Semisimplicial and simplicial homotopy and a resolution functor

Master Thesis

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Chapter 0

Introduction

0.1 Simplicial and semisimplicial sets

Let Δ be the category with objects the ordered sets $[n] = \{0, 1, \ldots, n\}$ and with morphism the monotone maps between them. A simplicial set is a functor X from Δ^{op} to Set. So $X \in \mathrm{Ob}(\mathrm{Set}^{\Delta^{\mathrm{op}}}) = \mathrm{Ob}(\mathrm{Simp}(\mathrm{Set}))$. The image X_n of $[n]$ is interpreted as the set of n-simplices. The image of the injective monotone map $\partial_i^n : [n-1] \to [n]$ that leaves out $i \in [n]$ is the map from X_n to X_{n-1} that sends an n-simplex to its i-th face, which is an $(n-1)$-simplex. The image of the surjective monotone map $\sigma_i^n : [n+1] \to [n]$ that doubles $i \in [n]$ is the map from X_n to X_{n+1} that sends an n-simplex to a degenerate $(n+1)$-simplex.

Let $\Delta_{\mathrm{inj}} \subseteq \Delta$ be the subcategory consisting of the injective monotone maps of Δ. A semisimplicial set is a functor X from $\Delta_{\mathrm{inj}}^{\mathrm{op}}$ to Set. So $X \in \mathrm{Ob}(\mathrm{Set}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}}) = \mathrm{Ob}(\mathrm{SemiSimp}(\mathrm{Set}))$. Again, the image X_n of $[n]$ is interpreted as the set of n-simplices. We still have the face maps at our disposal. But we no longer have the possibility of forming degenerate simplices.

0.2 Simplicial and semisimplicial objects

Let $\mathcal{C}$ be a category.

A simplicial object in $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. The functor category $\mathrm{Simp}(\mathcal{C}) := \mathcal{C}^{\Delta^{\mathrm{op}}}$ is called the category of simplicial objects in $\mathcal{C}$.

A semisimplicial object in $\mathcal{C}$ is a functor $X : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$. The functor category $\mathrm{SemiSimp}(\mathcal{C}) := \mathcal{C}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}}$ is called the category of semisimplicial objects in $\mathcal{C}$.

0.3 Simplicial resolutions

Consider a category $\mathcal{C}$ with finite limits. Suppose given a resolving subcategory $\mathcal{P} \subseteq \mathcal{C}$. That means that $\mathcal{P} \subseteq \mathcal{C}$ is a full subcategory and that for each $X \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $P \xrightarrow{p} X$ that is $\mathcal{P}$-epic, where $P \in \mathrm{Ob}(\mathcal{P})$. Here, a morphism $A \xrightarrow{f} B$ in $\mathcal{C}$ is called $\mathcal{P}$-epic if for each morphism $P \xrightarrow{g} B$ with $P \in \mathrm{Ob}(\mathcal{P})$ there exists a lift of g along f. For example, in $\mathcal{C} = R$-Mod we have the resolving subcategory of projective modules. Then $\mathcal{P}$-epic morphisms are just surjective R-linear maps.

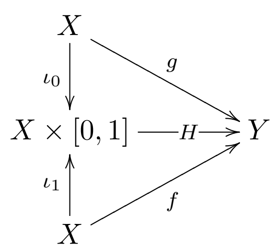
Given $X \in \mathrm{Ob}(\mathcal{C})$, there exists a simplicial resolution by an infinite process of alternately choosing $\mathcal{P}$-epic morphisms from an object in $\mathcal{P}$ and forming simplicial kernels, which is due to Tierney and Vogel; cf. [**2,** p. 3], see also [**1,** §3.2]. It is analogous to the process of projective resolution in the category R-Mod. Instead of obtaining a complex, we obtain a semisimplicial object in $\mathcal{P}$, called a semisimplicial resolution of X.

Consider the forgetful functor $\mathcal{V} : \mathrm{Simp}(\mathcal{P}) \to \mathrm{SemiSimp}(\mathcal{P})$. If $\mathcal{P}$ is required to have finite coproducts one can show that $\mathcal{V}$ has a left adjoint, which we call $\mathcal{F} : \mathrm{SemiSimp}(\mathcal{P}) \to \mathrm{Simp}(\mathcal{P})$, cf. [**1,** Proposition 65]. So applying $\mathcal{F}$ to our semisimplicial resolution yields a simplicial resolution of X.

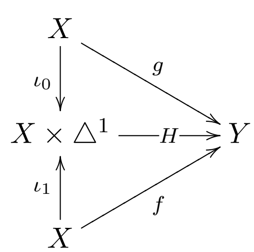
We aim to turn this construction into a functor from $\mathcal{C}$ to the homotopy category of $\mathrm{Simp}(\mathcal{P})$. The construction is done in two steps, the first due to Tierney and Vogel, cf. [**2,** Theorem 2.4], the second being taken care of here.

0.4 Homotopy categories

Recall that continuous maps $f, g : X \to Y$ between topological spaces are homotopic if there exists a continuous function $H : X \times [0,1] \to Y$, called homotopy, making the following diagram commutative.



This motivates a definition of elementary homotopy between morphisms in the category Simp(Set) as follows. An elementary homotopy from $(f : X \to Y) \in \mathrm{Mor}(\mathrm{Simp}(\mathrm{Set}))$ to $(g : X \to Y) \in \mathrm{Mor}(\mathrm{Simp}(\mathrm{Set}))$ is a simplicial map $H : X \times \triangle^1 \to Y$ making the diagram



commutative, where ι_0, ι_1 are suitably defined inclusions and where $\triangle^1 := {}_{\Delta}(-,[1])$ is the simplicial version of an interval. One can show that the existence of an elementary homotopy is equivalent to the existence of a tuple $((X_\ell \xrightarrow{h_i^\ell} Y_{\ell+1})_{i\in[0,\ell]})_{\ell \geqslant 0}$ satisfying certain relations that involve the face maps, the degeneracy maps and the components of f and g.

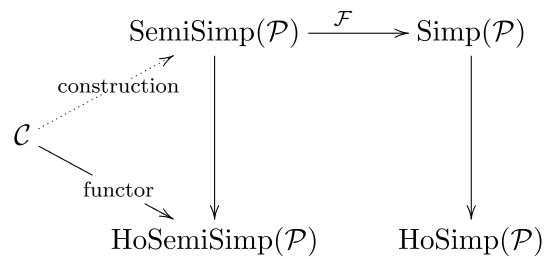
This can be cut down to a definition of elementary homotopy between morphisms of semisimplicial sets by leaving out the conditions involving degeneracy maps.

Such tuples are also useable to define elementary homotopy for $\mathrm{Simp}(\mathcal{C})$ and $\mathrm{SemiSimp}(\mathcal{C})$ for an arbitrary category $\mathcal{C}$.

The relation of elementary homotopy on $\mathrm{Simp}(\mathcal{C})$ generates a congruence, called homotopy, and thus yields a homotopy category $\mathrm{HoSimp}(\mathcal{C})$. Analogously, we obtain a homotopy category $\mathrm{HoSemiSimp}(\mathcal{C})$.

0.5 The simplicial resolution functor

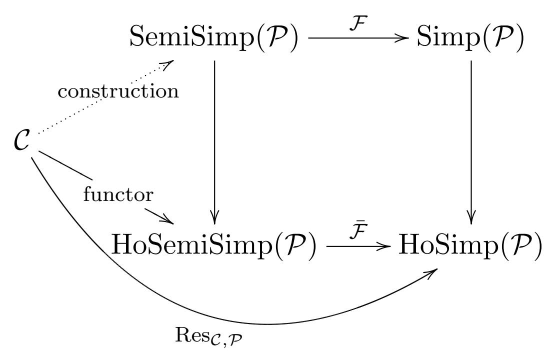
We keep the situation of §0.3. Tierney and Vogel show the existence of a semisimplicial resolution functor $\mathcal{C} \to \mathrm{HoSemiSimp}(\mathcal{P})$, cf. [**2,** p. 6], see also §4. So we are in the following situation.



We need a functor $\bar{\mathcal{F}} : \mathrm{HoSemiSimp}(\mathcal{P}) \to \mathrm{HoSimp}(\mathcal{P})$ on homotopy categories induced by $\mathcal{F}$. For this we need that $\mathcal{F}$ maps homotopic morphisms to homotopic morphisms.

This is shown in Proposition 37, its main ingredient being Proposition 25 for $\mathcal{C} = \mathrm{Set}$ and Proposition 30 for general $\mathcal{C}$. Whereas Proposition 30 has Proposition 25 as special case, we nonetheless kept the proof of Proposition 25, since it proceeds using elements. The proof of Proposition 30, in contrast, has to work with morphisms.

So we obtain the following diagram.



The composite functor $\mathrm{Res}_{\mathcal{C},\mathcal{P}}$ is the sought resolution functor.

0.6 Dold-Puppe-Kan

Let R be a ring. Let $\mathcal{A} := R$-Mod. Let $\mathrm{C}(\mathcal{A})_{\geqslant 0}$ be the category of complexes over $\mathcal{A}$ bounded above at position 0. The functor $\mathcal{N} : \mathrm{C}(\mathcal{A})_{\geqslant 0} \to \mathrm{SemiSimp}(\mathcal{A})$ adds, roughly speaking, to a complex a couple of zero maps, such that it becomes a semisimplicial R-module. The Moore complex functor $\mathcal{M} : \mathrm{SemiSimp}(\mathcal{A}) \to \mathrm{C}(\mathcal{A})_{\geqslant 0}$ replaces, roughly speaking, the modules in a semisimplicial R-module by the intersection of the kernels of all face maps starting in this module except one and takes the remaining face map as differential at the respective position to form a complex. The Dold-Puppe-Kan correspondence is the statement that $\mathcal{F} \circ \mathcal{N}$ and $\mathcal{M} \circ \mathcal{V}$ are mutually inverse equivalences of categories;

cf. [**5,** Ths. 8.1, 8.2], [**6,** Th. 1.5] and, for abelian categories, [**7,** Satz 3.6].

$$\mathrm{C}(\mathcal{A})_{\geqslant 0} \underset{\mathcal{M}}{\overset{\mathcal{N}}{\underset{\perp}{\rightleftarrows}}} \mathrm{SemiSimp}(\mathcal{A}) \underset{\mathcal{V}}{\overset{\mathcal{F}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Simp}(\mathcal{A}), \qquad \mathrm{C}(\mathcal{A})_{\geqslant 0} \underset{\mathcal{M}\circ\mathcal{V}}{\overset{\mathcal{F}\circ\mathcal{N}}{\underset{\sim}{\rightleftarrows}}} \mathrm{Simp}(\mathcal{A})$$

The isotransformations that arise are constructed directly, without reference to a stepwise procedure.

We show that $\mathcal{N}$ is not an equivalence of categories. Consequently, neither are $\mathcal{F}$, $\mathcal{M}$ nor $\mathcal{V}$.

We show that $\mathcal{N}$ is not compatible with homotopy and thus does not induce a functor from $\mathrm{K}(\mathcal{A})_{\geqslant 0}$ to $\mathrm{HoSemiSimp}(\mathcal{A})$.

Chapter 1

Conventions

- We use the notation of [**1**].
- All categories $\mathcal{C}$ under consideration are small. This means that $\mathrm{Mor}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{C})$ are sets. We choose universes $\mathcal{U} \in \mathcal{V}$ such that the categories Set of sets in $\mathcal{U}$ and R-Mod of left modules over a ring R in $\mathcal{U}$ become small, with respect to $\mathcal{V}$.
- Composition of morphisms is written naturally: the composite of morphisms $X \xrightarrow{a} Y \xrightarrow{b} Z$ is written $X \xrightarrow{a \cdot b} Y$, or sometimes $ab := a \cdot b$. Contrarily, composition of functors is written traditionally: the composite of functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is written $G \circ F$.
- Rings are unitary.
- For $i \in \mathbb{Z}$ we write the set $\{j \in \mathbb{Z} : 0 \leqslant j \leqslant i\} =: [0, i]$. We sometimes write $[i] := [0, i]$.
- For $i \in \mathbb{Z}$ we write the set $\{j \in \mathbb{Z} : j \geqslant i\} =: \mathbb{Z}_{\geqslant i}$.
- Suppose given a category $\mathcal{C}$.

  Suppose given $a, b \in \mathbb{Z}$ such that $a \leqslant b + 1$ and morphisms $(X_i \xrightarrow{\alpha_i} X_{i+1})$ in $\mathcal{C}$ for $i \in [a, b]$. We write
  $$ {}^{X_a}\prod_{i \in \lceil a\,,\,b \rceil}^{X_{b+1}} \alpha_i := \alpha_a \cdots \alpha_b : X_a \to X_{b+1} \qquad \text{if } a \leqslant b $$
  and
  $$ {}^{X_a}\prod_{i \in \lceil a\,,\,b \rceil}^{X_{b+1}} \alpha_i := \mathrm{id}_{X_a} : X_a \to X_a \qquad \text{if } a = b + 1\,. $$

  Suppose given $a, b \in \mathbb{Z}$ such that $a \leqslant b + 1$ and morphisms $(X_i \xleftarrow{\alpha_i} X_{i+1})$ in $\mathcal{C}$ for $i \in [a, b]$. We write
  $$ {}^{X_{b+1}}\prod_{i \in \lfloor b\,,\,a \rfloor}^{X_a} \alpha_i := \alpha_b \cdots \alpha_a : X_{b+1} \to X_a \qquad \text{if } a \leqslant b $$
  and
  $$ {}^{X_{b+1}}\prod_{i \in \lfloor b\,,\,a \rfloor}^{X_a} \alpha_i := \mathrm{id}_{X_a} : X_a \to X_a \qquad \text{if } a = b + 1\,. $$
- The category Δ has as objects intervals $[n] := [0, n] = \{\, k \in \mathbb{Z} \,:\, 0 \leqslant k \leqslant n \,\}$ for $n \geqslant 0$ and as morphisms monotone maps. The category Δ is called the *simplex category.*
- The category $\Delta_{\mathrm{inj}} \subseteq \Delta$ is the subcategory of injective monotone maps.

- Let $\mathcal{C}$ be category. We have $\operatorname{Simp}(\mathcal{C}) := \mathcal{C}^{\Delta^{\mathrm{op}}}$ the category of simplicial objects and morphisms in $\mathcal{C}$. We have $\operatorname{SemiSimp}(\mathcal{C}) := \mathcal{C}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}}$ the category of semisimplicial objects and morphisms in $\mathcal{C}$.

- We write $\operatorname{surj} := \{f \in \operatorname{Mor}(\Delta) : f \text{ is surjective}\} \subseteq \operatorname{Mor}(\Delta)$.

- For $n \geqslant 0$ and $i \in [0,n]$ we have a map

$$\begin{array}{rccl} \sigma_i^n : & [n+1] & \to & [n] \\ & j & \mapsto & \begin{cases} j & \text{for } j \leqslant i \\ j-1 & \text{for } j > i \end{cases} \end{array}$$

For a simplicial object $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ we write $s_i^{X,n} := X_{\sigma_i^n}$. We sometimes we just write $s_i := s_i^{X,n}$.

The following relations hold for $n \geqslant 0$:

$$\sigma_j^{n+1} \cdot \sigma_i^n = \sigma_i^{n+1} \cdot \sigma_{j-1}^n \text{ for } 0 \leqslant i < j \leqslant n+2$$

and

$$\begin{array}{ll} \partial_j^{n+1} \cdot \sigma_i^n = \sigma_{i-1}^{n-1} \cdot \partial_j^n & \text{for } i \in [0,n] \text{ and } j \in [0,i-1] \\ \partial_j^{n+1} \cdot \sigma_i^n = \mathrm{id}_{[n]} & \text{for } i \in [0,n] \text{ and } j \in [i,i+1] \\ \partial_j^{n+1} \cdot \sigma_i^n = \sigma_i^{n-1} \cdot \partial_{j-1}^n & \text{for } i \in [0,n] \text{ and } j \in [i+2,n+1] \end{array}$$

So for a simplicial object $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ the following relations hold for $n \geqslant 0$:

$$s_i^{X,n} \cdot s_j^{X,n+1} = s_{j-1}^{X,n} \cdot s_i^{X,n+1} \text{ for } 0 \leqslant i < j \leqslant n+2$$

and

$$\begin{array}{ll} s_i^{X,n} \cdot d_j^{X,n+1} = d_j^{X,n} \cdot s_{i-1}^{X,n-1} & \text{for } i \in [0,n] \text{ and } j \in [0,i-1] \\ s_i^{X,n} \cdot d_j^{X,n+1} = \mathrm{id}_{X_n} & \text{for } i \in [0,n] \text{ and } j \in [i,i+1] \\ s_i^{X,n} \cdot d_j^{X,n+1} = d_{j-1}^{X,n} \cdot s_i^{X,n-1} & \text{for } i \in [0,n] \text{ and } j \in [i+2,n+1] \end{array}$$

We will use these relations without reference.

- Let A be a statement. We set

$$\lfloor A \rfloor := \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false}. \end{cases}$$

- For a set A and $n \geqslant 0$, we write $A^{\times n} := A \times \cdots \times A$ for the n-fold cartesian product of A.

- Concerning indices, we sometimes write as $a_b =: a(b)$.

- Suppose given a category $\mathcal{C}$ and $X \in \operatorname{Ob}(\mathcal{C})$.

  We write the Hom-functor from $\mathcal{C}$ to Set mapping $(Y \xrightarrow{f} Z)$ to ${}_{\mathcal{C}}(X,Y) \to {}_{\mathcal{C}}(X,Z), g \mapsto gf$ as ${}_{\mathcal{C}}(X,-)$.

  We write the Hom-functor from $\mathcal{C}^{\mathrm{op}}$ to Set mapping $(Y \xrightarrow{f} Z)$ to ${}_{\mathcal{C}}(Y,X) \to {}_{\mathcal{C}}(Z,X), g \mapsto fg$ as ${}_{\mathcal{C}}(-,X)$.

- Transformations that are isomorphisms in the functor category are called isotransformations.

Chapter 2

Preliminaries

2.1 A remark on equivalence relations

Remark 1. Suppose given a set X. Suppose given a relation $(\rightsquigarrow) \subseteq X \times X$. Let $(\sim) \subseteq X \times X$ be the equivalence relation generated by $(\rightsquigarrow)$, i.e. the intersection of all equivalence relations containing $(\rightsquigarrow)$. Suppose given a map $f : X \to Y$ satisfying $xf = x'f$ for $x, x' \in X$ with $x \rightsquigarrow x'$. Then we have $xf = x'f$ whenever given $x, x' \in X$ such that $x \sim x'$. In particular, we have the map $\bar{f} : X/(\sim) \to Y$, $[x] \mapsto [x]\bar{f} := xf$.

Proof. Let the relation $(\underset{f}{\sim}) \subseteq X \times X$ be defined by $x \underset{f}{\sim} x' :\Leftrightarrow xf = x'f$, where $x, x' \in X$. We have $(\rightsquigarrow) \subseteq (\underset{f}{\sim})$ and $(\underset{f}{\sim})$ is an equivalence relation. Therefore we have $(\sim) \subseteq (\underset{f}{\sim})$. $\square$

2.2 Congruences and precongruences on categories

Let $\mathcal{C}$ and $\mathcal{D}$ be a categories.

Definition 2. A *precongruence* on $\mathcal{C}$ is a relation $(\rightsquigarrow)$ on $\mathrm{Mor}(\mathcal{C})$ having the properties (Con 1–2).

(Con 1) Given $f, g \in \mathrm{Mor}(\mathcal{C})$ such that $f \rightsquigarrow g$, we have $\mathrm{Source}(f) = \mathrm{Source}(g)$ and $\mathrm{Target}(f) = \mathrm{Target}(g)$.

(Con 2) Given

$$X' \xrightarrow{u} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{v} