The resolution equivalence for n-complexes

$$
\begin{tikzcd}
 & & & & & 0 \arrow[r, dotted] & {} & \\
 & & & & 0 \arrow[r] & X_{0^{+1}/3} \arrow[r] \arrow[u] & X_{1^{+1}/3} \arrow[r, dotted] \arrow[u, dotted] & {} \\
 & & & 0 \arrow[r] & X_{3/2} \arrow[r] \arrow[u] & X_{0^{+1}/2} \arrow[r] \arrow[u] & X_{1^{+1}/2} \arrow[r] \arrow[u] & 0 \arrow[u, dotted] \\
 & & 0 \arrow[r] & X_{2/1} \arrow[r] \arrow[u] & X_{3/1} \arrow[r] \arrow[u] & X_{0^{+1}/1} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] & X_{2/0} \arrow[r] \arrow[u] & X_{3/0} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
0 \arrow[r] & X_{0/3^{-1}} \arrow[r] \arrow[u] & X_{1/3^{-1}} \arrow[r] \arrow[u] & X_{2/3^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & & \\
{} \arrow[r, dotted] \arrow[u, dotted] & X_{0/2^{-1}} \arrow[r] \arrow[u] & X_{1/2^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & & & \\
 & {} \arrow[r, dotted] \arrow[u, dotted] & 0 \arrow[u] & & & & &
\end{tikzcd}
$$

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Contents

0 Introduction

0.1 Complexes and injective resolutions

Suppose given an abelian category $\mathcal{A}$ with enough injective objects. Suppose given an abelian category $\mathcal{B}$ and an additive functor

$$\mathcal{A} \xrightarrow{F} \mathcal{B}.$$

For $k \geqslant 0$, we get the k-th right derived functor $\mathcal{A} \xrightarrow{\mathrm{R}^k F} \mathcal{B}$ of F as follows.

In the category $(\mathbb{Z}, \mathcal{A})$ of functors from the poset category $\mathbb{Z}$ to $\mathcal{A}$, we have the full subcategory $\mathrm{C}(\mathcal{A})$ of complexes, that is, diagrams of the form

$$\cdots\cdots\to A^{-2} \xrightarrow{a^{-2}} A^{-1} \xrightarrow{a^{-1}} A^0 \xrightarrow{a^0} A^1 \xrightarrow{a^1} A^2 \cdots\cdots\to ,$$

where $a^i a^{i+1} = 0$ for every $i \in \mathbb{Z}$. We also have the full subcategory $\mathrm{C}^{(\text{sp ac})}(\mathcal{A})$ of split acyclic complexes, which are direct sums of complexes of the form

$$\begin{array}{l}
\ldots \xrightarrow{1} A^{-1} \longrightarrow 0 \longrightarrow A^2 \xrightarrow{1} A^2 \longrightarrow 0 \longrightarrow \ldots \\
\ldots \longrightarrow A^0 \xrightarrow{1} A^0 \longrightarrow 0 \longrightarrow A^3 \xrightarrow{1} A^3 \longrightarrow \ldots \\
\ldots \longrightarrow 0 \longrightarrow A^1 \xrightarrow{1} A^1 \longrightarrow 0 \longrightarrow A^4 \xrightarrow{1} \ldots
\end{array}$$

The factor category $\mathrm{C}(\mathcal{A})/\,\mathrm{C}^{(\text{sp ac})}(\mathcal{A}) =: \mathrm{K}(\mathcal{A})$ is the homotopy category of complexes.

We define the full subcategory $\mathrm{C}^{(\text{ires})}(\mathcal{A}) \subseteq \mathrm{C}(\mathcal{A})$ to consist of all complexes

$$\cdots\cdots\to A^{-2} \xrightarrow{a^{-2}} A^{-1} \xrightarrow{a^{-1}} A^0 \xrightarrow{a^0} A^1 \xrightarrow{a^1} A^2 \cdots\cdots\to$$

with A^i injective for $i \in \mathbb{Z}$, with $A^i \cong 0_{\mathcal{A}}$ for $i < 0$ and exact at position i for $i \geqslant 1$.

We have the functor

$$\begin{array}{c}
\mathrm{C}^{(\text{ires})}(\mathcal{A}) \xrightarrow{\mathrm{H}^0} \mathcal{A} \\
(\ldots \longrightarrow 0 \longrightarrow I^0 \xrightarrow{i^0} I^1 \xrightarrow{i^1} \ldots) \longmapsto I\,\mathrm{H}^0 = \mathrm{Ker}(I^0 \xrightarrow{i^0} I^1)
\end{array}$$

Define $\mathrm{K}^{(\text{ires})}(\mathcal{A}) := \mathrm{C}^{(\text{ires})}(\mathcal{A})/\,\mathrm{Ker}(\mathrm{H}^0)$. Then the induced functor

$$\mathrm{H}^0 \colon \mathrm{K}^{(\text{ires})}(\mathcal{A}) \to \mathcal{A}$$

is an equivalence. So we may choose a functor

$$\mathrm{IRes}\colon \mathcal{A} \to \mathrm{K}^{(\mathrm{ires})}(\mathcal{A})$$

such that $\mathrm{IRes}\,\mathrm{H}^0 \cong 1_{\mathcal{A}}$ and $\mathrm{H}^0\,\mathrm{IRes} \cong 1_{\mathrm{K}^{(\mathrm{ires})}(\mathcal{A})}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathrm{IRes}"] \arrow[r, phantom, "\sim"] & \mathrm{K}^{(\mathrm{ires})}(\mathcal{A}) \arrow[l, bend left, "\mathrm{H}^0"] \end{tikzcd}$$

So for $A \in \mathrm{Ob}(\mathcal{A})$, the complex $A\,\mathrm{IRes} \in \mathrm{Ob}(\mathrm{K}^{(\mathrm{ires})}(\mathcal{A}))$ is an injective resolution of A.

The inclusion functor $\mathrm{C}^{(\mathrm{ires})}(\mathcal{A}) \to \mathrm{C}(\mathcal{A})$ induces a full and faithful additive functor

$$\mathrm{K}^{(\mathrm{ires})}(\mathcal{A}) \to \mathrm{K}(\mathcal{A}).$$

We now have

$$\mathcal{A} \xrightarrow[\sim]{\mathrm{IRes}} \mathrm{K}^{(\mathrm{ires})}(\mathcal{A}) \longrightarrow \mathrm{K}(\mathcal{A}).$$

By applying F pointwise to objects and differentials of complexes and to the entries of complex morphisms, we get an additive functor $\mathrm{K}(\mathcal{A}) \xrightarrow{\mathrm{K}(F)} \mathrm{K}(\mathcal{B})$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathrm{IRes}", "\sim"'] & \mathrm{K}^{(\mathrm{ires})}(\mathcal{A}) \arrow[r] & \mathrm{K}(\mathcal{A}) \arrow[d, "\mathrm{K}(F)"] \\ & & \mathrm{K}(\mathcal{B}) \end{tikzcd}$$

We also have the functor $\mathrm{K}(\mathcal{B}) \xrightarrow{\mathrm{H}^k} \mathcal{B}$, where for $B \in \mathrm{Ob}(\mathrm{K}(\mathcal{B}))$ the homology object $B\,\mathrm{H}^k$ at position k can be constructed as follows: We have the induced monomorphism from an image of a differential to the kernel of the following differential. Then $B\,\mathrm{H}^k$ is a cokernel of this monomorphism.

Then we get the right derived functor $\mathrm{R}^k\,F$ as composite:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathrm{IRes}", "\sim"'] \arrow[d, "\mathrm{R}^k\,F"'] & \mathrm{K}^{(\mathrm{ires})}(\mathcal{A}) \arrow[r] & \mathrm{K}(\mathcal{A}) \arrow[d, "\mathrm{K}(F)"] \\ \mathcal{B} & & \mathrm{K}(\mathcal{B}) \arrow[ll, "\mathrm{H}^k"] \end{tikzcd}$$

For example,

$$\mathrm{R}^k\,{}_{\mathcal{A}}(X,-) =: \mathrm{Ext}^k_{\mathcal{A}}(X,-)$$

for $X \in \mathrm{Ob}(\mathcal{A})$, where $\mathcal{B} = \mathbb{Z}\text{-Mod}$.

0.2 Generalisation to n-complexes

Alternatively, we can write a complex as a commutative diagram

$$
\begin{tikzcd}
 & & & {} & \\
 & & 0 \arrow[r] & A^4 \arrow[u, dotted] \arrow[r, dotted] & {} \\
 & 0 \arrow[r] & A^2 \arrow[u] \arrow[r, "a^2"] & A^3 \arrow[u, "a^3"] \arrow[r] & 0 \arrow[u, dotted] \\
0 \arrow[r] & A^0 \arrow[u] \arrow[r, "a^0"] & A^1 \arrow[u, "a^1"] \arrow[r] & 0 \arrow[u] & \\
{} \arrow[u, dotted] \arrow[r, dotted] & A^{-1} \arrow[u, "a^{-1}"] \arrow[r] & 0 \arrow[u] & & \\
 & {} \arrow[u, dotted] & & &
\end{tikzcd}
$$

on the underlying poset $\bar{\Delta}_2^{\#}$:

$$
\begin{tikzcd}
 & & & {} & \\
 & & 2/2 \arrow[r] & 0^{+1}/2 \arrow[u, dotted] \arrow[r, dotted] & {} \\
 & 1/1 \arrow[r] & 2/1 \arrow[u] \arrow[r] & 0^{+1}/1 \arrow[u] \arrow[r] & 1^{+1}/1 \arrow[u, dotted] \\
0/0 \arrow[r] & 1/0 \arrow[u] \arrow[r] & 2/0 \arrow[u] \arrow[r] & 0^{+1}/0 \arrow[u] & \\
{} \arrow[u, dotted] \arrow[r, dotted] & 1/2^{-1} \arrow[u] \arrow[r] & 2/2^{-1} \arrow[u] & &
\end{tikzcd}
$$

E.g. the element $2/1 \in \bar{\Delta}_2^{\#}$ is to be read "2 mod 1".

So we call a diagram on $\bar{\Delta}_2^{\#}$ a 2-complex if the indices on the boundary

$$\{\ldots, 0/0, 1/1, 2/2, \ldots\} \cup \{\ldots, 2/2^{-1}, 0^{+1}/0, 1^{+1}/1, \ldots\}$$

are mapped to zero objects in $\mathcal{A}$. The category of 2-complexes is called $\mathrm{C}^{(2)}(\mathcal{A})$.

So up to indexing, 2-complexes are complexes:

$$\mathrm{C}^{(2)}(\mathcal{A}) = \mathrm{C}(\mathcal{A}),$$

cf. §0.1.

We generalize to n-complexes for $n \geqslant 0$. For example, a 3-complex looks as follows.

$$
\begin{tikzcd}
& & & & & 0 \arrow[r, dotted] & {} & \\
& & & & 0 \arrow[r] & X_{0^{+1}/3} \arrow[r] \arrow[u] & X_{1^{+1}/3} \arrow[r, dotted] \arrow[u, dotted] & {} \\
& & & 0 \arrow[r] & X_{3/2} \arrow[r] \arrow[u] & X_{0^{+1}/2} \arrow[r] \arrow[u] & X_{1^{+1}/2} \arrow[r] \arrow[u] & 0 \arrow[u, dotted] \\
& & 0 \arrow[r] & X_{2/1} \arrow[r] \arrow[u] & X_{3/1} \arrow[r] \arrow[u] & X_{0^{+1}/1} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
& 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] & X_{2/0} \arrow[r] \arrow[u] & X_{3/0} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
0 \arrow[r] & X_{0/3^{-1}} \arrow[r] \arrow[u] & X_{1/3^{-1}} \arrow[r] \arrow[u] & X_{2/3^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & & \\
{} \arrow[r, dotted] \arrow[u, dotted] & X_{0/2^{-1}} \arrow[r] \arrow[u] & X_{1/2^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & & & \\
& {} \arrow[r, dotted] \arrow[u, dotted] & 0 \arrow[u] & & & & &
\end{tikzcd}
$$

It is a diagram on the underlying poset $\bar{\Delta}_3^{\#}$:

$$
\begin{tikzcd}
& & & & & 0^{+1}/0^{+1} \arrow[r, dotted] & {} & \\
& & & & 3/3 \arrow[r] & 0^{+1}/3 \arrow[r] \arrow[u] & 1^{+1}/3 \arrow[r, dotted] \arrow[u, dotted] & {} \\
& & & 2/2 \arrow[r] & 3/2 \arrow[r] \arrow[u] & 0^{+1}/2 \arrow[r] \arrow[u] & 1^{+1}/2 \arrow[r] \arrow[u] & 2^{+1}/2 \arrow[u, dotted] \\
& & 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & 3/1 \arrow[r] \arrow[u] & 0^{+1}/1 \arrow[r] \arrow[u] & 1^{+1}/1 \arrow[u] & \\
& 0/0 \arrow[r] & 1/0 \arrow[r] \arrow[u] & 2/0 \arrow[r] \arrow[u] & 3/0 \arrow[r] \arrow[u] & 0^{+1}/0 \arrow[u] & & \\
3^{-1}/3^{-1} \arrow[r] & 0/3^{-1} \arrow[r] \arrow[u] & 1/3^{-1} \arrow[r] \arrow[u] & 2/3^{-1} \arrow[r] \arrow[u] & 3/3^{-1} \arrow[u] & & & \\
{} \arrow[r, dotted] \arrow[u, dotted] & 0/2^{-1} \arrow[r] \arrow[u] & 1/2^{-1} \arrow[r] \arrow[u] & 2/2^{-1} \arrow[u] & & & & \\
& {} \arrow[r, dotted] \arrow[u, dotted] & 1/1^{-1} \arrow[u] & & & & &
\end{tikzcd}
$$

The poset $\bar{\Delta}_3^{\#}$ is derived from the linearly ordered set

$$\bar{\Delta}_3 = \{\ldots, 3^{-1}, 0,\, 1,\, 2,\, 3,\, 0^{+1}, 1^{+1}, 2^{+1}, 3^{+1}, 0^{+2} \ldots\}.$$

Here the upper indices are mere indices.

We will also denote by $s+1$ the successor of s in $\bar{\Delta}_3$. So e.g. $2+1=3$, $3+1=0^{+1}$ and $0^{+1}+1=1^{+1}$.

More generally, we have the linearly ordered set $\bar{\Delta}_n$, yielding the poset $\bar{\Delta}_n^{\#}$.

Diagrams on $\bar{\Delta}_n^{\#}$ with zero objects on the boundary are called *n-complexes*. They form the category

$$\mathrm{C}^{(n)}(\mathcal{A})$$

of n-complexes.

Note that a 1-complex looks as follows.

$$\begin{tikzcd}
 & & & {} & {} \\
 & & 0 \arrow[r] & X_{0^{+1}/1} \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & X_{0/1^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

The shift on $\bar{\Delta}_n$ is defined by $(i^{+k})^{+1} = i^{k+1}$. A monotone map $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ is called *quasiperiodic* if $i^{+1}f = (if)^{+1}$ for every $i \in \bar{\Delta}_n$.

The shift on $\bar{\Delta}_n^{\#}$ is defined by $({}^t/{}_s)^{+1} = {}^{s^{+1}}/{}_t$. A monotone map $g\colon \bar{\Delta}_n^{\#} \to \bar{\Delta}_m^{\#}$ is called *quasiperiodic* if $({}^j/{}_i)^{+1}g = ({}^j/{}_ig)^{+1}$ for every ${}^j/{}_i \in \bar{\Delta}_n^{\#}$.

For $n, m \geqslant 0$, every quasiperiodic monotone map

$$f\colon \bar{\Delta}_n \to \bar{\Delta}_m$$

defines a quasiperiodic monotone map

$$\begin{aligned} f^{\#}\colon \bar{\Delta}_n^{\#} &\to \bar{\Delta}_m^{\#} \\ {}^j/{}_i &\mapsto {}^j/{}_if^{\#} \;:=\; {}^{jf}/{}_{if}. \end{aligned}$$

For every m-complex X and every quasiperiodic monotone map $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$, we get an n-complex $X^{(f)} \;:=\; f^{\#}X$ with

$$\begin{aligned} X^{(f)}_{j/i} &:= X_{j/if^{\#}} \\ ({}^j/{}_i, {}^{j'}/{}_{i'})X^{(f)} &:= ({}^j/{}_if^{\#}, {}^{j'}/{}_{i'}f^{\#})X. \end{aligned}$$

We obtain a functor

$$\begin{aligned}\mathrm{C}^{(m)}(\mathcal{A}) &\to \mathrm{C}^{(n)}(\mathcal{A})\\ X &\mapsto X^{(f)}.\end{aligned}$$

For example, for a 1-complex X and the quasiperiodic monotone map $f\colon \bar{\Delta}_2 \to \bar{\Delta}_1$ defined by

$$\begin{aligned}f\colon \bar{\Delta}_2 &\to \bar{\Delta}_1\\ 2 &\mapsto 1\\ 1 &\mapsto 0\\ 0 &\mapsto 0,\end{aligned}$$

we get the 2-complex $X^{(f)}$, displayed as follows.

$$\begin{tikzcd}
 & & & {} & {} & {} \\
 & & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u, dotted] & 0 \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r] & 0 \arrow[r] \arrow[u] & X_{1/0} \arrow[r] \arrow[u, "1"] & 0 \arrow[u] & \\
0 \arrow[r] & X_{0/1^{-1}} \arrow[r, "1"] \arrow[u] & X_{0/1^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

Cf. the definition of split acyclic complexes in §0.1.

A quadrangle

$$\begin{tikzcd}
C \arrow[r, "d"] & D \\
A \arrow[r, "a"] \arrow[u, "c"] & B \arrow[u, "b"']
\end{tikzcd}$$

in $\mathcal{A}$ is called a *weak square* and denoted

$$\begin{tikzcd}
C \arrow[r, "d"] & D \\
A \arrow[r, "a"] \arrow[u, "c"] \arrow[ur, phantom, "+"] & B \arrow[u, "b"']
\end{tikzcd}$$

if the diagonal sequence

$$A \xrightarrow{(a\ c)} B \oplus C \xrightarrow{\left(\begin{smallmatrix} b \\ -d \end{smallmatrix}\right)} D$$

is exact in the middle. The composite of weak squares is a weak square.

Weak squares are called exact squares by Schubert [**7,** §13.4.1]. He shows that they are closed under composition [**7,** §13.4.5].

Given $m \in [1,n]$, we let

$$\mathrm{C}^{(n,m)}(\mathcal{A}) \subseteq \mathrm{C}^{(n)}(\mathcal{A})$$

denote the full subcategory consisting of finite direct sums of n-complexes $X_i^{(f_i)}$, where X_i is an m-complex and where $\bar{\Delta}_n \xrightarrow{f_i} \bar{\Delta}_m$ is a quasiperiodic monotone map for $i \in I$, for some finite set I.

We obtain the homotopy category as factor category

$$\mathrm{K}^{(n/m)}(\mathcal{A}) \;:=\; \mathrm{C}^{(n)}(\mathcal{A}) / \,\mathrm{C}^{(n,m)}(\mathcal{A}).$$

E.g. the subcategory $\mathrm{C}^{(2,1)}(\mathcal{A})$ corresponds to the subcategory $\mathrm{C}^{(\mathrm{sp\ ac})}(\mathcal{A})$ of split acyclic complexes; cf. §0.1. So, up to indexing, $\mathrm{K}^{(2/1)}(\mathcal{A}) = \mathrm{K}(\mathcal{A})$.

Let

$$\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \subseteq \mathrm{C}^{(n)}(\mathcal{A})$$

be the full subcategory of n-complexes I satisfying the following conditions (i), (ii), (iii).

(i) All objects $I_{t/s}$ in I with $s < 0$ are zero objects.

(ii) For $0 \leqslant s < t < s^{+1}$, we have

$$\begin{tikzcd} I_{t/s+1} \arrow[r] & I_{t+1/s+1} \\ I_{t/s} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & I_{t+1/s} \arrow[u] \end{tikzcd} .$$

(iii) All objects in I are injective.

A 3-complex I in $\mathrm{C}^{(3,\mathrm{ires})}(\mathcal{A})$ consists of injective objects and looks as follows.

$$
\begin{tikzcd}
 & & & & & 0 \arrow[r, dotted] & & \\
 & & & & 0 \arrow[r] & I_{0+1/3} \arrow[r] \arrow[u] & I_{1+1/3} \arrow[r, dotted] \arrow[u, dotted] & \\
 & & & 0 \arrow[r] & I_{3/2} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{0+1/2} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{1+1/2} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u, dotted] \\
 & & 0 \arrow[r] & I_{2/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{3/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{0+1/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u] & \\
 & 0 \arrow[r] & I_{1/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{2/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{3/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & & \\
 \arrow[r, dotted] \arrow[u, dotted] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & & & \\
 & \arrow[r, dotted] \arrow[u, dotted] & 0 \arrow[u] & & & & &
\end{tikzcd}
$$

A 2-complex I in $\mathrm{C}^{(2,\mathrm{ires})}(\mathcal{A})$ consists of injective objects and looks as follows.

$$
\begin{tikzcd}
 & & & & \\
 & & 0 \arrow[r] & I_{0+1/2} \arrow[r, dotted] \arrow[u, dotted] & \\
 & 0 \arrow[r] & I_{2/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{0+1/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u, dotted] \\
0 \arrow[r] & I_{1/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{2/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u] & \\
 \arrow[r, dotted] \arrow[u, dotted] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
 & \arrow[u, dotted] & & &
\end{tikzcd}
$$

So up to indexing, $\mathrm{C}^{(2,\mathrm{ires})}(\mathcal{A}) = \mathrm{C}^{(\mathrm{ires})}(\mathcal{A})$; cf. §0.1.

Let $(\dot{\Delta}_{n-1}, \mathcal{A})$ be the category of functors from the poset $\dot{\Delta}_{n-1} = [1, n-1] \subseteq \mathbb{Z}$ to $\mathcal{A}$. That is, an object of $(\dot{\Delta}_{n-1}, \mathcal{A})$ is a diagram of the form

$$A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-2}} A_{n-1}.$$

We define a functor $\mathrm{Pb}^{(n)} \colon \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \to (\dot{\Delta}_{n-1}, \mathcal{A})$ by recursively adding pullbacks as shown in the case $n = 3$ for a 3-complex $I \in \mathrm{Ob}(\mathrm{C}^{(3,\mathrm{ires})}(\mathcal{A}))$:

$$\begin{tikzcd}
 & & {} & {} & {} & {} \\
 & I_{1/1} \arrow[r] & I_{2/1} \arrow[r] \arrow[u, dotted] & I_{3/1} \arrow[r] \arrow[u, dotted] & I_{0^{+1}/1} \arrow[r] \arrow[u, dotted] & I_{1^{+1}/1} \arrow[u, dotted] \\
I_{0/0} \arrow[r] & I_{1/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{2/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{3/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & I_{0^{+1}/0} \arrow[u] & \\
 & X_1 \arrow[r, "x_1"] \arrow[u, "b_1", tail] \arrow[ur, phantom, "\llcorner"] & X_2 \arrow[r] \arrow[u, "b_2", tail] \arrow[ur, phantom, "\llcorner"] & 0 \arrow[u, "b_3", tail] & &
\end{tikzcd}$$

Here we have $I\,\mathrm{Pb}^{(n)} = (X_1 \xrightarrow{x_1} X_2)$.

For morphisms we get unique induced morphisms between the resulting diagrams.

Note that in the case $n = 2$, we only have the pullback

$$\begin{tikzcd}
I_{1/0} \arrow[r] & I_{2/0} \\
X_1 \arrow[r] \arrow[u, "b_1"] \arrow[ur, phantom, "\llcorner"] & 0 \arrow[u]
\end{tikzcd}$$

where $X_1 \xrightarrow{b_1} I_{1/0}$ is a kernel of $I_{1/0} \longrightarrow I_{2/0}$. So $\mathrm{Pb}^{(2)} = \mathrm{H}^0$; cf. §0.1.

Let

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \ := \ \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) / \operatorname{Ker}(\mathrm{Pb}).$$

We have $\operatorname{Ker}(\mathrm{Pb}) = \mathrm{C}^{(n,1)}(\mathcal{A}) \cap \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$.

The induced functor

$$\overline{\mathrm{Pb}}^{(n)} \colon \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \to (\dot{\Delta}_{n-1}, \mathcal{A})$$

is an equivalence. So we may choose a functor

$$\mathrm{IRes}^{(n)} \colon (\dot{\Delta}_{n-1}, \mathcal{A}) \to \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$$

such that $\mathrm{IRes}^{(n)}\, \overline{\mathrm{Pb}}^{(n)} \cong 1_{(\dot{\Delta}_{n-1}, \mathcal{A})}$ and $\overline{\mathrm{Pb}}^{(n)}\, \mathrm{IRes}^{(n)} \cong 1_{\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})}$.

The functor $\mathrm{IRes}^{(n)}$ is called the *injective resolution equivalence*.

The inclusion functor $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{J} \mathrm{C}^{(n)}(\mathcal{A})$ induces a full and faithful additive functor

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\bar{J}} \mathrm{K}^{(n/1)}(\mathcal{A}).$$

We now have

$$(\dot{\Delta}_{n-1}, \mathcal{A}) \xrightarrow[\sim]{\mathrm{IRes}^{(n)}} \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\quad \bar{J} \quad} \mathrm{K}^{(n/1)}(\mathcal{A}).$$

By applying F pointwise to n-complexes and n-complex morphisms in $\mathrm{C}^{(n)}(\mathcal{A})$, we get n-complexes and n-complex morphisms in $\mathrm{C}^{(n)}(\mathcal{B})$. For $m \in [1, n]$, we get an induced additive functor $\mathrm{K}^{(n/m)}(\mathcal{A}) \xrightarrow{\mathrm{K}^{(n/m)}(F)} \mathrm{K}^{(n/m)}(\mathcal{B})$. For $m = 1$, this yields

$$\begin{tikzcd}
(\dot{\Delta}_{n-1}, \mathcal{A}) \arrow[r, "\mathrm{IRes}^{(n)}", "\sim"'] & \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \arrow[r, "\bar{J}"] & \mathrm{K}^{(n/1)}(\mathcal{A}) \arrow[d, "\mathrm{K}^{(n/1)}(F)"] \\
& & \mathrm{K}^{(n/1)}(\mathcal{B})\ .
\end{tikzcd}$$

There seem to be possibilities to generalise the homology functor as well. This might lead to generalized right derived functors.

$$\begin{tikzcd}
(\dot{\Delta}_{n-1}, \mathcal{A}) \arrow[r, "\mathrm{IRes}^{(n)}", "\sim"'] \arrow[d, dotted, "?"] & \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \arrow[r, "\bar{J}"] & \mathrm{K}^{(n/1)}(\mathcal{A}) \arrow[d, "\mathrm{K}^{(n/1)}(F)"] \\
(\,?\,, \mathcal{B}) & & \mathrm{K}^{(n/1)}(\mathcal{B}) \arrow[ll, dotted, "?"]
\end{tikzcd}$$

1 Conventions

We assume the reader to be familiar with elementary category theory. An introduction can be found in **[4]** and **[7]**. Some basic definitions and notations are given below. Concerning additive categories we essentially follow Mathias Ritter **[6]**.

Suppose given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

- All categories are supposed to be small (with respect to a sufficiently big universe).
- The set of objects of $\mathcal{A}$ is denoted by $\mathrm{Ob}(\mathcal{A})$, the set of morphisms by $\mathrm{Mor}(\mathcal{A})$. Given $A, B \in \mathrm{Ob}(\mathcal{A})$, we write the set of morphisms from A to B as ${}_{\mathcal{A}}(A, B)$. The identity morphism of $A \in \mathrm{Ob}(\mathcal{A})$ is written as 1_A. We often write $1 := 1_A$ if unambiguous.
- For a finite set M, we denote its cardinality by $|M|$.
- Given a set M and subsets $M_1, M_2 \subseteq M$. If $M_1 \cap M_2 = \emptyset$, we sometimes write $M_1 \dot{\cup} M_2 := M_1 \cup M_2$ for their union.
- A partially ordered set is also called a *poset* for short. A poset P will also be considered as a category, where $\mathrm{Ob}(P) = P$ and $\mathrm{Mor}(P) = \{(x, y) \in P \times P \colon x \leqslant y\}$. For details, see §3.1.
- Given a totally ordered set I and $a, b \in I$, we let $[a, b]_I := \{z \in I : a \leqslant z \leqslant b\}$.

  If $I = \mathbb{Z}$ we write $[a, b] := [a, b]_{\mathbb{Z}}$.
- Given morphisms $(A_k \xrightarrow{a_k} A_{k+1})_{k \in \mathbb{Z}}$ in $\mathcal{A}$ and given $i, j \in \mathbb{Z}$ with $i \leqslant j$, we write

  $$\prod_{k \in [i,j]} a_k := a_i a_{i+1} \cdots a_{j-1} a_j .$$

  We set

  $$\prod_{k \in [i,i-1]} a_k := 1_{A_i} .$$
- By Δ_n we denote the totally ordered set $[0, n] \subseteq \mathbb{Z}$.
- Let A, B be sets and $B_1 \subset B$. For a map $A \xrightarrow{f} B$, the *inverse image of* B_1 *under* f is $f^{-1}(B_1) := \{a \in A \colon af \in B_1\}$. If $B_1 = \{b\}$, we also write $f^{-1}(b) := f^{-1}(B_1)$.
- An object $A \in \mathrm{Ob}(\mathcal{A})$ is called *zero object* if it satisfies the condition

  $$|{}_{\mathcal{A}}(A, B)| = 1 = |{}_{\mathcal{A}}(B, A)|$$

  for $B \in \mathrm{Ob}(\mathcal{A})$.

- To indicate that a morphism $X \xrightarrow{f} Y$ is a monomorphism, we sometimes write

$$\begin{tikzcd} X \arrow[r, "f", "\bullet" marking] & Y \end{tikzcd} .$$

  We call a morphism that is a monomorphism *monic*.

  To indicate that a morphism $X \xrightarrow{f} Y$ is an epimorphism, we sometimes write

$$\begin{tikzcd} X \arrow[r, "f", "|" marking] & Y \end{tikzcd} .$$

  The property of being an epimorphism is called being *epic*.

  To indicate that a morphism $X \xrightarrow{f} Y$ is an isomorphism, we sometimes write

$$\begin{tikzcd} X \arrow[r, "f", "\sim"'] & Y \end{tikzcd} .$$

- A morphism $X \xrightarrow{f} Y$ in a category $\mathcal{A}$ is called a *split monomorphism* or *split monic* if there exists a morphism $Y \xrightarrow{g} X$ in $\mathcal{A}$ with $fg = 1_X$. To show that f is split monic, we sometimes write $\begin{tikzcd} X \arrow[r, tail, "f", "\bullet" marking] & Y \end{tikzcd}$.

  A morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ is called a *split epimorphism* or *split epic* if there exists a morphism $Y \xrightarrow{g} X$ in $\mathcal{A}$ with $gf = 1_Y$. To show that f is split epic, we sometimes write $\begin{tikzcd} X \arrow[r, tail, "f", "|" marking] & Y \end{tikzcd}$.

- We write the composition of morphisms on the right. That is, the composite of $X \xrightarrow{f} Y \xrightarrow{g} Z$ is written $X \xrightarrow{f \cdot g} Z$. Often, we write $fg := f \cdot g$.

  The same applies to functors. The composite of $\left(\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}\right)$ is written

$$\left(\mathcal{A} \xrightarrow{F \cdot G} \mathcal{C}\right) = \left(\mathcal{A} \xrightarrow{FG} \mathcal{C}\right) .$$

- A commutative rectangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

  in $\mathcal{A}$ is called a *pullback* if for all morphisms s, t with $sf' = ty$ there exists a unique morphism u with $uf = t$ and $ux = s$.

$$\begin{tikzcd} T \arrow[dr, dotted, "\exists! u"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ & X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

We sometimes indicate that a quadrangle is a pullback as follows.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

- The dual concept to the pullback is called pushout. We sometimes indicate that a quadrangle is a pushout as follows.

$$\begin{tikzcd}
X \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l, "f"'] \\
X' \arrow[u, "x"] & Y' \arrow[l, "f'"] \arrow[u, "y"']
\end{tikzcd}$$

- Suppose given a poset A and a subset $B \subseteq A$. An element $b_1 \in B$ is called *initial in* B if $b_1 \leqslant b$ for every $b \in B$. If B contains an initial element, we write it $\min B$.

  An element $b_2 \in B$ is called *terminal in* B if $b \leqslant b_2$ for every $b \in B$. If B contains a terminal element, we write it $\max B$.

  The idea of initial and terminal elements of a poset corresponds to initial and terminal objects in the poset category; cf. Remark 67.

- Let $F, G \colon \mathcal{A} \to \mathcal{B}$ be functors. Suppose given $XF \xrightarrow{X\alpha} XG$ for $X \in \mathrm{Ob}(\mathcal{A})$. The tuple $(X\alpha)_{X \in \mathrm{Ob}(\mathcal{A})}$ is called *natural* if the following diagram commutes for every $X \xrightarrow{f} Y$ in $\mathcal{A}$.

$$\begin{tikzcd}
XF \arrow[r, "fF"] \arrow[d, "X\alpha"] & YF \arrow[d, "Y\alpha"] \\
XG \arrow[r, "fG"] & YG
\end{tikzcd}$$

  Such a natural tuple is often called a *transformation* from F to G.

  We often write $\alpha_X := X\alpha$ for $X \in \mathrm{Ob}(\mathcal{A})$.

  A *isotransformation* is a transformation $(X\alpha)_{X \in \mathrm{Ob}(\mathcal{A})}$ where all morphisms are isomorphisms in $\mathcal{B}$.

- The functors between two categories $\mathcal{A}$ and $\mathcal{B}$ together with transformations between these functors form the category $(\mathcal{A}, \mathcal{B})$ where $\mathrm{Ob}(\mathcal{A}, \mathcal{B})$ is the set of functors from $\mathcal{A}$ to $\mathcal{B}$ and $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$ is the set of transformations between these functors.

  Knowing this we can define a transformation α as a functor from $\mathcal{A}$ to $(\Delta_1, \mathcal{B})$, where $(\Delta_1, \mathcal{B})$ is the category of diagrams on $\bullet \longrightarrow \bullet$, that maps $X \xrightarrow{f} Y$ to the commutative diagram

$$\begin{tikzcd}
XF \arrow[r, "fF"] \arrow[d, "X\alpha"] & YF \arrow[d, "Y\alpha"] \\
XG \arrow[r, "fG"] & YG
\end{tikzcd}$$

in $\mathcal{B}$.

- Given functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ we call F *left adjoint to* G and G *right adjoint to* F, sometimes denoted $F \dashv G$, if there exist transformations $1_{\mathcal{A}} \xrightarrow{\eta} FG$, called *unit*, and $GF \xrightarrow{\varepsilon} 1_{\mathcal{B}}$, called *counit*, such that for every $A \in \mathrm{Ob}(\mathcal{A})$ and for every $B \in \mathrm{Ob}(\mathcal{B})$ the following diagrams commute.

$$\begin{tikzcd} (A)F \arrow[r, "A\eta F"] \arrow[dr, "1_{(A)F}"'] & (A)FGF \arrow[d, "(A)F\varepsilon"] \\ & (A)F \end{tikzcd} \qquad \begin{tikzcd} (B)G \arrow[r, "BG\eta"] \arrow[dr, "1_{(B)G}"'] & (B)GFG \arrow[d, "B\varepsilon G"] \\ & (B)G \end{tikzcd}$$

These commutative diagrams are called the *triangle identities*.

Then for $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$ we get a bijection ${}_{\mathcal{A}}(A, BG) \xrightarrow{\sim} {}_{\mathcal{B}}(AF, B)$ by $f \mapsto (f)F \cdot (B)\varepsilon$ for $f \in {}_{\mathcal{A}}(A, BG)$ with inverse $g \mapsto (A)\eta \cdot (g)G$ for $g \in {}_{\mathcal{B}}(AF, B)$; cf. [**3,** §2.2.6].

- A *preadditive* category is a category $\mathcal{A}$, together with the structure of an abelian group on ${}_{\mathcal{A}}(A, B)$ for $A, B \in \mathrm{Ob}(\mathcal{A})$, written additively, such that the following property (1) holds.

  (1) The composition of morphisms is bilinear. This means, for

  $$A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \xrightarrow{h} D$$

  in $\mathcal{A}$ we have $f(g_1 + g_2)h = fg_1h + fg_2h$.

  The zero of ${}_{\mathcal{A}}(A, B)$ is denoted $0_{A,B} := 0_{{}_{\mathcal{A}}(A,B)}$. If unambiguous we often write $0 := 0_{A,B}$.

- Let $\mathcal{A}$ be a preadditive category. Suppose given a finite set I and a tuple $(A_i)_{i \in I}$ with $A_i \in \mathrm{Ob}(\mathcal{A})$ for $i \in I$. A *direct sum* of $(A_i)_{i\in I}$ is a tuple $(C, (\pi_i)_{i\in I}, (\iota_i)_{i\in I})$ with $C \in \mathrm{Ob}(\mathcal{A})$ and $\iota_i \colon A_i \to C$ as well as $\pi_i \colon C \to A_i$ for $i \in I$, that fulfil the following properties (i, ii, iii).

  (i) $\iota_i \pi_j = 0_{A_i, A_j}$ for $i, j \in I$ with $i \neq j$

  (ii) $\iota_i \pi_i = 1_{A_i}$ for $i \in I$

  (iii) $\sum\limits_{i \in I} \pi_i \iota_i = 1_C$

  We sometimes just write C for the direct sum $(C, (\pi_i)_{i\in I}, (\iota_i)_{i\in I})$.

- An object $Q \in \mathrm{Ob}(\mathcal{A})$ is called *injective* if for every monomorphism $x \colon X \to Y$ and morphism $f \colon X \to Q$ there exists a morphism $f' \colon Y \to Q$ with $xf' = f$.

  The category $\mathcal{A}$ is said to have *enough injective objects* if for every object $A \in \mathrm{Ob}(\mathcal{A})$, there exists a monomorphism $a \colon A \to B$ with B injective.

- A preadditive category $\mathcal{A}$ is called *additive*, if it fulfils the following conditions (1, 2).
  (1) For every finite set I and every $(A_i)_{i\in I}$ with $A_i \in \mathrm{Ob}(\mathcal{A})$ for $i \in I$ there exists a direct sum in $\mathcal{A}$.
  (2) There exists a zero object in $\mathcal{A}$.
- For an additive category $\mathcal{A}$, we choose a zero object $0_{\mathcal{A}} \in \mathrm{Ob}(\mathcal{A})$.
  For each $(A_i)_{i\in I}$ with I finite and $A_i \in \mathrm{Ob}(\mathcal{A})$ for $i \in I$ we choose a *standard direct sum*
  $$\left(\bigoplus_{i\in I} A_i,\ \left(\pi_i^{(A_j)_{j\in I}}\right)_{i\in I},\ \left(\iota_i^{(A_j)_{j\in I}}\right)_{i\in I}\right)$$
  in $\mathcal{A}$.
  If unambiguous, we often write $\pi_i := \pi_i^{(A_j)_{j\in I}}$ and $\iota_i := \iota_i^{(A_j)_{j\in I}}$ for $i \in I$.
  In particular, we choose
  $$\left(\bigoplus_{i\in\{x\}} A_i,\ (\pi_i)_{i\in\{x\}},\ (\iota_i)_{i\in\{x\}}\right) = (A_x,\ (1_{A_x}),\ (1_{A_x}))$$
  and
  $$\left(\bigoplus_{i\in\emptyset} A_i,\ (\pi_i)_{i\in\emptyset},\ (\iota_i)_{i\in\emptyset}\right) = (0_{\mathcal{A}},\ (),\ ())\,.$$
  If $I = \{i_1, \ldots, i_n\}$ is a totally ordered set with $i_1 < \cdots < i_n$, we often write
  $$A_{i_1} \oplus \cdots \oplus A_{i_n} := \bigoplus_{i\in I} A_i.$$
  Given (A, B) without an index set I, we assume I to be $[1, 2]$ with $1 \mapsto A$ and $2 \mapsto B$. So we have the standard direct sum $A \oplus B$ with morphisms
  $$\begin{aligned}\iota_1 &\colon A \to A \oplus B\\ \iota_2 &\colon B \to A \oplus B\\ \pi_1 &\colon A \oplus B \to A\\ \pi_2 &\colon A \oplus B \to B.\end{aligned}$$
  We also expand this to direct sums with n components by assuming $I = [1, n]$, if no other index set is mentioned.
- As every morphism $f\colon \bigoplus_{i\in I} A_i \to \bigoplus_{j\in J} B_j$ between direct sums $\left(\bigoplus_{i\in I} A_i,\ (\pi_i^A)_{i\in I},\ (\iota_i^A)_{i\in I}\right)$ of $(A_i)_{i\in I} =: A$ and $\left(\bigoplus_{j\in J} B_j,\ (\pi_j^B)_{j\in J},\ (\iota_j^B)_{j\in J}\right)$ of $(B_i)_{i\in I} =: B$ can be written as
  $$f = \sum_{(i,j)\in I\times J} \pi_i^A f_{i,j} \iota_j^B$$

where $f_{i,j} := \iota_i^A f \pi_j^B \in {}_{\mathcal{A}}(A_i, B_j)$ for $(i,j) \in I \times J$, we often write

$$f = (f_{i,j})_{i,j} = (f_{i,j})_{i \in I, j \in J}.$$

If $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_n\}$ are totally ordered sets with $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_n$, we often write

$$f = \begin{pmatrix} f_{i_1,j_1} & \cdots & f_{i_1,j_n} \\ \vdots & \ddots & \vdots \\ f_{i_m,j_1} & \cdots & f_{i_m,j_n} \end{pmatrix}$$

instead.

For $\big(A_1 \oplus \cdots \oplus A_n,\ (\pi_i)_{i \in [1,n]},\ (\iota_i)_{i \in [1,n]}\big)$ this yields

$$\pi_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow i\text{-th position} \\ \\ \\ \\ \end{matrix}$$

and

$$\iota_i = \begin{pmatrix} 0 & \ldots & 0 & \underset{\substack{\uparrow \\ i\text{-th position}}}{1} & 0 & \ldots & 0 \end{pmatrix}.$$

- A functor $F\colon \mathcal{A} \to \mathcal{B}$ between two preadditive categories $\mathcal{A}$ and $\mathcal{B}$ is called *additive*, if $(\varphi + \psi)F = \varphi F + \psi F$ for $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$ in $\mathcal{A}$.

- Given a functor $F\colon \mathcal{A} \to \mathcal{B}$, we denote its image by $\mathrm{Im}(F) := \mathcal{A}F \subseteq \mathcal{B}$. This is the subcategory of $\mathcal{B}$ with

$$\mathrm{Ob}(\mathrm{Im}(F)) = \{AF \colon A \in \mathrm{Ob}(\mathcal{A})\}$$
$$\mathrm{Mor}(\mathrm{Im}(F)) = \{\varphi F \colon \varphi \in \mathrm{Mor}(\mathcal{A})\}$$

The *full image* $\overline{\mathrm{Im}}(F)$ of F is the full subcategory of $\mathcal{B}$ with $\mathrm{Ob}(\overline{\mathrm{Im}}(F)) = \mathrm{Ob}(\mathrm{Im}(F))$.

- The *kernel* $\mathrm{Ker}(F)$ of an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is the full subcategory of $\mathcal{A}$ with

$$\mathrm{Ob}(\mathrm{Ker}(\mathcal{A})) = \{A \in \mathrm{Ob}(\mathcal{A}) \colon AF \text{ is a zero object in } \mathcal{B}\}.$$

2 Preliminaries

2.1 General categories

Suppose given a category $\mathcal{A}$.

Remark 1. Suppose given a morphisms f, g, h in $\mathcal{A}$ with $f = gh$.

- If f is monic, then g is monic.
- If f is split monic, then g is split monic.
- If f is epic, then h is epic.
- If f is split epic, then h is split epic.

Remark 2. Suppose given $I \xrightarrow{\alpha} B$ in $\mathcal{A}$ with α monic and I injective. Then α is split monic.

Suppose given $X \overset{i}{\rightarrowtail} I$ in $\mathcal{A}$ with i split monic and I injective. Then X is injective.

Suppose given $I \xrightarrow{r} X$ in $\mathcal{A}$ with r split epic and I injective. Then X is injective.

2.2 Additive categories

We first state a few properties of the direct sum and further basic notions. Suppose given a additive categories $\mathcal{A}$ and $\mathcal{B}$. Suppose given an additive functor $F\colon A \to B$.

Remark 3. Suppose given a finite set I and a tuple of objects $(A_i)_{i \in I}$ in $\mathcal{A}$. Suppose given a direct sum $(C,\ (\pi_i)_{i\in I},\ (\iota_i)_{i\in I})$ of $(A_i)_{i\in I}$ in $\mathcal{A}$. Suppose given an isomorphism $\varphi\colon C \to C'$ in $\mathcal{A}$. Then $(C',\ (\varphi^{-1}\pi_i)_{i\in I},\ (\iota_i\varphi)_{i\in I})$ is a direct sum of $(A_i)_{i\in I}$, too.

Remark 4. Suppose given a finite set I and a tuple of objecst $(A_i)_{i\in I}$ in $\mathcal{A}$. Suppose given a direct sum $(C, (\pi_i)_{i\in I}, (\iota_i)_{i\in I})$ of $(A_i)_{i\in I}$. The tuple $(C, (\iota_i)_{i\in I})$ is a coproduct and, the tuple $(C, (\pi_i)_{i\in I})$ a product of $(A_i)_{i\in I}$.

$$
\begin{tikzcd}
C \arrow[dr, dashed, "\exists! f"] & \\
A_i \arrow[u, "\iota_i"] \arrow[r, "f_i"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C \arrow[d, "\pi_i"'] & \\
A_i & B \arrow[l, "g_i"] \arrow[ul, dashed, "\exists! g"']
\end{tikzcd}
$$

Suppose given $B \in \operatorname{Ob}(\mathcal{A})$ and $(A_i \xrightarrow{f_i} B)_{i\in I}$, the morphism f with $\iota_i f = f_i$ for every $i \in I$ is given as $f = \sum\limits_{i\in I} \pi_i f_i$ with

$$\iota_j f = \iota_j \sum_{i\in I} \pi_i f_i = \sum_{i\in I} (\iota_j \pi_i) f_i = f_j.$$

Suppose given $C \xrightarrow{\hat{f}} B$ with $\iota_i \hat{f} = f_i$ for every $i \in I$. Then

$$\hat{f} = \sum_{i \in I} \pi_i \iota_i \hat{f} = \sum_{i \in I} \pi_i f_i = f$$

Therefore f is unique.

The same holds for given morphisms $B \xrightarrow{g_i} A_i$ and $g = \sum\limits_{i \in I} g_i \iota_i$. We have

$$g\pi_j = \sum_{i \in I} g_i \iota_i \cdot \pi_j = \sum_{i \in I} g_i(\iota_i \pi_j) = g_j$$

for $j \in I$. Suppose given $B \xrightarrow{\hat{g}} C$ with $\hat{g}\pi_i = g_i$ for $i \in I$. Then

$$\hat{g} = \sum_{i \in I} \hat{g}\pi_i \iota_i = \sum_{i \in I} g_i \iota_i = g.$$

Remark 5. Suppose given two direct sums $(C, (\pi_i)_{i \in I}, (\iota_i)_{i \in I})$ and $(C', (\pi'_i)_{i \in I}, (\iota'_i)_{i \in I})$ of $(A_i)_{i \in I}$. Then an isomorphism $C \xrightarrow{\varphi} C'$ is given by $\varphi := \sum\limits_{i \in I} \pi_i \iota'_i$ with inverse $\varphi^{-1} := \sum\limits_{i \in I} \pi'_i \iota_i$.

Remark 6. Additive functors preserve direct sums as follows:

Suppose given preadditive categories $\mathcal{C}, \mathcal{D}$. Suppose given an additive functor $G \colon \mathcal{C} \to \mathcal{D}$. Suppose given I finite and a tuple of objects $(A_i)_{i \in I}$ in $\mathrm{Ob}(\mathcal{C})$ and a direct sum $(C, (\pi_i)_{i \in I}, (\iota_i)_{i \in I})$ of $(A_i)_{i \in I}$.

- We have $\iota_i G \cdot \pi_i G = (\iota_i \pi_i)G = 1_{A_i} G = 1_{A_i G}$ for $i \in I$,
- We have $\iota_i G \cdot \pi_j G = (\iota_i \pi_j)G = 0_{A_i, A_j} G = 0_{A_i G, A_j G}$ for $i, j \in I$ with $i \neq j$,
- $\sum\limits_{i \in I} \pi_i G \cdot \iota_i G = \sum\limits_{i \in I} (\pi_i \iota_i) G = \left(\sum\limits_{i \in I} \pi_i \iota_i \right) G = 1_{CG}$.

Then $(CG, (\pi_i G)_{i \in I}, (\iota_i G)_{i \in I})$ is a direct sum of $(A_i G)_{i \in I}$ in $\mathcal{D}$.

Remark 7. Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$. Suppose given an additive functor $F \colon \mathcal{A} \to \mathcal{B}$. Remember that $0_\mathcal{A}$ is a direct sum of () in $\mathcal{A}$ and $0_\mathcal{B}$ is a the direct sum of () in $\mathcal{B}$. Therefore $0_\mathcal{A} F \cong 0_\mathcal{B}$.

Definition 8. A full subcategory $\mathcal{U}$ of $\mathcal{A}$ is called *full additive subcategory* if the following conditions (1), (2) hold:

(1) $\mathrm{Ob}(\mathcal{U})$ contains a zero object of $\mathcal{A}$.

(2) Given $A, B \in \mathrm{Ob}(\mathcal{U})$ there exists a direct sum $(C, (\pi_i)_{i \in [1,2]}, (\iota_i)_{i \in [1,2]})$ of (A, B) in $\mathcal{A}$ such that $C \in \mathrm{Ob}(\mathcal{U})$.

Lemma 9. *For a preadditive category $\mathcal{C}$ to be additive it is sufficient to fulfil the following conditions:*

- *There exists a zero object $0_{\mathcal{C}} \in \mathrm{Ob}(\mathcal{C})$.*
- *For any $A, B \in \mathrm{Ob}(\mathcal{C})$ there exists a direct sum $(C, (\pi_i)_{i\in[1,2]}, (\iota_i)_{i\in[1,2]})$ of A and B in $\mathcal{C}$.*

Proof. For any objects $U, V \in \mathrm{Ob}(\mathcal{C})$ we choose a direct sum

$$(U \oplus V, (\pi_i^{(U,V)})_{i\in[1,2]}, (\iota^{(U,V)})_{i\in[1,2]})$$

of (U, V) in $\mathcal{C}$. Given a finite set I and $(A_i)_{i\in I}$, we choose $\bigoplus\limits_{i\in I} A_i \in \mathrm{Ob}(\mathcal{C})$ as follows:

- Let $\bigoplus\limits_{i\in I} A_i \; := \; 0_{\mathcal{U}}$ if $I = \emptyset$.
- Let $\bigoplus\limits_{i\in I} A_i \; := \; A_j$ if $I = \{j\}$.
- If $|I| > 1$ choose $j \in I$ and $\bigoplus\limits_{i\in I} A_i \; := \; \left(\bigoplus\limits_{i\in I\setminus\{j\}} A_i\right) \oplus A_j$.

For $j \in I$ define $\tilde{I} := I \setminus \{j\}$. Let

$$\left(\bigoplus_{i\in\tilde{I}} A_i, (\tilde{\pi}_i)_{i\in\tilde{I}}, (\tilde{\iota}_i)_{i\in\tilde{I}}\right)$$

be a direct sum of $(A_i)_{i\in\tilde{I}}$. We consider the direct sum of $(\bigoplus\limits_{i\in\tilde{I}} A_i, A_j)$ given as

$$\left(\bigoplus_{i\in I} A_i = \left(\bigoplus_{i\in\tilde{I}} A_i\right) \oplus A_j, (\pi_k^2)_{k\in[1,2]}, (\iota_k^2)_{k\in[1,2]}\right).$$

Then $\left(\bigoplus\limits_{i\in I} A_i, (\pi_i)_{i\in I}, (\iota_i)_{i\in I}\right)$ with

- $\pi_i := \pi_1^2 \tilde{\pi}_i$ for $i \in \tilde{I}$ and $\pi_j := \pi_2^2$,
- $\iota_i := \tilde{\iota}_i \iota_1^2$ for $i \in \tilde{I}$ and $\iota_j := \iota_2^2$

is a direct sum of $(A_i)_{i\in I}$:

$\iota_i \pi_l = 0_{A_i, A_l}$ for $i, l \in I$ with $i \neq l$.

$\iota_i \pi_i = 1_{A_i}$ for $i \in I$.

$\sum\limits_{i\in I} \pi_i \iota_i = \sum\limits_{i\in\tilde{I}} \pi_1^2 \tilde{\pi}_i \tilde{\iota}_i \iota_1^2 + \pi_2^2 \iota_2^2 = \pi_1^2 \left(\sum\limits_{i\in\tilde{I}} \tilde{\pi}_i \tilde{\iota}_i\right) \iota_1^2 + \pi_2^2 \iota_2^2 = \pi_1^2 \iota_1^2 + \pi_2^2 \iota_2^2 = 1_{\bigoplus_{i\in I} A_i}.$ □

Lemma 10. *Suppose given a full additive subcategory $\mathcal{U}$ of $\mathcal{A}$. Then $\mathcal{U}$ is an additive category. The inclusion functor $I\colon \mathcal{U} \to \mathcal{A}$ is additive.*

Proof. The category $\mathcal{U}$ is preadditive due to being full. It contains a zero object. Thus $\mathcal{U}$ is additive according to Lemma 9.

The inclusion functor I fulfils $(\varphi + \psi)I = \varphi + \psi = \varphi I + \psi I$ and thus is additive. □

Lemma 11. *Let $G\colon \mathcal{A} \to \mathcal{C}$ be an additive functor and $\mathcal{C}$ a preadditive category. Then the subcategory $\overline{\mathrm{Im}}(G)$ of $\mathcal{C}$ is additive.*

Proof. The category $\overline{\mathrm{Im}}(G)$ is preadditive due to being a full subcategory of $\mathcal{C}$. It contains the zero object $0_{\mathcal{A}}G$. For all $A, B \in \overline{\mathrm{Im}}(G)$ there exist $A', B' \in \mathrm{Ob}(\mathcal{A})$ with $A'G = A$ and $B'G = B$. Then, following Remark 6, $(A' \oplus B')G$ is a direct sum of (A, B). Thus for every $A, B \in \mathrm{Ob}(\overline{\mathrm{Im}}(G)) \subseteq \mathrm{Ob}(\mathcal{C})$, there exists a direct sum in $\overline{\mathrm{Im}}(G)$ and following Lemma 9, $\overline{\mathrm{Im}}(G)$ is additive. □

Remark 12. Given a category $\mathcal{C}$, then $\llbracket \mathcal{C}, \mathcal{A} \rrbracket$ is additive.

We sketch that $\llbracket \mathcal{C}, \mathcal{A} \rrbracket$ is preadditive.

Suppose given $F, G \in \mathrm{Ob}\llbracket \mathcal{C}, \mathcal{A} \rrbracket$. For $\alpha, \beta \in {}_{\llbracket \mathcal{C}, \mathcal{A} \rrbracket}(F, G)$ we let

$$\alpha + \beta \ := \ (C\alpha + C\beta)_{C \in \mathrm{Ob}(\mathcal{C})} \colon F \to G.$$

We sketch that $\llbracket \mathcal{C}, \mathcal{A} \rrbracket$ is additive. It contains the zero object defined by $0_{\llbracket \mathcal{C}, \mathcal{A} \rrbracket} \colon C \mapsto 0_{\mathcal{A}}$ for $C \in \mathrm{Ob}(\mathcal{C})$.

Suppose given $F, G \in \mathrm{Ob}\llbracket \mathcal{C}, \mathcal{A} \rrbracket$.

We define a functor $F \oplus G$ as follows. Define

$$(C)(F \oplus G) \ := \ CF \oplus CG$$

for $C \in \mathrm{Ob}(\mathcal{C})$. Define

$$\begin{aligned} X\pi_1^{(F,G)} \ &:= \ \pi_1^{(XF,XG)} \\ X\pi_2^{(F,G)} \ &:= \ \pi_2^{(XF,XG)} \\ X\iota_1^{(F,G)} \ &:= \ \iota_1^{(XF,XG)} \\ X\iota_2^{(F,G)} \ &:= \ \iota_2^{(XF,XG)} \end{aligned}$$

for $X \in \mathrm{Ob}(\mathcal{C})$ and

$$\varphi(F \oplus G) \ := \ \begin{pmatrix} \varphi F & 0 \\ 0 & \varphi G \end{pmatrix} = \pi_1^{(CF,CG)} \varphi F \iota_1^{(DF,DG)} + \pi_2^{(CF,CG)} \varphi G \iota_2^{(DF,DG)}$$

for $\varphi \in \mathrm{Mor}(\mathcal{C})$ with $\varphi\colon C \to D$. Then

$$(F \oplus G, (\pi_1^{(F,G)}, \pi_2^{(F,G)}), (\iota_1^{(F,G)}, \iota_2^{(F,G)}))$$

is a direct sum of (F, G) in $\llbracket \mathcal{C}, \mathcal{A} \rrbracket$.

Lemma 13. *Suppose given morphisms $f, g\colon A \to B$ in $\mathcal{A}$. Suppose given a full additive subcategory $\mathcal{U}$ of $\mathcal{A}$. Suppose given $N \in \mathrm{Ob}(\mathcal{U})$ and morphisms $A \xrightarrow{f_1} N \xrightarrow{f_2} B$ with $f_1 f_2 = f$. Suppose given $N' \in \mathrm{Ob}(\mathcal{U})$ and morphisms $A \xrightarrow{g_1} N' \xrightarrow{g_2} B$ with $g_1 g_2 = g$. Then there exists an $U \in \mathrm{Ob}(\mathcal{U})$ and morphisms $A \xrightarrow{h_1} U \xrightarrow{h_2} B$ with $h_1 h_2 = f + g$.*

Proof. Set $U := N \oplus N' \in \mathrm{Ob}(\mathcal{U})$ as $\mathcal{U}$ is additive. Set $h_1 := (f_1\ g_1)\colon A \to N \oplus N'$ as well as $h_2 := \binom{f_2}{g_2}\colon N \oplus N' \to B$. Then $h_1 h_2 = (f_1\ g_1) \cdot \binom{f_2}{g_2} = f_1 f_2 + g_1 g_2 = f + g$. □

Lemma 14. *Suppose given $(A_i)_{i \in \mathbb{Z}_{\geqslant 0}}$ and $(B_i)_{i \in \mathbb{Z}_{\geqslant 1}}$ in $\mathcal{A}$ and direct sums $(A_{i-1}, (\pi_i, \tau_i), (\iota_i, \kappa_i))$ of (A_i, B_i) for every $i \in \mathbb{Z}_{\geqslant 1}$ with $A_i \underset{\pi_i}{\overset{\iota_i}{\rightleftarrows}} A_{i-1} \underset{\kappa_i}{\overset{\tau_i}{\rightleftarrows}} B_i$.*

Define $\hat{\pi}_k := \pi_1 \cdot \ldots \cdot \pi_k\colon A_0 \to A_k$ and $\hat{\iota}_k := \iota_k \cdot \ldots \cdot \iota_1\colon A_k \to A_0$ for $k \in \mathbb{Z}_{\geqslant 0}$. In particular, $\hat{\pi}_0 = 1_{A_0}$ and $\hat{\iota}_0 = 1_{A_0}$.

Then

$$\hat{\pi}_k \hat{\iota}_k + \sum_{i \in [1,k]} \hat{\pi}_{i-1} \cdot \tau_i \cdot \kappa_i \cdot \hat{\iota}_{i-1} = 1_{A^0}$$

for $k \in \mathbb{Z}_{\geqslant 1}$.

Proof. For $k = 1$ we have $\pi_1 \iota_1 + \tau_1 \iota_1 = 1_{A_0}$.

Now suppose the statement holds for $k \geqslant 1$. We show that it also holds for $k+1$:

$$\begin{aligned}
1_{A^0} &= \hat{\pi}_k \hat{\iota}_k + \sum_{i \in [1,k]} \hat{\pi}_{i-1} \cdot \tau_i \cdot \kappa_i \cdot \hat{\iota}_{i-1} \\
&= \hat{\pi}_k (\pi_{k+1} \iota_{k+1} + \tau_{k+1} \kappa_{k+1}) \hat{\iota}_k + \sum_{i \in [1,k]} \hat{\pi}_{i-1} \cdot \tau_i \cdot \kappa_i \cdot \hat{\iota}_{i-1} \\
&= \hat{\pi}_k \pi_{k+1} \cdot \iota_{k+1} \hat{\iota}_k + \hat{\pi}_k \tau_{k+1} \cdot \kappa_{k+1} \hat{\iota}_k + \sum_{i \in [1,k]} \hat{\pi}_{i-1} \cdot \tau_i \cdot \kappa_i \cdot \hat{\iota}_{i-1} \\
&= \hat{\pi}_{k+1} \cdot \hat{\iota}_{k+1} + \hat{\pi}_k \tau_{k+1} \cdot \kappa_{k+1} \hat{\iota}_k + \sum_{i \in [1,k]} \hat{\pi}_{i-1} \cdot \tau_i \cdot \kappa_i \cdot \hat{\iota}_{i-1} \\
&= \hat{\pi}_{k+1} \hat{\iota}_{k+1} + \sum_{i \in [1,k+1]} \hat{\pi}_{i-1} \cdot \tau_i \cdot \kappa_i \cdot \hat{\iota}_{i-1}
\end{aligned}$$

□

Remark 15. Given morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{A}$. Then (1) and (2) are equivalent.

(1) The morphism f is a kernel of g.

(2) The morphism f is a kernel of gh.

Remark 16. Given morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{A}$. Then (1) and (2) are equivalent.

(1) The morphism h is a cokernel of g.

(2) The morphism h is a cokernel of fg.

Definition 17. Suppose given a set I. Suppose given additive subcategories $\mathcal{D}_i \subseteq \mathcal{A}$ for $i \in I$. We define a full additive subcategory $\mathcal{D} \subseteq \mathcal{A}$ by

$$\begin{aligned} \mathrm{Ob}(\mathcal{D}) := \quad \{C \in \mathrm{Ob}(\mathcal{A}) : \; & \text{there exists a finite subset} I_0 \subseteq I \text{ and } (D_i)_{i \in I_0} \\ & \text{with } D_i \in \mathrm{Ob}(\mathcal{D}_i) \text{ such that } C \cong \bigoplus_{i \in I_0} D_i\}. \end{aligned}$$

This category is denoted by $\sum\limits_{i \in I} \mathcal{D}_i \; := \; \mathcal{D}$.

