Applications of the Mellin-Perron Formula in Number Theory: Addendum Number I: A problem from es.ciencia.matematicas

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1 Introduction

This addendum to my MSc. Thesis treats a problem that was posted on es.ciencia.matematicas on Mar. 11, 2009, by Francisco de León-Sotelo y Esteban. The problem statement goes like this:

> Para n entero positivo tomamos los números n+1,n+2,n+3,...,2n.Si a cada uno de estos números de la lista le calculamos su mayor divisor impar y los sumamos ¿que valor se obtiene? Generalizar.

The answer to this problem can be calculated by elementary means. Let

$$S_n = \sum_{k=1}^{n} q_k,$$

where q_k is the largest odd divisor of k then the problem asks for $S_{2n} - S_n$ and it is not difficult to see that

$$S_{2n} - S_n = n^2.$$

(This is explained in the original thread, which is listed among the external links.)

Now the sum S_n is a prime candidate for Mellin summation, since

$$q_k = \frac{k}{2^{v_2(k)}} \quad \text{where} \quad v_2(k) = \max\left\{ m \,\middle|\, 2^m | k \right\}.$$

(If this is not obvious there is a detailed explanation in the thread.) We will evaluate $S_{2n} - S_n$ by Mellin summation in three steps:

- Find a closed form of the Dirichlet series for q_k.
- Apply the Mellin-Perron formula for simple sums ($m = 1$) to S_n.
- Compute $S_{2n} - S_n$.
- Use generalized Mellin summation to evaluate a remainder integral.

References to my thesis will be provided throughout.

2 Closed form of the Dirichlet series

This computation extends section 2.7.1, page 76 of the thesis.

$$q(s) \quad = \quad \sum_{n\geq 1} \frac{n/2^{v_2(n)}}{n^s} = \sum_{n\geq 1} \frac{2^{-v_2(n)}}{n^{s-1}}$$

$$
\begin{aligned}
&= \sum_{m\geq 1} \frac{2^{-v_2(2m)}}{(2m)^{s-1}} + \sum_{m\geq 0} \frac{2^{-v_2(2m+1)}}{(2m+1)^{s-1}} \\
&= 1/2^s \sum_{m\geq 1} \frac{2^{-v_2(m)}}{m^{s-1}} + \sum_{m\geq 0} \frac{1}{(2m+1)^{s-1}} \\
&= 1/2^s q(s) + \sum_{m\geq 1} \frac{1}{m^{s-1}} - \sum_{m\geq 1} \frac{1}{(2m)^{s-1}} \\
&= \frac{1}{2^s} q(s) + \left(1 - \frac{1}{2^{s-1}}\right) \zeta(s-1)
\end{aligned}
$$

This implies that

$$
\left(1 - \frac{1}{2^s}\right) q(s) = \left(1 - \frac{1}{2^{s-1}}\right) \zeta(s-1) \quad \text{or} \quad (2^s - 1) q(s) = (2^s - 2)\zeta(s-1)
$$

and finally

$$
q(s) = \frac{2^s - 2}{2^s - 1} \zeta(s-1).
$$

3 Application of Mellin-Perron for simple sums

Considering that the poles of $q(s)$ are at $s = 0$, $s = \frac{2\pi i k}{\log 2}$ $(k \in Z \setminus \{0\})$ and $s = 2$, the form of the Mellin-Perron formula to use is the one from section 2.9.2, page 85, namely:

$$
\sum_{1\leq k\leq n} q_k = \frac{1}{2} q_n + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q(s) n^s \frac{ds}{s},
$$

where we take $c = 3$.

