Augmentation modules and cocycles

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$$
\begin{tikzcd}
G^{\times n} \arrow[r, "\substack{d\,:\ n\text{-cocycle} \\ \text{relative to } H \leqslant G}"] \arrow[d, "\substack{\check{\xi}_n\,:\ \text{universal } n\text{-cocycle} \\ \text{relative to } H \leqslant G}"'] & M \\
\mathrm{I}_R^{(n)}(G,H) \arrow[ur, dotted, "\exists!\ \tilde{d}\ RG\text{-linear}"'] &
\end{tikzcd}
$$

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Contents

0 Introduction

0.1 Cocycles and relative cocycles

Suppose given a group G, a subgroup $H \leqslant G$ and a commutative ring R.

Suppose given an RG-module M.

0.1.1 Cohomology and relative cohomology

The n-th cohomology group of G with values in M is defined as

$$\mathrm{H}^n(G, M) \;:=\; \mathrm{Ext}^n_{RG}(R, M)\;,$$

where $n \geqslant 0$. To calculate it, one can resolve the trivial RG-module R projectively in the standard way, as follows.

$$\ldots \;\to\; R(G^{\times 3}) \;\to\; R(G^{\times 2}) \;\to\; R(G^{\times 1}) \;\to\; R\;.$$

Letting

$$P \;:=\; (\,\ldots \;\to\; R(G^{\times 3}) \;\to\; R(G^{\times 2}) \;\to\; R(G^{\times 1}) \;\to\; 0 \;\to\; \ldots\,)$$

and applying the hom-functor ${}_{RG}(-, M)$ to the complex P, we obtain

$$\mathrm{H}^n(G, M) \;=\; \mathrm{Ext}^n_{RG}(R, M) \;=\; \mathrm{H}^n\big({}_{RG}(P, M)\big)\;.$$

Usage of the standard resolution yields a way of representing cohomology elements by n-cocycles, on which we shall concentrate.

There is a version of group cohomology relative to a subgroup, which is defined as follows.

We abbreviate $\check{G} := H\backslash G$, which is a G-set.

Adamson [**1,** §3] defined cohomology of G relative to H with values in M starting from the resolution

$$\ldots \;\to\; R(\check{G}^{\times 3}) \;\to\; R(\check{G}^{\times 2}) \;\to\; R(\check{G}^{\times 1}) \;\to\; R\;.$$

Hochschild [**5,** §1, 2, 3] developed a framework of relative homological algebra, covering in particular the relative cohomology as defined by Adamson in [**5,** §4] as

$$\mathrm{H}^n(G, H, M) \;:=\; \mathrm{Ext}^n_{(RG,RH)}(R, M)\;,$$

where $n \geqslant 0$. This relative Ext-module can be calculated using the resolution of Adamson: Letting

$$\check{P} \;:=\; (\,\ldots \;\to\; R(\check{G}^{\times 3}) \;\to\; R(\check{G}^{\times 2}) \;\to\; R(\check{G}^{\times 1}) \;\to\; 0 \;\to\; \ldots\,)$$

and applying the hom-functor ${}_{RG}(-, M)$ to the complex $\check{P}$, we obtain

$$\mathrm{H}^n(G, H, M) \;=\; \mathrm{Ext}^n_{(RG,RH)}(R, M) \;=\; \mathrm{H}^n\big({}_{RG}(\check{P}, M)\big)\;.$$

Cf. also [**3,** (10.20), (19.2)].

Usage of the standard resolution yields a way of representing relative cohomology elements by relative n-cocycles, on which we shall concentrate.

A different definition of relative cohomology groups is due to AUSLANDER in his thesis; cf. [**2,** §1]. He defined it as $\mathrm{Ext}^{n-1}_{RG}(\mathrm{I}^{(1)}_R(G,H),M)$, where $n \geqslant 1$; cf. [**2,** Prop. 5.4], Definiton 62.(1); cf. also TAKASU [**8,** §4].

0.1.2 An R-linear basis of the n-th augmentation module

In the standard resolution

$$\ldots \;\to\; R(G^{\times 3}) \;\to\; R(G^{\times 2}) \;\to\; R(G^{\times 1}) \;\to\; R$$

from §0.1.1, we define the n-th augmentation module by

$$\mathrm{I}^{(n)}_R(G) \;:=\; \mathrm{Im}\big(R(G^{\times(n+1)}) \to R(G^{\times n})\big) \;=\; \mathrm{Ker}\big(R(G^{\times n}) \to R(G^{\times(n-1)})\big) \;\subseteq\; R(G^{\times n})\;,$$

where $n \geqslant 1$.

For instance, $\mathrm{I}^{(1)}_R(G) \subseteq R(G^{\times 1}) = RG$ is the augmentation ideal.

An R-linear basis of $\mathrm{I}^{(1)}_R(G)$ is given by $\{\, g-1 \,|\, g \in G \smallsetminus \{1\}\,\}$.

We generalise this to an R-linear basis of $\mathrm{I}^{(n)}_R(G)$ for $n \geqslant 1$; cf. Proposition 29.

For instance,

$$\{\;(xy,y)-(xy,1)+(y,1)\;\mid\; x\in G,\; y\in G\setminus\{1\}\;\}\;\;\cup\;\;\{(1,1)\}$$

is an R-linear basis of $\mathrm{I}^{(2)}_R(G) \subseteq R(G^{\times 2})$; cf. Remark 38.

0.1.3 Universal n-cocycles

Suppose given $n \geqslant 0$.

A map $d : G^{\times n} \to M$ is called an *n-cocycle* of G with values in M if the condition

$$0 \;=\; (g_1,\ldots,g_n)d + \left(\sum_{k=1}^{n}(-1)^k(g_0,\ldots,g_{k-2},\, g_{k-1}\cdot g_k\,, g_{k+1},\ldots,g_n)d\right) + (-1)^{n+1}(g_0,\ldots,g_{n-1})d\cdot g_n$$

is satisfied for $g_0,\,\ldots,\,g_n \,\in\, G$.

The R-module of n-cocycles of G with values in M is written

$$\mathrm{Z}^n(G,M) \;\subseteq\; {}_{\mathrm{Map}}(G^{\times n},M)\;.$$

For instance, $d : G = G^{\times 1} \to M$ is a 1-cocycle if

$$0 \;=\; (g_1)d - (g_0\cdot g_1)d + (g_0)d\cdot g_1$$

for $g_0,\, g_1 \in G$. A 1-cocycle is classically also called a derivation.

For instance, $d : G^{\times 2} \to M$ is a 2-cocycle if

$$0 \;=\; (g_1, g_2)d - (g_0 \cdot g_1, g_2)d + (g_0, g_1 \cdot g_2)d - (g_0, g_1)d \cdot g_2$$

for $g_0,\, g_1,\, g_2 \in G$. A 1-cocycle is classically also called a factor system.

We shall present the functor $M \mapsto \mathrm{Z}^n(G, M)$ as follows.

We factor the differentials of the standard resolution, where the object $R(G^{\times(n+1)})$ is at position n:

$$\begin{tikzcd}
R(G^{\times(n+2)}) \arrow[rr, "\varepsilon_{n+2}"] \arrow[dr, "\bar{\varepsilon}_{n+2}"] & & R(G^{\times(n+1)}) \arrow[rr, "\varepsilon_{n+1}"] \arrow[dr, "\bar{\varepsilon}_{n+1}"] & & R(G^{\times n}) \\
& \mathrm{I}_R^{(n+1)}(G) \arrow[ur, hook] & & \mathrm{I}_R^{(n)}(G) \arrow[ur, hook] &
\end{tikzcd}$$

Applying the left-exact hom-functor ${}_{RG}(-, M)$ and inserting the cokernel $\mathrm{H}^n(G, M)$, we obtain the following diagram.

$$\begin{tikzcd}
{}_{RG}(R(G^{\times(n+2)}), M) & & {}_{RG}(R(G^{\times(n+1)}), M) \arrow[ll, "\varepsilon^*_{n+2}"'] \arrow[dl] & & {}_{RG}(R(G^{\times n}), M) \arrow[ll, "\varepsilon^*_{n+1}"'] \arrow[dl] \\
& {}_{RG}(\mathrm{I}_R^{(n+1)}(G), M) \arrow[ul, "\bar{\varepsilon}^*_{n+2}"] & & {}_{RG}(\mathrm{I}_R^{(n)}(G), M) \arrow[ul, "\bar{\varepsilon}^*_{n+1}"] \arrow[dl] & \\
& & \mathrm{H}^n(G, M) & &
\end{tikzcd}$$

We will make use of the R-module ${}_{RG}(\mathrm{I}_R^{(n)}(G), M)$. To this end, we define an n-cocycle

$$\xi_n \;:\; G^{\times n} \;\to\; \mathrm{I}_R^{(n)}(G)\ .$$

cf. Definition 20. For instance,

$$\begin{array}{rcll}
(x)\xi_1 & = & x - 1 & \text{for } x \in G \\
(x, y)\xi_2 & = & (xy, y) - (xy, 1) + (y, 1) & \text{for } x,\, y \in G \\
(x, y, z)\xi_3 & = & (xyz, yz, z) - (xyz, yz, 1) + (xyz, z, 1) - (yz, z, 1) & \text{for } x,\, y,\, z \in G\ .
\end{array}$$

Composition with ξ_n yields the isomorphism of R-modules

$$\begin{array}{rcrcl}
\xi_n \cdot (-) & : & {}_{RG}(\mathrm{I}_R^{(n)}(G), M) & \stackrel{\sim}{\to} & \mathrm{Z}^n(G, M) \\
& & \tilde{d} & \mapsto & \xi_n \cdot \tilde{d}\ .
\end{array}$$

In other words, ξ_n is the universal n-cocycle.

$$\begin{tikzcd}
G^{\times n} \arrow[rr, "{d \in \mathrm{Z}^n(G,M)}" above, "n\text{-cocycle}" below] \arrow[d, "{\xi_n \in \mathrm{Z}^n(G,\mathrm{I}_R^{(n)}(G))}" left, "\text{universal } n\text{-cocycle}" left] & & M \\
\mathrm{I}_R^{(n)}(G) \arrow[urr, dotted, "\exists!\ \tilde{d}\ \ RG\text{-linear}" below] & &
\end{tikzcd}$$

Cf. Lemma 48 and Proposition 49.

0.1.4 Universal relative n-cocycles

We shall develop a variant relative to a subgroup. To this end, we let

$$
\begin{array}{l}
{}_{\mathrm{Map},[H]}(G^{\times n}, M) \ := \\
\left\{ f \in {}_{\mathrm{Map}}(G^{\times n}, M) \;\middle|\; \begin{array}{rl} & (\,h_0 g_0\,,\, h_1 g_1\,,\, h_2 g_2\,,\,\dots\,,\, h_{n-2}g_{n-2}\,,\, h_{n-1}g_{n-1}\,)f \cdot h_n \\ = & (\,g_0 h_1\,,\, g_1 h_2\,,\, g_2 h_3\,,\,\dots,\, g_{n-2}h_{n-1}\,,\, g_{n-1}h_n\,)f \\ & \forall\ g_0,\dots,g_{n-1} \in G,\ h_0,\dots,h_n \in H \end{array} \right\}
\end{array}
$$

and

$$\mathrm{Z}^n(G,H,M) \ := \ {}_{\mathrm{Map},[H]}(G^{\times n}, M) \ \cap \ \mathrm{Z}^n(G,M) \ \subseteq \ {}_{\mathrm{Map}}(G^{\times n}, M)\ .$$

An element $d : G^{\times n} \to M$ in $\mathrm{Z}^n(G,H,M)$ is called an *n-cocycle relative to H with values in M* or a *relative n-cocycle.*

So a relative n-cocycle satisfies the usual n-cocycle condition mentioned in §0.1.3 and, in addition, the compatibility with H mentioned in the description of ${}_{\mathrm{Map},[H]}(G^{\times n}, M)$.

For instance,

$$
\begin{array}{l}
\mathrm{Z}^1(G,H,M) \\
= \ \left\{ G \xrightarrow{d} M \;\middle|\; \begin{array}{l} (g_0 \cdot g_1)d = (g_1)d + (g_0)d \cdot g_1 \quad \text{for } \ g_0,\, g_1 \in G \\ (h_0 \cdot g_0)d \cdot h_1 = (g_0 \cdot h_1)d \quad \text{for } g_1 \in G \ \text{ and } \ h_0,\, h_1 \in H \end{array} \right\} \\
= \ \left\{ G \xrightarrow{d} M \;\middle|\; \begin{array}{l} (g_0 \cdot g_1)d = (g_1)d + (g_0)d \cdot g_1 \quad \text{for } \ g_0,\, g_1 \in G \\ (h)d = 0 \quad \text{for } \ h \in H \end{array} \right\}
\end{array}
$$

$$
\begin{array}{l}
\mathrm{Z}^2(G,H,M) \\
= \ \left\{ G \times G \xrightarrow{d} M \;\middle|\; \begin{array}{r} (\,g_1\,,\,g_2\,)d - (\,g_0 \cdot g_1\,,\,g_2\,)d + (\,g_0\,,\,g_1 \cdot g_2)d - (\,g_0\,,\,g_1\,)d \cdot g_2 \ = \ 0 \\ \text{for } g_0,\, g_1,\, g_2 \in G \\ \\ (\,h_0 \cdot g_0,\, h_1 \cdot g_1\,)d \cdot h_2 \ = \ (\,g_0 \cdot h_1\,,\,g_1 \cdot h_2\,)d \\ \text{for } g_0,\, g_1 \in G \text{ and } h_0,\, h_1,\, h_2 \in H \end{array} \right\} .
\end{array}
$$

We shall present the functor $M \mapsto \mathrm{Z}^n(G,H,M)$ as follows.

We write the set of right cosets of H in G as

$$\check{G} \;:=\; H \setminus G \;=\; \{\, Hg \mid g \in G \,\}\;.$$

In the standard resolution

$$\ldots \;\to\; R(\check{G}^{\times 3}) \;\to\; R(\check{G}^{\times 2}) \;\to\; R(\check{G}^{\times 1}) \;\to\; R\;.$$

from §0.1.1, we define the n-th augmentation module relative to H by

$$\mathrm{I}_R^{(n)}(G,H) \;:=\; \mathrm{Im}\big(R(\check{G}^{\times(n+1)}) \to R(\check{G}^{\times n})\big) \;=\; \mathrm{Ker}\big(R(\check{G}^{\times n}) \to R(\check{G}^{\times(n-1)})\big) \;\subseteq\; R(\check{G}^{\times n})\;,$$

where $n \geqslant 1$.

We factor the differentials of the standard resolution, where the object $R(\check{G}^{\times(n+1)})$ is at position n:

$$\begin{tikzcd}
R(\check{G}^{\times(n+2)}) \arrow[rr, "\check{\varepsilon}_{n+2}"] \arrow[dr, "\bar{\check{\varepsilon}}_{n+2}"'] & & R(\check{G}^{\times(n+1)}) \arrow[rr, "\check{\varepsilon}_{n+1}"] \arrow[dr, "\bar{\check{\varepsilon}}_{n+1}"'] & & R(\check{G}^{\times n}) \\
& \mathrm{I}_R^{(n+1)}(G,H) \arrow[ur, hook] & & \mathrm{I}_R^{(n)}(G,H) \arrow[ur, hook] &
\end{tikzcd}$$

Applying the left-exact hom-functor ${}_{RG}(-,M)$ and inserting the cokernel $\mathrm{H}^n(G,H,M)$, we obtain the following diagram.

$$\begin{tikzcd}
{}_{RG}(R(\check{G}^{\times(n+2)}),M) & & {}_{RG}(R(\check{G}^{\times(n+1)}),M) \arrow[ll, "\check{\varepsilon}^*_{n+2}"'] \arrow[dl] & & {}_{RG}(R(\check{G}^{\times n}),M) \arrow[ll, "\check{\varepsilon}^*_{n+1}"'] \arrow[dl] \\
& {}_{RG}(\mathrm{I}_R^{(n+1)}(G,H),M) \arrow[ul, "\bar{\check{\varepsilon}}^*_{n+2}"] & & {}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) \arrow[ul, "\bar{\check{\varepsilon}}^*_{n+1}"] \arrow[dl] & \\
& & \mathrm{H}^n(G,H,M) & &
\end{tikzcd}$$

We will make use of the R-module ${}_{RG}(\mathrm{I}_R^{(n)}(G,H),M)$. To this end, we define a relative n-cocycle

$$\check{\xi}_n \;:\; G^{\times n} \;\to\; \mathrm{I}_R^{(n)}(G,H)\;.$$

For instance,

$$\begin{aligned}
(x)\check{\xi}_1 &= Hx - H1 && \text{for } x \in G \\
(x,y)\check{\xi}_2 &= (Hxy,Hy) - (Hxy,H1) + (Hy,H1) && \text{for } x,y \in G \\
(x,y,z)\check{\xi}_3 &= (Hxyz,Hyz,Hz) - (Hxyz,Hyz,H1) + (Hxyz,Hz,H1) - (Hyz,Hz,H1) && \text{for } x,y,z \in G\;.
\end{aligned}$$

Composition with $\check{\xi}_n$ yields the isomorphism of R-modules

$$\begin{array}{rcrcl}
\check{\xi}_n \cdot (-) & : & {}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) & \xrightarrow{\sim} & \mathrm{Z}^n(G,H,M) \\
& & \tilde{d} & \mapsto & \check{\xi}_n \cdot \tilde{d}\;.
\end{array}$$

In other words, $\check{\xi}_n$ is the universal relative n-cocycle.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
G^{\times n} \arrow[r, "\substack{d \in \mathrm{Z}^n(G,H,M) \\ n\text{-cocycle relative to } H}"] \arrow[d, "\substack{\xi_n \in \mathrm{Z}^n(G,H,\mathrm{I}_R^{(n)}(G,H)) \\ \text{universal } n\text{-cocycle relative to } H}"'] & M \\
\mathrm{I}_R^{(n)}(G,H) \arrow[ur, dotted, "\exists!\ \tilde{d}\ \ RG\text{-linear}"'] &
\end{tikzcd}
$$

Cf. Lemma 86 and Proposition 87.

0.2 The cotangent-square

We investigate the interrelation of the first relative augmentation modules when varying the subgroups.

Suppose given a commutative ring R.

Suppose given a group G and subgroups $L \leqslant K \leqslant H \leqslant G$.

We have the square of RG-modules

$$
\begin{tikzcd}
\mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG \arrow[r, "\kappa'_H \underset{RH}{\otimes} RG"] \arrow[d, "j'_L"'] \arrow[dr, phantom, "\Box"] & \mathrm{I}_R^{(1)}(H,K) \underset{RH}{\otimes} RG \arrow[d, "j'_K"] \\
\mathrm{I}_R^{(1)}(G,L) \arrow[r, "\kappa'_G"'] & \mathrm{I}_R^{(1)}(G,K)\ ,
\end{tikzcd}
$$

called the *cotangent-square*; cf. Proposition 94.(2).

Here κ'_G is induced by the map $L\backslash G \to K\backslash G : Lg \mapsto Kg$, and κ'_H is induced by the map $L\backslash H \to K\backslash H : Lh \mapsto Kh$.

Moreover, j'_L and j'_K are induced by inclusion maps.

Finally, being a square means having a short exact diagonal sequence, i.e. being a pullback and a pushout.

In particular, if $L \leqslant K = H \leqslant G$, we obtain the short exact cotangent-sequence

$$
\mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG \ \stackrel{j'_L}{\longrightarrow}\ \mathrm{I}_R^{(1)}(G,L) \ \stackrel{\kappa'_G}{\longrightarrow}\ \mathrm{I}_R^{(1)}(G,H)\ ;
$$

cf. Corollary 95.

The name is borrowed from a similar sequence in Commutative Algebra; cf. [**4,** Prop. 16.2].

0.3 The conormal-sequence

We investigate the interrelation of the first relative augmentation modules when passing to a factor group.

Suppose given a commutative ring R.

Suppose given a surjective group morphism $\varphi : H \to G$. Write $N := \operatorname{Ker}(\varphi) \trianglelefteqslant H$.

Suppose given subgroups $L \leqslant H$ and $K \leqslant G$ such that $L\varphi \leqslant K$.

$$\begin{tikzcd} N \arrow[r, hook] & H \arrow[r, "\varphi"] & G \\ & L \arrow[u, hook] \arrow[r, "\varphi|_L^K"'] & K \arrow[u, hook] \end{tikzcd}$$

We consider the surjective RH-linear map

$$\begin{array}{rcrcl} R\check{\varphi} & : & R(L\backslash H) & \to & R(K\backslash G) \\ & & Lh & \mapsto & K(h\varphi)\ . \end{array}$$

Let

$$\mathrm{I}_R^{(1)}(\varphi, L, K) \ := \ \operatorname{Ker}(R\check{\varphi}) \ \subseteq \ R(L\backslash H)\ ,$$

which is an RH-submodule.

We have the right exact sequence of RG-modules

$$\begin{array}{rcccl} \mathrm{I}_R^{(1)}(\varphi, L, K)/\,\mathrm{I}_R^{(1)}(\varphi, L, K)\,\mathrm{I}_R^{(1)}(N) & \to & \mathrm{I}_R^{(1)}(H, L)/\,\mathrm{I}_R^{(1)}(H, L)\,\mathrm{I}_R^{(1)}(N) & \to & \mathrm{I}_R^{(1)}(G, K) \\ x + \mathrm{I}_R^{(1)}(\varphi, L, K)\,\mathrm{I}_R^{(1)}(N) & \mapsto & x + \mathrm{I}_R^{(1)}(H, L)\,\mathrm{I}_R^{(1)}(N) & & \\ & & y + \mathrm{I}_R^{(1)}(H, L)\,\mathrm{I}_R^{(1)}(N) & \mapsto & (y)(R\check{\varphi}) \end{array}$$

It is called the *conormal-sequence*. Cf. Proposition 108.

In the particular case of $R = \mathbb{Z}$, $K = 1$ and $L = 1$, we deduce the *short conormal-sequence*

$$\begin{array}{rcccl} N/N^{(1)} & \to & \mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \to & \mathrm{I}_{\mathbb{Z}}^{(1)}(G) \\ nN^{(1)} & \mapsto & n - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & & \\ & & h - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \mapsto & h\varphi - 1 \end{array}$$

which is a short exact sequence of $\mathbb{Z}G$-modules. Cf. Proposition 121.

Here, $N^{(1)}$ is the commutator subgroup of N. The abelianised group $N/N^{(1)}$ becomes a $\mathbb{Z}G$-module via

$$nN^{(1)} \bullet g \ := \ n^h N^{(1)}\ ,$$

where $n \in N$, $g \in G$ and $h \in H$ such that $h\varphi = g$.

The name is borrowed from a similar sequence in Commutative Algebra; cf. [**4,** Prop. 16.3], cf. also Remark 107.(3).

For example, let $H := \mathrm{S}_3$, $N := \mathrm{C}_3 = \langle (1,2,3) \rangle \trianglelefteqslant H$ and $G := H/N$.

Moreover, let $\varphi : H \to H/N$ be the residue class morphism.

Then the short conormal-sequence

$$\mathrm{C}_3 \;\to\; \mathrm{I}^{(1)}_{\mathbb{Z}}(\mathrm{S}_3)/\,\mathrm{I}^{(1)}_{\mathbb{Z}}(\mathrm{S}_3)\,\mathrm{I}^{(1)}_{\mathbb{Z}}(\mathrm{C}_3) \;\to \mathrm{I}^{(1)}_{\mathbb{Z}}(\mathrm{S}_3/\mathrm{C}_3)$$

is isomorphic to

$$\mathbb{Z}/(3) \;\xrightarrow{(1\,0)}\; \mathbb{Z}/(3) \oplus \mathbb{Z} \;\xrightarrow{\binom{0}{1}}\; \mathbb{Z}\;;$$

cf. §7.3.

Conventions

Suppose given sets X and Y. Suppose given a commutative ring R. Suppose given a group G.

(1) Given morphisms $A \xrightarrow{u} B \xrightarrow{v} C$ in a category, the composite of u and v is written $A \xrightarrow{u\cdot v} C$. We often abbreviate $uv := u \cdot v$.

(2) For $a,\, b \in \mathbb{Z}$, we write $[a, b] := \{\, x \in \mathbb{Z} \,|\, a \leqslant x \leqslant b \,\}$.

(3) Suppose given a map $f : X \to Y$.

Suppose given $x \in X$. The image of x under f is written $xf \in Y$.

Suppose given $X' \subseteq X$ and $Y' \subseteq Y$ such that $X'f \subseteq Y'$.

Write $f|_{X'}^{Y'} : X' \to Y' : x \mapsto xf$ for the restriction of f to X' and Y'.

If $X' = X$, we also write $f|^{Y'} := f|_X^{Y'}$.

If $Y' = Y$, we also write $f|_{X'} := f|_{X'}^{Y}$.

(4) Suppose given subsets $U,\, V \subseteq X$. If $U \cap V = \emptyset$, then we write their disjoint union often as $U \,\dot{\cup}\, V := U \cup V \subseteq X$.

(5) For $m, n \in \mathbb{Z}$ and elements $x_m,\, x_{m+1},\, \ldots,\, x_{n-2}, x_{n-1}$ in X, we write the tuple containing these elements as

$$(x_i)_{i\in[m,n-1]} \;:=\; (x_m, x_{m+1}, \ldots, x_{n-2}, x_{n-1}) \,.$$

If $m > n - 1$, we have $(x_i)_{i\in[m,n-1]} \;:=\; (\,)$, the empty tuple.

In particular, for an element x, if $n \geqslant 0$, we have the tuple $(x)_{i\in[0,n-1]} = (x, x, \ldots, x)$ with n entries.

If $m = 0$, we often abbreviate $\underline{x} := (x_i)_{i\in[0,n-1]}$.

(6) Suppose given $m,\, n \geqslant 0$ and tuples $(x_i)_{i\in[0,m-1]}$ and $(y_j)_{j\in[0,n-1]}$ with entries in X.

We write their concatenation as

$$(x_i)_{i\in[0,m-1]} \,\sqcup\, (y_j)_{j\in[0,n-1]} \;=\; (z_k)_{k\in[0,m+n-1]} \;,$$

where

$$z_k \;:=\; \begin{cases} x_k & \text{for } k \in [0, m-1] \\ y_{k-m} & \text{for } k \in [m, m+n-1] \,. \end{cases}$$

That is,

$$(x_i)_{i\in[0,m-1]} \,\sqcup\, (y_j)_{j\in[0,n-1]} \;=\; (x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1}) \,.$$

(7) Suppose given $m \geqslant 0$ and a tuple $\underline{x} = (x_i)_{i\in[0,m-1]}$ with entries in X.

Given $j,\, k \in [0, m-1]$ with $j < k$, we write

$$(\underline{x}) \uparrow^k = (x_0, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{m-1}) \uparrow^k \,:=\; (x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{m-1})$$

$$(\underline{x}) \uparrow^{j,k} = (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{m-1}) = \Big((\underline{x}) \uparrow^k \Big) \uparrow^j = \Big((\underline{x}) \uparrow^j \Big) \uparrow^{k-1} \,.$$

(8) We write $g^x := x^{-1} \cdot g \cdot x$ for $g,\, x \in G$.

(9) Suppose given $m, n \in \mathbb{Z}$ and $g_i \in G$ for $i \in [m, n]$. We write

$$\prod_{k=m}^{n} g_k := g_m \cdot g_{m+1} \cdot g_{m+2} \cdot \ldots \cdot g_{n-1} \cdot g_n \in G.$$

If $m + 1 = n$ we have $\prod_{k=m}^{n} g_k := 1_G = 1$.

(10) The zero module is written $0 := \{0\}$.

(11) By RX, we denote the free R-module on X. So

$$RX \;=\; \left\{ \sum_{x \in X} r_x \, x \;\middle|\; r_x \in R \text{ for } x \in X, \text{ where } \{\, x \in X \mid r_x \neq 0 \,\} \text{ is finite} \right\} .$$

We often implicitly require $\{\, x \in X \mid r_x \neq 0 \,\}$ to be finite when writing an element

$$\sum_{x \in X} r_x \, x \;\in\; RX \;.$$

Suppose given a map $f : X \to Y$. We define the R-linear map

$$\begin{array}{rcll} Rf \;:\; RX & \to & RY & \\ \sum_{x \in X} r_x \, x & \mapsto & \sum_{x \in X} r_x \, xf & = \sum_{y \in Y} \Bigg(\sum_{\substack{x \in X \\ xf = y}} r_x \Bigg) \, y \end{array}$$

We sometimes write $f := Rf : RX \to RY$ by abuse of notation.

(12) Suppose given an R-module M. Suppose given a set I and a tuple of elements $(m_i)_{i \in I}$ of M. Writing down an R-linear combination

$$\sum_{i \in I} \lambda_i \, m_i$$

of $(m_i)_{i \in I}$, where $\lambda_i \in R$ for $i \in I$, we implicitly require that $\{\, i \in I \mid \lambda_i \neq 0 \,\}$ is finite.

(13) A G-set is, by convention, a right G-set.

An R-module is written, by choice, as an R-left module or as an R-right module.

By an R-basis of a free R-module we understand an R-linear basis.

An RG-module is, by convention, an RG-right module.

(14) Suppose given RG-modules A, B, M and a RG-linear map $f : A \to B$.

We write ${}_{RG}(A, M)$ for the R-module of RG-linear maps from A to M.

We often write $f^* := {}_{RG}(f, M) : {}_{RG}(B, M) \to {}_{RG}(A, M) : u \mapsto fu$, which is an R-linear map.

(15) For $n \in \mathbb{Z}_{\geqslant 0}$, we consider $R(G^{\times n})$ as an RG-module via

$$(g_0, g_1, \ldots, g_{n-1}) \cdot g \ := \ (g_0 \cdot g,\ g_1 \cdot g,\ \ldots,\ g_{n-1} \cdot g)\ ,$$

where $(g_0, g_1, \ldots, g_{n-1}) \in G^{\times n}$ and $g \in G$.

In particular, we obtain the regular RG-module $R(G^{\times 1}) = RG$ and the trivial RG-module $R(G^{\times 0}) = R$.

(16) Suppose given a ring S. Suppose given a commutative quadrangle of S-right modules as follows.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

The quadrangle (A, B, C, D) is called a *square* if the sequence $A \xrightarrow{(f\ g)} B \oplus C \xrightarrow{\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right)} D$ is short exact.

To indicate that this quadrangle is a square, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Box"] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D\ . \end{tikzcd}$$

1 Preliminaries on diagrams

Let S be a ring.

Remark 1 *Suppose given the following commutative diagram of S-right modules.*

$$\begin{tikzcd}
& C' \arrow[rr, "k'"] \arrow[dd, "c'"'] & & D' \arrow[dd, "d'"] \\
A' \arrow[ur, "g'"] \arrow[rr, "f'"] \arrow[dd, "a'"'] & & B' \arrow[ur, "h'"] \arrow[dd, "b'"] & \\
& C \arrow[rr, "k"] \arrow[dd, "c''"'] \arrow[dr, phantom, "\square"] & & D \arrow[dd, "d''"] \\
A \arrow[ur, "g"] \arrow[rr, "f"] \arrow[dd, "a''"'] & & B \arrow[ur, "h"] \arrow[dd, "b''"] & \\
& C'' \arrow[rr, "k''"] \arrow[dr, phantom, "\square"] & & D'' \\
A'' \arrow[ur, "g''"] \arrow[rr, "f''"] & & B'' \arrow[ur, "h''"] &
\end{tikzcd}$$

Suppose that the vertical sequences (A', A, A''), (B', B, B''), (C', C, C''), (D', D, D'') are short exact.

Then the top layer of the diagram is a square:

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & B' \arrow[d, "h'"] \\
C' \arrow[r, "k'"'] & D'
\end{tikzcd} .$$

Proof. We have the following commutative diagram.

$$\begin{tikzcd}
A' \arrow[r, "(f'\ g')"] \arrow[d, "a'"] & B' \oplus C' \arrow[r, "\left(\begin{smallmatrix} h' \\ -k' \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} b' & 0 \\ 0 & c' \end{smallmatrix}\right)"] & D' \arrow[d, "d'"] \\
A \arrow[r, "(f\ g)"] \arrow[d, "a''"] & B \oplus C \arrow[r, "\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} b'' & 0 \\ 0 & c'' \end{smallmatrix}\right)"] & D \arrow[d, "d''"] \\
A'' \arrow[r, "(f''\ g'')"] & B'' \oplus C'' \arrow[r, "\left(\begin{smallmatrix} h'' \\ -k'' \end{smallmatrix}\right)"] & D''
\end{tikzcd}$$

In the vertical direction we have short exact sequences (A', A, A''), $(B' \oplus C',\ B \oplus C,\ B'' \oplus C'')$ and (D', D, D'').

In fact, concerning $(B' \oplus C',\, B \oplus C,\, B'' \oplus C'')$ being short exact, we can argue as follows.

The map $\left(\begin{smallmatrix} b' & 0 \\ 0 & c' \end{smallmatrix}\right)$ is injective, as b' and c' are injective maps.

The map $\left(\begin{smallmatrix} b'' & 0 \\ 0 & c'' \end{smallmatrix}\right)$ is surjective, as b'' and c'' are surjective maps.

Moreover, we have $\left(\begin{smallmatrix} b' & 0 \\ 0 & c' \end{smallmatrix}\right)\left(\begin{smallmatrix} b'' & 0 \\ 0 & c'' \end{smallmatrix}\right) = \left(\begin{smallmatrix} b'b'' & 0 \\ 0 & c'c'' \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Finally, given $(x, y) \in B \oplus C$ such that $(0, 0) = (x, y)\left(\begin{smallmatrix} b'' & 0 \\ 0 & c'' \end{smallmatrix}\right) = (xb'', yc'')$, there exist elements $x' \in B'$ and $y' \in C'$ such that $x'b' = x$ and $y'c' = y$. Hence we have $(x', y') \in B' \oplus C'$ with $(x', y')\left(\begin{smallmatrix} b' & 0 \\ 0 & c' \end{smallmatrix}\right) = (x'b', y'c') = (x, y)$.

Since (A, B, C, D) is a square, the sequence $(A, B \oplus C, D)$ is short exact.

Since (A'', B'', C'', D'') is a square, the sequence $(A'', B'' \oplus C'', D'')$ is short exact.

We now apply the snake lemma to this diagram and get an S-linear map $\varphi : D' \to \operatorname{Coker}(a'')$ such that

$$0 \longrightarrow A' \xrightarrow{(f'\ g')} B' \oplus C' \xrightarrow{\left(\begin{smallmatrix} h' \\ -k' \end{smallmatrix}\right)} D' \xrightarrow{\varphi} \operatorname{Coker}(a'')$$

is an exact sequence. As a'' is surjective, we know that $\operatorname{Coker}(a'') = 0$. Hence the sequence $(A', B' \oplus C', D')$ is short exact. Thus the quadrangle (A', B', C', D') is a square. $\square$

Remark 2 *Suppose given a commutative quadrangle of S-right modules as follows.*

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "\sim", "k"'] & D \end{tikzcd}$$

Then

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] \arrow[d, "g"'] \arrow[dr, phantom, "\Box"] & B \arrow[d, "h"] \\ C \arrow[r, "\sim", "k"'] & D \end{tikzcd}.$$

Proof. We have to show that the sequence

$$A \xrightarrow{(f\ g)} B \oplus C \xrightarrow{\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right)} D$$

is short exact.

We show that $A \xrightarrow{(f\ g)} B \oplus C$ is injective. Suppose given $a \in A$ such that $a\,(f\ g) = (af, ag) = (0, 0)$. Then $af = 0$. Since f is injective, we conclude that $a = 0$.

We show that $B \oplus C \xrightarrow{\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right)} D$ is surjective. Suppose given $d \in D$. Then $(0, -dk^{-1})\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right) = dk^{-1}k = d$.

We show that $\mathrm{Im}((\,f\;g\,)) \subseteq \mathrm{Ker}(\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right))$. In fact, $(\,f\;g\,)\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right) = fh - gk = 0$.

We show that $\mathrm{Im}((\,f\;g\,)) \supseteq \mathrm{Ker}(\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right))$. Suppose given $(b,c) \in \mathrm{Ker}(\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right))$. Then $0 = (b,c)\left(\begin{smallmatrix} h \\ -k \end{smallmatrix}\right) = bh - ck$. So $bh = ck$. For the element $bf^{-1} \in A$, we obtain

$$bf^{-1}(\,f\;g\,) \;=\; (bf^{-1}f, bf^{-1}g) \;=\; (b, bf^{-1}gkk^{-1}) \;=\; (b, bf^{-1}fhk^{-1}) \;=\; (b, ckk^{-1}) \;=\; (b,c)\,.$$

□

Remark 3 *Suppose given the following commutative triangle of S-right modules.*

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "h"'] & & W \end{tikzcd} .$$

Let K_f, K_g and K_h denote the respective kernels, with inclusion maps i_f, i_g and i_h.

Let C_f, C_g and C_h denote the respective cokernels, with residue class maps r_f, r_g and r_h.

We obtain a commutative diagram of S-right modules as follows.

$$\begin{tikzcd}
 & \mathrm{K}_g \arrow[rr, "i_g r_f"] \arrow[dr, "i_g"] & & \mathrm{C}_f \arrow[dr, "\check{g}"] & \\
 & & V \arrow[ur, "r_f"] \arrow[dr, "g"] & & \\
\mathrm{K}_h \arrow[uur, "\check{f}"] \arrow[r, "i_h"] & U \arrow[ur, "f"] \arrow[rr, "h"'] & & W \arrow[r, "r_h"] \arrow[dr, "r_g"'] & \mathrm{C}_h \arrow[d, "v"] \\
\mathrm{K}_f \arrow[u, "u"] \arrow[ur, "i_f"'] & 0 \arrow[l] & & 0 & \mathrm{C}_g \arrow[l]
\end{tikzcd}$$

Therein, the S-linear maps u, $\check{f}$, $\check{g}$, v are uniquely determined by commutativity.

Then the sequence

$$0 \longrightarrow \mathrm{K}_f \xrightarrow{u} \mathrm{K}_h \xrightarrow{\check{f}} \mathrm{K}_g \xrightarrow{i_g \cdot r_f} \mathrm{C}_f \xrightarrow{\check{g}} \mathrm{C}_h \xrightarrow{v} \mathrm{C}_g \longrightarrow 0$$

is exact. It is called the circonference sequence *of the commutative triangle.*

Proof.

As $i_f h = i_f fg = 0$, there exists a unique S-linear map $u : \mathrm{K}_f \to \mathrm{K}_h$ such that $ui_h = i_f$.

As $i_h fg = i_h h = 0$, there exists a unique S-linear map $\check{f} : \mathrm{K}_h \to \mathrm{K}_g$ such that $\check{f} i_g = i_h f$.

As $fgr_h = hr_h = 0$, there exists a unique S-linear map $\check{g} : \mathrm{C}_f \to \mathrm{C}_h$ such that $r_f \check{g} = gr_h$.

As $hr_g = fgr_g = 0$, there exists a unique S-linear map $v : \mathrm{C}_h \to \mathrm{C}_g$ such that $r_h v = r_g$.

Exactness at K_f. Since $ui_h = i_f$ and since i_f is injective, the map u is injective.

Exactness at K_h. We have $u\check{f}i_g = ui_h f = i_f f = 0$, and thus $u\check{f} = 0$ since i_g is injective. Conversely, suppose given $x \in \mathrm{K}_h$ such that $x\check{f} = 0$. Then $0 = x\check{f}i_g = xi_h f = xf$. So $x \in \mathrm{K}_f$. Hence $xu \in \mathrm{K}_h$. We obtain

$$xu \;=\; xui_h \;=\; xi_f \;=\; x\,.$$

Exactness at K_g. We have $\check{f}(i_g r_f) = i_h f r_f = 0$. Conversely, suppose given $x \in \mathrm{K}_g$ such that $x(i_g r_f) = 0$. Then $x = xi_g \in Uf$. Choose $y \in U$ such that $yf = x$. We obtain

$$yh \;=\; yfg \;=\; xg \;=\; 0\,.$$

Thus $y \in \mathrm{K}_h$. We obtain

$$y\check{f} \;=\; y\check{f}i_g \;=\; yi_h f \;=\; yf \;=\; x\,.$$

Exactness at C_f. We have $(i_g r_f)\check{g} = i_g g r_h = 0$. Conversely, suppose given $x \in V$ such that $(x + Uf)\check{g} = 0$. Then $0 = (x + Uf)\check{g} = xr_f\check{g} = xgr_h$. So $xg \in Uh$. We choose $y \in U$ such that $xg = yh$. So $xg = yfg$. So $(x - yf)g = 0$, i.e. $x - yf \in \mathrm{K}_g$. We have

$$(x - yf)(i_g r_f) \;=\; (x - yf)r_f \;=\; (x - yf) + Uf \;=\; x + Uf\,.$$

Exactness at C_h. We have $r_f\check{g}v = gr_h v = gr_g = 0$, and thus $\check{g}v = 0$ since r_f is surjective. Conversely, suppose given $x \in W$ such that $(x + Uh)v = 0$. Then $0 = (x + Uh)v = xr_h v = xr_g$. So $x \in Vg$. Choose $y \in V$ such that $x = yg$. Then $y + Uf \in \mathrm{C}_f$ and

$$(y + Uf)\check{g} \;=\; yr_f\check{g} \;=\; ygr_h \;=\; xr_h \;=\; x + Uh\,.$$

Exactness at C_g. Since $r_h v = r_g$ and since r_g is surjective, the map v is surjective. ▫

2 The standard resolution

Let G be a group. Let R be a commutative ring.