Remark 18. The category $\sum\limits_{i \in I} \mathcal{D}_i \subseteq \mathcal{A}$ is closed under isomorphism.

2.3 Abelian categories

2.3.1 Definition of abelian categories

Remark 19. Consider the following situation in an additive category $\mathcal{A}$, where i is a kernel of f, where r is a cokernel of f, where $\tilde{c}$ is a cokernel of i and where $\tilde{i}$ is a kernel of r.

$$\begin{tikzcd} & & \tilde{C} & & \\ K \arrow[r, "i"] & X \arrow[ur, "\tilde{c}"] \arrow[rr, "f"] & & Y \arrow[r, "r"] & C \\ & & \tilde{K} \arrow[ur, "\tilde{i}"] & & \end{tikzcd}$$

Then there exists a unique morphism $\varphi \colon \tilde{C} \to \tilde{K}$ with $f = \tilde{c}\varphi\tilde{i}$.

$$\begin{tikzcd} & & \tilde{C} \arrow[dd, dotted, "\exists ! \varphi"] & & \\ K \arrow[r, "i"] & X \arrow[ur, "\tilde{c}"] \arrow[rr, "f"] & & Y \arrow[r, "r"] & C \\ & & \tilde{K} \arrow[ur, "\tilde{i}"] & & \end{tikzcd}$$

Definition 20. An additive category $\mathcal{A}$ is called *abelian* if it fulfils the following conditions (1) and (2).

(1) For every morphism in $\mathrm{Mor}(\mathcal{A})$ there exists a kernel and a cokernel.

(2) For every morphism in $\mathrm{Mor}(\mathcal{A})$, after adding kernels and cokernels as in Remark 19, the unique morphism φ is an isomorphism.

2.3.2 Properties of abelian categories

Suppose given an abelian category $\mathcal{A}$.

Definition 21. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is called a *short exact sequence* if f is a kernel of g and g is a cokernel of f.

Remark 22. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence in $\mathcal{A}$. Then (1), (2), and (3) are equivalent:

(1) The sequence is a short exact sequence.

(2) The morphism g is epic and f a kernel of g.

(3) The morphism f is monic and g a cokernel of f.

Lemma 23. *For every morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ there exists a factorisation*

$$X \xrightarrow{\bar{f}} I \xrightarrow{\dot{f}} Y$$

with $f = \bar{f}\dot{f}$. We call I an image *of f.*

Given two factorisations

$$X \xrightarrow{a} I \xrightarrow{b} Y$$

and

$$X \xrightarrow{a'} \hat{I} \xrightarrow{b'} Y$$

of $X \xrightarrow{f} Y$, there exists a unique isomorphism g such that $f = agb'$. That is, the diagram

$$\begin{tikzcd}
 & I \arrow[dr, "b"] \arrow[dd, "g", "\wr"'] & \\
X \arrow[ur, "a"] \arrow[rr, "f"] \arrow[dr, "a'"'] & & Y \\
 & \hat{I} \arrow[ur, "b'"'] &
\end{tikzcd}$$

commutes.

Proof. In an abelian category we can choose kernels and cokernels like in Remark 19. Then $\tilde{c}$ being epic, $\tilde{i}$ being monic and g being an isomorphism yields the wanted factorisation $f = \tilde{c}\varphi\tilde{i}$ with $\tilde{c}\varphi$ epic and $\tilde{i}$ monic.

Given two factorisations $f = ab = a'b'$ as above, we can choose a kernel $K \xrightarrow{i} X$ of f and a cokernel $Y \xrightarrow{r} C$ of f. Following Remark 15, i is a kernel of a so, according to Remark 22, a is a cokernel of i. Similarly, following Remark 16, r is a cokernel of b' and thus b' is a kernel of r.

Following Remark 19, there exists a unique isomorphism $I \xrightarrow{g} \hat{I}$ such that $agb' = f$. Because a is an epimorphism, $agb' = ab$ implies $gb' = b$. Because b' is a monomorphism, $agb' = a'b'$ implies $ag = a'$. This means the diagram is commutative. $\square$

Lemma 24. *Given a morphism $A \xrightarrow{f} B$ in $\mathcal{A}$. If $N_1 \longrightarrow A$ is a kernel of f and $B \longrightarrow N_2$ is a cokernel of f with $N_1, N_2 \cong 0_{\mathcal{A}}$ being zero objects in $\mathcal{A}$, then f is an isomorphism.*

Proof. Adding the kernel 1_B of $B \longrightarrow N_2$ and the cokernel 1_A of $N_1 \longrightarrow A$, we get the following commutative diagram.

$$\begin{tikzcd}
 & & A \arrow[dd, "\wr"', "f'"] & & \\
N_1 \arrow[r] & A \arrow[ur, "1"] \arrow[rr, "f"'] & & B \arrow[r] & N_2 \\
 & & B \arrow[ur, "1"'] & &
\end{tikzcd}$$

Thus $f = f'$ is an isomorphism. $\square$

Remark 25. Suppose given a category $\mathcal{C}$. The functor category $(\mathcal{C}, \mathcal{A})$ is abelian; cf. [**5,** §II.11]

Definition 26. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is called *exact in the middle* if for factorisations

$$\begin{tikzcd}
 & I \arrow[dr, tail, "b"] & & I' \arrow[dr, tail, "b'"] & \\
X \arrow[ur, two heads, "a"] \arrow[rr, "f"] & & Y \arrow[ur, two heads, "a'"] \arrow[rr, "g"] & & Z
\end{tikzcd}$$

the sequence $I \xrightarrow{b} Y \xrightarrow{a'} I'$ is short exact. Note that this is independent of the choice of I and I'.

Definition 27. A sequence $X_0 \xrightarrow{f_0} \dots \xrightarrow{f_{n-1}} X_n$ in $\mathcal{A}$ is called *exact* if

$$X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1}$$

is exact in the middle for every $i \in [1, n-1]$.

Remark 28. Suppose given a commutative quadrangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in an abelian category. Suppose given a kernel $K \xrightarrow{k} X$ of f and a kernel $K' \xrightarrow{k'} X'$ of f'. Suppose given a cokernel $Y \xrightarrow{c} C$ of f and a cokernel $Y' \xrightarrow{c'} C'$ of f'.

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[r, "c"] \arrow[d, "y"] & C \\ K' \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \arrow[r, "c'"] & C' \end{tikzcd}$$

There exist morphisms φ and ψ uniquely defined by the universal property of k' as kernel and c' as cokernel, respectively, that make the diagram commutative.

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\varphi"] & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[r, "c"] \arrow[d, "y"] & C \arrow[d, "\psi"] \\ K' \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \arrow[r, "c'"] & C' \end{tikzcd}$$

We call φ the induced morphism on these kernels. We call ψ the induced morphism on these cokernels.

Definition 29.

(i) A commutative quadrangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in $\mathcal{A}$ is called a *pullback* if for all morphisms $s, t \in \operatorname{Mor}(\mathcal{A})$ with $sf' = ty$, there exists exactly one morphism $u \in \operatorname{Mor}(\mathcal{A})$ with $uf = t$ and $ux = s$.

$$\begin{tikzcd} U \arrow[dr, dotted, "\exists! u"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ & X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

To indicate that the quadrangle is a pullback, we often write

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} .$$

(ii) A commutative quadrangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in $\mathcal{A}$ is called a *pushout* if for all morphisms $p, q \in \mathrm{Mor}(\mathcal{A})$ with $xp = fq$, there exists exactly one morphism $r \in \mathrm{Mor}\,\mathcal{A}$ with $f'r = p$ and $yr = q$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \arrow[ddr, bend left, "q"] & \\ X' \arrow[r, "f'"] \arrow[drr, bend right, "p"] & Y' \arrow[dr, dotted, "\exists! r"] & \\ & & R \end{tikzcd}$$

To indicate that the quadrangle is a pushout, we often write

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} .$$

Lemma 30. *For a commutative quadrangle*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in $\mathcal{A}$ the following assertions (1), (2) are equivalent

(1) The quadrangle is a pullback.

(2) In the diagonal sequence $X \xrightarrow{(f\ x)} Y \oplus X' \xrightarrow{\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)} Y'$, the morphism $(f\ x)$ is a kernel of $\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)$.

Proof. (1) $\Rightarrow$ (2):
Suppose given a morphism $M \xrightarrow{(m_1\ m_2)} Y \oplus X'$ with $(m_1\ m_2)\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right) = 0$. This is the case if and only if $m_1 y = m_2 f'$. As the quadrangle is a pullback, this means there exists exactly one $M \xrightarrow{m} X$ with $mf = m_1$ and $mx = m_2$. That is, there exists exactly one $M \xrightarrow{m} X$ with $m(f\ x) = (m_1\ m_2)$. This means that $(f\ x)$ is a kernel of $\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)$.

(2) $\Rightarrow$ (1):
Suppose given $M \xrightarrow{m_1} Y$ and $M \xrightarrow{m_2} X'$ with $m_1 y = m_2 f'$. That is, the following diagram

is commutative.

$$\begin{tikzcd}
M \arrow[rrd, bend left, "m_1"] \arrow[rdd, bend right, "m_2"'] & & \\
& X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
& X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

Thus $(m_1\ m_2)\binom{y}{-f'} = 0$. Hence there exists exactly one $M \xrightarrow{m} X$ with $m(f\ x) = (m_1\ m_2)$, that is, $mf = m_1$ and $mx = m_2$.

$$\begin{tikzcd}
M \arrow[rrd, bend left, "m_1"] \arrow[rdd, bend right, "m_2"'] \arrow[rd, dotted, "\exists! m"] & & \\
& X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
& X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

Thus the quadrangle is a pullback. $\square$

Lemma 31. *For a commutative quadrangle*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

in $\mathcal{A}$ the following assertions (1), (2) are equivalent

(1) The quadrangle is a pushout.

(2) In the diagonal sequence $X \xrightarrow{(f\ x)} Y \oplus X' \xrightarrow{\binom{y}{-f'}} Y'$, the morphism $\binom{y}{-f'}$ is a cokernel of $(f\ x)$.

Proof. The assertion is essentially dual to Lemma 30. $\square$

Remark 32. Suppose given the following commutative diagram in $\mathcal{A}$.

$$\begin{tikzcd}
& A \arrow[rr, "g"] \arrow[dl, "a"'] & & B \arrow[dl, "b"] \arrow[dd, "i_1"] \\
A' \arrow[rr, "g'"] \arrow[dd, "i_3"] & & B' \arrow[dd, "i_2"] & \\
& X \arrow[rr, "f"] \arrow[dl, "x"'] & & Y \arrow[dl, "y"] \\
X' \arrow[rr, "f'"] & & Y' &
\end{tikzcd}$$

Then there exists exactly one morphism $A \xrightarrow{i_4} X$ such that the resulting diagram

$$\begin{tikzcd}
& A \arrow[rr, "g"] \arrow[dl, "a"'] \arrow[dd, "i_4"] & & B \arrow[dl, "b"] \arrow[dd, "i_1"] \\
A' \arrow[rr, "g'"] \arrow[dd, "i_3"] & & B' \arrow[dd, "i_2"] & \\
& X \arrow[rr, "f"] \arrow[dl, "x"] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dl, "y"] \\
X' \arrow[rr, "f'"] & & Y' &
\end{tikzcd}$$

is commutative.

If the quadrangle (A, B, A', B') is a pullback and i_1, i_2, i_3 are isomorphisms, then i_4 is an isomorphism, too.

Lemma 33 (**[7,** §13.6.8]). *In $\mathcal{A}$, we suppose given a commutative triangle*

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "fg"] & & Z\,.
\end{tikzcd}$$

We add kernels and cokernels for every morphism and complete to the following commutative diagram in a unique way.

$$\begin{tikzcd}
& & M \arrow[rr] \arrow[dr] & & C \arrow[ddrr] & & \\
& & & Y \arrow[ur] \arrow[dr, "g"] & & & \\
L \arrow[uurr] \arrow[rr] & & X \arrow[ur, "f"] \arrow[rr, "fg"] & & Z \arrow[rr] \arrow[dr] & & D \arrow[dl] \\
& K \arrow[ul] \arrow[ur] & & & & E &
\end{tikzcd}$$

Then $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$ *is an exact sequence.*

We call this sequence the circumference sequence *of the triangle.*

Definition 34. A commutative quadrangle

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

in a category $\mathcal{A}$ is called a *square* if its diagonal sequence $X \xrightarrow{(f\ x)} Y \oplus X' \xrightarrow{\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)} Y'$ is a short exact sequence. A square is often denoted

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\Box"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' . \end{tikzcd}$$

So a commutative quadrangle is a square if and only if it is a pullback and a pushout.

Definition 35. A commutative quadrangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in $\mathcal{A}$ is called *weak square*, if its diagonal sequence $X \xrightarrow{(f\ x)} Y \oplus X' \xrightarrow{\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)} Y'$ is exact in the middle. A weak square is often denoted

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "+"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

In particular, each pullback and each pushout is a weak square.

Weak squares are called exact squares by Schubert [**7,** §13.4.1].

Lemma 36. *In $\mathcal{A}$, suppose given a weak square*

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "x"] \arrow[dr, phantom, "+"] & B \arrow[d, "y"] \\ C \arrow[r, "z"] & D . \end{tikzcd}$$

Then there exists a pushout

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "x"] & B \arrow[d, "p_1"] \\ C \arrow[r, "p_2"] \arrow[ur, phantom, "\lrcorner"] & P \end{tikzcd}$$

and a monomorphism $P \xrightarrow{p} D$ *such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "v"] \arrow[dd, "x"] & & B \arrow[dl, "p_1"] \arrow[dd, "y"] \\ & P \arrow[dr, "p"] & \\ C \arrow[ur, "p_2"] \arrow[rr, "z"] & & D \end{tikzcd}$$

Proof. We can choose a factorisation

$$\begin{tikzcd} A \arrow[r, "(v\ x)"] & B \oplus C \arrow[rr, "\left(\begin{smallmatrix} y \\ -z \end{smallmatrix}\right)"] \arrow[dr, "\left(\begin{smallmatrix} p_1 \\ -p_2 \end{smallmatrix}\right)"'] & & D \\ & & P \arrow[ur, "p"'] & \end{tikzcd}$$

As the diagonal sequence

$$A \xrightarrow{(v\ x)} B \oplus C \xrightarrow{\left(\begin{smallmatrix} y \\ -z \end{smallmatrix}\right)} D$$

is exact in the middle, $\left(\begin{smallmatrix} p_1 \\ -p_2 \end{smallmatrix}\right)$ is a cokernel of $A \xrightarrow{(v\ x)} B \oplus C$. Therefore

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "x"] & B \arrow[d, "p_1"] \\ C \arrow[r, "p_2"] & P \end{tikzcd}$$

is a pushout, p is a monomorphism and the diagram commutes. □

Lemma 37 ([**2,** §4.5.2 Lemma 134])**.** *Given a commutative quadrangle*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in $\mathcal{A}$. Suppose given a kernel $K \xrightarrow{k} X$ of f and a kernel $K' \xrightarrow{k'} X'$ of f'. Suppose given a cokernel $Y \xrightarrow{c} C$ of f and a cokernel $Y \xrightarrow{c'} C'$ of f'. Let φ be the induced morphism on the kernels and ψ the induced morphism on the cokernels.

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\varphi"] & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[r, "c"] \arrow[d, "y"] & C \arrow[d, "\psi"] \\ K' \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \arrow[r, "c'"] & C' \end{tikzcd}$$

Then the following assertions (1), (2), (3), (4) hold for the quadrangle (X, Y, X', Y').

(1) The quadrangle is a weak square if and only if φ is epic and ψ is monic.

(2) The quadrangle is a pullback if and only if φ is an isomorphism and ψ is monic.

(3) The quadrangle is a pushout if and only if φ is epic and ψ is an isomorphism.

(4) The quadrangle is a square if and only if φ and ψ are both isomorphisms.

Proof. We prove the following claims:

Claim (a): If the quadrangle is a weak square, then φ is epic and ψ is monic.

Suppose that the quadrangle is a weak square.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "+"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

That is in a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "(f\ x)"] \arrow[dr, "w"] & & Y \oplus X' \arrow[rr, "\binom{y}{-f'}"] \arrow[dr, "\binom{e}{-g'}"] & & Y' \\
& W \arrow[ur, "(g\ d)"] & & Z \arrow[ur, "z"] &
\end{tikzcd}$$

the sequence $W \xrightarrow{(g\ d)} Y \oplus X' \xrightarrow{\binom{e}{-g'}} Z$ is short exact.

We prove that φ is epic:

We consider the commutative triangle

$$\begin{tikzcd}
& Y \oplus X' \arrow[dr, "\binom{y}{f'}"] & \\
X' \arrow[ur, "(0\ 1)"] \arrow[rr, "-f'"] & & Y'
\end{tikzcd}$$

According to Lemma 33, in the following extension with kernels and cokernels

$$\begin{tikzcd}
& & W \arrow[rrrr, "g"] \arrow[dr, "(g\ d)"] & & & & Y \arrow[ddrr] & & \\
& & & Y \oplus X' \arrow[urrr, "\binom{1}{0}"] \arrow[drrr, "\binom{y}{-f'}"] & & & & & \\
K' \arrow[uurr, "i"] \arrow[rr, "k'"] & & X' \arrow[ur, "(0\ 1)"] \arrow[rrrr, "-f'"] & & & & Y' \arrow[rr] \arrow[dr] & & C' \arrow[dl] \\
& 0 \arrow[ul] \arrow[ur] & & & & & & E &
\end{tikzcd}$$

i is a kernel of g. We let $j := kw$ and expand the diagram as follows.

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[drr, "j"'] & X \arrow[dr, "w"] \arrow[drrr, "f"] & & & & & \\
& & W \arrow[rr, "g"] \arrow[dr, "(g\ d)"] & & Y \arrow[ddrr] & & \\
& & & Y \oplus X' \arrow[ur, "\binom{1}{0}"'] \arrow[dr, "\binom{y}{-f'}"] & & & \\
K' \arrow[uurr, "i"] \arrow[rr, "k'"] & & X' \arrow[ur, "(0\ 1)"] \arrow[rr, "-f'"] & & Y' \arrow[rr] \arrow[dr] & & C' \arrow[dl] \\
& 0 \arrow[ul] \arrow[ur] & & & & E &
\end{tikzcd}
$$

Then $jg = kf = 0$, so there exists exactly one morphism p with $pi = j$.

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[drr, "j"'] \arrow[ddd, "p"] & X \arrow[dr, "w"] \arrow[drrr, "f"] & & & & & \\
& & W \arrow[rr, "g"] \arrow[dr, "(g\ d)"] & & Y \arrow[ddrr] & & \\
& & & Y \oplus X' \arrow[ur, "\binom{1}{0}"'] \arrow[dr, "\binom{y}{-f'}"] & & & \\
K' \arrow[uurr, "i"] \arrow[rr, "k'"] & & X' \arrow[ur, "(0\ 1)"] \arrow[rr, "-f'"] & & Y' \arrow[rr] \arrow[dr] & & C' \arrow[dl] \\
& 0 \arrow[ul] \arrow[ur] & & & & E &
\end{tikzcd}
$$

This morphism also fulfils $pk'(0\ 1) = pi(g\ d) = j(g\ d)$ and therefore

$$
pk' = pk'(0\ 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = j(g\ d)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = jd = kwd = kx.
$$

Thus $p = \varphi$.

The morphism φ is part of the circumference sequence of

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "w"'] & & Y \\
& W \arrow[ur, "g"'] &
\end{tikzcd}
$$

as follows.

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[ddr, "\varphi"] & X \arrow[rr, "f"] \arrow[dr, "w"] & & Y \\
& & W \arrow[ur, "g"] \arrow[dr] & \\
& K' \arrow[ur, "i"] \arrow[rr] & & 0
\end{tikzcd}
$$

Thus φ is epic; cf. Lemma 33.

Dually, ψ is monic.

Claim (b): If the quadrangle is a pullback, then φ is an isomorphism and ψ is monic.

The quadrangle is a weak square as in

$$
\begin{tikzcd}
X \arrow[rr, "(f\ x)"] \arrow[dr, "1"'] & & Y \oplus X' \arrow[rr, "\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)"] \arrow[dr, "\left(\begin{smallmatrix} e \\ -g' \end{smallmatrix}\right)"] & & Y' \\
& X \arrow[ur, "(f\ x)"] & & Z \arrow[ur] &
\end{tikzcd}
$$

$(f\ x)$ is a kernel of $\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)$ and thus also a kernel of $\left(\begin{smallmatrix} e \\ -g' \end{smallmatrix}\right)$.

Then in the situation of (a) both $K \xrightarrow{k} X$ and $K' \xrightarrow{i} X$ are kernels of $X \xrightarrow{f} Y$ and thus φ is an isomorphism.

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[drr, "k"'] \arrow[ddd, "\varphi", "\wr"'] & X \arrow[dr, "1"] \arrow[drrr, "f"] & & & & & \\
& & X \arrow[rr, "f"] \arrow[dr, "(f\ x)"] & & Y \arrow[ddrr] & & \\
& & & Y \oplus X' \arrow[ur, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[dr, "\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)"] & & & \\
K' \arrow[uurr, "i"] \arrow[rr, "k'"] & & X' \arrow[ur, "(0\ 1)"] \arrow[rr, "-f'"] & & Y' \arrow[rr] \arrow[dr] & & C' \arrow[dl] \\
& 0 \arrow[ul] \arrow[ur] & & & & E &
\end{tikzcd}
$$

Claim (c): If the quadrangle is a pushout, then φ is epic and ψ is an isomorphism.

The assertion is dual to the previous Claim (b).

Claim (d): If φ is epic and ψ is monic, then the quadrangle is a weak square.

The quadrangle can be written as follows

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "x"'] & & Y \arrow[dl, "p_2"'] \arrow[dd, "y"] \\
& P \arrow[dr, "p"] & \\
X' \arrow[ur, "p_1"] \arrow[rr, "f'"] & & Y'
\end{tikzcd}$$

with (X, Y, X', P) being a pushout; cf. Lemma 36. It suffices to show that p is a monomorphism. From Claim (c) we know that the induced morphisms on kernels of f and p_1 is an epimorphism and the induced morphism on cokernels of f and p_1 is an isomorphism.

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\hat{\varphi}"'] & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[r, "c"] \arrow[d, "p_2"] & C \arrow[d, "\hat{\psi}", "\wr"'] \\
K_1 \arrow[r, "k_1"] & X' \arrow[r, "p_1"] & P \arrow[r, "c_1"] & C_1
\end{tikzcd}$$

We form the following commutative diagram containing a circumference sequence of (X', P, Y'):

$$\begin{tikzcd}
K \arrow[rr, "k"] \arrow[ddd, "\varphi"] \arrow[ddddr, "\hat{\varphi}"] & & X \arrow[rrr, "f"] \arrow[ddd, "x"] & & & Y \arrow[rr, "c"] \arrow[ddl, "p_2"'] \arrow[ddd, "y"] & & C \arrow[dl, "\hat{\psi}"', "\sim"] \arrow[ddd, "\psi"] \\
& & & K_p \arrow[rrr, "q"] \arrow[dr] & & & C_1 \arrow[ddr, "\hat{c}"] & \\
& & & & P \arrow[urr, "c_1"] \arrow[dr, "p"] & & & \\
K' \arrow[uurrr, "\hat{k}"] \arrow[rr, "k'"] & & X' \arrow[urr, "p_1"] \arrow[rrr, "f'"] & & & Y' \arrow[rr, "c'"] \arrow[dr] & & C' \arrow[dl] \\
& K_1 \arrow[ul, "k_1'"] \arrow[ur, "k_1"] & & & & & C_p &
\end{tikzcd}$$

In particular, we have $\hat{\varphi} k_1' k' = \varphi k'$, hence $\hat{\varphi} k_1' = \varphi$. Moreover, we have $c\hat{\psi}\hat{c} = c\psi$, hence $\hat{\psi}\hat{c} = \psi$.

Since ψ is monic and $\hat{\psi}$ is an isomorphism, $\hat{c}$ is monic, whence $q = 0$. Since $\varphi = \hat{\varphi} k_1'$ is an epimorphism, k_1' is epic and $\hat{k} = 0$. Then $K_p \cong 0_{\mathcal{A}}$ and p is monic.

$$
\begin{tikzcd}
K \arrow[rr, "k"] \arrow[ddd, "\varphi"] \arrow[ddddr, "\hat{\varphi}"] & & X \arrow[rr, "f"] \arrow[ddd, "x"] & & Y \arrow[rr, "c"] \arrow[ddl, "p_2"'] \arrow[ddd, "y"] & & C \arrow[dl, "\hat{\psi}"', "\sim"] \arrow[ddd, "\psi"] \\
& & & 0 \arrow[rr, "0"] \arrow[d, "0"] & & C_1 \arrow[ddr, "\hat{c}"] & \\
& & & P \arrow[urr, "c_1"] \arrow[dr, "p"] & & & \\
K' \arrow[rr, "k'"] \arrow[uurrr, "0"] & & X' \arrow[rr, "f'"] \arrow[ur, "p_1"] & & Y' \arrow[rr, "c'"] \arrow[dr] & & C' \arrow[dl] \\
& K_1 \arrow[ul, "k'_1"] \arrow[ur, "k_1"] & & & & C_p &
\end{tikzcd}
$$

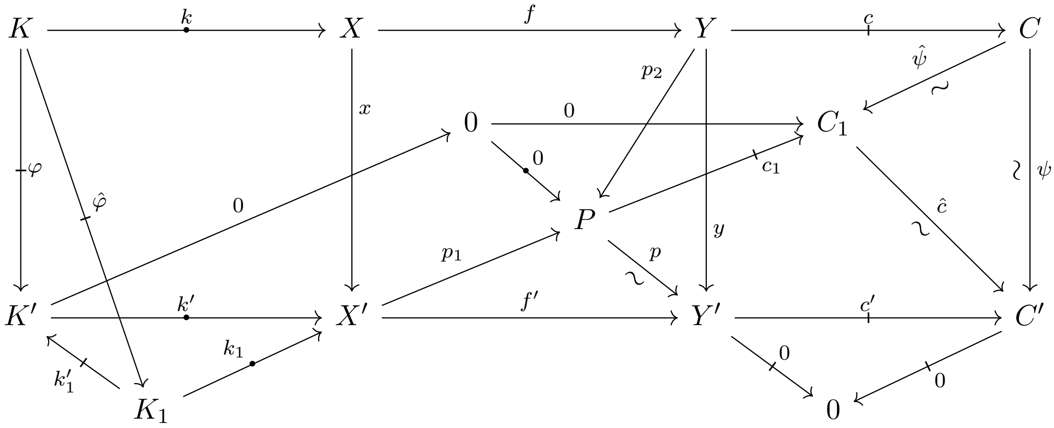
Thus (X, Y, X', Y') is a weak square.

Claim (e): If φ is epic and ψ is an isomorphism, then the quadrangle is a pushout.

We keep the notation of Claim (d) and obtain the following commutative diagram.

$$
\begin{tikzcd}
K \arrow[rr, "k"] \arrow[ddd, "\varphi"] \arrow[ddddr, "\hat{\varphi}"] & & X \arrow[rr, "f"] \arrow[ddd, "x"] & & Y \arrow[rr, "c"] \arrow[ddl, "p_2"'] \arrow[ddd, "y"] & & C \arrow[dl, "\hat{\psi}"', "\sim"] \arrow[ddd, "\psi", "\wr"'] \\
& & & 0 \arrow[rr, "0"] \arrow[d, "0"] & & C_1 \arrow[ddr, "\hat{c}"] & \\
& & & P \arrow[urr, "c_1"] \arrow[dr, "p"] & & & \\
K' \arrow[rr, "k'"] \arrow[uurrr, "0"] & & X' \arrow[rr, "f'"] \arrow[ur, "p_1"] & & Y' \arrow[rr, "c'"] \arrow[dr] & & C' \arrow[dl, "c'_1"] \\
& K_1 \arrow[ul, "k'_1"] \arrow[ur, "k_1"] & & & & C_p &
\end{tikzcd}
$$

Then $\hat{c}$ is an isomorphism, thus $C_p \cong 0_{\mathcal{A}}$. Then p is an isomorphism, cf. Lemma 24, and the quadrangle (X, Y, X', Y') is a pushout.



Claim (f): If φ is an isomorphism and ψ is a monomorphism, then the quadrangle is a pullback.

The assertion is dual to Claim (e).

Now the assertions (1), (2), (3) and (4) are deduced as follows:

(1) follows from Claims (a) and (d).

(2) follows from Claims (b) and (f).

(3) follows from Claims (c) and (e).

(4) follows from (2) and (3).

$\square$

Lemma 38. *Suppose given $N, N' \in \mathrm{Ob}(\mathcal{A})$ with $N, N' \cong 0_{\mathcal{A}}$. Then every quadrangle*

$$\begin{tikzcd} N \arrow[r, "0"] & B \\ A \arrow[u, "0"] \arrow[r, "0"] & N' \arrow[u, "0"'] \end{tikzcd}$$

in $\mathcal{A}$ is a weak square.

Proof. In the diagram

$$\begin{tikzcd} 0_{\mathcal{A}} \arrow[r, "0"] & N \arrow[r, "0"] & B \arrow[r, "1"] & B \\ A \arrow[u, "0"] \arrow[r, "1_A"] & A \arrow[u, "0"] \arrow[r, "0"] & N' \arrow[u, "0"] \arrow[r, "0"] & 0_{\mathcal{A}} \arrow[u, "0"] \end{tikzcd}$$

we have inserted kernels and cokernels of the horizontal morphisms and the induced morphisms on the kernels and cokernels. The induced morphism on the kernels is epic. The induced morphism on the cokernels is monic. Therefore the quadrangle is a weak square. $\square$

Lemma 39. *Every quadrangle*

$$
\begin{tikzcd}
B \arrow[r, "b", "\sim"'] & B' \\
A \arrow[u, "f"] \arrow[r, "a", "\sim"'] & A' \arrow[u, "f'"]
\end{tikzcd}
$$

in $\mathcal{A}$ is a square. In particular, the quadrangle is a weak square.

Proof. In the diagram

$$
\begin{tikzcd}
0_{\mathcal{A}} \arrow[r, "\bullet" marking] & B' \arrow[r, "b"] & B \arrow[r, "0", "|" marking] & 0_{\mathcal{A}} \\
0_{\mathcal{A}} \arrow[u, "\wr"'] \arrow[r, "\bullet" marking] & A \arrow[u, "f"] \arrow[r, "a"] & A' \arrow[u, "f'"] \arrow[r, "0", "|" marking] & 0_{\mathcal{A}} \arrow[u, "\wr"']
\end{tikzcd}
$$

we have inserted kernels and cokernels of the horizontal morphisms and the induced morphisms on the kernels and cokernels. The induced morphisms are isomorphisms. Therefore the quadrangle is a square and in particular a weak square. □

Lemma 40. *Given two weak squares*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "+"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "y"] \arrow[dr, phantom, "+"] & Z \arrow[d, "z"] \\
Y' \arrow[r, "g'"] & Z'
\end{tikzcd}
$$

in $\mathcal{A}$, their composite

$$
\begin{tikzcd}
X \arrow[r, "fg"] \arrow[d, "x"] & Z \arrow[d, "z"] \\
X' \arrow[r, "f'g'"] & Z'
\end{tikzcd}
$$

is a weak square.

Proof. We choose kernels and cokernels of x, y, z like in Remark 28. We get the following commutative diagram.

$$\begin{tikzcd}
K \arrow[r, "|" marking] \arrow[rr, bend left, "\varphi"] \arrow[d, "\bullet" marking] & \hat{K} \arrow[r, "|" marking] \arrow[d, "\bullet" marking] & \tilde{K} \arrow[d, "\bullet" marking] \\
X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[r, "g"] \arrow[d, "y"] & Z \arrow[d, "z"] \\
X' \arrow[r, "f'"] \arrow[d, "|" marking] & Y' \arrow[r, "g'"] \arrow[d, "|" marking] & Z \arrow[d, "|" marking] \\
C \arrow[r, "\bullet" marking] \arrow[rr, bend right, "\psi"'] & \hat{C} \arrow[r, "\bullet" marking] & \tilde{C}
\end{tikzcd}$$

Then the induced morphism on the kernels $K \xrightarrow{\varphi} \tilde{K}$ is an epimorphism and the induced morphism on the cokernels $C \xrightarrow{\psi} \tilde{C}$ is a monomorphism. Thus (X, Z, X', Z') is a weak square; cf. Lemma 37.

Another proof can be found in [**7,** §13.4.5]. $\square$

Lemma 41. *Given weak squares*

$$\begin{tikzcd}
C \arrow[r, "d"] & D \\
A \arrow[u, "c"] \arrow[r, "a"] \arrow[ur, phantom, "+"] & B \arrow[u, "b"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "y"] & Y \\
U \arrow[u, "x"] \arrow[r, "u"] \arrow[ur, phantom, "+"] & V \arrow[u, "v"']
\end{tikzcd}$$

in $\mathcal{A}$, their direct sum

$$\begin{tikzcd}[column sep=large]
C \oplus X \arrow[r, "\left(\begin{smallmatrix} d & 0 \\ 0 & y \end{smallmatrix}\right)"] & D \oplus Y \\
A \oplus U \arrow[u, "\left(\begin{smallmatrix} c & 0 \\ 0 & x \end{smallmatrix}\right)"] \arrow[r, "\left(\begin{smallmatrix} a & 0 \\ 0 & u \end{smallmatrix}\right)"] \arrow[ur, phantom, "+"] & B \oplus V \arrow[u, "\left(\begin{smallmatrix} b & 0 \\ 0 & v \end{smallmatrix}\right)"']
\end{tikzcd}$$

is a weak square.

Proof. By Lemma 37, we get that the induced morphism between kernels is an epimorphism.

$$\begin{tikzcd}
K' \arrow[r, "\bullet" marking, "k'"] & C \arrow[r, "d"] & D \\
K \arrow[u, "|" marking, "\varphi"] \arrow[r, "\bullet" marking, "k"] & A \arrow[u, "c"] \arrow[r, "a"] \arrow[ur, phantom, "+"] & B \arrow[u, "b"']
\end{tikzcd}$$

$$
\begin{tikzcd}
L' \arrow[r, "l'", "\bullet" marking] & X \arrow[r, "y"] & Y \\
L \arrow[u, "\hat{\varphi}", "-" marking] \arrow[r, "l", "\bullet" marking] & U \arrow[u, "x"] \arrow[r, "u"] \arrow[ur, phantom, "+"] & V \arrow[u, "v"]
\end{tikzcd}
$$

For the direct sum, we get

$$
\begin{tikzcd}[column sep=large]
K' \oplus L' \arrow[r, "\left(\begin{smallmatrix} k' & 0 \\ 0 & l' \end{smallmatrix}\right)", "\bullet" marking] & C \oplus X \arrow[r, "\left(\begin{smallmatrix} d & 0 \\ 0 & y \end{smallmatrix}\right)"] & D \oplus Y \\
K \oplus L \arrow[u, "\left(\begin{smallmatrix} \varphi & 0 \\ 0 & \hat{\varphi} \end{smallmatrix}\right)", "-" marking] \arrow[r, "\left(\begin{smallmatrix} k & 0 \\ 0 & l \end{smallmatrix}\right)", "\bullet" marking] & A \oplus U \arrow[u, "\left(\begin{smallmatrix} c & 0 \\ 0 & x \end{smallmatrix}\right)"] \arrow[r, "\left(\begin{smallmatrix} a & 0 \\ 0 & u \end{smallmatrix}\right)"] \arrow[ur, phantom, "+"] & B \oplus V \arrow[u, "\left(\begin{smallmatrix} b & 0 \\ 0 & v \end{smallmatrix}\right)"]
\end{tikzcd}
$$

with kernels $K' \oplus L' \xrightarrow{\left(\begin{smallmatrix} k' & 0 \\ 0 & l' \end{smallmatrix}\right)} C \oplus X$ and $K \oplus L \xrightarrow{\left(\begin{smallmatrix} k & 0 \\ 0 & l \end{smallmatrix}\right)} A \oplus U$, where the induced morphism is $\left(\begin{smallmatrix} \varphi & 0 \\ 0 & \hat{\varphi} \end{smallmatrix}\right)$, which is an epimorphism.

Dually we get that the induced morphism on the cokernels is a monomorphism.

By Lemma 37, the quadrangle is a weak square. $\square$

Lemma 42. *Suppose $\mathcal{A}$ to have enough injective objects. Then every diagram*

$$
\begin{tikzcd}
Y & \\
X \arrow[u, "x"] \arrow[r, "f"] & Z
\end{tikzcd}
$$

in $\mathcal{A}$ can be completed to a weak square

$$
\begin{tikzcd}
Y \arrow[r, "g"] & I \\
X \arrow[u, "x"] \arrow[r, "f"] \arrow[ur, phantom, "+"] & Z \arrow[u, "z"]
\end{tikzcd}
$$

with I injective.

Proof. We choose a pushout

$$
\begin{tikzcd}
Y \arrow[dr, "p_1"] & & \\
& P & \\
X \arrow[uu, "x"] \arrow[rr, "f"] & & Z \arrow[ul, "p_2"']
\end{tikzcd}
$$

and a monomorphism $P \xrightarrow{p} I$ with I injective. This exists, as we assumed $\mathcal{A}$ to have enough injective objects. Then we get the following weak square, letting $z := p_2 p$ and $g := p_1 p$.

$$
\begin{tikzcd}
Y \arrow[rr, "g"] \arrow[dr, "p_1"] & & I \\
& P \arrow[ur, "p"] & \\
X \arrow[uu, "x"] \arrow[rr, "f"] & & Z \arrow[uu, "z"'] \arrow[ul, "p_2"]
\end{tikzcd}
$$

□

Lemma 43. *Suppose given a pullback*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}
$$

in $\mathcal{A}$.

Then f is a monomorphism if and only if f' is a monomorphism.

Proof. Consider the following diagram as in Lemma 37 (2).

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\varphi", "\wr"'] & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[r, "c"] \arrow[d, "y"] & C \arrow[d, "\psi"] \\
K' \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \arrow[r, "c'"] & C'
\end{tikzcd}
$$

Then

$$f \text{ monic} \;\Leftrightarrow\; K \cong 0_{\mathcal{A}} \;\Leftrightarrow\; K' \cong 0_{\mathcal{A}} \;\Leftrightarrow\; f' \text{ monic}.$$

□

Corollary 44. *Suppose given a pullback*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}
$$

in $\mathcal{A}$ with f' monic and $Y \cong 0_{\mathcal{A}}$. Then $X \cong 0_{\mathcal{A}}$.

Proof. By Lemma 43, $f\colon X \to Y$ is monic. As $Y \cong 0_{\mathcal{A}}$, we have $X \cong 0_{\mathcal{A}}$. □

Lemma 45. *Suppose given a weak square in $\mathcal{A}$ with $X \xrightarrow{f} Y$ monic.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "+"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

Then f' is monic, too. Moreover, (X, Y, X', Y') is a pullback.

Proof. As f is a monomorphism, $0_{\mathcal{A}}$ is a kernel of f.

$$\begin{tikzcd}
0_{\mathcal{A}} \arrow[r, "k"] \arrow[d, "\varphi"] & X \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[r, "c"] \arrow[d, "y"] & C \arrow[d, "\psi"] \\
K' \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \arrow[r, "c'"] & C'
\end{tikzcd}$$

Due to φ being epic, $K' \cong 0_{\mathcal{A}}$ and thus f' is monic. Moreover, φ is an isomorphism. Hence, (X, Y, X', Y') is a pullback. $\square$

Lemma 46. *Suppose given a weak square in $\mathcal{A}$ with $X \cong 0_{\mathcal{A}}$.*

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, "0"] \arrow[dr, phantom, "+"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

Then f' and y are both monic.

Proof. Since $X \xrightarrow{0} Y$ is monic, f' is monic. Since $X \xrightarrow{0} X'$ is monic, y is monic; cf. Lemma 43. $\square$

Lemma 47. *Suppose given a commutative diagram*

$$\begin{tikzcd}
 & U' \arrow[rr, "g'"] & & V' \\
X' \arrow[ur, "a_1"] \arrow[rr, "f'"] & & Y' & \\
 & U \arrow[uu, "u"] \arrow[rr, "g"] & & V \arrow[uu, "v"] \\
X \arrow[uu, "x"] \arrow[ur, "a_2"] \arrow[rr, "f"] & & Y \arrow[uu, "y"] \arrow[ur, "a_3"] &
\end{tikzcd}$$

in $\mathcal{A}$ where

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "x"] \arrow[r, "f"] \arrow[ur, phantom, "+"] & Y \arrow[u, "y"'] \end{tikzcd}$$

is a weak square and V' is injective. Then there exists a morphism $Y' \xrightarrow{a_4} V'$ for which the following diagram is commutative.

$$\begin{tikzcd} & U' \arrow[rr, "g'"] & & V' \\ X' \arrow[ur, "a_1"] \arrow[rr, "f'"] & & Y' \arrow[ur, "a_4"] & \\ & U \arrow[uu, "u"] \arrow[rr, "g"] & & V \arrow[uu, "v"'] \\ X \arrow[uu, "x"] \arrow[ur, "a_2"] \arrow[rr, "f"] & & Y \arrow[uu, "y"] \arrow[ur, "a_3"] & \end{tikzcd}$$

Proof. We can insert a pushout, cf. Lemma 36, and obtain the following commutative diagram:

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr, "p_1"] & & Y' \\ & P \arrow[ur, "p"] & \\ X \arrow[uu, "x"] \arrow[rr, "f"] & & Y \arrow[uu, "y"'] \arrow[ul, "p_2"] \end{tikzcd}$$

Thus there exists a morphism $P \xrightarrow{\varphi} V'$ such that the following diagram commutes

$$\begin{tikzcd} & & V' \\ X' \arrow[urr, bend left, "a_1 g'"] \arrow[r, "p_1"] & P \arrow[ur, "\varphi"] & \\ X \arrow[u, "x"] \arrow[r, "f"] & Y \arrow[u, "p_2"] \arrow[uur, bend right, "a_3 v"'] & \end{tikzcd}$$

As p is monic and V' is injective, there exists a morphism $Y' \xrightarrow{a_4} V'$ with $pa_4 = \varphi$.

Then $a_1 g' = p_1 \varphi = p_1 p a_4 = f' a_4$ and $y a_4 = p_2 p a_4 = p_2 \varphi = a_3 v$, so that the resulting diagram is commutative.

$$\begin{tikzcd}
 & U' \arrow[rr, "g'"] & & V' \\
X' \arrow[ur, "a_1"] \arrow[rr, "f'"] & & Y' \arrow[ur, "a_4"] & \\
 & U \arrow[uu, "u"] \arrow[rr, "g"] & & V \arrow[uu, "v"'] \\
X \arrow[uu, "x"] \arrow[ur, "a_2"] \arrow[rr, "f"] & & Y \arrow[uu, "y"] \arrow[ur, "a_3"] &
\end{tikzcd}$$

□

Definition 48. Recall that a morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ is called a *split monomorphism* or *split monic* if there exists a morphism $Y \xrightarrow{g} X$ in $\mathcal{A}$ with $fg = 1_X$. To indicate that f is split monic, we sometimes write $\begin{tikzcd} X \arrow[r, tail, "f", "\bullet" marking] & Y \end{tikzcd}$.

A morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ is called a *split epimorphism* or *split epic* if there exists a morphism $Y \xrightarrow{g} X$ in $\mathcal{A}$ with $gf = 1_Y$. To indicate that f is split epic, we sometimes write $\begin{tikzcd} X \arrow[r, tail, "f", "|" marking] & Y \end{tikzcd}$.

A morphism $U \xrightarrow{u} V$ in an abelian category $\mathcal{A}$ is called *split* if there exist $U \xrightarrow{f} W$ split epic and $W \xrightarrow{g} V$ split monic with $u = fg$. To indicate that u is split, we sometimes write $\begin{tikzcd} U \arrow[r, tail, "u"] & V \end{tikzcd}$.

Remark 49. If $U \xrightarrow{u} V$ is split and $u = fg$ with $U \xrightarrow{f} W$ epic and $W \xrightarrow{g} V$ monic, then f is split epic, g is split monic and W is an image of u; cf. Lemma 23.

Lemma 50. *Given a short exact sequence* $\begin{tikzcd} A \arrow[r, "f", "\bullet" marking] & B \arrow[r, "g", "|" marking] & C \end{tikzcd}$ *in* $\mathcal{A}$*. Then the following assertions (1), (2), (3) are equivalent.*

(1) The morphism g is split epic.

(2) The morphism f is split monic.

(3) There exists an isomorphism $B \underset{\sim}{\xrightarrow{\varphi}} A \oplus C$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f", "\bullet" marking] \arrow[d, "1_A"] & B \arrow[r, "g", "|" marking] \arrow[d, "\varphi", "\wr"'] & C \arrow[d, "1_C"] \\
A \arrow[r, tail, "(1\ 0)", "\bullet" marking] & A \oplus C \arrow[r, tail, "\binom{0}{1}", "|" marking] & C
\end{tikzcd}$$

Proof. (1) $\Rightarrow$ (3):
The morphism g is split epic, thus there exists $C \xrightarrow{g'} B$ with $g'g = 1_C$. By applying the

universal property of the kernel to

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, tail, "g"] & C \\ B \arrow[ur, "1_B - gg'"'] & & \end{tikzcd}$$

we get $B \xrightarrow{p} A$ with $pf = 1_B - gg'$. Then the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1"] & B \arrow[r, tail, "g"] \arrow[d, "(p\ g)"] & C \arrow[d, "1"] \\ A \arrow[r, tail, "(1\ 0)"] & A \oplus C \arrow[r, tail, "\binom{0}{1}"] & C \end{tikzcd}$$

and $(p\ g)$ is an isomorphism with inverse $A \oplus C \xrightarrow{\binom{f}{g'}} B$:
We have $(p\ g)\binom{f}{g'} = pf + gg' = 1_B - gg' + gg' = 1_B$ and $\binom{f}{g'}(p\ g) = \begin{pmatrix} fp & fg \\ g'p & g'g \end{pmatrix}$ with

$$\begin{aligned} & fpf = f(1_B - gg') = f \overset{f \text{ monic}}{\Rightarrow} fp = 1_A \\ & fg = 0 \\ & g'pf = g'(1_B - gg') = g' - g' = 0 \overset{f \text{ monic}}{\Rightarrow} g'p = 0 \\ & g'g = 1_C. \end{aligned}$$

$(2) \Rightarrow (3)$:
Dual to $(1) \Rightarrow (3)$.
$(3) \Rightarrow (1), (2)$:
Suppose given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1_A"] & B \arrow[r, "g"] \arrow[d, "\varphi", "\wr"'] & C \arrow[d, "1_C"] \\ A \arrow[r, tail, "(1\ 0)"] & A \oplus C \arrow[r, tail, "\binom{0}{1}"] & C \end{tikzcd}$$

commutative. Then $f\left(\varphi\binom{1}{0}\right) = 1_A$ and $\left((0\ 1)\varphi^{-1}\right) g = 1_C$, thus f is split monic and g is split epic. $\square$

Lemma 51. *Suppose given a split monomorphism* $A \overset{f}{\rightarrowtail} B$ *in* $\mathcal{A}$*. Suppose given a cokernel* $B \xrightarrow{g} C$ *of* f *and suppose given* $B \xrightarrow{f'} A$ *with* $ff' = 1_A$.

$$\begin{tikzcd} A \arrow[r, tail, shift left, "f"] & B \arrow[l, shift left, "f'"] \arrow[r, tail, "g"] & C \end{tikzcd}$$

Then there exists a morphism $C \xrightarrow{g'} B$ such that $(B, (f', g), (f, g'))$ is a direct sum of (A, C).

Proof. Like in the proof of Lemma 50 we get g' by the universal property of the cokernel

$$\begin{tikzcd} A \arrow[r, tail, "f"] & B \arrow[r, tail, "g"] \arrow[dr, "1_B - f'f"'] & C \arrow[d, "g'"] \\ & & B \end{tikzcd}$$

and $\binom{f}{g'}(f' \; g) = \begin{pmatrix} ff' & fg \\ g'f' & g'g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

By definition g' fulfils $1_B = gg' + f'f$, thus $(B, (f', g), (f, g'))$ is a direct sum of (A, C). □

Lemma 52.

(1) A morphism $U \xrightarrow{u} V$ in $\mathcal{A}$ is split epic if and only if there exist $K \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $U \xrightarrow{\varphi} K \oplus V$ such that

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "\varphi"] & V \arrow[d, "1_V"] \\ K \oplus V \arrow[r, "\binom{0}{1}"] & V \end{tikzcd}$$

is commutative.

(2) A morphism $U \xrightarrow{u} V$ in $\mathcal{A}$ is split monic if and only if there exist $C \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $U \xrightarrow{\psi} U \oplus C$ such that

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "1_U"'] & V \arrow[d, "\psi"] \\ U \arrow[r, "(1\;0)"] & U \oplus C \end{tikzcd}$$

is commutative.

Proof. Ad (1): Suppose given $U \xrightarrow{u} V$ split epic. Choose a kernel $K \xrightarrow{v} U$ of u. Then we get an isomorphism φ with the wanted property by Lemma 50.

Suppose given an isomorphism φ with

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "\varphi"] & V \arrow[d, "1_V"] \\ K \oplus V \arrow[r, "\binom{0}{1}"] & V \end{tikzcd}$$

commutative. Then $(0 \; 1)\varphi^{-1}u = 1_V$ and u is split epic.

Ad (2): This is dual to (1). □

Lemma 53. *A morphism* $U \xrightarrow{u} V$ *in* $\mathcal{A}$ *is split if and only if there exist* $X, Y, Z \in \mathrm{Ob}(\mathcal{A})$ *with* $U \xrightarrow{u} V$ *isomorphic to* $X \oplus Y \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} Y \oplus Z$ *in* $(\Delta_1, \mathcal{A})$.

Proof. Suppose that $U \xrightarrow{u} V$ is split. We have $U \overset{f}{\rightarrowtail} W \overset{g}{\rightarrowtail} V$ with $fg = u$. As f is split epic and g is split monic, we get $K, C \in \mathrm{Ob}(\mathcal{A})$ and isomorphisms $U \xrightarrow{\varphi} K \oplus V$ and $V \xrightarrow{\psi} U \oplus C$ such that

$$\begin{tikzcd}
U \arrow[r, tail, "f"] \arrow[rr, bend left, tail, "u"] \arrow[d, "\varphi", "\wr"'] & W \arrow[r, tail, "g"] \arrow[d, "1_W", "\wr"'] & V \arrow[d, "\psi", "\wr"'] \\
K \oplus W \arrow[r, tail, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[rr, bend right, tail, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"'] & W \arrow[r, tail, "(1\ 0)"] & W \oplus C
\end{tikzcd}$$

commutes; Lemma 52 (1), Remark 52 (2).

Suppose that $U \xrightarrow{u} V$ is isomorphic to $X \oplus Y \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} Y \oplus Z$. So we have the following commutative quadrangle.

$$\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, "\varphi", "\wr"'] & V \arrow[d, "\psi", "\wr"'] \\
X \oplus Y \arrow[r, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] & Y \oplus Z
\end{tikzcd}$$

We can expand the quadrangle as follows

$$\begin{tikzcd}
U \arrow[r, "\varphi\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d, "\varphi", "\wr"'] & Y \arrow[r, "(1\ 0)\psi^{-1}"] \arrow[d, "1_Y", "\wr"'] & V \arrow[d, "\psi", "\wr"'] \\
X \oplus Y \arrow[r, tail, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & Y \arrow[r, tail, "(1\ 0)"] & Y \oplus Z
\end{tikzcd}$$

Then $\varphi\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is split epic, $(1\ 0)\psi^{-1}$ is split monic and $\varphi\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)(1\ 0)\psi^{-1} = u$. Therefore u is split. $\square$

Remark 54. For any commutative quadrangle

$$\begin{tikzcd}
B \arrow[r, "w"] & D \\
A \arrow[r, "v"] \arrow[u, "u"] & C \arrow[u, "z"']
\end{tikzcd}$$

in $\mathcal{A}$ we can choose an image I_w of w and an image I_v of v as well as kernels of w and v

as follows

$$
\begin{tikzcd}
K_w \arrow[r, "k_w"] & B \arrow[r, "w_1"] \arrow[rr, bend left, "w"] & I_w \arrow[r, "w_2"] & D \\
K_v \arrow[r, "k_v"] & A \arrow[u, "u"] \arrow[r, "v_1"] \arrow[rr, bend right, "v"'] & I_v \arrow[r, "v_2"] & C \arrow[u, "z"']
\end{tikzcd}
$$

We have $0 = k_v vz = k_v uw$ and, as w_2 is a monomorphism, $k_v uw_1 = 0$. As v_1 is a cokernel of k_v, there exists exactly one $I_v \xrightarrow{i} I_w$ with $v_1 i = uw_1$. It makes the diagram commutative as $v_1 iw_2 = uw_1 w_2 = uw = vz = v_1 v_2 z$, whence, because v_1 is an epimorphism, it follows that $iw_2 = v_2 z$.

$$
\begin{tikzcd}
B \arrow[r, "w_1"] \arrow[rr, bend left, "w"] & I_w \arrow[r, "w_2"] & D \\
A \arrow[u, "u"] \arrow[r, "v_1"] \arrow[rr, bend right, "v"'] & I_v \arrow[u, "i"] \arrow[r, "v_2"] & C \arrow[u, "z"']
\end{tikzcd}
$$

Applying the same by choosing images of u, i and z yields

$$
\begin{tikzcd}
B \arrow[r, "w_1"] & I_w \arrow[r, "w_2"] & D \\
I_u \arrow[u, "u_2"] \arrow[r, "i_u"] & I \arrow[u, "i_w"] \arrow[r, "i_z"] & I_z \arrow[u, "z_2"] \\
A \arrow[u, "u_1"] \arrow[r, "v_1"] & I_v \arrow[u, "i_v"] \arrow[r, "v_2"] & C \arrow[u, "z_1"]
\end{tikzcd}
$$

commutative with $u_1 u_2 = u$, $i_v i_w = i$ and $z_1 z_2 = z$. The morphism i_u is epic as $v_1 i_v = u_1 i_u$ with $v_1 i_v$ epic. The morphism i_z is monic as $i_w w_2 = i_z z_2$ with $i_w w_2$ monic; cf. Remark 1.

Lemma 55. *For every weak square*

$$
\begin{tikzcd}
B \arrow[r, "w"] & D \\
A \arrow[u, "u"] \arrow[r, "v"] \arrow[ur, phantom, "+"] & C \arrow[u, "z"']
\end{tikzcd}
$$

in $\mathcal{A}$ there exists a commutative diagram

$$
\begin{tikzcd}
B \arrow[r, "w_1"] & I_w \arrow[r, "w_2"] & D \\
I_u \arrow[u, "u_2"] \arrow[r, "i_u"] & I \arrow[u, "i_w"] \arrow[r, "i_z"] & I_z \arrow[u, "z_2"] \\
A \arrow[u, "u_1"] \arrow[r, "v_1"] & I_v \arrow[u, "i_v"] \arrow[r, "v_2"] & C \arrow[u, "z_1"]
\end{tikzcd} .
$$

For every such diagram we have

$$\begin{tikzcd}
B \arrow[r, "\shortmid" marking] & I_w \arrow[r, "\bullet" marking] & D \\
I_u \arrow[u, "\bullet" marking] \arrow[r, "\shortmid" marking] \arrow[ur, phantom, "\square"] & I \arrow[u, "\bullet" marking] \arrow[r, "\bullet" marking] \arrow[ur, phantom, "\llcorner"] & I_z \arrow[u, "\bullet" marking] \\
A \arrow[u, "\shortmid" marking] \arrow[r, "\shortmid" marking] \arrow[ur, phantom, "\urcorner"] & I_v \arrow[u, "\shortmid" marking] \arrow[r, "\bullet" marking] \arrow[ur, phantom, "\square"] & C \arrow[u, "\shortmid" marking]
\end{tikzcd} .$$

Proof. The diagram exists by Remark 54.

We choose kernels and cokernels of $w, i := i_u i_z$ and v. Note that these are kernels of w_1, i_u, v_1 and cokernels of w_2, i_z, v_2 respectively; cf. Remarks 15,16. We obtain the following commutative diagram.

$$\begin{tikzcd}
K_w \arrow[r, "k_w", "\bullet" marking] & B \arrow[r, "w_1", "\shortmid" marking] & I_w \arrow[r, "w_2", "\bullet" marking] & D \arrow[r, "c_w", "\shortmid" marking] & C_w \\
K_i \arrow[u, "k_2"'] \arrow[r, "k_i", "\bullet" marking] & I_u \arrow[u, "u_2", "\bullet" marking] \arrow[r, "i_u", "\shortmid" marking] & I \arrow[u, "i_w", "\bullet" marking] \arrow[r, "i_z", "\bullet" marking] & I_z \arrow[u, "z_2", "\bullet" marking] \arrow[r, "c_i", "\shortmid" marking] & C_i \arrow[u, "c_2"] \\
K_v \arrow[u, "k_1"'] \arrow[uu, bend left, "k", "\shortmid" marking] \arrow[r, "k_v", "\bullet" marking] & A \arrow[u, "u_1", "\shortmid" marking] \arrow[r, "v_1", "\shortmid" marking] & I_v \arrow[u, "i_v", "\shortmid" marking] \arrow[r, "v_2", "\bullet" marking] & C \arrow[u, "z_1", "\shortmid" marking] \arrow[r, "c_v", "\shortmid" marking] & C_v \arrow[u, "c_1"] \arrow[uu, bend right, "c"', "\bullet" marking]
\end{tikzcd}$$

As k is epic and $k = k_1 k_2$, it follows that k_2 is epic; cf. Remark 1. We also have $k_i u_2 = k_2 k_w$ monic, thus k_2 is also monic. Because $\mathcal{A}$ is abelian, it follows that k_2 is an isomorphism. Then $k_1 = k k_2^{-1}$ is epic. The dual argument applies to the cokernels and yields c_1 isomorphism and c_2 monic.

$$\begin{tikzcd}
K_w \arrow[r, "k_w", "\bullet" marking] & B \arrow[r, "w_1", "\shortmid" marking] & I_w \arrow[r, "w_2", "\bullet" marking] & D \arrow[r, "c_w", "\shortmid" marking] & C_w \\
K_i \arrow[u, "k_2"', "\wr"] \arrow[r, "k_i", "\bullet" marking] & I_u \arrow[u, "u_2", "\bullet" marking] \arrow[r, "i_u", "\shortmid" marking] & I \arrow[u, "i_w", "\bullet" marking] \arrow[r, "i_z", "\bullet" marking] & I_z \arrow[u, "z_2", "\bullet" marking] \arrow[r, "c_i", "\shortmid" marking] & C_i \arrow[u, "c_2", "\bullet" marking] \\
K_v \arrow[u, "k_1"', "\shortmid" marking] \arrow[uu, bend left, "k", "\shortmid" marking] \arrow[r, "k_v", "\bullet" marking] & A \arrow[u, "u_1", "\shortmid" marking] \arrow[r, "v_1", "\shortmid" marking] & I_v \arrow[u, "i_v", "\shortmid" marking] \arrow[r, "v_2", "\bullet" marking] & C \arrow[u, "z_1", "\shortmid" marking] \arrow[r, "c_v", "\shortmid" marking] & C_v \arrow[u, "c_1", "\wr"'] \arrow[uu, bend right, "c"', "\bullet" marking]
\end{tikzcd}$$

Thus by Lemma 37 we get

$$\begin{tikzcd}
B \arrow[r, "w_1", "\shortmid" marking] & I_w \arrow[r, "w_2", "\bullet" marking] & D \\
I_u \arrow[u, "u_2", "\bullet" marking] \arrow[r, "i_u", "\shortmid" marking] \arrow[ur, phantom, "\square"] & I \arrow[u, "i_w", "\bullet" marking] \arrow[r, "i_z", "\bullet" marking] \arrow[ur, phantom, "\llcorner"] & I_z \arrow[u, "z_2", "\bullet" marking] \\
A \arrow[u, "u_1", "\shortmid" marking] \arrow[r, "v_1", "\shortmid" marking] \arrow[ur, phantom, "\urcorner"] & I_v \arrow[u, "i_v", "\shortmid" marking] \arrow[r, "v_2", "\bullet" marking] \arrow[ur, phantom, "\square"] & C \arrow[u, "z_1", "\shortmid" marking]
\end{tikzcd}$$

$\square$

Lemma 56. *In $\mathcal{A}$, suppose given*

$$\begin{tikzcd} B \arrow[r, "w"] & D \\ A \arrow[u, tail, "u"] \arrow[r, "v"] \arrow[ur, phantom, "\lrcorner"] & C \arrow[u, "z"'] \end{tikzcd} .$$

Then v is split epic if and only if w is split epic.

Proof. We choose a kernel $K \xrightarrow{k} A$ of v. As the quadrangle is a pullback, $K \xrightarrow{ku} B$ is a kernel of w.

$$\begin{tikzcd} K \arrow[r, "ku"] & B \arrow[r, "w"] & D \\ K \arrow[u, equal] \arrow[r, "k"] & A \arrow[u, tail, "u"] \arrow[r, "v"] \arrow[ur, phantom, "\lrcorner"] & C \arrow[u, "z"'] \end{tikzcd}$$

Then k is split monic if and only if ku is split monic:

$\Rightarrow$: Suppose that k is split monic. Then ku is split monic.

$\Leftarrow$: Suppose that ku is split monic. Then there exists a morphism x with $kux = 1_K$. Then $k(ux) = 1_K$ and k is split monic.

According to Lemma 50 v is split epic if and only if k is split monic, and w is split epic if and only if ku is split monic. This proves the assertion. $\square$

Lemma 57. *In $\mathcal{A}$, suppose given*

$$\begin{tikzcd} B \arrow[r, "w"] & D \\ A \arrow[u, "u"] \arrow[r, "v"] \arrow[ur, phantom, "\urcorner"] & C \arrow[u, tail, "z"'] \end{tikzcd} .$$

Then v is split monic if and only if w is split monic.

