Elementary divisors of Gram matrices of certain Specht modules

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Notation

$\mathcal{S}_n$	symmetric group
$\lambda \vdash n$	partition of n
λ'	transpose of λ
S^λ	integral Specht module over $\mathbf{Z}\mathcal{S}_n$
$n_\lambda := \mathrm{rk}_{\mathbf{Z}}\, S^\lambda$	
$S^{\lambda,*}$	$\mathbf{Z}$-linear dual
$S^\lambda_A := A \otimes_{\mathbf{Z}} S^\lambda$	for a commutative ring A
$(=,-)$	the $\mathcal{S}_n$-invariant nondegenerate $\mathbf{Z}$-bilinear form on S^λ

Problem

$\mathbf{Z}\mathcal{S}_n$-linear map:

$$\begin{array}{ccc} S^\lambda & \overset{\eta}{\hookrightarrow} & S^{\lambda,*} \\ \xi & \longmapsto & (\xi,-) \end{array}$$

Determine quotient $S^{\lambda,*}/S^\lambda$ as a finite abelian group.

Equivalently, determine el. div. of Gram matrix of $(=,-)$.

Motivation

I) Simple modules

Ordinary:

$$\{S^\lambda_{\mathbf{C}} \mid \lambda \vdash n\} \;=\; \{\text{simple } \mathbf{C}\mathcal{S}_n\text{-modules}\}$$

Modular: p prime.

$$D^\lambda_{\mathbf{F}_p} \;:=\; \operatorname{Im}\Big((S^\lambda \overset{\eta}{\hookrightarrow} S^{\lambda,*}) \otimes_{\mathbf{Z}} \mathbf{F}_p\Big)$$

James proves:

$$\{D^\lambda_{\mathbf{F}_p} \mid \lambda \vdash n \; p\text{-regular}\} \;=\; \{\text{simple } \mathbf{F}_p\mathcal{S}_n\text{-modules}\}$$

So

$$\dim_{\mathbf{F}_p} D^\lambda_{\mathbf{F}_p} \;=\; \#\{\text{el. div. of } S^\lambda \text{ not divisible by } p\}$$

However, we will make extensive use of known such dimensions rather than derive them.

II) Quasiblocks

How to understand the integral group ring $\mathbf{Z}\mathcal{S}_n$?

Wedderburn isomorphism:

$$\mathbf{Q}\mathcal{S}_n \xrightarrow{\sim} \prod_{\lambda \vdash n} \underbrace{\operatorname{End}_{\mathbf{Q}} S^{\lambda}_{\mathbf{Q}}}_{\simeq\, \mathbf{Q}^{n_\lambda \times n_\lambda}}$$

Restrict to Wedderburn embedding.

$$\mathbf{Z}\mathcal{S}_n \hookrightarrow \prod_{\lambda \vdash n} \underbrace{\operatorname{End}_{\mathbf{Z}} S^{\lambda}}_{\simeq\, \mathbf{Z}^{n_\lambda \times n_\lambda}}$$

Describe $\mathbf{Z}\mathcal{S}_n$ as image of the Wedderburn embedding.

Example:

$$\begin{tikzcd} \mathbf{Z}\mathcal{S}_3 \simeq & \mathbf{Z} \arrow[r, dash, "3"] \arrow[rr, dash, bend right, "2"'] & \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 3\mathbf{Z} & \mathbf{Z} \end{pmatrix} \arrow[r, dash, "3"] & \mathbf{Z} \end{tikzcd}$$

Call $Q^{(2,1)} = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 3\mathbf{Z} & \mathbf{Z} \end{pmatrix}$ a *quasiblock* of $\mathbf{Z}\mathcal{S}_3$.

Orthogonal decomposition into rational primitive central idempotents:

$$1_{\mathbf{Q}\mathcal{S}_n} \;=\; \sum_{\lambda \vdash n} \varepsilon_\lambda$$

General definition of a *quasiblock* :

$$Q^\lambda \;:=\; \mathbf{Z}\mathcal{S}_n\varepsilon_\lambda \;\left(\;\hookrightarrow\quad \mathrm{End}_{\mathbf{Z}}\, S^\lambda \;\simeq\; \mathbf{Z}^{n_\lambda \times n_\lambda}\;\right)$$

Unknown: index of this inclusion (as abelian groups).

Necessary condition on $\varphi \in \mathrm{End}_{\mathbf{Z}}\, S^\lambda$ to lie in image:

$$\eta^{-1}\varphi\,\eta \quad \text{has to be integral}$$

(η^{-1} = rational inverse of η). Not sufficient in general.

The Smith form of $S^{(2,1)}$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)$, so for $Q^{(2,1)} = \left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ 3\mathbf{Z} & \mathbf{Z} \end{smallmatrix}\right)$ this necessary condition is in fact sufficient.

In general, elementary divisors of S^λ partially describe Q^λ.

Method

I) Jantzen filtration

Let p be a prime, let $i \geqslant 0$. Define $S^\lambda_{\mathbf{F}_p}(i)$ by

$$\begin{tikzcd}
\mathrm{Kern} \arrow[r, hook] \arrow[d] & S^\lambda \arrow[r, "\eta"] \arrow[d] & S^{\lambda,*}/p^i S^{\lambda,*} \\
S^\lambda_{\mathbf{F}_p}(i) \arrow[r, hook] & S^\lambda_{\mathbf{F}_p} &
\end{tikzcd}$$

The *Jantzen filtration* is the finite filtration

$$S^\lambda_{\mathbf{F}_p} \;=\; S^\lambda_{\mathbf{F}_p}(0) \;\supseteq\; S^\lambda_{\mathbf{F}_p}(1) \;\supseteq\; S^\lambda_{\mathbf{F}_p}(2) \;\supseteq\; \cdots \;,$$

We have $S^\lambda_{\mathbf{F}_p}(0)/S^\lambda_{\mathbf{F}_p}(1) \simeq D^\lambda_{\mathbf{F}_p}$.

The Jantzen filtration gives $\mathbf{Z}_{(p)}$-linear elementary divisors:

$$S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}} \;\simeq\; \bigoplus_{i\geqslant 0} (\mathbf{Z}/p^i\mathbf{Z})^{\dim_{\mathbf{F}_p} S^\lambda_{\mathbf{F}_p}(i)/S^\lambda_{\mathbf{F}_p}(i+1)} \;.$$

So to get elementary divisors:

- Determine composition factors of Jantzen subquotients (using Jantzen's Lemma and Schaper's Theorem).
- Use results of James et al. on the dimensions of simple modules.

II) Jantzen's Lemma

p prime, $\lambda \vdash n$ arbitrary, $\mu \vdash n$ p-regular.

$$\vartheta_i \ := \ [S^\lambda_{\mathbf{F}_p}(i)/S^\lambda_{\mathbf{F}_p}(i+1) : D^\mu_{\mathbf{F}_p}] \ .$$

Jantzen's Lemma:

$$\begin{array}{lcl} [S^\lambda_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}] & = & \vartheta_0 \ + \ \ \vartheta_1 \ + \ \ \vartheta_2 \ + \ \cdots \\ [S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}} : D^\mu_{\mathbf{F}_p}] & = & 0\vartheta_0 \ + \ 1\vartheta_1 \ + \ 2\vartheta_2 \ + \ \cdots \end{array}$$

If $\lambda = \mu$, then $\vartheta_0 = 1$, and $\vartheta_i = 0$ for $i > 0$.

If $\lambda \neq \mu$, we have $\vartheta_0 = 0$. Then :

(i) If $[S^\lambda_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}] = 1$, then

$$\vartheta_i = \begin{cases} 1 \ \text{ for } i = [S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}} : D^\mu_{\mathbf{F}_p}] \\ 0 \ \text{ else.} \end{cases}$$

(ii) If $[S^\lambda_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}] = 2$, then :

If $[S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}} : D^\mu_{\mathbf{F}_p}] = 2$, then $\vartheta_1 = 2$, others $= 0$.

If $[S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}} : D^\mu_{\mathbf{F}_p}] = 3$, then $\vartheta_1 = 1$, $\vartheta_2 = 1$, others $= 0$.

If $[S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}} : D^\mu_{\mathbf{F}_p}] = 4$, then ($\vartheta_1 = 1$, $\vartheta_3 = 1$, others $= 0$) or ($\vartheta_2 = 2$, others $= 0$).

Etc. So there is still ambiguity.

III) Schaper's Theorem

Even if all occurring decomposition numbers $[S^\lambda_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}]$ are known, we still need $[S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}} : D^\mu_{\mathbf{F}_p}]$.

Schaper's Theorem is a formula in the Grothendieck group:

$$[S^{\lambda,*}_{\mathbf{Z}_{(p)}}/S^\lambda_{\mathbf{Z}_{(p)}}] \;=\; \sum_{\lambda \trianglelefteq \nu} \alpha_\nu [S^\nu_{\mathbf{F}_p}]\;,$$

with combinatorially determined coefficients $\alpha_\nu \in \mathbf{Z}$.

So for our purposes, knowledge of further decomposition numbers $[S^\nu_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}]$ is required.

IV) Transposition

Jantzen filtration of S^λ $\quad\longleftrightarrow\quad$ Jantzen filtration of $S^{\lambda'}$

Some results

Table of calculated Jantzen filtrations.

Partially depending on conjectures on dec. numbers.

Partially with ambiguity.

λ	Remarks
$(n-m, 1^{n-m})$	hooks, two methods: Jantzen-Schaper and directly; independently done by James-Mathas
$(n-m, m)$	two-row; independently done by Fayers and by Murphy
$(n-3, 2, 1)$	
$(n-4, 2^2)$	if $p=2$: depending on a conj. on $[S_{\mathbf{F}_2}^{(n-4,2^2)} : D_{\mathbf{F}_2}^{(n)}]$; if $p=3$: with ambiguity
$(n-4, 3, 1)$	if $p=2$: with ambiguity
$(n-4, 2, 1^2)$	if $p=2$: depending heavily on conj. on dec. nos., and with ambiguity; if $p=3$: depending on conj. on dec. nos.
Case $n-\lambda_1 < p$	using Kleshchev's Modular Branching and Carter-Payne

How to proceed further?

Let e^μ be a primitive idempotent of $\mathbf{Z}_{(p)}\mathcal{S}_n$ belonging to $D^\mu_{\mathbf{F}_p}$ and let ε^λ be the primitive central idempotent of $\mathbf{Q}\mathcal{S}_n$ belonging to $S^\lambda_{\mathbf{Q}}$.

The embedding η decomposes into a direct sum of embeddings

$$S^\lambda_{\mathbf{Z}_{(p)}} e^\mu \xhookrightarrow{\; e^\mu \eta e^\mu \;} S^{\lambda,*}_{\mathbf{Z}_{(p)}} e^\mu \ ,$$

so we get a block diagonalisation of the Gram matrix, with blocks of edge length

$$\mathrm{rk}_{\mathbf{Z}_{(p)}} S^\lambda e^\mu \;=\; [S^\lambda_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}] \ .$$

It remains to

calculate the Smith form of $e^\mu \eta e^\mu$.

Schaper's Theorem ess. allows to calculate the determinant:

$$v_p(\det(e^\mu \eta e^\mu)) \;=\; [S^{\lambda,*}_{\mathbf{Z}_{(p)}} / S^\lambda_{\mathbf{Z}_{(p)}} : D^\mu_{\mathbf{F}_p}]$$

Assume $[S^\lambda_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}] \geqslant 1$. Let

$$Q^{\lambda:\mu}_{(p)} \;:=\; e^\mu Q^\lambda_{\mathbf{Z}_{(p)}} e^\mu \;\left(\;\hookrightarrow\;\; \mathbf{Z}_{(p)}^{[S^\lambda_{\mathbf{F}_p}:D^\mu_{\mathbf{F}_p}]\times[S^\lambda_{\mathbf{F}_p}:D^\mu_{\mathbf{F}_p}]}\right).$$

- $\mathbf{Q}\otimes_{\mathbf{Z}_{(p)}} Q^{\lambda:\mu}_{(p)}$ is simple.
- $Q^{\lambda:\mu}_{(p)}$ is local (for $e^\mu Q^\lambda_{\mathbf{Z}_{(p)}}$ is indecomposable).
- $e^\mu \eta e^\mu$ is an inclusion of simple $Q^{\lambda:\mu}_{(p)}$-lattices.

If we could classify all inclusions of simple lattices over $Q^{\lambda:\mu}_{(p)}$, we would know the Smith form of $e^\mu\eta e^\mu$ to be one of the Smith forms of these inclusions (moreover, one of a given determinant).

Example: $Q^{(3,1^2):(5)}_{(2)} \;\simeq\; \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathbf{Z}^{2\times 2}_{(2)} \;\middle|\; b \equiv_2 0\,,\, a\equiv_2 d\right\}$, with simple lattices $X := \begin{pmatrix}\mathbf{Z}_{(2)}\\ \mathbf{Z}_{(2)}\end{pmatrix}$ and $Y := \begin{pmatrix}2\mathbf{Z}_{(2)}\\ \mathbf{Z}_{(2)}\end{pmatrix}$.

Inclusions of determinant 4: only the scalar inclusions $X \overset{2}{\hookrightarrow} X$ and $Y \overset{2}{\hookrightarrow} Y$, both with Smith form $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$.

Need: lattices over $Q^{\lambda:\mu}_{(p)}$. Note: $\mathrm{rk}_{\mathbf{Z}_{(p)}} Q^{\lambda:\mu}_{(p)} = [S^\lambda_{\mathbf{F}_p} : D^\mu_{\mathbf{F}_p}]^2$.
So to ask for $Q^{\lambda:\mu}_{(p)}$ can be viewed a refined version of the question for the decomposition numbers.

Elementary divisors over $\mathbf{Z}[q,q^{-1}]$?

For the Specht module over the Hecke algebra $\mathcal{H}$ with ground ring $\mathbf{Z}[q,q^{-1}]$, we may also consider the Gram matrix of the invariant bilinear form.

Distinguish (also over $\mathbf{Z}_{(p)}[q,q^{-1}]$):

- divisibly diagonalizable matrices (each diagonal entry divides successor)
- diagonalizable, but not divisibly diagonalizable matrices
- non-diagonalizable matrices

Andersen remarked that not every Gram matrix of a Specht module is diagonalizable.

For $\lambda = (n-l, 1^l)$: bases divisibly diagonalizing the Gram matrix (the q causing difficulties).

For $\lambda = (3,3,2)$: not diagonalizable over $\mathbf{Z}_{(2)}[q,q^{-1}]$.

For $\lambda = (4,2,1,1)$: neither diagonalizable over $\mathbf{Z}_{(2)}[q,q^{-1}]$ nor over $\mathbf{Z}_{(3)}[q,q^{-1}]$.

Method: if diagonalizable, then there is a connection between the elementary divisors over $\mathbf{Q}[q,q^{-1}]$ and $\mathbf{F}_p[q,q^{-1}]$.

We do not know (not even conjecturally) a criterion for λ to decide on diagonalizability.