Gr-catégories

Summary

The purpose of these notes is to study the Gr-categories and give some applications of them. Below is a brief description of the organization of the work.

Chapter I gives some definitions and results, which are used continually in the sequel, on ⊗-categories one can find in [2], [6], [11], [14], [15], the terminology employed in this chapter being of Neantro Saavedra Rivano [14]. A <u>⊗-category</u> is a category $\underline{C}$ together with a <u>law ⊗</u>, i.e. a covariant bifunctor

$$\begin{aligned} \otimes : \underline{C} \times \underline{C} &\longrightarrow \underline{C} \\ (X, Y) &\longmapsto X \otimes Y \end{aligned}$$

An <u>associativity constraint</u> for a ⊗-category $\underline{C}$ is an isomorphism of trifunctors

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z \qquad X, Y, Z \in \mathrm{Ob}\,\underline{C}$$

satisfying the <u>pentagon axiom</u>, i.e all the pentagonal diagrams

$$\begin{tikzcd}
 & (X\otimes Y)\otimes(Z\otimes T) \arrow[dr, "a_{X\otimes Y, Z, T}"] & \\
X\otimes(Y\otimes(Z\otimes T)) \arrow[ur, "a_{X,Y,Z\otimes T}"] \arrow[d, "id_X \otimes a_{Y,Z,T}"'] & & ((X\otimes Y)\otimes Z)\otimes T \\
X\otimes((Y\otimes Z)\otimes T) \arrow[rr, "a_{X, Y\otimes Z, T}"] & & (X\otimes(Y\otimes Z))\otimes T \arrow[u, "a_{X,Y,Z}\otimes id_T"']
\end{tikzcd}$$

are commutative. A ⊗-category together with an associativity constraint is called a <u>⊗-associativite category</u>.

A <u>commutativity constraint</u> for a ⊗-category $\underline{C}$ is an isomorphism

of bifunctors

$$c_{X,Y} : X\otimes Y \xrightarrow{\sim} Y\otimes X \quad , \quad X,Y\in \mathrm{Ob}\,\underline{C} .$$

verifying the relation

$$c_{Y,X}\circ c_{X,Y} = id_{X\otimes Y}$$

The commutativity constraint c is said to be <u>strict</u> if $c_{X,X} = id_{X\otimes X}$ for all $X\in \mathrm{Ob}\,\underline{C}$. A $\otimes$-category together with a commutativity constraint is a <u>$\otimes$-commutative category</u>. A $\otimes$-commutative category is <u>strict</u> if its commutativity constraint is strict.

An <u>unity constraint</u> for a $\otimes$-category $\underline{C}$ is a triple $(\underline{1}, g, d)$ where $\underline{1}$ is an object of $\underline{C}$, g and d natural isomorphisms

$$g_X : X \xrightarrow{\sim} \underline{1}\otimes X \quad , \quad d_X : X \xrightarrow{\sim} X\otimes \underline{1} \quad , \quad X\in \mathrm{Ob}\,\underline{C}$$

such that $g_{\underline{1}} = d_{\underline{1}}$. A $\otimes$-category together with an unity constraint is a <u>$\otimes$-unifer category</u>.

A $\otimes$-category $\underline{C}$ together with an associativity constraint a and a commutativity constraint c is a <u>$\otimes$-AC category</u> if the <u>hexagon axiom</u> is fulfilled, i.e all the hexagonal diagrams commute

$$\begin{tikzcd}
 & (X\otimes Y)\otimes Z \arrow[r, "c_{X\otimes Y,Z}"] & Z\otimes (X\otimes Y) \arrow[dr, "a_{Z,X,Y}"] & \\
X\otimes (Y\otimes Z) \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "id_X\otimes c_{Y,Z}"'] & & & (Z\otimes X)\otimes Y \\
 & X\otimes (Z\otimes Y) \arrow[r, "a_{X,Z,Y}"] & (X\otimes Z)\otimes Y \arrow[ur, "c_{X,Z}\otimes id_Y"'] &
\end{tikzcd}$$