Proof. This is dual to Lemma 56. $\square$

Lemma 58. *Given*

$$\begin{tikzcd} B \arrow[r, "w"] & D \\ A \arrow[u, tail, "u"] \arrow[r, tail, "v"] \arrow[ur, phantom, "+"] & C \arrow[u, tail, "z"'] \end{tikzcd}$$

in $\mathcal{A}$ with B injective. Suppose given a commutative diagram as provided by Lemma 55 and Remark 49.

$$
\begin{tikzcd}
B \arrow[r, "w_1", "\shortmid" marking] & I_w \arrow[r, "w_2", "\bullet" marking] & D \\
I_u \arrow[u, tail, "u_2", "\bullet" marking] \arrow[r, "i_u", "\shortmid" marking] \arrow[ur, phantom, "\square"] & I \arrow[u, "i_w", "\bullet" marking] \arrow[r, "i_z", "\bullet" marking] \arrow[ur, phantom, "\llcorner"] & I_z \arrow[u, tail, "z_2", "\bullet" marking] \\
A \arrow[u, tail, "u_1", "\shortmid" marking] \arrow[r, tail, "v_1", "\shortmid" marking] \arrow[ur, phantom, "\urcorner"] & I_v \arrow[u, "i_v", "\shortmid" marking] \arrow[r, tail, "v_2", "\bullet" marking] \arrow[ur, phantom, "\square"] & C \arrow[u, tail, "z_1", "\shortmid" marking]
\end{tikzcd}
$$

Here, $u = u_1u_2$, $v = v_1v_2$, $w = w_1w_2$ and $z = z_1z_2$.

Then all morphisms in this diagram are split.

$$
\begin{tikzcd}
B \arrow[r, tail, "w_1", "\shortmid" marking] & I_w \arrow[r, tail, "w_2", "\bullet" marking] & D \\
I_u \arrow[u, tail, "u_2", "\bullet" marking] \arrow[r, tail, "i_u", "\shortmid" marking] \arrow[ur, phantom, "\square"] & I \arrow[u, tail, "i_w", "\bullet" marking] \arrow[r, tail, "i_z", "\bullet" marking] \arrow[ur, phantom, "\llcorner"] & I_z \arrow[u, tail, "z_2", "\bullet" marking] \\
A \arrow[u, tail, "u_1", "\shortmid" marking] \arrow[r, tail, "v_1", "\shortmid" marking] \arrow[ur, phantom, "\urcorner"] & I_v \arrow[u, tail, "i_v", "\shortmid" marking] \arrow[r, tail, "v_2", "\bullet" marking] \arrow[ur, phantom, "\square"] & C \arrow[u, tail, "z_1", "\shortmid" marking]
\end{tikzcd}
$$

Proof. Applying Lemma 56 and 57 to (I_v, I, C, I_z) yields i_z split monic and i_v split epic. As $i_zz_2 = i_ww_2$ is split monic, i_w is split monic. As $v_1i_v = u_1i_u$ is split epic, i_u is split epic; cf. Remark 1.

$$
\begin{tikzcd}
B \arrow[r, "w_1", "\shortmid" marking] & I_w \arrow[r, "w_2", "\bullet" marking] & D \\
I_u \arrow[u, tail, "u_2", "\bullet" marking] \arrow[r, tail, "i_u", "\shortmid" marking] \arrow[ur, phantom, "\square"] & I \arrow[u, tail, "i_w", "\bullet" marking] \arrow[r, tail, "i_z", "\bullet" marking] \arrow[ur, phantom, "\llcorner"] & I_z \arrow[u, tail, "z_2", "\bullet" marking] \\
A \arrow[u, tail, "u_1", "\shortmid" marking] \arrow[r, tail, "v_1", "\shortmid" marking] \arrow[ur, phantom, "\urcorner"] & I_v \arrow[u, tail, "i_v", "\shortmid" marking] \arrow[r, tail, "v_2", "\bullet" marking] \arrow[ur, phantom, "\square"] & C \arrow[u, tail, "z_1", "\shortmid" marking]
\end{tikzcd}
$$

Applying Lemma 56 and 57 to (I_u, B, I, I_w) yields w_1 split epic. Since B is injective, I_w is injective and thus w_2 is split monic; cf. Remark 2.

□

2.4 Factor categories

Definition 59. Suppose given an full additive subcategory $\mathcal{N}$ that is closed under isomorphy. We define the *factor category* $\mathcal{A}/\mathcal{N}$ as follows:

$$\mathrm{Ob}(\mathcal{A}/\mathcal{N}) \ := \ \mathrm{Ob}(\mathcal{A})$$

$$ {}_{\mathcal{A}/\mathcal{N}}(A, B) \ := \ {}_{\mathcal{A}}(A, B)/{}_{\mathcal{A},\mathcal{N}}(A, B) \text{ for } A, B \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$$

with

$$
{}_{\mathcal{A},\mathcal{N}}(A,B) \;:=\; \{A \xrightarrow{f} B\colon \;\; \text{there exists } N \in \mathrm{Ob}(\mathcal{N}) \text{ and } A \xrightarrow{\varphi} N \xrightarrow{\psi} B \text{ in } \mathcal{A} \text{ with } f = \varphi\psi\,\}.
$$

The equivalence class of a morphism $(\varphi\colon A \to B) \in \mathrm{Mor}(\mathcal{A})$ in ${}_{\mathcal{A}/\mathcal{N}}(A,B)$ is denoted as $[\varphi]_{\mathcal{A},\mathcal{N}}$ or $[\varphi]$ or even φ if unambiguous.

For $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ in $\mathcal{A}$, the composite in $\mathcal{A}/\mathcal{N}$ is defined by $[\varphi][\psi] = [\varphi\psi]$.

Remark 60. The factor category is in fact a category:

For $A, B \in \mathrm{Ob}(\mathcal{A})$ the morphism set ${}_{\mathcal{A},\mathcal{N}}(A,B)$ is a subgroup of ${}_{\mathcal{A}}(A,B)$ because for $\varphi, \psi \in {}_{\mathcal{A}}(A,B)$ and $N_1, N_2 \in \mathrm{Ob}(\mathcal{N})$ like in the following commutative diagrams

$$
\begin{tikzcd}
A \arrow[r, "\varphi_1"] \arrow[rr, bend right, "\varphi"'] & N_1 \arrow[r, "\varphi_2"] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "\psi_1"'] \arrow[rr, bend left, "\psi"] & N_2 \arrow[r, "\psi_2"'] & B
\end{tikzcd}
$$

we have

$$
\begin{tikzcd}
A \arrow[r, "(\varphi_1\ \psi_1)"] \arrow[rr, bend right, "\varphi+\psi"'] & N_1 \oplus N_2 \arrow[r, "\binom{\varphi_2}{\psi_2}"] & B.
\end{tikzcd}
$$

Thus $\varphi + \psi \in {}_{\mathcal{A},\mathcal{N}}(A,B)$.

The composition is well-defined: Given $A \overset{\alpha}{\underset{\alpha'}{\rightrightarrows}} B \overset{\beta}{\underset{\beta'}{\rightrightarrows}} C$ with $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, we have

$$
[\alpha'][\beta'] = [\alpha'\beta'] = [\alpha\beta + \underbrace{(\alpha' - \alpha)\beta + \alpha'(\beta' - \beta)}_{\in {}_{\mathcal{A},\mathcal{N}}(A,C)}] = [\alpha\beta] = [\alpha][\beta].
$$

The factor category $\mathcal{A}/\mathcal{N}$ is preadditive.

The functor $R_{\mathcal{A}/\mathcal{N}}\colon \mathcal{A} \to \mathcal{A}/\mathcal{N}$ with $A \mapsto A$ for $A \in \mathrm{Ob}(\mathcal{A})$ and $\varphi \mapsto [\varphi]$ for $\varphi \in \mathrm{Mor}(\mathcal{A})$ is additive. Thus by Lemma 11 the category $\mathcal{A}/\mathcal{N}$ is additive.

Remark 61. Every object of $\mathrm{Ob}(\mathcal{N})$ is a zero object in $\mathcal{A}/\mathcal{N}$. In fact all morphisms $A \xrightarrow{\varphi} N \xrightarrow{\psi} B$ in $\mathcal{A}$ factor like

$$
\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[rr, bend left, "\varphi"] & N \arrow[r, "1_N"] \arrow[rr, bend right, "\psi"'] & N \arrow[r, "\psi"] & B.
\end{tikzcd}
$$

Lemma 62 (Universal property of the factor category)**.** *Suppose given an additive category $\mathcal{A}$ and a full additive subcategory $\mathcal{N} \subseteq \mathcal{A}$. Suppose given an additive category $\mathcal{B}$. Suppose given an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ with $NF \cong 0_{\mathcal{B}}$ for all $N \in \mathrm{Ob}(\mathcal{N})$. Then there exists a unique additive functor $\hat{F}\colon \mathcal{A}/\mathcal{N} \to \mathcal{B}$ with $R_{\mathcal{A}/\mathcal{N}}\hat{F} = F$.*

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "F"'] \arrow[d, "R_{\mathcal{A}/\mathcal{N}}"] \\ \mathcal{B} & \mathcal{A}/\mathcal{N} \arrow[l, dotted, "\exists!\hat{F}"] \end{tikzcd}$$

This functor $\hat{F}$ is given by $A \mapsto AF$ for $A \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$ and $[\varphi] \mapsto \varphi F$ for $[\varphi] \in \mathrm{Mor}(\mathcal{A}/\mathcal{N})$ with representative $\varphi \in \mathrm{Mor}(\mathcal{A})$.

Proof. The functor $R_{\mathcal{A}/\mathcal{N}}$ is surjective on morphisms and objects. This means if $\hat{F}$ is well-defined, it is unique.

For every morphism $A \xrightarrow{\varphi} B$ in $\mathcal{A}$ with $[\varphi] = [0_{A,B}]$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$ we have

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\varphi_1"'] & & B \\ & N \arrow[ur, "\varphi_2"'] & \end{tikzcd} \quad \overset{F}{\curvearrowright} \quad \begin{tikzcd} AF \arrow[rr, "\varphi F"] \arrow[dr, "\varphi_1 F"'] & & BF \\ & NF \cong 0_{\mathcal{B}} \arrow[ur, "\varphi_2 F"'] & \end{tikzcd}$$

The functor F maps N to a zero object in $\mathcal{B}$, thus $\varphi F = 0_{AF,BF}$. Because F is additive, this is sufficient and we get

$$[\psi] = [\tilde{\psi}] \;\Rightarrow\; \psi F = \tilde{\psi} F.$$

Therefore $\hat{F}$ is well-defined.

We have $[1_A]\hat{F} = 1_A F = 1_{AF}$ for every $A \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$ and

$$[\alpha\beta]\hat{F} = (\alpha\beta)F = \alpha F \cdot \beta F = [\alpha]\hat{F} \cdot [\beta]\hat{F}$$

for $A \xrightarrow{[\alpha]} B \xrightarrow{[\beta]} C$ in $\mathcal{A}/\mathcal{N}$.

The functor $\hat{F}$ is additive, as we have

$$[\alpha + \beta]\hat{F} = (\alpha + \beta)F = \alpha F + \beta F = [\alpha]\hat{F} + [\beta]\hat{F}$$

for $A \overset{[\alpha]}{\underset{[\beta]}{\rightrightarrows}} B$ in $\mathcal{A}/\mathcal{N}$.

We have $R_{\mathcal{A}/\mathcal{N}}\hat{F} = F$ as we have $XR_{\mathcal{A}/\mathcal{N}}\hat{F} = X = XF$ for $X \in \mathrm{Ob}(\mathcal{A})$ and $\varphi R_{\mathcal{A}/\mathcal{N}}\hat{F} = [\varphi]\hat{F} = \varphi F$ for $\varphi \in \mathrm{Mor}(\mathcal{A})$. $\square$

Lemma 63. *Suppose given additive categories $\mathcal{A}, \mathcal{B}$ and full additive subcategories $\mathcal{M} \subseteq \mathcal{A}$ and $\mathcal{N} \subseteq \mathcal{B}$. Suppose given additive functors $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ and a transformation $\alpha\colon F \to G$.*

Suppose that $\mathrm{Ob}(\mathcal{M})F \subseteq \mathrm{Ob}(\mathcal{N})$ *and* $\mathrm{Ob}(\mathcal{N})G \subseteq \mathrm{Ob}(\mathcal{M})$.

We get induced additive functors $\mathcal{A}/\mathcal{M} \underset{\bar{G}}{\overset{\bar{F}}{\rightrightarrows}} \mathcal{B}/\mathcal{N}$, *mapping* $A \xrightarrow{[a]_\mathcal{M}} A'$ *to*

$$(A \xrightarrow{[a]_\mathcal{M}} A')\bar{F} = (AF \xrightarrow{[aF]_\mathcal{N}} A'F)$$

respective to

$$(A \xrightarrow{[a]_\mathcal{M}} A')\bar{G} = (AG \xrightarrow{[aG]_\mathcal{N}} A'G).$$

Define

$$\alpha' \;:=\; ([A\alpha]_\mathcal{N})_{A\in\mathrm{Ob}(\mathcal{A}/\mathcal{M})}.$$

Then α' *is a transformation from* $\bar{F}$ *to* $\bar{G}$.

Proof. By the universal property of the factor category, cf. Lemma 62, applied to

$$\begin{tikzcd} \mathcal{A} \arrow[d, "R_{\mathcal{A}/\mathcal{M}}"'] \arrow[dr, "FR_{\mathcal{B}/\mathcal{N}}"] & \\ \mathcal{A}/\mathcal{M} & \mathcal{B}/\mathcal{N} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A} \arrow[d, "R_{\mathcal{A}/\mathcal{M}}"'] \arrow[dr, "GR_{\mathcal{B}/\mathcal{N}}"] & \\ \mathcal{A}/\mathcal{M} & \mathcal{B}/\mathcal{N} \end{tikzcd}$$

we get the wanted functors $\bar{F}, \bar{G} \colon \mathcal{A}/\mathcal{M} \to \mathcal{B}/\mathcal{N}$.

Suppose given $A \xrightarrow{[a]} A'$ in $\mathcal{A}/\mathcal{M}$. We have

$$\begin{tikzcd} A\bar{F} \arrow[r, "{[a]_\mathcal{M}\bar{F}}"] \arrow[d, "A\alpha'"] & A'\bar{F} \arrow[d, "A'\alpha'"] \\ A\bar{G} \arrow[r, "{[a]_\mathcal{M}\bar{G}}"] & A'\bar{G} \end{tikzcd} \quad = \quad \begin{tikzcd} AF \arrow[r, "{[aF]_\mathcal{N}}"] \arrow[d, "{[A\alpha]_\mathcal{N}}"] & A'F \arrow[d, "{[A'\alpha]_\mathcal{N}}"] \\ AG \arrow[r, "{[aG]_\mathcal{N}}"] & A'G \end{tikzcd}$$

with $[aF]_\mathcal{N} \cdot [A'\alpha]_\mathcal{N} = [aF \cdot A'\alpha]_\mathcal{N} = [A\alpha \cdot aG]_\mathcal{N} = [A\alpha]_\mathcal{N} \cdot [aG]_\mathcal{N}$.

Therefore α' is a transformation. □

Lemma 64. *Suppose given equivalent additive categories* $\mathcal{A}$ *and* $\mathcal{A}'$ *with full additive subcategories* $\mathcal{N} \subseteq \mathcal{A}$ *and* $\mathcal{N}' \subseteq \mathcal{A}'$. *The equivalence be given as* $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{A}'$ *with* $\alpha \colon 1_\mathcal{A} \to FG$ *and* $\beta \colon GF \to 1_{\mathcal{A}'}$ *isotransformations. If* $\mathcal{N}F \subseteq \mathcal{N}'$ *and* $\mathcal{N}'G \subseteq \mathcal{N}$, *then we get induced functors* $F' \colon \mathcal{A}/\mathcal{N} \to \mathcal{A}'/\mathcal{N}'$ *and* $G' \colon \mathcal{A}'/\mathcal{N}' \to \mathcal{A}/\mathcal{N}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[d, "R_{\mathcal{A}/\mathcal{N}}"'] & \mathcal{A}' \arrow[l, shift left, "G"] \arrow[d, "R_{\mathcal{A}'/\mathcal{N}'}"] \\ \mathcal{A}/\mathcal{N} \arrow[r, shift left, dotted, "F'"] & \mathcal{A}'/\mathcal{N}' \arrow[l, shift left, dotted, "G'"] \end{tikzcd}$$

Then $\mathcal{A}/\mathcal{N} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{A}'/\mathcal{N}'$ *is an equivalence of categories.*

Proof. First note that every equivalence is additive; cf. [**7,** §16.5.10.(b)].

We get commutative triangles with the functors F' and G' by the universal property

$$\begin{tikzcd} \mathcal{A} \arrow[d, "R_{\mathcal{A}/\mathcal{N}}"'] \arrow[dr, "FR_{\mathcal{A}'/\mathcal{N}'}"] & \\ \mathcal{A}/\mathcal{N} \arrow[r, dotted, "\exists! F'"'] & \mathcal{A}'/\mathcal{N}' \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A}' \arrow[d, "R_{\mathcal{A}'/\mathcal{N}'}"] \arrow[dl, "GR_{\mathcal{A}/\mathcal{N}}"'] \\ \mathcal{A}/\mathcal{N} & \mathcal{A}'/\mathcal{N}' \arrow[l, dotted, "\exists! G'"] \end{tikzcd}$$

with

$$\begin{aligned} F'\colon A &\longmapsto AF \\ [\varphi]_{\mathcal{N}} &\longmapsto [\varphi F]_{\mathcal{N}'} \end{aligned} \qquad \begin{aligned} G'\colon A' &\longmapsto A'G \\ [\varphi']_{\mathcal{N}'} &\longmapsto [\varphi G]_{\mathcal{N}} \end{aligned}$$

where $[\varphi']_{\mathcal{N}'}$ denotes the equivalence class of $\varphi' \in \mathrm{Mor}(\mathcal{A}')$ in $\mathcal{A}'/\mathcal{N}'$.

For the composite this yields

$$\begin{aligned} F'G'\colon A &\longmapsto AFG \\ [\varphi]_{\mathcal{N}} &\longmapsto [\varphi FG]_{\mathcal{N}} \end{aligned} \qquad \begin{aligned} G'F'\colon A' &\longmapsto A'GF \\ [\varphi']_{\mathcal{N}'} &\longmapsto [\varphi GF]_{\mathcal{N}'} \end{aligned}$$

We define new isotransformations by

$$\begin{aligned} \alpha' &:= ([A\alpha]_{\mathcal{N}})_{A\in\mathrm{Ob}(\mathcal{A}/\mathcal{N})} \\ \beta' &:= ([B\beta]_{\mathcal{N}'})_{B\in\mathrm{Ob}(\mathcal{A}'/\mathcal{N}')} \end{aligned}$$

These are in fact isotransformations, as they are transformations by Lemma 63 and isomorphisms yield isomorphisms in the factor category. □

Lemma 65. *Suppose given additive categories $\mathcal{A}, \mathcal{B}$ and full additive subcategories $\mathcal{M} \subseteq \mathcal{A}$ and $\mathcal{N} \subseteq \mathcal{B}$. Suppose given additive functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ with $F \dashv G$ via unit $1_{\mathcal{A}} \xrightarrow{\alpha} FG$ and via counit $GF \xrightarrow{\beta} 1_{\mathcal{B}}$.*

Suppose that $\mathrm{Ob}(\mathcal{M})F \subseteq \mathrm{Ob}(\mathcal{N})$ and $\mathrm{Ob}(\mathcal{N})G \subseteq \mathrm{Ob}(\mathcal{M})$.

We get induced additive functors $\mathcal{A}/\mathcal{M} \underset{\bar{G}}{\overset{\bar{F}}{\rightleftarrows}} \mathcal{B}/\mathcal{N}$.

Using Lemma 63, we define transformations

$$\alpha' := ([A\alpha]_{\mathcal{M}})_{A\in\mathrm{Ob}(\mathcal{A}/\mathcal{M})}$$

and

$$\beta' := ([B\beta]_{\mathcal{N}})_{B\in\mathrm{Ob}(\mathcal{B}/\mathcal{N})}.$$

Then $\bar{F} \dashv \bar{G}$ via unit $1_{\mathcal{A}/\mathcal{M}} \xrightarrow{\alpha'} \bar{F}\bar{G}$ and via counit $\bar{G}\bar{F} \xrightarrow{\beta'} 1_{\mathcal{B}/\mathcal{N}}$.

Proof. We have $A\bar{F} = AF$ for $A \in \mathrm{Ob}(\mathcal{A}/\mathcal{M}) = \mathrm{Ob}(\mathcal{A})$ and $[a]_{\mathcal{M}}\bar{F} = [aF]_{\mathcal{N}}$ for $a \in \mathrm{Mor}(\mathcal{A})$.

We have $B\bar{G} = BG$ for $B \in \mathrm{Ob}(\mathcal{B}/\mathcal{N}) = \mathrm{Ob}(\mathcal{B})$ and $[b]_{\mathcal{N}}\bar{G} = [bG]_{\mathcal{M}}$ for $b \in \mathrm{Mor}(\mathcal{B})$.

We need to show that the triangles

$$\begin{tikzcd} A\bar{F} \arrow[r, "A\alpha'\bar{F}"] \arrow[dr, "1_{A\bar{F}}"'] & A\bar{F}\bar{G}\bar{F} \arrow[d, "A\bar{F}\beta'"] \\ & A\bar{F} \end{tikzcd} \qquad \begin{tikzcd} B\bar{G} \arrow[r, "B\bar{G}\alpha'"] \arrow[dr, "1_{B\bar{G}}"'] & B\bar{G}\bar{F}\bar{G} \arrow[d, "B\beta'\bar{G}"] \\ & B\bar{G} \end{tikzcd}$$

commute for every $A \in \mathrm{Ob}(\mathcal{A}/\mathcal{M})$ and for every $B \in \mathrm{Ob}(\mathcal{B}/\mathcal{N})$.

We have

$$\begin{aligned} A\alpha'\bar{F} \cdot A\bar{F}\beta' &= [A\alpha]_{\mathcal{M}}\bar{F} \cdot [AF\beta]_{\mathcal{N}} \\ &= [A\alpha F]_{\mathcal{N}} \cdot [AF\beta]_{\mathcal{N}} \\ &= [A\alpha F \cdot AF\beta]_{\mathcal{N}} \\ &= [1_{AF}]_{\mathcal{N}} \\ &= 1_{A\bar{F}} \end{aligned}$$

and

$$\begin{aligned} B\bar{G}\alpha' \cdot B\beta'\bar{G} &= [BG\alpha]_{\mathcal{M}} \cdot [B\beta]_{\mathcal{N}}\bar{G} \\ &= [BG\alpha]_{\mathcal{M}} \cdot [B\beta G]_{\mathcal{M}} \\ &= [BG\alpha \cdot B\beta G]_{\mathcal{M}} \\ &= [1_{BG}]_{\mathcal{M}} \\ &= 1_{B\bar{G}}. \end{aligned}$$

Therefore $\bar{F} \dashv \bar{G}$ via α' and β'. $\square$

Definition 66. A *complex* in $\mathcal{A}$ is a functor $F\colon \mathbb{Z} \to \mathcal{A}$ with $(z, z+2)F = 0$ for every $z \in \mathbb{Z}$.

It can be given as a sequence $(A^i)_{i\in\mathbb{Z}}$ of objects in $\mathcal{A}$ with corresponding morphisms $(d^i)_{i\in\mathbb{Z}}$ with $d^i\colon A^i \to A^{i+1}$ and which fulfil $d^i \cdot d^{i+1} = 0$ for every $i \in \mathbb{Z}$. Then we get F by $iF := A^i$ and $(i, i+1)F := d^i$.

The category $\mathrm{C}(\mathcal{A})$ is defined as the full subcategory of $(\mathbb{Z}, \mathcal{A})$ whose objects are the complexes.

We consider the full subcategory $\mathrm{C}^{(\mathrm{sp\ ac})}(\mathcal{A}) \subseteq \mathrm{C}(\mathcal{A})$, whose objects are complexes isomorphic to a complex of the form

$$\ldots \longrightarrow A^{-1} \oplus A^0 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} A^0 \oplus A^1 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} A^1 \oplus A^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} A^2 \oplus A^3 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} A^3 \oplus A^4 \longrightarrow \ldots$$

with $A^i \in \mathrm{Ob}(\mathcal{A})$ for $i \in \mathbb{Z}$. Objects of $\mathrm{C}^{(\mathrm{sp\ ac})}(\mathcal{A})$ are called *split acyclic complexes.* Note that these can be obtained as direct sums of complexes

$$
\begin{gathered}
\ldots \xrightarrow{1} A^{-1} \longrightarrow 0 \longrightarrow A^2 \xrightarrow{1} A^2 \longrightarrow 0 \longrightarrow \ldots \\
\ldots \longrightarrow A^0 \xrightarrow{1} A^0 \longrightarrow 0 \longrightarrow A^3 \xrightarrow{1} A^3 \longrightarrow \ldots \\
\ldots \longrightarrow 0 \longrightarrow A^1 \xrightarrow{1} A^1 \longrightarrow 0 \longrightarrow A^4 \xrightarrow{1} \ldots
\end{gathered}
$$

which are split acyclic, too.

The factor category $\mathrm{C}(\mathcal{A})/\,\mathrm{C}^{(\mathrm{sp\ ac})}(\mathcal{A}) =: \mathrm{K}(\mathcal{A})$ is the homotopy category of complexes.

3 Posets and adjoints

3.1 Posets as categories

Remark 67. Given a poset $A = (A, \leqslant)$ we have the category $\mathcal{C}_A$ with $\operatorname{Ob}\mathcal{C}_A = A$ and $\operatorname{Mor}\mathcal{C}_A = \{(a, b) \mid a, b \in A,\ a \leqslant b\}$. A morphism $(a, b) \in \operatorname{Mor}\mathcal{C}_A$ has source a and target b, i.e. $a \xrightarrow{(a,b)} b$.

For morphisms $a \xrightarrow{(a,b)} b \xrightarrow{(b,c)} c$, the composite is defined as $(a, b) \cdot (b, c) \ := \ (a, c)$.

For every object $a \in \operatorname{Ob}\mathcal{C}_A$ we have the identity morphism $1_a = (a, a)$.

We sometimes just write A for the category $\mathcal{C}_A$.

Given two posets A and B, a monotone map $f\colon A \to B$ defines a functor $\mathcal{C}_f\colon \mathcal{C}_A \to \mathcal{C}_B$ with $a\mathcal{C}_f := af$ for $a \in A$ and $(a, b)\mathcal{C}_f := (af, bf)$ for $(a, b) \in \operatorname{Mor}(\mathcal{C}_A)$.

Given a functor $F\colon \mathcal{C}_A \to \mathcal{C}_B$ we get a monotone map $f\colon A \to B$ by $f \ := \ \operatorname{Ob} F$; cf. Notation 70.

As for every source and target there exists only one morphism, every diagram in $\mathcal{C}_A$ commutes.

The set of monotone maps $A \to B$ is denoted $(A, B)_{\mathrm{m}}$. We define

$$f \leqslant g \ :\Leftrightarrow \ (af \leqslant ag \text{ for every } a \in A)$$

for $f, g \in (A, B)_{\mathrm{m}}$. Then $((A, B)_{\mathrm{m}}, \leqslant)$ is a poset.

Lemma 68. *Given two posets A and B and functors $\mathcal{C}_f, \mathcal{C}_g\colon \mathcal{C}_A \to \mathcal{C}_B$. There exists a transformation $\alpha\colon \mathcal{C}_f \to \mathcal{C}_g$ if and only if $f \leqslant g$ in $(A, B)_{\mathrm{m}}$. In this case there exists exactly one tranformation from $\mathcal{C}_f$ to $\mathcal{C}_g$, that is $(a\alpha)_{a\in A}$ with $a\alpha = (af, ag)$ for $a \in A$. This transformation is denoted $\mathcal{C}_(f, g)$.*

Proof. Suppose given a transformation $\alpha\colon \mathcal{C}_f \to \mathcal{C}_g$, that is for every $a \in \operatorname{Ob}\mathcal{C}_A = A$ we get a morphism (af, ag) in $\mathcal{C}_B$. The existence of a morphism in $\mathcal{C}_B$ implies $af \leqslant ag$ for every $a \in A$, therefore $f \leqslant g$ in $(A, B)_{\mathrm{m}}$.

Suppose given $f, g \in (A, B)_{\mathrm{m}}$ with $f \leqslant g$. Then for every $a \in A$ there exists exactly one morphism $af \xrightarrow{a\alpha} ag$ in $\operatorname{Mor}\mathcal{C}_B$, namely $a\alpha = (af, ag)$. The tuple $(a\alpha)_{a\in A}$ is natural due to $\mathcal{C}_B$ being a poset category. $\square$

Lemma 69. *Suppose given posets A and B and functors $\mathcal{C}_A \underset{\mathcal{C}_g}{\overset{\mathcal{C}_f}{\rightleftarrows}} \mathcal{C}_B$. Then $\mathcal{C}_f$ is left adjoint to $\mathcal{C}_g$ if and only if $1_A \leqslant fg$ in $(A, A)_{\mathrm{m}}$ and $gf \leqslant 1_B$ in $(B, B)_{\mathrm{m}}$.*

Proof. There exist transformations $1_{\mathcal{C}_A} \to \mathcal{C}_f\mathcal{C}_g$ and $\mathcal{C}_g\mathcal{C}_f \to 1_{\mathcal{C}_B}$ if and only if $1_A \leqslant fg$ and $gf \leqslant 1_B$. The triangle identities automatically hold as A and B are posets. $\square$

Notation 70. From now on we will usually identify the concepts of a poset and its category and write $A := \mathcal{C}_A$, $f := \mathcal{C}_f$ as well as $(f,g) := \mathcal{C}_{(f,g)}$.

Remark 71. Given posets A, B, we can translate any monotone map f to a functor $\mathcal{C}_f$ and vice versa. Also every morphism $(f,g) \in \operatorname{Mor}(\mathcal{C}_{(A,B)_\mathrm{m}})$ implies $f \leqslant g$ and thus defines a unique transformation from $\mathcal{C}_f$ to $\mathcal{C}_g$ in $\operatorname{Mor}\big(\mathcal{C}_A, \mathcal{C}_B\big)$. Every transformation $\mathcal{C}_f \xrightarrow{\alpha} \mathcal{C}_g$ in $\operatorname{Mor}\big(\mathcal{C}_A, \mathcal{C}_B\big)$ implies $f \leqslant g$, which translates to a morphism $(f,g) \in \operatorname{Mor}\mathcal{C}_{(A,B)_\mathrm{m}}$.

Lemma 72. *Suppose given posets A, B, C and monotone maps*

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \underset{\tilde{g}}{\overset{\tilde{f}}{\rightleftarrows}} C$$

with $f \dashv g$ and $\tilde{f} \dashv \tilde{g}$. Then $f\tilde{f} \dashv \tilde{g}g$.

Proof. We have $1_A \leqslant fg$, $gf \leqslant 1_B$, $1_B \leqslant \tilde{f}\tilde{g}$ and $\tilde{g}\tilde{f} \leqslant 1_C$.

We show that $1_A \leqslant f\tilde{f}\tilde{g}g$:

Suppose given $i \in A$. Then $i \leqslant ifg$. As $if \in B$, we get $if = if1_B \leqslant (if)\tilde{f}\tilde{g}$. As g is monotone, we get $ifg \leqslant if\tilde{f}\tilde{g}g$. Together this yields $i \leqslant ifg \leqslant if\tilde{f}\tilde{g}g$ for every $i \in A$.

We show that $\tilde{g}gf\tilde{f} \leqslant 1_C$:

Suppose given $j \in C$. Then $j = j1_C \geqslant j\tilde{g}\tilde{f}$. For $j\tilde{g} \in B$ we get $j\tilde{g} \geqslant (j\tilde{g})gf$ and as $\tilde{f}$ is monotone, we get $j\tilde{g}\tilde{f} \geqslant j\tilde{g}gf\tilde{f}$. Together this yields $j \geqslant j\tilde{g}\tilde{f} \geqslant j\tilde{g}gf\tilde{f}$.

Therefore $f\tilde{f} \dashv \tilde{g}g$. □

3.2 Adjoints of monotone maps

Lemma 73. *Let A, B be posets.*

(1) Suppose given a monotone map $g\colon B \to A$ such that for every $i \in A$ the set $\{j \in (B)g\colon j \geqslant i\}$ contains an initial element and such that for every $i, j \in (B)g$ with $i \leqslant j$ the set $g^{-1}(\{i,j\})$ contains an initial element.

Let

$$\begin{aligned} e\colon A &\to A \\ i &\mapsto \min\{j \in (B)g\colon j \geqslant i\}. \end{aligned}$$

Let

$$\begin{aligned} f\colon A &\to B \\ i &\mapsto \min g^{-1}(ie). \end{aligned}$$

So $if = \min g^{-1}(\min\{j \in (B)g\colon j \geqslant i\})$ for $i \in A$.

Then f is left adjoint to g, i.e. $f \dashv g$.

If g is injective, then $gf = 1_B$. If g is surjective, then $fg = 1_A$.

(2) *Suppose given a monotone map* $f\colon A \to B$ *such that for every* $i \in B$ *the set* $\{j \in (A)f\colon j \leqslant i\}$ *contains a terminal element and for every* $i, j \in (A)f$ *with* $i \leqslant j$ *the set* $f^{-1}(\{i, j\})$ *contains a terminal element.*

Let

$$\begin{aligned} l\colon B &\to B \\ i &\mapsto \max\{j \in (A)f\colon j \leqslant i\}. \end{aligned}$$

Let

$$\begin{aligned} g\colon B &\to A \\ i &\mapsto \max f^{-1}(il). \end{aligned}$$

So $ig = \max f^{-1}(\max\{j \in (A)f\colon j \leqslant i\})$ *for* $i \in B$.

Then g *is right adjoint to* f, *i.e.* $f \dashv g$.

If f *is injective, then* $fg = 1_A$. *If* f *is surjective, then* $gf = 1_B$.

Proof. (1): We show that f *is monotone:*

Let $i, j \in A$ with $i \leqslant j$. Then $\{x \in (B)g\colon x \geqslant j\} \subseteq \{x \in (B)g\colon x \geqslant i\}$ and therefore $ie \leqslant je$, that is, e is monotone.

We need to show that this implies $if \leqslant jf$.

If $ie = je$, then $if = jf$ and we are done. So suppose that $ie < je$. Then the sets $g^{-1}(ie) = \{b \in B\colon bg = ie\}$ and $g^{-1}(je) = \{b \in B\colon bg = je\}$ are disjoint. We know $ifg = ie < je = jfg$. By requirement there exists an initial element

$$d \;:=\; \min\{b \in B\colon bg \in \{ie, je\}\} = \min(g^{-1}(ie) \cup g^{-1}(je)),$$

so $d \leqslant if$ and $d \leqslant jf$. Either $d \in g^{-1}(ie)$ or $d \in g^{-1}(je)$.

Assume $d \in g^{-1}(je)$. Then d and jf are both initial elements in $g^{-1}(je)$, therefore $d = jf$. This implies $jf = d \leqslant if$ and as g is monotone, $jfg \leqslant ifg$, *contradicting* $ifg = ie < je = jfg$.

So $d \in g^{-1}(ie)$ as $if = d$ and if and d are both initial elements in $g^{-1}(ie)$. This implies $if = d \leqslant jf$. Therefore f is monotone.

We show that $1_A \leqslant fg$*:*

Suppose given $i \in A$. We have $if \in g^{-1}(ie)$ and thus $ifg = ie$.

Note that $ie \in \{j \in (B)g\colon j \geqslant i\}$ and therefore $ie \geqslant i$. We have $i1_A = i \leqslant ie = ifg$. Therefore $1_A \leqslant fg$.

If g is surjective, then

$$\begin{aligned} ie &= \min\{j \in (B)g\colon j \geqslant i\} \\ &= \min\{j \in A\colon j \geqslant i\} \\ &= i \end{aligned}$$

and therefore $ifg = (\min g^{-1}(i))g = i$ for every $i \in A$. Therefore $fg = 1_A$.

We show that $gf \leqslant 1_B$:

Suppose given $i \in B$.

$$\begin{aligned} igf &= \min g^{-1}(ige) \\ &= \min g^{-1}(\min\{j \in (B)g \colon j \geqslant ig\}) \\ &= \min g^{-1}(ig) \\ &\leqslant i. \end{aligned}$$

So $1_A \leqslant fg$ and $gf \leqslant 1_B$. By Lemma 69, f is left adjoint to g.

If g is injective, then $g^{-1}(ig) = \{i\}$ and therefore $igf = \min g^{-1}(ige) = \min g^{-1}(ig) = i$ for every $i \in B$.

(2): This is dual to the previous case. ☐

Lemma 74. *Suppose given posets A, B and monotone maps* $A \underset{g}{\overset{f}{\rightleftarrows}} B$ *with $f \dashv g$.*

(1) For every $i \in A$ the element ifg is initial in $\{j \in (B)g \colon j \geqslant i\}$.

(2) For every $i, j \in (B)g$ with $i \leqslant j$, the element if is initial in $g^{-1}(\{i, j\})$.

(3) For every $i \in B$ the element igf is terminal in $\{j \in (A)f \colon j \leqslant i\}$.

(4) For every $i, j \in (A)f$ with $i \leqslant j$, the element jg is terminal in $f^{-1}(\{i, j\})$.

In particular,

$$if = ifgf = \min g^{-1}(\min\{j \in (B)g \colon j \geqslant i\})$$

for every $i \in A$. Moreover, if f is injective, we get $fg = 1_A$. If g is surjective, we get $gf = 1_B$.

$$ig = igfg = \max f^{-1}(\max\{j \in (A)f \colon j \leqslant i\}$$

for every $i \in B$. Moreover, if g is surjective, we get $fg = 1_A$. If g is injective, we get $gf = 1_B$.

Proof. From $f \dashv g$ we know that $1_A \leqslant fg$ and $gf \leqslant 1_B$; cf. Lemma 69.

Suppose given $i \in A$. Then $i \leqslant ifg$. As f is monotone, we get $if \leqslant ifgf$. From $gf \leqslant 1_B$ we get $if \geqslant (if)gf$. Therefore $f = fgf$.

Suppose given $i \in B$. Then $igf \leqslant i$. As g is monotone, we get $igfg \leqslant ig$. From $1_A \leqslant fg$ we get $ig \leqslant (ig)fg$. Therefore $g = gfg$.

(1):

Suppose given $i \in A$.

By $1_A \leqslant fg$ we know that $i \leqslant ifg$. Therefore $ifg \in \{j \in (B)g \colon j \geqslant i\}$. We need to show that $ifg \leqslant x$ for every $x \in \{j \in (B)g \colon j \geqslant i\}$. Let $x \in \{j \in (B)g \colon j \geqslant i\}$.

As $x \geqslant i$ and fg is monotone, we get $ifg \leqslant xfg$.

As $x \in (B)g$, there is a $b \in B$ with $x = bg$.

Then $ifg \leqslant xfg = bgfg = bg = x$ yields the wanted result.

(2):

Suppose given $i, j \in A$ with $i \leqslant j$.

We first need to show that $if \in g^{-1}(\{i, j\})$. As $i \in (B)g$, there is a $b \in B$ with $bg = i$. Thus $(if)g = bgfg = bg = i \in \{i, j\}$.

Suppose given $x \in g^{-1}(\{i, j\}$. We need to show that $if \leqslant x$.

If $xg = i$, then due to f being monotone, $if = xgf \leqslant x$.

If $xg = j$, thanks to $i \leqslant j$ we get $if \leqslant jf = xgf \leqslant x$.

(3):

Suppose given $i \in B$. We know that $igf \in (A)f$ and $igf \leqslant i$ thanks to $1_B \geqslant gf$. Thus $igf \in \{j \in (A)f \colon j \leqslant i\}$. We need to show that $igf \geqslant x$ for every $x \in \{j \in (A)f \colon j \leqslant i\}$. So suppose given $x = af \leqslant i$, where $a \in A$. Then $igf \geqslant xgf = afgf = af = x$.

(4):

Suppose given $i, j \in (A)f$ with $i \leqslant j$.

As $j \in (A)f$, there is an $a \in A$ with $af = j$. Thus $jgf = afgf = af = j$. Therefore $jg \in f^{-1}(\{i, j\})$.

Suppose given $x \in f^{-1}(\{i, j\})$. We need to show that $x \leqslant jg$.

If $xf = i$, we get $jg \geqslant ig = xfg \geqslant x$.

If $xf = j$, we get $jg = xfg \geqslant x$.

We show that $if = ifgf = \min g^{-1}(\min\{j \in (B)g \colon j \geqslant i\})$ for $i \in A$:

By (1), ifg is initial in $\{j \in (B)g \colon j \geqslant i\}$ for every $i \in A$. That is

$$ifg = \min\{j \in (B)g \colon j \geqslant i\}.$$

By (2), for every $j \in (B)g$ we know that jf is initial in $g^{-1}(j)$. As $ifg \in (B)g$ and $if = ifgf$, we get

$$if = ifgf = \min g^{-1}(ifg) = \min g^{-1}(\min\{j \in (B)g \colon j \geqslant i\})$$

We show that $ig = igfg = \max f^{-1}(\max\{j \in (A)f \colon j \leqslant i\})$ for $i \in B$:

As igf is terminal in $\{j \in (A)f : j \leqslant i\}$ for every $i \in B$ and because of $igf \in (A)f$, $igfg$ is terminal in $f^{-1}(igf)$, we get

$$ig = igfg = \max f^{-1}(\max\{j \in (A)f : j \leqslant i\})$$

□

Corollary 75. *Suppose given posets A, B.*

(1) For a monotone map $g\colon B \to A$, a monotone left adjoint $f\colon A \to B$, i.e. $f \dashv g$, exists if and only if conditions (i), (ii) hold.

(i) For every $i \in A$ the set $\{j \in (B)g : j \geqslant i\}$ contains an initial element.

(ii) For every $i, j \in A$ with $i \leqslant j$ the set $g^{-1}(\{i, j\})$ contains an initial element.

In this case the left adjoint is unique.

(2) For a monotone map $f\colon A \to B$, a monotone right adjoint $g\colon B \to A$, i.e. $f \dashv g$, exists if and only if conditions (i),(ii) hold.

(i) For every $i \in B$ the set $\{j \in (A)f : j \leqslant i\}$ contains a terminal element.

(ii) For every $i, j \in B$ with $i \leqslant j$ the set $f^{-1}(\{i, j\})$ contains a terminal element.

In this case, the right adjoint is unique.

Proof. This follows by Lemma 73 and Lemma 74. Note that the formulas in Lemma 74 determine the respective adjoint uniquely. □

3.3 Posets with shift

Definition 76. Suppose given a poset A. A *shift operator* on A is a bijective, monotone map

$$\begin{aligned} A &\to A \\ a &\mapsto a^{+1} \end{aligned}$$

Its inverse is denoted $A \to A,\ a \mapsto a^{-1}$.

Suppose given $a \in A$. For $k \in \mathbb{Z}_{\geqslant 0}$, we let $a^{+(k+1)} := (a^{+k})^{+1}$, recursively.

For $k \in \mathbb{Z}_{\leqslant 0}$, we let $a^{+0} := a$ and $a^{+(k-1)} := (a^{+k})^{-1}$, recursively.

So a^{+k} is defined for $\mathbb{Z}$, and we have $(a^{+k})^{+l} = a^{+k+l}$ for $k, l \in \mathbb{Z}$.

For $k > 0$ and $a \in A$ we sometimes write $a^{+(-k)} =: a^{-k}$.

A poset A together with a shift operator on A is called a *poset with shift*.

Lemma 77. *Suppose given a poset A with shift. The following defines an equivalence relation on A.*

$$a \sim_{\mathrm{s}} b \;:\Leftrightarrow\; a^{+k} = b \text{ for a } k \in \mathbb{Z}$$

The equivalence class of $a \in A$ regarding $(\sim_{\mathrm{s}})$ is denoted $[a]_{\sim_{\mathrm{s}}}$.

Proof. *Reflexive:* We have $a^{+0} = a$.

Symmetric: If $a^{+k} = b$, then $a = b^{-k}$.

Transitive: For $a^{+k} = b$ and $b^{+l} = c$ with $k, l \in \mathbb{Z}$, we get $c = (a^{+k})^{+l} = a^{+k+l}$. □

Definition 78. Suppose given posets with shift A, B.

A monotone map $f\colon A \to B$ is called *quasiperiodic*, if it satisfies the condition

$$a^{+1} f = (af)^{+1}$$

for every $a \in A$.

The set of quasiperiodic monotone maps $A \to B$ is denoted $(A, B)_{\text{q.p.}}$. The subset $(A, B)_{\text{q.p.}} \subseteq (A, B)_{\text{m}}$ inherits the structure of a poset. So for $f, g \in (A, B)_{\text{q.p.}}$ we have $f \leqslant g$ if $af \leqslant ag$ for every $a \in A$. The poset category of $(A, B)_{\text{q.p.}}$ is a full subcategory of $\boldsymbol{(}A, B\boldsymbol{)}$.

Lemma 79. *Let A, B be posets with shift. Suppose that $b^{+k} \neq b$ for every $b \in B$ and every $k \in \mathbb{Z}$ with $k \neq 0$.*

Suppose given $f\colon A \to B$ quasiperiodic monotone. Suppose given $a \in A$ and $b \in B$. Then

$$|f^{-1}(b) \cap [a]_{\sim_{\mathrm{s}}}| \leqslant 1.$$

Proof. Suppose given $x, y \in A$ with $x \sim_{\mathrm{s}} y$ and $xf = yf = b$. Then $x \sim_{\mathrm{s}} y$ means there is a $k \in \mathbb{Z}$ with $x^{+k} = y$. As f is quasiperiodic monotone, this means $xf = yf = x^{+k} f = xf^{+k}$ and therefore $k = 0$. Then $y = x^{+k} = x$. □

Lemma 80. *Let A, B be posets with shift operators. Suppose given a quasiperiodic monotone map $f\colon A \to B$. Then for every $a \in A$ we have*

$$[a]_{\sim_{\mathrm{s}}} f = [af]_{\sim_{\mathrm{s}}}$$

Proof. *Ad* $\subseteq$: Suppose given $a' \in [a]_{\sim_{\mathrm{s}}}$. Then $a' = a^{+k}$ for a $k \in \mathbb{Z}$. Then

$$a' f = a^{+k} f = (af)^{+k} \in [af]_{\sim_{\mathrm{s}}}.$$

Ad $\supseteq$: Suppose given $b \in [af]_{\sim_{\mathrm{s}}}$. Then $b = (af)^{+k}$ for a $k \in \mathbb{Z}$. Therefore

$$b = (af)^{+k} = a^{+k} f \in [a]_{\sim_{\mathrm{s}}} f.$$

□

4 Specific posets

Suppose given $n, m \in \mathbb{Z}_{\geqslant 0}$

4.1 The poset $\bar{\Delta}_n$ and quasiperiodic monotone maps

Definition 81. Let $\bar{\Delta}_n$ be defined as

$$\bar{\Delta}_n \ := \ \{(k,z) : \ k \in [0,n], \ z \in \mathbb{Z}\}.$$

On $\bar{\Delta}_n$ we define a total order by

$$(k,z) \leqslant (k',z') \ :\Leftrightarrow \ z < z' \vee (z = z' \wedge k \leqslant k'),$$

where $(k,z), (k',z') \in \bar{\Delta}_n$. This is sometimes called the colexicographical order.

- We usually write $k^{+z} := (k,z)$ for $(k,z) \in \bar{\Delta}_n$.
- Given $z > 0$ and $k \in [0,n]$, we often write $k^{+(-z)} =: k^{-z}$, e.g. $4^{+(-3)} =: 4^{-3}$.
- Given $k \in [0,n]$, we often write $k^{+0} =: k$. Accordingly we get $[0,n] \subseteq \bar{\Delta}_n$.

Define a map

$$\mathrm{b}_n \colon \bar{\Delta}_n \to [0,n], \ k^{+z} \mapsto k.$$

Define a shift operator as $(k^{+z})^{+1} := k^{+(z+1)}$ for $(k,z) \in \bar{\Delta}_n$.

Remark 82. We can give a bijection $\bar{\Delta}_n \to \mathbb{Z}$ that preserves the ordering:

$$\varphi_n \colon \bar{\Delta}_n \to \mathbb{Z}, \ (k,z) \mapsto k + z \cdot (n+1).$$

Thus we can write $k^{+z} + 1 \ := \ ((k^{+z})\varphi_n + 1)\varphi_n^{-1}$.

E.g. if $n = 5$, then we have $4^{-3} + 1 = 5^{-3}$ and $5^{-3} + 1 = 0^{-2}$.

This will also be expanded to any $i \in \mathbb{Z}$ by $k^{+z} + i \ := \ ((k^{+z})\varphi_n + i)\varphi_n^{-1}$.

In particular, $x + n + 1 = x^{+1}$ for $x \in \bar{\Delta}_n$.

Remark 83. Given quasiperiodic monotone maps $\bar{\Delta}_n \underset{g}{\overset{f}{\rightleftarrows}} \bar{\Delta}_m$, f is left adjoint to g if and only if $1_{\bar{\Delta}_n} \leqslant fg$ and $gf \leqslant 1_{\bar{\Delta}_m}$ in $(\bar{\Delta}_n, \bar{\Delta}_m)_{\text{q.p.}}$; cf. Lemma 69.

Remark 84. To define a quasiperiodic monotone map $f \colon \bar{\Delta}_n \to \bar{\Delta}_m$ it is sufficient to give the values on $[0,n]$. These values have to satisfy $if \leqslant jf$ for $0 \leqslant i \leqslant j \leqslant n$ and $nf \leqslant (0f)^{+1}$ for the resulting quasiperiodic map to be monotone. This is also sufficient. Equivalently it is sufficient to give the values on an interval $[x, x+n] = [x, x^{+1} - 1] \subset \bar{\Delta}_n$ for some $x \in \bar{\Delta}_n$.

Example 85. An example for a quasiperiodic monotone map $\bar{\Delta}_2 \to \bar{\Delta}_3$:

$$\begin{aligned} 0 &\longmapsto 3 \\ 1 &\longmapsto 1^{+1} \\ 2 &\longmapsto 1^{+1} \end{aligned}$$

Then $0^{+1} \mapsto 3^{+1}$, $1^{-2} \mapsto 1^{-1}$ etc.

Lemma 86.

(1) Suppose given $g\colon \bar{\Delta}_m \to \bar{\Delta}_n$ quasiperiodic monotone. Then there exists a unique quasiperiodic monotone map $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ such that f is left adjoint to g, i.e. $f \dashv g$.

(2) Suppose given $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ quasiperiodic monotone. Then there exists a unique quasiperiodic monotone map $g\colon \bar{\Delta}_m \to \bar{\Delta}_n$ such that f is left adjoint to g, i.e. $f \dashv g$.

Proof. (1): The map g fulfils the requirements for Lemma 75 (1) as $\bar{\Delta}_m$ is isomorphic to $\mathbb{Z}$ as a poset and all mentioned sets are bounded from below. Therefore Lemma 75 (1) gives a unique monotone map f that is left adjoint to g. We show that f is quasiperiodic:

As g is quasiperiodic and thus $j \in (\bar{\Delta}_m)g \;\Leftrightarrow\; j^{+1} \in (\bar{\Delta}_m)g$ for $j \in \bar{\Delta}_n$, therefore $i^{+1}e = (ie)^{+1}$. We also have $j \in g^{-1}(ie) \;\Leftrightarrow\; j^{+1} \in g^{-1}((ie)^{+1}) = g^{-1}(i^{+1}e)$ for $j \in \bar{\Delta}_m$ and therefore $i^{+1}f = (if)^{+1}$. So f is quasiperiodic.

(2): As f is quasiperiodic, it fulfils the requirements of Lemma 75 (2) as all required sets are bounded from above. Thus there exists a unique monotone map $g\colon \bar{\Delta}_m \to \bar{\Delta}_n$. Then arguments dual to (1) apply and thus g is quasiperiodic. □

Remark 87. *For a quasiperiodic monotone map $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ there exists a $k \leqslant m$ and quasiperiodic monotone maps $\bar{\Delta}_n \xrightarrow{g} \bar{\Delta}_k$ and $\bar{\Delta}_k \xrightarrow{h} \bar{\Delta}_m$ with g surjective, h injective and $gh = f$.*

Proof. Define $I := [0,n]f\,\mathrm{b}_m$ and $k := |I|-1$. Let $f_0 < f_1 < \cdots < f_k$ be the elements of I. Define the bijective monotone map $\sigma\colon \bar{\Delta}_n f \to \bar{\Delta}_k$ by $f_i^{+j} \mapsto i^{+j}$ for $f_i \in I$ and $j \in \mathbb{Z}$. Define the injective monotone map $h\colon \bar{\Delta}_k \to \bar{\Delta}_m$ by $i^{+j} \mapsto f_i^{+j}$ for $i \in [0,k]$ and $j \in \mathbb{Z}$.

Now define $g := f|^{\bar{\Delta}_n f}\sigma$.

We show that g is quasiperiodic monotone:

Suppose given $i \in \bar{\Delta}_n$. Let $f_j^{+l} = if$, where $l \in \mathbb{Z}$ and $j \in [0,k]$. Then $ig = if\sigma = f_j^{+l}\sigma = j^{+l}$ and $i^{+1}g = i^{+1}f\sigma = (if)^{+1}\sigma = f_j^{+l+1}\sigma = j^{+l+1}$. Thus g is quasiperiodic monotone.

We show that h is quasiperiodic monotone:

We get $i^{+j+1}h = f_i^{+j+1} = (f_i^{+j})^{+1} = i^{+j}h$, thus h is quasiperiodic monotone.

The composite of g and h is f:

Suppose given $i \in \bar{\Delta}_n$. Let $f_j^{+l} = if$ where $l \in \mathbb{Z}$ and $j \in [0,k]$. Then

$$igh = if\sigma h = f_j^{+l}\sigma h = j^{+l}h = f_j^{+l} = if.$$

$\square$

4.2 The poset with shift $\bar{\Delta}_n^{\#}$

Suppose given $n, m \in \mathbb{Z}_{\geqslant 1}$.

Definition 88. We define a partially ordered set $\bar{\Delta}_n^{\#}$ as

$$\bar{\Delta}_n^{\#} \;:=\; \{(t,s) :\; t, s \in \bar{\Delta}_n,\; s \leqslant t \leqslant s^{+1}\}$$

with

$$(t,s) \leqslant (t',s') \;:\Leftrightarrow\; t \leqslant t' \;\wedge\; s \leqslant s'$$

for $(t,s), (t',s') \in \bar{\Delta}_n^{\#}$.

We also define a subset

$$\bar{\Delta}_n^{\#,\circ} \;:=\; \{(t,s) :\; t, s \in \bar{\Delta}_n,\; s < t < s^{+1}\} \subseteq \bar{\Delta}_n^{\#}$$

We abbreviate ${}^{t}\!/\!{}_{s} := (t,s)$ for $(t,s) \in \bar{\Delta}_n^{\#}$, which reads “t mod s”.

Define a shift operator as $({}^{t}\!/\!{}_{s})^{+1} \;:=\; {}^{s^{+1}}\!/\!{}_{t}$.

Note that $({}^{t}\!/\!{}_{s})^{-1} = {}^{s}\!/\!{}_{t^{-1}}$ and that $({}^{t}\!/\!{}_{s})^{+2} = {}^{t^{+1}}\!/\!{}_{s^{+1}}$.

For $k \in \mathbb{Z}$, define ${}^{t}\!/\!{}_{s} + k \;:=\; {}^{t+k}\!/\!{}_{s+k}$.

For ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#}$, define

$$\begin{aligned} \lceil {}^{t}\!/\!{}_{s} \rceil \;&:=\; ({}^{t}\!/\!{}_{s})^{+1} - 1 = {}^{s^{+1}-1}\!/\!{}_{t-1} \\ \lfloor {}^{t}\!/\!{}_{s} \rfloor \;&:=\; ({}^{t}\!/\!{}_{s})^{-1} + 1 = {}^{s+1}\!/\!{}_{t^{-1}+1}. \end{aligned}$$

Define subsets $B_n^{\circ} \subseteq \bar{\Delta}_n^{\#,\circ}$ and $B_n \subseteq \bar{\Delta}_n^{\#}$ by

$$\begin{aligned} B_n \;&:=\; \{{}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#} : 0 \leqslant s \leqslant t \leqslant n\} \\ B_n^{\circ} \;&:=\; \{{}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#} : 0 \leqslant s < t \leqslant n\} \\ &= B_n \cap \bar{\Delta}_n^{\#,\circ}. \end{aligned}$$

We also define the map

$$\begin{aligned} \mathrm{b}_n^{\mathrm{s}} \colon \bar{\Delta}_n^{\#} &\to B_n \\ {}^{j}\!/\!{}_{i} &\mapsto \begin{cases} {}^{j\,\mathrm{b}_n}\!/\!{}_{i\,\mathrm{b}_n} & \text{if } i\,\mathrm{b}_n \leqslant j\,\mathrm{b}_n \\ {}^{i\,\mathrm{b}_n}\!/\!{}_{j\,\mathrm{b}_n} & \text{else.} \end{cases} \end{aligned}$$

Example 89. The poset $\bar{\Delta}_3^{\#}$:

$$\begin{tikzcd}
 & & & & & {}^{0^{+1}}\!/_{0^{+1}} \arrow[r, dotted] & {} & \\
 & & & & {}^{3}\!/_{3} \arrow[r] & {}^{0^{+1}}\!/_{3} \arrow[r] \arrow[u] & {}^{1^{+1}}\!/_{3} \arrow[r, dotted] \arrow[u, dotted] & {} \\
 & & & {}^{2}\!/_{2} \arrow[r] & {}^{3}\!/_{2} \arrow[r] \arrow[u] & {}^{0^{+1}}\!/_{2} \arrow[r] \arrow[u] & {}^{1^{+1}}\!/_{2} \arrow[r] \arrow[u] & {}^{2^{+1}}\!/_{2} \arrow[u, dotted] \\
 & & {}^{1}\!/_{1} \arrow[r] & {}^{2}\!/_{1} \arrow[r] \arrow[u] & {}^{3}\!/_{1} \arrow[r] \arrow[u] & {}^{0^{+1}}\!/_{1} \arrow[r] \arrow[u] & {}^{1^{+1}}\!/_{1} \arrow[u] & \\
 & {}^{0}\!/_{0} \arrow[r] & {}^{1}\!/_{0} \arrow[r] \arrow[u] & {}^{2}\!/_{0} \arrow[r] \arrow[u] & {}^{3}\!/_{0} \arrow[r] \arrow[u] & {}^{0^{+1}}\!/_{0} \arrow[u] & & \\
{}^{3^{-1}}\!/_{3^{-1}} \arrow[r] & {}^{0}\!/_{3^{-1}} \arrow[r] \arrow[u] & {}^{1}\!/_{3^{-1}} \arrow[r] \arrow[u] & {}^{2}\!/_{3^{-1}} \arrow[r] \arrow[u] & {}^{3}\!/_{3^{-1}} \arrow[u] & & & \\
{} \arrow[r, dotted] \arrow[u, dotted] & {}^{0}\!/_{2^{-1}} \arrow[r] \arrow[u] & {}^{1}\!/_{2^{-1}} \arrow[r] \arrow[u] & {}^{2}\!/_{2^{-1}} \arrow[u] & & & & \\
 & {} \arrow[r, dotted] \arrow[u, dotted] & {}^{1}\!/_{1^{-1}} \arrow[u] & & & & &
\end{tikzcd}$$

Here, we have only depicted morphisms of the form ${}^{t}\!/_{s} \to {}^{t+1}\!/_{s}$ and ${}^{t}\!/_{s} \to {}^{t}\!/_{s+1}$, whenever existent.

Remark 90. *For $k \in \mathbb{Z}$, the map*

$$\begin{aligned}\bar{\Delta}_n^{\#} &\to \bar{\Delta}_n^{\#} \\ {}^{t}\!/_{s} &\mapsto {}^{t}\!/_{s} + k\end{aligned}$$

is bijective and quasiperiodic monotone.

Proof. We have the inverse map ${}^{t}\!/_{s} \mapsto {}^{t}\!/_{s} + (-k)$.

For ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}$ we get

$$\begin{aligned}({}^{t}\!/_{s})^{+1} + k &= {}^{s^{+1}}\!/_{t} + k \\ &= {}^{s^{+1}+k}\!/_{t+k} \\ &= {}^{(s+k)^{+1}}\!/_{t+k} \\ &= \left({}^{t+k}\!/_{s+k}\right)^{+1} \\ &= \left({}^{t}\!/_{s} + k\right)^{+1}.\end{aligned}$$

□

Lemma 91. *Suppose given elements ${}^{t}\!/\!_{s+k}, {}^{t+l}\!/\!_{s}$ in $\bar{\Delta}_n^{\#}$ with $s, t \in \bar{\Delta}_n$ and $k, l \geqslant 0$. Then for every $i, j \in \bar{\Delta}_n$ with $s \leqslant i \leqslant s+k$ and $t \leqslant j \leqslant t+l$ the element ${}^{j}\!/\!_{i}$ is contained in $\bar{\Delta}_n^{\#}$.*

Proof. We have that $s + k \leqslant t \leqslant (s+k)^{+1}$ and $s \leqslant t + l \leqslant s^{+1}$. Using $s \leqslant i \leqslant s + k$ and $t \leqslant j \leqslant t + l$ we get

$$s \leqslant i \leqslant s + k \leqslant t \leqslant j \leqslant t + l \leqslant s^{+1} \leqslant i^{+1} \leqslant (s+k)^{+1}.$$

In particular, $i \leqslant j \leqslant i^{+1}$, i.e. ${}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#}$. □

Lemma 92. *Suppose given ${}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#}$. Then*

(i) $\lfloor \lceil {}^{j}\!/\!_{i} \rceil \rfloor = {}^{j}\!/\!_{i}$

(ii) $\lceil \lfloor {}^{j}\!/\!_{i} \rfloor \rceil = {}^{j}\!/\!_{i}$.

Proof. Ad (i): We have

$$\begin{aligned} \lfloor \lceil {}^{j}\!/\!_{i} \rceil \rfloor &= \lfloor ({}^{j}\!/\!_{i})^{+1} - 1 \rfloor \\ &= (({}^{j}\!/\!_{i})^{+1} - 1)^{-1} + 1 \\ &\overset{\text{R.90}}{=} (({}^{j}\!/\!_{i})^{+1})^{-1} \\ &= {}^{j}\!/\!_{i}. \end{aligned}$$

Ad (ii): We have

$$\begin{aligned} \lceil \lfloor {}^{j}\!/\!_{i} \rfloor \rceil &= \lfloor ({}^{j}\!/\!_{i})^{-1} + 1 \rfloor \\ &= (({}^{j}\!/\!_{i})^{-1} + 1)^{+1} - 1 \\ &\overset{\text{R.90}}{=} (({}^{j}\!/\!_{i})^{-1})^{+1} \\ &= {}^{j}\!/\!_{i}. \end{aligned}$$

□

Definition 93. For ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#}$ we define

$$\begin{aligned} \mathrm{u}_n^{t/s} &:= \{ {}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#} : {}^{t}\!/\!_{s} \leqslant {}^{j}\!/\!_{i} \leqslant \lceil {}^{t}\!/\!_{s} \rceil \} \\ \mathrm{d}_n^{t/s} &:= \{ {}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#} : \lfloor {}^{t}\!/\!_{s} \rfloor \leqslant {}^{j}\!/\!_{i} \leqslant {}^{t}\!/\!_{s} \} \end{aligned}$$

For ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$ we have $\mathrm{u}_n^{t/s} = \mathrm{d}_n^{t/s} = \emptyset$.

