+\frac{2}{\exp(2z)-1}\right)$$
$$= \frac{4}{\exp(4z)-1} - \frac{2(\exp(2z)+1)}{\exp(4z)-1}$$
$$= -\frac{2(\exp(2z)-1)}{\exp(4z)-1} = -\frac{2}{\exp(2z)+1}$$

and we have the claim.

This was from H.W. Goulds *Explicit formulas for Bernoulli numbers*, page 49, [Gou72b, Eq. 18,19].

1.137 From the Saalschütz text

1.137.1 Bernoulli numbers I

We seek to show that

$$\sum_{k=1}^{n}\binom{2n+1}{2k-1}\left(2^{n+1-k}-\frac{1}{2^{n+1-k}}\right)^2 B_{2n+2-2k} = (2n+1)\left(\frac{1}{4}+\frac{1}{2^{2n+1}}\right).$$

We get for the LHS

$$\sum_{k=1}^{n}\binom{2n+1}{2k-1}\left(2^{2n+2-2k}-2+\frac{1}{2^{2n+2-2k}}\right)B_{2n+2-2k}.$$

Using the fact that odd-index Bernoulli numbers are zero except for $B_1 = -\frac{1}{2}$ this becomes

$$\sum_{k=1}^{2n} \binom{2n+1}{k-1} \left(2^{2n+2-k} - 2 + \frac{1}{2^{2n+2-k}}\right) B_{2n+2-k}$$
$$= \sum_{k=0}^{2n-1} \binom{2n+1}{k} \left(2^{2n+1-k} - 2 + \frac{1}{2^{2n+1-k}}\right) B_{2n+1-k}.$$

Raising the upper range we find (the contribution from $2n+1$ is zero):

$$(2n+1) \times \frac{1}{4} + \sum_{k=0}^{2n+1} \binom{2n+1}{k} \left(2^{2n+1-k} - 2 + \frac{1}{2^{2n+1-k}}\right) B_{2n+1-k}.$$

Hence we have reduced the claim to

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} \left(2^{2n+1-k} - 2 + \frac{1}{2^{2n+1-k}}\right) B_{2n+1-k} = \frac{2n+1}{2^{2n+1}}.$$

We have by convolution of EGFs that the LHS is

$$(2n+1)![z^{2n+1}] \exp(z) \left(\frac{2z}{\exp(2z)-1} - \frac{2z}{\exp(z)-1} + \frac{z/2}{\exp(z/2)-1}\right)$$
$$= (2n+1)![z^{2n}] \exp(z) \left(\frac{2}{\exp(2z)-1} - \frac{2}{\exp(z)-1} + \frac{1/2}{\exp(z/2)-1}\right).$$

Working with the parenthesized term we find

$$\frac{2}{\exp(2z)-1} - \frac{2(\exp(z)+1)}{\exp(2z)-1} + \frac{1/2 \times (\exp(z/2)+1)}{\exp(z)-1}$$
$$= -\frac{2\exp(z)}{\exp(2z)-1} + \frac{1/2 \times (\exp(z/2)+1)(\exp(z)+1)}{\exp(2z)-1}$$
$$= \frac{1}{2} \frac{1 + \exp(z/2) - 3\exp(z) + \exp(3z/2)}{\exp(2z)-1}.$$

We thus have to show that

$$(2n)![z^{2n}] \exp(z) \frac{1 + \exp(z/2) - 3\exp(z) + \exp(3z/2)}{\exp(2z)-1} = \frac{1}{2^{2n}}.$$

As we are only using even index coefficients we prepare to eliminate the odd index ones:

$$\exp(-z) \frac{1 + \exp(-z/2) - 3\exp(-z) + \exp(-3z/2)}{\exp(-2z)-1}$$

$$= \exp(z)\frac{1+\exp(-z/2)-3\exp(-z)+\exp(-3z/2)}{1-\exp(2z)}.$$

Add and multiply by 1/2 to get

$$\frac{1}{2}(2n)![z^{2n}]\frac{\exp(z)}{\exp(2z)-1}$$
$$\times(\exp(z/2)-\exp(-z/2)-3\exp(z)+3\exp(-z)+\exp(3z/2)-\exp(-3z/2))$$
$$= \frac{1}{2}(2n)![z^{2n}]\exp(-z/2)(\exp(z)-3\exp(z/2)+1).$$

With $n \ge 1$ this becomes

$$\frac{1}{2}(2n)![z^{2n}](\exp(z/2)+\exp(-z/2)) = \frac{1}{2}\left(\frac{1}{2^{2n}}+\frac{1}{2^{2n}}\right) = \frac{1}{2^{2n}}.$$

This is the claim.

This was page 47 eqn. 8 from Saalschütz on Bernoulli, [Saa93].

1.137.2 Bernoulli numbers II

We seek to show that

$$\sum_{k=1}^{n}\binom{2n}{2k}2^{4k}B_{2k} = 4n-1-(2^{2n}-2)B_{2n}.$$

Now using the fact that odd-index Bernoulli numbers are zero except for $B_1 = -1/2$ the LHS becomes

$$-1+4n+\sum_{k=0}^{2n}\binom{2n}{k}2^{2k}B_k$$

so we require

$$\sum_{k=0}^{2n}\binom{2n}{k}2^{2k}B_k = (2-2^{2n})B_{2n}.$$

By convolution of EGFs we have for the LHS

$$(2n)![z^{2n}]\exp(z)\frac{4z}{\exp(4z)-1}.$$

As we are using only even index coefficients we prepare to eliminate the odd ones:

$$\exp(-z)\frac{-4z}{\exp(-4z)-1} = \exp(3z)\frac{-4z}{1-\exp(4z)} = \exp(3z)\frac{4z}{\exp(4z)-1}$$

to obtain

$$\frac{1}{2} 4z \frac{\exp(z)+\exp(3z)}{\exp(4z)-1} = 2z\frac{\exp(z)}{\exp(2z)-1}.$$

We get for the RHS

$$(2n)![z^{2n}] \left(\frac{2z}{\exp(z)-1} - \frac{2z}{\exp(2z)-1}\right)$$
$$= 2(2n)![z^{2n}] \frac{z\exp(z)}{\exp(2z)-1}.$$

We have equality and hence the claim.

This was page 50 eqn. XLIX from Saalschütz on Bernoulli, [Saa93].

1.137.3 Bernoulli numbers III

We seek to show that

$$\sum_{k=1}^{m} \frac{1}{k+1} (-1)^k k! \left\{ m \atop k \right\} = B_m$$

and

$$\sum_{k=1}^{m} \frac{1}{k(k+1)} (-1)^{k+1} k! \left\{ m \atop k \right\} = B_m.$$

First identity

We get for the LHS

$$(m)![z^m] \sum_{k=1}^{m} \frac{1}{k+1} (-1)^k (\exp(z)-1)^k.$$

Now here we can extend to infinity because the coefficient extractor enforces the upper limit owing to $\exp(z) - 1 = z + \cdots$, and we also may include $k = 0$ because it does not contribute to the extractor. We find

$$(m)![z^m] \sum_{k\ge 0} \frac{1}{k+1} (-1)^k (\exp(z)-1)^k$$
$$= -(m)![z^m] \frac{1}{\exp(z)-1} \sum_{k\ge 0} \frac{1}{k+1} (-1)^{k+1} (\exp(z)-1)^{k+1}$$
$$= -(m)![z^m] \frac{1}{\exp(z)-1} \log \frac{1}{1+\exp(z)-1}$$
$$= (m)![z^m] \frac{z}{\exp(z)-1} = B_m.$$

Second identity

With

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

we get two pieces, the second is B_m so we must show the first is zero.

We get

$$(m)![z^m] \sum_{k=1}^{m} \frac{1}{k} (-1)^{k+1} (\exp(z) - 1)^k.$$

We may once more extend k to infinity and obtain

$$-(m)![z^m] \sum_{k \ge 1} \frac{1}{k} (-1)^k (\exp(z) - 1)^k$$

$$= -(m)![z^m] \log \frac{1}{1 + \exp(z) - 1} = (m)![z^m] z = 0$$

and we have the claim.

This was page 83 eqn. LXIV and LXV from Saalschütz on Bernoulli, [Saa93].

1.137.4 Bernoulli numbers IV

We seek to show that

$$\sum_{k=1}^{m} \left[{2m \atop 2k} \right] B_{2k} = \frac{(2m-1)(2m-1)!}{2(2m+1)}$$

and

$$\sum_{k=1}^{m} \left[{2m+1 \atop 2k} \right] B_{2k} = \frac{m(2m)!}{2m+2}.$$

First identity

With odd-index Bernoulli numbers being zero except for $B_1 = -1/2$ we get for the LHS

$$\frac{1}{2}(2m-1)! + \sum_{k=1}^{2m} \left[{2m \atop k} \right] B_k.$$

Continuing with the sum we have

$$(2m)![w^{2m}] \sum_{k=1}^{2m} \frac{1}{k!} \left(\log \frac{1}{1-w} \right)^k k![z^k] \frac{z}{\exp(z) - 1}$$

$$= (2m)![w^{2m}] \sum_{k\ge 0} \left(\log\frac{1}{1-w}\right)^k [z^k]\frac{z}{\exp(z)-1}.$$

Here we have included $k=0$ and $k>2m$ because they do not contribute due to the coefficient extractor in w. We get

$$(2m)![w^{2m}] \log\frac{1}{1-w}\frac{1}{\exp\log\frac{1}{1-w}-1}$$

$$= (2m)![w^{2m}] \log\frac{1}{1-w}\frac{1}{\frac{1}{1-w}-1}$$

$$= (2m)![w^{2m}] \log\frac{1}{1-w}\frac{1-w}{w} = (2m)![w^{2m+1}] \log\frac{1}{1-w}(1-w)$$

$$= (2m)!\left(\frac{1}{2m+1}-\frac{1}{2m}\right).$$

Collecting everything we find

$$(2m-1)! \times \left(\frac{2m}{2m+1} - 1 + \frac{1}{2}\right) = (2m-1)!\frac{1}{2m+1}(m-1/2)$$

which is the claim.

Second identity

Incorporating B_1 we have as before

$$\frac{1}{2}(2m)! + \sum_{k=1}^{2m} \left[{2m+1 \atop k}\right] B_k.$$

We may raise to $2m+1$ because this is an odd-index Bernoulli number:

$$\frac{1}{2}(2m)! + \sum_{k=1}^{2m+1} \left[{2m+1 \atop k}\right] B_k.$$

Recycling the earlier computation we obtain

$$(2m+1)!\left(\frac{1}{2m+2}-\frac{1}{2m+1}\right).$$

Collecting everything we find

$$(2m)!\left(\frac{2m+1}{2m+2} - 1 + \frac{1}{2}\right) = (2m)!\frac{m}{2m+2}.$$

again as claimed.

This was page 90 eqn. LXVI and LXVII from Saalschütz on Bernoulli, [Saa93].

1.137.5 Bernoulli and Eulerian numbers I

We seek to show that as given

$$B_{2m} = \frac{1}{2m(2m+1)} \sum_{k=0}^{m-1} (-1)^k (2m-1-2k) {2m-1 \choose k}^{-1} \left\langle {2m \atop k} \right\rangle.$$

Note that

$$\sum_{k=m}^{2m-1} (-1)^k (2m-1-2k) {2m-1 \choose k}^{-1} \left\langle {2m \atop k} \right\rangle$$

$$= \sum_{k=0}^{m-1} (-1)^{2m-1-k} (2m-1-2(2m-1-k)) {2m-1 \choose k}^{-1} \left\langle {2m \atop 2m-1-k} \right\rangle$$

$$= \sum_{k=0}^{m-1} (-1)^{k+1} (-2m+1+2k) {2m-1 \choose k}^{-1} \left\langle {2m \atop k} \right\rangle.$$

This is the same as the original sum. Therefore it will suffice to show that

$$B_{2m} = \frac{1}{4m(2m+1)} \sum_{k=0}^{2m-1} (-1)^k (2m-1-2k) {2m-1 \choose k}^{-1} \left\langle {2m \atop k} \right\rangle.$$

Considering this alternate form as it appears in the source we can do better and actually prove that with $n \ge 2$

$$\boxed{B_n = \frac{1}{2n(n+1)} \sum_{k=0}^{n-1} (-1)^k (n-1-2k) {n-1 \choose k}^{-1} \left\langle {n \atop k} \right\rangle.}$$

Here we get two pieces, where we subtract the second from the first,

$$\frac{1}{2} \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k {n-1 \choose k}^{-1} \left\langle {n \atop k} \right\rangle$$

and

$$\frac{1}{n+1} \sum_{k=0}^{n-1} (-1)^k {n \choose k+1}^{-1} \left\langle {n \atop k} \right\rangle.$$

We suppose as given that

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle t^k = n! [z^n] \frac{t-1}{t - \exp((t-1)z)}.$$

First piece

We find using the Beta function

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n-1}(-1)^k \mathrm{B}(n-k,k+1) \left\langle {n \atop k} \right\rangle \\
&= \frac{1}{2}\int_0^1 \sum_{k=0}^{n-1}(-1)^k u^{n-1-k}(1-u)^k \left\langle {n \atop k} \right\rangle \; du \\
&= \frac{1}{2}\int_0^1 u^{n-1} \sum_{k=0}^{n-1}(-1)^k u^{-k}(1-u)^k \left\langle {n \atop k} \right\rangle \; du.
\end{aligned}
$$

Now we may include n in the range of the sum because the Eulerian number is zero there i.e. $\left\langle {n \atop n} \right\rangle = 0$. Continuing,

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 u^{n-1} n![z^n] \frac{(u-1)/u-1}{(u-1)/u-\exp((u-1)/u-1)z)} \; du \\
&= \frac{1}{2}\int_0^1 u^{n-1} n![z^n] \frac{1}{1-u+u\exp((u-1)/u-1)z)} \; du \\
&= \frac{1}{2}\int_0^1 \frac{1}{u} n![z^n] \frac{1}{1-u+u\exp(-z)} \; du \\
&= \frac{(-1)^n}{2}\int_0^1 \frac{1}{u} n![z^n] \frac{1}{1-u(1-\exp(z))} \; du.
\end{aligned}
$$

Taking into account the possible contributions to the coefficient extractor in the limits of the sum to appear we find without the scalar $\frac{(-1)^n}{2}$

$$
\int_0^1 \frac{1}{u} n! \sum_{q=1}^{n} u^q [z^n](1-\exp(z))^q \; du = n![z^n] \sum_{q=1}^{n} \frac{1}{q}(1-\exp(z))^q.
$$

We may extend to infinity due to the coefficient extractor and get

$$
n![z^n] \log \frac{1}{1-(1-\exp(z))} = n![z^n](-z) = 0.
$$

because we said that $n \ge 2$. The first piece vanishes.

Second piece

This piece differs from the first one in that there is a term $\mathrm{B}(n-k,k+2)$ instead of $\mathrm{B}(n-k,k+1)$ which produces an extra $(1-u)$ and there is no scalar like the $1/2$ in the first piece. This gives

$$
(-1)^n \int_0^1 \frac{1-u}{u} n![z^n] \frac{1}{1-u(1-\exp(z))} \; du.
$$

Again taking into account the possible contributions to the coefficient extractor in the limits of the sum to appear we find without the scalar $(-1)^n$

$$\int_0^1 \frac{1-u}{u} n! \sum_{q=1}^n u^q [z^n](1-\exp(z))^q \; du$$

$$= n![z^n] \sum_{q=1}^n \left(\frac{1}{q} - \frac{1}{q+1}\right) (-1)^q (\exp(z)-1)^q$$

$$= \sum_{q=1}^n \frac{(q-1)!(-1)^q}{q+1} \left\{ n \atop q \right\}.$$

This is an interesting intermediate form. Note that we may extend q to infinity due to the Stirling number.

We get as our first subpiece the multiple of $1/q$ which is the same as the first piece and evaluates to zero. Our second subpiece has a minus on it and is

$$n![z^n] \sum_{q\ge 1} (-1)^{q+1} \frac{(\exp(z)-1)^q}{q+1}$$

$$= n![z^n] \frac{1}{\exp(z)-1} \sum_{q\ge 1} (-1)^{q+1} \frac{(\exp(z)-1)^{q+1}}{q+1}$$

$$= n![z^n] \frac{1}{\exp(z)-1} \left(\exp(z) - 1 + \log \frac{1}{1+\exp(z)-1} \right)$$

$$= n![z^n] \left(1 - \frac{z}{\exp(z)-1} \right) = -B_n.$$

Observe that multiplication by $(-1)^n$ does not affect this value since the odd Bernoulli numbers with index $n \ge 2$ are zero.

Conclusion

The conclusion is quite simply that we subtract $-B_n$ from a zero value and hence get B_n as claimed.

This was page 94 eqn. LXVIII from Saalschütz on Bernoulli, [Saa93].

1.137.6 Bernoulli and Eulerian numbers II

We seek to show that as given

$$\frac{1-2^{2m}}{2^{2m-1}} \frac{B_{2m}}{2m} = \binom{m-\frac{1}{2}}{2m} \left\langle {2m-1 \atop m-1} \right\rangle + 2 \sum_{k=1}^{m-1} \binom{k-\frac{1}{2}}{2m} \left\langle {2m-1 \atop k-1} \right\rangle.$$

The LHS is

$$\binom{m-\frac{1}{2}}{2m}\left\langle {2m-1 \atop m-1} \right\rangle + 2\sum_{k=0}^{m-2}\binom{k+\frac{1}{2}}{2m}\left\langle {2m-1 \atop k} \right\rangle.$$

Observe that

$$\sum_{k=m}^{2m-2}\binom{k+\frac{1}{2}}{2m}\left\langle {2m-1 \atop k} \right\rangle = \sum_{k=0}^{m-2}\binom{2m-k-\frac{3}{2}}{2m}\left\langle {2m-1 \atop 2m-2-k} \right\rangle.$$

But we have $\binom{2m-k-\frac{3}{2}}{2m} = \binom{k+\frac{1}{2}}{2m}$ by upper negation which means that this sum is the same as the double one from the source. This gives for the RHS

$$\sum_{k=0}^{2m-2}\binom{k+\frac{1}{2}}{2m}\left\langle {2m-1 \atop k} \right\rangle.$$

We now extend this formula to include odd index Bernoulli numbers and set out to prove that for $n \ge 2$

$$\boxed{\frac{1-2^n}{2^{n-1}}\frac{B_n}{n} = \sum_{k=0}^{n-2}\binom{k+\frac{1}{2}}{n}\left\langle {n-1 \atop k} \right\rangle.}$$

Observe that

$$\left\langle {n \atop k} \right\rangle = \sum_{q=0}^{n} \left\{ {n \atop q} \right\} \binom{n-q}{k}(-1)^{n-q-k}q!$$

$$= n![w^n]\sum_{q=0}^{n}\binom{n-q}{k}(-1)^{n-q-k}(\exp(w)-1)^q.$$

We write for the RHS

$$[z^n]\sqrt{1+z}\sum_{k=0}^{n-2}(1+z)^k\left\langle {n-1 \atop k} \right\rangle.$$

We can raise k to infinity because again there is zero contribution due to the Eulerian number. We find

$$(n-1)![w^{n-1}][z^n]\sqrt{1+z}\sum_{k\ge 0}(1+z)^k\sum_{q=0}^{n-1}\binom{n-1-q}{k}(-1)^{n-1-q-k}(\exp(w)-1)^q$$

$$= (n-1)![w^{n-1}][z^n]\sqrt{1+z}\sum_{q=0}^{n-1}(-1)^{n-1-q}(\exp(w)-1)^q$$

$$\times \sum_{k\ge 0} \binom{n-1-q}{k} (-1)^k (1+z)^k$$

$$= (n-1)![w^{n-1}][z^n]\sqrt{1+z}\sum_{q=0}^{n-1} (-1)^{n-1-q}(\exp(w)-1)^q (-1)^{n-1-q} z^{n-1-q}$$

$$= (n-1)![w^{n-1}][z^n]\sqrt{1+z}\sum_{q=0}^{n-1} (\exp(w)-1)^{n-1-q} z^q.$$

Continuing,

$$(n-1)![w^{n-1}](\exp(w)-1)^{n-1}[z^n]\sqrt{1+z}\frac{1-z^n/(\exp(w)-1)^n}{1-z/(\exp(w)-1)}.$$

The first piece here is

$$(n-1)![w^{n-1}](\exp(w)-1)^{n-1}\sum_{q=0}^{n} \binom{1/2}{q} \frac{1}{(\exp(w)-1)^{n-q}}$$

$$= (n-1)![w^{n-1}]\sum_{q=0}^{n} \binom{1/2}{q} (\exp(w)-1)^{q-1}.$$

Once more we may raise to infinity and we get

$$(n-1)![w^{n-1}]\frac{\exp(w/2)}{\exp(w)-1} = (n-1)![w^n]\frac{w\exp(w/2)}{\exp(w)-1}.$$

The second piece is

$$(n-1)![w^{n-1}][z^n]\sqrt{1+z}\frac{z^n/(\exp(w)-1)}{1-z/(\exp(w)-1)}$$

$$= (n-1)![w^{n-1}][z^0]\sqrt{1+z}\frac{1/(\exp(w)-1)}{1-z/(\exp(w)-1)}$$

$$= (n-1)![w^{n-1}]\frac{1}{\exp(w)-1} = (n-1)![w^n]\frac{w}{\exp(w)-1}.$$

We have brought it down to

$$(n-1)![w^n]\frac{w}{\exp(w/2)+1}.$$

On the other hand the LHS of the closed form is

$$\frac{1-2^n}{2^{n-1}}\frac{B_n}{n} = \frac{2}{2^n}\frac{B_n}{n} - 2\frac{B_n}{n}$$

$$= 2(n-1)![w^n]\left(\frac{w/2}{\exp(w/2)-1} - \frac{w}{\exp(w)-1}\right)$$

$$= 2(n-1)![w^n]\left(\frac{w/2\times(\exp(w/2)+1)}{\exp(w)-1} - \frac{w}{\exp(w)-1}\right)$$

$$= 2(n-1)![w^n]\frac{w/2\times(\exp(w/2)-1)}{\exp(w)-1}$$

$$= (n-1)![w^n]\frac{w}{\exp(w/2)+1}.$$

This is the claim.

Proof of auxiliary identity

This identity is from section 6.2 of *Concrete Mathematics* [GKP89]. In trying to show that

$$\left\langle {n \atop k} \right\rangle = n![w^n]\sum_{q=0}^{n}\binom{n-q}{k}(-1)^{n-q-k}(\exp(w)-1)^q.$$

we expand the powered term to get

$$n![w^n]\sum_{q=0}^{n}\binom{n-q}{k}(-1)^{n-q-k}\sum_{p=0}^{q}\binom{q}{p}(-1)^{q-p}\exp(pw)$$

$$= (-1)^{n-k}n![w^n]\sum_{p=0}^{n}(-1)^p\exp(pw)\sum_{q=p}^{n}\binom{q}{p}\binom{n-q}{k}.$$

We find for the inner sum

$$\sum_{q=0}^{n-p}\binom{q+p}{p}\binom{n-p-q}{k} = \sum_{q\ge 0}\binom{q+p}{p}\operatorname*{res}_z \frac{1}{z^{n-p-q-k+1}}\frac{1}{(1-z)^{k+1}}.$$

Here we have extended to infinity because the residue vanishes when q moves beyond $n-p$. We find

$$\operatorname*{res}_z \frac{1}{z^{n-p-k+1}}\frac{1}{(1-z)^{k+1}}\sum_{q\ge 0}\binom{q+p}{p}z^q$$

$$= \operatorname*{res}_z \frac{1}{z^{n-p-k+1}}\frac{1}{(1-z)^{p+k+2}} = \binom{n-p-k+p+k+1}{p+k+1} = \binom{n+1}{p+k+1}.$$

Returning to the outer sum we have

$$(-1)^{n-k}n![w^n]\sum_{p=0}^{n}(-1)^p\exp(pw)\binom{n+1}{p+k+1}.$$

With the indices on the binomial coefficient being positive we require $n+1 \ge p+k+1$ or $n-k\ge p$ to get

$$(-1)^{n-k} n! [w^n] \sum_{p=0}^{n-k} (-1)^p \exp(pw) \binom{n+1}{p+k+1}$$

$$= (-1)^{n-k} n! [w^n] \exp(-(k+1)w) \sum_{p=k+1}^{n+1} (-1)^{p-(k+1)} \exp(pw) \binom{n+1}{p}.$$

Taking all terms on the sum i.e. from $p = 0$ to $p = n + 1$ we get

$$(-1)^{n+1} n! [w^n] \exp(-(k+1)w)(1 - \exp(w))^{n+1} = 0$$

owing to the remaining coefficient extractor. That means for our sum that it is given by

$$(-1)^n n! [w^n] \exp(-(k+1)w) \sum_{p=0}^{k} (-1)^p \exp(pw) \binom{n+1}{p}$$

$$= n! [w^n] \exp((k+1)w) \sum_{p=0}^{k} (-1)^p \exp(-pw) \binom{n+1}{p}$$

$$= \sum_{p=0}^{k} (-1)^p (k+1-p)^n \binom{n+1}{p}.$$

We have obtained the defining sum which concludes the argument.

This was page 95 eqn. LXIX from Saalschütz on Bernoulli, [Saa93].

1.137.7 Bernoulli numbers V

We seek to show that as given with Euler numbers

$$E_{2m} = \frac{1}{2m+1} \left\{ 4m + 1 - \sum_{k=1}^{m} \binom{2m+1}{2k} B_{2k} 2^{4k} \right\}.$$

Using the fact that odd-index Bernoulli numbers vanish the parenthesized term becomes

$$\sum_{k=0}^{2m+1} \binom{2m+1}{k} B_k 2^{2k} - 1 + (2m+1) \times \frac{1}{2} \times 4$$

so that we have

$$E_{2m} = -\frac{1}{2m+1} \sum_{k=0}^{2m+1} \binom{2m+1}{k} B_k 2^{2k}.$$

We hence seek to evaluate with n even

$$-\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}B_k 2^{2k}.$$

We have by convolution of EGFs for the RHS

$$-\frac{1}{n+1}(n+1)![z^{n+1}]\frac{4z}{\exp(4z)-1}\exp(z) = -4n![z^n]\frac{\exp(z)}{\exp(4z)-1}.$$

We prepare to restrict to the even coefficients

$$-4n![z^n]\frac{\exp(-z)}{\exp(-4z)-1} = -4n![z^n]\frac{\exp(3z)}{1-\exp(4z)}.$$

Add and halve to get

$$2n![z^n]\frac{\exp(3z)-\exp(z)}{\exp(4z)-1} = 2n![z^n]\exp(z)\frac{1}{\exp(2z)+1}$$
$$= n![z^n]\frac{2}{\exp(z)+\exp(-z)} = n![z^n]\frac{1}{\cosh(z)}.$$

This is the claim.

This was page 96 eqn. LXX from Saalschütz on Bernoulli, [Saa93].

1.137.8 Bernoulli numbers and roots of unity

We seek to show that as given

$$B_{2m} = \frac{(2m)!}{2^{4m}(2^{2m}-1)}[z^{2m}]\prod_{q=0}^{2m-1}(\exp(\zeta_{2m}^q z)+\exp(-\zeta_{2m}^q z))$$
$$\text{where}\quad \zeta_{2m} = \exp(2\pi i/(2m))$$

We will prove the more general (here $n \ge 2$)

$$B_n = \frac{n!}{2^{2n}(2^n-1)}[z^n]\prod_{q=0}^{n-1}(\exp(\zeta_n^q z)+\exp(-\zeta_n^q z))$$
$$\text{where}\quad \zeta_n = \exp(2\pi i/n).$$

We introduce

$$f_n(z) = \prod_{q=0}^{n-1}(\exp(\zeta_n^q z)+\exp(-\zeta_n^q z)).$$

We have

$$\log f_n(z) = \sum_{q=0}^{n-1} \log(\exp(\zeta_n^q z) + \exp(-\zeta_n^q z))$$

$$= n \sum_{k\ge 0} z^{kn} [w^{kn}] \log(\exp(w) + \exp(-w)).$$

Exponentiating

$$f_n(z) = \sum_{p\ge 0} \frac{1}{p!} \left[n \sum_{k\ge 0} z^{kn} [w^{kn}] \log(\exp(w) + \exp(-w)) \right]^p .$$

But we are extracting the coefficient on $[z^n]$ so the only contribution comes from $k = 0, 1$.

$$[z^n] \sum_{p\ge 0} \frac{1}{p!} [n \log 2 + n z^n [w^n] \log(\exp(w) + \exp(-w))]^p$$

$$= 2^n [z^n] \exp(n z^n [w^n] \log(\exp(w) + \exp(-w)))$$

$$= 2^n [z^n] \sum_{p\ge 0} \frac{1}{p!} (n z^n [w^n] \log(\exp(w) + \exp(-w)))^p.$$

Here we may again restrict due to the coefficient extractor, this time to $p = 0, 1$. We get

$$2^n [z^n](1 + n z^n [w^n] \log(\exp(w) + \exp(-w))) = 2^n n [w^n] \log(\exp(w) + \exp(-w)).$$

Collecting everything we get as our task

$$B_n = \frac{n!}{2^n(2^n-1)} n [w^n] \log(\exp(w) + \exp(-w))$$

$$= \frac{n!}{2^n(2^n-1)} [w^{n-1}] \frac{\exp(w) - \exp(-w)}{\exp(w) + \exp(w)}.$$

We re-write this as

$$B_n 2^n(2^n-1) = n! [w^n] \frac{w(\exp(w) - \exp(-w))}{\exp(w) + \exp(-w)}$$

$$= n! [w^n] \left(w - \frac{2w \exp(-w)}{\exp(w) + \exp(-w)} \right).$$

The term w does not contribute with $n \ge 2$ and we get

$$-n! [w^n] \frac{2w}{\exp(2w) + 1}.$$

But we also have for the LHS

$$n![w^n]\left(\frac{4w}{\exp(4w)-1} - \frac{2w}{\exp(2w)-1}\right) = n![w^n]\frac{4w - 2w(\exp(2w)+1)}{\exp(4w)-1}$$

$$= n![w^n]\frac{2w(1-\exp(2w))}{\exp(2w)-1} = -n![w^n]\frac{2w}{\exp(2w)+1}.$$

This is the claim.

This was page 108 eqn. LXXXI from Saalschütz on Bernoulli, [Saa93].

1.138 Worpitzky's identity

We seek to show that

$$x^n = \sum_{k=0}^{n-1} \binom{x+k}{n} \left\langle {n \atop k} \right\rangle.$$

We write for the RHS

$$[z^n](1+z)^x \sum_{k=0}^{n-1} (1+z)^k \left\langle {n \atop k} \right\rangle.$$

Observe that as shown in section 1.137.6

$$\left\langle {n \atop k} \right\rangle = n![w^n] \sum_{q=0}^{n} \binom{n-q}{k} (-1)^{n-q-k} (\exp(w)-1)^q.$$

We thus obtain

$$n![w^n][z^n](1+z)^x \sum_{k\ge 0} (1+z)^k \sum_{q=0}^{n} \binom{n-q}{k} (-1)^{n-q-k} (\exp(w)-1)^q.$$

Here we have raised k to infinity owing to the Eulerian number being zero. Continuing,

$$n![w^n][z^n](1+z)^x \sum_{q=0}^{n} (-1)^{n-q} (\exp(w)-1)^q \sum_{k\ge 0} \binom{n-q}{k} (-1)^k (1+z)^k$$

$$= n![w^n][z^n](1+z)^x \sum_{q=0}^{n} (-1)^{n-q} (\exp(w)-1)^q (-1)^{n-q} z^{n-q}$$

$$= n![w^n][z^n](1+z)^x \sum_{q=0}^{n} (\exp(w)-1)^{n-q} z^q$$

$$= n![w^n](\exp(w)-1)^n[z^n](1+z)^x \frac{1-z^{n+1}/(\exp(w)-1)^{n+1}}{1-z/(\exp(w)-1)}.$$

The first piece is

$$n![w^n](\exp(w)-1)^n \sum_{q=0}^{n} \binom{x}{q} \frac{1}{(\exp(w)-1)^{n-q}}$$

$$= n![w^n] \sum_{q=0}^{n} \binom{x}{q} (\exp(w)-1)^q.$$

Here we may raise q to infinity owing to the coefficient extractor in w and the fact that $\exp(w) - 1 = w + \cdots$, getting

$$n![w^n]\exp(wx) = x^n.$$

We have the claim if we can show that the second piece is zero. But this is

$$n![w^n](\exp(w)-1)^n[z^n](1+z)^x \frac{z^{n+1}}{(\exp(w)-1)^{n+1}} = 0$$

owing to the coefficient extractor in z. This concludes the argument.
This is from the canonical Worpitzky paper [Wor83].

1.139 MSE 4627726: Quadruple binomial coefficient

We seek to verify as given that with $N \geq 1$ and $0 \leq n \leq N$ we have

$$2^N = \sum_{m=0}^{N} \sum_{r=0}^{n} \sum_{s=0}^{m} (-1)^{n+m} (-2)^{r+s} \binom{n}{r} \binom{m}{s} \binom{N-r}{m} \binom{N-s}{n}.$$

We can re-write the RHS as

$$\sum_{m=0}^{N} (-1)^{n+m} \sum_{r=0}^{n} (-2)^r \binom{n}{r} \binom{N-r}{m} \sum_{s=0}^{m} (-2)^s \binom{m}{s} \binom{N-s}{n}.$$

Working with the inner sum we get

$$[z^n](1+z)^N \sum_{s=0}^{m} (-2)^s \binom{m}{s} (1+z)^{-s}$$

$$= [z^n](1+z)^N(1-2/(1+z))^m = [z^n](1+z)^{N-m}(z-1)^m.$$

The middle sum yields

$$[w^m](1+w)^N \sum_{r=0}^{n} (-2)^r \binom{n}{r} (1+w)^{-r}$$

$$= [w^m](1+w)^N(1-2/(1+w))^n = [w^m](1+w)^{N-n}(w-1)^n.$$

Combining these in the outer sum we have

$$(-1)^n[z^n](1+z)^N \sum_{m=0}^{N} (-1)^m(1+z)^{-m}(z-1)^m[w^m](1+w)^{N-n}(w-1)^n$$

$$= (-1)^n[z^n](1+z)^N \sum_{m=0}^{N} (-1)^{N-m}(1+z)^{-N+m}(z-1)^{N-m}[w^{N-m}](1+w)^{N-n}(w-1)^n$$

$$= (-1)^{n+N}[z^n](z-1)^N \sum_{m=0}^{N} (-1)^m(1+z)^m(z-1)^{-m}[w^{N-m}](1+w)^{N-n}(w-1)^n$$

$$= (-1)^{n+N}[z^n](z-1)^N[w^N](1+w)^{N-n}(w-1)^n \sum_{m=0}^{N} (-1)^m(1+z)^m(z-1)^{-m}w^m.$$

We may extend the sum to infinity owing to the coefficient extractor in w :

$$(-1)^{n+N}[z^n](z-1)^N[w^N](1+w)^{N-n}(w-1)^n \frac{1}{1+(1+z)w/(z-1)}$$

$$= (-1)^{n+N}[z^n](z-1)^{N+1}[w^N](1+w)^{N-n}(w-1)^n \frac{1}{z-1+(1+z)w}$$

$$= (-1)^{n+N+1}[z^n](z-1)^{N+1}[w^N](1+w)^{N-n}(w-1)^n \frac{1}{1-z-w-wz}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{N+1}}(1+w)^{N-n}(w-1)^n \frac{1}{1-z-w-wz}.$$

Now put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2\, dv$ and $1+w = 1/(1-v)$ to get

$$\operatorname*{res}_{v} \frac{1}{v^{N+1}}(1-v)(2v-1)^n \frac{1}{1-z-v/(1-v)-vz/(1-v)} \frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_{v} \frac{1}{v^{N+1}}(2v-1)^n \frac{1}{(1-z)(1-v)-v-vz}$$

$$= \operatorname*{res}_{v} \frac{1}{v^{N+1}}(2v-1)^n \frac{1}{1-z-2v}$$

$$= (-1)^n \operatorname*{res}_{v} \frac{1}{v^{N+1}}(1-2v)^n \frac{1}{1-z-2v}$$

$$= (-1)^n \operatorname*{res}_v \frac{1}{v^{N+1}} (1-2v)^{n-1} \frac{1}{1-z/(1-2v)}.$$

Extracting the coefficient in z we get

$$(-1)^{N+1} \sum_{q=0}^{n} \binom{N+1}{q} (-1)^{N+1-q} \operatorname*{res}_v \frac{1}{v^{N+1}} (1-2v)^{n-1} \frac{1}{(1-2v)^{n-q}}$$

$$= (-1)^{N+1} \sum_{q=0}^{n} \binom{N+1}{q} (-1)^{N+1-q} \operatorname*{res}_v \frac{1}{v^{N+1}} (1-2v)^{q-1}$$

$$= (-1)^{N+1} \sum_{q=0}^{n} \binom{N+1}{q} (-1)^{N+1-q} \binom{q-1}{N} 2^N (-1)^N$$

$$= 2^N (-1)^N \sum_{q=0}^{n} \binom{N+1}{q} (-1)^q \binom{q-1}{N}$$

$$= 2^N + 2^N (-1)^N \sum_{q=1}^{n} \binom{N+1}{q} (-1)^q \binom{q-1}{N}$$

We almost have the claim. We get for the remaining sum

$$2^N (-1)^N (N+1) \sum_{q=1}^{n} \frac{1}{q} (-1)^q \binom{N}{q-1} \binom{q-1}{N}.$$

The only non-zero pair of binomial coefficients is when $q = N+1$. But $q \le n$ and $n \le N$ as per the original problem, so we get zero and may conclude. We also see that when $n > N$ the sum contributes with one term for a total of

$$2^N + 2^N (-1)^N (N+1) \frac{(-1)^{N+1}}{N+1} = 0.$$

Remark. For the case when $N = 0$ there is only one possible value for n, which is zero also. Recall the closed form

$$(-1)^{n+N+1} [z^n] (z-1)^{N+1} [w^N] (1+w)^{N-n} (w-1)^n \frac{1}{1-z-w-wz}.$$

In this case we are extracting the constant coefficients. We get starting with w

$$(-1)^{n+N+1} [z^n] (z-1)^{N+1} \frac{(-1)^n}{1-z} = (-1)^{N+1} (-1)^{N+1} = 1.$$

This means that the formula holds for $N = 0$ as well.

This was math.stackexchange.com problem 4627726.

1.140 MSE 4627918: Alternating power sum

We seek to evaluate

$$\sum_{k=0}^{m} k^{\ell} (-1)^k \binom{n}{k}.$$

This is

$$[z^m]\frac{1}{1-z}\sum_{k\ge 0} z^k k^{\ell} (-1)^k \binom{n}{k}$$
$$= [z^m]\frac{1}{1-z}\ell![w^{\ell}]\sum_{k\ge 0} z^k \exp(kw)(-1)^k \binom{n}{k}$$
$$= \ell![w^{\ell}][z^m]\frac{1}{1-z}(1-z\exp(w))^n$$
$$= \ell![w^{\ell}][z^m]\frac{1}{1-z}(1-z-z(\exp(w)-1))^n$$
$$= \ell![w^{\ell}][z^m]\frac{1}{1-z}\sum_{k=0}^{n}\binom{n}{k}(1-z)^{n-k}(-1)^k z^k (\exp(w)-1)^k$$
$$= \sum_{k=0}^{n}\binom{n}{k}\binom{n-k-1}{m-k}(-1)^m k! \left\{ {\ell \atop k} \right\}.$$

As a remark observe carefully that the coefficient extractor construction for the second binomial coefficient will produce zero when $k > m$ without any kind of singularity. Continuing we note that the problem statement says that $\ell < n$. That means we get zero from the Stirling number when $\ell < k \le n$. Hence we may set the upper limit to ℓ and get the formula (Stirling number is zero when $k = 0$):

$$\boxed{(-1)^m \sum_{k=1}^{\ell}\binom{n}{k}\binom{n-k-1}{m-k} k! \left\{ {\ell \atop k} \right\}.}$$

Actually we have for $k \le m$

$$\binom{n}{k}\binom{n-k-1}{m-k} = \frac{n}{n-k}\binom{n-1}{k}\binom{n-k-1}{m-k}$$
$$= \frac{n}{n-k}\frac{(n-1)!}{k! \times (m-k)! \times (n-m-1)!} = \frac{n}{n-k}\binom{n-1}{m}\binom{m}{k}$$

so the first boxed formula simplifies to

$$(-1)^m n \binom{n-1}{m} \sum_{k=1}^{\ell} \frac{1}{n-k} \binom{m}{k} k! \left\{ \ell \atop k \right\}.$$

Note that $\binom{m}{k} k! = m^{\underline{k}}$ so this is

$$\boxed{(-1)^m n \binom{n-1}{m} \sum_{k=1}^{\ell} \frac{1}{n-k} m^{\underline{k}} \left\{ \ell \atop k \right\}}$$

This will also correctly produce a zero contribution when $k > m$.
We get for $\ell = 3$ and $n > 3$

$$(-1)^m n \binom{n-1}{m} \left[\frac{m}{n-1} \left\{ 3 \atop 1 \right\} + \frac{m(m-1)}{n-2} \left\{ 3 \atop 2 \right\} + \frac{m(m-1)(m-2)}{n-3} \left\{ 3 \atop 3 \right\} \right].$$

This was math.stackexchange.com problem 4627918.

1.141 MSE 4227433: Squared power sum

We can evaluate the following general sum:

$$\sum_{k=1}^{n} k^{\ell} \binom{n}{k}^2 .$$

We get

$$\ell! [w^{\ell}] \sum_{k=0}^{n} \exp(kw) \binom{n}{k} \binom{n}{n-k} = \ell! [w^{\ell}][z^n](1 + z\exp(w))^n (1+z)^n$$

$$= \ell! [w^{\ell}][z^n](1 + z + z(\exp(w) - 1))^n (1+z)^n$$

$$= \ell! [w^{\ell}][z^n] \sum_{k=0}^{n} \binom{n}{k} z^k (\exp(w) - 1)^k (1+z)^{2n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n-k} k! \left\{ \ell \atop k \right\}.$$

Note that when $n > \ell$ we may lower to ℓ due to the Stirling number. On the other hand when $n < \ell$ we may raise to ℓ due to the first binomial coefficient. We get

$$\boxed{\sum_{k=1}^{\ell} \binom{2n-k}{n} n^{\underline{k}} \left\{ \ell \atop k \right\}.}$$

We find e.g. for $\ell = 3$

$$\binom{2n-1}{n} n \left\{ {3 \atop 1} \right\} + \binom{2n-2}{n} n(n-1) \left\{ {3 \atop 2} \right\} + \binom{2n-3}{n} n(n-1)(n-2) \left\{ {3 \atop 3} \right\}.$$

This was math.stackexchange.com problem 4227433.

1.142 MSE 4428892: Ordinary power sum

We seek to simplify

$$\sum_{q=1}^{n} \left\{ {k \atop q} \right\} q! \binom{n+1}{q+1}.$$

This is

$$k![z^k] \sum_{q=1}^{n} \binom{n+1}{q+1} (\exp(z)-1)^q$$

$$= k![z^k] \frac{1}{\exp(z)-1} \sum_{q=1}^{n} \binom{n+1}{q+1} (\exp(z)-1)^{q+1}$$

$$= k![z^k] \frac{1}{\exp(z)-1} \sum_{q=2}^{n+1} \binom{n+1}{q} (\exp(z)-1)^q.$$

Observe that with $k \ge 1$ this is

$$-k![z^{k+1}] \frac{z}{\exp(z)-1} + k![z^{k+1}] \frac{z \exp((n+1)z)}{\exp(z)-1}$$

$$= k![z^{k+1}] \frac{z(\exp((n+1)z)-1)}{\exp(z)-1}$$

$$= k![z^k] \sum_{q=0}^{n} \exp(qz) = \sum_{q=1}^{n} q^k.$$

Note that when we regard k as a parameter we can write the initial sum as

$$\sum_{q=1}^{k} \left\{ {k \atop q} \right\} q! \binom{n+1}{q+1}.$$

This is because if $n < k$ we may raise to k because the binomial coefficient is zero on the added range. If $n > k$ we may lower to k due to the Stirling number. We then have

$$\boxed{(n+1)n\sum_{q=1}^{k} \left\{ {k \atop q} \right\} \frac{(n-1)^{\underline{q-1}}}{q+1}.}$$

We find e.g. letting $k=3$ that

$$\sum_{q=1}^{n} q^3 = (n+1)n\left[\left\{ {3 \atop 1} \right\}\frac{1}{2} + \left\{ {3 \atop 2} \right\}\frac{n-1}{3} + \left\{ {3 \atop 3} \right\}\frac{(n-1)(n-2)}{4}\right].$$

This was math.stackexchange.com problem 4428892.

1.143 MSE 3932757: Stirling numbers and a tree-function like term

We seek to prove the following identity:

$$\sum_{k=0}^{m} \binom{m}{k} \left\{ {n+k+1 \atop k+1} \right\} k! = \sum_{k=0}^{m} \binom{m}{k} (-k)^{m-k}(k+1)^{n+k}.$$

For the RHS we introduce the polynomial

$$\sum_{k=0}^{m} \binom{m}{k} (-k)^{m-k}(x+k)^{n+k}.$$

and extract the coefficient on $[x^q]$ where $0 \le q \le n+m$ to get

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} k^{m-k} \binom{n+k}{q} k^{n+k-q}$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \binom{n+k}{q} k^{n+m-q}$$

$$= (n+m-q)![z^{n+m-q}] \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \binom{n+k}{q} \exp(kz)$$

$$= (n+m-q)![z^{n+m-q}][w^q](1+w)^n \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (1+w)^k \exp(kz).$$

Working with the inner extractor,

$$[w^q](1+w)^n((1+w)\exp(z)-1)^m$$
$$= [w^q](1+w)^n((1+w)(\exp(z)-1)+w)^m.$$

Expanding the power,

$$(n+m-q)![z^{n+m-q}][w^q](1+w)^n \sum_{k=0}^{m} \binom{m}{k} (1+w)^k (\exp(z)-1)^k w^{m-k}.$$

This is (here the second binomial coefficient is zero by construction if $q < m-k$)

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{k+q-m} k! \left\{ n+m-q \atop k \right\}.$$

We require the value at $x=1$ which means we must sum over all q from the range. We get for the component that is dependent on q:

$$\sum_{q=0}^{n+m} \binom{n+k}{n+m-q} (n+m-q)![z^{n+m-q}] \frac{1}{k!} (\exp(z)-1)^k$$

$$= \sum_{q=0}^{n+m} \binom{n+k}{q} q![z^q] \frac{1}{k!} (\exp(z)-1)^k.$$

Here we may lower to $n+k$ due to the binomial coefficient:

$$\sum_{q=0}^{n+k} \binom{n+k}{q} q![z^q] \frac{1}{k!} (\exp(z)-1)^k.$$

We have by convolution of EGFs that this is

$$(n+k)![z^{n+k}] \exp(z) \frac{1}{k!} (\exp(z)-1)^k$$

$$= (n+k+1)![z^{n+k+1}] \frac{1}{(k+1)!} (\exp(z)-1)^{k+1} = \left\{ n+k+1 \atop k+1 \right\}$$

as required. This concludes the argument.
This was math.stackexchange.com problem 3932757.

1.144 MSE 4641290: A vanishing variable

We seek to prove the following identity where $1 \le m \le n$:

$$\sum_{k=1}^{n} \frac{\prod_{1\le r\le n, r\ne m} (x+k-r)}{\prod_{1\le r\le n, r\ne k} (k-r)} = 1.$$

Observe that the denominator is

$$\frac{1}{(k-1)!(n-k)!(-1)^{n-k}}$$

so that the identity becomes

$$\sum_{k=1}^{n}\binom{n}{k}k(-1)^{n-k}\prod_{\substack{r=1\\r\neq m}}^{n}(x+k-r)=n!.$$

The LHS is

$$\sum_{k=1}^{n}\binom{n}{k}k(-1)^{n-k}(x+k-1)^{\underline{m-1}}(x+k-1-m)^{\underline{n-m}}.$$

This is

$$\sum_{k=1}^{n}\binom{n}{k}k(-1)^{n-k}\binom{x+k-1}{m-1}(m-1)!\binom{x+k-m-1}{n-m}(n-m)!.$$

Hence an alternate form is

$$n\sum_{k=1}^{n}\binom{n-1}{k-1}(-1)^{n-k}\binom{x+k-1}{m-1}\binom{x+k-m-1}{n-m}=m\binom{n}{m}.$$

We will prove it for x being an integer, equality for complex x then follows by equality of polynomials. The LHS is

$$n[z^{m-1}](1+z)^{x-1}[w^{n-m}](1+w)^{x-m-1}\sum_{k=1}^{n}\binom{n-1}{k-1}(-1)^{n-k}(1+w)^{k}(1+z)^{k}$$

$$=n[z^{m-1}](1+z)^{x}[w^{n-m}](1+w)^{x-m}\sum_{k=1}^{n}\binom{n-1}{k-1}(-1)^{n-1-(k-1)}(1+w)^{k-1}(1+z)^{k-1}.$$

Working with the sum term,

$$((1+w)(1+z)-1)^{n-1}=(w+z+wz)^{n-1}=(w+z(1+w))^{n-1},$$

restoring the extractors,

$$n[z^{m-1}](1+z)^{x}[w^{n-m}](1+w)^{x-m}\sum_{q=0}^{n-1}\binom{n-1}{q}w^{n-1-q}z^{q}(1+w)^{q}.$$

Here we may lower the upper limit to $m-1$ owing to the coefficient extractor in z :

$$n\sum_{q=0}^{m-1}\binom{n-1}{q}\binom{x-m+q}{q+1-m}\binom{x}{m-1-q}.$$

Note very carefully that the middle binomial coefficient is zero when $q+1 < m$. This is because the coefficient extractor in w yields zero when $n-1-q > n-m$ or $m-1 > q$. This means that the only non-zero contribution to the sum originates with $q = m-1$ and we get

$$n\binom{n-1}{m-1} \times 1 \times 1 = m\binom{n}{m}$$

as claimed.

This was math.stackexchange.com problem 4641290.

1.145 MSE 4644963: From trigonometric to rational

We seek to show that

$$S_{N,m} = \sum_{k=1}^{N} (-1)^k (\cos\frac{k\pi}{N})^{N-m} (\sin\frac{k\pi}{N})^m = \frac{1+(-1)^m}{2} (-1)^{m/2} \frac{N}{2^{N-1}}.$$

Observe that

$$\sum_{k=N+1}^{2N} (-1)^k (\cos\frac{k\pi}{N})^{N-m} (\sin\frac{k\pi}{N})^m$$

$$= \sum_{k=1}^{N} (-1)^{N+k} (-1)^{N-m} (\cos\frac{k\pi}{N})^{N-m} (-1)^m (\sin\frac{k\pi}{N})^m = S_{N,m}$$

so that

$$S_{N,m} = \frac{1}{2} \sum_{k=1}^{2N} (-1)^k (\cos\frac{k\pi}{N})^{N-m} (\sin\frac{k\pi}{N})^m.$$

With $\rho_k = \exp(ki\pi/N)$ the roots of $z^{2N} - 1 = 0$ this becomes

$$\frac{1}{2} \sum_{k=1}^{2N} (-1)^k \frac{1}{2^{N-m}} (\rho_k + 1/\rho_k)^{N-m} \frac{1}{2^m i^m} (\rho_k - 1/\rho_k)^m.$$

Now introduce

$$f(z) = \frac{1}{2^{N+1} i^m} z^N (z+1/z)^{N-m} (z-1/z)^m \frac{2N/z}{z^{2N}-1}.$$

We then have by inspection that the sum is given by the residues due to the rational term at $\rho_k = \exp(ki\pi/N)$ with $1 \le k \le 2N$. Here we use that

$$\lim_{z\to\rho_k} \frac{z-\rho_k}{z^{2N}-1} = \frac{1}{2N\rho_k^{2N-1}}.$$

Note that should $z+1/z$ or $z-1/z$ be zero the corresponding trigonometric sum term would have been zero as well. In that case the simple pole from the rational term is canceled, making for a zero contribution and everything is in order. Continuing,

$$f(z) = \frac{N}{2^N i^m}\frac{1}{z}(z^2+1)^{N-m}(z^2-1)^m\frac{1}{z^{2N}-1}.$$

With residues adding to zero we must compute minus the residue at zero and minus the residue at infinity. We get for the former (including the switched sign)

$$\frac{N}{2^N i^m}(-1)^m.$$

and the latter (the residue at infinity is $-\mathrm{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)$)

$$\frac{N}{2^N i^m}\mathrm{Res}_{z=0}\frac{1}{z^2}z(1/z^2+1)^{N-m}(1/z^2-1)^m\frac{1}{1/z^{2N}-1}$$
$$= \frac{N}{2^N i^m}\mathrm{Res}_{z=0}\frac{1}{z}\frac{(z^2+1)^{N-m}}{z^{2N-2m}}\frac{(1-z^2)^m}{z^{2m}}\frac{z^{2N}}{1-z^{2N}}$$
$$= \frac{N}{2^N i^m}\mathrm{Res}_{z=0}\frac{1}{z}(z^2+1)^{N-m}(1-z^2)^m\frac{1}{1-z^{2N}} = \frac{N}{2^N i^m}.$$

Therefore we have

$$S_{N,m} = \frac{N}{2^N i^m}(-1)^m + \frac{N}{2^N i^m} = \frac{1+(-1)^m}{2}\frac{1}{i^m}\frac{N}{2^{N-1}}.$$

This is zero when m is odd as claimed. When m is even the term $1/i^m$ simplfies and we have at last

$$\frac{1+(-1)^m}{2}(-1)^{m/2}\frac{N}{2^{N-1}}$$

as desired.