A $\otimes$-category $\underline{C}$ together with an associativity constraint a and an unity constraint $(\underline{1}, g, d)$ is a <u>$\otimes$-AU category</u> if all the following triangles commute

$$\begin{tikzcd}
X\otimes(\underline{1}\otimes Y) \arrow[rr, "a_{X,\underline{1},Y}"] & & (X\otimes \underline{1})\otimes Y \\
 & X\otimes Y \arrow[ul, "id_X\otimes g_Y"] \arrow[ur, "d_X\otimes id_Y"'] &
\end{tikzcd}$$

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A <u>$\otimes$- ACU category</u> is a $\otimes$- AC and AU category. An object X of a $\otimes$- ACU category $\underline{C}$ is <u>invertible</u> if there are two objects $X', X'' \in \mathrm{Ob}\,\underline{C}$ such that $X' \otimes X \simeq X \otimes X'' \simeq \underline{1}$.

A <u>$\otimes$ - functor</u> from a $\otimes$- category $\underline{C}$ to a $\otimes$- category $\underline{C}'$ is a pair $(F, \check{F})$ where F is a functor $\underline{C} \to \underline{C}'$ and $\check{F}$ an isomorphism of bifunctors

$$\check{F}_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y) \qquad X, Y \in \mathrm{Ob}\,\underline{C}$$

A $\otimes$- functor $(F, \check{F})$ from a $\otimes$- associative category $\underline{C}$ to a $\otimes$- associative category $\underline{C}'$ is <u>associative</u> if the following diagram commutes:

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "id \otimes \check{F}"] \arrow[d, "a'"'] & FX \otimes F(Y \otimes Z) \arrow[r, "\check{F}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "Fa"] \\
(FX \otimes FY) \otimes FZ \arrow[r, "\check{F} \otimes id"] & F(X \otimes Y) \otimes FZ \arrow[r, "\check{F}"] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

where a is the associativity constraint of $\underline{C}$ and a' of $\underline{C}'$.

A $\otimes$- functor $(F, \check{F})$ from a $\otimes$- commutative category $\underline{C}$ to a $\otimes$- commutative category $\underline{C}'$ is <u>commutative</u> if the following diagram commutes

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\check{F}"] \arrow[d, "c'"'] & F(X \otimes Y) \arrow[d, "Fc"] \\
FY \otimes FX \arrow[r, "\check{F}"] & F(Y \otimes X)
\end{tikzcd}$$

c and c' being the commutativity constraints of $\underline{C}$ and $\underline{C}'$ respectively.

A $\otimes$- functor $(F, \check{F})$ from a $\otimes$- category $\underline{C}$ with an unity constraint $(\underline{1}', g', d')$ is a $(\underline{1}, g, d)$ to a $\otimes$- category $\underline{C}'$ with an unity constraint $(\underline{1}', g', d')$ is a <u>$\otimes$- unifer functor</u> if there exists an isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$ such that the following diagrams commute:

$$\begin{tikzcd}
\underline{1}' \otimes FX \arrow[r, "\hat{F} \otimes id_{FX}"] & F\underline{1} \otimes FX \arrow[d, "\check{F}"] \\
FX \arrow[u, "g'_{FX}"] \arrow[r, "Fg_X"'] & F(\underline{1} \otimes X)
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes \underline{1}' \arrow[r, "id_{FX} \otimes \hat{F}"] & FX \otimes F\underline{1} \arrow[d, "\check{F}"] \\
FX \arrow[u, "d'_{FX}"] \arrow[r, "Fd_X"'] & F(X \otimes \underline{1})
\end{tikzcd}$$

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It follows from the definition that the isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$, if it exists, is unique.

A *⊗- AC functor* is an ⊗- associative and commutative functor. A *⊗- ACU functor* is a ⊗- associative, commutative and unifer functor.