Remark 94. For ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#,\circ}$ we have $\mathrm{u}_n^{t/s} \subseteq \bar{\Delta}_n^{\#,\circ}$ and $\mathrm{d}_n^{t/s} \subseteq \bar{\Delta}_n^{\#,\circ}$.

Example 95. A depiction of $\mathrm{u}_5^{3/1} \subset \bar{\Delta}_5^{\#}$:

$$\begin{tikzcd}
 & & & & & & & & & \\
 & & & 3/3 \arrow[r] & 4/3 \arrow[r] \arrow[u, dotted] & 5/3 \arrow[r] \arrow[u, dotted] & 0^{+1}/3 \arrow[r] \arrow[u, dotted] & 1^{+1}/3 \arrow[r] \arrow[u, dotted] & 2^{+1}/3 \arrow[r] \arrow[u, dotted] & 3^{+1}/3 \arrow[u, dotted] \\
 & & 2/2 \arrow[r] & 3/2 \arrow[r] \arrow[u] & 4/2 \arrow[r] \arrow[u] & 5/2 \arrow[r] \arrow[u] & 0^{+1}/2 \arrow[r] \arrow[u] & 1^{+1}/2 \arrow[r] \arrow[u] & 2^{+1}/2 \arrow[u] & \\
 & 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & 3/1 \arrow[r] \arrow[u] & 4/1 \arrow[r] \arrow[u] & 5/1 \arrow[r] \arrow[u] & 0^{+1}/1 \arrow[r] \arrow[u] & 1^{+1}/1 \arrow[u] & & \\
0/0 \arrow[r] & 1/0 \arrow[r] \arrow[u] & 2/0 \arrow[r] \arrow[u] & 3/0 \arrow[r] \arrow[u] & 4/0 \arrow[r] \arrow[u] & 5/0 \arrow[r] \arrow[u] & 0^{+1}/0 \arrow[u] & & & \\
\arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & & & &
\end{tikzcd}$$

Lemma 96. *Let $t/s, t'/s' \in \bar{\Delta}_n^{\#}$ Then $t/s \in \mathrm{d}_n^{t'/s'}$ if and only if $t'/s' \in \mathrm{u}_n^{t/s}$.*

Proof. We have

$$\begin{aligned}
t/s \in \mathrm{d}_n^{t'/s'} &\Leftrightarrow\ t/s \leqslant t'/s' \ \wedge\ (t'/s')^{-1} + 1 \leqslant t/s \\
&\Leftrightarrow\ t/s \leqslant t'/s' \ \wedge\ t'/s' \leqslant (t/s - 1)^{+1} \\
&\Leftrightarrow\ t/s \leqslant t'/s' \ \wedge\ t'/s' \leqslant (t'/s')^{+1} - 1 \\
&\Leftrightarrow\ t'/s' \in \mathrm{u}_n^{t/s}\,.
\end{aligned}$$

□

Lemma 97. *For every quasiperiodic monotone map $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ we get the quasiperiodic monotone map*

$$\begin{aligned}
f^{\#}\colon \bar{\Delta}_n^{\#} &\to \bar{\Delta}_m^{\#} \\
t/s &\mapsto tf/sf.
\end{aligned}$$

Suppose given $f, g\colon \bar{\Delta}_n \to \bar{\Delta}_m$ quasiperiodic monotone. Then $f \leqslant g$ implies $f^{\#} \leqslant g^{\#}$. In fact

$$\begin{aligned}
(\bar{\Delta}_n, \bar{\Delta}_m)_{\text{q.p.}} &\to \boldsymbol{(}\bar{\Delta}_n^{\#}, \bar{\Delta}_m^{\#}\boldsymbol{)} \\
(f \xrightarrow{(f,g)} g) &\mapsto (f^{\#} \xrightarrow{(f^{\#},g^{\#})} g^{\#})
\end{aligned}$$

is a functor.

For a transformation $\alpha = (f, g)\colon f \to g$ we write $\alpha^{\#} := (f^{\#}, g^{\#})\colon f^{\#} \to g^{\#}$.

Proof. Suppose given ${}^{t}\!/_{s} \in \bar{\Delta}_n^\#$, that is $s \leqslant t \leqslant s^{+1}$. Then $sf \leqslant tf \leqslant s^{+1}f = (sf)^{+1}$ and thus ${}^{tf}\!/_{sf} \in \bar{\Delta}_m^\#$.

The map $f^\#$ is monotone as ${}^{t}\!/_{s} \leqslant {}^{t'}\!/_{s'}$ implies $t \leqslant t'$ and $s \leqslant s'$, whence $tf \leqslant t'f$ and $sf \leqslant s'f$ and so ${}^{tf}\!/_{sf} \leqslant {}^{t'f}\!/_{s'f}$.

It is quasiperiodic monotone as we have

$$({}^{t}\!/_{s})^{+1} f^\# = ({}^{s^{+1}}\!/_{t}) f^\# = {}^{s^{+1}f}\!/_{tf} = {}^{sf^{+1}}\!/_{tf} = ({}^{tf}\!/_{sf})^{+1}.$$

For the identity transformation $1_f = (f, f)$ we put the identity transformation $(1_f)^\# = (f^\#, f^\#) = 1_{f^\#}$.

Suppose given a quasiperiodic monotone map $h \colon \bar{\Delta}_n \to \bar{\Delta}_m$ and the transformation $g \xrightarrow{(g,h)} h$. Then we get

$$((f, g) \cdot (g, h))^\# = (f, h^\# = (f^\#, h^\#) = (f^\#, g^\#) \cdot (g^\#, h^\#).$$

Thus $(f \xrightarrow{(f,g)} g) \mapsto (f^\# \xrightarrow{(f^\#,g^\#)} g^\#)$ defines a functor. □

Remark 98. *Suppose given $f, \tilde{f} \colon \bar{\Delta}_n \to \bar{\Delta}_m$, $g \colon \bar{\Delta}_m \to \bar{\Delta}_p$ and $h \colon \bar{\Delta}_m \to \bar{\Delta}_n$ quasiperiodic monotone. Then*

(i) $(fg)^\# = f^\# g^\#$,

(ii) f *injective if and only if* $f^\#$ *injective,*

(iii) $f \leqslant \tilde{f}$ *if and only if* $f^\# \leqslant \tilde{f}^\#$,

(iv) $f \dashv h$ *if and only if* $f^\# \dashv h^\#$.

Proof. *(i)* We have ${}^{t}\!/_{s} f^\# g^\# = {}^{tfg}\!/_{sfg} = {}^{t}\!/_{s} (fg)^\#$ for every ${}^{t}\!/_{s} \in \bar{\Delta}_n^\#$.

(ii) Suppose that f is injective. Suppose given ${}^{t}\!/_{s}, {}^{t'}\!/_{s'} \in \bar{\Delta}_n^\#$ with ${}^{t}\!/_{s} \neq {}^{t'}\!/_{s'}$, that is $t \neq t'$ or $s \neq s'$. Then $tf \neq t'f$ or $sf \neq s'f$ and therefore ${}^{t}\!/_{s} f^\# = {}^{tf}\!/_{sf} \neq {}^{t'f}\!/_{s'f} = {}^{t'}\!/_{s'} f^\#$.

Suppose that $f^\#$ is injective. Suppose given $i, j \in \bar{\Delta}_n$ with $i \neq j$. We know that ${}^{i}\!/_{i} f^\# \neq {}^{j}\!/_{j} f^\#$ and therefore $if \neq jf$.

(iii) By Lemma 97, we have $f \leqslant \tilde{f} \;\Rightarrow\; f^\# \leqslant \tilde{f}^\#$.

Suppose $f^\# \leqslant \tilde{f}^\#$. Then ${}^{i}\!/_{i} f^\# = {}^{if}\!/_{if} \leqslant {}^{i\tilde{f}}\!/_{i\tilde{f}} = {}^{i}\!/_{i} \tilde{f}^\#$ and thus $if \leqslant i\tilde{f}$ for every $i \in \bar{\Delta}_n$. Therefore $f \leqslant \tilde{f}$.

(iv) By *(iii)* we have $1_{\bar{\Delta}_n} \leqslant fh$ if and only if $1_{\bar{\Delta}_n^\#} \leqslant (fh)^\# = f^\# h^\#$. We also have $hf \leqslant 1_{\bar{\Delta}_m}$ if and only $h^\# f^\# \leqslant 1_{\bar{\Delta}_m^\#}$. Therefore $f \dashv h$ if and only if $f^\# \dashv h^\#$; cf. Lemma 69, Remark 83. □

Definition 99. Every $k \in \mathbb{Z}$ can be written in a unique way as $k = an + b$ with $a \in \mathbb{Z}$ and $b \in [1, n]$. We use this to define

$$\rho_n \colon \mathbb{Z} \to \bar{\Delta}_n^{\#,\circ},\ k \mapsto {}^{(a+b)\varphi_n^{-1}}/_{a\varphi_n^{-1}}$$

for $k = an + b$ with $a \in \mathbb{Z}$ and $b \in [1, n]$. The map ρ_n is bijective.

The inverse map is given by

$$\rho_n^{-1} \colon \bar{\Delta}_n^{\#,\circ} \to \mathbb{Z},\ {}^{t}/_{s} \mapsto s\varphi_n \cdot (n-1) + t\varphi_n$$

using the bijection $\varphi_n \colon \bar{\Delta}_n \to \mathbb{Z}$ from Remark 82. It respects the partial order in the sense that for $i, j \in \mathbb{Z}$, $i\rho_n \leqslant j\rho_n$ implies $i \leqslant j$, i.e. ρ_n^{-1} is monotone.

If n is clear from context, we sometimes only write $\rho := \rho_n$ and $\rho^{-1} := \rho_n^{-1}$ for short.

Example 100. E.g. $\rho_3 \colon \mathbb{Z} \to \bar{\Delta}_3^{\#,\circ}$ maps

$$\begin{aligned} -2 &\mapsto {}^{0}/_{3^{-1}} \\ 0 &\mapsto {}^{2}/_{3^{-1}} \\ 1 &\mapsto {}^{1}/_{0} \\ 2 &\mapsto {}^{2}/_{0} \\ 4 &\mapsto {}^{2}/_{1} \end{aligned}$$

A depiction of $\bar{\Delta}_3^{\#,\circ}$.

$$\begin{tikzcd}
 & & & {} & {} & {} & {} \\
 & & {}^{1}/_{1} \arrow[r] & {}^{2}/_{1} \arrow[r] \arrow[u, dotted] & {}^{3}/_{1} \arrow[r] \arrow[u, dotted] & {}^{0^{+1}}/_{1} \arrow[r] \arrow[u, dotted] & {}^{1^{+1}}/_{1} \arrow[u, dotted] \\
 & {}^{0}/_{0} \arrow[r] & {}^{1}/_{0} \arrow[r] \arrow[u] & {}^{2}/_{0} \arrow[r] \arrow[u] & {}^{3}/_{0} \arrow[r] \arrow[u] & {}^{0^{+1}}/_{0} \arrow[u] & \\
{}^{3^{-1}}/_{3^{-1}} \arrow[r] & {}^{0}/_{3^{-1}} \arrow[r] \arrow[u] & {}^{1}/_{3^{-1}} \arrow[r] \arrow[u] & {}^{2}/_{3^{-1}} \arrow[r] \arrow[u] & {}^{3}/_{3^{-1}} \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

4.3 Quasiperiodic monotone maps on $\bar{\Delta}_1$

Suppose given $n \in \mathbb{Z}_{\geqslant 1}$.

Definition 101. Suppose given $s, t \in \bar{\Delta}_n$ such that $s \leqslant t \leqslant s^{+1}$, i.e. such that ${}^{t}/_{s} \in \bar{\Delta}_n^{\#}$. We define the quasiperiodic monotone map $f_n^{t/s} \colon \bar{\Delta}_1 \to \bar{\Delta}_n$ by

$$\begin{aligned} f_n^{t/s} \colon \bar{\Delta}_1 &\to \bar{\Delta}_n \\ 0 &\mapsto s \\ 1 &\mapsto t. \end{aligned}$$

If n is clear from context, we usually write $f^{t/s} := f_n^{t/s}$.

The map $f^{t/s}\colon \bar{\Delta}_1 \to \bar{\Delta}_n$ is injective if and only if $t/s \in \bar{\Delta}_n^{\#,\circ}$.

Every $f^{t/s}\colon \bar{\Delta}_1 \to \bar{\Delta}_n$ defines a quasiperiodic monotone map $(f^{t/s})^{\#}\colon \bar{\Delta}_1^{\#} \to \bar{\Delta}_n^{\#}$; cf. Lemma 97. We usually write $f^{t/s,\#} := (f^{t/s})^{\#}$.

We define $\hat{f}^{t/s}\colon \bar{\Delta}_n \to \bar{\Delta}_1$ to be the right adjoint of $f^{t/s}$, cf. Lemma 86 and Lemma 74. The map $\hat{f}^{t/s}$ is given by

$$i\hat{f}^{t/s} = \begin{cases} 0 & \text{for } s \leqslant i < t \\ 1 & \text{for } t \leqslant i < s^{+1}. \end{cases}$$

We define $\check{f}^{t/s}\colon \bar{\Delta}_n \to \bar{\Delta}_1$ to be the left adjoint of $f^{t/s}$; cf. Lemma 86. The map $\check{f}^{t/s}$ is given by

$$i\check{f}^{t/s} = \begin{cases} 0 & \text{for } t^{-1} < i \leqslant s \\ 1 & \text{for } s < i \leqslant t. \end{cases}$$

So for every $t/s \in \bar{\Delta}_n^{\#}$ we have $\check{f}^{t/s} \dashv f^{t/s} \dashv \hat{f}^{t/s}$.

We denote the according transformations

$$\begin{aligned}
\check{\eta}^{t/s} &:= (1_{\bar{\Delta}_n}, \check{f}^{t/s} f^{t/s})\colon 1_{\bar{\Delta}_n} \to \check{f}^{t/s} f^{t/s} \\
\check{\varepsilon}^{t/s} &:= (f^{t/s} \check{f}^{t/s}, 1_{\bar{\Delta}_1})\colon f^{t/s} \check{f}^{t/s} \to 1_{\bar{\Delta}_1} \\
\hat{\eta}^{t/s} &:= (1_{\bar{\Delta}_1}, f^{t/s} \hat{f}^{t/s})\colon 1_{\bar{\Delta}_1} \to f^{t/s} \hat{f}^{t/s} \\
\hat{\varepsilon}^{t/s} &:= (\hat{f}^{t/s} f^{t/s}, 1_{\bar{\Delta}_n})\colon \hat{f}^{t/s} f^{t/s} \to 1_{\bar{\Delta}_n}.
\end{aligned}$$

For $j/i \in \bar{\Delta}_n^{\#}$ and $t/s \in \bar{\Delta}_n^{\#}$, we define

$$\begin{aligned}
\lfloor j/i \rfloor_{t/s} &:= j/i \hat{f}^{t/s,\#} f^{t/s,\#} \\
\lceil j/i \rceil_{t/s} &:= j/i \check{f}^{t/s,\#} f^{t/s,\#}.
\end{aligned}$$

Remark 102. *Suppose given $j/i \in \bar{\Delta}_n^{\#,\circ}$. Then*

(i) $\lfloor j/i \rfloor = \lfloor j/i \rfloor_{j/i+1}$

(ii) $\lceil j/i \rceil = \lceil j/i \rceil_{j/i-1}$

Cf. Definition 88, Definition 101.

Proof. Suppose given ${}^{j}\!/_{i} \in \bar{\Delta}_n^{\#,\circ}$. Then we have *(i)*

$$\begin{aligned}
\lfloor {}^{j}\!/_{i} \rfloor_{{}^{j}\!/_{i}\,+1} &= ({}^{j}\!/_{i})\hat{f}^{{}^{j}\!/_{i}\,+1,\#} f^{{}^{j}\!/_{i}\,+1,\#} \\
&= ({}^{0}\!/_{1^{-1}})f^{{}^{j}\!/_{i}\,+1,\#} \\
&= (({}^{1}\!/_{0})^{-1})f^{{}^{j}\!/_{i}\,+1,\#} \\
&= ({}^{j}\!/_{i} + 1)^{-1} \\
&= ({}^{j}\!/_{i})^{-1} + 1 \\
&= \lfloor {}^{j}\!/_{i} \rfloor
\end{aligned}$$

and *(ii)*

$$\begin{aligned}
\lceil {}^{j}\!/_{i} \rceil_{{}^{j}\!/_{i}\,-1} &= ({}^{j}\!/_{i})\check{f}^{{}^{j}\!/_{i}\,-1,\#} f^{{}^{j}\!/_{i}\,-1,\#} \\
&= ({}^{0^{+1}}\!/_{1})f^{{}^{j}\!/_{i}\,-1,\#} \\
&= (({}^{1}\!/_{0})^{+1})f^{{}^{j}\!/_{i}\,-1,\#} \\
&= ({}^{j}\!/_{i} - 1)^{+1} \\
&= ({}^{j}\!/_{i})^{+1} - 1 \\
&= \lceil {}^{j}\!/_{i} \rceil .
\end{aligned}$$

□

Lemma 103. *Suppose given* ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#,\circ}$. *Then*

$$\hat{f}^{{}^{t}\!/_{s}} = \check{f}^{\lceil {}^{t}\!/_{s} \rceil}$$

Proof. Suppose given ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#,\circ}$. Suppose given $i \in \bar{\Delta}_n$. We have

$$i\hat{f}^{{}^{t}\!/_{s}} = \max\Big((f^{{}^{t}\!/_{s}})^{-1}(\max\{j \in (\bar{\Delta}_1)f^{{}^{t}\!/_{s}} \colon j \leqslant i\})\Big)$$

and thus

$$il \;:=\; i\hat{f}^{{}^{t}\!/_{s}} f^{{}^{t}\!/_{s}} = \max\{j \in (\bar{\Delta}_1)f^{{}^{t}\!/_{s}} \colon j \leqslant i\} \in (\bar{\Delta}_1)f^{{}^{t}\!/_{s}}.$$

Case $il = t^{+k}$ *for* $k \in \mathbb{Z}$*:*

Suppose that $il = t^{+k}$ for some $k \in \mathbb{Z}$.

Then

$$\begin{aligned}
i\hat{f}^{{}^{t}\!/_{s}} &= \max\Big((f^{{}^{t}\!/_{s}})^{-1}(\max\{j \in (\bar{\Delta}_1)f^{{}^{t}\!/_{s}} \colon j \leqslant i\})\Big) \\
&= \max\Big((f^{{}^{t}\!/_{s}})^{-1}(t^{+k})\Big) \\
&= \max\Big(\{1^{+k}\}\Big) \\
&= 1^{+k}
\end{aligned}$$

For $il = t^{+k}$ we have

$$s^{+k} < t^{+k} = il \leqslant i < s^{+k+1} \leqslant t^{+k+1}$$

and thus

$$(s-1)^{+k} < (t-1)^{+k} < i \leqslant (s-1)^{+k+1} < (t-1)^{+k+1}.$$

Therefore

$$\min\{j \in (\bar{\Delta}_1)f^{\lceil t/s \rceil} \colon j \geqslant i\} = (s-1)^{+k+1}.$$

We get

$$\begin{aligned} i\check{f}^{\lceil t/s \rceil} &= \min\Big((f^{\lceil t/s \rceil})^{-1}(\min\{j \in (\bar{\Delta}_1)f^{\lceil t/s \rceil} \colon j \geqslant i\})\Big) \\ &= \min\Big((f^{\lceil t/s \rceil})^{-1}((s-1)^{+k+1})\Big) \\ &= \min\Big(\{1^{+k}\}\Big) \\ &= 1^{+k} \end{aligned}$$

Case $il = s^{+k}$ for $k \in \mathbb{Z}$:

Suppose that $il = s^{+k}$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} i\hat{f}^{t/s} &= \max\Big((f^{t/s})^{-1}(\max\{j \in (\bar{\Delta}_1)f^{t/s} \colon j \leqslant i\})\Big) \\ &= \max\Big((f^{t/s})^{-1}(s^{+k})\Big) \\ &= \max\Big(\{0^{+k}\}\Big) \\ &= 0^{+k} \end{aligned}$$

We have

$$t^{+k-1} < s^{+k} = il \leqslant i < t^{+k} < s^{+k+1}$$

and thus

$$(t-1)^{+k-1} < (s-1)^{+k} < i \leqslant (t-1)^{+k} < (s-1)^{+k+1}$$

Therefore

$$\min\{j \in (\bar{\Delta}_1)f^{\lceil t/s \rceil} \colon j \geqslant i\} = (t-1)^{+k}.$$

We get

$$\begin{aligned} i\check{f}^{\lceil t/s \rceil} &= \min\Big((f^{\lceil t/s \rceil})^{-1}(\min\{j \in (\bar{\Delta}_1)f^{\lceil t/s \rceil} \colon j \geqslant i\})\Big) \\ &= \min\Big((f^{\lceil t/s \rceil})^{-1}((t-1)^{+k})\Big) \\ &= \min\Big(\{0^{+k}\}\Big) \\ &= 0^{+k} \end{aligned}$$

Therefore $\hat{f}^{t/s} = \check{f}^{\lceil t/s \rceil}$. $\square$

Lemma 104. *Suppose given $i, j \in \bar{\Delta}_n$ and $t/s, t'/s' \in \bar{\Delta}_n^{\#}$ with $t/s \sim_{\mathrm{s}} t'/s'$. Then*

$$(i\hat{f}^{t/s} = j\hat{f}^{t/s}) \Leftrightarrow (i\hat{f}^{t'/s'} = j\hat{f}^{t'/s'}).$$

In particular, we get

$$(j/i\hat{f}^{t/s,\#} = j'/i'\hat{f}^{t/s,\#}) \Leftrightarrow (j/i\hat{f}^{t'/s',\#} = j'/i'\hat{f}^{t'/s',\#}).$$

for $j/i, j'/i' \in \bar{\Delta}_n^{\#}$.

Proof. The fibers of $\hat{f}^{t/s}$ on $\bar{\Delta}_n^{\#}$ are given by $[s^{+l}, t^{+l} - 1]$ and $[t^{+l}, s^{+l+1} - 1]$ for $l \in \mathbb{Z}$; cf. Definition 101.

The fibers of $\hat{f}^{(t/s)^{+1}} = \hat{f}^{s^{+1}/t}$ are $[t^{+l}, s^{+l+1} - 1]$ and $[s^{+l+1}, t^{+l+1}]$ for $l \in \mathbb{Z}$; cf. Definition 101.

So the fibers are the same.

Moreover, we have $j/i\hat{f}^{t/s,\#} = j'/i'\hat{f}^{t/s,\#}$ if and only if $j\hat{f}^{t/s} = j'\hat{f}^{t/s}$ and $i\hat{f}^{t/s} = i'\hat{f}^{t/s}$. $\square$

Lemma 105. *Suppose given $j/i \in \bar{\Delta}_n^{\#,\circ}$. Suppose given $j'/i' \in \bar{\Delta}_n^{\#,\circ}$ with $j/i \sim_{\mathrm{s}} j'/i'$. Then*

$$\begin{aligned} \mathrm{u}_n^{j/i} &= (\hat{f}^{j'/i',\#})^{-1}(j/i\hat{f}^{j'/i',\#}) \\ \mathrm{d}_n^{j/i} &= (\check{f}^{j'/i',\#})^{-1}(j/i\check{f}^{j'/i',\#}) \end{aligned}$$

In particular, $t/s\hat{f}^{j'/i',\#} = j/i\hat{f}^{j'/i',\#}$ for every $t/s \in \mathrm{u}_n^{j/i}$ and $t/s\check{f}^{j'/i',\#} = j/i\check{f}^{j'/i',\#}$ for every $t/s \in \mathrm{d}_n^{j/i}$.

Proof. We first show that $\mathrm{u}_n^{j/i} = (\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#})$.

Recall that $\lfloor\lceil j/i \rceil\rfloor = j/i$ and $\lceil\lfloor j/i \rfloor\rceil = j/i$ for every $j/i \in \bar{\Delta}_n^{\#}$; Lemma 92.

We know that $\mathrm{u}_n^{j/i} = \mathrm{d}_n^{\lceil j/i \rceil} = \{t/s \in \bar{\Delta}_n^{\#} : j/i \leqslant t/s \leqslant \lceil j/i \rceil\}$; cf. Definition 93.

We have $f^{j/i,\#} \dashv \hat{f}^{j/i,\#} = \check{f}^{\lceil j/i \rceil,\#} \dashv f^{\lceil j/i \rceil,\#}$; cf. Remark 98.

According to Lemma 74 (2), we get the initial element of $(\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#})$ by

$$\min(\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#}) = j/i\hat{f}^{j/i,\#}f^{j/i,\#} = 1/0f^{j/i,\#} = j/i$$

and according to Lemma 74 (4) the terminal element of $(\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#})$ by

$$\max\Big((\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#})\Big) \overset{\text{L.103}}{=} j/i\hat{f}^{j/i,\#}f^{\lceil j/i \rceil,\#} = 1/0f^{\lceil j/i \rceil,\#} = \lceil j/i \rceil.$$

Therefore we know that $(\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#}) \subseteq \mathrm{u}_n^{j/i}$.

For every element $t/s \in \mathrm{u}_n^{j/i}$ we have $j/i \leqslant t/s \leqslant \lceil j/i \rceil$. Therefore

$$j/i\hat{f}^{j/i,\#} \leqslant t/s\hat{f}^{j/i,\#} \leqslant \lceil j/i \rceil\hat{f}^{j/i,\#} = 1/0 = j/i\hat{f}^{j/i,\#}$$

and thus

$$t/s \in (\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#}).$$

Thus $\mathrm{u}_n^{j/i} = (\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#})$.

Suppose that $j'/i' = (j/i)^{+k}$ for some $k \in \mathbb{Z}$. We get

$$\begin{aligned}
\mathrm{u}_n^{j/i} &= (\hat{f}^{j/i,\#})^{-1}(j/i\hat{f}^{j/i,\#}) \\
&= \{b/a \in \bar{\Delta}_n^{\#} : b/a\hat{f}^{j/i,\#} = j/i\hat{f}^{j/i,\#}\} \\
&\overset{\text{L.104}}{=} \{b/a \in \bar{\Delta}_n^{\#} : b/a\hat{f}^{j'/i',\#} = j/i\hat{f}^{j'/i',\#}\} \\
&= (\hat{f}^{j'/i',\#})^{-1}(j/i\hat{f}^{j'/i',\#}).
\end{aligned}$$

For $\mathrm{d}_n^{j/i}$ we get

$$\begin{aligned}
\mathrm{d}_n^{j/i} &= \mathrm{u}_n^{\lfloor j/i \rfloor} \\
&= (\hat{f}^{\lfloor j/i \rfloor,\#})^{-1}(\lfloor j/i \rfloor \hat{f}^{\lfloor j/i \rfloor,\#}) \\
&= \left(\hat{f}^{\lfloor j/i \rfloor,\#}\right)^{-1}(1/0) \\
&= (\hat{f}^{\lfloor j/i \rfloor,\#})^{-1}(j/i\hat{f}^{\lfloor j/i \rfloor,\#}) \\
&\overset{\text{L.104}}{=} (\hat{f}^{(\lfloor j/i \rfloor)^{+k},\#})^{-1}(j/i\hat{f}^{(\lfloor j/i \rfloor)^{+k},\#}) \\
&= (\hat{f}^{(\lfloor (j/i)^{+k} \rfloor),\#})^{-1}(j/i\hat{f}^{(\lfloor (j/i)^{+k} \rfloor),\#}) \\
&= (\hat{f}^{(\lfloor j'/i' \rfloor),\#})^{-1}(j/i\hat{f}^{(\lfloor j'/i' \rfloor),\#}) \\
&\overset{\text{L.103}}{=} (\check{f}^{j'/i',\#})^{-1}(j/i\check{f}^{j'/i',\#}).
\end{aligned}$$

□

Remark 106. Suppose given $t/s \in \bar{\Delta}_n^{\#,\circ}$. Then $f^{t/s}$ is injective and therefore

$$f^{t/s}\hat{f}^{t/s} = 1_{\bar{\Delta}_1}$$

and

$$f^{t/s}\hat{f}^{\lfloor t/s \rfloor} = 1_{\bar{\Delta}_1}.$$

Cf. Lemma 74.

Lemma 107. *Suppose given $j/i \in \bar{\Delta}_1^{\#}$. Then the following assertions (1), (2), (3) are equivalent.*

(1) $j/i \in \bar{\Delta}_1^{\#,\circ}$

(2) $j/i = (1/0)^{+k}$ *for some* $k \in \mathbb{Z}$

(3) $j = i + 1$

In particular, $\bar{\Delta}_1^{\#,\circ} = [{}^1\!/\!{}_0]_{\sim_s}$.

Proof. (2) $\Rightarrow$ (1) : We show that

$$[{}^1\!/\!{}_0]_{\sim_s} = \{{}^t\!/\!{}_s \in \bar{\Delta}_1^{\#} : {}^t\!/\!{}_s = ({}^1\!/\!{}_0)^{+k} \text{ for some } k \in \mathbb{Z}\} \overset{!}{\subseteq} \bar{\Delta}_1^{\#,\circ}.$$

Then in particular ${}^j\!/\!{}_i \in \bar{\Delta}_1^{\#,\circ}$.

For every ${}^t\!/\!{}_s \in \bar{\Delta}_1^{\#,\circ}$ we have

$$s < t < s^{+1}$$

and therefore

$$t < s^{+1} < t^{+1}$$

as well as

$$t^{-1} < s < t.$$

Therefore for every ${}^t\!/\!{}_s \in \bar{\Delta}_1^{\#,\circ}$, we also have $({}^t\!/\!{}_s)^{+1} \in \bar{\Delta}_1^{\#,\circ}$ and $({}^t\!/\!{}_s)^{-1} \in \bar{\Delta}_1^{\#,\circ}$.

Starting with ${}^1\!/\!{}_0 \in \bar{\Delta}_1^{\#,\circ}$, we get $({}^1\!/\!{}_0)^{+k} \in \bar{\Delta}_1^{\#,\circ}$ for every $k \in \mathbb{Z}$ via induction and thus $[{}^1\!/\!{}_0]_{\sim_s} \subseteq \bar{\Delta}_1^{\#,\circ}$.

(1) $\Rightarrow$ (3) : Let ${}^j\!/\!{}_i \in \bar{\Delta}_1^{\#,\circ}$. This means, $i < j < i^{+1}$. As $i^{+1} = i+2$, this implies $j = i+1$.

(3) $\Rightarrow$ (2) : Let $j = i+1$. Suppose that $i = 0^{+k}$ for some $k \in \mathbb{Z}$. Then

$${}^j\!/\!{}_i = {}^{1^{+k}}\!/\!{}_{0^{+k}} = ({}^1\!/\!{}_0)^{+2k}.$$

Suppose that $i = 1^{+k}$ for some $k \in \mathbb{Z}$. Then

$${}^j\!/\!{}_i = {}^{0^{+k+1}}\!/\!{}_{1^{+k}} = ({}^{0^{+1}}\!/\!{}_1)^{+2k} = ({}^1\!/\!{}_0)^{+2k+1}.$$

□

Lemma 108. *Suppose given* ${}^t\!/\!{}_s \in \bar{\Delta}_n^{\#}$. *Then*

$$[{}^t\!/\!{}_s]_{\sim_s} = \bar{\Delta}_1^{\#,\circ} f^{t/s,\#}.$$

In particular, $\bar{\Delta}_1^{\#,\circ} f^{t/s,\#} = \bar{\Delta}_1^{\#,\circ} f^{t'/s',\#}$ *if and only if* ${}^t\!/\!{}_s \sim_s {}^{t'}\!/\!{}_{s'}$.

Proof. As by Lemma 107 we have $\bar{\Delta}_1^{\#,\circ} = [{}^1\!/\!{}_0]_{\sim_s}$, this follows from Lemma 80. □

Lemma 109. *Suppose given* ${}^t\!/\!{}_s \in \bar{\Delta}_n^{\#}$.

(1) Define quasiperiodic monotone maps

$$\begin{aligned} f_k \colon \bar{\Delta}_1 &\to \bar{\Delta}_1 \\ i &\mapsto i + k \end{aligned}$$

for $k \in \mathbb{Z}$. Then

$$f^{t/s} = f_{-k} \cdot f^{(t/s)^{+k}}$$

and

$$\hat{f}^{t/s} = \hat{f}^{(t/s)^{+k}} \cdot f_k$$

for every $k \in \mathbb{Z}$.

(2) Suppose given $t/s \in \bar{\Delta}_n^{\#}$ and $j/i \in \bar{\Delta}_1^{\#,\circ}$. Then

$$j/i f^{(t/s)^{+k},\#} = \left((j/i) f^{t/s,\#}\right)^{+k}$$

for $k \in \mathbb{Z}$. Note that in general this doesn't hold for $j/i \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$.

Proof. Ad (1):

The map $f^{t/s}$ is characterised by the values $0f^{t/s} = s$ and $1f^{t/s} = t$. It is sufficient to show that $f^{t/s} = f_{-1}f^{(t/s)^{+1}}$. Then we get $f_{+1}f^{t/s} = f_{+1}f_{-1}f^{(t/s)^{+1}} = f^{(t/s)^{+1}}$. The rest follows via induction.

We have

$$0f_{-1}f^{(t/s)^{+1}} = 1^{-1}f^{(t/s)^{+1}} = 1^{-1}f^{s^{+1}/t} = s$$

and

$$1f_{-1}f^{(t/s)^{+1}} = 0f^{(t/s)^{+1}} = 0f^{s^{+1}/t} = t.$$

Therefore $f^{t/s} = f_{-1}f^{(t/s)^{+1}}$.

The map f_{+k} is the inverse of f_{-k} for every $k \in \mathbb{Z}$. In particular, $f_{-k} \dashv f_{+k}$ for every $k \in \mathbb{Z}$. Together with $f^{(t/s)^{+k}} \dashv \hat{f}^{(t/s)^{+k}}$ for $t/s \in \bar{\Delta}_n^{\#}$, we get

$$f^{t/s} = f_{-k}f^{(t/s)^{+k}} \dashv \hat{f}^{(t/s)^{+k}} f_k;$$

cf. Lemma 72. As the adjoint is unique, this yields $\hat{f}^{t/s} = \hat{f}^{(t/s)^{+k}} f_k$ for $k \in \mathbb{Z}$.

Ad (2):

We have

$$(j/i)f_{+1}^{\#} = jf_{+1}/if_{+1} = j{+}1/i{+}1 = i^{+1}/j = (j/i)^{+1}$$

and

$$(j/i)f_{-1}^{\#} = j{-}1/i{-}1 = i/j^{-1} = (j/i)^{-1}.$$

Then

$$j/i f^{(t/s)^{+1},\#} \overset{(1)}{=} j/i f_{+1}^{\#} \cdot f^{t/s,\#} = (j/i)^{+1} f^{t/s,\#} = \left(j/i f^{t/s,\#}\right)^{+1}$$

and

$$j/i f^{(t/s)^{-1},\#} \overset{(1)}{=} j/i f_{-1}^{\#} \cdot f^{t/s,\#} = (j/i)^{-1} f^{t/s,\#} = \left(j/i f^{t/s,\#}\right)^{-1}.$$

□

Lemma 110. *Suppose given* ${}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'} \in \bar{\Delta}_n^{\#}$. *The following assertions (1), (2), (3) are equivalent.*

(1) ${}^{t}\!/\!_{s} \sim_{\mathrm{s}} {}^{t'}\!/\!_{s'}$

(2) ${}^{t}\!/\!_{s}\, \mathrm{b}_n^{\mathrm{s}} = {}^{t'}\!/\!_{s'}\, \mathrm{b}_n^{\mathrm{s}}$

(3) $\{t\,\mathrm{b}_n, s\,\mathrm{b}_n\} = \{t'\,\mathrm{b}_n, s'\,\mathrm{b}_n\}$

In particular, ${}^{t}\!/\!_{s} \sim_{\mathrm{s}} {}^{t}\!/\!_{s}\, \mathrm{b}_n^{\mathrm{s}}$.

Proof. (1) $\Rightarrow$ (3):

Suppose given ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#}$.

We have $({}^{t}\!/\!_{s})^{+1} = {}^{s^{+1}}\!/\!_{t}$ and therefore $\{t\,\mathrm{b}_n, s\,\mathrm{b}_n\} = \{s^{+1}\,\mathrm{b}_n, t\,\mathrm{b}_n\}$. We have $({}^{t}\!/\!_{s})^{-1} = {}^{s}\!/\!_{t^{-1}}$ and $\{t\,\mathrm{b}_n, s\,\mathrm{b}_n\} = \{s\,\mathrm{b}_n, t^{-1}\,\mathrm{b}_n\}$.

Via induction we get $\{t\,\mathrm{b}_n, s\,\mathrm{b}_n\} = \{s'\,\mathrm{b}_n, t'\,\mathrm{b}_n\}$ for every $k \in \mathbb{Z}$ and ${}^{t'}\!/\!_{s'} = ({}^{t}\!/\!_{s})^{+k}$.

(3) $\Rightarrow$ (2) :

Suppose given ${}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'} \in \bar{\Delta}_n^{\#}$ with $\{t\,\mathrm{b}_n, s\,\mathrm{b}_n\} = \{t'\,\mathrm{b}_n, s'\,\mathrm{b}_n\}$. Recall that

$$\begin{aligned} \mathrm{b}_n^{\mathrm{s}} \colon \bar{\Delta}_n^{\#} &\to B_n \\ {}^{j}\!/\!_{i} &\mapsto \begin{cases} {}^{j\,\mathrm{b}_n}\!/\!_{i\,\mathrm{b}_n} & \text{if } i\,\mathrm{b}_n \leqslant j\,\mathrm{b}_n \\ {}^{i\,\mathrm{b}_n}\!/\!_{j\,\mathrm{b}_n} & \text{else.} \end{cases} \end{aligned}$$

Therefore ${}^{t}\!/\!_{s}\, \mathrm{b}_n^{\mathrm{s}} = {}^{t'}\!/\!_{s'}\, \mathrm{b}_n^{\mathrm{s}}$.

(2) $\Rightarrow$ (1) :

It is sufficient to show ${}^{t}\!/\!_{s} \sim_{\mathrm{s}} {}^{t}\!/\!_{s}\, \mathrm{b}_n^{\mathrm{s}}$, for then ${}^{t}\!/\!_{s} \sim_{\mathrm{s}} {}^{t}\!/\!_{s}\, \mathrm{b}_n^{\mathrm{s}} = {}^{t'}\!/\!_{s'}\, \mathrm{b}_n^{\mathrm{s}} \sim_{\mathrm{s}} {}^{t'}\!/\!_{s'}$.

Recall again that

$$\begin{aligned} \mathrm{b}_n^{\mathrm{s}} \colon \bar{\Delta}_n^{\#} &\to B_n \\ {}^{j}\!/\!_{i} &\mapsto \begin{cases} {}^{j\,\mathrm{b}_n}\!/\!_{i\,\mathrm{b}_n} & \text{if } i\,\mathrm{b}_n \leqslant j\,\mathrm{b}_n \\ {}^{i\,\mathrm{b}_n}\!/\!_{j\,\mathrm{b}_n} & \text{else.} \end{cases} \end{aligned}$$

Case $t = s$ or $t\,\mathrm{b}_n \neq s\,\mathrm{b}_n$ and ${}^{t}\!/\!_{s}\, \mathrm{b}_n^{\mathrm{s}} = {}^{t\,\mathrm{b}_n}\!/\!_{s\,\mathrm{b}_n}$:

We get $t = (t\,\mathrm{b}_n)^{+l}$ for some $l \in \mathbb{Z}$ and $s = (s\,\mathrm{b}_n)^{+k}$ for some $k \in \mathbb{Z}$. If $t = s$, we get $l = k$. If $t\,\mathrm{b}_n \neq s\,\mathrm{b}_n$ we know that $s < t < s^{+1}$ and $s\,\mathrm{b}_n < t\,\mathrm{b}_n$, therefore

$$(s\,\mathrm{b}_n)^{+k} < (t\,\mathrm{b}_n)^{+l} < (s\,\mathrm{b}_n)^{+k+1},$$

which implies $k = l$. Then

$$\begin{aligned} ({}^{t}\!/\!_{s}\, \mathrm{b}_n^{\mathrm{s}})^{+2k} &= ({}^{t\,\mathrm{b}_n}\!/\!_{s\,\mathrm{b}_n})^{+2k} \\ &= {}^{(t\,\mathrm{b}_n)^{+k}}\!/\!_{(s\,\mathrm{b}_n)^{+k}} \\ &= {}^{t}\!/\!_{s}. \end{aligned}$$

Case $t \neq s$ and ${}^{t}\!/\!_{s}\,\mathrm{b}_n^{\mathrm{s}} = {}^{s\,\mathrm{b}_n}\!/\!_{t\,\mathrm{b}_n}$:

We get $t = (t\,\mathrm{b}_n)^{+l}$ for some $l \in \mathbb{Z}$ and $s = (s\,\mathrm{b}_n)^{+k}$ for some $k \in \mathbb{Z}$. As $s \leqslant t \leqslant s^{+1}$ and $t\,\mathrm{b}_n \leqslant s\,\mathrm{b}_n$ and therefore

$$(s\,\mathrm{b}_n)^{+k} \leqslant (t\,\mathrm{b}_n)^{+l} \leqslant (s\,\mathrm{b}_n)^{+k+1},$$

we get $l = k+1$. We now have

$$\begin{aligned} ({}^{t}\!/\!_{s}\,\mathrm{b}_n^{\mathrm{s}})^{+2k+1} &= ({}^{s\,\mathrm{b}_n}\!/\!_{t\,\mathrm{b}_n})^{+2k+1} \\ &= {}^{(t\,\mathrm{b}_n)^{+k+1}}\!/\!_{(s\,\mathrm{b}_n)^{+k}} \\ &= {}^{t}\!/\!_{s}. \end{aligned}$$

□

Corollary 111. *The following equations (1) and (2) hold.*

(1)

$$\bar{\Delta}_n^{\#} = \dot{\bigcup_{{}^{t}\!/\!_{s} \in B_n}} [{}^{t}\!/\!_{s}]_{\sim_{\mathrm{s}}} = \dot{\bigcup_{{}^{t}\!/\!_{s} \in B_n}} \bar{\Delta}_1^{\#,\circ} f^{{}^{t}\!/\!_{s},\#}$$

(2)

$$\bar{\Delta}_n^{\#,\circ} = \dot{\bigcup_{{}^{t}\!/\!_{s} \in B_n^{\circ}}} [{}^{t}\!/\!_{s}]_{\sim_{\mathrm{s}}} = \dot{\bigcup_{{}^{t}\!/\!_{s} \in B_n^{\circ}}} \bar{\Delta}_1^{\#,\circ} f^{{}^{t}\!/\!_{s},\#}$$

Proof. Ad (1):

Suppose given ${}^{t'}\!/\!_{s'} \in \bar{\Delta}_n^{\#}$. Let ${}^{t}\!/\!_{s} := {}^{t'}\!/\!_{s'}\,\mathrm{b}_n^{\mathrm{s}} \in B_n$. Then ${}^{t'}\!/\!_{s'} \in [{}^{t}\!/\!_{s}]_{\sim_{\mathrm{s}}} = \bar{\Delta}_1^{\#,\circ} f^{{}^{t}\!/\!_{s},\#}$; cf. Lemma 110, Lemma 108.

The union is disjoint, as for ${}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'} \in B_n$, the relation ${}^{t}\!/\!_{s} \sim_{\mathrm{s}} {}^{t'}\!/\!_{s'}$ implies ${}^{t}\!/\!_{s}\,\mathrm{b}_n^{\mathrm{s}} = {}^{t'}\!/\!_{s'}\,\mathrm{b}_n^{\mathrm{s}}$ and therefore ${}^{t}\!/\!_{s} = {}^{t}\!/\!_{s}\,\mathrm{b}_n^{\mathrm{s}} = {}^{t'}\!/\!_{s'}\,\mathrm{b}_n^{\mathrm{s}} = {}^{t'}\!/\!_{s'}$.

Ad (2):

For ${}^{t}\!/\!_{s} \in B_n^{\circ}$, we have $\bar{\Delta}_1^{\#,\circ} f^{{}^{t}\!/\!_{s},\#} \subseteq \bar{\Delta}_n^{\#,\circ}$.

For every ${}^{t'}\!/\!_{s'} \in \bar{\Delta}_n^{\#,\circ}$ we have $s' < t' < (s')^{+1}$. Therefore we get $s'\,\mathrm{b}_n \neq t'\,\mathrm{b}_n$ and thus ${}^{t'}\!/\!_{s'}\,\mathrm{b}_n^{\mathrm{s}} \in B_n^{\circ}$. □

Lemma 112. *Suppose given ${}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'} \in \bar{\Delta}_n^{\#}$ with ${}^{t}\!/\!_{s} \sim_{\mathrm{s}} {}^{t'}\!/\!_{s'}$. Then*

$$\bar{\Delta}_1^{\#} f^{{}^{t}\!/\!_{s},\#} = \bar{\Delta}_1^{\#} f^{{}^{t'}\!/\!_{s'},\#}.$$

Proof. We show that $\bar{\Delta}_1^{\#} f^{{}^{t}\!/\!_{s},\#} = \bar{\Delta}_1^{\#} f^{{}^{t}\!/\!_{s}\,\mathrm{b}_n^{\mathrm{s}},\#}$ holds for every ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#}$. Then $\bar{\Delta}_1^{\#} f^{{}^{t}\!/\!_{s},\#} = \bar{\Delta}_1^{\#} f^{{}^{t'}\!/\!_{s'},\#}$ by Lemma 110.

By definition we have

$$({}^{t}\!/\!{}_{s})\,\mathrm{b}_n^{\mathrm{s}} = \begin{cases} {}^{t\,\mathrm{b}_n}\!/\!{}_{s\,\mathrm{b}_n} & \text{if } s\,\mathrm{b}_n \leqslant t\,\mathrm{b}_n \\ {}^{s\,\mathrm{b}_n}\!/\!{}_{t\,\mathrm{b}_n} & \text{if } t\,\mathrm{b}_n < s\,\mathrm{b}_n\,. \end{cases}$$

Note that

$$\begin{aligned} ([{}^{0}\!/\!{}_{0}]_{\sim_{\mathrm{s}}})\, f^{({}^{t}\!/\!{}_{s})\,\mathrm{b}_n^{\mathrm{s}},\#} &= \begin{cases} [{}^{s\,\mathrm{b}_n}\!/\!{}_{s\,\mathrm{b}_n}]_{\sim_{\mathrm{s}}} & \text{if } s\,\mathrm{b}_n \leqslant t\,\mathrm{b}_n \\ [{}^{t\,\mathrm{b}_n}\!/\!{}_{t\,\mathrm{b}_n}]_{\sim_{\mathrm{s}}} & \text{if } t\,\mathrm{b}_n < s\,\mathrm{b}_n \end{cases} \\ &= \begin{cases} [{}^{s}\!/\!{}_{s}]_{\sim_{\mathrm{s}}} & \text{if } s\,\mathrm{b}_n \leqslant t\,\mathrm{b}_n \\ [{}^{t}\!/\!{}_{t}]_{\sim_{\mathrm{s}}} & \text{if } t\,\mathrm{b}_n < s\,\mathrm{b}_n \end{cases} \end{aligned}$$

and

$$\begin{aligned} ([{}^{1}\!/\!{}_{1}]_{\sim_{\mathrm{s}}})\, f^{({}^{t}\!/\!{}_{s})\,\mathrm{b}_n^{\mathrm{s}},\#} &= \begin{cases} [{}^{t\,\mathrm{b}_n}\!/\!{}_{t\,\mathrm{b}_n}]_{\sim_{\mathrm{s}}} & \text{if } s\,\mathrm{b}_n \leqslant t\,\mathrm{b}_n \\ [{}^{s\,\mathrm{b}_n}\!/\!{}_{s\,\mathrm{b}_n}]_{\sim_{\mathrm{s}}} & \text{if } t\,\mathrm{b}_n < s\,\mathrm{b}_n \end{cases} \\ &= \begin{cases} [{}^{t}\!/\!{}_{t}]_{\sim_{\mathrm{s}}} & \text{if } s\,\mathrm{b}_n \leqslant t\,\mathrm{b}_n \\ [{}^{s}\!/\!{}_{s}]_{\sim_{\mathrm{s}}} & \text{if } t\,\mathrm{b}_n < s\,\mathrm{b}_n\,. \end{cases} \end{aligned}$$

From Lemma 111, we have $\bar{\Delta}_1^{\#} = [{}^{0}\!/\!{}_{0}]_{\sim_{\mathrm{s}}} \cup [{}^{1}\!/\!{}_{0}]_{\sim_{\mathrm{s}}} \cup [{}^{1}\!/\!{}_{1}]_{\sim_{\mathrm{s}}}$. Therefore

$$\begin{aligned} \bar{\Delta}_1^{\#} f^{{}^{t}\!/\!{}_{s},\#} &= ([{}^{0}\!/\!{}_{0}]_{\sim_{\mathrm{s}}} \cup [{}^{1}\!/\!{}_{0}]_{\sim_{\mathrm{s}}} \cup [{}^{1}\!/\!{}_{1}]_{\sim_{\mathrm{s}}}) f^{{}^{t}\!/\!{}_{s},\#} \\ &\overset{\text{L.80}}{=} [{}^{s}\!/\!{}_{s}]_{\sim_{\mathrm{s}}} \cup [{}^{t}\!/\!{}_{s}]_{\sim_{\mathrm{s}}} \cup [{}^{t}\!/\!{}_{t}]_{\sim_{\mathrm{s}}} \\ &\overset{\text{L.110}}{=} [{}^{s}\!/\!{}_{s}\,\mathrm{b}_n^{\mathrm{s}}]_{\sim_{\mathrm{s}}} \cup [{}^{t}\!/\!{}_{s}\,\mathrm{b}_n^{\mathrm{s}}]_{\sim_{\mathrm{s}}} \cup [{}^{t}\!/\!{}_{t}\,\mathrm{b}_n^{\mathrm{s}}]_{\sim_{\mathrm{s}}} \\ &= ([{}^{0}\!/\!{}_{0}]_{\sim_{\mathrm{s}}} \cup [{}^{1}\!/\!{}_{0}]_{\sim_{\mathrm{s}}} \cup [{}^{1}\!/\!{}_{1}]_{\sim_{\mathrm{s}}}) f^{({}^{t}\!/\!{}_{s})\,\mathrm{b}_n^{\mathrm{s}},\#} \\ &= \bar{\Delta}_1^{\#} f^{({}^{t}\!/\!{}_{s})\,\mathrm{b}_n^{\mathrm{s}},\#} \end{aligned}$$

□

Lemma 113. *Suppose given ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#,\circ}$. Suppose given $k \in \mathbb{Z}$. Then*

(1)

$$\hat{f}^{({}^{t}\!/\!{}_{s})^{+k},\#} f^{({}^{t}\!/\!{}_{s})^{+k},\#} = \hat{f}^{{}^{t}\!/\!{}_{s},\#} f^{{}^{t}\!/\!{}_{s},\#},$$

(2)

$$\check{f}^{({}^{t}\!/\!{}_{s})^{+k},\#} f^{({}^{t}\!/\!{}_{s})^{+k},\#} = \check{f}^{{}^{t}\!/\!{}_{s},\#} f^{{}^{t}\!/\!{}_{s},\#}.$$

Proof. Ad (1): Due to $f^{{}^{t}\!/\!{}_{s},\#} \dashv \hat{f}^{{}^{t}\!/\!{}_{s},\#}$ we know that for every ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#}$, the element ${}^{j}\!/\!{}_{i}\hat{f}^{{}^{t}\!/\!{}_{s},\#} f^{{}^{t}\!/\!{}_{s},\#}$ is terminal in $\{{}^{j'}\!/\!{}_{i'} \in \bar{\Delta}_1^{\#} f^{{}^{t}\!/\!{}_{s},\#} : {}^{j'}\!/\!{}_{i'} \leqslant {}^{j}\!/\!{}_{i}\}$; cf. Lemma 74 (3). Due to $f^{({}^{t}\!/\!{}_{s})^{+k},\#} \dashv \hat{f}^{({}^{t}\!/\!{}_{s})^{+k},\#}$, we know that ${}^{j}\!/\!{}_{i}\hat{f}^{({}^{t}\!/\!{}_{s})^{+k},\#} f^{({}^{t}\!/\!{}_{s})^{+k},\#}$ is terminal in

$$\{{}^{j'}\!/\!{}_{i'} \in \bar{\Delta}_1^{\#} f^{({}^{t}\!/\!{}_{s})^{+k},\#} : {}^{j'}\!/\!{}_{i'} \leqslant {}^{j}\!/\!{}_{i}\}.$$

As Lemma 112 gives $\bar{\Delta}_1^{\#} f^{t/s,\#} = \bar{\Delta}_1^{\#} f^{(t/s)^{+k},\#}$, both are terminal in the same set and therefore equal.

Ad (2): Due to $\check{f}^{t/s,\#} \dashv f^{t/s,\#}$, we know that for every $j/i \in \bar{\Delta}_n^{\#}$, the element j/i is initial in $\{j'/i' \in \bar{\Delta}_1^{\#} f^{t/s,\#} : j'/i' \geqslant j/i\}$; cf. Lemma 74 (1). Due to $\check{f}^{(t/s)^{+k},\#} \dashv f^{(t/s)^{+k},\#}$, we know that $j/i \check{f}^{(t/s)^{+k},\#} f^{(t/s)^{+k},\#}$ is initial in $\{j'/i' \in \bar{\Delta}_1^{\#} f^{(t/s)^{+k},\#} : j'/i' \geqslant j/i\}$. As Lemma 112 gives $\bar{\Delta}_1^{\#} f^{t/s,\#} = \bar{\Delta}_1^{\#} f^{(t/s)^{+k},\#}$, both are initial in the same set and therefore equal. □

Lemma 114. *Suppose given $t/s, t'/s' \in \bar{\Delta}_n^{\#}$ with $t/s \sim_{\mathrm{s}} t'/s'$. Then*

$$\lfloor j/i \rfloor_{t/s} = \lfloor j/i \rfloor_{t'/s'}$$

and

$$\lceil j/i \rceil_{t/s} = \lceil j/i \rceil_{t'/s'}$$

for every $j/i \in \bar{\Delta}_n^{\#}$.

Proof. For $t'/s' = (t/s)^{+k}$ for $k \in \mathbb{Z}$ we get

$$\lfloor j/i \rfloor_{t/s} \overset{\text{D.101}}{=} j/i \hat{f}^{t/s,\#} f^{t/s,\#} \overset{\text{L.113 (1)}}{=} j/i \hat{f}^{t'/s',\#} f^{t'/s',\#} \overset{\text{D.101}}{=} \lfloor j/i \rfloor_{t'/s'}$$

and

$$\lceil j/i \rceil_{t/s} \overset{\text{D.101}}{=} j/i \check{f}^{t/s,\#} f^{t/s,\#} \overset{\text{L.113 (2)}}{=} j/i \check{f}^{t'/s',\#} f^{t'/s',\#} \overset{\text{D.101}}{=} \lceil j/i \rceil_{t'/s'}.$$

□

Lemma 115. *Suppose given $j/i, t/s \in \bar{\Delta}_n^{\#}$. Then $\lfloor j/i \rfloor_{t/s} \in \bar{\Delta}_n^{\#,\circ}$ if and only if $j/i \in \mathrm{u}_n^{\lfloor j/i \rfloor_{t/s}}$.*

Proof. $\Leftarrow$: If $j/i \in \mathrm{u}_n^{\lfloor j/i \rfloor_{t/s}}$, then $\mathrm{u}_n^{\lfloor j/i \rfloor_{t/s}} \neq \emptyset$ and therefore $\lfloor j/i \rfloor_{t/s} \in \bar{\Delta}_n^{\#,\circ}$; cf. Remark 94.

$\Rightarrow$: Suppose that $\lfloor j/i \rfloor_{t/s} \in \bar{\Delta}_n^{\#,\circ}$.

According to Lemma 105, we have

$$\mathrm{u}_n^{\lfloor j/i \rfloor_{t/s}} = (\hat{f}^{\lfloor j/i \rfloor_{t/s},\#})^{-1}(\lfloor j/i \rfloor_{t/s} \hat{f}^{\lfloor j/i \rfloor_{t/s},\#})$$

By Definition 101 and Lemma 108 we have $\lfloor j/i \rfloor_{t/s} \sim_{\mathrm{s}} t/s$.

Then

$$\begin{aligned} j/i \hat{f}^{\lfloor j/i \rfloor_{t/s},\#} &\overset{\text{R.106}}{=} j/i \hat{f}^{\lfloor j/i \rfloor_{t/s},\#} f^{\lfloor j/i \rfloor_{t/s},\#} \hat{f}^{\lfloor j/i \rfloor_{t/s},\#} \\ &\overset{\text{L.113}}{=} j/i \hat{f}^{t/s,\#} f^{t/s,\#} \hat{f}^{\lfloor j/i \rfloor_{t/s},\#} \\ &= \lfloor j/i \rfloor_{t/s} \hat{f}^{\lfloor j/i \rfloor_{t/s},\#} \end{aligned}$$

and therefore $j/i \in \mathrm{u}_n^{\lfloor j/i \rfloor_{t/s}}$. □

Lemma 116. *Suppose given ${}^{j}\!/\!_{i}, {}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#}$. Then*

$$|\bar{\Delta}_1^{\#} f^{t/s,\#} \cap \mathrm{d}_n^{j/i}| \leqslant 1$$

Proof. If ${}^{j}\!/\!_{i} \notin \bar{\Delta}_n^{\#,\circ}$, we have $\mathrm{d}_n^{j/i} = \emptyset$ and therefore $|\bar{\Delta}_1^{\#} \cap \mathrm{d}_n^{j/i}| = 0$.

Suppose that ${}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#,\circ}$. Then $\mathrm{d}_n^{j/i} \subseteq \bar{\Delta}_n^{\#,\circ}$; cf. Remark 94. For ${}^{b}\!/\!_{a} \in \bar{\Delta}_1^{\#} \setminus \bar{\Delta}_1^{\#,\circ}$, we have ${}^{b}\!/\!_{a} f^{t/s,\#} \notin \bar{\Delta}_n^{\#,\circ}$. Therefore

$$\bar{\Delta}_1^{\#} f^{t/s,\#} \cap \mathrm{d}_n^{j/i} = \bar{\Delta}_1^{\#,\circ} f^{t/s,\#} \cap \mathrm{d}_n^{j/i}.$$

If ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$ we have $\bar{\Delta}_1^{\#,\circ} f^{t/s,\#} \subseteq \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$. Therefore $\bar{\Delta}_1^{\#,\circ} f^{t/s,\#} \cap \mathrm{d}_n^{j/i} = \emptyset$.

So suppose that ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#,\circ}$. Then $\bar{\Delta}_1^{\#,\circ} f^{t/s,\#} = [{}^{t}\!/\!_{s}]_{\sim_\mathrm{s}}$ and $\mathrm{d}_n^{j/i} = (\check{f}^{j/i,\#})^{-1}({}^{j}\!/\!_{i}\check{f}^{j/i,\#})$; cf. Lemma 105. Then by Lemmas 79 and 108, $|\bar{\Delta}_1^{\#,\circ} f^{t/s,\#} \cap \mathrm{d}_n^{j/i}| \leqslant 1$. □

Lemma 117. *Suppose given ${}^{t}\!/\!_{s} \in B_n^{\circ}$ and ${}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#}$.*

The following assertions (1), (2), (3), (4), (5) are equivalent.

(1) There exists an element ${}^{t'}\!/\!_{s'} \in \mathrm{d}_n^{j/i}$ with ${}^{t'}\!/\!_{s'} \sim_\mathrm{s} {}^{t}\!/\!_{s}$.

(2) We have $\mathrm{d}_n^{j/i} \cap [{}^{t}\!/\!_{s}]_{\sim_\mathrm{s}} = \{\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s}\}$.

(3) We have $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} \in \bar{\Delta}_n^{\#,\circ}$.

(4) We have $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i}$.

(5) We have $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} \sim_\mathrm{s} {}^{t}\!/\!_{s}$.

Proof. If ${}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$, then $\mathrm{d}_n^{j/i} = \emptyset$ and $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$, so we are done; cf. Definition 93. So suppose that ${}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#,\circ}$.

Ad (1) ⇒ (2): Suppose given ${}^{t'}\!/\!_{s'} \in \mathrm{d}_n^{j/i}$ with ${}^{t'}\!/\!_{s'} \sim_\mathrm{s} {}^{t}\!/\!_{s}$. Since ${}^{t'}\!/\!_{s'} \in \mathrm{d}_n^{j/i}$, we have ${}^{j}\!/\!_{i} \in \mathrm{u}_n^{t'/s'}$ with $\mathrm{u}_n^{t'/s'} = \{{}^{j'}\!/\!_{i'} \in \bar{\Delta}_n^{\#} : {}^{t'}\!/\!_{s'} \leqslant {}^{j'}\!/\!_{i'} \leqslant \lceil {}^{t'}\!/\!_{s'} \rceil\}$; cf. Lemma 96 (2). As ${}^{t'}\!/\!_{s'} \in \bar{\Delta}_n^{\#,\circ}$, we get $\mathrm{u}_n^{t'/s'} \overset{\text{L.105}}{=} (\hat{f}^{t/s,\#})^{-1}({}^{t'}\!/\!_{s'}\hat{f}^{t/s,\#})$ and thus ${}^{j}\!/\!_{i}\hat{f}^{t/s,\#} = {}^{t'}\!/\!_{s'}\hat{f}^{t/s,\#}$. By Lemma 74 (2), the element $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} = {}^{j}\!/\!_{i}\hat{f}^{t/s,\#} f^{t/s,\#}$ is initial in

$$(\hat{f}^{t/s,\#})^{-1}({}^{j}\!/\!_{i}\hat{f}^{t/s,\#}) = (\hat{f}^{t/s,\#})^{-1}({}^{t'}\!/\!_{s'}\hat{f}^{t/s,\#}) = \{{}^{j'}\!/\!_{i'} \in \bar{\Delta}_n^{\#} : {}^{t'}\!/\!_{s'} \leqslant {}^{j'}\!/\!_{i'} \leqslant \lceil {}^{t'}\!/\!_{s'} \rceil\}.$$

Therefore ${}^{t'}\!/\!_{s'} = \lfloor {}^{j}\!/\!_{i} \rfloor_{t/s}$.