Introduce

$$
w(s) = q(s) \frac{n^s}{s} = \frac{2^s - 2}{2^s - 1} \zeta(s-1) \frac{n^s}{s} \quad \text{and} \quad \rho_k = \frac{2\pi i k}{\log 2} \quad (k \in Z \setminus \{0\}).
$$

Shifting the contour to the left, we obtain

$$
\frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} q(s) n^s \frac{ds}{s} = \mathrm{Res}\,[w(s);\ s = 2] + \mathrm{Res}\,[w(s);\ s = 0] + \sum_{k\in Z\setminus\{0\}} \mathrm{Res}\,[w(s);\ s = \rho_k] + \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} q(s) n^s \frac{ds}{s}.
$$

The values of the residues are

$$
\begin{aligned}
\mathrm{Res}\,[w(s);\ s = 2] &= \frac{n^2}{3} \\
\mathrm{Res}\,[w(s);\ s = 0] &= -\frac{1}{8} + \frac{1}{12} \log_2(n) - \frac{\zeta'(1)}{\log 2} \\
\mathrm{Res}\,[w(s);\ s = \rho_k] &= -\frac{1}{\log 2} \zeta(\rho_k - 1) \frac{n^{\rho_k}}{\rho_k}.
\end{aligned}
$$

This yields the following expression for S_n:

$$
S_n = \frac{1}{2} q_n + \frac{n^2}{3} - \frac{1}{8} + \frac{1}{12} \log_2(n) - \frac{\zeta'(1)}{\log 2} - \sum_{k\in Z\setminus\{0\}} \frac{1}{\log 2} \zeta(\rho_k - 1) \frac{n^{\rho_k}}{\rho_k} + \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} q(s) n^s \frac{ds}{s}.
$$

Note that the sum in ρ_k is a Fourier series in $\log_2 n$, because $n^{\rho_k} = e^{\log_2 n\, 2\pi i k}$.

4 Computation of $S_{2n} - S_n$

By simple substitution, we find

$$\begin{aligned} S_{2n} - S_n &= \frac{1}{2}q_{2n} - \frac{1}{2}q_n + \frac{4n^2}{3} - \frac{n^2}{3} + \frac{1}{12}\log_2(2n) - \frac{1}{12}\log_2(n) \\ &- \sum_{k \in Z\setminus\{0\}} \frac{1}{\log 2}\zeta(\rho_k - 1)\frac{(2n)^{\rho_k} - n^{\rho_k}}{\rho_k} + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} q(s)\left((2n)^s - n^s\right)\frac{ds}{s}. \end{aligned}$$

Now we have $q_{2n} = q_n$, $\frac{1}{12}\log_2(2n) - \frac{1}{12}\log_2(n) = \frac{1}{12}$ and $(2n)^{\rho_k} - n^{\rho_k} = 2^{\rho_k} n^{\rho_k} - n^{\rho_k} = 0$, so this simplifies to

$$S_{2n} - S_n = n^2 + \frac{1}{12} + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} q(s)\left((2n)^s - n^s\right)\frac{ds}{s}.$$

We are done if we can show that the integral has the value $-\frac{1}{12}$.

5 Generalized Mellin summation applied to the remainder integral

First note that

$$q(s)\left((2n)^s - n^s\right) = q(s)(2^s - 1)n^s = (2^s - 2)\zeta(s-1)n^s.$$

Let $\omega(s) = (2^s - 2)\zeta(s-1)\frac{n^s}{s}$. Shifting the integral to the right, we find that

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} (2^s - 2)\zeta(s-1)n^s\frac{ds}{s} = -\mathrm{Res}\left[\omega(s);\, s=0\right] - \mathrm{Res}\left[\omega(s);\, s=2\right] + \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} (2^s - 2)\zeta(s-1)n^s\frac{ds}{s}.$$

The values of the residues are

$$\begin{aligned} \mathrm{Res}\left[w(s);\, s=0\right] &= \frac{1}{12} \\ \mathrm{Res}\left[w(s);\, s=2\right] &= n^2, \end{aligned}$$

so we have

$$-\frac{1}{12} - n^2 + \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} (2^s - 2)\zeta(s-1)n^s\frac{ds}{s}$$

for the remainder integral.