Let $(F, \breve{F})$ and $(G, \breve{G})$ be ⊗- functors from a ⊗- category $\underline{C}$ to a ⊗- category $\underline{C}'$. A *⊗- morphism* from the ⊗- functor $(F, \breve{F})$ to the ⊗- functor $(G, \breve{G})$ is a morphism of functors $\lambda : F \to G$ such that the following diagram commutes

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{\breve{F}} & F(X \otimes Y) \\ \lambda_X \otimes \lambda_Y \big\downarrow & & \big\downarrow \lambda_{X \otimes Y} \\ GX \otimes GY & \xrightarrow{\breve{G}} & G(X \otimes Y) \end{array} \qquad X, Y \in \mathrm{Ob}\, \underline{C}$$

Chapter II is a study of Gr- categories and Pic- categories. A *Gr- category* is a ⊗- AU category, the objects of which are all invertible, and the base category a groupoïd (i.e. all arrows are isomorphisms). Thus a Gr- category is like a group. We obtain from this definition that if $\underline{P}$ is a Gr- category, the set $\Pi_0(\underline{P})$ of the classes up to isomorphism of objects of $\underline{P}$, together with the operation induced by the law ⊗ of $\underline{P}$, is a group; the group $\mathrm{Aut}(\underline{1}) = \Pi_1(\underline{P})$ is a commutative group; and for all $X \in \mathrm{Ob}\, \underline{P}$

$$\gamma_X : u \longmapsto u \otimes id_X = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}\,(X)$$

$$\delta_X : u \longmapsto id_X \otimes u = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}\,(X)$$

We attribute thus to a Gr- category $\underline{P}$ two groups $\Pi_0(\underline{P})$ and $\Pi_1(\underline{P})$ when $\Pi_1(\underline{P})$ is commutative. Furthermore we can define an action of $\Pi_0(\underline{P})$ on $\Pi_1(\underline{P})$ by the formula

$$su = \delta_X^{-1}\, \gamma_X\, (u)$$

for $s \in \Pi_0(\underline{P})$ represented by X and $u \in \Pi_1(\underline{P})$. The commutative group $\Pi_1(\underline{P})$ together with this action is a left $\Pi_0(\underline{P})$- module.

Let M be a group, N a left M- module. A *preeping lager* of

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of type (M,N) for a Gr-category $\underline{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \Pi_0(\underline{P}) \quad , \quad \varepsilon_1 : N \xrightarrow{\sim} \Pi_1(\underline{P}).$$

compatible with the action of M on N, $\Pi_0(\underline{P})$ on $\Pi_1(\underline{P})$. A Gr-category <u>prepingled</u> of type (M,N) is a Gr-category $\underline{P}$ together with a prepinglage. Finally, an <u>arrow</u> of Gr-categories prepingled of type (M,N) $(\underline{P}, \varepsilon) \to (\underline{P}', \varepsilon')$ is a $\otimes$-associative functor such that the following triangles commute:

$$\begin{tikzcd} \Pi_0(\underline{P}) \arrow[rr] & & \Pi_0(\underline{P}') \\ & M \arrow[ul, "\varepsilon_0"] \arrow[ur, "\varepsilon'_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \Pi_1(\underline{P}) \arrow[rr] & & \Pi_1(\underline{P}') \\ & N \arrow[ul, "\varepsilon_1"] \arrow[ur, "\varepsilon'_1"'] & \end{tikzcd}$$

It follows from this definition that a such arrow is a $\otimes$-equivalence. Thus the set of the equivalence classes of Gr-categories prepingled of type (M,N) is equal to the set of the connected components of the category of Gr-categories prepingled of type (M,N).

If we consider the cohomology group $H^3(M,N)$ of the group M with coefficients N (in the sense of the group cohomology [12]) we obtain a canonical bijection between the set $H^3(M,N)$ and the set of the equivalence classes of Gr-categories prepingled of type (M,N).

A <u>Pic-category</u> is a Gr-category together with a commutativity constraint which is compatible with its associativity constraint, i.e the hexagon axiom is satisfied. Thus a Pic-category is like a commutative group. We verify immediately that a necessary condition for the existence of a Pic-category structure on a Gr-category is that $\Pi_0(\underline{P})$ must be commutative and act trivially on $\Pi_1(\underline{P})$. A Pic-category is <u>strict</u> if its commutativity constraint is strict.

Let M, N be abelian groups. A <u>prepinglage</u> of type (M,N) for a Pic-category $\underline{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \Pi_0(\underline{P}) \quad , \quad \varepsilon_1 : N \xrightarrow{\sim} \Pi_1(\underline{P})$$

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A Pic-category preepingled of type (M,N) is a Pic-category together with a preepinglage. We define the arrow of such objects in the same way as for Gr-categories.