By Lemma 116 it follows that $\mathrm{d}_n^{j/i} \cap [{}^{t}\!/\!_{s}]_{\sim_\mathrm{s}} = \{\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s}\}$.

Ad (2) ⇒ (1): We have $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i}$ with $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} \in [{}^{t}\!/\!_{s}]_{\sim_\mathrm{s}}$, that is $\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s} \sim_\mathrm{s} {}^{t}\!/\!_{s}$.

Ad (2) ⇒ (3): We have $\{\lfloor {}^{j}\!/\!_{i} \rfloor_{t/s}\} = \mathrm{d}_n^{j/i} \cap [{}^{t}\!/\!_{s}]_{\sim_\mathrm{s}} \subseteq \bar{\Delta}_n^{\#,\circ}$.

Ad (3) $\Rightarrow$ (5): We have

$$\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \overset{\text{D.101}}{=} {}^{j}\!/\!{}_{i}\hat{f}^{t/s,\#} f^{t/s,\#} \in \bar{\Delta}_n^{\#,\circ} \cap \bar{\Delta}_1^{\#} f^{t/s,\#} \subseteq \bar{\Delta}_1^{\#,\circ} f^{t/s,\#} \overset{\text{L.108}}{=} [{}^{t}\!/\!{}_{s}]_{\sim_\mathrm{s}}$$

and therefore $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in [{}^{t}\!/\!{}_{s}]_{\sim_\mathrm{s}}$.

Ad (5) $\Rightarrow$ (4): We get

$$\begin{aligned} \lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i} &\overset{\text{L.96}}{\Leftrightarrow} {}^{j}\!/\!{}_{i} \in \mathrm{u}_n^{\lfloor j/i \rfloor_{t/s}} \\ &\overset{\text{L.105}}{\Leftrightarrow} {}^{j}\!/\!{}_{i}\hat{f}^{t/s,\#} = \lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s}\hat{f}^{t/s,\#}. \end{aligned}$$

We have $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s}\hat{f}^{t/s,\#} \overset{\text{D.101}}{=} {}^{j}\!/\!{}_{i}\hat{f}^{t/s,\#} f^{t/s,\#}\hat{f}^{t/s,\#} \overset{\text{L.106}}{=} {}^{j}\!/\!{}_{i}\hat{f}^{t/s,\#}$ and therefore $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i}$.

Ad (4) $\Rightarrow$ (3): We have $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i} \subseteq \bar{\Delta}_n^{\#,\circ}$.

Ad (5) $\Rightarrow$ (1): If $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \sim_\mathrm{s} {}^{t}\!/\!{}_{s}$, then in particular $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i}$ as we already proved *(5) $\Rightarrow$ (4)*. Therefore we have the element $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i}$ with $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \sim_\mathrm{s} {}^{t}\!/\!{}_{s}$. □

Lemma 118. *Suppose given ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#,\circ}$. Define $I := \{{}^{t}\!/\!{}_{s} \in B_n^\circ \colon {}^{j}\!/\!{}_{i}\hat{f}^{t/s,\#} \in \bar{\Delta}_1^{\#,\circ}\}$. Then*

$$\begin{aligned} d\colon I &\to \mathrm{d}_n^{j/i} \\ {}^{t}\!/\!{}_{s} &\mapsto \lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \end{aligned}$$

is a bijection with inverse $\mathrm{b}_n^\mathrm{s}|^{I}_{\mathrm{d}_n^{j/i}}$.

Proof. The map d and its inverse are well-defined:

Suppose given ${}^{t}\!/\!{}_{s} \in B_n^\circ$ with ${}^{j}\!/\!{}_{i}\hat{f}^{t/s,\#} \in \bar{\Delta}_n^{\#,\circ}$. Then

$$\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} = {}^{j}\!/\!{}_{i}\hat{f}^{t/s,\#} f^{t/s,\#} \in \bar{\Delta}_n^{\#,\circ} f^{t/s,\#} = [{}^{t}\!/\!{}_{s}]_{\sim_\mathrm{s}};$$

cf. Lemma 108. By Lemma 117 we get $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i}$.

Suppose given ${}^{t'}\!/\!{}_{s'} \in \mathrm{d}_n^{j/i}$. Then ${}^{t}\!/\!{}_{s} := {}^{t'}\!/\!{}_{s'}\,\mathrm{b}_n^\mathrm{s} \in [{}^{t'}\!/\!{}_{s'}]_{\sim_\mathrm{s}} = \bar{\Delta}_1^{\#,\circ} f^{t'/s',\#} \subseteq \bar{\Delta}_n^{\#,\circ}$. Therefore ${}^{t}\!/\!{}_{s} \in B_n^\circ$.

We also have ${}^{t}\!/\!{}_{s}\hat{f}^{t/s,\#} \in [{}^{t}\!/\!{}_{s}]_{\sim_\mathrm{s}}\hat{f}^{t/s,\#} = \bar{\Delta}_1^{\#,\circ} f^{t/s,\#}\hat{f}^{t/s,\#} \overset{\text{R.106}}{=} \bar{\Delta}_1^{\#,\circ}$. Therefore ${}^{t}\!/\!{}_{s} \in I$.

We show that ${}^{t}\!/\!{}_{s}d\,\mathrm{b}_n^\mathrm{s} = {}^{t}\!/\!{}_{s}$ for ${}^{t}\!/\!{}_{s} \in I$:

By Lemma 117 and $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \in \mathrm{d}_n^{j/i}$, we know that ${}^{t}\!/\!{}_{s} \sim_\mathrm{s} \lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s}$.

Therefore $\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s}\,\mathrm{b}_n^\mathrm{s} = {}^{t}\!/\!{}_{s}\,\mathrm{b}_n^\mathrm{s} = {}^{t}\!/\!{}_{s}$; cf. Lemma 110.

We show that ${}^{t'}\!/\!{}_{s'}\,\mathrm{b}_n^\mathrm{s}\,d = {}^{t'}\!/\!{}_{s'}$ for every ${}^{t'}\!/\!{}_{s'} \in \mathrm{d}_n^{j/i}$:

Suppose given ${}^{t'}\!/\!{}_{s'} \in \mathrm{d}_n^{j/i}$. Let ${}^{t}\!/\!{}_{s} := {}^{t'}\!/\!{}_{s'}\,\mathrm{b}_n^\mathrm{s}$, in particular ${}^{t'}\!/\!{}_{s'} \sim_\mathrm{s} {}^{t}\!/\!{}_{s}$. Then

$${}^{t'}\!/\!{}_{s'}\,\mathrm{b}_n^\mathrm{s}\,d = \lfloor {}^{j}\!/\!{}_{i} \rfloor_{t'/s'} \overset{\text{L.114}}{=} \lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s} \overset{\text{L.117}}{=} {}^{t'}\!/\!{}_{s'}.$$

□

5 Definition of n-complexes

5.1 Definition of $\mathrm{C}^{(n)}(\mathcal{A})$

Suppose given $n \in \mathbb{Z}_{\geqslant 0}$ and an additive category $\mathcal{A}$.

Definition 119. An *n-complex* over the additive category $\mathcal{A}$ is a functor $X \colon \bar{\Delta}_n^{\#} \to \mathcal{A}$ with $({}^{t}/_{t})X \cong 0_{\mathcal{A}}$ and $({}^{t^{+1}}/_{t})X \cong 0_{\mathcal{A}}$ for every $t \in \bar{\Delta}_n$.

Suppose given n-complexes X and Y. A *morphism of n-complexes* from X to Y, also called *n-complex morphism*, is a transformation from X to Y. The n-complexes together with these morphisms form the category $\mathrm{C}^{(n)}(\mathcal{A})$ of n-complexes over $\mathcal{A}$. For readability, given two n-complexes $X, Y \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ we sometimes abbreviate the set of morphisms from X to Y by

$$ {}_{\mathrm{C}^{(n)}}(X, Y) \; := \; {}_{\mathrm{C}^{(n)}(\mathcal{A})}(X, Y) $$

or sometimes $(X, Y) \; := \; {}_{\mathrm{C}^{(n)}(\mathcal{A})}(X, Y)$.

For a given n-complex X, we often write $X_{t/s} := ({}^{t}/_{s})X$ and $x_{t/s, t'/s'} := ({}^{t}/_{s}, {}^{t'}/_{s'})X$ for ${}^{t}/_{s}, {}^{t'}/_{s'} \in \bar{\Delta}_n^{\#}$ or even $x := ({}^{t}/_{s}, {}^{t'}/_{s'})X$, if source and target are apparent from context.

Suppose given an n-complex morphism $f \colon X \to Y$. We often write $f_{t/s} \; := \; ({}^{t}/_{s})f$.

Remark 120. As a diagram, an n-complex is commutative due to $\bar{\Delta}_n^{\#}$ being a poset.

Example 121. A 3-complex X can be depicted as follows.

$$
\begin{tikzcd}
 & & & & & 0 \arrow[r, dotted] & {} & \\
 & & & & 0 \arrow[r] & X_{0^{+1}/3} \arrow[u] \arrow[r, "x"] & X_{1^{+1}/3} \arrow[u, dotted] \arrow[r, dotted] & {} \\
 & & & 0 \arrow[r] & X_{3/2} \arrow[u] \arrow[r, "x"] & X_{0^{+1}/2} \arrow[u, "x"] \arrow[r, "x"] & X_{1^{+1}/2} \arrow[u, "x"] \arrow[r] & 0 \arrow[u, dotted] \\
 & & 0 \arrow[r] & X_{2/1} \arrow[u] \arrow[r, "x"] & X_{3/1} \arrow[u, "x"] \arrow[r, "x"] & X_{0^{+1}/1} \arrow[u, "x"] \arrow[r] & 0 \arrow[u] & \\
 & 0 \arrow[r] & X_{1/0} \arrow[u] \arrow[r, "x"] & X_{2/0} \arrow[u, "x"] \arrow[r, "x"] & X_{3/0} \arrow[u, "x"] \arrow[r] & 0 \arrow[u] & & \\
0 \arrow[r] & X_{0/3^{-1}} \arrow[u] \arrow[r, "x"] & X_{1/3^{-1}} \arrow[u, "x"] \arrow[r, "x"] & X_{2/3^{-1}} \arrow[u, "x"] \arrow[r] & 0 \arrow[u] & & & \\
{} \arrow[u, dotted] \arrow[r, dotted] & X_{0/2^{-1}} \arrow[u, "x"] \arrow[r, "x"] & X_{1/2^{-1}} \arrow[u, "x"] \arrow[r] & 0 \arrow[u] & & & & \\
 & {} \arrow[u, dotted] \arrow[r, dotted] & 0 \arrow[u] & & & & &
\end{tikzcd}
$$

Of course there is also the e.g. the morphism $X_{2/0} \xrightarrow{x} X_{3/1}$ in the diagram, which we have not depicted. Likewise we omit the labelling of arrows that obviously describe the respective zero morphism.

We sometimes denote positions containing a zero object by 0. Different positions with 0 can contain different zero objects.

Notation 122. Suppose given a diagram, e.g. an n-complex $X \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$. We sometimes refer to a quadrangle in the diagram by specifying the vertices. E.g. for the n-complex X we write $(X_{t/s}, X_{t+l/s}, X_{t/s+k}, X_{t+l/s+k})$ for the quadrangle

$$\begin{tikzcd} X_{t/s+k} \arrow[r, "x"] & X_{t+l/s+k} \\ X_{t/s} \arrow[u, "x"] \arrow[r, "x"] & X_{t+l/s} \arrow[u, "x"] \end{tikzcd} .$$

Lemma 123. *An n-complex is determined up to isomorphism by its values on $\bar{\Delta}_n^{\#,\circ}$.*

Proof. Every n-complex X is isomorphic to the n-complex defined by

$$^{t}/_{s}X' = \begin{cases} ^{t}/_{s}X & \text{for } ^{t}/_{s} \in \bar{\Delta}_n^{\#,\circ} \\ 0_{\mathcal{A}} & \text{else} \end{cases}$$

with $X' \,|_{\mathrm{Mor}(\bar{\Delta}_n^{\#,\circ})} := X \,|_{\mathrm{Mor}(\bar{\Delta}_n^{\#,\circ})}$ and zero else.

The isomorphism is $\varphi\colon X \to X'$ with

$$^{t}/_{s}\varphi = \begin{cases} 1_{^{t}/_{s}X} & \text{for } ^{t}/_{s} \in \bar{\Delta}_n^{\#,\circ} \\ 0 & \text{else.} \end{cases}$$

It is a morphism of n-complexes, because the quadrangles

$$\begin{tikzcd} ^{t}/_{s}X \arrow[r, "(^{t}/_{s},\, ^{t'}/_{s'})X"] \arrow[d, "1_{^{t}/_{s}X}"] & ^{t'}/_{s'}X \arrow[d, "1_{^{t'}/_{s'}X}"] \\ ^{t}/_{s}X' \arrow[r, "(^{t}/_{s},\, ^{t'}/_{s'})X'"] & ^{t'}/_{s'}X' \end{tikzcd}$$

for $(^{t}/_{s}, ^{t'}/_{s'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#,\circ})$ as well as

$$\begin{tikzcd} ^{t}/_{s}X \arrow[r, "0"] \arrow[d, "1_{^{t}/_{s}X}"] & ^{t'}/_{s'}X \arrow[d, "0", "\wr"'] \\ ^{t}/_{s}X' \arrow[r, "0"] & 0_{\mathcal{A}} \end{tikzcd}$$

for $({}^{t}\!/_{s}, {}^{t'}\!/_{s'}) \in \operatorname{Mor}(\bar{\Delta}_n^{\#}) \setminus \operatorname{Mor}(\bar{\Delta}_n^{\#,\circ})$ with ${}^{t'}\!/_{s'} \notin \bar{\Delta}_n^{\#,\circ}$ and

$$\begin{tikzcd} {}^{t}\!/_{s}X \arrow[r, "0"] \arrow[d, "\wr", "0"'] & {}^{t'}\!/_{s'}X \arrow[d, "1_{{}^{t'}\!/_{s'}X}"] \\ 0_{\mathcal{A}} \arrow[r, "0"] & {}^{t'}\!/_{s'}X' \end{tikzcd}$$

for $({}^{t}\!/_{s}, {}^{t'}\!/_{s'}) \in \operatorname{Mor}(\bar{\Delta}_n^{\#}) \setminus \operatorname{Mor}(\bar{\Delta}_n^{\#,\circ})$ with ${}^{t}\!/_{s} \notin \bar{\Delta}_n^{\#,\circ}$ commute. □

Definition 124.

(1) We consider the subset

$$\begin{aligned} S \ :=\ & \{({}^{s}\!/_{t}, {}^{s+1}\!/_{t}) \mid {}^{s}\!/_{t} \in \bar{\Delta}_n^{\#},\ t \leqslant s < t^{+1}\} \cup \{({}^{s}\!/_{t}, {}^{s}\!/_{t+1}) \mid {}^{s}\!/_{t} \in \bar{\Delta}_n^{\#},\ t < s \leqslant t^{+1}\} \\ & \subseteq \operatorname{Mor}(\bar{\Delta}_n^{\#}). \end{aligned}$$

Pictorially, the set S consists of the morphisms in $\bar{\Delta}_n^{\#}$ going one step to the right or one step upwards.

We choose maps

$$\check{X}_{\mathrm{Ob}} \colon \bar{\Delta}_n^{\#} \to \operatorname{Ob}(\mathcal{A})$$
$$\check{X}_{\mathrm{Mor}} \colon S \to \operatorname{Mor}(\mathcal{A})$$

with ${}^{t}\!/_{t}\check{X}_{\mathrm{Ob}} \cong 0_{\mathcal{A}}$ and ${}^{t}\!/_{t^{+1}}\check{X}_{\mathrm{Ob}} \cong 0_{\mathcal{A}}$ for $t \in \bar{\Delta}_n$ and $({}^{s}\!/_{t}, {}^{s'}\!/_{t'})\check{X}_{\mathrm{Mor}} \in {}_{\mathcal{A}}({}^{s}\!/_{t}\check{X}_{\mathrm{Ob}}, {}^{s'}\!/_{t'}\check{X}_{\mathrm{Ob}})$ for $({}^{s}\!/_{t}, {}^{s'}\!/_{t'}) \in S$ and for which for every

$$\begin{tikzcd} {}^{s}\!/_{t+1} \arrow[r] & {}^{s+1}\!/_{t+1} \\ {}^{s}\!/_{t} \arrow[r] \arrow[u] & {}^{s+1}\!/_{t} \arrow[u] \end{tikzcd}$$

in $\bar{\Delta}_n^{\#}$, i.e. ${}^{s}\!/_{t} \in \bar{\Delta}_n^{\#,\circ}$, the quadrangle

$$\begin{tikzcd} {}^{s}\!/_{t+1}\check{X}_{\mathrm{Ob}} \arrow[r, "({}^{s}\!/_{t+1}, {}^{s+1}\!/_{t+1})\check{X}_{\mathrm{Mor}}"] & {}^{s+1}\!/_{t+1}\check{X}_{\mathrm{Ob}} \\ {}^{s}\!/_{t}\check{X}_{\mathrm{Ob}} \arrow[r, "({}^{s}\!/_{t}, {}^{s+1}\!/_{t})\check{X}_{\mathrm{Mor}}"] \arrow[u, "({}^{s}\!/_{t}, {}^{s}\!/_{t+1})\check{X}_{\mathrm{Mor}}"] & {}^{s+1}\!/_{t}\check{X}_{\mathrm{Ob}} \arrow[u, "({}^{s+1}\!/_{t}, {}^{s+1}\!/_{t+1})\check{X}_{\mathrm{Mor}}"'] \end{tikzcd}$$

commutes.

We call $\check{X} := (\check{X}_{\mathrm{Ob}}, \check{X}_{\mathrm{Mor}})$ a *one-step n-complex.* The morphisms on S are called *one-step morphisms.*

(2) Suppose given a one-step n-complex $\check{X} = (\check{X}_{\mathrm{Ob}}, \check{X}_{\mathrm{Mor}})$. We can expand this definition to $\mathrm{Mor}(\bar{\Delta}_n^{\#})$ by defining

$$X_{\mathrm{Ob}} \;:=\; \check{X}_{\mathrm{Ob}}$$

and

$$\begin{aligned}
({}^{s}\!/\!{}_{t}, {}^{s}\!/\!{}_{t})X_{\mathrm{Mor}} \;&:=\; 1_{{}^{s}\!/\!{}_{t}X_{\mathrm{Mor}}} \quad \text{for } {}^{s}\!/\!{}_{t} \in \bar{\Delta}_n^{\#} \\
({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t})X_{\mathrm{Mor}} \;&:=\; ({}^{s}\!/\!{}_{t}, {}^{s+1}\!/\!{}_{t})\check{X}_{\mathrm{Mor}} \cdot ({}^{s+1}\!/\!{}_{t}, {}^{s+2}\!/\!{}_{t})\check{X}_{\mathrm{Mor}} \cdot \ldots \cdot ({}^{s'-1}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t})\check{X}_{\mathrm{Mor}} \\
&\qquad \text{for } ({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#}) \\
({}^{s}\!/\!{}_{t}, {}^{s}\!/\!{}_{t'})X_{\mathrm{Mor}} \;&:=\; ({}^{s}\!/\!{}_{t}, {}^{s}\!/\!{}_{t+1})\check{X}_{\mathrm{Mor}} \cdot ({}^{s}\!/\!{}_{t+1}, {}^{s}\!/\!{}_{t+2})\check{X}_{\mathrm{Mor}} \cdot \ldots \cdot ({}^{s}\!/\!{}_{t'-1}, {}^{s}\!/\!{}_{t'})\check{X}_{\mathrm{Mor}} \\
&\qquad \text{for } ({}^{s}\!/\!{}_{t}, {}^{s}\!/\!{}_{t'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#}).
\end{aligned}$$

Moreover, for $({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$, we let

$$({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'})X_{\mathrm{Mor}} \;:=\; \begin{cases} ({}^{s}\!/\!{}_{t}, {}^{s}\!/\!{}_{t'})X_{\mathrm{Mor}} \cdot ({}^{s}\!/\!{}_{t'}, {}^{s'}\!/\!{}_{t'})X_{\mathrm{Mor}} & \text{if } t' \leqslant s' \leqslant (t')^{+1}, \text{ i.e. } {}^{s}\!/\!{}_{t'} \in \bar{\Delta}_n^{\#} \\ 0_{{}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'}} & \text{else.} \end{cases}$$

Often we just write $({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'})X \;:=\; ({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'})X_{\mathrm{Mor}}$ and ${}^{s}\!/\!{}_{t}X \;:=\; {}^{s}\!/\!{}_{t}X_{\mathrm{Ob}}$ for $({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$.

We note that

$$({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'})X \cdot ({}^{s'}\!/\!{}_{t'}, {}^{s''}\!/\!{}_{t'})X = ({}^{s}\!/\!{}_{t}, {}^{s''}\!/\!{}_{t'})X$$

for $({}^{s}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$ and

$$({}^{s'}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'})X \cdot ({}^{s'}\!/\!{}_{t'}, {}^{s''}\!/\!{}_{t''})X = ({}^{s'}\!/\!{}_{t}, {}^{s''}\!/\!{}_{t''})X$$

for $({}^{s'}\!/\!{}_{t}, {}^{s'}\!/\!{}_{t'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$.

Lemma 125. *Let $\check{X}$ be a one-step n-complex. Define $(X_{\mathrm{Ob}}, X_{\mathrm{Mor}})$ as in Definition 124 (2). Suppose given the quadrangle*

$$\begin{tikzcd}
{}^{s}\!/\!{}_{t+\ell} \arrow[r] & {}^{s+1}\!/\!{}_{t+\ell} \\
{}^{s}\!/\!{}_{t} \arrow[r] \arrow[u] & {}^{s+1}\!/\!{}_{t} \arrow[u]
\end{tikzcd}$$

in $\bar{\Delta}_n^{\#}$, where $\ell \geqslant 0$. Then $({}^{s}\!/\!{}_{t}, {}^{s+1}\!/\!{}_{t})X \cdot ({}^{s+1}\!/\!{}_{t}, {}^{s+1}\!/\!{}_{t+\ell})X = ({}^{s}\!/\!{}_{t}, {}^{s+1}\!/\!{}_{t+\ell})X$.

Proof. For $\ell = 0$ this holds by Definition 124 (2).

Assume that $l \geqslant 1$ and that this already holds for $\ell - 1$, that is

$$({}^{s}\!/\!{}_{t}, {}^{s+1}\!/\!{}_{t})X \cdot ({}^{s+1}\!/\!{}_{t}, {}^{s+1}\!/\!{}_{t+\ell-1})X = ({}^{s}\!/\!{}_{t}, {}^{s+1}\!/\!{}_{t+\ell-1})X.$$

Then we have

$$\begin{aligned}
({}^{s}/_{t}, {}^{s+1}/_{t})X \cdot ({}^{s+1}/_{t}, {}^{s+1}/_{t+\ell})X &= ({}^{s}/_{t}, {}^{s+1}/_{t})X \cdot ({}^{s+1}/_{t}, {}^{s+1}/_{t+\ell-1})X \cdot ({}^{s+1}/_{t+\ell-1}, {}^{s+1}/_{t+\ell})X \\
&= ({}^{s}/_{t}, {}^{s+1}/_{t+\ell-1})X \cdot ({}^{s+1}/_{t+\ell-1}, {}^{s+1}/_{t+\ell})X \\
&\overset{\text{L.91}}{=} ({}^{s}/_{t}, {}^{s}/_{t+\ell-1})X \cdot ({}^{s}/_{t+\ell-1}, {}^{s+1}/_{t+\ell-1})X \cdot ({}^{s+1}/_{t+\ell-1}, {}^{s+1}/_{t+\ell})X \\
&\overset{\text{D.124(1)}}{=} ({}^{s}/_{t}, {}^{s}/_{t+\ell-1})X \cdot ({}^{s}/_{t+\ell-1}, {}^{s}/_{t+\ell})X \cdot ({}^{s}/_{t+\ell}, {}^{s+1}/_{t+\ell})X \\
&= ({}^{s}/_{t}, {}^{s}/_{t+\ell-1})X \cdot ({}^{s}/_{t+\ell-1}, {}^{s+1}/_{t+\ell})X \\
&= ({}^{s}/_{t}, {}^{s+1}/_{t+\ell})X
\end{aligned}$$

$\square$

Lemma 126. *Let $\check{X}$ be a one-step n-complex and $(X_{\mathrm{Ob}}, X_{\mathrm{Mor}})$ be defined as in Definition 124 (2). For a quadrangle*

$$\begin{tikzcd}
{}^{s}/_{t+l} \arrow[r] & {}^{s+m}/_{t+l} \\
{}^{s}/_{t} \arrow[r] \arrow[u] & {}^{s+m}/_{t} \arrow[u]
\end{tikzcd}$$

in $\bar{\Delta}_n^{\#}$, where $l, m \geqslant 0$, it holds that

$$({}^{s}/_{t}, {}^{s+m}/_{t})X \cdot ({}^{s+m}/_{t}, {}^{s+m}/_{t+l})X = ({}^{s}/_{t}, {}^{s+m}/_{t+l})X.$$

This means the quadrangle

$$\begin{tikzcd}
({}^{s}/_{t+l})X \arrow[r, "({}^{s}/_{t+l}, {}^{s+m}/_{t+l})X"] & ({}^{s+m}/_{t+l})X \\
({}^{s}/_{t})X \arrow[r, "({}^{s}/_{t}, {}^{s+m}/_{t})X"] \arrow[u, "({}^{s}/_{t}, {}^{s}/_{t+l})X"] & ({}^{s+m}/_{t})X \arrow[u, "({}^{s+m}/_{t}, {}^{s+m}/_{t+l})X"']
\end{tikzcd}$$

commutes.

Proof. If the difference $(s+m) - s = m = 0$, the assumption holds by Definition 124 (2). Assume that $m \geqslant 1$ and that the quadrangle

$$\begin{tikzcd}
({}^{s}/_{t+l})X \arrow[r, "({}^{s}/_{t+l}, {}^{s+m}/_{t+l})X"] & ({}^{s+m}/_{t+l})X \\
({}^{s}/_{t})X \arrow[r, "({}^{s}/_{t}, {}^{s+m}/_{t})X"] \arrow[u, "({}^{s}/_{t}, {}^{s}/_{t+l})X"] & ({}^{s+m}/_{t})X \arrow[u, "({}^{s+m}/_{t}, {}^{s+m}/_{t+l})X"']
\end{tikzcd}$$

commutes in which we have $(s+m)-(s+1)=m-1$. Then we have

$$\begin{aligned}
({}^{s}\!/\!_{t}, {}^{s+m}\!/\!_{t})X \cdot ({}^{s+m}\!/\!_{t}, {}^{s+m}\!/\!_{t+l})X &= ({}^{s}\!/\!_{t}, {}^{s+1}\!/\!_{t})X \cdot ({}^{s+1}\!/\!_{t}, {}^{s+m}\!/\!_{t})X \cdot ({}^{s+m}\!/\!_{t}, {}^{s+m}\!/\!_{t+l})X \\
&= ({}^{s}\!/\!_{t}, {}^{s+1}\!/\!_{t})X \cdot ({}^{s+1}\!/\!_{t}, {}^{s+m}\!/\!_{t+l})X \\
&= ({}^{s}\!/\!_{t}, {}^{s+1}\!/\!_{t})X \cdot ({}^{s+1}\!/\!_{t}, {}^{s+1}\!/\!_{t+l})X \cdot ({}^{s+1}\!/\!_{t+l}, {}^{s+m}\!/\!_{t+l})X \\
&\overset{\text{L. }125}{=} ({}^{s}\!/\!_{t}, {}^{s}\!/\!_{t+l})X \cdot ({}^{s}\!/\!_{t+l}, {}^{s+1}\!/\!_{t+l})X \cdot ({}^{s+1}\!/\!_{t+l}, {}^{s+m}\!/\!_{t+l})X \\
&= ({}^{s}\!/\!_{t}, {}^{s+m}\!/\!_{t+l})X
\end{aligned}$$

□

Lemma 127. *Suppose given a one-step n-complex $\check{X}$. Define $(X_{\mathrm{Ob}}, X_{\mathrm{Mor}})$ as in Definition 124 (2). Then $X := (X_{\mathrm{Ob}}, X_{\mathrm{Mor}})$ is an n-complex.*

Proof. We already have from the definition that ${}_{\bar{\Delta}_n^\#}({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'})X \subseteq {}_{\mathcal{A}}({}^{s}\!/\!_{t}X, {}^{s'}\!/\!_{t'}X)$ and $1_{s/t}X = 1_{s/tX}$.

We still need to show that for given morphisms $({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'}), ({}^{s'}\!/\!_{t'}, {}^{s''}\!/\!_{t''}) \in \mathrm{Mor}(\bar{\Delta}_n^\#)$ we have $({}^{s}\!/\!_{t}, {}^{s''}\!/\!_{t''})X = ({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'})X \cdot ({}^{s'}\!/\!_{t'}, {}^{s''}\!/\!_{t''})X$.

Let $({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'})$ and $({}^{s'}\!/\!_{t'}, {}^{s''}\!/\!_{t''})$ be morphisms in $\bar{\Delta}_n^\#$.

If $s < t'$, then $({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'})X = 0$. Now $t' \leqslant t''$ implies $s < t''$. This implies that $({}^{s}\!/\!_{t}, {}^{s''}\!/\!_{t''})X = 0$. So $({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'})X \cdot ({}^{s'}\!/\!_{t'}, {}^{s''}\!/\!_{t''})X = 0 = ({}^{s}\!/\!_{t}, {}^{s''}\!/\!_{t''})X$.

If $s' < t''$ then $({}^{s'}\!/\!_{t'}, {}^{s''}\!/\!_{t''})X = 0$. Now $s \leqslant s'$ implies $s < t''$ and thus $({}^{s}\!/\!_{t}, {}^{s''}\!/\!_{t''})X = 0$. So $({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'})X \cdot ({}^{s'}\!/\!_{t'}, {}^{s''}\!/\!_{t''})X = 0 = ({}^{s}\!/\!_{t}, {}^{s''}\!/\!_{t''})X$.

For the following we suppose that ${}^{s}\!/\!_{t'}, {}^{s'}\!/\!_{t''} \in \bar{\Delta}_n^\#$.

Case $t'' \leqslant s \leqslant (t'')^{+1}$, so ${}^{s}\!/\!_{t''} \in \bar{\Delta}_n^\#$. We have

$$\begin{aligned}
({}^{s}\!/\!_{t}, {}^{s'}\!/\!_{t'})X \cdot ({}^{s'}\!/\!_{t'}, {}^{s''}\!/\!_{t''})X &= ({}^{s}\!/\!_{t}, {}^{s}\!/\!_{t'})X \cdot ({}^{s}\!/\!_{t'}, {}^{s'}\!/\!_{t'})X \cdot ({}^{s'}\!/\!_{t'}, {}^{s'}\!/\!_{t''})X \cdot ({}^{s'}\!/\!_{t''}, {}^{s''}\!/\!_{t''})X \\
&\overset{\text{L. }126}{=} ({}^{s}\!/\!_{t}, {}^{s}\!/\!_{t'})X \cdot ({}^{s}\!/\!_{t'}, {}^{s}\!/\!_{t''})X \cdot ({}^{s}\!/\!_{t''}, {}^{s'}\!/\!_{t''})X \cdot ({}^{s'}\!/\!_{t''}, {}^{s''}\!/\!_{t''})X \\
&= ({}^{s}\!/\!_{t}, {}^{s}\!/\!_{t''})X \cdot ({}^{s}\!/\!_{t''}, {}^{s''}\!/\!_{t''})X \\
&= ({}^{s}\!/\!_{t}, {}^{s''}\!/\!_{t''})X
\end{aligned}$$

Depiction for $n = 4$ with $s/t = 3/1$, $s'/t' = 0^{+1}/2$ and $s''/t'' = 1^{+1}/3$:

$$
\begin{tikzcd}
 & & & & & & & & \\
 & & & X_{3/3} \arrow[r] & X_{4/3} \arrow[r] \arrow[u, dotted] & X_{0^{+1}/3} \arrow[r] \arrow[u, dotted] & X_{1^{+1}/3} \arrow[r, dotted] \arrow[u, dotted] & X_{2^{+1}/3} \arrow[r, dotted] \arrow[u, dotted] & X_{3^{+1}/3} \arrow[u, dotted] \\
 & & X_{2/2} \arrow[r, dotted] & X_{3/2} \arrow[r] \arrow[u] & X_{4/2} \arrow[r] \arrow[u, dotted] & X_{0^{+1}/2} \arrow[r, dotted] \arrow[u] \arrow[ur] & X_{1^{+1}/2} \arrow[r, dotted] \arrow[u, dotted] & X_{2^{+1}/2} \arrow[u, dotted] & \\
 & X_{1/1} \arrow[r, dotted] & X_{2/1} \arrow[r, dotted] \arrow[u, dotted] & X_{3/1} \arrow[r, dotted] \arrow[u] \arrow[urr] & X_{4/1} \arrow[r, dotted] \arrow[u, dotted] & X_{0^{+1}/1} \arrow[r, dotted] \arrow[u, dotted] & X_{1^{+1}/1} \arrow[u, dotted] & & \\
X_{0/0} \arrow[r, dotted] & X_{1/0} \arrow[r, dotted] \arrow[u, dotted] & X_{2/0} \arrow[r, dotted] \arrow[u, dotted] & X_{3/0} \arrow[r, dotted] \arrow[u, dotted] & X_{4/0} \arrow[r, dotted] \arrow[u, dotted] & X_{0^{+1}/0} \arrow[u, dotted] & & & \\
\arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & & & &
\end{tikzcd}
$$

So here, the composite results to be zero, since $X_{3/3} \cong 0_{\mathcal{A}}$.

Case $s < t''$. In this case we have $(s/t, s''/t'')X = 0_{s/tX, s''/t''X}$. The quadrangle with diagonal $(t''/t', s'/t'')$ is in $\bar{\Delta}_n^{\#}$ and $t''/t''X \cong 0_{\mathcal{A}}$. Thus $(t''/t', s'/t'')X = 0_{t''/t'X, s'/t''X}$.
This yields

$$
\begin{aligned}
(s/t, s'/t')X \cdot (s'/t', s''/t'')X &= (s/t, t''/t')X \cdot (t''/t', s'/t')X \cdot (s'/t', s'/t'')X \cdot (s'/t'', s''/t'')X \\
&\overset{\text{L. }126}{=} (s/t, t''/t')X \cdot (t''/t', s'/t'')X \cdot (s'/t'', s''/t'')X \\
&= 0_{s/tX, s''/t''X}
\end{aligned}
$$

Then X is a functor from $\bar{\Delta}_n^{\#}$ to $\mathcal{A}$ with $(t/t)X \cong 0_{\mathcal{A}}$ and $(t^{+1}/t)X \cong 0_{\mathcal{A}}$ for $t \in \bar{\Delta}_n$. I.e. X is an n-complex. $\square$

Lemma 128. *Every n-complex X in $\mathrm{C}^{(n)}(\mathcal{A})$ can be obtained from an one-step n-complex via Definition 124 (2).*

Proof. Define an one-step n-complex $\check{X}$ as the restriction of X to $(\bar{\Delta}_n^{\#}, S)$; cf. Definition 124 (1). Then the n-complex defined by $\check{X}$ is X. $\square$

Definition 129. An n-complex X in $\mathrm{C}^{(n)}(\mathcal{A})$ is called *split* if aX is split for all $a \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$.

Definition 130. An n-complex X in $\mathrm{C}^{(n)}(\mathcal{A})$ is called *acyclic* if for every quadrangle

$$
\begin{tikzcd}
t/s' \arrow[r] & t'/s' \\
t/s \arrow[r] \arrow[u] & t'/s \arrow[u]
\end{tikzcd}
$$

in $\bar{\Delta}_n^{\#}$, the quadrangle

$$\begin{tikzcd} X_{t/s'} \arrow[r] & X_{t'/s'} \\ X_{t/s} \arrow[u] \arrow[r] & X_{t'/s} \arrow[u] \end{tikzcd}$$

in X is a weak square.

Remark 131. Every 1-complex in $\mathrm{C}^{(1)}(\mathcal{A})$ is acyclic; cf. Lemma 38.

Lemma 132. *Suppose given an n-complex X and $t/s, t'/s' \in \bar{\Delta}_n^{\#}$ with $t/s \leqslant t'/s'$. Then $t'/s' \notin \mathrm{u}_n^{t/s}$ implies $(t/s, t'/s')X = 0$. Cf. Definition 93.*

Proof. If $t/s \notin \bar{\Delta}_n^{\#,\circ}$, then $X_{t/s} \cong 0_{\mathcal{A}}$ and $(t/s, t'/s')X = 0$. So suppose that $t/s \in \bar{\Delta}_n^{\#,\circ}$ and $t'/s' \notin \mathrm{u}_n^{t/s}$. This implies $t' > s + n = s^{+1} - 1$ or $s' > t - 1$, i.e. $s^{+1} \leqslant t'$ or $t \leqslant s'$.

Suppose that $s^{+1} \leqslant t'$. Then $t/s \leqslant s^{+1}/s \leqslant t'/s'$ and we can write

$$(t/s, t'/s')X = (t/s, s^{+1}/s)X \cdot (s^{+1}/s, t'/s')X = 0.$$

Suppose that $t \leqslant s'$. Then $t/s \leqslant t/t \leqslant t'/s'$ and we can write

$$(t/s, t'/s')X = (t/s, t/t)X \cdot (t/t, t'/s')X = 0.$$

□

Lemma 133.

1. *The category $\mathrm{C}^{(n)}(\mathcal{A})$ is additive.*
2. *If $\mathcal{A}$ is abelian, the category $\mathrm{C}^{(n)}(\mathcal{A})$ is abelian.*

Proof. Ad (1): The functor category $(\bar{\Delta}_n^{\#}, \mathcal{A})$ is additive; cf. Remark 12. The category $\mathrm{C}^{(n)}(\mathcal{A})$ is a full subcategory of $(\bar{\Delta}_n^{\#}, \mathcal{A})$. We show that $\mathrm{C}^{(n)}(\mathcal{A})$ is a full additive subcategory of $(\bar{\Delta}_n^{\#}, \mathcal{A})$; cf. Definition 8. It contains the zero object $N \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ with $N_{t/s} := 0_{\mathcal{A}}$ for every $t/s \in \bar{\Delta}_n^{\#}$. Given $X, Y \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$, we get a direct sum $(C, (\pi_i)_{i\in[1,2]}, (\iota_i)_{i\in[1,2]})$ in $(\bar{\Delta}_n^{\#}, \mathcal{A})$ with $C_{t/s} := X_{t/s} \oplus Y_{t/s}$ for $t/s \in \bar{\Delta}_n^{\#}$. We have $C_{t/s} \cong 0_{\mathcal{A}}$ for $t/s \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$. Therefore $C \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ and $(C, (\pi_i)_{i\in[1,2]}, (\iota_i)_{i\in[1,2]})$ is a direct sum in $\mathrm{C}^{(n)}(\mathcal{A})$. So $\mathrm{C}^{(n)}(\mathcal{A})$ is an additive category; cf. Lemma 10.

Ad (2): Let $\mathcal{A}$ be abelian. Then the functor category $(\bar{\Delta}_n^{\#}, \mathcal{A})$ is abelian, where kernels and cokernels can be formed pointwise; cf. Remark 25.

As $\mathrm{C}^{(n)}(\mathcal{A})$ is a full subcategory therein, it is sufficient to show that $\mathrm{C}^{(n)}(\mathcal{A})$ is closed under taking kernels and cokernels.

Let $X \xrightarrow{\alpha} Y$ be a morphism in $\mathrm{C}^{(n)}(\mathcal{A})$ and let $K \xrightarrow{k} X$ be a kernel of α. Then $K_{t/s} \xrightarrow{k_{t/s}} X_{t/s}$ is a kernel of $X_{t/s} \xrightarrow{\alpha_{t/s}} Y_{t/s}$ in $\mathcal{A}$ for $t/s \in \bar{\Delta}_n^{\#}$. For $t/s \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$ we have $X_{t/s} \cong 0_{\mathcal{A}}$ and $Y_{t/s} \cong 0_{\mathcal{A}}$, therefore $K_{t/s} \cong 0_{\mathcal{A}}$ for every $t/s \in \bar{\Delta}_n \setminus \bar{\Delta}_n^{\#,\circ}$. Therefore $K \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$.

Dually for cokernels.

Thus $\mathrm{C}^{(n)}(\mathcal{A})$ is abelian. $\square$

5.2 2-complexes are complexes in the classical sense

Suppose given an additive category $\mathcal{A}$.

Remark 134. By Definition 99, we have a bijection

$$\begin{aligned} \rho_2 \colon \mathbb{Z} &\to \bar{\Delta}_2^{\#,\circ} \\ k &\mapsto (a+b)\varphi_2^{-1}/a\varphi_2^{-1} \end{aligned}$$

with $k = 2a + b$ for $a \in \mathbb{Z}$ and $b \in [1, 2]$; cf. Remark 82.

The inverse is given by

$$\begin{aligned} \rho_2^{-1} \colon \bar{\Delta}_2^{\#,\circ} &\to \mathbb{Z} \\ t/s &\mapsto (s)\varphi_2 + (t)\varphi_2. \end{aligned}$$

The map ρ_2 is monotone:

For $k \in \mathbb{Z}$ and $k\rho_2 =: j/i \in \bar{\Delta}_2^{\#,\circ}$, we either have $(k+1)\rho_2 = j{+}1/i$ or $(k+1)\rho_2 = j/i{+}1$. In both cases we get $k\rho_2 < (k+1)\rho_2$.

Depiction of $\bar{\Delta}_2^{\#,\circ}$:

$$\begin{tikzcd}
 & & & {} & \\
 & & 2/2 \arrow[r] & 0^{+1}/2 \arrow[u, dotted] \arrow[r, dotted] & {} \\
 & 1/1 \arrow[r] & 2/1 \arrow[u] \arrow[r] & 0^{+1}/1 \arrow[u] \arrow[r] & 1^{+1}/1 \arrow[u, dotted] \\
0/0 \arrow[r] & 1/0 \arrow[u] \arrow[r] & 2/0 \arrow[u] \arrow[r] & 0^{+1}/0 \arrow[u] & \\
{} \arrow[u, dotted] \arrow[r, dotted] & 1/2^{-1} \arrow[u] \arrow[r] & 2/2^{-1} \arrow[u] & &
\end{tikzcd}$$

Remark 135. Every complex A defines a 2-complex X_A:

Suppose given a complex A over $\mathcal{A}$, i.e. a functor $\mathbb{Z} \to \mathcal{A}$ with $(i, i+2)A = 0$ for all $i \in \mathbb{Z}$. Consider the subset

$$\begin{aligned} S \ :=\ & \{({}^{t}\!/\!_{s}, {}^{t+1}\!/\!_{s}) \colon {}^{t}\!/\!_{s} \in \bar{\Delta}_2^{\#}, s \leqslant t < s^{+1}\} \cup \{({}^{t}\!/\!_{s}, {}^{t}\!/\!_{s+1}) \colon {}^{t}\!/\!_{s} \in \bar{\Delta}_2^{\#}, s < t \leqslant s^{+1}\} \\ & \subseteq \operatorname{Mor}(\bar{\Delta}_2^{\#}) \end{aligned}$$

with $S \cap \operatorname{Mor}(\bar{\Delta}_2^{\#,\circ}) = \{(i\rho_2, (i+1)\rho_2) \colon i \in \mathbb{Z}\}$.

We define a one-step 2-complex $\check{X}_A$ by

$${}^{t}\!/\!_{s}\check{X}_A = \begin{cases} {}^{t}\!/\!_{s}\rho_2^{-1}A & \text{for } {}^{t}\!/\!_{s} \in \bar{\Delta}_2^{\#,\circ} \\ 0_{\mathcal{A}} & \text{else} \end{cases}$$

and

$$({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})\check{X}_A = \begin{cases} (i, i+1)A & \text{if } ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'}) = (i\rho_2, (i+1)\rho_2) \in S \cap \operatorname{Mor}(\bar{\Delta}_2^{\#,\circ}) \text{ for some } i \in \mathbb{Z} \\ 0 & \text{if } ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'}) \in S \setminus \operatorname{Mor}(\bar{\Delta}_2^{\#,\circ}). \end{cases}$$

Then $\check{X}_A$ is an one-step 2-complex because all elements in $\bar{\Delta}_2^{\#,\circ}$ are of the form ${}^{s+1}\!/\!_{s}$ or ${}^{s+2}\!/\!_{s}$ for an $s \in \bar{\Delta}_2$ and

$$\begin{tikzcd} {}^{s+1}\!/\!_{s+1} \arrow[r] & {}^{s+2}\!/\!_{s+1} & & 0_{\mathcal{A}} \arrow[r] & (2s\varphi_2+3)A \\ {}^{s+1}\!/\!_{s} \arrow[r] \arrow[u] & {}^{s+2}\!/\!_{s} \arrow[u] \arrow[r, phantom, "\overset{\check{X}_A}{\longmapsto}"] & & (2s\varphi_2+1)A \arrow[r] \arrow[u] & (2s\varphi_2+2)A \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd} {}^{s+2}\!/\!_{s+1} \arrow[r] & {}^{s^{+1}}\!/\!_{s+1} & & (2s\varphi_2+3)A \arrow[r] & (2s\varphi_2+4)A \\ {}^{s+2}\!/\!_{s} \arrow[r] \arrow[u] & {}^{s^{+1}}\!/\!_{s} \arrow[u] \arrow[r, phantom, "\overset{\check{X}_A}{\longmapsto}"] & & (2s\varphi_2+2)A \arrow[r] \arrow[u] & 0_{\mathcal{A}} \arrow[u] \end{tikzcd}$$

commute for every $s \in \bar{\Delta}_2^{\#,\circ}$.

This defines an 2-complex X_A as in Definition 124 (2); cf. Lemma 127.

Lemma 136. *The categories* $\mathrm{C}(\mathcal{A})$ *and* $\mathrm{C}^{(2)}(\mathcal{A})$ *are equivalent.*

In particular, they are equivalent via $\mathrm{C}(\mathcal{A}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{C}^{(2)}(\mathcal{A})$ *where*

$$\begin{aligned} G \colon \mathrm{C}^{(2)}(\mathcal{A}) &\to \mathrm{C}(\mathcal{A}) \\ X &\mapsto \rho_2 X \quad \text{for } X \in \operatorname{Ob}(\mathrm{C}^{(2)}(\mathcal{A})) \\ \tau &\mapsto \rho_2 \tau \quad\ \text{for } \tau \in \operatorname{Mor}(\mathrm{C}^{(2)}(\mathcal{A})) \end{aligned}$$

and

$$\begin{array}{rll} F\colon \mathrm{C}(\mathcal{A}) & \to \mathrm{C}^{(2)}(\mathcal{A}) & \\ A & \mapsto X_A & \text{for } A \in \mathrm{Ob}(\mathrm{C}(\mathcal{A})) \\ \tau & \mapsto \hat{\tau} & \text{for } \tau \in \mathrm{Mor}(\mathrm{C}(\mathcal{A})) \end{array}$$

with X_A defined as in Remark 135 and

$$^{t}\!/_{s}\hat{\tau} = \begin{cases} ^{t}\!/_{s}\rho_2^{-1}\tau & \text{for } ^{t}\!/_{s} \in \bar{\Delta}_2^{\#,\circ} \\ 0 & \text{else.} \end{cases}$$

Proof. We have $FG = 1_{\mathrm{C}(\mathcal{A})}$.

For every $X \in \mathrm{Ob}(\mathrm{C}^{(2)}(\mathcal{A}))$, let $\hat{X}$ be the 2-complex with $\hat{X}\mid_{\bar{\Delta}_2^{\#,\circ}} = X \mid_{\bar{\Delta}_2^{\#,\circ}}$ and $a\hat{X} = 0_{\mathcal{A}}$ for $a \in \mathrm{Ob}(\bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ})$. Then $XGF = \hat{X}$. Analogously for $\tau\colon X \to Y$ we define $\hat{\tau}\colon \hat{X} \to \hat{Y}$ as the transformation with $\hat{\tau}\mid_{\bar{\Delta}_2^{\#,\circ}} := \tau \mid_{\bar{\Delta}_2^{\#,\circ}}$.

Define $X\sigma\colon X \to \hat{X}$ by

$$aX\sigma \;:= \begin{cases} 1_{aX} & \text{for } a \in \bar{\Delta}_2^{\#,\circ} \\ 0 & \text{else.} \end{cases}$$

for $X \in \mathrm{Ob}(\mathrm{C}^{(2)}(\mathcal{A}))$ and $a \in \bar{\Delta}_2^{\#}$.

Then we get an isotransformation $\sigma\colon 1_{\mathrm{C}^{(2)}(\mathcal{A})} \to GF$ by

$$(X \xrightarrow{\tau} Y) \quad \overset{\sigma}{\mapsto} \quad \begin{array}{ccc} X & \xrightarrow{\tau} & Y \\ \downarrow{\scriptstyle X\sigma} & & \downarrow{\scriptstyle Y\sigma} \\ \hat{X} & \xrightarrow{\hat{\tau}} & \hat{Y}\,. \end{array}$$

The quadrangle commutes, as for every $a \in \bar{\Delta}_n^{\#}$ either both morphisms $aX\sigma$ and $aY\sigma$ are the identity morphism and $a\tau = a\hat{\tau}$, or $a \in \bar{\Delta}_2^{\#} \setminus \bar{\Delta}_2^{\#,\circ}$ and every morphism is zero. $\square$

Example 137. A 2-complex X:

$$
\begin{tikzcd}
 & & & {} & \\
 & & 0 \arrow[r] & X_{0+{}^{1}/_{2}} \arrow[r, dotted] \arrow[u, dotted] & {} \\
 & 0 \arrow[r] & X_{2/1} \arrow[r, "x"] \arrow[u] & X_{0+{}^{1}/_{1}} \arrow[r] \arrow[u, "x"] & 0 \arrow[u, dotted] \\
0 \arrow[r] & X_{1/0} \arrow[r, "x"] \arrow[u] & X_{2/0} \arrow[r] \arrow[u, "x"] & 0 \arrow[u] & \\
{} \arrow[u, dotted] \arrow[r, dotted] & X_{1/2-1} \arrow[r] \arrow[u, "x"] & 0 \arrow[u] & & \\
 & {} \arrow[u, dotted] & & &
\end{tikzcd}
$$

The complex of X in usual notation:

$$
\cdots\rightarrow \underbrace{X_{1/2-1}}_{0A=A^0} \longrightarrow \underbrace{X_{1/0}}_{1A=A^1} \longrightarrow \underbrace{X_{2/0}}_{2A=A^2} \longrightarrow \underbrace{X_{2/1}}_{3A=A^3} \longrightarrow \underbrace{X_{0+{}^{1}/_{1}}}_{4A=A^4} \longrightarrow \underbrace{X_{0+{}^{1}/_{2}}}_{5A=A^5} \cdots\rightarrow
$$

6 Functors defined from quasiperiodic monotone maps

Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$. Suppose given an additive category $\mathcal{A}$.

Lemma 138. *Suppose given a quasiperiodic monotone map $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$. We define the additive functor*

$$\mathrm{C}^{(f)}(\mathcal{A})\colon\ \mathrm{C}^{(m)}(\mathcal{A}) \to \mathrm{C}^{(n)}(\mathcal{A})$$

by

$$\begin{aligned} \mathrm{C}^{(f)}(\mathcal{A})\colon\ \mathrm{C}^{(m)}(\mathcal{A}) &\to \mathrm{C}^{(n)}(\mathcal{A}) \\ X &\mapsto f^{\#} \cdot X \quad \text{for } X \in \mathrm{Ob}(\mathrm{C}^{(m)}(\mathcal{A})) \\ \tau &\mapsto f^{\#} \cdot \tau \quad\ \text{for } \tau \in \mathrm{Mor}(\mathrm{C}^{(m)}(\mathcal{A})). \end{aligned}$$

When $\mathcal{A}$ is clear from context, we sometimes write $\mathrm{C}^{(f)} := \mathrm{C}^{(f)}(\mathcal{A})$. We sometimes write

$$X^{(f)} := X\,\mathrm{C}^{(f)}(\mathcal{A})$$

for $X \in \mathrm{Ob}(\mathrm{C}^{(m)}(\mathcal{A}))$ and

$$\tau^{(f)} := \tau\,\mathrm{C}^{(f)}(\mathcal{A})$$

for $\tau \in \mathrm{Mor}(\mathrm{C}^{(m)}(\mathcal{A}))$.

Proof. The resulting $X^{(f)}$ is indeed an n-complex:

The composite of the functors $f^{\#}$ and X is a functor again.

For ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}$ we have ${}^{t}\!/_{s}X^{(f)} = ({}^{t}\!/_{s}f^{\#})X = ({}^{tf}\!/_{sf})X \cong 0_{\mathcal{A}}$ due to the quasiperiodicity of f, both if $t = s^{+1}$ or if $t = s$.

Suppose given $X \xrightarrow{\sigma} Y \xrightarrow{\tau} Z$ in $\mathrm{C}^{(m)}(\mathcal{A})$. Then

$$\begin{aligned} {}^{t}\!/_{s}\Big((\sigma \cdot \tau)\,\mathrm{C}^{(f)}\Big) &= (\sigma \cdot \tau)_{{}^{t}\!/_{s}f^{\#}} \\ &= \sigma_{{}^{t}\!/_{s}f^{\#}} \cdot \tau_{{}^{t}\!/_{s}f^{\#}} \\ &= {}^{t}\!/_{s}\Big(\sigma\,\mathrm{C}^{(f)}\Big) \cdot {}^{t}\!/_{s}\Big(\tau\,\mathrm{C}^{(f)}\Big) \\ &= {}^{t}\!/_{s}(\sigma\,\mathrm{C}^{(f)} \cdot \tau\,\mathrm{C}^{(f)}) \end{aligned}$$

for ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}$. So $(\sigma \cdot \tau)\,\mathrm{C}^{(f)} = \sigma\,\mathrm{C}^{(f)} \cdot \tau\,\mathrm{C}^{(f)}$.

For $\varphi, \psi \in \mathrm{Mor}(\mathrm{C}^{(m)}(\mathcal{A}))$ we have

$$\begin{aligned} (\varphi + \psi)\,\mathrm{C}^{(f)}(\mathcal{A}) &= \Big({}^{t}\!/_{s}f^{\#}(\varphi + \psi)\Big)_{{}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}} \\ &= \Big({}^{t}\!/_{s}f^{\#}\varphi\Big)_{{}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}} + \Big({}^{t}\!/_{s}f^{\#}\psi\Big)_{{}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}} \\ &= \varphi\,\mathrm{C}^{(f)}(\mathcal{A}) + \psi\,\mathrm{C}^{(f)}(\mathcal{A}). \end{aligned}$$

Thus $\mathrm{C}^{(f)}(\mathcal{A})$ is additive. $\square$

Remark 139. *Suppose given* $p \in \mathbb{Z}_{\geqslant 1}$ *as well as* $f \in (\bar{\Delta}_n, \bar{\Delta}_m)_{\text{q.p.}}$ *and* $g \in (\bar{\Delta}_p, \bar{\Delta}_n)_{\text{q.p.}}$*.*

(i) We have $\mathrm{C}^{(g \cdot f)}(\mathcal{A}) = \mathrm{C}^{(f)}(\mathcal{A}) \cdot \mathrm{C}^{(g)}(\mathcal{A})$*.*

(ii) We have $\mathrm{C}^{(1_{\bar{\Delta}_n})}(\mathcal{A}) = 1_{\mathrm{C}^{(n)}(\mathcal{A})}$*.*

(iii) If f *is an isomorphism, then* $\mathrm{C}^{(f)}(\mathcal{A})$ *is an isomorphism, too.*

Proof. Ad (i):
For every $X \xrightarrow{\tau} Y$ in $\mathrm{C}^{(m)}(\mathcal{A})$ we have

$$\begin{aligned}
(X \xrightarrow{\tau} Y)\,\mathrm{C}^{(gf)}(\mathcal{A}) &= (gf)^{\#}X \xrightarrow{(gf)^{\#}\tau} (gf)^{\#}Y \\
&\overset{\text{R.98(1)}}{=} (g^{\#}f^{\#})X \xrightarrow{(g^{\#}f^{\#})\tau} (g^{\#}f^{\#})Y \\
&= g^{\#}(f^{\#}X) \xrightarrow{g^{\#}(f^{\#}\tau)} g^{\#}(f^{\#}Y) \\
&= (X \xrightarrow{\tau} Y)\,\mathrm{C}^{(f)}(\mathcal{A})\,\mathrm{C}^{(g)}(\mathcal{A})
\end{aligned}$$

Ad (ii):
For $X \xrightarrow{\tau} Y$ in $\mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ we have

$$\begin{aligned}
(X \xrightarrow{\tau} Y)\,\mathrm{C}^{(1_{\bar{\Delta}_n})}(\mathcal{A}) &= 1_{\bar{\Delta}_n^{\#}}X \xrightarrow{1_{\bar{\Delta}_n^{\#}}\tau} 1_{\bar{\Delta}_n^{\#}}Y \\
&= X \xrightarrow{\tau} Y
\end{aligned}$$

Ad (iii): This follows from *(i)* and *(ii)*. □

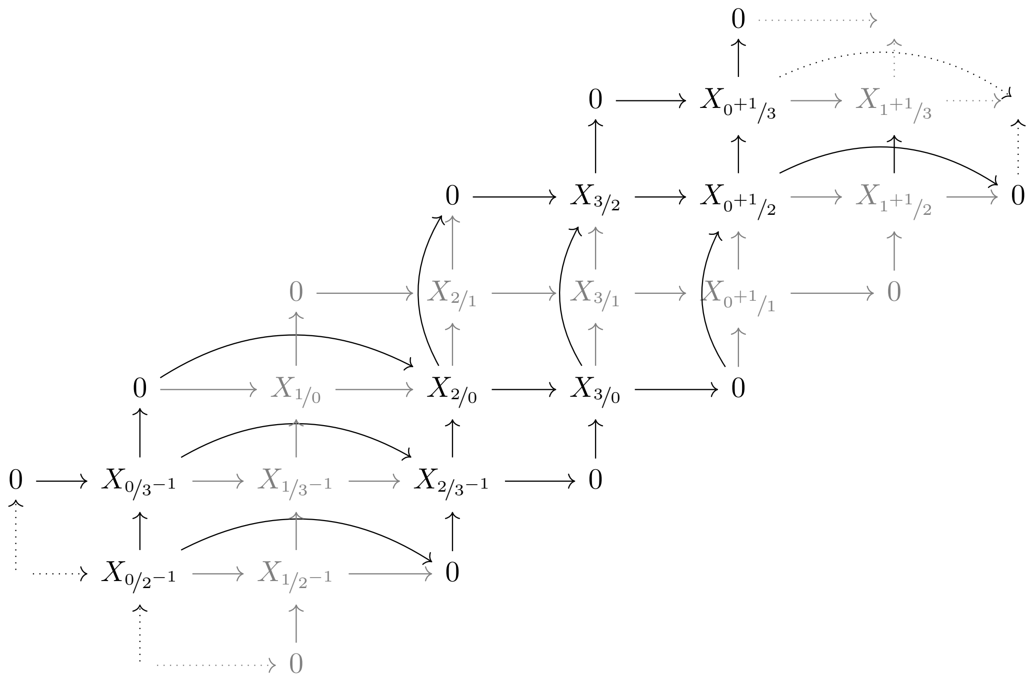
Example 140.

(1) If f is injective, then $\mathrm{C}^{(f)}(\mathcal{A})$ is the operation that removes rows and columns in the diagram containing indices not in the image of f, as can be seen in the following example.

Consider the monotone quasiperiodic map $f \colon \bar{\Delta}_2 \to \bar{\Delta}_3$ with

$$\begin{aligned}
0 &\longmapsto 0 \\
1 &\longmapsto 2 \\
2 &\longmapsto 3
\end{aligned}$$

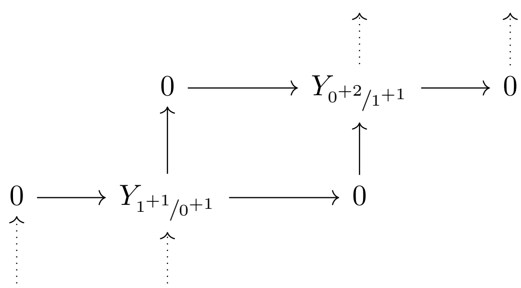
and the 3-complex X from Example 121. Then $X^{(f)}$ is the complex, where all columns and rows with index containing 1^{+z} for some $z \in \mathbb{Z}$ are removed as pictured in the following diagram. The resulting complex $X^{(f)}$ is displayed in black.



(2) Consider the monotone quasiperiodic map $g\colon \bar{\Delta}_2 \to \bar{\Delta}_1$ with

$$\begin{aligned} 0 &\mapsto 1^{+1} \\ 1 &\mapsto 1^{+1} \\ 2 &\mapsto 0^{+2} \end{aligned}$$

and the 1-complex Y



the resulting complex $Y^{(g)}$ is

$$
\begin{tikzcd}
 & & & {} & {} & {} \\
 & & 0 \arrow[r] & Y_{0^{+2}/1^{+1}} \arrow[r] \arrow[u, dotted] & 0 \arrow[r, equals] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r, equals] & 0 \arrow[r] \arrow[u, equals] & Y_{0^{+2}/1^{+1}} \arrow[r] \arrow[u, equals] & 0 \arrow[u, equals] & \\
0 \arrow[r] & Y_{1^{+1}/0^{+1}} \arrow[r, equals] \arrow[u] & Y_{1^{+1}/0^{+1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}
$$

This is to be read as a diagram on

$$
\begin{tikzcd}
 & & & {} & {} & {} \\
 & & 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u, dotted] & 0^{+1}/1 \arrow[r] \arrow[u, dotted] & 1^{+1}/1 \arrow[u, dotted] \\
 & 0/0 \arrow[r] & 1/0 \arrow[r] \arrow[u] & 2/0 \arrow[r] \arrow[u] & 0^{+1}/0 \arrow[u] & \\
2^{-1}/2^{-1} \arrow[r] & 0/2^{-1} \arrow[r] \arrow[u] & 1/2^{-1} \arrow[r] \arrow[u] & 2/2^{-1} \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}
$$

For instance, $Y^{(g)}_{2/1} = Y_{2/1g^{\#}} = Y_{2g/1g} = Y_{0^{+2}/1^{+1}}$.