The integral that has appeared can be evaluated using generalized Mellin summation (harmonic sums) as described in section 2.10, page 87 of the thesis. We will show that the integral is an inverse Mellin transform and equal to a simple sum that we can compute. We take $F(x) = \sum_k \lambda_k f(\mu_k x)$ and choose the parameters such that the inverse Mellin transform of the Mellin transform of $F(x)$ is given by the integral that we want to evaluate, and hence equal to $F(x)$.

Take $x = \frac{1}{nr}$, with $n, r \in \mathbb{N}$. Furthermore take $\mu_k = k$ and $\lambda_k = k$, $k \in N$. Finally, take $f(x) = H_0(x)$, with H_0 being the Heaviside step function that is one on $[0,1)$ and zero elsewhere.

We thus have

$$F(x) = \sum_{k\in\mathbb{N}} kH_0(kx) \quad \text{and} \quad F(x)|_{x=\frac{1}{nr}} = \sum_{k\in\mathbb{N}} kH_0\left(k\frac{1}{nr}\right) = \sum_{\substack{k\in\mathbb{N} \\ 0 \le \frac{k}{nr} < 1}} k = \sum_{k=1}^{nr-1} k = \frac{1}{2}(nr-1)nr.$$

and the Mellin transform of $F(x)$ is

$$F^*(s) = \left(\sum_{k\in\mathbb{N}} \frac{k}{k^s}\right) f^*(s) = \zeta(s-1)\frac{1}{s}.$$

Applying Mellin inversion, we find

$$\frac{1}{2}(nr-1)n = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} r^{s-1}\zeta(s-1)\frac{n^s}{s}ds.$$

This means that

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} 2(2^{s-1}-1)\zeta(s-1)n^s\frac{ds}{s} = 2\left(\frac{1}{2}(2n-1)n - \frac{1}{2}(n-1)n\right) = n^2.$$

For the remainder integral, this yields

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}(2^s-2)\zeta(s-1)n^s\frac{ds}{s} = -\frac{1}{12} - n^2 + \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}(2^s-2)\zeta(s-1)n^s\frac{ds}{s} = -\frac{1}{12} - n^2 + n^2 = -\frac{1}{12},$$

as desired.

The conclusion is that indeed

$$S_{2n} - S_n = n^2,$$

which we set out to prove.

6 Concluding remark (omitting the Fourier series)

It is worth noting that this computation can be simplified quite a bit if we drop the Fourier series, i.e. only shift the first integral to $c = \frac{1}{2}$ instead of $c = -\frac{1}{2}$. Then there is only one pole to take into account, the one at $s = 2$. This gives

$$S_n = \frac{1}{2}q_n + \frac{n^2}{3} + \frac{1}{2\pi i}\int_{+\frac{1}{2}-i\infty}^{+\frac{1}{2}+i\infty} q(s)n^s\frac{ds}{s}.$$

Computing the difference, we have

$$S_{2n} - S_n = n^2 + \frac{1}{2\pi i}\int_{+\frac{1}{2}-i\infty}^{+\frac{1}{2}+i\infty} q(s)\left((2n)^s - n^s\right)\frac{ds}{s} = n^2 + \frac{1}{2\pi i}\int_{+\frac{1}{2}-i\infty}^{+\frac{1}{2}+i\infty}(2^s-2)\zeta(s-1)n^s\frac{ds}{s}.$$

Shifting this last integral to $c = 3$, we find

$$S_{2n} - S_n = n^2 - n^2 + \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}(2^s-2)\zeta(s-1)n^s\frac{ds}{s} = n^2 - n^2 + n^2 = n^2.$$

7 External links

- Marko Riedel

  http://pnp.mathematik.uni-stuttgart.de/iadm/Riedel/index.html Applications of the Mellin-Perron Formula in Number Theory.

- Francisco de León-Sotelo y Esteban

  http://groups.google.com/group/es.ciencia.matematicas/browse_thread/thread/a49404f68e6bda64# Gran Non Divisor.