For next propositions, let us consider two complexes of free abelian groups

$$L.(M) : L_3(M) \xrightarrow{d_3} L_2(M) \xrightarrow{d_2} L_1(M) \xrightarrow{d_1} L_0(M) \longrightarrow M$$

$$'L.(M) : {}'L_3(M) \xrightarrow{'d_3} {}'L_2(M) \xrightarrow{'d_2} {}'L_1(M) \xrightarrow{'d_1} {}'L_0(M) \longrightarrow M$$

where

$$L_0(M) = {}'L_0(M) = \mathbb{Z}[M]$$

$$L_1(M) = {}'L_1(M) = \mathbb{Z}[M \times M]$$

$$L_2(M) = {}'L_2(M) = \mathbb{Z}[M \times M \times M] + \mathbb{Z}[M \times M]$$

$$L_3(M) = {}'L_3(M) + \mathbb{Z}[M]$$

$$'L_3(M) = \mathbb{Z}[M \times M \times M \times M] + \mathbb{Z}[M \times M \times M] + \mathbb{Z}[M \times M]$$

$$d_1[x,y] = {}'d_1[x,y] = [y] - [x+y] + [x]$$

$$d_2[x,y] = {}'d_2[x,y] = [x,y] - [y,x]$$

$$d_2[x,y,z] = {}'d_2[x,y,z] = [y,z] - [x+y,z] + [x,y+z] - [x,y]$$

$$d_3[x,y,z,t] = {}'d_3[x,y,z,t] = [y,z,t] - [x+y,z,t] + [x,y+z,t] - [x,y,z+t] + [x,y,z]$$

$$d_3[x,y,z] = {}'d_3[x,y,z] = [x,y,z] - [x,z,y] + [z,x,y] - [y,z] + [x+y,z] - [x,z].$$

$$d_3[x,y] = [x,y] + [y,x] = {}'d_3[x,y]$$

$$d_3[x] = [x,x],$$

so that $L.(M)$ is a truncated resolution of M. One obtains a canonical bijection between the set of the equivalence classes of

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Pic-categories prepingled of type (M,N) and the set $H^2(\mathrm{Hom}({}'L.(M),N))$. The exactitude of the complex $L.(M)$ gives us the triviality of the classification of Pic-categories prepingled of type (M,N) which are strict, i.e all Pic-categories prepingled of type (M,N) which are strict, are equivalent. Determine $H^2(\mathrm{Hom}({}'L.(M),N))$ i.e. the

Finally chapter III gives us the construction of the solution of two universal problems : problem of making objects "unity objects" and problem of reversing objects.

Let $\underline{A}$ be a $\otimes$-AC category, $\underline{A}'$ another $\otimes$-AC category whose base category is a groupoid, and $(T,\check{T}) : \underline{A}' \to \underline{A}$ a $\otimes$-AC functor. We try to make the objects TA' of $\underline{A}$, $A' \in \mathrm{Ob}\,\underline{A}'$, "unity object", i.e we try to get :

A with unit?

1° A $\otimes$-ACU category $\underline{P}$

2° A $\otimes$-AC functor $(D,\check{D}) : \underline{A} \to \underline{P}$

3° A $\otimes$-isomorphism

$$\lambda : (D,\check{D}) \circ (T,\check{T}) \xrightarrow{\sim} (\underline{I}_P, \check{\underline{I}}_P).$$

where $(\underline{I}_P, \check{\underline{I}}_P)$ is the $\otimes$-constant functor 1_P from $\underline{A}'$ to $\underline{P}$. The triple $(\underline{P}, (D,\check{D}), \lambda)$ must be universal for triples $(\underline{Q}, (E,\check{E}), \mu)$ satisfying 1°, 2°, 3°.

For the description of the triple $(\underline{P}, (D,\check{D}), \lambda)$, we introduce a quotient category of a $\otimes$-AC category as follows :

Let $\underline{A}$ be a $\otimes$-AC category, $\mathcal{S}$ a multiplicative subset of $\underline{A}$ (that means a subset of the set of all ~~endomorphisms~~ arrows of $\underline{A}$ such that $id_X \in \mathcal{S}$ for all $X \in \mathrm{Ob}\,\underline{A}$ and the tensor product of two arrows of $\mathcal{S}$ belongs to $\mathcal{S}$). The $\otimes$-AC category quotient $A^{\mathcal{S}}$ of $\underline{A}$ with respect to $\mathcal{S}$ is the solution of the universal problem

what about composition?

$$(K,\check{K}) : \underline{A} \longrightarrow \underline{B} \quad , \quad K(u) = id \text{ for all } u \in \mathcal{S}$$

where B is a $\otimes$-AC category and $(K,\check{K})$ a $\otimes$-AC functor.

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Now let us give an idea of the construction of the triple $(\underline{P}, (D, \check{D}), \lambda)$ for $\underline{A}' \neq \phi$:

1° $\mathrm{Ob}\,\underline{P} = \mathrm{Ob}\,\underline{A}$

2° $\mathrm{Hom}_{\underline{P}}(A, B) = \Phi(A, B)/\mathcal{R}_{A,B}$, $A, B \in \mathrm{Ob}\,\underline{P}$

$\Phi(A, B)$ being the set of all triples (A', B', u) where $A', B' \in \mathrm{Ob}\,\underline{A}'$, $u \in \mathrm{Fl}\,\underline{A}$ $u : A \otimes TA' \longrightarrow B \otimes TB'$; $\mathcal{R}_{A,B}$ the equivalence relation defined in $\Phi(A,B)$ as follows

$$(A'_1, B'_1, u)\ \mathcal{R}_{A,B}\ (A'_2, B'_2, u)$$

if and only if there are objects C'_1, C'_2 and isomorphisms

$$u' : A'_1 \otimes C'_1 \xrightarrow{\sim} A'_2 \otimes C'_2 \ , \quad v' : B'_1 \otimes C'_1 \xrightarrow{\sim} B'_2 \otimes C'_2$$

of $\underline{A}'$ such that the following diagram commutes in $\underline{A}^{S}$, $\otimes$-AC quotient category of $\underline{A}$ with respect to the multiplicative subset of $\underline{A}$ generated by the endomorphisms of the form $T(c_{A', A'})$;

$$\begin{tikzcd}
A \otimes T(A'_1 \otimes C'_1) \arrow[r, "id \otimes \check{T}^{-1}"] \arrow[d, "id \otimes Tu'"] & A \otimes (TA'_1 \otimes TC'_1) \arrow[r, "a"] & (A \otimes TA'_1) \otimes TC'_1 \arrow[r, "u_1 \otimes id"] & (B \otimes TB'_1) \otimes TC'_1 \arrow[d, "a^{-1}"'] \\
A \otimes T(A'_2 \otimes C'_2) \arrow[d, "id \otimes \check{T}^{-1}"] & & & B \otimes (TB'_1 \otimes TC'_1) \arrow[d, "id \otimes \check{T}"'] \\
A \otimes (TA'_2 \otimes TC'_2) \arrow[d, "a"] & & & B \otimes T(B'_1 \otimes C'_1) \arrow[d, "id \otimes Tv'"'] \\
(A \otimes TA'_2) \otimes TC'_2 \arrow[r, "u_2 \otimes id"] & (B \otimes TB'_2) \otimes TC'_2 \arrow[r, "a^{-1}"] & B \otimes (TB'_2 \otimes TC'_2) \arrow[r, "id \otimes \check{T}^{-1}"] & B \otimes T(B'_2 \otimes C'_2)
\end{tikzcd}$$

We denote by $[A', B', u]$ the class which has (A', B', u) as representative

3° Composition of arrows in $\underline{P}$. Let $[A', B', u] : A \to B$, $[B'', C'', v] : B \to C$ be arrows in $\underline{P}$. We define

$$[B'', C'', v] \circ [A', B', u] = [A' \otimes B'', B' \otimes C'', w] : A \longrightarrow C$$