Moreover let $g' \colon \bar{\Delta}_1 \to \bar{\Delta}_2$ be the quasiperiodic monotone map defined by

$$
\begin{aligned}
0 &\mapsto 2^{-2} \\
1 &\mapsto 0^{-1}.
\end{aligned}
$$

Then we have $g \cdot g' = 1_{\bar{\Delta}_1}$ and thus

$$\left(Y^{(g)}\right)^{(g')} = Y^{(g' \cdot g)} = Y^{(1_{\bar{\Delta}_1})} = Y.$$

Definition 141. Suppose given two quasiperiodic monotone maps $f, g \colon \bar{\Delta}_n \to \bar{\Delta}_m$ such that $f \leqslant g$. Consider the transformation $f \xrightarrow{\alpha} g$.

(1) For every $X \in \mathrm{Ob}(\mathrm{C}^{(m)}(\mathcal{A}))$ we define a morphism $X\,\mathrm{C}^{(\alpha)}(\mathcal{A}) \colon X^{(f)} \to X^{(g)}$ by

$$X\,\mathrm{C}^{(\alpha)}(\mathcal{A}) \;:=\; \alpha^{\#} X;$$

cf. Lemma 97.

(2) We have a transformation $\mathrm{C}^{(\alpha)}(\mathcal{A}) = (X\,\mathrm{C}^{(\alpha)}(\mathcal{A}))_{X \in \mathrm{Ob}(\mathrm{C}^{(m)})}$ from $\mathrm{C}^{(f)}(\mathcal{A})$ to $\mathrm{C}^{(g)}(\mathcal{A})$.

We often write $\mathrm{C}^{(\alpha)}\colon \mathrm{C}^{(f)} \to \mathrm{C}^{(g)}$ instead of $\mathrm{C}^{(\alpha)}(\mathcal{A})\colon \mathrm{C}^{(f)}(\mathcal{A}) \to \mathrm{C}^{(g)}(\mathcal{A})$ if $\mathcal{A}$ is clear from context.

We sometimes write $X^{(\alpha)} := X\,\mathrm{C}^{(\alpha)}$ for $X \in \mathrm{Ob}(\mathrm{C}^{(m)}(\mathcal{A}))$.

We also write $\mathrm{C}^{(f,g)} := \mathrm{C}^{(\alpha)}$ for the unique transformation $\alpha = (f,g)\colon f \to g$.

Concerning (2), we show that $(X\,\mathrm{C}^{(\alpha)})_{X \in \mathrm{Ob}(\mathrm{C}^{(m)})}$ is natural.

Suppose given $X \xrightarrow{\varphi} Y$ in $\mathrm{C}^{(m)}(\mathcal{A})$.

For every $a \in \bar{\Delta}_n^{\#}$, the quadrangle

$$\begin{tikzcd}
a(X\,\mathrm{C}^{(f)}) \arrow[r, "a\varphi\,\mathrm{C}^{(f)}"] \arrow[d, "aX\,\mathrm{C}^{(\alpha)}"] & a(Y\,\mathrm{C}^{(f)}) \arrow[d, "aY\,\mathrm{C}^{(\alpha)}"] \\
a(X\,\mathrm{C}^{(g)}) \arrow[r, "a\varphi\,\mathrm{C}^{(g)}"] & a(Y\,\mathrm{C}^{(g)})
\end{tikzcd}$$

has to commute.

Using the definitions of $\mathrm{C}^{(f)}, \mathrm{C}^{(g)}$ and $\mathrm{C}^{(\alpha)}$ this means

$$\begin{tikzcd}
af^{\#}X \arrow[r, "af^{\#}\varphi"] \arrow[d, "(af^{\#}{,}ag^{\#})X"'] & af^{\#}Y \arrow[d, "(af^{\#}{,}ag^{\#})Y"] \\
ag^{\#}X \arrow[r, "ag^{\#}\varphi"] & ag^{\#}Y
\end{tikzcd}$$

has to commute. This holds because φ is a morphism of n-complexes.

Lemma 142. *For every $n, m \in \mathbb{Z}_{\geqslant 0}$,*

$$\begin{aligned}
(\bar{\Delta}_n, \bar{\Delta}_m) &\longrightarrow (\mathrm{C}^{(m)}, \mathrm{C}^{(n)}) \\
(f \xrightarrow{\alpha} g) &\longmapsto (\mathrm{C}^{(f)} \xrightarrow{\mathrm{C}^{(\alpha)}} \mathrm{C}^{(g)})
\end{aligned}$$

is a functor.

Proof. For $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ and the identity transformation $1_f = (f,f)$, we get

$$aX\,\mathrm{C}^{(f,f)} = (af^{\#}, af^{\#})X = 1_{aX\,\mathrm{C}^{(f)}}$$

for every $X \in \mathrm{Ob}(\mathrm{C}^{(m)})$ and every $a \in \bar{\Delta}_n^{\#}$. Therefore $\mathrm{C}^{(1_f)} = 1_{\mathrm{C}^{(f)}}$.

For $f, g, h\colon \bar{\Delta}_n \to \bar{\Delta}_m$ and transformations (f,g) and (g,h), we have

$$aX\,\mathrm{C}^{(f,g)} \cdot aX\,\mathrm{C}^{(g,h)} = (af^{\#}, ag^{\#})X \cdot (ag^{\#}, ah^{\#})X = (af^{\#}, ah^{\#})X = aX\,\mathrm{C}^{(f,h)}$$

for every $X \in \mathrm{Ob}(\mathrm{C}^{(m)})$ and every $a \in \bar{\Delta}_n^{\#}$. Therefore $\mathrm{C}^{(f,g)}\,\mathrm{C}^{(g,h)} = \mathrm{C}^{(f,h)}$. $\square$

Lemma 143. *Suppose given quasiperiodic monotone maps* $\bar{\Delta}_n \underset{g}{\overset{f}{\rightleftarrows}} \bar{\Delta}_m$ *with f left adjoint to g via unit $\alpha := (1_{\bar{\Delta}_n}, fg)$ and counit $\beta := (gf, 1_{\bar{\Delta}_m})$. Then $\mathrm{C}^{(g)}(\mathcal{A})$ is left adjoint to $\mathrm{C}^{(f)}(\mathcal{A})$ via unit*

$$1_{\mathrm{C}^{(n)}(\mathcal{A})} \xrightarrow{\mathrm{C}^{(\alpha)}(\mathcal{A})} \mathrm{C}^{(g)}(\mathcal{A})\,\mathrm{C}^{(f)}(\mathcal{A})$$

and counit

$$\mathrm{C}^{(f)}(\mathcal{A})\,\mathrm{C}^{(g)}(\mathcal{A}) \xrightarrow{\mathrm{C}^{(\beta)}(\mathcal{A})} 1_{\mathrm{C}^{(m)}(\mathcal{A})}.$$

Proof. We need to show that for every $X \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ and for every $Y \in \mathrm{Ob}(\mathrm{C}^{(m)}(\mathcal{A}))$ the triangle identities

$$\begin{tikzcd} (X)\,\mathrm{C}^{(g)} \arrow[r, "(X)\,\mathrm{C}^{(\alpha)}\,\mathrm{C}^{(g)}"] \arrow[dr, "1_{(X)\,\mathrm{C}^{(g)}}"'] & (X)\,\mathrm{C}^{(g)}\,\mathrm{C}^{(f)}\,\mathrm{C}^{(g)} \arrow[d, "(X)\,\mathrm{C}^{(g)}\,\mathrm{C}^{(\beta)}"] \\ & (X)\,\mathrm{C}^{(g)} \end{tikzcd} \qquad \begin{tikzcd} (Y)\,\mathrm{C}^{(f)} \arrow[r, "(Y)\,\mathrm{C}^{(f)}\,\mathrm{C}^{(\alpha)}"] \arrow[dr, "1_{(Y)\,\mathrm{C}^{(f)}}"'] & (Y)\,\mathrm{C}^{(f)}\,\mathrm{C}^{(g)}\,\mathrm{C}^{(f)} \arrow[d, "Y\,\mathrm{C}^{(\beta)}\,\mathrm{C}^{(f)}"] \\ & (Y)\,\mathrm{C}^{(f)} \end{tikzcd}$$

hold. Evaluated, they look as follows.

$$\begin{tikzcd} g^{\#}X \arrow[r, "g^{\#}\alpha^{\#}X"] \arrow[dr, "1_{g^{\#}X}"'] & g^{\#}f^{\#}g^{\#}X \arrow[d, "\beta^{\#}g^{\#}X"] \\ & g^{\#}X \end{tikzcd} \qquad \begin{tikzcd} f^{\#}Y \arrow[r, "\alpha^{\#}f^{\#}Y"] \arrow[dr, "1_{f^{\#}Y}"'] & f^{\#}g^{\#}f^{\#}Y \arrow[d, "f^{\#}\beta^{\#}Y"] \\ & f^{\#}Y \end{tikzcd}$$

We know that $f^{\#}$ is left adjoint to $g^{\#}$; cf. Remark 98, necessarily via unit $\alpha^{\#} = (1_{\bar{\Delta}_n^{\#}}, f^{\#}g^{\#})$ and counit $\beta^{\#} = (g^{\#}f^{\#}, 1_{\bar{\Delta}_m^{\#}})$. Therefore the following diagrams commute for every $i \in \bar{\Delta}_n^{\#}$ and every $j \in \bar{\Delta}_m^{\#}$.

$$\begin{tikzcd} if^{\#} \arrow[r, "i\alpha^{\#}f^{\#}"] \arrow[dr, "1_{if^{\#}}"'] & if^{\#}g^{\#}f^{\#} \arrow[d, "if^{\#}\beta^{\#}"] \\ & if^{\#} \end{tikzcd} \qquad \begin{tikzcd} jg^{\#} \arrow[r, "jg^{\#}\alpha^{\#}"] \arrow[dr, "1_{jg^{\#}}"'] & jg^{\#}f^{\#}g^{\#} \arrow[d, "j\beta^{\#}g^{\#}"] \\ & jg^{\#} \end{tikzcd}$$

Now applying X and Y, respectively, yields the wanted identities for every $i \in \bar{\Delta}_n^{\#}$ and every $j \in \bar{\Delta}_m^{\#}$.

$$\begin{tikzcd} if^{\#}Y \arrow[r, "i\alpha^{\#}f^{\#}Y"] \arrow[dr, "1_{if^{\#}Y}"'] & if^{\#}g^{\#}f^{\#}Y \arrow[d, "if^{\#}\beta^{\#}Y"] \\ & if^{\#}Y \end{tikzcd} \qquad \begin{tikzcd} jg^{\#}X \arrow[r, "jg^{\#}\alpha^{\#}X"] \arrow[dr, "1_{jg^{\#}X}"'] & jg^{\#}f^{\#}g^{\#}X \arrow[d, "j\beta^{\#}g^{\#}X"] \\ & jg^{\#}X \end{tikzcd}$$

$\square$

Corollary 144. *Suppose given $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ quasiperiodic monotone. Then there exist quasiperiodic monotone maps $g, h\colon \bar{\Delta}_m \to \bar{\Delta}_n$ such that*

$$g \dashv f \dashv h$$

cf. Lemma 86.

We have $\mathrm{C}^{(g)}(\mathcal{A}), \mathrm{C}^{(h)}(\mathcal{A})\colon\ \mathrm{C}^{(n)}(\mathcal{A}) \to \mathrm{C}^{(m)}(\mathcal{A})$ with

$$\mathrm{C}^{(h)}(\mathcal{A}) \dashv \mathrm{C}^{(f)}(\mathcal{A}) \dashv \mathrm{C}^{(g)}(\mathcal{A}),$$

cf. Lemma 143.

Remark 145. Suppose given ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}$. We abbreviate

$$\begin{aligned} f &:= f^{t/s}\colon \bar{\Delta}_1 \to \bar{\Delta}_n \\ \hat{f} &:= \hat{f}^{t/s}\colon \bar{\Delta}_n \to \bar{\Delta}_1 \\ \check{f} &:= \check{f}^{t/s}\colon \bar{\Delta}_n \to \bar{\Delta}_1, \end{aligned}$$

cf. Definition 101. So $\check{f} \dashv f \dashv \hat{f}$.

Let

$$\begin{aligned} \hat{\eta} &:= (1_{\bar{\Delta}_1}, f^{t/s}\hat{f}^{t/s}) = (1_{\bar{\Delta}_1}, f\hat{f}), & \check{\eta} &:= (1_{\bar{\Delta}_1}, \check{f}^{t/s} f^{t/s}) = (1_{\bar{\Delta}_1}, \check{f} f), \\ \hat{\varepsilon} &:= (\hat{f}^{t/s} f^{t/s}, 1_{\bar{\Delta}_n}) = (\hat{f} f, 1_{\bar{\Delta}_n}), & \check{\varepsilon} &:= (f^{t/s}\check{f}^{t/s}, 1_{\bar{\Delta}_n}) = (f\check{f}, 1_{\bar{\Delta}_n}). \end{aligned}$$

By Lemma 143, we have

$$\mathrm{C}^{(\hat{f})} \dashv \mathrm{C}^{(f)} \dashv \mathrm{C}^{(\check{f})},$$

where $\mathrm{C}^{(\hat{f})} \dashv \mathrm{C}^{(f)}$ via unit $\mathrm{C}^{(\hat{\eta})}\colon 1_{\mathrm{C}^{(1)}} \to \mathrm{C}^{(\hat{f})}\,\mathrm{C}^{(f)}$ and counit $\mathrm{C}^{(\hat{\varepsilon})}\colon\ \mathrm{C}^{(f)}\,\mathrm{C}^{(\hat{f})} \to 1_{\mathrm{C}^{(n)}}$ and where $\mathrm{C}^{(f)} \dashv \mathrm{C}^{(\check{f})}$ via unit $\mathrm{C}^{(\check{\eta})}\colon 1 \to \mathrm{C}^{(f)}\,\mathrm{C}^{(\check{f})}$ and counit $\mathrm{C}^{(\check{\varepsilon})}\colon\ \mathrm{C}^{(\check{f})}\,\mathrm{C}^{(f)}$.

Then for every $X \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$ and every $Y \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ we get the following two bijections,

$$\begin{aligned} {}_{\mathrm{C}^{(1)}}(X, Y^{(f)}) &\longrightarrow {}_{\mathrm{C}^{(n)}}(X^{(\hat{f})}, Y) \\ \sigma &\longmapsto (\sigma)\,\mathrm{C}^{(\hat{f})} \cdot Y\,\mathrm{C}^{(\hat{\varepsilon})} \\ X\,\mathrm{C}^{(\hat{\eta})} \cdot (\tau)\,\mathrm{C}^{(f)} &\longleftarrow\!\shortmid\ \tau \end{aligned}$$

as well as

$$\begin{aligned} {}_{\mathrm{C}^{(n)}}(X, Y^{(\check{f})}) &\longrightarrow {}_{\mathrm{C}^{(1)}}(X^{(f)}, Y) \\ \tau &\longmapsto (\tau)\,\mathrm{C}^{(\check{f})} \cdot Y\,\mathrm{C}^{(\check{\varepsilon})} \\ X\,\mathrm{C}^{(\check{\eta})} \cdot (\sigma)\,\mathrm{C}^{(f)} &\longleftarrow\!\shortmid\ \sigma \end{aligned}$$

Lemma 146. *Suppose given an injective quasiperiodic monotone map $g\colon \bar{\Delta}_m \to \bar{\Delta}_n$. Let $\hat{f}$ be left adjoint to g and f left adjoint to $\hat{f}$, that is $f \dashv \hat{f} \dashv g$. Note that $g\hat{f} = 1_{\bar{\Delta}_m}$; cf. Lemma 74.*

$$\begin{tikzcd} \bar{\Delta}_m \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bar{\Delta}_n \arrow[l, "\hat{f}"'] \end{tikzcd}$$

Let $\varepsilon := (\hat{f}f, 1_{\bar{\Delta}_n})\colon \hat{f}f \to 1_{\bar{\Delta}_n}$ and $\eta' := (1_{\bar{\Delta}_n}, \hat{f}g)\colon 1_{\bar{\Delta}_n} \to \hat{f}g$.

Suppose given an m-complex $V \in \mathrm{Ob}(\mathrm{C}^{(m)}(\mathcal{A}))$, an n-complex $X \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ and an m-complex morphism $\alpha\colon V \to X^{(f)}$.

Define

$$\alpha' := (\alpha)\,\mathrm{C}^{(\hat{f})} \cdot (X)\,\mathrm{C}^{(\varepsilon)}\colon V^{(\hat{f})} \to X$$

and

$$\bar{\alpha} := (\alpha')\,\mathrm{C}^{(g)}\colon V^{(g\hat{f})} = V \to X^{(g)}.$$

Then $\bar{\alpha} = \alpha \cdot (X)\,\mathrm{C}^{(\tau)}$ with $\tau := (f, g)\colon f \to g$.

Suppose given $\beta\colon X^{(g)} \to V$ with $\bar{\alpha}\beta = 1_V$. We define

$$\beta' := (X)\,\mathrm{C}^{(\eta')} \cdot (\beta)\,\mathrm{C}^{(\hat{f})}\colon X \to V^{(\hat{f})}.$$

Then $\alpha'\beta' = 1_{V^{(\hat{f})}}$.

In particular, if $\bar{\alpha}$ is split monic, then α' is split monic, too.

Proof. Existence of τ: We have $\hat{f}f \leqslant 1_{\bar{\Delta}_n} \leqslant \hat{f}g$ and thus $f = f\hat{f}f \leqslant g\hat{f}f = g$.

We have

$$\begin{aligned} \bar{\alpha} &= \alpha'^{(g)} \\ &= \alpha^{(\hat{f})(g)} \cdot X^{(\varepsilon)(g)} \\ &= \alpha^{(f\hat{f})} \cdot X^{(g\varepsilon)} \\ &= \alpha \cdot X^{(\tau)}. \end{aligned}$$

Suppose given ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_1^{\#}$. We obtain

$$
\begin{array}{rcl}
1_{V_{t/s}} = ({}^{t}\!/\!{}_{s})(\bar{\alpha} \cdot \beta) & = & ({}^{t}\!/\!{}_{s})\bar{\alpha} \cdot ({}^{t}\!/\!{}_{s})\beta \\
& = & ({}^{t}\!/\!{}_{s})(\alpha')\,\mathrm{C}^{(g)} \cdot ({}^{t}\!/\!{}_{s})\beta \\
& = & ({}^{t}\!/\!{}_{s}g^{\#})\alpha' \cdot ({}^{t}\!/\!{}_{s})\beta \\
& = & ({}^{t}\!/\!{}_{s}g^{\#})\Big((\alpha)\,\mathrm{C}^{(\hat{f})}\Big) \cdot ({}^{t}\!/\!{}_{s}g^{\#})\Big((X)\,\mathrm{C}^{(\varepsilon)}\Big) \cdot ({}^{t}\!/\!{}_{s})\beta \\
& = & ({}^{t}\!/\!{}_{s})g^{\#}\hat{f}^{\#}\alpha \cdot ({}^{t}\!/\!{}_{s})g^{\#}\varepsilon^{\#}X \cdot ({}^{t}\!/\!{}_{s})\beta \\
& = & ({}^{t}\!/\!{}_{s})g^{\#}\hat{f}^{\#}\alpha \cdot (({}^{t}\!/\!{}_{s})g^{\#}\hat{f}^{\#}f^{\#}, ({}^{t}\!/\!{}_{s})g^{\#})X \cdot ({}^{t}\!/\!{}_{s})\beta \\
& \overset{g^{\#}\hat{f}^{\#}=1}{=} & ({}^{t}\!/\!{}_{s})\alpha \cdot (({}^{t}\!/\!{}_{s})f^{\#}, ({}^{t}\!/\!{}_{s})g^{\#})X \cdot ({}^{t}\!/\!{}_{s})\beta.
\end{array}
$$

For ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#}$, we get

$$
\begin{array}{rcl}
({}^{j}\!/\!{}_{i})(\alpha' \cdot \beta') & = & ({}^{j}\!/\!{}_{i})\alpha' \cdot ({}^{j}\!/\!{}_{i})\beta' \\
& = & ({}^{j}\!/\!{}_{i})\Big((\alpha)\,\mathrm{C}^{(\hat{f})}\Big) \cdot ({}^{j}\!/\!{}_{i})\Big((X)\,\mathrm{C}^{(\varepsilon)}\Big) \cdot ({}^{j}\!/\!{}_{i})\Big((X)\,\mathrm{C}^{(\eta')}\Big) \cdot ({}^{j}\!/\!{}_{i})\Big((\beta)\,\mathrm{C}^{(\hat{f})}\Big) \\
& = & ({}^{j}\!/\!{}_{i})\hat{f}^{\#}\alpha \cdot ({}^{j}\!/\!{}_{i})\varepsilon^{\#}X \cdot ({}^{j}\!/\!{}_{i})\eta'^{\#}X \cdot ({}^{j}\!/\!{}_{i})\hat{f}^{\#}\beta \\
& = & ({}^{j}\!/\!{}_{i})\hat{f}^{\#}\alpha \cdot ({}^{j}\!/\!{}_{i}\hat{f}^{\#}f^{\#}, {}^{j}\!/\!{}_{i})X \cdot ({}^{j}\!/\!{}_{i}, {}^{j}\!/\!{}_{i}\hat{f}^{\#}g^{\#})X \cdot ({}^{j}\!/\!{}_{i})\hat{f}^{\#}\beta \\
& = & ({}^{j}\!/\!{}_{i})\hat{f}^{\#}\alpha \cdot ({}^{j}\!/\!{}_{i}\hat{f}^{\#}f^{\#}, {}^{j}\!/\!{}_{i}\hat{f}^{\#}g^{\#})X \cdot ({}^{j}\!/\!{}_{i})\hat{f}^{\#}\beta \\
& \overset{\text{see above}}{=} & 1_{V_{j/i\hat{f}^{\#}}} \\
& = & 1_{V^{(\hat{f})}_{j/i}}.
\end{array}
$$

Thus $\alpha' \cdot \beta' = 1_{V^{(\hat{f})}}$. $\square$

7 The homotopy category $\mathrm{K}^{(n/m)}(\mathcal{A})$

Suppose given an additive category $\mathcal{A}$.

7.1 Definition of $\mathrm{K}^{(n/m)}(\mathcal{A})$

Suppose given $n, m, \tilde{n}, \tilde{m} \in \mathbb{Z}_{\geqslant 0}$.

Definition 147. We define the homotopy category $\mathrm{K}^{(n/m)}(\mathcal{A})$ by

$$\mathrm{K}^{(n/m)}(\mathcal{A}) \ := \ \mathrm{C}^{(n)}(\mathcal{A}) / \, \mathrm{C}^{(n,m)}(\mathcal{A})$$

with

$$\mathrm{C}^{(n,m)}(\mathcal{A}) \ := \sum_{f \in (\bar{\Delta}_n, \bar{\Delta}_m)_{\text{q.p.}}} \overline{\mathrm{Im}}(\mathrm{C}^{(f)}(\mathcal{A})) \subseteq \mathrm{C}^{(n)}(\mathcal{A}).$$

Cf. Lemma 11, Definition 17.

For a morphism φ in $\mathrm{C}^{(n)}(\mathcal{A})$ we denote its equivalence class in $\mathrm{K}^{(n/m)}(\mathcal{A})$ by $[\varphi]_{n,m}$.

Lemma 148. *Suppose given* ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#,\circ}$. *Then*

$$\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})) = \overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{(t/s)^{+1}})}(\mathcal{A})).$$

If ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$, *then every object in* $\mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})))$ *is a zero object. In particular,*

$$\mathrm{C}^{(n,1)}(\mathcal{A}) = \sum_{t/s \in B_n^\circ} \overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})).$$

Proof. Suppose given ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#,\circ}$. Define quasiperiodic monotone maps

$$\begin{aligned} f_+ \colon \bar{\Delta}_1 &\to \bar{\Delta}_1 \\ i &\mapsto i+1 \end{aligned}$$

and

$$\begin{aligned} f_- \colon \bar{\Delta}_1 &\to \bar{\Delta}_1 \\ i &\mapsto i-1. \end{aligned}$$

We have

$$({}^{t}\!/_{s}) f_+^{\#} = {}^{tf_+}\!/_{sf_+} = {}^{t+1}\!/_{s+1} = {}^{s^{+1}}\!/_{t} = ({}^{t}\!/_{s})^{+1}$$

and

$$({}^{t}\!/_{s}) f_-^{\#} = {}^{t-1}\!/_{s-1} = {}^{s}\!/_{t^{-1}} = ({}^{t}\!/_{s})^{-1}$$

for every ${}^{t}\!/_{s} \in \bar{\Delta}_1^{\#,\circ}$.

We have $\hat{f}^{t/s} = \hat{f}^{(t/s)^{+1}} \cdot f_+$ and $\hat{f}^{(t/s)^{+1}} = \hat{f}^{t/s} \cdot f_-$; cf. Lemma 109 (1).

We show that $\mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A}))) \subseteq \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{(t/s)^{+1}})}(\mathcal{A})))$:

Suppose given $X \in \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})))$, that is $X = A^{(\hat{f}^{t/s})}$ for some $A \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A})$. Then $X = A^{(\hat{f}^{t/s})} = A^{(\hat{f}^{(t/s)^{+1}} \cdot f_+)} = \big(A^{(f_+)}\big)^{(\hat{f}^{(t/s)^{+1}})} \in \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{(t/s)^{+1}})}(\mathcal{A})))$.

We show that $\mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A}))) \supseteq \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{(t/s)^{+1}})}(\mathcal{A})))$:

Suppose given $X \in \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{(t/s)^{+1}})}(\mathcal{A})))$, that is $X = A^{(\hat{f}^{(t/s)^{+1}})}$ for some $A \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$. Then $X = A^{(\hat{f}^{(t/s)^{+1}})} = A^{(\hat{f}^{t/s} f_-)} = \big(A^{(f_-)}\big)^{(\hat{f}^{t/s})} \in \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})))$.

Suppose given $t/s \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$. Then $t/s = s/s$ or $t/s = s^{+1}/s = (s/s)^{+1}$.

Case $t/s = s/s$: Suppose given $X \in \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{s/s})}(\mathcal{A})))$, that is $X = A^{(\hat{f}^{s/s})}$ for some $A \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$. Suppose given $j/i \in \bar{\Delta}_n^{\#}$. We have $X_{j/i} = A_{(j/i)\hat{f}^{s/s,\#}} = A_{1^{+k}/1^{+l}} \cong 0_{\mathcal{A}}$ for some $k, l \in \mathbb{Z}$.

Case $t/s = s^{+1}/s$: Suppose given $X \in \mathrm{Ob}(\overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{s^{+1}/s})}))$, that is $X = A^{(\hat{f}^{s^{+1}/s})}$ for some $A \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$. Suppose given $j/i \in \bar{\Delta}_n^{\#}$. We have $X_{j/i} = A_{(j/i)\hat{f}^{s^{+1}/s,\#}} = A_{0^{+k}/0^{+l}} \cong 0_{\mathcal{A}}$ for some $k, l \in \mathbb{Z}$.

Suppose given $\hat{f} \colon \bar{\Delta}_n^{\#} \to \bar{\Delta}_1^{\#}$ quasiperiodic monotone. Let $f \colon \bar{\Delta}_1^{\#} \to \bar{\Delta}_n^{\#}$ be the left adjoint of $\hat{f}$; cf. Lemma 86. Then $f = f^{1f/0f}$. The right adjoint of f is $\hat{f}^{1f/0f}$. As adjoints are unique, we get $\hat{f} = \hat{f}^{1f/0f}$.

As for every quasiperiodic monotone map f, the image $\overline{\mathrm{Im}}(\mathrm{C}^{(f)}(\mathcal{A}))$ is closed under direct sums, we get

$$\mathrm{C}^{(n,1)}(\mathcal{A}) = \sum_{t/s \in \bar{\Delta}_n^{\#,\circ}} \overline{\mathrm{Im}}(\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})).$$

□

Remark 149. To unambiguously define a quasiperiodic monotone map $f \colon \bar{\Delta}_n \to \bar{\Delta}_m$ it is sufficient to know its values on $[0, n]_{\bar{\Delta}_n}$:

Suppose given

$$a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant a_0 + m + 1$$

in $\bar{\Delta}_m$, then there exists a unique quasiperiodic monotone map $f \colon \bar{\Delta}_n \to \bar{\Delta}_m$ such that $if = a_i$ for $i \in [0, n]$.

The condition $0f = 0$ means $[0, n]_{\bar{\Delta}_n} f \subseteq [0, 0^{+1}]_{\bar{\Delta}_m}$.

There are $\binom{m+n+1}{n}$ options to choose the remaining values:

The number of monotone maps $[1, n]_{\bar{\Delta}_n} \to [0, 0^{+1}]_{\bar{\Delta}_m}$ is the same as the number of strictly monotone maps $[1, n]_{\bar{\Delta}_n} \to [0, 0^{+1} + (n-1)]_{\bar{\Delta}_m}$. A bijection is given by $g \mapsto \hat{g}$ for a monotone map $g \colon [1, n]_{\bar{\Delta}_n} \to [0, 0^{+1}]_{\bar{\Delta}_m}$ with $k\hat{g} := kg + k - 1$. The strictly monotone maps correspond to the n-element subsets of $[0, 0^{+1} + (n-1)]_{\bar{\Delta}_m}$ as a map $\hat{g}$ gives the subset $\mathrm{Im}(\hat{g})$ with n elements and any n-element subset defines exactly one strictly monotone map by ordering it. The amount of n-element subsets of $[0, 0^{+1} + (n-1)]_{\bar{\Delta}_m}$ is $\binom{m+n+1}{n}$.

This means there exist $\binom{m+n+1}{n}$ quasiperiodic monotone maps $\bar{\Delta}_n \to \bar{\Delta}_m$ with $0f = 0$.

Remark 150. As we are only interested in the full images $\overline{\mathrm{Im}}(\mathrm{C}^{(n,m)}(\mathcal{A}))$, it is enough to consider quasiperiodic monotone maps $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ with $0f = 0$. Any other quasiperiodic monotone map $g\colon \bar{\Delta}_n \to \bar{\Delta}_m$ where $0g = 0 + z$ for some $z \in \mathbb{Z}$ can be written as $g = f s_z$ where

$$\begin{aligned} s_z\colon \bar{\Delta}_m &\longrightarrow \bar{\Delta}_m \\ t &\longmapsto t + z \end{aligned}$$

is an isomorphism. In particular, $f = g s_{-z}$.

Then $\mathrm{C}^{(g)}(\mathcal{A}) = \mathrm{C}^{(s_z)}(\mathcal{A})\,\mathrm{C}^{(f)}(\mathcal{A})$ where $\mathrm{C}^{(s_z)}(\mathcal{A})\colon \mathrm{C}^{(m)}(\mathcal{A}) \to \mathrm{C}^{(m)}(\mathcal{A})$ is an isomorphism. Therefore $\overline{\mathrm{Im}}(\mathrm{C}^{(g)}) = \overline{\mathrm{Im}}(\mathrm{C}^{(f)})$.

The images $\overline{\mathrm{Im}}(\mathrm{C}^{(f)}(\mathcal{A})$ are closed under direct sums.

This means for $\mathrm{C}^{(n,m)}(\mathcal{A})$ we have the finite sum

$$\mathrm{C}^{(n,m)}(\mathcal{A}) = \sum_{\substack{f \in (\bar{\Delta}_n, \bar{\Delta}_m)_{\text{q.p.}} \\ 0f = 0}} \overline{\mathrm{Im}}(\mathrm{C}^{(f)}(\mathcal{A})).$$

Lemma 151. *Suppose given $f \in (\bar{\Delta}_{\tilde{n}}, \bar{\Delta}_n)_{\text{q.p.}}$. Suppose given $0 \leqslant m \leqslant \tilde{m}$. Then we have a unique functor $\mathrm{K}^{(f,m,\tilde{m})}(\mathcal{A})$ that makes the following diagram commutative.*

$$\begin{tikzcd} \mathrm{C}^{(n)}(\mathcal{A}) \arrow[r, "\mathrm{C}^{(f)}(\mathcal{A})"] \arrow[d, "R_{n,m}"] & \mathrm{C}^{(\tilde{n})}(\mathcal{A}) \arrow[d, "R_{\tilde{n},\tilde{m}}"] \\ \mathrm{K}^{(n/m)}(\mathcal{A}) \arrow[r, dotted, "\mathrm{K}^{(f,m,\tilde{m})}(\mathcal{A})"] & \mathrm{K}^{(\tilde{n}/\tilde{m})}(\mathcal{A}) \end{tikzcd}$$

This functor is given by

$$\begin{aligned} \mathrm{K}^{(f,m,\tilde{m})}(\mathcal{A})\colon \mathrm{K}^{(n/m)}(\mathcal{A}) &\longrightarrow \mathrm{K}^{(\tilde{n}/\tilde{m})}(\mathcal{A}) \\ (X \xrightarrow{[\varphi]_{n,m}} Y) &\longmapsto (X^{(f)} \xrightarrow{[\varphi^{(f)}]_{\tilde{n},\tilde{m}}} Y^{(f)}). \end{aligned}$$

We abbreviate $\mathrm{K}^{(f,1,1)} =: \mathrm{K}^{(f)}$.

Proof. Suppose given $g \in (\bar{\Delta}_n, \bar{\Delta}_m)_{\text{q.p.}}$. We have to show that $\overline{\mathrm{Im}}(\mathrm{C}^{(g)})\,\mathrm{C}^{(f)} \subseteq \mathrm{C}^{(\tilde{n},\tilde{m})}$.

Choose $h_1 \in (\bar{\Delta}_m, \bar{\Delta}_{\tilde{m}})_{\text{q.p.}}$ and $h_2 \in (\bar{\Delta}_{\tilde{m}}, \bar{\Delta}_m)_{\text{q.p.}}$ with $h_1 h_2 = 1_{\bar{\Delta}_m}$.

Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{(m)})$. Then

$$\begin{aligned}\left(X\,\mathrm{C}^{(g)}\right)\mathrm{C}^{(f)} &= \mathrm{C}^{(fg)} \\ &= X\,\mathrm{C}^{(fgh_1h_2)} \\ &= \left(X\,\mathrm{C}^{(h_2)}\right)\mathrm{C}^{(fgh_1)} \\ &\in \mathrm{Ob}\left(\mathrm{C}^{(\tilde{m},\tilde{n})}\right)\end{aligned}$$

since $fgh_1 \in (\bar{\Delta}_{\tilde{n}}, \bar{\Delta}_{\tilde{m}})_{\text{q.p.}}$. Therefore $\left(\mathrm{Ob}(\overline{\mathrm{Im}}\mathrm{C}^{(g)})\right)\mathrm{C}^{(f)} \subseteq \mathrm{Ob}(\mathrm{C}^{(\tilde{n},\tilde{m})})$.

As $\mathrm{C}^{(\tilde{n},\tilde{m})}$ is closed under direct sums and $\mathrm{C}^{(f)}$ is additive, we get

$$\mathrm{Ob}(\mathrm{C}^{(n,m)})\,\mathrm{C}^{(f)} \subseteq \mathrm{Ob}(\mathrm{C}^{(\tilde{n},\tilde{m})}).$$

By the universal property of the factor category, we get a unique additive functor

$$\begin{aligned}\mathrm{K}^{(f,m,\tilde{m})}(\mathcal{A})\colon\ \mathrm{K}^{(n/m)}(\mathcal{A}) &\longrightarrow \mathrm{K}^{(\tilde{n}/\tilde{m})}(\mathcal{A}) \\ (X \xrightarrow{[\varphi]_{n,m}} Y) &\longmapsto (X^{(f)} \xrightarrow{[\varphi^{(f)}]_{\tilde{n},\tilde{m}}} Y^{(f)}).\end{aligned}$$

□

Corollary 152. *Suppose given* $\bar{\Delta}_{\tilde{n}} \underset{g}{\overset{f}{\rightleftarrows}} \bar{\Delta}_n$ *quasiperiodic monotone with* $f \dashv g$. *Suppose given* $0 \leqslant m$. *Then*

$$\mathrm{K}^{(g,m,m)}(\mathcal{A}) \dashv \mathrm{K}^{(f,m,m)}(\mathcal{A}).$$

Proof. We know that $\mathrm{C}^{(g)}(\mathcal{A}) \dashv \mathrm{C}^{(f)}(\mathcal{A})$; cf. Lemma 143.

By Lemma 151, we get unique induced functors $\mathrm{K}^{(f,m,m)}(\mathcal{A})$ and $\mathrm{K}^{(g,m,m)}(\mathcal{A})$. By Lemma 65 we get $\mathrm{K}^{(g,m,m)}(\mathcal{A}) \dashv \mathrm{K}^{(f,m,m)}(\mathcal{A})$. □

Corollary 153. *Suppose given* $f\colon \bar{\Delta}_n \to \bar{\Delta}_{\tilde{n}}$ *quasiperiodic monotone. Then there exist quasiperiodic monotone maps* $g, h\colon \bar{\Delta}_{\tilde{n}} \to \bar{\Delta}_n$ *such that*

$$g \dashv f \dashv h$$

cf. Lemma 86.

For $m \geqslant 0$, *we have* $\mathrm{K}^{(f,m,m)}(\mathcal{A}), \mathrm{K}^{(g,m,m)}(\mathcal{A})\colon\ \mathrm{K}^{(n/m)}(\mathcal{A}) \to \mathrm{K}^{(\tilde{n}/m)}(\mathcal{A})$ *with*

$$\mathrm{K}^{(h,m,m)}(\mathcal{A}) \dashv \mathrm{K}^{(f,m,m)}(\mathcal{A}) \dashv \mathrm{K}^{(g,m,m)}(\mathcal{A})$$

cf. Corollary 152.

Lemma 154. *Suppose given an additive category $\mathcal{B}$ and an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$. Then*

$$\begin{aligned} \mathrm{C}^{(n)}(\mathcal{A}) &\xrightarrow{\mathrm{C}^{(n)}(F)} \mathrm{C}^{(n)}(\mathcal{B}) \\ (X \xrightarrow{u} Y) &\longmapsto (XF \xrightarrow{uF} YF) \end{aligned}$$

gives the induced additive functor

$$\mathrm{K}^{(n/m)}(\mathcal{A}) \xrightarrow{\mathrm{K}^{(n/m)}(F)} \mathrm{K}^{(n/m)}(\mathcal{B}).$$

Proof. Let $R_{\mathcal{A}}\colon \mathrm{C}^{(n)}(\mathcal{A}) \to \mathrm{K}^{(n/m)}(\mathcal{A})$ and $R_{\mathcal{B}}\colon \mathrm{C}^{(n)}(\mathcal{B}) \to \mathrm{K}^{(n/m)}(\mathcal{B})$ be the respective residue class functors.

By the universal property of the factor category, we obtain an additive induced functor as claimed if $\mathrm{C}^{(n)}(F)R_{\mathcal{B}}$ maps every object of $\mathrm{C}^{(n,m)}(\mathcal{A})$ to a zero object. It is sufficient to show this for every $X^{(f)} \in \mathrm{Ob}(\mathrm{C}^{(n,m)}(\mathcal{A}))$ with $X \in \mathrm{Ob}(\mathrm{C}^{(m)}(\mathcal{A}))$ and $f\colon \bar{\Delta}_n \to \bar{\Delta}_m$ a quasiperiodic monotone map.

In fact, we have

$$\begin{aligned} X^{(f)}\,\mathrm{C}^{(n)}(F) &= f^{\#}XF \\ &= (XF)\,\mathrm{C}^{(f)}(B) \in \mathrm{Ob}(\mathrm{C}^{(n,m)}(\mathcal{B})), \end{aligned}$$

which is a zero object in $\mathrm{K}^{(n/m)}(\mathcal{B})$. □

7.2 $\mathrm{K}^{(2/1)}(\mathcal{A})$

Suppose given an additive category $\mathcal{A}$.

The category $\mathrm{K}^{(2/1)}(\mathcal{A})$ is defined as

$$\mathrm{K}^{(2/1)}(\mathcal{A}) \;:=\; \mathrm{C}^{(2)}(\mathcal{A}) / \,\mathrm{C}^{(2,1)}(\mathcal{A})$$

with

$$\mathrm{C}^{(2,1)}(\mathcal{A}) \;:=\; \sum_{(\bar{\Delta}_2,\bar{\Delta}_1)_{\text{q.p.}}} \overline{\mathrm{Im}}(\mathrm{C}^{(f)}(\mathcal{A})) \overset{\text{R.150}}{=} \sum_{\substack{(\bar{\Delta}_2,\bar{\Delta}_1)_{\text{q.p.}} \\ 0f=0}} \overline{\mathrm{Im}}(\mathrm{C}^{(f)}(\mathcal{A}))$$

The quasiperiodic monotone maps $\bar{\Delta}_2 \to \bar{\Delta}_1$ that map 0 to 0 are given by

	f_1		f_2	f_3	
$0 \mapsto 0$	$0 \mapsto 0$	$0 \mapsto 0$	$0 \mapsto 0$	$0 \mapsto 0$	$0 \mapsto 0$
$1 \mapsto 0$	$1 \mapsto 0$	$1 \mapsto 0$	$1 \mapsto 1$	$1 \mapsto 1$	$1 \mapsto 0^{+1}$
$2 \mapsto 0$	$2 \mapsto 1$	$2 \mapsto 0^{+1}$	$2 \mapsto 1$	$2 \mapsto 0^{+1}$	$2 \mapsto 0^{+1}$

The maps with $\operatorname{Im}(f) \subseteq \{0^{+k} : \ k \in \mathbb{Z}\}$ can be ignored, as their corresponding functors map everything to zero.

For a 1-complex X

$$\begin{tikzcd}
 & & & {} & {} \\
 & & 0 \arrow[r] & X_{0^{+1}/1} \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & X_{0/1^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

this yields the following images:

$X^{(f_1)}$:

$$\begin{tikzcd}
 & & & {} & {} & {} \\
 & & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u, dotted] & 0 \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r] & 0 \arrow[r] \arrow[u] & X_{1/0} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & X_{0/1^{-1}} \arrow[r] \arrow[u] & X_{0/1^{-1}} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

$X^{(f_2)}$:

$$\begin{tikzcd}
 & & & {} & {} & {} \\
 & & 0 \arrow[r] & 0 \arrow[r] \arrow[u, dotted] & X_{0^{+1}/1} \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] & X_{1/0} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & X_{0/1^{-1}} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

$X^{(f_3)}$:

$$\begin{tikzcd}
 & & & {} & {} & {} \\
 & & 0 \arrow[r] & X_{0^{+1}/1} \arrow[r] \arrow[u, dotted] & X_{0^{+1}/1} \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & X_{1/0} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

Here the second position on the second row is position $1/0$.

Note that every choice of objects $X_{0^{+1}/1}, X_{1/0}, X_{0/0^{-1}}, \ldots$ in $\operatorname{Ob}(\mathcal{A})$ yields a 1-complex, as all the morphism in X are zero.

Thus $\mathrm{C}^{(2,1)}(\mathcal{A}) = \sum_{i=1}^{3} \overline{\mathrm{Im}}(\mathrm{C}^{(f_i)}(\mathcal{A})$ is the full subcategory of $\mathrm{C}^{(2)}(\mathcal{A})$ consisting, up to isomorphism, of 2-complexes of the form

$$
\begin{tikzcd}
 & & & {} & {} & {} \\
 & & 0 \arrow[r] & A_3 \oplus A_4 \arrow[r, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"] \arrow[u, dotted] & A_4 \oplus A_5 \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & 0 \arrow[r] & A_1 \oplus A_2 \arrow[r, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"] \arrow[u] & A_2 \oplus A_3 \arrow[r] \arrow[u, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"'] & 0 \arrow[u] & \\
0 \arrow[r] & A_{-1} \oplus A_0 \arrow[r, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"] \arrow[u] & A_0 \oplus A_1 \arrow[r] \arrow[u, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"'] & 0 \arrow[u] & & \\
{} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] & & &
\end{tikzcd}
$$

for $A_i \in \mathrm{Ob}(\mathcal{A})$.

By Lemma 136, we know that $\mathrm{C}(\mathcal{A})$ and $\mathrm{C}^{(2)}(\mathcal{A})$ are equivalent via $\mathrm{C}(\mathcal{A}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{C}^{(2)}(\mathcal{A})$.

Recall that F maps a complex A to the 2-complex X_A with $(X_A)_{j/i} = {}^{j}/_{i}\rho^{-1}A$ for ${}^{j}/_{i} \in \bar{\Delta}_2^{\#,\circ}$ and $(X_A)_{j/i} = 0_{\mathcal{A}}$ else. Recall that G maps a 2-complex X to the complex $\hat{X} := \rho X$.

For every complex $B \in \mathrm{Ob}(\mathrm{C}^{(\text{sp ac})}(\mathcal{A})) \subseteq \mathrm{Ob}(\mathrm{C}(\mathcal{A}))$, that is, B being isomorphic to a complex of the form

$$\ldots \longrightarrow A^{-1} \oplus A^0 \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} A^0 \oplus A^1 \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} A^1 \oplus A^2 \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} A^2 \oplus A^3 \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} A^3 \oplus A^4 \longrightarrow \ldots,$$

we have $BF \in \mathrm{Ob}(\mathrm{C}^{(2,1)}(\mathcal{A}))$.

For every 2-complex $X \in \mathrm{Ob}(\mathrm{C}^{(2,1)}(\mathcal{A}))$ we have $XG \in \mathrm{Ob}(\mathrm{C}^{(\text{sp ac})}(\mathcal{A}))$.

By Lemma 64, we get that $\mathrm{K}^{(2/1)}(\mathcal{A})$ is equivalent to $\mathrm{K}(\mathcal{A})$.

8 The pullback functor

Suppose given an abelian category $\mathcal{A}$ that has enough injectives. Suppose given $n \in \mathbb{Z}_{\geqslant 1}$.

8.1 The category $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$

Definition 155. We define the subcategory $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ of $\mathrm{C}^{(n)}(\mathcal{A})$ as the full subcategory whose objects are the n-complexes X fulfilling the conditions (i), (ii), (iii).

(i) $X_{t/s} \cong 0_{\mathcal{A}}$ for $s < 0$

(ii) For every $0 \leqslant s < t < s^{+1}$ in $\bar{\Delta}_n$, we have

$$\begin{tikzcd} X_{t/s+1} \arrow[r] & X_{t+1/s+1} \\ X_{t/s} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{t+1/s} \arrow[u] \end{tikzcd} ,$$

i.e. this quadrangle is a weak square.

(iii) For every $t/s \in \bar{\Delta}_n^{\#}$, the object $X_{t/s}$ is injective.

An n-complex $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ is called *injectively resolving.*

Remark 156. Suppose given a one-step n-complex $\check{X}$ fulfilling conditions (i),(ii) and (iii) of Definition 155. Then the n-complex X obtained from $\check{X}$ is in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ as the definition only contains conditions for objects and one-step morphisms; cf. Lemma 127.

Remark 157. For a 1-complex $X \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$ to be in $\mathrm{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$, it is sufficient to check conditions (i) and (iii) of Definition 155, as 1-complexes are acyclic; cf. Lemma 131.

Lemma 158. *The category* $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ *is additive.*

Proof. We show that the category $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ is a full additive subcategory of $\mathrm{C}^{(n)}(\mathcal{A})$, which is additive; cf. Lemma 133 (1).

The n-complex X with $X_{t/s} := 0_{\mathcal{A}}$ for every $t/s \in \bar{\Delta}_n^{\#}$ is a zero object in $\mathrm{C}^{(n)}(\mathcal{A})$ and lies in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$.

Given two n-complexes $X, Y \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$, we need to check the three conditions from Definition 155 for the n-complex $X \oplus Y \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ to be in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$. Condition (i) holds because the direct sum of two zero objects is a zero object. Condition (ii) holds by Lemma 41. Condition (iii) holds as the direct sum of two injective objects is injective again.

Therefore $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ is additive by Lemma 10. $\square$

Lemma 159. *Suppose given an n-complex $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$.*
Then for every ${}^{t}/_{s+k}, {}^{t+l}/_{s} \in \bar{\Delta}_n^{\#}$ with $k, l \geqslant 0$ and $s \geqslant 0$ the quadrangle

$$\begin{tikzcd}
X_{{}^{t}/_{s+k}} \arrow[r] & X_{{}^{t+l}/_{s+k}} \\
X_{{}^{t}/_{s}} \arrow[r] \arrow[u] & X_{{}^{t+l}/_{s}} \arrow[u]
\end{tikzcd}$$

is a weak square.

Proof. Suppose given ${}^{t}/_{s+k}, {}^{t+l}/_{s} \in \bar{\Delta}_n^{\#}$ with $k, l \geqslant 0$ and $s \geqslant 0$.
We show the case $k = 1$ by induction on l.
For $l = 0$, the quadrangle is a weak square by Lemma 39.
Suppose given $l \geqslant 1$. We consider the diagram

$$\begin{tikzcd}
X_{{}^{t}/_{s+1}} \arrow[r] & X_{{}^{t+l-1}/_{s+1}} \arrow[r] & X_{{}^{t+l}/_{s+1}} \\
X_{{}^{t}/_{s}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{{}^{t+l-1}/_{s}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{{}^{t+l}/_{s}} \arrow[u]
\end{tikzcd} .$$

Then we have

$$\begin{tikzcd}
X_{{}^{t}/_{s+1}} \arrow[r] & X_{{}^{t+l}/_{s+1}} \\
X_{{}^{t}/_{s}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{{}^{t+l}/_{s}} \arrow[u]
\end{tikzcd}$$

weak square by Lemma 40.
We show the general case by induction on k:
For $k = 0$, the quadrangle is a weak square by Lemma 39.
Suppose given $k \geqslant 1$. We consider the diagram

$$\begin{tikzcd}
X_{{}^{t}/_{s+k}} \arrow[r] & X_{{}^{t+l}/_{s+k}} \\
X_{{}^{t}/_{s+k-1}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{{}^{t+l}/_{s+k-1}} \arrow[u] \\
X_{{}^{t}/_{s}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{{}^{t+l}/_{s}} \arrow[u]
\end{tikzcd} .$$

Then we get

$$\begin{tikzcd} X_{t/s+k} \arrow[r] & X_{t+l/s+k} \\ X_{t/s} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{t+l/s} \arrow[u] \end{tikzcd} .$$

weak square by Lemma 40. □

Lemma 160. *Suppose given a sequence*

$$A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} A_n$$

in $\mathcal{A}$ with $A_1, A_2, \dots, A_n$ injective. Then there exists an n-complex $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ with $X_{i/0} = A_i$ for $i \in [1,n]$ and $x_{i/0,i+1/0} = a_i$ for $i \in [1, n-1]$.

Proof. We prove the existence by constructing a one-step n-complex.

We set $X_{i/j} = 0_{\mathcal{A}}$ for $j < 0$. We also set $X_{i/0} = A_i$ for $i \in [1,n]$ and $x_{i/0,i+1/0} = a_i$ for $i \in [1, n-1]$.

Claim: Let $\bar{\Delta}_n^{\#,k} := \{t/s \in \bar{\Delta}_n^{\#} | s \leqslant k\} \subset \bar{\Delta}_n^{\#}$ for $k \in \bar{\Delta}_n$. If all values are defined on $\bar{\Delta}_n^{\#,k}$, we can expand the definition to $\bar{\Delta}_n^{\#,k+1}$.

We set $X_{k+1/k+1} := 0_{\mathcal{A}}$ and consider the following diagram.

$$\begin{tikzcd} 0_{\mathcal{A}} \\ X_{k+1/k} \arrow[r, "x"] \arrow[u] & X_{k+2/k} \arrow[r, "x"] & \dots \arrow[r, "x"] & X_{k^{+1}-1/k} \arrow[r] & 0_{\mathcal{A}} \end{tikzcd}$$

For every $i \in [1,n]$ we complete

$$\begin{tikzcd} X_{k+i/k+1} \\ X_{k+i/k} \arrow[r, "x"] \arrow[u, "x"] & X_{k+i+1/k} \end{tikzcd}$$

to a weak square

$$\begin{tikzcd} X_{k+i/k+1} \arrow[r, "x"] & X_{k+i+1/k+1} \\ X_{k+i/k} \arrow[r, "x"] \arrow[u, "x"] \arrow[ur, phantom, "+"] & X_{k+i+1/k} \arrow[u, "x"] \end{tikzcd}$$

with $X_{k+i+1/k+1}$ injective; cf. Lemma 42. Then we set $X_{(k+1)^{+1}/k+1} := 0_{\mathcal{A}}$.

As the diagram on $\bar{\Delta}_n^{\#,0}$ is already defined, this yields a construction of a one-step n-complex fulfilling the conditions of Definition 155. Thus it defines the wanted n-complex; cf. Remark 156. $\square$

Lemma 161. *Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ with $({}^{s'}\!/_0, {}^{s}\!/_0)X$ split for $0 < s' < s < 0^{+1}$. Then X is split.*

Proof. For ${}^{s}\!/_t \in \bar{\Delta}_n^{\#}$ let $M_{{}^{s}\!/_t} = \{({}^{s'}\!/_{t'}, {}^{s}\!/_t) : {}^{s'}\!/_{t'} \in \bar{\Delta}_n^{\#},\ {}^{s'}\!/_{t'} < {}^{s}\!/_t\} \subseteq \mathrm{Mor}(\bar{\Delta}_n^{\#})$ the set of all morphisms in $\bar{\Delta}_n^{\#}$ with target ${}^{s}\!/_t$. We show that for every ${}^{s}\!/_t \in \bar{\Delta}_n^{\#}$ all morphisms in $M_{{}^{s}\!/_t}X$ are split.

For $s \in \{t, t^{+1}\}$ all morphisms in $M_{{}^{s}\!/_t}X$ are zero and thus split.

For $t \leqslant 0$ all morphisms in $M_{{}^{s}\!/_t}X$ for $s \in \bar{\Delta}_n$ are split by assumption or zero and thus split.

Let ${}^{s}\!/_t \in \bar{\Delta}_n^{\#,\circ}$ and $t > 0$. Suppose that for ${}^{u}\!/_v < {}^{s}\!/_t$ all morphisms in $M_{{}^{u}\!/_v}X$ are split. Suppose given $({}^{s'}\!/_{t'}, {}^{s}\!/_t) \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$. We show that $({}^{s'}\!/_{t'}, {}^{s}\!/_t)X$ is split:

If $s' \leqslant t$ or $s \geqslant (t')^{+1}$, the morphism is zero and thus split.

Suppose that $t < s'$ and $s < (t')^{+1}$.

Case $t = t'$:

We get the following weak square in X with $X_{{}^{t}\!/_t} \cong 0_{\mathcal{A}}$ a zero object.

$$\begin{tikzcd}
X_{{}^{t}\!/_t} \arrow[r, tail, "0"] \arrow[dr, phantom, "+"] & X_{{}^{s}\!/_t} \\
X_{{}^{t}\!/_{t-1}} \arrow[r, tail] \arrow[u, two heads, "0"] & X_{{}^{s}\!/_{t-1}} \arrow[u]
\end{tikzcd}$$

By Lemma 58, $({}^{s}\!/_{t-1}, {}^{s}\!/_t)X$ is split. Applying Lemma 58 to

$$\begin{tikzcd}
X_{{}^{s'}\!/_t} \arrow[r] \arrow[dr, phantom, "+"] & X_{{}^{s}\!/_t} \\
X_{{}^{s'}\!/_{t-1}} \arrow[r, tail] \arrow[u, two heads] & X_{{}^{s}\!/_{t-1}} \arrow[u, two heads]
\end{tikzcd}$$

yields that $({}^{s'}\!/_t, {}^{s}\!/_t)X$ is split.

Case $s = s'$:

Applying Lemma 58 to

$$\begin{tikzcd}
X_{{}^{t}\!/_t} \arrow[r, tail, "0"] \arrow[dr, phantom, "+"] & X_{{}^{s}\!/_t} \\
X_{{}^{t}\!/_{t'}} \arrow[r, tail] \arrow[u, two heads, "0"] & X_{{}^{s}\!/_{t'}} \arrow[u]
\end{tikzcd}$$

yields that $({}^{s}\!/\!{}_{t'}, {}^{s}\!/\!{}_{t})X$ is split.

Case $s' < s$, $t' < t$:

We get

$$\begin{tikzcd} X_{t/t} \arrow[r, tail, "0"] & X_{s/t} \\ X_{t/t'} \arrow[u, tail, "0"] \arrow[r, tail] \arrow[ur, phantom, "+"] & X_{s/t'} \arrow[u] \end{tikzcd}$$

and by Lemma 58, $({}^{s}\!/\!{}_{t'}, {}^{s}\!/\!{}_{t})X$ is split. Applying Lemma 58 to

$$\begin{tikzcd} X_{s'/t} \arrow[r] & X_{s/t} \\ X_{s'/t'} \arrow[u, tail] \arrow[r, tail] \arrow[ur, phantom, "+"] & X_{s/t'} \arrow[u, tail] \end{tikzcd}$$

yields that $({}^{s'}\!/\!{}_{t'}, {}^{s}\!/\!{}_{t})X$ is split. □

Corollary 162. *Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$. If $({}^{s'}\!/\!{}_{0}, {}^{s}\!/\!{}_{0})X$ is monic for every $0 \leqslant s' < s < 0^{+1}$, then X is split.*

Proof. As $X_{j/i}$ is injective for every ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#}$, the morphism $({}^{s'}\!/\!{}_{0}, {}^{s}\!/\!{}_{0})X$ is split. Therefore X is split by Lemma 161. □

Lemma 163. *Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$. Then $X^{(\hat{f}^{t/s})} \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ for every ${}^{t}\!/\!{}_{s} \in B_n^{\circ}$.*

Proof. We check the three conditions of Definition 155.

Ad (i): $X^{(\hat{f}^{t/s})}_{j/i} = 0_{\mathcal{A}}$ for every ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#}$ with $i < 0$:

Suppose given ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#}$ with $i < 0$. That is $i \leqslant n^{-1}$. Note that $i \leqslant s - 1$. We have $X^{(\hat{f}^{t/s})}_{j/i} = X_{(j/i)\hat{f}^{t/s,\#}} = X_{j\hat{f}^{t/s}/i\hat{f}^{t/s}}$ with $i\hat{f}^{t/s} \leqslant (s-1)\hat{f}^{t/s} = 1^{-1} < 0$; cf. Definition 101. As $X \in \mathrm{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$, we have $X_{j'/i'} \cong 0_{\mathcal{A}}$ for ${}^{j'}\!/\!{}_{i'} \in \bar{\Delta}_1^{\#}$ with $i' < 0$. Therefore $X^{(\hat{f}^{t/s})}_{j/i} \cong 0_{\mathcal{A}}$.

Ad (ii): For every ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#}$ with $0 \leqslant i < j < i^{+1}$ the quadrangle

$$\begin{tikzcd} X^{(\hat{f}^{t/s})}_{j+1/i} \arrow[r] & X^{(\hat{f}^{t/s})}_{j+1/i+1} \\ X^{(\hat{f}^{t/s})}_{j/i} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & X^{(\hat{f}^{t/s})}_{j+1/i} \arrow[u] \end{tikzcd}$$

is a weak square:

The quadrangle

$$\begin{array}{ccc} X_{(j+1)\hat{f}^{t/s}/i\hat{f}^{t/s}} & \longrightarrow & X_{(j+1)\hat{f}^{t/s}/(i+1)\hat{f}^{t/s}} \\ \uparrow & + & \uparrow \\ X_{j\hat{f}^{t/s}/i\hat{f}^{t/s}} & \longrightarrow & X_{(j+1)\hat{f}^{t/s}/i\hat{f}^{t/s}} \end{array}$$

is a quadrangle in X and X is acyclic; cf. Lemma 131.

(iii): $X^{(\hat{f}^{t/s})}_{j/i}$ is injective for $j/i \in \bar{\Delta}^{\#}_n$:

Suppose given $j/i \in \bar{\Delta}^{\#}_n$. Then $X^{(\hat{f}^{t/s})}_{j/i} = X_{j\hat{f}^{t/s}/i\hat{f}^{t/s}}$ is injective. □

Lemma 164. *Let $X, Y, Z \in \mathrm{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ with $(X, (\pi^1, \pi^2), (\iota^1, \iota^2))$ a direct sum of (Y, Z). Then $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ if and only if $Y, Z \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$.*

Proof. $\Leftarrow$: The category $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ is additive; cf. Lemma 158.

$\Rightarrow$: Suppose that $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$. We show that $Y \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$.

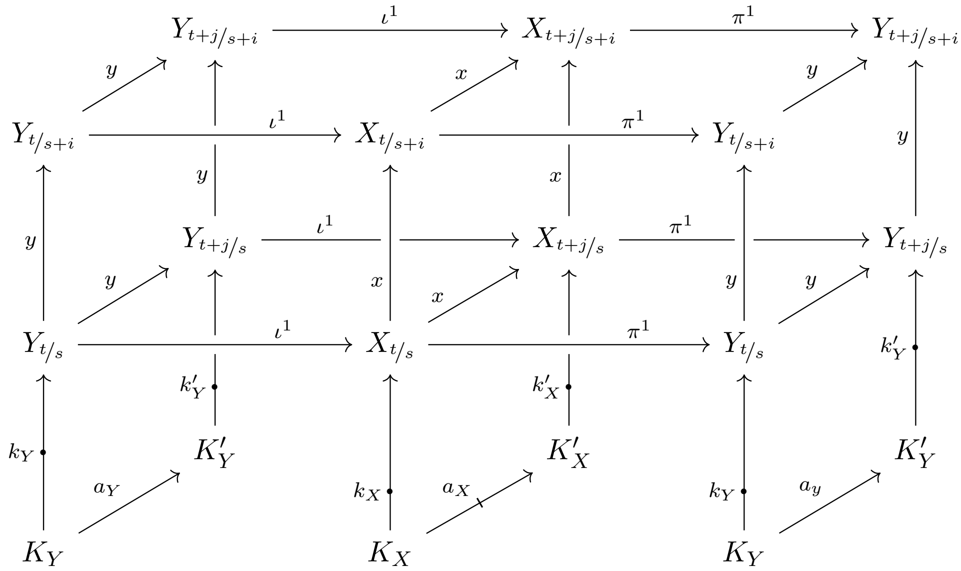
We have $X_{t/s} \cong Y_{t/s} \oplus Z_{t/s}$ in $\mathcal{A}$ for $t/s \in \bar{\Delta}^{\#}_n$. Therefore $Y_{t/s}$ is injective. For $t/s \in \bar{\Delta}^{\#}_n$ with $s < 0$ we have $Y_{t/s} \oplus Z_{t/s} \cong 0_{\mathcal{A}}$. Therefore $Y_{t/s} \cong 0_{\mathcal{A}}$.