Definition 4

(1) Suppose given a G-set M.

We write RM for the free R-module with basis M. Then RM is an RG-module via

$$\left(\sum_{m\in M} r_m\, m\right)\cdot\left(\sum_{g\in G} s_g\, g\right) \;:=\; \sum_{m\in M}\sum_{g\in G} r_m\, s_g\, m\cdot g \;=\; \sum_{n\in M}\left(\sum_{\substack{g\in G\\ m\cdot g=n}} r_m\, s_g\right) n$$

where $r_m \in R$ and $s_g \in R$.

(2) Suppose given G-sets M and N. Suppose given a G-map $u : M \to N$.

We have the RG-linear map

$$\begin{array}{rcl} Ru \;:\; RM & \to & RN \\ \sum_{m\in M} r_m\, m & \mapsto & \sum_{m\in M} r_m\, mu \;=\; \sum_{n\in N}\left(\sum_{\substack{m\in M\\ mu=n}} r_m\right) n \end{array}$$

We sometimes write $u := Ru : RM \to RN$ by abuse of notation.

Definition 5 Suppose given $n,\, k \in \mathbb{Z}_{\geqslant 0}$. We define the R-linear map

$$\begin{array}{rcl} R(G^{\times n}) & \xrightarrow{\chi_{n,k}} & R(G^{\times(n+k)}) \\ (g_i)_{i\in[0,n-1]} & \longmapsto & (g_i)_{i\in[0,n-1]} \,\sqcup\, (1)_{i\in[0,k-1]}\,. \end{array}$$

Additionally, we define the R-linear map $\chi_{-1,k} : 0 \to R(G^{\times(k-1)}) : 0 \mapsto 0$.

Remark 6 *Suppose given $n \in \mathbb{Z}_{\geqslant -1}$ and $k, m \in \mathbb{Z}_{\geqslant 0}$. Then we have $\chi_{n,k}\chi_{n+k,m} = \chi_{n,k+m}$.*

Proof.

Case $n \geqslant 0$. Suppose given $(g_i)_{i\in[0,n-1]} \in G^{\times n}$. Then

$$\begin{array}{rcl} (g_i)_{i\in[0,n-1]}\chi_{n,k}\chi_{n+k,m} & = & \big((g_i)_{i\in[0,n-1]} \,\sqcup\, (1)_{i\in[0,k-1]}\big) \,\sqcup\, (1)_{i\in[0,m-1]} \\ & = & (g_i)_{i\in[0,n-1]} \,\sqcup\, (1)_{i\in[0,k+m-1]} \\ & = & (g_i)_{i\in[0,n-1]}\chi_{n,k+m}\,. \end{array}$$

Case $n = -1$. We have $(0)\chi_{-1,k}\chi_{k-1,m} = (0)\chi_{k-1,m} = 0 = (0)\chi_{-1,k+m}$. □

Remark 7 *Suppose given $n,\, k \in \mathbb{Z}_{\geqslant 0}$. Then the map $\chi_{n,k}$ is injective.*

Proof. For $R(G^{\times n})$ we can choose the elements of $G^{\times n}$ as an R-basis. Similarly we can choose the elements of $G^{\times (n+k)}$ as an R-basis of $R(G^{\times (n+k)})$. This way $\chi_{n,k}$ maps basis elements of $R(G^{\times n})$ to basis elements of $R(G^{\times (n+k)})$. Hence it is sufficient to show injectivity on basis elements to show that $\chi_{n,k}$ is injective.

Suppose given $\underline{g}, \underline{h} \in G^{\times n}$ with $\underline{g}\chi_{n,k} = \underline{h}\chi_{n,k}$. As $\underline{g}\chi_{n,k} = \underline{g} \sqcup (1)_{i\in[0,k-1]}$ and $\underline{h}\chi_{n,k} = \underline{h} \sqcup (1)_{i\in[0,k-1]}$ we obtain that $(\underline{g} \sqcup (1)_{i\in[0,k-1]})_j = (\underline{h} \sqcup (1)_{i\in[0,k-1]})_j$ for all $j \in [0, n-1]$. From here it follows that $\underline{g} = \underline{h}$. ▫

Definition 8 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. We define the RG-linear map ε_n^G as follows.

If $n \geqslant 1$, we let

$$\begin{aligned} \varepsilon_n^G : R(G^{\times n}) &\to R(G^{\times (n-1)}) \\ \underline{g} &\mapsto \sum_{k=0}^{n-1} (-1)^k \underline{g}\uparrow^k . \end{aligned}$$

If $n = 0$, we let

$$\begin{aligned} \varepsilon_0^G : R &\to 0 \\ r &\mapsto 0. \end{aligned}$$

Note that for $n = 1$, we obtain

$$\begin{array}{ccccccccc} \varepsilon_1^G & : & RG & = & R(G^{\times 1}) & \to & R(G^{\times 0}) & = & R \\ & & g & = & (g) & \mapsto & (\,) & = & 1 \end{array}$$

by identification.

We often write $\varepsilon_n := \varepsilon_n^G$.

Remark 9 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.*

(1) *We have* $\chi_{n,1}\varepsilon_{n+1} = \varepsilon_n\chi_{n-1,1} + (-1)^n \operatorname{id}_{R(G^{\times n})}$.

(2) *We have* $\chi_{n,2}\varepsilon_{n+2} = \varepsilon_n\chi_{n-1,2}$.

Proof. Suppose given $\underline{g} \in G^{\times n}$.

Ad (1). If $n \geqslant 1$, then we obtain

$$\begin{aligned}
\underline{g}\chi_{n,1}\varepsilon_{n+1} &= \sum_{k=0}^{n}(-1)^k \underline{g}\chi_{n,1}\uparrow^k \\
&= \sum_{k=0}^{n-1}(-1)^k \underline{g}\chi_{n,1}\uparrow^k + (-1)^n(\underline{g}\chi_{n,1})\uparrow^n \\
&= \sum_{k=0}^{n-1}(-1)^k \underline{g}\uparrow^k \chi_{n-1,1} + (-1)^n \underline{g} \\
&= \underline{g}\big(\varepsilon_n\chi_{n-1,1} + (-1)^n \operatorname{id}\big) .
\end{aligned}$$

If $n = 0$, then we obtain

$$\begin{aligned}
(\,)\chi_{0,1}\varepsilon_1 &= (1)\varepsilon_1 \\
&= (\,) \\
&= 0\chi_{-1,1} + (\,) \\
&= (\,)\big(\varepsilon_0\chi_{-1,1} + \operatorname{id}\big) .
\end{aligned}$$

Ad (2). If $n \geqslant 1$, then we obtain

$$\begin{aligned}
(\underline{g}\chi_{n,2})\varepsilon_{n+2} &= \sum_{k=0}^{n+1}(-1)^k(\underline{g}\chi_{n,2})\uparrow^k \\
&= \sum_{k=0}^{n-1}(-1)^k(\underline{g}\chi_{n,2})\uparrow^k \\
&\quad + (-1)^n(\underline{g}\chi_{n,2})\uparrow^n + (-1)^{n+1}(\underline{g}\chi_{n,2})\uparrow^{n+1} \\
&= \sum_{k=0}^{n-1}(-1)^k(\underline{g}\uparrow^k)\chi_{n-1,2} \\
&\quad + (-1)^n\underline{g}\chi_{n,1} + (-1)^{n+1}\underline{g}\chi_{n,1} \\
&= \sum_{k=0}^{n-1}(-1)^k(\underline{g}\uparrow^k)\chi_{n-1,2} \\
&= (\underline{g}\varepsilon_n)\chi_{n-1,2} .
\end{aligned}$$

If $n = 0$, then we obtain

$$\begin{aligned}
(\,)\chi_{0,2}\varepsilon_2 &= (1,1)\varepsilon_2 \\
&= (1) - (1) \\
&= 0 \\
&= 0\chi_{-1,2} \\
&= (\,)\varepsilon_0\chi_{-1,2} .
\end{aligned}$$

$\square$

Lemma 10 *We have the following acyclic complex of RG-modules.*

$$\ldots \longrightarrow R(G^{\times 4}) \xrightarrow{\varepsilon_4} R(G^{\times 3}) \xrightarrow{\varepsilon_3} R(G^{\times 2}) \xrightarrow{\varepsilon_2} RG \xrightarrow{\varepsilon_1} R \xrightarrow{\varepsilon_0} 0 \longrightarrow \ldots$$

Proof. Suppose given $n \geqslant 1$.

Ad $\mathrm{Im}(\varepsilon_n) \overset{!}{\subseteq} \mathrm{Ker}(\varepsilon_{n-1})$.

Case $n = 1$. We have $\varepsilon_1 \varepsilon_0 = 0$.

Case $n \geqslant 2$. Suppose given $\underline{g} = (g_0, g_1, g_2, \ldots, g_{n-1}) \in G^{\times n}$.

For $k \in [0, n-1]$, we get

$$\begin{aligned}
(\underline{g}\uparrow^k)\varepsilon_{n-1} &= \sum_{i=0}^{n-2}(-1)^i \Big(\underline{g}\uparrow^k\Big)\uparrow^i \\
&= \sum_{i=0}^{k-1}(-1)^i \Big(\underline{g}\uparrow^k\Big)\uparrow^i + \sum_{i=k}^{n-2}(-1)^i \Big(\underline{g}\uparrow^k\Big)\uparrow^i \\
&= \sum_{i=0}^{k-1}(-1)^i \Big(\underline{g}\uparrow^k\Big)\uparrow^i + \sum_{i=k+1}^{n-1}(-1)^{i-1} \Big(\underline{g}\uparrow^k\Big)\uparrow^{i-1} \\
&= \sum_{i=0}^{k-1}(-1)^i \underline{g}\uparrow^{i,k} + \sum_{i=k+1}^{n-1}(-1)^{i-1} \underline{g}\uparrow^{k,i} .
\end{aligned}$$

We conclude that

$$\begin{aligned}
&((\underline{g})\varepsilon_n)\varepsilon_{n-1} \\
&= \sum_{k=0}^{n-1}(-1)^k \Big(\sum_{i=0}^{k-1}(-1)^i \underline{g}\uparrow^{i,k} + \sum_{i=k+1}^{n-1}(-1)^{i-1}\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{k=0}^{n-1}(-1)^k \sum_{i=0}^{k-1}(-1)^i \underline{g}\uparrow^{i,k}\Big) + \Big(\sum_{k=0}^{n-1}(-1)^k \sum_{i=k+1}^{n-1}(-1)^{i-1}\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{k=0}^{n-1}\sum_{i=0}^{k-1}(-1)^{k+i} \underline{g}\uparrow^{i,k}\Big) + \Big((-1)^{-1}\sum_{k=0}^{n-1}\sum_{i=k+1}^{n-1}(-1)^{k+i}\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{\substack{i,k\in[0,n-1]\\ i<k}}(-1)^{k+i} \underline{g}\uparrow^{i,k}\Big) + \Big((-1)^{-1}\sum_{\substack{i,k\in[0,n-1]\\ k<i}}(-1)^{k+i}\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{\substack{i,k\in[0,n-1]\\ k<i}}(-1)^{k+i} \underline{g}\uparrow^{k,i}\Big) - \Big(\sum_{\substack{i,k\in[0,n-1]\\ k<i}}(-1)^{k+i}\underline{g}\uparrow^{k,i}\Big) = 0 .
\end{aligned}$$

Ad $\mathrm{Ker}(\varepsilon_{n-1}) \overset{!}{\subseteq} \mathrm{Im}(\varepsilon_n)$. Suppose given $x \in \mathrm{Ker}(\varepsilon_{n-1})$. We obtain

$$\begin{array}{rcl} x & = & (-1)^{n-1} x((-1)^{n-1}\,\mathrm{id}) \\ & \overset{\text{R. 9.(1)}}{=} & (-1)^{n-1} x(\chi_{n-1,1}\varepsilon_n - \varepsilon_{n-1}\chi_{n-2,1}) \\ & = & (-1)^{n-1}(\underbrace{x\chi_{n-1,1}\varepsilon_n}_{\in \mathrm{Im}(\varepsilon_n)} - \underbrace{x\varepsilon_{n-1}\chi_{n-2,1}}_{=0}) \\ & \in & \mathrm{Im}(\varepsilon_n) \end{array}$$

□

Definition 11 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

(1) We define the n-th *augmentation module* as

$$\mathrm{I}_R^{(n)}(G) \;:=\; \mathrm{Ker}(\varepsilon_n) \;\subseteq\; R(G^{\times n}) \,.$$

Then $\mathrm{I}_R^{(n)}(G)$ is an RG-submodule of $R(G^{\times n})$.

(2) We write $\bar{\varepsilon}_n := \varepsilon_n|^{\mathrm{I}_R^{(n-1)}(G)} : R(G^{\times n}) \to \mathrm{I}_R^{(n-1)}(G)$.

Remark 12 *Suppose given $n \in \mathbb{Z}_{\geqslant 2}$. Then*

$$\mathrm{I}_R^{(n)}(G) \;\cap\; R(G^{\times(n-1)})\chi_{n-1,1} \;\;=\;\; \mathrm{I}_R^{(n-2)}(G)\chi_{n-2,2} \,.$$

Proof.

Ad $(\supseteq)$. Suppose given $y \in \mathrm{I}_R^{(n-2)}(G)$.

We have to show that $y\chi_{n-2,2} \overset{!}{\in} \mathrm{I}_R^{(n)}(G) \;\cap\; R(G^{\times(n-1)})\chi_{n-1,1}$.

It suffices to show that $y\chi_{n-2,2} \overset{!}{\in} \mathrm{I}_R^{(n)}(G)$; cf. Remark 6. In fact, we have

$$y\chi_{n-2,2}\varepsilon_n \;\overset{\text{R. 9.(2)}}{=}\; y\varepsilon_{n-2}\chi_{n-3,2} \;=\; 0\chi_{n-3,2} \;=\; 0 \,.$$

Ad $(\subseteq)$. Suppose given $x \in R(G^{\times(n-1)})$ such that $x\chi_{n-1,1} \in \mathrm{I}_R^{(n)}(G)$.

We have to show that $x\chi_{n-1,1} \overset{!}{\in} \mathrm{I}_R^{(n-2)}(G)\chi_{n-2,2}$.

Since $x\chi_{n-1,1} \in \mathrm{I}_R^{(n)}(G)$, we have $x\chi_{n-1,1}\varepsilon_n = 0$.

We have

$$\begin{array}{rcl} 0 & = & x\chi_{n-1,1}\varepsilon_n \\ & \overset{\text{R. 9.(1)}}{=} & x\varepsilon_{n-1}\chi_{n-2,1} + (-1)^{n-1}x \end{array}$$

So

$$x \;=\; (-1)^n x\varepsilon_{n-1}\chi_{n-2,1} \,.$$

Hence

$$x\chi_{n-1,1} \;=\; (-1)^n x\varepsilon_{n-1}\chi_{n-2,1}\chi_{n-1,1} \;\overset{\text{R. 6}}{=}\; \big((-1)^n x\varepsilon_{n-1}\big)\chi_{n-2,2} \,.$$

Moreover, $(-1)^n x\varepsilon_{n-1} \in \mathrm{I}_R^{(n-2)}(G)$ since $((-1)^n x\varepsilon_{n-1})\varepsilon_{n-2} \overset{\text{L. 10}}{=} 0$. □

3 An R-linear basis of $\mathrm{I}_R^{(n)}(G)$

Suppose given a finite group G. Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

Definition 13 (bracket notation)

We define the map ι_n as follows.

$$\begin{array}{rcl} G^{\times n} & \xrightarrow{\iota_n} & G^{\times n} \\ (g_0, g_1, \ldots, g_{n-1}) & \mapsto & \left(\prod_{j=k}^{n-1} g_j : k \in [0, n-1] \right) \\ & & = \; (\, g_0 \cdot \ldots \cdot g_{n-1} \,,\; g_1 \cdot \ldots \cdot g_{n-1} \,,\; \ldots \,,\; g_{n-2} \cdot g_{n-1} \,,\; g_{n-1} \,) \\ & & =: \; [g_0, g_1, \ldots, g_{n-1}] \end{array}$$

Remark 14 The map $\iota_0 : G^{\times 0} \to G^{\times 0}$ maps the empty tuple to the empty tuple, i.e.

$$(\,)\iota_0 \;=\; (\,)\,.$$

In particular, ι_0 is bijective.

Note that $[\,] = (\,) \in G^{\times 0}$.

Example 15

(1) Given $(g_0) \in G^{\times 1}$, we have

$$[g_0] \;=\; (g_0)\iota_1 \;=\; (g_0)\,.$$

(2) Given $(g_0, g_1) \in G^{\times 2}$, we have

$$[g_0, g_1] \;=\; (g_0, g_1)\iota_2 \;=\; (g_0 \cdot g_1,\; g_1)\,.$$

(3) Given $(g_0, g_1, g_2) \in G^{\times 3}$, we have

$$[g_0, g_1, g_2] \;=\; (g_0, g_1, g_2)\iota_3 \;=\; (g_0 \cdot g_1 \cdot g_2,\; g_1 \cdot g_2,\; g_2)\,.$$

Remark 16 *Suppose that $n \geqslant 1$.*

We define the map $\tilde{\iota}_n$ as follows.

$$\begin{array}{rcl} G^{\times n} & \xrightarrow{\tilde{\iota}_n} & G^{\times n} \\ (h_0, h_1, \ldots, h_{n-1}) & \mapsto & (h_0 h_1^{-1}, h_1 h_2^{-1}, \ldots, h_{n-2} h_{n-1}^{-1}, h_{n-1}) \end{array}$$

The map $\iota_n : G^{\times n} \to G^{\times n}$ is bijective, with inverse map $\tilde{\iota}_n$.

Proof. For $(g_0, g_1, \ldots, g_{n-1}) \in G^{\times n}$, we obtain

$$
\begin{aligned}
& ((g_0, g_1, \ldots, g_{n-1})\iota_n)\tilde{\iota}_n \\
= \; & \Big((\prod_{j=0}^{n-1} g_j, \prod_{j=1}^{n-1} g_j, \prod_{j=2}^{n-1} g_j, \ldots, g_{n-2}g_{n-1}, g_{n-1}) \Big) \tilde{\iota}_n \\
= \; & \Big((\prod_{j=0}^{n-1} g_j)(\prod_{j=1}^{n-1} g_j)^{-1}, (\prod_{j=1}^{n-1} g_j)(\prod_{j=2}^{n-1} g_j)^{-1}, (\prod_{j=2}^{n-1} g_j)(\prod_{j=3}^{n-1} g_j)^{-1}, \ldots, g_{n-2}g_{n-1}g_{n-1}^{-1}, g_{n-1} \Big) \\
= \; & \Big(g_0(\prod_{j=1}^{n-1} g_j)(\prod_{j=1}^{n-1} g_j)^{-1}, g_1(\prod_{j=2}^{n-1} g_j)(\prod_{j=2}^{n-1} g_j)^{-1}, g_2(\prod_{j=3}^{n-1} g_j)(\prod_{j=3}^{n-1} g_j)^{-1}, \ldots, g_{n-2}g_{n-1}g_{n-1}^{-1}, g_{n-1} \Big) \\
= \; & (g_0, g_1, g_2, \ldots, g_{n-2}, g_{n-1}) \, .
\end{aligned}
$$

Hence $\iota_n \cdot \tilde{\iota}_n = \mathrm{id}_{G^{\times n}}$.

For $(h_0, h_1, \ldots, h_{n-1}) \in G^{\times n}$, we obtain

$$
\begin{aligned}
& ((h_0, h_1, \ldots, h_{n-1})\tilde{\iota}_n)\iota_n \\
= \; & (h_0h_1^{-1}, h_1h_2^{-1}, \ldots, h_{n-2}h_{n-1}^{-1}, h_{n-1})\iota_n \\
= \; & \Big((\prod_{j=0}^{n-2}(h_jh_{j+1}^{-1}))h_{n-1}, (\prod_{j=1}^{n-2}(h_jh_{j+1}^{-1}))h_{n-1}, (\prod_{j=2}^{n-2}(h_jh_{j+1}^{-1}))h_{n-1}, \ldots, \big(h_{n-2}h_{n-1}^{-1}\big)h_{n-1}, h_{n-1} \Big) \\
= \; & (h_0, h_1, h_2, \ldots, h_{n-2}, h_{n-1})
\end{aligned}
$$

Hence $\tilde{\iota}_n \cdot \iota_n = \mathrm{id}_{G^{\times n}}$. ▫

Corollary 17 *The set*

$$\{\, [g_0, g_1, \ldots, g_{n-1}] \mid g_0, \ldots, g_{n-1} \in G \,\} \;=\; G^{\times n}$$

is an R-linear basis of $R(G^{\times n})$.

Proof. By Remarks 14 and 16, the map $\iota_n : G^{\times n} \to G^{\times n} : (g_0, g_1, \ldots, g_{n-1}) \mapsto [g_0, g_1, \ldots, g_{n-1}]$ is bijective. ▫

Remark 18 *We abbreviate $\chi_{n,k} := \chi_{n,k}|_{G^{\times n}}^{G^{\times (n+k)}}$.*

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. We have

$$\iota_n \chi_{n,k} \;=\; \chi_{n,k}\iota_{n+k} \;:\; G^{\times n} \;\to\; G^{\times (n+k)} \, .$$

Proof. Suppose given $(g_0, g_1, \dots, g_{n-1}) \in G^{\times n}$. We obtain

$$
\begin{aligned}
& (g_0, g_1, \dots, g_{n-1})\iota_n \chi_{n,k} \\
= \;& [g_0, g_1, \dots, g_{n-1}]\chi_{n,k} \\
= \;& \left(\prod_{j=0}^{n-1} g_j, \prod_{j=1}^{n-1} g_j, \prod_{j=2}^{n-1} g_j, \dots, g_{n-2}g_{n-1}, g_{n-1}, 1, \dots, 1 \right) \\
= \;& \left((\prod_{j=0}^{n-1} g_j) \cdot 1^k, (\prod_{j=1}^{n-1} g_j) \cdot 1^k, (\prod_{j=2}^{n-1} g_j) \cdot 1^k, \dots, g_{n-2}g_{n-1}1^k, g_{n-1}1^k, 1^k, 1^{k-1}, \dots, 1^2, 1 \right) \\
= \;& [g_0, g_1, \dots, g_{n-1}, 1, \dots, 1] \\
= \;& (g_0, g_1, \dots, g_{n-1}, 1, \dots, 1)\iota_{n+k} \\
= \;& (g_0, g_1, \dots, g_{n-1})\chi_{n,k}\iota_{n+k} \; .
\end{aligned}
$$

□

Remark 19 Suppose given $(g_0, g_1, \dots, g_{n-1}) \in G^{\times n}$. Suppose given $k \in [0, n-1]$.

We have

$$
\begin{aligned}
[g_0, g_1, \dots, g_{n-1}] \uparrow^k &= \left(\prod_{j=0}^{n-1} g_j, \prod_{j=1}^{n-1} g_j, \prod_{j=2}^{n-1} g_j, \dots, g_{n-2}g_{n-1}, g_{n-1} \right) \uparrow^k \\
&= \left(\prod_{j=0}^{n-1} g_j, \prod_{j=1}^{n-1} g_j, \prod_{j=2}^{n-1} g_j, \dots, \prod_{j=k-1}^{n-1} g_j, \prod_{j=k+1}^{n-1} g_j, \dots, g_{n-2}g_{n-1}, g_{n-1} \right) \\
&= \begin{cases} [g_1, \dots, g_{n-1}] & \text{if } k = 0 \\ [g_0, g_1, \dots, g_{k-2},\; g_{k-1} \cdot g_k\,, g_{k+1}, \dots, g_{n-1}] & \text{if } k \in [1, n-1]. \end{cases}
\end{aligned}
$$

Definition 20 We abbreviate $\varepsilon_n := \varepsilon_n|_{G^{\times n}}$.

We define the map

$$
\xi_n \;:=\; \xi_n^G \;:=\; (\iota_n + (-1)^n \iota_n \varepsilon_n \chi_{n-1,1})|^{\mathrm{I}_R^{(n)}(G)} \;:\; G^{\times n} \;\to\; \mathrm{I}_R^{(n)}(G) \;\subseteq\; R(G^{\times n}) \; .
$$

If $n \geqslant 1$, we in fact obtain

$$
\begin{aligned}
(\iota_n + (-1)^n \iota_n \varepsilon_n \chi_{n-1,1})\varepsilon_n \quad &= \quad \iota_n\varepsilon_n + (-1)^n \iota_n \varepsilon_n \chi_{n-1,1} \varepsilon_n \\
&\overset{\text{R. 9.(1)}}{=} \quad \iota_n\varepsilon_n + (-1)^n \iota_n \varepsilon_n (\varepsilon_{n-1}\chi_{n-2,1} + (-1)^{n-1} \operatorname{id}_{R(G^{\times(n-1)})}) \\
&= \quad \iota_n\varepsilon_n + (-1)^n \iota_n \varepsilon_n \varepsilon_{n-1}\chi_{n-2,1} + (-1)^{n+(n-1)} \iota_n\varepsilon_n \operatorname{id}_{R(G^{\times(n-1)})} \\
&\overset{\text{L. 10}}{=} \quad \iota_n\varepsilon_n - \iota_n\varepsilon_n \\
&= \quad 0 \; .
\end{aligned}
$$

So $\iota_n + (-1)^n \iota_n \varepsilon_n \chi_{n-1,1}$ actually maps to $\mathrm{I}_R^{(n)}(G) = \operatorname{Ker}(\varepsilon_n)$. Hence ξ_n is well-defined.

Note that given $\underline{g} = (g_0, g_1, \ldots, g_{n-2}, g_{n-1}) \in G^{\times n}$, we have, using Remark 19,

$$\begin{array}{rcl}
(\underline{g})\xi_n & = & [\underline{g}] + (-1)^n([\underline{g}])\varepsilon_n\chi_{n-1,1} \\
& = & [\underline{g}] \\
& & +(-1)^{n+(n-1)}[g_0, g_1, \ldots, g_{n-3}, g_{n-2} \cdot g_{n-1}, 1] \\
& & +(-1)^{n+(n-2)}[g_0, g_1, \ldots, g_{n-4}, g_{n-3} \cdot g_{n-2}, g_{n-1}, 1] \\
& & + \ldots \\
& & +(-1)^{n+1}[g_0 \cdot g_1, g_2, \ldots, g_{n-3}, g_{n-2}, g_{n-1}, 1] \\
& & +(-1)^{n+0}[g_1, \ldots, g_{n-3}, g_{n-2}, g_{n-1}, 1] \; .
\end{array}$$

Remark 21

(0) We have $(\,)\xi_0 = [\,] + (-1)^0([\,])\varepsilon_0\chi_{-1,1} = [\,] + 0 = (\,) \in \mathrm{I}_R^{(0)}(G) = R$.

(1) We have $(g_0)\xi_1 = [g_0] + (-1)^1([g_0])\varepsilon_1\chi_{0,1} = (g_0) - (1) = g_0 - 1 \in \mathrm{I}_R^{(1)}(G)$ for $g_0 \in G$.

Cf. Remark 33 below.

(2) We have

$$\begin{array}{rcl}
(g_0, g_1)\xi_2 & = & [g_0, g_1] + (-1)^2([g_0, g_1])\varepsilon_2\chi_{1,1} \\
& = & [g_0, g_1] + (-1)^{2+1}[g_0 \cdot g_1, 1] + (-1)^{2+0}[g_1, 1] \\
& = & [g_0, g_1] - [g_0 \cdot g_1, 1] + [g_1, 1] \\
& = & (g_0 \cdot g_1,\ g_1) - (g_0 \cdot g_1,\ 1) + (g_1,\ 1)
\end{array}$$

for $(g_0, g_1) \in G^{\times 2}$.

Cf. Remark 37 below.

(3) We have

$$\begin{array}{rcl}
(g_0, g_1, g_2)\xi_3 & = & [g_0, g_1, g_2] + (-1)^3([g_0, g_1, g_2])\varepsilon_3\chi_{2,1} \\
& = & [g_0, g_1, g_2] + (-1)^{3+2}[g_0, g_1 \cdot g_2, 1] + (-1)^{3+1}[g_0 \cdot g_1, g_2, 1] + (-1)^{3+0}[g_1, g_2, 1] \\
& = & [g_0, g_1, g_2] - [g_0, g_1 \cdot g_2, 1] + [g_0 \cdot g_1, g_2, 1] - [g_1, g_2, 1] \\
& = & (g_0 \cdot g_1 \cdot g_2,\ g_1 \cdot g_2,\ g_2) - (g_0 \cdot g_1 \cdot g_2,\ g_1 \cdot g_2,\ 1) \\
& & \quad + (g_0 \cdot g_1 \cdot g_2,\ g_2,\ 1) - (g_1 \cdot g_2,\ g_2,\ 1) \; .
\end{array}$$

for $(g_0, g_1, g_2) \in G^{\times 3}$.

Remark 22 *Suppose given $\underline{g} \in G^{\times n}$. Suppose given $k \in \mathbb{Z}_{\geqslant 0}$.*

We have

$$(\underline{g})\xi_n\chi_{n,2k} \;=\; (\underline{g})\chi_{n,2k}\xi_{n+2k} \; .$$

Proof. By Remark 6, it suffices to consider the case $k = 1$. We have to show that

$$(\underline{g})\xi_n\chi_{n,2} \overset{!}{=} (\underline{g})\chi_{n,2}\xi_{n+2}\ .$$

In fact, we obtain

$$\begin{array}{rcl}
(\underline{g})\xi_n\chi_{n,2} & = & ([\underline{g}] + (-1)^n([\underline{g}])\varepsilon_n\chi_{n-1,1})\chi_{n,2} \\
& \overset{\text{R. 6}}{=} & [\underline{g}]\chi_{n,2} + (-1)^n([\underline{g}])\varepsilon_n\chi_{n-1,2}\chi_{n+1,1} \\
& \overset{\text{R. 9.(2)}}{=} & [\underline{g}]\chi_{n,2} + (-1)^n([\underline{g}])\chi_{n,2}\varepsilon_{n+2}\chi_{n+1,1} \\
& = & (\underline{g})\iota_n\chi_{n,2} + (-1)^{n+2}(\underline{g})\iota_n\chi_{n,2}\varepsilon_{n+2}\chi_{n+1,1} \\
& \overset{\text{R. 18}}{=} & (\underline{g})\chi_{n,2}\iota_{n+2} + (-1)^{n+2}(\underline{g})\chi_{n,2}\iota_{n+2}\varepsilon_{n+2}\chi_{n+1,1} \\
& = & [\underline{g}\chi_{n,2}] + (-1)^{n+2}[\underline{g}\chi_{n,2}]\varepsilon_{n+2}\chi_{n+1,1} \\
& = & (\underline{g}\chi_{n,2})\xi_{n+2}\ .
\end{array}$$

□

Definition 23 We define the following subsets of $G^{\times n}$.

If $n \geqslant 1$, we let

$$\begin{array}{rcccl}
G^{\times n}_{=1} & := & \{(g_0,\dots,g_{n-1}) \in G^{\times n} \mid g_{n-1} = 1\} & \subseteq & G^{\times n} \\
G^{\times n}_{\neq 1} & := & \{(g_0,\dots,g_{n-1}) \in G^{\times n} \mid g_{n-1} \neq 1\} & \subseteq & G^{\times n}
\end{array}$$

We note that $G^{\times n}_{=1} \,\dot{\cup}\, G^{\times n}_{\neq 1} = G^{\times n}$. It follows that $R(G^{\times n}_{=1}) \oplus R(G^{\times n}_{\neq 1}) = R(G^{\times n})$.

In addition, we let $G^{\times 0}_{=1} := \emptyset$ and $G^{\times 0}_{\neq 1} := \{(\,)\}$.

If $n \geqslant 2$, we let

$$\begin{array}{rcccl}
G^{\times n}_{=(1,1)} & := & \{(g_0,\dots,g_{n-1}) \in G^{\times n} \mid g_{n-2} = 1 \wedge g_{n-1} = 1\} & \subseteq & G^{\times n} \\
G^{\times n}_{\neq(1,1)} & := & \{(g_0,\dots,g_{n-1}) \in G^{\times n} \mid g_{n-2} \neq 1 \vee g_{n-1} \neq 1\} & \subseteq & G^{\times n}
\end{array}$$

We note that $G^{\times n}_{=(1,1)} \,\dot{\cup}\, G^{\times n}_{\neq(1,1)} = G^{\times n}$. It follows that $R(G^{\times n}_{=(1,1)}) \oplus R(G^{\times n}_{\neq(1,1)}) = R(G^{\times n})$.

Remark 24 *Suppose that* $n \geqslant 1$.

The map $\xi_n|_{G^{\times n}_{n\neq 1}} : G^{\times n}_{n\neq 1} \to \mathrm{I}^{(n)}_R(G)$ *is injective. Its image does not contain* 0.

Proof. Suppose given $\underline{g}$, $\underline{h} \in G^{\times n}_{\neq 1}$ such that $(\underline{g})\xi_n = (\underline{h})\xi_n$. We have to show that $\underline{g} \overset{!}{=} \underline{h}$.

We have

$$[\underline{g}] + (-1)^n([\underline{g}])\varepsilon_n\chi_{n-1,1} = (\underline{g})\xi_n = (\underline{h})\xi_n = [\underline{h}] + (-1)^n([\underline{h}])\varepsilon_n\chi_{n-1,1}\ .$$

Hence

$$\underbrace{[\underline{g}] - [\underline{h}]}_{\in R(G^{\times n}_{\neq 1})} = \underbrace{-(-1)^n([\underline{g}])\varepsilon_n\chi_{n-1,1} + (-1)^n([\underline{h}])\varepsilon_n\chi_{n-1,1}}_{\in R(G^{\times n}_{=1})}\ .$$

Since $R(G^{\times n}) = R(G^{\times n}_{=1}) \oplus R(G^{\times n}_{\neq 1})$, we obtain $[\underline{g}] - [\underline{h}] = 0$. Thus $[\underline{g}] = [\underline{h}]$. Thus $\underline{g} = \underline{h}$; cf. Remark 16.

Assume given $\underline{g} \in G^{\times n}_{\neq 1}$ such that $(\underline{g})\xi_n = 0$. Then

$$\underbrace{[\underline{g}]}_{\in R(G^{\times n}_{\neq 1})} \;=\; \underbrace{-(-1)^n([\underline{g}])\varepsilon_n\chi_{n-1,1}}_{\in R(G^{\times n}_{=1})} \;.$$

Since $R(G^{\times n}) = R(G^{\times n}_{=1}) \oplus R(G^{\times n}_{\neq 1})$, we obtain $[\underline{g}] = 0$. *Contradiction.* $\square$

Definition 25 Let

$$D_n \;:=\; \{(\underline{g})\xi_n \mid \underline{g} \in G^{\times n}_{\neq 1}\} \;\subseteq\; \mathrm{I}^{(n)}_R(G)\;.$$

Note that D_n does not contain 0 ; cf. Remark 24, Remark 21.(0).

If n is even, we define the subset $B_n \subseteq \mathrm{I}^{(n)}_R(G)$ as follows.

$$\begin{aligned}
B_n \;&:=\; \bigcup_{k=0}^{n/2} D_{n-2k}\chi_{n-2k,2k} \\
&\overset{\text{R. 22}}{=} \;\; \{(\underline{g})\xi_n \mid \underline{g} \in G^{\times n}_{\neq 1}\} \\
&\qquad \cup\{(\underline{g}\chi_{n-2,2})\xi_n \mid \underline{g} \in G^{\times(n-2)}_{\neq 1}\} \\
&\qquad \cup\{(\underline{g}\chi_{n-4,4})\xi_n \mid \underline{g} \in G^{\times(n-4)}_{\neq 1}\} \\
&\qquad \dots \\
&\qquad \cup\{(\underline{g}\chi_{2,n-2})\xi_n \mid \underline{g} \in G^{\times 2}_{\neq 1}\} \\
&\qquad \cup\{(\underline{g}\chi_{0,n})\xi_n \mid \underline{g} \in G^{\times 0}_{\neq 1}\}
\end{aligned}$$

Note that $G^{\times 0}_{\neq 1} = \{(\,)\}$ and that $(\,)\chi_{0,n} = (1)_{i\in[0,n-1]}$.

If n is odd, we define the subset $B_n \subseteq \mathrm{I}^{(n)}_R(G)$ as follows.

$$\begin{aligned}
B_n \;&:=\; \bigcup_{k=0}^{(n-1)/2} D_{n-2k}\chi_{n-2k,2k} \\
&\overset{\text{R. 22}}{=} \;\; \{(\underline{g})\xi_n \mid \underline{g} \in G^{\times n}_{\neq 1}\} \\
&\qquad \cup\{(\underline{g}\chi_{n-2,2})\xi_n \mid \underline{g} \in G^{\times(n-2)}_{\neq 1}\} \\
&\qquad \cup\{(\underline{g}\chi_{n-4,4})\xi_n \mid \underline{g} \in G^{\times(n-4)}_{\neq 1}\} \\
&\qquad \dots \\
&\qquad \cup\{(\underline{g}\chi_{3,n-3})\xi_n \mid \underline{g} \in G^{\times 3}_{\neq 1}\} \\
&\qquad \cup\{(\underline{g}\chi_{1,n-1})\xi_n \mid \underline{g} \in G^{\times 1}_{\neq 1}\}
\end{aligned}$$

Remark 26

(0) We have $B_0 = D_0 = \{(\,)\xi_0\} = \{[\,]\} = \{(\,)\}$; cf. Remark 14.

(1) We have $B_1 = D_1 = \{\, g - 1 \mid g \in G \setminus \{1\}\}$.

(2) We have $B_2 = D_2 \cup D_0\chi_{0,2}$. We will consider this set in detail in Remark 38 below, for sake of illustration.

Remark 27 *The set $B_1 = \{\, g-1 \mid g \in G \setminus \{1\}\}$ is an R-linear basis of $\mathrm{I}_R^{(1)}(G)$.*

Proof. We need to show that B_1 is R-linearly independent and R-linearly generates $\mathrm{I}_R^{(1)}(G)$.

Linearly independent set. Suppose given $\lambda_g \in R$ for $g \in G\setminus\{1\}$ such that we obtain the R-linear combination

$$\sum_{g\in G\setminus\{1\}} \lambda_g(g-1) \;=\; 0\,.$$

Then

$$\underbrace{\sum_{g\in G\setminus\{1\}} \lambda_g}_{\in\langle 1\rangle_R} \;=\; \underbrace{\sum_{g\in G\setminus\{1\}} \lambda_g g}_{\in\langle g\,|\,g\in G\setminus\{1\}\rangle_R}$$

Therefore $\lambda_g = 0$ for $g \in G \setminus \{1\}$.

Generating set. Suppose given $x \in \mathrm{I}_R^{(1)}(G)$. We can write $x = \sum_{g\in G} r_g g$ with $0 = x\varepsilon_1 = \sum_{g\in G} r_g$, as $x \in \mathrm{I}_R^{(1)}(G) = \mathrm{Ker}(\varepsilon_1)$. It follows that

$$\begin{aligned} x &= \sum_{g\in G} r_g g - \sum_{g\in G} r_g \\ &= \sum_{g\in G} (r_g g - r_g) \\ &= \sum_{g\in G} r_g(g-1) \\ &= \sum_{g\in G\setminus\{1\}} r_g(g-1) \;\in\; \langle g-1 \,|\, g\in G\setminus\{1\}\rangle_R \;=\; \langle B_1\rangle_R\,. \end{aligned}$$

$\square$

Remark 28 *Suppose that $n \geqslant 2$. We have $B_n = D_n \,\dot\cup\, B_{n-2}\chi_{n-2,2}$.*

Proof. If n is even, we obtain

$$\begin{aligned} B_n &= D_n \cup \bigcup_{k=1}^{n/2} D_{n-2k}\chi_{n-2k,2k} \\ &= D_n \cup \bigcup_{k=0}^{n/2-1} D_{n-2-2k}\chi_{n-2-2k,2k+2} \\ &\overset{\text{R. 6}}{=} D_n \cup \bigcup_{k=0}^{(n-2)/2} D_{(n-2)-2k}\chi_{(n-2)-2k,2k}\chi_{n-2,2} \\ &= D_n \cup B_{n-2}\chi_{n-2,2}\,. \end{aligned}$$

If n is odd, we obtain

$$\begin{aligned}
B_n &= D_n \cup \bigcup_{k=1}^{(n-1)/2} D_{n-2k}\chi_{n-2k,2k} \\
&= D_n \cup \bigcup_{k=0}^{(n-1)/2-1} D_{n-2-2k}\chi_{n-2-2k,2k+2} \\
&\overset{\text{R. 6}}{=} D_n \cup \bigcup_{k=0}^{((n-2)-1)/2} D_{(n-2)-2k}\chi_{(n-2)-2k,2k}\chi_{n-2,2} \\
&= D_n \cup B_{n-2}\chi_{n-2,2}\,.
\end{aligned}$$

In both cases, we have $D_n \subseteq R(G^{\times n}_{\neq(1,1)})$ and $B_{n-2}\chi_{n-2,2} \subseteq R(G^{\times n}_{=(1,1)})$.

Since $0 \notin D_n$, we obtain $B_n = D_n \,\dot\cup\, B_{n-2}\chi_{n-2,2}\,$. ▫

Proposition 29 *The set B_n is an R-linear basis of $\mathrm{I}_R^{(n)}(G) = \mathrm{Ker}(\varepsilon_n)$; cf. Definition 25.*

Proof. We proceed by induction on $n \geqslant 0$. As the base of induction, we use the cases $n = 0$ and $n = 1$.

Case $n = 0$. As $\varepsilon_0 : R \to 0$, we know that $\mathrm{Ker}(\varepsilon_0) = R = R(G^{\times 0})$. Thus $D_0 = B_0 = \{(\,)\}$ is an R-linear basis of $\mathrm{I}_R^{(0)}(G)$.

Case $n = 1$. As we have seen in Remark 27, the set $B_1 = \{g - 1 \mid g \in G \setminus \{1\}\}$ is a basis of $\mathrm{I}_R^{(1)}(G)$.

Induction step. Suppose given $n \geqslant 2$.

By induction hypothesis, B_{n-2} is an R-linear basis of $\mathrm{I}_R^{(n-2)}(G)$.

Linearly independent set. Suppose given an R-linear combination of B_n that is equal to 0, with coefficients $\lambda_x \in R$ for $x \in B_n$:

$$\begin{aligned}
0 &= \sum_{x \in B_n} \lambda_x x \\
&= \sum_{x \in D_n} \lambda_x x + \sum_{x \in B_n \setminus D_n} \lambda_x x \\
&\overset{\text{R. 28}}{=} \sum_{x \in D_n} \lambda_x x + \sum_{x \in B_{n-2}\chi_{n-2,2}} \lambda_x x
\end{aligned}$$

So we obtain

$$\underbrace{-\sum_{x \in D_n} \lambda_x x}_{\in R(G^{\times n}_{\neq(1,1)})} = \underbrace{\sum_{x \in B_{n-2}\chi_{n-2,2}} \lambda_x x}_{\in R(G^{\times n}_{=(1,1)})}$$

Since $R(G^{\times n}) = R(G^{\times n}_{=(1,1)}) \oplus R(G^{\times n}_{\neq(1,1)})$, we obtain $\sum_{x \in D_n} \lambda_x x = 0$ and $\sum_{x \in B_{n-2}\chi_{n-2,2}} \lambda_x x = 0$.

By induction hypothesis, B_{n-2} is R-linearly independent. Hence so is $B_{n-2}\chi_{n-2,2}$; cf. Remark 7. So $\lambda_x = 0$ for $x \in B_{n-2}\chi_{n-2,2}$.

Moreover,

$$\begin{aligned} 0 &= \sum_{x \in D_n} \lambda_x x \\ &\overset{\text{R. 24}}{=} \sum_{\underline{g} \in G^{\times n}_{\neq 1}} \lambda_{(\underline{g})\xi_n} (\underline{g})\xi_n \\ &= \sum_{\underline{g} \in G^{\times n}_{\neq 1}} \lambda_{(\underline{g})\xi_n} [\underline{g}] + (-1)^n \lambda_{(\underline{g})\xi_n} [\underline{g}] \varepsilon_n \chi_{n-1,1} \ . \end{aligned}$$

So we obtain

$$\underbrace{-\sum_{\underline{g} \in G^{\times n}_{\neq 1}} \lambda_{(\underline{g})\xi_n} [\underline{g}]}_{\in R(G^{\times n}_{\neq 1})} = \underbrace{\sum_{\underline{g} \in G^{\times n}_{\neq 1}} (-1)^n \lambda_{(\underline{g})\xi_n} ([\underline{g}]) \varepsilon_n \chi_{n-1,1}}_{\in R(G^{\times n}_{=1})} \ .$$

Since $R(G^{\times n}) = R(G^{\times n}_{=1}) \oplus R(G^{\times n}_{\neq 1})$, we obtain $\sum_{\underline{g} \in G^{\times n}_{\neq 1}} \lambda_{(\underline{g})\xi_n} [\underline{g}] = 0$. Hence $\lambda_{(\underline{g})\xi_n} = 0$ for $\underline{g} \in G^{\times n}_{\neq 1}$, i.e. $\lambda_x = 0$ for $x \in D_n$.

Altogether, $\lambda_x = 0$ for $x \in D_n \,\dot{\cup}\, B_{n-2}\chi_{n-2,2} \overset{\text{R. 24}}{=} B_n$.

Generating set. Suppose given $x \in \mathrm{I}^{(n)}_R(G)$.

We have to show that x is an R-linear combination of B_n .

Write $x = \sum_{\underline{g} \in G^{\times n}} \lambda_{\underline{g}} [\underline{g}]$, where $\lambda_{\underline{g}} \in R$. We calculate as follows.

$$\begin{aligned} x &= \sum_{\underline{g} \in G^{\times n}} \lambda_{\underline{g}} [\underline{g}] \\ &= \sum_{\underline{g} \in G^{\times n}_{\neq 1}} \lambda_{\underline{g}} [\underline{g}] + \sum_{\underline{g} \in G^{\times n}_{= 1}} \lambda_{\underline{g}} [\underline{g}] \\ &= \sum_{\underline{g} \in G^{\times n}_{\neq 1}} \lambda_{\underline{g}} (\underline{g})\xi_n - (-1)^n [\underline{g}] \varepsilon_n \chi_{n-1,1} + \sum_{\underline{g} \in G^{\times n}_{= 1}} \lambda_{\underline{g}} [\underline{g}] \\ &= \underbrace{\sum_{\underline{g} \in G^{\times n}_{\neq 1}} \lambda_{\underline{g}} (\underline{g})\xi_n}_{\in \mathrm{I}^{(n)}_R(G) \cap \langle D_n \rangle_R} - \sum_{\underline{g} \in G^{\times n}_{\neq 1}} (-1)^n [\underline{g}] \varepsilon_n \chi_{n-1,1} + \sum_{\underline{g} \in G^{\times n}_{= 1}} \lambda_{\underline{g}} [\underline{g}] \end{aligned}$$

Hence also

$$- \sum_{\underline{g} \in G^{\times n}_{\neq 1}} (-1)^n [\underline{g}] \varepsilon_n \chi_{n-1,1} \ + \sum_{\underline{g} \in G^{\times n}_{= 1}} \lambda_{\underline{g}} [\underline{g}] \ \in \ \mathrm{I}^{(n)}_R(G) \ .$$

We conclude that

$$\begin{aligned} - \sum_{\underline{g} \in G^{\times n}_{\neq 1}} (-1)^n [\underline{g}] \varepsilon_n \chi_{n-1,1} \ + \sum_{\underline{g} \in G^{\times n}_{= 1}} \lambda_{\underline{g}} [\underline{g}] \quad &\in \quad \mathrm{I}^{(n)}_R(G) \cap R(G^{\times (n-1)}) \chi_{n-1,1} \\ &\overset{\text{R. 12}}{=} \quad \mathrm{I}^{(n-2)}_R(G) \chi_{n-2,2} \\ &\overset{\text{I.H.}}{=} \quad \langle B_{n-2} \chi_{n-2,2} \rangle_R \ . \end{aligned}$$

Altogether, we have $x \in \langle D_n \rangle_R + \langle B_{n-2}\chi_{n-2,2} \rangle_R \overset{\text{R. 28}}{=} \langle B_n \rangle_R$.

So we have shown that B_n is an R-linear independent generating set of $\mathrm{I}_R^{(n)}(G)$, i.e. an R-linear basis of $\mathrm{I}_R^{(n)}(G)$. $\square$

4 The n-cocycles of G

Let R be a commutative ring. Let G be a finite group. Let M be an RG-module.

4.1 A universal 1-cocycle

In this §4.1, we want to recall the interplay of the augmentation ideal and derivations. We use direct methods. Further below, when successively generalising this interplay, we will then make use of functorial methods.

Definition 30 A map $d : G \to M$ is called a 1-*cocycle* or a *derivation* of G with values in M if

$$(h)d - (g \cdot h)d + (g)d \cdot h \;=\; 0$$

for $g,\, h \,\in\, G$.

The set of 1-cocycles of G with values in M is written $\mathrm{Z}^1(G, M)$.

Note that $\mathrm{Z}^1(G, M)$ is an R-submodule of ${}_{\mathrm{Map}}(G, M)$.

Remark 31 Suppose given $d \in \mathrm{Z}^1(G, M)$. Then

$$0 \;=\; (1)d - (1 \cdot 1)d + (1)d \cdot 1 \;=\; (1)d\,.$$

Remark 32 Note that the R-linear map

$$\begin{array}{rcl} RG & \xrightarrow{\varepsilon_1} & R \\ \sum\limits_{g \in G} r_g g & \mapsto & \sum\limits_{g \in G} r_g \end{array}$$

actually is an R-algebra morphism.

Thus its kernel, the augmention module $\mathrm{I}_R^{(1)}(G)$, is an ideal of RG, and is also called the *augmentation ideal*; cf. Definition 11.(1).

Remark 33 *The map*

$$\begin{array}{rcl} G & \xrightarrow{\xi_1} & \mathrm{I}_R^{(1)}(G) \\ g & \longmapsto & g - 1 \end{array}$$

from Definition 20 *is a* 1*-cocycle; cf. Remark* 21.(1).

Proof. Suppose given $g, h \in G$. We then have

$$\begin{aligned} (g)\xi_1 \cdot h - (gh)\xi_1 + (h)\xi_1 &= (g-1) \cdot h - (gh - 1) + (h - 1) \\ &= gh - h - gh + 1 + h - 1 \\ &= 0. \end{aligned}$$

□

Remark 34

(1) *Suppose given a 1-cocycle* $G \xrightarrow{d} M$.

Then there exists a unique RG-linear map $\mathrm{I}_R^{(1)}(G) \xrightarrow{f} M$ *such that the following triangle of maps commutes.*

$$\begin{tikzcd} G \arrow[r, "d"] \arrow[d, "\xi_1"'] & M \\ \mathrm{I}_R^{(1)}(G) \arrow[ur, "f"'] & \end{tikzcd}$$

Moreover, $(g-1)f = (g)d$ *for* $g \in G$*; cf. Remark* 27.

(2) *We have the bijective R-linear map*

$$\begin{array}{rcl} {}_{RG}(\mathrm{I}_R^{(1)}(G), M) & \to & \mathrm{Z}^1(G, M) \\ f & \mapsto & \xi_1 \cdot f \end{array}$$

Proof.

Ad (1).

Existence. Using the basis B_1 from Remark 27, we define the R-linear map

$$\begin{array}{rcl} \mathrm{I}_R^{(1)}(G) & \xrightarrow{f} & M \\ g-1 & \longmapsto & (g-1)f \;:=\; (g)d \quad \text{for } g \in G \smallsetminus \{1\}\ . \end{array}$$

Note that for $g = 1$, we also have $(g-1)f = (0)f = 0 \overset{\text{R. 31}}{=} (1)d = (g)d$, so that $(g\xi_1)f = (g-1)f = (g)d$ holds for $g \in G$. In other words, we have $\xi_1 \cdot f = d$.

The map f is RG-linear because

$$\begin{array}{rcl} ((g-1)\cdot x)f & = & (gx - x)f \\ & = & ((gx-1)-(x-1))f \\ & = & (gx-1)f - (x-1)f \\ & = & (gx)d - (x)d \\ & \overset{\text{1-cocyc.}}{=} & (g)d\cdot x + (x)d - (x)d \\ & = & (g)d \cdot x \\ & = & (g-1)f \cdot x \end{array}$$

for $g,\, x \in G$.

Uniqueness. By Remark 27, the image of ξ_1 is an R-linear basis of $\mathrm{I}_R^{(1)}(G)$. Uniqueness of f with respect to $\xi_1 \cdot f = d$ follows.

Ad (2). Suppose given $f \in {}_{RG}(\mathrm{I}_R^{(1)}(G), M)$. We have to show that $\xi_1 \cdot f : G \to M$ is a 1-cocycle.

Suppose given $g,\,h\in G$. We obtain

$$\begin{aligned}(h)(\xi_1\cdot f)-(gh)(\xi_1\cdot f)+(g)(\xi_1\cdot f)\cdot h &= ((h)\xi_1)f-((gh)\xi_1)f+((g)\xi_1)f\cdot h\\ &= ((h)\xi_1-(gh)\xi_1+(g)\xi_1\cdot h)f\\ &\overset{\text{R. 33}}{=} (0)f \;=\; 0\,.\end{aligned}$$

So the claimed map ${}_{RG}(\mathrm{I}_R^{(1)}(G),M)\to \mathrm{Z}^1(G,M)\,:\,f\mapsto \xi_1\cdot f$ exists. By (1), it is bijective. □

4.2 A universal 2-cocycle

In this section, we construct a universal 2-cocycle using an R-linear basis of $\mathrm{I}_R^{(2)}(G)$. This will be the case $n=2$ of a more general statement for $n\geqslant 1$; cf. Proposition 49. The case $n=2$ can still be treated with direct methods.

Definition 35 A map

$$G^{\times 2}\;=\;G\times G\;\;\xrightarrow{d}\;\;M$$

is called a 2-*cocycle* of G with values in M if

$$0\;=\;(h,k)d-(g\cdot h,k)d+(g,h\cdot k)d-(g,h)d\cdot k$$

for $g,\,h,\,k\in G$.

The set of 2-cocycles of G with values in M is written $\mathrm{Z}^2(G,M)$.

Note that $\mathrm{Z}^2(G,M)$ is an R-submodule of ${}_{\mathrm{Map}}(G^{\times 2},M)$.

Remark 36 *Suppose given a* 2*-cocycle* $d\in \mathrm{Z}^2(G,M)$.

We have $(g,1)d=(1,1)d$ *for* $g\in G$.

We have $(1,g)d=(1,1)d\cdot g$ *for* $g\in G$.

Proof. We have

$$0\;=\;(1,1)d-(g\cdot 1,1)d+(g,1\cdot 1)d-(g,1)d\cdot 1\;=\;(1,1)d-(g,1)d$$

and

$$0\;=\;(1,g)d-(1\cdot 1,g)d+(1,1\cdot g)d-(1,1)d\cdot g\;=\;(1,g)d-(1,1)d\cdot g\,.$$

□

We recall the R-linear map

$$\begin{array}{rcl} R(G^{\times 2}) & \xrightarrow{\varepsilon_2} & RG\\ \displaystyle\sum_{(g,h)\in G^{\times 2}} r_{g,h}(g,h) & \longmapsto & \displaystyle\sum_{(g,h)\in G^{\times 2}} r_{g,h}(h-g)\,;\end{array}$$

cf. Definition 8. We also recall the RG-submodule $\mathrm{I}_R^{(2)}(G)\;=\;\mathrm{Ker}(\varepsilon_2)\;\subseteq\;R(G^{\times 2})$; cf. Definition 11.(1).

Remark 37 *The map*

$$
\begin{array}{rcl}
G^{\times 2} & \xrightarrow{\xi_2} & \mathrm{I}_R^{(2)}(G) \\
(g,h) & \mapsto & (g,h)\xi_2 \;=\; (gh,h) - (gh,1) + (h,1)
\end{array}
$$

is a 2-cocycle; cf. Remark 21.(2)*.*

Proof. Suppose given $g,h,k \in G$. We get

$$
\begin{array}{rl}
& (h,k)\xi_2 - (gh,k)\xi_2 + (g,hk)\xi_2 - (g,h)\xi_2 \cdot k \\
= & \quad (hk,k) - (hk,1) + (k,1) \\
& - (ghk,k) + (ghk,1) - (k,1) \\
& + (ghk,hk) - (ghk,1) + (hk,1) \\
& - (ghk,hk) + (ghk,k) - (hk,k) \\
= & 0\,.
\end{array}
$$

$\square$

Remark 38 Recall from Definition 25 that

$$
\begin{array}{rcl}
B_2 & = & \{\,(g,h)\xi_2 \,|\, g \in G,\; h \in G \setminus \{1\}\} \cup \{(1,1)\xi_2\} \\
& = & \{\,(gh,h) - (gh,1) + (h,1) \,|\, g \in G,\; h \in G \setminus \{1\}\,\} \cup \{(1,1)\}\,.
\end{array}
$$

As we have seen in Proposition 29, the set B_2 is an R-linear basis of $\mathrm{I}_R^{(2)}(G)$.

For sake of illustration, we will give a direct proof of this fact.

Proof.

Linearly independent set. Suppose given $\lambda_{g,h} \in R$ for $g \in G$, $h \in G \setminus \{1\}$ and $\lambda_{1,1}$ such that we obtain the R-linear combination

$$
0 \;=\; \lambda_{1,1}(1,1) \quad + \sum_{g\in G,\; h\in G\setminus\{1\}} \lambda_{g,h}((gh,h) - (gh,1) + (h,1))\,.
$$

Then

$$
\underbrace{-\lambda_{1,1}(1,1) \quad + \sum_{g\in G,\; h\in G\setminus\{1\}} \lambda_{g,h}((gh,1) - (h,1))}_{\in\langle\, (g,1) \,|\, g\in G \,\rangle_R} \quad = \quad \underbrace{\sum_{g\in G,\; h\in G\setminus\{1\}} \lambda_{g,h}(gh,h)}_{\in\langle\, (g,h) \,|\, g\in G,\; h\in G\setminus\{1\} \,\rangle_R}
$$

Therefore, $\lambda_{g,h} = 0$ for $g \in G$, $h \in G \setminus \{1\}$.

Then it follows that $\lambda_{1,1} = 0$.

Altogether we have shown that B_2 is linearly independent.

Generating set. We have to show that each element of $\mathrm{I}_R^{(2)}(G)$ is an R-linear combination of B_2.

Using $\mathrm{I}_R^{(2)}(G) = \mathrm{Ker}(\varepsilon_2) = (R(G^{\times 3}))\varepsilon_3$, it suffices to show that given $(a,b,c) \in G^{\times 3}$, the element

$$(a,b,c)\varepsilon_3 \;=\; (b,c) - (a,c) + (a,b)$$

is an R-linear combination of B_2; cf. Lemma 10.

Case $b \neq 1$ and $c \neq 1$. We obtain

$$\begin{aligned}
+(bc^{-1},c)\xi_2 - (ac^{-1},c)\xi_2 + (ab^{-1},b)\xi_2 &= + \big((bc^{-1})c,c) - ((bc^{-1})c,1) + (c,1)\big) \\
&\quad - \big((ac^{-1})c,c) - ((ac^{-1})c,1) + (c,1)\big) \\
&\quad + \big((ab^{-1})b,b) - ((ab^{-1})b,1) + (b,1)\big) \\
&= + (b,c) - (b,1) + (c,1) \\
&\quad - (a,c) + (a,1) - (c,1) \\
&\quad + (a,b) - (a,1) + (b,1) \\
&= + (b,c) - (a,c) + (a,b)\ .
\end{aligned}$$

Case $b \neq 1$, $c = 1$. We obtain

$$\begin{aligned}
+(ab^{-1},b)\xi_2 &= + \big((ab^{-1})b,b) - ((ab^{-1})b,1) + (b,1)\big) \\
&= + (a,b) - (a,1) + (b,1) \\
&= + (b,c) - (a,c) + (a,b)\ .
\end{aligned}$$

Case $b = 1$, $c \neq 1$. We obtain

$$\begin{aligned}
-(ac^{-1},c)\xi_2 + (c^{-1},c)\xi_2 + (1,1)\xi_2 &= - \big((ac^{-1})c,c) - ((ac^{-1})c,1) + (c,1)\big) \\
&\quad + \big((c^{-1})c,c) - ((c^{-1})c,1) + (c,1)\big) \\
&\quad + (1,1) \\
&= - (a,c) + (a,1) - (c,1) \\
&\quad + (1,c) - (1,1) + (c,1) \\
&\quad + (1,1) \\
&= + (b,c) - (a,c) + (a,b)\ .
\end{aligned}$$

Case $b = 1$, $c = 1$. We obtain

$$\begin{aligned}
+(1,1)\xi_2 &= +(1,1) - (a,1) + (a,1) \\
&= +(b,c) - (a,c) + (a,b)
\end{aligned}$$

Hence in all cases, $(a,b,c)\varepsilon_3$ can be written as an R-linear combination of elements in B_2. $\square$

Proposition 39

(1) *Suppose given a 2-cocycle* $G^{\times 2} \xrightarrow{d} M$.

Then there exists a unique RG*-linear map* $\mathrm{I}_R^{(2)}(G) \xrightarrow{f} M$ *such that the following triangle of maps commutes.*

$$\begin{tikzcd} G^{\times 2} \arrow[r, "d"] \arrow[d, "\xi_2"'] & M \\ \mathrm{I}_R^{(2)}(G) \arrow[ur, "f"'] & \end{tikzcd}$$

Moreover, $((gh,h) - (gh,1) + (h,1))f = (g,h)d$ *for* $(g,h) \in G^{\times 2}$*; cf. Remark* 38.

(2) *We have the bijective* R*-linear map*

$$\begin{array}{rcl} {}_{RG}(\mathrm{I}_R^{(2)}(G), M) & \to & \mathrm{Z}^2(G, M) \\ f & \mapsto & \xi_2 \cdot f \end{array}$$

Proof.

Ad (1).

Existence. Using the basis B_2 from Remark 38, we define the R-linear map

$$\begin{array}{rcl} \mathrm{I}_R^{(2)}(G) & \xrightarrow{f} & M \\ (gh,h) - (gh,1) + (h,1) & \longmapsto & ((gh,h) - (gh,1) + (h,1))f \;:=\; (g,h)d \quad \text{for } (g,h) \in G \times (G \smallsetminus \{1\}) \\ (1,1) & \longmapsto & (1,1)d\,. \end{array}$$

For $g \in G$, we obtain

$$((g,1)\xi_2)f \;=\; ((g\cdot 1,1) - (g\cdot 1,1) + (1,1))f \;=\; (1,1)f \;=\; (1,1)d \;\overset{\text{R. 36}}{=}\; (g,1)d\,.$$

Hence

$$((g,h)\xi_2)f \;=\; ((gh,h) - (gh,1) + (h,1))f \;=\; (g,h)d$$

holds for $(g,h) \in G^{\times 2}$. In other words, we have $\xi_2 \cdot f = d$.

To show RG-linearity of f, it suffices to show that

$$((g,h)\xi_2)f \cdot x \;\overset{!}{=}\; ((g,h)\xi_2 \cdot x)f$$

for $g,\ h,\ x\ \in\ G$. We calculate.

$$
\begin{array}{cl}
& ((g,h)\xi_2)f\cdot x \\
= & (g,h)d\cdot x \\
\overset{\text{D. 35}}{=} & (h,x)d-(gh,x)d+(g,hx)d \\
= & ((h,x)\xi_2)f-((gh,x)\xi_2)f+((g,hx)\xi_2)f \\
\overset{f\ \text{R-lin.}}{=} & ((h,x)\xi_2-(gh,x)\xi_2+(g,hx)\xi_2)f \\
= & ((hx,x)-(hx,1)+(x,1)-(ghx,x)+(ghx,1)-(x,1)+(ghx,hx)-(ghx,1)+(hx,1))f \\
= & ((hx,x)-(ghx,x)+(ghx,hx))f \\
= & (((h,1)-(gh,1)+(gh,h))\cdot x)f \\
= & ((g,h)\xi_2\cdot x)f
\end{array}
$$

Uniqueness. By Remark 38, we have the R-linear basis B_2 of $\mathrm{I}_R^{(2)}(G)$ which consists of element in the image of ξ_2. Uniqueness of f with respect to $\xi_2\cdot f=d$ follows.

Ad (2). Suppose given $f\in\ _{RG}(\mathrm{I}_R^{(2)}(G),M)$. We have to show that $\xi_2\cdot f:G^{\times 2}\to M$ is a 2-cocycle.

Suppose given $g,\ h,\ k\ \in\ G$. We obtain

$$
\begin{array}{cl}
& (h,k)(\xi_2\cdot f)-(gh,k)(\xi_2\cdot f)+(g,hk)(\xi_2\cdot f)-(g,h)(\xi_2\cdot f)\cdot k \\
= & ((h,k)\xi_2)f-((gh,k)\xi_2)f+((g,hk)\xi_2)f-((g,h)\xi_2)f\cdot k \\
= & ((h,k)\xi_2-(gh,k)\xi_2+(g,hk)\xi_2-(g,h)\xi_2\cdot k)f \\
\overset{\text{R. 37}}{=} & (0)f\ =\ 0\,.
\end{array}
$$

□

4.3 A universal n-cocycle

Let G be a group. Let M be an RG-module.

Suppose given $n\in\mathbb{Z}_{\geqslant 0}$.

Definition 40 (n-cocycles) A map $d:G^{\times n}\to M$ is called an *n-cocycle of G with values in M* or, for short, an *n-cocycle*, if

$$
(g_0,\ldots,g_n)\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1}d+(-1)^{n+1}(g_0,\ldots,g_{n-1})d\cdot g_n\ =\ 0
$$

for $g_0,\ g_1,\ \ldots,\ g_{n-1},\ g_n\ \in\ G$.

Explicitly, a map $d : G^{\times n} \to M$ is an n-cocycle if

$$
\begin{array}{cl}
 & 0 \\
= & (g_0,\ldots,g_n)\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1}d + (-1)^{n+1}(g_0,\ldots,g_{n-1})d\cdot g_n \\
= & [g_0,\ldots,g_n]\varepsilon_{n+1}\iota_n^{-1}d + (-1)^{n+1}(g_0,\ldots,g_{n-1})d\cdot g_n \\
= & \left(\sum_{k=0}^{n}(-1)^k[g_0,\ldots,g_n]\uparrow^k \iota_n^{-1}d\right) + (-1)^{n+1}(g_0,\ldots,g_{n-1})d\cdot g_n \\
\overset{\text{R. 19}}{=} & \left([g_1,\ldots,g_n]\iota_n^{-1}d + \sum_{k=1}^{n}(-1)^k[g_0,\ldots,g_{k-2},\ g_{k-1}\cdot g_k\,,g_{k+1},\ldots,g_n]\iota_n^{-1}d\right) + (-1)^{n+1}(g_0,\ldots,g_{n-1})d\cdot g_n \\
= & (-1)^0(g_1,\ldots,g_n)d + \left(\sum_{k=1}^{n}(-1)^k(g_0,\ldots,g_{k-2},\ g_{k-1}\cdot g_k\,,g_{k+1},\ldots,g_n)d\right) + (-1)^{n+1}(g_0,\ldots,g_{n-1})d\cdot g_n \\
= & \quad (g_1,\ldots,g_{n-1},g_n)d \\
 & -\ (g_0\cdot g_1\,,g_2,\ldots,g_{n-1},g_n)d \\
 & +\ (g_0,\ g_1\cdot g_2\,,g_3,\ldots,g_{n-1},g_n)d \\
 & \pm\ \ldots \\
 & +\ (-1)^k(g_0,\ldots,g_{k-2},\ g_{k-1}\cdot g_k\,,g_{k+1},\ldots,g_{n-1},g_n)d \\
 & \pm\ \ldots \\
 & +\ (-1)^n(g_1,\ldots,g_{n-2},\ g_{n-1}\cdot g_n)d \\
 & +\ (-1)^{n+1}(g_0,\ldots,g_{n-2},g_{n-1})d\cdot g_n
\end{array}
$$

for $g_0,\ \ldots,\ g_n\ \in\ G$.

The set of n-cocycles of G with values in M is written

$$\mathrm{Z}^n(G,M)\ .$$

Note that $\mathrm{Z}^n(G,M)$ is an R-submodule of ${}_{\mathrm{Map}}(G^{\times n},M)$.

Remark 41

(0) Suppose that $n=0$. Then a map d from $\{(\,)\} = G^{\times 0}$ to M is a 0-cocycle if

$$0\ =\ (\,)d - (\,)d\cdot g_0$$

for $g_0\ \in\ G$. So 0-cocycles can be identified with the elements in M that are fix under G.

(1) Suppose that $n=1$. Then a map d from $G = G^{\times 1}$ to M is a 1-cocycle if

$$0\ =\ (g_1)d - (g_0\cdot g_1)d + (g_0)d\cdot g_1$$

for $g_0,\ g_1\ \in\ G$. Cf. Definition 30.

(2) Suppose that $n=2$. Then a map d from $G\times G = G^{\times 2}$ to M is a 2-cocycle if

$$0\ =\ (g_1,g_2)d - (g_0\cdot g_1,g_2)d + (g_0,g_1\cdot g_2)d - (g_0,g_1)d\cdot g_2$$

for $g_0,\ g_1,\ g_2\ \in\ G$. Cf. Definition 35.

Lemma 42 *Recall that*

$$\xi_n \;=\; (\iota_n + (-1)^n \iota_n \varepsilon_n \chi_{n-1,1})\big|^{\mathrm{I}_R^{(n)}(G)} \;:\; G^{\times n} \;\to\; \mathrm{I}_R^{(n)}(G) \;\subseteq\; R(G^{\times n})\;;$$

cf. Definition 20.

The map $\xi_n : G^{\times n} \to \mathrm{I}_R^{(n)}(G)$ *is an* n*-cocycle.*

Proof. Suppose given $\underline{g} = (g_0, g_1, \ldots, g_{n-1}) \in G^{\times n}$ and $x \in G$.

We write $\underline{h} := \underline{g} \sqcup (x) = (g_0, g_1, \ldots, g_{n-1}, x) \in G^{\times(n+1)}$.

According to Definition 40, we have to show that

$$\underline{h}\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1}\xi_n + (-1)^{n+1}\underline{g}\xi_n \cdot x \;\overset{!}{=}\; 0\;,$$

i.e. that

$$\underline{h}\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1}\xi_n \;\overset{!}{=}\; (-1)^n \underline{g}\xi_n \cdot x\;.$$

On the left hand side of that equation we get

$$\begin{array}{rcl}
\underline{h}\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1}\xi_n & = & \underline{h}\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1}(\iota_n + (-1)^n\iota_n\varepsilon_n\chi_{n-1,1}) \\
& = & \underline{h}\iota_{n+1}\varepsilon_{n+1} + (-1)^n\underline{h}\iota_{n+1}\varepsilon_{n+1}\varepsilon_n\chi_{n-1,1} \\
& \overset{\text{L. 10}}{=} & \underline{h}\iota_{n+1}\varepsilon_{n+1} \\
& = & [\underline{h}]\varepsilon_{n+1} \\
& \overset{\text{D. 13}}{=} & ([\underline{g}]\chi_{n,1} \cdot x)\varepsilon_{n+1} \\
& = & [\underline{g}]\chi_{n,1}\varepsilon_{n+1} \cdot x.
\end{array}$$

On the right hand side we get

$$\begin{aligned}
(-1)^n\underline{g}\xi_n \cdot x &= (-1)^n\underline{g}(\iota_n + (-1)^n\iota_n\varepsilon_n\chi_{n-1,1}) \cdot x \\
&= (-1)^n(\underline{g}\iota_n + (-1)^n\underline{g}\iota_n\varepsilon_n\chi_{n-1,1}) \cdot x \\
&= ((-1)^n\underline{g}\iota_n + \underline{g}\iota_n\varepsilon_n\chi_{n-1,1}) \cdot x \\
&= ((-1)^n[\underline{g}] + [\underline{g}]\varepsilon_n\chi_{n-1,1}) \cdot x\;.
\end{aligned}$$

It remains to show that

$$[\underline{g}]\chi_{n,1}\varepsilon_{n+1} \;\overset{!}{=}\; (-1)^n[\underline{g}] + [\underline{g}]\varepsilon_n\chi_{n-1,1}\;.$$

But by Remark 9.(1) we already know that

$$\chi_{n,1}\varepsilon_{n+1} = \varepsilon_n\chi_{n-1,1} + (-1)^n \,\mathrm{id}\;.$$

□

We can already observe a connection between n-cocycles and RG-linear maps via ξ_n .

Remark 43 *Suppose given an n-cocycle $d : G^{\times n} \to M$.*

Suppose given an R-linear map $f : \mathrm{I}_R^{(n)}(G) \to M$ that fulfills

$$((\underline{g})\xi_n)f \;=\; (\underline{g})d$$

for $\underline{g} \in G^{\times n}$. Then f is RG-linear.

$$\begin{tikzcd} G^{\times n} \arrow[r, "d"] \arrow[d, "\xi_n"'] & M \\ \mathrm{I}_R^{(n)}(G) \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. Suppose given $\underline{g} \in G^{\times n}$ and $x \in G$. We need to show that $((\underline{g})\xi_n \cdot x)f \stackrel{!}{=} ((\underline{g})\xi_n)f \cdot x$. Writing $\underline{h} := \underline{g} \sqcup (x)$, we obtain

$$\begin{aligned} ((\underline{g})\xi_n)f \cdot x &= (\underline{g})d \cdot x \\ &\stackrel{\text{D. 40}}{=} (-1)^n(\underline{h}\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1})d \\ &= (-1)^n((\underline{h}\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1})\xi_n)f \\ &= (-1)^n([\underline{h}]\varepsilon_{n+1}\iota_n^{-1}(\iota_n + (-1)^n\iota_n\varepsilon_n\chi_{n-1,1}))f \\ &= (-1)^n([\underline{h}]\varepsilon_{n+1} + (-1)^n[\underline{h}]\varepsilon_{n+1}\varepsilon_n\chi_{n-1,1})f \\ &\stackrel{\text{L. 10}}{=} (-1)^n([\underline{h}]\varepsilon_{n+1})f \\ &\stackrel{\text{D. 13}}{=} (-1)^n(([\underline{g}]\chi_{n,1} \cdot x)\varepsilon_{n+1})f \\ &= ((-1)^n[g]\chi_{n,1}\varepsilon_{n+1} \cdot x)f \\ &\stackrel{\text{R. 9.(1)}}{=} ((-1)^n((-1)^n[\underline{g}] + [\underline{g}]\varepsilon_n\chi_{n-1,1}) \cdot x)f \\ &= (([\underline{g}] + (-1)^n[\underline{g}]\varepsilon_n\chi_{n-1,1}) \cdot x)f \\ &= ((\underline{g})\xi_n \cdot x)f \end{aligned}$$

□

Such a map f is uniquely determined by d; cf. Definition 25, Proposition 29.

The problem is the existence of f for a given n-cocycle d. We shall solve this problem in Proposition 49 below.

Lemma 44 *Suppose that $n \geqslant 1$.*

We have the R-linear isomorphism

$$\begin{array}{rcccl} \rho_n & : & {}_{RG}(R(G^{\times n}), M) & \stackrel{\sim}{\to} & {}_{\mathrm{Map}}(G^{\times(n-1)}, M) \\ & & f & \mapsto & \big((\chi_{n-1,1} \cdot f)|_{G^{\times(n-1)}} : (g_0, \dots, g_{n-2}) \mapsto (g_0, \dots, g_{n-2}, 1)f\big) \,. \end{array}$$

Its inverse is given by

$$\begin{array}{rrcl} \rho_n^{-1} : & {}_{\mathrm{Map}}(G^{\times(n-1)}, M) & \xrightarrow{\sim} & {}_{RG}(R(G^{\times n}), M) \\ & \mu & \mapsto & \big([g_0, \dots, g_{n-1}] \mapsto [g_0, \dots, g_{n-2}]\mu \cdot g_{n-1}\big) \ . \end{array}$$

Cf. Corollary 17.

As formulas, for $g_0, \dots, g_{n-2} \in G$, *we have*

$$\begin{array}{rcl} (g_0, \dots, g_{n-2})(f\rho_n) & = & (g_0, \dots, g_{n-2}, 1)f \ , \\ \textit{and so} \quad (g_0, \dots, g_{n-2}, 1)(\mu\rho_n^{-1}) & = & (g_0, \dots, g_{n-2})\mu \ . \end{array}$$

Note that also

$$[g_0, \dots, g_{n-2}](f\rho_n) \ = \ [g_0, \dots, g_{n-2}](\chi_{n-1,1} \cdot f) \ = \ [g_0, \dots, g_{n-2}, 1]f \ .$$

Proof. We already use the notation ρ_n^{-1} for the claimed inverse of ρ_n .

Ad $\rho_n\rho_n^{-1} = \mathrm{id}$. Suppose given $f \in {}_{RG}(R(G^{\times n}), M)$.

First, we obtain

$$f\rho_n \ = \ \Big([g_0, \dots, g_{n-2}] \mapsto [g_0, \dots, g_{n-2}, 1]f\Big) \ .$$

Then we get

$$\begin{array}{rcl} (f\rho_n)\rho_n^{-1} & = & \Big([g_0, \dots, g_{n-2}, g_{n-1}] \mapsto [g_0, \dots, g_{n-2}](f\rho_n) \cdot g_{n-1}\Big) \\ & = & \Big([g_0, \dots, g_{n-2}, g_{n-1}] \mapsto [g_0, \dots, g_{n-2}, 1]f \cdot g_{n-1}\Big) \\ & \overset{f\ RG\text{-lin.}}{=} & \Big([g_0, \dots, g_{n-2}, g_{n-1}] \mapsto [g_0, \dots, g_{n-2}, g_{n-1}]f\Big) \\ & = & f \ . \end{array}$$

Ad $\rho_n^{-1}\rho_n = \mathrm{id}$. Suppose given $\mu \in {}_{\mathrm{Map}}(G^{\times(n-1)}, M)$.

First, we obtain

$$\mu\rho_n^{-1} \ = \ \Big([g_0, \dots, g_{n-1}] \mapsto [g_0, \dots, g_{n-2}]\mu \cdot g_{n-1}\Big) \ .$$

Then we get

$$\begin{array}{rcl} (\mu\rho_n^{-1})\rho_n & = & \Big([g_0, \dots, g_{n-2}] \mapsto [g_0, \dots, g_{n-2}, 1](\mu\rho_n^{-1})\Big) \\ & = & \Big([g_0, \dots, g_{n-2}] \mapsto [g_0, \dots, g_{n-2}]\mu \cdot 1\Big) \\ & = & \mu \ . \end{array}$$

□

Definition 45

(1) Suppose given RG-modules X and Y and an RG-linear map $u : X \to Y$.

We often abbreviate

$$u^* \;:=\; {}_{RG}(u, M) \;:\; {}_{RG}(Y, M) \;\to\; {}_{RG}(X, M)$$

So u^* is an R-linear map.

Given an R-linear map $w : Y \to M$, we have $(w)u^* = u \cdot w : X \to M$.

(2) We define the R-linear map

$$\alpha_n \;:=\; \rho_{n+1}^{-1} \cdot \varepsilon_{n+2}^* \cdot \rho_{n+2} \;:\; {}_{\mathrm{Map}}(G^{\times n}, M) \;\to\; {}_{\mathrm{Map}}(G^{\times (n+1)}, M)$$

cf. Lemma 44.

$${}_{\mathrm{Map}}(G^{\times n}, M) \xrightarrow{\rho_{n+1}^{-1}} {}_{RG}(R(G^{\times(n+1)}), M) \xrightarrow{\varepsilon_{n+2}^*} {}_{RG}(R(G^{\times(n+2)}), M) \xrightarrow{\rho_{n+2}} {}_{\mathrm{Map}}(G^{\times(n+1)}, M)$$

(3) For $j \in \mathbb{Z}_{\geqslant 1}$, we have the surjective RG-linear map

$$\bar{\varepsilon}_j \;:=\; \varepsilon_j|^{\mathrm{I}_R^{(j-1)}(G)} \;:\; R(G^{\times j}) \to \mathrm{I}_R^{(j-1)}(G)$$

as $\mathrm{Im}(\varepsilon_j) = \mathrm{Ker}(\varepsilon_{j-1}) = \mathrm{I}_R^{(j-1)}(G)$; cf. Lemma 10.

(4) For $j \in \mathbb{Z}_{\geqslant 0}$, we have the injective RG-linear map

$$\begin{array}{rccc} \dot{\varepsilon}_j & : & \mathrm{I}_R^{(j)}(G) & \to & R(G^{\times j}) \\ & & x & \mapsto & x\,. \end{array}$$

Note that $\varepsilon_j = \bar{\varepsilon}_j \cdot \dot{\varepsilon}_{j-1} : R(G^{\times j}) \to R(G^{\times (j-1)})$ for $j \in \mathbb{Z}_{\geqslant 1}$.

Proposition 46 *We have the isomorphism*

$$(\bar{\varepsilon}_{n+1}^* \rho_{n+1})|^{\mathrm{Ker}(\alpha_n)} \;:\; {}_{RG}(\mathrm{I}_R^{(n)}(G), M) \;\xrightarrow{\sim}\; \mathrm{Ker}(\alpha_n)$$

of R-modules.

Proof. By Lemma 10, we have the acyclic complex

$$\dots \longrightarrow R(G^{\times(n+2)}) \xrightarrow{\varepsilon_{n+2}} R(G^{\times(n+1)}) \xrightarrow{\varepsilon_{n+1}} R(G^{\times n}) \longrightarrow \dots\,.$$

Adding the augmention modules we get the commutative diagram

$$\begin{tikzcd}
{} \arrow[dr, dotted] & & R(G^{\times(n+2)}) \arrow[rr, "\varepsilon_{n+2}"] \arrow[dr, "\bar{\varepsilon}_{n+2}"] & & R(G^{\times(n+1)}) \arrow[rr, "\varepsilon_{n+1}"] \arrow[dr, "\bar{\varepsilon}_{n+1}"] & & R(G^{\times n}) \arrow[dr, dotted] & \\
& \mathrm{I}_R^{(n+2)}(G) \arrow[ur, hook, "\dot{\varepsilon}_{n+2}"] & & \mathrm{I}_R^{(n+1)}(G) \arrow[ur, hook, "\dot{\varepsilon}_{n+1}"] & & \mathrm{I}_R^{(n)}(G) \arrow[ur, hook, "\dot{\varepsilon}_n"] & & {}
\end{tikzcd}$$

We apply the functor ${}_{RG}(-,M)$ to obtain the commutative diagram

$$\begin{tikzcd}
{} & {}_{RG}(R(G^{\times(n+2)}),M) \arrow[dl, "\dot{\varepsilon}^*_{n+2}"'] & & {}_{RG}(R(G^{\times(n+1)}),M) \arrow[ll, "\varepsilon^*_{n+2}"'] \arrow[dl, "\dot{\varepsilon}^*_{n+1}"'] & & {}_{RG}(R(G^{\times n}),M) \arrow[ll, "\varepsilon^*_{n+1}"'] \arrow[dl, "\dot{\varepsilon}^*_{n}"'] & \\
{}_{RG}(\mathrm{I}_R^{(n+2)}(G),M) \arrow[u, dotted] & & {}_{RG}(\mathrm{I}_R^{(n+1)}(G),M) \arrow[ul, "\bar{\varepsilon}^*_{n+2}"'] & & {}_{RG}(\mathrm{I}_R^{(n)}(G),M) \arrow[ul, "\bar{\varepsilon}^*_{n+1}"'] & & {} \arrow[ul, dotted]
\end{tikzcd}$$

Since the sequence

$$R(G^{\times(n+2)}) \xrightarrow{\varepsilon_{n+2}} R(G^{\times(n+1)}) \xrightarrow{\bar{\varepsilon}_{n+1}} \mathrm{I}_R^{(n)}(G)$$

is right exact, and since the functor ${}_{RG}(-,M)$ is contravariant and left exact, it follows that the sequence

$$\begin{tikzcd}
{}_{RG}(R(G^{\times(n+2)}),M) & {}_{RG}(R(G^{\times(n+1)}),M) \arrow[l, "\varepsilon^*_{n+2}"'] & \\
& & {}_{RG}(\mathrm{I}_R^{(n)}(G),M) \arrow[ul, "\bar{\varepsilon}^*_{n+1}"']
\end{tikzcd}$$

is left exact. Now we apply Lemma 44 to ${}_{RG}(R(G^{\times(n+2)}),M)$ and ${}_{RG}(R(G^{\times(n+1)}),M)$ to get the following commutative diagram.

$$\begin{tikzcd}
{}_{RG}(R(G^{\times(n+2)}),M) \arrow[dd, "\rho_{n+2}"', "\wr"] & {}_{RG}(R(G^{\times(n+1)}),M) \arrow[l, "\varepsilon^*_{n+2}"'] \arrow[dd, "\rho_{n+1}"', "\wr"] & \\
& & {}_{RG}(\mathrm{I}_R^{(n)}(G),M) \arrow[ul, "\bar{\varepsilon}^*_{n+1}"'] \arrow[dd, "(\bar{\varepsilon}^*_{n+1}\rho_{n+1})|^{\mathrm{Ker}(\alpha_n)}", "\wr"'] \\
{}_{\mathrm{Map}}(G^{\times(n+1)},M) & {}_{\mathrm{Map}}(G^{\times n},M) \arrow[l, "\alpha_n"'] & \\
& & \mathrm{Ker}(\alpha_n) \arrow[ul, hook']
\end{tikzcd}$$

□

Lemma 47 *Suppose given* $\mu \in {}_{\mathrm{Map}}(G^{\times n},M)$. *We have*

$$\mu \in \mathrm{Ker}(\alpha_n) \qquad \Longleftrightarrow \qquad \iota_n\mu \in \mathrm{Z}^n(G,M)\ .$$

Proof. Suppose given $\mu \in {}_{\mathrm{Map}}(G^{\times n},M)$ and $(g_0,\ldots,g_n) \in G^{\times(n+1)}$. We have

$$\begin{array}{cl}
& [g_0,\ldots,g_n](\mu\alpha_n) \\
= & [g_0,\ldots,g_n](\mu\rho_{n+1}^{-1}\varepsilon^*_{n+2}\rho_{n+2}) \\
= & [g_0,\ldots,g_n]\chi_{n+1,1}(\mu\rho_{n+1}^{-1}\varepsilon^*_{n+2}) \\
= & [g_0,\ldots,g_n]\chi_{n+1,1}\varepsilon_{n+2}(\mu\rho_{n+1}^{-1}) \\
\overset{\mathrm{R.\,9.(1)}}{=} & [g_0,\ldots,g_n](\varepsilon_{n+1}\chi_{n,1} + (-1)^{n+1}\,\mathrm{id})(\mu\rho_{n+1}^{-1}) \\
= & [g_0,\ldots,g_n]\varepsilon_{n+1}\chi_{n,1}(\mu\rho_{n+1}^{-1}) + (-1)^{n+1}[g_0,\ldots,g_n](\mu\rho_{n+1}^{-1}) \\
= & ([g_0,\ldots,g_n]\varepsilon_{n+1})\mu + (-1)^{n+1}[g_0,\ldots,g_{n-1}]\mu\cdot g_n \\
= & ((g_0,\ldots,g_n)\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1})\iota_n\mu + (-1)^{n+1}(g_0,\ldots,g_{n-1})\iota_n\mu\cdot g_n\ .
\end{array}$$

Hence

$$
\begin{array}{rl}
& \mu\alpha_n \;=\; 0 \\
\Leftrightarrow & ((g_0,\ldots,g_n)\iota_{n+1}\varepsilon_{n+1}\iota_n^{-1})(\iota_n\mu) + (-1)^{n+1}(g_0,\ldots,g_{n-1})(\iota_n\mu)\cdot g_n = 0 \text{ for } (g_0,\ldots,g_n) \in G^{\times(n+1)}
\end{array}
$$

This means that $\mu \in \operatorname{Ker}(\alpha_n)$ if and only if $\iota_n\mu$ is an n-cocycle in the sense of Definition 40. ▫

Lemma 48 *We have the isomorphism*

$$
\begin{array}{rcrcl}
\xi_n\cdot(-) & : & {}_{RG}(\mathrm{I}_R^{(n)}(G), M) & \to & \mathrm{Z}^n(G,M) \\
 & & \tilde{d} & \mapsto & \xi_n\cdot\tilde{d}
\end{array}
$$

of R-modules, where ξ_n has been introduced in Definition 20*.*

We get the following commutative diagram of R-modules.

$$
\begin{tikzcd}
{}_{RG}(\mathrm{I}_R^{(n)}(G), M) \arrow[r, "((-1)^n\bar{\varepsilon}^*_{n+1}\cdot\rho_{n+1})|^{\operatorname{Ker}(\alpha_n)}", "\sim"'] \arrow[dr, "\sim", "\xi_n\cdot(-)"'] & \operatorname{Ker}(\alpha_n) \arrow[r, hook] \arrow[d, "(\iota_n\cdot(-))|^{\mathrm{Z}^n(G,M)}_{\operatorname{Ker}(\alpha_n)}", "\wr"'] & {}_{\mathrm{Map}}(G^{\times n}, M) \arrow[d, "\iota_n\cdot(-)", "\wr"'] \\
 & \mathrm{Z}^n(G,M) \arrow[r, hook] & {}_{\mathrm{Map}}(G^{\times n}, M).
\end{tikzcd}
$$

Proof. By Lemma 47, the isomorphism

$$
\begin{array}{rcrcl}
\iota_n\cdot(-) & : & {}_{\mathrm{Map}}(G^{\times n}, M) & \xrightarrow{\sim} & {}_{\mathrm{Map}}(G^{\times n}, M) \\
 & & \mu & \mapsto & \iota_n\mu
\end{array}
$$

restricts to an isomorphism

$$
\begin{array}{rcrcl}
(\iota_n\cdot(-))|^{\mathrm{Z}^n(G,M)}_{\operatorname{Ker}(\alpha_n)} & : & \operatorname{Ker}(\alpha_n) & \xrightarrow{\sim} & \mathrm{Z}^n(G,M) \\
 & & \mu & \mapsto & \iota_n\mu\ .
\end{array}
$$

By Proposition 46, we have the isomorphism $(\bar{\varepsilon}^*_{n+1}\cdot\rho_{n+1})|^{\operatorname{Ker}(\alpha_n)} : {}_{RG}(\mathrm{I}_R^{(n)}(G), M) \xrightarrow{\sim} \operatorname{Ker}(\alpha_n)$, which we multiply with $(-1)^n$.

It remains to show the commutativity of the triangle in the diagram.

Suppose given $f \in {}_{RG}(\mathrm{I}_R^{(n)}(G), M)$ and $(g_0, \dots, g_{n-1}) \in G^{\times n}$. We obtain

$$
\begin{aligned}
& (g_0, \dots, g_{n-1})\Big((f)((-1)^n \bar{\varepsilon}_{n+1}^* \cdot \rho_{n+1})(\iota_n \cdot (-))\Big) \\
= \quad & (g_0, \dots, g_{n-1})\Big(\iota_n \cdot (f)((-1)^n \bar{\varepsilon}_{n+1}^* \cdot \rho_{n+1})\Big) \\
= \quad & [g_0, \dots, g_{n-1}]\Big((f)((-1)^n \bar{\varepsilon}_{n+1}^* \cdot \rho_{n+1})\Big) \\
= \quad & (-1)^n [g_0, \dots, g_{n-1}]\Big((f)(\bar{\varepsilon}_{n+1}^* \cdot \rho_{n+1})\Big) \\
= \quad & (-1)^n [g_0, \dots, g_{n-1}]\Big((\bar{\varepsilon}_{n+1} \cdot f)\rho_{n+1}\Big) \\
= \quad & (-1)^n [g_0, \dots, g_{n-1}]\Big((\chi_{n,1} \cdot (\bar{\varepsilon}_{n+1} \cdot f))|_{G^{n+1}}\Big) \\
= \quad & (-1)^n (([g_0, \dots, g_{n-1}]\chi_{n,1})\varepsilon_{n+1})f \\
\overset{\text{R. 9.(1)}}{=} \quad & (-1)^n ([g_0, \dots, g_{n-1}]\varepsilon_n \chi_{n-1,1} + (-1)^n [g_0, \dots, g_{n-1}])f \\
= \quad & ((-1)^n [g_0, \dots, g_{n-1}]\varepsilon_n \chi_{n-1,1} + [g_0, \dots, g_{n-1}])f \\
= \quad & (g_0, \dots, g_{n-1})((-1)^n \iota_n \varepsilon_n \chi_{n-1,1} + \iota_n)f \\
\overset{\text{D. 20}}{=} \quad & (g_0, \dots, g_{n-1})\xi_n f \\
= \quad & (g_0, \dots, g_{n-1})\Big((f)(\xi_n \cdot (-))\Big) .
\end{aligned}
$$

Hence

$$
(f)((-1)^n \bar{\varepsilon}_{n+1}^* \cdot \rho_{n+1})(\iota_n \cdot (-)) \;=\; (f)(\xi_n \cdot (-))
$$

for $f \in {}_{RG}(\mathrm{I}_R^{(n)}(G), M)$, as was to be shown. ▫

Proposition 49 (ξ_n is a universal n-cocycle)

(1) *We have $\xi_n \in \mathrm{Z}^n(G, \mathrm{I}_R^{(n)}(G))$.*

(2) *For a given n-cocycle $d \in \mathrm{Z}^n(G, M)$, there exists a unique RG-linear map $\tilde{d}\colon \mathrm{I}_R^{(n)}(G) \to M$ with $d = \xi_n \cdot \tilde{d}$.*

$$
\begin{tikzcd}
G^{\times n} \arrow[r, "d \in \mathrm{Z}^n(G,M)"] \arrow[d, "\xi_n \in \mathrm{Z}^n(G, \mathrm{I}_R^{(n)}(G))"'] & M \\
\mathrm{I}_R^{(n)}(G) \arrow[ur, "\exists!\, \tilde{d} \;\; RG\text{-linear}"'] &
\end{tikzcd}
$$

Proof.

Ad (1)*.* This is Lemma 42.

Ad (2)*.* This follows from Lemma 48, as for every element $d \in \mathrm{Z}^n(G, M)$ there is a unique element $\tilde{d} \in {}_{RG}(\mathrm{I}_R^{(n)}(G), M)$ mapping to d via $\xi_n \cdot (-)$. ▫

Remark 50 We summarize to the following commutative diagram of R-modules.

$$\begin{tikzcd}
{}_{RG}(R(G^{\times(n+2)}), M) \arrow[r, "\rho_{n+2}", "\sim"'] & {}_{\mathrm{Map}}(G^{\times(n+1)}, M) & \\
{}_{RG}(R(G^{\times(n+1)}), M) \arrow[u, "\varepsilon^*_{n+2}"] \arrow[r, "\rho_{n+1}", "\sim"'] & {}_{\mathrm{Map}}(G^{\times n}, M) \arrow[u, "\alpha_n"'] \arrow[r, "\iota_n(-)", "\sim"'] & {}_{\mathrm{Map}}(G^{\times n}, M) \\
{}_{RG}(\mathrm{I}_R^{(n)}(G), M) \arrow[u, "\bar{\varepsilon}^*_{n+1}"] \arrow[r, "(\bar{\varepsilon}^*_{n+1}\cdot\rho_{n+1})|^{\mathrm{Ker}(\alpha_n)}", "\sim"'] \arrow[rr, bend right, "\sim", "(-1)^n\xi_n\cdot(-)"'] & \mathrm{Ker}(\alpha_n) \arrow[u, hook] \arrow[r, "(\iota_n\cdot(-))|^{\mathrm{Z}^n(G,M)}_{\mathrm{Ker}(\alpha_n)}", "\sim"'] & \mathrm{Z}^n(G, M) \arrow[u, hook]
\end{tikzcd}$$

5 The n-cocycles of G relative to H

Let G be a group. Let $H \leqslant G$ be a subgroup. Let M be an RG-module.

5.1 Preparations

Definition 51 We write the set of right cosets of H in G as

$$\check{G} \;:=\; H \setminus G \;=\; \{\, Hg \,|\, g \in G \,\} \;.$$

We choose a subset $T \subseteq G$ such that

$$\dot{\bigcup_{t\in T}} Ht = G \;.$$

So T is a set of representing elements of the right cosets of H in G.

Definition 52 For $k \in \mathbb{Z}_{\geqslant 0}$, we consider $R(\check{G}^{\times k})$ as an RG-module via

$$(Hg_0, Hg_1, \ldots, Hg_{k-1}) \cdot g \;:=\; (Hg_0 \cdot g,\; Hg_1 \cdot g,\; \ldots,\; Hg_{k-1} \cdot g) \;,$$

where $(Hg_0, Hg_1, \ldots, Hg_{k-1}) \in \check{G}^{\times k}$ and $g \in G$.

Definition 53 Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. We define the surjective RG-linear map

$$\begin{array}{rcl} R(G^{\times k}) & \xrightarrow{\sigma_k} & R(\check{G}^{\times k}) \\ (g_i)_{i\in[0,k-1]} & \longmapsto & (Hg_i)_{i\in[0,k-1]} \;. \end{array}$$

Note that this map, which we have constructed as an R-linear map, is in fact RG-linear, since

$$((g_i)_{i\in[0,k-1]})\sigma_k \cdot g \;=\; (Hg_i)_{i\in[0,k-1]} \cdot g \;=\; (Hg_i \cdot g)_{i\in[0,k-1]} \;=\; ((g_i \cdot g)_{i\in[0,k-1]})\sigma_k$$

for $(g_i)_{i\in[0,k-1]} \in G^{\times k}$ and $g \in G$.

Notation 54 Suppose given $\underline{g} = (g_0, g_1, \ldots, g_{n-1}) \in G^{\times n}$. We often write

$$H\underline{g} \;:=\; (Hg_0, Hg_1, \ldots, Hg_{n-1}) \;\in\; \check{G}^{\times n} \;.$$

Note that for $k \in [0, n-1]$, we have $(H\underline{g}) \uparrow^k = H(\underline{g} \uparrow^k) =: H\underline{g} \uparrow^k$, where we allow to omit the parentheses.

Definition 55 Suppose given $n,\, k \,\in\, \mathbb{Z}_{\geqslant 0}$. We define the RH-linear map

$$\begin{array}{rcl} R(\check{G}^{\times n}) & \xrightarrow{\check{\chi}_{n,k}} & R(\check{G}^{\times (n+k)}) \\ (Hg_i)_{i\in[0,n-1]} & \longmapsto & (Hg_i)_{i\in[0,n-1]} \sqcup (H1)_{i\in[0,k-1]} \;. \end{array}$$

Note that in fact

$$\big(((Hg_i)_{i\in[0,n-1]})\cdot h\big)\check{\chi}_{n,k} = \big((Hg_ih)_{i\in[0,n-1]}\big)\check{\chi}_{n,k} \;=\; (Hg_ih)_{i\in[0,n-1]} \;\sqcup\; (H1)_{i\in[0,k-1]}$$

and

$$\begin{aligned}\big((Hg_i)_{i\in[0,n-1]}\big)\check{\chi}_{n,k}\cdot h &= ((Hg_i)_{i\in[0,n-1]} \;\sqcup\; (H1)_{i\in[0,k-1]})\cdot h \\ &= (Hg_ih)_{i\in[0,n-1]} \;\sqcup\; (Hh)_{i\in[0,k-1]} \\ &= (Hg_ih)_{i\in[0,n-1]} \;\sqcup\; (H1)_{i\in[0,k-1]}\end{aligned}$$

for $(Hg_i)_{i\in[0,n-1]} \in \check{G}^{\times n}$ and $h \in H$, which is the same.

Additionally, we define the RH-linear map $\check{\chi}_{-1,k} : 0 \to R(\check{G}^{\times(k-1)})$: $0 \mapsto 0$.

Remark 56 *Suppose given* $n,\, k \in \mathbb{Z}_{\geqslant 0}$ *.*

We have the following commutative quadrangle.

$$\begin{tikzcd} R(G^{\times n}) \arrow[r, "\chi_{n,k}"] \arrow[d, "\sigma_n"'] & R(G^{\times(n+k)}) \arrow[d, "\sigma_{n+k}"'] \\ R(\check{G}^{\times n}) \arrow[r, "\check{\chi}_{n,k}"] & R(\check{G}^{\times(n+k)}) \end{tikzcd}$$

Proof. For $\underline{g} = (g_i)_{i\in[0,n-1]}$, we obtain

$$\begin{aligned}\underline{g}\chi_{n,k}\sigma_n &= ((g_i)_{i\in[0,n-1]} \;\sqcup\; (1)_{i\in[0,k-1]})\sigma_{n+k} \\ &= (Hg_i)_{i\in[0,n-1]} \;\sqcup\; (H1)_{i\in[0,k-1]}) \\ &= (Hg_i)_{i\in[0,n-1]}\check{\chi}_{n,k} \\ &= \underline{g}\sigma_n\check{\chi}_{n,k}\,.\end{aligned}$$

Cf. Definitions 5 and 55. ▫

Remark 57 *Suppose given* $n \in \mathbb{Z}_{\geqslant -1}$ *and* $k,\, m \in \mathbb{Z}_{\geqslant 0}$*. Then we have* $\check{\chi}_{n,k}\check{\chi}_{n+k,m} = \check{\chi}_{n,k+m}$ *.*

Proof.

Case $n \geqslant 0$. Suppose given $(Hg_i)_{i\in[0,n-1]} \in \check{G}^{\times n}$. Then

$$\big((Hg_i)_{i\in[0,n-1]} \;\sqcup\; (H1)_{i\in[0,k-1]}\big) \;\sqcup\; (H1)_{i\in[0,m-1]} \;=\; (Hg_i)_{i\in[0,n-1]} \;\sqcup\; (H1)_{i\in[0,k+m-1]}.$$

Case $n = -1$. We have $(0)\check{\chi}_{-1,k}\check{\chi}_{k-1,m} = (0)\check{\chi}_{k-1,m} = 0 = (0)\check{\chi}_{-1,k+m}$. ▫

Remark 58 *Suppose given* $n, k \in \mathbb{Z}_{\geqslant 0}$*. Then the map* $\check{\chi}_{n,k}$ *is injective.*

Proof. We use the R-linear bases $\check{G}^{\times n}$ of $R(\check{G}^{\times n})$ and $\check{G}^{\times(n+k)}$ of $R(\check{G}^{\times(n+k)})$ and remark that $\check{\chi}_{n,k}$ maps basis elements to basis elements. So it suffices to show injectivity on basis elements.

Suppose given $H\underline{g}$, $H\underline{\tilde{g}} \in \check{G}^{\times n}$ with $(H\underline{g})\check{\chi}_{n,k} = (H\underline{\tilde{g}})\check{\chi}_{n,k}$. As $(H\underline{g})\check{\chi}_{n,k} = H\underline{g} \;\sqcup\; (H1)_{i\in[0,k-1]}$ and $(H\underline{\tilde{g}})\check{\chi}_{n,k} = H\underline{\tilde{g}} \;\sqcup\; (H1)_{i\in[0,k-1]}$, we obtain $(H\underline{g} \;\sqcup\; (H1)_{i\in[0,k-1]})_j = (H\underline{\tilde{g}} \;\sqcup\; (H1)_{i\in[0,k-1]})_j$ for $j \in [0, n-1]$. Thus $H\underline{g} = H\underline{\tilde{g}}$. ▫

Definition 59 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. We define the RG-linear map $\check{\varepsilon}_n^{G,H}$ as follows.

If $n \geqslant 1$, we let

$$\begin{aligned} \check{\varepsilon}_n^{G,H} : R(\check{G}^{\times n}) &\to R(\check{G}^{\times (n-1)}) \\ H\underline{g} &\mapsto \sum_{k=0}^{n-1} (-1)^k H\underline{g}\uparrow^k . \end{aligned}$$

If $n = 0$, we let

$$\begin{aligned} \check{\varepsilon}_0^{G,H} : R &\to 0 \\ r &\mapsto 0. \end{aligned}$$

Note that for $n = 1$, we obtain

$$\begin{array}{ccccccccc} \check{\varepsilon}_1^{G,H} & : & R\check{G} & = & R(\check{G}^{\times 1}) & \to & R(\check{G}^{\times 0}) & = & R \\ & & Hg & = & (Hg) & \mapsto & (\,) & = & 1 \end{array}$$

by identification.

We often write $\check{\varepsilon}_n = \check{\varepsilon}_n^{G,H}$.

Remark 60 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.*

(1) *We have $\check{\chi}_{n,1}\check{\varepsilon}_{n+1} = \check{\varepsilon}_n\check{\chi}_{n-1,1} + (-1)^n \operatorname{id}_{R(\check{G}^{\times n})}$.*

(2) *We have $\check{\chi}_{n,2}\check{\varepsilon}_{n+2} = \check{\varepsilon}_n\check{\chi}_{n-1,2}$.*

Proof. Suppose given $H\underline{g} \in \check{G}^{\times n}$.

Ad (1). If $n \geqslant 1$, then we obtain

$$\begin{aligned} H\underline{g}\check{\chi}_{n,1}\check{\varepsilon}_{n+1} &= \sum_{k=0}^{n} (-1)^k H\underline{g}\check{\chi}_{n,1}\uparrow^k \\ &= \sum_{k=0}^{n-1} (-1)^k H\underline{g}\check{\chi}_{n,1}\uparrow^k + (-1)^n (H\underline{g}\check{\chi}_{n,1})\uparrow^n \\ &= \sum_{k=0}^{n-1} (-1)^k H\underline{g}\uparrow^k \check{\chi}_{n-1,1} + (-1)^n H\underline{g} \\ &= H\underline{g}\big(\check{\varepsilon}_n\check{\chi}_{n-1,1} + (-1)^n \operatorname{id}\big) . \end{aligned}$$

If $n = 0$, then we obtain

$$\begin{aligned} (\,)\check{\chi}_{0,1}\check{\varepsilon}_1 &= (H1)\check{\varepsilon}_1 \\ &= (\,) \\ &= 0\check{\chi}_{-1,1} + (\,) \\ &= (\,)\big(\check{\varepsilon}_0\check{\chi}_{-1,1} + \operatorname{id}\big) . \end{aligned}$$

Ad (2). If $n \geqslant 1$, then we obtain

$$\begin{aligned}
(H\underline{g}\check{\chi}_{n,2})\check{\varepsilon}_{n+2} &= \sum_{k=0}^{n+1}(-1)^k(H\underline{g}\check{\chi}_{n,2})\uparrow^k \\
&= \sum_{k=0}^{n-1}(-1)^k(H\underline{g}\check{\chi}_{n,2})\uparrow^k \\
&\quad +(-1)^n(H\underline{g}\check{\chi}_{n,2})\uparrow^n +(-1)^{n+1}(H\underline{g}\check{\chi}_{n,2})\uparrow^{n+1} \\
&= \sum_{k=0}^{n-1}(-1)^k(H\underline{g}\uparrow^k)\check{\chi}_{n-1,2} \\
&\quad +(-1)^n H\underline{g}\check{\chi}_{n,1} + (-1)^{n+1}H\underline{g}\check{\chi}_{n,1} \\
&= \sum_{k=0}^{n-1}(-1)^k(H\underline{g}\uparrow^k)\check{\chi}_{n-1,2} \\
&= (H\underline{g}\check{\varepsilon}_n)\check{\chi}_{n-1,2}\ .
\end{aligned}$$

If $n = 0$, then we obtain

$$\begin{aligned}
(\,)\check{\chi}_{0,2}\check{\varepsilon}_2 &= (H1, H1)\check{\varepsilon}_2 \\
&= (H1) - (H1) \\
&= 0 \\
&= 0\check{\chi}_{-1,2} \\
&= (\,)\check{\varepsilon}_0\check{\chi}_{-1,2}\ .
\end{aligned}$$

Lemma 61 *We have the following acyclic complex of RG-modules.*

$$\ldots \longrightarrow R(\check{G}^{\times 4}) \xrightarrow{\check{\varepsilon}_4} R(\check{G}^{\times 3}) \xrightarrow{\check{\varepsilon}_3} R(\check{G}^{\times 2}) \xrightarrow{\check{\varepsilon}_2} R\check{G} \xrightarrow{\check{\varepsilon}_1} R \xrightarrow{\check{\varepsilon}_0} 0 \longrightarrow \ldots$$

Proof. Suppose given $n \geqslant 1$.

Ad $\operatorname{Im}(\check{\varepsilon}_n) \overset{!}{\subseteq} \operatorname{Ker}(\check{\varepsilon}_{n-1})$.

Case $n = 1$. We have $\check{\varepsilon}_1\check{\varepsilon}_0 = 0$.

Case $n \geqslant 2$. Suppose given $H\underline{g} = (Hg_0, Hg_1, Hg_2, \ldots, Hg_{n-1}) \in \check{G}^{\times n}$.

For $k \in [0, n-1]$, we get

$$
\begin{aligned}
(H\underline{g}\uparrow^k)\check{\varepsilon}_{n-1} &= \sum_{i=0}^{n-2}(-1)^i\Big(H\underline{g}\uparrow^k\Big)\uparrow^i \\
&= \sum_{i=0}^{k-1}(-1)^i\Big(H\underline{g}\uparrow^k\Big)\uparrow^i + \sum_{i=k}^{n-2}(-1)^i\Big(H\underline{g}\uparrow^k\Big)\uparrow^i \\
&= \sum_{i=0}^{k-1}(-1)^i\Big(H\underline{g}\uparrow^k\Big)\uparrow^i + \sum_{i=k+1}^{n-1}(-1)^{i-1}\Big(H\underline{g}\uparrow^k\Big)\uparrow^{i-1} \\
&= \sum_{i=0}^{k-1}(-1)^i H\underline{g}\uparrow^{i,k} + \sum_{i=k+1}^{n-1}(-1)^{i-1}H\underline{g}\uparrow^{k,i} \quad .
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&((H\underline{g})\check{\varepsilon}_n)\check{\varepsilon}_{n-1} \\
&= \sum_{k=0}^{n-1}(-1)^k\Big(\sum_{i=0}^{k-1}(-1)^i H\underline{g}\uparrow^{i,k} + \sum_{i=k+1}^{n-1}(-1)^{i-1}H\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{k=0}^{n-1}(-1)^k\sum_{i=0}^{k-1}(-1)^i H\underline{g}\uparrow^{i,k}\Big) + \Big(\sum_{k=0}^{n-1}(-1)^k\sum_{i=k+1}^{n-1}(-1)^{i-1}H\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{k=0}^{n-1}\sum_{i=0}^{k-1}(-1)^{k+i}H\underline{g}\uparrow^{i,k}\Big) + \Big((-1)^{-1}\sum_{k=0}^{n-1}\sum_{i=k+1}^{n-1}(-1)^{k+i}H\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{\substack{i,k\in[0,n-1]\\ i<k}}(-1)^{k+i}H\underline{g}\uparrow^{i,k}\Big) + \Big((-1)^{-1}\sum_{\substack{i,k\in[0,n-1]\\ k<i}}(-1)^{k+i}H\underline{g}\uparrow^{k,i}\Big) \\
&= \Big(\sum_{\substack{i,k\in[0,n-1]\\ k<i}}(-1)^{k+i}H\underline{g}\uparrow^{k,i}\Big) - \Big(\sum_{\substack{i,k\in[0,n-1]\\ k<i}}(-1)^{k+i}H\underline{g}\uparrow^{k,i}\Big) = 0
\end{aligned}
$$

.

Ad $\operatorname{Ker}(\check{\varepsilon}_{n-1}) \overset{!}{\subseteq} \operatorname{Im}(\check{\varepsilon}_n)$. Suppose given $x \in \operatorname{Ker}(\check{\varepsilon}_{n-1})$. We obtain

$$
\begin{aligned}
x &= (-1)^{n-1}x((-1)^{n-1}\,\mathrm{id}) \\
&\overset{\text{R. 60.(1)}}{=} (-1)^{n-1}x(\check{\chi}_{n-1,1}\check{\varepsilon}_n - \check{\varepsilon}_{n-1}\check{\chi}_{n-2,1}) \\
&= (-1)^{n-1}(\underbrace{x\check{\chi}_{n-1,1}\check{\varepsilon}_n}_{\in\operatorname{Im}(\check{\varepsilon}_n)} - \underbrace{x\check{\varepsilon}_{n-1}\check{\chi}_{n-2,1}}_{=0}) \\
&\in \operatorname{Im}(\check{\varepsilon}_n)\ .
\end{aligned}
$$

□

Definition 62 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

(1) We define the n-th *augmentation module relative to H* or *relative augmentation module* as

$$\mathrm{I}_R^{(n)}(G,H) \;:=\; \mathrm{Ker}(\check{\varepsilon}_n) \;\subseteq\; R(\check{G}^{\times n})\;.$$

Then $\mathrm{I}_R^{(n)}(G,H)$ is an RG-submodule of $R(\check{G}^{\times n})$.

(2) We write

$$\bar{\check{\varepsilon}}_{n+1} \;:=\; \check{\varepsilon}_{n+1}|^{\mathrm{I}_R^{(n)}(G,H)} \;:\; R(\check{G}^{\times(n+1)}) \;\to\; \mathrm{I}_R^{(n)}(G,H)\;,$$

which is a surjective RG-linear map.

(3) For $j \in \mathbb{Z}_{\geqslant 0}$, we have the injective RG-linear map

$$\begin{array}{rcccl} \dot{\check{\varepsilon}}_j & : & \mathrm{I}_R^{(j)}(G,H) & \to & R(\check{G}^{\times j}) \\ & & x & \mapsto & x\;. \end{array}$$

Note that $\check{\varepsilon}_j = \bar{\check{\varepsilon}}_j \cdot \dot{\check{\varepsilon}}_{j-1} : R(\check{G}^{\times j}) \to R(\check{G}^{\times(j-1)})$ for $j \in \mathbb{Z}_{\geqslant 1}$.

Remark 63 *The maps σ_k form the following morphism of complexes.*

$$\begin{tikzcd} \ldots \arrow[r] & R(G^{\times 2}) \arrow[r, "\varepsilon_2"] \arrow[d, "\sigma_2"] & R(G^{\times 1}) \arrow[r, "\varepsilon_1"] \arrow[d, "\sigma_1"] & R(G^{\times 0}) \arrow[r, "\varepsilon_0"] \arrow[d, "\sigma_0"] & 0 \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & R(\check{G}^{\times 2}) \arrow[r, "\check{\varepsilon}_2"] & R(\check{G}^{\times 1}) \arrow[r, "\check{\varepsilon}_1"] & R(\check{G}^{\times 0}) \arrow[r, "\check{\varepsilon}_0"] & 0 \arrow[r] & \ldots \end{tikzcd}$$

Proof. Suppose given $n \geqslant 1$. We have to show that $\varepsilon_n\sigma_{n-1} \overset{!}{=} \sigma_n\check{\varepsilon}_n$.

Suppose given $\underline{g} \in G^{\times n}$. On the left hand side we get

$$\underline{g}\varepsilon_n\sigma_{n-1} = \Big(\sum_{k=0}^{n-1}(-1)^k \underline{g}\uparrow^k\Big)\sigma_{n-1} = \sum_{k=0}^{n-1}(-1)^k H\underline{g}\uparrow^k\;.$$

On the right hand side we have

$$\underline{g}\sigma_n\check{\varepsilon}_n = H\underline{g}\check{\varepsilon}_n = \sum_{k=0}^{n-1}(-1)^k H\underline{g}\uparrow^k\;.$$

In both cases, we obtain the same result. ▫

Definition 64 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

Using the commutativity of the quadrangles appearing in the morphism of complexes in Remark 63, we may define the RG-linear map

$$\bar{\sigma}_n \;:=\; \bar{\sigma}_n^{G,H} \;:=\; \sigma_n\big|_{\mathrm{I}_R^{(n)}(G)}^{\mathrm{I}_R^{(n)}(G,H)} \;:\; \mathrm{I}_R^{(n)}(G) \;\to\; \mathrm{I}_R^{(n)}(G,H)\;.$$

We obtain the following commutative diagram.

$$\begin{tikzcd}
R(G^{\times(n+1)}) \arrow[rr, "\varepsilon_{n+1}"] \arrow[dr, "\bar{\varepsilon}_{n+1}"'] \arrow[dd, "\sigma_{n+1}"] & & R(G^{\times n}) \arrow[dd, "\sigma_n"] \\
& \mathrm{I}_R^{(n)}(G) \arrow[ur, hook, "\dot{\varepsilon}_n"] \arrow[dd, "\bar{\sigma}_n"] & \\
R(\check{G}^{\times(n+1)}) \arrow[rr, "\check{\varepsilon}_{n+1}"] \arrow[dr, "\bar{\check{\varepsilon}}_{n+1}"'] & & R(\check{G}^{\times n}) \\
& \mathrm{I}_R^{(n)}(G,H) \arrow[ur, hook, "\dot{\check{\varepsilon}}_n"] &
\end{tikzcd}$$

Definition 65 We have the surjective RG-linear map $\sigma_n : R(G^{\times n}) \to R(\check{G}^{\times n}) : \underline{g} \mapsto H\underline{g}$, where $n \in \mathbb{Z}_{\geqslant 0}$; cf. Definition 53.

By composition we get the injective R-linear map

$$\begin{array}{rcl}
\sigma_n^* := \sigma_n \cdot (-) \;\;:\;\; {}_{RG}(R(\check{G}^{\times n}), M) & \to & {}_{RG}(R(G^{\times n}), M) \\
\check{f} & \mapsto & (\sigma_n \cdot \check{f} : \underline{g} \mapsto \underline{g}\sigma_n \check{f}).
\end{array}$$

Remark 66 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. We have*

$$\mathrm{Im}(\sigma_n^*) \;=\; \left\{ f \in {}_{RG}(R(G^{\times n}), M) \;\middle|\; \begin{array}{l} (h_0 g_0, \dots h_{n-1} g_{n-1}) f = (g_0, \dots, g_{n-1}) f \\ \forall g_0, \dots, g_{n-1} \in G, h_0, \dots, h_{n-1} \in H \end{array} \right\}.$$

Proof.

Ad $\subseteq$. Suppose given $\check{f} \in {}_{RG}(R(\check{G}^{\times n}), M)$.

Suppose given $h_0, \dots, h_{n-1} \in H$ and $g_0, \dots, g_{n-1} \in G$. We get

$$(h_0 g_0, \dots, h_{n-1} g_{n-1})(\sigma_n \cdot \check{f}) = (Hh_0 g_0, \dots, Hh_{n-1} g_{n-1})f = (Hg_0, \dots, Hg_{n-1})f = (g_0, \dots, g_{n-1})(\sigma_n \cdot \check{f})\;.$$

Ad $\supseteq$. Suppose given $f \in {}_{RG}(R(G^{\times n}), M)$ with

$$(h_0 g_0, \dots, h_{n-1} g_{n-1}) f \;=\; (g_0, \dots, g_{n-1}) f$$

for $h_0, \dots, h_{n-1} \in H$, $g_0, \dots, g_{n-1} \in G$.

We define $\check{f} \in {}_{RG}(R(\check{G}^{\times n}), M)$ as $(H\underline{g})\check{f} := (\underline{g})f$ for $H\underline{g} \in \check{G}^{\times n}$.

This is a well-defined R-linear map, as f fulfills $(h_0g_0,\ldots,h_{n-1}g_{n-1})f = (g_0,\ldots,g_{n-1})f$ by assumption.

Moreover, $\check{f}$ is RG-linear, as for $H\underline{g} \in \check{G}^{\times n}$ and $x \in G$, we obtain

$$(H\underline{g})\check{f}\cdot x \;=\; (\underline{g})f\cdot x \;=\; (\underline{g}\cdot x)f \;=\; (H\underline{g}\cdot x)\check{f}\;.$$

Finally, we get $f = \sigma\cdot\check{f}$. ▫

Definition 67 Suppose given $n \geqslant 0$.

(1) We have the RG-linear map $\check{\varepsilon}_{n+2} : R(\check{G}^{\times(n+2)}) \to R(\check{G}^{\times(n+1)})$; cf. Definition 59.

By composition, we get the R-linear map

$$\begin{array}{rcl} \check{\varepsilon}^*_{n+2} := \check{\varepsilon}_{n+2}\cdot(-) \;:\; {}_{RG}(R(\check{G}^{\times(n+1)}),M) & \to & {}_{RG}(R(\check{G}^{\times(n+2)}),M) \\ \check{f} & \mapsto & (\check{\varepsilon}_{n+2}\cdot\check{f} : \underline{g}\mapsto \underline{g}\check{\varepsilon}_{n+2}\check{f})\;. \end{array}$$

(2) We have the surjective RG-linear map $\bar{\check{\varepsilon}}_{n+1} : R(\check{G}^{\times(n+1)}) \to \mathrm{I}_R^{(n)}(G,H)$; cf. Definition 62.(2).

By composition, we get the injective R-linear map

$$\begin{array}{rcl} \bar{\check{\varepsilon}}^*_{n+1} := \bar{\check{\varepsilon}}_{n+1}\cdot(-) \;:\; {}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) & \to & {}_{RG}(R(\check{G}^{\times(n+1)}),M) \\ \check{f} & \mapsto & (\bar{\check{\varepsilon}}_{n+1}\cdot\check{f} : \underline{g}\mapsto \underline{g}\bar{\check{\varepsilon}}_{n+1}\check{f} = \underline{g}\check{\varepsilon}_{n+1}\check{f})\;. \end{array}$$

Remark 68 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

(1) We have the following commutative quadrangle.

$$\begin{tikzcd} {}_{RG}(R(G^{\times(n+2)}),M) & {}_{RG}(R(G^{\times(n+1)}),M) \arrow[l, "\varepsilon^*_{n+2}"'] \\ {}_{RG}(R(\check{G}^{\times(n+2)}),M) \arrow[u, "\sigma^*_{n+2}"] & {}_{RG}(R(\check{G}^{\times(n+1)}),M) \arrow[l, "\check{\varepsilon}^*_{n+2}"'] \arrow[u, "\sigma^*_{n+1}"'] \end{tikzcd}$$

This follows by applying the functor ${}_{RG}(-,M)$ to a commutative quadrangle of the morphism of complexes in Remark 63.

(2) We have the following commutative quadrangle.

$$\begin{tikzcd} {}_{RG}(R(G^{\times(n+1)}),M) & {}_{RG}(\,\mathrm{I}_R^{(n)}(G),M) \arrow[l, "\bar{\varepsilon}^*_{n+1}"'] \\ {}_{RG}(R(\check{G}^{\times(n+1)}),M) \arrow[u, "\sigma^*_{n+1}"] & {}_{RG}(\,\mathrm{I}_R^{(n)}(G,H),M) \arrow[l, "\bar{\check{\varepsilon}}^*_{n+1}"'] \arrow[u, "\bar{\sigma}^*_n"'] \end{tikzcd}$$

This follows by applying the functor ${}_{RG}(-,M)$ to the left-hand side commutative quadrangle from Definition 64.

Definition 69 For $n \in \mathbb{Z}_{\geqslant 1}$, we have, in Lemma 44, derived the R-linear isomorphism

$$\begin{array}{rrcl} \rho_n : & {}_{RG}(R(G^{\times n}), M) & \xrightarrow{\sim} & {}_{\mathrm{Map}}(G^{\times (n-1)}, M) \\ & f & \mapsto & \big((g_0, \ldots, g_{n-2}) \mapsto ((g_0, \ldots, g_{n-2}, 1)f\big) = (\chi_{n-1,1} \cdot f)|_{G^{\times (n-1)}} \ . \end{array}$$

For $n \in \mathbb{Z}_{\geqslant 0}$, we let

$$ {}_{\mathrm{Map},H}(G^{\times n}, M) \ := \ (\mathrm{Im}(\sigma^*_{n+1}))\rho_{n+1} \ \subseteq \ {}_{\mathrm{Map}}(G^{\times n}, M) \ .$$

Definition 70 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

Since ${}_{\mathrm{Map},H}(G^{\times n}, M) = (\mathrm{Im}(\sigma^*_{n+1}))\rho_{n+1}$, we have the R-linear isomorphism

$$\check{\rho}_{n+1} \ := \ (\sigma^*_{n+1}\rho_{n+1})|^{{}_{\mathrm{Map},H}(G^{\times n}, M)} \ : \ {}_{RG}(R(\check{G}^{\times (n+1)}), M) \ \xrightarrow{\sim} \ {}_{\mathrm{Map},H}(G^{\times n}, M)$$

We get the commutative diagram

$$\begin{tikzcd} {}_{RG}(R(G^{\times (n+1)}), M) \arrow[r, "\rho_{n+1}", "\sim"'] & {}_{\mathrm{Map}}(G^{\times n}, M) \\ {}_{RG}(R(\check{G}^{\times (n+1)}), M) \arrow[r, "\check{\rho}_{n+1}", "\sim"'] \arrow[u, "\sigma^*_{n+1}"] & {}_{\mathrm{Map},H}(G^{\times n}, M) \arrow[u, hook] \end{tikzcd}$$

Remark 71 *Suppose given* $n \in \mathbb{Z}_{\geqslant 0}$.

We have

$${}_{\mathrm{Map},H}(G^{\times n}, M) = \left\{ f \in {}_{\mathrm{Map}}(G^{\times n}, M) \ \middle| \ \begin{array}{cl} & (\, h_0 g_0 h_n \, , \ h_1 g_1 h_n \, , \ \ldots \, , \ h_{n-1} g_{n-1} h_n \,)f \\ = & (\, g_0 \, , \ g_1 \, , \ldots , g_{n-1} \,)f \cdot h_n \\ & \forall \ g_0, \ldots, g_{n-1} \in G, \ h_0, \ldots, h_n \in H \end{array} \right\}$$

Proof.

Ad $\subseteq$. Suppose given $q \in \mathrm{Im}(\sigma^*_{n+1}) \subseteq {}_{RG}(R(G^{\times n}), M)$.

Suppose given $h_0, \ldots, h_n \in H$ and $g_0, \ldots, g_{n-1} \in G$. We obtain

$$\begin{array}{rcl} (h_0 g_0 h_n, \ldots, h_{n-1} g_{n-1} h_n)(q\rho_{n+1}) & \overset{\text{L. 44}}{=} & (h_0 g_0 h_n, \ldots, h_{n-1} g_{n-1} h_n, 1)q \\ & = & (h_0 g_0 h_n, \ldots, h_{n-1} g_{n-1} h_n, h_n^{-1} h_n)q \\ & \overset{RG\text{-lin.}}{=} & (h_0 g_0, \ldots, h_{n-1} g_{n-1}, h_n^{-1})q \cdot h_n \\ & \overset{\text{R. 66}}{=} & (g_0, \ldots, g_{n-1}, 1)q \cdot h_n \\ & \overset{\text{L. 44}}{=} & (g_0, \ldots, g_{n-1})(q\rho_{n+1}) \cdot h_n \end{array}$$

Ad $\supseteq$. Suppose given a map $f \in {}_{\mathrm{Map}}(G^{\times n}, M)$ that fulfills

$$(h_0 g_0 h_n, \ldots, h_{n-1} g_{n-1} h_n)f \ = \ (g_0, \ldots, g_{n-1}) f h_n$$

for $g_0, \ldots, g_{n-1} \in G$ and $h_0, \ldots, h_n \in H$.

We get

$$\begin{aligned}
& (h_0 g_0, \dots, h_{n-1} g_{n-1}, h_n g_n)(f\rho_{n+1}^{-1}) \\
= \; & (h_0 g_0 (g_n^{-1} h_n^{-1} h_n g_n), \dots, h_{n-1} g_{n-1} (g_n^{-1} h_n^{-1} h_n g_n), h_n g_n)(f\rho_{n+1}^{-1}) \\
\overset{f\rho_{n+1}^{-1} \text{ is } RG\text{-linear}}{=} \; & (h_0 g_0 g_n^{-1} h_n^{-1}, \dots, h_{n-1} g_{n-1} g_n^{-1} h_n^{-1}, 1)(f\rho_{n+1}^{-1}) \cdot h_n g_n \\
\overset{\text{L. 44}}{=} \; & (h_0 g_0 g_n^{-1} h_n^{-1}, \dots, h_{n-1} g_{n-1} g_n^{-1} h_n^{-1}) f \cdot h_n g_n \\
\overset{\text{ass. on f}}{=} \; & (g_0 g_n^{-1}, \dots, g_{n-1} g_n^{-1}) f \cdot h_n^{-1} \cdot h_n g_n \\
\overset{\text{L. 44}}{=} \; & (g_0 g_n^{-1}, \dots, g_{n-1} g_n^{-1}, 1)(f\rho_{n+1}^{-1}) \cdot g_n \\
\overset{f\rho_{n+1}^{-1} \text{ is } RG\text{-linear}}{=} \; & (g_0, \dots, g_{n-1}, g_n)(f\rho_{n+1}^{-1})
\end{aligned}$$

So by Remark 66 we know that $f\rho_{n+1}^{-1} \in \operatorname{Im}(\sigma_{n+1}^*)$, hence

$$f \;\in\; (\operatorname{Im}(\sigma_{n+1}^*))\rho_{n+1} \;=\; {}_{\mathrm{Map},H}(G^{\times n}, M)\,.$$

$\square$

Definition 72 Suppose given $n \geqslant 0$.

In Remarks 14 and 16, we have obtained the bijection $\iota_n : G^{\times n} \xrightarrow{\sim} G^{\times n}$, yielding the R-linear isomorphism

$$\begin{array}{rcccl}
\iota_n^* \;:=\; \iota_n \cdot (-) & : & {}_{\mathrm{Map}}(G^{\times n}, M) & \xrightarrow{\sim} & {}_{\mathrm{Map}}(G^{\times n}, M) \\
 & & f & \mapsto & \iota_n \cdot f\,.
\end{array}$$

Let

$${}_{\mathrm{Map},[H]}(G^{\times n}, M) \;:=\; \iota_n^* \left({}_{\mathrm{Map},H}(G^{\times n}, M) \right) \;\subseteq\; {}_{\mathrm{Map}}(G^{\times n}, M)\,.$$

Definition 73 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

Since ${}_{\mathrm{Map},[H]}(G^{\times n}, M) \;:=\; \iota_n^* \left({}_{\mathrm{Map},H}(G^{\times n}, M) \right)$, we have the R-linear isomorphism

$$\breve{\iota}_n^* \;:=\; \iota_n^*\Big|_{{}_{\mathrm{Map},H}(G^{\times n},M)}^{{}_{\mathrm{Map},[H]}(G^{\times n},M)} \;:\; {}_{\mathrm{Map},H}(G^{\times n}, M) \xrightarrow{\sim} {}_{\mathrm{Map},[H]}(G^{\times n}, M)\,.$$

We get the following commutative quadrangle.

$$\begin{tikzcd}
{}_{\mathrm{Map}}(G^{\times n}, M) \arrow[r, "\iota_n^*", "\sim"'] & {}_{\mathrm{Map}}(G^{\times n}, M) \\
{}_{\mathrm{Map},H}(G^{\times n}, M) \arrow[u, hook] \arrow[r, "\breve{\iota}_n^*", "\sim"'] & {}_{\mathrm{Map},[H]}(G^{\times n}, M) \arrow[u, hook]
\end{tikzcd}$$

Remark 74 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.*

We have

$${}_{\mathrm{Map},[H]}(G^{\times n}, M) \;=$$

$$\left\{ f \in {}_{\mathrm{Map}}(G^{\times n}, M) \;\middle|\; \begin{array}{ll}
 & (\,h_0 g_0\,,\; h_1 g_1\,,\; h_2 g_2\,,\; \dots\,,\; h_{n-2} g_{n-2}\,,\; h_{n-1} g_{n-1}\,) f \cdot h_n \\
= & (\,g_0 h_1\,,\; g_1 h_2\,,\; g_2 h_3\,,\; \dots\,,\; g_{n-2} h_{n-1}\,,\; g_{n-1} h_n\,) f \\
 & \forall\; g_0, \dots, g_{n-1} \in G,\; h_0, \dots, h_n \in H
\end{array} \right\}$$

Proof.

Ad $\subseteq$. Suppose given $q \in {}_{\mathrm{Map},H}(G^{\times n}, M)$, whence $q\iota_n^* = \iota_n \cdot q \in {}_{\mathrm{Map},[H]}(G^{\times n}, M)$. Suppose given $g_0, \ldots, g_{n-1} \in G$ and $h_0, \ldots, h_n \in H$. Then

$$\begin{array}{cl}
& (\,h_0 g_0\,,\,h_1 g_1\,,\,h_2 g_2\,,\,\ldots\,,\,h_{n-2}g_{n-2}\,,\,h_{n-1}g_{n-1}\,)(\iota_n \cdot q)\cdot h_n \\
= & [\,h_0 g_0\,,\,h_1 g_1\,,\,h_2 g_2\,,\,\ldots\,,\,h_{n-2}g_{n-2}\,,\,h_{n-1}g_{n-1}\,]q \cdot h_n \\
= & (\,h_0 g_0 h_1 g_1 h_2 g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}\,,\; h_1 g_1 h_2 g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}\,,\;\ldots\;,\,h_{n-2}g_{n-2}h_{n-1}g_{n-1}\,,\,h_{n-1}g_{n-1}\,)q \cdot h_n \\
\overset{\text{R. 71}}{=} & (\,g_0 h_1 g_1 h_2 g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}\,,\; g_1 h_2 g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}\,,\;\ldots\;,\,g_{n-2}h_{n-1}g_{n-1}\,,\,g_{n-1}\,)q \cdot h_n \\
\overset{\text{R. 71}}{=} & (\,g_0 h_1 g_1 h_2 g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}h_n\,,\; g_1 h_2 g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}h_n\,,\;\ldots\;,\,g_{n-2}h_{n-1}g_{n-1}h_n\,,\,g_{n-1}h_n\,)q \\
= & [\,g_0 h_1\,,\,g_1 h_2\,,\,\ldots\,,g_{n-2}h_{n-1}\,,g_{n-1}h_n\,]q \\
= & (\,g_0 h_1\,,\,g_1 h_2\,,\,\ldots\,,g_{n-2}h_{n-1}\,,g_{n-1}h_n\,)(\iota_n \cdot q)\;.
\end{array}$$

Ad $\supseteq$. Suppose given a map f that fulfills the property in the description of the set in the statement. We have to show that there exists a map $q \in {}_{\mathrm{Map},H}(G^{\times n}, M)$ with $q\iota_n^* = \iota_n \cdot q \overset{!}{=} f$. So we need to show that $\iota_n^{-1} \cdot f \overset{!}{\in} {}_{\mathrm{Map},H}(G^{\times n}, M)$, which means that $\iota_n^{-1} \cdot f$ fulfills the property describing the set in Remark 71.

Suppose given $g_0, \ldots, g_{n-1} \in G$ and $h_0, \ldots, h_n \in H$. Then

$$\begin{array}{cl}
& (\,h_0 g_0 h_n\,,\;h_1 g_1 h_n\,,\;h_2 g_2 h_n\,,\,\ldots\,,\,h_{n-1}g_{n-1}h_n\,)(\iota_n^{-1}\cdot f) \\
\overset{\text{R. 16}}{=} & (\,(h_0 g_0 h_n)(h_1 g_1 h_n)^{-1}\,,\;(h_1 g_1 h_n)(h_2 g_2 h_n)^{-1}\,,\,\ldots\,,\,(h_{n-2}g_{n-2}h_n)(h_{n-1}g_{n-1}h_n)^{-1},\,(h_{n-1}g_{n-1}h_n)\,)f \\
= & (\,(h_0 g_0)(h_1 g_1)^{-1}\,,\;(h_1 g_1)(h_2 g_2)^{-1}\,,\,\ldots\,,\,(h_{n-2}g_{n-2})(h_{n-1}g_{n-1})^{-1},\,(h_{n-1}g_{n-1}h_n)\,)f \\
= & (\,h_0 g_0 g_1^{-1} h_1^{-1}\,,\;h_1 g_1 g_2^{-1} h_2^{-1}\,,\,\ldots\,,\,h_{n-2}g_{n-2}g_{n-1}^{-1}h_{n-1}^{-1},\,h_{n-1}g_{n-1}h_n\,)f \\
\overset{\text{property of } f}{=} & (\,g_0 g_1^{-1}\,,\;g_1 g_2^{-1}\,,\,\ldots\,,\,g_{n-2}g_{n-1}^{-1},\,g_{n-1})f \cdot h_n \\
\overset{\text{R. 16}}{=} & (g_0, g_1, g_2, \ldots, g_{n-1})(\iota_n^{-1}\cdot f)\cdot h_n\;.
\end{array}$$

□

5.2 A universal relative n-cocycle

Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

Definition 75 (relative n-cocycles) A map $d : G^{\times n} \to M$ is called an *n-cocycle of G relative to H with values in M* or, for short, a *relative n-cocycle*, if d is an n-cocycle of G with values in M that is contained in ${}_{\mathrm{Map},[H]}(G^{\times n}, M)$.

Let

$$\mathrm{Z}^n(G,H,M) \;:=\; \mathrm{Z}^n(G,M) \cap {}_{\mathrm{Map},[H]}(G^{\times n}, M) \;\subseteq\; {}_{\mathrm{Map}}(G^{\times n}, M)\;.$$

be the set of n-cocycles of G relative to H with values in M. Cf. Definitions 40 and 73, Remark 74.

Note that $\mathrm{Z}^n(G,H,M)$ is an R-submodule of ${}_{\mathrm{Map}}(G^{\times n}, M)$.

Explicitly, $d \in {}_{\mathrm{Map}}(G^{\times n}, M)$ is contained in $\mathrm{Z}^n(G,H,M)$ if and only if it satisfies the following conditions $(1, 2)$.

(1) For $g_0, \ldots, g_n \in G$, we have

$$\begin{array}{rcl} 0 & = & (-1)^0(\, g_1\, , \,\ldots\, , \, g_n\,)d \\ & & + \left(\sum_{k=1}^{n} (-1)^k(\, g_0\, , \,\ldots\, , \, g_{k-2}\, , \, g_{k-1}\cdot g_k\, , \, g_{k+1}\, , \,\ldots\, , \, g_n\,)d \right) \\ & & + (-1)^{n+1}(\, g_0\, , \,\ldots\, , \, g_{n-1}\,)d \cdot g_n\, . \end{array}$$

(2) For $g_0, \ldots, g_{n-1} \in G$ and $h_0, \ldots, h_n \in H$, we have

$$\begin{array}{rl} & (\, h_0 g_0\, , \, h_1 g_1\, , \, h_2 g_2\, , \,\ldots\, , \, h_{n-2} g_{n-2}\, , \, h_{n-1} g_{n-1}\,)d \cdot h_n \\ = & (\, g_0 h_1\, , \, g_1 h_2\, , \, g_2 h_3\, , \,\ldots, \, g_{n-2} h_{n-1}\, , \, g_{n-1} h_n\,)d\, . \end{array}$$

Remark 76

We consider $\mathrm{Z}^n(G, H, M)$ in the cases $n \in \{0, 1, 2\}$; cf. Remark 41.

(0) To obtain an element $d \in \mathrm{Z}^0(G, M)$, we need an element $m \in M$ such that $m \cdot g = m$ for $g \in G$. Then we may let $(\,)d := m$. In fact, condition (1) of Definition 75 reads

$$(\,)d - (\,)d \cdot g_0 \;=\; 0 \qquad \text{for } g_0 \in G\, .$$

Condition (2) is then satisfied as well.

So $\mathrm{Z}^0(G, H, M)$ can be identified with the R-submodule

$$\{\, m \in M \mid m \cdot g = m \text{ for } g \in G\,\} \;\subseteq\; M\, .$$

(1) An element of $\mathrm{Z}^1(G, M)$ is a derivation, i.e. a map $d : G \to M$ such that

$$(g_0 \cdot g_1)d \;=\; (g_1)d + (g_0)d \cdot g_1$$

for $g_0, g_1 \in G$. So such an element satisfies condition (1) of Definition 75.

Condition (2) reads

$$(h_0 \cdot g_0)d \cdot h_1 \;=\; (g_0 \cdot h_1)d$$

for $g_0 \in G$ and $h_0, h_1 \in H$.

Suppose given a derivation $d : G \to M$. Note that $(1 \cdot 1)d = (1)d + (1)d \cdot 1$, whence $(1)d = 0$.

We *claim* that condition (2) is equivalent to $(h)d = 0$ for $h \in H$.

Suppose that d satisfies condition (2). Then $(h)d = (h \cdot 1)d \cdot 1 = (1 \cdot 1)d = 0$.

Suppose that $(h)d = 0$ for $h \in H$. Then

$$\begin{array}{rcl} (h_0 \cdot g_0)d \cdot h_1 & = & (g_0)d \cdot h_1 + (h_0)d \cdot g_0 \cdot h_1 \\ & = & (g_0)d \cdot h_1 \\ & = & (h_1)d + (g_0)d \cdot h_1 \\ & = & (g_0 \cdot h_1)d \end{array}$$

for $g_0 \in G$ and $h_0,\, h_1 \,\in\, H$. This proves the *claim*.

So

$$\begin{array}{rl} & \mathrm{Z}^1(G,H,M) \\ = & \{\, G \overset{d}{\to} M \mid (g_0 \cdot g_1)d = (g_1)d + (g_0)d \cdot g_1 \ \text{ and } \ (h)d = 0 \quad \text{for } \ g_0,\, g_1 \,\in\, G \ \text{ and } \ h \in H \,\}\ . \end{array}$$

(2) An element of $\mathrm{Z}^2(G,M)$ is a map $d : G \times G \to M$ such that

$$(\, g_1 \,,\, g_2 \,)d - (\, g_0 \cdot g_1 \,,\, g_2 \,)d + (\, g_0 \,,\, g_1 \cdot g_2 \,)d - (\, g_0 \,,\, g_1 \,)d \cdot g_2 \;=\; 0$$

for $g_0,\, g_1,\, g_2 \,\in\, G$. So such an element satisfies condition (1) of Definition 75.

Condition (2) reads

$$(\, h_0 \cdot g_0,\, h_1 \cdot g_1 \,)d \cdot h_2 \;=\; (\, g_0 \cdot h_1 \,,\, g_1 \cdot h_2 \,)d$$

for $g_0,\, g_1 \,\in\, G$ and $h_0,\, h_1,\, h_2 \,\in\, H$.

So

$$\begin{array}{rl} & \mathrm{Z}^2(G,H,M) \\ = & \left\{ G \times G \overset{d}{\to} M \;\middle|\; \begin{array}{r} (\, g_1 \,,\, g_2 \,)d - (\, g_0 \cdot g_1 \,,\, g_2 \,)d + (\, g_0 \,,\, g_1 \cdot g_2 \,)d - (\, g_0 \,,\, g_1 \,)d \cdot g_2 \;=\; 0 \\ \text{for } g_0,\, g_1,\, g_2 \,\in\, G \\ \\ (\, h_0 \cdot g_0,\, h_1 \cdot g_1 \,)d \cdot h_2 \;=\; (\, g_0 \cdot h_1 \,,\, g_1 \cdot h_2 \,)d \\ \text{for } g_0,\, g_1 \,\in\, G \text{ and } h_0,\, h_1,\, h_2 \,\in\, H \end{array} \right\} . \end{array}$$

Definition 77 (The map $\check{\alpha}_n$) In Definition 45.(2) we defined

$$\alpha_n \;=\; \rho_{n+1}^{-1} \cdot \varepsilon_{n+2}^{*} \cdot \rho_{n+2} \;:\; {}_{\mathrm{Map}}(G^{\times n}, M) \to {}_{\mathrm{Map}}(G^{\times (n+1)}, M)\ .$$

Similarly, we define

$$\check{\alpha}_n \;=\; \check{\rho}_{n+1}^{-1} \cdot \check{\varepsilon}_{n+2}^{*} \cdot \check{\rho}_{n+2} \;:\; {}_{\mathrm{Map},H}(G^{\times n}, M) \to {}_{\mathrm{Map},H}(G^{\times (n+1)}, M)\ .$$

Cf. Definitions 70, 68 and 59.

$${}_{\mathrm{Map},H}(G^{\times n}, M) \;\xrightarrow{\check{\rho}_{n+1}^{-1}}\; {}_{RG}(R(\check{G}^{\times (n+1)}), M) \;\xrightarrow{\check{\varepsilon}_{n+2}^{*}}\; {}_{RG}(R(\check{G}^{\times (n+2)}), M) \;\xrightarrow{\check{\rho}_{n+2}}\; {}_{\mathrm{Map},H}(G^{\times (n+1)}, M)$$

Note that

$$\check{\rho}_{n+1} \cdot \check{\alpha}_n \;=\; \check{\varepsilon}_{n+2}^{*} \cdot \check{\rho}_{n+2}\ .$$

Remark 78 *We have* $\check{\alpha}_n = \alpha_n|_{{}_{\mathrm{Map},H}(G^{\times n},M)}^{{}_{\mathrm{Map},H}(G^{\times (n+1)},M)}$.

Proof. Suppose given $f \in {}_{\mathrm{Map},H}(G^{\times n}, M)$. We get

$$
\begin{aligned}
f\check{\alpha}_n &= f\check{\rho}_{n+1}^{-1}\check{\varepsilon}_{n+2}^{*}\check{\rho}_{n+2} \\
&\overset{\text{Def. 70}}{=} f\check{\rho}_{n+1}^{-1}\check{\varepsilon}_{n+2}^{*}\sigma_{n+2}^{*}\rho_{n+2} \\
&\overset{\text{Rem. 68.(1)}}{=} f\check{\rho}_{n+1}^{-1}\sigma_{n+1}^{*}\varepsilon_{n+2}^{*}\rho_{n+2} \\
&\overset{\text{Def. 70}}{=} f\rho_{n+1}^{-1}\varepsilon_{n+2}^{*}\rho_{n+2} \\
&= f\alpha_n \, .
\end{aligned}
$$

□

Remark 79 Since the functor ${}_{RG}(-, M)$ is left exact, the rows in the commutative diagram of R-modules

$$
\begin{tikzcd}
{}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) \arrow[r, "\bar{\check{\varepsilon}}^{*}_{n+1}"] & {}_{RG}(R(\check{G}^{\times(n+1)}),M) \arrow[r, "\check{\varepsilon}^{*}_{n+2}"] \arrow[d, "\check{\rho}_{n+1}"', "\wr"] & {}_{RG}(R(\check{G}^{\times(n+2)}),M) \arrow[d, "\check{\rho}_{n+2}"', "\wr"] \\
\mathrm{Ker}(\check{\alpha}_n) \arrow[r, hook] & {}_{\mathrm{Map},H}(G^{\times n},M) \arrow[r, "\check{\alpha}_n"] & {}_{\mathrm{Map},H}(G^{\times(n+1)},M)
\end{tikzcd}
$$

are left exact; cf. Lemma 61, Definition 70. Thus we may complete to the following diagram of R-modules.

$$
\begin{tikzcd}
{}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) \arrow[r, "\bar{\check{\varepsilon}}^{*}_{n+1}"] \arrow[d, "(\bar{\check{\varepsilon}}^{*}_{n+1}\cdot\check{\rho}_{n+1})|^{\mathrm{Ker}(\check{\alpha}_n)}"', "\wr"] & {}_{RG}(R(\check{G}^{\times(n+1)}),M) \arrow[r, "\check{\varepsilon}^{*}_{n+2}"] \arrow[d, "\check{\rho}_{n+1}"', "\wr"] & {}_{RG}(R(\check{G}^{\times(n+2)}),M) \arrow[d, "\check{\rho}_{n+2}"', "\wr"] \\
\mathrm{Ker}(\check{\alpha}_n) \arrow[r, hook] & {}_{\mathrm{Map},H}(G^{\times n},M) \arrow[r, "\check{\alpha}_n"] & {}_{\mathrm{Map},H}(G^{\times(n+1)},M)
\end{tikzcd}
$$

Remark 80 Since in the diagram

$$
\begin{tikzcd}
 & {}_{RG}(R(G^{\times(n+1)}),M) \arrow[rr, "\rho_{n+1}", "\sim"'] & & {}_{\mathrm{Map}}(G^{\times n},M) \\
{}_{RG}(\mathrm{I}_R^{(n)}(G),M) \arrow[ur, "\bar{\varepsilon}^{*}_{n+1}"] \arrow[rr, "(\bar{\varepsilon}^{*}_{n+1}\cdot\rho_{n+1})|^{\mathrm{Ker}(\alpha_n)}", "\sim"'] & & \mathrm{Ker}(\alpha_n) \arrow[ur, hook] & \\
 & {}_{RG}(R(\check{G}^{\times(n+1)}),M) \arrow[uu, "\sigma^{*}_{n+1}"'] \arrow[rr, "\check{\rho}_{n+1}", "\sim"'] & & {}_{\mathrm{Map},H}(G^{\times n},M) \arrow[uu, hook] \\
{}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) \arrow[uu, "\bar{\sigma}^{*}_{n}"] \arrow[ur, "\bar{\check{\varepsilon}}^{*}_{n+1}"] \arrow[rr, "(\bar{\check{\varepsilon}}^{*}_{n+1}\cdot\check{\rho}_{n+1})|^{\mathrm{Ker}(\check{\alpha}_n)}", "\sim"'] & & \mathrm{Ker}(\check{\alpha}_n) \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

all faces except possibly the front face commute and since we have the injective inclusion map $\mathrm{Ker}(\alpha_n) \hookrightarrow {}_{\mathrm{Map},H}(G^{\times n}, M)$, it is a commutative diagram, including the front face. Cf. Definition 70, Remarks 68.(2), 79, 50, 78.

Remark 81 Since

$$\mathrm{Ker}(\breve{\alpha}_n) \;=\; \mathrm{Ker}(\alpha_n) \cap {}_{\mathrm{Map},H}(G^{\times n}, M)$$

by Remark 78 and since

$$\mathrm{Z}^n(G, H, M) \;=\; \mathrm{Z}^n(G, M) \cap {}_{\mathrm{Map},[H]}(G^{\times n}, M)$$

by Definition 75, we may complete the commutative diagram of R-modules

$$\begin{tikzcd}
& {}_{\mathrm{Map}}(G^{\times n}, M) \arrow[rr, "\iota_n^*", "\sim"'] & & {}_{\mathrm{Map}}(G^{\times n}, M) \\
\mathrm{Ker}(\alpha_n) \arrow[ur, hook] \arrow[rr, "\iota_n^*|^{\mathrm{Z}^n(G,M)}_{\mathrm{Ker}(\alpha_n)}", "\sim"'] & & \mathrm{Z}^n(G, M) \arrow[ur, hook] & \\
& {}_{\mathrm{Map},H}(G^{\times n}, M) \arrow[uu, hook] \arrow[rr, "\breve{\iota}_n^*", "\sim"'] & & {}_{\mathrm{Map},[H]}(G^{\times n}, M) \arrow[uu, hook] \\
\mathrm{Ker}(\breve{\alpha}_n) \arrow[uu, hook] \arrow[ur, hook] & & \mathrm{Z}^n(G, H, M) \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

to the following commutative diagram of R-modules.

$$\begin{tikzcd}
& {}_{\mathrm{Map}}(G^{\times n}, M) \arrow[rr, "\iota_n^*", "\sim"'] & & {}_{\mathrm{Map}}(G^{\times n}, M) \\
\mathrm{Ker}(\alpha_n) \arrow[ur, hook] \arrow[rr, "\iota_n^*|^{\mathrm{Z}^n(G,M)}_{\mathrm{Ker}(\alpha_n)}", "\sim"'] & & \mathrm{Z}^n(G, M) \arrow[ur, hook] & \\
& {}_{\mathrm{Map},H}(G^{\times n}, M) \arrow[uu, hook] \arrow[rr, "\breve{\iota}_n^*", "\sim"'] & & {}_{\mathrm{Map},[H]}(G^{\times n}, M) \arrow[uu, hook] \\
\mathrm{Ker}(\breve{\alpha}_n) \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, "\breve{\iota}_n^*|^{\mathrm{Z}^n(G,H,M)}_{\mathrm{Ker}(\breve{\alpha}_n)}", "\sim"'] & & \mathrm{Z}^n(G, H, M) \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Cf. Definition 73, Lemma 48.

Definition 82 We define the map

$$\check{\xi}_n \;:=\; \check{\xi}_n^{G,H} \;:=\; \xi_n^G \cdot \bar{\sigma}_n^{G,H} \;:\; G^{\times n} \;\to\; \mathrm{I}_R^{(n)}(G,H)\;.$$

Cf. Definitions 20 and 64.

Lemma 83 *The map*

$$\check{\xi}_n \;:\; G^{\times n} \;\to\; \mathrm{I}_R^{(n)}(G,H)$$

is an n-cocycle of G relative to H with values in $\mathrm{I}_R^{(n)}(G,H)$. Cf. Definitions 82 *and* 75*. That is, we have $\check{\xi}_n \in \mathrm{Z}^n(G,H,\mathrm{I}_R^{(n)}(G,H))$.*

Proof. Since $\xi_n \in \mathrm{Z}^n(G,\mathrm{I}_R^{(n)}(G))$ and since $\bar{\sigma}_n : \mathrm{I}_R^{(n)}(G) \to \mathrm{I}_R^{(n)}(G,H)$ is RG-linear, the map $\xi_n \cdot \bar{\sigma}_n = \check{\xi}_n$ is in $\mathrm{Z}^n(G,\mathrm{I}_R^{(n)}(G,H))$; cf. Lemma 42, Definition 64.

We need to show that $\check{\xi}_n \overset{!}{\in} {}_{\mathrm{Map},[H]}(G^{\times n},\mathrm{I}_R^{(n)}(G,H)) \subseteq {}_{\mathrm{Map}}(G^{\times n},\mathrm{I}_R^{(n)}(G,H))$; cf. Remark 74.

First, we remark that for $\underline{g} \in G^{\times n}$, we have

$$\begin{array}{rcl}
\underline{g}\check{\xi}_n & = & \underline{g}\xi_n \cdot \bar{\sigma}_n \\
& \overset{\text{D. 20}}{=} & \underline{g}(\iota_n + (-1)^n \iota_n \varepsilon_n \chi_{n-1,1})\bar{\sigma}_n \\
& = & \underline{g}\iota_n(\mathrm{id} + (-1)^n \varepsilon_n \chi_{n-1,1})\bar{\sigma}_n \\
& = & \underline{g}\iota_n(\mathrm{id} + (-1)^n \varepsilon_n \chi_{n-1,1})\sigma_n \\
& \overset{\text{R. 56}}{=} & \underline{g}\iota_n(\sigma_n + (-1)^n \varepsilon_n \sigma_{n-1} \check{\chi}_{n-1,1}) \\
& \overset{\text{R. 63}}{=} & \underline{g}\iota_n(\sigma_n + (-1)^n \sigma_n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \\
& = & \underline{g}\iota_n \sigma_n(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1})\;.
\end{array}$$

For $g_0,\,\ldots,\,g_{n-1} \in G$ and $h_0,\,\ldots,\,h_n \in H$, we obtain

$$\begin{array}{rl}
& (h_0 g_0\,,\, h_1 g_1\,,\,\ldots\,,\, h_{n-1} g_{n-1})\check{\xi}_n \cdot h_n \\
= & (h_0 g_0\,,\, h_1 g_1\,,\,\ldots\,,\, h_{n-1} g_{n-1})\iota_n \sigma_n(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \cdot h_n \\
= & (h_0 g_0 \cdot\ldots\cdot h_{n-1} g_{n-1}\,,\, h_1 g_1 \cdot\ldots\cdot h_{n-1} g_{n-1}\,,\,\ldots\,,\, h_{n-1} g_{n-1})\sigma_n(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \cdot h_n \\
= & (H h_0 g_0 \cdot\ldots\cdot h_{n-1} g_{n-1}\,,\, H h_1 g_1 \cdot\ldots\cdot h_{n-1} g_{n-1}\,,\,\ldots\,,\, H h_{n-1} g_{n-1})(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \cdot h_n
\end{array}$$

$$
\begin{array}{cl}
= & (Hg_0h_1g_1 \cdot \ldots \cdot h_{n-1}g_{n-1}\,,\; Hg_1h_2g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}\,,\; \ldots\,,\; Hg_{n-1})(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \cdot h_n \\
\overset{\check{\varepsilon}_n,\, \check{\chi}_{n-1,1}\ RH\text{-lin.}}{=} & (Hg_0h_1g_1 \cdot \ldots \cdot h_{n-1}g_{n-1}h_n\,,\; Hg_1h_2g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}h_n\,,\; \ldots\,,\; Hg_{n-1}h_n)(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \\
= & (g_0h_1g_1 \cdot \ldots \cdot h_{n-1}g_{n-1}h_n\,,\; g_1h_2g_2 \cdot \ldots \cdot h_{n-1}g_{n-1}h_n\,,\; \ldots\,,\; g_{n-1}h_n)\sigma_n(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \\
= & (g_0h_1\,,\; g_1h_2\,,\; \ldots\,,\; g_{n-1}h_n)\iota_n\sigma_n(\mathrm{id} + (-1)^n \check{\varepsilon}_n \check{\chi}_{n-1,1}) \\
= & (g_0h_1\,,\; g_1h_2\,,\; \ldots\,,\; g_{n-1}h_n)\check{\xi}_n\,.
\end{array}
$$

□

Remark 84 We have the following R-linear map.

$$
\begin{array}{rcl}
\check{\xi}_n \cdot (-) \quad : \quad {}_{RG}(\mathrm{I}^{(n)}_R(G,H),M) & \to & \mathrm{Z}^n(G,H,M) \\
\check{f} & \mapsto & \check{\xi}_n \cdot \check{f}\,.
\end{array}
$$

Proof. We have to show that $\check{\xi}_n \cdot \check{f} \overset{!}{\in} \mathrm{Z}^n(G,H,M) = \mathrm{Z}^n(G,M) \cap {}_{\mathrm{Map},[H]}(G^{\times n},M)$.

Since $\check{\xi}_n \in \mathrm{Z}^n(G,H,\mathrm{I}^{(n)}_R(G,H)) \subseteq \mathrm{Z}^n(G,\mathrm{I}^{(n)}_R(G,H))$ and $\check{f}$ is RG-linear, we have $\check{\xi}_n \cdot \check{f} \in \mathrm{Z}^n(G,M)$; cf. Lemma 83.

It remains to show that $\check{\xi}_n \cdot \check{f} \overset{!}{\in} {}_{\mathrm{Map},[H]}(G^{\times n},M)$.

For $g_0, \ldots, g_{n-1} \in G$ and $h_0, \ldots, h_n \in H$, we obtain

$$
\begin{array}{cl}
& (h_0g_0\,,\; h_1g_1\,,\; \ldots\,,\; h_{n-1}g_{n-1})(\check{\xi}_n \cdot \check{f}) \cdot h_n \\
\overset{\check{f}\text{ is }RG\text{-lin.}}{=} & ((h_0g_0\,,\; h_1g_1\,,\; \ldots\,,\; h_{n-1}g_{n-1})\check{\xi}_n \cdot h_n)\check{f} \\
\overset{\mathrm{L.\,83}}{=} & ((g_0h_1\,,\; g_1h_2\,,\; \ldots\,,\; g_{n-1}h_n)\check{\xi}_n)\check{f} \\
= & (g_0h_1\,,\; g_1h_2\,,\; \ldots\,,\; g_{n-1}h_n)(\check{\xi}_n \cdot \check{f})\,.
\end{array}
$$

□

Remark 85 We have the following commutative quadrangle.

$$
\begin{tikzcd}
{}_{RG}(\mathrm{I}^{(n)}_R(G),M) \arrow[r, "\xi_n\cdot(-)"] & \mathrm{Z}^n(G,M) \\
{}_{RG}(\mathrm{I}^{(n)}_R(G,H),M) \arrow[u, "\bar{\sigma}^*_n"] \arrow[r, "\check{\xi}_n\cdot(-)"] & \mathrm{Z}^n(G,H,M) \arrow[u, hook]
\end{tikzcd}
$$

In fact, for $\check{f} \in {}_{RG}(\mathrm{I}^{(n)}_R(G,H),M)$, we obtain

$$
(\check{f})(\check{\xi}_n \cdot (-))(\mathrm{id}_{\mathrm{Z}^n(G,M)}|_{\mathrm{Z}^n(G,H,M)}) \;=\; \check{\xi}_n \cdot \check{f} \;=\; \xi_n \cdot \bar{\sigma}_n \cdot \check{f} \;=\; (\check{f})\bar{\sigma}^*_n(\xi_n \cdot (-))\,.
$$

Lemma 86 We have the following commutative diagram.

$$
\begin{tikzcd}[column sep=small]
 & & {}_{RG}(R(G^{\times(n+2)}),M) \arrow[rr, "\rho_{n+2}", "\sim"'] & & {}_{\mathrm{Map}}(G^{\times(n+1)},M) & \\
 & {}_{RG}(R(G^{\times(n+1)}),M) \arrow[ur, "\bar{\varepsilon}^*_{n+2}"] \arrow[rr, "\rho_{n+1}", "\sim"'] & & {}_{\mathrm{Map}}(G^{\times n},M) \arrow[ur, "\alpha_n"'] \arrow[rr, "\iota_n^*", "\sim"'] & & {}_{\mathrm{Map}}(G^{\times n},M) \\
{}_{RG}(\mathrm{I}_R^{(n)}(G),M) \arrow[ur, "\bar{\varepsilon}^*_{n+1}"] \arrow[rr, "(\bar{\varepsilon}^*_{n+1}\cdot\rho_{n+1})|^{\mathrm{Ker}(\alpha_n)}", "\sim"'] \arrow[rrrr, bend right, "\sim", "(-1)^n\xi_n\cdot(-)"'] & & \mathrm{Ker}(\alpha_n) \arrow[ur, hook] \arrow[rr, "\iota_n^*|^{\mathrm{Z}^n(G,M)}_{\mathrm{Ker}(\alpha_n)}", "\sim"'] & & \mathrm{Z}^n(G,M) \arrow[ur, hook] & \\
 & & {}_{RG}(R(\check{G}^{\times(n+2)}),M) \arrow[uuu, "\sigma^*_{n+2}"] \arrow[rr, dashed, "\check{\rho}_{n+2}", "\sim"'] & & {}_{\mathrm{Map},H}(G^{\times(n+1)},M) \arrow[uuu, hook] & \\
 & {}_{RG}(R(\check{G}^{\times(n+1)}),M) \arrow[uuu, "\sigma^*_{n+1}"] \arrow[ur, "\check{\varepsilon}^*_{n+2}"] \arrow[rr, "\check{\rho}_{n+1}", "\sim"'] & & {}_{\mathrm{Map},H}(G^{\times n},M) \arrow[uuu, hook] \arrow[ur, "\check{\alpha}_n"'] \arrow[rr, "\check{\iota}_n^*", "\sim"'] & & {}_{\mathrm{Map},[H]}(G^{\times n},M) \arrow[uuu, hook] \\
{}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) \arrow[uuu, "\bar{\sigma}^*_n"] \arrow[ur, "\bar{\check{\varepsilon}}^*_{n+1}"] \arrow[rr, "(\bar{\check{\varepsilon}}^*_{n+1}\cdot\check{\rho}_{n+1})|^{\mathrm{Ker}(\check{\alpha}_n)}", "\sim"'] \arrow[rrrr, bend right, "\sim", "(-1)^n\check{\xi}_n\cdot(-)"'] & & \mathrm{Ker}(\check{\alpha}_n) \arrow[uuu, hook] \arrow[ur, hook] \arrow[rr, "\check{\iota}_n^*|^{\mathrm{Z}^n(G,H,M)}_{\mathrm{Ker}(\check{\alpha}_n)}", "\sim"'] & & \mathrm{Z}^n(G,H,M) \arrow[uuu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

In particular, the R-linear map

$$
\begin{array}{rcrcl}
\check{\xi}_n \cdot (-) & : & {}_{RG}(\mathrm{I}_R^{(n)}(G,H),M) & \to & \mathrm{Z}^n(G,H,M) \\
 & & \tilde{d} & \mapsto & \check{\xi}_n \cdot \tilde{d}
\end{array}
$$

is bijective.

Proof. The top layer is the commutative diagram from Remark 50, having horizontal isomorphisms from left to right.

The vertical quadrangles on the left commute by Remark 68.$(1,2)$.

The vertical quadrangles left to the middle in the back commute by Definition 70.

The vertical quadrangle left to the middle in the front commutes by Remark 80.

The left quadrangles in the lower layer commute and have horizontal isomorphisms by Remark 79.

The vertical quadrangle in the middle in the back commutes by Remark 78.

The cuboid right to the middle commutes and has horizontal isomorphisms by Remark 81.

The big quadrangle in the front commutes by Remark 85.

Since the inclusion map $\mathrm{Z}^n(G, H, M) \hookrightarrow \mathrm{Z}^n(G, M)$, appearing vertically on the right in the front, is injective, we conclude that the triangle in the front of the lower layer commutes, i.e. that

$$(-1)^n \check{\xi}_n \cdot (-) \;=\; (\bar{\bar{\varepsilon}}^*_{n+1} \cdot \check{\rho}_{n+1})\big|^{\mathrm{Ker}(\check{\alpha}_n)} \;\cdot\; \check{\iota}^*_n\big|^{\mathrm{Z}^n(G,H,M)}_{\mathrm{Ker}(\check{\alpha}_n)} \;.$$

In particular, the R-linear map $\check{\xi}_n \cdot (-)$ is bijective. ▫

Proposition 87 ($\check{\xi}_n$ is a universal relative n-cocycle)

(1) *We have* $\check{\xi}_n \in \mathrm{Z}^n(G, H, \mathrm{I}_R^{(n)}(G, H))$.

(2) *For a given relative n-cocycle* $d \in \mathrm{Z}^n(G, H, M)$, *there exists a unique RG-linear map* $\tilde{d}\colon \mathrm{I}_R^{(n)}(G, H) \to M$ *with* $d = \check{\xi}_n \cdot \tilde{d}$.

$$\begin{tikzcd}
G^{\times n} \arrow[rr, "d \in \mathrm{Z}^n(G,H,M)"] \arrow[d, "\check{\xi}_n \in \mathrm{Z}^n(G,H,\mathrm{I}_R^{(n)}(G,H))"'] & & M \\
\mathrm{I}_R^{(n)}(G, H) \arrow[urr, "\exists!\ \tilde{d}\ \ RG\text{-linear}"'] & &
\end{tikzcd}$$

Proof.

Ad (1). This is Lemma 83.

Ad (2). This follows from Lemma 86, as for every element $d \in \mathrm{Z}^n(G, H, M)$ there is a unique element $\tilde{d} \in {}_{RG}(\mathrm{I}_R^{(n)}(G, H), M)$ mapping to d via $\check{\xi}_n \cdot (-)$. ▫

6 The cotangent-square and the conormal-sequence

6.1 The cotangent-square

Suppose given a commutative ring R.

Suppose given groups $L \leqslant K \leqslant H \leqslant G$.

Notation 88 Suppose given a set I. Suppose given an R-module M_i for $i \in I$.

For $j \in I$, we have the R-linear map

$$\begin{array}{rcl} M_j & \xrightarrow{\iota_j} & \bigoplus\limits_{i\in I} M_i \\ m & \mapsto & m\iota_j \ , \end{array}$$

where $m\iota_j$ has entry m at position j and entry 0 elsewhere.

Remark 89 Suppose given an RH-module M.

Suppose given a subset $S \subseteq G$ such that $\dot{\bigcup\limits_{s\in S}} Hs = G$.

We have the isomorphism of R-modules

$$\begin{array}{rcl} \bigoplus\limits_{s\in S} M & \xrightarrow{\sim} & M \underset{RH}{\otimes} RG \\ (m_s)_{s\in S} & \mapsto & \sum\limits_{s\in S} m_s \otimes s \ . \end{array}$$

Proof. Since we have the isomorphism $M \xrightarrow{\sim} M \underset{RH}{\otimes} RH : m \mapsto m \otimes 1$ of R-modules, we have the following isomorphism of R-modules.

$$\begin{array}{rcl} \bigoplus\limits_{s\in S} M & \overset{\nu}{\underset{\sim}{\to}} & \bigoplus\limits_{s\in S} (M \underset{RH}{\otimes} RH) \\ (m_s)_{s\in S} & \mapsto & (m_s \otimes 1)_{s\in S} \end{array}$$

Since the functor $M \underset{RH}{\otimes} - : RH\text{-Mod} \to R\text{-Mod}$ is a left adjoint, we have the following isomorphism of R-modules.

$$\begin{array}{rcl} \bigoplus\limits_{s\in S} (M \underset{RH}{\otimes} RH) & \overset{\mu}{\underset{\sim}{\to}} & M \underset{RH}{\otimes} (\bigoplus\limits_{s\in S} RH) \\ (m \otimes h)\iota_s & \mapsto & m \otimes (h\iota_s) \end{array}$$

We have the following isomorphism of left RH-modules.

$$\begin{array}{rcl} \bigoplus\limits_{s\in S} RH & \overset{\psi}{\underset{\sim}{\to}} & \bigoplus\limits_{s\in S} RHs \ = \ RG \\ (x_s)_s & \mapsto & \sum\limits_{s\in S} x_s\, s \ . \end{array}$$

Thus we have the following isomorphism of R-modules.

$$M \underset{RH}{\otimes} (\bigoplus_{s\in S} RH) \xrightarrow[\sim]{\lambda := M \underset{RH}{\otimes} \psi} M \underset{RH}{\otimes} RG$$

Altogether, we compose to obtain the following isomorphism of R-modules.

$$\begin{array}{rcl}
\bigoplus\limits_{s\in S} M & \overset{\nu\cdot\mu\cdot\lambda}{\underset{\sim}{\longrightarrow}} & M \underset{RH}{\otimes} RG \\
(m_s)_{s\in S} & \mapsto & ((m_s)_{s\in S})(\nu\cdot\mu\cdot\lambda) \\
& = & ((m_s\otimes 1)_{s\in S})(\mu\cdot\lambda) \\
& = & (\sum_{s\in S}(m_s\otimes 1)\iota_s)(\mu\cdot\lambda) \\
& = & \sum_{s\in S}((m_s\otimes 1)\iota_s)(\mu\cdot\lambda) \\
& = & \sum_{s\in S}(m_s\otimes (1\iota_s))\lambda \\
& = & \sum_{s\in S} m_s\otimes (1\iota_s)\psi \\
& = & \sum_{s\in S} m_s\otimes s
\end{array}$$

□

Remark 90 The maps

$$\begin{array}{rcl}
j_L : R(L\backslash H) \underset{RH}{\otimes} RG & \to & R(L\backslash G) \\
Lh\otimes g & \mapsto & Lhg \\[1ex]
j_K : R(K\backslash H) \underset{RH}{\otimes} RG & \to & R(K\backslash G) \\
Kh\otimes g & \mapsto & Khg
\end{array}$$

are isomorphisms of RG-modules.

Proof. We show the statement for j_L .

As we have $(Lh\tilde{h})g = Lh(\tilde{h}g)$ for $h, \tilde{h} \in H$ and $g \in G$, we obtain the well-defined R-linear map j_L as given above.

Since $((Lh\otimes g)\cdot x)j_L = (Lh\otimes gx)j_L = Lhgx = (Lh\otimes g)j_L\cdot x$ for $h \in H$ and $g,\ x \in G$, the map j_L is RG-linear.

We choose a subset $S \subseteq G$ such that $\dot{\bigcup\limits_{s\in S}} Hs = G$.

So for $g \in G$, there is a unique pair $(h_g, s_g) \in H\times S$ such that $g = h_g\cdot s_g$.

We choose a subset $T \subseteq H$ such that $\dot{\bigcup\limits_{t\in T}} Lt = H$.

So for $h \in H$, there is a unique pair $(\ell_h, t_h) \in L\times T$ such that $h = \ell_h\cdot t_h$.

We construct an R-linear basis of $R(L\backslash H) \underset{RH}{\otimes} RG$.

By Remark 89, we have the isomorphism of R-modules

$$\begin{array}{rcl}
\bigoplus\limits_{s\in S} R(L\backslash H) & \overset{\sim}{\to} & R(L\backslash H) \underset{RH}{\otimes} RG \\
(m_s)_{s\in S} & \mapsto & \sum\limits_{s\in S} m_s\otimes s\ .
\end{array}$$

Since $(\, Lt \,|\, t\in T\,)$ is an R-linear basis of $R(L\backslash H)$, the tuple

$$(\,(Lt)\iota_s \,|\, t\in T,\ s\in S\,)$$

is an R-linear basis of $\bigoplus\limits_{s\in S} R(L\backslash H)$.

Applying our isomorphism to $(Lt)\iota_s$, we obtain the element $Lt \otimes s \in R(L\backslash H) \underset{RH}{\otimes} RG$. So we obtain the R-linear basis

$$(Lt \otimes s \,|\, t \in T,\ s \in S)$$

of $R(L\backslash H) \underset{RH}{\otimes} RG$.

We construct an R-linear basis of $R(L\backslash G)$.

We have

$$G \;=\; \dot{\bigcup_{s\in S}} Hs \;=\; \dot{\bigcup_{s\in S}} (\dot{\bigcup_{t\in T}} Lt)s \;=\; \dot{\bigcup_{(s,t)\,\in S\times T}} Lts\ .$$

So

$$(Lts \,|\, (s,t) \;\in\; S \times T)$$

is an R-linear basis of $R(L\backslash G)$.

For $s \in S$ and $t \in T$, we obtain

$$(Lt \otimes s)j_L \;=\; Lts\ .$$

Hence j_L maps our R-linear basis of $R(L\backslash H) \underset{RH}{\otimes} RG$ to our R-linear basis of $R(L\backslash G)$. So j_L is an isomorphism of R-modules. $\square$

Remark 91 *Suppose given $U \subseteq G$ such that $1 \in U$ and such that $G = \dot{\bigcup\limits_{u\in U}} Lu$.*

The set $B_1^L := \{\, Lu - L1 \mid u \in U \setminus \{1\}\}$ is an R-linear basis of $\mathrm{I}_R^{(1)}(G, L)$.

Proof. We need to show that B_1^L is R-linearly independent and R-linearly generates $\mathrm{I}_R^{(1)}(G, L)$.

Linearly independent set. Suppose given $\lambda_u \in R$ for $u \in U \setminus \{1\}$ such that we have the R-linear combination

$$\sum_{u\in U\setminus\{1\}} \lambda_u(Lu - L1) \;=\; 0\ .$$

Then

$$\underbrace{\sum_{u\in U\setminus\{1\}} \lambda_u\, L1}_{\in\langle L1\rangle_R} \;=\; \underbrace{\sum_{u\in U\setminus\{1\}} \lambda_u\, Lu}_{\in\langle Lu\,|\,u\in U\setminus\{1\}\rangle_R}$$

Therefore $\lambda_u = 0$ for $u \in U \setminus \{1\}$.

Generating set. Suppose given $x \in \mathrm{I}_R^{(1)}(G, L)$. We can write $x = \sum\limits_{u\in U} r_u\, Lu$ with

$$0 \;=\; x\check{\varepsilon}_1^{G,L} \;=\; \sum_{u\in U} r_u\ ,$$

as $x \in \mathrm{I}_R^{(1)}(G, L) = \mathrm{Ker}(\check{\varepsilon}_1^{G,L})$; cf. Definitions 59 and 62. It follows that

$$\begin{aligned} x &= \sum_{u \in U} r_u \, Lu \\ &= \sum_{u \in U} r_u \, Lu - \sum_{u \in U} r_u \, L1 \\ &= \sum_{u \in U} (r_u \, Lu - r_u \, L1) \\ &= \sum_{u \in U \setminus \{1\}} r_u (Lu - L1) \;\in\; \langle Lu - L1 \,|\, u \in U \setminus \{1\} \rangle_R \;=\; \langle B_1^L \rangle_R \, . \end{aligned}$$

□

Remark 92

(1) *We have the RG-linear map*

$$\begin{array}{ccccc} R & \underset{RH}{\otimes} & RG & \xrightarrow{j''} & R \\ 1 & \otimes & g & \mapsto & 1 \, . \end{array}$$

(2) *We obtain the following commutative diagram of RG-modules and RG-linear maps.*

$$\begin{tikzcd} \mathrm{I}_R^{(1)}(H, L) \underset{RH}{\otimes} RG \arrow[r, "j'_L"] \arrow[d, "\dot{\check{\varepsilon}}_1^{H,L} \underset{RH}{\otimes} RG"'] & \mathrm{I}_R^{(1)}(G, L) \arrow[d, "\dot{\check{\varepsilon}}_1^{G,L}"] \\ R(L \backslash H) \underset{RH}{\otimes} RG \arrow[r, "j_L", "\sim"'] \arrow[d, "\check{\varepsilon}_1^{H,L} \underset{RH}{\otimes} RG"'] & R(L \backslash G) \arrow[d, "\check{\varepsilon}_1^{G,L}"] \\ R \underset{RH}{\otimes} RG \arrow[r, "j''"] & R \end{tikzcd}$$

The map j'_L is uniquely determined by commutativity, with

$$((Lh - L1) \otimes g) j'_L \;=\; ((Lh - L1) \otimes g) j_L \;=\; Lhg - Lg$$

for $h \in H$ and $g \in G$; cf. Remark 91.

(3) *We obtain the following commutative diagram of RG-modules and RG-linear maps.*

$$\begin{tikzcd} \mathrm{I}_R^{(1)}(H, K) \underset{RH}{\otimes} RG \arrow[r, "j'_K"] \arrow[d, "\dot{\check{\varepsilon}}_1^{H,K} \underset{RH}{\otimes} RG"'] & \mathrm{I}_R^{(1)}(G, K) \arrow[d, "\dot{\check{\varepsilon}}_1^{G,K}"] \\ R(K \backslash H) \underset{RH}{\otimes} RG \arrow[r, "j_K", "\sim"'] \arrow[d, "\check{\varepsilon}_1^{H,K} \underset{RH}{\otimes} RG"'] & R(K \backslash G) \arrow[d, "\check{\varepsilon}_1^{G,K}"] \\ R \underset{RH}{\otimes} RG \arrow[r, "j''"] & R \end{tikzcd}$$

The map j'_K is uniquely determined by commutativity, with

$$((Kh - K1) \otimes g)j'_K \;=\; ((Kh - K1) \otimes g)j_K \;=\; Khg - Kg$$

for $h \in H$ and $g \in G$; cf. Remark 91.

In particular, $(x \otimes 1)j'_K = x$ for $x \in \mathrm{I}_R^{(1)}(G, H)$.

Proof.

Ad (1). The R-linear map j'' is well-defined.

Since $((1 \otimes g) \cdot x)j'' = (1 \otimes gx)j'' = 1 = 1 \cdot x = (1 \otimes g)j'' \cdot x$, the map j'' is RG-linear.

Ad (2). We show that the lower quadrangle commutes.

For $h \in H$ and $g \in G$, we obtain

$$\begin{array}{rcl} ((Lh \otimes g)j_L)\check{\varepsilon}_1^{G,L} & = & (Lhg)\check{\varepsilon}_1^{G,L} \\ & \overset{\text{D. 59}}{=} & 1 \\ & = & (1 \otimes g)j'' \\ & \overset{\text{D. 59}}{=} & ((Lh \otimes g)(\check{\varepsilon}_1^{H,L} \underset{RH}{\otimes} RG))j'' \,. \end{array}$$

The sequence $(\mathrm{I}_R^{(1)}(G, L), R(L\backslash G), R)$ in the diagram is short exact; cf. Lemma 61, Definition 62.

We have the composite $(\dot{\varepsilon}_1^{H,L} \underset{RH}{\otimes} RG) \cdot (\check{\varepsilon}_1^{H,L} \underset{RH}{\otimes} RG) = (\dot{\varepsilon}_1^{H,L} \cdot \check{\varepsilon}_1^{H,L}) \underset{RH}{\otimes} RG = 0$.

Therefore, we get a unique RG-linear map $j'_L : \mathrm{I}_R^{(1)}(H, L) \underset{RH}{\otimes} RG \to \mathrm{I}_R^{(1)}(G, L)$ such that the upper quadrangle commutes.

We have

$$\begin{array}{rcl} ((Lh - L1) \otimes g)j'_L & = & (((Lh - L1) \otimes g)j'_L)\dot{\varepsilon}_1^{G,L} \\ & = & (((Lh - L1) \otimes g)(\dot{\varepsilon}_1^{H,K} \underset{RH}{\otimes} RG))j_L \\ & = & ((Lh - L1) \otimes g)j_L \;=\; Lhg - Lg \end{array}$$

for $h \in H$ and $g \in G$. □

Definition 93

(1) We have the RG-linear map

$$\begin{array}{rrl} \kappa_G : & R(L\backslash G) \to & R(K\backslash G) \\ & Lg \mapsto & Kg \end{array}$$

We have the following commutative diagram of RG-linear maps, where $\kappa'_G := \kappa_G|_{\mathrm{I}_R^{(1)}(G,L)}^{\mathrm{I}_R^{(1)}(G,K)}$

is the induced map on the kernels.

$$\begin{tikzcd}
\mathrm{I}_R^{(1)}(G,L) \arrow[r, "\kappa'_G"] \arrow[d, "\dot{\check{\varepsilon}}_1^{G,L}"'] & \mathrm{I}_R^{(1)}(G,K) \arrow[d, "\dot{\check{\varepsilon}}_1^{G,K}"] \\
R(L\backslash G) \arrow[r, "\kappa_G"] \arrow[d, "\check{\varepsilon}_1^{G,L}"'] & R(K\backslash G) \arrow[d, "\check{\varepsilon}_1^{G,K}"] \\
R \arrow[r, "\mathrm{id}"] & R
\end{tikzcd}$$

(2) We have the RH-linear map

$$\begin{aligned} \kappa_H : \quad R(L\backslash H) &\to \quad R(K\backslash H) \\ Lh &\mapsto \quad Kh \end{aligned}$$

We have the following commutative diagram of RH-linear maps, where $\kappa'_H := \kappa_H|_{\mathrm{I}_R^{(1)}(H,L)}^{\mathrm{I}_R^{(1)}(H,K)}$ is the induced map on the kernels.

$$\begin{tikzcd}
\mathrm{I}_R^{(1)}(H,L) \arrow[r, "\kappa'_H"] \arrow[d, "\dot{\check{\varepsilon}}_1^{H,L}"'] & \mathrm{I}_R^{(1)}(H,K) \arrow[d, "\dot{\check{\varepsilon}}_1^{H,K}"] \\
R(L\backslash H) \arrow[r, "\kappa_H"] \arrow[d, "\check{\varepsilon}_1^{H,L}"'] & R(K\backslash H) \arrow[d, "\check{\varepsilon}_1^{H,K}"] \\
R \arrow[r, "\mathrm{id}"] & R
\end{tikzcd}$$

Proposition 94

(1) *We have the following commutative diagram of RG-modules and RG-linear maps.*

$$\begin{tikzcd}
 & \mathrm{I}_R^{(1)}(G,L) \arrow[rr, "\kappa'_G"] \arrow[dd, "\dot{\check{\varepsilon}}_1^{G,L}"] & & \mathrm{I}_R^{(1)}(G,K) \arrow[dd, "\dot{\check{\varepsilon}}_1^{G,K}"] \\
\mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG \arrow[ur, "j'_L"'] \arrow[rr, "\kappa'_H \underset{RH}{\otimes} RG"] \arrow[dd, "\dot{\check{\varepsilon}}_1^{H,L} \underset{RH}{\otimes} RG"'] & & \mathrm{I}_R^{(1)}(H,K) \underset{RH}{\otimes} RG \arrow[ur, "j'_K"'] \arrow[dd, "\dot{\check{\varepsilon}}_1^{H,K} \underset{RH}{\otimes} RG"] & \\
 & R(L\backslash G) \arrow[rr, "\kappa_G"] \arrow[dd, "\check{\varepsilon}_1^{G,L}"] & & R(K\backslash G) \arrow[dd, "\check{\varepsilon}_1^{G,K}"] \\
R(L\backslash H) \underset{RH}{\otimes} RG \arrow[ur, "\sim", "j_L"'] \arrow[rr, "\kappa_H \underset{RH}{\otimes} RG"] \arrow[dd, "\check{\varepsilon}_1^{H,L} \underset{RH}{\otimes} RG"'] & & R(K\backslash H) \underset{RH}{\otimes} RG \arrow[ur, "\sim", "j_K"'] \arrow[dd, "\check{\varepsilon}_1^{H,K} \underset{RH}{\otimes} RG"] & \\
 & R \arrow[rr, "\mathrm{id}", "="'] & & R \\
R \underset{RH}{\otimes} RG \arrow[ur, "j''"'] \arrow[rr, "\mathrm{id} \underset{RH}{\otimes} RG", "="'] & & R \underset{RH}{\otimes} RG \arrow[ur, "j''"'] &
\end{tikzcd}$$

(2) *The top layer quadrangle is a square:*

$$\begin{tikzcd}
\mathrm{I}_R^{(1)}(H,L)\underset{RH}{\otimes}RG \arrow[r, "\kappa'_H\underset{RH}{\otimes}RG"] \arrow[d, "j'_L"'] \arrow[dr, phantom, "\square"] & \mathrm{I}_R^{(1)}(H,K)\underset{RH}{\otimes}RG \arrow[d, "j'_K"] \\
\mathrm{I}_R^{(1)}(G,L) \arrow[r, "\kappa'_G"'] & \mathrm{I}_R^{(1)}(G,K)\ .
\end{tikzcd}$$

It is called the cotangent-square*.*

Proof.

Ad (1)*.* The front face is the commutative diagram from Remark 93.(2), to which $-\underset{RH}{\otimes}RG$ has been applied.

The back face is the commutative diagram from Remark 93.(1).

The left face is the commutative diagram from Remark 92.(2).

The right face is the commutative diagram from Remark 92.(3).

The bottom layer quadrangle is commutative.

The middle layer quadrangle is commutative since for $h\in H$, we have

$$(Lh\otimes 1)j_L\kappa_G \;=\; (Lh)\kappa_G \;=\; Kh \;=\; (Kh\otimes 1)j_K \;=\; (Lh\otimes 1)(\kappa_H\underset{RH}{\otimes}RG)j_K\ .$$

The top layer quadrangle is commutative since all other quadrangles in the diagram are commutative and the map $\dot{\check{\varepsilon}}_1^{G,K}$ is injective.

Ad (2)*.* The bottom and the middle layers are squares by Remark 2.

The vertical sequences in the back are short exact by Lemma 61.

The vertical sequences in the front are short exact by Lemma 61 and by exactness of the functor $-\underset{RH}{\otimes}RG$, as RG is a free RH-module; cf. Remark 89.

So the top layer quadrangle is a square by Remark 1. ▫

Corollary 95 *Suppose that $L\leqslant H=K\leqslant G$.*

The sequence

$$\mathrm{I}_R^{(1)}(H,L)\underset{RH}{\otimes}RG \;\xrightarrow{j'_L}\; \mathrm{I}_R^{(1)}(G,L) \;\xrightarrow{\kappa'_G}\; \mathrm{I}_R^{(1)}(G,H)$$

is short exact. It is called the cotangent-sequence*.*

Proof. By Proposition 94.(2), it suffices to show that $\mathrm{I}_R^{(1)}(H,H)\underset{RH}{\otimes}RG \overset{!}{=} 0$.

We have the bijective RH-linear map $\check{\varepsilon}_1^{H,H} : R(H\backslash H)\to R : H1\mapsto 1$.

Its kernel is $\mathrm{I}_R^{(1)}(H,H)=0$. ▫

6.2 The cotangent-pullback for relative 1-cocycles

Suppose given a commutative ring R.

Suppose given groups $L \leqslant K \leqslant H \leqslant G$.

Suppose given an RG-module M.

Definition 96

(1) We have the R-linear map

$$\begin{array}{ccc} \mathrm{Z}^1(G,L,M) & \xrightarrow{\eta_{G,H,L,M}} & \mathrm{Z}^1(H,L,M|_H) \\ (d : G \to M) & \mapsto & (d|_H : H \to M|_H)\ . \end{array}$$

Cf. Remark 76.(1).

(2) We have the inclusion map

$$\begin{array}{ccc} \mathrm{Z}^1(G,K,M) & \xrightarrow{\vartheta_{G,K,L,M}} & \mathrm{Z}^1(G,L,M) \\ (d : G \to M) & \mapsto & (d : G \to M)\ . \end{array}$$

Cf. Remark 76.(1).

Remark 97 *We have the following pullback of R-modules and R-linear maps.*

$$\begin{tikzcd} \mathrm{Z}^1(H,L,M|_H) & \mathrm{Z}^1(H,K,M|_H) \arrow[l, "\vartheta_{H,K,L,M|_H}"'] \\ \mathrm{Z}^1(G,L,M) \arrow[u, "\eta_{G,H,L,M}"] & \mathrm{Z}^1(G,K,M) \arrow[l, "\vartheta_{G,K,L,M}"] \arrow[u, "\eta_{G,H,K,M}"'] \arrow[ul, phantom, "\lrcorner" very near start] \end{tikzcd}$$

Proof. Since the horizontal maps $\vartheta_{H,K,L,M|_H}$ and $\vartheta_{G,K,L,M}$ are inclusion maps, we have to show that

$$\mathrm{Z}^1(G,K,M) \overset{!}{=} \eta^{-1}_{G,H,L,M}\big(\mathrm{Z}^1(H,K,M|_H)\big)$$

as subsets of $\mathrm{Z}^1(G,L,M)$.

Suppose given $d \in \mathrm{Z}^1(G,L,M)$.

We have $d \in \eta^{-1}_{G,H,L,M}\big(\mathrm{Z}^1(H,K,M|_H)\big)$ if and only if $d|_H \in \mathrm{Z}^1(H,K,M|_H)$, which holds if and only if $kd = 0$ for $k \in K$.

We have $d \in \mathrm{Z}^1(G,K,M)$ if and only if $kd = 0$ for $k \in K$.

So the required equality holds. $\square$

We shall apply the functor ${}_{RG}(-,M)$ to the cotangent-square of Proposition 94.(2) and reinterpret the resulting pullback in terms of relative 1-cocycles. We will obtain the pullback of Remark 97 again; cf. Remark 102 below.

Remark 98 *Suppose given RH-modules X and Y and an RH-linear map $f : X \to Y$.*

We write $f^* := {}_{RH}(f, M|_H) : {}_{RH}(Y, M|_H) \to {}_{RH}(X, M|_H)$.

We write $(f \underset{RH}{\otimes} RG)^* := {}_{RG}(f \underset{RH}{\otimes} RG, M) : {}_{RG}(Y \underset{RH}{\otimes} RG, M) \to {}_{RG}(X \underset{RH}{\otimes} RG, M)$.

We make use of the following bijective R-linear map.

$$\begin{array}{rcl} {}_{RG}(X \underset{RH}{\otimes} RG, M) & \xrightarrow[\sim]{\varphi_X} & {}_{RH}(X, M|_H) \\ u & \overset{\varphi_X}{\mapsto} & (x \mapsto (x \otimes 1)u) \\ (x \otimes g \mapsto xv \cdot g) & \overset{\varphi_X^{-1}}{\mapsfrom} & v \end{array}$$

We have the following commutative quadrangle of R-linear maps.

$$\begin{tikzcd} {}_{RH}(X, M|_H) & {}_{RH}(Y, M|_H) \arrow[l, "f^*"'] \\ {}_{RG}(X \underset{RH}{\otimes} RG, M) \arrow[u, "\wr", "\varphi_X"'] & {}_{RG}(Y \underset{RH}{\otimes} RG, M) \arrow[l, "(f \underset{RH}{\otimes} RG)^*"'] \arrow[u, "\wr", "\varphi_Y"'] \end{tikzcd}$$

Proof. Suppose given $u \in {}_{RG}(Y \underset{RH}{\otimes} RG, M)$.

Suppose given $x \in X$.

On the one hand, we have

$$\begin{array}{rcl} x(u\varphi_Y f^*) & = & xf(u\varphi_Y) \\ & = & (xf \otimes 1)u \ . \end{array}$$

On the other hand, we have

$$\begin{array}{rcl} x(u(f \underset{RH}{\otimes} RG)^* \varphi_X) & = & x(((f \underset{RH}{\otimes} RG) \cdot u)\varphi_X) \\ & = & (x \otimes 1)((f \underset{RH}{\otimes} RG) \cdot u) \\ & = & (xf \otimes 1)u \ . \end{array}$$

The results are the same. $\square$

Notation 99

We write $\varphi_L := \varphi_{\mathrm{I}_R^{(1)}(H,L)} : {}_{RG}(\mathrm{I}_R^{(1)}(H, L) \underset{RH}{\otimes} RG, M) \xrightarrow{\sim} {}_{RH}(\mathrm{I}_R^{(1)}(H, L), M|_H)$.

We write $\varphi_K := \varphi_{\mathrm{I}_R^{(1)}(H,K)} : {}_{RG}(\mathrm{I}_R^{(1)}(H, K) \underset{RH}{\otimes} RG, M) \xrightarrow{\sim} {}_{RH}(\mathrm{I}_R^{(1)}(H, K), M|_H)$.

Remark 100 *We have the following commutative diagram.*

$$\begin{tikzcd}
\mathrm{Z}^1(H,L,M|_H) & {}_{RH}(\mathrm{I}_R^{(1)}(H,L),M|_H) \arrow[l, "\check{\xi}_1^{H,L}\cdot(-)"', "\sim"] \\
 & {}_{RG}(\mathrm{I}_R^{(1)}(H,L)\underset{RH}{\otimes} RG, M) \arrow[u, "\varphi_L"', "\wr"] \\
\mathrm{Z}^1(G,L,M) \arrow[uu, "\eta_{G,H,L,M}"] & {}_{RG}(\mathrm{I}_R^{(1)}(G,L),M) \arrow[l, "\check{\xi}_1^{G,L}\cdot(-)"', "\sim"] \arrow[u, "j_L'^*"']
\end{tikzcd}$$

Concerning the horizontal maps, cf. Definition 82*, Lemma* 86*.*

Proof. Suppose given $u \in {}_{RG}(\mathrm{I}_R^{(1)}(G,L),M)$. We have to show that

$$u j_L'^* \varphi_L(\check{\xi}_1^{H,L}\cdot(-)) \overset{!}{=} u(\check{\xi}_1^{G,L}\cdot(-))\eta_{G,H,L,M} \; .$$

I.e. we have to show that

$$\check{\xi}_1^{H,L}\cdot(j_L'\cdot u)\varphi_L \overset{!}{=} (\check{\xi}_1^{G,L}\cdot u)|_H$$

as elements of $\mathrm{Z}^1(H,L,M|_H)$.

Note that $(j_L'\cdot u)\varphi_L$ maps $x \in \mathrm{I}_R^{(1)}(H,L)$ to $(x\otimes 1)(j_L'\cdot u) = ((x\otimes 1)j_L')u = xu$; cf. Remarks 98 and 92.(3). So $(j_L'\cdot u)\varphi_L = u|_{\mathrm{I}_R^{(1)}(H,L)}$.

So we have to show that

$$\check{\xi}_1^{H,L}\cdot u|_{\mathrm{I}_R^{(1)}(H,L)} \overset{!}{=} (\check{\xi}_1^{G,L}\cdot u)|_H$$

as elements of $\mathrm{Z}^1(H,L,M|_H)$.

Suppose given $h \in H$.

On the one hand, we obtain

$$h(\check{\xi}_1^{H,L}\cdot u|_{\mathrm{I}_R^{(1)}(H,L)}) = (h\check{\xi}_1^{H,L})u \overset{\mathrm{D.}\,82}{=} ((h\xi_1^H)\bar{\sigma}_1^{H,L})u \overset{\mathrm{R.}\,33}{=} ((h-1)\bar{\sigma}_1^{H,L})u \overset{\mathrm{D.}\,64,\,53}{=} (Lh - L1)u \; .$$

On the other hand, we obtain

$$h((\check{\xi}_1^{G,L}\cdot u)|_H) = (h\check{\xi}_1^{G,L})u \overset{\mathrm{D.}\,82}{=} ((h\xi_1^G)\bar{\sigma}_1^{G,L})u \overset{\mathrm{R.}\,33}{=} ((h-1)\bar{\sigma}_1^{G,L})u \overset{\mathrm{D.}\,64,\,53}{=} (Lh - L1)u \; .$$

The results are the same. ▫

Remark 101 *We have the following commutative diagram.*

$$\begin{tikzcd}
{}_{RG}(\mathrm{I}_R^{(1)}(G,L),M) \arrow[d, "\check{\xi}_1^{G,L}\cdot(-)"', "\wr"] & {}_{RG}(\mathrm{I}_R^{(1)}(G,K),M) \arrow[l, "\kappa_G'^*"'] \arrow[d, "\check{\xi}_1^{G,K}\cdot(-)", "\wr"'] \\
\mathrm{Z}^1(G,L,M) & \mathrm{Z}^1(G,K,M) \arrow[l, "\vartheta_{G,K,L,M}"']
\end{tikzcd}$$

Concerning the vertical maps, cf. Definition 82*, Lemma* 86*.*

Proof. Suppose given $u \in {}_{RG}(\mathrm{I}_R^{(1)}(G,K), M)$. We have to show that

$$u\kappa_G'^*(\check{\xi}_1^{G,L} \cdot (-)) \overset{!}{=} u(\check{\xi}_1^{G,K} \cdot (-))\vartheta_{G,K,L,M} \; .$$

I.e. we have show that

$$\check{\xi}_1^{G,L} \cdot \kappa_G' \cdot u \overset{!}{=} \check{\xi}_1^{G,K} \cdot u$$

as elements of $\mathrm{Z}^1(G,L,M)$.

Suppose given $g \in G$.

On the one hand, we obtain

$$\begin{aligned} g(\check{\xi}_1^{G,L} \cdot \kappa_G' \cdot u) &\overset{\text{D. 82}}{=} (((g\xi_1^G)\bar{\sigma}_1^{G,L})\kappa_G')u \overset{\text{R. 33}}{=} (((g-1)\bar{\sigma}_1^{G,L})\kappa_G')u \\ &\overset{\text{D. 64, 53}}{=} ((Lg - L1)\kappa_G')u \overset{\text{D. 93.(1)}}{=} (Kg - K1)u \; . \end{aligned}$$

On the one hand, we obtain

$$\begin{aligned} g(\check{\xi}_1^{G,K} \cdot u) &\overset{\text{D. 82}}{=} ((g\xi_1^G)\bar{\sigma}_1^{G,K})u \overset{\text{R. 33}}{=} ((g-1)\bar{\sigma}_1^{G,K})u \\ &\overset{\text{D. 64, 53}}{=} (Kg - K1)u \; . \end{aligned}$$

The results are the same. $\square$

Remark 102 *We have the following commutative diagram of R-modules and R-linear maps.*

$$\begin{tikzcd}
\mathrm{Z}^1(H,L,M|_H) & & & \mathrm{Z}^1(H,K,M|_H) \arrow[lll, "\vartheta_{H,K,L,M|_H}"'] \\
& {}_{RH}(\mathrm{I}_R^{(1)}(H,L), M|_H) \arrow[ul, "\check{\xi}_1^{H,L}\cdot(-)", "\sim"'] & {}_{RH}(\mathrm{I}_R^{(1)}(H,K), M|_H) \arrow[l, "\kappa_H'^*"] \arrow[ur, "\check{\xi}_1^{H,K}\cdot(-)"', "\sim"] & \\
& {}_{RG}(\mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG, M) \arrow[u, "\varphi_L"', "\wr"] & {}_{RG}(\mathrm{I}_R^{(1)}(H,K) \underset{RH}{\otimes} RG, M) \arrow[l, "(\kappa_H' \underset{RH}{\otimes} RG)^*"] \arrow[u, "\varphi_K", "\wr"'] & \\
& {}_{RG}(\mathrm{I}_R^{(1)}(G,L), M) \arrow[u, "j_L'^*"'] \arrow[dl, "\check{\xi}_1^{G,L}\cdot(-)"', "\sim"] & {}_{RG}(\mathrm{I}_R^{(1)}(G,K), M) \arrow[l, "\kappa_G'^*"] \arrow[u, "j_K'^*"] \arrow[ul, phantom, "\lrcorner"] \arrow[dr, "\check{\xi}_1^{G,K}\cdot(-)", "\sim"'] & \\
\mathrm{Z}^1(G,L,M) \arrow[uuuu, "\eta_{G,H,L,M}"] & & & \mathrm{Z}^1(G,K,M) \arrow[lll, "\vartheta_{G,K,L,M}"] \arrow[uuuu, "\eta_{G,H,K,M}"']
\end{tikzcd}$$

In particular, we obtain the pullback

$$\begin{tikzcd}
\mathrm{Z}^1(H,L,M|_H) & \mathrm{Z}^1(H,K,M|_H) \arrow[l, "\vartheta_{H,K,L,M|_H}"'] \\
\mathrm{Z}^1(G,L,M) \arrow[u, "\eta_{G,H,L,M}"] & \mathrm{Z}^1(G,K,M) \arrow[l, "\vartheta_{G,K,L,M}"] \arrow[u, "\eta_{G,H,K,M}"'] \arrow[ul, phantom, "\lrcorner"]
\end{tikzcd}$$

again; cf. Remark 97.

Proof. The left and right pentagons commute, due to Remark 100.

The top and bottom quadrangle commute, as shown in Remark 101.

The upper central quadrangle commutes due to Remark 98; cf. Notation 99 for the vertical maps and Definition 93 for the horizontal maps.

The lower central quadrangle is a pullback, as it is the square of Proposition 94.(2) with the contravariant left exact functor ${}_{RG}(-, M)$ applied to it. ▫

6.3 The conormal-sequence

6.3.1 The conormal-sequence in general

Suppose given a commutative ring R.

Suppose given groups G and H.

Suppose given a surjective group morphism $\varphi : H \to G$.

Write $N := \mathrm{Ker}(\varphi) \trianglelefteqslant H$.

Suppose given subgroups $L \leqslant H$ and $K \leqslant G$ such that $L\varphi \leqslant K$.

$$\begin{tikzcd} N \arrow[r, hook] & H \arrow[r, "\varphi"] & G \\ & L \arrow[u, hook] \arrow[r, "\varphi|_L^K"'] & K \arrow[u, hook] \end{tikzcd}$$

Note that G is an H-set by $g \bullet h := g \cdot h\varphi$ for $g \in G$ and $h \in H$.

Definition 103

(1) We have the surjective H-map

$$\begin{array}{rcrcl} \check{\varphi} & : & L\backslash H & \to & K\backslash G \\ & & Lh & \mapsto & K(h\varphi) \end{array}$$

In fact, we have $K((\ell \cdot h)\varphi) = K(\ell\varphi \cdot h\varphi) = K(h\varphi)$ for $\ell \in L$ and $h \in H$, so that this is a well-defined H-map.

(2) We deduce from (1) the surjective RH-linear map

$$\begin{array}{rcrcl} R\check{\varphi} & : & R(L\backslash H) & \to & R(K\backslash G) \\ & & Lh & \mapsto & K(h\varphi) \; . \end{array}$$

(3) Let

$$\mathrm{I}_R^{(1)}(\varphi, L, K) \;=\; \mathrm{I}_R^{(1)}(H \xrightarrow{\varphi} G, L, K) \;:=\; \mathrm{Ker}(R\check{\varphi}) \;\subseteq\; R(L\backslash H) \; ,$$

which is an RH-submodule.

Remark 104

Suppose given $T \subseteq H$ *such that* $H = \dot{\bigcup}_{t\in T} Lt$.

Note that $G = H\varphi = \bigcup_{t\in T} (L\varphi)(t\varphi) \subseteq \bigcup_{t\in T} K(t\varphi) \subseteq G$, *whence* $G = \bigcup_{t\in T} K(t\varphi)$.

Suppose given $S \subseteq T\varphi \subseteq G$ *such that* $G = \dot{\bigcup}_{s\in S} Ks$.

Note that given $t \in T$, *we have* $K(t\varphi) = Ks$ *for a unique element* $s \in S$. *Write* $t\tilde{\varphi} := s$. *So* $K(t\varphi) = K(t\tilde{\varphi})$ *for* $t \in T$. *We have* $t\tilde{\varphi} \in S$ *for* $t \in T$, *i.e. we have a map* $\tilde{\varphi} : T \to S$.

Suppose given a map $\psi : S \to T$ *such that* $K(s\psi\varphi) = Ks$ *for* $s \in S$.

Then

$$B \;:=\; \Big\{ Lt - L(t\tilde{\varphi}\psi) \;:\; t \in T \setminus S\psi \Big\}$$

is an R*-linear basis of* $\mathrm{I}_R^{(1)}(\varphi, L, K)$.

Proof. Recall that $\mathrm{I}_R^{(1)}(\varphi, L, K) = \mathrm{Ker}(R\check{\varphi})$. Note that

$$(Lt - L(t\tilde{\varphi}\psi))(R\check{\varphi}) \;=\; K(t\varphi) - K(t\tilde{\varphi}\psi\varphi) \;=\; K(t\varphi) - K(t\tilde{\varphi}) \;=\; 0$$

for $t \in T \setminus S\psi$. So the elements of B are contained in $\mathrm{Ker}(R\check{\varphi})$.

The set B *is* R*-linearly generating.* We have the R-linear map

$$\begin{array}{rcrcl} \hat{\psi} & : & R(K\backslash G) & \to & R(L\backslash H) \\ & & Ks & \mapsto & L(s\psi)\ . \end{array}$$

Suppose given $\sum_{t\in T} Lt \cdot r_t \in \mathrm{I}_R^{(1)}(\varphi, L, K) = \mathrm{Ker}(R\check{\varphi}) \subseteq R(L\backslash H)$, where $r_t \in R$ for $t \in T$.

Note that ψ is injective.

We obtain

$$\begin{array}{rcl} \sum_{t\in T} Lt \cdot r_t & = & \sum_{t\in T} Lt \cdot r_t - (\sum_{t\in T} Lt \cdot r_t)(R\check{\varphi})\hat{\psi} \\ & = & \sum_{t\in T} Lt \cdot r_t - (\sum_{t\in T} K(t\varphi) \cdot r_t)\hat{\psi} \\ & = & \sum_{t\in T} Lt \cdot r_t - (\sum_{t\in T} K(t\tilde{\varphi}) \cdot r_t)\hat{\psi} \\ & = & \sum_{t\in T} Lt \cdot r_t - \sum_{t\in T} L(t\tilde{\varphi}\psi) \cdot r_t \\ & = & \sum_{t\in T} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t \\ & = & \sum_{t\in T\setminus S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t + \sum_{t\in S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t \\ & = & \sum_{t\in T\setminus S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t + \sum_{s\in S} (L(s\psi) - L(s\psi\tilde{\varphi}\psi)) \cdot r_{s\psi} \\ & = & \sum_{t\in T\setminus S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t + (\sum_{s\in S} (Ks - K(s\psi\tilde{\varphi})) \cdot r_{s\psi})\hat{\psi} \\ & = & \sum_{t\in T\setminus S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t + (\sum_{s\in S} (Ks - K(s\psi\varphi)) \cdot r_{s\psi})\hat{\psi} \\ & = & \sum_{t\in T\setminus S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t + (\sum_{s\in S} (Ks - Ks) \cdot r_{s\psi})\hat{\psi} \\ & = & \sum_{t\in T\setminus S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t\ . \end{array}$$

The set B is R-linearly independent. Suppose given $r_t \in R$ for $t \in T \setminus S\psi$ such that

$$\sum_{t \in T \setminus S\psi} (Lt - L(t\tilde{\varphi}\psi)) \cdot r_t \;=\; 0 \,.$$

Then for $t \in T \setminus S\psi$, the coefficient of the left hand side at the basis element Lt of $R(L\backslash H)$ equals r_t, as $t'\tilde{\varphi}\psi \neq t$ and thus $L(t'\tilde{\varphi}\psi) \neq Lt$ for all $t' \in T$. So $r_t = 0$ for $t \in T \setminus S\psi$. $\square$

Remark 105 We consider Remark 104 in the particular case $L = 1$ and $K = 1$.

Then $T = H$ and $S = G$. Thus $\varphi = \tilde{\varphi}$.

Suppose given a map $\psi : G \to H$ such that $\psi\varphi = \mathrm{id}_G$.

Then the set

$$B \;=\; \{\, h - h\varphi\psi \,|\, h \in H \setminus G\psi \,\}$$

is an R-linear basis of

$$\mathrm{Ker}(RH \xrightarrow{R\varphi} RG) \;=\; \mathrm{I}_R^{(1)}(\varphi, 1, 1) \,.$$

In particular,

$$\mathrm{Ker}(RH \xrightarrow{R\varphi} RG) \;=\; \mathrm{I}_R^{(1)}(\varphi, 1, 1) \;=\; \langle\, h - h\varphi\psi \;:\; h \in H \,\rangle_R \,.$$

Remark 106

(1) *Suppose given an RG-module Y. Then Y is an RH-module via*

$$y \bullet h \;:=\; y \cdot h\varphi$$

for $y \in Y$ and $h \in H$.

(2) *Suppose given an RH-module X with $xn = x$ for $n \in N$. Then X is an RG-module via*

$$x \bullet g \;:=\; x \cdot h$$

for $x \in X$ and $g \in G$, where $h \in H$ is such that $h\varphi = g$.

Proof.

Ad (1). We may compose the module-defining R-algebra morphism $RG \to \mathrm{End}_R(Y)$ with $RH \xrightarrow{R\varphi} RG$ from the left in order to obtain the R-algebra morphism $RH \to \mathrm{End}_R(Y)$, which defines the sought RH-module structure on Y.

Ad (2). We have the module-defining R-algebra morphism

$$\mu \;:\; RH \to \mathrm{End}_R(X) \;:\; h \mapsto (x \mapsto x \cdot h) \,.$$

Now $\mathrm{Ker}(R\varphi) = \mathrm{I}_R^{(1)}(\varphi, 1, 1) = \langle\, h - h\varphi\psi \;:\; h \in H \,\rangle_R$, where $\psi : H \to G$ is a map such that $\psi\varphi = \mathrm{id}_G$.

Suppose given $h \in H$. Since $h\varphi = (h\varphi\psi)\varphi$, there exists a unique $n \in N$ such that $nh = h\varphi\psi$. Now $h - h\varphi\psi = h - nh$ is mapped under μ to the R-linear endomorphism

$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & x \cdot (h - nh) \;=\; xh - xnh \;=\; xh - xh \;=\; 0 \end{array}$$

Therefore, there exist a unique R-algebra morphism $\bar{\mu} : RG \to \mathrm{End}_R(X)$ such that $\mu = (R\varphi)\bar{\mu}$.

$$\begin{tikzcd} \mathrm{Ker}(R\varphi) \arrow[d, hook] \arrow[dr, "0"] & \\ RH \arrow[r, "\mu"] \arrow[d, "R\varphi"'] & \mathrm{End}_R(X) \\ RG \arrow[ur, "\bar{\mu}"'] & \end{tikzcd}$$

□

The following considerations are in the spirit of [**7,** §13].

Remark 107 *Suppose given an* RH*-module* X*.*

Let

$$X\,\mathrm{I}_R^{(1)}(N) \;:=\; \langle\, xu \,:\, x \in X,\; u \in \mathrm{I}_R^{(1)}(N)\,\rangle_R \;\overset{\text{R. 27}}{=}\; \langle\, x(n-1) \,:\, x \in X,\; n \in N\,\rangle_R \;\subseteq\; X$$

(1) *We have the* RH*-submodule* $X\,\mathrm{I}_R^{(1)}(N) \subseteq X$*.*

(2) *We have the* RG*-module* $X/X\,\mathrm{I}_R^{(1)}(N)$ *via*

$$(x + X\,\mathrm{I}_R^{(1)}(N)) \cdot g \;:=\; xh + X\,\mathrm{I}_R^{(1)}(N)\;,$$

where $h \in H$ *is such that* $h\varphi = g$*.*

(3) *We have the* RG*-linear isomorphism*

$$\begin{array}{rcl} X \underset{RH}{\otimes} RG & \xrightarrow[\sim]{\eta_X} & X/X\,\mathrm{I}_R^{(1)}(N) \\ x \otimes 1 & \mapsto & x + X\,\mathrm{I}_R^{(1)}(N)\;. \end{array}$$

(4) *If* X *is an* RG*-module, considered as an* RH*-module via Remark* 106.(1)*, then we have the* RG*-linear isomorphism*

$$\begin{array}{rcl} X \underset{RH}{\otimes} RG & \xrightarrow[\sim]{\eta_X} & X \\ x \otimes 1 & \mapsto & x\;. \end{array}$$

Proof.

Ad (1). Suppose given $x \in X$ and $n \in N$.

Suppose given $h \in H$. We have to show that $x(n-1) \cdot h \overset{!}{\in} X\,\mathrm{I}_R^{(1)}(N)$.

In fact, we have

$$x(n-1) \cdot h \;=\; xhh^{-1}(n-1) \cdot h \;=\; xh(n^h - 1) \;\in\; X\,\mathrm{I}_R^{(1)}(N)\,.$$

Ad (2). By Remark 106.(2), we have to show that $(x + X\,\mathrm{I}_R^{(1)}(N)) \cdot n = x + X\,\mathrm{I}_R^{(1)}(N)$ for $x \in X$ and $n \in N$. In fact,

$$(x + X\,\mathrm{I}_R^{(1)}(N)) \cdot n - (x + X\,\mathrm{I}_R^{(1)}(N)) \;=\; x \cdot (n-1) + X\,\mathrm{I}_R^{(1)}(N) \;=\; 0 + X\,\mathrm{I}_R^{(1)}(N)\,.$$

Ad (3). In order to have the RG-linear map η_X as claimed, we have to show that the elements $xh \otimes 1$ and $x \otimes (h \bullet 1) = x \otimes h\varphi$ yield the same element in $X/X\,\mathrm{I}_R^{(1)}(N)$.

On the one hand, $xh \otimes 1$ yields $xh + X\,\mathrm{I}_R^{(1)}(N)$.

On the other hand, $x \otimes h\varphi = (x \otimes 1) \cdot h\varphi$ yields

$$(x + X\,\mathrm{I}_R^{(1)}(N)) \bullet h\varphi \;\overset{\text{R. 106.(2)}}{=}\; (x + X\,\mathrm{I}_R^{(1)}(N)) \cdot h \;=\; xh + X\,\mathrm{I}_R^{(1)}(N)\,.$$

So the results are in fact the same.

In order to show that η_X is an RG-linear isomorphism, we show that we have the RG-linear map

$$\begin{array}{rcl} X \underset{RH}{\otimes} RG & \longleftarrow & X/X\,\mathrm{I}_R^{(1)}(N) \\ x \otimes 1 & \leftarrow\!\shortmid & x + X\,\mathrm{I}_R^{(1)}(N)\,, \end{array}$$

which then inverts η_X from the left and from the right.

So we have to show that for $x \in X$ and $n \in N$, the element $(x \cdot (n-1)) \otimes 1$ is zero. In fact,

$$(x \cdot (n-1)) \otimes 1 \;=\; x \otimes ((n-1) \bullet 1) \;=\; x \otimes ((n\varphi - 1\varphi) \cdot 1) \;=\; x \otimes (1-1) \;=\; 0\,.$$

Ad (4). This follows from (3), once we have shown that $X\,\mathrm{I}_R^{(1)}(N)$ is zero in this case.

Suppose given $x \in X$ and $n \in N$. We obtain

$$x \bullet (n-1) \;=\; x \cdot (n\varphi - 1\varphi) \;=\; x \cdot (1-1) \;=\; 0\,.$$

□

Proposition 108 *We have the following right exact sequence of RG-modules and RG-linear maps.*

$$\begin{array}{rcccl} \mathrm{I}_R^{(1)}(\varphi, L, K)/\,\mathrm{I}_R^{(1)}(\varphi, L, K)\,\mathrm{I}_R^{(1)}(N) & \overset{\alpha}{\to} & \mathrm{I}_R^{(1)}(H, L)/\,\mathrm{I}_R^{(1)}(H, L)\,\mathrm{I}_R^{(1)}(N) & \overset{\beta}{\to} & \mathrm{I}_R^{(1)}(G, K) \\ x + \mathrm{I}_R^{(1)}(\varphi, L, K)\,\mathrm{I}_R^{(1)}(N) & \mapsto & x + \mathrm{I}_R^{(1)}(H, L)\,\mathrm{I}_R^{(1)}(N) & & \\ & & y + \mathrm{I}_R^{(1)}(H, L)\,\mathrm{I}_R^{(1)}(N) & \mapsto & (y)(R\check{\varphi}) \end{array}$$

It is called the conormal-sequence*.*

Note that for $h \in H$ and $y = Lh - L1$, we have $(y)(R\check{\varphi}) = (Lh - L1)(R\check{\varphi}) = K(h\varphi) - K1$; cf. Definition 103.(3)*, Remark* 91.

Proof. Using the maps from Definition 59, we obtain the commutative triangle of RH-modules

$$\begin{tikzcd}
& R(K\backslash G) \arrow[dr, "\check{\varepsilon}_1^{G,K}"] & \\
R(L\backslash H) \arrow[ur, "R\check{\varphi}"] \arrow[rr, "\check{\varepsilon}_1^{H,L}"] & & R
\end{tikzcd} ,$$

as $(Lh)(R\check{\varphi})\check{\varepsilon}_1^{G,K} = (K(h\varphi))\check{\varepsilon}_1^{G,K} = 1 = (Lh)\check{\varepsilon}_1^{H,L}$ for $h \in H$.

We can apply Remark 3 to this triangle and get the following commutative diagram of RH-modules, in which the outer sequence is exact. Cf. Definition 62.

$$\begin{tikzcd}
& \mathrm{I}_R^{(1)}(G,K) \arrow[rr] \arrow[dr, "\dot{\check{\varepsilon}}_1^{G,L}"] & & 0 \arrow[ddr] & \\
& & R(K\backslash G) \arrow[ur] \arrow[dr, "\check{\varepsilon}_1^{G,K}"] & & \\
\mathrm{I}_R^{(1)}(H,L) \arrow[uur, "\tilde{\varphi}"] \arrow[r, "\dot{\check{\varepsilon}}_1^{H,L}"] & R(L\backslash H) \arrow[ur, "R\check{\varphi}"] \arrow[rr, "\check{\varepsilon}_1^{H,L}"] & & R \arrow[r] \arrow[dr] & 0 \arrow[d] \\
\mathrm{I}_R^{(1)}(\varphi,L,K) \arrow[u, "i"] \arrow[ur] & 0 \arrow[l] & & 0 & 0 \arrow[l]
\end{tikzcd}$$

Here $\tilde{\varphi} := (R\check{\varphi})|_{\mathrm{I}_R^{(1)}(H,L)}^{\mathrm{I}_R^{(1)}(G,K)}$ is obtained by restriction. Moreover, $i : \mathrm{I}_R^{(1)}(\varphi,L,K) \to \mathrm{I}_R^{(1)}(H,L)$ is the inclusion map.

All the cokernels are 0, as $R\check{\varphi}$, $\check{\varepsilon}_1^{H,L}$ and $\check{\varepsilon}_1^{G,K}$ are surjective.

Hence we have the short exact sequence

$$0 \longrightarrow \mathrm{I}_R^{(1)}(\varphi,L,K) \xrightarrow{\;i\;} \mathrm{I}_R^{(1)}(H,L) \xrightarrow{\;\tilde{\varphi}\;} \mathrm{I}_R^{(1)}(G,K) \longrightarrow 0 .$$

An application of the right exact functor $- \underset{RH}{\otimes} RG$ yields the right exact sequence

$$\mathrm{I}_R^{(1)}(\varphi,L,K) \underset{RH}{\otimes} RG \xrightarrow{i \underset{RH}{\otimes} RG} \mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG \xrightarrow{\tilde{\varphi} \underset{RH}{\otimes} RG} \mathrm{I}_R^{(1)}(G,K) \underset{RH}{\otimes} RG \longrightarrow 0 .$$

We substitute isomorphically, using Remark 107.$(3,4)$.

$$\begin{tikzcd}
\mathrm{I}_R^{(1)}(\varphi,L,K) \underset{RH}{\otimes} RG \arrow[r, "\eta_{\mathrm{I}_R^{(1)}(\varphi,L,K)}", "\sim"'] \arrow[d, "i \underset{RH}{\otimes} RG"'] & \mathrm{I}_R^{(1)}(\varphi,L,K) / \mathrm{I}_R^{(1)}(\varphi,L,K)\, \mathrm{I}_R^{(1)}(N) \arrow[d, "\alpha"] & x + \mathrm{I}_R^{(1)}(\varphi,L,K)\, \mathrm{I}_R^{(1)}(N) \arrow[d, mapsto] & \\
\mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG \arrow[r, "\eta_{\mathrm{I}_R^{(1)}(H,L)}", "\sim"'] \arrow[d, "\tilde{\varphi} \underset{RH}{\otimes} RG"'] & \mathrm{I}_R^{(1)}(H,L) / \mathrm{I}_R^{(1)}(H,L)\, \mathrm{I}_R^{(1)}(N) \arrow[d, "\beta"] & x + \mathrm{I}_R^{(1)}(H,L)\, \mathrm{I}_R^{(1)}(N) & y + \mathrm{I}_R^{(1)}(H,L)\, \mathrm{I}_R^{(1)}(N) \arrow[d, mapsto] \\
\mathrm{I}_R^{(1)}(G,K) \underset{RH}{\otimes} RG \arrow[r, "\eta_{\mathrm{I}_R^{(1)}(G,K)}", "\sim"'] & \mathrm{I}_R^{(1)}(G,K) & & (y)(R\check{\varphi})
\end{tikzcd}$$

Its quadrangles commute, as we shall calculate using Remark 107.(3, 4) and the definitions of i and $\tilde{\varphi}$.

For $x \in \mathrm{I}_R^{(1)}(\varphi, L, K)$, the upper quadrangle maps as follows.

$$\begin{tikzcd}
x \otimes 1 \arrow[r, mapsto, "\eta_{\mathrm{I}_R^{(1)}(\varphi,L,K)}", "\sim"'] \arrow[d, mapsto, "i \underset{RH}{\otimes} RG"'] & x + \mathrm{I}_R^{(1)}(\varphi, L, K)\, \mathrm{I}_R^{(1)}(N) \arrow[d, mapsto, "\alpha"] \\
x \otimes 1 \arrow[r, mapsto, "\eta_{\mathrm{I}_R^{(1)}(H,L)}", "\sim"'] & x + \mathrm{I}_R^{(1)}(H, L)\, \mathrm{I}_R^{(1)}(N)
\end{tikzcd}$$

For $y \in \mathrm{I}_R^{(1)}(H, L)$, the lower quadrangle maps as follows.

$$\begin{tikzcd}
y \otimes 1 \arrow[r, mapsto, "\eta_{\mathrm{I}_R^{(1)}(H,L)}", "\sim"'] \arrow[d, mapsto, "\tilde{\varphi} \underset{RH}{\otimes} RG"'] & y + \mathrm{I}_R^{(1)}(H, L)\, \mathrm{I}_R^{(1)}(N) \arrow[d, mapsto, "\beta"] \\
(y)(R\check{\varphi}) \otimes 1 \arrow[r, "\eta_{\mathrm{I}_R^{(1)}(G,K)}", "\sim"'] & (y)(R\check{\varphi})
\end{tikzcd}$$

□

6.3.2 The conormal-sequence in particular

In the setup of §6.3.1, we specialise to

$$\begin{aligned} R &:= \mathbb{Z} \\ K &:= 1 \\ L &:= 1\,. \end{aligned}$$

So $H \xrightarrow{\varphi} G$ is a surjective group morphism with kernel $N = \mathrm{Ker}(\varphi)$.

Let $N^{(1)} := \langle\, n^{-1}m^{-1}nm \,:\, n,\, m \,\in\, N \,\rangle \trianglelefteqslant N$ denote the commutator subgroup.

We have the $\mathbb{Z}$-module $N/N^{(1)}$, which we write multiplicatively. It is a $\mathbb{Z}H$-module via

$$nN^{(1)} \bullet h \;:=\; n^h N^{(1)}$$

for $n \in N$ and $h \in H$. In fact, $n \mapsto n^h$ maps $N^{(1)}$ to $N^{(1)}$. Moreover, $(n \cdot \tilde{n})^h = n^h \cdot \tilde{n}^h$ and $n^1 = n$ and $n^{h \cdot \tilde{h}} = (n^h)^{\tilde{h}}$ for $n,\, \tilde{n} \,\in\, N$ and $h,\, \tilde{h} \,\in\, H$.

Since N acts trivially on $N/N^{(1)}$, it is also a $\mathbb{Z}G$-module via

$$nN^{(1)} \bullet g \;:=\; n^h N^{(1)}$$

for $n \in N$ and $g \in G$, where $h \in H$ is such that $h\varphi = g$.

We shall consider the $\mathbb{Z}H$-submodule $\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi, 1, 1) \subseteq \mathbb{Z}H$.

Let $\psi : G \to H$ be a map such that $\psi \cdot \varphi = \mathrm{id}$.

By Remark 105, the set

$$\{\, h - h\varphi\psi \,|\, h \in H \setminus G\psi \,\}$$

is a $\mathbb{Z}$-linear basis of $\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi, 1, 1)$.

Definition 109 We define the following $\mathbb{Z}$-linear map.

$$\begin{array}{rcl} \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1) & \stackrel{\tau}{\to} & N/N^{(1)} \\ h - h\varphi\psi & \mapsto & (h\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}N^{(1)} \quad \text{for } h\in H\setminus G\psi \end{array}$$

Remark 110 *For $n\in N$ and $h\in H$, we have $nh-h\in \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)$ and*

$$(nh-h)\tau \;=\; n^h N^{(1)}\;.$$

Proof. We have $n\varphi = 1$ and thus $(nh)\varphi\psi = h\varphi\psi =: \tilde{h}$. Thus

$$nh-h \;=\; (nh-(nh)\varphi\psi)-(h-h\varphi\psi) \;\in\; \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\;.$$

We have to show that $(nh-h)\tau \stackrel{!}{=} n^h N^{(1)}$.

First, we remark that if $h\in G\psi$, then we may choose $g\in G$ with $h=g\psi$ and obtain

$$(h-h\varphi\psi)\tau \;=\; (g\psi-g\psi\varphi\psi)\tau \;=\; 0\tau \;=\; 1N^{(1)}$$

and

$$(h\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}N^{(1)} \;=\; (g\psi\cdot(g\psi\varphi\psi)^{-1})^{g\psi\varphi\psi}N^{(1)} \;=\; 1^{g\psi\varphi\psi}N^{(1)} \;=\; 1N^{(1)}\;,$$

which is the same.

Thus

$$h-h\varphi\psi \;=\; (h\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}N^{(1)}$$

for $h\in H$.

Now, we obtain

$$\begin{array}{rcl} (nh-h)\tau & = & ((nh-(nh)\varphi\psi)-(h-h\varphi\psi))\tau \\ & = & (nh-(nh)\varphi\psi)\tau\cdot((h-h\varphi\psi)\tau)^{-1} \\ & = & (nh\cdot((nh)\varphi\psi)^{-1})^{(nh)\varphi\psi}N^{(1)}\cdot((h\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}N^{(1)})^{-1} \\ & = & (nh\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}N^{(1)}\cdot((h\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}N^{(1)})^{-1} \\ & = & (nh\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}N^{(1)}\cdot(h\varphi\psi\cdot h^{-1})^{h\varphi\psi}N^{(1)} \\ & = & (nh\cdot(h\varphi\psi)^{-1})^{h\varphi\psi}\cdot(h\varphi\psi\cdot h^{-1})^{h\varphi\psi}N^{(1)} \\ & = & (nh\cdot(h\varphi\psi)^{-1}\cdot h\varphi\psi\cdot h^{-1})^{h\varphi\psi}N^{(1)} \\ & = & n^{h\varphi\psi}N^{(1)}\;. \end{array}$$

We have $(h\varphi\psi\cdot h^{-1})\varphi = h\varphi\psi\varphi\cdot(h\varphi)^{-1} = 1$, whence $h\varphi\psi = h\tilde{n}$ for some $\tilde{n}\in N$. We have

$$n^{h\varphi\psi}\cdot(n^h)^{-1} \;=\; n^{h\tilde{n}}\cdot(n^h)^{-1} \;=\; \tilde{n}^{-1}\cdot n^h\cdot\tilde{n}\cdot(n^h)^{-1} \;\in\; N^{(1)}\;.$$

Thus

$$(nh-h)\tau \;=\; n^{h\varphi\psi}N^{(1)} \;=\; n^h N^{(1)}\;.$$

$\square$

Remark 111 *The set*

$$\{\, nh - h \,|\, n \in N,\ h \in H \,\}$$

$\mathbb{Z}$-linearly generates $\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1)$.

Proof. We have

$$\{\, nh - h \,|\, n \in N,\ h \in H \,\} \ \subseteq\ \mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1)\ ;$$

cf. Remark 110. Moreover,

$$\{\, h - h\varphi\psi \,|\, h \in H \setminus G\psi \,\} \ \subseteq\ \{\, nh - h \,|\, n \in N,\ h \in H \,\}\ ,$$

and the former set is a $\mathbb{Z}$-linear basis of $\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1)$; cf. Remark 105. ▫

Remark 112 *The map*

$$\begin{array}{rcll} \mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1) & \xrightarrow{\tau} & N/N^{(1)} & \\ nh - h & \mapsto & n^h N^{(1)} & \textit{for } n \in N \textit{ and } h \in H \end{array}$$

is $\mathbb{Z}H$-linear; cf. Definition 109 *and Remarks* 110*,* 111*.*

Proof. By construction, τ is $\mathbb{Z}$-linear. Moreover, given $n \in N$ and $h,\, \tilde{h} \in H$, we have

$$((nh - h)\cdot \tilde{h})\tau \ =\ (nh\tilde{h} - h\tilde{h})\tau \ =\ n^{h\tilde{h}} N^{(1)} \ =\ n^h N^{(1)} \bullet \tilde{h} \ =\ (nh - h)\tau \bullet \tilde{h}\ .$$

Remark 113

(1) *We have the $\mathbb{Z}H$-submodule $\mathrm{I}^{(1)}_{\mathbb{Z}}(H)\,\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1) \subseteq \mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1)$.*

(2) *We have $\mathrm{I}^{(1)}_{\mathbb{Z}}(H)\,\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1) = \mathrm{I}^{(1)}_{\mathbb{Z}}(H)\,\mathrm{I}^{(1)}_{\mathbb{Z}}(N)$.*

(3) *We have $\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1) \subseteq \mathrm{I}^{(1)}_{\mathbb{Z}}(H)$.*

Proof. We have

$$\mathrm{I}^{(1)}_{\mathbb{Z}}(H)\,\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1) \ =\ {}_{\mathbb{Z}}\langle\, (\tilde{h} - 1)(nh - h)\ :\ \tilde{h} \in H,\, n \in N,\, h \in H \,\rangle\ ;$$

cf. Remarks 111 and 27.

Ad (1). Given $\tilde{h},\, h,\, h' \in H$ and $n \in N$, we obtain

$$(\tilde{h} - 1)(nh - h)\cdot h' \ =\ (\tilde{h} - 1)(nhh' - hh') \ \in\ \mathrm{I}^{(1)}_{\mathbb{Z}}(H)\,\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1)\ .$$

Ad (2).

Ad ($\supseteq$). We have $\mathrm{I}^{(1)}_{\mathbb{Z}}(\varphi, 1, 1) \supseteq \mathrm{I}^{(1)}_{\mathbb{Z}}(N)$; cf. Remarks 111 and 27.

Ad ($\subseteq$). Given $\tilde{h},\, h \in H$ and $n \in N$, we obtain

$$(\tilde{h} - 1)(nh - h) \ =\ (\tilde{h} - 1)h(n^h - 1) \ =\ ((\tilde{h}h - 1) - (h - 1))(n^h - 1) \ \in\ \mathrm{I}^{(1)}_{\mathbb{Z}}(H)\,\mathrm{I}^{(1)}_{\mathbb{Z}}(N)\ ;$$

cf. Remarks 111 and 27.

Ad (3). Given $n \in N$ and $h \in H$, we have $nh - h \in \mathrm{I}^{(1)}_{\mathbb{Z}}(H)$; cf. Remark 111. ▫

Remark 114 *We have* $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\cdot\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \overset{\mathrm{R.}\,113.(2)}{=} \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1) \subseteq \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)$.
We have $(\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\cdot\mathrm{I}_{\mathbb{Z}}^{(1)}(N))\tau = 1$.
Thus we have the $\mathbb{Z}H$*-linear map*

$$\begin{array}{rcl} \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \xrightarrow{\bar{\tau}} & N/N^{(1)} \\ nh-h+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \mapsto & n^h N^{(1)}\ , \end{array}$$

where $n\in N$ *and* $h\in H$*; cf. Remark* 112.

Proof. The set

$$\{\,(h-1)(n-1)\,|\,h\in H,\ n\in N\,\}$$

is $\mathbb{Z}$-linearly generating $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)$; cf. Remark 27. We obtain

$$\begin{array}{rcl} ((h-1)(n-1))\tau & = & ((hn-h)-(n-1))\tau \\ & = & ((n^{h^{-1}}h-h)-(n-1))\tau \\ & = & (n^{h^{-1}}h-h)\tau\cdot((n\cdot 1-1)\tau)^{-1} \\ & = & (n^{h^{-1}h}N^{(1)})\cdot(nN^{(1)})^{-1} \\ & = & (nN^{(1)})\cdot(nN^{(1)})^{-1} \\ & = & 1N^{(1)}\ . \end{array}$$

□

Definition 115 We define the following map; cf. Remark 110.

$$\begin{array}{rcl} N & \xrightarrow{\tau'} & \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ n & \mapsto & n-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)\ . \end{array}$$

Remark 116

(1) *The map* τ' *is a group morphism; cf. Definition* 115.

(2) *Since* $\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)$ *is an abelian group, by* (1) *we obtain the group morphism*

$$\begin{array}{rcl} N/N^{(1)} & \xrightarrow{\hat{\tau}'} & \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ nN^{(1)} & \mapsto & n-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)\ . \end{array}$$

(3) *Since* $\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\subseteq\mathrm{I}_{\mathbb{Z}}^{(1)}(H)$ *by Remark* 113.(3)*, by* (2) *we obtain the group morphism*

$$\begin{array}{rcl} N/N^{(1)} & \xrightarrow{\bar{\tau}'} & \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ nN^{(1)} & \mapsto & n-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)\ . \end{array}$$

Then $\bar{\tau}'$ *is* $\mathbb{Z}H$*-linear.*

Proof.

Ad (1). We have to show that τ' is a group morphism.

Suppose given n, $\tilde{n} \in N$. We have to show that

$$(n \cdot \tilde{n})\tau' \overset{!}{=} n\tau' + \tilde{n}\tau' .$$

We obtain

$$\begin{aligned}
(n \cdot \tilde{n})\tau' - (n\tau' + \tilde{n}\tau') &= (n\tilde{n} - 1) - (n-1) - (\tilde{n} - 1) + \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi, 1, 1)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\
&= (n\tilde{n} - n - \tilde{n} + 1) + \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi, 1, 1)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\
&= (n-1)(\tilde{n} - 1) + \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi, 1, 1)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\
&= 0 + \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi, 1, 1)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) .
\end{aligned}$$

Ad (3). Suppose given $n \in N$ and $h \in H$. We have

$$(nN^{(1)} \bullet h)\bar{\tau}' = (n^h N^{(1)})\bar{\tau}' = n^h - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N)$$

and

$$(nN^{(1)})\bar{\tau}' \cdot h = nh - h + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) .$$

The difference of the results is

$$\begin{aligned}
(n^h - 1) - (nh - h) + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) &= h^{-1}(nh - h) - (nh - h) + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\
&= (h^{-1} - 1)(nh - h) + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\
&\overset{\text{R. 113.(2)}}{=} 0 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N)
\end{aligned}$$

□

Remark 117 *We have* $\bar{\tau} \cdot \bar{\tau}' = \mathrm{id}_{\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)}$ *and* $\bar{\tau}' \cdot \bar{\tau} = \mathrm{id}_{N/N^{(1)}}$ *.*

So $\bar{\tau}$ *is a* $\mathbb{Z}H$*-linear isomorphism and* $\bar{\tau}^{-1} = \bar{\tau}'$*.*

Altogether, we have the $\mathbb{Z}H$*-linear isomorphism*

$$\begin{aligned}
\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi, 1, 1)/\, \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) &\overset{\bar{\tau}}{\xrightarrow{\sim}} N/N^{(1)} \\
nh - h + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) &\overset{\bar{\tau}}{\mapsto} n^h N^{(1)} \\
n - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) &\overset{\bar{\tau}^{-1}}{\hookleftarrow} nN^{(1)} ,
\end{aligned}$$

where $n \in N$ *and* $h \in H$*.*

Proof.

We show that $\bar{\tau} \cdot \bar{\tau}' \overset{!}{=} \mathrm{id}$. Suppose given $n \in N$ and $h \in H$. We have to show that

$$(nh - h + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N))\bar{\tau} \cdot \bar{\tau}' \overset{!}{=} nh - h + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\, \mathrm{I}_{\mathbb{Z}}^{(1)}(N) ;$$

cf. Remark 111. We obtain

$$\begin{array}{rl} & (nh - h + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N))\bar{\tau} \cdot \bar{\tau}' - (nh - h + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)) \\ = & (n^h N^{(1)})\bar{\tau}' - (nh - h + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)) \\ = & (n^h - 1) - (nh - h) + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ = & (h^{-1}nh - 1 - nh + h) + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ = & -(h-1)(h^{-1}nh - 1) + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ = & 0 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)\ ; \end{array}$$

cf. Remark 27.

We show that $\bar{\tau}' \cdot \bar{\tau} \overset{!}{=} \mathrm{id}$. Suppose given $n \in N$. We obtain

$$(nN^{(1)})\bar{\tau}'\bar{\tau} \;=\; (n - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N))\bar{\tau} \;=\; nN^{(1)}$$

□

Definition 118

(1) We have the surjective $\mathbb{Z}H$-linear map

$$\begin{array}{rcl} \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \xrightarrow{\bar{\alpha}} & \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ x + \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \mapsto & x + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)\ . \end{array}$$

(2) We have the injective $\mathbb{Z}H$-linear map

$$\begin{array}{rcl} \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \xrightarrow{\dot{\alpha}} & \mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \\ x + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \mapsto & x + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)\ . \end{array}$$

Cf. Remark 113.(3).

Remark 119 *We have the following commutative triangle of $\mathbb{Z}H$-linear maps.*

$$\begin{tikzcd} \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[dr, "\bar{\alpha}"] \arrow[dd, "\alpha"'] & \\ & \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[dl, "\dot{\alpha}"] \\ \mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \end{tikzcd}$$

Proof. The maps α, $\bar{\alpha}$ and $\dot{\alpha}$ act identically on representatives; cf. Proposition 108, Definition 118. □

Remark 120 *We have the following commutative triangle.*

$$\begin{tikzcd}
\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[d, "\bar{\alpha}"'] & \\
\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & N/N^{(1)} \arrow[l, "\sim", "\bar{\tau}'"'] \arrow[ul, "\hat{\tau}'"']
\end{tikzcd}$$

Here, $\bar{\alpha}$ and $\bar{\tau}'$ are $\mathbb{Z}H$-linear, but $\hat{\tau}'$ is $\mathbb{Z}$-linear.

Proof. Given $n \in N$, we obtain

$$(nN^{(1)})\hat{\tau}'\bar{\alpha} \;=\; ((n-1) + \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N))\bar{\alpha} \;=\; (n-1) + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \;=\; (nN^{(1)})\bar{\tau}' \,.$$

□

Proposition 121

(1) *We have the following commutative diagram of $\mathbb{Z}H$-linear maps.*

$$\begin{tikzcd}
\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[dd, "\alpha"'] \arrow[dr, "\bar{\alpha}"] & & \\
& \mathrm{I}_{\mathbb{Z}}^{(1)}(\varphi,1,1)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[dl, "\dot{\alpha}"] \arrow[r, bend left, "\bar{\tau}"] \arrow[r, phantom, "\sim"] & N/N^{(1)} \arrow[l, bend left, "\bar{\tau}'"] \\
\mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[d, "\beta"] & & \\
\mathrm{I}_{\mathbb{Z}}^{(1)}(G) & &
\end{tikzcd}$$

Here, $\bar{\alpha}$ is surjective, $\dot{\alpha}$ is injective and the sequence $(\dot{\alpha}, \beta)$ is short exact.

Since N acts trivially on the $\mathbb{Z}H$-modules in this diagram, it can also be considered as a diagram of $\mathbb{Z}G$-modules and $\mathbb{Z}G$-linear maps.

(2) *We have the following short exact sequence of $\mathbb{Z}G$-modules and $\mathbb{Z}G$-linear maps.*

$$\begin{array}{ccccc}
N/N^{(1)} & \xrightarrow{\bar{\tau}'\cdot\dot{\alpha}} & \mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \xrightarrow{\beta} & \mathrm{I}_{\mathbb{Z}}^{(1)}(G) \\
nN^{(1)} & \mapsto & n-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & & \\
 & & h-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \mapsto & h\varphi - 1
\end{array}$$

It is called the short conormal-sequence.

Proof.

Ad (1). The sequence (α, β) is right exact; cf. Proposition 108. We have $\alpha = \bar{\alpha}\cdot\dot{\alpha}$, with $\bar{\alpha}$ surjective and $\dot{\alpha}$ injective; cf. Definition 118, Remark 119. Thus the sequence $(\dot{\alpha}, \beta)$ is short exact.

The $\mathbb{Z}H$-linear maps $\bar{\tau}'$ and $\bar{\tau}$ are mutually inverse; cf. Remark 117.

Ad (2). This follows from (1); cf. Remark 116, Definition 118.(2), Proposition 108. □

7 Examples

7.1 Example for a cotangent-square

We consider the commutative ring $\mathbb{Z}$.

Suppose given $s, t, u, v \in \mathbb{Z}_{\geqslant 1}$.

We consider the cyclic group $\mathrm{C}_{s\cdot t\cdot u\cdot v} := \langle\, a \,|\, a^{s\cdot t\cdot u\cdot v} \,\rangle$.

In $\mathrm{C}_{s\cdot t\cdot u\cdot v}$, we define the subgroups

$$
\begin{array}{lcll}
\mathrm{C}_s & := & \langle a^{t\cdot u\cdot v}\rangle & \leqslant \ \mathrm{C}_{s\cdot t\cdot u\cdot v} \\
\mathrm{C}_{s\cdot t} & := & \langle a^{u\cdot v}\rangle & \leqslant \ \mathrm{C}_{s\cdot t\cdot u\cdot v} \\
\mathrm{C}_{s\cdot t\cdot u} & := & \langle a^{v}\rangle & \leqslant \ \mathrm{C}_{s\cdot t\cdot u\cdot v} \ .
\end{array}
$$

So we get

$$
\mathrm{C}_s \ \leqslant \ \mathrm{C}_{s\cdot t} \ \leqslant \ \mathrm{C}_{s\cdot t\cdot u} \ \leqslant \ \mathrm{C}_{s\cdot t\cdot u\cdot v} \ .
$$

Suppose given a $\mathbb{Z}\mathrm{C}_{s\cdot t\cdot u\cdot v}$-module M.

We write $M' := M|_{\mathrm{C}_{s\cdot t\cdot u}}$.

We can apply Remark 102 to get the following pullback of $\mathbb{Z}$-modules.

$$
\begin{tikzcd}
 & d & d \arrow[l, mapsto] & \\
d|_{\mathrm{C}_{s\cdot t\cdot u}} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_s, M') \arrow[dr, phantom, "\lrcorner", very near end] & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s\cdot t}, M') \arrow[l, "\vartheta_{\mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s\cdot t}, \mathrm{C}_s, M'}"'] & d|_{\mathrm{C}_{t\cdot u\cdot v}} \\
d \arrow[u, mapsto] & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_s, M) \arrow[u, "\eta_{\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_s, M}"] & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t}, M) \arrow[l, "\vartheta_{\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t}, \mathrm{C}_s, M}"] \arrow[u, "\eta_{\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s\cdot t}, M}"'] & d \arrow[u, mapsto] \\
 & d & d \arrow[l, mapsto] &
\end{tikzcd}
$$

Cf. also Remark 97.

First, we consider absolute 1-cocycles.

Each 1-cocycle in $\mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, M)$ is determined by its image of the generator a, as

$$
(a^k)d \ = \ (a^{k-1}\cdot a)d \ = \ (a)d + (a^{k-1})d\cdot a \ = \ (a)d + (a)d\cdot a + (a^{k-2})d\cdot a^2 \ = \ \ldots \ = \ (a)d\cdot\left(\sum_{i=0}^{k-1} a^i\right)
$$

for $k \in \mathbb{Z}_{\geqslant 0}$. In particular, we obtain

$$
0 \ = \ (1)d \ = \ (a^{s\cdot t\cdot u\cdot v})d \ = \ (a)d\cdot\left(\sum_{i=0}^{s\cdot t\cdot u\cdot v-1} a^i\right) \ .
$$

Conversely, given

$$m \;\in\; M_{(s\cdot t\cdot u\cdot v)} \;:=\; \left\{ m \in M \;\middle|\; m \cdot \left(\sum_{i=0}^{s\cdot t\cdot u\cdot v-1} a^i \right) = 0 \right\} .$$

Then we may define the map

$$\begin{array}{rcrcl} d_m & : & \mathrm{C}_{s\cdot t\cdot u\cdot v} & \to & M \\ & & a^k & \mapsto & m \cdot \left(\sum_{i=0}^{k-1} a^i \right) \quad \text{for } k \in \mathbb{Z}_{\geqslant 0} \;. \end{array}$$

In fact, the element a^k is supposed to be mapped to $m \cdot \left(\sum_{i=0}^{k-1} a^i \right)$. The element $a^{k+s\cdot t\cdot u\cdot v}$ is supposed to be mapped to

$$\begin{array}{rcl} m \cdot \left(\sum_{i=0}^{k+s\cdot t\cdot u\cdot v-1} a^i \right) & = & m \cdot \left(\sum_{i=0}^{k-1} a^i \right) + m \cdot \left(\sum_{i=k}^{k+s\cdot t\cdot u\cdot v-1} a^i \right) \\ & = & m \cdot \left(\sum_{i=0}^{k-1} a^i \right) + m \cdot \left(\sum_{i=0}^{s\cdot t\cdot u\cdot v-1} a^i \right) \cdot a^k \\ & = & m \cdot \left(\sum_{i=0}^{k-1} a^i \right) + 0 \cdot a^k \;, \end{array}$$

which is the same. Thus d_m is a well-defined as a map.

We verify that d_m is a 1-cocycle. Suppose given $k,\, \ell \,\in\, \mathbb{Z}_{\geqslant 0}$. Then

$$\begin{array}{rcl} (a^\ell)d_m + (a^k)d_m \cdot a^\ell & = & m \cdot \left(\sum_{i=0}^{\ell-1} a^i \right) + m \cdot \left(\sum_{i=0}^{k-1} a^i \right) \cdot a^\ell \\ & = & m \cdot \left(\sum_{i=0}^{\ell-1} a^i \right) + m \cdot \left(\sum_{i=\ell}^{k+\ell-1} a^i \right) \\ & = & m \cdot \left(\sum_{i=0}^{k+\ell-1} a^i \right) \\ & = & (a^k \cdot a^\ell)d_m \end{array}$$

Altogether, with

$$M_{(s\cdot t\cdot u\cdot v)} \;=\; \left\{ m \in M \;\middle|\; m \cdot \left(\sum_{i=0}^{s\cdot t\cdot u\cdot v-1} a^i \right) = 0 \right\} ,$$

we have the isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} M_{(s\cdot t\cdot u\cdot v)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, M) \\ m & \mapsto & (d_m : a^k \mapsto m \cdot (\sum_{i=0}^{k-1} a^i)) \\ (a)d & \leftarrow\!\shortmid & d \;. \end{array}$$

Similarly, letting

$$M'_{(s\cdot t\cdot u)} \;:=\; \left\{ m \in M \;\middle|\; m \cdot \left(\sum_{i=0}^{s\cdot t\cdot u-1} (a^v)^i \right) = 0 \right\} ,$$

we have the isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} M'_{(s\cdot t\cdot u)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u}, M') \\ m & \mapsto & (d'_m : (a^v)^k \mapsto m \cdot (\sum_{i=0}^{k-1} (a^v)^i)) \\ (a^v)d' & \leftarrow\!\shortmid & d' \;. \end{array}$$

Now we consider relative 1-cocycles.

By Remark 76.(1), we know that relative 1-cocycles have to have value 0 on the subgroup. In this case this means that they have value 0 on the generator of said subgroup.

We have $\mathrm{C}_{s\cdot t} = \langle a^{u\cdot v} \rangle$. For $m \in M_{(s\cdot t\cdot u\cdot v)}$, we have

$$(a^{u\cdot v})d_m \;=\; m\cdot \left(\sum_{i=0}^{u\cdot v-1} a^i \right) .$$

We have $\mathrm{C}_{s} = \langle a^{t\cdot u\cdot v} \rangle$. For $m \in M_{(s\cdot t\cdot u\cdot v)}$, we have

$$(a^{t\cdot u\cdot v})d_m \;=\; m\cdot \left(\sum_{i=0}^{t\cdot u\cdot v-1} a^i \right) .$$

Let

$$\begin{aligned} M_{(u\cdot v)} \;&:=\; \left\{ m\in M \;\middle|\; m\cdot \left(\textstyle\sum_{i=0}^{u\cdot v-1} a^i\right) = 0 \right\} \\ M_{(t\cdot u\cdot v)} \;&:=\; \left\{ m\in M \;\middle|\; m\cdot \left(\textstyle\sum_{i=0}^{t\cdot u\cdot v-1} a^i\right) = 0 \right\} . \end{aligned}$$

Then

$$M_{(u\cdot v)} \;\subseteq\; M_{(t\cdot u\cdot v)} \;\subseteq\; M_{(s\cdot t\cdot u\cdot v)} \;.$$

We have the isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} M_{(u\cdot v)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t}, M) \\ m & \mapsto & (d_m : a^k \mapsto m\cdot (\sum_{i=0}^{k-1} a^i)) \\ (a)d & \leftarrow\!\shortmid & d\;. \end{array}$$

We have the isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} M_{(t\cdot u\cdot v)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s}, M) \\ m & \mapsto & (d_m : a^k \mapsto m\cdot (\sum_{i=0}^{k-1} a^i)) \\ (a)d & \leftarrow\!\shortmid & d\;. \end{array}$$

Substituting isomorphically, $\vartheta_{\mathrm{C}_{s\cdot t\cdot u\cdot v},\mathrm{C}_{s\cdot t},\mathrm{C}_s,M}$ becomes the inclusion map of $M_{(u\cdot v)}$ into $M_{(t\cdot u\cdot v)}$.

We have $\mathrm{C}_{s\cdot t} = \langle a^{u\cdot v} \rangle$. For $m \in M'_{(s\cdot t\cdot u)}$, we have

$$((a^v)^u)d'_m \;=\; m\cdot \left(\sum_{i=0}^{u-1} (a^v)^i \right) .$$

We have $\mathrm{C}_{s} = \langle a^{t\cdot u\cdot v} \rangle$. For $m \in M'_{(s\cdot t\cdot u)}$, we have

$$((a^v)^{t\cdot u})d'_m \;=\; m\cdot \left(\sum_{i=0}^{t\cdot u-1} (a^v)^i \right) .$$

Let

$$\begin{array}{rcl} M'_{(u)} & := & \left\{ m \in M \;\middle|\; m \cdot \left(\sum_{i=0}^{u-1} (a^v)^i\right) = 0 \right\} \\ M'_{(t\cdot u)} & := & \left\{ m \in M \;\middle|\; m \cdot \left(\sum_{i=0}^{t\cdot u-1} (a^v)^i\right) = 0 \right\} . \end{array}$$

Then

$$M'_{(u)} \;\subseteq\; M'_{(t\cdot u)} \;\subseteq\; M'_{(s\cdot t\cdot u)} \;.$$

We have the isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} M'_{(u)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s\cdot t}, M') \\ m & \mapsto & (d'_m : (a^v)^k \mapsto m \cdot (\sum_{i=0}^{k-1} (a^v)^i)) \\ (a^v)d' & \leftarrow\!\shortmid & d' \;. \end{array}$$

We have the isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} M'_{(t\cdot u)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s}, M') \\ m & \mapsto & (d'_m : (a^v)^k \mapsto m \cdot (\sum_{i=0}^{k-1} (a^v)^i)) \\ (a^v)d' & \leftarrow\!\shortmid & d' \;. \end{array}$$

Substituting isomorphically, $\vartheta_{\mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s\cdot t}, \mathrm{C}_s, M'}$ becomes the inclusion map of $M'_{(u)}$ into $M'_{(t\cdot u)}$.

Now we substitute the vertical maps isomorphically.

We obtain

$$\begin{array}{ccccccc} M_{(u\cdot v)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t}, M) & \xrightarrow{\eta_{\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s\cdot t}, M}} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s\cdot t}, M') & \xrightarrow{\sim} & M'_{(u)} \\ m & \mapsto & d_m & \mapsto & d_m|_{\mathrm{C}_{s\cdot t\cdot u}} & \mapsto & (a^v)d_m \;=\; m \cdot \sum_{i=0}^{v-1} a^i \;. \end{array}$$

We obtain

$$\begin{array}{ccccccc} M_{(t\cdot u\cdot v)} & \xrightarrow{\sim} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s}, M) & \xrightarrow{\eta_{\mathrm{C}_{s\cdot t\cdot u\cdot v}, \mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s}, M}} & \mathrm{Z}^1(\mathrm{C}_{s\cdot t\cdot u}, \mathrm{C}_{s}, M') & \xrightarrow{\sim} & M'_{(t\cdot u)} \\ m & \mapsto & d_m & \mapsto & d_m|_{\mathrm{C}_{s\cdot t\cdot u}} & \mapsto & (a^v)d_m \;=\; m \cdot \sum_{i=0}^{v-1} a^i \;. \end{array}$$

Altogether, our pullback gets isomorphically substituted to the following pullback of $\mathbb{Z}$-modules and $\mathbb{Z}$-linear maps.

$$\begin{tikzcd}
& m & m \arrow[l, mapsto] & \\
m \cdot \left(\sum_{i=0}^{v-1} a^i\right) & M'_{(t\cdot u)} & M'_{(u)} \arrow[l, hook'] & m \cdot \left(\sum_{i=0}^{v-1} a^i\right) \\
m \arrow[u, mapsto] & M_{(t\cdot u\cdot v)} \arrow[u] & M_{(u\cdot v)} \arrow[l, hook'] \arrow[u] \arrow[ul, phantom, "\lrcorner"] & m \arrow[u, mapsto] \\
& m & m \arrow[l, mapsto] &
\end{tikzcd}$$

We verify directly that this commutative quadrangle is a pullback.

Suppose given $m \in M_{(t\cdot u\cdot v)}$ such that $m \cdot \left(\sum_{i=0}^{v-1} a^i\right) \in M'_{(u)}$. We have to show that $m \overset{!}{\in} M_{(u\cdot v)}$.

In fact, we obtain

$$0 \;=\; \left(m \cdot \left(\sum_{i=0}^{v-1} a^i\right)\right) \cdot \left(\sum_{j=0}^{u-1} (a^v)^j\right) \;=\; m \cdot \left(\sum_{i=0}^{u\cdot v-1} a^i\right) .$$

So $m \in M_{(u\cdot v)}$.

7.2 1-cocycles of S_3

Let $R = \mathbb{Z}$ be our ground ring.

To calculate the following example, we make use of Magma [**6**].

In order to do so, Magma uses the first cohomology group as follows.

Let G be a group. Let M be a $\mathbb{Z}G$-module.

For $m \in M$, we have the 1-cocycle

$$\begin{array}{rcrcl} d^m & : & G & \to & M \\ & & g & \mapsto & m \cdot g - m \,. \end{array}$$

We define the $\mathbb{Z}$-submodule

$$\mathrm{B}^1(G, M) \;:=\; \{\, d^m \mid m \in M \,\} \;\subseteq\; \mathrm{Z}^1(G, M) .$$

Then the first cohomology group of G with coefficients in M is defined by

$$\mathrm{H}^1(G, M) \;:=\; \mathrm{Z}^1(G, M) / \, \mathrm{B}^1(G, M) ,$$

Magma calculates a set C of representing 1-cocycles of the elements of $\mathrm{H}^1(G, M)$.

So

$$\mathrm{Z}^1(G, M) \;=\; \{\, c + d^m \mid c \in C,\; m \in M \,\} .$$

Example 122 We calculate $\mathrm{Z}^1(\mathrm{S}_3, \mathbb{F}_2)$, where $\mathbb{F}_2$ is the $\mathbb{Z}\mathrm{S}_3$-module with trivial S_3-action.

Note that $\mathrm{B}^1(\mathrm{S}_3, \mathbb{F}_2) = 0$.

Magma calculates $\mathrm{H}^1(G, M) = \mathrm{Z}^1(G, M)$ as follows.

```
G := SymmetricGroup(3);
M := PermutationModule(G, G, GF(2));
CM := CohomologyModule(G, M);
H1 := CohomologyGroup(CM, 1);
```

Then `H1` $= \mathrm{H}^1(\mathrm{S}_3, \mathbb{F}_2)$ has order 2:

```
> H1;
Full Vector space of degree 1 over GF(2)
```

So we have the following nonzero cocycle

```
c := [x : x in H1][2];
cc := OneCocycle(CM, c);
cc_values := [<x,<x>@cc>  : x in G];
```

It has the following values.

```
> cc_values;
[
    <Id(G), (0)>,
    <(1, 2, 3), (0)>,
    <(1, 3, 2), (0)>,
    <(2, 3), (1)>,
    <(1, 2), (1)>,
    <(1, 3), (1)>
]
```

So the cocycle $c \in \mathrm{Z}^1(\mathrm{S}_3, \mathbb{F}_2)$ is the map $c : \mathrm{S}_3 \to \mathbb{F}_2$ that sends every element of $\mathrm{A}_3 = \langle (1, 2, 3) \rangle$ to 0 and every element of $\mathrm{S}_3 \smallsetminus \mathrm{A}_3$ to 1. We have

$$\mathrm{Z}^1(\mathrm{S}_3, \mathbb{F}_2) \;=\; \{\, 0,\, c \,\} \,.$$

It corresponds to the signum morphism $\mathrm{S}_3 \to \{-1, +1\}$.

Note that $c \in \mathrm{Z}^1(\mathrm{S}_3, \mathrm{A}_3, \mathbb{F}_2)$, i.e. that c is a 1-cocycle of S_3 relative to A_3 with values in $\mathbb{F}_2$. In fact, c vanishes on A_3 ; cf. Remark 76.(1). So

$$\mathrm{Z}^1(\mathrm{S}_3, \mathrm{A}_3, \mathbb{F}_2) \;=\; \{\, 0,\, c \,\} \,.$$

Example 123 We calculate $\mathrm{Z}^1(\mathrm{S}_3, M)$, where $M := \mathbb{Z}^{1\times 2}$ is the $\mathbb{Z}\,\mathrm{S}_3$-module on which $(1,2,3)$ acts via $\left(\begin{smallmatrix}-1 & 1\\ -1 & 0\end{smallmatrix}\right)$ and on which $(1,2)$ acts via $\left(\begin{smallmatrix}-1 & 0\\ -1 & 1\end{smallmatrix}\right)$.

```
G := SymmetricGroup(3);
g1 := Matrix([[-1,1],[-1,0]]);
g2 := Matrix([[-1,0],[-1,1]]);
M := GModule(G,[g1,g2]);
CM := CohomologyModule(G, M);
H1 := CohomologyGroup(CM, 1);
c := [x : x in H1][2];
cc := OneCocycle(CM, c);
cc_values := [<x,M!(<x>@cc)>  : x in G];
```

Magma in fact chooses `G.1` $= (1,2,3)$ and `G.2` $= (1,2)$.

This gives the following 1-cocycle $c :=$ `cc_values`.

```
[
    <Id(G), M: (0 0)>,
    <(1, 2, 3), M: (-1  0)>,
    <(1, 3, 2), M: ( 0 -1)>,
    <(1, 2), M: (-1  0)>,
    <(1, 3), M: ( 0 -1)>,
    <(2, 3), M: (0 0)>
]
```

Magma gives the set of representing elements $C := \{0 \cdot c,\ 1 \cdot c,\ (-1) \cdot c\}$.

Moreover, we get the following generators of $\mathrm{B}^1(G, M)$.

```
m1 := M!Vector([1,0]);
dm1_values := [<x,m1*x - m1> : x in G];
dm1_values;
[
    <Id(G), M: (0 0)>,
    <(1, 2, 3), M: (-2  1)>,
    <(1, 3, 2), M: (-1 -1)>,
    <(1, 2), M: (-2  0)>,
    <(1, 3), M: ( 0 -1)>,
    <(2, 3), M: (-1  1)>
]
m2 := M!Vector([0,1]);
dm2_values := [<x,m2*x - m2> : x in G];
dm2_values;
```

```
[
    <Id(G), M: (0 0)>,
    <(1, 2, 3), M: (-1 -1)>,
    <(1, 3, 2), M: ( 1 -2)>,
    <(1, 2), M: (-1  0)>,
    <(1, 3), M: ( 0 -2)>,
    <(2, 3), M: ( 1 -1)>
]
```

So we have obtained c, d^{m_1}, $d^{m_2} \in \mathrm{Z}^1(G, M)$, where

$$\begin{array}{rcl} G & \to & M \\[1ex] (1,2,3) & \overset{c}{\mapsto} & \begin{pmatrix} -1 & 0 \end{pmatrix} \\ (1,2) & \overset{c}{\mapsto} & \begin{pmatrix} -1 & 0 \end{pmatrix} \\[1ex] (1,2,3) & \overset{d^{m_1}}{\longmapsto} & \begin{pmatrix} -2 & 1 \end{pmatrix} \\ (1,2) & \overset{d^{m_1}}{\longmapsto} & \begin{pmatrix} -2 & 0 \end{pmatrix} \\[1ex] (1,2,3) & \overset{d^{m_2}}{\longmapsto} & \begin{pmatrix} -1 & -1 \end{pmatrix} \\ (1,2) & \overset{d^{m_2}}{\longmapsto} & \begin{pmatrix} -1 & 0 \end{pmatrix} . \end{array}$$

Note that

$$3c \;=\; d^{m_1} + d^{m_2} \,.$$

Moreover, $\mathrm{B}^1(G, M) = {}_{\mathbb{Z}}\langle\, d^{m_1},\, d^{m_2}\,\rangle$. Altogether, we obtain

$$\mathrm{Z}^1(G, M) \;=\; \{\, uc + v_1 d^{m_1} + v_2 d^{m_2} \mid u \in \{-1, 0, 1\},\; v_1,\, v_2 \,\in\, \mathbb{Z} \,\} \,.$$

Let $H := \langle (2,3) \rangle \leqslant G$. We want to calculate $\mathrm{Z}^1(G, H, M)$, using Remark 76.(1).

Note that c maps $(2,3)$ to $\begin{pmatrix} 0 & 0 \end{pmatrix}$, whence $c \in \mathrm{Z}^1(G, H, M)$.

Moreover, $v_1 d^{m_1} + v_2 d^{m_2}$ maps $(2,3)$ to $v_1 \begin{pmatrix} -1 & 1 \end{pmatrix} + v_2 \begin{pmatrix} 1 & -1 \end{pmatrix}$, which is $\begin{pmatrix} 0 & 0 \end{pmatrix}$ if and only if $v_1 = v_2$. Hence

$$\begin{array}{rcl} \mathrm{Z}^1(G, H, M) & = & \{\, uc + v_1(d^{m_1} + d^{m_2}) \mid u \in \{-1, 0, 1\},\; v_1 \in \mathbb{Z} \,\} \\ & = & \{\, uc + v_1 \cdot 3c \mid u \in \{-1, 0, 1\},\; v_1 \in \mathbb{Z} \,\} \\ & = & \{\, uc \mid u \in \mathbb{Z} \,\} \,. \end{array}$$

Let $\tilde{H} := \langle (1,2) \rangle \leqslant G$. We want to calculate $\mathrm{Z}^1(G, \tilde{H}, M)$, using Remark 76.(1).

A 1-cocycle of the form described above vanishes on $(1,2)$ if and only if

$$\begin{pmatrix} 0 & 0 \end{pmatrix} \;=\; (1,2)(uc + v_1 d^{m_1} + v_2 d^{m_2}) \;=\; u \begin{pmatrix} -1 & 0 \end{pmatrix} + v_1 \begin{pmatrix} -2 & 0 \end{pmatrix} + v_2 \begin{pmatrix} -1 & 0 \end{pmatrix} \;,$$

i.e. if $v_2 = -u - 2v_1$.

Note that $3(c - d^{m_2}) = d^{m_1} + d^{m_2} - 3d^{m_2} = d^{m_1} - 2d^{m_2}$.

We obtain

$$
\begin{array}{rcl}
\mathrm{Z}^1(G,\tilde{H},M) & = & \{\, uc + v_1 d^{m_1} + (-u-2v_1)d^{m_2} \,|\, u\in\{-1,0,1\},\ v_1\in\mathbb{Z}\,\} \\
& = & \{\, u(c-d^{m_2}) + v_1(d^{m_1}-2d^{m_2}) \,|\, u\in\{-1,0,1\},\ v_1\in\mathbb{Z}\,\} \\
& = & \{\, u(c-d^{m_2}) + v_1\cdot 3(c-d^{m_2}) \,|\, u\in\{-1,0,1\},\ v_1\in\mathbb{Z}\,\} \\
& = & \{\, u(c-d^{m_2}) \,|\, u\in\mathbb{Z}\,\}\ .
\end{array}
$$

7.3 The short conormal-sequence for S_3

Let $R=\mathbb{Z}$ be our ground ring.

Let $H:=\mathrm{S}_3$. Let $N:=\mathrm{C}_3=\langle(1,2,3)\rangle\trianglelefteq H$. Let $G:=H/N$. Let $\varphi:H\to H/N$ be the residue class morphism. Then $N=\mathrm{Ker}(\varphi)$. Cf. §6.3.1, §6.3.2.

In this situation, we shall calculate the short conormal-sequence; cf. Proposition 121.(2).

We write $a:=(1,2,3)$ and $b:=(2,3)$. Then $\mathrm{S}_3=\{1,a,a^2,b,ab,a^2b\}$.

We write $c:=bN\in H/N=G$.

We have the $\mathbb{Z}$-linear basis

$$(a-1,\ a^2-1,\ b-1,\ ab-1,\ a^2b-1)$$

of $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)$; cf. Remark 27.

We have $\mathrm{I}_{\mathbb{Z}}^{(1)}(N) := {}_{\mathbb{Z}}\langle a-1,\ a^2-1\rangle$. Hence the $\mathbb{Z}$-submodule $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)$ of $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)$ is $\mathbb{Z}$-linearly generated by the following list of elements.

$$
\begin{array}{rcl}
(a-1)(a-1) & = & -2(a-1)+(a^2-1) \\
(a^2-1)(a-1) & = & -(a-1)-(a^2-1) \\
(b-1)(a-1) & = & -(a-1)-(b-1)+(a^2b-1) \\
(ab-1)(a-1) & = & -(a-1)+(b-1)-(ab-1) \\
(a^2b-1)(a-1) & = & -(a-1)+(ab-1)-(a^2b-1) \\
(a-1)(a^2-1) & & \text{redundant} \\
(a^2-1)(a^2-1) & = & (a-1)-2(a^2-1) \\
(b-1)(a^2-1) & = & -(a^2-1)-(b-1)+(ab-1) \\
(ab-1)(a^2-1) & = & -(a^2-1)-(ab-1)+(a^2b-1) \\
(a^2b-1)(a^2-1) & = & -(a^2-1)+(b-1)-(a^2b-1)
\end{array}
$$

So in the chosen basis of $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)$, the $\mathbb{Z}$-submodule $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)$ of $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)$ is $\mathbb{Z}$-linearly generated by the rows of the following matrix.

$a-1$	a^2-1	$b-1$	$ab-1$	a^2b-1
-2	1	0	0	0
-1	-1	0	0	0
-1	0	-1	0	1
-1	0	1	-1	0
-1	0	0	1	-1
1	-2	0	0	0
0	-1	-1	1	0
0	-1	0	-1	1
0	-1	1	0	-1

We use MAGMA [6] for elementary divisor calculations. We obtain the following.

```
IHIN :=
Matrix([
[-2, 1, 0, 0, 0],
[-1,-1, 0, 0, 0],
[-1, 0,-1, 0, 1],
[-1, 0, 1,-1, 0],
[-1, 0, 0, 1,-1],
[ 1,-2, 0, 0, 0],
[ 0,-1,-1, 1, 0],
[ 0,-1, 0,-1, 1],
[ 0,-1, 1, 0,-1]
]);
D,S,T := SmithForm(IHIN);

D;
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 3 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
```

```
S;
[ 1  0  0  0  0  0  0  0  0]
[ 0  0  0  1  0  0  0  0  0]
[ 0  0  1  0  0  0  0  0  0]
[ 1  0  0  3  0 -1  0  0  0]
[ 1 -1  0  0  0  1  0  0  0]
[ 0 -1  0  1  0  0  1  0  0]
[-1  0  0  1  1  0  0  0 -1]
[ 0 -1  0  0  1  0  0  1  0]
[ 0 -1  1  0  0  0  0  0  1]
```

```
T;
[-1 -1  0  1  0]
[-1 -2  0  2  0]
[ 0  0  0  0  1]
[ 1  0  0 -1  1]
[-1 -1  1  1  1]
```

So

$$\mathtt{S}\cdot\mathtt{IHIN}\cdot\mathtt{T} \;=\; \mathtt{D}\,.$$

Side remark. Note that $(a-1)(b-1) \in \mathrm{I}_{\mathbb{Z}}^{(1)}(N)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)$, which gives

$$(a-1)(b-1) \;=\; -(a-1)-(b-1)+(ab-1)\;,$$

corresponds to the row $(-1\;\;0\;-1\;\;1\;\;0)$. Adding this row to the matrix `IHIN` above, we get the following.

```
SmithForm(Matrix([
[-2, 1, 0, 0, 0],
[-1,-1, 0, 0, 0],
[-1, 0,-1, 0, 1],
[-1, 0, 1,-1, 0],
[-1, 0, 0, 1,-1],
[ 1,-2, 0, 0, 0],
[ 0,-1,-1, 1, 0],
[ 0,-1, 0,-1, 1],
[ 0,-1, 1, 0,-1],
[-1, 0,-1, 1, 0]
]));
```

```
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
```

Since the elementary divisor 3 does not appear here, the added row is not contained in the $\mathbb{Z}$-linear span of the other rows. Hence

$$\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \;\neq\; \mathrm{I}_{\mathbb{Z}}^{(1)}(N)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\;.$$

This concludes the *side remark*.

Note that $N^{(1)} = 1$ since N is abelian.

So we have the following short conormal-sequence; cf. Proposition 121.(2).

$$\begin{array}{ccccc}
N & \longrightarrow & \mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \longrightarrow & \mathrm{I}_{\mathbb{Z}}^{(1)}(G) \\
a & \longmapsto & a - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \longmapsto & a\varphi - 1 \;=\; 0 \\
 & & b - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \longmapsto & b\varphi - 1 \;=\; c - 1
\end{array}$$

We shall verify short exactness directly, by an isomorphic substitution.

We have the $\mathbb{Z}$-linear isomorphism

$$\begin{array}{rcl}
N & \overset{t}{\underset{\sim}{\to}} & \mathbb{Z}/(3) \\
c & \mapsto & 1 + (3)\;.
\end{array}$$

We denote the $\mathbb{Z}$-linear row span of `IHIN` in $\mathbb{Z}^{1\times 5}$ by R_1.

By our usage of the chosen $\mathbb{Z}$-linear basis of $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)$, we have the following $\mathbb{Z}$-linear isomorphism.

$$\begin{array}{rcl}
\mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) & \overset{u_1}{\underset{\sim}{\to}} & \mathbb{Z}^{1\times 5}/R_1 \\
a - 1 & \mapsto & (1\;0\;0\;0\;0) + R_1 \\
b - 1 & \mapsto & (0\;0\;1\;0\;0) + R_1
\end{array}$$

We denote the $\mathbb{Z}$-linear row span of `D` in $\mathbb{Z}^{1\times 5}$ by R_2.

By our elementary divisor calculation, we have the following commutative diagram with right exact rows.

$$\begin{tikzcd}
\mathbb{Z}^{1\times 9} \arrow[r, "(-)\cdot\texttt{IHIN}"] & \mathbb{Z}^{1\times 5} \arrow[r, "\text{residue class map}"] \arrow[d, "(-)\cdot\texttt{T}", "\wr"'] & \mathbb{Z}^{1\times 5}/R_1 \arrow[d, "u_2", "\wr"'] \\
\mathbb{Z}^{1\times 9} \arrow[u, "(-)\cdot\texttt{S}"', "\wr"] \arrow[r, "(-)\cdot\texttt{D}"] & \mathbb{Z}^{1\times 5} \arrow[r, "\text{residue class map}"] & \mathbb{Z}^{1\times 5}/R_2
\end{tikzcd}$$

In particular, we have

$$\begin{array}{rcl}
\mathbb{Z}^{1\times 5}/R_1 & \overset{u_2}{\underset{\sim}{\to}} & \mathbb{Z}^{1\times 5}/R_2 \\
(1\ 0\ 0\ 0\ 0) + R_1 & \mapsto & (-1\ -1\ \ 0\ \ 1\ \ 0) + R_2 \\
(0\ 0\ 1\ 0\ 0) + R_1 & \mapsto & (0\ \ 0\ \ 0\ \ 0\ \ 1) + R_2
\end{array}$$

Moreover, we have the following $\mathbb{Z}$-linear isomorphism.

$$\begin{array}{rcl}
\mathbb{Z}^{1\times 5}/R_2 & \overset{u_3}{\underset{\sim}{\to}} & \mathbb{Z}/(3) \oplus \mathbb{Z} \\
(x_1\ x_2\ x_3\ x_4\ x_5) + R_2 & \mapsto & (x_4 + (3), x_5)
\end{array}$$

By Remark 27, we have the $\mathbb{Z}$-linear isomorphism

$$\begin{array}{rcl}
\mathrm{I}_{\mathbb{Z}}^{(1)}(G) & \overset{v}{\underset{\sim}{\to}} & \mathbb{Z} \\
c - 1 & \mapsto & 1\,.
\end{array}$$

Altogether, we obtain the following commutative diagram by isomorphic substitution, in which the upper row is the short conormal-sequence.

$$\begin{tikzcd}
N \arrow[r] \arrow[ddd, "t", "\wr"'] & \mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[r] \arrow[d, "u_1", "\wr"'] & \mathrm{I}_{\mathbb{Z}}^{(1)}(G) \arrow[ddd, "v", "\wr"'] \\
& \mathbb{Z}^{1\times 5}/R_1 \arrow[d, "u_2", "\wr"'] & \\
& \mathbb{Z}^{1\times 5}/R_2 \arrow[d, "u_3", "\wr"'] & \\
\mathbb{Z}/(3) \arrow[r, "(1\,0)"] & \mathbb{Z}/(3) \oplus \mathbb{Z} \arrow[r, "\binom{0}{1}"] & \mathbb{Z}
\end{tikzcd}$$

The left side commutes, since

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[ddd, mapsto, "t"] & a - 1 + \mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[d, mapsto, "u_1"] \\
& (1\ 0\ 0\ 0\ 0) + R_1 \arrow[d, mapsto, "u_2"] \\
& (-1\ -1\ \ 0\ \ 1\ \ 0) + R_2 \arrow[d, mapsto, "u_3"] \\
1 + (3) \arrow[r, mapsto, "(1\,0)"] & (1 + (3), 0)\ .
\end{tikzcd}$$

The right side of the diagram commutes, since

$$\begin{tikzcd}
a-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[r, mapsto] \arrow[d, mapsto, "u_1"] & 0 \arrow[ddd, "v"] \\
(1\ 0\ 0\ 0\ 0)+R_1 \arrow[d, mapsto, "u_2"] & \\
(-1\ {-1}\ \ 0\ \ 1\ \ 0)+R_2 \arrow[d, mapsto, "u_3"] & \\
(1+(3),0) \arrow[r, "\binom{0}{1}"] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
b-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \arrow[r, mapsto] \arrow[d, mapsto, "u_1"] & c-1 \arrow[ddd, "v"] \\
(0\ 0\ 1\ 0\ 0)+R_1 \arrow[d, mapsto, "u_2"] & \\
(0\ \ 0\ \ 0\ \ 0\ \ 1)+R_2 \arrow[d, mapsto, "u_3"] & \\
(0+(3),1) \arrow[r, "\binom{0}{1}"] & 1\,.
\end{tikzcd}$$

Note that $a-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)$ and $b-1+\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)$ do $\mathbb{Z}$-linearly generate $\mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N)$, as their images under $u_1\cdot u_2\cdot u_3$ generate in $\mathbb{Z}/(3)\oplus\mathbb{Z}$.

We see that the lower row of our diagram is short exact, as was to be verified.

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Zusammenfassung

Sei R ein kommutativer Ring.

Cozykel

Seien gegeben eine Gruppe G, eine Untergruppe $H \leqslant G$ und ein RG-Modul M. Wir haben die Standard-Auflösung

$$\ldots \;\to\; R(G^{\times 3}) \;\to\; R(G^{\times 2}) \;\to\; R(G^{\times 1}) \;\to\; R\;.$$

Für $n \geqslant 1$ definieren wir den n-ten Augmentationsmodul als

$$\mathrm{I}_R^{(n)}(G) \;:=\; \mathrm{Im}\big(R(G^{\times(n+1)}) \to R(G^{\times n})\big) \;=\; \mathrm{Ker}\big(R(G^{\times n}) \to R(G^{\times(n-1)})\big) \;\subseteq\; R(G^{\times n})\;.$$

Wir geben eine R-lineare Basis von $\mathrm{I}_R^{(n)}(G)$ an; vgl. Proposition 29.

Für $n \geqslant 0$ ist eine Abbildung $d : G^{\times n} \to M$ ein n-Cozykel mit Werten in M, falls

$$0 \;=\; (g_1, \ldots, g_n)d + \left(\sum_{k=1}^{n} (-1)^k (g_0, \ldots, g_{k-2},\, g_{k-1} \cdot g_k\,, g_{k+1}, \ldots, g_n)d\right) + (-1)^{n+1}(g_0, \ldots, g_{n-1})d \cdot g_n$$

für $g_0,\, \ldots,\, g_n \,\in\, G$ gilt.

Wir schreiben die Menge dieser n-Cozykeln als $\mathrm{Z}^n(G, M) \;\subseteq\; {}_{\mathrm{Map}}(G^{\times n}, M)$.

Wir konstruieren einen n-Cozykel $\xi_n : G^{\times n} \to \mathrm{I}_R^{(n)}(G)$, der folgende universelle Eigenschaft erfüllt; vgl. Lemma 48, Proposition 49.

$$\begin{tikzcd}
G^{\times n} \arrow[rr, "{\substack{d \,\in\, \mathrm{Z}^n(G,M) \\ n\text{-Cozykel}}}"] \arrow[d, "{\substack{\xi_n \,\in\, \mathrm{Z}^n(G,\mathrm{I}_R^{(n)}(G)) \\ \text{universeller } n\text{-Cozykel}}}"'] & & M \\
\mathrm{I}_R^{(n)}(G) \arrow[urr, dotted, "{\exists!\ \tilde{d}\quad RG\text{-linear}}"'] & &
\end{tikzcd}$$

Relative Cozykel

Für die Einführung von relativen n-Cozykeln schreiben wir die Menge der Rechtsnebenklassen von H in G als $\check{G} \;:=\; H \setminus G \;=\; \{\, Hg \mid g \in G \,\}$ und erhalten die zugehörige Standard-Auflösung

$$\ldots \;\to\; R(\check{G}^{\times 3}) \;\to\; R(\check{G}^{\times 2}) \;\to\; R(\check{G}^{\times 1}) \;\to\; R\;.$$

Für $n \geqslant 0$ ist eine Abbildung $d : G^{\times n} \to M$ ein n-Cozykel relativ zu H mit Werten in M, falls zusätzlich zur obengenannten n-Cozykelbedingung auch noch

$$\begin{aligned} & (\, h_0 g_0\,,\, h_1 g_1\,,\, h_2 g_2\,,\, \ldots\,,\, h_{n-2} g_{n-2}\,,\, h_{n-1} g_{n-1}\,)f \cdot h_n \\ =\;\; & (\, g_0 h_1\,,\, g_1 h_2\,,\, g_2 h_3\,,\, \ldots\,,\, g_{n-2} h_{n-1}\,,\, g_{n-1} h_n\,)f \end{aligned}$$

gilt für $g_0, \ldots, g_{n-1} \in G$ und $h_0, \ldots, h_n \in H$. Vgl. Remark 74, Definition 75.

Wir schreiben die Menge dieser relativen n-Cozykeln als $\mathrm{Z}^n(G, H, M) \;\subseteq\; \mathrm{Z}^n(G, M)$.

Wir konstruieren einen relativen n-Cozykel $\check{\xi}_n : G^{\times n} \to \mathrm{I}_R^{(n)}(G,H)$, der folgende universelle Eigenschaft erfüllt; vgl. Proposition 87.

$$\begin{tikzcd} G^{\times n} \arrow[rr, "{\substack{d \in \mathrm{Z}^n(G,H,M)\\ \text{relativer } n\text{-Cozykel}}}"] \arrow[d, "{\substack{\check{\xi}_n \in \mathrm{Z}^n(G,H,\mathrm{I}_R^{(n)}(G,H))\\ \text{universeller relativer } n\text{-Cozykel}}}"'] & & M \\ \mathrm{I}_R^{(n)}(G,H) \arrow[urr, dotted, "\exists!\ \tilde{d}\ \ RG\text{-linear}"'] & & \end{tikzcd}$$

Cotangentialsequenz

Sei G eine Gruppe. Seien $L \leqslant K \leqslant H \leqslant G$ Untergruppen.

In dieser Situation leiten wir das Cotangential-Quadrat

$$\begin{tikzcd} \mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & \mathrm{I}_R^{(1)}(H,K) \underset{RH}{\otimes} RG \arrow[d] \\ \mathrm{I}_R^{(1)}(G,L) \arrow[r] & \mathrm{I}_R^{(1)}(G,K) \end{tikzcd}$$

her, welches also eine kurz exakte Diagonalsequenz besitzt; vgl. Proposition 94.(2).

Im Spezialfall $H = K$ wird dies zur kurz exakten Cotangentialsequenz

$$\mathrm{I}_R^{(1)}(H,L) \underset{RH}{\otimes} RG \;\to\; \mathrm{I}_R^{(1)}(G,L) \;\to\; \mathrm{I}_R^{(1)}(G,H)\;;$$

vgl. Corollary 95.

Conormalsequenz

Sei ein surjektiver Gruppenmorphismus $\varphi : H \to G$ gegeben. Wir schreiben $N := \mathrm{Ker}(\varphi) \trianglelefteqslant H$.

Seien Untergruppen $L \leqslant H$ und $K \leqslant G$ gegeben mit $L\varphi \leqslant K$.

$$\begin{tikzcd} N \arrow[r, hook] & H \arrow[r, "\varphi"] & G \\ & L \arrow[u, hook] \arrow[r, "\varphi|_L^K"'] & K \arrow[u, hook] \end{tikzcd}$$

Wir definieren ausgehend von φ den RG-Modul $\mathrm{I}_R^{(1)}(\varphi,L,K)$.

Wir erhalten die rechtsexakte Conormalsequenz

$$\mathrm{I}_R^{(1)}(\varphi,L,K)/\,\mathrm{I}_R^{(1)}(\varphi,L,K)\,\mathrm{I}_R^{(1)}(N) \;\to\; \mathrm{I}_R^{(1)}(H,L)/\,\mathrm{I}_R^{(1)}(H,L)\,\mathrm{I}_R^{(1)}(N) \;\to\; \mathrm{I}_R^{(1)}(G,K)\;;$$

vgl. Proposition 108.

Im Spezialfall $R = \mathbb{Z}$ und $K = L = 1$ folgern wir hieraus die kurz exakte Conormalsequenz

$$N/N^{(1)} \;\to\; \mathrm{I}_{\mathbb{Z}}^{(1)}(H)/\,\mathrm{I}_{\mathbb{Z}}^{(1)}(H)\,\mathrm{I}_{\mathbb{Z}}^{(1)}(N) \;\to\; \mathrm{I}_{\mathbb{Z}}^{(1)}(G)\;;$$

vgl. Proposition 121.

Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Masterarbeit selbstständig verfasst und keine anderen als die angegebenen Quellen benutzt habe. Alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen habe ich als solche gekennzeichnet. Meine eingereichte Arbeit ist weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen. Das elektronische Exemplar stimmt mit den gedruckten Exemplaren überein.

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