This was math.stackexchange.com problem 4644963.

1.146 MSE 4657112: Triple combinatorial numbers to constant

We seek to show that with $0 \le q < n$

$$1 = \sum_{p=0}^{n}(-1)^p\binom{n+q}{n-p-1}\binom{n+p}{n-q-1}\binom{p+q}{p}.$$

We see that $p=n$ does not really contribute owing to the first binomial coefficient. We write

$$[z^{n-1}](1+z)^{n+q}\sum_{p\ge 0}(-1)^p z^p \binom{n+p}{n-q-1}\binom{p+q}{p}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$[z^{n-1}](1+z)^{n+q}[w^{n-1-q}](1+w)^n \sum_{p\ge 0}(-1)^p z^p (1+w)^p \binom{p+q}{p}$$

$$= [z^{n-1}](1+z)^{n+q}[w^{n-1-q}](1+w)^n \frac{1}{(1+z(1+w))^{q+1}}$$

$$= [z^{n-1}](1+z)^{n-1}[w^{n-1-q}](1+w)^n \frac{1}{(1+zw/(1+z))^{q+1}}$$

$$= [z^{n-1}](1+z)^{n-1}[w^{n-1-q}](1+w)^n \sum_{p=0}^{n-1}\binom{p+q}{p}(-1)^p w^p \frac{z^p}{(1+z)^p}.$$

The upper limit on this sum is due to the factor z^p and the coefficient extractor in z. Note that $\binom{n-1-p}{n-1-p}=1$ so this becomes

$$[w^{n-1-q}](1+w)^n \sum_{p=0}^{n-1}\binom{p+q}{p}(-1)^p w^p.$$

At this point we are now permitted to raise to infinity again, this time due to the coefficient extractor in w:

$$[w^{n-1-q}](1+w)^n \frac{1}{(1+w)^{q+1}} = [w^{n-1-q}](1+w)^{n-1-q} = 1.$$

This is the claim. Note that we have used the condition $0 \le q < n$ in the coefficient extractor on w.

A companion identity

We also have with $0 \le q < n$

$$1 = (-1)^q \sum_{p=0}^{n}(-1)^p \binom{n+q}{n-p-1}\binom{n+p}{n-q-1}\left[{p+q \atop p}\right].$$

We again see that $p=n$ does not really contribute owing to the first binomial coefficient. We write

$$(-1)^q[z^{n-1}](1+z)^{n+q}\sum_{p\ge 0}(-1)^p z^p \binom{n+p}{n-q-1}\left[{p+q \atop p}\right].$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$(-1)^q [z^{n-1}](1+z)^{n+q} [w^{n-1-q}](1+w)^n \sum_{p\ge 0} (-1)^p z^p (1+w)^p \left[\begin{matrix} p+q \\ p \end{matrix}\right].$$

We first evaluate this for $q = 0$:

$$[z^{n-1}](1+z)^n [w^{n-1}](1+w)^n \sum_{p\ge 0} (-1)^p z^p (1+w)^p$$
$$= [z^{n-1}](1+z)^n [w^{n-1}](1+w)^n \frac{1}{1+z(1+w)}.$$

But this is precisely the case $q = 0$ of the hypergeometric from the previous section, and hence equal to one. Continung with $q \ge 1$ we have

$$(-1)^q [z^{n-1}](1+z)^{n+q} [w^{n-1-q}](1+w)^n \sum_{p\ge 1} (-1)^p z^p (1+w)^p \left[\begin{matrix} p+q \\ p \end{matrix}\right]$$

$$= (-1)^{q+1} [z^{n-2}](1+z)^{n+q} [w^{n-1-q}](1+w)^{n+1} \sum_{p\ge 0} (-1)^p z^p (1+w)^p \left[\begin{matrix} p+q+1 \\ p+1 \end{matrix}\right].$$

Recall from **??** that

$$Q_r(z) = \sum_{n\ge 0} \left[\begin{matrix} n+r+1 \\ n+1 \end{matrix}\right] z^n = \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle \begin{matrix} r \\ r-k \end{matrix} \right\rangle\!\!\right\rangle z^k.$$

We thus have

$$(-1)^{q+1} [z^{n-2}](1+z)^{n+q} [w^{n-1-q}](1+w)^{n+1} \frac{1}{(1+z(1+w))^{2q+1}}$$
$$\times \sum_{k=0}^{q} \left\langle\!\!\left\langle \begin{matrix} q \\ q-k \end{matrix} \right\rangle\!\!\right\rangle (-1)^k z^k (1+w)^k.$$

Working with the term in front of the evaluated Eulerian polynomial we obtain

$$(-1)^{q+1} [z^{n-2}](1+z)^{n-1-q} [w^{n-1-q}](1+w)^{n+1} \frac{1}{(1+wz/(1+z))^{2q+1}}$$

or alternatively

$$(-1)^{q+1}[z^{n-2}](1+z)^{n-1-q}[w^{n-1-q}](1+w)^{n+1}\sum_{p=0}^{n-1-q}\binom{p+2q}{p}(-1)^p\frac{w^p z^p}{(1+z)^p}.$$

The upper limit was due to the coefficient extractor in w. Continuing with entire expression,

$$(-1)^{q+1}[z^{n-2}](1+z)^{n-1-q}\sum_{p=0}^{n-1-q}\binom{p+2q}{p}(-1)^p\frac{z^p}{(1+z)^p}$$
$$\times\sum_{k=0}^{q}\left\langle\!\!\left\langle {q \atop q-k} \right\rangle\!\!\right\rangle(-1)^k z^k\binom{n+1+k}{n-1-q-p}.$$

For the remaining coefficient extractor we require

$$n-2-p-k \le n-1-q-p$$

because $(1+z)^{n-1-q-p}$ is a polynomial. This simplifies to $q-1 \le k$.

Hence only the values for $k=q$ and $k=q-1$ contribute. But the Eulerian number is zero for the former and one for the latter. This yields

$$(-1)^{q+1}[z^{n-2}](1+z)^{n-1-q}\sum_{p=0}^{n-1-q}\binom{p+2q}{p}(-1)^p\frac{z^p}{(1+z)^p}(-1)^{q-1}z^{q-1}\binom{n+q}{n-1-q-p}.$$

The binomial coefficient from the extractor is $\binom{n-1-q-p}{n-2-(q-1)-p}=1$ which leaves us with

$$\sum_{p=0}^{n-1-q}\binom{p+2q}{p}(-1)^p\binom{n+q}{n-1-q-p}$$
$$=[w^{n-1-q}](1+w)^{n+q}\sum_{p\ge 0}\binom{p+2q}{p}(-1)^p w^p = [w^{n-1-q}](1+w)^{n+q}\frac{1}{(1+w)^{2q+1}}$$
$$=[w^{n-1-q}](1+w)^{n-1-q}=1.$$

This is the claim.

This was math.stackexchange.com problem 4657112.

1.147 Computer search

1.147.1 OEIS A106800

We seek to prove with $n \ge 1$ and $0 \le m \le n$

$$\left\{ {n \atop n-m} \right\} = (-1)^{n-m} \sum_{k=0}^n (-1)^k \binom{m-1+k}{m-n+k} \binom{2n-m}{n-m-k} \left\{ {n \atop k} \right\}.$$

Here we define the first binomial coefficient to be $\binom{m-1+k}{n-1}$ so as not to have a negative lower index. With the second binomial coefficient we will use a coefficient extractor that produces zero for negative lower indices. We find

$$(-1)^{n-m} \sum_{k=0}^n (-1)^k \binom{m-1+k}{n-1} \binom{2n-m}{n-m-k} \left\{ {n \atop k} \right\}$$

$$= (-1)^{n-m} [z^{n-1}](1+z)^{m-1} [w^{n-m}](1+w)^{2n-m} \sum_{k=0}^n (-1)^k (1+z)^k w^k \left\{ {n \atop k} \right\}$$

$$= (-1)^{n-m} [z^{n-1}](1+z)^{m-1} [w^{n-m}](1+w)^{2n-m}$$

$$\times n![v^n] \sum_{k=0}^n (-1)^k (1+z)^k w^k \frac{(\exp(v)-1)^k}{k!}.$$

Now with v enforcing the upper limit of the sum due to $\exp(v) - 1 = v + \cdots$ this becomes (recall $m \le n$)

$$(-1)^{n-m} [z^{n-1}](1+z)^{m-1} [w^{n-m}](1+w)^{2n-m} n![v^n] \exp(-(1+z)w(\exp(v)-1))$$

$$= (-1)^{n-m} [w^{n-m}](1+w)^{2n-m} n![v^n] \exp(-w(\exp(v)-1))$$

$$\times \sum_{p=0}^{m-1} \binom{m-1}{p} \frac{(-1)^{n-1-p}}{(n-1-p)!} w^{n-1-p} (\exp(v)-1)^{n-1-p}.$$

With the functional terms in w all being FPS, we must have $n-1-p \le n-m$ or $m-1 \le p$ which means just one term contributes which is $p = m-1$, producing

$$(-1)^{n-m} [w^{n-m}](1+w)^{2n-m}$$

$$\times n![v^n] \exp(-w(\exp(v)-1)) \frac{(-1)^{n-m}}{(n-m)!} w^{n-m} (\exp(v)-1)^{n-m}$$

$$= (-1)^{n-m} [w^0](1+w)^{2n-m}$$

$$\times n![v^n] \exp(-w(\exp(v)-1)) \frac{(-1)^{n-m}}{(n-m)!} (\exp(v)-1)^{n-m}$$

$$= n![v^n] \frac{(\exp(v)-1)^{n-m}}{(n-m)!} = \left\{ {n \atop n-m} \right\}.$$

This is the claim.

This identity was found by a computer search which pointed to [OEIS A106800 (Stirling numbers of the second kind)}(https://oeis.org/A106800).

1.147.2 OEIS A079901

We seek to prove with $n \ge 1$

$$n^m = \sum_{k=0}^{n} (-1)^k \left[n \atop n-k \right] \left\{ n-k+m \atop n \right\}.$$

We get from basic EGFs that the sum is

$$\sum_{k=0}^{n} (-1)^k \left[n \atop n-k \right] (n-k+m)! [w^{n-k+m}] \frac{(\exp(w)-1)^n}{n!}$$
$$= \sum_{k=0}^{n} (-1)^k \frac{1}{n!} \left[n \atop n-k \right] (n-k+m)! [w^{n-k+m}] \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} \exp(qw)$$
$$= \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} \sum_{k=0}^{n} (-1)^k \frac{1}{n!} \left[n \atop n-k \right] q^{n-k+m}$$
$$= \sum_{q=0}^{n} \binom{n}{q} (-1)^{q} \sum_{k=0}^{n} (-1)^k \frac{1}{n!} \left[n \atop k \right] q^{k+m}$$
$$= \frac{1}{n!} \sum_{q=0}^{n} \binom{n}{q} (-1)^{q} q^m \sum_{k=0}^{n} (-1)^k \left[n \atop k \right] q^{k}$$
$$= \frac{1}{n!} \sum_{q=0}^{n} \binom{n}{q} (-1)^{q} q^m \prod_{p=0}^{n-1} (-q+p).$$

Now observe that for all $-q$ there is a matching p to get a zero factor in the product except for $q = n$. We thus have

$$\frac{1}{n!} \binom{n}{n} (-1)^n n^m \prod_{p=0}^{n-1} (-n+p) = n^m$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A079901, triangle of powers.

1.147.3 OEIS A104684

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$

$$\binom{n}{m} \binom{2n-m}{n} = (-1)^{n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n+k}{n}^2 \binom{2n-m}{n-m-k}.$$

We have for the RHS

$$(-1)^{n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n+k}{n} \binom{n+k}{n} \binom{2n-m}{n-m-k}$$

$$= (-1)^{n-m} [z^n](1+z)^n \sum_{k=0}^{n-m} (-1)^k \binom{n+k}{n} (1+z)^k \binom{2n-m}{n-m-k}$$

$$= (-1)^{n-m} [z^n](1+z)^n [w^{n-m}](1+w)^{2n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n+k}{n} (1+z)^k w^k.$$

Here we can certainly extend the sum to infinity owing to the coefficient extractor in w and we obtain

$$(-1)^{n-m} [z^n](1+z)^n [w^{n-m}](1+w)^{2n-m} \frac{1}{(1+(1+z)w)^{n+1}}$$

$$= (-1)^{n-m} [z^n](1+z)^n [w^{n-m}](1+w)^{2n-m} \frac{1}{(1+w+wz)^{n+1}}$$

$$= (-1)^{n-m} [z^n](1+z)^n [w^{2n-m+1}](1+w)^{2n-m} \frac{1}{(z+(1+w)/w)^{n+1}}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^n \frac{1}{(z+(1+w)/w)^{n+1}}.$$

Fortunately here the residue at infinity is zero, so we can evaluate by computing minus the residue at $z = -(1+w)/w$ since residues sum to zero. This requires the Leibniz rule:

$$\frac{1}{n!} \left(\frac{1}{z^{n+1}} (1+z)^n \right)^{(n)} = \frac{1}{n!} \sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q (n+1)^{\overline{q}}}{z^{n+1+q}} n^{\underline{n-q}} (1+z)^q$$

$$= \frac{1}{n!} \sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q}{z^{n+1+q}} q! \binom{n+q}{q} (n-q)! \binom{n}{n-q} (1+z)^q$$

$$= \sum_{q=0}^{n} \binom{n}{q} \binom{n+q}{q} \frac{(-1)^q}{z^{n+1+q}} (1+z)^q.$$

Evaluate at the pole and flip the sign:

$$-(-1)^{n+1} \sum_{q=0}^{n} \binom{n}{q} \binom{n+q}{q} \frac{w^{n+1+q}}{(1+w)^{n+1+q}} \frac{(-1)^q}{w^q}.$$

Substitute into the extractor for w:

$$(-1)^m \sum_{q=0}^{n} \binom{n}{q} \binom{n+q}{q} (-1)^q [w^{n-m}] (1+w)^{n-1-m-q}$$
$$= (-1)^m \sum_{q=0}^{n} \binom{n}{q} \binom{n+q}{q} (-1)^q \binom{n-1-m-q}{n-m}.$$

With upper negation this becomes

$$(-1)^n \sum_{q=0}^{n} \binom{n}{q} \binom{n+q}{q} (-1)^q \binom{q}{n-m}.$$

Note that with $q \ge n-m$

$$\binom{n}{q} \binom{q}{n-m} = \frac{n!}{(n-q)! \times (n-m)! \times (q+m-n)!}$$
$$= \binom{n}{n-m} \binom{m}{n-q} = \binom{n}{m} \binom{m}{n-q}.$$

This will correctly produce zero when $q < n-m$. Returning to the sum we now have

$$(-1)^n \binom{n}{m} \sum_{q=0}^{n} \binom{n+q}{q} (-1)^q \binom{m}{n-q}.$$

This at last becomes

$$(-1)^n \binom{n}{m} [z^n] (1+z)^m \sum_{q \ge 0} \binom{n+q}{q} (-1)^q z^q.$$

Here we have again extended to infinity owing to the coefficient extractor. Continuing,

$$(-1)^n \binom{n}{m} [z^n] (1+z)^m \frac{1}{(1+z)^{n+1}} = -(-1)^n \binom{n}{m} [z^n] \frac{1}{(1+z)^{n-m+1}}$$
$$= (-1)^n \binom{n}{m} \binom{n+n-m}{n} (-1)^n = \binom{n}{m} \binom{2n-m}{n}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A104684, lattice paths.

1.147.4 OEIS A003056

We seek to prove with $n \ge 1$ and $n \ge m \ge 1$

$$n = (-1)^{n+1} \sum_{k=0}^{n} (-1)^k \binom{m+k}{m} \binom{m+n}{n-k-1} \binom{m-1+k}{m-n+k}.$$

Here the third binomial coefficient could go negative on the lower index so we re-write as

$$(-1)^{n+1} \sum_{k=0}^{n} (-1)^k \binom{m+k}{m} \binom{m+n}{n-k-1} \binom{m-1+k}{n-1}.$$

We then have

$$(-1)^{n+1} [z^{n-1}] (1+z)^{m+n} \sum_{k\ge 0} (-1)^k \binom{m+k}{m} z^k \binom{m-1+k}{n-1}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$(-1)^{n+1} [z^{n-1}] (1+z)^{m+n} [w^{n-1}] (1+w)^{m-1} \sum_{k\ge 0} (-1)^k \binom{m+k}{m} z^k (1+w)^k$$

$$= (-1)^{n+1} [z^{n-1}] (1+z)^{m+n} [w^{n-1}] (1+w)^{m-1} \frac{1}{(1+z(1+w))^{m+1}}$$

$$= (-1)^{n+1} [z^{n+m}] (1+z)^{m+n} [w^{n-1}] (1+w)^{m-1} \frac{1}{(w+(1+z)/z)^{m+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^n} (1+w)^{m-1} \frac{1}{(w+(1+z)/z)^{m+1}}.$$

Fortunately here the residue at infinity is zero so using the fact that residues sum to zero we can evaluate with minus the residue at $w = -(1+z)/z$. This requires the Leibniz rule:

$$\frac{1}{m!} \left(\frac{1}{w^n} (1+w)^{m-1} \right)^{(m)} = \frac{1}{m!} \sum_{q=0}^{m} \binom{m}{q} \frac{(-1)^q n^{\overline{q}}}{w^{n+q}} (m-1)^{\underline{m-q}} (1+w)^{q-1}$$

$$= \frac{1}{m!} \sum_{q=0}^{m} \binom{m}{q} \frac{(-1)^q}{w^{n+q}} q! \binom{n+q-1}{q} (m-q)! \binom{m-1}{m-q} (1+w)^{q-1}$$

$$= \sum_{q=0}^{m} \frac{(-1)^q}{w^{n+q}} \binom{n+q-1}{q} \binom{m-1}{m-q} (1+w)^{q-1}.$$

Substitute the value of the pole to get

$$\sum_{q=0}^{m} \frac{(-1)^n z^{n+q}}{(1+z)^{n+q}} \binom{n+q-1}{q} \binom{m-1}{m-q} \frac{(-1)^{q-1}}{z^{q-1}}.$$

Restore the coefficient extractor in z (flip sign):

$$-[z^{n+m}](1+z)^{m+n} \sum_{q=0}^{m} \frac{z^{n+q}}{(1+z)^{n+q}} \binom{n+q-1}{q} \binom{m-1}{m-q} \frac{(-1)^{q}}{z^{q-1}}.$$

Now from z we get $[z^{m-1}](1+z)^{m-q} = \binom{m-q}{m-1}$ so that we have

$$-\sum_{q=0}^{m} (-1)^q \binom{n+q-1}{q} \binom{m-1}{m-q} \binom{m-q}{m-1}.$$

The only way for the product of the two mutually flipped binomial coefficients to be non-zero is to have $m-1 = m-q$ or $q = 1$. This yields at last

$$-(-1)^1 \binom{n}{1} = n$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A003056, inverse of triangular numbers.

1.147.5 OEIS A155865

We seek to prove with $n \ge 1$ and $n > m \ge 1$

$$\binom{n-2}{m-1} = \frac{(-1)^{m-1}}{n-1} \sum_{k=0}^{n} (-1)^k \binom{n+k}{n-m-1} \binom{n+1}{k+1} \binom{2k}{m-1}.$$

The sum is

$$\sum_{k=1}^{n+1} (-1)^{k-1} \binom{n-1+k}{n-m-1} \binom{n+1}{k} \binom{2k-2}{m-1}$$

$$= -[z^{n-m-1}](1+z)^{n-1}[w^{m-1}](1+w)^{-2} \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^k (1+z)^k (1+w)^{2k}.$$

Merging in the value for $k = 0$ we have

$$\binom{n-1}{m}(-1)^{m-1}m - [z^{n-m-1}](1+z)^{n-1}[w^{m-1}](1+w)^{-2}(1-(1+z)(1+w^2))^{n+1}.$$

Working with the coefficient extractor including the sign we have

$$(-1)^n [z^{n-m-1}](1+z)^{n-1}[w^{m-1}](1+w)^{-2}(z+w^2+zw^2)^{n+1}$$

$$= (-1)^n [z^{n-m-1}](1+z)^{n-1}[w^{m-1}](1+w)^{-2} \sum_{q=0}^{n+1} \binom{n+1}{q} (1+z)^q w^{2q} z^{n+1-q}.$$

Now from the coefficient extractors we must have $0 \le n-m-1-(n+1-q) \le n-1+q$ (extracting from a polynomial in z) and $2q \le m-1$. The first of these works out to the pair $m+2 \le q$ and $-m-(n+1) \le 0$ which holds trivially. Putting these together we have $m+2 \le q \le (m-1)/2$ which is the empty set given that $m \ge 1$. There is no contribution from any q and the sum vanishes.

We are left with

$$\binom{n-1}{m}(-1)^{m-1} m = (n-1)\binom{n-2}{m-1}(-1)^{m-1}$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A155865, Leibniz harmonic triangle.

1.147.6 OEIS A033276

We seek to prove with $n \ge 1$ and $n \ge m \ge 0$

$$\binom{n+m}{m-1}\binom{n-1}{m-1}\frac{1}{m} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{k}{m} \frac{1}{k+1}\binom{2k}{k}\binom{n+k}{n-k}.$$

For the sum start by re-writing the Catalan number:

$$\frac{1}{k+1}\binom{2k}{k}\binom{n+k}{n-k} = \frac{1}{k+1}\frac{(n+k)!}{k! \times k! \times (n-k)!}$$

$$= \frac{1}{k+1}\binom{n}{k}\binom{n+k}{k} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+k}{k}.$$

We get for the sum

$$\frac{(-1)^n}{n+1} \sum_{k=0}^{n} (-1)^k \binom{k}{m}\binom{n+1}{k+1}\binom{n+k}{k}$$

$$= -\frac{(-1)^n}{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{k-1}{m}\binom{n+1}{k}\binom{n+k-1}{n}.$$

Here we may include $k=0$ because the third binomial coefficient is zero there when $n \ge 1$. We obtain

$$-\frac{(-1)^n}{n+1}[w^m]\frac{1}{1+w}[z^n](1+z)^{n-1}\sum_{k=0}^{n+1}(-1)^k(1+w)^k(1+z)^k\binom{n+1}{k}$$

$$= -\frac{(-1)^n}{n+1}[w^m]\frac{1}{1+w}[z^n](1+z)^{n-1}(1-(1+w)(1+z))^{n+1}$$

$$= \frac{1}{n+1}[w^m]\frac{1}{1+w}[z^n](1+z)^{n-1}(w+z+wz)^{n+1}$$

$$= \frac{1}{n+1}[w^m]\frac{1}{1+w}[z^n](1+z)^{n-1}\sum_{q=0}^{n+1}\binom{n+1}{q}z^q(1+w)^q w^{n+1-q}$$

$$= \frac{1}{n+1}\sum_{q=0}^{n}\binom{n+1}{q}\binom{n-1}{n-q}\binom{q-1}{m+q-n-1}.$$

Note here that the middle binomial coefficient is zero when $q=0$ so we may skip this:

$$\sum_{q=1}^{n}\frac{1}{q}\binom{n}{q-1}\binom{n-1}{n-q}\binom{q-1}{m+q-n-1}.$$

Now we have

$$\binom{n}{q-1}\binom{q-1}{m+q-n-1} = \frac{n!}{(n+1-q)!\times(m+q-n-1)!\times(n-m)!}$$

$$= \binom{n}{n-m}\binom{m}{n+1-q} = \binom{n}{m}\binom{m}{n+1-q}.$$

We get for our sum

$$\binom{n}{m}\sum_{q=1}^{n}\frac{1}{q}\binom{n-1}{n-q}\binom{m}{n+1-q} = \frac{1}{n}\binom{n}{m}\sum_{q=0}^{n}\binom{n}{q}\binom{m}{n+1-q}.$$

Here we have lowered to include $q=0$ due to the range of m, which is non-negative and at most n. We finally have

$$\frac{1}{n}\binom{n}{m}[z^{n+1}](1+z)^m\sum_{q=0}^{n}\binom{n}{q}z^q$$

$$= \frac{1}{n}\binom{n}{m}[z^{n+1}](1+z)^{m+n} = \frac{1}{n}\binom{n}{m}\binom{m+n}{n+1}.$$

To conclude and match the data from the OEIS we re-write the binomial coefficients and get

$$\binom{n-1}{m-1}\frac{1}{m}\binom{m+n}{m-1}$$

which is the claim.

This identity was found by a computer search which pointed to [OEIS A033276/7, diagonal dissections}(https://oeis.org/A033276).

1.147.7 OEIS A051162

We seek to prove with $n \ge 1$ and $n \ge m \ge 0$

$$n+m = \sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k-1}\binom{m-1+k}{k}\binom{n+m+k}{n}.$$

We have for the sum using basic relations that it is

$$[z^{n-1}](1+z)^{n+m}[w^n](1+w)^{n+m}\sum_{k\ge 0}\binom{m-1+k}{k} z^k(1+w)^k.$$

Here we have raised to infinity because the coefficient extractor in z enforces $k \le n-1$. We find

$$[z^{n-1}](1+z)^{n+m}[w^n](1+w)^{n+m}\frac{1}{(1+z+zw)^m}$$
$$= [z^{n+m-1}](1+z)^{n+m}[w^n](1+w)^{n+m}\frac{1}{(w+(1+z)/z)^m}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}}(1+w)^{n+m}\frac{1}{(w+(1+z)/z)^m}.$$

Observe carefully that here the residue at infinity does not vanish. Hence we evaluate using minus the residues at $w=-(1+z)/z$ and at infinity. We get for the former using the Leibniz rule

$$\frac{1}{(m-1)!}\left(\frac{1}{w^{n+1}}(1+w)^{n+m}\right)^{(m-1)}$$
$$= \frac{1}{(m-1)!}\sum_{q=0}^{m-1}\binom{m-1}{q}\frac{(-1)^q (n+1)^{\overline{q}}}{w^{n+1+q}}(n+m)^{\underline{m-1-q}}(1+w)^{n+1+q}$$
$$= \frac{1}{(m-1)!}\sum_{q=0}^{m-1}\binom{m-1}{q} q!\binom{n+q}{q}\frac{(-1)^q}{w^{n+1+q}}(m-1-q)!\binom{n+m}{m-1-q}(1+w)^{n+1+q}$$

$$= \sum_{q=0}^{m-1} \binom{n+q}{q} \frac{(-1)^q}{w^{n+1+q}} \binom{n+m}{m-1-q} (1+w)^{n+1+q}.$$

Substitute $w = -(1+z)/z$ and flip the sign to get

$$\sum_{q=0}^{m-1} \binom{n+q}{q} \frac{(-1)^n z^{n+1+q}}{(1+z)^{n+1+q}} \binom{n+m}{m-1-q} (-1)^{n+1+q} \frac{1}{z^{n+1+q}}.$$

Apply the coefficient extractor in z to obtain

$$\sum_{q=0}^{m-1} \binom{n+q}{q} (-1)^{q+1} \binom{n+m}{m-1-q} [z^{n+m-1}](1+z)^{m-1-q} = 0.$$

We have for minus the residue at infinity that it is

$$\begin{aligned}
&\operatorname*{res}_w \frac{1}{w^2} w^{n+1} (1+1/w)^{n+m} \frac{1}{(1/w+(1+z)/z)^m} \\
&= \operatorname*{res}_w \frac{1}{w^2} w^{n+1} (1+w)^{n+m} \frac{1}{w^{n+m}} \frac{w^m}{(1+w(1+z)/z)^m} \\
&= \operatorname*{res}_w \frac{1}{w} (1+w)^{n+m} \frac{1}{(1+w(1+z)/z)^m} = 1.
\end{aligned}$$

The coefficient extractor in z now yields

$$[z^{n+m-1}](1+z)^{n+m} = \binom{n+m}{n+m-1} = n+m$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A051162, a triangular array defined by the relation $T(n,k) = n+k$.

1.147.8 OEIS A122899

We seek to prove with $n \geq 0$ and $n \geq m \geq 0$

$$\binom{n}{m} \binom{m+1}{n-m} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{m} \binom{k+1}{m+1}.$$

We get for the sum from basic principles that it is

$$\begin{aligned}
&(-1)^n [z^m][w^{m+1}](1+w) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+z)^k (1+w)^k \\
&= (-1)^n [z^m][w^{m+1}](1+w)(1-(1+z)(1+w))^n \\
&= [z^m][w^{m+1}](1+w)(z+w+wz)^n
\end{aligned}$$

$$= [z^m][w^{m+1}](1+w)\sum_{q=0}^{n}\binom{n}{q}z^q(1+w)^q w^{n-q}$$

$$= [w^{m+1}](1+w)\binom{n}{m}(1+w)^m w^{n-m} = \binom{n}{m}\binom{m+1}{2m+1-n}.$$

By construction the second binomial coefficient is zero when $n-m > m+1$ or $n > 2m+1$. We therefore set the non-zero range to $m \le (n-1)/2$. With that range we may simplify to

$$\binom{n}{m}\binom{m+1}{n-m}$$

which is the claim. We also get the correct zero values when m is out of range.

This identity was found by a computer search which pointed to OEIS A122899, a triangular array counting directed animals.

1.147.9 OEIS A110555

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$

$$\binom{n-1}{m} = (-1)^m \sum_{k=0}^{n}(-1)^k\binom{m+k}{m}\binom{2n}{n+k}\binom{n-1+k}{k}.$$

We get for the sum that it is

$$(-1)^m \sum_{k=0}^{n}(-1)^k\binom{m+k}{m}\binom{2n}{n-k}\binom{n-1+k}{n-1}$$

$$= (-1)^m [z^n](1+z)^{2n}[w^{n-1}](1+w)^{n-1}\sum_{k\ge 0}\binom{m+k}{m}(-1)^k z^k(1+w)^k.$$

Here we have extended to infinity because the extractor in z enforces the upper range. Continuing,

$$(-1)^m [z^n](1+z)^{2n}[w^{n-1}](1+w)^{n-1}\frac{1}{(1+z+zw)^{m+1}}$$

$$= (-1)^m [z^{n+m+1}](1+z)^{2n}[w^{n-1}](1+w)^{n-1}\frac{1}{(w+(1+z)/z)^{m+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^n}(1+w)^{n-1}\frac{1}{(w+(1+z)/z)^{m+1}}.$$

Fortunately we see that the residue at infinity is zero here so we may evaluate using minus the residue at $w = -(1+z)/z$. We use the Leibniz rule:

$$\frac{1}{m!}\left(\frac{1}{w^n}(1+w)^{n-1}\right)^{(m)} = \frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\frac{(-1)^q n^{\overline{q}}}{w^{n+q}}(n-1)^{\underline{m-q}}(1+w)^{n-1-m+q}$$

$$= \frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\frac{(-1)^q}{w^{n+q}}q!\binom{n+q-1}{q}(m-q)!\binom{n-1}{m-q}(1+w)^{n-1-m+q}$$

$$= \sum_{q=0}^{m}\frac{(-1)^q}{w^{n+q}}\binom{n+q-1}{q}\binom{n-1}{m-q}(1+w)^{n-1-m+q}.$$

Substitute the location of the pole and flip the sign to get

$$\sum_{q=0}^{m}\frac{(-1)^{n-1}z^{n+q}}{(1+z)^{n+q}}\binom{n+q-1}{q}\binom{n-1}{m-q}(-1)^{n-1-m+q}\frac{1}{z^{n-1-m+q}}.$$

Restore the extractor in z

$$\sum_{q=0}^{m}\binom{n+q-1}{q}\binom{n-1}{m-q}(-1)^q[z^{n+m+1}](1+z)^{2n}z^{m+1}\frac{1}{(1+z)^{n+q}}$$

$$= \sum_{q=0}^{m}\binom{n+q-1}{q}\binom{n-1}{m-q}(-1)^q\binom{n-q}{n}.$$

Due to the third binomial coefficient we have that the only non-zero contribution originates with $q=0$ and we get

$$1\times\binom{n-1}{m}\times(-1)^0\binom{n}{n} = \binom{n-1}{m}$$

as claimed.

This identity was found by a computer search which pointed to OEIS A110555, partial sums of alternating binomial coefficients.

1.147.10 OEIS A141662

We seek to prove with $n \ge 2$ and $n \ge m \ge 1$

$$n - m^2 = (-1)^m\sum_{k=0}^{n}(-1)^k\binom{n+m}{n-k-1}\binom{m-1+k}{k}\binom{m-1+k}{m}.$$

We get for the sum

$$(-1)^m [z^{n-1}](1+z)^{n+m} [w^m](1+w)^{m-1} \sum_{k\ge 0} \binom{m-1+k}{k} z^k (1+w)^k.$$

Here we have extended to infinity due to the coefficient extractor in z. Continuing,

$$(-1)^m [z^{n-1}](1+z)^{n+m} [w^m](1+w)^{m-1} \frac{1}{(1+z+zw)^m}$$
$$= (-1)^m [z^{n+m-1}](1+z)^{n+m} [w^m](1+w)^{m-1} \frac{1}{(w+(1+z)/z)^m}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{m+1}}(1+w)^{m-1} \frac{1}{(w+(1+z)/z)^m}.$$

Fortunately here the residue at infinity is zero, so by residues adding to zero we may evaluate using minus the residue at $w = -(1+z)/z$. This requires the Leibniz rule:

$$\frac{1}{(m-1)!} \left(\frac{1}{w^{m+1}} (1+w)^{m-1} \right)^{(m-1)}$$
$$= \frac{1}{(m-1)!} \sum_{q=0}^{m-1} \binom{m-1}{q} \frac{(-1)^q (m+1)^{\overline{q}}}{w^{m+1+q}} (m-1)^{\underline{m-1-q}} (1+w)^q$$
$$= \frac{1}{(m-1)!} \sum_{q=0}^{m-1} \binom{m-1}{q} \frac{(-1)^q}{w^{m+1+q}} q! \binom{m+q}{q} (m-1-q)! \binom{m-1}{m-1-q} (1+w)^q$$
$$= \sum_{q=0}^{m-1} \frac{(-1)^q}{w^{m+1+q}} \binom{m+q}{q} \binom{m-1}{q} (1+w)^q.$$

Substitute the value of the pole and flip the sign to get

$$\sum_{q=0}^{m-1} \frac{(-1)^m z^{m+1+q}}{(1+z)^{m+1+q}} \binom{m+q}{q} \binom{m-1}{q} (-1)^q \frac{1}{z^q}.$$

Restore the extractor in z including the scalar:

$$\sum_{q=0}^{m-1} \binom{m+q}{q} \binom{m-1}{q} (-1)^q [z^{n-2}](1+z)^{n-1-q}$$
$$= \sum_{q=0}^{m-1} \binom{m+q}{q} \binom{m-1}{q} (-1)^q \binom{n-1-q}{n-2}.$$

Note that the upper and lower index of the third binomial coefficient are non-negative since $n \ge m$ and $m \ge 1$. That means only $q = 0$ and $q = 1$ actually contribute. We get

$$\binom{m}{0}\binom{m-1}{0}(-1)^0 \times (n-1) + \binom{m+1}{1}\binom{m-1}{1}(-1)^1 \times 1$$
$$= n - 1 - (m+1)(m-1) = n - m^2$$

and we have the claim.

This identity was found by a computer search which pointed to OEIS A141662, absolute value of $n - m^2$.

1.147.11 OEIS A046816

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ and a variable p where $m \ge p$

$$\binom{n}{m}\binom{m}{p} = (-1)^m \sum_{k=0}^{n} (-1)^k \binom{m}{k}\binom{k}{p}\binom{k-p+n}{k}.$$

First note that

$$\binom{m}{k}\binom{k}{p} = \frac{m!}{(m-k)! \times p! \times (k-p)!} = \binom{m}{p}\binom{m-p}{k-p}$$

and our sum becomes

$$(-1)^m \binom{m}{p} \sum_{k=p}^{n} \binom{m-p}{k-p}\binom{k-p+n}{k}$$
$$= (-1)^m \binom{m}{p} \sum_{k=p}^{n} \binom{m-p}{m-k}\binom{k-p+n}{n-p}$$
$$= (-1)^m \binom{m}{p} [z^m](1+z)^{m-p}[w^{n-p}](1+w)^{n-p} \sum_{k\ge 0} (-1)^k z^k (1+w)^k.$$

Here we have extended to infinity due to the coefficient extractor in z. Continuing,

$$(-1)^m \binom{m}{p} [z^m](1+z)^{m-p}[w^{n-p}](1+w)^{n-p} \frac{1}{1+z+zw}$$
$$= (-1)^m \binom{m}{p} [z^{m+1}](1+z)^{m-p}[w^{n-p}](1+w)^{n-p} \frac{1}{w+(1+z)/z}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n-p+1}}(1+w)^{n-p} \frac{1}{w+(1+z)/z}.$$

Fortunately here the residue at infinity is zero (just barely) so we may evaluate using minus the residue at $w = -(1+z)/z$ to get

$$-(-1)^m \binom{m}{p} [z^{m+1}](1+z)^{m-p}(-1)^{n-p+1} \frac{z^{n-p+1}}{(1+z)^{n-p+1}} (-1)^{n-p} \frac{1}{z^{n-p}}$$

$$= (-1)^m \binom{m}{p} [z^m] \frac{1}{(1+z)^{n-m+1}}$$

$$= (-1)^m \binom{m}{p} (-1)^m \binom{n}{n-m} = \binom{n}{m}\binom{m}{p}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A046816, Pascal's tetrahedron.

1.147.12 OEIS A001700

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ and a variable p where $p \le n-m$ with $p \ge 1$

$$\binom{2p-1}{p-1} = (-1)^p \sum_{k=0}^{n-m} (-1)^k \binom{2n-m}{n-m-k} \binom{n-1+k}{k} \binom{k-p+n}{p}.$$

We have for the sum

$$(-1)^p [z^{n-m}](1+z)^{2n-m} [w^p](1+w)^{n-p} \sum_{k\ge 0} (-1)^k \binom{n-1+k}{k} z^k (1+w)^k.$$

Here we have extended to infinity due to the extractor in z. Continuing,

$$(-1)^p [z^{n-m}](1+z)^{2n-m} [w^p](1+w)^{n-p} \frac{1}{(1+z+zw)^n}$$

$$= (-1)^p [z^{2n-m}](1+z)^{2n-m} [w^p](1+w)^{n-p} \frac{1}{(w+(1+z)/z)^n}.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{p+1}} (1+w)^{n-p} \frac{1}{(w+(1+z)/z)^n}.$$

Here the residue at infinity is zero under the stated preconditions and we may evaluate at minus the residue at $w = -(1+z)/z$ as residues sum to zero. This requires the Leibniz rule:

$$\frac{1}{(n-1)!}\left(\frac{1}{w^{p+1}}(1+w)^{n-p}\right)^{(n-1)}$$

$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(-1)^q(p+1)^{\overline{q}}}{w^{p+1+q}}(n-p)^{\underline{n-1-q}}(1+w)^{q+1-p}$$

$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}q!\binom{p+q}{q}\frac{(-1)^q}{w^{p+1+q}}(n-1-q)!\binom{n-p}{n-1-q}(1+w)^{q+1-p}$$

$$= \sum_{q=0}^{n-1}\binom{p+q}{q}\frac{(-1)^q}{w^{p+1+q}}\binom{n-p}{n-1-q}(1+w)^{q+1-p}.$$

Substitute the location of the pole and flip the sign,

$$\sum_{q=0}^{n-1}\binom{p+q}{q}\frac{(-1)^p z^{p+1+q}}{(1+z)^{p+1+q}}\binom{n-p}{n-1-q}(-1)^{q+1-p}\frac{1}{z^{q+1-p}}.$$

Restore the outer term,

$$\sum_{q=0}^{n-1}\binom{p+q}{q}[z^{2n-m}](1+z)^{2n-m}\frac{z^{p+1+q}}{(1+z)^{p+1+q}}\binom{n-p}{n-1-q}(-1)^{q+1-p}\frac{1}{z^{q+1-p}}$$

$$= \sum_{q=0}^{n-1}\binom{p+q}{q}\binom{n-p}{n-1-q}(-1)^{q+1-p}[z^{2n-m-2p}](1+z)^{2n-m-p-1-q}.$$

Note that by the preconditions $2n-m-2p \ge 0$ so all is in order. Next,

$$\sum_{q=0}^{n-1}\binom{p+q}{q}\binom{n-p}{n-1-q}(-1)^{q+1-p}\binom{2n-m-p-1-q}{2n-m-2p}.$$

Note that with $2n-m-2p$ not being negative we must have $2n-m-p-1-q \ge 2n-m-2p$ or $p \ge 1+q$. Also with $n-1-q$ not being negative we require from the second binomial coefficient that $n-p \ge n-1-q$ or $q+1 \ge p$. The only way to fullfill these two is that $q = p-1$. This will produce

$$\binom{p+p-1}{p-1}\binom{n-p}{n-p}(-1)^0\binom{2n-m-2p}{2n-m-2p} = \binom{2p-1}{p-1}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A001700, C(2n+1,n+1).

1.147.13 OEIS A007318

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ and a variable p such that $m \le n-p$ or $p \le n-m$

$$\binom{n-1}{p} = (-1)^p \sum_{k=0}^{n-m} (-1)^k \binom{p+k}{k} \binom{2n-m}{n-m-k} \binom{n-1+k}{k}.$$

We have for the sum

$$(-1)^p [z^{n-m}](1+z)^{2n-m} [w^{n-1}](1+w)^{n-1} \sum_{k\ge 0} (-1)^k \binom{p+k}{k} z^k (1+w)^k.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$(-1)^p [z^{n-m}](1+z)^{2n-m} [w^{n-1}](1+w)^{n-1} \frac{1}{(1+z+zw)^{p+1}}$$
$$= (-1)^p [z^{n-m+p+1}](1+z)^{2n-m} [w^{n-1}](1+w)^{n-1} \frac{1}{(w+(1+z)/z)^{p+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^n}(1+w)^{n-1} \frac{1}{(w+(1+z)/z)^{p+1}}.$$

Here fortunately the residue at infinity is zero (note the case $n=0$) so we may evaluate using minus the residue at $w = -(1+z)/z$ which requires the Leibniz rule:

$$\frac{1}{p!}\left(\frac{1}{w^n}(1+w)^{n-1}\right)^{(p)} = \frac{1}{p!}\sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^q n^{\overline{q}}}{w^{n+q}} (n-1)^{\underline{p-q}} (1+w)^{n+q-1-p}$$

$$= \frac{1}{p!}\sum_{q=0}^{p} \binom{p}{q} q! \binom{n+q-1}{q} \frac{(-1)^q}{w^{n+q}} (p-q)! \binom{n-1}{p-q} (1+w)^{n+q-1-p}$$

$$= \sum_{q=0}^{p} \binom{n+q-1}{q} \frac{(-1)^q}{w^{n+q}} \binom{n-1}{p-q} (1+w)^{n+q-1-p}.$$

Substitute the valie of the pole and flip the sign:

$$\sum_{q=0}^{p} \binom{n+q-1}{q} \frac{(-1)^{n+1} z^{n+q}}{(1+z)^{n+q}} \binom{n-1}{p-q} (-1)^{n+q-1-p} \frac{1}{z^{n+q-1-p}}.$$

Restore the extractor in z:

$$(-1)^p \sum_{q=0}^{p} \binom{n+q-1}{q} [z^{n-m+p+1}](1+z)^{2n-m} \frac{z^{n+q}}{(1+z)^{n+q}} \binom{n-1}{p-q} (-1)^{q-p} \frac{1}{z^{n+q-1-p}}$$

$$= \sum_{q=0}^{p} \binom{n+q-1}{q} [z^{n-m}](1+z)^{n-m-q} \binom{n-1}{p-q} (-1)^q$$

$$= \sum_{q=0}^{p} \binom{n+q-1}{q} \binom{n-m-q}{n-m} \binom{n-1}{p-q} (-1)^q.$$

Now with $n-m$ not being negative we have that only $q=0$ contributes to the middle binomial coefficient, yielding

$$\binom{n-1}{0} \binom{n-m}{n-m} \binom{n-1}{p} (-1)^0 = \binom{n-1}{p}.$$

This is the claim. Observe carefully that here we have used the precondition $p \le n-m$ which prevents the upper index of the middle binomial coefficient from going negative, producing a non-zero value for $q > n-m$.

We can use this to construct identities for fixed values of p beyond $n-m$. For example with $p = n-m+1$ we get an additional term which is

$$\binom{n-1+n-m+1}{n-1} \binom{-1}{n-m} \binom{n-1}{0} (-1)^{n-m+1} = -\binom{2n-m}{n-1}$$

and we have shown that

$$\binom{n-1}{m-2} - \binom{2n-m}{n-1}$$

$$= (-1)^{n-m+1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m+1+k}{k} \binom{2n-m}{n-m-k} \binom{n-1+k}{k}.$$

This identity was found by a computer search which pointed to OEIS A007318, Pascal's triangle.

1.147.14 OEIS A144484

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$

$$\binom{3n+1-m}{2n+1} = (-1)^{n-m} \sum_{k=0}^{n-m} (-1)^k \binom{n-m+k}{k} \binom{2n-k}{n-m-k} \binom{2n+1}{k}.$$

We have for the sum that it is

$$(-1)^{n-m}[z^{n-m}](1+z)^{n-m}[w^{n-m}](1+w)^{2n}\sum_{k\geq 0}\binom{2n+1}{k}(-1)^k(1+z)^k w^k(1+w)^{-k}.$$

Here we have extended to infinity due to the coefficient extractor in w. Continuing,

$$(-1)^{n-m}[z^{n-m}](1+z)^{n-m}[w^{n-m}](1+w)^{2n}\left[1-\frac{(1+z)w}{1+w}\right]^{2n+1}$$
$$=(-1)^{n-m}[z^{n-m}](1+z)^{n-m}[w^{n-m}]\frac{1}{1+w}[1-wz]^{2n+1}$$
$$=(-1)^{n-m}[z^{n-m}](1+z)^{n-m}\sum_{q=0}^{n-m}\binom{2n+1}{q}(-1)^q z^q(-1)^{n-m-q}$$
$$=[z^{n-m}](1+z)^{n-m}\sum_{q\geq 0}\binom{2n+1}{q}z^q.$$

Here we have once more raised to infinity owing to the remaining coefficient extractor. Concluding,

$$[z^{n-m}](1+z)^{n-m}(1+z)^{2n+1}=[z^{n-m}](1+z)^{3n+1-m}=\binom{3n+1-m}{2n+1}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A144484, binomial coefficient $\binom{3n+1-k}{n-k}$.

1.147.15 OEIS A130595

We seek to prove with $n\geq 0$ and $n\geq m\geq 0$

$$\binom{n}{m}=(-1)^{n+m}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n-m+k}{k}\binom{m-k+n}{m}.$$

The sum is

$$(-1)^{n+m}[z^{n-m}](1+z)^{n-m}[w^m](1+w)^{n+m}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+z)^k(1+w)^{-k}$$
$$=(-1)^{n+m}[z^{n-m}](1+z)^{n-m}[w^m](1+w)^{n+m}\left[1-\frac{1+z}{1+w}\right]^n$$

$$= (-1)^{n+m}[z^{n-m}](1+z)^{n-m}[w^m](1+w)^m\,[w-z]^n$$

$$= (-1)^{n+m}[z^{n-m}](1+z)^{n-m}[w^m](1+w)^m\sum_{q=0}^{n}\binom{n}{q}w^q(-1)^{n-q}z^{n-q}$$

$$= (-1)^{n+m}[z^{n-m}](1+z)^{n-m}\sum_{q=0}^{m}\binom{n}{q}\binom{m}{m-q}(-1)^{n-q}z^{n-q}.$$

Now the only q that contributes here is $q=m$ and we get

$$(-1)^{n+m}[z^{n-m}](1+z)^{n-m}\binom{n}{m}\binom{m}{0}(-1)^{n-m}z^{n-m} = \binom{n}{m}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A130595, inverse Pascal's triangle.

1.147.16 OEIS A011973

We seek to prove with $n \ge 1$ and $n \ge m \ge 1$

$$\binom{n-m-1}{m-1} = \sum_{k=0}^{n}(-1)^k\binom{n-1+k}{n-m+k}\binom{2m}{m+k}\binom{m-1+k}{k}.$$

We start by re-writing the sum as follows:

$$\sum_{k=0}^{m}(-1)^k\binom{n-1+k}{m-1}\binom{2m}{m-k}\binom{m-1+k}{k}$$

$$= [z^{m-1}](1+z)^{n-1}[w^m](1+w)^{2m}\sum_{k\ge 0}\binom{m-1+k}{k}(-1)^k(1+z)^k w^k.$$

Here we have extended to infinity due to the extractor in w. Continuing,

$$[z^{m-1}](1+z)^{n-1}[w^m](1+w)^{2m}\frac{1}{(1+w+wz)^m}$$

$$= [z^{m-1}](1+z)^{n-m-1}[w^m](1+w)^{2m}\frac{1}{(w+1/(1+z))^m}.$$

The contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^{m+1}}(1+w)^{2m}\frac{1}{(w+1/(1+z))^m}.$$

Here we see that the residue at infinity is not zero, so we compute using minus the residues at $w=-1/(1+z)$ and infinity. We get for the latter,

$$\operatorname*{res}_{w}\frac{1}{w^2}w^{m+1}(1+1/w)^{2m}\frac{1}{(1/w+1/(1+z))^m}$$

$$= \operatorname*{res}_{w} \frac{1}{w^2} w^{m+1}(1+w)^{2m} \frac{1}{w^{2m}} \frac{w^m}{(1+w/(1+z))^m}$$

$$= \operatorname*{res}_{w} \frac{1}{w}(1+w)^{2m} \frac{1}{(1+w/(1+z))^m} = 1.$$

Substitute this into the extractor for z to obtain

$$[z^{m-1}](1+z)^{n-m-1} = \binom{n-m-1}{m-1}.$$

This is the claim. Now we just have to show that the contribution from the finite pole is zero. This requires the Leibniz rule:

$$\frac{1}{(m-1)!} \left(\frac{1}{w^{m+1}} (1+w)^{2m} \right)^{(m-1)}$$

$$= \frac{1}{(m-1)!} \sum_{q=0}^{m-1} \binom{m-1}{q} \frac{(-1)^q (m+1)^{\overline{q}}}{w^{m+1+q}} (2m)^{\underline{m-1-q}} (1+w)^{m+1+q}$$

$$= \frac{1}{(m-1)!} \sum_{q=0}^{m-1} \binom{m-1}{q} \binom{m+q}{q} q! \frac{(-1)^q}{w^{m+1+q}} \binom{2m}{m-1-q} (m-1-q)! (1+w)^{m+1+q}$$

$$= \sum_{q=0}^{m-1} \binom{m+q}{q} \frac{(-1)^q}{w^{m+1+q}} \binom{2m}{m+1+q} (1+w)^{m+1+q}.$$

Evaluating $(1+w)/w$ at $w = -1/(1+z)$ gives $-z$ so we have from the extractor in z the result (flip sign)

$$-\sum_{q=0}^{m-1} \binom{m+q}{q} (-1)^q \binom{2m}{m+1+q} (-1)^{m+1+q} [z^{m-1}](1+z)^{n-m-1} z^{m+1+q}$$

$$= (-1)^m \sum_{q=0}^{m-1} \binom{m+q}{q} \binom{2m}{m+1+q} [z^0](1+z)^{n-m-1} z^{2+q} = 0.$$

The coefficient extractor has produced a zero value as required.