Suppose given a quadrangle

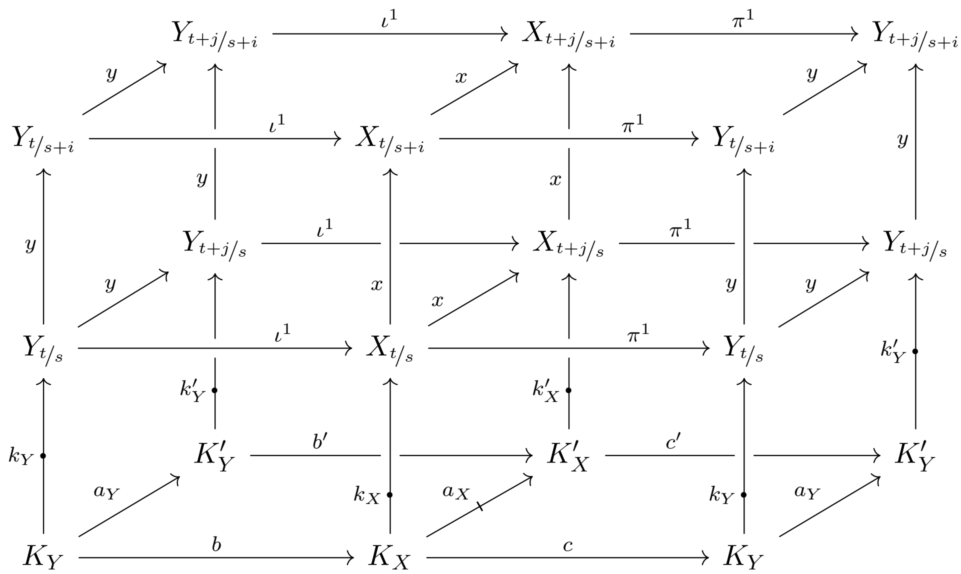
$$\begin{array}{ccc} Y_{t/s+i} & \longrightarrow & Y_{t+j/s+i} \\ \uparrow & & \uparrow \\ Y_{t/s} & \longrightarrow & Y_{t+j/s} \end{array}$$

in Y. We show that this quadrangle is a weak square:

We add kernels and the induced morphisms to the diagram; cf. Remark 28. By Lemma 37, $K_X \xrightarrow{a_X} K'_X$ is an epimorphism.



Because of $k'_Y \iota^1_{t+j/s} x_{t+j/s,t+j/s+i} = k'_Y y_{t+j/s,t+j/s+i} \iota^1_{t+j/s+i} = 0$, there is exactly one morphism $K'_Y \xrightarrow{b'} K'_X$ with $b'k'_X = k'_Y \iota^1_{t+j/s}$. Similarly, there is exactly one morphism $K_Y \xrightarrow{b} K_X$ with $bk_X = k_Y \iota^1_{t/s}$, exactly one morphism $K_X \xrightarrow{c} K_Y$ with $ck_Y = k_X \pi^1_{t/s}$ and exactly one morphism $K'_X \xrightarrow{c'} K'_Y$ with $c'k'_Y = k'_X \pi^1_{t+j/s}$. Because k'_X and k'_Y are monomorphisms, it follows that $a_Y b' = b a_X$ and $a_X c' = c a_Y$ and the resulting diagram is commutative.



As there is also only one morphism $K'_Y \xrightarrow{d'} K'_Y$ with $d'k'_Y = k'_Y$, it follows that $d' = 1_{K'_Y} = b'c'$

and thus c' is epic. Then $a_X c' = c a_Y$ is epic and thus a_Y is epic too; cf. Remark 1.

Dually it follows that the induced morphism between the cokernels is monic.

Thus the quadrangle under consideration is a weak square. $\square$

Lemma 165. *Suppose given* $A \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$ *and* ${}^{t}\!/\!{}_{s} \in B_n^\circ$.

If $(A)\,\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A}) \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$, *then* $A \in \mathrm{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$.

Proof. As $\hat{f}^{t/s}$ is surjective, every object $A_{j/i}$ for $j/i \in \bar{\Delta}_1^{\#}$ can be found in the n-complex $(A)\,\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A}) \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ and is therefore injective.

Suppose given $j/i \in \bar{\Delta}_1^{\#}$ with $i < 0$. We show that $A_{j/i} \cong 0_{\mathcal{A}}$.

As $\hat{f}^{t/s}$ is surjective, we have $(\hat{f}^{t/s})^{-1}(i) \neq \emptyset$.

We show that there is an element $i' \in (\hat{f}^{t/s})^{-1}(i)$ with $i' < 0$. Then $i'\hat{f}^{t/s} = i$ and $A_{j/i} = X_{j'/i'} \cong 0_{\mathcal{A}}$ for some $j' \in \bar{\Delta}_n^{\#}$.

If $i = 1^{-1}$, then $(\hat{f}^{t/s})^{-1}(1^{-1}) = [t^{-1}, s-1]$ with $t^{-1} \leqslant 1^{-1} < 0$; cf. Definition 101.

If $i < 1^{-1}$, then $i' \leqslant t^{-1} < 0$ for every $i' \in (\hat{f}^{t/s})^{-1}(i)$; cf. Definition 101.

Therefore $A \in \mathrm{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$; cf. Remark 157. $\square$

8.2 Definition of the pullback functor

Suppose that $n \geqslant 2$.

Definition 166. For every diagram $X' \xrightarrow{f'} Y' \xleftarrow{y} Y$ in $\mathcal{A}$ we choose a pullback

$$\begin{tikzcd} P_{y,f'} \arrow[r, "\pi_{1,y,f'}"] \arrow[d, "\pi_{2,y,f'}"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd},$$

called the *standard pullback.*

Definition 167. For $n \geqslant 2$, we define a functor $\mathrm{Pb}^{(n)}\colon \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \to (\dot{\Delta}_{n-1}, \mathcal{A})$ as follows:

First, we define the functor on objects.

Given an n-complex $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$, its zeroth row is

$$0 \longrightarrow X_{1/0} \xrightarrow{a_1} X_{2/0} \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} X_{n/0} \longrightarrow 0$$

with $a_i := ({}^{i}\!/\!{}_{0}, {}^{i+1}\!/\!{}_{0})X$ for $i \in [0, n]$.

We define

$$X\,\mathrm{Pb}^{(n)} \ := \ \hat{X} \ := \ (\ \hat{X}_1 \xrightarrow{\hat{x}_1} \hat{X}_2 \xrightarrow{\hat{x}_2} \ldots \xrightarrow{\hat{x}_{n-2}} \hat{X}_{n-1})$$

recursively by defining

$$\begin{aligned} \hat{X}_n \ &:= \ 0_{\mathcal{A}} \\ b_{X,n} \ &:= \ 0\colon 0_{\mathcal{A}} \longrightarrow X_{n/0} \end{aligned}$$

and

$$\begin{aligned} \hat{X}_i \ &:= \ P_{a_i, b_{X,i+1}} \\ \hat{x}_i \ &:= \ \pi_{2,a_i,b_{X,i+1}} \\ b_{X,i} \ &:= \ \pi_{1,a_i,b_{X,i+1}} \end{aligned}$$

for $i \in [1, n-1]$, using the standard pullbacks defined in Definition 166. As $b_{X,n}$ is a monomorphism and pullbacks preserve monomorphisms, cf. Lemma 43, $b_{X,i}$ is a monomorphism for every $i \in [1, n]$.

One step of the recursion:

$$\begin{tikzcd} X_{i/0} \arrow[r, "a_i"] & X_{i+1/0} \\ P_{a_i,b_{i+1}} \arrow[u, tail, "\pi_{1,a_i,b_{i+1}}"] \arrow[r, "\pi_{2,a_i,b_{i+1}}"'] \arrow[ur, phantom, "\llcorner"] & \hat{X}_{i+1} \arrow[u, tail, "b_{i+1}"'] \end{tikzcd}$$

For illustration we show a diagram in the case $n = 3$:

$$\begin{tikzcd} & {} & {} & {} & {} \\ X_{0/0} \arrow[r] & X_{1/0} \arrow[r, "a_1"] \arrow[u, dotted] & X_{2/0} \arrow[r, "a_2"] \arrow[u, dotted] & X_{3/0} \arrow[r] \arrow[u, dotted] & X_{0+1/0} \arrow[u, dotted] \\ & \hat{X}_1 \arrow[u, tail, "b_{X,1}"] \arrow[r, "\hat{x}_1"] \arrow[ur, phantom, "\llcorner"] & \hat{X}_2 \arrow[u, tail, "b_{X,2}"] \arrow[r] \arrow[ur, phantom, "\llcorner"] & 0 \arrow[u, tail, "b_{X,3}"] & \end{tikzcd}$$

So $X\,\mathrm{Pb}^{(n)} = (\hat{X}_1 \xrightarrow{\hat{x}_1} \hat{X}_2)$.

Suppose given two n-complexes X and Y with

$$X\,\mathrm{Pb}^{(n)} = \hat{X} = (\hat{X}_1 \xrightarrow{\hat{x}_1} \hat{X}_2 \xrightarrow{\hat{x}_2} \ldots \xrightarrow{\hat{x}_{n-1}} \hat{X}_{n-1}) \text{ and}$$
$$Y\,\mathrm{Pb}^{(n)} = \hat{Y} = (\hat{Y}_1 \xrightarrow{\hat{y}_1} \hat{Y}_2 \xrightarrow{\hat{y}_2} \ldots \xrightarrow{\hat{y}_{n-1}} \hat{Y}_{n-1})$$

and a morphism of n-complexes $X \xrightarrow{\sigma} Y$.

In the following situation

$$
\begin{tikzcd}[column sep=small]
 & & & & & & & & & & & & & \\
 & 0 \arrow[rr] & & Y_{1/0} \arrow[u, dotted] \arrow[rr, dotted] & & \dots \arrow[rr, dotted] & & Y_{n-2/0} \arrow[u, dotted] \arrow[rr] & & Y_{n-1/0} \arrow[u, dotted] \arrow[rr] & & Y_{n/0} \arrow[u, dotted] \arrow[rr, dashed] & & 0 \arrow[u, dotted] \\
0 \arrow[ur, "\sigma_{0/0}"] \arrow[rr] & & X_{1/0} \arrow[uu, dotted] \arrow[ur, "\sigma_{1/0}"] \arrow[rr, dotted] & & \dots \arrow[rr, dotted] & & X_{n-2/0} \arrow[uu, dotted] \arrow[ur, "\sigma_{n-2/0}"] \arrow[rr] & & X_{n-1/0} \arrow[uu, dotted] \arrow[ur, "\sigma_{n-1/0}"] \arrow[rr] & & X_{n/0} \arrow[uu, dotted] \arrow[ur, "\sigma_{n/0}"] \arrow[rr] & & 0 \arrow[uu, dotted] \arrow[ur] & \\
 & & & \hat{Y}_1 \arrow[uu, "b_{Y,1}"] \arrow[rr, dotted] & & \dots \arrow[rr, dotted] & & \hat{Y}_{n-2} \arrow[uu, "b_{Y,n-2}"] \arrow[rr] & & \hat{Y}_{n-1} \arrow[uu, "b_{Y,n-1}"] \arrow[rr] & & 0 \arrow[uu, "b_{Y,n}"] & & \\
 & & \hat{X}_1 \arrow[uu, "b_{X,1}"] \arrow[rr, dotted] & & \dots \arrow[rr, dotted] & & \hat{X}_{n-2} \arrow[uu, "b_{X,n-2}"] \arrow[rr] & & \hat{X}_{n-1} \arrow[uu, "b_{X,n-1}"] \arrow[rr] & & 0 \arrow[uu, "b_{X,n}"] \arrow[ur, "\hat{\sigma}_n"] & & &
\end{tikzcd}
$$

there exists exactly one $\hat{\sigma} = (\hat{\sigma}_0, \ldots, \hat{\sigma}_{n-1})\colon \hat{X} \to \hat{Y}$ with $\hat{\sigma}_i\colon \hat{X}_i \to \hat{Y}_i$ for $i \in [0, n-1]$ that makes the diagram commutative. This follows by iterated application of the universal property of the pullbacks

$$
\begin{tikzcd}
Y_{i/0} \arrow[r] & Y_{i+1/0} \\
\hat{Y}_i \arrow[u] \arrow[r] & \hat{Y}_{i+1} \arrow[u]
\end{tikzcd}
$$

for $i \in [0, n-1]$. The morphism $\hat{\sigma}_i$ is already characterised by $\hat{\sigma}_i b_{Y,i} = b_{X,i}\sigma_{i/0}$ since $b_{Y,i}$ is monic for $i \in [0, n]$. Therefore $\hat{\sigma}$ is unique. We define $\sigma \operatorname{Pb}^{(n)} = \hat{\sigma}$.

Suppose that $\sigma = 1_X\colon X \to X$. Then $1_{\hat{X}}$ fulfils $1_{\hat{X}_i} b_{X,i} = b_{X,i} 1_{X_{i/0}}$ and thus we have $1_X \operatorname{Pb}^{(n)} = 1_{X \operatorname{Pb}^{(n)}}$.

Suppose given n-complexes X, Y, Z with n-complex morphisms $X \xrightarrow{\sigma} Y \xrightarrow{\tau} Z$.

The morphism $(\sigma\tau) \operatorname{Pb}^{(n)}$ is characterised by $((\sigma\tau) \operatorname{Pb}^{(n)})_i b_{Z,i} = b_{X,i}\sigma_i\tau_i$ for $i \in [0, n-1]$.

$$
\begin{tikzcd}
X_{i/0} \arrow[r, "\sigma_i\tau_i"] & Z_{i/0} \\
\hat{X}_i \arrow[u, "b_{X,i}"] \arrow[r, "((\sigma\tau)\operatorname{Pb}^{(n)})_i"'] & \hat{Z}_i . \arrow[u, "b_{Z,i}"']
\end{tikzcd}
$$

By definition this is fulfilled by $\sigma \operatorname{Pb}^{(n)} \tau \operatorname{Pb}^{(n)}$.

$$
\begin{tikzcd}
X_i \arrow[r, "\sigma_i"] & Y_i \arrow[r, "\tau_i"] & Z_i \\
\hat{X} \arrow[u, "b_{X,i}"] \arrow[r, "\sigma \operatorname{Pb}_i^{(n)}"] & \hat{Y} \arrow[u, "b_{Y,i}"] \arrow[r, "\tau \operatorname{Pb}_i^{(n)}"] & \hat{Z} \arrow[u, "b_{Z,i}"]
\end{tikzcd}
$$

As $b_{Z,i}$ is a monomorphism, $(\sigma\tau) \operatorname{Pb}^{(n)} = \sigma \operatorname{Pb}^{(n)} \tau \operatorname{Pb}^{(n)}$.

If n is clear from context, we often write $\operatorname{Pb} := \operatorname{Pb}^{(n)}$.

8.3 The pullback functor and its kernel

Suppose that $n \geqslant 2$.

Lemma 168. *The pullback functor* Pb *is additive, full and dense.*

Proof. We show that Pb *is additive:*

Suppose given n-complex morphisms $\sigma, \tau \colon X \to Y$ between n-complexes X and Y. The image $(\sigma + \tau) \operatorname{Pb}$ of $\sigma + \tau$ is characterised by fulfilling

$$((\sigma + \tau) \operatorname{Pb})_i b_{Y,i} = b_{X,i}(\sigma_i + \tau_i)$$

for $i \in [0, n-1]$. By definition of $\sigma \operatorname{Pb}$ and $\tau \operatorname{Pb}$, we have

$$b_{X,i}(\sigma_i + \tau_i) = b_{X,i}\sigma_i + b_{X,i}\tau_i = (\sigma \operatorname{Pb})_i b_{Y,i} + (\tau \operatorname{Pb})_i b_{Y,i} = ((\sigma \operatorname{Pb})_i + (\tau \operatorname{Pb})_i) b_{Y,i}$$

for $i \in [0, n-1]$. Therefore, $(\sigma + \tau) \operatorname{Pb} = \sigma \operatorname{Pb} + \tau \operatorname{Pb}$.

We show that Pb *is dense:*

Suppose given a sequence $(X'_1 \xrightarrow{x'_1} X_2 \longrightarrow \dots \xrightarrow{x'_{n-2}} X'_{n-1}) \in \operatorname{Ob}((\dot{\Delta}_{n-1}, \mathcal{A}))$. We construct an n-complex as follows. Set $X'_n := 0_{\mathcal{A}}$ and $x'_{n-1} := 0$. For X'_1 we choose an injective object $X_{1/0} \in \operatorname{Ob}(\mathcal{A})$ and a monomorphism $X'_1 \xrightarrow{b_1} X_{1/0}$. Such a choice exists as we assumed $\mathcal{A}$ to have enough injective objects.

For every $i \in [0, n-1]$ we complete

$$
\begin{tikzcd}
X_{i/0} & \\
X'_i \arrow[u, "b_i"] \arrow[r, "x'_i"] & X'_{i+1}
\end{tikzcd}
$$

to a weak square with $X_{i+1/0}$ injective; cf. Lemma 42.

$$
\begin{tikzcd}
X_{i/0} \arrow[r, "x_{i/0,i+1/0}"] & X_{i+1/0} \\
X'_i \arrow[u, "b_i"] \arrow[r, "x'_i"'] \arrow[ur, phantom, "+"] & X'_{i+1} \arrow[u, "b_{i+1}"']
\end{tikzcd}
$$

As b_i is monic, b_{i+1} is monic, too; cf. Lemma 45. The quadrangle is a pullback; cf. Lemma 45.

$$
\begin{tikzcd}
X_{i/0} \arrow[r, "x_{i/0,i+1/0}"] & X_{i+1/0} \\
X'_i \arrow[u, "b_i"] \arrow[r, "x'_i"'] \arrow[ur, phantom, "\llcorner", very near start] & X'_{i+1} \arrow[u, "b_{i+1}"']
\end{tikzcd}
$$

Finally we set $X_{0+1/0} := 0_{\mathcal{A}}$. According to Lemma 160 the sequence

$$
X_{1/0} \longrightarrow X_{2/0} \longrightarrow \ldots \longrightarrow X_{n/0}
$$

can be completed to a n-complex $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})})$.

Applying the pullback functor to X yields

$$
\begin{tikzcd}
X_{1/0} \arrow[r] & X_{2/0} \arrow[r] & \ldots \arrow[r] & X_{n-1/0} \arrow[r] & X_{n/0} \\
\hat{X}_1 \arrow[u, "b'_1"] \arrow[r, "\hat{x}_1"] \arrow[ur, phantom, "\llcorner", very near start] & \hat{X}_2 \arrow[u, "b'_2"] \arrow[r] \arrow[ur, phantom, "\llcorner", very near start] & \ldots \arrow[r] & \hat{X}_{n-1} \arrow[u, "b'_{n-1}"] \arrow[r, "\hat{x}_{n-1}"] \arrow[ur, phantom, "\llcorner", very near start] & 0 \arrow[u]
\end{tikzcd}
$$

Applying the pullback property we inductively get morphisms $\hat{X}_{n-j} \xrightarrow{a_{n-j}} X'_{n-j}$ for $j \in [1, n-1]$ that make the following diagram commutative.

$$
\begin{tikzcd}
X_{1/0} \arrow[rr, "x"] & & X_{2/0} \arrow[rr] & & \ldots \arrow[rr] & & X_{n-1/0} \arrow[rr, "x"] & & X_{n/0} \\
& X'_1 \arrow[ul, "b_1"'] \arrow[rr, "x'_1"] & & X'_2 \arrow[ul, "b_2"'] \arrow[rr] & & \ldots \arrow[rr] & & X'_{n-1} \arrow[ul, "b_{n-1}"'] \arrow[rr, "x'_{n-1}"] & & 0 \arrow[ul] \\
\hat{X}_1 \arrow[uu, "b'_1"] \arrow[ur, "a_1"] \arrow[rr, "\hat{x}_1"] & & \hat{X}_2 \arrow[uu, "b'_2"] \arrow[ur, "a_2"] \arrow[rr] & & \ldots \arrow[rr] & & \hat{X}_{n-1} \arrow[uu, "b'_{n-1}"] \arrow[ur, "a_{n-1}"] \arrow[rr, "\hat{x}_{n-1}"] & & 0 \arrow[uu] \arrow[ur]
\end{tikzcd}
$$

By Lemma 32, a_i is an isomorphism for $i \in [1, n-1]$. Thus the image of X under the pullback functor is isomorphic to $(X'_1 \xrightarrow{x'_1} X_2 \longrightarrow \dots \xrightarrow{x'_{n-2}} X'_{n-1})$. Thus the pullback functor is dense.

We show that Pb *is full:*

Suppose given n-complexes $X, Y \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})})$ and a morphism $\hat{X} \xrightarrow{\hat{a}} \hat{Y}$ with $\hat{X} = X\,\mathrm{Pb}$ and $\hat{Y} = Y\,\mathrm{Pb}$.

$$\begin{tikzcd}
& Y_{1/0} \arrow[rr] & & Y_{2/0} \arrow[rr] & & \dots \arrow[rr] & & Y_{n-1/0} \arrow[rr] & & Y_{n/0} \\
X_{1/0} \arrow[rr] & & X_{2/0} \arrow[rr] & & \dots \arrow[rr] & & X_{n-1/0} \arrow[rr] & & X_{n/0} & \\
& \hat{Y}_1 \arrow[uu, "b'_1"] \arrow[rr] & & \hat{Y}_2 \arrow[uu, "b'_2"] \arrow[rr] & & \dots \arrow[rr] & & \hat{Y}_{n-1} \arrow[uu, "b'_{n-1}"] \arrow[rr] & & 0 \arrow[uu, "b'_n"] \\
\hat{X}_1 \arrow[uu, "b_1"] \arrow[ur, "\hat{a}_1"] \arrow[rr] & & \hat{X}_2 \arrow[uu, "b_2"] \arrow[ur, "\hat{a}_2"] \arrow[rr] & & \dots \arrow[rr] & & \hat{X}_{n-1} \arrow[uu, "b_{n-2}"] \arrow[ur, "\hat{a}_{n-1}"] \arrow[rr] & & 0 \arrow[uu, "b_n"] \arrow[ur] &
\end{tikzcd}$$

We construct a morphism $X \xrightarrow{a} Y$ by inductively defining its components $X_{s/t} \xrightarrow{a_{s/t}} Y_{s/t}$ for $s/t \in \bar{\Delta}_n^{\#}$.

For $s/t \in \bar{\Delta}_n^{\#}$ with $t < 0$ we set $a_{s/t} := 0$. We set $a_{0/0} := 0$.

As b_1 is a monomorphism and $Y_{1/0}$ is injective, there exists a morphism $X_{1/0} \xrightarrow{a_{1/0}} Y_{1/0}$ with $b_1 a_{1/0} = \hat{a}_1 b'_1$. Then for $i \in [1, n-1]$ there exists a morphism $X_{i+1/0} \xrightarrow{a_{i+1/0}} Y_{i+1/0}$ with $\hat{a}_{i+1} b'_{i+1} = b_{i+1} a_{i+1/0}$ and $x_{i/0,i+1/0} a_{i+1/0} = a_{i/0} y_{i/0,i+1/0}$; cf. Lemma 47. We set $a_{0+1/0} := 0$.

$$\begin{tikzcd}
& Y_{i/0} \arrow[rr, "y"] & & Y_{i+1/0} \\
X_{i/0} \arrow[ur, "a_{i/0}"] \arrow[rr, "x"] & & X_{i+1/0} \arrow[ur, "a_{i+1/0}"] & \\
& \hat{Y}_i \arrow[uu, "b'_i"] \arrow[rr, "\hat{y}_i"] & & \hat{Y}_{i+1} \arrow[uu, "b'_{i+1}"] \\
\hat{X}_i \arrow[uu, "b_i"] \arrow[ur, "\hat{a}_i"] \arrow[rr, "\hat{x}_i"] & & \hat{X}_{i+1} \arrow[uu, "b_{i+1}"] \arrow[ur, "\hat{a}_{i+1}"] &
\end{tikzcd}$$

For $j \geqslant 1$ we inductively set $a_{j/j} = 0$, we choose $a_{j+i/j}$ following Lemma 47 for $i \in [1, n-1]$ and we set $a_{j+1/_j} := 0$.

This defines an n-complex morphism $X \xrightarrow{a} Y$.

Since $a\,\mathrm{Pb}$ is characterised by $(a\,\mathrm{Pb})_i b'_i = b_i a_{i/0}$ for $i \in [1, n-1]$ we have $a\,\mathrm{Pb} = \hat{a}$.

□

Definition 169. We define the category $\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$ as

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \;:=\; \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) / \operatorname{Ker}(\mathrm{Pb}).$$

Lemma 170. *Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$. S.g. $k \in \mathbb{Z}_{\geqslant 1}$ such that $X_{l\rho} \cong 0_{\mathcal{A}}$ for all $l < k$. Suppose that $({}^{i}/_{0}, {}^{j}/_{0})X$ is monic for $0 \leqslant i \leqslant j \leqslant n$. Let ${}^{t}/_{s} = k\rho$. Let $V^1 \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$ be the 1-complex with $V^1_{1/0} = X_{t/s}$ and $V^1_{j/i} = 0_{\mathcal{A}}$ for ${}^{j}/_{i} \neq {}^{1}/_{0}$. Let $V := (V^1)\,\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})$. Then $V \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ and there exist $X' \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ and morphisms*

$$V \underset{c_V}{\overset{\alpha_V}{\rightleftarrows}} X \underset{\alpha_{X'}}{\overset{c_{X'}}{\rightleftarrows}} X'$$

such that $(X,\ (c_{X'}, c_V),\ (\alpha_{X'}, \alpha_V))$ is a direct sum of (X', V) in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$. Furthermore, $X'_{l\rho} \cong 0_{\mathcal{A}}$ for $l < k+1$ and $({}^{i}/_{0}, {}^{j}/_{0})X'$ is monic for $0 \leqslant i \leqslant j \leqslant n$.

Proof. We show that there is a split monomorphism $V \overset{\alpha_V}{\rightarrowtail} X$ in $\mathrm{C}^{(n)}(\mathcal{A})$.

The n-complex V looks as follows: We have $V_{j/i} = X_{t/s}$ for all ${}^{j}/_{i} \in \mathrm{u}_n^{t/s}$ with the identity morphism $1_{X_{t/s}}$ between them, and $V_{j/i} = 0_{\mathcal{A}}$ otherwise; cf. Definitions 101, 93.

Example of V for $n = 4$ and ${}^{t}/_{s} = {}^{3}/_{1}$:

$$\begin{tikzcd}
 & & & & & & & & \\
 & & & 0 \arrow[r] & 0 \arrow[r] \arrow[u, dotted] & 0 \arrow[r] \arrow[u, dotted] & 0 \arrow[r] \arrow[u, dotted] & 0 \arrow[r] \arrow[u, dotted] & 0 \arrow[u, dotted] \\
 & & 0 \arrow[r] & X_{3/1} \arrow[r, equal] \arrow[u] & X_{3/1} \arrow[r, equal] \arrow[u] & X_{3/1} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[r] & 0 \arrow[r] \arrow[u] & X_{3/1} \arrow[r, equal] \arrow[u, equal] & X_{3/1} \arrow[r, equal] \arrow[u, equal] & X_{3/1} \arrow[r] \arrow[u, equal] & 0 \arrow[u] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & & \\
\arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & \arrow[u, dotted] & & & &
\end{tikzcd}$$

We define a morphism of 1-complexes $V^1 \xrightarrow{\alpha^1} X^{(f^{t/s})}$ by

$$({}^{j}\!/\!{}_{i})\alpha^1 \;:=\; \begin{cases} 1_{X_{t/s}} & \text{for } {}^{j}\!/\!{}_{i} = {}^{1}\!/\!{}_{0} \\ 0 & \text{else.} \end{cases}$$

Then we get an n-complex morphism $V \xrightarrow{\alpha_V} X$ via $\mathrm{C}^{(\hat{f}^{t/s})} \dashv \mathrm{C}^{(f^{t/s})}$ and $\hat{\varepsilon} \;:=\; \hat{\varepsilon}^{t/s} \colon \hat{f}^{t/s,\#} \to f^{t/s,\#}$; cf. Corollary 145:

$$\alpha_V \;:=\; \alpha^1\, \mathrm{C}^{(\hat{f}^{t/s})} \cdot X\, \mathrm{C}^{(\hat{\varepsilon})}$$

The morphism looks as follows

$$\begin{aligned} ({}^{j}\!/\!{}_{i})\alpha_V &= ({}^{j}\!/\!{}_{i})(\alpha^1\, \mathrm{C}^{(\hat{f}^{t/s})}) \cdot ({}^{j}\!/\!{}_{i})X\, \mathrm{C}^{(\hat{\varepsilon})} \\ &= ({}^{j}\!/\!{}_{i})\hat{f}^{t/s,\#}\alpha^1 \cdot ({}^{j}\!/\!{}_{i})\hat{\varepsilon}^{\#}X \\ &= ({}^{j}\!/\!{}_{i})\hat{f}^{t/s,\#}\alpha^1 \cdot (({}^{j}\!/\!{}_{i})\hat{f}^{t/s,\#} f^{t/s,\#}, {}^{j}\!/\!{}_{i})X \\ &= ({}^{j}\!/\!{}_{i})\hat{f}^{t/s,\#}\alpha^1 \cdot (\lfloor {}^{j}\!/\!{}_{i} \rfloor_{t/s}, {}^{j}\!/\!{}_{i})X \end{aligned}$$

for ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n$, that is

$$({}^{j}\!/\!{}_{i})\alpha_V = \begin{cases} ({}^{t}\!/\!{}_{s}, {}^{j}\!/\!{}_{i})X & \text{for } {}^{j}\!/\!{}_{i} \in \mathrm{u}_n^{t/s} \\ 0 & \text{else.} \end{cases}$$

We also get a morphism $\bar{\alpha}^1 \colon V^1 \to X^{(f^{\lceil t/s \rceil})}$ by $\bar{\alpha}^1 \;:=\; (\alpha_V)\,\mathrm{C}^{(f^{\lceil t/s \rceil})} \colon V^{(f^{\lceil t/s \rceil})} \to X^{(f^{\lceil t/s \rceil})}$ and

$$V^{(\lceil t/s \rceil)} = \left((V^1)^{(\hat{f}^{t/s})}\right)^{(f^{\lceil t/s \rceil})} = (V^1)^{(f^{\lceil t/s \rceil}\hat{f}^{t/s})} \overset{\text{L.92}}{=} (V^1)^{(f^{\lceil t/s \rceil}\hat{f}^{\lfloor \lceil t/s \rceil \rfloor})} \overset{\text{R.106}}{=} V^1.$$

Then $\bar{\alpha}^1$ looks as follows

$$({}^{j}\!/\!{}_{i})\bar{\alpha}^1 = \begin{cases} ({}^{t}\!/\!{}_{s}, \lceil {}^{t}\!/\!{}_{s} \rceil)X & \text{for } {}^{j}\!/\!{}_{i} = {}^{1}\!/\!{}_{0} \\ 0 & \text{for } {}^{j}\!/\!{}_{i} \in \bar{\Delta}_1^{\#} \text{ with } {}^{j}\!/\!{}_{i} \neq {}^{1}\!/\!{}_{0}. \end{cases}$$

As $V^1_{j/i} = 0_{\mathcal{A}}$ for ${}^{j}\!/\!{}_{i} \neq {}^{1}\!/\!{}_{0}$, the morphism $({}^{j}\!/\!{}_{i})\bar{\alpha}^1$ is split monic for every ${}^{j}\!/\!{}_{i} \neq {}^{1}\!/\!{}_{0}$.

We show that $({}^{1}\!/\!{}_{0})\bar{\alpha}^1 = ({}^{t}\!/\!{}_{s}, \lceil {}^{t}\!/\!{}_{s} \rceil)X$ is split monic. As $X_{t/s}$ is injective, it suffices to show that $({}^{1}\!/\!{}_{0})\bar{\alpha}^1$ is monic; cf. Lemma 2.

The morphism can be written as

$$({}^{1}\!/\!{}_{0})\bar{\alpha}^1 = ({}^{t}\!/\!{}_{s}, \lceil {}^{t}\!/\!{}_{s} \rceil)X = ({}^{t}\!/\!{}_{s}, {}^{s+n}\!/\!{}_{t-1})X = ({}^{t}\!/\!{}_{s}, {}^{s+n}\!/\!{}_{s})X \cdot ({}^{s+n}\!/\!{}_{s}, {}^{s+n}\!/\!{}_{t-1})X.$$

We show that $({}^{t}\!/\!{}_{s}, {}^{s+n}\!/\!{}_{s})X$ is monic:

If $s = 0$, then the morphism is monic by assumption and we are done.

For $s > 0$ consider the following quadrangle in X. It is a weak square due to $s > 0$.

$$\begin{tikzcd}
X_{t/s} \arrow[r] & X_{s+n/s} \\
X_{t/s-1} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & X_{s+n/s-1} \arrow[u]
\end{tikzcd}$$

Then $X_{t/s-1} \cong 0_{\mathcal{A}}$, so by Lemma 46, $({t/s}, {s+n/s})X$ is monic.

We show that $({s+n/s}, {s+n/t-1})X$ *is monic:*

The following quadrangle is a weak square and $X_{t-1/s} \cong 0_{\mathcal{A}}$. Thus $({t/s}, {t/t-1})X$ is monic; Lemma 45.

$$\begin{tikzcd}
X_{t-1/t-1} \arrow[r] & X_{t/t-1} \\
X_{t-1/s} \arrow[u, "0"] \arrow[r] \arrow[ur, phantom, "+"] & X_{t/s} \arrow[u]
\end{tikzcd}$$

Then we get the following weak square

$$\begin{tikzcd}
X_{t/t-1} \arrow[r] & X_{s+n/t-1} \\
X_{t/s} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & X_{s+n/s} \arrow[u]
\end{tikzcd}$$

and by Lemma 45, $({s+n/s}, {s+n/t-1})X$ is monic.

Then $({t/s}, {s+n/s})X \cdot ({s+n/s}, {s+n/t-1})X = ({t/s}, \lceil {t/s} \rceil)X$ is monic.

Now we know that every morphism in

$$({j/i})\bar{\alpha}^1 = \begin{cases} ({t/s}, \lceil {t/s} \rceil)X & \text{for } {j/i} = {1/0} \\ 0 & {j/i} \in \bar{\Delta}_1^{\#} \text{ with } {j/i} \neq {1/0}. \end{cases}$$

is split monic. This means there exists a 1-complex morphism $c^1 \colon X^{(f^{\lceil t/s \rceil})} \to V^1$ with $\bar{\alpha}^1 c^1 = 1_{V^1}$.

Note that by Lemma 103 $f^{t/s} \dashv \hat{f}^{t/s} \dashv f^{\lceil t/s \rceil}$. Recall that

$$\alpha^1 \colon V^1 \to X^{(f^{t/s})},$$

that

$$\alpha_V = \alpha^1 \, \mathrm{C}^{(\hat{f}^{t/s})} \cdot X \, \mathrm{C}^{(\hat{\varepsilon})}$$

and that

$$\bar{\alpha}_1 = \alpha \, \mathrm{C}^{(f^{\lceil t/s \rceil})} .$$

Thus, according to Lemma 146 we get an n-complex morphism $c_V \colon X \to V$ with $\alpha_V c_V = 1_V$.

As $\mathrm{C}^{(n)}(\mathcal{A})$ is abelian, cf. Lemma 133 (2), we can choose a cokernel $X \xrightarrow{c_{X'}} X'$ of $V \xrightarrow{\alpha_V} X$ in $\mathrm{C}^{(n)}(\mathcal{A})$. By Lemma 51, we get $X' \xrightarrow{\alpha_{X'}} X$ such that $(X, (\alpha_V, \alpha_{X'}), (c_V, c_{X'}))$ is a direct sum of (X', V) in $\mathrm{C}^{(n)}(\mathcal{A})$. By Lemma 164, it is a direct sum in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$.

As $V_{l\rho} \cong 0_{\mathcal{A}}$ and $X_{l\rho} \cong 0_{\mathcal{A}}$ for $l < k$ as well as $(\alpha_V)_{k\rho} = 1_{X_{k\rho}}$, we get $X'_{l\rho} \cong 0_{\mathcal{A}}$ for $l < k+1$ for the cokernel X'.

We show that $({}^{i}\!/\!{}_{0}, {}^{j}\!/\!{}_{0})X'$ *is monic for* $0 \leqslant i \leqslant j \leqslant n$*:*

We have $(\alpha_{X'})_{i/0} \cdot ({}^{i}\!/\!{}_{0}, {}^{j}\!/\!{}_{0})X = ({}^{i}\!/\!{}_{0}, {}^{j}\!/\!{}_{0})X' \cdot (\alpha_{X'})_{j/0}$ with $(\alpha_{X'})_{i/0} \cdot ({}^{i}\!/\!{}_{0}, {}^{j}\!/\!{}_{0})X$ monic. Therefore $({}^{i}\!/\!{}_{0}, {}^{j}\!/\!{}_{0})X'$ is monic; cf. Remark 1. $\square$

Construction 171. Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ with $({}^{t}\!/\!{}_{0}, {}^{t'}\!/\!{}_{0})X$ monic for every $0 \leqslant t \leqslant t' \leqslant n$. Applying Lemma 170 to X with $k = 1$ gives an n-complex V^1, obtained from a 1-complex, and an n-complex R^1 with $R^1_{1/0} \cong 0_{\mathcal{A}}$ with $(X, (c_{V^1}, c_{R^1}), (\alpha_{V^1}, \alpha_{R^1}))$ direct sum of (V^1, R^1). As R^1 fulfils all necessary conditions, now with $k = 2$, we can apply Lemma 170 and get $(R^1, (c_{V^2}, c_{R^2}), (\alpha_{V^2}, \alpha_{R^2}))$ as a direct sum of (V^2, R^2). By repeatedly applying Lemma 170 to the remainders, we get sequences $(V^i)_{i \in \mathbb{Z}_{\geqslant 1}}$, $(R^i)_{i \in \mathbb{Z}_{\geqslant 1}}$ and direct sums $(R^i, (c_{V^{i+1}}, c_{R^{i+1}}), (\alpha_{V^{i+1}}, \alpha_{R^{i+1}}))$ of (V^{i+1}, R^{i+1}) for every $i \in \mathbb{Z}_{\geqslant 1}$.

For $i \geqslant 1$, we obtain $R^i \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ and $R^i_{l\rho} \cong 0_{\mathcal{A}}$ for $l \leqslant i$.

$$\begin{tikzcd}
X \arrow[r, shift left, "c_{R^1}"] \arrow[d, shift left, "c_{V^1}"] & R^1 \arrow[l, shift left, "\alpha_{R^1}"] \arrow[r, shift left, "c_{R^2}"] \arrow[d, shift left, "c_{V^2}"] & R^2 \arrow[l, shift left, "\alpha_{R^2}"] \arrow[r, shift left, "c_{R^3}"] \arrow[d, shift left, "c_{V^3}"] & R^3 \arrow[l, shift left, "\alpha_{R^3}"] \arrow[r, shift left, "c_{R^4}"] \arrow[d, shift left, "c_{V^4}"] & \cdots \arrow[l, shift left, "\alpha_{R^4}"] \\
V^1 \arrow[u, shift left, "\alpha_{V^1}"] & V^2 \arrow[u, shift left, "\alpha_{V^2}"] & V^3 \arrow[u, shift left, "\alpha_{V^3}"] & V^4 \arrow[u, shift left, "\alpha_{V^4}"] &
\end{tikzcd}$$

Define a sequence $(V_{i\rho})_{i \in \mathbb{Z}_{\geqslant 1}}$ of objects in $\mathcal{A}$ by $V_{i\rho} := V^i_{i\rho}$.

Recall $B^\circ_n := \{{}^{j}\!/\!{}_{i} \in \bar{\Delta}^{\#,\circ}_n : 0 \leqslant i < j \leqslant n\}$. For every ${}^{t}\!/\!{}_{s} \in B^\circ_n$ define a 1-complex $T^{t/s} \in \mathrm{Ob}(\mathrm{C}^{(1)}(\mathcal{A}))$ by

$$T^{t/s}_{j/i} := \begin{cases} V_{j/i f^{t/s,\#}} & \text{for } {}^{j}\!/\!{}_{i} \in \bar{\Delta}^{\#,\circ}_1 \text{ with } i \geqslant 0 \\ 0_{\mathcal{A}} & \text{else.} \end{cases}$$

Now define $\bar{S}^{t/s} := (T^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})}$ and

$$\bar{S} := \bigoplus_{t/s \in B^\circ_n} \bar{S}^{t/s}$$

The objects $T^{t/s}_{j/i}$ are injective for ${}^{j}\!/\!{}_{i} \in \bar{\Delta}^{\#}_n$ and $T^{t/s}_{j/i} = 0_{\mathcal{A}}$ for $i < 0$.

Therefore $T^{t/s} \in \mathrm{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$; cf. Remark 157.

By Lemma 163, $\bar{S}^{t/s} \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$.

Lemma 172. *Suppose given* $X \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ *and* $\bar{S}$ *as in Construction 171. Then* $X \cong \bar{S}$.

Proof. We use the notation of Construction 171. We show that X is a direct sum of $(\bar{S}^{t/s})_{t/s \in B_n^\circ}$ by constructing the according morphisms.

Recall that we have $f^{t/s} \dashv \hat{f}^{t/s} \dashv f^{\lceil t/s \rceil}$ with $f^{t/s}\hat{f}^{t/s} = 1_{\bar{\Delta}_1}$ and $f^{\lceil t/s \rceil}\hat{f}^{t/s} = 1_{\bar{\Delta}_1}$ for every $t/s \in B_n^\circ$; cf. Remark 106.

For every $t/s \in B_n^\circ$ define a 1-complex morphism $\hat{\gamma}^{t/s} \colon T^{t/s} \to X^{(f^{t/s})}$ by

$$\hat{\gamma}^{t/s}_{j/i} \;:=\; (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1})_{j/i f^{t/s,\#}}$$

for $j/i \in \bar{\Delta}_1^{\#}$ and $k \;:=\; (j/i f^{t/s,\#})\rho^{-1}$.

For every $t/s \in B_n^\circ$, define a 1-complex morphism $\hat{\delta}^{t/s} \colon X^{(f^{\lceil t/s \rceil})} \to T^{t/s}$ by

$$\hat{\delta}^{t/s}_{j/i} \;:=\; (c_{R^1} \cdot \ldots \cdot c_{R^{k-1}} \cdot c_{V^k})_{j/i f^{\lceil t/s \rceil,\#}}$$

for $j/i \in \bar{\Delta}_1^{\#}$.

Note that $f^{t/s}\hat{f}^{t/s} = 1_{\bar{\Delta}_1^{\#}} = f^{\lceil t/s \rceil}\hat{f}^{t/s}$, cf. Remark 106, and therefore we have $T^{t/s} = (\bar{S}^{t/s})^{(f^{t/s})} = (\bar{S}^{t/s})^{(f^{\lceil t/s \rceil})}$.

Suppose given $j/i \in \bar{\Delta}_1^{\#} \setminus \bar{\Delta}_1^{\#,\circ}$. Then $T^{t/s}_{j/i} \cong 0_{\mathcal{A}}$, thus every morphism $T^{t/s}_{j/i} \to T^{t/s}_{j/i}$ is the identity morphism $1_{T^{t/s}_{j/i}}$.

Suppose given $j/i \in \bar{\Delta}_1^{\#,\circ}$. Let $t'/s' \;:=\; j/i f^{t/s,\#}$. Let $k \;:=\; t'/s'\rho^{-1}$. Let

$$\tau \;:=\; (f^{t/s}, f^{\lceil t/s \rceil}) \colon f^{t/s} \to f^{\lceil t/s \rceil}.$$

Because $\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1} \colon V^k \to X$ is an n-complex morphism and because of $(t'/s', \lceil t'/s' \rceil)V^k = 1_{V^k_{t'/s'}}$, we have

$$\begin{aligned}
\hat{\gamma}^{t/s}_{j/i} \cdot (j/i)(X)\,\mathrm{C}^{(\tau)} &= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1})_{t'/s'} \cdot (j/i f^{t/s,\#}, j/i f^{\lceil t/s \rceil,\#})X \\
&= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1})_{t'/s'} \cdot (t'/s', \lceil t'/s' \rceil)X \\
&= (t'/s', \lceil t'/s' \rceil)V^k \cdot (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1})_{\lceil t'/s' \rceil} \\
&= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1})_{\lceil t'/s' \rceil}
\end{aligned}$$

and therefore

$$\begin{aligned}
\hat{\gamma}^{t/s}_{j/i} \cdot (j/i)(X)\,\mathrm{C}^{(\tau)} \cdot \hat{\delta}^{t/s}_{j/i} &= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1})_{\lceil t'/s' \rceil}(c_{R^1} \cdot \ldots \cdot c_{R^{k+1}} \cdot c_{V^k})_{j/i f^{\lceil t/s \rceil,\#}} \\
&= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \ldots \cdot \alpha_{R^1})_{\lceil t'/s' \rceil}(c_{R^1} \cdot \ldots \cdot c_{R^{k+1}} \cdot c_{V^k})_{\lceil t'/s' \rceil} \\
&= 1_{V^k_{t/s}}
\end{aligned}$$

with $V^k_{t'/s'} = V^k_{k\rho} = V_{k\rho} = V_{t'/s'} = T^{t/s}_{j/i}$.

Recall that

$$f^{t/s} \dashv \hat{f}^{t/s} \dashv f^{\lceil t/s \rceil}$$

and that

$$\hat{\gamma}^{t/s} \colon T^{t/s} \to X^{(f^{t/s})}.$$

Let

$$\gamma^{t/s} \;:=\; (\hat{\gamma}^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})} \cdot (X)\,\mathrm{C}^{(\hat{\varepsilon}^{t/s})} \colon \bar{S}^{t/s} = (T^{t/s})^{(\hat{f}^{t/s})} \to X.$$

Recall that $(\hat{\gamma}^{t/s} \cdot X\,\mathrm{C}^{(\tau)}) \cdot \hat{\delta}^{t/s} = 1_{T^{t/s}}$.

Let

$$\delta^{t/s} \;:=\; (X)\,\mathrm{C}^{(\check{\eta}^{\lceil t/s \rceil})} \cdot (\hat{\delta}^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})} \colon X \to (T^{t/s})^{(\hat{f}^{t/s})} = \bar{S}^{t/s}.$$

By Lemma 146, we now have $\gamma^{t/s}\delta^{t/s} = 1_{\bar{S}^{t/s}}$ for every $t/s \in B^\circ_n$.

We need to show that $\gamma^{t/s}\delta^{r/q} = 0$ for $t/s, r/q \in B^\circ_n$ with $t/s \neq r/q$. We do this by showing $\gamma^{t/s}_{j/i}\delta^{r/q}_{j/i} = 0$ for every $j/i \in \bar{\Delta}^{\#}_n$. For all $j/i \in \bar{\Delta}^{\#}_n \setminus \bar{\Delta}^{\#,\circ}_n$, the morphisms are already zero.

So suppose given $j/i \in \bar{\Delta}^{\#,\circ}_n$. Let $t'/s' \;:=\; j/i \hat{f}^{t/s,\#} f^{t/s,\#}$ and $r'/q' \;:=\; j/i \hat{f}^{r/q,\#} f^{r/q,\#}$. If $t'/s' \notin \bar{\Delta}^{\#,\circ}_n$, then $\gamma^{t/s}_{j/i} = 0$ and we are done. If $s' < 0$, then

$$\begin{aligned} 0_{\mathcal{A}} &\cong \bar{S}^{t/s}_{t'/s'} \\ &= T^{t/s}_{(t'/s')\hat{f}^{t/s,\#}} \\ &= T^{t/s}_{(j/i)\hat{f}^{t/s,\#}} \\ &= \bar{S}^{t/s}_{j/i} \end{aligned}$$

and we are done. So suppose that $t'/s' \in \bar{\Delta}^{\#,\circ}_n$ with $s' \geqslant 0$. The same holds for r'/q'.

Let $k \;:=\; t'/s'\rho^{-1}$ and $k' \;:=\; r'/q'\rho^{-1}$.

Case $k > k'$:

We get

$$\begin{aligned} \gamma^{t/s}_{j/i} \cdot \delta^{r/q}_{j/i} &= \hat{\gamma}^{t/s}_{j/i\hat{f}^{t/s},\#} \cdot (t'/s', j/i)X \cdot (j/i, q'+n/r'-1)X \cdot \hat{\delta}^{r/q}_{j/i\hat{f}^{r/q,\#}} \\ &= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \dots \cdot \alpha_{R^1})_{t'/s'} \cdot (t'/s', j/i)X \cdot (j/i, q'+n/r'-1)X \cdot (c_{R^1} \cdot \dots \cdot c_{R^{k'-1}} \cdot c_{V^{k'}})_{r'/q'} \\ &= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \dots \cdot \alpha_{R^1})_{j/i} \cdot (c_{R^1} \cdot \dots \cdot c_{R^{k'-1}} \cdot c_{V^{k'}})_{j/i} \\ &= (\alpha_{V^k} \cdot \alpha_{R^{k-1}} \cdot \dots \cdot \alpha_{R^{k'}})_{j/i} \cdot (c_{V^{k'}})_{j/i} \\ &= 0 \end{aligned}$$

Case $k < k'$:

We get

$$(\alpha_{V^k})_{j/i} \cdot (c_{R^k} \cdot \dots \cdot c_{R^{k'-1}} \cdot c_{V^{k'}})_{j/i} = 0.$$

We finally need to show that $\sum\limits_{t/s \in B_n^\circ} \delta^{t/s}\gamma^{t/s} = 1_X$ by showing this equation holds for every $j/i \in \bar{\Delta}_n^{\#}$. For $j/i \notin \bar{\Delta}_n^{\#,\circ}$ or $i \leqslant 0$ we know that $X_{j/i} \cong 0$ and are done.

So suppose given $j/i \in \bar{\Delta}_n^{\#,\circ}$ with $i \geqslant 0$. For every $t/s \in \bar{\Delta}_n^{\#,\circ}$ let

$$t'/s' := j/i \hat{f}^{t/s,\#} f^{t/s,\#} = \lfloor j/i \rfloor_{t/s}.$$

Let

$$l\hat{c} := c_{R^1} \cdot \dots \cdot c_{R^{l-1}} \cdot c_{V^l} \colon X \to V^l$$

and

$$l\hat{\alpha} := \alpha_{V^l} \cdot \alpha_{R^{l-1}} \cdot \dots \cdot \alpha_{R^1} \colon V^l \to X$$

for $l \in \mathbb{Z}_{\geqslant 1}$.

Note that for every $l \in \mathbb{Z}_{\geqslant 1}$ with $i\rho \notin \mathrm{d}_n^{j/i}$, we have $V^l_{j/i} = 0_{\mathcal{A}}$ and therefore $(l\hat{c} \cdot l\hat{\alpha})_{j/i} = 0$.

Define $I := \{t/s \in B_n^\circ \colon j/i \hat{f}^{t/s} \in \bar{\Delta}_1^{\#,\circ}\}$. For $t/s \in B_n^\circ \setminus I$, we have $t'/s' \notin \bar{\Delta}_n^{\#,\circ}$ and therefore $\gamma^{t/s}_{j/i} = 0$.

Let $k := j/i \rho^{-1}$. By Lemma 14, we get

$$\begin{aligned}
1_{X_{j/i}} &\overset{\text{L.14}}{=} \underbrace{(c_{R^1} \cdot \dots \cdot c_{R^{R-1}} \cdot c_{R^k} \cdot \alpha_{R^k} \cdot \alpha_{R^{k-1}} \cdot \dots \cdot \alpha_{R^1})_{j/i}}_{=0 \text{ because } R^k_{j/i} \cong 0_{\mathcal{A}}} + \sum_{l \in [1,k]} (l\hat{c} \cdot l\hat{\alpha})_{j/i} \\
&= \sum_{l \in [1,k]} (l\hat{c} \cdot l\hat{\alpha})_{j/i} \\
&= \sum_{t'/s' \in \mathrm{d}_n^{j/i}} \big((t'/s')\rho^{-1}\hat{c} \cdot (t'/s')\rho^{-1}\hat{\alpha}\big)_{j/i} \\
&\overset{\text{L.118}}{=} \sum_{t/s \in I} \big((\lfloor j/i \rfloor_{t/s})\rho^{-1}\hat{c}\big)_{j/i} \cdot \big((\lfloor j/i \rfloor_{t/s})\rho^{-1}\hat{\alpha}\big)_{j/i} \\
&= \sum_{t/s \in I} \big((\lfloor j/i \rfloor_{t/s})\rho^{-1}\hat{c}\big)_{j/i} \cdot \underbrace{(j/i, \lceil \lfloor j/i \rfloor_{t/s} \rceil) V^{(\lfloor j/i \rfloor_{t/s}\rho^{-1})}}_{=1_{V^l_{\lfloor j/i \rfloor_{t/s}}} \text{ with } l = (\lfloor j/i \rfloor_{t/s}\rho^{-1})} \cdot \big((\lfloor j/i \rfloor_{t/s})\rho^{-1}\hat{\alpha}\big)_{j/i} \\
&= \sum_{t/s \in I} (j/i, \lfloor j/i \rfloor_{t/s})X \cdot \big((\lfloor j/i \rfloor_{t/s})\rho^{-1}\hat{c}\big)_{\lceil t'/s' \rceil} \cdot \big((\lfloor j/i \rfloor_{t/s})\rho^{-1}\hat{\alpha}\big)_{\lceil t'/s' \rceil} \cdot (\lfloor j/i \rfloor_{t/s}, j/i)X \\
&= \sum_{t/s \in I} \delta^{t/s}_{j/i} \cdot \gamma^{t/s}_{j/i} \\
&= \sum_{t/s \in B_n^\circ} \delta^{t/s}_{j/i} \cdot \gamma^{t/s}_{j/i}
\end{aligned}$$

□

Lemma 173. *Suppose given an n-complex $X \in \operatorname{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$.*

The following assertions (1), (2), (3), (4) are equivalent.

(1) $X \in \operatorname{Ob}(\operatorname{Ker}(\operatorname{Pb}))$

(2) The morphism $({}^{i'}\!/_0, {}^{i}\!/_0)X$ *is monic for* $0 \leqslant i' < i < 0^{+1}$

(3) The morphism $({}^{i'}\!/_0, {}^{i}\!/_0)X$ *is monic for* $0 \leqslant i' < i < 0^{+1}$ *and* X *is split.*

(4) There exist 1*-complexes* $T^{t/s} \in \operatorname{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$ *for* ${}^{t}\!/_{s} \in B_n^\circ$ *such that*

$$X \cong \bigoplus_{t/s \in B_n^\circ} (T^{t/s})\, \mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A}).$$

In particular, $\operatorname{Ker}(\operatorname{Pb}) = \mathrm{C}^{(n,1)}(\mathcal{A}) \cap \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$.

Proof. We proceed as follows.

$$\begin{tikzcd}
(1) \arrow[r, Rightarrow, bend left, "\text{now}"] & (2) \arrow[l, Rightarrow, "\text{now}"] \arrow[r, Rightarrow, bend left, "\text{Corollary 162}"] \arrow[d, Rightarrow, "\text{Lemma 172}"] & (3) \arrow[l, Rightarrow, "\text{a fortiori}"] \\
& (4) \arrow[u, Rightarrow, bend left, "\text{now}"] &
\end{tikzcd}$$

(1) $\Rightarrow$ (2):

Suppose given $X \in \operatorname{Ob}(\operatorname{Ker}(\operatorname{Pb}))$. We have

$$\begin{tikzcd}
X_{0/0} \arrow[r] & X_{1/0} \arrow[r] & \dots \arrow[r] & X_{n-2/0} \arrow[r] & X_{n-1/0} \arrow[r] & X_{n/0} \arrow[r] & X_{0^{+1}/0} \\
& \hat{X}_1 \arrow[u, tail] \arrow[r, "\sim"'] \arrow[ur, phantom, "\llcorner"] & \dots \arrow[r, "\sim"'] & \hat{X}_{n-2} \arrow[u, tail] \arrow[r, "\sim"'] \arrow[ur, phantom, "\llcorner"] & \hat{X}_{n-1} \arrow[u, tail] \arrow[r, "\sim"'] \arrow[ur, phantom, "\llcorner"] & 0_{\mathcal{A}} \arrow[u, tail] &
\end{tikzcd}$$

with $\hat{X}_i \cong 0_{\mathcal{A}}$ for $i \in [0, n-1]$. Then $\hat{X}_i \cong 0_{\mathcal{A}}$ is kernel of $({}^{i}\!/_0, {}^{i+1}\!/_0)X$ for $i \in [1, n-1]$ and therefore $({}^{i}\!/_0, {}^{i+1}\!/_0)X$ is monic for $i \in [1, n-1]$. Then $({}^{i'}\!/_0, {}^{i}\!/_0)X$ is monic for every $0 \leqslant i' < i < 0^{+1}$ as a composite of monomorphisms.

(4) $\Rightarrow$ (2):

Suppose given ${}^{t}\!/_{s} \in B_n^\circ$ and $T \in \operatorname{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$.

If $s > 0$, we get $T^{(\hat{f}^{t/s})}_{i/0} = T_{i\hat{f}^{t/s}/0\hat{f}^{t/s}} = T_{i\hat{f}^{t/s}/1^{-1}} \cong 0_{\mathcal{A}}$ for $i \in [0, 0^{+1}]$; cf. Definition 101.

Then $({}^{i'}\!/_0, {}^{i}\!/_0)T^{(\hat{f}^{t/s})}$ is monic for $0 \leqslant i' < i < 0^{+1}$.

If $s = 0$, we get

$$T_{i/0}^{(\hat{f}^{t/s})} = \begin{cases} T_{0/0} & \text{for } i \in [0, t-1] \\ T_{1/0} & \text{for } i \in [t, n] \\ T_{0^{+1}/0} & \text{for } i = 0^{+1} \end{cases}$$

Then $({}^{i}\!/\!{}_{0}, {}^{i+1}\!/\!{}_{0})T^{(\hat{f}^{t/s})}$ is monic for $i \in [1, n-1]$, and therefore $({}^{i'}\!/\!{}_{0}, {}^{i}\!/\!{}_{0})T^{(\hat{f}^{t/s})}$ monic for $0 \leqslant i' < i < 0^{+1}$.

$$\begin{tikzcd}
T_{0/0} \arrow[r, equal] & T_{0/0} \arrow[r, equal] \arrow[dr, phantom, "\llcorner"] & \ldots \arrow[r, equal] & T_{0/0} \arrow[r, tail] \arrow[dr, phantom, "\llcorner"] & T_{1/0} \arrow[r, equal] \arrow[dr, phantom, "\llcorner"] & \ldots \arrow[r, equal] & T_{1/0} \arrow[r, equal] \arrow[dr, phantom, "\llcorner"] & T_{1/0} \arrow[r] & T_{0^{+1}/0} \\
 & \hat{T}_1 \arrow[u, tail] \arrow[r] & \ldots \arrow[r] & \hat{T}_{t-1} \arrow[u, tail] \arrow[r] & \hat{T}_t \arrow[u, tail] \arrow[r] & \ldots \arrow[r] & \hat{T}_{n-1} \arrow[u, tail] \arrow[r] & 0_{\mathcal{A}} \arrow[u, "0"] &
\end{tikzcd}$$

(2) $\Rightarrow$ (1): As kernel of $X_{i/0} \rightarrowtail X_{n/0}$, we have $\hat{X}_i \cong 0$ for $i \in [1, n-1]$.

We show that $\mathrm{Ker}(\mathrm{Pb}) = \mathrm{C}^{(n,1)}(\mathcal{A}) \cap \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$*:*

Suppose given $N \in \mathrm{Ob}(\mathrm{Ker}(\mathrm{Pb})) \subseteq \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$. There exist 1-complexes $(T^{t/s})_{t/s \in B_n^\circ}$ in $\mathrm{C}^{(1,\mathrm{ires})} \subseteq \mathrm{C}^{(1)}(\mathcal{A})$ with $N \cong \bigoplus_{t/s \in B_n^\circ} (T^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})$. Therefore $N \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A}))$.

Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A}) \cap \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$. Then there exist 1-complexes $(A^{t/s})_{t/s \in B_n^\circ}$ such that $X \cong \bigoplus_{t/s \in B_n^\circ} (A^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A})$; cf. Lemma 148. For every $t/s \in B_n^\circ$, we have $(A^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A}) \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$; cf. Lemma 164. By Lemma 165, $A^{t/s} \in \mathrm{Ob}(\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A}))$ for every $t/s \in B_n^\circ$. Then $X \in \mathrm{Ob}(\mathrm{Ker}(\mathrm{Pb}))$. $\square$

8.4 Homotopies of n-complex morphisms

Suppose that $n \geqslant 2$.

Definition 174. Suppose given a morphism $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(n)}(\mathcal{A})$. We say α *allows a homotopy*, if there exist morphisms $h^{t/s} \colon X_{\lceil t/s \rceil} \to Y_{t/s}$ in $\mathcal{A}$ for every $t/s \in \bar{\Delta}_n^{\#}$ such that

$$\alpha_{j/i} = \sum_{t/s \in \mathrm{d}_n^{j/i}} (j/i, \lceil t/s \rceil)X \cdot h^{t/s} \cdot (t/s, j/i)Y$$

for every $j/i \in \bar{\Delta}_n^{\#}$. Note that $h^{t/s} = 0$ for $t/s \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$.

Remark 175. In the case $n = 2$ for a morphism $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(2)}(\mathcal{A})$, this corresponds to a homotopy of complex morphisms:

Illustration of a 2-complex, where $\bar{\Delta}_2^{\#,\circ}$ is indexed via $i\rho$ with $i \in \mathbb{Z}$:

$$\begin{tikzcd}
 & & & {} & \\
 & & 0 \arrow[r] & X_{5\rho} \arrow[u, dotted] \arrow[r, dotted] & {} \\
 & 0 \arrow[r] & X_{3\rho} \arrow[r] \arrow[u] & X_{4\rho} \arrow[r] \arrow[u] & 0 \arrow[u, dotted] \\
0 \arrow[r] & X_{1\rho} \arrow[r] \arrow[u] & X_{2\rho} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
{} \arrow[u, dotted] \arrow[r, dotted] & X_{0\rho} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
 & {} \arrow[u, dotted] & & &
\end{tikzcd}$$

Suppose given ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#,\circ}$. Then ${}^{t}\!/\!{}_{s} = (t+s)\rho$ and $\lceil {}^{t}\!/\!{}_{s} \rceil = {}^{s+2}\!/\!{}_{t-1} = (t+s+1)\rho$; cf. Definition 99.