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where ω is such that the following diagram commutes :

$$\begin{tikzcd}
A\otimes T(A'\otimes B'') \arrow[r, "id\otimes \check{T}^{-1}"] \arrow[dddddd, "\omega"] & A\otimes(TA'\otimes TB'') \arrow[r, "a"] & (A\otimes TA')\otimes TB'' \arrow[r, "u\otimes id"] & (B\otimes TB')\otimes TB'' \arrow[d, "a^{-1}"] \\
 & & & B\otimes(TB'\otimes TB'') \arrow[d, "id\otimes c"] \\
 & & & B\otimes(TB''\otimes TB') \arrow[d, "a"] \\
 & & & (B\otimes TB'')\otimes TB' \arrow[d, "v\otimes id"] \\
 & & & (C\otimes TC'')\otimes TB' \arrow[dd, phantom] \\
 & & & \\
C\otimes T(B'\otimes C'') & C\otimes(TB'\otimes TC'') \arrow[l, "id\otimes \check{T}"'] & C\otimes(TC''\otimes TB') \arrow[l, "id\otimes c"'] & (C\otimes TC'')\otimes TB' \arrow[l, "a^{-1}"']
\end{tikzcd}$$

4° $\otimes$- Structure on $\underline{P}$

$$A\otimes E \ (\text{in } \underline{P}) = A\otimes E \ (\text{in } \underline{A})$$

$$[A', B', u]\otimes[E', F', v] = [A'\otimes E', B'\otimes F', w]$$

where w is defined by the commutative diagram (1).

5° ACU constraint in $\underline{P}$

$([A', A', a\otimes id], [A', A', c\otimes id], (\underline{1}_{P} = TA'_0, g_A = [A'_0\otimes A', A', t_A], d_A = [A'_0\otimes A', A', r_A]))$

where A'_0 is a fixed object of $\underline{A'}$, A' an arbitrary object of $\underline{A'}$, g_A and d_A natural isomorphisms

$$g_A : A \longrightarrow \underline{1}_{P}\otimes A \quad , \quad d_A : A \longrightarrow A\otimes \underline{1}_{P}$$

with t_A and r_A defined by the commutative diagrams (2)

6° $(D, \check{D})$ is defined by

$$DA = A \quad , \quad Du = [A', A', u\otimes id_{TA'}] \ , \quad \check{D}_{A,B} = id_{A\otimes B}$$

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(1)
$$\begin{tikzcd}
(A\otimes TA')\otimes(E\otimes TE') \arrow[r, "u\otimes v"] \arrow[d, "a"] & (B\otimes TB')\otimes(F\otimes TF') \arrow[d, "a"] \\
((A\otimes TA')\otimes E)\otimes TE' \arrow[d, "a^{-1}\otimes id"] & ((B\otimes TB')\otimes F)\otimes TF' \arrow[d, "a^{-1}\otimes id"] \\
(A\otimes(TA'\otimes E))\otimes TE' \arrow[d, "(id\otimes c)\otimes id"] & (B\otimes(TB'\otimes F))\otimes TF' \arrow[d, "(id\otimes c)\otimes id"] \\
(A\otimes(E\otimes TA'))\otimes TE' \arrow[d, "a\otimes id"] & (B\otimes(F\otimes TB'))\otimes TF' \arrow[d, "a\otimes id"] \\
((A\otimes E)\otimes TA')\otimes TE' \arrow[d, "a^{-1}"] & ((B\otimes F)\otimes TB')\otimes TF' \arrow[d, "a^{-1}"] \\
(A\otimes E)\otimes(TA'\otimes TE') \arrow[d, "id\otimes\check{T}"] & (B\otimes F)\otimes(TB'\otimes TF') \arrow[d, "id\otimes\check{T}"] \\
(A\otimes E)\otimes T(A'\otimes E') \arrow[r, "W"] & (B\otimes F)\otimes T(B'\otimes F')
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
A\otimes(TA'_0\otimes TA') \arrow[r, "id\otimes\check{T}"] \arrow[d, "a"] & A\otimes T(A'_0\otimes A') \arrow[d, "t_A"] \\
(A\otimes TA'_0)\otimes TA' \arrow[r, "c\otimes id"] & (TA'_0\otimes A)\otimes TA'
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes(TA'_0\otimes TA') \arrow[r, "id\otimes\check{T}"] \arrow[d, "a"] & A\otimes T(A'_0\otimes A') \arrow[d, "r_A"] \\
(A\otimes TA'_0)\otimes TA' \arrow[r, equal] & (A\otimes TA'_0)\otimes TA'
\end{tikzcd}$$

7° The $\otimes$-isomorphism

$$\lambda : (D,\check{D})\circ(T,\check{T}) \simeq (I_{\underline{P}}, \check{I}_{\underline{P}})$$

is defined by natural isomorphisms

$$DTA' = TA' \xrightarrow{\lambda_{A'} = [A'_0,\, A';\, c_{TA',\,TA'_0}]} I_{\underline{P}}\, A' = TA'_0 \qquad A'\in \mathrm{ob}\,\underline{A}'$$

? $\underline{P}$ is called the $\otimes$-ACU category of the $\otimes$-AC category $\underline{A}$ with respect to $(A', (T,\check{T}))$. examples?

For the problem of reversing objects, let us consider a $\otimes$-category

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$\underline{C}$ with a ACU constraint $(a, c, (\underline{1}, g, d))$, a $\otimes$-category $\underline{C}'$ with a ACU constraint $(a', c', (\underline{1}', g', d'))$, the base category of which is a groupoid, and a $\otimes$-ACU functor $(F, \breve{F}) : \underline{C}' \longrightarrow \underline{C}$. We try to find a $\otimes$-ACU category $\underline{P}$ and a $\otimes$-ACU functor $(\mathcal{D}, \breve{\mathcal{D}}) : \underline{C} \longrightarrow \underline{P}$ having the following properties :

1° $\mathcal{D}FX'$ is invertible in $\underline{P}$ for all $X' \in \mathrm{Ob}\,\underline{C}'$.

2° For all $\otimes$-ACU functor $(\mathcal{Y}, \breve{\mathcal{Y}})$ from $\underline{C}$ to a $\otimes$-ACU category $\underline{Q}$ such that $\mathcal{Y}FX'$ is invertible in $\underline{Q}$ for all $X' \in \mathrm{Ob}\,\underline{C}'$, there exists a $\otimes$-ACU functor $(E', \breve{E}')$, unique up to $\otimes$-isomorphism, from $\underline{P}$ to $\underline{Q}$ such that $(\mathcal{Y}, \breve{\mathcal{Y}}) \simeq (E', \breve{E}') \circ (\mathcal{D}, \breve{\mathcal{D}})$.

give the def of equivalence of $\otimes$ categories

This problem is reduced by the first by putting $\underline{A}' = \underline{C}'$, $\underline{A} = \underline{C} \times \underline{C}'$, $TX' = (FX', X')$ and by remarking that if $\underline{C}$, $\underline{C}'$, $\underline{Q}$ are $\otimes$-ACU categories, $\underline{\mathrm{Hom}}^{\otimes, ACU}(\underline{C}, \underline{Q})$ the category of all $\otimes$-ACU functor from $\underline{C}$ to $\underline{Q}$, then there is a canonical equivalence of categories

$$\underline{\mathrm{Hom}}^{\otimes, ACU}(\underline{C} \times \underline{C}', \underline{Q}) \longrightarrow \underline{\mathrm{Hom}}^{\otimes, ACU}(\underline{C}, \underline{Q}) \times \underline{\mathrm{Hom}}^{\otimes, ACU}(\underline{C}', \underline{Q})$$

The $\otimes$-ACU category $\underline{P}$ thus defined is called the <u>$\otimes$-category of fractions</u> of the category $\underline{C}$ with respect to $(\underline{C}', (F, \breve{F}))$. The $\otimes$-category of fractions of $\underline{C}^{is}$ with respect to $(\underline{C}^{is}, (id_{\underline{C}^{is}}, \breve{id}))$ is a Pic-category which is called the <u>Pic-envelope</u> of the category $\underline{C}$, and denoted by $\mathrm{Pic}(\underline{C})$.

For an application of the Pic-envelope, we take $\underline{C} = \mathcal{P}(R)$, category of all finitely generated R-modules (R a ring) and $\underline{P} = \mathrm{Pic}(\mathcal{P}(R))$, then one obtains

$$\pi_0(\underline{P}) \simeq K^0(R)$$

$$\pi_1(\underline{P}) \simeq K^1(R)$$

this Pic-envelope is <u>not</u> a strict Pic-category

where $K^0(R)$ is the Grothendieck group and $K^1(R)$ the Whitehead group [1]

The use of the $\otimes$-category of fractions of a $\otimes$-ACU category gives us the following result :

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Let $\underline{C}$ be a $\otimes$- ACU category, Z an arbitrary object of $\underline{C}$ different from the unity object $\underline{1}$, S the functor from $\underline{C}$ to $\underline{C}$ defined by 12

$$X \longmapsto X \otimes Z .$$

The <u>suspension category</u> of the $\otimes$- ACU category $\underline{C}$ defined by the object Z is the triple $(\underline{P}, i, p)$ which solves the universal problem for triples $(\underline{Q}, j, q)$ where $\underline{Q}$ is a category, j a functor from $\underline{C}$ to $\underline{Q}$, and q an equivalence of categories from $\underline{Q}$ to $\underline{Q}$, so that the following diagram commutes

$$\begin{array}{ccc} \underline{C} & \xrightarrow{S} & \underline{C} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ \underline{Q} & \xrightarrow{q} & \underline{Q} \end{array}$$

up to natural isomorphism. In the case when $\underline{C}$ is the homotopy category of pointed topological spaces $\underline{Htp}_*$ together with the smash $\wedge$ (the smash $X \wedge Y$ of two spaces X and Y, with the base points x_0 and y_0, is obtained from the product $X \times Y$ by shrinking the subset $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ to a single point which is taken as the base point of $X \wedge Y$), and the usual ACU constraint; and Z is the 1- sphere S^1 hence S is the suspension functor, we get the well-known definition of the suspension category.

Let $\underline{C}'$ be the $\otimes$- stable subcategory of $\underline{C}$ generated by Z and $\underline{P}$ the $\otimes$- category of fractions of $\underline{C}$ with respect to $(\underline{C}', (F, id))$ where $F : \underline{C}' \to \underline{C}$ is the inclusion functor. One obtains a functor $G : \underline{P} \to \underline{P}$ from the suspension category to the $\otimes$- category of fractions $\underline{P}$. If G is not faithful, that is the case of the homotopy category of pointed topological spaces $\underline{Htp}_*$ together with the smash $\wedge$ and the 1- sphere S^1; then it is impossible to construct in $\underline{P}$ a law $\otimes$ such that $\underline{P}$ together with this law is a $\otimes$- ACU category, iZ invertible in $\underline{P}$, and i imbedded in a pair $(i, \breve{i})$ which is a $\otimes$- ACU functor from $\underline{C}$ to $\underline{P}$.

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References

[1] Bass, H. : K-theory and stable algebra. Publ. math. de l'IHES, n° 22.

[2] Bénabou, J. : Thèse, Paris 1966.

[3] Bourbaki : Théorie des ensembles.

[4] ——— : Algèbre commutative.

[5] ——— : Algèbre multilinéaire.

[6] Deligne, P. : Champs de Picard strictement commutatifs. SGA 4 XVIII

[7] Eilenberg, S. and Kelly, G.M : Closed categories. Proceedings of the conference on categorical algebra (421-561). Springer-Verlag 1965.

[8] Freyd, P. : Stable homotopy. Proceedings of the conference on categorical algebra (121-176). Springer-Verlag 1965.

[9] Grothendieck, A. : Biextensions de faisceaux de groupes. SGA 7 VII.

[10] ——— : Catégories cofibrées additives et complexe cotangent relatif. Lecture notes in mathematics N° 79, Springer-Verlag 1968.

[11] Mac-Lane, S. : Homology. Springer-Verlag 1967.

[12] ——— : Categorical algebra. Bull. Amer. Mat. Soc. 71 (1965) (40-106)

[13] Mitchell, B. : Theory of categories. Academic Press 1965.

[14] Neantro Saavedra Rivano : Thèse, Paris (1970 ?)

[15] ——— : Catégories tannakiennes. Lecture notes in mathematics N° 265. Springer-Verlag 1972.

[16] Spanier, E : Algebraic topology. Mc Graw-Hill Inc. 1966.

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