Identities by C. Schneider, Egorychev method

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We apply Egorychev method to some identities proved by C. Schneider.

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$$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{n+r}{r}\binom{r}{s}^3=\sum_{r=0}^{n}\binom{n}{r}^2\binom{n+r}{r}^2$$

section 2

$$\sum_{r=0}^{n}\sum_{s=0}^{n}\binom{r+s}{r}^2\binom{4n-2r-2s}{2n-2r}=(2n+1)\binom{2n}{n}^2$$

section 3

With $0\le x<1$

$$\binom{m}{n-1}\sum_{k\ge m+1}\binom{k}{n}^{-1}\sum_{j=0}^{m}\binom{k}{j}x^{k-j}(1-x)^j=\begin{cases}\log\frac{1}{1-x} & n=1\\ \frac{n}{n-1}(1-(1-x)^{n-1}) & n\ge 2\end{cases}$$

1 Apery-Schmidt-Strehl identity

We seek

$$\boxed{\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{n+r}{r}\binom{r}{s}^3=\sum_{r=0}^{n}\binom{n}{r}^2\binom{n+r}{r}^2.}$$

We start with the inner sum on the LHS

$$[z^r](1+z)^r[w^r](1+w)^r\sum_{s=0}^{r}\binom{r}{s}z^sw^s=[z^r](1+z)^r[w^r](1+w)^r(1+wz)^r.$$

We then get for the outer sum

$$\begin{aligned}[v^n](1+v)^{2n}\sum_{r=0}^{n}\binom{n}{r}(1+v)^{-r}[z^{n-r}](1+z)^{n-r}[w^{n-r}](1+w)^{n-r}(1+wz)^{n-r}\\ =[v^n](1+v)^{2n}[z^n](1+z)^n[w^n](1+w)^n(1+wz)^n\\ \times(1+zw/(1+wz)/(1+z)/(1+w)/(1+v))^n\\ =[v^n](1+v)^n[z^n][w^n]((1+wz)(1+z)(1+w)(1+v)+wz)^n.\end{aligned}$$

In terms of FLS,

$$\operatorname*{res}_v\frac{1}{v^{n+1}}(1+v)^n[z^n][w^n]((1+wz)(1+z)(1+w)(1+v)+wz)^n.$$

Letting $v/(1+v)=u$,

$$\begin{aligned}&\operatorname*{res}_u\frac{1}{u^{n+1}}(1-u)[z^n][w^n]((1+wz)(1+z)(1+w)/(1-u)+wz)^n\frac{1}{(1-u)^2}\\ &=\operatorname*{res}_u\frac{1}{u^{n+1}}\frac{1}{(1-u)^{n+1}}[z^n][w^n]((1+wz)(1+z)(1+w)+wz(1-u))^n.\end{aligned}$$

Extracting the coefficient on u,

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n+r}{n}[z^n][w^n]\binom{n}{r}(-1)^{n-r}w^{n-r}z^{n-r}((1+wz)(1+z)(1+w)+wz)^r\\ &=\sum_{r=0}^{n}\binom{n}{r}\binom{n+r}{n}[z^r][w^r](-1)^{n-r}((1+wz)(1+z)(1+w)+wz)^r\\ &=\sum_{r=0}^{n}\binom{n}{r}\binom{n+r}{n}[z^r][w^r](-1)^{n-r}(1+w(1+z))^r(1+z+wz)^r\end{aligned}$$

$$= \sum_{r=0}^{n} \binom{n}{r} \binom{n+r}{n} (-1)^{n-r} [z^r] \sum_{s=0}^{r} \binom{r}{r-s} (1+z)^{r-s} \binom{r}{s} z^s (1+z)^{r-s}$$

$$= \sum_{r=0}^{n} \binom{n}{r} \binom{n+r}{n} (-1)^{n-r} \sum_{s=0}^{r} \binom{r}{s}^2 \binom{2r-2s}{r-s}.$$

We have for the inner sum,

$$[z^r] \frac{1}{\sqrt{1-4z}} \sum_{s=0}^{r} \binom{r}{s}^2 z^s$$

$$= [z^r] \frac{1}{\sqrt{1-4z}} [w^r] (1+w)^r \sum_{s=0}^{r} \binom{r}{s} w^s z^s$$

$$= [z^r] \frac{1}{\sqrt{1-4z}} [w^r] (1+w)^r (1+wz)^r.$$

This yields for the outer sum,

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n-r}{n} (-1)^r [z^{n-r}] \frac{1}{\sqrt{1-4z}} [w^{n-r}] (1+w)^{n-r} (1+wz)^{n-r}$$

$$= [z^n] \frac{1}{\sqrt{1-4z}} [v^n] (1+v)^{2n} [w^n] (1+w)^n (1+wz)^n$$

$$\times (1 - wz/(1+v)/(1+w)/(1+wz))^n$$

$$= [z^n] \frac{1}{\sqrt{1-4z}} [v^n] (1+v)^n [w^n]$$

$$\times ((1+w)(1+wz)(1+v) - wz)^n$$

$$= [z^n] \frac{1}{\sqrt{1-4z}} [v^n] (1+v)^n [w^n] \sum_{r=0}^{n} \binom{n}{r} (1+w)^{n-r} (1+v)^{n-r} (wz(v+w+vw))^r$$

$$= [v^n] (1+v)^n [w^n] \sum_{r=0}^{n} \binom{n}{r} \binom{2n-2r}{n-r} (1+w)^{n-r} (1+v)^{n-r} (w(v+w+vw))^r$$

$$= [v^n] (1+v)^n [w^n] \sum_{r=0}^{n} \binom{n}{r} \binom{2n-2r}{n-r} (1+w)^{n-r} (1+v)^{n-r}$$

$$\times w^r \sum_{s=0}^{r} \binom{r}{s} v^s (1+w)^s w^{r-s}$$

$$= [w^n] \sum_{r=0}^{n} \binom{n}{r} \binom{2n-2r}{n-r} (1+w)^{n-r}$$

$$\times w^r \sum_{s=0}^{r} \binom{r}{s} \binom{2n-r}{n-s} (1+w)^s w^{r-s}$$

$$= \sum_{r=0}^{n} \binom{n}{r} \binom{2n-2r}{n-r} \sum_{s=0}^{r} \binom{r}{s} \binom{2n-r}{n-s} \binom{n+s-r}{n+s-2r}.$$

Now we have

$$\binom{2n-r}{n-s} \binom{n+s-r}{n+s-2r} = \binom{2n-r}{r} \binom{2n-2r}{n+s-2r}$$

which yields

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n-2r}{n-r} \binom{2n-r}{r} \sum_{s=0}^{r} \binom{r}{s} \binom{2n-2r}{n+s-2r}.$$

Working with the inner sum,

$$[z^n](1+z)^{2n-2r} \sum_{r=0}^{s} \binom{r}{s} z^s = [z^n](1+z)^{2n-2r}(1+z)^r = \binom{2n-r}{n}.$$

We have the closed form

$$\boxed{\sum_{r=0}^{n} \binom{n}{r} \binom{2n-2r}{n-r} \binom{2n-r}{r} \binom{2n-r}{n}.}$$

Re-indexing the sum,

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2r}{r} \binom{n+r}{n-r} \binom{n+r}{n}$$

but we also have $\binom{2r}{r}\binom{n+r}{n-r} = \binom{n}{r}\binom{n+r}{r}$ which concludes the argument and the identity is proved.

This identity is from [PS25].

2 Blodgett-Andrews-Paule sum

We seek

$$\boxed{\sum_{r=0}^{n} \sum_{s=0}^{n} \binom{r+s}{r}^2 \binom{4n-2r-2s}{2n-2r} = (2n+1)\binom{2n}{n}^2.}$$

Re-write the sum as or use symmetry

$$\sum_{r=0}^{n} \sum_{s=0}^{n} \binom{r+s}{r}^2 \binom{4n-2r-2s}{2n-2s}$$

and work with the inner sum where we see that the third binomial coefficient enforces the upper range through its extractor:

$$[z^{2n}](1+z)^{4n-2r}\sum_{s\ge 0}\binom{r+s}{r}^2\frac{z^{2s}}{(1+z)^{2s}}$$

$$=[w^r](1+w)^r[z^{2n}](1+z)^{4n-2r}\sum_{s\ge 0}\binom{r+s}{r}(1+w)^s\frac{z^{2s}}{(1+z)^{2s}}$$

$$=[w^r](1+w)^r[z^{2n}](1+z)^{4n-2r}\frac{1}{(1-(1+w)z^2/(1+z)^2)^{r+1}}.$$

Here the contribution from z is in FLS

$$\operatorname*{res}_z \frac{1}{z^{2n+1}}(1+z)^{4n-2r}\frac{1}{(1-(1+w)z^2/(1+z)^2)^{r+1}}.$$

We put $z/(1+z)=u$ or $z=u/(1-u)$ to get

$$\operatorname*{res}_u \frac{1}{u^{2n+1}}\frac{1}{(1-u)^{2n-2r-1}}\frac{1}{(1-(1+w)u^2)^{r+1}}\frac{1}{(1-u)^2}$$

$$=\operatorname*{res}_u \frac{1}{u^{2n+1}}\frac{1}{(1-u)^{2n-2r+1}}\frac{1}{(1-(1+w)u^2)^{r+1}}.$$

At this point we see that we will need contour integration with $|u|=\varepsilon$ and $|w|=\gamma$

$$\frac{1}{(2\pi i)^2}\int_{|u|=\varepsilon}\frac{1}{u^{2n+1}}\frac{1}{(1-u)^{2n-2r+1}}\int_{|w|=\gamma}\frac{(1+w)^r}{w^{r+1}}\frac{1}{(1-(1+w)u^2)^{r+1}}\,dw\,du.$$

For this to hold we need $\varepsilon<1$ and $|(1-u^2)/u^2|>\gamma$. The image of $|u|=\varepsilon$ under $1/u^2-1$ is a circle of radius $1/\varepsilon^2$ centered at minus one making two turns. Therefore if $\varepsilon=1/5$ and $\gamma=3$ the image is completely outside $|w|=\gamma$.

Now summing the outer geometric series we get

$$\frac{1-((1-u)^2(1+w)/w/(1-(1+w)u^2))^{n+1}}{1-(1-u)^2(1+w)/w/(1-(1+w)u^2)}.$$

Multiply by the pending factors in w to obtain

$$\frac{1-((1-u)^2(1+w)/w/(1-(1+w)u^2))^{n+1}}{w(1-(1+w)u^2)-(1+w)(1-u)^2}$$

$$=-\frac{1-((1-u)^2(1+w)/w/(1-(1+w)u^2))^{n+1}}{(1-u(1+w))^2}.$$

From the one constant in front of the numerator the pole at $w=0$ has been canceled and the moving pole at $w=(1-u)/u$ is a circle of radius $1/\varepsilon$ centered at minus one. This too is completely outside $|w|=\gamma$, making for a zero contribution.

This means our double sum is given by

$$\frac{1}{(2\pi i)^2}\int_{|u|=\varepsilon}\frac{1}{u^{2n+1}}\frac{1}{(1-u)^{2n+1}}$$

$$\times\int_{|w|=\gamma}((1-u)^2(1+w)/w/(1-(1+w)u^2))^{n+1}\frac{1}{(1-(1+w)u)^2}\,dw\,du$$

$$=\frac{1}{(2\pi i)^2}\int_{|u|=\varepsilon}\frac{1}{u^{2n+1}}(1-u)$$

$$\times\int_{|w|=\gamma}\frac{(1+w)^{n+1}}{w^{n+1}}\frac{1}{(1-(1+w)u^2)^{n+1}}\frac{1}{(1-(1+w)u)^2}\,dw\,du.$$

Now the pole at $u=1/(1+w)$ has lower bound on the norm $1/4>1/5$ and hence is not in the contour in u. Similarly the poles at $u=\pm 1/\sqrt{1+w}$ are not inside the contour either. This leaves the contribution from $u=0$ which produces two pieces through the Cauchy product:

$$\sum_{q=0}^{n}\binom{n+q}{n}[w^n](1+w)^{n+1+q}[u^{2n-2q}]\frac{1}{(1-(1+w)u)^2}$$

$$=\sum_{q=0}^{n}\binom{n+q}{n}[w^n](1+w)^{n+1+q}(2n-2q+1)(1+w)^{2n-2q}$$

$$=\sum_{q=0}^{n}\binom{n+q}{n}(2n-2q+1)\binom{3n+1-q}{n}$$

as well as

$$\sum_{q=0}^{n}\binom{n+q}{n}(2n-2q)\binom{3n-q}{n}.$$

Evaluation of auxiliary sums

For the first piece,

$$\sum_{q=0}^{n}\binom{2n-q}{n-q}(2q+1)\binom{2n+1+q}{n}$$

$$=[z^n](1+z)^{2n}\sum_{q\geq 0}\frac{z^q}{(1+z)^q}(2q+1)\binom{2n+1+q}{n}$$

$$=[z^n](1+z)^{2n}[w^n](1+w)^{2n+1}\sum_{q\geq 0}\frac{z^q}{(1+z)^q}(2q+1)(1+w)^q$$

$$=[z^n](1+z)^{2n}[w^n](1+w)^{2n+1}\frac{1+z(1+w)/(1+z)}{(1-z(1+w)/(1+z))^2}$$

$$= [z^n](1+z)^{2n+1}[w^n](1+w)^{2n+1}\frac{1+z+z(1+w)}{(1-wz)^2}.$$

Repeat for the second piece,

$$[z^n](1+z)^{2n+1}[w^n](1+w)^{2n}\frac{2z(1+w)}{(1-wz)^2}.$$

Subtract to get

$$[z^n](1+z)^{2n+1}[w^n](1+w)^{2n+1}\frac{1+wz}{(1-wz)^2}.$$

Basic binomial coefficients

To show that this is indeed $Q_n = (2n+1)\binom{2n}{n}^2$ we observe that the extractors will give

$$[w^n](1+w)^{2n+1}\sum_{q=0}^{n}\binom{2n+1}{n-q}[z^q]\frac{1+wz}{(1-wz)^2}$$

$$= [w^n](1+w)^{2n+1}\sum_{q=0}^{n}\binom{2n+1}{n-q}\left[(q+1)w^q + qww^{q-1}\right]$$

$$= \sum_{q=0}^{n}\binom{2n+1}{n-q}^2(2q+1).$$

Introducing $F(n,q)$ we require that

$$Q_n = \sum_{q=0}^{n}\binom{2n+1}{n+1+q}^2(2q+1)\left[\frac{2q+3}{2q+1}\frac{(n-q)^2}{(n+2+q)^2}F(n,q+1) - F(n,q)\right]$$

Here we must have

$$F(n,0) = -\frac{(n+1)^2}{2n+1}$$

as well as for $q < n$

$$F(n,q+1) = (1+F(n,q))\frac{2q+1}{2q+3}\frac{(n+2+q)^2}{(n-q)^2}.$$

Computing the first few values we see that

$$F(n,q) = -\frac{1}{2q+1}\frac{(n+q+1)^2}{2n+1}$$

which is readily verified by substitution. This completes the proof. Here we take the closed form as the value for all q which ensures no singularity from $n = q$.

This identity is from [PS25] by C. Schneider and P. Paule.

3 A truncated sum

We seek to evaluate with $0 \le n-1 \le m$ and $0 \le x < 1/6$ (actually we will do $n \ge 2$)

$$\boxed{\binom{m}{n-1} \sum_{k\ge m+1} \binom{k}{n}^{-1} \sum_{j=0}^{m} \binom{k}{j} x^{k-j}(1-x)^j.}$$

We get from first principles

$$\binom{m}{n-1}[z^m]\frac{1}{1-z} \sum_{k\ge m+1} \binom{k}{n}^{-1} \sum_{j\ge 0} \binom{k}{j} x^{k-j} z^j (1-x)^j$$

$$= \binom{m}{n-1}[z^m]\frac{1}{1-z} \sum_{k\ge m+1} \binom{k}{n}^{-1} (z(1-x)+x)^k.$$

Now recall the OJAC identity [Mar25]

$$\frac{1}{n}\binom{k}{n}^{-1} = [v^k] \log \frac{1}{1-v}(v-1)^{k-n}$$

which yields

$$n\binom{m}{n-1}[z^m]\frac{1}{1-z} \sum_{k\ge m+1} (z(1-x)+x)^k [v^k] \log \frac{1}{1-v}(v-1)^{k-n}.$$

Here we are going to need contour integration and we get without the scalar in front,

$$\frac{1}{(2\pi i)^2} \int_{|z|=\varepsilon} \frac{1}{z^{m+1}} \frac{1}{1-z} \int_{|v|=\gamma} \frac{1}{v} \log \frac{1}{1-v}(v-1)^{-n}$$

$$\times \sum_{k\ge m+1} (z(1-x)+x)^k \frac{(v-1)^k}{v^k}\, dv\, dz.$$

This holds with $\varepsilon, \gamma < 1$ but we also need convergence of the geometric series. Now here $z(1-x)+x$ is on the line segment connecting the point 1 to the point z. For convergence we need $|z(1-x)+x| < |v/(v-1)|$ which holds when $\varepsilon(1-x)+x < \gamma/(1+\gamma)$. We now take $\varepsilon = x/(1-x)$ which is less than one and $\gamma = 3x$, which is also less than one. We get from convergence the requirement that $\varepsilon(1-x)+x = 2x < \frac{3x}{1+3x}$ which holds because $2(1+3x) < 3$ when $x < 1/6$ and we have convergence.

Summing the geometric series yields

$$\frac{(z(1-x)+x)^{m+1}(v-1)^{m+1}/v^{m+1}}{1-(z(1-x)+x)(v-1)/v}.$$

We get two pieces from the simple poles, the residue at infinity in z vanishes.

First piece, residue at $z = 1$

We obtain from this pole, which is outside our contour,

$$\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{1}{v}\log\frac{1}{1-v}(v-1)^{-n}\frac{(v-1)^{m+1}/v^{m+1}}{1-(v-1)/v}\,dv$$

$$=\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{1}{v^{m+1}}\log\frac{1}{1-v}(v-1)^{m+1-n}\,dv=\frac{1}{n-1}\binom{m}{n-1}^{-1}.$$

Multiplying by the scalar in front gives a value of $n/(n-1)$.

Second piece, the other simple pole

We rewrite the geometric term,

$$\frac{(z(1-x)+x)^{m+1}(v-1)^{m+1}/v^m}{v-(z(1-x)+x)(v-1)}$$

$$=\frac{(z(1-x)+x)^{m+1}(v-1)^{m+1}/v^m}{(v-1)(x-1)z-((v-1)(x-1)-1)}.$$

Introducing $f(v)=(v-1)(x-1)$ we obtain

$$\frac{1}{f(v)}\frac{(z(1-x)+x)^{m+1}(v-1)^{m+1}/v^m}{z-(f(v)-1)/f(v)}.$$

The pole at $z=(f(v)-1)/f(v)=1-1/f(v)$ is also outside the contour $|z|=\varepsilon$. To see this note that

$$\frac{vx-v-x}{(v-1)(x-1)}$$

is a Moebius transformation with real coefficients which maps the real axis to itself. Therefore by preservation of symmetry the image of $|v|=\gamma$ is another circle with a diameter on the real axis. The images of the end points of the diameter of the preimage are $(3x^2-4x)/(3x-1)/(x-1)$ or $x(3x-4)/(3x-1)/(x-1)$ and $(-3x^2+3x-x)/(-3x-1)/(x-1)$ or $x(3x-2)/(3x+1)/(x-1)$. Scaling by $\varepsilon=x/(1-x)$ we get $|(4-3x)/(1-3x)|>1$ and $|(2-3x)/(1+3x)|>1$ which means the image circle completely encloses $|z|=\varepsilon$ and the pole is outside the contour. Here we have used the fact that the diameter of the image circle passes through the origin because the end points have opposite signs.

Flipping the sign and substituting,

$$-\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{1}{v}\log\frac{1}{1-v}\frac{f(v)^{m+1}}{(f(v)-1)^{m+1}}\frac{1}{1-(f(v)-1)/f(v)}(v-1)^{-n}$$

$$\times\frac{1}{f(v)}((f(v)-1)(1-x)/f(v)+x)^{m+1}(v-1)^{m+1}/v^m\,dv$$

$$= -\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v}\log\frac{1}{1-v}\frac{f(v)^{m+1}}{(f(v)-1)^{m+1}}(v-1)^{-n}$$

$$\times((f(v)-1)(1-x)/f(v)+x)^{m+1}(v-1)^{m+1}/v^m\ dv$$

$$= -\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v^{m+1}}\log\frac{1}{1-v}\frac{1}{(f(v)-1)^{m+1}}(v-1)^{-n}$$

$$\times((f(v)-1)(1-x)+xf(v))^{m+1}(v-1)^{m+1}\ dv$$

$$= -(x-1)^{m+1}\frac{1}{2\pi i}\int_{|v|=\gamma} \log\frac{1}{1-v}\frac{1}{(f(v)-1)^{m+1}}(v-1)^{m+1-n}\ dv$$

$$= -(x-1)^{m+1}\frac{1}{2\pi i}\int_{|v|=\gamma} \log\frac{1}{1-v}\frac{1}{(vx-x-v)^{m+1}}(v-1)^{m+1-n}\ dv$$

$$= -\frac{1}{2\pi i}\int_{|v|=\gamma} \log\frac{1}{1-v}\frac{1}{(v-x/(x-1))^{m+1}}(v-1)^{m+1-n}\ dv.$$

Here the potential poles are at $v=1$, which does not happen however as $m \geq n-1$. The pole at zero has been canceled. The pole at $x/(x-1)$ has norm $|x|/|1-x|$ which gives the requirement $x/(1-x) < 3x$ which only needs $x < 2/3$ but we have $x < 1/6$. This means the pole is inside the contour and hence constitutes the contribution from the integral.

Now here we require the Leibniz rule which gives

$$-\frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\left[\log\frac{1}{1-v}\right]^{(q)}\left[(v-1)^{m+1-n}\right]^{(m-q)}\Bigg|_{v=x/(x-1)}$$

$$= -\frac{1}{m!}\log\frac{1}{1-x/(x-1)}(x/(x-1)-1)^{-(n-1)}(m+1-n)^{\underline{m}}$$

$$-\frac{1}{m!}\sum_{q=1}^{m}\binom{m}{q}\frac{(q-1)!}{(1-x/(x-1))^q}(x/(x-1)-1)^{m+1-n-(m-q)}(m+1-n)^{\underline{m-q}}.$$

Note that from the logarithmic term taking the mth derivative of $(v-1)^{m-(n-1)}$ gives a zero contribution as implemented in the falling factorial, which leaves just the rational polynomial terms. We get

$$-\sum_{q=1}^{m}\frac{1}{q}\frac{1}{(1-x/(x-1))^q}(x/(x-1)-1)^{q-(n-1)}\binom{m-(n-1)}{m-q}$$

$$= -\sum_{q=1}^{m}\frac{1}{q}(-1)^q(x-1)^q\frac{1}{(x-1)^{q-(n-1)}}\binom{m-(n-1)}{m-q}$$

$$= -(x-1)^{n-1}\sum_{q=1}^{m}\frac{1}{q}(-1)^q\binom{m-(n-1)}{m-q}.$$

Working with the sum term,

$$(-1)^m \sum_{q=0}^{m-1} \frac{1}{m-q} (-1)^q \binom{m-(n-1)}{q}$$

$$= (-1)^m [w^m] \log \frac{1}{1-w} \sum_{q\geq 0} w^q (-1)^q \binom{m-(n-1)}{q}$$

$$= (-1)^m [w^m] \log \frac{1}{1-w} (1-w)^{m-(n-1)} = (-1)^{n-1} \frac{1}{n-1} \binom{m}{n-1}^{-1}.$$

Collecting everything we conclude the argument with the answer

$$\boxed{\frac{n}{n-1}(1-(1-x)^{n-1}).}$$

The case $n = 1$

With $\binom{k}{n}^{-1} = \frac{1}{k}$ we get the integral

$$\frac{1}{(2\pi i)^2} \int_{|z|=\varepsilon} \frac{1}{z^{m+1}} \frac{1}{1-z} \int_{|v|=\gamma} \frac{1}{v} \log \frac{1}{1-v} \sum_{k\geq m+1} (z(1-x)+x)^k \frac{1}{v^k} \, dv \, dz.$$

For this to hold we need $\varepsilon, \gamma < 1$ and for the convergence of the geometric series $|z(1-x)+x| < |v|$ or $\varepsilon(1-x)+x < \gamma$. With the same choice of contour as before we get $2x < \gamma = 3x$ and we have convergence. Summing the geometric series

$$\frac{(z(1-x)+x))^{m+1}/v^{m+1}}{1-(z(1-x)+x)/v}.$$

The integral simplifies to

$$\frac{1}{(2\pi i)^2} \int_{|z|=\varepsilon} \frac{1}{z^{m+1}} \frac{1}{1-z} \int_{|v|=\gamma} \frac{1}{v^{m+1}} \log \frac{1}{1-v} \frac{(z(1-x)+x))^{m+1}}{v-(z(1-x)+x)} \, dv \, dz.$$

We will once more use the fact that residues in z sum to zero and the residue at infinity vanishes by inspection, so we may evaluate with minus the residues from outside the contour. The contribution from $z = 1$ which is outside the contour evaluates to the harmonic number $-H_m$. For the contribution from the other simple pole we write flipping the sign

$$\frac{1}{1-x} \frac{1}{(2\pi i)^2} \int_{|z|=\varepsilon} \frac{1}{z^{m+1}} \frac{1}{1-z} \int_{|v|=\gamma} \frac{1}{v^{m+1}} \log \frac{1}{1-v} \frac{(z(1-x)+x))^{m+1}}{z-(v-x)/(1-x)} \, dv \, dz.$$

The pole at $z = (v-x)/(1-x) = \frac{x}{x-1} + \frac{1}{1-x} v$ is also outside the contour in z. This is a circle centered at $x/(x-1)$ which is negative with radius $\frac{\gamma}{1-x} = \frac{3x}{1-x}$ so

the closest to comes to the origin is $\frac{2x}{1-x} > \frac{x}{1-x} = \varepsilon$. Evaluating the contribution we find

$$\frac{1}{1-x}\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v^{m+1}}\log\frac{1}{1-v} v^{m+1}\frac{(1-x)^{m+1}}{(v-x)^{m+1}}\frac{1-x}{1-v}\,dv$$
$$= \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{1-v}\log\frac{1}{1-v}\frac{(1-x)^{m+1}}{(v-x)^{m+1}}\,dv.$$

The pole at $v=1$ and the branch cut are obviously not inside this contour but the pole at $v=x$ is. We once more require the Leibniz rule

$$\frac{1}{m!}\sum_{q=0}^{m}\binom{m}{q}\left[\log\frac{1}{1-v}\right]^{(q)}\left[\frac{1}{1-v}\right]^{(m-q)}\Bigg|_{v=x}$$
$$= \frac{1}{m!}\log\frac{1}{1-x}\frac{1}{(1-x)^{1+m}}(1)^{\overline{m}}$$
$$+\frac{1}{m!}\sum_{q=1}^{m}\binom{m}{q}\frac{(q-1)!}{(1-x)^q}\frac{1}{(1-x)^{1+m-q}}(1)^{\overline{m-q}}.$$

Multiply by $(1-x)^{m+1}$ to get

$$\log\frac{1}{1-x} + \sum_{q=1}^{m}\frac{1}{q}.$$

Colllecting everything the Harmonic numbers cancel and we are left with just

$$\boxed{\log\frac{1}{1-x}.}$$

This identity is from [PS07] by C. Schneider and P. Paule.

References

[Mar25] Marko Riedel, Markus Scheuer, and Hosam Mahmoud. Inverse binomial coefficients in Egorychev method. *OJAC*, 2025.

[PS07] P. Paule and C. Schneider. Truncating Binomial Series with Symbolic Summation. *INTEGERS. Electronic Journal of Combinatorial Number Theory*, 7(A22):1–9, 2007.

[PS25] P. Paule and C. Schneider. Creative Telescoping for Hypergeometric Double Sums. *J. Symb. Comput.*, 128(102394):1–30, 2025.