This identity was found by a computer search which pointed to OEIS A011973, coefficients of Fibonacci polynomials.

1.147.17 OEIS A253909

We seek to prove with $n \ge 1$

$$n^2 = \sum_{k=0}^{n} (-1)^k \binom{k-n}{n}^2 \binom{2n+1}{k}.$$

We have for the sum by upper negation,

$$\sum_{k=0}^{n} (-1)^k \binom{2n-k-1}{n}^2 \binom{2n+1}{k}$$

which is

$$\sum_{k=0}^{n} (-1)^k \binom{2n-k-1}{n-1-k}^2 \binom{2n+1}{k}$$

This is

$$[z^{n-1}](1+z)^{2n-1}[w^{n-1}](1+w)^{2n-1} \sum_{k\ge 0} (-1)^k \binom{2n+1}{k} \frac{z^k}{(1+z)^k} \frac{w^k}{(1+w)^k}.$$

Here we have extended to infinity owing to the two coefficient extractors. Continuing,

$$\begin{aligned}
&[z^{n-1}](1+z)^{2n-1}[w^{n-1}](1+w)^{2n-1} \left[1 - \frac{wz}{(1+w)(1+z)}\right]^{2n+1} \\
&= [z^{n-1}]\frac{1}{(1+z)^2}[w^{n-1}]\frac{1}{(1+w)^2}(1+w+z)^{2n+1} \\
&= [z^{n-1}]\frac{1}{(1+z)^2}[w^{n-1}]\frac{1}{(1+w)^2} \sum_{q=0}^{2n+1} \binom{2n+1}{q} (1+w)^{2n+1-q} z^q \\
&= \sum_{q=0}^{n-1} \binom{2n+1}{q} \binom{2n-1-q}{n-1} (-1)^{n-1-q} \binom{n-q}{1} \\
&= n \sum_{q=0}^{n-1} \binom{2n+1}{q} \binom{2n-1-q}{n} (-1)^{n-1-q} \\
&= n \sum_{q=0}^{n-1} \binom{2n+1}{q} \binom{2n-1-q}{n-1-q} (-1)^{n-1-q}.
\end{aligned}$$

This is

$$(-1)^{n-1} n [z^{n-1}](1+z)^{2n-1} \sum_{q\ge 0} \binom{2n+1}{q} \frac{z^q}{(1+z)^q} (-1)^q.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$(-1)^{n-1} n [z^{n-1}](1+z)^{2n-1} \left[1 - \frac{z}{1+z}\right]^{2n+1} = (-1)^{n-1} n [z^{n-1}] \frac{1}{(1+z)^2}$$

$$= (-1)^{n-1} n \binom{n-1+1}{1} (-1)^{n-1} = n^2.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A253909, positive squares.

1.147.18 OEIS A142150

We seek to prove with $n \ge 0$

$$(-1)^{(n+1)/2} \frac{n+1}{2} \times \frac{1+(-1)^{n+1}}{2} = \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} \binom{2k-n}{n} \binom{2k}{n-1}.$$

We have from the first binomial coefficient that with n non-negative and $0 \le k \le n$ that we must have $n-k \ge k$ or $n \ge 2k$. The third binomial coefficient requires $2k \ge n-1$. Together we have $n \ge 2k \ge n-1$. Now if n is even this implies $n = 2k$ and the middle binomial coefficient produces a zero value. If n is odd we get $2k = n-1$ and the coefficients yield

$$(-1)^k \binom{2k+1-k}{k} \binom{2k-2k-1}{2k+1} \binom{2k}{2k} = (-1)^k (k+1)(-1)^{2k+1}$$
$$= (-1)^{k+1}(k+1).$$

Now we have $k = (n-1)/2$ and so $k+1 = (n+1)/2$ and we have the claim.

This identity was found by a computer search which pointed to OEIS A142150, integers interleaved with zeroes.

1.147.19 OEIS A001147

We seek to prove with $n \ge 0$

$$\frac{(2n)!}{2^n n!} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \left[{n+k \atop k} \right].$$

We get for the sum

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{k} \left[{2n-k \atop n-k} \right]$$
$$= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (2n-k)! [z^{2n-k}] \frac{1}{(n-k)!} \left(\log \frac{1}{1-z} \right)^{n-k}$$
$$= \frac{(2n)!}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} [z^{2n-k}] \left(\log \frac{1}{1-z} \right)^{n-k}$$

$$= \frac{(2n)!}{n!}[z^{2n}] \left(\log\frac{1}{1-z}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k \left(\log\frac{1}{1-z}\right)^{-k}$$

$$= \frac{(2n)!}{n!}[z^{2n}] \left(\log\frac{1}{1-z}\right)^n \left[1 - z\left(\log\frac{1}{1-z}\right)^{-1}\right]^n$$

$$= \frac{(2n)!}{n!}[z^{2n}] \left[\log\frac{1}{1-z} - z\right]^n.$$

Note that $\log\frac{1}{1-z} - z = \frac{1}{2}z^2 + \cdots$ so only the first term of the power contributes to $[z^{2n}]$ with a value of $1/2^n$ and we have

$$\frac{(2n)!}{2^n n!}$$

as claimed.

This identity was found by a computer search which pointed to OEIS A001147, double factorial of odd numbers.

1.147.20 OEIS A243594

We claim that wth $n \ge 0$ and $n \ge m \ge 0$ and p and r real numbers

$$n^m \binom{n}{m} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{p+k}{n-m} \binom{kn+r}{m}.$$

We get for the sum

$$(-1)^n [z^{n-m}](1+z)^p \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+z)^k \binom{kn+r}{m}$$

$$= (-1)^n [z^{n-m}](1+z)^p [w^m](1+w)^r \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+z)^k (1+w)^{kn}$$

$$= [z^{n-m}](1+z)^p [w^m](1+w)^r \left[(1+z)(1+w)^n - 1\right]^n$$

$$= [z^{n-m}](1+z)^p [w^m](1+w)^r \left[z(1+w)^n + (1+w)^n - 1\right]^n$$

$$= [w^m](1+w)^r \sum_{q=0}^{n-m} \binom{p}{n-m-q} \binom{n}{q} (1+w)^{qn} ((1+w)^n - 1)^{n-q}.$$

Now since $(1+w)^n - 1 = nw + \cdots$ we must have per the coefficient extractor in w that $n-q \le m$ or $n-m \le q$. Hence only $q = n-m$ can possibly contribute and we get

$$[w^m](1+w)^r \binom{p}{0} \binom{n}{m} (1+w)^{n^2-nm} ((1+w)^n - 1)^m$$

$$= [w^m]\binom{n}{m}(1+w)^{n^2-nm+r}(w^m n^m + \cdots) = n^m\binom{n}{m}.$$

We have the claim.

This identity was found by a computer search which pointed to OEIS A013612, expansion of coefficients of $(1+5x)^n$.

1.147.21 OEIS A013609

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{n}{m}2^m = (-1)^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{2k}{m}\binom{n-m+k}{k}.$$

We get for the sum

$$(-1)^n [w^m][z^{n-m}](1+z)^{n-m} \sum_{k=0}^{n} \binom{n}{k}(-1)^k (1+w)^{2k}(1+z)^k$$
$$= [w^m][z^{n-m}](1+z)^{n-m}((1+w)^2(1+z)-1)^n$$
$$= [w^m][z^{n-m}](1+z)^{n-m}(w(2+w)(1+z)+z)^n$$
$$= [w^m][z^{n-m}](1+z)^{n-m} \sum_{q=0}^{n} \binom{n}{q} w^q (2+w)^q (1+z)^q z^{n-q}.$$

Now from the coefficient extractor in w we get $q \le m$ while the coefficient extractor in z yields $n-q \le n-m$ or $m \le q$. The only q to fit these is $q=m$ and we get

$$[w^m][z^{n-m}](1+z)^{n-m}\binom{n}{m} w^m (2+w)^m (1+z)^m z^{n-m}$$
$$= [w^m][z^0](1+z)^{n-m}\binom{n}{m} w^m (2+w)^m (1+z)^m$$
$$= [w^0]\binom{n}{m}(2+w)^m = \binom{n}{m}2^m$$

as claimed. Compare also 1.165.18.

This identity was found by a computer search which pointed to OEIS A013609, coefficients of $(1+2x)^n$.

1.147.22 OEIS A181543

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{n}{m}^3 = \sum_{k=0}^{n} \binom{n}{k}\binom{k}{m}\binom{m}{k-m}\binom{n-m+k}{n}.$$

First observe that

$$\binom{n}{k}\binom{k}{m} = \frac{n!}{(n-k)! \times m! \times (k-m)!} = \binom{n}{m}\binom{n-m}{n-k}$$

so that we seek to prove

$$\binom{n}{m}^2 = \sum_{k=0}^{n} \binom{n-m}{n-k}\binom{m}{k-m}\binom{n-m+k}{n}.$$

The sum is

$$\sum_{k=0}^{n} \binom{n-m}{k}\binom{m}{n-k-m}\binom{2n-m-k}{n}$$

$$= [z^{n-m}](1+z)^m [w^n](1+w)^{2n-m} \sum_{k=0}^{n} \binom{n-m}{k} \frac{z^k}{(1+w)^k}$$

$$= [z^{n-m}](1+z)^m [w^n](1+w)^{2n-m} \left[1 + \frac{z}{1+w}\right]^{n-m}$$

$$= [z^{n-m}](1+z)^m [w^n](1+w)^n (1+w+z)^{n-m}$$

$$= [z^{n-m}](1+z)^m [w^n](1+w)^n \sum_{q=0}^{n-m} \binom{n-m}{q} (1+z)^q w^{n-m-q}$$

$$= \sum_{q=0}^{n-m} \binom{n-m}{q}\binom{m+q}{n-m}\binom{n}{m+q}.$$

Next observe that

$$\binom{m+q}{n-m}\binom{n}{m+q} = \frac{n!}{(n-m)! \times (2m+q-n)! \times (n-m-q)!}$$

$$= \binom{n}{m}\binom{m}{n-m-q}.$$

It therefore remains to prove that

$$\binom{n}{m} = \sum_{q=0}^{n-m} \binom{n-m}{q}\binom{m}{n-m-q}.$$

The sum is

$$[z^{n-m}](1+z)^m \sum_{q=0}^{n-m} \binom{n-m}{q} z^q = [z^{n-m}](1+z)^m (1+z)^{n-m}$$

$$= [z^{n-m}](1+z)^n = \binom{n}{n-m} = \binom{n}{m}$$

and we have the claim.

This identity was found by a computer search which pointed to OEIS A181543, cubed binomial coefficients.

1.147.23 OEIS A008279

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{n}{m} m! = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{n-m+k}{n-m} k^m.$$

We get for the sum

$$(-1)^n [z^{n-m}](1+z)^{n-m} m! [w^m] \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+z)^k \exp(kw)$$
$$= (-1)^n m! [z^{n-m}](1+z)^{n-m} [w^m] (1-(1+z)\exp(w))^n$$
$$= m! [z^{n-m}](1+z)^{n-m} [w^m] (z\exp(w) + \exp(w) - 1)^n$$
$$= m! [z^{n-m}](1+z)^{n-m} [w^m] \sum_{q=0}^{n} \binom{n}{q} z^q \exp(qw)(\exp(w)-1)^{n-q}.$$

Now the coefficient extractor in z requires $q \le n-m$ and we have for the term related to the Stirling number EGF that we need $n-q \le m$ or $n-m \le q$. Hence only $q = n-m$ can possibly contribute. We find

$$m! [z^{n-m}](1+z)^{n-m} [w^m] \binom{n}{n-m} z^{n-m} \exp((n-m)w)(\exp(w)-1)^m$$

$$= m! [z^0](1+z)^{n-m} [w^m] \binom{n}{m} \exp((n-m)w)(\exp(w)-1)^m$$

$$= \binom{n}{m} m! [w^m] \exp((n-m)w) \left(w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots\right)^m = \binom{n}{m} m!.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A008279, number of permutations.

1.147.24 OEIS A100100

We seek to prove with $n \ge 1$ and $n \ge m \ge 0$ that

$$\binom{2n-m-1}{n-1} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \binom{m-1+k}{k-n+m} \binom{n-1+k}{n}.$$

We have for the sum

$$(-1)^n [z^{n-1}](1+z)^{m-1} [w^n](1+w)^{n-1} \sum_{k=0}^{n} (-1)^k \binom{2n}{n-k} (1+z)^k (1+w)^k.$$

Here we may extend to infinity owing to the coefficient extractor for the remaining binomial coefficient:

$$(-1)^n [z^{n-1}](1+z)^{m-1} [w^n](1+w)^{n-1} [v^n](1+v)^{2n} \sum_{k\ge 0} (-1)^k v^k (1+z)^k (1+w)^k$$

$$= (-1)^n [z^{n-1}](1+z)^{m-1} [w^n](1+w)^{n-1} [v^n](1+v)^{2n} \frac{1}{1+v(1+z)(1+w)}$$

$$= (-1)^n [z^{n-1}](1+z)^{m-2} [w^n](1+w)^{n-1} [v^{n+1}](1+v)^{2n} \frac{1}{1/v/(1+z)+1+w}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} (1+w)^{n-1} \frac{1}{1/v/(1+z)+1+w}.$$

Here the residue at infinity is zero and we may evaluate using minus the residue at $w = -1 - 1/v/(1+z)$.

We get

$$-\frac{1}{(-1-1/v/(1+z))^{n+1}} \frac{(-1)^{n-1}}{v^{n-1}(1+z)^{n-1}} = -\frac{v^2(1+z)^2}{(1+v(1+z))^{n+1}}.$$

Substitute into the remaining extractors to obtain

$$-(-1)^n [z^{n-1}](1+z)^{m-2} [v^{n+1}](1+v)^{2n} \frac{v^2(1+z)^2}{(1+v+vz)^{n+1}}$$

$$= (-1)^{n+1} [z^{n-1}](1+z)^m [v^{n-1}](1+v)^{2n} \frac{1}{(1+v+vz)^{n+1}}$$

$$= (-1)^{n+1} [z^{n-1}](1+z)^m [v^{2n}](1+v)^{2n} \frac{1}{((1+v)/v+z)^{n+1}}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^n}(1+z)^m \frac{1}{(z+(1+v)/v)^{n+1}}.$$

Now with $n \ge 1$ and $n \ge m \ge 0$ the residue at infinity is zero and we may evaluate with minus the residue at $z = -(1+v)/v$. This requires the Leibniz rule:

$$\frac{1}{n!}\left(\frac{1}{z^n}(1+z)^m\right)^{(n)} = \frac{1}{n!}\sum_{q=0}^{n}\binom{n}{q}\frac{(-1)^q n^{\overline{q}}}{z^{n+q}} m^{\underline{n-q}}(1+z)^{q+m-n}$$

$$= \frac{1}{n!}\sum_{q=0}^{n}\binom{n}{q} q!\binom{n+q-1}{q}\frac{(-1)^q}{z^{n+q}}(n-q)!\binom{m}{n-q}(1+z)^{q+m-n}$$

$$= \sum_{q=0}^{n}\binom{n+q-1}{q}\frac{(-1)^q}{z^{n+q}}\binom{m}{n-q}(1+z)^{q+m-n}.$$

Evaluate at $z = -(1+v)/v$ so that $1+z = -1/v$ and flip the sign to get

$$\sum_{q=0}^{n}\binom{n+q-1}{q}\frac{(-1)^{n+1}v^{n+q}}{(1+v)^{n+q}}\binom{m}{n-q}\frac{(-1)^{q+m-n}}{v^{q+m-n}}.$$

Restore the remaining coefficient extractor in v:

$$(-1)^{n+1}[v^{2n}](1+v)^{2n}\sum_{q=0}^{n}\binom{n+q-1}{q}\frac{v^{2n-m}}{(1+v)^{n+q}}\binom{m}{n-q}(-1)^{q+m+1}$$

$$= (-1)^{n+1}\sum_{q=0}^{n}\binom{n+q-1}{q}\binom{n-q}{m}\binom{m}{n-q}(-1)^{q+m+1}.$$

Now with $n-q$ and m not being negative for the right two binomial coefficients to be non-zero at the same time requires $n-q=m$ or $q=n-m$, yielding

$$(-1)^{n+1}\binom{2n-m-1}{n-m}(-1)^{n+1} = \binom{2n-m-1}{n-1}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A100100, binomial $\binom{2n-m-1}{n-1}$.

1.147.25 OEIS A076756

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{2n-m}{m} = (-1)^{n+m} \sum_{k=0}^{n} \binom{2n+1}{k} (-1)^k \binom{2m-k}{m} \binom{2n-k}{n}.$$

We have for the sum

$$(-1)^{n+m} \sum_{k=0}^{n} \binom{2n+1}{k} (-1)^k \binom{2m-k}{m} \binom{2n-k}{n-k}$$
$$= (-1)^{n+m} [z^n] (1+z)^{2n} \sum_{k\ge 0} \binom{2n+1}{k} (-1)^k \binom{2m-k}{m} \frac{z^k}{(1+z)^k}.$$

The upper limit on k is gone due to the coefficient extractor. Continuing,

$$(-1)^{n+m} [z^n] (1+z)^{2n} [w^m] (1+w)^{2m} \sum_{k\ge 0} \binom{2n+1}{k} \frac{(-1)^k}{(1+w)^k} \frac{z^k}{(1+z)^k}$$
$$= (-1)^{n+m} [z^n] (1+z)^{2n} [w^m] (1+w)^{2m} \left[1 - \frac{z}{(1+w)(1+z)}\right]^{2n+1}$$
$$= (-1)^{n+m} [z^n] \frac{1}{1+z} [w^m] \frac{1}{(1+w)^{2n-2m+1}} (1+w+wz)^{2n+1}.$$

Extracting the coefficient in z,

$$(-1)^{n+m} [w^m] \frac{1}{(1+w)^{2n-2m+1}} \sum_{q=0}^{n} (-1)^{n-q} \binom{2n+1}{q} w^q (1+w)^{2n+1-q}$$
$$= (-1)^m [w^m] (1+w)^{2m} \sum_{q=0}^{n} (-1)^q \binom{2n+1}{q} w^q (1+w)^{-q}$$
$$= (-1)^m \sum_{q=0}^{m} \binom{2n+1}{q} (-1)^q \binom{2m-q}{m-q}$$
$$= \sum_{q=0}^{m} \binom{2n+1}{q} \binom{-m-1}{m-q}$$
$$= [w^m] (1+w)^{-m-1} \sum_{q\ge 0} \binom{2n+1}{q} w^q = [w^m] (1+w)^{-m-1} (1+w)^{2n+1}$$
$$= [w^m] (1+w)^{2n-m} = \binom{2n-m}{m}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A076756, coefficients of a characteristic polynomial.

1.147.26 OEIS A130595

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{n}{m} = (-1)^{n+m} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{nm-k+m}{m} \binom{n-m+k}{k}.$$

We obtain for the sum

$$(-1)^{n+m}[z^m](1+z)^{nm+m}[w^{n-m}](1+w)^{n-m} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{(1+w)^k}{(1+z)^k}$$

$$= (-1)^{n+m}[z^m](1+z)^{nm+m}[w^{n-m}](1+w)^{n-m} \left[1 - \frac{1+w}{1+z}\right]^n$$

$$= (-1)^{n+m}[z^m](1+z)^{nm+m-n}[w^{n-m}](1+w)^{n-m}(z-w)^n$$

$$= (-1)^{n+m}[z^m](1+z)^{nm+m-n}[w^{n-m}](1+w)^{n-m} \sum_{q=0}^{n} \binom{n}{q} z^q (-1)^{n-q} w^{n-q}.$$

We see from the coefficient extractors that we must have $q \le m$ and $n-q \le n-m$ or $m \le q$. Hence only $q = m$ can contribute and we obtain

$$(-1)^{n+m}[z^m](1+z)^{nm+m-n}[w^{n-m}](1+w)^{n-m} \binom{n}{m} z^m (-1)^{n-m} w^{n-m}$$

$$= [z^0](1+z)^{nm+m-n}[w^0](1+w)^{n-m} \binom{n}{m} = \binom{n}{m}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A130595, inverse of Pascal's triangle.

1.147.27 OEIS A206735

We seek to prove with $n \ge 1$ and $n \ge m \ge 1$ that

$$\binom{n}{m} = (-1)^{n+m} \sum_{k=0}^{n} \binom{n+m}{m+k} (-1)^k \binom{m-1+k}{k-n+m} \binom{n+m+k}{m}.$$

We get for the sum

$$(-1)^{n+m}\sum_{k=0}^{n}\binom{n+m}{n-k}(-1)^k\binom{m-1+k}{n-1}\binom{n+m+k}{m}$$
$$= (-1)^{n+m}[z^n](1+z)^{n+m}\sum_{k\ge 0} z^k(-1)^k\binom{m-1+k}{n-1}\binom{n+m+k}{m}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$(-1)^{n+m}[z^n](1+z)^{n+m}[w^{n-1}](1+w)^{m-1}\sum_{k\ge 0}(-1)^k z^k(1+w)^k\binom{n+m+k}{m}$$
$$= (-1)^{n+m}[z^n](1+z)^{n+m}[w^{n-1}](1+w)^{m-1}[v^m](1+v)^{n+m}$$
$$\times\sum_{k\ge 0}(-1)^k z^k(1+w)^k(1+v)^k$$
$$= (-1)^{n+m}[z^n](1+z)^{n+m}[w^{n-1}](1+w)^{m-1}[v^m](1+v)^{n+m}$$
$$\times\frac{1}{1+z(1+w)(1+v)}$$
$$= (-1)^{n+m}[z^n](1+z)^{n+m}[w^{n-1}](1+w)^{m-1}[v^m](1+v)^{n+m}$$
$$\times\frac{1}{1+z(1+v)+wz(1+v)}$$
$$= (-1)^{n+m}[z^{n+1}](1+z)^{n+m}[w^{n-1}](1+w)^{m-1}[v^m](1+v)^{n+m-1}$$
$$\times\frac{1}{w+1/z/(1+v)+1}.$$

The contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^n}(1+w)^{m-1}\frac{1}{w+1/z/(1+v)+1}.$$

Here the residue at infinity is zero and we may use minus the residue at $w=-1-1/z/(1+v)$ as residues sum to zero. We get

$$-(-1)^n\frac{1}{(1+1/z/(1+v))^n}(-1)^{m-1}\frac{1}{z^{m-1}(1+v)^{m-1}}$$
$$= (-1)^{n+m}\frac{z^{n+1-m}(1+v)^{n+1-m}}{(1+z+zv)^n}.$$

Restore the remaining two extractors to get

$$[z^{n+1}](1+z)^{n+m}[v^m](1+v)^{n+m-1}\frac{z^{n+1-m}(1+v)^{n+1-m}}{(1+z+zv)^n}$$

$$= [z^m](1+z)^{n+m}[v^m](1+v)^{2n}\frac{1}{(1+z+zv)^n}$$

$$= [z^m](1+z)^{n+m}[v^m](1+v)^n\frac{1}{(z+1/(1+v))^n}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{m+1}}(1+z)^{n+m}\frac{1}{(z+1/(1+v))^n}.$$

Here the residue at infinity is not zero and it must be evaluated. We find (flip sign)

$$\operatorname*{res}_z \frac{1}{z^2}z^{m+1}(1+1/z)^{n+m}\frac{1}{(1/z+1/(1+v))^n}$$

$$= \operatorname*{res}_z z^{m-1}(1+z)^{n+m}\frac{1}{z^{n+m}}\frac{z^n}{(1+z/(1+v))^n}$$

$$= \operatorname*{res}_z \frac{1}{z}(1+z)^{n+m}\frac{1}{(1+z/(1+v))^n} = 1.$$

Substitute into the remaining coefficient extractor in v to obtain

$$[v^m](1+v)^n = \binom{n}{m}.$$

This is the claim. Now we just have to verify that the contribution from the pole at $z = -1/(1+v)$ is zero. This requires the Leibniz rule:

$$\frac{1}{(n-1)!}\left(\frac{1}{z^{m+1}}(1+z)^{n+m}\right)^{(n-1)}$$

$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(-1)^q(m+1)^{\overline{q}}}{z^{m+1+q}}(n+m)^{\underline{n-1-q}}(1+z)^{m+1+q}$$

$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}q!\binom{m+q}{q}\frac{(-1)^q}{z^{m+1+q}}(n-1-q)!\binom{n+m}{n-1-q}(1+z)^{m+1+q}$$

$$= \sum_{q=0}^{n-1}\binom{m+q}{q}(-1)^q\binom{n+m}{m+1+q}\left(\frac{1+z}{z}\right)^{m+1+q}.$$

Evaluate at $z = -1/(1+v)$ and restore the coefficient extractor in front (flip sign)

$$-[v^m](1+v)^n\sum_{q=0}^{n-1}\binom{m+q}{q}(-1)^q\binom{n+m}{m+1+q}(-1)^{m+1+q}v^{m+1+q}$$

$$= (-1)^m [v^m](1+v)^n \sum_{q=0}^{n-1} \binom{m+q}{q} \binom{n+m}{m+1+q} v^{m+1+q}.$$

Note however that $m+1+q > m$ and we get a zero contribution as required. This concludes the argument.

This identity was found by a computer search which pointed to OEIS A206735, a triangle like Pascal's obtained from a DELTA operator.

1.147.28 OEIS A318107

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{n}{m}\binom{2n-m}{n}\binom{3n-2m}{n-m} = \sum_{k=0}^{n} \binom{2n-m}{n-m-k}^2 \binom{m+k}{k}\binom{n+k}{m+k}.$$

We get for the sum

$$[z^{n-m}](1+z)^{2n-m}[w^{n-m}](1+w)^{2n-m}[v^{n-m}](1+v)^n \sum_{k\ge 0} \binom{m+k}{k} z^k w^k (1+v)^k.$$

Here we have extended to infinity due to the extractors in z and in w. Continuing,

$$[z^{n-m}](1+z)^{2n-m}[w^{n-m}](1+w)^{2n-m}[v^{n-m}](1+v)^n \frac{1}{(1-zw-zwv)^{m+1}}$$

$$= (-1)^{m+1}[z^{n+1}](1+z)^{2n-m}[w^{n+1}](1+w)^{2n-m}[v^{n-m}](1+v)^n \frac{1}{(v+1-1/z/w)^{m+1}}.$$

The contribution from v is

$$\operatorname*{res}_v \frac{1}{v^{n-m+1}}(1+v)^n \frac{1}{(v+1-1/z/w)^{m+1}}.$$

Here the residue at infinity is zero so we may evaluate using minus the residue at $v = -1 + 1/z/w$. This requires the Leibniz rule:

$$\frac{1}{m!}\left(\frac{1}{v^{n-m+1}}(1+v)^n\right)^{(m)}$$

$$= \frac{1}{m!}\sum_{q=0}^{m} \binom{m}{q} \frac{(-1)^q (n-m+1)^{\overline{q}}}{v^{n-m+1+q}} n^{\underline{m-q}} (1+v)^{n-m+q}$$

$$= \frac{1}{m!}\sum_{q=0}^{m} \binom{m}{q} q! \binom{n-m+q}{q} \frac{(-1)^q}{v^{n-m+1+q}} (m-q)! \binom{n}{m-q} (1+v)^{n-m+q}$$

$$= \sum_{q=0}^{m} \binom{n-m+q}{q} \frac{(-1)^q}{v^{n-m+1+q}} \binom{n}{m-q} (1+v)^{n-m+q}.$$

Observe that

$$\binom{n-m+q}{q}\binom{n}{m-q} = \frac{n!}{q! \times (n-m)! \times (m-q)!} = \binom{n}{n-m}\binom{m}{q}$$

so this becomes

$$\binom{n}{m} \sum_{q=0}^{m} \binom{m}{q} \frac{(-1)^q}{v^{n-m+1+q}} (1+v)^{n-m+q}$$

$$= \binom{n}{m} \frac{(1+v)^{n-m}}{v^{n-m+1}} \left[1 - \frac{1+v}{v}\right]^m = (-1)^m \binom{n}{m} \frac{(1+v)^{n-m}}{v^{n+1}}.$$

Instantiate the residue and flip the sign:

$$(-1)^{m+1} \binom{n}{m} \frac{1}{z^{n-m} w^{n-m}} \frac{1}{(-1+1/z/w)^{n+1}}$$

$$= (-1)^{m+1} \binom{n}{m} z^{m+1} w^{m+1} \frac{1}{(1-wz)^{n+1}}.$$

Substitute into the remaining coefficient extractors to get

$$\binom{n}{m} [z^{n-m}](1+z)^{2n-m} [w^{n-m}](1+w)^{2n-m} \frac{1}{(1-wz)^{n+1}}.$$

Do the extraction

$$\binom{n}{m} [z^{n-m}](1+z)^{2n-m} \sum_{q=0}^{n-m} \binom{2n-m}{n-m-q} z^q \binom{q+n}{n}$$

$$= \binom{n}{m} \sum_{q=0}^{n-m} \binom{2n-m}{n-m-q}^2 \binom{q+n}{n}.$$

Next observe that

$$\binom{2n-m}{n-m-q}\binom{q+n}{n} = \frac{(2n-m)!}{(n-m-q)! \times n! \times q!} = \binom{2n-m}{n}\binom{n-m}{q}$$

so we obtain

$$\binom{n}{m}\binom{2n-m}{n} \sum_{q=0}^{n-m} \binom{2n-m}{n-m-q}\binom{n-m}{q}.$$

The sum is Vandermonde but we can compute it like so:

$$[z^{n-m}](1+z)^{2n-m}\sum_{q=0}^{n-m}\binom{n-m}{q}z^q = [z^{n-m}](1+z)^{3n-2m} = \binom{3n-2m}{n-m}.$$

Putting everything together we get

$$\binom{n}{m}\binom{2n-m}{n}\binom{3n-2m}{n-m}$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A318107, multinomial $(3n-2k)!/(n-k)!^3/k!$.

1.147.29 OEIS A123110

We seek to prove with $n \ge 1$ and $n \ge m \ge 1$ that

$$1 = (-1)^m \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k}\binom{n-1+k}{n-m+k}\binom{n-1+k}{n}.$$

We get for the sum

$$(-1)^m \sum_{k=0}^{n} (-1)^k \binom{2n}{n-k}\binom{n-1+k}{m-1}\binom{n-1+k}{n}.$$

This is

$$(-1)^m [z^n](1+z)^{2n} \sum_{k\ge 0} (-1)^k z^k \binom{n-1+k}{m-1}\binom{n-1+k}{n}.$$

Here we have extended to infinity due to the coefficient extractor in z. Continuing,

$$(-1)^m [z^n](1+z)^{2n}[w^{m-1}](1+w)^{n-1}[v^n](1+v)^{n-1}\sum_{k\ge 0}(-1)^k z^k (1+w)^k (1+v)^k$$

$$= (-1)^m [z^n](1+z)^{2n}[w^{m-1}](1+w)^{n-1}[v^n](1+v)^{n-1}\frac{1}{1+z(1+w)(1+v)}$$

$$= (-1)^m [z^{n+1}](1+z)^{2n}[w^{m-1}](1+w)^{n-2}[v^n](1+v)^{n-1}\frac{1}{v+1+1/z/(1+w)}.$$

The contribution from v is

$$\operatorname*{res}_{v} \frac{1}{v^{n+1}}(1+v)^{n-1}\frac{1}{v+1+1/z/(1+w)}.$$

Here the residue at infinity is zero and we may evaluate using minus the residue at $v = -1 - 1/z/(1+w)$. We get

$$\frac{(-1)^{n+1}}{(1+1/z/(1+w))^{n+1}} \frac{(-1)^{n-1}}{z^{n-1}(1+w)^{n-1}} = \frac{z^2(1+w)^2}{(1+z+zw)^{n+1}}.$$

Substitute into the remaining extractors and flip sign to get

$$(-1)^{m+1}[z^{n-1}](1+z)^{2n}[w^{m-1}](1+w)^n \frac{1}{(1+z+zw)^{n+1}}$$

$$= (-1)^{m+1}[z^{2n}](1+z)^{2n}[w^{m-1}](1+w)^n \frac{1}{(w+(1+z)/z)^{n+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^m}(1+w)^n \frac{1}{(w+(1+z)/z)^{n+1}}.$$

With $m \ge 1$ we see that the residue at infinity is zero and we may evaluate using minus the residue at $w = -(1+z)/z$ using the Leibniz rule:

$$\frac{1}{n!}\left(\frac{1}{w^m}(1+w)^n\right)^{(n)} = \frac{1}{n!}\sum_{q=0}^n \binom{n}{q} \frac{(-1)^q m^{\overline{q}}}{w^{m+q}} n^{\underline{n-q}}(1+w)^q$$

$$= \frac{1}{n!}\sum_{q=0}^n \binom{n}{q} q! \binom{m+q-1}{q} \frac{(-1)^q}{w^{m+q}} (n-q)! \binom{n}{n-q} (1+w)^q$$

$$= \sum_{q=0}^n \binom{m+q-1}{q} \frac{(-1)^q}{w^{m+q}} \binom{n}{q} (1+w)^q.$$

Substitute to evaluate the residue in w and flip sign

$$\sum_{q=0}^n \binom{m+q-1}{q} \frac{(-1)^{m+1} z^{m+q}}{(1+z)^{m+q}} \binom{n}{q} \frac{(-1)^q}{z^q}.$$

Restore the coefficient extractor in z including the scalar:

$$[z^{2n}](1+z)^{2n} \sum_{q=0}^n \binom{m+q-1}{q} \frac{z^m}{(1+z)^{m+q}} \binom{n}{q} (-1)^q$$

$$= \sum_{q=0}^n \binom{m+q-1}{q} \binom{2n-m-q}{2n-m} \binom{n}{q} (-1)^q.$$

Now with $2n-m-q$ and $2n-m$ not being negative the only case where the second binomial coefficient is not zero in the given range is $q = 0$ and we get

$$\binom{m-1}{0}\binom{2n-m}{2n-m}\binom{n}{0}(-1)^0 = 1$$

which is the claim. Observe carefully that we have made use of the fact that $n \ge m$ because if not we get a non-zero contribution from q when $2n-m-q < 0$ or $q > 2n-m$ which then includes a contribution from $2n-m+1 \le q \le n$. This contribution is

$$\sum_{q=2n-m+1}^{n} \binom{m+q-1}{q} \binom{2n-m-q}{2n-m} \binom{n}{q} (-1)^q.$$

Note also that when $m > 2n$ we have two types of binomial coefficients, under the first interpretation $\binom{2n-m-q}{2n-m}$ is zero because $2n-m < 0$ and everything sums to zero, under the second we replace it by $\binom{2n-m-q}{-q}$ which is non-zero when $q = 0$ and everything sums to one. We point out however that the coefficient extractor that was responsible for this binomial coefficient implements the first interpretation (zero when $m > 2n$) so that is the one we should choose.

This identity was found by a computer search which pointed to OEIS A123110, a product of the DELTA operator.

1.147.30 OEIS A202409

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\frac{1}{n-m+1} \binom{n+1}{m+1} \binom{n+1}{m} \left[(n+1) \binom{n+1}{m+1} - m \binom{n}{m+1} \right]$$
$$= \sum_{k=0}^{n} \binom{n+1}{k+1} \binom{n-k}{m-k} \binom{n+k+1}{k+1} \binom{n-m}{k}.$$

The sum is

$$\sum_{k=0}^{n-m} \binom{n-m}{k} \binom{n+1}{n-k} \binom{n-k}{m-k} \binom{n+k+1}{n}.$$

Observe that

$$\binom{n+1}{n-k} \binom{n-k}{m-k} = \frac{(n+1)!}{(k+1)! \times (n-m)! \times (m-k)!} = \binom{n+1}{n-m} \binom{m+1}{k+1}$$

and we find

$$\binom{n+1}{m+1} \sum_{k=0}^{n-m} \binom{n-m}{k} \binom{m+1}{k+1} \binom{n+k+1}{n}.$$

Note that

$$\binom{n-m}{k} \binom{n+k+1}{n} = \binom{n-m}{k} \frac{n+1}{k+1} \binom{n+k+1}{n+1}$$

$$= \binom{n-m+1}{k+1} \frac{k+1}{n-m+1} \frac{n+1}{k+1} \binom{n+k+1}{n+1}$$

We have for our sum

$$\frac{n+1}{n-m+1} \binom{n+1}{m+1} \sum_{k=0}^{n-m} \binom{n-m+1}{n-m-k} \binom{m+1}{m-k} \binom{n+k+1}{n+1}.$$

Working with the sum only we have

$$[z^{n-m}](1+z)^{n-m+1} [w^m](1+w)^{m+1} \sum_{k\ge 0} \binom{n+k+1}{n+1} z^k w^k.$$

Here we have extended to infinity owing to the extractor in z. Continuing,

$$[z^{n-m}](1+z)^{n-m+1} [w^m](1+w)^{m+1} \frac{1}{(1-wz)^{n+2}}$$
$$= (-1)^n [z^{2n+2-m}](1+z)^{n-m+1} [w^m](1+w)^{m+1} \frac{1}{(w-1/z)^{n+2}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{m+1}} (1+w)^{m+1} \frac{1}{(w-1/z)^{n+2}}.$$

Here the residue at infinity is zero so we may evaluate with minus the residue at $w = 1/z$, which requires the Leibniz rule:

$$\frac{1}{(n+1)!} \left(\frac{1}{w^{m+1}} (1+w)^{m+1} \right)^{(n+1)}$$
$$= \frac{1}{(n+1)!} \sum_{q=0}^{n+1} \binom{n+1}{q} \frac{(-1)^q (m+1)^{\overline{q}}}{w^{m+1+q}} (m+1)^{\underline{n+1-q}} (1+w)^{m+q-n}$$
$$= \frac{1}{(n+1)!} \sum_{q=0}^{n+1} \binom{n+1}{q} q! \binom{m+q}{q} \frac{(-1)^q}{w^{m+1+q}} (n+1-q)! \binom{m+1}{n+1-q} (1+w)^{m+q-n}$$
$$= \sum_{q=0}^{n+1} \binom{m+q}{q} \frac{(-1)^q}{w^{m+1+q}} \binom{m+1}{n+1-q} (1+w)^{m+q-n}.$$

Evaluate at $w = 1/z$ and restore the extractor in z (flip sign):

$$(-1)^{n+1} [z^{2n+2-m}](1+z)^{n-m+1} \sum_{q=0}^{n+1} \binom{m+q}{q} (-1)^q z^{m+1+q} \binom{m+1}{n+1-q} (1+1/z)^{m+q-n}$$

$$= (-1)^{n+1}[z^{2n+2-m}](1+z)^{n-m+1}\sum_{q=0}^{n+1}\binom{m+q}{q}(-1)^q z^{n+1}\binom{m+1}{n+1-q}(1+z)^{m+q-n}$$

$$= (-1)^{n+1}\sum_{q=0}^{n+1}\binom{m+q}{q}(-1)^q\binom{m+1}{n+1-q}\binom{q+1}{n+1-m}.$$

Note that

$$\binom{m+q}{q}\binom{q+1}{n+1-m} = (q+1)\frac{(m+q)!}{m!\times(n+1-m)!\times(q+m-n)!}$$

$$= (q+1)\binom{n+1}{m}\frac{1}{n+1}\binom{m+q}{n}.$$

Just to re-capitulate where we are at this point, we have for the sum from the start

$$\frac{(-1)^{n+1}}{n-m+1}\binom{n+1}{m+1}\binom{n+1}{m}\sum_{q=0}^{n+1}(q+1)(-1)^q\binom{m+q}{n}\binom{m+1}{n+1-q}.$$

Focus on the remaining sum and write

$$[z^{n+1}](1+z)^{m+1}\sum_{q\ge 0}(q+1)(-1)^q\binom{m+q}{n}z^q.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$[z^{n+1}](1+z)^{m+1}[w^n](1+w)^m\sum_{q\ge 0}(q+1)(-1)^q(1+w)^q z^q$$

$$= [z^{n+1}](1+z)^{m+1}[w^n](1+w)^m\frac{1}{(1+z+zw)^2}$$

$$= [z^{n+3}](1+z)^{m+1}[w^n](1+w)^m\frac{1}{(w+(1+z)/z)^2}.$$

The contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^{n+1}}(1+w)^m\frac{1}{(w+(1+z)/z)^2}.$$

Once more the residue at infinity is zero and we may evaluate using minus the residue at $w=-(1+z)/z$. Differentiating once we have

$$-(n+1)\frac{1}{w^{n+2}}(1+w)^m + m\frac{1}{w^{n+1}}(1+w)^{m-1}.$$

Substitute and flip the sign

$$(n+1)(-1)^n \frac{z^{n+2}}{(1+z)^{n+2}} \frac{(-1)^m}{z^m} - m(-1)^{n+1} \frac{z^{n+1}}{(1+z)^{n+1}} \frac{(-1)^{m-1}}{z^{m-1}}$$
$$= (n+1)(-1)^{n+m} \frac{z^{n-m+2}}{(1+z)^{n+2}} - m(-1)^{n+m} \frac{z^{n-m+2}}{(1+z)^{n+1}}.$$

Apply the remaining extractor to get

$$(n+1)(-1)^{n+m}[z^{m+1}] \frac{1}{(1+z)^{n-m+1}} - m(-1)^{n+m}[z^{m+1}] \frac{1}{(1+z)^{n-m}}$$
$$= (n+1)(-1)^{n+1} \binom{n+1}{m+1} - m(-1)^{n+1} \binom{n}{m+1}.$$

We have achieved the closed form

$$\frac{1}{n-m+1} \binom{n+1}{m+1} \binom{n+1}{m} \left[(n+1) \binom{n+1}{m+1} - m \binom{n}{m+1} \right]$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A202409, a multiple of a third power of a binomial coefficient.

1.147.31 OEIS A001498

We seek to prove with $n \geq 0$ and $n \geq m \geq 0$ that

$$\frac{(n+m)!}{2^m \times (n-m)! \times m!} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+m}{m+k} \left[m+k \atop k \right] \binom{k-m}{n-m}.$$

We have for the sum,

$$(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k} \left[m+k \atop k \right] \binom{k-m}{n-m}$$
$$= \sum_{k=0}^{n} (-1)^k \binom{n+m}{k} \left[n+m-k \atop n-k \right] \binom{n-m-k}{n-m}$$
$$= [z^{n-m}](1+z)^{n-m} \sum_{k=0}^{n} (-1)^k \binom{n+m}{k} \left[n+m-k \atop n-k \right] (1+z)^{-k}$$
$$= [z^{n-m}](1+z)^{n-m} \sum_{k=0}^{n} (-1)^k \binom{n+m}{k} (1+z)^{-k}$$

$$\times(n+m-k)![v^{n+m-k}]\frac{1}{(n-k)!}\left(\log\frac{1}{1-v}\right)^{n-k}$$

$$= \frac{(n+m)!}{n!}[z^{n-m}](1+z)^{n-m}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+z)^{-k}[v^{n+m-k}]\left(\log\frac{1}{1-v}\right)^{n-k}$$

$$= \frac{(n+m)!}{n!}[z^{n-m}](1+z)^{n-m}[v^{n+m}]\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+z)^{-k}v^k\left(\log\frac{1}{1-v}\right)^{n-k}$$

$$= \frac{(n+m)!}{n!}[z^{n-m}](1+z)^{n-m}[v^{n+m}]\left[\log\frac{1}{1-v}-\frac{v}{1+z}\right]^n$$

$$= \frac{(n+m)!}{n!}[z^{n-m}]\frac{1}{(1+z)^m}[v^{n+m}]\left[(1+z)\log\frac{1}{1-v}-v\right]^n$$

$$= \frac{(n+m)!}{n!}[z^{n-m}]\frac{1}{(1+z)^m}[v^{n+m}]\left[z\log\frac{1}{1-v}+\log\frac{1}{1-v}-v\right]^n$$

$$= \frac{(n+m)!}{n!}[v^{n+m}]\sum_{q=0}^{n-m}(-1)^{n-m-q}\binom{n-1-q}{m-1}\binom{n}{q}\left[\log\frac{1}{1-v}\right]^q\left[\log\frac{1}{1-v}-v\right]^{n-q}.$$

Observe that the coefficient extractor together with the series $\log\frac{1}{1-v} = \sum_{p\ge1}\frac{v^p}{p}$ produces the requirement that

$$n+m \ge q+2(n-q) = 2n-q$$

which is $q+m\ge n$. There is only one value in the range of q that works here which is $q=n-m$. We thus obtain

$$\frac{(n+m)!}{n!}[v^{n+m}]\binom{n-1-n+m}{m-1}\binom{n}{n-m}\left[\log\frac{1}{1-v}\right]^{n-m}\left[\log\frac{1}{1-v}-v\right]^m$$

$$= \frac{(n+m)!}{n!}\binom{n}{m}[v^{n+m}]\left[\log\frac{1}{1-v}\right]^{n-m}\left[\log\frac{1}{1-v}-v\right]^m.$$

The product of the two logarithmic terms starts at the power $n-m+2m = n+m$. This is minimal and uses the first terms of the two series which are v^{n-m} and $(v^2/2)^m$. Therefore everything simplifies to

$$\frac{(n+m)!}{n!}\binom{n}{m}\frac{1}{2^m}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A001498, coefficients of Bessel polynomials.

1.147.32 OEIS A086810

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that with Catalan numbers

$$\frac{1}{n+1}\binom{n+m}{m}\binom{n-1}{m-1} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{k}{m}\binom{n+k}{2k}\frac{1}{k+1}\binom{2k}{k}.$$

Note first that

$$\binom{n+k}{2k}\binom{2k}{k} = \frac{(n+k)!}{(n-k)! \times k! \times k!} = \binom{n+k}{k}\binom{n}{k}$$

so we get for the sum

$$(-1)^n \sum_{k=0}^{n} (-1)^k \binom{k}{m}\frac{1}{k+1}\binom{n+k}{k}\binom{n}{k}$$

$$= \frac{(-1)^n}{n+1}\sum_{k=0}^{n} (-1)^k \binom{k}{m}\binom{n+k}{k}\binom{n+1}{k+1}$$

$$= \frac{(-1)^n}{n+1}[w^m][z^n](1+z)^{n+1}\sum_{k\ge 0}(-1)^k(1+w)^k z^k \binom{n+k}{k}$$

$$= \frac{(-1)^n}{n+1}[w^m][z^n](1+z)^{n+1}\frac{1}{(1+z+zw)^{n+1}}$$

$$= \frac{(-1)^n}{n+1}[w^m][z^{2n+1}](1+z)^{n+1}\frac{1}{(w+(1+z)/z)^{n+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{m+1}}\frac{1}{(w+(1+z)/z)^{n+1}}.$$

Now here the residue at infinity is clearly zero, we may use minus the residue at $w = -(1+z)/z$. This requires

$$-\frac{1}{n!}\left(\frac{1}{w^{m+1}}\right)^{(n)}\Bigg|_{w=-(1+z)/z} = -\frac{1}{n!}\frac{(-1)^n}{w^{n+m+1}}(m+1)^{\overline{n}}\Bigg|_{w=-(1+z)/z}$$

$$= \binom{m+n}{n}(-1)^m \frac{z^{n+m+1}}{(1+z)^{n+m+1}}.$$

Applying the coefficient extractor in z we find

$$\frac{(-1)^{n+m}}{n+1}\binom{n+m}{n}[z^{n-m}]\frac{1}{(1+z)^m}$$

$$= \frac{(-1)^{n+m}}{n+1}\binom{n+m}{n}(-1)^{n-m}\binom{n-m+m-1}{m-1}$$
$$= \frac{1}{n+1}\binom{n+m}{m}\binom{n-1}{m-1}.$$

This is the claim. An alternate approach without residues uses

$$\frac{(-1)^n}{n+1}[w^m][z^n]\frac{1}{(1+zw/(1+z))^{n+1}}$$
$$= \frac{(-1)^n}{n+1}[z^n]\binom{n+m}{m}(-1)^m\frac{z^m}{(1+z)^m}$$
$$= \frac{(-1)^n}{n+1}\binom{n+m}{m}(-1)^m(-1)^{n-m}\binom{n-m+m-1}{m-1}$$
$$= \frac{1}{n+1}\binom{n+m}{m}\binom{n-1}{m-1}.$$

Again we have the claim.

This identity was found by a computer search which pointed to OEIS A086810, a triangle related to OEIS A033282.

1.147.33 OEIS A008459

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{n}{m}^2 = \sum_{k=0}^{n}\binom{m+k}{m-k}\binom{2k}{k}\binom{n}{m+k}.$$

Start with

$$[z^{n-m}](1+z)^n\sum_{k\ge 0}\binom{m+k}{m-k}\binom{2k}{k}z^k.$$

Here we have extended to infinity due to the coefficient extractor. Note that

$$\binom{m+k}{m-k}\binom{2k}{k} = \frac{(m+k)!}{(m-k)!\times k!\times k!} = \binom{m+k}{k}\binom{m}{k}.$$

This yields

$$[z^{n-m}](1+z)^n[w^m](1+w)^m\sum_{k\ge 0}\binom{m+k}{k}w^k z^k$$
$$= [z^{n-m}](1+z)^n[w^m](1+w)^m\frac{1}{(1-wz)^{m+1}}$$
$$= (-1)^{m+1}[z^{n-m}](1+z)^n[w^{2m+1}](1+w)^m\frac{1}{(z-1/w)^{m+1}}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n-m+1}}(1+z)^n \frac{1}{(z-1/w)^{m+1}}.$$

Fortunately here the residue at infinity is zero (just barely) and we may evaluate using minus the residue at $z = 1/w$, which requires the Leibniz rule:

$$\frac{1}{m!}\left(\frac{1}{z^{n-m+1}}(1+z)^n\right)^{(m)}$$

$$= \frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\frac{(-1)^q \times (n-m+1)^{\overline{q}}}{z^{n-m+1+q}} n^{\underline{m-q}}(1+z)^{n-(m-q)}$$

$$= \frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\frac{(-1)^q}{z^{n-m+1+q}}\binom{n-m+q}{q} q! \binom{n}{m-q}(m-q)!(1+z)^{n-m+q}$$

$$= \sum_{q=0}^{m}\frac{(-1)^q}{z^{n-m+1+q}}\binom{n-m+q}{q}\binom{n}{n-m+q}(1+z)^{n-m+q}.$$

Now observe that

$$\binom{n-m+q}{q}\binom{n}{n-m+q} = \frac{n!}{q! \times (n-m)! \times (m-q)!} = \binom{n}{m}\binom{m}{q}.$$

We have one of the desired coefficients and are left with (flip sign)

$$\binom{n}{m}(-1)^m [w^{2m+1}](1+w)^m \sum_{q=0}^{m}\binom{m}{q}(-1)^q w^{n-m+1+q}\frac{(1+w)^{n-m+q}}{w^{n-m+q}}$$

$$= \binom{n}{m}(-1)^m [w^{2m}](1+w)^m \sum_{q=0}^{m}\binom{m}{q}(-1)^q (1+w)^{n-m+q}$$

$$= \binom{n}{m}(-1)^m \sum_{q=0}^{m}\binom{m}{q}(-1)^q \binom{n+q}{2m}.$$

We find for the remaining sum

$$[w^{2m}](1+w)^n \sum_{q=0}^{m}\binom{m}{q}(-1)^q (1+w)^q = [w^{2m}](1+w)^n (1-(1+w))^m$$

$$= (-1)^m [w^m](1+w)^n = (-1)^m \binom{n}{m}.$$

Joining the two pieces we have the claim.

This identity was found by a computer search which pointed to OEIS A008459, square the entries of Pascal's triangle.

1.147.34 OEIS 130321

We seek to prove with $n \ge 0$ and $n \ge m \ge 0$ that

$$2^{n-m} = \sum_{k=0}^{n} (-1)^k \binom{n+m-k}{k} \binom{2n-2k}{n-k}.$$

We start with

$$[z^n](1+z)^{2n} \sum_{k\ge 0} (-1)^k \binom{n+m-k}{n+m-2k} \frac{z^k}{(1+z)^{2k}}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$[z^n](1+z)^{2n}[w^{n+m}](1+w)^{n+m} \sum_{k\ge 0} (-1)^k \frac{w^{2k}}{(1+w)^k} \frac{z^k}{(1+z)^{2k}}$$

$$= [z^n](1+z)^{2n}[w^{n+m}](1+w)^{n+m} \frac{1}{1+w^2 z/(1+w)/(1+z)^2}$$

$$= [z^n](1+z)^{2n+2}[w^{n+m}](1+w)^{n+m+1} \frac{1}{(1+w)(1+z)^2 + w^2 z}$$

$$= [z^n](1+z)^{2n+2}[w^{n+m}](1+w)^{n+m+1} \frac{1}{(w+1+z)(wz+1+z)}$$

$$= [z^{n+1}](1+z)^{2n+2}[w^{n+m}](1+w)^{n+m+1} \frac{1}{(w+1+z)(w+(1+z)/z)}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+m+1}} (1+w)^{n+m+1} \frac{1}{(w+1+z)(w+(1+z)/z)}.$$

Here the residue at infinity is zero (just barely) so we may use minus the residues at $w = -(1+z)$ and $w = -(1+z)/z$. We get for the former,

$$-[z^{n+1}](1+z)^{2n+2} \frac{(-1)^{n+m+1}}{(1+z)^{n+m+1}} (-1)^{n+m+1} z^{n+m+1} \frac{1}{-(1+z)+(1+z)/z}$$

$$= -[z^{n+1}](1+z)^{n-m+1} z^{n+m+1} \frac{z}{-z(1+z)+(1+z)}$$

$$= -[z^{n+1}](1+z)^{n-m+1} z^{n+m+2} \frac{1}{1-z^2} = 0$$

due to the coefficient extractor and $n+m+2 > n+1$. We get for the second residue

$$\begin{aligned}
&-[z^{n+1}](1+z)^{2n+2}\frac{(-1)^{n+m+1}z^{n+m+1}}{(1+z)^{n+m+1}}\frac{(-1)^{n+m+1}}{z^{n+m+1}}\frac{1}{-(1+z)/z+1+z}\\
&= -[z^{n+1}](1+z)^{n-m+1}\frac{z}{-(1+z)+(1+z)z}\\
&= [z^n](1+z)^{n-m+1}\frac{1}{1-z^2} = [z^n](1+z)^{n-m}\frac{1}{1-z}\\
&= \sum_{q=0}^{n}\binom{n-m}{q} = 2^{n-m}.
\end{aligned}$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A130321, triangle with 2^{n-m}.

1.148 MSE 4667102: Two different representations of a coefficient

We will show that

$$\sum_{k=0}^{r}\binom{n}{2k}\binom{n-2k}{r-k} = \sum_{k=r}^{n}\binom{n}{k}\binom{2k}{2r}\left(\frac{3}{4}\right)^{n-k}\left(\frac{1}{2}\right)^{2k-2r}.$$

Computation for LHS

We get for LHS

$$\begin{aligned}
&\sum_{k=0}^{r}\binom{n}{n-2k}\binom{n-2k}{r-k}\\
&= [z^n](1+z)^n[w^r](1+w)^n\sum_{k\ge 0}z^{2k}\frac{w^k}{(1+w)^{2k}}.
\end{aligned}$$

Here we have extended to infinity due to the coefficient extractor in w. We obtain

$$\begin{aligned}
&[z^n](1+z)^n[w^r](1+w)^n\frac{1}{1-z^2w/(1+w)^2}\\
&= [z^n](1+z)^n[w^r](1+w)^{n+2}\frac{1}{(1+w)^2-wz^2}\\
&= [z^n](1+z)^n[w^{2r}](1+w^2)^{n+2}\frac{1}{(1+w^2)^2-w^2z^2}
\end{aligned}$$

$$= [z^n](1+z)^n [w^{2r}](1+w^2)^{n+2} \frac{1}{1+wz+w^2} \frac{1}{1-wz+w^2}$$

$$= -[z^n](1+z)^n [w^{2r+2}](1+w^2)^{n+2} \frac{1}{z+(1+w^2)/w} \frac{1}{z-(1+w^2)/w}.$$

The contribution from z is

$$\operatorname{res}_z \frac{1}{z^{n+1}}(1+z)^n \frac{1}{z+(1+w^2)/w} \frac{1}{z-(1+w^2)/w}.$$

Here the residue at infinity is zero so we may use minus the residues at $z = \pm(1+w^2)/w$. We get first

$$[w^{2r+2}](1+w^2)^{n+2}(-1)^{n+1} \frac{w^{n+1}}{(1+w^2)^{n+1}} \frac{(-1+w-w^2)^n}{w^n} \frac{1}{-2(1+w^2)/w}$$

$$= \frac{1}{2}[w^{2r}](w^2-w+1)^n = \frac{1}{2}(-1)^{2r}[w^{2r}](w^2-w+1)^n$$

$$= \frac{1}{2}[w^{2r}](w^2+w+1)^n.$$

and second

$$[w^{2r+2}](1+w^2)^{n+2} \frac{w^{n+1}}{(1+w^2)^{n+1}} \frac{(w^2+w+1)^n}{w^n} \frac{1}{2(1+w^2)/w}$$

$$= \frac{1}{2}[w^{2r}](w^2+w+1)^n.$$

Collecting everything,

$$[w^{2r}](w^2+w+1)^n.$$

Computation for RHS

We have for RHS

$$\left(\frac{3}{4}\right)^n \left(\frac{1}{2}\right)^{-2r} \sum_{k=r}^{n} \binom{n}{n-k} \binom{2k}{2r} \left(\frac{4}{3}\right)^k \left(\frac{1}{2}\right)^{2k}$$

$$= \left(\frac{3}{4}\right)^n \left(\frac{1}{2}\right)^{-2r} [z^n](1+z)^n [w^{2r}] \sum_{k=r}^{n} z^k (1+w)^{2k} \left(\frac{1}{3}\right)^k$$

We may lower k to zero owing to the coefficient extractor in w and raise to infinity due to the extractor in z to get

$$\left(\frac{3}{4}\right)^n \left(\frac{1}{2}\right)^{-2r} [z^n](1+z)^n [w^{2r}] \frac{1}{1-z(1+w)^2/3}$$

$$= 3^n 2^{2r-2n} [z^n](1+z)^n [w^{2r}] \frac{1}{1-z(1+w)^2/3}$$
$$= 2^{2r-2n} [z^n](1+3z)^n [w^{2r}] \frac{1}{1-z(1+w)^2}$$
$$= -2^{2r-2n} [z^n](1+3z)^n [w^{2r}] \frac{1}{(1+w)^2} \frac{1}{z-1/(1+w)^2}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}} (1+3z)^n \frac{1}{z-1/(1+w)^2}.$$

Here the residue at infinity is zero so we may use minus the residue at $z = 1/(1+w)^2$ to get

$$2^{2r-2n} [w^{2r}] \frac{1}{(1+w)^2} (1+w)^{2n+2} (1+3/(1+w)^2)^n$$
$$= 2^{2r-2n} [w^{2r}] (4+2w+w^2)^n = 2^{-2n} [w^{2r}] (4+4w+4w^2)^n$$
$$= [w^{2r}] (w^2+w+1)^n.$$

We have equality and thus the claim. Here we had some help from OEIS A005714.

This was math.stackexchange.com problem 4667102.

1.149 MSE 4675665: Rational term of constant degree

We seek to show that

$$\sum_{k=0}^n k^2 \binom{n+k}{k} = \frac{n(n+1)^3}{(n+2)(n+3)} \binom{2n+1}{n+1}.$$

We get two pieces

$$\sum_{k=0}^n \left[2\binom{k}{2} + \binom{k}{1} \right] \binom{n+k}{k}.$$

We start by evaluating the general sum

$$\sum_{k=q}^n \binom{n+k}{k} \binom{k}{q}.$$

We have

$$\binom{n+k}{k} \binom{k}{q} = \frac{(n+k)!}{n! \times q! \times (k-q)!} = \binom{n+q}{q} \binom{n+k}{n+q}$$

and find for the sum

$$\binom{n+q}{q} \sum_{k=q}^{n} \binom{n+k}{n+q}.$$

The sum component is

$$\sum_{k=0}^{n-q} \binom{n+q+k}{n+q}$$
$$= [w^{n-q}] \frac{1}{1-w} \sum_{k\ge 0} \binom{n+q+k}{n+q} w^k = [w^{n-q}] \frac{1}{1-w} \frac{1}{(1-w)^{n+q+1}}$$
$$= [w^{n-q}] \frac{1}{(1-w)^{n+q+2}} = \binom{n-q+n+q+1}{n-q} = \binom{2n+1}{n-q}.$$

We thus have the closed form

$$\binom{2n+1}{n-q}\binom{n+q}{q}.$$

Adding the two pieces we find

$$2\binom{2n+1}{n-2}\binom{n+2}{2} + \binom{2n+1}{n-1}\binom{n+1}{1}$$
$$= \frac{n-1}{2n+2}\binom{2n+2}{n-1}(n+2)(n+1) + \binom{2n+2}{n-1}\frac{n+3}{2n+2}(n+1)$$
$$= \frac{1}{2}(n+1)^2\binom{2n+2}{n-1}.$$

This was math.stackexchange.com problem 4675665.

1.150 MSE 4666141: Double square root

We seek to show that

$$2^m \sum_{k=0}^{m} \sum_{j=0}^{p} (-1)^j \binom{k}{j}\binom{m-k}{p-j}\binom{m}{k}\binom{1/2(m+k-1)}{m} = (-1)^p \binom{m}{p}^2.$$

We will suppose $m \ge p$. Re-ordering we find

$$2^m \sum_{k=0}^{m} \binom{m}{k}\binom{1/2(m+k-1)}{m} \sum_{j=0}^{p} (-1)^j \binom{k}{j}\binom{m-k}{p-j}.$$

We get for the inner sum

$$[z^p](1+z)^{m-k}\sum_{j\ge 0}(-1)^j\binom{k}{j}z^j.$$

Here we have extended to infinity because of the coeffient extractor. Continuing,

$$[z^p](1+z)^{m-k}(1-z)^k.$$

With the outer sum we have

$$2^m[z^p](1+z)^m\sum_{k=0}^{m}\binom{m}{k}\binom{1/2(m+k-1)}{m}(1+z)^{-k}(1-z)^k$$

$$= 2^m[z^p](1+z)^m[w^m](1+w)^{1/2(m-1)}\sum_{k=0}^{m}\binom{m}{k}(1+w)^{1/2k}(1+z)^{-k}(1-z)^k$$

$$= 2^m[z^p](1+z)^m[w^m](1+w)^{1/2(m-1)}\left[1+\frac{\sqrt{1+w}(1-z)}{1+z}\right]^m$$

$$= 2^m[z^p][w^m](1+w)^{1/2(m-1)}[1+z+\sqrt{1+w}(1-z)]^m$$

$$= 2^m[z^p][w^m](1+w)^{1/2(m-1)}[1+\sqrt{1+w}+z(1-\sqrt{1+w})]^m$$

$$= 2^m\binom{m}{p}[w^m](1+w)^{1/2(m-1)}(1-\sqrt{1+w})^p(1+\sqrt{1+w})^{m-p}.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{m+1}}(1+w)^{1/2(m-1)}(1-\sqrt{1+w})^p(1+\sqrt{1+w})^{m-p}.$$

Now we put $1-\sqrt{1+w}=v$ so that $w=v(v-2)$ and $dw=2(v-1)\;dv$ (map takes zero to zero) to get

$$\operatorname*{res}_v \frac{1}{v^{m+1}(v-2)^{m+1}}(1-v)^{m-1}v^p(2-v)^{m-p}2(v-1)$$

$$= 2(-1)^{m-p+1}\operatorname*{res}_v \frac{1}{v^{m-p+1}(v-2)^{p+1}}(1-v)^m$$

$$= 2^{-p}(-1)^m\operatorname*{res}_v \frac{1}{v^{m-p+1}(1-v/2)^{p+1}}(1-v)^m$$

$$= 2^{-p}(-1)^m\sum_{q=0}^{m-p}\binom{m}{q}(-1)^q\binom{m-q}{p}2^{-(m-p-q)}$$

$$= 2^{-m}(-1)^m\sum_{q=0}^{m-p}\binom{m}{q}(-1)^q\binom{m-q}{p}2^q.$$

Next observe that

$$\binom{m}{q}\binom{m-q}{p} = \frac{m!}{q! \times p! \times (m-p-q)!} = \binom{m}{p}\binom{m-p}{q}.$$

Collecting everything we find

$$2^m \binom{m}{p} 2^{-m} (-1)^m \binom{m}{p} \sum_{q=0}^{m-p} \binom{m-p}{q} (-1)^q 2^q$$

$$= \binom{m}{p}^2 (-1)^m (1-2)^{m-p} = (-1)^p \binom{m}{p}^2.$$

This is the claim. (Also goes through for $m = p$.)
This was math.stackexchange.com problem 4666141.

1.151 MSE 4699857: Four auxiliary parameters

We seek with $0 \le c \le a$ and $0 \le d \le b$

$$\binom{n+c}{a}\binom{n+d}{b} = \sum_{q=0}^{a+b} \binom{a-c+d}{q-c}\binom{b-d+c}{q-d}\binom{n+q}{a+b}.$$

We get for the RHS

$$[z^{a+b}]\frac{1}{1-z}\sum_{q\ge 0} z^q \binom{a-c+d}{a+d-q}\binom{b-d+c}{b+c-q}\binom{n+q}{a+b}$$

$$= [z^{a+b}]\frac{1}{1-z}[w^{a+d}](1+w)^{a-c+d}[v^{b+c}](1+v)^{b-d+c}\sum_{q\ge 0} z^q w^q v^q \binom{n+q}{a+b}$$

$$= [z^{a+b}]\frac{1}{1-z}[w^{a+d}](1+w)^{a-c+d}[v^{b+c}](1+v)^{b-d+c}[u^{a+b}](1+u)^n$$

$$\times \sum_{q\ge 0} z^q w^q v^q (1+u)^q$$

$$= [z^{a+b}]\frac{1}{1-z}[w^{a+d}](1+w)^{a-c+d}[v^{b+c}](1+v)^{b-d+c}[u^{a+b}](1+u)^n \frac{1}{1-zwv(1+u)}$$

$$= -[z^{a+b+1}]\frac{1}{1-z}[w^{a+d+1}](1+w)^{a-c+d}[v^{b+c}](1+v)^{b-d+c}$$

$$\times [u^{a+b}](1+u)^{n-1}\frac{1}{v-1/z/w/(1+u)}.$$

The contribution from v is

$$\operatorname*{res}_{v} \frac{1}{v^{b+c+1}}(1+v)^{b-d+c}\frac{1}{v-1/z/w/(1+u)}.$$

Here we see that the residue at infinity is zero, so we may use minus the residue at $v = 1/z/w/(1+u)$ (residues sum to zero):

$$-(1+u)^{b+c+1} z^{b+c+1} w^{b+c+1} \frac{(zw(1+u)+1)^{b-d+c}}{z^{b-d+c} w^{b-d+c} (1+u)^{b-d+c}}$$
$$= -(1+u)^{d+1} z^{d+1} w^{d+1} (zw(1+u)+1)^{b-d+c}.$$

Substitute into the remaining extractors,

$$[z^{a+b-d}] \frac{1}{1-z} [w^a](1+w)^{a-c+d} [u^{a+b}](1+u)^{n+d} (zw(1+u)+1)^{b-d+c}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{a+b-d+1}} \frac{1}{1-z} (zw(1+u)+1)^{b-d+c}.$$

Using the the inequalities as stated in the preliminaries we find again that the residue at infinity is zero, and we may use minus the residue at $z = 1$. We also note that there is no pole at $z = -1/w/(1+u)$. Collecting the remaining two extractors, we get

$$[w^a](1+w)^{a-c+d} [u^{a+b}](1+u)^{n+d} (w(1+u)+1)^{b-d+c}$$
$$= [w^a](1+w)^{a-c+d} [u^{a+b}](1+u)^{n+d} (1+w+uw)^{b-d+c}$$
$$= [w^a](1+w)^{a-c+d} [u^{a+b}](1+u)^{n+d} \sum_{q=0}^{b-d+c} \binom{b-d+c}{q} (1+w)^{b-d+c-q} u^q w^q$$
$$= [u^{a+b}](1+u)^{n+d} \sum_{q=0}^{b-d+c} \binom{b-d+c}{q} \binom{a+b-q}{a-q} u^q$$
$$= \sum_{q=0}^{b-d+c} \binom{b-d+c}{q} \binom{a+b-q}{a-q} \binom{n+d}{a+b-q}.$$

Here we have by construction that the rightmost two binomial coefficients are zero when the lower index goes negative. This means that we can replace the upper range by a. If $a < b-d+c$ we get for $q > a$ that the second binomial coefficient is zero. If $b-d+c < a$ we get for $q > b-d+c$ that the first binomial coefficient is zero. We thus have

$$\sum_{q=0}^{a} \binom{b-d+c}{q} \binom{a+b-q}{a-q} \binom{n+d}{a+b-q}.$$

Now observe that

$$\binom{a+b-q}{a-q}\binom{n+d}{a+b-q} = \frac{(n+d)!}{(a-q)! \times b! \times (n+d+q-a-b)!}$$
$$= \binom{n+d}{b}\binom{n+d-b}{a-q}.$$

We thus obtain

$$\binom{n+d}{b}\sum_{q=0}^{a}\binom{b-d+c}{q}\binom{n+d-b}{a-q}.$$

Here we may use Vandermonde which yields

$$\binom{n+d}{b}\binom{n+c}{a}.$$

This is the claim. It is interesting to try this sum with the major CAS. This was math.stackexchange.com problem 4699857.

1.152 MSE 4703564: A family of odd polynomials

We seek to show that

$$[x^{2p}]P_{m,n}(x) = [x^{2p}]\sum_{k=0}^{m}\binom{2x+2k}{2k+1}\binom{n+m-k-x-1/2}{m-k} = 0$$

i.e. that these polynomials are odd and moreover that the coefficients are positive. We first try to simplify the sum. We get

$$-[z^m](1+z)^{n+m-x-1/2}\sum_{k\ge 0}\binom{-2x}{2k+1}z^k(1+z)^{-k}.$$

Here we have extended to infinity due to the coefficient extractor. Next,

$$-[z^m](1+z)^{n+m-x-1/2}\sum_{k\ge 0}z^k(1+z)^{-k}[w^{2k+1}](1+w)^{-2x}$$
$$= -[z^{2m}](1+z^2)^{n+m-x-1/2}\sum_{k\ge 0}z^{2k}(1+z^2)^{-k}[w^{2k+1}](1+w)^{-2x}$$
$$= -[z^{2m+1}](1+z^2)^{n+m-x-1/2}\sqrt{1+z^2}\sum_{k\ge 0}z^{2k+1}(1+z^2)^{-k-1/2}[w^{2k+1}](1+w)^{-2x}$$
$$= -[z^{2m+1}](1+z^2)^{n+m-x}\frac{1}{2}((1+z/\sqrt{1+z^2})^{-2x} - (1-z/\sqrt{1+z^2})^{-2x})$$
$$= -[z^{2m+1}](1+z^2)^{n+m}\frac{1}{2}((z+\sqrt{1+z^2})^{-2x} - (-z+\sqrt{1+z^2})^{-2x}).$$

We know from the initial representation that this is a polynomial in x. Extract the coefficient on $[x^q]$

$$[x^q]\exp(-2x\log(\pm z+\sqrt{1+z^2})) = \frac{(-1)^q 2^q}{q!}\log^q(\pm z+\sqrt{1+z^2}).$$

The logarithms are both valid formal power series because the arguments are formal power series that start with constant coefficient equal to one. Note however that

$$\log(-z+\sqrt{1+z^2}) = -\log\frac{1}{-z+\sqrt{1+z^2}}$$
$$= -\log\frac{z+\sqrt{1+z^2}}{1+z^2-z^2} = -\log(z+\sqrt{1+z^2}).$$

This yields

$$[x^q]P_{m,n}(x) = -[z^{2m+1}](1+z^2)^{n+m}\frac{(-1)^q 2^q}{q!}\frac{1}{2}\log^q(z+\sqrt{1+z^2})(1-(-1)^q).$$

This is zero when q is even (inspect last term) and we have the claim. We get for q odd the coefficient

$$\frac{2^q}{q!}[z^{2m+1}](1+z^2)^{n+m}\log^q(z+\sqrt{1+z^2}).$$

The powered logarithm starts at z^q so we get zero when $q > 2m+1$. To see positivity note that an alternate representation is

$$(-1)^{m+1}[z^m](1+z)^{x-1/2-n}\sum_{k\ge 0}\binom{-2x}{2k+1}(-1)^k z^k$$
$$= (-1)^{m+1}[z^m](1+z)^{x-1/2-n}\sum_{k\ge 0}(-1)^k z^k [w^{2k+1}](1+w)^{-2x}$$
$$= (-1)^{m+1}[z^{2m+1}](1+z^2)^{x-1/2-n}\sum_{k\ge 0}(-1)^k z^{2k+1}[w^{2k+1}](1+w)^{-2x}$$
$$= i(-1)^m[z^{2m+1}](1+z^2)^{x-1/2-n}\sum_{k\ge 0} i^{2k+1} z^{2k+1}[w^{2k+1}](1+w)^{-2x}$$
$$= \frac{i}{2}(-1)^m[z^{2m+1}](1+z^2)^{x-1/2-n}((1+iz)^{-2x}-(1-iz)^{-2x})$$
$$= \frac{i}{2}(-1)^m[z^{2m+1}](1+z^2)^{-1/2-n}\left[\left(\frac{1-iz}{1+iz}\right)^x - \left(\frac{1+iz}{1-iz}\right)^x\right].$$

Extract the coefficient on $[x^q]$ to get

$$\frac{i}{2q!}(-1)^m[z^{2m+1}](1+z^2)^{-1/2-n}\left[\log^q\frac{1-iz}{1+iz}-\log^q\frac{1+iz}{1-iz}\right]$$
$$=\frac{i}{2q!}(-1)^m[z^{2m+1}](1+z^2)^{-1/2-n}\log^q\frac{1-iz}{1+iz}(1-(-1)^q).$$

Zero again for q even, and for q odd,

$$\frac{i}{q!}(-1)^m[z^{2m+1}](1+z^2)^{-1/2-n}\log^q\frac{1-iz}{1+iz}$$
$$=\frac{1}{q!}[z^{2m+1}](1-z^2)^{-1/2-n}\log^q\frac{1+z}{1-z}.$$

The two terms in the product both have series in positive terms only. To see this, note that the first one is

$$\sum_{p\ge 0}\binom{-1/2-n}{p}(-1)^p z^{2p}=\sum_{p\ge 0}\binom{p+n-1/2}{p}z^{2p}.$$

The coefficients are positive because $(p+n-1/2)^{\underline{p}}$ is. Note also that

$$\log\frac{1+z}{1-z}=\log\left(1+\frac{2z}{1-z}\right)=\sum_{p\ge 1}(-1)^{p-1}\frac{2^p z^p}{p(1-z)^p}.$$

This means the coefficient on $[z^r]$ is

$$\sum_{p=1}^{r}(-1)^{p-1}2^p[z^{r-p}]\frac{1}{p(1-z)^p}=\sum_{p=1}^{r}(-1)^{p-1}2^p\binom{r-1}{p-1}\frac{1}{p}$$
$$=\frac{1}{r}\sum_{p=1}^{r}(-1)^{p-1}2^p\binom{r}{p}=-\frac{1}{r}((1-2)^r-1)=-\frac{1}{r}((-1)^r-1).$$

This is zero when r is even and $\frac{2}{r}$ when r is odd. We have established the desired positivity of the coefficients, as we have the product of a series with positive coefficients times a power of a series that also has positive coefficients.

This was math.stackexchange.com problem 4703564.

1.153 MSE 4713851: A sum equals zero

We seek to prove with $n\ge 1$ and $2n\le m$ that

$$\sum_{k=0}^{n}(-1)^k\frac{1}{m-k}\binom{m-k}{k}\frac{1}{m+2n-2k}\binom{m+2n-2k}{n-k}=0.$$

Observe that

$$\frac{1}{m-k}\binom{m-k}{k}=\frac{1}{m}\left[\binom{m-k}{k}+\binom{m-1-k}{k-1}\right]$$

and

$$\frac{1}{m+2n-2k}\binom{m+2n-2k}{n-k} = \frac{1}{m}\left[\binom{m+2n-2k}{n-k} - 2\binom{m+2n-1-2k}{n-k-1}\right].$$

Therefore it is sufficient to prove that

$$\sum_{k=0}^{n}(-1)^k\left[\binom{m-k}{k}+\binom{m-1-k}{k-1}\right]$$
$$\times\left[\binom{m+2n-2k}{n-k} - 2\binom{m+2n-1-2k}{n-k-1}\right] = 0.$$

We get four pieces, the first is

$$\sum_{k=0}^{n}(-1)^k\binom{m-k}{k}\binom{m+2n-2k}{n-k}$$
$$= [z^n](1+z)^{m+2n}\sum_{k\ge 0}(-1)^k\binom{m-k}{m-2k}\frac{z^k}{(1+z)^{2k}}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$[z^n](1+z)^{m+2n}[w^m](1+w)^m\sum_{k\ge 0}(-1)^k\frac{w^{2k}}{(1+w)^k}\frac{z^k}{(1+z)^{2k}}$$
$$= [z^n](1+z)^{m+2n}[w^m](1+w)^m\frac{1}{1+w^2z/(1+w)/(1+z)^2}.$$

We get for the next piece

$$\sum_{k=0}^{n}(-1)^k\binom{m-1-k}{k-1}\binom{m+2n-2k}{n-k}$$
$$= [z^n](1+z)^{m+2n}[w^m](1+w)^{m-1}\frac{1}{1+w^2z/(1+w)/(1+z)^2}.$$

The third piece is

$$-2\sum_{k=0}^{n}(-1)^k\binom{m-k}{k}\binom{m+2n-1-2k}{n-1-k}$$
$$= -2[z^{n-1}](1+z)^{m+2n-1}[w^m](1+w)^m\frac{1}{1+w^2z/(1+w)/(1+z)^2}.$$

The fourth and last piece is

$$-2\sum_{k=0}^{n}(-1)^k\binom{m-1-k}{k-1}\binom{m+2n-1-2k}{n-1-k}$$

$$= -2[z^{n-1}](1+z)^{m+2n-1}[w^m](1+w)^{m-1}\frac{1}{1+w^2z/(1+w)/(1+z)^2}.$$

Adding the four pieces we obtain

$$[z^n](1+z)^{m+2n-1}[w^m](1+w)^{m-1}\frac{1}{1+w^2z/(1+w)/(1+z)^2}$$
$$\times[(1+z)(1+w)+(1+z)-2z(1+w)-2z]$$
$$= [z^n](1+z)^{m+2n+1}[w^m](1+w)^m\frac{(1-z)(2+w)}{(1+w)(1+z)^2+w^2z}$$
$$= [z^n](1+z)^{m+2n+1}[w^m](1+w)^m\frac{(1-z)(2+w)}{(w+1+z)(wz+1+z)}$$
$$= [z^{n+1}](1+z)^{m+2n+1}[w^m](1+w)^m\frac{(1-z)(2+w)}{(w+1+z)(w+(1+z)/z)}.$$

The contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^{m+1}}(1+w)^m\frac{2+w}{(w+1+z)(w+(1+z)/z)}.$$

Here the residue at infinity is zero and we may use minus the residues at $w=-(1+z)$ and $w=-(1+z)/z$. We get for the former

$$-[z^{n+1}](1+z)^{m+2n+1}\frac{(-1)^{m+1}}{(1+z)^{m+1}}(-1)^m z^m\frac{(1-z)^2}{-(1+z)+(1+z)/z}$$

$$= [z^{n+1}](1+z)^{2n}z^{m+1}\frac{(1-z)^2}{1-z^2} = [z^n](1+z)^{2n-1}z^m(1-z) = 0.$$

This is zero because $m > n$ due to the prerequisites. In fact we even have $m \ge 2n$. The second residue yields

$$-[z^{n+1}](1+z)^{m+2n+1}\frac{(-1)^{m+1}z^{m+1}}{(1+z)^{m+1}}\frac{(-1)^m}{z^m}\frac{(1-z)(2-(1+z)/z)}{-(1+z)/z+1+z}$$

$$= [z^{n+1}](1+z)^{2n}z\frac{(1-z)^2}{1-z^2} = [z^n](1+z)^{2n-1}(1-z)$$
$$= \binom{2n-1}{n}-\binom{2n-1}{n-1} = 0.$$

This concludes the argument.

This was math.stackexchange.com problem 4713851.

1.154 MSE 4722503: Euler numbers and Stirling numbers

First identity

We seek to prove that with Euler numbers

$$E_n = 2^{2n-1} \sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell+1} \left\{ n \atop \ell \right\} \left(3 \left(\frac{1}{4}\right)^{\overline{\ell}} - \left(\frac{3}{4}\right)^{\overline{\ell}} \right).$$

With the usual generating functions we have for the RHS

$$2^{2n-1} n! [z^n] \sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell+1} (\exp(z)-1)^\ell \left(3\binom{-3/4+\ell}{\ell} - \binom{-1/4+\ell}{\ell} \right)$$

$$= 2^{2n-1} n! [z^n] \sum_{\ell=1}^{n} \frac{1}{\ell+1} (\exp(z)-1)^\ell \left(3\binom{-1/4}{\ell} - \binom{-3/4}{\ell} \right)$$

$$= 2^{2n-1} n! [z^n] \frac{1}{\exp(z)-1} \sum_{\ell\ge 0} \frac{1}{\ell+1} (\exp(z)-1)^{\ell+1}$$

$$\times [w^\ell] \left(3(1+w)^{-1/4} - (1+w)^{-3/4} \right).$$

We have extended the range both ways due to the coefficient extractor. Continuing,

$$2^{2n-1} n! [z^n] \frac{1}{\exp(z)-1} (4\exp(z)^{3/4} - 4\exp(z)^{1/4})$$

$$= 2^{2n+1} n! [z^n] \frac{1}{\exp(z)-1} (\exp(z)^{3/4} - \exp(z)^{1/4})$$

$$= 2 \times n! [z^n] \frac{1}{\exp(4z)-1} (\exp(3z) - \exp(z))$$

$$= 2 \times n! [z^n] \frac{1}{\exp(2z)+1} \exp(z) = 2 \times n! [z^n] \frac{1}{\exp(z)+\exp(-z)}$$

$$= n! [z^n] \frac{1}{\cosh(z)}.$$

This is the claim.

Second identity

We seek to prove that with Euler numbers

$$E_{2n} = -4^{2n} \sum_{\ell=1}^{2n} \frac{(-1)^\ell}{\ell+1} \left\{ 2n \atop \ell \right\} \left(\frac{3}{4}\right)^{\overline{\ell}}.$$

We again have using standard generating functions for the RHS

$$\begin{aligned}
&-4^{2n}(2n)![z^{2n}]\sum_{\ell=1}^{2n}\frac{(-1)^\ell}{\ell+1}(\exp(z)-1)^\ell\binom{-1/4+\ell}{\ell} \\
&= -4^{2n}(2n)![z^{2n}]\sum_{\ell\ge 0}\frac{1}{\ell+1}(\exp(z)-1)^\ell\binom{-3/4}{\ell}
\end{aligned}$$

Here we have extended the range both ways due to the coefficient extractor. Continuing,

$$\begin{aligned}
&-4^{2n}(2n)![z^{2n}]\frac{1}{\exp(z)-1}\sum_{\ell\ge 0}\frac{1}{\ell+1}(\exp(z)-1)^{\ell+1}\binom{-3/4}{\ell} \\
&= -4^{2n}(2n)![z^{2n}]\frac{1}{\exp(z)-1}\sum_{\ell\ge 0}\frac{1}{\ell+1}(\exp(z)-1)^{\ell+1}[w^\ell](1+w)^{-3/4} \\
&= -4^{2n+1}(2n)![z^{2n}]\frac{1}{\exp(z)-1}(\exp(z)^{1/4}-1) \\
&= -4(2n)![z^{2n}]\frac{1}{\exp(4z)-1}(\exp(z)-1).
\end{aligned}$$

Here we are extracting even coefficients so this is

$$\begin{aligned}
&-2(2n)![z^{2n}]\left[\frac{\exp(z)-1}{\exp(4z)-1}+\frac{\exp(-z)-1}{\exp(-4z)-1}\right] \\
&= -2(2n)![z^{2n}]\left[\frac{\exp(z)-1}{\exp(4z)-1}+\frac{\exp(4z)-\exp(3z)}{\exp(4z)-1}\right] \\
&= -2(2n)![z^{2n}]\frac{\exp(z)-1}{\exp(4z)-1}(1+\exp(3z)).
\end{aligned}$$

Now as this is for $n\ge 1$ we should get a constant difference with the Euler number EGF, and indeed we have

$$\begin{aligned}
\frac{2}{\exp(z)+\exp(-z)} &= 2\frac{\exp(z)}{\exp(2z)+1} = 2\frac{\exp(z)(\exp(2z)-1)}{\exp(4z)-1} \\
&= 2\frac{\exp(z)(\exp(z)-1)(\exp(z)+1)}{\exp(4z)-1}
\end{aligned}$$

Substract to get

$$2\frac{\exp(z)-1}{\exp(4z)-1}(\exp(2z)+\exp(z)+1+\exp(3z)) = 2.$$

This is the claim and we may conclude.

This was math.stackexchange.com problem 4722503.

1.155 MSE 495371: Even-index binomial coefficient convolution

We seek to prove that

$$\sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} = \frac{1}{2} \frac{n}{n-k} 2^{n-2k} \binom{n-k}{k}$$

where $0 \le k \le n/2$. We have for the LHS

$$[z^n](1+z)^n \sum_{j \ge k} z^{2j} \binom{j}{k}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$\begin{aligned}
[z^n] z^{2k} (1+z)^n \sum_{j \ge 0} z^{2j} \binom{j+k}{k} &= [z^{n-2k}](1+z)^n \frac{1}{(1-z^2)^{k+1}} \\
&= [z^{n-2k}](1+z)^{n-k-1} \frac{1}{(1-z)^{k+1}} \\
&= \sum_{q=0}^{n-2k} \binom{n-k-1}{q} \binom{n-k-q}{k} \\
&= \frac{1}{n-k} \sum_{q=0}^{n-2k} (n-k-q) \binom{n-k}{q} \binom{n-k-q}{k}.
\end{aligned}$$

Now observe that

$$\binom{n-k}{q} \binom{n-k-q}{k} = \frac{(n-k)!}{q! \times k! \times (n-2k-q)!} = \binom{n-k}{k} \binom{n-2k}{q}.$$

This yields

$$\frac{1}{n-k} \binom{n-k}{k} \sum_{q=0}^{n-2k} (n-k-q) \binom{n-2k}{q}.$$

The sum produces two pieces, the first is

$$(n-k) 2^{n-2k}$$

by inspection. The second is

$$- \sum_{q=0}^{n-2k} q \binom{n-2k}{q} = -(n-2k) \sum_{q=1}^{n-2k} \binom{n-2k-1}{q-1}$$

$$= -(n-2k)2^{n-2k-1}.$$

Collecting everything we have

$$\frac{1}{n-k}\binom{n-k}{k}2^{n-2k}(n-k-\frac{1}{2}(n-2k)) = \frac{1}{n-k}\binom{n-k}{k}2^{n-2k}\frac{1}{2}n.$$

This is the claim.

This was math.stackexchange.com problem 495371.

1.156 MSE 4731417: Kravchuk polynomials

Starting from

$$K_k = \sum_{j=0}^{k}(-1)^j(q-1)^{k-j}\binom{X}{j}\binom{n-X}{k-j}$$

we first obtain

$$K_k = [z^k](1+z)^{n-X}\sum_{j\ge 0}(-1)^j(q-1)^{k-j}z^j\binom{X}{j}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$(q-1)^k[z^k](1+z)^{n-X}\sum_{j\ge 0}(-1)^j(q-1)^{-j}z^j\binom{X}{j}$$
$$= (q-1)^k[z^k](1+z)^{n-X}(1-z/(q-1))^X$$
$$= [z^k](1+(q-1)z)^{n-X}(1-z)^X.$$

First identity

We seek to verify that

$$K_k = \sum_{j=0}^{k}(-q)^j(q-1)^{k-j}\binom{n-j}{k-j}\binom{X}{j}.$$

We get for the sum

$$[z^k](1+z)^n\sum_{j\ge 0}(-q)^j(q-1)^{k-j}(1+z)^{-j}z^j\binom{X}{j}.$$

Here the coefficient extractor has once more enforced the range. Continuing,

$$(q-1)^k[z^k](1+z)^n\sum_{j\ge 0}(-q)^j(q-1)^{-j}(1+z)^{-j}z^j\binom{X}{j}$$

$$= (q-1)^k [z^k](1+z)^n \left[1 - \frac{qz}{(q-1)(1+z)}\right]^X$$

$$= [z^k](1+(q-1)z)^n \left[1 - \frac{qz}{1+(q-1)z}\right]^X$$

$$= [z^k](1+(q-1)z)^{n-X}[1+(q-1)z-qz]^X$$

$$= [z^k](1+(q-1)z)^{n-X}(1-z)^X.$$

This is the claim.

Second identity

Here we set out to simplify

$$K_k = \sum_{j=0}^{k} (-1)^j q^{k-j} \binom{n-k+j}{j} \binom{n-X}{k-j}.$$

We get for the sum

$$[z^k](1+z)^{n-X} \sum_{j\ge 0} (-1)^j q^{k-j} \binom{n-k+j}{j} z^j.$$

This is the third time with the extractor enforcing the range. Continuing,

$$q^k [z^k](1+z)^{n-X} \sum_{j\ge 0} (-1)^j q^{-j} \binom{n-k+j}{j} z^j$$

$$= q^k [z^k](1+z)^{n-X} \frac{1}{(1+z/q)^{n-k+1}} = [z^k](1+qz)^{n-X} \frac{1}{(1+z)^{n-k+1}}.$$

This is

$$\operatorname*{res}_z \frac{1}{z^{k+1}} (1+z)^{k+1} (1+qz)^{n-X} \frac{1}{(1+z)^{n+2}}.$$

Now we put $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2 \; dw$ to get

$$\operatorname*{res}_w \frac{1}{w^{k+1}} (1+qw/(1-w))^{n-X} \frac{1}{(1+w/(1-w))^{n+2}} \frac{1}{(1-w)^2}$$

$$= \operatorname*{res}_w \frac{1}{w^{k+1}} (1+qw/(1-w))^{n-X} (1-w)^n$$

$$= \operatorname*{res}_w \frac{1}{w^{k+1}} (1-w+qw)^{n-X} (1-w)^X$$

$$= [w^k](1+(q-1)w)^{n-X}(1-w)^X.$$

Once more we have the claim.

This was math.stackexchange.com problem 4731417.

1.157 MSE 4762542: Binomial-Bernoulli convolution

We seek

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{\alpha^{2k+1}} \frac{B_{2k+2}}{(n-2k)!(2k+2)!}$$

We first try to evaluate

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{\alpha^{2k+1}} \frac{B_{2k+2}}{(n-2k)!(2k+2)!}$$

and we note that this is

$$\sum_{k=0}^{n} \frac{1}{\alpha^{k+1}} \frac{B_{k+2}}{(n-k)!(k+2)!}$$

We get with the EGF of the Bernoulli numbers

$$\sum_{k=0}^{n} \frac{1}{\alpha^{k+1}} \frac{1}{(n-k)!} [z^{k+2}] \frac{z}{\exp(z)-1}$$
$$= [w^n] \exp(w) \sum_{k\ge 0} \frac{1}{\alpha^{k+1}} w^k [z^{k+2}] \frac{z}{\exp(z)-1}.$$

Here we have extended to infinity due to the coefficient extractor. Continuing,

$$\alpha [w^{n+2}] \exp(w) \sum_{k\ge 0} \frac{1}{\alpha^{k+2}} w^{k+2} [z^{k+2}] \frac{z}{\exp(z)-1}$$
$$= \alpha [w^{n+2}] \exp(w) \sum_{k\ge 0} w^{k+2} [z^{k+2}] \frac{z/\alpha}{\exp(z/\alpha)-1}$$
$$= [w^{n+2}] \exp(w) \sum_{k\ge 0} w^{k+2} [z^{k+2}] \frac{z}{\exp(z/\alpha)-1}$$
$$= [w^{n+2}] \exp(w) \left[\frac{w}{\exp(w/\alpha)-1} - \alpha + \frac{1}{2} w\right]$$
$$= -\frac{\alpha}{(n+2)!} + \frac{1}{2} \frac{1}{(n+1)!} + [w^{n+2}] \exp(w) \frac{w}{\exp(w/\alpha)-1}.$$

Now observe that

$$\sum_{j=0}^{\alpha-1} j^{n+1} = (n+1)! \sum_{j=0}^{\alpha-1} [w^{n+1}] \exp(jw)$$

$$= (n+1)! [w^{n+1}] \frac{\exp(w\alpha) - 1}{\exp(w) - 1}$$

$$= (n+1)! [w^{n+2}] \frac{w \exp(w\alpha) - w}{\exp(w) - 1}$$

$$= -\frac{B_{n+2}}{n+2} + (n+1)! [w^{n+2}] \frac{w \exp(w\alpha)}{\exp(w) - 1}$$

$$= -\frac{B_{n+2}}{n+2} + \alpha^{n+1} (n+1)! [w^{n+2}] \frac{w \exp(w)}{\exp(w/\alpha) - 1}.$$

Merge the two to get

$$-\frac{\alpha}{(n+2)!} + \frac{1}{2} \frac{1}{(n+1)!} + \frac{1}{\alpha^{n+1}(n+1)!} \sum_{j=0}^{\alpha-1} j^{n+1} + \frac{B_{n+2}}{\alpha^{n+1}(n+2)!}.$$

To conclude recall that OP asks for the upper limit $\lfloor (n-1)/2 \rfloor$ rather than $\lfloor n/2 \rfloor$. These are equal when n is odd but when n is even the top term is missing. This term is (substitute and simplify)

$$\frac{1}{\alpha^{n+1}} \frac{B_{n+2}}{(n+2)!}$$

so that we have

$$-\frac{\alpha}{(n+2)!} + \frac{1}{2} \frac{1}{(n+1)!} + \frac{1}{\alpha^{n+1}(n+1)!} \sum_{j=0}^{\alpha-1} j^{n+1}$$
$$+ \frac{B_{n+2}}{\alpha^{n+1}(n+2)!} (1 - [[n \text{ even}]]).$$

However when n is odd we get zero from the Bernoulli number and when n is even we get zero from the parenthesized term containing the Iverson bracket. We have shown that

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{\alpha^{2k+1}} \frac{B_{2k+2}}{(n-2k)!(2k+2)!}$$
$$= \frac{1}{2} \frac{n+2-2\alpha}{(n+2)!} + \frac{1}{\alpha^{n+1}(n+1)!} \sum_{j=0}^{\alpha-1} j^{n+1}.$$

Remark. For the initial segment of the series about zero of $z/(\exp(z/\alpha)-1)$ we write

$$\alpha\frac{z/\alpha}{\exp(z/\alpha)-1} = \alpha\frac{B_0}{0!} + \alpha\frac{B_1}{1!}\frac{z}{\alpha} + \cdots = \alpha - \frac{1}{2}z + \cdots$$

This was math.stackexchange.com problem 4762542.

1.158 MSE 4774167: Two probabilities

We seek to prove that with $1 \le q \le n$ and $0 \le p \le 1$ that

$$\sum_{k=q}^{n} \binom{k-1}{q-1} p^q (1-p)^{k-q} = \sum_{k=q}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

We start on the LHS with

$$p^q \sum_{k=0}^{n-q} \binom{k+q-1}{q-1} (1-p)^k$$

$$= p^q [z^{n-q}] \frac{1}{1-z} \sum_{k\ge 0} \binom{k+q-1}{q-1} (1-p)^k z^k$$

$$= p^q [z^{n-q}] \frac{1}{1-z} \frac{1}{(1-(1-p)z)^q}.$$

This is

$$(-1)^q \frac{p^q}{(1-p)^q} \operatorname*{res}_{z} \frac{1}{z^{n-q+1}} \frac{1}{1-z} \frac{1}{(z-1/(1-p))^q}.$$

Now residues sum to zero and the residue at infinity is zero so this is minus the residue at $z=1$ plus minus the residue at $z=1/(1-p)$. The former is one by inspection. For the latter we need the Leibniz rule:

$$\frac{1}{(q-1)!} \left(\frac{1}{z^{n-q+1}} \frac{1}{1-z} \right)^{(q-1)}$$

$$= \frac{1}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{(-1)^k}{z^{n-q+1+k}} (n-q+1)^{\overline{k}} \frac{1}{(1-z)^{1+q-1-k}} 1^{\overline{q-1-k}}$$

$$= \frac{1}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{(-1)^k}{z^{n-q+1+k}} \binom{n-q+k}{k} k! \frac{1}{(1-z)^{q-k}} (q-1-k)!$$

$$= \sum_{k=0}^{q-1} \frac{(-1)^k}{z^{n-q+1+k}} \binom{n-q+k}{k} \frac{1}{(1-z)^{q-k}}.$$

Evaluate at $z = 1/(1-p)$ to get

$$\sum_{k=0}^{q-1}(-1)^k(1-p)^{n-q+1+k}\binom{n-q+k}{k}\frac{(1-p)^{q-k}}{(-1)^{q-k}p^{q-k}}.$$

Collecting the two contributions we find

$$\begin{aligned}
&1-(1-p)^{n-(q-1)}\sum_{k=0}^{q-1}\binom{n-q+k}{k}p^k \\
&= 1-(1-p)^{n-(q-1)}[z^{q-1}]\frac{1}{1-z}\sum_{k\ge 0}\binom{n-q+k}{k}p^k z^k \\
&= 1-(1-p)^{n-(q-1)}[z^{q-1}]\frac{1}{1-z}\frac{1}{(1-pz)^{n-q+1}}
\end{aligned}$$

On the other hand we get for the RHS

$$1-\sum_{k=0}^{q-1}\binom{n}{k}p^k(1-p)^{n-k}$$

so we need to show that the coefficient extractor term is equal to the sum, which yields

$$[z^{q-1}]\frac{1}{1-z}\sum_{k\ge 0}\binom{n}{k}p^k(1-p)^{n-k}z^k = \operatorname*{res}_{z}\frac{1}{z^q}\frac{1}{1-z}(pz+1-p)^n.$$

Now put $z/(pz+1-p) = w$ so that $z = (1-p)w/(1-pw)$ and $dz = (1-p)/(1-pw)^2\, dw$ as well as $pz+1-p = (1-p)/(1-pw)$ to obtain

$$\begin{aligned}
&\operatorname*{res}_{w}\frac{1}{w^q}\frac{1-pw}{1-w}\frac{(1-p)^{n-q}}{(1-pw)^{n-q}}\frac{1-p}{(1-pw)^2} \\
&= (1-p)^{n-(q-1)}\operatorname*{res}_{w}\frac{1}{w^q}\frac{1}{1-w}\frac{1}{(1-pw)^{n-q+1}}.
\end{aligned}$$

This is the claim.

This was math.stackexchange.com problem 4774167.

1.159 MSE 4791957: Motzkin numbers

We seek to prove that the recurrence

$$a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k a_{n-k}$$

where $a_0 = a_1 = 1$ is solved by

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n}{2k}.$$

Starting from the recurrence we have for the generating function $M(z)$ that

$$[z^{n+2}]M(z) = [z^{n+1}]M(z) + [z^n]M(z)^2.$$

Multiply by w^n and sum to infinity to get

$$\sum_{n\ge 0} w^n [z^{n+2}]M(z) = \sum_{n\ge 0} w^n [z^{n+1}]M(z) + M(w)^2$$

or

$$w^{-2} \sum_{n\ge 0} w^{n+2} [z^{n+2}]M(z) = w^{-1} \sum_{n\ge 0} w^{n+1} [z^{n+1}]M(z) + M(w)^2$$

which is

$$w^{-2}(M(w) - w - 1) = w^{-1}(M(w) - 1) + M(w)^2.$$

Solve this to obtain

$$M(w) = \frac{1 - w - \sqrt{1 - 2w - 3w^2}}{2w^2}$$

which is indeed OEIS A001006, Motzkin numbers. Note that we get

$$\frac{1}{M(w)} = \frac{2w^2}{1 - w - \sqrt{1 - 2w - 3w^2}}$$

$$= 2w^2 \frac{1 - w + \sqrt{1 - 2w - 3w^2}}{(1-w)^2 - (1 - 2w - 3w^2)} = -w^2 \frac{-1 + w - \sqrt{1 - 2w - 3w^2}}{2w^2}$$

$$= -w^2 \left(M(w) - \frac{1}{w^2} + \frac{1}{w} \right) = 1 - w - w^2 M(w).$$

On the other hand we have working with

$$\widetilde{M}(w) = \sum_{n\ge 0} w^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n}{2k}$$

that

$$\binom{n}{2k} \binom{2k}{k} = \frac{n!}{(n-2k)! \times k! \times k!} = \binom{n}{k} \binom{n-k}{n-2k}$$

and obtain

$$\widetilde{M}(w) = \sum_{n\ge 0} w^n \sum_{k=0}^{\lfloor n/2\rfloor} \frac{1}{k+1}\binom{n}{k}\binom{n-k}{n-2k}$$

$$= \sum_{n\ge 0} w^n [z^n](1+z)^n \sum_{k\ge 0} \frac{1}{k+1}\binom{n}{k}(1+z)^{-k} z^{2k}.$$

Here we have extended to infinity due to the coefficient extractor in z. Continuing,

$$\widetilde{M}(w) = \sum_{n\ge 0} \frac{w^n}{n+1} [z^n](1+z)^n \sum_{k\ge 0} \binom{n+1}{k+1}(1+z)^{-k} z^{2k}$$

$$= \sum_{n\ge 0} \frac{w^n}{n+1} [z^n](1+z)^{n+1} z^{-2} \sum_{k\ge 0} \binom{n+1}{k+1}(1+z)^{-k-1} z^{2(k+1)}$$

$$= \sum_{n\ge 0} \frac{w^n}{n+1} [z^{n+2}](1+z)^{n+1} \left(-1 + \left[1 + \frac{z^2}{1+z}\right]^{n+1}\right).$$

The contribution from the minus one term is zero and we find for the inner term

$$[z^{n+2}](1+z)^{n+1} \left[1 + \frac{z^2}{1+z}\right]^{n+1} = [z^{n+2}][1+z+z^2]^{n+1}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^2}\frac{1}{z^{n+1}}[1+z+z^2]^{n+1}.$$

Now put $z = vM(v)$ so that $z/(1+z+z^2) = v$ and $dz = (M(v)+vM'(v))\; dv$ to get

$$\operatorname*{res}_v \frac{1}{v^2 M(v)^2}\frac{1}{v^{n+1}}(M(v)+vM'(v)) = -\operatorname*{res}_v \frac{1}{v^{n+1}}\left(\frac{1}{vM(v)}\right)'.$$

We thus have

$$\widetilde{M}(w) = -\sum_{n\ge 0} \frac{w^n}{n+1} \operatorname*{res}_v \frac{1}{v^{n+1}}\left(\frac{1}{vM(v)}\right)'$$

so that

$$w\widetilde{M}(w) = -\sum_{n\ge 0} \frac{w^{n+1}}{n+1}[v^n]\left(\frac{1}{vM(v)}\right)'$$

$$= -\sum_{n\ge 0} \frac{w^{n+1}}{n+1}[v^n]\left(\frac{1}{v} - 1 - vM(v)\right)' = wM(w)$$

and we finally have

$$\widetilde{M}(w) = M(w)$$

as desired.
This was math.stackexchange.com problem 4791957.

1.160 MSE 4821034: An inverse binomial coefficient

Supposing we start from

$$\sum_{q=0}^{n} \binom{n}{q} \frac{q}{q+1} \binom{2n}{q+1}^{-1} = \frac{1}{n+1}.$$

The LHS is

$$n \sum_{q=1}^{n} \binom{n-1}{q-1} \frac{1}{q+1} \binom{2n}{q+1}^{-1}.$$

Recall from 1.89 the following identity which was proved there: with $1 \le k \le n$

$$\frac{1}{k} \binom{n}{k}^{-1} = [z^n] \log \frac{1}{1-z} (z-1)^{n-k}.$$

In the present case we get

$$n \sum_{q=1}^{n} \binom{n-1}{q-1} [z^{2n}] \log \frac{1}{1-z} (z-1)^{2n-q-1}$$

$$= n[z^{2n}] \log \frac{1}{1-z} (z-1)^{2n-2} \sum_{q=1}^{n} \binom{n-1}{q-1} \frac{1}{(z-1)^{q-1}}$$

$$= n[z^{2n}] \log \frac{1}{1-z} (z-1)^{2n-2} \left[1 + \frac{1}{z-1}\right]^{n-1}$$

$$= n[z^{2n}] \log \frac{1}{1-z} (z-1)^{n-1} z^{n-1} = n[z^{n+1}] \log \frac{1}{1-z} (z-1)^{n-1}.$$

Apply the identity again (with $k = 2$) to get

$$n \frac{1}{2} \binom{n+1}{2}^{-1} = \frac{1}{2} n \frac{2}{(n+1)n} = \frac{1}{n+1}.$$

This is the claim.
This was math.stackexchange.com problem 4821034.

1.161 MSE 4830342: Euler numbers, Stirling numbers and Touchard polynomials

We seek in terms of Euler numbers

$$E_n = -\sqrt{2} \sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{k!}{\sqrt{2}^k} \cos(3\pi(k+1)/4) = 2 \int_0^\infty \exp(-t) \cos(t) T_n(-t) \; dt.$$

The first of these is

$$E_n = -\sqrt{2} \; \Re \sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{k!}{\sqrt{2}^k} \exp(3\pi i(k+1)/4).$$

Using the Stirling set number EGF this becomes

$$-\Re \left[n![z^n] \; (-1+i) \sum_{k=0}^{n} (\exp(z)-1)^k \frac{1}{\sqrt{2}^k} \exp(3\pi i k/4) \right].$$

We may extend the inner sum to infinity due to the coefficient extractor and the fact that $\exp(z) - 1 = z + \cdots$, getting

$$\Re \left[n![z^n] \; (1-i) \frac{1}{1 - \exp(3\pi i/4)(\exp(z)-1)/\sqrt{2}} \right] = \Re \; n![z^n] f(z).$$

We have that $f(z)$ simplifies to

$$\frac{2(1-i)}{1+i+\exp(z)(1-i)} = \frac{2}{i+\exp(z)}.$$

For the real part we need the EGF of the conjugates. Using Mittag-Leffler we start with (poles of $f(z)$ are simple with residue $2i$)

$$g(z) = \sum_k \frac{2i}{z - (-\pi i/2 + 2\pi i k)} = \sum_k \frac{i(2z+\pi i)}{(z+\pi i/2)^2 - (2\pi i k)^2}$$

(the latter is convergent) and we can evaluate the sum through the residues in w using the function

$$h(z,w) = -\frac{1}{w+(z+\pi i/2)} \frac{1}{w-(z+\pi i/2)} i(2z+\pi i) \cot(-iw/2) \frac{1}{2i}.$$

We use the fact that the residues in w of $h(z,w)$ sum to zero and the residue at infinity is zero so that the residue at the two simple poles $w = \pm(z+\pi i/2)$ produces for $g(z)$ (flip sign due to residue sum)

$$\begin{aligned} g(z) &= -\frac{1}{2}\cot(-i(-(z+\pi i/2))/2) + \frac{1}{2}\cot(-i(+(z+\pi i/2))/2) \\ &= -\frac{1}{2}\cot(iz/2-\pi/4) + \frac{1}{2}\cot(-iz/2+\pi/4) \\ &= \cot(-iz/2+\pi/4). \end{aligned}$$

We get for $g(z)$

$$\begin{aligned} &\frac{\cos(-iz/2+\pi/4)}{\sin(-iz/2+\pi/4)} \\ &= i\frac{\exp(z/2)\exp(\pi i/4)+\exp(-z/2)\exp(-\pi i/4)}{\exp(z/2)\exp(\pi i/4)-\exp(-z/2)\exp(-\pi i/4)} \\ &= i\frac{\exp(z)\exp(\pi i/4)+\exp(-\pi i/4)}{\exp(z)\exp(\pi i/4)-\exp(-\pi i/4)} = i\frac{\exp(z)i+1}{\exp(z)i-1} = \frac{\exp(z)i+1}{i+\exp(z)}. \end{aligned}$$

The difference between $f(z)$ and $g(z)$ is exactly $-i$ and we finally have

$$f(z) = -i + \sum_k \frac{2i}{z-(-\pi i/2+2\pi ik)}.$$

We seek the generating function of the conjugates which can now be obtained by inspection and is seen to be (expand terms into a series about zero)

$$f_C(z) = i + \sum_k \frac{-2i}{z-(\pi i/2-2\pi ik)} = i - \sum_k \frac{2i}{z-(\pi i/2-2\pi ik)}.$$

Here we have applied conjugation to

$$\frac{C}{z-\rho} = -\frac{C}{\rho}\frac{1}{1-z/\rho} = -\frac{C}{\rho}\sum_{q\ge 0}\frac{z^q}{\rho^q}.$$

We claim this is $-f(z-\pi i)$ and check

$$-f(z-\pi i) = i - \sum_k \frac{2i}{z-\pi i-(-\pi i/2+2\pi ik)} = i - \sum_k \frac{2i}{z-(\pi i/2+2\pi ik)}.$$

We are iterating over k in two possible directions. Returning to the original problem we have that our answer is given by

$$\Re\, n![z^n]f(z) = \frac{1}{2}n![z^n](f(z)+f_C(z)).$$

Using the expression for $f_C(z)$ in terms of $f(z)$ this becomes

$$\frac{1}{i+\exp(z)} - \frac{1}{i+\exp(z-\pi i)} = \frac{1}{i+\exp(z)} - \frac{1}{i-\exp(z)}$$

$$= \frac{i - \exp(z) - i - \exp(z)}{(-1) - \exp(2z)} = \frac{2\exp(z)}{\exp(2z)+1} = \frac{1}{\cosh(z)}.$$

We have obtained the EGF of the Euler numbers and may conclude.

Remark. The computation of the residue at infinity being zero may be seen at the following MSE link.

For the second one OP proposes in terms of Touchard polynomials

$$E_n = 2\int_0^\infty \exp(-t)\cos(t) T_n(-t)\; dt.$$

This is

$$2\sum_{k=0}^{n} {n \brace k} \Re \int_0^\infty \exp(-t(1-i))(-1)^k t^k \; dt.$$

Now put $(1-i)t = u$ so that $\frac{1}{2}(1+i)u = t$ and we get

$$\sum_{k=0}^{n} {n \brace k} \frac{1}{2^k}(-1)^k \; \Re \; (1+i)^{k+1} \int_0^{\infty(1-i)} \exp(-u) u^k \; du.$$

Evaluating the integral with the Gamma function we get

$$\sum_{k=0}^{n} {n \brace k} \frac{1}{2^k}(-1)^k \; \Re \; (1+i)^{k+1} k!.$$

This has EGF

$$n![z^n]\Re\left[(1+i)\sum_{k\ge 0}(\exp(z)-1)^k \frac{1}{2^k}(-1)^k(1+i)^k\right]$$

where we have extended to infinity due to $\exp(z) - 1 = z + \cdots$. This is

$$n![z^n] \; \Re \; \frac{1+i}{1 + (\exp(z)-1)(1+i)/2} = n![z^n] \; \Re \; f(z).$$

We have that $f(z)$ simplifies to

$$\frac{2(1+i)}{1 - i + (1+i)\exp(z)} = \frac{2}{-i + \exp(z)} = -\frac{2}{i - \exp(z)}.$$

Note that we learned in the companion answer that the EGF of the conjugates of $f(z)$ is $\frac{2}{i+\exp(z)}$. We thus obtain one more time that

$$\frac{1}{2}(f(z) + f_C(z)) = \frac{1}{\cosh(z)}$$

and may conclude.

Remark. For the Gamma function evaluation we use a pizza slice contour with an angle of $-\pi/4$ which contains no poles. So to apply the Gamma function we just need to show that the contribution from the arc Q vanishes in the limit.

We get with $z = R\exp(i\theta)$ and $dz = iR\exp(i\theta)\; d\theta$

$$\int_Q \exp(-z) z^k \; dz = \int_{-\pi/4}^0 \exp(-R\exp(i\theta)) R^k \exp(ki\theta) iR\exp(i\theta)\; d\theta.$$

We have as an upper bound on the norm of this integral

$$\begin{aligned}\int_{-\pi/4}^0 \exp(-R\cos(\theta)) R^{k+1} \; d\theta &< R^{k+1} \exp(-R/\sqrt{2}) \int_{-\pi/4}^0 1 \; d\theta \\ &= \frac{\pi}{4} R^{k+1} \exp(-R/\sqrt{2}) \to 0\end{aligned}$$

as $R \to \infty$.

This was math.stackexchange.com problem 4830342.

1.162 MSE 4832009: A triple binomial

We seek to prove that

$$\sum_{k=j}^{\lfloor n/2 \rfloor} \frac{1}{4^k} {n \choose 2k} {k \choose j} {2k \choose k} = \frac{1}{2^n} {2n-2j \choose n-j} {n-j \choose j}.$$

Observe that

$${n \choose 2k} {2k \choose k} = \frac{n!}{(n-2k)! \times k! \times k!} = {n \choose k} {n-k \choose n-2k}$$

so we get for the LHS

$$\sum_{k=j}^{\lfloor n/2 \rfloor} {n \choose k} \frac{1}{4^k} {k \choose j} {n-k \choose n-2k}.$$

The coefficient extractor for rightmost binomial coefficient enforces the upper range:

$$\begin{aligned}&[z^n](1+z)^n \sum_{k \ge j} {n \choose k} \frac{z^{2k}}{(1+z)^k} \frac{1}{4^k} {k \choose j} \\ &= [z^n](1+z)^n [w^j] \sum_{k \ge j} {n \choose k} \frac{z^{2k}}{(1+z)^k} \frac{1}{4^k} (1+w)^k.\end{aligned}$$

Now when $k < j$ we get zero from the coefficient extractor in w so it enforces the lower range:

$$[z^n](1+z)^n[w^j]\sum_{k\ge 0}\binom{n}{k}\frac{z^{2k}}{(1+z)^k}\frac{1}{4^k}(1+w)^k$$

$$= [z^n](1+z)^n[w^j]\left[1+\frac{z^2(1+w)}{4(1+z)}\right]^n$$

$$= \frac{1}{2^{2n}}[z^n][w^j][4+4z+z^2(1+w)]^n$$

$$= \frac{1}{2^{2n}}[z^n][w^j][(z+2)^2+z^2w]^n$$

$$= \frac{1}{2^{2n}}[z^n]\binom{n}{j}z^{2j}(z+2)^{2n-2j} = \frac{1}{2^{2n}}[z^{n-2j}]\binom{n}{j}(z+2)^{2n-2j}$$

$$= \frac{1}{2^{2n}}\binom{n}{j}\binom{2n-2j}{n-2j}2^n = \frac{1}{2^n}\binom{n}{j}\binom{2n-2j}{n-2j}.$$

Use

$$\binom{n}{j}\binom{2n-2j}{n-2j} = \frac{(2n-2j)!}{(n-j)!\times j!\times(n-2j)!} = \binom{2n-2j}{n-j}\binom{n-j}{j}$$

to conclude.
This was math.stackexchange.com problem 4832009.

1.163 MSE 4843051: Double sum with an absolute value

We seek a closed form of

$$\sum_{p=0}^{n-1}\sum_{q=0}^{n}|n-p-q|\binom{n+p-q}{p}\binom{n-p+q-1}{q}.$$

First part

We get for the argument to the absolute value being positive the contribution

$$\sum_{p=0}^{n-1}\sum_{q=0}^{n-1-p}(n-p-q)\binom{n+p-q}{p}\binom{n-p+q-1}{q}.$$

This is

$$\sum_{p=0}^{n-1}\sum_{q=0}^{n-1-p}(q+1)\binom{n+p-(n-1-p-q)}{p}\binom{n-p+(n-1-p-q)-1}{n-1-p-q}$$

$$= \sum_{p=0}^{n-1} \sum_{q=0}^{n-1-p} (q+1) \binom{2p+q+1}{p} \binom{2n-2p-q-2}{n-p-q-1}$$

$$= \sum_{p=0}^{n-1} \sum_{q=0}^{p} (q+1) \binom{2(n-1-p)+q+1}{n-1-p} \binom{2n-2(n-1-p)-q-2}{n-(n-1-p)-q-1}$$

$$= \sum_{p=0}^{n-1} \sum_{q=0}^{p} (q+1) \binom{2n-2p+q-1}{n-1-p} \binom{2p-q}{p-q}$$

$$= [z^{n-1}](1+z)^{2n-1} \sum_{p=0}^{n-1} z^p (1+z)^{-2p} \sum_{q=0}^{p} (q+1)(1+z)^q \binom{2p-q}{p-q}$$

$$= [z^{n-1}](1+z)^{2n-1} \sum_{p=0}^{n-1} z^p (1+z)^{-2p} [w^p](1+w)^{2p}$$

$$\times \sum_{q=0}^{p} (q+1)(1+z)^q w^q (1+w)^{-q}.$$

We may extend the inner sum to infinity due to the coefficient extractor in w, getting

$$[z^{n-1}](1+z)^{2n-1} \sum_{p=0}^{n-1} z^p (1+z)^{-2p} [w^p](1+w)^{2p} \frac{1}{(1-(1+z)w/(1+w))^2}$$

$$= [z^{n-1}](1+z)^{2n-1} \sum_{p=0}^{n-1} z^p (1+z)^{-2p} [w^p](1+w)^{2p+2} \frac{1}{(1+w-(1+z)w)^2}$$

$$= [z^{n-1}](1+z)^{2n-1} \sum_{p\ge 0} z^p (1+z)^{-2p} [w^p](1+w)^{2p+2} \frac{1}{(1-wz)^2}.$$

Here we have extended to infinity due to the extractor in z. The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{p+1}} (1+w)^{2p+2} \frac{1}{(1-wz)^2}.$$

Now put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2 \; dv$ to get

$$\operatorname*{res}_{v} \frac{1}{v^{p+1}} \frac{1}{(1-v)^{p+1}} \frac{1}{(1-vz/(1-v))^2} \frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_{v} \frac{1}{v^{p+1}} \frac{1}{(1-v)^{p+1}} \frac{1}{(1-v-vz)^2}.$$

Next put $v = (1 - \sqrt{1-4u})/2$ so that $v(1-v) = u$ and $dv = 1/\sqrt{1-4u}\, du$ to obtain

$$\operatorname*{res}_u \frac{1}{u^{p+1}} \frac{1}{(1-(1+z)(1-\sqrt{1-4u})/2)^2} \frac{1}{\sqrt{1-4u}}.$$

Evaluate at $u = z/(1+z)^2$ to cancel the remaining sum (substitution rule), here we get

$$\sqrt{1-4u} = \sqrt{1-4z/(1+z)^2} = \frac{1}{1+z}\sqrt{(1+z)^2-4z} = \frac{1-z}{1+z}$$

and obtain

$$\begin{aligned}
&[z^{n-1}](1+z)^{2n-1} \frac{1}{(1-(1+z)(1-(1-z)/(1+z))/2)^2} \frac{1+z}{1-z} \\
&= [z^{n-1}](1+z)^{2n-1} \frac{1}{(1-(1+z-(1-z))/2)^2} \frac{1+z}{1-z} \\
&= [z^{n-1}](1+z)^{2n} \frac{1}{(1-z)^3}.
\end{aligned}$$

Second part

We get for the argument to the absolute value being negative the contribution

$$\begin{aligned}
&-\sum_{p=0}^{n-1} \sum_{q=n-p}^{n} (n-p-q) \binom{n+p-q}{p} \binom{n-p+q-1}{q} \\
&= -\sum_{p=0}^{n-1} \sum_{q=0}^{p} (n-p-(n-q)) \binom{n+p-(n-q)}{p} \binom{n-p+n-q-1}{n-q} \\
&= \sum_{p=0}^{n-1} \sum_{q=0}^{p} (p-q) \binom{p+q}{p} \binom{2n-p-q-1}{n-q} \\
&= \sum_{p=0}^{n-1} \sum_{q=0}^{p} q \binom{2p-q}{p} \binom{2n-2p+q-1}{n-p+q} \\
&= \sum_{p=0}^{n-1} \sum_{q=0}^{p} q \binom{2p-q}{p-q} \binom{2n-2p+q-1}{n-1-p}.
\end{aligned}$$

We have the first part with q replaced by $q+1$ and adjust the infinite series accordingly, getting:

$$[z^{n-1}](1+z)^{2n-1} \sum_{p=0}^{n-1} z^p (1+z)^{-2p} [w^p](1+w)^{2p} \frac{(1+z)w/(1+w)}{(1-(1+z)w/(1+w))^2}$$

$$= [z^{n-1}](1+z)^{2n} \sum_{p\ge 0} z^p (1+z)^{-2p} [w^{p-1}](1+w)^{2p+1} \frac{1}{(1-wz)^2}.$$

We extended p to infinity due to the extractor in z. Unfortunately this isn't quite a repeat so we need to do the residues one more time. We have

$$\operatorname*{res}_{w} \frac{1}{w^p}(1+w)^{2p+1} \frac{1}{(1-wz)^2}.$$

Now again put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = 1/(1-v)^2\; dv$ to get

$$\operatorname*{res}_{v} \frac{1}{v^p} \frac{1}{(1-v)^{p+1}} \frac{1}{(1-vz/(1-v))^2} \frac{1}{(1-v)^2}$$
$$= \operatorname*{res}_{v} \frac{1}{v^p} \frac{1}{(1-v)^p} \frac{1}{(1-v-vz)^2} \frac{1}{1-v}.$$

Next put $v = (1-\sqrt{1-4u})/2$ so that $v(1-v) = u$ and $dv = 1/\sqrt{1-4u}\; du$ to obtain

$$\operatorname*{res}_{u} \frac{1}{u^p} \frac{1}{(1-(1+z)(1-\sqrt{1-4u})/2)^2} \frac{1}{\sqrt{1-4u}} \frac{1-\sqrt{1-4u}}{2u}.$$

Evaluate at $u = z/(1+z)^2$ to cancel the remaining sum (substitution rule, same as before),

$$[z^{n-1}](1+z)^{2n} \frac{1}{(1-(1+z)(1-(1-z)/(1+z))/2)^2} \frac{1+z}{1-z} \frac{1-(1-z)/(1+z)}{2}$$
$$= [z^{n-1}](1+z)^{2n} \frac{1}{(1-(1+z-(1-z))/2)^2} \frac{z}{1-z}$$
$$= [z^{n-1}](1+z)^{2n} \frac{z}{(1-z)^3} = [z^{n-2}](1+z)^{2n} \frac{1}{(1-z)^3}.$$

Conclusion

It remains to add up the two pieces. We have

$$[z^{n-1}](1+z)^{2n} \frac{1}{(1-z)^3} + [z^{n-1}](1+z)^{2n} \frac{z}{(1-z)^3}$$
$$= [z^{n-1}](1+z)^{2n+1} \frac{1}{(1-z)^3}.$$

This is

$$S = \operatorname*{res}_{z} \frac{1}{z^n}(1+z)^{2n+1} \frac{1}{(1-z)^3}.$$

Residues sum to zero so we may evaluate using minus the residues at $z = 1$ and at infinity. We require for the former (flip sign one more time)

$$\frac{1}{2}\left[\frac{1}{z^n}(1+z)^{2n+1}\right]''\Bigg|_{z=1}$$

$$= \frac{1}{2}\left[-\frac{n}{z^{n+1}}(1+z)^{2n+1} + \frac{1}{z^n}(2n+1)(1+z)^{2n}\right]'\Bigg|_{z=1}$$

$$= \frac{1}{2}\left[n(n+1)2^{2n+1} - n(2n+1)2^{2n} - n(2n+1)2^{2n} + (2n+1)(2n)2^{2n-1}\right]$$

$$= \frac{1}{2}n4^n.$$

We get from the residue at infinity with the sign flipped

$$\operatorname*{res}_z \frac{1}{z^2} z^n (1+1/z)^{2n+1} \frac{1}{(1-1/z)^3}$$

$$= \operatorname*{res}_z \frac{1}{z^2} z^n (1+z)^{2n+1} \frac{1}{z^{2n+1}} \frac{z^3}{(z-1)^3} = -\operatorname*{res}_z \frac{1}{z^n}(1+z)^{2n+1}\frac{1}{(1-z)^3} = -S.$$

We have shown that $S = \frac{1}{2}n4^n - S$ or alternatively

$$S = n4^{n-1}.$$

This was math.stackexchange.com problem 4843051.

1.164 MSE 4850609: Inverse central binomial coefficient in sum

Supposing we start from

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+r}{k}\binom{2n}{2k}^{-1}.$$

Write

$$\binom{n+r}{k} = \frac{(n+r)!}{k! \times (n+r-k)!} = \binom{n+r}{n}\frac{n! \times r!}{k! \times (n+r-k)!}$$

$$= \binom{n+r}{n}\binom{n}{k}\binom{n-k+r}{r}^{-1}.$$

Furthermore

$$\binom{n}{k}^2\binom{2n}{2k}^{-1} = \binom{2n}{n}^{-1}\frac{(2k)! \times (2n-2k)!}{(n-k)!^2 \times k!^2}$$

$$= \binom{2n}{n}^{-1} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

We thus have for our sum

$$\binom{n+r}{n} \binom{2n}{n}^{-1} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \binom{n-k+r}{r}^{-1}.$$

Recall from 1.89 the following identity which was proved there: with $1 \le k \le n$

$$\frac{1}{k} \binom{n}{k}^{-1} = [v^n] \log \frac{1}{1-v} (v-1)^{n-k}.$$

We can re-write this as

$$\binom{n-1}{k-1}^{-1} = n[v^n] \log \frac{1}{1-v} (v-1)^{n-k}.$$

We have for the sum without the scalar in front

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (n+1-k+r)[v^{n+1-k+r}] \log \frac{1}{1-v} (v-1)^{n-k}.$$

The contribution from v is

$$\operatorname*{res}_{v} \frac{1}{v^{n+2-k+r}} \log \frac{1}{1-v} (v-1)^{n-k}.$$

Now put $v/(1-v) = z$ so that $v = z/(1+z)$ and $dv = 1/(1+z)^2 \; dz$ to obtain

$$\operatorname*{res}_{z} \frac{1}{z^{n+2-k+r}} (-1)^{n-k} (1+z)^{r+2} \log(1+z) \frac{1}{(1+z)^2}.$$

We thus find for the sum

$$\begin{aligned}
&(-1)^n [z^{n+1+r}] \log(1+z)(1+z)^r \\
&\times \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (n+1-k+r)(-1)^k z^k \\
&= (-1)^r [z^{n+1+r}] \log \frac{1}{1-z} (1-z)^r \\
&\times \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (n+1-k+r) z^k \\
&= [z^{n+1+r}] \log \frac{1}{1-z} (z-1)^r
\end{aligned}$$

$$\times \sum_{k=0}^n \binom{2k}{k}\binom{2n-2k}{n-k}(n+1-k+r)z^k.$$

We now get two pieces.

First piece

This is

$$\begin{aligned}(n+1+r)[z^{n+1+r}] \log\frac{1}{1-z}(z-1)^r \\ \times \sum_{k=0}^n \binom{2k}{k}\binom{2n-2k}{n-k} z^k \\ = (n+1+r)[z^{n+1+r}] \log\frac{1}{1-z}(z-1)^r [w^n]\frac{1}{\sqrt{1-4wz}}\frac{1}{\sqrt{1-4w}}.\end{aligned}$$

The square root yields

$$\begin{aligned}[w^n]\frac{1}{\sqrt{1-4w-4w(z-1)}}\frac{1}{\sqrt{1-4w}} \\ = [w^n]\frac{1}{\sqrt{1-4w(z-1)/(1-4w)}}\frac{1}{1-4w} \\ = [w^n]\sum_{p=0}^n \binom{2p}{p}\frac{w^p(z-1)^p}{(1-4w)^{p+1}} = \sum_{p=0}^n \binom{2p}{p}(z-1)^p 4^{n-p}\binom{n}{p}.\end{aligned}$$

Applying the logarithm

$$\sum_{p=0}^n \binom{2p}{p} 4^{n-p}\binom{n}{p}\binom{n+r}{n-p}^{-1}.$$

Next observe that

$$\begin{aligned}\binom{n}{p}\binom{n+r}{n-p}^{-1} = \frac{n!\times(r+p)!}{p!\times(n+r)!} \\ = \binom{n+r}{n}^{-1}\binom{r+p}{r}\end{aligned}$$

which produces

$$\binom{n+r}{n}^{-1}\sum_{p=0}^n \binom{2p}{p}4^{n-p}\binom{r+p}{r}.$$

Second piece

This is

$$-[z^{n+1+r}] \log \frac{1}{1-z} (z-1)^r$$

$$\times \sum_{k=0}^{n} k \binom{2k}{k} \binom{2n-2k}{n-k} z^k$$

$$= -[z^{n+1+r}] \log \frac{1}{1-z} (z-1)^r [w^n] \frac{2wz}{\sqrt{1-4wz}^3} \frac{1}{\sqrt{1-4w}}.$$

The square root yields

$$[w^n] \frac{2wz}{\sqrt{1-4w-4w(z-1)}^3} \frac{1}{\sqrt{1-4w}}$$

$$= z[w^n] \frac{2w}{\sqrt{1-4w(z-1)/(1-4w)}^3} \frac{1}{(1-4w)^2}$$

$$= \frac{z}{z-1} [w^n] \frac{2w(z-1)/(1-4w)}{\sqrt{1-4w(z-1)/(1-4w)}^3} \frac{1}{1-4w}$$

$$= \frac{z}{z-1} [w^n] \sum_{p=0}^{n} p \binom{2p}{p} \frac{w^p (z-1)^p}{(1-4w)^{p+1}} = \frac{z}{z-1} \sum_{p=0}^{n} p \binom{2p}{p} (z-1)^p 4^{n-p} \binom{n}{p}.$$

Applying the logarithm and the sign

$$-\frac{1}{n+r} \sum_{p=0}^{n} p \binom{2p}{p} 4^{n-p} \binom{n}{p} \binom{n+r-1}{n-p}^{-1}.$$

Next observe that

$$\frac{1}{n+r} \binom{n}{p} \binom{n+r-1}{n-p}^{-1} = \frac{n! \times (r+p-1)!}{p! \times (n+r)!}$$

$$= \frac{1}{r+p} \binom{n+r}{n}^{-1} \binom{r+p}{r}$$

which produces

$$-\binom{n+r}{n}^{-1} \sum_{p=0}^{n} \frac{p}{r+p} \binom{2p}{p} 4^{n-p} \binom{r+p}{r}.$$

Join the two pieces

We join the two pieces and activate the scalars to get

$$\binom{2n}{n}^{-1} \sum_{p=0}^{n} \frac{r}{r+p} \binom{2p}{p} 4^{n-p} \binom{r+p}{r}.$$

For this to hold we need $r \ge 1$. We further obtain

$$\binom{2n}{n}^{-1} \sum_{p=0}^{n} \binom{2p}{p} 4^{n-p} \binom{r+p-1}{r-1}.$$

Working with the sum

$$4^n [z^n] \frac{1}{1-z} \sum_{p\ge 0} z^p \binom{2p}{p} 4^{-p} \binom{r+p-1}{r-1}$$

$$= 4^n [z^n] \frac{1}{1-z} [w^{r-1}] (1+w)^{r-1} \sum_{p\ge 0} z^p \binom{2p}{p} 4^{-p} (1+w)^p$$

$$= 4^n [z^n] \frac{1}{1-z} [w^{r-1}] (1+w)^{r-1} \frac{1}{\sqrt{1-z(1+w)}}$$

$$= 4^n [z^n] \frac{1}{(1-z)^{3/2}} [w^{r-1}] (1+w)^{r-1} \frac{1}{\sqrt{1-wz/(1-z)}}$$

$$= 4^n [z^n] \frac{1}{(1-z)^{3/2}} \sum_{p=0}^{r-1} \binom{r-1}{r-1-p} \binom{2p}{p} 4^{-p} \frac{z^p}{(1-z)^p}$$

$$= 4^n \sum_{p=0}^{r-1} \binom{r-1}{p} \binom{2p}{p} 4^{-p} \binom{n+1/2}{n-p}.$$

The last binomial coefficient is zero for a negative lower index by construction. We have for $r \ge 1$ the closed form

$$\boxed{4^n \binom{2n}{n}^{-1} \sum_{p=0}^{r-1} \binom{r-1}{p} \binom{2p}{p} 4^{-p} \binom{n+1/2}{n-p}.}$$

This gives e.g. for $r = 1$

$$4^n \binom{2n}{n}^{-1} \binom{n+1/2}{n}.$$

We get for $r = 2$

$$4^n \binom{2n}{n}^{-1} \left[\binom{n+1/2}{n} + \frac{1}{2} \binom{n+1/2}{n-1} \right].$$

One more example is $r = 3$ which yields

$$4^n \binom{2n}{n}^{-1} \left[\binom{n+1/2}{n} + \binom{n+1/2}{n-1} + \frac{3}{8} \binom{n+1/2}{n-2} \right].$$

Last example is $r = 4$

$$4^n \binom{2n}{n}^{-1} \left[\binom{n+1/2}{n} + \frac{3}{2}\binom{n+1/2}{n-1} + \frac{9}{8}\binom{n+1/2}{n-2} + \frac{5}{16}\binom{n+1/2}{n-3} \right].$$

The case of $r = 0$

We have from the introduction

$$\binom{n+r}{n}\binom{2n}{n}^{-1} \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{n-k+r}{r}^{-1}.$$

Evaluate at $r = 0$ to get

$$\binom{2n}{n}^{-1} \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$$

$$= \binom{2n}{n}^{-1} [z^n] \frac{1}{\sqrt{1-4z}} \frac{1}{\sqrt{1-4z}} = \binom{2n}{n}^{-1} [z^n] \frac{1}{1-4z} = 4^n \binom{2n}{n}^{-1}.$$

This was math.stackexchange.com problem 4850609.

1.165 Computer search II

1.165.1 OEIS A122366

We have with $n \ge 1$ and $m \ge p$ and $n \ge m - p$ that

$$\binom{2n-p}{m-p} = (-1)^{n+m+p} \sum_{k=0}^{n} \binom{2n+1}{k} (-1)^k \binom{2n-k}{n} \binom{m-k}{m-p}.$$

We have for the RHS that it is

$$(-1)^{n+m+p} \sum_{k=0}^{n} \binom{2n+1}{k} (-1)^k \binom{2n-k}{n-k} \binom{m-k}{m-p}$$

$$= (-1)^{n+m+p} [z^n] (1+z)^{2n} \sum_{k\ge 0} \binom{2n+1}{k} (-1)^k \frac{z^k}{(1+z)^k} \binom{m-k}{m-p}.$$

Here we have extended the range of k to infinity due to the coefficient extractor in z. Continuing,

$$(-1)^{n+m+p} [z^n] (1+z)^{2n} [w^{m-p}] (1+w)^m \sum_{k\ge 0} \binom{2n+1}{k} (-1)^k \frac{z^k}{(1+z)^k} \frac{1}{(1+w)^k}$$

$$= (-1)^{n+m+p}[z^n](1+z)^{2n}[w^{m-p}](1+w)^m \left[1 - \frac{z}{(1+z)(1+w)}\right]^{2n+1}$$

$$= (-1)^{n+m+p}[z^n]\frac{1}{1+z}[w^{m-p}]\frac{1}{(1+w)^{2n+1-m}}[1+w(1+z)]^{2n+1}$$

$$= (-1)^{n+m+p}[z^n]\frac{1}{1+z}[w^{m-p}]\frac{1}{(1+w)^{2n+1-m}}\sum_{k=0}^{2n+1}\binom{2n+1}{k}w^k(1+z)^k$$

$$= (-1)^{n+m+p}[w^{m-p}]\frac{1}{(1+w)^{2n+1-m}}\sum_{k=0}^{2n+1}\binom{2n+1}{k}w^k\binom{k-1}{n}.$$

We get from $k = 0$ the term

$$(-1)^{m+p}[w^{m-p}]\frac{1}{(1+w)^{2n+1-m}} = (-1)^{m+p}(-1)^{m-p}\binom{m-p+2n-m}{m-p}$$

$$= \binom{2n-p}{m-p}.$$

This is the claim. It remains to show that the non-zero terms from the sum vanish. We get zero from the second binomial coefficient when $1 \le k \le n$. This leaves

$$(-1)^{n+m+p}[w^{m-p}]\frac{1}{(1+w)^{2n+1-m}}\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}w^k\binom{k-1}{n}$$

$$= (-1)^{n+m+p}[w^{m-p}]\frac{1}{(1+w)^{2n+1-m}}\sum_{k=0}^{n}\binom{2n+1}{k+n+1}w^{k+n+1}\binom{k+n}{n}.$$

Here we see that owing to $n \ge m - p$ we have a zero contribution from the coefficient extractor in w (the term under the sum starts at $n+1$) which concludes the argument.

This identity was found by a computer search which pointed to OEIS A122366, triangle of $\binom{2n+1}{k}$.

1.165.2 OEIS A100100

We claim that with $p \ge 0$ and $1 \le m \le n+1-p$ the following holds:

$$\binom{n-m-1}{p} = (-1)^{m+1}\sum_{k=0}^{n}(-1)^k\binom{m-1+k}{k}^2\binom{n+m}{m+k}\binom{n-1+k}{p}.$$

The first binomial coefficient that we turn into an extractor also enforces the range and we obtain:

$$(-1)^{m+1}[z^n](1+z)^{n+m}\sum_{k\ge 0}(-1)^k\binom{m-1+k}{k}^2 z^k\binom{n-1+k}{p}.$$

Continuing,

$$\begin{aligned}
&(-1)^{m+1}[z^n](1+z)^{n+m}[w^{m-1}](1+w)^{m-1}\\
&\times\sum_{k\ge 0}(-1)^k\binom{m-1+k}{k}z^k(1+w)^k\binom{n-1+k}{p}\\
&=(-1)^{m+1}[z^n](1+z)^{n+m}[w^{m-1}](1+w)^{m-1}[v^p](1+v)^{n-1}\\
&\times\sum_{k\ge 0}(-1)^k\binom{m-1+k}{k}z^k(1+w)^k(1+v)^k\\
&=(-1)^{m+1}[z^n](1+z)^{n+m}[w^{m-1}](1+w)^{m-1}[v^p](1+v)^{n-1}\\
&\times\frac{1}{(1+z(1+w)(1+v))^m}\\
&=(-1)^{m+1}[z^{n+m}](1+z)^{n+m}[w^{m-1}](1+w)^{-1}[v^p](1+v)^{n-1}\\
&\times\frac{1}{(v+1+1/z/(1+w))^m}.
\end{aligned}$$

The fractional term is

$$\begin{aligned}
&\frac{1}{(1+1/z/(1+w))^m}\frac{1}{(1+v/(1+1/z/(1+w)))^m}\\
&=\frac{z^m(1+w)^m}{(1+z(1+w))^m}\frac{1}{(1+vz(1+w)/(1+z(1+w)))^m}.
\end{aligned}$$

This leaves for the extractors

$$[z^n](1+z)^{n+m}[w^{m-1}](1+w)^{m-1}.$$

The contribution from v is

$$\begin{aligned}
&\frac{1}{(1+z(1+w))^m}\sum_{q=0}^{p}\binom{n-1}{p-q}\binom{m-1+q}{q}\frac{(-1)^q z^q(1+w)^q}{(1+z(1+w))^q}\\
&=\sum_{q=0}^{p}\binom{n-1}{p-q}\binom{m-1+q}{q}(-1)^q\sum_{r=0}^{n}\binom{m-1+q+r}{r}(-1)^r z^{q+r}(1+w)^{q+r}.
\end{aligned}$$

Doing the extraction

$$(-1)^{m+1}\sum_{q=0}^{p}\binom{n-1}{p-q}\binom{m-1+q}{q}(-1)^q$$
$$\times\sum_{r=0}^{n-q}\binom{m-1+q+r}{r}(-1)^r\binom{n+m}{n-q-r}\binom{m-1+q+r}{m-1}.$$

We got from the fourth binomial coefficient by construction that the upper range must be set to $n-q$. Continuing,

$$(-1)^{m+1}\sum_{q=0}^{p}\binom{n-1}{p-q}\binom{m-1+q}{q}(-1)^q$$
$$\times\sum_{r=0}^{n-q}\binom{m-1+n-r}{n-q-r}(-1)^{n-q-r}\binom{n+m}{r}\binom{m-1+n-r}{m-1}.$$

Inner sum

Working with the inner sum we obtain

$$(-1)^{n-q}[z^{n-q}](1+z)^{m-1+n}[w^{m-1}](1+w)^{m-1+n}$$
$$\times\sum_{r\ge 0}(-1)^r\binom{n+m}{r}\frac{z^r}{(1+z)^r}\frac{1}{(1+w)^r}.$$

Here we have extended to infinity due to the coefficient extractor in z. Continuing,

$$(-1)^{n-q}[z^{n-q}](1+z)^{m-1+n}[w^{m-1}](1+w)^{m-1+n}[1-z/(1+z)/(1+w)]^{n+m}$$
$$=(-1)^{n-q}[z^{n-q}](1+z)^{-1}[w^{m-1}](1+w)^{-1}[1+w+wz]^{n+m}$$
$$=(-1)^{n-q}[z^{n-q}](1+z)^{-1}\sum_{r=0}^{m-1}(-1)^{m-1-r}\binom{n+m}{r}(1+z)^r$$
$$=(-1)^{n-q}\sum_{r=0}^{m-1}(-1)^{m-1-r}\binom{n+m}{r}\binom{r-1}{n-q}$$
$$=(-1)^{m-1}+(-1)^{n-q}\sum_{r=1}^{m-1}(-1)^{m-1-r}\binom{n+m}{r}\binom{r-1}{n-q}.$$

Note however that $r-1$ is not negative and $r-1<n-q$. This is because the most $r-1$ can be is $m-2$ and the least that $n-q$ can be is $n-p$ and we have $m-2<n-p$ as per the initial conditions stated in the introduction. Hence the remaining terms vanish and our inner sum is $(-1)^{m+1}$. The case $m=1$ goes through with the sum as expected.

Outer sum

Returning to the outer sum we are left with

$$\sum_{q=0}^{p} \binom{n-1}{p-q} \binom{m-1+q}{q} (-1)^q$$

$$= [w^p](1+w)^{n-1} \sum_{q\ge 0} w^q \binom{m-1+q}{q} (-1)^q = [w^p](1+w)^{n-1} \frac{1}{(1+w)^m}$$

$$= [w^p](1+w)^{n-m-1} = \binom{n-m-1}{p}.$$

We have our claim and may conclude.

This identity was found by a computer search which pointed to OEIS A100100, a binomial coefficient triangle.

1.165.3 OEIS A010854

We claim that with $p \ge 1$, $n \ge m$ and $m \ge p$ the following holds:

$$\binom{4p-2}{p} = (-1)^m \sum_{k=0}^{n} \binom{n+p}{k+p} (-1)^k \binom{k+p-1}{k+p-m} \binom{2k+1-p}{p}.$$

We use the first binomial coefficient to enforce the upper range and may write

$$(-1)^m [z^n](1+z)^{n+p} \sum_{k\ge 0} z^k (-1)^k \binom{k+p-1}{m-1} \binom{2k+1-p}{p}$$

$$= (-1)^m [z^n](1+z)^{n+p} [w^{m-1}](1+w)^{p-1} [v^p] \frac{1}{(1+v)^{p-1}}$$

$$\times \sum_{k\ge 0} z^k (-1)^k (1+w)^k (1+v)^{2k}$$

$$= (-1)^m [z^n](1+z)^{n+p} [w^{m-1}](1+w)^{p-1} [v^p] \frac{1}{(1+v)^{p-1}}$$

$$\times \frac{1}{1+z(1+w)(1+v)^2}.$$

It is convenient to re-order the extractors,

$$(-1)^m [v^p] \frac{1}{(1+v)^{p-1}} [w^{m-1}](1+w)^{p-1} [z^n](1+z)^{n+p}$$

$$\times \frac{1}{1+z(1+w)(1+v)^2}$$

$$= (-1)^m [v^p](1+v)^{n+1-p}[w^{m-1}](1+w)^{p-1}[z^n](1+z/(1+v))^{n+p}$$
$$\times \frac{1}{1+z(1+w)(1+v)}$$
$$= (-1)^m [v^p]\frac{1}{(1+v)^{2p-1}}[w^{m-1}](1+w)^{p-1}[z^n](1+v+z)^{n+p}$$
$$\times \frac{1}{1+z(1+w)(1+v)}$$
$$= (-1)^m [v^p]\frac{1}{(1+v)^{2p}}[w^{m-1}](1+w)^{p-1}[z^{n+1}](1+v+z)^{n+p}$$
$$\times \frac{1}{w+1+1/z/(1+v)}.$$

Here the contribution from w is

$$\operatorname*{res}_w \frac{1}{w^m}(1+w)^{p-1}\frac{1}{w+1+1/z/(1+v)}.$$

Now we see that the residue at infinity is zero by the boundary conditions (just barely) and we can evaluate using minus the residue at $w = -1-1/z/(1+v)$, which is a simple pole, getting

$$-(-1)^m \frac{z^m(1+v)^m}{(1+z+vz)^m}(-1)^{p-1}\frac{1}{z^{p-1}(1+v)^{p-1}}.$$

Substitute into the remaining extractors to get

$$(-1)^p [v^p]\frac{1}{(1+v)^{3p-1-m}}[z^{n-m+p}](1+v+z)^{n+p}\frac{1}{(1+z+vz)^m}.$$

Here the contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n-m+p+1}}(1+v+z)^{n+p}\frac{1}{(1+z+vz)^m}.$$

Now put $z/(1+v+z) = u$ so that $z = (1+v)u/(1-u)$ and $dz = (1+v)/(1-u)^2\, du$ to get

$$\operatorname*{res}_u \frac{1}{u^{n-m+p+1}}\frac{(1+v)^{m-1}}{(1-u)^{m-1}}\frac{1}{(1+(1+v)^2u/(1-u))^m}\frac{1+v}{(1-u)^2}.$$

Here is what we have:

$$(-1)^p [v^p]\frac{1}{(1+v)^{3p-1-2m}} \operatorname*{res}_u \frac{1}{u^{n-m+p+1}}\frac{1}{1-u}\frac{1}{(1+uv(2+v))^m}.$$

The residue at infinity in u is zero and we may evaluate using minus the residues at $u=1$ and $u = -1/v/(2+v)$. We get from the former,

$$(-1)^p [v^p] \frac{1}{(1+v)^{3p-1-2m}} \frac{1}{(1+v)^{2m}} = (-1)^p [v^p] \frac{1}{(1+v)^{3p-1}} = \binom{4p-2}{p}.$$

Good, we have the claim. Now we need to show that the contribution from the other residue vanishes. We write

$$(-1)^p [v^p] \frac{1}{(1+v)^{3p-1-2m}} \frac{1}{v^m (2+v)^m} \operatorname*{res}_u \frac{1}{u^{n-m+p+1}} \frac{1}{1-u} \frac{1}{(u+1/v/(2+v))^m}.$$

We require the Leibniz rule,

$$\frac{1}{(m-1)!} \left(\frac{1}{u^{n-m+p+1}} \frac{1}{(1-u)^1} \right)^{(m-1)}$$

$$= \frac{1}{(m-1)!} \sum_{q=0}^{m-1} \binom{m-1}{q} \frac{(n-m+p+1)^{\overline{q}}}{u^{n-m+p+1+q}} (-1)^q \frac{1^{\overline{m-1-q}}}{(1-u)^{1+m-1-q}}$$

$$= \sum_{q=0}^{m-1} \frac{1}{u^{n-m+p+1+q}} (-1)^q \binom{n-m+p+q}{q} \frac{1}{(1-u)^{m-q}}.$$

Evaluate at $u = -1/v/(2+v)$ and flip sign to get

$$(-1)^{n-m+p} \sum_{q=0}^{m-1} (v(2+v))^{n-m+p+1+q} \binom{n-m+p+q}{q} \frac{v^{m-q}(2+v)^{m-q}}{(1+v)^{2m-2q}}$$

$$= (-1)^{n-m+p} \sum_{q=0}^{m-1} (v(2+v))^{n+p+1} \binom{n-m+p+q}{q} \frac{1}{(1+v)^{2m-2q}}.$$

Now we are extracting a coefficient on $[v^p]$ but collecting everything the exponent on v is $n+p+1-m$. This makes for a zero contribution because of the boundary condition that $n \ge m$.

This identity was found by a computer search which pointed to OEIS A010854, the constant sequence with value fifteen i.e. six-choose-two.

1.165.4 OEIS A001498

We claim that the following holds with $0 \le m \le n$

$$\frac{(n+m)!}{2^m (n-m)! m!} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+m}{m+k} \binom{n-m+k}{k} \left\{ {m+k+1 \atop k+1} \right\}.$$

The RHS is

$$\sum_{k=0}^{n}(-1)^k \binom{n+m}{k}\binom{2n-m-k}{n-k}\left\{ {n+m-k+1 \atop n-k+1} \right\}.$$

We can use the middle binomial coefficient to enforce the upper range of the sum and obtain

$$[z^n](1+z)^{2n-m}\sum_{k\ge 0}(-1)^k\binom{n+m}{k}\frac{z^k}{(1+z)^k}\left\{ {n+m-k+1 \atop n-k+1} \right\}$$

$$= [z^n](1+z)^{2n-m}\sum_{k\ge 0}(-1)^k\binom{n+m}{k}\frac{z^k}{(1+z)^k}$$

$$\times(n+m-k+1)![w^{n+m-k+1}]\frac{1}{(n-k+1)!}(\exp(w)-1)^{n-k+1}.$$

Observe that

$$\binom{n+m}{k}(n+m-k+1)!\frac{1}{(n-k+1)!}$$

$$= (n+m-k+1)\frac{(n+m)!}{k!\times(n-k+1)!}$$

$$= (n+m-k+1)\frac{(n+m)!}{(n+1)!}\binom{n+1}{k}.$$

We get for our sum

$$\frac{(n+m)!}{(n+1)!}[z^n](1+z)^{2n-m}[w^{n+m+1}](\exp(w)-1)^{n+1}$$

$$\times\sum_{k\ge 0}\binom{n+1}{k}(n+m-k+1)(-1)^k\frac{z^k}{(1+z)^k}\frac{w^k}{(\exp(w)-1)^k}.$$

We now obtain two pieces, the first is

$$\frac{(n+m+1)!}{(n+1)!}[z^n](1+z)^{2n-m}[w^{n+m+1}](\exp(w)-1)^{n+1}$$

$$\times\left[1-\frac{wz}{(1+z)(\exp(w)-1)}\right]^{n+1}$$

$$= \frac{(n+m+1)!}{(n+1)!}[z^n](1+z)^{n-m-1}[w^{n+m+1}]\left[(1+z)(\exp(w)-1)-wz\right]^{n+1}$$

$$= \frac{(n+m+1)!}{(n+1)!}[z^n](1+z)^{n-m-1}[w^{n+m+1}]\left[(1+z)(\exp(w)-1-w)+w\right]^{n+1}.$$

The power expands to

$$\sum_{q=0}^{n+1} \binom{n+1}{q} (1+z)^q (\exp(w)-1-w)^q w^{n+1-q}.$$

Applying the extractor in z we get

$$\sum_{q=0}^{n+1} \binom{n+1}{q} \binom{n-m-1+q}{n} (\exp(w)-1-w)^q w^{n+1-q}.$$

This means we only get a contribution when $n-m-1+q \ge n$ or $q \ge m+1$. But from the extractor in w we get that $2q+n+1-q \le n+m+1$ or $q \le m$. The intersection of the two ranges is empty and hence the first piece contributes zero.

Continuing with the second piece we have

$$\begin{aligned}
&\frac{(n+m)!}{(n+1)!} [z^n](1+z)^{2n-m} [w^{n+m+1}](\exp(w)-1)^{n+1} \\
&\quad \times \sum_{k\ge 1} \binom{n+1}{k} k(-1)^k \frac{z^k}{(1+z)^k} \frac{w^k}{(\exp(w)-1)^k} \\
&= \frac{(n+m)!}{n!} [z^n](1+z)^{2n-m} [w^{n+m+1}](\exp(w)-1)^{n+1} \\
&\quad \times \sum_{k\ge 1} \binom{n}{k-1} (-1)^k \frac{z^k}{(1+z)^k} \frac{w^k}{(\exp(w)-1)^k} \\
&= -\frac{(n+m)!}{n!} [z^{n-1}](1+z)^{2n-m-1} [w^{n+m}](\exp(w)-1)^n \\
&\quad \times \sum_{k\ge 0} \binom{n}{k} (-1)^k \frac{z^k}{(1+z)^k} \frac{w^k}{(\exp(w)-1)^k} \\
&= -\frac{(n+m)!}{n!} [z^{n-1}](1+z)^{n-m-1} \\
&\quad \times [w^{n+m}][(1+z)(\exp(w)-1-w)+w]^n.
\end{aligned}$$

Once more expanding the power we have

$$\sum_{q=0}^{n} \binom{n}{q} (1+z)^q (\exp(w)-1-w)^q w^{n-q}.$$

Apply the extractor in z to get

$$\sum_{q=0}^{n} \binom{n}{q} \binom{n-m-1+q}{n-1} (\exp(w)-1-w)^q w^{n-q}.$$

This yields the constraints on the range, we have from z that $n-m-1+q \ge$

$n-1$ or $q \ge m$. From w we obtain $2q+n-q \le n+m$ or $q \le m$. This means the only term that contributes is $q=m$ and we find collecting everything

$$-\frac{(n+m)!}{n!}\binom{n}{m}\binom{n-1}{n-1}[w^{n+m}](\exp(w)-1-w)^m w^{n-m}$$
$$= -\frac{(n+m)!}{n!}\binom{n}{m}[w^{2m}](\exp(w)-1-w)^m.$$

Note however that $\exp(w)-1-w = \frac{1}{2}w^2 + \cdots$ so we finally have

$$-\frac{(n+m)!}{n!}\binom{n}{m} \times \frac{1}{2^m}.$$

Subtract the second piece from the first to get

$$\frac{(n+m)!}{n!}\frac{n!}{(n-m)!m!} \times \frac{1}{2^m} = \frac{(n+m)!}{2^m(n-m)!m!}$$

and we have the claim. This also proves the companion identity

$$\frac{(n+m)!}{2^m(n-m)!m!} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+m}{m+k}\binom{n-m+k}{k}\left[{m+k+1 \atop k+1}\right].$$

This identity was found by a computer search which pointed to OEIS A001498, coefficients of Bessel polynomials.

1.165.5 OEIS A178300

We claim that the following holds with $0 \le p \le m \le n$

$$\binom{n+m-1-p}{n-1} = (-1)^{n+p+1} \sum_{k=0}^{n} \binom{m-k}{p-k}(-1)^k \binom{2m}{k-n+m}\binom{-n+k}{m}.$$

We leave the case $n=0$ or $m=0$ as well as $p=m=0$ to the reader. We start by re-writing the third binomial coefficient:

$$(-1)^{n+p+m+1} \sum_{k=0}^{n} \binom{m-k}{p-k}(-1)^k \binom{2m}{k-n+m}\binom{m+n-k-1}{m}$$
$$= (-1)^{n+p+m+1} \sum_{k=0}^{n} \binom{m-k}{p-k}(-1)^k \binom{2m}{k-n+m}\binom{m+n-k-1}{n-k-1}.$$

We can use it to enforce the upper range, it will truncate at $n-1$ but we are not losing anything at $k=n$ where it would have been zero anyway. Extending k to infinity we get

$$(-1)^{n+p+m+1}[z^{n-1}](1+z)^{n+m-1}\sum_{k\ge 0}\binom{m-k}{m-p}(-1)^k\binom{2m}{m+n-k}\frac{z^k}{(1+z)^k}$$

$$= (-1)^{n+p+m+1}[z^{n-1}](1+z)^{n+m-1}[w^{n+m}](1+w)^{2m}\sum_{k\ge 0}\binom{m-k}{m-p}(-1)^k w^k\frac{z^k}{(1+z)^k}$$

$$= (-1)^{n+p+m+1}[z^{n-1}](1+z)^{n+m-1}[w^{n+m}](1+w)^{2m}[v^{m-p}](1+v)^m$$
$$\times\sum_{k\ge 0}(-1)^k w^k\frac{1}{(1+v)^k}\frac{z^k}{(1+z)^k}$$
$$= (-1)^{n+p+m+1}[z^{n-1}](1+z)^{n+m-1}[w^{n+m}](1+w)^{2m}[v^{m-p}](1+v)^m$$
$$\times\frac{1}{1+wz/(1+z)/(1+v)}$$
$$= (-1)^{n+p+m+1}[z^{n}](1+z)^{n+m}[w^{n+m}](1+w)^{2m}[v^{m-p}](1+v)^{m+1}$$
$$\times\frac{1}{w+(1+z)(1+v)/z}.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{n+m+1}}(1+w)^{2m}\frac{1}{w+(1+z)(1+v)/z}.$$

Per the initial conditions we have zero for the residue at infinity and may evaluate using minus the contribution from the pole at $-(1+z)(1+v)/z$, getting

$$(-1)^{n+m+1}\frac{z^{n+m+1}}{(1+z)^{n+m+1}(1+v)^{n+m+1}}\frac{(z-(1+z)(1+v))^{2m}}{z^{2m}}.$$

Substitute into the extractors to get (remember to flip signs)

$$(-1)^{p+1}[z^{m-1}]\frac{1}{1+z}[v^{m-p}]\frac{1}{(1+v)^n}(z-(1+z)(1+v))^{2m}$$
$$= (-1)^{p+1}[z^{m-1}]\frac{1}{1+z}[v^{m-p}]\frac{1}{(1+v)^n}(1+v+vz)^{2m}$$
$$= (-1)^{p+1}[v^{m-p}]\frac{1}{(1+v)^n}\sum_{q=0}^{m-1}(-1)^{m-1-q}\binom{2m}{q}v^q(1+v)^{2m-q}$$
$$= (-1)^{p+1}\sum_{q=0}^{m-1}(-1)^{m-1-q}\binom{2m}{q}\binom{2m-n-q}{m-p-q}$$
$$= (-1)^{p+1}\binom{2m}{m}\binom{m-n}{-p} + (-1)^{p+1}\sum_{q=0}^{m}(-1)^{m-1-q}\binom{2m}{q}\binom{2m-n-q}{m-p-q}.$$

Observe very carefully that the binomial coefficient that we obtained from the variable v is zero when q grows beyond $m-p$. With $m-p \le m$ we may thus write

$$-\binom{2m}{m}[[p=0]] + (-1)^{p+1} \sum_{q=0}^{m-p} (-1)^{m-1-q} \binom{2m}{q} \binom{2m-n-q}{m-p-q}$$

$$= -\binom{2m}{m}[[p=0]] + (-1)^{p+1}[z^{m-p}](1+z)^{2m-n} \sum_{q=0}^{m-p} (-1)^{m-1-q} \binom{2m}{q} \frac{z^q}{(1+z)^q}.$$

Extending the sum without the Iverson bracket term in front we have

$$(-1)^{p+m}[z^{m-p}](1+z)^{2m-n} \sum_{q\ge 0} (-1)^q \binom{2m}{q} \frac{z^q}{(1+z)^q}$$

$$= (-1)^{p+m}[z^{m-p}](1+z)^{2m-n} \left[1 - \frac{z}{1+z}\right]^{2m}$$

$$= (-1)^{p+m}[z^{m-p}] \frac{1}{(1+z)^n}$$

$$= (-1)^{p+m}(-1)^{m-p} \binom{m-p+n-1}{n-1} = \binom{n+m-p-1}{n-1}.$$

We have the end result in closed form which is

$$\binom{n+m-p-1}{n-1} - \binom{2m}{m}[[p=0]].$$

This identity was found by a computer search which pointed to OEIS A178300,a certain type of binomial coefficient.

1.165.6 OEIS A052553

We claim that the following holds with n, m, p non-negative values and $m \le 2n+1$:

$$\binom{n-m-p}{m} = \sum_{k=0}^{n} (-1)^k \binom{p+m}{p+k} \binom{n-m+k}{m} \binom{p-1+k}{k}.$$

Start by re-writing the RHS as

$$(-1)^n \sum_{k=0}^{n} (-1)^k \binom{p+m}{p+n-k} \binom{2n-m-k}{m} \binom{p-1+n-k}{n-k}.$$

Now we may use the third binomial coefficient to enforce the upper range and extend to infinity, getting

$$(-1)^n[z^n](1+z)^{p-1+n}\sum_{k\ge 0}(-1)^k\binom{p+m}{p+n-k}\binom{2n-m-k}{m}\frac{z^k}{(1+z)^k}$$

$$= (-1)^n[z^n](1+z)^{p-1+n}[w^m](1+w)^{2n-m}$$

$$\times\sum_{k\ge 0}(-1)^k\binom{p+m}{p+n-k}\frac{1}{(1+w)^k}\frac{z^k}{(1+z)^k}$$

$$= (-1)^n[z^n](1+z)^{p-1+n}[w^m](1+w)^{2n-m}[v^{p+n}](1+v)^{p+m}$$

$$\times\sum_{k\ge 0}(-1)^k v^k\frac{1}{(1+w)^k}\frac{z^k}{(1+z)^k}$$

$$= (-1)^n[z^n](1+z)^{p-1+n}[w^m](1+w)^{2n-m}[v^{p+n}](1+v)^{p+m}$$

$$\times\frac{1}{1+vz/(1+w)/(1+z)}$$

$$= (-1)^n[z^n](1+z)^{p+n}[w^m](1+w)^{2n+1-m}[v^{p+n}](1+v)^{p+m}$$

$$\times\frac{1}{(1+w)(1+z)+vz}$$

$$= (-1)^n[z^n](1+z)^{p+n}[w^m](1+w)^{2n+1-m}[v^{p+n}](1+v)^{p+m}$$

$$\times\frac{1}{w(1+z)+1+z(1+v)}$$

$$= (-1)^n[z^n](1+z)^{p+n-1}[w^m](1+w)^{2n+1-m}[v^{p+n}](1+v)^{p+m}$$

$$\times\frac{1}{w+1+zv/(1+z)}.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{m+1}}(1+w)^{2n+1-m}\frac{1}{w+1+zv/(1+z)}.$$

Now put $w/(1+w)=u$ so that $w=u/(1-u)$ and $dw = 1/(1-u)^2\ du$ to get

$$\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m}}\frac{1}{u/(1-u)+1+zv/(1+z)}\frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m+1}}\frac{1}{u+1-u+z(1-u)v/(1+z)}$$

$$= \operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m+1}}\frac{1}{1+z(1-u)v/(1+z)}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}}(1+z)^{p+n-1}\frac{1}{1+z(1-u)v/(1+z)}.$$

Now put $z/(1+z) = x$ so that $z = x/(1-x)$ and $dz = 1/(1-x)^2\, dx$ to get

$$\operatorname*{res}_x \frac{1}{x^{n+1}}\frac{1}{(1-x)^{p-2}}\frac{1}{1+x(1-u)v}\frac{1}{(1-x)^2}$$
$$= \operatorname*{res}_x \frac{1}{x^{n+1}}\frac{1}{(1-x)^{p}}\frac{1}{1+x(1-u)v}.$$

Re-capitulating what we have,

$$(-1)^n[v^{p+n+1}](1+v)^{p+m}\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m+2}}$$
$$\times \operatorname*{res}_x \frac{1}{x^{n+1}}\frac{1}{(1-x)^{p}}\frac{1}{x+1/v/(1-u)}.$$

We will evaluate this using the fact that residues in x sum to zero, there is no residue at infinity. We start with the contribution from $x = -1/v/(1-u)$ and obtain

$$-(-1)^n[v^{p+n+1}](1+v)^{p+m}\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m+2}}$$
$$\times(-1)^{n+1}v^{n+1}(1-u)^{n+1}\frac{1}{(1+1/v/(1-u))^p}$$
$$= [v^{p+n+1}](1+v)^{p+m}\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m+2}}$$
$$\times v^{n+1}(1-u)^{n+1}\frac{v^p(1-u)^p}{(1+v(1-u))^p}$$
$$= [v^0](1+v)^{p+m}\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m+2}}$$
$$\times(1-u)^{n+p+1}\frac{1}{(1+v(1-u))^p}.$$

We see that only the constant term of the series contributes. This yields

$$\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{2n-2m+2}}(1-u)^{n+p+1}$$
$$= \binom{2n-2m+1-n-p-1+m}{m} = \binom{n-m-p}{m}.$$

This is the claim. It remains to show that the residue at $x = 1$ makes a zero contribution. This requires the Leibniz rule:

$$(-1)^p \frac{1}{(p-1)!} \left(\frac{1}{x^{n+1}} \frac{1}{(x+1/v/(1-u))^1}\right)^{(p-1)}$$

$$= (-1)^p \frac{1}{(p-1)!} \sum_{q=0}^{p-1} \binom{p-1}{q} \frac{(n+1)^{\overline{q}}}{x^{n+1+q}} (-1)^q \frac{1^{\overline{p-1-q}}}{(x+1/v/(1-u))^{1+p-1-q}} (-1)^{p-1-q}$$

$$= -\sum_{q=0}^{p-1} \frac{1}{x^{n+1+q}} \binom{n+q}{q} \frac{1}{(x+1/v/(1-u))^{p-q}}.$$

Evaluate at $x = 1$ and restore the extractors to get

$$-[v^{p+n+1}](1+v)^{p+m} \operatorname*{res}_u \frac{1}{u^{m+1}} \frac{1}{(1-u)^{2n-2m+2}}$$

$$\times \sum_{q=0}^{p-1} \binom{n+q}{q} \frac{1}{(1+1/v/(1-u))^{p-q}}.$$

Expanding the series in the sum we obtain a sequence of decreasing powers of v starting with v^0. Therefore if $p+m < p+n+1$ we get zero from the extractor. This is $m-1 < n$ or $m \le n$. We now work with the remaining case which is $n < m$. This yields without the sign as we are looking for zero:

$$\operatorname*{res}_u \frac{1}{u^{m+1}} \frac{1}{(1-u)^{2n-2m+2}}$$

$$\times \sum_{r=p+n+1}^{p+m} \binom{p+m}{r} \sum_{q=0}^{p-1} \binom{n+q}{q} \binom{p-1-q+r-(p+n+1)}{r-(p+n+1)} \frac{(-1)^{r-(p+n+1)}}{(1-u)^{r-(p+n+1)}}$$

$$= \operatorname*{res}_u \frac{1}{u^{m+1}} \frac{1}{(1-u)^{2n-2m+2}}$$

$$\times \sum_{r=0}^{m-1-n} \binom{p+m}{r+p+n+1} \sum_{q=0}^{p-1} \binom{n+q}{q} \binom{p-1-q+r}{r} \frac{(-1)^r}{(1-u)^r}$$

$$= \sum_{r=0}^{m-1-n} \binom{p+m}{r+p+n+1} (-1)^r \sum_{q=0}^{p-1} \binom{n+q}{q} \binom{p-1-q+r}{r} \binom{2n-m+1+r}{m}.$$

Note however that in the rightmost binomial coefficient the maximum upper index is $2n-m+1+m-1-n = n < m$ so it vanishes, and with it the entire sum, which concludes the argument. This is supposing the upper index is non-negative. For a negative upper index and a non-zero contribution we require $2n-m+1+r < 0$ or $2n+1-m < -r$. But per the initial conditions $2n+1-m$ is non-negative, so this cannot happen.

This identity was found by a computer search which pointed to OEIS A052553, binomial coefficients read by antidiagonals.

1.165.7 OEIS A114607

We claim that with $m \le n-1-p$ or $p \le n-1-m$

$$(2n+1)^p = (-1)^{n+m} \sum_{k=0}^{n} \binom{2n}{k-1} \binom{2n-k}{n-m-k} (-1)^k k^p.$$

Start by re-writing the RHS to get

$$\sum_{k=0}^{n} \binom{2n}{2n+1-k} \binom{-n-m-1}{n-m-k} k^p.$$

We use the second binomial coefficient which enforces an upper range of $n-m$ to get

$$[z^{n-m}] \frac{1}{(1+z)^{n+m+1}} \sum_{k\ge 0} \binom{2n}{2n+1-k} z^k k^p$$

$$= [z^{n-m}] \frac{1}{(1+z)^{n+m+1}} [w^{2n+1}](1+w)^{2n} \sum_{k\ge 0} w^k z^k k^p$$

$$= p![v^p][z^{n-m}] \frac{1}{(1+z)^{n+m+1}} [w^{2n+1}](1+w)^{2n} \sum_{k\ge 0} w^k z^k \exp(kv)$$

$$= p![v^p][z^{n-m}] \frac{1}{(1+z)^{n+m+1}} [w^{2n+1}](1+w)^{2n} \frac{1}{1-wz\exp(v)}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{2n+2}} (1+w)^{2n} \frac{1}{1-wz\exp(v)}.$$

Now put $w/(1+w) = u$ so that $w = u/(1-u)$ and $dw = 1/(1-u)^2\, du$ to get

$$\operatorname*{res}_{u} \frac{1}{u^{2n+2}} (1-u)^2 \frac{1}{1-uz\exp(v)/(1-u)} \frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_{u} \frac{1}{u^{2n+2}} \frac{1-u}{1-u(1+z\exp(v))}.$$

We get two pieces, namely

$$p![v^p][z^{n-m}] \frac{1}{(1+z)^{n+m+1}} (1+z\exp(v))^{2n+1}$$

and

$$p![v^p][z^{n-m}] \frac{1}{(1+z)^{n+m+1}} (1+z\exp(v))^{2n}.$$

Hence we set ourselves the task to evaluate where $a \in \{0, 1\}$

$$p![v^p][z^{n-m}]\frac{1}{(1+z)^{n+m+1}}(1+z\exp(v))^{2n+a}.$$

This is

$$p![v^p][z^{n-m}]\frac{1}{(1+z)^{n+m+1}}\sum_{q=0}^{2n+a}\binom{2n+a}{q}(1+z)^{2n+a-q}z^q(\exp(v)-1)^q$$

$$= p![v^p]\sum_{q=0}^{2n+a}\binom{2n+a}{q}\binom{n+a-q-m-1}{n-m-q}(\exp(v)-1)^q.$$

The second binomial coefficient is zero when $n-m-q$ goes negative by construction. It will retain this property in future manipulations. We evaluate this coefficient by the falling factorial and get

$$p![v^p]\sum_{q=0}^{2n+a}\binom{2n+a}{q}(-1)^{n-m-q}\binom{-a}{n-m-q}(\exp(v)-1)^q,$$

The term with the exponential has $\exp(v)-1 = v+\cdots$ so it starts at v^q. Note that from the second binomial coefficient everything vanishes when $a=0$ except perhaps $q=n-m$ which would require $p \ge n-m$ which cannot happen by the initial conditions. The entire piece two contributes zero.

We get for the binomial cofficient when $a=-1$ that it is

$$[[q \le n-m]] \times \binom{-1}{n-m-q} = [[q \le n-m]](-1)^{n-m-q}$$

$$= (-1)^{n-m}[z^q]\sum_{r=0}^{n-m}(-z)^r = (-1)^{n-m}[z^q]\frac{1-(-1)^{n-m+1}z^{n-m+1}}{1+z}.$$

We see that $a=-1$ has two sub-cases corresponding to the two terms in the numerator. We get from the first one

$$p![v^p]\sum_{q=0}^{2n+1}\binom{2n+1}{q}(-1)^q(\exp(v)-1)^q[z^q]\frac{1}{1+z}$$

$$= p![v^p]\sum_{q=0}^{2n+1}\binom{2n+1}{q}(\exp(v)-1)^q$$

$$= p![v^p]\exp(v)^{2n+1} = (2n+1)^p.$$

This is our claim. The remaining subcase is

$$p![v^p]\sum_{q=0}^{2n+1}\binom{2n+1}{q}(-1)^{n-m-q}(\exp(v)-1)^q[z^q]\frac{z^{n-m+1}}{1+z}$$

$$=\sum_{q=0}^{2n+1}\binom{2n+1}{q}(-1)^{n-m-q}q!\left\{ p \atop q \right\}[[q\ge n-m+1]](-1)^{q-(n-m+1)}.$$

Note that when $q \ge n-m+1$ we also have $q > p$ because $n-m+1 > n-m-1 \ge p$ by the initial conditions. But when $q > p$ the Stirling number vanishes so the total contribution is zero. The term $\left\{ 0 \atop 0 \right\}$ does not contribute here. This concludes the argument.

This identity was found by a computer search which pointed to OEIS A114607, a fractal binary pattern.

1.165.8 OEIS A000984

We claim that with $n \ge p$

$$\binom{2p}{p}=\sum_{k=0}^{n}(-1)^k\binom{2p}{k}\binom{m-k+p}{p}\binom{n+p-k}{p}.$$

We start by using the third binomial coefficient to enforce the upper range,

$$\sum_{k=0}^{n}(-1)^k\binom{2p}{k}\binom{m-k+p}{p}\binom{n+p-k}{n-k}$$

$$=[z^n](1+z)^{n+p}\sum_{k\ge 0}(-1)^k\binom{2p}{2p-k}\binom{m-k+p}{p}\frac{z^k}{(1+z)^k}$$

$$=[z^n](1+z)^{n+p}[w^{2p}](1+w)^{2p}\sum_{k\ge 0}(-1)^k w^k\binom{m-k+p}{p}\frac{z^k}{(1+z)^k}$$

$$=[z^n](1+z)^{n+p}[w^{2p}](1+w)^{2p}[v^p](1+v)^{m+p}\sum_{k\ge 0}(-1)^k w^k\frac{1}{(1+v)^k}\frac{z^k}{(1+z)^k}$$

$$=[z^n](1+z)^{n+p}[w^{2p}](1+w)^{2p}[v^p](1+v)^{m+p}\frac{1}{1+wz/(1+z)/(1+v)}.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{2p+1}}(1+w)^{2p}\frac{1}{1+wz/(1+z)/(1+v)}.$$

Now put $w/(1+w)=u$ so that $w=u/(1-u)$ and $dw = 1/(1-u)^2\, du$ to get

$$\operatorname*{res}_u \frac{1}{u^{2p+1}}(1-u)\frac{1}{1+uz/(1-u)/(1+z)/(1+v)}\frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_{u} \frac{1}{u^{2p+1}} \frac{1}{1-u+uz/(1+z)/(1+v)}$$

$$= \operatorname*{res}_{u} \frac{1}{u^{2p+1}} \frac{1}{1-u(1-z/(1+z)/(1+v))} = \left[1 - \frac{z}{(1+z)(1+v)}\right]^{2p}.$$

Re-capitulating what we have,

$$[z^n](1+z)^{n-p}[v^p](1+v)^{m-p}[1+v+vz]^{2p}$$

$$= [z^n](1+z)^{n-p}[v^p](1+v)^{m-p}\sum_{q=0}^{2p}\binom{2p}{q}v^q(1+z)^q$$

$$= \sum_{q=0}^{2p}\binom{2p}{q}\binom{m-p}{p-q}\binom{n+q-p}{n} = \binom{2p}{p}.$$

This is because we have by construction of the binomial coefficients that we have for the second binomial coefficient that it is zero when $p < q$. Furthermore the initial conditions say that $n \ge p$ so $n+q-p$ is a non-negative number which means the third binomial coefficient is zero when $n+q-p < n$ or $q < p$. The only non-zero coefficient that is left is $q = p$ which gives the result.

This identity was found by a computer search which pointed to OEIS A000984, central binomial coefficients.

1.165.9 OEIS A101688

We claim that with $\lfloor n/2 \rfloor + p \le m \le n$ and $p \ge 1$

$$1 = (-1)^{n+m+p+1}\sum_{k=0}^{m-p}(-1)^k\binom{m-1+k}{n-1}\binom{m-1+k}{p-1}\binom{2m-p}{m-p-k}.$$

We use the third binomial coefficient to enforce the upper range:

$$(-1)^{n+m+p+1}[z^{m-p}](1+z)^{2m-p}\sum_{k\ge 0}(-1)^k\binom{m-1+k}{n-1}\binom{m-1+k}{p-1}z^k$$

$$= (-1)^{n+m+p+1}[z^{m-p}](1+z)^{2m-p}[w^{n-1}](1+w)^{m-1}[v^{p-1}](1+v)^{m-1}$$
$$\times\sum_{k\ge 0}(-1)^k(1+w)^k(1+v)^k z^k$$

$$= (-1)^{n+m+p+1}[z^{m-p}](1+z)^{2m-p}[w^{n-1}](1+w)^{m-1}[v^{p-1}](1+v)^{m-1}$$
$$\times\frac{1}{1+z(1+w)(1+v)}$$

$$= (-1)^{n+m+p+1}[z^{m-p+1}](1+z)^{2m-p}[w^{n-1}](1+w)^{m-1}[v^{p-1}](1+v)^{m-2}$$

$$\times \frac{1}{w+1+1/z/(1+v)}.$$

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^n}(1+w)^{m-1}\frac{1}{w+1+1/z/(1+v)}.$$

The residue at infinity is zero because we stipulated that $m \le n$. Hence we may evaluate using minus the residue at $w = -1 - 1/z/(1+v)$, getting

$$(-1)^{n+1}\frac{z^n(1+v)^n}{(1+z(1+v))^n}(-1)^{m-1}\frac{1}{z^{m-1}(1+v)^{m-1}}.$$

Merging into the remaining extractors,

$$(-1)^{p+1}[z^{2m-p-n}](1+z)^{2m-p}[v^{p-1}](1+v)^{n-1}\frac{1}{(1+z(1+v))^n}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{2m-p-n+1}}(1+z)^{2m-p}\frac{1}{(1+z(1+v))^n}.$$

Now put $z/(1+z) = u$ so that $z = u/(1-u)$ and $dz = 1/(1-u)^2\, du$ to get

$$\operatorname*{res}_u \frac{1}{u^{2m-p-n+1}}\frac{1}{(1-u)^{n-1}}\frac{1}{(1+u(1+v)/(1-u))^n}\frac{1}{(1-u)^2}$$
$$= \operatorname*{res}_u \frac{1}{u^{2m-p-n+1}}\frac{1}{1-u}\frac{1}{(1+uv)^n}.$$

We now get from the extractor in v

$$\sum_{q=0}^{p-1}\binom{n-1}{p-1-q}\binom{n-1+q}{q}(-1)^q u^q.$$

With the residue in u this becomes

$$(-1)^{p+1}\sum_{q=0}^{p-1}\binom{n-1}{p-1-q}\binom{n-1+q}{q}(-1)^q[[2m-p-n-q \ge 0]].$$

But we have $2m-p-n \ge p-1$ as per the initial conditions which yield $2m-2p \ge 2\lfloor n/2\rfloor \ge n-1$ so we may drop the Iverson bracket. This leaves

$$(-1)^{p+1}[w^{p-1}](1+w)^{n-1}\sum_{q\ge 0}\binom{n-1+q}{q}(-1)^q w^q$$
$$= (-1)^{p+1}[w^{p-1}](1+w)^{n-1}\frac{1}{(1+w)^n}$$

$$= (-1)^{p+1}[w^{p-1}]\frac{1}{1+w} = (-1)^{p+1}(-1)^{p-1} = 1.$$

This concludes the argument.

This identity was found by a computer search which pointed to OEIS A101688, a recursively constructed binary sequence.

1.165.10 OEIS A008459

We claim that with $n+1-p \le m \le n$ and $n \ge p \ge 1$ that

$$\binom{n-1}{p-1}^2 = (-1)^m \sum_{k=0}^{n} \binom{p-1+k}{k} (-1)^k \binom{n-1+k}{m} \binom{2n}{n+k} \binom{k+n-p}{k}.$$

We use the binomial coefficient with upper index $2n$ to enforce the upper range of the sum writing $\binom{2n}{n-k} = (-1)^{n-k}\binom{-n-1-k}{n-k}$:

$$(-1)^{n+m}[z^n]\frac{1}{(1+z)^{n+1}} \sum_{k=0}^{n} \binom{p-1+k}{k} \binom{n-1+k}{m} \frac{z^k}{(1+z)^k} \binom{k+n-p}{k}$$

$$= (-1)^{n+m}[z^n]\frac{1}{(1+z)^{n+1}}[w^m](1+w)^{n-1}$$

$$\times \sum_{k=0}^{n} \binom{p-1+k}{p-1} (1+w)^k \frac{z^k}{(1+z)^k} \binom{k+n-p}{k}$$

$$= (-1)^{n+m}[z^n]\frac{1}{(1+z)^{n+1}}[w^m](1+w)^{n-1}[v^{p-1}](1+v)^{p-1}$$

$$\times \sum_{k=0}^{n} (1+w)^k \frac{z^k}{(1+z)^k} (1+v)^k \binom{k+n-p}{k}$$

$$= (-1)^{n+m}[z^n]\frac{1}{(1+z)^{n+1}}[w^m](1+w)^{n-1}[v^{p-1}](1+v)^{p-1}$$

$$\times \frac{1}{(1-z(1+w)(1+v)/(1+z))^{n-p+1}}$$

$$= (-1)^{m+p-1}[z^n]\frac{1}{(1+z)^{p}}[w^m](1+w)^{n-1}[v^{p-1}](1+v)^{p-1}$$

$$\times \frac{1}{(zv(1+w)+zw-1)^{n-p+1}}.$$

The contribution from v is

$$\operatorname*{res}_{v} \frac{1}{v^p}(1+v)^{p-1}\frac{1}{(zv(1+w)+zw-1)^{n-p+1}}.$$

Now put $v/(1+v) = u$ so that $v = u/(1-u)$ and $dv = 1/(u-1)^2 \; du$ to get

$$\begin{aligned}
&\operatorname*{res}_u \frac{1}{u^p}(1-u)\frac{1}{(zu(1+w)/(1-u)+zw-1)^{n-p+1}}\frac{1}{(1-u)^2}\\
&= \operatorname*{res}_u \frac{1}{u^p}\frac{(1-u)^{n-p}}{(zu(1+w)+(zw-1)(1-u))^{n-p+1}}\\
&= \operatorname*{res}_u \frac{1}{u^p}\frac{(1-u)^{n-p}}{(u(1+z)+wz-1)^{n-p+1}}\\
&= \frac{1}{(wz-1)^{n-p+1}} \operatorname*{res}_u \frac{1}{u^p}\frac{(1-u)^{n-p}}{(1+u(1+z)/(wz-1))^{n-p+1}}\\
&= \frac{1}{(wz-1)^{n-p+1}}\sum_{q=0}^{p-1}\binom{n-p}{p-1-q}(-1)^{p-1-q}\binom{q+n-p}{q}(-1)^q\frac{(1+z)^q}{(wz-1)^q}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
\binom{n-p}{p-1-q}\binom{q+n-p}{q} &= \frac{(q+n-p)!}{(p-1-q)!\times(n+1+q-2p)!\times q!}\\
&= \binom{p-1}{q}\binom{q+n-p}{p-1}.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\frac{1}{(wz-1)^{n-p+1}}(-1)^{p-1}\sum_{q=0}^{p-1}\binom{p-1}{q}\binom{q+n-p}{p-1}\frac{(1+z)^q}{(wz-1)^q}\\
&= [v^{p-1}]\frac{(1+v)^{n-p}}{(wz-1)^{n-p+1}}(-1)^{p-1}\sum_{q=0}^{p-1}\binom{p-1}{q}\frac{(1+z)^q}{(wz-1)^q}(1+v)^q\\
&= [v^{p-1}]\frac{(1+v)^{n-p}}{(wz-1)^{n-p+1}}(-1)^{p-1}\left[1+\frac{(1+z)(1+v)}{wz-1}\right]^{p-1}\\
&= [v^{p-1}]\frac{(1+v)^{n-p}}{(wz-1)^{n}}(-1)^{p-1}\left[z(1+w+v)+v\right]^{p-1}.
\end{aligned}$$

Let us re-capitulate what we now have

$$\begin{aligned}
&(-1)^{n+m}[w^m](1+w)^{n-1}[v^{p-1}](1+v)^{n-p}\\
&\times[z^n]\frac{1}{(1+z)^p}\frac{1}{(1-wz)^n}\left[z(1+w)+v(1+z)\right]^{p-1},
\end{aligned}$$

With the extractor in z we write

$$[z^n]\frac{1}{(1+z)^p}\frac{1}{(1-wz)^n}\sum_{q=0}^{p-1}\binom{p-1}{q}z^q(1+w)^q v^{p-1-q}(1+z)^{p-1-q}.$$

The extractor in v now yields

$$[z^n]\frac{1}{(1-wz)^n}\sum_{q=0}^{p-1}\binom{p-1}{q}z^q(1+w)^q\frac{1}{(1+z)^{q+1}}[v^q](1+v)^{n-p}$$
$$=[z^n]\frac{1}{(1-wz)^n}\sum_{q=0}^{p-1}\binom{p-1}{q}\binom{n-p}{q}z^q(1+w)^q\frac{1}{(1+z)^{q+1}}$$
$$=\sum_{q=0}^{p-1}\binom{p-1}{q}\binom{n-p}{q}(1+w)^q\sum_{r=0}^{n-q}\binom{n-1+r}{r}w^r(-1)^{n-q-r}\binom{n-r}{n-q-r}.$$

Switching sums,

$$(-1)^m\sum_{r=0}^{n}\binom{n-1+r}{r}w^r(-1)^r\sum_{q=0}^{n-r}\binom{p-1}{q}\binom{n-p}{q}(-1)^q\binom{n-r}{q}(1+w)^q.$$

Working with the inner sum

$$[z^{p-1}](1+z)^{p-1}[v^{n-p}](1+v)^{n-p}\sum_{q=0}^{n-r}\binom{n-r}{q}(-1)^q(1+w)^q z^q v^q$$
$$=[z^{p-1}](1+z)^{p-1}[v^{n-p}](1+v)^{n-p}[1-vz(1+w)]^{n-r}.$$

Activating the outer sum,

$$(-1)^m[z^{p-1}](1+z)^{p-1}[v^{n-p}](1+v)^{n-p}\frac{[1-vz(1+w)]^n}{(1+w/(1-vz(1+w)))^n}$$
$$=(-1)^m[z^{p-1}](1+z)^{p-1}[v^{n-p}](1+v)^{n-p}\frac{[1-vz(1+w)]^{2n}}{(1-vz(1+w)+w))^n}.$$

The extractor in w is still pending in front,

$$(-1)^{n+m}[w^m]\frac{1}{1+w}[z^{p-1}](1+z)^{p-1}[v^{n-p}](1+v)^{n-p}\frac{[1-vz-wvz]^{2n}}{(1-vz)^n}$$

$$= (-1)^{n+m} \sum_{q=0}^{m} (-1)^{m-q} \binom{2n}{q} [z^{p-1}](1+z)^{p-1} [v^{n-p}](1+v)^{n-p} (-vz)^q (1-vz)^{n-q}$$

$$= (-1)^n \sum_{q=0}^{m} \binom{2n}{q} \operatorname*{res}_z \frac{1}{z^{p-q}} (1+z)^{p-1} \operatorname*{res}_v \frac{1}{v^{n-p-q+1}} (1+v)^{n-p} (1-vz)^{n-q}.$$

We make an important observation. As per the initial conditions $m \ge n-p+1$ which means that the residue in v vanishes in the upper range. Hence we may lower m to $n-p$ and the independence of the closed form of the value of m is justified. Continuing,

$$(-1)^n [w^{n-p}] \frac{1}{1-w} \sum_{q\ge 0} \binom{2n}{q} w^q \operatorname*{res}_z \frac{(1+z)^{p-1}}{z^{p-q}} \operatorname*{res}_v \frac{(1+v)^{n-p}}{v^{n-p-q+1}} (1-vz)^{n-q}$$

$$= (-1)^n [w^{n-p}] \frac{1}{1-w} \operatorname*{res}_z \frac{(1+z)^{p-1}}{z^p} \operatorname*{res}_v \frac{(1+v)^{n-p}}{v^{n-p+1}} (1-vz)^n \left[1 + \frac{wzv}{1-vz}\right]^{2n}$$

$$= (-1)^n [w^{n-p}] \frac{1}{1-w} \operatorname*{res}_z \frac{(1+z)^{p-1}}{z^p} \operatorname*{res}_v \frac{(1+v)^{n-p}}{v^{n-p+1}} \frac{1}{(1-vz)^n} [1 - vz + wzv]^{2n}.$$

Summing the residues from w we get from the residue at $w=1$ the contribution

$$\operatorname*{res}_z \frac{(1+z)^{p-1}}{z^p} \operatorname*{res}_v \frac{(1+v)^{n-p}}{v^{n-p+1}} \frac{1}{(1-vz)^n}$$

$$= \operatorname*{res}_z \frac{1}{z^p} \frac{1}{(1+z)^{n+1-p}} \operatorname*{res}_v \frac{(1+v)^{n-p}}{v^{n-p+1}} \frac{1}{(1-(1+v)z/(1+z))^n}$$

$$= \sum_{q=0}^{p-1} \binom{n-1+q}{q} \operatorname*{res}_z \frac{1}{z^p} \frac{1}{(1+z)^{n+1-p}} \operatorname*{res}_v \frac{(1+v)^{n-p}}{v^{n-p+1}} \frac{z^q (1+v)^q}{(1+z)^q}$$

$$= \sum_{q=0}^{p-1} \binom{n-1+q}{q} \binom{p-1-q+n-p+q}{p-1-q} (-1)^{p-1-q} \binom{n-p+q}{n-p}$$

$$= (-1)^{p-1} \sum_{q=0}^{p-1} \binom{n-1+q}{q} \binom{n-1}{n-p+q} (-1)^q \binom{n-p+q}{n-p}.$$

Observe that

$$\binom{n-1}{n-p+q} \binom{n-p+q}{n-p} = \frac{(n-1)!}{(p-1-q)! \times (n-p)! \times q!} = \binom{n-1}{p-1} \binom{p-1}{q}.$$

We have established one of the factors to make the square. For the remaining factor we have

$$
\begin{aligned}
&(-1)^{p-1}\sum_{q=0}^{p-1}\binom{n-1+q}{q}(-1)^q\binom{p-1}{q}\\
&= (-1)^{p-1}[z^{n-1}](1+z)^{n-1}\sum_{q\ge 0}\binom{p-1}{q}(-1)^q(1+z)^q\\
&= (-1)^{p-1}[z^{n-1}](1+z)^{n-1}(1-(1+z))^{p-1} = [z^{n-p}](1+z)^{n-1}\\
&= \binom{n-1}{n-p} = \binom{n-1}{p-1}.
\end{aligned}
$$

We have the second factor and may conclude the argument. For the sake of rigour we will now check the residue at infinity. We get

$$
\begin{aligned}
&-\operatorname*{res}_{w}\frac{1}{w^2}w^{n+1-p}\frac{1}{1-1/w}\frac{1}{w^{2n}}[w(1-vz)+zv]^{2n}\\
&= \operatorname*{res}_{w}\frac{1}{w^{n+p}}\frac{1}{1-w}[w(1-vz)+zv]^{2n}\\
&= \sum_{q=0}^{n+p-1}\binom{2n}{q}(1-vz)^q z^{2n-q}v^{2n-q}.
\end{aligned}
$$

We then obtain from the two remaining residues,

$$
\sum_{q=0}^{n+p-1}\binom{2n}{q}\operatorname*{res}_{z}\frac{(1+z)^{p-1}}{z^p}\operatorname*{res}_{v}\frac{(1+v)^{n-p}}{v^{n-p+1}}z^{2n-q}v^{2n-q}\frac{1}{(1-vz)^{n-q}}.
$$

Note however that $2n-q-(n-p+1)\ge 2n-(n+p-1)-(n-p+1)=0$ so the residue in v cancels those contributions.

This identity was found by a computer search which pointed to OEIS A008459, square the entries of Pascal's triangle.

1.165.11 OEIS A053126

We claim that with $0\le m\le n-p$ and $n\ge p$ that

$$
\binom{2n-1}{p} = (-1)^{m+p}\sum_{k=0}^{n}\binom{n-1+k}{k}(-1)^k\binom{n-1+k}{m}\binom{2n}{n+k}\binom{2k+p}{p}.
$$

We use the third binomial coefficient to enforce the upper range of k at the start,

$$(-1)^{n+m+p}[z^n](1+z)^{2n}\sum_{k=0}^{n}\binom{n-1+k}{k}(-1)^k\binom{n-1+k}{m}z^k\binom{2k+p}{p}$$

$$= (-1)^{m+p}[z^n](1+z)^{2n}[w^m](1+w)^{n-1}$$

$$\times \sum_{k\ge 0}\binom{n-1+k}{k}(-1)^k(1+w)^k z^k\binom{2k+p}{p}$$

$$= (-1)^{m+p}[z^n](1+z)^{2n}[w^m](1+w)^{n-1}[v^p](1+v)^p$$

$$\times \sum_{k\ge 0}\binom{n-1+k}{k}(-1)^k(1+w)^k z^k(1+v)^{2k}$$

$$= (-1)^{m+p}[z^n](1+z)^{2n}[w^m](1+w)^{n-1}[v^p](1+v)^p$$

$$\times \frac{1}{(1+z(1+w)(1+v)^2)^n}.$$

It is convenient to re-order the extractors,

$$(-1)^{m+p}[v^p](1+v)^p[w^m](1+w)^{n-1}[z^n](1+z)^{2n}$$

$$\times \frac{1}{(1+z(1+w)(1+v)^2)^n}$$

$$= (-1)^{m+p}[v^p](1+v)^{n+p}[w^m](1+w)^{n-1}[z^n](1+z/(1+v))^{2n}$$

$$\times \frac{1}{(1+z(1+w)(1+v))^n}$$

$$= (-1)^{m+p}[v^p]\frac{1}{(1+v)^{n-p}}[w^m](1+w)^{n-1}[z^n](1+z+v)^{2n}$$

$$\times \frac{1}{(1+z(1+w)(1+v))^n}$$

$$= (-1)^{m+p}[v^p]\frac{1}{(1+v)^{n-p}}[w^m]\frac{1}{1+w}[z^{2n}](1+z+v)^{2n}$$

$$\times \frac{1}{(v+1+1/z/(1+w))^n}$$

$$= (-1)^{m+p}[v^p]\frac{1}{(1+v)^{n-p}}[w^m]\frac{1}{1+w}[z^{2n}](1+z+v)^{2n}\frac{1}{(1+1/z/(1+w))^n}$$

$$\times \frac{1}{(1+v/(1+1/z/(1+w)))^n}$$

The contribution from v is

$$\sum_{q=0}^p (-1)^{p-q} \binom{p-q+n-p-1}{p-q} [v^q] \frac{(1+z+v)^{2n}}{(1+v/(1+1/z/(1+w)))^n}$$

$$= \sum_{q=0}^p (-1)^{p-q} \binom{n-q-1}{p-q}$$

$$\times \sum_{r=0}^q \binom{2n}{q-r} (1+z)^{2n-q+r} (-1)^r \binom{r+n-1}{r} \frac{1}{(1+1/z/(1+w))^r}.$$

Restoring the fractional term in z and $1+w$

$$\sum_{q=0}^p (-1)^{p-q} \binom{n-q-1}{p-q}$$

$$\times \sum_{r=0}^q \binom{2n}{q-r} (1+z)^{2n-q+r} (-1)^r \binom{r+n-1}{r} \frac{1}{(1+1/z/(1+w))^{n+r}}.$$

Now note that $2n-q+r \ge 0$ as per the initial conditions and the series from the rightmost term only produces negative powers of z. That means with the coefficient extractor being $[z^{2n}]$ only $q=r$ contributes and we get

$$(-1)^{m+p} [w^m] \frac{1}{1+w} \sum_{q=0}^p (-1)^{p-q} \binom{n-q-1}{p-q} (-1)^q \binom{q+n-1}{q}$$

$$= \sum_{q=0}^p \binom{n-q-1}{p-q} \binom{q+n-1}{q}$$

$$= [u^p](1+u)^{n-1} \sum_{q\ge 0} \binom{q+n-1}{q} \frac{u^q}{(1+u)^q}$$

$$= [u^p](1+u)^{n-1} \frac{1}{(1-u/(1+u))^n}$$

$$= [u^p](1+u)^{n-1}(1+u)^n = [u^p](1+u)^{2n-1} = \binom{2n-1}{p}.$$

This is the claim. An alternate computation uses expansions about zero:

$$\sum_{q=0}^p (-1)^{p-q} \binom{n-q-1}{p-q}$$

$$\times \sum_{r=0}^q \binom{2n}{q-r} (1+z)^{2n-q+r} (-1)^r \binom{r+n-1}{r} \frac{z^{n+r}(1+w)^{n+r}}{(1+z(1+w))^{n+r}}$$

$$= \sum_{q=0}^{p} (-1)^{p-q} \binom{n-q-1}{p-q}$$

$$\times \sum_{r=0}^{q} \binom{2n}{q-r} (1+z)^{n-q} (-1)^r \binom{r+n-1}{r} \frac{z^{n+r}(1+w)^{n+r}}{(1+wz/(1+z))^{n+r}}.$$

Due to the coefficient extractor in w this becomes

$$\sum_{q=0}^{p} (-1)^{p-q} \binom{n-q-1}{p-q}$$

$$\times \sum_{r=0}^{q} \binom{2n}{q-r} (1+z)^{n-q} (-1)^r \binom{r+n-1}{r} z^{n+r}(1+w)^{n+r}$$

$$\times \sum_{s=0}^{m} \binom{s+n+r-1}{s} (-1)^s \frac{z^s w^s}{(1+z)^s}.$$

With the extractor in z:

$$\sum_{q=0}^{p} (-1)^{p-q} \binom{n-q-1}{p-q}$$

$$\times \sum_{r=0}^{q} \binom{2n}{q-r} (-1)^r \binom{r+n-1}{r} (1+w)^{n+r}$$

$$\times \sum_{s=0}^{m} \binom{s+n+r-1}{s} (-1)^s \binom{n-q-s}{n-r-s} w^s.$$

Note that $n-q-s$ is still a non-negative number, we have as a lower bound $n-p-m \ge 0$ as per the initial conditions. Now with the power on z we have $2n-(n+r+s) \ge n-q-s$ since $q \ge r$. Equality can only occur when $r=q$ and we obtain

$$(-1)^{m+p}[w^m] \frac{1}{1+w} \sum_{q=0}^{p} (-1)^{p-q} \binom{n-q-1}{p-q} (-1)^q \binom{q+n-1}{q} (1+w)^{n+q}$$

$$\times \sum_{s=0}^{m} \binom{s+n+q-1}{s} (-1)^s w^s.$$

We may extend s to infinity due to the coefficient extractor in w:

$$(-1)^m [w^m] \frac{1}{1+w} \sum_{q=0}^{p} \binom{n-q-1}{p-q} \binom{q+n-1}{q} (1+w)^{n+q} \frac{1}{(1+w)^{n+q}}$$

$$= \sum_{q=0}^{p} \binom{n-q-1}{p-q} \binom{q+n-1}{q}.$$

This is the same as before and we may once more conclude the argument.

This identity was found by a computer search which pointed to OEIS A053126, a subsection of binomial coefficients.

1.165.12 OEIS A356546

We claim that

$$\binom{2n}{n}\binom{n}{m} = \sum_{k=0}^{n} \binom{n+1}{2k+1}\binom{n+k}{m+k}\binom{m+k}{m}.$$

We use the first binomial coefficient to enforce the upper range of the sum:

$$\sum_{k=0}^{n} \binom{n+1}{n-2k}\binom{n+k}{m+k}\binom{m+k}{m}$$
$$= [z^n](1+z)^{n+1} \sum_{k\ge 0} z^{2k} \binom{n+k}{n-m}\binom{m+k}{m}$$
$$= [z^n](1+z)^{n+1}[w^{n-m}](1+w)^n \sum_{k\ge 0} z^{2k}(1+w)^k \binom{m+k}{m}$$
$$= [z^n](1+z)^{n+1}[w^{n-m}](1+w)^n \frac{1}{(1-z^2(1+w))^{m+1}}$$
$$= (-1)^{m+1}[z^{n+2m+2}](1+z)^{n+1}[w^{n-m}](1+w)^n \frac{1}{(w-(1-z^2)/z^2)^{m+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n-m+1}}(1+w)^n \frac{1}{(w-(1-z^2)/z^2)^{m+1}}.$$

Here the residue at infinity is zero and residues sum to zero so we may evaluate using minus the residue at $w = (1-z^2)/z^2$, which requires the Leibniz rule:

$$\frac{1}{m!}\left(\frac{1}{w^{n-m+1}}(1+w)^n\right)^{(m)}$$
$$= \frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\frac{1}{w^{n-m+1+q}}(-1)^q (n-m+1)^{\overline{q}}(1+w)^{n-(m-q)} n^{\underline{m-q}}$$
$$= \sum_{q=0}^{m} \frac{(-1)^q}{w^{n-m+1+q}} \binom{n-m+q}{q} (1+w)^{n-m+q} \binom{n}{m-q}.$$

Evaluate at $w = (1-z^2)/z^2$ to get

$$\frac{z^2}{1-z^2}\sum_{q=0}^{m}(-1)^q\binom{n-m+q}{q}\frac{1}{(1-z^2)^{n-m+q}}\binom{n}{m-q}.$$

Now observe that

$$\binom{n-m+q}{q}\binom{n}{m-q} = \frac{n!}{q!\times(n-m)!\times(m-q)!} = \binom{n}{n-m}\binom{m}{q}.$$

We thus obtain

$$\binom{n}{m}\frac{z^2}{(1-z^2)^{n-m+1}}\sum_{q=0}^{m}\binom{m}{q}(-1)^q\frac{1}{(1-z^2)^q}$$
$$= \binom{n}{m}\frac{z^2}{(1-z^2)^{n-m+1}}\frac{(-z^2)^m}{(1-z^2)^m}.$$

Restoring the coefficient extractor in z and flipping the sign yields

$$\binom{n}{m}[z^n](1+z)^{n+1}\frac{1}{(1-z^2)^{n+1}} = \binom{n}{m}[z^n]\frac{1}{(1-z)^{n+1}} = \binom{n}{m}\binom{2n}{n}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A356546, a central binomial coefficient times a binomial coefficient from Pascal's triangle.

1.165.13 OEIS A370232

We claim that with $0 \le m \le n$

$$\binom{n+m}{2m}^2 = (-1)^n\sum_{k=0}^{n}\binom{n+k}{n-k}\binom{2k}{2m}\binom{2k-2m}{k-m}(-1)^k.$$

To start observe that

$$\binom{2k}{2m}\binom{2k-2m}{k-m} = \frac{(2k)!}{(2m)!\times(k-m)!\times(k-m)!} = \binom{2k}{k-m}\binom{k+m}{2m}.$$

We re-index the sum

$$\sum_{k=0}^{n}\binom{2n-k}{k}\binom{2n-2k}{n-k-m}\binom{n+m-k}{2m}(-1)^k.$$

We use the middle binomial coefficient to enforce the upper range of the sum:

$$[z^{n-m}](1+z)^{2n} \sum_{k=0}^{n} \binom{2n-k}{k} \frac{z^k}{(1+z)^{2k}} \binom{n+m-k}{2m} (-1)^k$$

$$= [z^{n-m}](1+z)^{2n}[w^{2m}](1+w)^{n+m} \sum_{k=0}^{n} \binom{2n-k}{2n-2k} \frac{z^k}{(1+z)^{2k}} \frac{1}{(1+w)^k} (-1)^k.$$

Here we see that the remaining binomial coefficient also enforces the range. Continuing,

$$[z^{n-m}](1+z)^{2n}[w^{2m}](1+w)^{n+m}[v^{2n}](1+v)^{2n}$$
$$\times \sum_{k=0}^{n} \frac{v^{2k}}{(1+v)^k} \frac{z^k}{(1+z)^{2k}} \frac{1}{(1+w)^k} (-1)^k$$
$$= [z^{n-m}](1+z)^{2n}[w^{2m}](1+w)^{n+m}[v^{2n}](1+v)^{2n}$$
$$\times \frac{1}{1+v^2z/(1+v)/(1+z)^2/(1+w)}$$
$$= [z^{n-m}][w^{2m}](1+w)^{n+m}[v^{2n}](1+v+vz)^{2n}$$
$$\times \frac{1}{1+v^2z/(1+v(1+z))/(1+w)}$$
$$= [z^{n-m}][w^{2m}](1+w)^{n+m+1}[v^{2n}](1+v+vz)^{2n}$$
$$\times \frac{1+v+vz}{(1+w)(1+v+vz)+v^2z}$$
$$= [z^{n-m}][w^{2m}](1+w)^{n+m+1}[v^{2n}](1+v+vz)^{2n}$$
$$\times \frac{1+v+vz}{w(1+v+vz)+(1+v)(1+vz)}$$
$$= [z^{n-m}][w^{2m}](1+w)^{n+m+1}[v^{2n}](1+v+vz)^{2n}$$
$$\times \frac{1}{w+(1+v)(1+vz)/(1+v+vz)}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{2m+1}}(1+w)^{n+m+1} \frac{1}{w+(1+v)(1+vz)/(1+v+vz)}.$$

Now put $w/(1+w) = u$ so that $w = u/(1-u)$ and $dw = 1/(1-u)^2\, du$ to get

$$\operatorname*{res}_{w} \frac{(1-u)^{2m+1}}{u^{2m+1}} \frac{1}{(1-u)^{n+m+1}} \frac{1}{u/(1-u)+(1+v)(1+vz)/(1+v+vz)} \frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_{w} \frac{1}{u^{2m+1}} \frac{1}{(1-u)^{n-m+1}} \frac{1}{u+(1-u)(1+v)(1+vz)/(1+v+vz)}.$$

Here the residue at infinity is zero so we may evaluate using minus the residue at $u = 1$ and at $u = (1+v)(1+vz)/v^2/z$.

We require the Leibniz rule for the former:

$$(-1)^{n+m} \frac{1}{(n-m)!} \left(\frac{1}{u^{2m+1}} \frac{1}{(u+(1-u)(1+v)(1+vz)/(1+v+vz))^1} \right)^{(n-m)}$$

$$= (-1)^{n+m} \frac{1}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} \frac{1}{u^{2m+1+q}} (-1)^q (2m+1)^{\overline{q}}$$

$$\times \frac{1}{(u+(1-u)(1+v)(1+vz)/(1+v+vz))^{1+n-m-q}} 1^{\overline{n-m-q}} \left(\frac{v^2 z}{1+v+vz} \right)^{n-m-q}.$$

Put $u = 1$ to get

$$(-1)^{n+m} \frac{1}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q (2m+1)^{\overline{q}} 1^{\overline{n-m-q}} \left(\frac{v^2 z}{1+v+vz} \right)^{n-m-q}$$

$$= (-1)^{n+m} \sum_{q=0}^{n-m} \binom{2m+q}{q} (-1)^q \left(\frac{v^2 z}{1+v+vz} \right)^{n-m-q}.$$

With the extractors in z and v

$$(-1)^{n+m} [v^{2n}] (1+v+vz)^{2n} \sum_{q=0}^{n-m} \binom{2m+q}{q} (-1)^q [z^q] \left(\frac{v^2}{1+v+vz} \right)^{n-m-q}$$

$$= (-1)^{n+m} \sum_{q=0}^{n-m} \binom{2m+q}{q} (-1)^q [z^q] [v^{2m+2q}] (1+v+vz)^{n+m+q}$$

$$= (-1)^{n+m} \sum_{q=0}^{n-m} \binom{2m+q}{q} (-1)^q [v^{2m+2q}] \binom{n+m+q}{q} v^q (1+v)^{n+m}$$

$$= (-1)^{n+m} \sum_{q=0}^{n-m} \binom{2m+q}{q} (-1)^q \binom{n+m+q}{q} \binom{n+m}{2m+q}.$$

Now note that

$$\binom{n+m}{2m+q} \binom{2m+q}{q} = \frac{(n+m)!}{(n-m-q)! \times q! \times (2m)!} = \binom{n+m}{2m} \binom{n-m}{q}.$$

Good, we have the first factor of the square. This leaves

$$(-1)^{n+m} \sum_{q=0}^{n-m} \binom{n+m+q}{q} (-1)^q \binom{n-m}{q}$$

$$= (-1)^{n+m} [w^{n+m}] (1+w)^{n+m} \sum_{q=0}^{n-m} (-1)^q (1+w)^q \binom{n-m}{q}$$

$$= (-1)^{n+m} [w^{n+m}] (1+w)^{n+m} (1-(1+w))^{n-m}$$

$$= [w^{n+m}] (1+w)^{n+m} w^{n-m} = \binom{n+m}{2m}.$$

We have the second factor as well and may conclude. It still remains however to show that the contribution from $u = (1+v)(1+vz)/v^2/z$ is zero. With this in mind we write

$$\operatorname*{res}_{w} \frac{1}{u^{2m+1}} \frac{1}{(1-u)^{n-m+1}} \frac{1}{u+(1-u)(1+v)(1+vz)/(1+v+vz)}$$

$$= \operatorname*{res}_{w} \frac{1}{u^{2m+1}} \frac{1}{(1-u)^{n-m+1}} \frac{1}{(1+v)(1+vz)/(1+v+vz) - uv^2z/(1+v+vz)}$$

$$= \operatorname*{res}_{w} \frac{1}{u^{2m+1}} \frac{1}{(1-u)^{n-m+1}} \frac{1+v+vz}{(1+v)(1+vz) - uv^2z}$$

$$= -\frac{1}{v^2z} \operatorname*{res}_{w} \frac{1}{u^{2m+1}} \frac{1}{(1-u)^{n-m+1}} \frac{1+v+vz}{u-(1+v)(1+vz)/v^2/z}.$$

Do the substitution and flip the sign to get

$$\frac{1}{v^2z} \frac{(1+v+vz)(v^2z)^{2m+1}}{((1+v)(1+vz))^{2m+1}} (-1)^{n-m+1} \frac{(v^2z)^{n-m+1}}{(1+v+vz)^{n-m+1}}.$$

Restore the extractor in v:

$$[v^{2n}](1+v+vz)^{2n} \frac{(1+v+vz)(-1)^{n-m+1}}{((1+v)(1+vz))^{2m+1}} \frac{(v^2z)^{n+m+1}}{(1+v+vz)^{n-m+1}}$$

$$= [v^{2n}](1+v+vz)^{n+m} \frac{(-1)^{n-m+1}}{((1+v)(1+vz))^{2m+1}} (v^2z)^{n+m+1}.$$

We have however that $2n+2m+2 > 2n$ so this extractor returns zero and there is no contribution from the second residue.

This identity was found by a computer search which pointed to OEIS A370232, a squared binomial term.

1.165.14 OEIS A046899

We claim that with $0 \le m \le n$ and p a real number

$$\binom{n+m}{m} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+k}{n} \binom{n+m}{m+k} \binom{p+k}{m}.$$

We use the middle coefficient to enforce the upper range of the sum, writing $\binom{n+m}{n-k}$ to obtain

$$(-1)^n [z^n] (1+z)^{n+m} \sum_{k\ge 0} (-1)^k z^k \binom{n+k}{k} \binom{p+k}{m}$$

$$= (-1)^n [z^n] (1+z)^{n+m} [w^m] (1+w)^p \sum_{k\ge 0} (-1)^k z^k (1+w)^k \binom{n+k}{k}$$

$$= (-1)^n [z^n] (1+z)^{n+m} [w^m] (1+w)^p \frac{1}{(1+z(1+w))^{n+1}}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^{n+m} \frac{1}{(1+z(1+w))^{n+1}}.$$

Now put $z/(1+z) = u$ so that $z = u/(1-u)$ and $dz = 1/(1-u)^2 \; du$ to obtain

$$\operatorname*{res}_u \frac{(1-u)^{n+1}}{u^{n+1}} \frac{1}{(1-u)^{n+m}} \frac{1}{(1+u(1+w)/(1-u))^{n+1}} \frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_u \frac{1}{u^{n+1}} (1-u)^{n-m} \frac{1}{(1+uw)^{n+1}}$$

$$= \frac{1}{w^{n+1}} \operatorname*{res}_u \frac{1}{u^{n+1}} (1-u)^{n-m} \frac{1}{(u+1/w)^{n+1}}.$$

Here the residues sum to zero, there is no pole at $u = 1$ due to the initial conditions and the residue at infinity is zero by inspection. Hence we may evaluate using minus the residue at $u = -1/w$, which requires the Leibniz rule:

$$\frac{1}{n!} \left(\frac{1}{u^{n+1}} (1-u)^{n-m} \right)^{(n)}$$

$$= \frac{1}{n!} \sum_{q=0}^{n} \binom{n}{q} \frac{1}{u^{n+1+q}} (-1)^q (n+1)^{\overline{q}} (1-u)^{n-m-(n-q)} (-1)^{n-q} (n-m)^{\underline{n-q}}$$

$$= (-1)^n \sum_{q=0}^{n} \frac{1}{u^{n+1+q}} \binom{n+q}{q} (1-u)^{q-m} \binom{n-m}{n-q}.$$

Substitute $u = -1/w$ and flip the sign to get (double $(-1)^n$ cancels)

$$-[w^m](1+w)^p \frac{1}{w^{n+1}} \sum_{q=0}^{n} (-1)^{n+1+q} w^{n+1+q} \binom{n+q}{q} \frac{(1+w)^{q-m}}{w^{q-m}} \binom{n-m}{n-q}$$

$$= (-1)^n [w^0](1+w)^p \sum_{q=0}^{n} (-1)^q \binom{n+q}{q} (1+w)^{q-m} \binom{n-m}{n-q}$$

$$= (-1)^n \sum_{q=0}^{n} (-1)^q \binom{n+q}{q} \binom{n-m}{n-q}$$

$$= (-1)^n [v^n](1+v)^{n-m} \sum_{q\ge 0} (-1)^q \binom{n+q}{q} v^q = (-1)^n [v^n](1+v)^{n-m} \frac{1}{(1+v)^{n+1}}$$

$$= (-1)^n [v^n] \frac{1}{(1+v)^{m+1}} = \binom{n+m}{m}.$$

We have the claim.

This identity was found by a computer search which pointed to OEIS A046899, binomial $\binom{n+m}{m}$.

1.165.15 OEIS A094527

We claim that with $0 \le m \le n$

$$\binom{2n}{n+m} = \sum_{k=0}^{n} (-1)^k \binom{n-m+k}{k} \binom{2n-k}{n-m-k} \binom{2n}{n+m-k}.$$

We can use the middle binomial coefficient to enforce the upper range of the sum:

$$(-1)^{n-m} \sum_{k=0}^{n} \binom{n-m+k}{k} \binom{-n-m-1}{n-m-k} \binom{2n}{n+m-k}$$

$$= (-1)^{n-m} [z^{n-m}] \frac{1}{(1+z)^{n+m+1}} \sum_{k\ge 0} \binom{n-m+k}{k} z^k \binom{2n}{n+m-k}$$

$$= (-1)^{n-m} [z^{n-m}] \frac{1}{(1+z)^{n+m+1}} [w^{n+m}](1+w)^{2n} \sum_{k\ge 0} \binom{n-m+k}{k} z^k w^k$$

$$= (-1)^{n-m} [z^{n-m}] \frac{1}{(1+z)^{n+m+1}} [w^{n+m}](1+w)^{2n} \frac{1}{(1-zw)^{n-m+1}}$$

$$= -[z^{2n-2m+1}] \frac{1}{(1+z)^{n+m+1}} [w^{n+m}](1+w)^{2n} \frac{1}{(w-1/z)^{n-m+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+m+1}} (1+w)^{2n} \frac{1}{(w-1/z)^{n-m+1}}.$$

Here the residue at infinity is zero (just barely) and we may evaluate using minus the residue at $w = 1/z$. This requires the Leibniz rule:

$$\frac{1}{(n-m)!} \left(\frac{1}{w^{n+m+1}} (1+w)^{2n} \right)^{(n-m)}$$

$$= \frac{1}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} \frac{1}{w^{n+m+1+q}} (-1)^q (n+m+1)^{\overline{q}}$$

$$\times (1+w)^{2n-(n-m-q)} (2n)^{\underline{n-m-q}}$$

$$= \sum_{q=0}^{n-m} \frac{1}{w^{n+m+1+q}} (-1)^q \binom{n+m+q}{q} (1+w)^{n+m+q} \binom{2n}{n-m-q}.$$

Substitute $w = 1/z$ to get

$$[z^{2n-2m+1}] \frac{1}{(1+z)^{n+m+1}}$$

$$\times \sum_{q=0}^{n-m} z^{n+m+1+q} (-1)^q \binom{n+m+q}{q} \frac{(1+z)^{n+m+q}}{z^{n+m+q}} \binom{2n}{n-m-q}$$

$$= [z^{2n-2m}] \sum_{q=0}^{n-m} (-1)^q \binom{n+m+q}{q} (1+z)^{q-1} \binom{2n}{n-m-q}$$

$$= \sum_{q=0}^{n-m} (-1)^q \binom{n+m+q}{q} \binom{q-1}{2n-2m} \binom{2n}{n-m-q}.$$

Now supposing that $q \ge 1$ we require $q-1 \ge 2n-2m$ but $q-1$ is at most $n-m-1 < 2n-2m$. This leaves $q=0$ only and we get

$$(-1)^0 \binom{n+m}{0} (-1)^{2n-2m} \binom{2n}{n-m} = \binom{2n}{n+m}$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A094527, triangle of $\binom{2n}{n+m}$.

1.165.16 OEIS A008459

We claim that with $0 \le m \le n$

$$\binom{n}{m}^2 = (-1)^m \sum_{k=0}^{n} (-1)^k \binom{k}{m} \binom{2n-k}{k} \binom{2n-2k}{n-k}.$$

The third binomial coefficient serves to enforce the upper range of the sum where we write $(-1)^{n-k}\binom{-n-1+k}{n-k}$

$$(-1)^{n+m}[z^n]\frac{1}{(1+z)^{n+1}}\sum_{k=0}^{n}\binom{k}{m}\binom{2n-k}{2n-2k}z^k(1+z)^k$$

$$= (-1)^{n+m}[z^n]\frac{1}{(1+z)^{n+1}}[w^{2n}](1+w)^{2n}\sum_{k\ge 0}\binom{k}{m}\frac{w^{2k}}{(1+w)^k}z^k(1+z)^k$$

$$= (-1)^{n+m}[z^n]\frac{1}{(1+z)^{n+1}}[w^{2n}](1+w)^{2n}\sum_{k\ge m}\binom{k}{m}\frac{w^{2k}}{(1+w)^k}z^k(1+z)^k$$

$$= (-1)^{n+m}[z^{n-m}]\frac{1}{(1+z)^{n-m+1}}[w^{2n-2m}](1+w)^{2n-m}$$

$$\times\sum_{k\ge 0}\binom{k+m}{m}\frac{w^{2k}}{(1+w)^k}z^k(1+z)^k$$

$$= (-1)^{n+m}[z^{n-m}]\frac{1}{(1+z)^{n-m+1}}[w^{2n-2m}](1+w)^{2n-m}$$

$$\times\frac{1}{(1-z(1+z)w^2/(1+w))^{m+1}}$$

$$= (-1)^{n+m}[z^{n-m}]\frac{1}{(1+z)^{n-m+1}}[w^{2n-2m}](1+w)^{2n+1}$$

$$\times\frac{1}{(1-wz)^{m+1}(1+w+wz)^{m+1}}.$$

We find for the coefficient on w

$$\operatorname*{res}_{w}\frac{1}{w^{2n-2m+1}}(1+w)^{2n+1}\frac{1}{(1-wz)^{m+1}(1+w+wz)^{m+1}}.$$

Now put $w/(1+w)=v$ so that $w=v/(1-v)$ and $dw=1/(1-v)^2\ dv$ to get

$$\operatorname*{res}_{v}\frac{1}{v^{2n-2m+1}}\frac{1}{(1-v)^{2m}}\frac{1}{(1-vz/(1-v))^{m+1}(1+v(1+z)/(1-v))^{m+1}}\frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_{v}\frac{1}{v^{2n-2m+1}}\frac{1}{(1-v(1+z))^{m+1}(1+vz)^{m+1}}.$$

Restoring the extractor in z,

$$(-1)^{n+1}[z^{n+1}]\frac{1}{(1+z)^{n+2}}\operatorname*{res}_{v}\frac{1}{v^{2n-2m+1}}\frac{1}{(v-1/(1+z))^{m+1}(v+1/z)^{m+1}}.$$

Here residues sum to zero and the residue at infinity is zero, so we may evaluate using minus the residues at $v = 1/(1+z)$ and $v = -1/z$. We get from the former with the Leibniz rule,

$$\frac{1}{m!}\left(\frac{1}{v^{2n-2m+1}}\frac{1}{(v+1/z)^{m+1}}\right)^{(m)}$$

$$= \frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\frac{1}{v^{2n-2m+1+q}}(-1)^q(2n-2m+1)^{\overline{q}}$$

$$\times\frac{1}{(v+1/z)^{m+1+(m-q)}}(-1)^{m-q}(m+1)^{\overline{m-q}}$$

$$= (-1)^m\sum_{q=0}^{m}\frac{1}{v^{2n-2m+1+q}}\binom{2n-2m+q}{q}$$

$$\times\frac{1}{(v+1/z)^{2m+1-q}}\binom{2m-q}{m-q}.$$

Evaluate at $v = 1/(1+z)$ to get

$$(-1)^m\sum_{q=0}^{m}(1+z)^{2n-2m+1+q}\binom{2n-2m+q}{q}$$

$$\times\frac{z^{2m+1-q}(1+z)^{2m+1-q}}{(1+2z)^{2m+1-q}}\binom{2m-q}{m-q}.$$

Collect everything and flip sign,

$$(-1)^{n+m}[z^{n+1}](1+z)^n\sum_{q=0}^{m}\binom{2n-2m+q}{q}\binom{2m-q}{m-q}\frac{z^{2m+1-q}}{(1+2z)^{2m+1-q}}$$

$$= (-1)^{n+m}[z^{n+1}](1+z)^n\sum_{q=0}^{m}\binom{2n-m-q}{m-q}\binom{m+q}{q}\frac{z^{m+q+1}}{(1+2z)^{m+q+1}}.$$

Note that we need $m+q+1 \le n+1$ here or $q \le n-m$. Now when $n-m < m$ we may lower the upper range to $n-m$ because with $q > n-m$ we get an exponent $> n+1$ on the power of z. On the other hand when $n-m > m$ we may raise to $n-m$ due to the first binomial coefficient vanishing in the added range.

Expanding the term in z and resetting the upper limit on q we get

$$[z^{n+1}](1+z)^{n-m-q-1}z^{m+q+1}\sum_{p=0}^{n-m-q}\binom{m+q+p}{p}(-1)^p\frac{z^p}{(1+z)^p}$$

$$= \sum_{p=0}^{n-m-q} \binom{m+q+p}{p} (-1)^p \binom{n-m-q-1-p}{n-m-q-p}.$$

Now $n-m-q-p$ is non-negative hence the second binomial coefficient vanishes except when $p = n-m-q$ and we obtain

$$(-1)^{n+m} \sum_{q=0}^{n-m} \binom{2n-m-q}{m-q} \binom{m+q}{q} \binom{n}{n-m-q} (-1)^{n-m-q}.$$

Next observe that

$$\binom{m+q}{q} \binom{n}{n-m-q} = \frac{n!}{m! \times q! \times (n-m-q)!} = \binom{n}{m} \binom{n-m}{q}.$$

Good, we have the first binomial coefficient of our closed form. The remainder is

$$\sum_{q=0}^{n-m} \binom{2n-m-q}{m-q} \binom{n-m}{q} (-1)^q$$

$$= [w^m](1+w)^{2n-m} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \frac{w^q}{(1+w)^q}$$

$$= [w^m](1+w)^{2n-m} \left[1 - \frac{w}{1+w}\right]^{n-m} = [w^m](1+w)^n = \binom{n}{m}.$$

This is the second binomial coefficient and we may conclude. For the sake of being rigorous we must also check that the contribution from $v = -1/z$ is zero. We get from the Leibniz rule,

$$(-1)^m \sum_{q=0}^{m} \frac{1}{v^{2n-2m+1+q}} \binom{2n-2m+q}{q}$$

$$\times \frac{1}{(v-1/(1+z))^{2m+1-q}} \binom{2m-q}{m-q}.$$

Evaluate at $v = -1/z$ to get

$$(-1)^m \sum_{q=0}^{m} (-1)^{q+1} z^{2n-2m+1+q} \binom{2n-2m+q}{q}$$

$$\times (-1)^{q+1} \frac{z^{2m+1-q}(1+z)^{2m+1-q}}{(1+2z)^{2m+1-q}} \binom{2m-q}{m-q}.$$

Note however that the exponent on z is $2n+2 > n+1$ so there is no contribution as everything vanishes due to the extractor in z.

This identity was found by a computer search which pointed to OEIS A008459, square the entries of Pascal's triangle.

1.165.17 OEIS A123970

We claim that with $0 \le m \le n$ and p a real number,

$$\binom{n+m}{2m} = (-1)^m \sum_{k=0}^{n} \binom{n+m}{m+k} (-1)^k \binom{p-k}{n-m} \binom{2m+k}{k}.$$

We can use the first binomial coefficient to enforce the upper range to get

$$(-1)^m [z^n](1+z)^{n+m} \sum_{k\ge 0} z^k (-1)^k \binom{p-k}{n-m} \binom{2m+k}{2m}$$

$$= (-1)^m [z^n](1+z)^{n+m} [w^{2m}](1+w)^{2m} \sum_{k\ge 0} z^k (-1)^k \binom{p-k}{n-m} (1+w)^k$$

$$= (-1)^m [z^n](1+z)^{n+m} [w^{2m}](1+w)^{2m} [v^{n-m}](1+v)^p \\ \times \sum_{k\ge 0} z^k (-1)^k (1+w)^k (1+v)^{-k}$$

$$= (-1)^m [z^n](1+z)^{n+m} [w^{2m}](1+w)^{2m} [v^{n-m}](1+v)^p \\ \times \frac{1}{1+z(1+w)/(1+v)}$$

$$= (-1)^m [z^n](1+z)^{n+m} [w^{2m}](1+w)^{2m} [v^{n-m}](1+v)^{p+1} \\ \times \frac{1}{1+v+z(1+w)}$$

$$= (-1)^m [z^{n+1}](1+z)^{n+m} [w^{2m}](1+w)^{2m} [v^{n-m}](1+v)^{p+1} \\ \times \frac{1}{w+(1+v+z)/z}.$$

Now here the contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{2m+1}} (1+w)^{2m} \frac{1}{w+(1+v+z)/z}.$$

Here the residue at infinity is zero (just barely) and we can evaluate using minus the residue at $w = -(1+v+z)/z$ which is a simple pole:

$$\frac{z^{2m+1}}{(1+v+z)^{2m+1}} \frac{(1+v)^{2m}}{z^{2m}}.$$

We find

$$(-1)^m [z^n](1+z)^{n+m}[v^{n-m}](1+v)^{p+2m+1}\frac{1}{(1+v+z)^{2m+1}}.$$

Now here the contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}}(1+z)^{n+m}\frac{1}{(1+v+z)^{2m+1}}.$$

Here we again have a residue of zero at infinity and may evaluate using minus the residue at $z = -(1+v)$, which requires the Leibniz rule:

$$\frac{1}{(2m)!}\left(\frac{1}{z^{n+1}}(1+z)^{n+m}\right)^{(2m)}$$

$$= \frac{1}{(2m)!}\sum_{q=0}^{2m}\binom{2m}{q}\frac{1}{z^{n+1+q}}(-1)^q(n+1)^{\overline{q}}(1+z)^{n+m-(2m-q)}(n+m)^{\underline{2m-q}}$$

$$= \sum_{q=0}^{2m}\frac{1}{z^{n+1+q}}(-1)^q\binom{n+q}{q}(1+z)^{n-m+q}\binom{n+m}{2m-q}.$$

Evaluate at $z = -(1+v)$ and flip sign to get

$$(-1)^{n+m}\sum_{q=0}^{2m}\frac{1}{(1+v)^{n+1+q}}\binom{n+q}{q}(-v)^{n-m+q}\binom{n+m}{2m-q}.$$

Note however that we have a coefficient extractor on $[v^{n-m}]$ so only $q = 0$ can possibly contribute and we find

$$(-1)^{n+m}[v^0](-1)^{n-m}\binom{n+m}{2m}\frac{1}{(1+v)^{n-p-2m}} = \binom{n+m}{2m}$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A123970, a multiple of $\binom{n+m}{2m}$.

1.165.18 OEIS A013609

We claim that with $0 \le m \le n$ and p a real number

$$\binom{n}{m}2^m = (-1)^m\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{2k}{m}\binom{p-k}{n-m}.$$

We find for the sum

$$(-1)^m[w^m]\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1+w)^{2k}\binom{p-k}{n-m}$$

$$= (-1)^m [w^m][z^{n-m}](1+z)^p \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+w)^{2k} (1+z)^{-k}$$

$$= (-1)^m [w^m][z^{n-m}](1+z)^p \left[1 - \frac{(1+w)^2}{1+z}\right]^n$$

$$= (-1)^m [w^m][z^{n-m}](1+z)^{p-n} [z - 2w - w^2]^n$$

$$= (-1)^m [w^m][z^{n-m}](1+z)^{p-n} \sum_{q=0}^{n} \binom{n}{q} z^{n-q} (-1)^q w^q (2+w)^q.$$

Due to the coefficient extractor in w we need $q \le m$ and due to the coefficient extractor in z we need $n - q \le n - m$ or $m \le q$ hence only $q = m$ can possibly contribute. We find

$$(-1)^m [w^m][z^{n-m}](1+z)^{p-n} \binom{n}{m} z^{n-m} (-1)^m w^m (2+w)^m$$

$$= \binom{n}{m} 2^m [z^0](1+z)^{p-n} = \binom{n}{m} 2^m.$$

We have the claim. Compare also 1.147.21.

This identity was found by a computer search which pointed to OEIS A013609, coefficients of $(1+2x)^n$.

1.165.19 OEIS A001813

We claim that with n, m, p non-negative integers and $m \ge p$ and r a real number,

$$\frac{(2p)!}{p!} = (-1)^{p+m} \sum_{k=0}^{p+m} (n-k)^p (-1)^k \binom{2p}{p+m-k} \binom{r-k}{p}.$$

We can use the first binomial coefficient to enforce the upper range,

$$(-1)^{p+m} [z^{p+m}](1+z)^{2p} \sum_{k\ge 0} z^k (n-k)^p (-1)^k \binom{r-k}{p}$$

$$= (-1)^{p+m} [z^{p+m}](1+z)^{2p} p! [w^p] \sum_{k\ge 0} z^k \exp((n-k)w) (-1)^k \binom{r-k}{p}$$

$$= (-1)^{p+m} [z^{p+m}](1+z)^{2p} p! [w^p] \exp(nw) [v^p](1+v)^r \times \frac{1}{1 + z/\exp(w)/(1+v)}$$

$$= (-1)^{p+m} [z^{p+m}](1+z)^{2p} p! [w^p] \exp((n+1)w) [v^p](1+v)^{r+1} \times \frac{1}{z + \exp(w)(1+v)}.$$

Here the contribution from z is

$$\operatorname*{res}_{z} \frac{1}{z^{p+m+1}}(1+z)^{2p}\frac{1}{z+\exp(w)(1+v)}.$$

As per the boundary conditions the residue at infinity is zero here and we may evaluate using minus the residue at $z = -\exp(w)(1+v)$, which is a simple pole:

$$p![w^p]\exp((n+1)w)[v^p](1+v)^{r+1}$$
$$\times\frac{(1-\exp(w)(1+v))^{2p}}{\exp(w(p+m+1))(1+v)^{p+m+1}}$$
$$= p![w^p]\exp((n+1)w)[v^p](1+v)^{r-p-m}$$
$$\times\frac{(\exp(w)(1+v)-1)^{2p}}{\exp(w(p+m+1))}$$
$$= p![w^p]\exp((n+1)w)[v^p](1+v)^{r-p-m}$$
$$\times\frac{1}{\exp(w(p+m+1))}\sum_{q=0}^{2p}\binom{2p}{q}(\exp(w)-1)^q\exp(w(2p-q))v^{2p-q}.$$

Now from the extractor in v we get $p \ge 2p-q$ or $q \ge p$. From the extractor in w we get using $\exp(w)-1 = w+\cdots$ that $q \le p$. Hence only $q = p$ contributes and we get

$$p![w^p]\exp((n+1)w)[v^p](1+v)^{r-p-m}$$
$$\times\frac{1}{\exp(w(p+m+1))}\binom{2p}{p}(\exp(w)-1)^p\exp(pw)v^p.$$

The coefficient extractor in v reduces to the constant term which leaves

$$p![w^p]\exp((n+1)w)\frac{1}{\exp(w(m+1))}\binom{2p}{p}(\exp(w)-1)^p.$$

Lastly the extractor in w also reduces to the constant term since $(\exp(w)-1)^p = w^p+\cdots$ and we get

$$p!\binom{2p}{p}$$

which is the claim.

This identity was found by a computer search which pointed to OEIS A001813, quadruple factorial numbers $(2n)!/n!$.

1.165.20 OEIS A043302

We claim that with $n \ge 0$ and $n \ge m \ge 0$ and p a real number for Stirling set numbers

$$2^{n-2m}\frac{(n+m)!}{(n-m)!m!} = (-1)^m \sum_{k=0}^n \binom{n+m}{m+k}(-1)^k \binom{p-2k}{n-m} \left\{ {m+k+1 \atop k+1} \right\}.$$

We have for the LHS

$$(-1)^m \sum_{k=0}^n \binom{n+m}{m+k}(-1)^k \binom{p-2k}{n-m}(m+k+1)![z^{m+k+1}]\frac{(\exp(z)-1)^{k+1}}{(k+1)!}$$

$$= (-1)^m \sum_{k=0}^n \binom{n+m}{m+k}(-1)^k \binom{p-2k}{n-m}(m+k)![z^{m+k}]\exp(z)\frac{(\exp(z)-1)^{k}}{k!}$$

$$= (-1)^m \frac{(n+m)!}{n!} \sum_{k=0}^n \binom{n}{k}(-1)^k \binom{p-2k}{n-m}[z^{m+k}]\exp(z)(\exp(z)-1)^{k}.$$

Without the scalar in front,

$$[z^m]\exp(z) \sum_{k=0}^n \binom{n}{k}(-1)^k \binom{p-2k}{n-m}\frac{(\exp(z)-1)^{k}}{z^k}$$

$$= [z^m]\exp(z)[w^{n-m}](1+w)^p$$

$$\times \sum_{k=0}^n \binom{n}{k}(-1)^k \frac{(\exp(z)-1)^{k}}{z^k}\frac{1}{(1+w)^{2k}}$$

$$= [z^m]\exp(z)[w^{n-m}](1+w)^p \left[1 - \frac{\exp(z)-1}{z(1+w)^2}\right]^n$$

$$= [z^{n+m}]\exp(z)[w^{n-m}](1+w)^{p-2n} \left[z(1+w)^2 - \exp(z)+1\right]^n$$

$$= [z^{n+m}]\exp(z)[w^{n-m}](1+w)^{p-2n}$$

$$\times \sum_{q=0}^n \binom{n}{q} z^q w^q (2+w)^q (-1)^{n-q}(\exp(z)-z-1)^{n-q}.$$

Now from the coefficient extractor in w we require $q \le n-m$. From the extractor in z we get $2n-2q+q \le n+m$ or $n-m \le q$. Hence only $q = n-m$ can possibly contribute. We obtain

$$[z^{n+m}]\exp(z)[w^{n-m}](1+w)^{p-2n}\binom{n}{m} z^{n-m}w^{n-m}(2+w)^{n-m}(-1)^m(\exp(z)-1-z)^m$$

$$= [z^{2m}]\exp(z)[w^0](1+w)^{p-2n}\binom{n}{m}(2+w)^{n-m}(-1)^m(\exp(z)-1-z)^m$$

$$= [z^{2m}] \exp(z) 2^{n-m} \binom{n}{m} (-1)^m (\exp(z) - 1 - z)^m$$
$$= \binom{n}{m} (-1)^m \frac{2^{n-m}}{2^m}.$$

Collecting everything including the pending scalar in front yields

$$\binom{n}{m} \frac{(n+m)!}{n!} 2^{n-2m}$$

which is the claim. Note that this will also prove the companion identity for Stirling cycle numbers

$$2^{n-2m} \frac{(n+m)!}{(n-m)!m!} = (-1)^m \sum_{k=0}^{n} \binom{n+m}{m+k} (-1)^k \binom{p-2k}{n-m} \left[{m+k+1 \atop k+1} \right].$$

Consult also 1.165.35.

This identity was found by a computer search which pointed to OEIS A043302, table of $2^n(n+k)!/(n-k)!/k!/4^k$.

1.165.21 OEIS A059304

We claim that wth $n \ge 0$ and $0 \le p \le m \le n$ non-negative integers

$$2^p \binom{2p}{p} = (-1)^m \sum_{k=0}^{n} \binom{n-k+p}{p} (-1)^k \binom{2k}{p} \binom{2p}{m-k+p}.$$

We can use the first binomial coefficient to enforce the range of the sum, writing $\binom{n-k+p}{n-k}$:

$$(-1)^m [z^n] (1+z)^{n+p} \sum_{k \ge 0} \frac{z^k}{(1+z)^k} (-1)^k \binom{2k}{p} \binom{2p}{m-k+p}$$
$$= (-1)^m [z^n] (1+z)^{n+p} [w^p] \sum_{k \ge 0} \frac{z^k}{(1+z)^k} (-1)^k (1+w)^{2k} \binom{2p}{m-k+p}$$
$$= (-1)^m [z^n] (1+z)^{n+p} [w^p] [v^{m+p}] (1+v)^{2p} \sum_{k \ge 0} \frac{z^k}{(1+z)^k} (-1)^k (1+w)^{2k} v^k$$
$$= (-1)^m [z^n] (1+z)^{n+p} [w^p] [v^{m+p}] (1+v)^{2p}$$
$$\times \frac{1}{1 + zv(1+w)^2/(1+z)}$$
$$= (-1)^m [z^n] (1+z)^{n+p} [w^p] (1+w)^{m+p} [v^{m+p}] (1 + v/(1+w))^{2p}$$
$$\times \frac{1}{1 + zv(1+w)/(1+z)}$$

$$= (-1)^m [z^n](1+z)^{n+p} [w^p](1+w)^{m-p} [v^{m+p}](1+w+v)^{2p}$$
$$\times \frac{1}{1+zv(1+w)/(1+z)}$$
$$= (-1)^m [z^{n+1}](1+z)^{n+p+1} [w^p](1+w)^{m-p-1} [v^{m+p}](1+w+v)^{2p}$$
$$\times \frac{1}{v+(1+z)/z/(1+w)}.$$

Now the contribution from v is

$$\operatorname*{res}_v \frac{1}{v^{m+p+1}}(1+w+v)^{2p} \frac{1}{v+(1+z)/z/(1+w)}.$$

Due to the boundary conditions the residue at infinity is zero here and we may evaluate using minus the residue at $v = -(1+z)/z/(1+w)$ which is a simple pole:

$$(-1)^{m+p} \frac{(z(1+w))^{m+p+1}}{(1+z)^{m+p+1}}(1+w-(1+z)/z/(1+w))^{2p}$$
$$= (-1)^{m+p} \frac{(z(1+w))^{m-p+1}}{(1+z)^{m+p+1}}(z(1+w)^2-(1+z))^{2p}.$$

Activating the extractors in z and in w:

$$(-1)^p [z^{n-m+p}](1+z)^{n-m} [w^p](1+w)^{2m-2p}(z(1+w)^2-(1+z))^{2p}.$$

Now the contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n-m+p+1}}(1+z)^{n-m}(z(1+w)^2-(1+z))^{2p}.$$

Put $z/(1+z) = u$ so that $z = u/(1-u)$ and $dz = 1/(1-u)^2\; du$ to get

$$\operatorname*{res}_u \frac{1}{u^{n-m+p+1}}(1-u)^{p+1}(u(1+w)^2/(1-u)-(1+u/(1-u)))^{2p} \frac{1}{(1-u)^2}$$
$$= \operatorname*{res}_u \frac{1}{u^{n-m+p+1}} \frac{1}{(1-u)^{p+1}}(u(1+w)^2-1)^{2p}.$$

Here the residue at infinity is once more zero (just barely) and we may evaluate using minus the residue at $u = 1$. which requires the Leibniz rule:

$$\frac{1}{p!}\left(\frac{1}{u^{n-m+p+1}}(u(1+w)^2-1)^{2p}\right)^{(p)}$$
$$= \frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q}\frac{1}{u^{n-m+p+1+q}}(-1)^q(n-m+p+1)^{\overline{q}}$$

$$\times(u(1+w)^2-1)^{2p-(p-q)}(1+w)^{2p-2q}(2p)^{\underline{p-q}}$$

$$= \sum_{q=0}^{p} \frac{1}{u^{n-m+p+1+q}}(-1)^q \binom{n-m+p+q}{q}$$

$$\times(u(1+w)^2-1)^{p+q}(1+w)^{2p-2q}\binom{2p}{p-q}.$$

Now put $u=1$, restore the extractor in w, and adjust signs (recall that we had a term $1/(1-u)^{p+1}$ in the residue in u):

$$[w^p](1+w)^{2m-2p}\sum_{q=0}^{p}(-1)^q\binom{n-m+p+q}{q}$$

$$\times(w(2+w))^{p+q}(1+w)^{2p-2q}\binom{2p}{p-q}.$$

Note that we have an extractor of power p and an exponent of $p+q$ on w. Hence only $q=0$ can possibly contribute and we get

$$[w^p](1+w)^{2m-2p}(w(2+w))^p(1+w)^{2p}\binom{2p}{p}$$

$$= [w^0](1+w)^{2m-2p}(2+w)^p(1+w)^{2p}\binom{2p}{p} = 2^p\binom{2p}{p}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A059304, sequence $2^n\binom{2n}{n}$.

1.165.22 OEIS A079901

We claim that with $n \ge 0$ and $n \ge m \ge 0$

$$n^m = (-1)^m \sum_{k=0}^{n}\binom{n-1+k}{k}(-1)^k k^m \binom{2n}{n+k}.$$

We can use the second binomial coefficient to enforce the upper range of the sum, writing $\binom{2n}{n-k}$:

$$(-1)^m[z^n](1+z)^{2n}\sum_{k\ge 0}\binom{n-1+k}{k}(-1)^k z^k k^m$$

$$= (-1)^m m![w^m][z^n](1+z)^{2n}\sum_{k\ge 0}\binom{n-1+k}{k}(-1)^k z^k \exp(kw)$$

$$= (-1)^m m![w^m][z^n](1+z)^{2n}\frac{1}{(1+z\exp(w))^n}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}}(1+z)^{2n}\frac{1}{(1+z\exp(w))^n}.$$

Now put $z/(1+z) = u$ so that $z = u/(1-u)$ and $dz = 1/(1-u)^2\, du$ to get

$$\begin{aligned}&\operatorname*{res}_u \frac{1}{u^{n+1}}\frac{1}{(1-u)^{n-1}}\frac{1}{(1+u\exp(w)/(1-u))^n}\;\frac{1}{(1-u)^2}\\ &= \operatorname*{res}_u \frac{1}{u^{n+1}}\frac{1}{1-u}\frac{1}{(1-u+u\exp(w))^n}.\end{aligned}$$

Here the residue at infinity is zero so we may evaluate using minus the residues at $u = 1$ and $u = 1/(1-\exp(w))$ to get for the former (orient as $1/(u-1)$ for the sign)

$$(-1)^m m![w^m]\exp(-nw) = n^m.$$

This is the claim. Now we just need to show that the contribution from the latter at $u = 1/(1-\exp(w))$ is zero. We write

$$\begin{aligned}&\operatorname*{res}_u \frac{1}{u^{n+1}}\frac{1}{1-u}\frac{1}{(1-u(1-\exp(w)))^n}\\ &= (-1)^n\frac{1}{(1-\exp(w))^n}\frac{1}{u^{n+1}}\frac{1}{1-u}\frac{1}{(u-1/(1-\exp(w)))^n}.\end{aligned}$$

We require the Leibniz rule:

$$\begin{aligned}&\frac{1}{(n-1)!}\left(\frac{1}{u^{n+1}}\frac{1}{(1-u)^1}\right)^{(n-1)}\\ &= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{1}{u^{n+1+q}}(-1)^q(n+1)^{\overline{q}}\frac{1}{(1-u)^{1+n-1-q}}1^{\overline{n-1-q}}\\ &= \sum_{q=0}^{n-1}\frac{1}{u^{n+1+q}}(-1)^q\binom{n+q}{q}\frac{1}{(1-u)^{n-q}}.\end{aligned}$$

Evaluate at $u = 1/(1-\exp(w))$ to get

$$\sum_{q=0}^{n-1}(1-\exp(w))^{n+1+q}(-1)^q\binom{n+q}{q}(-1)^{n-q}(1-\exp(w))^{n-q}\frac{1}{\exp(w)^{n-q}}.$$

Collecting everything,

$$(1-\exp(w))^{n+1}\sum_{q=0}^{n-1}\binom{n+q}{q}\frac{1}{\exp(w)^{n-q}}.$$

To conclude observe that the coefficient extractor in w is $[w^m]$, we have

$(1-\exp(w))^{n+1} = (-1)^{n+1}w^{n+1} + \cdots$ and $n+1 > m$, making for a zero contribution.

This identity was found by a computer search which pointed to OEIS A079901, triangle of powers n^k.

1.165.23 OEIS A090181

We claim that with $n \ge 0$ and $n \ge m \ge 0$ featuring Catalan and Narayana numbers

$$\binom{n}{m}\binom{n-1}{n-m}\frac{1}{n-m+1} = (-1)^{n+m}\sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}(-1)^k\binom{n-k}{m}\binom{n+k}{n-k}.$$

Start by re-indexing the sum

$$(-1)^m\sum_{k=0}^{n}\frac{1}{n-k+1}\binom{2n-2k}{n-k}(-1)^k\binom{k}{m}\binom{2n-k}{k}.$$

We may use the first binomial coefficient to enforce the upper range,

$$(-1)^m[z^n](1+z)^{2n}\sum_{k\ge 0}\frac{1}{n-k+1}\frac{z^k}{(1+z)^{2k}}(-1)^k\binom{k}{m}\binom{2n-k}{k}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^m]\sum_{k\ge 0}\frac{1}{n-k+1}\frac{z^k}{(1+z)^{2k}}(-1)^k(1+w)^k\binom{2n-k}{2n-2k}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^m][v^{2n}](1+v)^{2n}$$

$$\times\sum_{k\ge 0}\frac{1}{n-k+1}\frac{z^k}{(1+z)^{2k}}(-1)^k(1+w)^k\frac{v^{2k}}{(1+v)^k}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^m][v^{2n}](1+v)^{2n}[u^{n+1}]\log\frac{1}{1-u}$$

$$\times\sum_{k\ge 0}\frac{z^k}{(1+z)^{2k}}(-1)^k(1+w)^k\frac{v^{2k}}{(1+v)^k}u^k$$

$$= (-1)^m[z^n](1+z)^{2n}[w^m][v^{2n}](1+v)^{2n}[u^{n+1}]\log\frac{1}{1-u}$$

$$\times\frac{1}{1+(1+w)uzv^2/(1+z)^2/(1+v)}$$

$$= (-1)^m[z^n][w^m][v^{2n}](1+v+vz)^{2n}[u^{n+1}]\log\frac{1}{1-u}$$

$$\times \frac{1}{1+(1+w)uzv^2/(1+v+vz)}$$

$$= (-1)^m [z^n][w^m][v^{2n}](1+v+vz)^{2n+1}[u^{n+1}] \log \frac{1}{1-u}$$

$$\times \frac{1}{1+v+vz+(1+w)uzv^2}$$

$$= (-1)^m [z^{n+1}][w^m][v^{2n+2}](1+v+vz)^{2n+1}[u^{n+2}] \log \frac{1}{1-u}$$

$$\times \frac{1}{w+1+(1+v+vz)/u/z/v^2}.$$

Here the contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{m+1}} \frac{1}{w+1+(1+v+vz)/u/z/v^2}$$

where the residue at infinity is zero and we can directly substitute the value of the other pole:

$$[z^{n+1}][v^{2n+2}](1+v+vz)^{2n+1}[u^{n+2}] \log \frac{1}{1-u} \frac{1}{(1+(1+v+vz)/u/z/v^2)^{m+1}}$$

$$= [z^{n-m}][v^{2n-2m}](1+v+vz)^{2n+1}[u^{n-m+1}] \log \frac{1}{1-u} \frac{1}{(1+v+vz+uzv^2)^{m+1}}$$

$$= [z^{n-m}][v^{2n-2m}](1+v+vz)^{2n-m}[u^{n-m+1}] \log \frac{1}{1-u} \frac{1}{(1+uzv^2/(1+v+vz))^{m+1}}$$

$$= [z^{n-m}][v^{2n-2m}](1+v+vz)^{2n-m}$$

$$\times \sum_{q=0}^{n-m} \frac{1}{n-m+1-q} \binom{m+q}{q} (-1)^q \frac{z^q v^{2q}}{(1+v+vz)^q}$$

$$= [z^{n-m}][v^{2n-2m}] \sum_{q=0}^{n-m} \frac{1}{n-m+1-q} \binom{m+q}{q} (-1)^q z^q$$

$$\times v^{2q}(1+v+vz)^{2n-m-q}$$

$$= [z^{n-m}] \sum_{q=0}^{n-m} \frac{1}{n-m+1-q} \binom{m+q}{q} (-1)^q z^q$$

$$\times \binom{2n-m-q}{2n-2m-2q} (1+z)^{2n-2m-2q}$$

$$= \sum_{q=0}^{n-m} \frac{1}{n-m+1-q} \binom{m+q}{q} (-1)^q$$

$$\times\binom{2n-m-q}{2n-2m-2q}\binom{2n-2m-2q}{n-m-q}.$$

These last two coefficients are equal to

$$\frac{(2n-m-q)!}{(m+q)! \times (n-m-q)! \times (n-m-q)!} = \binom{2n-m-q}{n-m-q}\binom{n}{m+q}.$$

We also have

$$\binom{m+q}{q}\binom{n}{m+q} = \frac{n!}{q! \times m! \times (n-m-q)!} = \binom{n}{m}\binom{n-m}{q}.$$

This gives for our sum

$$\binom{n}{m}\sum_{q=0}^{n-m}\frac{1}{n-m+1-q}(-1)^q\binom{2n-m-q}{n-m-q}\binom{n-m}{q}$$

$$= \frac{1}{n}\binom{n}{m}\sum_{q=0}^{n-m}(-1)^q\binom{2n-m-q}{n-m+1-q}\binom{n-m}{q}$$

$$= \frac{1}{n}\binom{n}{m}[z^{n-m+1}](1+z)^{2n-m}\sum_{q=0}^{n-m}(-1)^q\frac{z^q}{(1+z)^q}\binom{n-m}{q}$$

$$= \frac{1}{n}\binom{n}{m}[z^{n-m+1}](1+z)^{2n-m}\left[1-\frac{z}{1+z}\right]^{n-m} = \frac{1}{n}\binom{n}{m}[z^{n-m+1}](1+z)^{n}$$

$$= \frac{1}{n}\binom{n}{m}\binom{n}{n-m+1} = \frac{1}{n-m+1}\binom{n}{m}\binom{n-1}{n-m}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A090181, Narayana triangle.

1.165.24 OEIS A128908

We claim that with $n \ge 0$ and $n \ge m \ge 0$ with x and y variable parameters:

$$\binom{y-x}{n} = (-1)^n\sum_{k=0}^{n}\binom{x-k}{n-k}\binom{x+1}{k}(-1)^k\binom{y+1-k}{n}.$$

We prove it first for x and y positive integers. We may then conclude that the identity holds for x and y as variables because both sides are polynomials in x and y and agree at an infinite number of points. We can use the first binomial coefficient to enforce the range:

$$(-1)^n[z^n](1+z)^x \sum_{k\ge 0} \binom{x+1}{k} (-1)^k \frac{z^k}{(1+z)^k} \binom{y+1-k}{n}$$

$$= (-1)^n[z^n](1+z)^x [w^n](1+w)^{y+1}$$

$$\times \sum_{k\ge 0} \binom{x+1}{k} (-1)^k \frac{z^k}{(1+z)^k} \frac{1}{(1+w)^k}$$

$$= (-1)^n[z^n](1+z)^x [w^n](1+w)^{y+1} \left[1 - \frac{z}{(1+z)(1+w)}\right]^{x+1}$$

$$= (-1)^n[z^n] \frac{1}{1+z} [w^n](1+w)^{y-x} [1+w+wz]^{x+1}$$

$$= [w^n](1+w)^{y-x} \sum_{q=0}^{n} (-1)^q \binom{x+1}{q} w^q (1+w)^{x+1-q}$$

$$= \sum_{q=0}^{n} (-1)^q \binom{x+1}{q} \binom{y+1-q}{n-q}$$

$$= [v^n](1+v)^{y+1} \sum_{q\ge 0} (-1)^q \binom{x+1}{q} \frac{v^q}{(1+v)^q}$$

$$= [v^n](1+v)^{y+1} \left[1 - \frac{v}{1+v}\right]^{x+1} = [v^n](1+v)^{y-x} = \binom{y-x}{n}.$$

This identity was found by a computer search which pointed to OEIS A128908, Riordan array $(1, x/(1-x)^2)$.

1.165.25 OEIS A088617

We claim that with $n \ge 0$ and $n \ge m \ge 0$ featuring Catalan numbers

$$\binom{n+m}{n} \binom{n}{m} \frac{1}{m+1} = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{1}{k+1} \binom{2k}{k} \binom{k+1}{m+1} \binom{n+k}{n-k}.$$

First factor the Catalan number,

$$\frac{1}{m+1} (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{k}{m} \binom{n+k}{n-k}.$$

This is establishes the third term of the closed form, which may thus be omitted from the remaining calculation. Re-index the sum where the first binomial coefficient enforces the upper range:

$$\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{n-k}{m}\binom{2n-k}{k}$$

$$= [z^n](1+z)^{2n}\sum_{k\ge 0}(-1)^k\frac{z^k}{(1+z)^{2k}}\binom{n-k}{m}\binom{2n-k}{k}$$

$$= [z^n](1+z)^{2n}[w^m](1+w)^n\sum_{k\ge 0}(-1)^k\frac{z^k}{(1+z)^{2k}}\frac{1}{(1+w)^k}\binom{2n-k}{2n-2k}$$

$$= [z^n](1+z)^{2n}[w^m](1+w)^n[v^{2n}](1+v)^{2n}$$

$$\times\sum_{k\ge 0}(-1)^k\frac{z^k}{(1+z)^{2k}}\frac{1}{(1+w)^k}\frac{v^{2k}}{(1+v)^k}$$

$$= [z^n](1+z)^{2n}[w^m](1+w)^n[v^{2n}](1+v)^{2n}$$

$$\times\frac{1}{1+zv^2/(1+z)^2/(1+w)/(1+v)}$$

$$= [z^n][w^m](1+w)^n[v^{2n}](1+v+vz)^{2n}$$

$$\times\frac{1}{1+zv^2/(1+w)/(1+v+vz)}.$$

Expanding the initial segment of the fractional term up to the limit imposed by the extractor in z,

$$[z^n][w^m](1+w)^n[v^{2n}](1+v+vz)^{2n}$$

$$\times\sum_{q=0}^{n}(-1)^q\frac{z^qv^{2q}}{(1+w)^q(1+v+vz)^q}$$

$$= [z^n][w^m](1+w)^n[v^{2n}]$$

$$\times\sum_{q=0}^{n}(-1)^q\frac{z^qv^{2q}}{(1+w)^q}(1+v+vz)^{2n-q}$$

$$= [z^n][w^m](1+w)^n$$

$$\times\sum_{q=0}^{n}(-1)^q\frac{z^q}{(1+w)^q}\binom{2n-q}{2n-2q}(1+z)^{2n-2q}$$

$$= \sum_{q=0}^{n}(-1)^q\binom{n-q}{m}\binom{2n-q}{2n-2q}\binom{2n-2q}{n-q}.$$

Now note that

$$\binom{2n-q}{2n-2q}\binom{2n-2q}{n-q} = \frac{(2n-q)!}{q! \times (n-q)! \times (n-q)!} = \binom{n}{q}\binom{2n-q}{n}.$$

Note also that

$$\binom{n-q}{m}\binom{n}{q} = \frac{n!}{m! \times (n-q-m)! \times q!} = \binom{n}{m}\binom{n-m}{q}.$$

We get for our sum

$$\binom{n}{m}\sum_{q=0}^{n}(-1)^q\binom{n-m}{q}\binom{2n-q}{n}.$$

We have the second factor from the closed form. We may lower the upper range in the sum to $n-m$ because $n-m$ is non-negative due to the initial conditions and hence the corresponding coefficient is zero when $q > n-m$. We are left with

$$\sum_{q=0}^{n-m}(-1)^q\binom{n-m}{q}\binom{2n-q}{n}$$

$$= [z^n](1+z)^{2n}\sum_{q=0}^{n-m}(-1)^q\binom{n-m}{q}\frac{1}{(1+z)^q}$$

$$= [z^n](1+z)^{2n}\left[1-\frac{1}{1+z}\right]^{n-m} = [z^n](1+z)^{n+m}z^{n-m} = \binom{n+m}{m}.$$

This is the third factor and we may conclude the argument.

This identity was found by a computer search which pointed to OEIS A088617, triangle $\binom{n+m}{n}\binom{n}{m}\frac{1}{m+1}$.

1.165.26 OEIS A090802

We claim that with $n \ge 0$ and $n \ge m \ge 0$ and p, r real numbers and q a positive integer that

$$\binom{n}{m}m!q^{n-m} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{p-qk}{n-m}(r-k)^m.$$

We first prove it for p an integer. It then follows because the sum is a polynomial in p and we will have established equality at an infinite number of points. We have for the sum

$$[z^{n-m}](1+z)^p\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(1+z)^{qk}}(r-k)^m$$

$$= m![w^m] \exp(rw)[z^{n-m}](1+z)^p \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(1+z)^{qk}} \exp(-kw)$$

$$= m![w^m] \exp(rw)[z^{n-m}](1+z)^p$$

$$\times \left[1 - \frac{1}{\exp(w)(1+z)^q}\right]^n$$

$$= m![w^m] \exp((r-n)w)[z^{n-m}](1+z)^{p-qn}[(1+z)^q \exp(w) - 1]^n$$

$$= m![w^m] \exp((r-n)w)[z^{n-m}](1+z)^{p-qn}[\exp(w) - 1 + ((1+z)^q - 1)\exp(w)]^n$$

$$= m![w^m] \exp((r-n)w)[z^{n-m}](1+z)^{p-qn}$$

$$\times \sum_{k=0}^{n} \binom{n}{k} (\exp(w) - 1)^k ((1+z)^q - 1)^{n-k} \exp((n-k)w).$$

Now since $\exp(w) - 1 = w + \cdots$ the coefficient extractor in w enforces $k \le m$. Similarly, since $(1+z)^q - 1 = qz + \cdots$ the coefficient extractor in z enforces $n - k \le n - m$ or $m \le k$. Hence only $k = m$ can possibly contribute and we get

$$m![w^m] \exp((r-n)w)[z^{n-m}](1+z)^{p-qn}$$

$$\times \binom{n}{m} (\exp(w) - 1)^m ((1+z)^q - 1)^{n-m} \exp((n-m)w)$$

$$= m![w^m][z^{n-m}](1+z)^{p-qn}$$

$$\times \binom{n}{m} (\exp(w) - 1)^m \exp((r-m)w)((1+z)^q - 1)^{n-m}$$

$$= \binom{n}{m} m![z^{n-m}](1+z)^{p-qn}(qz + \cdots + z^q)^{n-m} = \binom{n}{m} m! q^{n-m}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A090802, walks in a Boolean algebra.

1.165.27 OEIS A013620

We claim that with $n \ge 0$ and $n \ge m \ge 0$ and x and y variables as well as two positive integer parameters p and q that

$$p^{n-m} q^m \binom{n}{m} = (-1)^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{x - pk}{n - m} \binom{y + qk}{m}.$$

We prove it for x and y integers and it then holds for all x, y because the sum produces a polynomial in those variables. Making the usual substitutions we find

$$(-1)^m [z^{n-m}](1+z)^x \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(1+z)^{pk}} \binom{y+qk}{m}$$

$$= (-1)^m [z^{n-m}](1+z)^x [w^m](1+w)^y \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(1+z)^{pk}} (1+w)^{qk}$$

$$= (-1)^m [z^{n-m}](1+z)^x [w^m](1+w)^y \left[1 - \frac{(1+w)^q}{(1+z)^p}\right]^n$$

$$= (-1)^m [z^{n-m}](1+z)^{x-pn} [w^m](1+w)^y [(1+z)^p - (1+w)^q]^n.$$

Expanding the sum,

$$(-1)^m [z^{n-m}](1+z)^{x-pn} [w^m](1+w)^y$$

$$\times \sum_{k=0}^{n} \binom{n}{k} z^{n-k} (p + \cdots + z^{p-1})^{n-k} (-1)^k w^k (q + \cdots + w^{q-1})^k.$$

We have from the coefficient extractor in z that we need $n-k \le n-m$ or $k \ge m$. We get from the coefficient extractor in w that $k \le m$. This leaves just $k = m$ and we obtain

$$(-1)^m [z^{n-m}](1+z)^{x-pn} [w^m](1+w)^y$$

$$\times \binom{n}{m} z^{n-m} (p + \cdots + z^{p-1})^{n-m} (-1)^m w^m (q + \cdots + w^{q-1})^m$$

$$= [z^0](1+z)^{x-pn} [w^0](1+w)^y \binom{n}{m} (p + \cdots + z^{p-1})^{n-m} (q + \cdots + w^{q-1})^m$$

$$= \binom{n}{m} p^{n-m} q^m.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A013620, coefficients of $(2+3x)^n$.

1.165.28 OEIS A000332

We claim that with $n \ge 1$ and $n \ge m \ge 1$ that

$$\binom{n-1}{m-1} = (-1)^{m+1} \sum_{k=0}^{n} \binom{2n}{n+k} (-1)^k \binom{m-1+k}{k} \binom{n-1+k}{k}.$$

If we would allow $m = 0$ we get zero on the LHS and $-\binom{2n}{n}$ on the RHS.

We can use the first binomial coefficient to enforce the upper range, writing $\binom{2n}{n-k}$:

$$(-1)^{m+1}[z^n](1+z)^{2n}\sum_{k=0}^{n}(-1)^k z^k\binom{m-1+k}{k}\binom{n-1+k}{k}$$

$$= (-1)^{m+1}[z^n](1+z)^{2n}[w^{m-1}](1+w)^{m-1}\sum_{k=0}^{n}(-1)^k z^k(1+w)^k\binom{n-1+k}{k}$$

$$= (-1)^{m+1}[z^n](1+z)^{2n}[w^{m-1}](1+w)^{m-1}\frac{1}{(1+z(1+w))^n}.$$

The contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^m}(1+w)^{m-1}\frac{1}{(1+z+zw)^n}.$$

Now put $w/(1+w)=v$ so that $w=v/(1-v)$ and $dw=1/(1-v)^2\;dv$ to get

$$\operatorname*{res}_{v}\frac{1}{v^m}(1-v)\frac{1}{(1+z+zv/(1-v))^n}\frac{1}{(1-v)^2}$$

$$= \operatorname*{res}_{v}\frac{1}{v^m}(1-v)^{n-1}\frac{1}{(1+z-v)^n}.$$

We evaluate this residue with the Leibniz rule, getting

$$\frac{1}{(m-1)!}\left((1-v)^{n-1}\frac{1}{(1+z-v)^n}\right)^{(m-1)}$$

$$= \frac{1}{(m-1)!}\sum_{q=0}^{m-1}\binom{m-1}{q}(1-v)^{n-1-q}(-1)^q(n-1)^{\underline{q}}\frac{1}{(1+z-v)^{n+m-1-q}}n^{\overline{m-1-q}}$$

$$= \sum_{q=0}^{m-1}(1-v)^{n-1-q}(-1)^q\binom{n-1}{q}\frac{1}{(1+z-v)^{n+m-1-q}}\binom{n+m-2-q}{m-1-q}.$$

Next put $v=0$ and apply the remaining extractor in z to get

$$(-1)^{m+1}\sum_{q=0}^{m-1}(-1)^q\binom{n-1}{q}\binom{n-m+1+q}{n}\binom{n+m-2-q}{m-1-q}.$$

From the boundary conditions we have that the upper index of the middle binomial coefficient does not go negative. Hence for it to be non-zero we must have $n-m+1+q\ge n$ or $q\ge m-1$. Only $q=m-1$ fits here and we find

$$(-1)^{m+1}(-1)^{m-1}\binom{n-1}{m-1}\binom{n}{n}\binom{n-1}{0}=\binom{n-1}{m-1}$$

and we have the claim.

This identity was found by a computer search which pointed to OEIS A000332, binomial coefficient $\binom{n}{4}$.

1.165.29 OEIS A370232

We claim that with $n \ge 0$ and $n \ge m \ge 0$ that

$$\binom{n+m}{2m}^2 = (-1)^n \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{m+k} (-1)^k \binom{k+m}{k-m}.$$

We re-factor the first two binomial coefficients:

$$\frac{(n+k)!}{(n-k)! \times (k+m)! \times (k-m)!} = \binom{n+k}{n-m} \binom{n-m}{n-k}.$$

We use the second coefficient to enforce the upper range and get

$$(-1)^n [z^n] (1+z)^{n-m} \sum_{k \ge 0} \binom{n+k}{n-m} z^k (-1)^k \binom{k+m}{2m}$$

$$= (-1)^n [z^n] (1+z)^{n-m} [w^{n-m}] (1+w)^n \sum_{k \ge 0} (1+w)^k z^k (-1)^k \binom{k+m}{2m}$$

$$= (-1)^n [z^n] (1+z)^{n-m} [w^{n-m}] (1+w)^n [v^{2m}] (1+v)^m \times \sum_{k \ge 0} (1+w)^k z^k (-1)^k (1+v)^k$$

$$= (-1)^n [z^n] (1+z)^{n-m} [w^{n-m}] (1+w)^n [v^{2m}] (1+v)^m \times \frac{1}{1+z(1+w)(1+v)}$$

$$= (-1)^n [z^{n+1}] (1+z)^{n-m} [w^{n-m}] (1+w)^{n-1} [v^{2m}] (1+v)^m \times \frac{1}{v+1+1/z/(1+w)}.$$

The contribution from v is

$$\operatorname*{res}_{v} \frac{1}{v^{2m+1}} (1+v)^m \frac{1}{v+1+1/z/(1+w)}.$$

Here the residue at infinity is zero by inspection and we can evaluate by substituting the simple pole:

$$-(-1)^{2m+1} \frac{z^{2m+1}(1+w)^{2m+1}}{(1+z+zw)^{2m+1}} \frac{(-1)^m}{z^m (1+w)^m}.$$

Restore the extractors in z and in w to get

$$(-1)^{n+m}[z^{n-m}](1+z)^{n-m}[w^{n-m}](1+w)^{n+m}\frac{1}{(1+z+wz)^{2m+1}}$$

$$= (-1)^{n+m}[z^{n+m+1}](1+z)^{n-m}[w^{n-m}](1+w)^{n+m}\frac{1}{(w+(1+z)/z)^{2m+1}}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n-m+1}}(1+w)^{n+m}\frac{1}{(w+(1+z)/z)^{2m+1}}.$$

Once more the residue at infinity is zero by inspection and we may evaluate with minus the pole at $w = -(1+z)/z$, which requires the Leibniz rule:

$$\frac{1}{(2m)!}\left(\frac{1}{w^{n-m+1}}(1+w)^{n+m}\right)^{(2m)}$$

$$= \frac{1}{(2m)!}\sum_{q=0}^{2m}\binom{2m}{q}\frac{(-1)^q(n-m+1)^{\overline{q}}}{w^{n-m+1+q}}(1+w)^{n+m-(2m-q)}(n+m)^{\underline{2m-q}}$$

$$= \sum_{q=0}^{2m}\frac{(-1)^q}{w^{n-m+1+q}}\binom{n-m+q}{q}(1+w)^{n-m+q}\binom{n+m}{2m-q}.$$

Note that

$$\binom{n-m+q}{q}\binom{n+m}{2m-q} = \frac{(n+m)!}{q!\times(n-m)!\times(2m-q)!} = \binom{n+m}{n-m}\binom{2m}{q}.$$

We have obtained the first instance of the squared binomial coefficient from the closed form. This leaves

$$\frac{1}{w}\sum_{q=0}^{2m}\binom{2m}{q}(-1)^q((1+w)/w)^{n-m+q}.$$

Evaluating at the pole we have for $(1+w)/w$ the value $1/(1+z)$ so this yields

$$(-1)^{n+m}[z^{n+m+1}](1+z)^{n-m}\frac{z}{1+z}\sum_{q=0}^{2m}\binom{2m}{q}(-1)^q\frac{1}{(1+z)^{n-m+q}}$$

$$= (-1)^{n+m}[z^{n+m}]\frac{1}{1+z}\sum_{q=0}^{2m}\binom{2m}{q}(-1)^q\frac{1}{(1+z)^q}$$

$$= (-1)^{n+m}[z^{n+m}]\frac{1}{1+z}\left[1-\frac{1}{1+z}\right]^{2m}$$
$$= (-1)^{n+m}[z^{n+m}]\frac{1}{(1+z)^{2m+1}}z^{2m}$$
$$= (-1)^{n+m}[z^{n-m}]\frac{1}{(1+z)^{2m+1}} = (-1)^{n+m}(-1)^{n-m}\binom{n-m+2m}{2m} = \binom{n+m}{2m}.$$

This is the second instance and concludes the argument.

This identity was found by a computer search which pointed to OEIS A370232, binomial coefficient squared $\binom{n+m}{2m}^2$.

1.165.30 OEIS A002299

We claim that with n and m and p non-negative integers where $p \le n-1$ and $m \ge 1$ that

$$\binom{2m-1}{p} = (-1)^{p+1}\sum_{k=0}^{n}\binom{n+m}{m+k}(-1)^k\binom{m-1+k}{m}\binom{k-m+p}{p}.$$

We can use the first binomial coefficient to enforce the upper range, writing $\binom{n+m}{n-k}$,

$$(-1)^{p+1}[z^n](1+z)^{n+m}\sum_{k=0}^{n}z^k(-1)^k\binom{m-1+k}{m}\binom{k-m+p}{p}$$
$$= (-1)^{p+1}[z^n](1+z)^{n+m}[w^m](1+w)^{m-1}\sum_{k\ge 0}z^k(-1)^k(1+w)^k\binom{k-m+p}{p}$$
$$= (-1)^{p+1}[z^n](1+z)^{n+m}[w^m](1+w)^{m-1}[v^p](1+v)^{p-m}$$
$$\times\sum_{k\ge 0}z^k(-1)^k(1+w)^k(1+v)^k$$
$$= (-1)^{p+1}[z^n](1+z)^{n+m}[w^m](1+w)^{m-1}[v^p](1+v)^{p-m}$$
$$\times\frac{1}{1+z(1+w)(1+v)}$$
$$= (-1)^{p+1}[z^{n+1}](1+z)^{n+m}[w^m](1+w)^{m-1}[v^p](1+v)^{p-m-1}$$
$$\times\frac{1}{w+1+1/z/(1+v)}.$$

The contribution from w is

$$\operatorname*{res}_{w}\frac{1}{w^{m+1}}(1+w)^{m-1}\frac{1}{w+1+1/z/(1+v)}.$$

Here the residue at infinity is zero and we may evaluate with minus the residue at the simple pole (no pole at $w=-1$ since $m \ge 1$) , getting

$$-(-1)^{m+1}\frac{z^{m+1}(1+v)^{m+1}}{(1+z+zv)^{m+1}}(-1)^{m-1}\frac{1}{z^{m-1}(1+v)^{m-1}}.$$

Substituting into the remaining extractors,

$$(-1)^p[z^{n-1}](1+z)^{n+m}[v^p](1+v)^{p-m+1}\frac{1}{(1+z+zv)^{m+1}}$$

$$=(-1)^p[z^{n-1}](1+z)^{n+m}[v^p](1+v)^{p-2m}\frac{1}{(z+1/(1+v))^{m+1}}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^n}(1+z)^{n+m}\frac{1}{(z+1/(1+v))^{m+1}}.$$

We evaluate this using the Leibniz rule:

$$\frac{1}{(n-1)!}\left((1+z)^{n+m}\frac{1}{(z+1/(1+v))^{m+1}}\right)^{(n-1)}$$

$$=\frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}(1+z)^{n+m-q}(n+m)^{\underline{q}}\frac{(-1)^{n-1-q}(m+1)^{\overline{n-1-q}}}{(z+1/(1+v))^{m+1+n-1-q}}$$

$$=\sum_{q=0}^{n-1}(1+z)^{n+m-q}\binom{n+m}{q}\frac{(-1)^{n-1-q}}{(z+1/(1+v))^{n+m-q}}\binom{n+m-1-q}{n-1-q}.$$

Let $z=0$ and re-activate the extractor in v to get

$$(-1)^p\sum_{q=0}^{n-1}\binom{n+m}{q}(-1)^{n-1-q}\binom{n+p-q-m}{p}\binom{n+m-1-q}{n-1-q}.$$

Preparing for another round,

$$(-1)^p[z^{n-1}](1+z)^{n+m-1}\sum_{q\ge 0}\binom{n+m}{q}(-1)^{n-1-q}\binom{n+p-q-m}{p}\frac{z^q}{(1+z)^q}$$

$$=(-1)^p[z^{n-1}](1+z)^{n+m-1}[w^p](1+w)^{n+p-m}$$
$$\times\sum_{q\ge 0}\binom{n+m}{q}(-1)^{n-1-q}\frac{1}{(1+w)^q}\frac{z^q}{(1+z)^q}$$

$$=(-1)^{p+n-1}[z^{n-1}](1+z)^{n+m-1}[w^p](1+w)^{n+p-m}\left[1-\frac{z}{(1+w)(1+z)}\right]^{n+m}$$

$$= (-1)^{p+n-1}[z^{n-1}]\frac{1}{1+z}[w^p](1+w)^{p-2m}[1+w+wz]^{n+m}.$$

Doing the extraction starting with z,

$$(-1)^{p+n-1}[w^p](1+w)^{p-2m}\sum_{q=0}^{n-1}(-1)^{n-1-q}\binom{n+m}{q}w^q(1+w)^{n+m-q}.$$

Note that by construction the sum term goes zero when $q > p$. But $p \le n-1$. Therefore we may raise the upper range to $n+m$ as we are only adding in zero values. Continuing,

$$(-1)^p[w^p](1+w)^{n+p-m}\sum_{q=0}^{n+m}(-1)^q\binom{n+m}{q}w^q(1+w)^{-q}$$

$$= (-1)^p[w^p](1+w)^{n+p-m}\left[1-\frac{w}{1+w}\right]^{n+m}$$

$$= (-1)^p[w^p](1+w)^{p-2m} = (-1)^p\binom{p-2m}{p} = \binom{2m-1}{p}.$$

This is the claim. Note that for $m = 0$ we get on the LHS the value $(-1)^p$ and on the RHS

$$(-1)^{p+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{k+p}{p}$$

$$= (-1)^{p+1}[v^p](1+v)^p\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1+v)^k$$

$$= (-1)^{p+1}[v^p](1+v)^p(-v)^n = 0$$

since $n > p$.

This identity was found by a computer search which pointed to OEIS A002299, binomial coefficients $\binom{2n+5}{5}$.

1.165.31 OEIS A135278

We claim that with $n \ge m \ge p$ non-negative integers and x a variable that

$$\binom{n}{m} = (-1)^{m+p}\sum_{k=0}^{n-p}(-1)^k\binom{n}{p+k}\binom{x-k}{n-m}\binom{m+k}{k}.$$

We prove it for x an integer and then have it for all x since the sum is a polynomial in x. We can use the first binomial coefficient to enforce the upper range:

$$(-1)^{m+p}[z^{n-p}](1+z)^n \sum_{k\ge 0} (-1)^k z^k \binom{x-k}{n-m}\binom{m+k}{k}$$

$$= (-1)^{m+p}[z^{n-p}](1+z)^n [w^{n-m}](1+w)^x \sum_{k\ge 0} (-1)^k z^k \frac{1}{(1+w)^k}\binom{m+k}{k}$$

$$= (-1)^{m+p}[z^{n-p}](1+z)^n [w^{n-m}](1+w)^x \frac{1}{(1+z/(1+w))^{m+1}}$$

$$= (-1)^{m+p}[z^{n-p}](1+z)^n [w^{n-m}](1+w)^{x+m+1} \frac{1}{(1+w+z)^{m+1}}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n-p+1}}(1+z)^n \frac{1}{(z+1+w)^{m+1}}.$$

Careful application of the boundary conditions now reveals that the residue at infinity is zero and we can evaluate using minus the residue at $z = -(1+w)$. This requires the Leibniz rule:

$$\frac{1}{m!}\left(\frac{1}{z^{n-p+1}}(1+z)^n\right)^{(m)}$$

$$= \frac{1}{m!}\sum_{q=0}^{m} \binom{m}{q} \frac{1}{z^{n-p+1+q}} (-1)^q (n-p+1)^{\overline{q}} (1+z)^{n-(m-q)} n^{\underline{m-q}}$$

$$= \sum_{q=0}^{m} \frac{1}{z^{n-p+1+q}} (-1)^q \binom{n-p+q}{q} (1+z)^{n-m+q} \binom{n}{m-q}.$$

Instantiate to $z = -(1+w)$ and flip sign to get

$$\sum_{q=0}^{m} \frac{1}{(1+w)^{n-p+1+q}} (-1)^{n-p} \binom{n-p+q}{q} w^{n-m+q} (-1)^{n-m+q} \binom{n}{m-q}.$$

Note however that the remaining coefficient extractor is $[w^{n-m}]$ and we have a term w^{n-m+q} in the sum, hence only $q=0$ can possibly contribute. We obtain at last

$$(-1)^{m+p}[w^{n-m}](1+w)^{x+m+1} \frac{1}{(1+w)^{n-p+1}} (-1)^{n-p} w^{n-m} (-1)^{n-m} \binom{n}{m}$$

$$= [w^0](1+w)^{x+m+p-n}\binom{n}{m} = \binom{n}{m}$$

and we have the claim.

This identity was found by a computer search which pointed to OEIS A135278, Pascal's triangle ${n+1 \choose m+1}$.

1.165.32 OEIS A076112

We claim that with $n \ge m$ non-negative integers

$$(n+1)^m = (-1)^{n+m} \sum_{k=0}^{n} (-1)^k {2n-k \choose n} {2n+1 \choose k} (n-k)^m.$$

We can use the first term to enforce the upper range, writing ${2n-k \choose n-k}$:

$$(-1)^{n+m} [z^n] (1+z)^{2n} \sum_{k\ge 0} (-1)^k \frac{z^k}{(1+z)^k} {2n+1 \choose k} (n-k)^m$$

$$= (-1)^{n+m} m! [z^n] (1+z)^{2n} [w^m] \exp(nw) \sum_{k\ge 0} (-1)^k \frac{z^k}{(1+z)^k} {2n+1 \choose k} \exp(-kw)$$

$$= (-1)^{n+m} m! [z^n] (1+z)^{2n} [w^m] \exp(nw)$$
$$\times \left[1 - \frac{z}{(1+z)\exp(w)}\right]^{2n+1}$$

$$= (-1)^{n+m} m! [z^n] \frac{1}{1+z} [w^m] \exp(-(n+1)w) [\exp(w) + z\exp(w) - z]^{2n+1}.$$

Here the contribution in z is

$$\mathrm{res}_z \frac{1}{z^{n+1}} \frac{1}{1+z} [\exp(w) + z\exp(w) - z]^{2n+1}.$$

We evaluate using minus the residues at $z=-1$ and at infinity. We get from the former,

$$-(-1)^{n+m} m! [w^m] \exp(-(n+1)w) (-1)^{n+1} 1^{2n+1}$$
$$= (-1)^m m! [w^m] \exp(-(n+1)w) = (n+1)^m.$$

This is the claim. We have from the residue at infinity

$$\mathrm{res}_z \frac{1}{z^2} z^{n+1} \frac{1}{1+1/z} [\exp(w) + \exp(w)/z - 1/z]^{2n+1}$$

$$= \mathrm{res}_z \frac{1}{1+z} \frac{1}{z^{n+1}} [z\exp(w) + \exp(w) - 1]^{2n+1}$$

$$= \sum_{q=0}^{n} (-1)^{n-q} {2n+1 \choose q} \exp(wq) (\exp(w)-1)^{2n+1-q}.$$

Note however that with $\exp(w) - 1 = w + \cdots$ the lowest power the last term can get in its series about $w = 0$ is w^{n+1}. Now the extractor is on $[w^m]$ with

$m \leq n$, making for a zero contribution.

This identity was found by a computer search which pointed to OEIS A076112, triangle of geometric progressions.

1.165.33 OEIS A367270

We claim that with $n > m \geq 0$ non-negative integers

$$\binom{n}{m}\binom{n-1}{m} = (-1)^{n+1}\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^k\binom{n+k-m-1}{n-m-1}\binom{m-1+k}{m}\binom{n-1+k}{n-1}.$$

We may use the first binomial coefficient to enforce the upper range,

$$(-1)^{n+1}[z^n](1+z)^{2n}\sum_{k\geq 0} z^k(-1)^k\binom{n+k-m-1}{n-m-1}\binom{m-1+k}{m}\binom{n-1+k}{n-1}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^{n-m-1}]\frac{1}{1+w}$$

$$\times \sum_{k\geq 0} z^k(-1)^k\frac{1}{(1+w)^k}\binom{m-1+k}{m}\binom{n-1+k}{n-1}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^{n-m-1}]\frac{1}{1+w}[v^m](1+v)^{m-1}$$

$$\times \sum_{k\geq 0} z^k(-1)^k\frac{1}{(1+w)^k}(1+v)^k\binom{n-1+k}{n-1}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^{n-m-1}]\frac{1}{1+w}[v^m](1+v)^{m-1}$$

$$\times \frac{1}{(1+z(1+v)/(1+w))^n}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^{n-m-1}](1+w)^{n-1}[v^m](1+v)^{m-1}$$

$$\times \frac{1}{(w+1+z(1+v))^n}$$

$$= (-1)^m[z^n](1+z)^{2n}[w^{n-m-1}](1+w)^{n-1}[v^m](1+v)^{m-1-n}$$

$$\times \frac{1}{(z+(1+w)/(1+v))^n}.$$

The contribution from z is

$$\operatorname*{res}_{z} \frac{1}{z^{n+1}}(1+z)^{2n}\frac{1}{(z+(1+w)/(1+v))^n}.$$

Now put $z/(1+z) = u$ so that $z = u/(1-u)$ and $dz = 1/(1-u)^2\, du$ to get

$$\operatorname*{res}_u \frac{1}{u^{n+1}} \frac{1}{(1-u)^{n-1}} \frac{1}{(u/(1-u)+(1+w)/(1+v))^n} \frac{1}{(1-u)^2}$$
$$= \operatorname*{res}_u \frac{1}{u^{n+1}} \frac{1}{1-u} \frac{1}{(u+(1-u)(1+w)/(1+v))^n}.$$

We will evaluate this using the residues at $u=1$ and at $u=(1+w)/(w-v)$. We get from $u=1$ a contribution of one, which then yields

$$(-1)^m [w^{n-m-1}](1+w)^{n-1}[v^m](1+v)^{m-1-n} = (-1)^m \binom{n-1}{n-m-1} \binom{m-1-n}{m}$$
$$= \binom{n-1}{m} \binom{n}{m}.$$

This is the claim. Now we just have to verify that the contribution from the other pole is zero. We write

$$\operatorname*{res}_u \frac{1}{u^{n+1}} \frac{1}{1-u} \frac{1}{((1+w)/(1+v)-u((1+w)/(1+v)-1))^n}$$
$$= \operatorname*{res}_u \frac{1}{u^{n+1}} \frac{1}{1-u} \frac{1}{((1+w)/(1+v)-u(w-v)/(1+v))^n}$$
$$= \frac{(1+v)^n}{(w-v)^n} \operatorname*{res}_u \frac{1}{u^{n+1}} \frac{1}{1-u} \frac{1}{((1+w)/(w-v)-u)^n}$$
$$= (-1)^n \frac{(1+v)^n}{(w-v)^n} \operatorname*{res}_u \frac{1}{u^{n+1}} \frac{1}{1-u} \frac{1}{(u-(1+w)/(w-v))^n}.$$

We require the Leibniz rule:

$$\frac{1}{(n-1)!} \left(\frac{1}{u^{n+1}} \frac{1}{(1-u)^1} \right)^{(n-1)}$$
$$= \frac{1}{(n-1)!} \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{1}{u^{n+1+q}} (-1)^q (n+1)^{\overline{q}} \frac{1}{(1-u)^{1+n-1-q}} 1^{\overline{n-1-q}}$$
$$= \sum_{q=0}^{n-1} \frac{1}{u^{n+1+q}} (-1)^q \binom{n+q}{q} \frac{1}{(1-u)^{n-q}}.$$

Instantiate at $u=(1+w)/(w-v)$ to get

$$\sum_{q=0}^{n-1} \frac{(w-v)^{n+1+q}}{(1+w)^{n+1+q}} (-1)^q \binom{n+q}{q} \frac{(v-w)^{n-q}}{(1+v)^{n-q}}$$
$$= (-1)^n (w-v)^{2n+1} \sum_{q=0}^{n-1} \frac{1}{(1+w)^{n+1+q}} \binom{n+q}{q} \frac{1}{(1+v)^{n-q}}.$$

Next observe that

$$\frac{1}{(w-v)^n}(w-v)^{2n+1} = \sum_{p=0}^{n+1} \binom{n+1}{p} (-1)^p v^p w^{n+1-p}.$$

We have from the coefficient extractor in v that we require $p \le m$ and from the one in w that $n+1-p \le n-m-1$ which is $m+2 \le p$. Hence the intersection of these two ranges is empty and we get a zero contribution, concluding the argument.

This identity was found by a computer search which pointed to OEIS A367270, triangle of $\binom{n}{k}\binom{n-1}{k}$.

1.165.34 OEIS A001725

We claim that with $n \ge 0$ and $n \ge m \ge 0$ and with r a real number

$$\frac{(n+m)!}{n!} = (-1)^n \sum_{k=0}^{n} (r+k)^m \binom{n+k}{n} (-1)^k \binom{n+m}{m+k}.$$

We may use the third binomial coefficient to enforce the range:

$$(-1)^n [z^n](1+z)^{n+m} \sum_{k\ge 0} (r+k)^m \binom{n+k}{n} (-1)^k z^k$$

$$= (-1)^n m! [z^n](1+z)^{n+m} [w^m] \exp(rw) \sum_{k\ge 0} \exp(kw) \binom{n+k}{n} (-1)^k z^k$$

$$= (-1)^n m! [z^n](1+z)^{n+m} [w^m] \exp(rw)$$
$$\times \frac{1}{(1+z\exp(w))^{n+1}}$$
$$= (-1)^n m! [z^n](1+z)^{n+m} [w^m] \exp((r-n-1)w)$$
$$\times \frac{1}{(z+\exp(-w))^{n+1}}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^{n+m} \frac{1}{(z+\exp(-w))^{n+1}}.$$

Here the residue at infinity is zero due to the boundary conditions. Residues sum to zero and we may thus evaluate using minus the residue at $z = -\exp(-w)$. We require the Leibniz rule:

$$\frac{1}{n!} \left(\frac{1}{z^{n+1}} (1+z)^{n+m} \right)^{(n)}$$

$$= \frac{1}{n!} \sum_{q=0}^{n} \binom{n}{q} \frac{1}{z^{n+1+q}} (-1)^q (n+1)^{\overline{q}} (1+z)^{n+m-(n-q)} (n+m)^{\underline{n-q}}$$

$$= \sum_{q=0}^{n} \frac{1}{z^{n+1+q}} (-1)^q \binom{n+q}{q} (1+z)^{m+q} \binom{n+m}{n-q}.$$

Evaluate at $z = -\exp(-w)$ and flip sign to get

$$(-1)^n \sum_{q=0}^{n} \exp((n+1+q)w) \binom{n+q}{q} (1-\exp(-w))^{m+q} \binom{n+m}{n-q}.$$

Observe that $1-\exp(-w) = w \pm \cdots$ so that $(1-\exp(-w))^{m+q} = w^{m+q} + \cdots$. We have a coefficient extractor on $[w^m]$ however, hence only $q = 0$ contributes:

$$m![w^m] \exp((n+1)w) \exp((r-n-1)w) \binom{n}{0} (w^m + \cdots) \binom{n+m}{n}$$

$$= m![w^0] \exp(rw) \binom{n+m}{n} = \frac{(n+m)!}{n!}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A001725, $n!/5!$.

1.165.35 OEIS A094305

We claim that with $n \ge m \ge p$ featuring the Stirling set numbers that

$$\frac{(n+p)!}{2^p (n-p)! p!} \binom{n-p}{m-p} = (-1)^m \sum_{k=0}^{m} \binom{n-k}{m-k} (-1)^k \left\{ {p+k+1 \atop k+1} \right\} \binom{m+k}{p+k} \binom{p+n}{p+k}.$$

We shall see that a single proof actually produces four identities. We re-index the sum,

$$\sum_{k=0}^{m} \binom{n-m+k}{k} (-1)^k \left\{ {p+m+1-k \atop m+1-k} \right\} \binom{2m-k}{p+m-k} \binom{p+n}{p+m-k}.$$

Using the standard EGF

$$\sum_{k=0}^{m} \binom{n-m+k}{k} (-1)^k (p+m+1-k)! [z^{p+m+1-k}] \frac{(\exp(z)-1)^{m+1-k}}{(m+1-k)!}$$

$$\times \binom{2m-k}{p+m-k}\binom{p+n}{p+m-k}$$

$$= \sum_{k=0}^{m} \binom{n-m+k}{k} (-1)^k (p+m-k)! [z^{p+m-k}] \exp(z) \frac{(\exp(z)-1)^{m-k}}{(m-k)!}$$

$$\times \binom{2m-k}{p+m-k}\binom{p+n}{p+m-k}$$

$$= \frac{m!}{(m-p)!} \sum_{k=0}^{m} \binom{n-m+k}{k} (-1)^k [z^{p+m}] \exp(z) z^k (\exp(z)-1)^{m-k}$$

$$\times \binom{2m-k}{m-k}\binom{p+n}{p+m-k}.$$

We see that we can use the middle binomial coefficient to enforce the upper range of the sum:

$$\frac{m!}{(m-p)!} [z^{p+m}] \exp(z)(\exp(z)-1)^m [w^m](1+w)^{2m}$$

$$\times \sum_{k\ge 0} \binom{n-m+k}{k} (-1)^k z^k (\exp(z)-1)^{-k} \frac{w^k}{(1+w)^k} \binom{p+n}{p+m-k}$$

$$= \frac{m!}{(m-p)!} [z^{p+m}] \exp(z)(\exp(z)-1)^m [w^m](1+w)^{2m} [v^{p+m}](1+v)^{p+n}$$

$$\times \sum_{k\ge 0} \binom{n-m+k}{k} (-1)^k z^k (\exp(z)-1)^{-k} \frac{w^k}{(1+w)^k} v^k$$

$$= \frac{m!}{(m-p)!} [z^{p+m}] \exp(z)(\exp(z)-1)^m [w^m](1+w)^{2m} [v^{p+m}](1+v)^{p+n}$$

$$\times \frac{1}{(1+zwv/(\exp(z)-1)/(1+w))^{n-m+1}}.$$

The contribution from v is

$$\operatorname*{res}_v \frac{1}{v^{p+m+1}} (1+v)^{p+n} \frac{1}{(1+zwv/(\exp(z)-1)/(1+w))^{n-m+1}}.$$

Now put $v/(1+v) = u$ so that $v = u/(1-u)$ and $dv = 1/(1-u)^2\, du$ to get

$$\operatorname*{res}_v \frac{1}{u^{p+m+1}} \frac{1}{(1-u)^{n-m-1}} \frac{1}{(1+zwu/(1-u)/(\exp(z)-1)/(1+w))^{n-m+1}} \frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_v \frac{1}{u^{p+m+1}} \frac{1}{(1-u+zwu/(\exp(z)-1)/(1+w))^{n-m+1}}$$

$$= \binom{p+n}{n-m} \left[1 - \frac{zw}{(\exp(z)-1)(1+w)}\right]^{p+m}.$$

Taking extra care not to leave the domain of formal power series we can re-capitulate what we have,

$$\frac{m!}{(m-p)!}\binom{p+n}{n-m}[z^{2p+m}]\exp(z)\frac{z^p}{(\exp(z)-1)^p}[w^m](1+w)^{m-p} \\ \times[(\exp(z)-1)(1+w)-zw]^{p+m}.$$

The powered term is

$$\sum_{q=0}^{p+m}\binom{p+m}{q}(\exp(z)-z-1)^q w^q(\exp(z)-1)^{p+m-q}.$$

From the extractor in w applying the boundary condition $m \ge p$ we get the requirement $m-q \le m-p$ or $p \le q$. From the extractor in z we get $2p+m \ge 2q+p+m-q$ or $p \ge q$. Hence only $q = p$ can possibly contribute. This leaves

$$\frac{m!}{(m-p)!}\binom{p+n}{n-m}[z^{2p+m}]\exp(z)\frac{z^p}{(\exp(z)-1)^p}[w^m](1+w)^{m-p} \\ \times\binom{p+m}{p}(\exp(z)-z-1)^p w^p(\exp(z)-1)^m$$

$$= \frac{m!}{(m-p)!}\binom{p+n}{n-m}[z^{2p+m}]\exp(z)\frac{z^p}{(\exp(z)-1)^p} \\ \times\binom{p+m}{p}(\exp(z)-z-1)^p(\exp(z)-1)^m$$

$$= \frac{m!}{(m-p)!}\binom{p+n}{n-m}[z^{2p+m}]\exp(z)(1+\cdots)^p \\ \times\binom{p+m}{p}\left(\frac{1}{2}z^2+\cdots\right)^p(z+\cdots)^m$$

$$= \frac{m!}{(m-p)!}\binom{p+n}{n-m}\binom{p+m}{p}\frac{1}{2^p}.$$

Expanding the binomial coefficients then yields at last

$$\frac{1}{(m-p)!}\frac{(p+n)!}{(n-m)!}\frac{1}{p!}\frac{1}{2^p}$$

$$= \frac{(n+p)!}{(n-p)!p!2^p}\binom{n-p}{n-m} = \frac{(n+p)!}{(n-p)!p!2^p}\binom{n-p}{m-p}$$

which is the claim. This also proves the same closed form for

$$(-1)^m \sum_{k=0}^{m} \binom{n-k}{m-k} (-1)^k \left\{ p+k \atop k \right\} \binom{m+k}{p+k} \binom{p+n}{p+k}.$$

as well as the Stirling cycle number pair (with $\frac{1}{1-z} = \left(\log \frac{1}{1-z}\right)'$ taking the place of $\exp(z) = (\exp(z)-1)'$ which appeared during differentiation):

$$(-1)^m \sum_{k=0}^{m} \binom{n-k}{m-k} (-1)^k \left[p+k+1 \atop k+1 \right] \binom{m+k}{p+k} \binom{p+n}{p+k}$$

and

$$(-1)^m \sum_{k=0}^{m} \binom{n-k}{m-k} (-1)^k \left[p+k \atop k \right] \binom{m+k}{p+k} \binom{p+n}{p+k}.$$

Alternate proof

Starting over we can observe that

$$\binom{n-k}{m-k} \binom{p+n}{p+k} = \frac{(p+n)!}{(n-m)! \times (m-k)! \times (p+k)!} = \binom{p+n}{n-m} \binom{p+m}{p+k}$$

so that the sum becomes

$$(-1)^m \binom{p+n}{n-m} \sum_{k=0}^{m} (-1)^k \left\{ p+k+1 \atop k+1 \right\} \binom{m+k}{p+k} \binom{p+m}{p+k}.$$

Re-indexing the sum

$$\binom{p+n}{n-m} \sum_{k=0}^{m} (-1)^k \left\{ p+m+1-k \atop m+1-k \right\} \binom{2m-k}{p+m-k} \binom{p+m}{p+m-k}$$

$$= \binom{p+n}{n-m} \sum_{k=0}^{m} (-1)^k (p+m+1-k)! [z^{p+m+1-k}] \frac{(\exp(z)-1)^{m+1-k}}{(m+1-k)!}$$

$$\times \binom{2m-k}{p+m-k} \binom{p+m}{k}$$

$$= \binom{p+n}{n-m} \sum_{k=0}^{m} (-1)^k (p+m-k)! [z^{p+m-k}] \exp(z) \frac{(\exp(z)-1)^{m-k}}{(m-k)!}$$

$$\times \binom{2m-k}{p+m-k} \binom{p+m}{k}$$

$$= \binom{p+n}{n-m} \frac{m!}{(m-p)!} \sum_{k=0}^{m} (-1)^k [z^{p+m}] z^k \exp(z)(\exp(z)-1)^{m-k}$$

$$\times \binom{2m-k}{m-k} \binom{p+m}{k}.$$

Here the middle binomial coefficient enforces the range of the sum:

$$\binom{p+n}{n-m} \frac{m!}{(m-p)!} [z^{p+m}] \exp(z)(\exp(z)-1)^m [w^m](1+w)^{2m}$$

$$\times \sum_{k\ge 0} (-1)^k z^k (\exp(z)-1)^{-k} \frac{w^k}{(1+w)^k} \binom{p+m}{k}$$

$$= \binom{p+n}{n-m} \frac{m!}{(m-p)!} [z^{p+m}] \exp(z)(\exp(z)-1)^m [w^m](1+w)^{2m}$$

$$\times \left[1 - \frac{zw}{(\exp(z)-1)(1+w)}\right]^{p+m}.$$

At this point the continuation of the proof to conclusion merges with the first version.

This identity was found by a computer search which pointed to OEIS A094305, $\binom{n+2}{2}\binom{n}{k}$.

1.165.36 OEIS A143219

We claim that with $n, m, p \ge 0$ and $n, m \ge p$ that

$$\binom{n-1}{p-1}\binom{m}{p} = (-1)^p \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1}\binom{n+k}{p+k}\binom{k-m+p}{p}.$$

Here we can use the first binomial coefficient to enforce the upper range, using $\binom{n+1}{n-k}$ to get

$$(-1)^p [z^n](1+z)^{n+1} \sum_{k\ge 0} (-1)^k z^k \binom{n+k}{n-p}\binom{k-m+p}{p}$$

$$= (-1)^p [z^n](1+z)^{n+1} [w^{n-p}](1+w)^n \sum_{k\ge 0} (-1)^k z^k (1+w)^k \binom{k-m+p}{p}$$

$$= (-1)^p [z^n](1+z)^{n+1} [w^{n-p}](1+w)^n [v^p](1+v)^{p-m}$$

$$\times \sum_{k\ge 0} (-1)^k z^k (1+w)^k (1+v)^k$$

$$= (-1)^p [z^n](1+z)^{n+1} [w^{n-p}](1+w)^n [v^p](1+v)^{p-m}$$

$$\times \frac{1}{1+z(1+w)(1+v)}$$

$$= (-1)^p [z^{n+1}](1+z)^{n+1} [w^{n-p}](1+w)^n [v^p] \frac{1}{(1+v)^{m-p+1}}$$

$$\times \frac{1}{w+1+1/z/(1+v)}.$$

The contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n-p+1}} (1+w)^n \frac{1}{w+1+1/z/(1+v)}.$$

Now put $w/(1+w) = u$ so that $w = u/(1-u)$ and $dw = 1/(1-u)^2 \; du$ to get

$$\operatorname*{res}_{u} \frac{1}{u^{n-p+1}} \frac{1}{(1-u)^{p-1}} \frac{1}{u/(1-u)+1+1/z/(1+v)} \frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_{u} \frac{1}{u^{n-p+1}} \frac{1}{(1-u)^p} \frac{1}{u+(1-u)(1+1/z/(1+v))}$$

$$= \operatorname*{res}_{u} \frac{1}{u^{n-p+1}} \frac{1}{(1-u)^p} \frac{1}{1+(1-u)(1/z/(1+v))}$$

$$= \operatorname*{res}_{u} \frac{1}{u^{n-p+1}} \frac{1}{(1-u)^p} \frac{z(1+v)}{z(1+v)+(1-u)}.$$

Let us re-capitulate what we now have

$$[z^n](1+z)^{n+1} [v^p] \frac{1}{(1+v)^{m-p}} \operatorname*{res}_{u} \frac{1}{u^{n-p+1}} \frac{1}{(u-1)^p} \frac{1}{1+z(1+v)-u}.$$

Here the residue at infinity is zero and we can evaluate using minus the residues at $u=1$ and $u = 1+z(1+v)$. We evaluate the first residue using the Leibniz rule:

$$\frac{1}{(p-1)!} \left(\frac{1}{u^{n-p+1}} \frac{1}{(1+z(1+v)-u)^1} \right)^{(p-1)}$$

$$= \frac{1}{(p-1)!} \sum_{q=0}^{p-1} \binom{p-1}{q} \frac{(n-p+1)^{\overline{q}}}{u^{n-p+1+q}} (-1)^q \frac{1^{\overline{p-1-q}}}{(1+z(1+v)-u)^{1+p-1-q}}$$

$$= \sum_{q=0}^{p-1} \frac{1}{u^{n-p+1+q}} (-1)^q \binom{n-p+q}{q} \frac{1}{(1+z(1+v)-u)^{p-q}}.$$

Put $u=1$ and restore the remaining extractors flipping the sign:

$$-[z^n](1+z)^{n+1}[v^p]\frac{1}{(1+v)^{m-p}}\sum_{q=0}^{p-1}(-1)^q\binom{n-p+q}{q}\frac{1}{(z(1+v))^{p-q}}$$

$$=\sum_{q=0}^{p-1}(-1)^{q+1}\binom{n-p+q}{q}\binom{n+1}{n+p-q}\binom{q-m}{p}.$$

Now with $n+1$ being positive the middle binomial coefficient requires $n+p-q\le n+1$ or $p-1\le q$. Hence only $q=p-1$ will contribute and we get

$$(-1)^p\binom{n-1}{p-1}\binom{n+1}{n+1}\binom{p-1-m}{p}=\binom{n-1}{p-1}\binom{m}{p}.$$

This is the claim. Now it remains to verify that the contribution from the residue at $u=1+z(1+v)$ is zero. We get

$$[z^n](1+z)^{n+1}[v^p]\frac{1}{(1+v)^{m-p}}\frac{1}{(1+z(1+v))^{n-p+1}}\frac{1}{(z(1+v))^p}$$

$$=[z^{n+p}](1+z)^{n+1}[v^p]\frac{1}{(1+v)^m}\frac{1}{(1+z(1+v))^{n-p+1}}$$

$$=[z^{n+p}](1+z)^{n+1}[v^p]\frac{1}{(1+v)^{n-p+1+m}}\frac{1}{(z+1/(1+v))^{n-p+1}}.$$

Here the contribution from z is

$$\operatorname*{res}_z\frac{1}{z^{n+p+1}}(1+z)^{n+1}\frac{1}{(z+1/(1+v))^{n-p+1}}.$$

We once more evaluate with the Leibniz rule as the residue at infinity is zero:

$$\frac{1}{(n-p)!}\left(\frac{1}{z^{n+p+1}}(1+z)^{n+1}\right)^{(n-p)}$$

$$=\frac{1}{(n-p)!}\sum_{q=0}^{n-p}\binom{n-p}{q}\frac{(-1)^q(n+p+1)^{\overline{q}}}{z^{n+p+1+q}}(1+z)^{n+1-(n-p-q)}(n+1)^{\underline{n-p-q}}$$

$$=\sum_{q=0}^{n-p}\frac{(-1)^q}{z^{n+p+1+q}}\binom{n+p+q}{q}(1+z)^{p+q+1}\binom{n+1}{n-p-q}.$$

Evaluate at $z=-1/(1+v)$ with $1+z=v/(1+v)$

$$(-1)^{n+p}\sum_{q=0}^{n-p}(1+v)^{n+p+1+q}\binom{n+p+q}{q}\frac{v^{p+q+1}}{(1+v)^{p+q+1}}\binom{n+1}{n-p-q}.$$

But note that the extractor on v is $[v^p]$ and the sum argument starts at v^{p+q+1}, hence there is a zero contribution (just barely) and we may conclude.

This identity was found by a computer search which pointed to OEIS A143219, $n\binom{k+1}{2}$.

1.165.37 OEIS A033918

We claim that with $n, m \ge 1$ and $n \ge m$ that

$$m^m = (-1)^n \sum_{k=0}^{n} \binom{n+m}{m+k} (-1)^k k^m \binom{m-1+k}{m-n+k}.$$

We can use the first binomial coefficient to enforce the upper range of the sum, writing $\binom{n+m}{n-k}$:

$$(-1)^n [z^n](1+z)^{n+m} \sum_{k\ge 0} z^k (-1)^k k^m \binom{m-1+k}{m-n+k}$$

$$= (-1)^n m! [z^n](1+z)^{n+m} [w^m] \sum_{k\ge 0} z^k (-1)^k \exp(kw) \binom{m-1+k}{n-1}$$

$$= (-1)^n m! [z^n](1+z)^{n+m} [w^m][v^{n-1}](1+v)^{m-1} \times \sum_{k\ge 0} z^k (-1)^k \exp(kw)(1+v)^k$$

$$= (-1)^n m! [z^n](1+z)^{n+m} [w^m][v^{n-1}](1+v)^{m-1} \times \frac{1}{1+z\exp(w)(1+v)}$$

$$= (-1)^n m! [z^{n+1}](1+z)^{n+m} [w^m] \exp(-w) [v^{n-1}](1+v)^{m-1} \times \frac{1}{v+1+1/z/\exp(w)}.$$

Here the contribution from v is

$$\operatorname*{res}_v \frac{1}{v^n}(1+v)^{m-1} \frac{1}{v+1+1/z/\exp(w)}.$$

We see that the residue at infinity is zero (just barely) owing to the boundary conditions and we may evaluate with minus the residue from the simple pole at $v = -1 - 1/z/\exp(w)$, getting

$$-\frac{(-1)^n}{(1+1/z/\exp(w))^n} \frac{(-1)^{m-1}}{(z\exp(w))^{m-1}}$$

$$= (-1)^{n+m} \frac{z^{n+1-m}\exp((n+1-m)w)}{(1+z\exp(w))^n}.$$

Restoring the two extractors,

$$(-1)^m m![z^m](1+z)^{n+m}[w^m]\frac{\exp((n-m)w)}{(1+z\exp(w))^n}.$$

Here the contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{m+1}}(1+z)^{n+m}\frac{1}{(1+z\exp(w))^n}.$$

Now put $z/(1+z) = u$ so that $z = u/(1-u)$ and $dz = 1/(1-u)^2\, du$ to get

$$\begin{aligned}
&\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{(1-u)^{n-1}}\frac{1}{(1+u\exp(w)/(1-u))^n}\frac{1}{(1-u)^2}\\
&= \operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{1-u}\frac{1}{(1-u(1-\exp(w)))^n}\\
&= \frac{(-1)^n}{(1-\exp(w))^n}\operatorname*{res}_u \frac{1}{u^{m+1}}\frac{1}{1-u}\frac{1}{(u-1/(1-\exp(w)))^n}.
\end{aligned}$$

Here the residue at infinity is zero and we may evaluate using minus the residues at $u = 1$ and $u = 1/(1-\exp(w))$. We get for the former

$$(-1)^m m![w^m]\exp((n-m)w)\exp(-nw) = m^m.$$

This is the claim. Now we just need to prove that the contribution from the other pole is zero which requires the Leibniz rule:

$$\begin{aligned}
&\frac{1}{(n-1)!}\left(\frac{1}{u^{m+1}}\frac{1}{(1-u)^1}\right)^{(n-1)}\\
&= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(-1)^q(m+1)^{\overline{q}}}{u^{m+1+q}}\frac{1^{\overline{n-1-q}}}{(1-u)^{1+n-1-q}}\\
&= \sum_{q=0}^{n-1}\binom{m+q}{q}\frac{(-1)^q}{u^{m+1+q}}\frac{1}{(1-u)^{n-q}}.
\end{aligned}$$

Substitute $u = 1/(1-\exp(w))$ so that $1-u = -\exp(w)/(1-\exp(w))$ restoring both terms in front

$$\begin{aligned}
&(-1)^m m![w^m]\exp((n-m)w)\\
&\times\frac{(-1)^n}{(1-\exp(w))^n}\sum_{q=0}^{n-1}\binom{m+q}{q}(-1)^q(1-\exp(w))^{m+1+q}\\
&\times(-1)^{n-q}\exp(-(n-q)w)(1-\exp(w))^{n-q}\\
&= (-1)^m m!\,[w^m](1-\exp(w))^{m+1}\sum_{q=0}^{n-1}\binom{m+q}{q}\exp((q-m)w).
\end{aligned}$$

Note however that $(1-\exp(w))^{m+1} = (-1)^{m+1} w^{m+1} + \cdots$ so the coefficient extractor $[w^m]$ returns zero (just barely) and we may conclude.

This identity was found by a computer search which pointed to OEIS A033918, array of $1^1, 2^2, \ldots, n^n$.

1.165.38 OEIS A000984

We claim that with $n \ge m \ge p \ge 0$ that

$$\binom{2p}{p} = (-1)^{m+p} \sum_{k=0}^{n} \binom{n-k+p}{p} \binom{m-k}{p} (-1)^k \binom{2p}{k-m+p}.$$

We start by re-indexing the sum,

$$(-1)^{m+p+n} \sum_{k=0}^{n} \binom{k+p}{p} \binom{m-n+k}{p} (-1)^k \binom{2p}{n-m+p-k}.$$

Observe carefully that $n-m+p \le n$ due to the boundary conditions and hence the third binomial coefficient enforces the upper range of the sum:

$$(-1)^{m+p+n} [z^{n-m+p}] (1+z)^{2p} \sum_{k\ge 0} \binom{k+p}{p} \binom{m-n+k}{p} (-1)^k z^k$$

$$= (-1)^{m+p+n} [z^{n-m+p}] (1+z)^{2p} [w^p] (1+w)^{m-n} \times \sum_{k\ge 0} \binom{k+p}{p} (-1)^k z^k (1+w)^k$$

$$= (-1)^{m+p+n} [z^{n-m+p}] (1+z)^{2p} [w^p] \frac{1}{(1+w)^{n-m}} \times \frac{1}{(1+z(1+w))^{p+1}}$$

$$= (-1)^{m+p+n} [z^{n-m+p}] (1+z)^{2p} [w^p] \frac{1}{(1+w)^{n-m+p+1}} \times \frac{1}{(z+1/(1+w))^{p+1}}.$$

Here the contribution from z is

$$\operatorname*{res}_{z} \frac{1}{z^{n-m+p+1}} (1+z)^{2p} \frac{1}{(z+1/(1+w))^{p+1}}.$$

We see that the residue at infinity is zero and we may evaluate using minus the residue at $z = -1/(1+w)$. We get using the Leibniz rule

$$\frac{1}{p!}\left(\frac{1}{z^{n-m+p+1}}(1+z)^{2p}\right)^{(p)}$$

$$= \frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^q(n-m+p+1)^{\overline{q}}}{z^{n-m+p+1+q}}(1+z)^{2p-(p-q)}(2p)^{\underline{p-q}}$$

$$= \sum_{q=0}^{p}\frac{(-1)^q}{z^{n-m+p+1+q}}\binom{n-m+p+q}{q}(1+z)^{p+q}\binom{2p}{p-q}.$$

Now put $z = -1/(1+w)$ to obtain

$$-\sum_{q=0}^{p}(-1)^q(-1)^{n-m+p+1}(1+w)^{n-m+p+1+q}\binom{n-m+p+q}{q}\frac{w^{p+q}}{(1+w)^{p+q}}\binom{2p}{p-q}.$$

Applying the extractor in w and accounting for the signs,

$$[w^p]\frac{1}{(1+w)^p}\sum_{q=0}^{p}(-1)^q\binom{n-m+p+q}{q}w^{p+q}\binom{2p}{p-q}.$$

Due to the extractor only $q = 0$ makes a non-zero contribution and we have at last

$$[w^0]\frac{1}{(1+w)^p}(-1)^0\binom{n-m+p}{0}\binom{2p}{p} = \binom{2p}{p}.$$

This is the claim.

This identity was found by a computer search which pointed to OEIS A000984, central binomial coefficients $\binom{2n}{n}$.

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