That is, for ${}^{t}\!/\!{}_{s} = i\rho$, we have

$$h^{i\rho} \colon X_{(i+1)\rho} \to Y_{i\rho}$$

with

$$\begin{aligned}
\alpha_{j\rho} &= \sum_{i \in \{j-1,j\}} (j\rho, (i+1)\rho)X \cdot h^{i\rho} \cdot (i\rho, j\rho)Y \\
&= (j\rho, j\rho)X \cdot h^{(j-1)\rho} \cdot ((j-1)\rho, j\rho)Y + (j\rho, (j+1)\rho)X \cdot h^{j\rho} \cdot (j\rho, j\rho)Y \\
&= h^{(j-1)\rho} \cdot ((j-1)\rho, j\rho)Y + (j\rho, (j+1)\rho)X \cdot h^{j\rho}.
\end{aligned}$$

So for complexes $A := \rho X|_{\bar{\Delta}_2^{\#,\circ}}$ and $B := \rho Y|\bar{\Delta}_2^{\#,\circ}$ in $\mathrm{C}(\mathcal{A})$ with $d^i := (i, i+1)A$ and $\hat{d}^i := (i, i+1)B$ for $i \in \mathbb{Z}$ and the complex morphism $\hat{\alpha} \colon A \to B$ with $\hat{\alpha}_i := \alpha_{\rho_i}$ as well as $D^i := h^{(i-1)\rho}$ for $i \in \mathbb{Z}$, this yields

$$\hat{\alpha}_i = D^i \cdot \hat{d}^{i-1} + d^i \cdot D^{i+1}$$

for $i \in \mathbb{Z}$, i.e. a homotopy in the classical sense.

Lemma 176. *Suppose given $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(n)}(\mathcal{A})$ allowing a homotopy. Suppose given n-complex morphisms $U \xrightarrow{\beta} X$ and $Y \xrightarrow{\gamma} V$. Then $\beta\alpha\gamma$ allows a homotopy.*

Proof. Suppose given morphisms $h^{t/s} \colon X_{\lceil t/s \rceil} \to Y_{t/s}$ for every ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#}$ with

$$\alpha_{j/i} = \sum_{t/s \in \mathrm{d}_n^{j/i}} ({}^{j}\!/\!{}_{i}, \lceil {}^{t}\!/\!{}_{s} \rceil)X \cdot h^{t/s} \cdot ({}^{t}\!/\!{}_{s}, {}^{j}\!/\!{}_{i})Y.$$

Then

$$
\begin{aligned}
(\beta\alpha\gamma)_{j/i} &= \beta_{j/i}\alpha_{j/i}\gamma_{j/i} \\
&= \beta_{j/i} \cdot \left(\sum_{t/s \in \mathrm{d}_n^{j/i}} (j/i, \lceil t/s \rceil)X \cdot h^{t/s} \cdot (t/s, j/i)Y \right) \cdot \gamma_{j/i} \\
&= \sum_{t/s \in \mathrm{d}_n^{j/i}} \beta_{j/i} \cdot (j/i, \lceil t/s \rceil)X \cdot h^{t/s} \cdot (t/s, j/i)Y \cdot \gamma_{j/i} \\
&= \sum_{t/s \in \mathrm{d}_n^{j/i}} (j/i, \lceil t/s \rceil)U \cdot \beta_{\lceil t/s \rceil} \cdot h^{t/s} \cdot \gamma_{t/s} \cdot (t/s, j/i)V \\
&= \sum_{t/s \in \mathrm{d}_n^{j/i}} (j/i, \lceil t/s \rceil)U \cdot \left(\beta_{\lceil t/s \rceil} \cdot h^{t/s} \cdot \gamma_{t/s} \right) \cdot (t/s, j/i)V.
\end{aligned}
$$

We get morphisms $\hat{h}^{t/s} := \beta_{\lceil t/s \rceil} \cdot h^{t/s} \cdot \gamma_{t/s} \colon U_{\lceil t/s \rceil} \to V_{t/s}$ with the wanted property. Therefore $\beta\alpha\gamma$ allows a homotopy. □

Lemma 177.

(1) Suppose given morphisms $A \xrightarrow{\alpha} B$ and $A \xrightarrow{\beta} B$ in $\mathrm{C}^{(n)}(\mathcal{A})$ allowing a homotopy. Then $A \xrightarrow{\alpha-\beta} B$ allows a homotopy. In particular, $\{A \xrightarrow{\alpha} B \colon \alpha \text{ allows a homotopy}\}$ is a subgroup of ${}_{\mathcal{A}}(A, B)$.

(2) Suppose given a finite set I and morphisms $(A_i \xrightarrow{\alpha_i} B_i)_{i \in I}$ in $\mathrm{C}^{(n)}(\mathcal{A})$ allowing a homotopy. Then $\bigoplus_{i \in I} A_i \xrightarrow{\sum_{i \in I} \pi_i^A \alpha_i \iota_i^B} \bigoplus_{i \in I} B_i$ allows a homotopy.

Proof. Ad (1): We have morphisms $(g^{t/s})_{t/s \in \bar{\Delta}_n^\#}$ and $(h^{t/s})_{t/s \in \bar{\Delta}_n^\#}$ with

$$\alpha_{j/i} = \sum_{t/s \in \mathrm{d}_n^{j/i}} (j/i, \lceil t/s \rceil)A \cdot g^{t/s} \cdot (t/s, j/i)B$$

and

$$\beta_{j/i} = \sum_{t/s \in \mathrm{d}_n^{j/i}} (j/i, \lceil t/s \rceil)A \cdot h^{t/s} \cdot (t/s, j/i)B$$

for $j/i \in \bar{\Delta}_n^\#$. Then

$$
\begin{aligned}
\alpha_{j/i} - \beta_{j/i} &= \sum_{t/s \in \mathrm{d}_n^{j/i}} \Big((j/i, \lceil t/s \rceil)A \cdot g^{t/s} \cdot (t/s, j/i)B - (j/i, \lceil t/s \rceil)A \cdot h^{t/s} \cdot (t/s, j/i)B \Big) \\
&= \sum_{t/s \in \mathrm{d}_n^{j/i}} (j/i, \lceil t/s \rceil)A \cdot (g^{t/s} - h^{t/s}) \cdot (t/s, j/i)B
\end{aligned}
$$

for ${}^{j}\!/_{i} \in \bar{\Delta}_n^{\#}$.

Ad (2): By Lemma 176, we get that

$$\bigoplus_{i \in I} A_i \xrightarrow{\pi_j^A} A_j \xrightarrow{\alpha_j} B_j \xrightarrow{\iota_j^B} \bigoplus_{i \in I} B_i$$

allows a homotopy for every $j \in I$. Then by (1), the morphism $\sum\limits_{i \in I} \pi_i^A \alpha_i \iota_i^B$ allows a homotopy. □

Lemma 178. *Suppose given a morphism $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(n)}(\mathcal{A})$.*

(1) If α allows a homotopy, then there exist morphisms

$$X \xrightarrow{\alpha_1} \bigoplus_{t/s \in B_n^\circ} Y^{(\hat{f}^{t/s} f^{t/s})} \xrightarrow{\alpha_2} Y$$

such that $\alpha = \alpha_1 \alpha_2$ with $Y^{(\hat{f}^{t/s} f^{t/s})} \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A}))$.

If $X \xrightarrow{\alpha} Y$ is in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$, then $Y^{(\hat{f}^{t/s} f^{t/s})} \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A}) \cap \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$ for ${}^{t}\!/_{s} \in B_n^\circ$.

(2) The following assertions (i), (ii) are equivalent.

(i) The morphism α allows a homotopy.

(ii) There exist morphisms $X \xrightarrow{\alpha_1} N \xrightarrow{\alpha_2} Y$ in $\mathrm{C}^{(n)}(\mathcal{A})$ with $N \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A}))$ such that $\alpha_1 \alpha_2 = \alpha$.

Proof. Ad (1): Suppose given $\big(X_{\lceil j/i \rceil} \xrightarrow{h^{j/i}} Y_{j/i}\big)_{j/i \in \bar{\Delta}_n^{\#}}$ with

$$\alpha_{t/s} = \sum_{j/i \in \mathrm{d}_n^{t/s}} ({}^{t}\!/_{s}, \lceil {}^{j}\!/_{i} \rceil)X \cdot h^{j/i} \cdot ({}^{j}\!/_{i}, {}^{t}\!/_{s})Y$$

for every ${}^{t}\!/_{s} \in \bar{\Delta}_n^{\#}$.

For every ${}^{t}\!/_{s} \in B_n^\circ$, define a 1-complex morphism $\hat{\beta}^{t/s} \colon X^{(f^{\lceil t/s \rceil})} \to Y^{(f^{t/s})}$ by

$$\hat{\beta}^{t/s}_{j/i} := \begin{cases} h^{j/i f^{t/s,\#}} & \text{for } {}^{j}\!/_{i} \in \bar{\Delta}_1^{\#,\circ} \\ 0 & \text{else.} \end{cases}$$

and an n-complex morphism

$$\beta^{t/s} := (X)\,\mathrm{C}^{(\check{\eta}^{\lceil t/s \rceil})} \cdot (\hat{\beta}^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})} \cdot (Y)\,\mathrm{C}^{(\hat{\varepsilon}^{t/s})} \colon X \to Y.$$

We show that $\alpha = \sum\limits_{{}^{t}\!/\!{}_{s} \in B_n^\circ} \beta^{t/s}$. It suffices to verify this at $j/i \in \bar{\Delta}_n^{\#,\circ}$ with $i \geqslant 0$ as else both sides are zero.

Suppose given $j/i \in \bar{\Delta}_n^{\#,\circ}$ with $i \geqslant 0$.

Define $I := \{t/s \in B_n^\circ \colon j/i\hat{f}^{t/s,\#} \in \bar{\Delta}_1^{\#,\circ}\}$. We get $(j/i\hat{f}^{t/s})\hat{\beta}^{t/s} = 0$ for $t/s \in B_n^\circ \setminus I$. For $t/s \in I$ we get $(j/i\hat{f}^{t/s})\hat{\beta}^{t/s} = h^{\lfloor j/i \rfloor_{t/s}}$.

$$\begin{aligned}
\sum_{t/s \in B_n^\circ} \beta^{t/s}_{j/i} &= \sum_{t/s \in B_n^\circ} \Big(\big(j/i, \lceil j/i \rceil_{t/s}\big)\, X \cdot (j/i\hat{f}^{t/s,\#})\hat{\beta}^{t/s} \cdot \big(\lfloor j/i \rfloor_{t/s}, j/i\big)\, Y \Big) \\
&= \sum_{t/s \in B_n^\circ} (j/i, \lceil j/i \rceil_{t/s})X \cdot \hat{\beta}^{t/s}_{j/i\hat{f}^{t/s,\#}} \cdot (\lfloor j/i \rfloor_{t/s}, j/i)Y \\
&= \sum_{t/s \in I} (j/i, \lceil j/i \rceil_{t/s})X \cdot h^{\lfloor j/i \rfloor_{t/s}} \cdot (\lfloor j/i \rfloor_{t/s}, j/i)Y \\
&\overset{\text{L.118}}{=} \sum_{t'/s' \in \mathrm{d}_n^{j/i}} (j/i, \lceil t'/s' \rceil)X \cdot h^{t'/s'} \cdot (t'/s', j/i)Y \\
&= \alpha_{j/i}
\end{aligned}$$

For every $t/s \in B_n^\circ$ we have

$$\beta^{t/s} = (X)\,\mathrm{C}^{(\check{\eta}^{\lceil t/s \rceil})} \cdot (\hat{\beta}^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})} \cdot (Y)\,\mathrm{C}^{(\hat{\varepsilon}^{t/s})}$$

with

$$X \xrightarrow{(X)\,\mathrm{C}^{(\check{\eta}^{\lceil t/s \rceil})}} X^{(\hat{f}^{t/s} f^{\lceil t/s \rceil})} \xrightarrow{(\hat{\beta}^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})}} Y^{(\hat{f}^{t/s} f^{t/s})} \xrightarrow{(Y)\,\mathrm{C}^{(\hat{\varepsilon}^{t/s})}} Y.$$

By Lemma 13, we get the wanted morphisms

$$X \xrightarrow{\alpha_1} \bigoplus_{t/s \in B_n^\circ} Y^{(\hat{f}^{t/s} f^{t/s})} \xrightarrow{\alpha_2} Y$$

with $\bigoplus\limits_{t/s \in B_n^\circ} Y^{(\hat{f}^{t/s} f^{t/s})} = \bigoplus\limits_{t/s \in B_n^\circ} \Big(Y^{(f^{t/s})}\Big)^{(\hat{f}^{t/s})} \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A}))$.

Suppose that $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$. In particular, $X, Y \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$.

Suppose given $t/s \in B_n^\circ$. Suppose given $j/i \in \bar{\Delta}_1^{\#}$ with $i < 0$, that is, $i \leqslant 1^{-1}$. Then $if^{t/s} \leqslant 1^{-1}f^{t/s} = t^{-1} < 0$. As $Y \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$, this yields $Y^{(f^{t/s})}_{j/i} = Y_{jf^{t/s}/if^{t/s}} \cong 0_{\mathcal{A}}$. As $Y \in \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}))$, we get $Y^{(f^{t/s})}_{j/i}$ injective for every $j/i \in \bar{\Delta}_n^{\#}$. Then $X^{(f^{t/s})}$ is in $\mathrm{C}^{(1,\mathrm{ires})}(\mathcal{A})$; cf. Lemma 157. Then

$$Y^{(\hat{f}^{t/s} f^{t/s})} = (Y^{(f^{t/s})})\,\mathrm{C}^{(\hat{f}^{t/s})}(\mathcal{A}) \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A})) \cap \mathrm{Ob}(\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}));$$

cf. Lemma 163.

Ad (2):

(i) $\Rightarrow$ (ii):

This follows from (1).

(ii) $\Rightarrow$ (i):

Suppose given $X \xrightarrow{\alpha_1} N \xrightarrow{\alpha_2} Y$ in $\mathrm{C}^{(n)}(\mathcal{A})$ with $N \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A}))$ such that $\alpha_1\alpha_2 = \alpha$.

We have objects $(A^{t/s})_{t/s \in B_n^\circ}$ in $\mathrm{C}^{(1)}(\mathcal{A})$ such that $N \cong \bigoplus\limits_{t/s \in B_n^\circ} (A^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})}$; cf. Lemma 148.

Let $N \xrightarrow{\varphi} \bigoplus\limits_{t/s \in B_n^\circ} (A^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})} =: \tilde{N}$ be an isomorphism.

S.g. $t/s \in B_n^\circ$. Define $N^{t/s} := (A^{t/s})\,\mathrm{C}^{(\hat{f}^{t/s})}$.

Note that for $t'/s' \in \bar{\Delta}_n^\#$ with $t'/s' \sim_{\mathrm{s}} t/s$ we have $j/i\hat{f}^{t/s,\#} = t'/s'\hat{f}^{t/s,\#}$ for $t'/s' \leqslant j/i \leqslant \lceil t'/s' \rceil$; cf. Lemma 105. Therefore

$$N^{t/s}_{j/i} = A^{t/s}_{(j/i)\hat{f}^{t/s,\#}} = A^{t/s}_{(t'/s')\hat{f}^{t/s,\#}} = N^{t/s}_{t'/s'}$$

for every $t'/s' \sim_{\mathrm{s}} t/s$ and every $j/i \in \mathrm{u}_n^{t'/s'}$.

For $t/s \in B_n^\circ$ and $t'/s' \in \bar{\Delta}_n^\#$, define $h^{t'/s'} \colon N^{t/s}_{\lceil t'/s' \rceil} \to N^{t/s}_{t'/s'}$ by

$$h^{t'/s'} := \begin{cases} 1_{N^{t/s}_{t'/s'}} & \text{if } t/s \sim_{\mathrm{s}} t'/s' \\ 0 & \text{else.} \end{cases}$$

Suppose given $j/i \in \bar{\Delta}_n^\#$. We want to show that

$$1_{N^{t/s}_{j/i}} = \sum_{t'/s' \in \mathrm{d}_n^{j/i}} (j/i, \lceil t'/s' \rceil) N^{t/s} \cdot h^{t'/s'} \cdot (t'/s', j/i) N^{t/s}.$$

If $j/i\hat{f}^{t/s,\#} \in \bar{\Delta}_n^\# \setminus \bar{\Delta}_n^{\#,\circ}$, then $N^{t/s}_{j/i} = A^{t/s}_{j/i\hat{f}^{t/s,\#}} \cong 0_{\mathcal{A}}$. In this case the sum is zero and we are done.

So suppose given $j/i \in \bar{\Delta}_n^\#$ with $j/i\hat{f}^{t/s,\#} \in \bar{\Delta}_1^{\#,\circ}$. In particular, $j/i \in \bar{\Delta}_n^{\#,\circ}$. As $f^{t/s,\#}$ is injective, we get $\lfloor j/i \rfloor_{t/s} = j/i\hat{f}^{t/s,\#} f^{t/s,\#} \in \bar{\Delta}_n^{\#,\circ}$.

We get

$$
\begin{aligned}
&\sum_{{}^{t'}\!/{}_{s'} \in \mathrm{d}_n^{j/i}} ({}^{j}\!/{}_{i}, \lceil {}^{t'}\!/{}_{s'} \rceil) N^{t/s} \cdot h^{t'/s'} \cdot ({}^{t'}\!/{}_{s'}, {}^{j}\!/{}_{i}) N^{t/s} \\
\overset{\text{L.117}}{=}\ & ({}^{j}\!/{}_{i}, \lceil \lfloor {}^{j}\!/{}_{i} \rfloor_{t/s} \rceil) N^{t/s} \cdot 1_{N^{t/s}_{\lfloor j/i \rfloor_{t/s}}} \cdot (\lfloor {}^{j}\!/{}_{i} \rfloor_{t/s}, {}^{j}\!/{}_{i}) N^{t/s} \\
=\ & ({}^{j}\!/{}_{i}\hat{f}^{t/s,\#}, \lceil \lfloor {}^{j}\!/{}_{i} \rfloor_{t/s} \rceil \hat{f}^{t/s,\#}) A^{t/s} \cdot (\lfloor {}^{j}\!/{}_{i} \rfloor_{t/s} \hat{f}^{t/s,\#}, {}^{j}\!/{}_{i}\hat{f}^{t/s,\#}) A^{t/s} \\
\overset{\text{L.105}}{=}\ & 1_{A^{t/s}_{j/i\hat{f}^{t/s,\#}}} \\
=\ & 1_{N^{t/s}_{j/i}}.
\end{aligned}
$$

Therefore $1_{N^{t/s}}$ allows a homotopy. By Lemma 177 (2), $1_{\tilde{N}}$ allows a homotopy. By Lemma 176, we get that

$$\alpha = \alpha_1 \cdot 1_N \cdot \alpha_2 = (\alpha_1 \cdot \varphi) \cdot 1_{\tilde{N}} \cdot (\varphi^{-1} \cdot \alpha_2)$$

allows a homotopy. $\square$

Lemma 179. *Suppose given the following commutative diagram in an abelian category where A_7, B_7, A_8, B_8 are zero objects. Suppose that B_2 is injective.*

$$
\begin{tikzcd}
 & B_7 \arrow[rr] & & B_4 \arrow[rr, "b_4"] & & B_5 \\
A_7 \arrow[ur] \arrow[rr] & & A_4 \arrow[ur, "\gamma_4"] \arrow[rr, "a_4"] & & A_5 \arrow[ur, "\gamma_5"] & \\
 & B_1 \arrow[uu] \arrow[rr, "b_1"] & & B_2 \arrow[uu, "b_2"] \arrow[rr, "b_3"] & & B_3 \arrow[uu, "b_5"'] \\
A_1 \arrow[uu] \arrow[ur, "0"] \arrow[rr, "a_1"] & & A_2 \arrow[uu, "a_2"] \arrow[ur, "\gamma_2"] \arrow[rr, "a_3"] & & A_3 \arrow[uu, "a_5"] \arrow[ur, "\gamma_3"] & \\
 & & & B_6 \arrow[uu, "b_6"] \arrow[rr] & & B_8 \arrow[uu] \\
 & & A_6 \arrow[uu, "a_6"] \arrow[ur, "0"] \arrow[rr] & & A_8 \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

Suppose that (A_1, A_2, A_7, A_4), (A_2, A_3, A_4, A_5) and (A_6, A_8, A_2, A_3) are weak squares.

$$\begin{tikzcd}
A_7 \arrow[r, "0"] & A_4 \arrow[r, "a_4"] & A_5 \\
A_1 \arrow[u, "0"] \arrow[r, "a_1"] \arrow[ur, phantom, "+"] & A_2 \arrow[u, "a_2"] \arrow[r, "a_3"] \arrow[ur, phantom, "+"] & A_3 \arrow[u, "a_5"'] \\
 & A_6 \arrow[u, "a_6"] \arrow[r, "0"] \arrow[ur, phantom, "+"] & A_8 \arrow[u, "0"']
\end{tikzcd}$$

Then there exists a morphism $h\colon A_5 \to B_2$ with $\gamma_2 = a_3 a_5 h = a_2 a_4 h$.

Proof. We can add an image of a_3 as

$$\begin{tikzcd}
 & B_2 & & \\
A_2 \arrow[ur, "\gamma_2"] \arrow[rrr, "a_3"] \arrow[drr, "\bar{c}"] & & & A_3 \\
 & & C \arrow[ur, "\dot{c}"] & \\
 & B_6 \arrow[uuu, "b_6"] \arrow[r, phantom, "+"] & & \\
A_6 \arrow[uuu, "a_6"] \arrow[ur, "0"] \arrow[rrr, "0"] & & & A_8 \arrow[uuu, "0"']
\end{tikzcd}$$

As (A_6, A_8, A_2, A_3) is a weak square with $A_8 \cong 0_{\mathcal{A}}$, $A_2 \xrightarrow{\bar{c}} C$ is a cokernel of a_6. As $a_6\gamma_2 = 0 \cdot b_6 = 0$, there exists a morphism $C \xrightarrow{\hat{c}} B_2$ with $\gamma_2 = \bar{c}\hat{c}$. As $\dot{c}$ is a monomorphism and B_2 is injective, there further exists a morphism $h_1\colon A_3 \to B_2$ with $\dot{c}h_1 = \hat{c}$.

$$\begin{tikzcd}
 & B_2 & & \\
A_2 \arrow[ur, "\gamma_2"] \arrow[rrr, "a_3"] \arrow[drr, "\bar{c}"] & & & A_3 \arrow[ull, "h_1"'] \\
 & & C \arrow[ur, "\dot{c}"] \arrow[uul, "\hat{c}"] & \\
 & B_6 \arrow[uuu, "b_6"] \arrow[r, phantom, "+"] & & \\
A_6 \arrow[uuu, "a_6"] \arrow[ur, "0"] \arrow[rrr, "0"] & & & A_8 \arrow[uuu, "0"']
\end{tikzcd}$$

Then $\gamma_2 = \bar{c}\hat{c} = \bar{c}\dot{c}h_1 = a_3 h_1$.

Symmetrically, we get a morphism $h_2 \colon A_4 \to B_2$ with $\gamma_2 = a_2 h_2$.

By adding a pushout to

$$\begin{tikzcd}
A_4 \arrow[r, "a_4"] & A_5 \\
A_2 \arrow[u, "a_2"] \arrow[r, "a_3"] \arrow[ur, phantom, "+"] & A_3 \arrow[u, "a_5"']
\end{tikzcd}$$

we get

$$\begin{tikzcd}
A_4 \arrow[rr, "a_4"] \arrow[dr, "p_2"] & & A_5 \\
& P \arrow[ur, "p"] & \\
A_2 \arrow[uu, "a_2"] \arrow[rr, "a_3"] & & A_3 \arrow[uu, "a_5"'] \arrow[ul, "p_1"]
\end{tikzcd}$$

as well as a morphism $\hat{p} \colon P \to B_2$ such that the following diagram is commutative.

$$\begin{tikzcd}
& & B_2 \\
A_4 \arrow[r, "p_2"] \arrow[urr, bend left, "h_2"] & P \arrow[ur, "\hat{p}"] & \\
A_2 \arrow[u, "a_2"] \arrow[r, "a_3"] & A_3 \arrow[u, "p_1"] \arrow[uur, bend right, "h_1"'] &
\end{tikzcd}$$

As p is monic and B_2 is injective, we get a morphism $A_5 \xrightarrow{h} B_2$ with $ph = \hat{p}$.

Then $\gamma_2 = a_3 h_1 = a_3 p_1 \hat{p} = a_3 p_1 p h = a_3 a_5 h$. $\square$

Lemma 180. *Suppose given a morphism $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ with $\alpha \,\mathrm{Pb} = 0$ and $k \in \mathbb{Z}_{\geqslant 1}$ such that $\alpha_{l\rho} = 0$ for every $l \in \mathbb{Z}_{\geqslant 1}$ with $l < k$. Let $t/s := k\rho$. Then there exist n-complex morphisms $X \xrightarrow{\alpha'} Y$ and $X \xrightarrow{\beta} Y$ and a morphism $h^{t/s} \colon X_{s+n/t-1} \to Y_{t/s}$ in $\mathcal{A}$ such that $\alpha'_{l\rho} = 0$ for every $l \leqslant k$ and such that*

$$\beta_{j/i} = \begin{cases} (j/i, \lceil t/s \rceil)X \cdot h^{t/s} \cdot (t/s, j/i)Y & \text{for } j/i \in \mathrm{u}_n^{t/s} \\ 0 & \text{else.} \end{cases}$$

and

$$\alpha = \alpha' + \beta.$$

Proof. Recall that $t/s = k\rho$. Let $\hat{X} := X\,\mathrm{Pb}$ and $\hat{Y} := Y\,\mathrm{Pb}$.

If $s = 0$, then we get the following commutative diagram

$$\begin{tikzcd}
 & Y_{t-1/t-1} \arrow[rr, "y"] & & Y_{t/t-1} \arrow[rr, "y"] & & Y_{n/t-1} \\
X_{t-1/t-1} \arrow[ur, "\alpha_{t-1/t-1}"] \arrow[rr, "x"] & & X_{t/t-1} \arrow[ur, "\alpha_{t/t-1}"] \arrow[rr, "x"] & & X_{n/t-1} \arrow[ur, "\alpha_{n/t-1}"] & \\
 & Y_{t-1/0} \arrow[uu, "y"] \arrow[rr, "y"] & & Y_{t/0} \arrow[uu, "y"] \arrow[rr, "y"] & & Y_{n/0} \arrow[uu, "y"] \\
X_{t-1/0} \arrow[uu, "x"] \arrow[ur, "0"] \arrow[rr, "x"] & & X_{t/0} \arrow[uu, "x"] \arrow[ur, "\alpha_{t/0}"] \arrow[rr, "x"] & & X_{n/0} \arrow[uu, "x"] \arrow[ur, "\alpha_{n/0}"] & \\
 & & & \hat{Y}_t \arrow[uu, "b_{Y,t}"] \arrow[rr, "\hat{y}"] & & 0_{\mathcal{A}} \arrow[uu, "b_{Y,n}"] \\
 & & \hat{X}_t \arrow[uu, "b_{X,t}"] \arrow[ur, "0"] \arrow[rr, "\hat{x}"] & & 0_{\mathcal{A}} \arrow[uu, "b_{X,n}"] \arrow[ur, "0"] &
\end{tikzcd}$$

with

$$\begin{tikzcd}
X_{t-1/t-1} \arrow[r, "x"] & X_{t/t-1} \arrow[r, "x"] & X_{n/t-1} \\
X_{t-1/0} \arrow[u, "x"] \arrow[r, "x"] \arrow[ur, phantom, "+"] & X_{t/0} \arrow[u, "x"] \arrow[r, "x"] \arrow[ur, phantom, "+"] & X_{n/0} \arrow[u, "x"] \\
 & \hat{X}_t \arrow[u, "b_{X,t}"] \arrow[r, "\hat{x}"] \arrow[ur, phantom, "\llcorner", very near start] & 0_{\mathcal{A}} \arrow[u, "b_{X,n}"]
\end{tikzcd} .$$

Note that $X_{t-1/0} \xrightarrow{\alpha_{t-1/0}} Y_{t-1/0}$ is zero because of either $t-1=0$ or $\alpha_{(k-1)\rho} = \alpha_{t-1/0} = 0$.
By Lemma 179, we get a morphism $h^{t/0}\colon X_{\lceil t/0 \rceil} \to Y_{t/0}$ with $\alpha_{t/0} = xh^{t/0}$.
If $s > 0$, then we get the following commutative diagram

$$\begin{tikzcd}
& Y_{t-1/t-1} \arrow[rr, "y"] & & Y_{t/t-1} \arrow[rr, "y"] & & Y_{s+n/t-1} \\
X_{t-1/t-1} \arrow[ur, "\alpha_{t-1/t-1}"] \arrow[rr, "x"] & & X_{t/t-1} \arrow[ur, "\alpha_{t/t-1}"] \arrow[rr, "x"] & & X_{s+n/t-1} \arrow[ur, "\alpha_{s+n/t-1}"] & \\
& Y_{t-1/s} \arrow[uu, "y"] \arrow[rr, "y"] & & Y_{t/s} \arrow[uu, "y"] \arrow[rr, "y"] & & Y_{s+n/s} \arrow[uu, "y"] \\
X_{t-1/s} \arrow[uu, "x"] \arrow[ur, "0"] \arrow[rr, "x"] & & X_{t/s} \arrow[uu, "x"] \arrow[ur, "\alpha_{t/s}"] \arrow[rr, "x"] & & X_{s+n/s} \arrow[uu, "x"] \arrow[ur, "\alpha_{s+n/s}"] & \\
& & & Y_{t/s-1} \arrow[uu, "b_{Y,t}"] \arrow[rr, "y"] & & Y_{s+n/s-1} \arrow[uu, "y"] \\
& & X_{t/s-1} \arrow[uu, "x"] \arrow[ur, "\alpha_{t/s-1}"] \arrow[rr, "x"] & & X_{s+n/s-1} \arrow[uu, "x"] \arrow[ur, "0"] &
\end{tikzcd}$$

with

$$\begin{tikzcd}
X_{t-1/t-1} \arrow[r, "x"] & X_{t/t-1} \arrow[r, "x"] & X_{s+n/t-1} \\
X_{t-1/s} \arrow[u, "x"] \arrow[r, "x"] \arrow[ur, phantom, "+"] & X_{t/s} \arrow[u, "x"] \arrow[r, "x"] \arrow[ur, phantom, "+"] & X_{s+n/s} \arrow[u, "x"] \\
& X_{t/s-1} \arrow[u, "x"] \arrow[r, "0"] \arrow[ur, phantom, "+"] & X_{s+n/s-1} \arrow[u, "0"]
\end{tikzcd} .$$

Again, $X_{t-1/s} \xrightarrow{\alpha_{t-1/s}} Y_{t-1/s}$ is zero because of either $t-1 = s$ or $\alpha_{(k-1)\rho} = \alpha_{t-1/s} = 0$.

By Lemma 179, we get a morphism $h^{t/s}\colon X_{\lceil t/s \rceil} \to Y_{t/s}$ with $\alpha_{t/s} = x h^{t/s}$.

Now define an n-complex morphism $X \xrightarrow{\beta} Y$ as follows.

First we define a 1-complex morphism $X^{(f^{\lceil t/s \rceil})} \xrightarrow{\beta'} Y^{(f^{t/s})}$ by

$$\beta'_{j/i} = \begin{cases} h^{t/s} & \text{for } j/i = 1/0 \\ 0 & \text{else.} \end{cases}$$

With $\hat{\varepsilon}^{t/s}\colon \hat{f}^{t/s} f^{t/s} \to 1_{\bar{\Delta}_n^{\#}}$ and $\check{\eta}^{\lceil t/s \rceil}\colon 1_{\bar{\Delta}_n^{\#}} \to \hat{f}^{t/s} f^{\lceil t/s \rceil}$, we get an n-complex morphism $X \xrightarrow{\beta} Y$ by

$$\beta \ := \ (X)\,\mathrm{C}^{(\check{\eta}^{\lceil t/s \rceil})} \cdot (\beta')\,\mathrm{C}^{(\hat{f}^{t/s})} \cdot (Y)\,\mathrm{C}^{(\hat{\varepsilon}^{t/s})} \colon X \to Y$$

Then

$$\begin{aligned}\beta_{j/i} &= \left({}^{j}\!/\!_{i}, ({}^{j}\!/\!_{i})\hat{f}^{t/s\#} f^{\lceil t/s\rceil,\#}\right) X \cdot ({}^{j}\!/\!_{i}\hat{f}^{t/s,\#})\beta' \cdot \left({}^{j}\!/\!_{i}\hat{f}^{t/s,\#} f^{t/s,\#}, {}^{j}\!/\!_{i}\right) Y \\ &= \begin{cases} ({}^{j}\!/\!_{i}, \lceil {}^{t}\!/\!_{s}\rceil)X \cdot h^{t/s} \cdot ({}^{t}\!/\!_{s}, {}^{j}\!/\!_{i})Y & \text{for } ({}^{j}\!/\!_{i})\hat{f}^{t/s,\#} = {}^{1}\!/\!_{0}, \text{ i.e. } {}^{j}\!/\!_{i} \in \mathrm{u}_n^{t/s} \\ 0 & \text{for } {}^{j}\!/\!_{i} \in \bar{\Delta}_n^{\#} \text{ with } ({}^{j}\!/\!_{i})\hat{f}^{t/s,\#} \neq {}^{1}\!/\!_{0}. \end{cases}\end{aligned}$$

Therefore

$$\beta_{t/s} = ({}^{t}\!/\!_{s}, \lceil {}^{t}\!/\!_{s}\rceil)X \cdot h^{t/s} = \alpha_{t/s}$$

and

$$\beta_{l\rho} = 0 \text{ for every } l \in \mathbb{Z}_{\geqslant 1} \text{ with } l < k.$$

We can define

$$\alpha' \;:=\; \alpha - \beta.$$

Then α' is an n-complex morphism with $\alpha'_{l\rho} = 0$ for every $l \in \mathbb{Z}_{\geqslant 1}$ with $l \leqslant k$. $\square$

Lemma 181. *Suppose given a morphism $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$. Then the following assertions (1), (2), (3) are equivalent.*

(1) We have $\alpha\,\mathrm{Pb} = 0$

(2) The morphism α allows a homotopy

(3) There exist morphisms $X \xrightarrow{\alpha^1} N \xrightarrow{\alpha^2} Y$ in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ with $N \in \mathrm{Ob}(\mathrm{Ker}(\mathrm{Pb}))$ such that $\alpha = \alpha^1\alpha^2$.

Proof. (1) $\Rightarrow$ (2): Suppose given $X \xrightarrow{\alpha} Y$ in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ with $\alpha\,\mathrm{Pb} = 0$. Suppose given ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#,\circ}$ with $s \geqslant 0$ and ${}^{t}\!/\!_{s} = k\rho$. By repeatedly applying Lemma 180 at positions $l\rho$ for $l \in \mathbb{Z}_{\geqslant 1}$, we get $(h^{l\rho})_{l\in\mathbb{Z}_{\geqslant 1}}$, $(\beta^l)_{l\in\mathbb{Z}_{\geqslant 1}}$ as well as $(\alpha^l)_{l\in\mathbb{Z}_{\geqslant 1}}$ with

$$\beta^l_{j/i} = \begin{cases} ({}^{j}\!/\!_{i}, \lceil l\rho\rceil)X \cdot h^{l\rho} \cdot (l\rho, {}^{j}\!/\!_{i})Y & \text{for } {}^{j}\!/\!_{i} \in \mathrm{u}_n^{l\rho} \\ 0 & \text{else.} \end{cases}$$

for every $l \in \mathbb{Z}_{\geqslant 1}$ and with

$$\alpha = \beta^1 + \beta^2 + \cdots + \beta^k + \alpha^k$$

for every $k \in \mathbb{Z}_{\geqslant 1}$, where $\alpha^k_{l\rho} = 0$ for $l \leqslant k$.

Set $h^{t/s} = 0$ for ${}^{t}\!/\!_{s} \in \bar{\Delta}_n^{\#}$ with ${}^{t}\!/\!_{s} \notin \bar{\Delta}_n^{\#,\circ}$ or $s < 0$. Set $\beta^l = 0$ for every $l < 0$. Note that this yields

$$\beta^l_{j/i} = \begin{cases} ({}^{j}\!/\!_{i}, \lceil l\rho\rceil)X \cdot h^{l\rho} \cdot (l\rho, {}^{j}\!/\!_{i})Y & \text{for } {}^{j}\!/\!_{i} \in \mathrm{u}_n^{l\rho} \\ 0 & \text{else.} \end{cases}$$

for every $l \in \mathbb{Z}$.

Suppose given ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#}$. If ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$, we have $\mathrm{d}_n^{t/s} = \emptyset$ and thus

$$\alpha_{t/s} = 0 = \sum_{j/i \in \mathrm{d}_n^{t/s}} ({}^{t}\!/\!{}_{s}, \lceil {}^{j}\!/\!{}_{i} \rceil)X \cdot h^{j/i} \cdot ({}^{j}\!/\!{}_{i}, {}^{t}\!/\!{}_{s})Y$$

So suppose that ${}^{t}\!/\!{}_{s} \in \bar{\Delta}_n^{\#}$. Set $k := {}^{t}\!/\!{}_{s}\rho^{-1}$. If $k \geqslant 1$, we have $\alpha_{t/s}^{k} = 0$.

For ${}^{j}\!/\!{}_{i} \in \bar{\Delta}_n^{\#}$, if $i < 0$ and therefore ${}^{j}\!/\!{}_{i}\rho^{-1} < 1$, we have $\beta^{(j/i)\rho^{-1}} = 0$.

As ${}^{j}\!/\!{}_{i} \in \mathrm{u}_n^{l\rho}$ if and only if $l\rho \in \mathrm{d}_n^{j/i}$, we get

$$\begin{aligned}
\alpha_{t/s} &= \sum_{l \in [1,k]} \beta_{t/s}^{l} \\
&= \sum_{\substack{l \in [1,k] \\ l\rho \in \mathrm{d}_n^{t/s}}} \beta_{t/s}^{l} \\
&= \sum_{j/i \in \mathrm{d}_n^{t/s}} \beta_{t/s}^{(j/i)\rho^{-1}} \\
&= \sum_{j/i \in \mathrm{d}_n^{t/s}} ({}^{t}\!/\!{}_{s}, \lceil {}^{j}\!/\!{}_{i} \rceil)X \cdot h^{j/i} \cdot ({}^{j}\!/\!{}_{i}, {}^{t}\!/\!{}_{s})Y
\end{aligned}$$

Therefore α allows a homotopy.

(2) $\Rightarrow$ (3): Recall that $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ is full. This follows by Lemma 178.

(3) $\Rightarrow$ (1): Suppose given $N \in \mathrm{Ob}(\mathrm{Ker}(\mathrm{Pb}))$ and morphisms $X \xrightarrow{\alpha} Y$ and $X \xrightarrow{\alpha^1} N \xrightarrow{\alpha^2} Y$ in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ with $\alpha = \alpha^1\alpha^2$. As $N\,\mathrm{Pb} \cong 0_{\mathcal{A}}$, we have $\alpha^1 = 0$ and $\alpha^2 = 0$.

Then $\alpha\,\mathrm{Pb} = \alpha^1\,\mathrm{Pb} \cdot \alpha^2\,\mathrm{Pb} = 0$. $\square$

Lemma 182. *Let $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{J} \mathrm{C}^{(n)}(\mathcal{A})$ be the full inclusion functor. We have the induced functor*

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\bar{J}} \mathrm{K}^{(n/1)}(\mathcal{A}).$$

It is additive, full and faithful.

Proof. The induced functor $\bar{J}$ is well-defined:

Let $R\colon \mathrm{C}^{(n)}(\mathcal{A}) \to \mathrm{K}^{(n/1)}(\mathcal{A})$ be the residue class functor.

By the universal property of the factor category, cf. Remark 62, we obtain an additive induced functor if JR maps each object of $\mathrm{Ker}(\mathrm{Pb})$ to a zero object. This holds by

$$\mathrm{Ker}(\mathrm{Pb}) \overset{\mathrm{L.173}}{=} \mathrm{C}^{(n,1)}(\mathcal{A}) \cap \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \subseteq \mathrm{C}^{(n,1)}(\mathcal{A}).$$

The functor $\bar{J}$ is faithful:

Suppose given $X \xrightarrow{[\alpha]} Y$ in $\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$. Suppose that $[\alpha]\bar{J} = 0$. We have

$$\begin{aligned} [\alpha]\bar{J} = 0 \;\Rightarrow\; & \exists\,(X \xrightarrow{\alpha_1} N \xrightarrow{\alpha_2} Y) \text{ in } \mathrm{C}^{(n)}(\mathcal{A}) \text{ with } N \in \mathrm{Ob}(\mathrm{C}^{(n,1)}(\mathcal{A})) \text{ and } \alpha_1\alpha_2 = \alpha \\ \overset{\text{L.178 (2)}}{\Rightarrow} & \alpha \text{ allows a homotopy} \\ \overset{\text{L.181}}{\Rightarrow} & \exists\,(X \xrightarrow{\alpha_1} N \xrightarrow{\alpha_2} Y) \text{ in } \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \text{ with } N \in \mathrm{Ob}(\mathrm{Ker}(\mathrm{Pb})) \text{ and } \alpha_1\alpha_2 = \alpha \\ \Rightarrow\; & [\alpha] = [0] \end{aligned}$$

The functor $\bar{J}$ is full:

A representative of a morphism in $\mathrm{K}^{(n/1)}(\mathcal{A})$ between objects of $\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$ can be used as a representative of an inverse image in $\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$. □

8.5 The resolution equivalence

Suppose that $n \geqslant 2$.

Lemma 183 ([**1,** §A, Lemma A.1])**.** *Let $\mathcal{B}$ and $\mathcal{C}$ be additive categories, and let $\mathcal{B} \xrightarrow{F} \mathcal{C}$ be a full and dense additive functor.*

Suppose that for each morphism $B \xrightarrow{b_0} B'$ in $\mathcal{B}$ such that $b_0F = 0$, there exists $B \xrightarrow{b_0'} N_0 \xrightarrow{b_0''} B'$ with $b_0'b_0'' = b_0$ and $N_0 \in \mathrm{Ob}(\mathrm{Ker}(F))$. Then the induced functor

$$\begin{aligned} \mathcal{B}/\operatorname{Ker}(F) &\xrightarrow{\bar{F}} \mathcal{C} \\ (B \xrightarrow{[b]} B') &\longmapsto (BF \xrightarrow{bF} B'F) \end{aligned}$$

is an equivalence.

Theorem 184. *For $n \geqslant 2$, the induced functor*

$$\begin{aligned} \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) &\xrightarrow{\overline{\mathrm{Pb}}} (\dot{\Delta}_{n-1}, \mathcal{A}) \\ (X \xrightarrow{[\varphi]} X') &\longmapsto (X\,\mathrm{Pb} \xrightarrow{\varphi\,\mathrm{Pb}} X'\,\mathrm{Pb}) \end{aligned}$$

is an equivalence.

Proof. We check the conditions for Lemma 183 for $\mathcal{B} := \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ and $\mathcal{C} := (\dot{\Delta}_{n-1}, \mathcal{A})$. Both categories are additive, cf. Lemmas 158, 12.

The functor $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\mathrm{Pb}} (\dot{\Delta}_{n-1}, \mathcal{A})$ is a full and dense additive functor; cf. Lemma 168.

We have $\mathcal{M} = \mathrm{Ker}(\mathrm{Pb})$.

For every morphism $B \xrightarrow{b_0} B'$ in $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A})$ with $b_0 \,\mathrm{Pb} = 0$, there exists a factorisation $B \xrightarrow{b_0'} M_0 \xrightarrow{b_0''} B'$ with $b_0' b_0'' = b_0$ and $M_0 \in \mathrm{Ob}(\mathrm{Ker}(\mathrm{Pb}))$; cf. Lemma 181.

By Lemma 183, the induced functor $\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\overline{\mathrm{Pb}}} (\dot{\Delta}_{n-1}, \mathcal{A})$ is an equivalence. $\square$

Corollary 185. *We now may choose an equivalence of categories*

$$(\dot{\Delta}_{n-1}, \mathcal{A}) \xrightarrow{\mathrm{IRes}^{(n)}} \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$$

such that $\mathrm{IRes}^{(n)}\, \overline{\mathrm{Pb}}^{(n)} \cong 1_{(\dot{\Delta}_{n-1}, \mathcal{A})}$ *and* $\overline{\mathrm{Pb}}^{(n)}\, \mathrm{IRes}^{(n)} \cong 1_{\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})}$. *This functor is called the injective resolution equivalence.*

9 Conclusion

Suppose given an abelian category $\mathcal{A}$ with enough injective objects, an abelian category $\mathcal{B}$ and $n \geqslant 2$. Suppose given an additive functor $F \colon \mathcal{A} \to \mathcal{B}$. We have an equivalence of categories

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\overline{\mathrm{Pb}}^{(n)}} (\dot{\Delta}_{n-1}, \mathcal{A})$$

by Theorem 184 and an equivalence

$$(\dot{\Delta}_{n-1}, \mathcal{A}) \xrightarrow{\mathrm{IRes}^{(n)}} \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$$

with $\mathrm{IRes}^{(n)}\,\overline{\mathrm{Pb}}^{(n)} \cong 1_{(\dot{\Delta}_{n-1},\mathcal{A})}$ and $\overline{\mathrm{Pb}}^{(n)}\,\mathrm{IRes}^{(n)} \cong 1_{\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})}$ by Corollary 185.
By Lemma 182, we get a full and faithful additive functor

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\bar{J}} \mathrm{K}^{(n/1)}(\mathcal{A}).$$

By Lemma 154, we get an additive functor

$$\mathrm{K}^{(n/1)}(\mathcal{A}) \xrightarrow{\mathrm{K}^{(n/1)}(F)} \mathrm{K}^{(n/1)}(\mathcal{B}).$$

Altogether, we now have

$$\begin{tikzcd}
(\dot{\Delta}_{n-1}, \mathcal{A}) \arrow[r, "\mathrm{IRes}^{(n)}", "\sim"'] & \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \arrow[l, bend left, "\overline{\mathrm{Pb}}^{(n)}"] \arrow[r, "\bar{J}"] & \mathrm{K}^{(n/1)}(\mathcal{A}) \arrow[d, "\mathrm{K}^{(n/1)}(F)"] \\
& & \mathrm{K}^{(n/1)}(\mathcal{B})\ .
\end{tikzcd}$$

10 Appendix: A diagonal complex

Lemma 186. *Suppose given an additive category $\mathcal{A}$. Suppose given $n \in \mathbb{Z}_{\geqslant 1}$.*

Let $\rho = \rho_n \colon \mathbb{Z} \to \bar{\Delta}_n^{\#,\circ}$ the bijection from Definition 99.

(1) Suppose given n-complexes $(S^i)_{i\in\mathbb{Z}}$ with $S^i \in \operatorname{Ob}(\mathrm{C}^{(n)}(\mathcal{A}))$ and n-complex morphisms $(S^i \xrightarrow{\psi^i} S^{i+1})_{i\in\mathbb{Z}}$. We can define an n-complex S^{D} as follows. Let

$$S^{\mathrm{D}}_{t/s} := S^i_{t/s}$$

for $t/s \in \bar{\Delta}_n^{\#,\circ}$ with $i\rho = t/s$ and $S^{\mathrm{D}}_{t/s} := 0_{\mathcal{A}}$ else.

For morphisms $(t/s, t'/s')$ with both $t/s, t'/s' \in \bar{\Delta}_n^{\#,\circ}$ and with $i\rho = t/s$ and $j\rho = t'/s'$ we define

$$(t/s, t'/s')S^{\mathrm{D}} := (t/s, t'/s')S^i \cdot \prod_{k\in[i,j-1]} \psi^k_{t'/s'}.$$

We set

$$(t/s, t'/s')S^{\mathrm{D}} := 0$$

else.

(2) Suppose given an n-complex X together with morphisms $(S^i \xrightarrow{\varphi^i} X)_{i\in\mathbb{Z}}$ that fulfil $\varphi^i = \psi^i\varphi^{i+1}$ for $i \in \mathbb{Z}$. Then $\varphi \colon S^{\mathrm{D}} \to X$ with

$$\varphi_{t/s} := \begin{cases} \varphi^{(t/s)\rho^{-1}}_{t/s} & \text{for } t/s \in \bar{\Delta}_n^{\#,\circ} \\ 0 & \text{else.} \end{cases}$$

is an n-complex morphism.

(3) If $\varphi^i \colon S^i \to X$ is an isomorphism for every $i \in \mathbb{Z}$, then φ is an isomorphism.

Proof. Ad (1): We prove that S^{D} is an n-complex.

By definition, we have $S^{\mathrm{D}}_{t/s} = 0_{\mathcal{A}}$ for $t/s \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$.

Suppose given $t/s \in \bar{\Delta}_n^{\#,\circ}$. Let $i := t/s\rho^{-1}$. We have

$$(t/s, t/s)S^{\mathrm{D}} = (t/s, t/s)S^i \cdot \prod_{k\in[i,i-1]} \psi^k_{t'/s'} = 1_{S^i_{t/s}} = 1_{S^{\mathrm{D}}_{t/s}}.$$

For $t/s \in \bar{\Delta}_n^{\#} \setminus \bar{\Delta}_n^{\#,\circ}$, we have $S^{\mathrm{D}}_{t/s} \cong 0_{\mathcal{A}}$ and therefore $(t/s, t/s)S^{\mathrm{D}} = 0 = 1_{S^{\mathrm{D}}_{t/s}}$.

Suppose given $(t/s, t'/s'), (t'/s', t''/s'') \in \operatorname{Mor}(\bar{\Delta}_n^{\#})$.

We show that $(t/s, t'/s')S^{\mathrm{D}} \cdot (t'/s', t''/s'')S^{\mathrm{D}} = (t/s, t''/s'')S^{\mathrm{D}}$.

Case ${}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'}, {}^{t''}\!/\!_{s''} \in \bar{\Delta}_n^{\#,\circ}$:

Let $i, j, l \in \mathbb{Z}$ with $i\rho = {}^{t}\!/\!_{s}$, $j\rho = {}^{t'}\!/\!_{s'}$ and $l\rho = {}^{t''}\!/\!_{s''}$. Then

$$\begin{aligned}
({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{\mathrm{D}} \cdot ({}^{t'}\!/\!_{s'}, {}^{t''}\!/\!_{s''})S^{\mathrm{D}} &= ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{i} \cdot \left(\prod_{k\in[i,j-1]} \psi^{k}_{{}^{t'}\!/\!_{s'}}\right) \cdot ({}^{t'}\!/\!_{s'}, {}^{t''}\!/\!_{s''})S^{j} \cdot \left(\prod_{k\in[j,l-1]} \psi^{k}_{{}^{t''}\!/\!_{s''}}\right) \\
&= ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{i} \cdot ({}^{t'}\!/\!_{s'}, {}^{t''}\!/\!_{s''})S^{i} \cdot \left(\prod_{k\in[i,j-1]} \psi^{k}_{{}^{t''}\!/\!_{s''}}\right) \cdot \left(\prod_{k\in[j,l-1]} \psi^{k}_{{}^{t''}\!/\!_{s''}}\right) \\
&= ({}^{t}\!/\!_{s}, {}^{t''}\!/\!_{s''})S^{i} \cdot \left(\prod_{k\in[i,l-1]} \psi^{k}_{{}^{t''}\!/\!_{s''}}\right) \\
&= ({}^{t}\!/\!_{s}, {}^{t''}\!/\!_{s''})S^{\mathrm{D}}.
\end{aligned}$$

Case ${}^{t}\!/\!_{s} \notin \bar{\Delta}_n^{\#,\circ}$ or ${}^{t''}\!/\!_{s''} \notin \bar{\Delta}_n^{\#,\circ}$:

We have $({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{\mathrm{D}} \cdot ({}^{t'}\!/\!_{s'}, {}^{t''}\!/\!_{s''})S^{\mathrm{D}} = 0 = ({}^{t}\!/\!_{s}, {}^{t''}\!/\!_{s''})S^{\mathrm{D}}$.

Case ${}^{t}\!/\!_{s}, {}^{t''}\!/\!_{s''} \in \bar{\Delta}_n^{\#,\circ}$ and ${}^{t'}\!/\!_{s'} \notin \bar{\Delta}_n^{\#,\circ}$:

Let $i, l \in \mathbb{Z}$ with $i\rho = {}^{t}\!/\!_{s}$ and $l\rho = {}^{t''}\!/\!_{s''}$. We have

$$({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{\mathrm{D}} \cdot ({}^{t'}\!/\!_{s'}, {}^{t''}\!/\!_{s''})S^{\mathrm{D}} = 0$$

and

$$({}^{t}\!/\!_{s}, {}^{t''}\!/\!_{s''})S^{\mathrm{D}} = ({}^{t}\!/\!_{s}, {}^{t''}\!/\!_{s''})S^{i} \cdot \prod_{k\in[i,l-1]} \psi^{k}_{{}^{t''}\!/\!_{s''}} = 0.$$

Ad (2):

We show that φ is a morphism of n-complexes. For this we need to show that

$$({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{\mathrm{D}}\varphi_{{}^{t'}\!/\!_{s'}} = \varphi_{{}^{t}\!/\!_{s}}({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})X$$

for every $({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$.

Suppose given $({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'}) \in \mathrm{Mor}(\bar{\Delta}_n^{\#})$.

If ${}^{t}\!/\!_{s} \notin \bar{\Delta}_n^{\#,\circ}$ or ${}^{t'}\!/\!_{s'} \notin \bar{\Delta}_n^{\#,\circ}$, then both sides are zero and we are done.

So suppose that ${}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'} \in \bar{\Delta}_n^{\#,\circ}$. Suppose given $i, j \in \mathbb{Z}$ with ${}^{t}\!/\!_{s} = i\rho$ and ${}^{t'}\!/\!_{s'} = j\rho$. Then

$$\begin{aligned}
({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{\mathrm{D}}\varphi_{{}^{t'}\!/\!_{s'}} &= ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{i} \cdot \prod_{k\in[i,j-1]} \psi^{k}_{{}^{t'}\!/\!_{s'}} \cdot \varphi^{j}_{{}^{t'}\!/\!_{s'}} \\
&= ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})S^{i} \cdot \varphi^{i}_{{}^{t'}\!/\!_{s'}} \\
&= \varphi^{i}_{{}^{t}\!/\!_{s}} \cdot ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})X \\
&= \varphi_{{}^{t}\!/\!_{s}} \cdot ({}^{t}\!/\!_{s}, {}^{t'}\!/\!_{s'})X
\end{aligned}$$

Thus φ is a morphism of n-complexes.

Ad (3):

If φ^i is an isomorphism for every $i \in \mathbb{Z}$, then $\varphi_{t/s}$ is an isomorphism for every $t/s \in \bar{\Delta}_n^{\#,\circ}$. For $t/s \in \bar{\Delta}_n^{\#} \backslash \bar{\Delta}_n^{\#,\circ}$, we have $S^{\mathrm{D}}_{t/s} = 0_{\mathcal{A}}$ and $X_{t/s} \cong 0_{\mathcal{A}}$. Thus $\varphi_{t/s} = 0$ is an isomorphism. $\square$

Remark 187. Note that we do not have a canonical choice of morphisms $S^i \to S^{\mathrm{D}}$ for $i \geqslant 1$.

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Zusammenfassung

Sei $\mathcal{A}$ eine abelsche Kategorie mit genügend Injektiven. Sei $n \geqslant 1$.

Wir bilden das Poset $\bar{\Delta}_n^{\#}$. Zum Beispiel hat $\bar{\Delta}_3^{\#}$ die Gestalt

$$
\begin{tikzcd}
 & & & & & 0^{+1}/0^{+1} \arrow[r, dotted] & {} & \\
 & & & & 3/3 \arrow[r] & 0^{+1}/3 \arrow[r] \arrow[u] & 1^{+1}/3 \arrow[r, dotted] \arrow[u, dotted] & {} \\
 & & & 2/2 \arrow[r] & 3/2 \arrow[r] \arrow[u] & 0^{+1}/2 \arrow[r] \arrow[u] & 1^{+1}/2 \arrow[r] \arrow[u] & 2^{+1}/2 \arrow[u, dotted] \\
 & & 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & 3/1 \arrow[r] \arrow[u] & 0^{+1}/1 \arrow[r] \arrow[u] & 1^{+1}/1 \arrow[u] & \\
 & 0/0 \arrow[r] & 1/0 \arrow[r] \arrow[u] & 2/0 \arrow[r] \arrow[u] & 3/0 \arrow[r] \arrow[u] & 0^{+1}/0 \arrow[u] & & \\
3^{-1}/3^{-1} \arrow[r] & 0/3^{-1} \arrow[r] \arrow[u] & 1/3^{-1} \arrow[r] \arrow[u] & 2/3^{-1} \arrow[r] \arrow[u] & 3/3^{-1} \arrow[u] & & & \\
{} \arrow[u, dotted] \arrow[r, dotted] & 0/2^{-1} \arrow[r] \arrow[u] & 1/2^{-1} \arrow[r] \arrow[u] & 2/2^{-1} \arrow[u] & & & & \\
 & {} \arrow[u, dotted] \arrow[r, dotted] & 1/1^{-1} \arrow[u] & & & & &
\end{tikzcd}
$$

Ein n-Komplex ist ein Diagramm auf $\bar{\Delta}_n^{\#}$, welches an beiden Rändern Nullobjekte aufweist. Die Kategorie der n-Komplexe über $\mathcal{A}$ wird als

$$\mathrm{C}^{(n)}(\mathcal{A})$$

bezeichnet.

Sei nun $n \geqslant 2$.

Wir bilden die volle Teilkategorie $\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \subseteq \mathrm{C}^{(n)}(\mathcal{A})$ der n-Komplexe, die aus injektiven Objekten bestehen, die unterhalb der Zeile 0 aus Nullobjekten bestehen und die oberhalb der Zeile 0 eine Exaktheitsbedingung erfüllen.

Sei $(\dot{\Delta}_{n-1}, \mathcal{A})$ die Kategorie der Diagramme der Form

$$A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-2}} A_{n-1}.$$

Der Funktor

$$\mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\mathrm{Pb}^{(n)}} \big(\dot{\Delta}_{n-1}, \mathcal{A}\big)$$

bilde wie folgt ab. Z.B. im Fall $n = 3$ bilden wir ausgehend von $I \in \mathrm{Ob}(\mathrm{C}^{(3,\mathrm{ires})}(\mathcal{A}))$ folgendes Diagramm

$$\begin{tikzcd}
& & {} & {} & {} & {} \\
& I_{1/1} \arrow[r] & I_{2/1} \arrow[u, dotted] \arrow[r] & I_{3/1} \arrow[u, dotted] \arrow[r] & I_{{}^{0+1}/1} \arrow[u, dotted] \arrow[r] & I_{{}^{1+1}/1} \arrow[u, dotted] \\
I_{0/0} \arrow[r] & I_{1/0} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & I_{2/0} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & I_{3/0} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & I_{{}^{0+1}/0} \arrow[u] & \\
& X_1 \arrow[u, "b_1"] \arrow[r, "x_1"] \arrow[ur, phantom, "\llcorner"] & X_2 \arrow[u, "b_2"] \arrow[r] \arrow[ur, phantom, "\llcorner"] & 0_{\mathcal{A}} \arrow[u, "b_3"] & &
\end{tikzcd}$$

In diesem sind in der unteren Zeile Pullbacks eingefügt worden. Wir setzen

$$I\,\mathrm{Pb}^{(3)} \;:=\; (X_1 \xrightarrow{x_1} X_2).$$

Analog für unser allgemeines n.

Sei

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \;:=\; \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}) / \operatorname{Ker}(\mathrm{Pb}^{(n)}).$$

Wir bekommen eine induzierte Äquivalenz

$$\overline{\mathrm{Pb}}^{(n)}\colon\; \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \to \big(\dot{\Delta}_{n-1}, \mathcal{A}\big).$$

Sei

$$\mathrm{IRes}^{(n)}\colon \big(\dot{\Delta}_{n-1}, \mathcal{A}\big) \to \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A})$$

eine dazu bis auf Isotransformationen inverse Äquivalenz, genannt *injektive Auflösungsäquivalenz.*

Sei $\mathrm{C}^{(n,1)}(\mathcal{A}) \subseteq \mathrm{C}^{(n)}(\mathcal{A})$ die volle Teilkategorie, die aus endlichen direkten Summen von n-Komplexen besteht, die über kombinatorisch definierte Funktoren aus $\mathrm{C}^{(1)}(\mathcal{A})$ stammen.

Wir haben

$$\operatorname{Ker}(\mathrm{Pb}^{(n)}) = \mathrm{C}^{(n,1)}(\mathcal{A}) \cap \mathrm{C}^{(n,\mathrm{ires})}(\mathcal{A}).$$

Sei

$$\mathrm{K}^{(n/1)}(\mathcal{A}) \;:=\; \mathrm{C}^{(n)}(\mathcal{A}) / \mathrm{C}^{(n,1)}(\mathcal{A}).$$

Wir bekommen einen vollen, treuen und additiven Funktor

$$\mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\bar{J}} \mathrm{K}^{(n/1)}(\mathcal{A}).$$

Insgesamt haben wir

$$(\dot{\Delta}_{n-1}, \mathcal{A}) \xrightarrow[\sim]{\mathrm{IRes}^{(n)}} \mathrm{K}^{(n,\mathrm{ires})}(\mathcal{A}) \xrightarrow{\bar{J}} \mathrm{K}^{(n/1)}(\mathcal{A})$$

erhalten.

Im Fall $n = 2$ spezialisiert dies zu in der klassischen homologischen Algebra benötigten Funktoren.

Erklärung

Ich versichere hiermit, dass ich die Arbeit selbstständig und nur mit den angegebenen Hilfsmitteln angefertigt habe und dass alle Stellen, die dem Wortlaut oder dem Sinne nach anderen Werken entnommen sind, durch Angabe der Quellen als Entlehnungen kenntlich gemacht worden sind. Die eingereichte Arbeit ist weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen. Das elektronische Exemplar stimmt mit den anderen Exemplaren überein.

Ort, Datum

Unterschrift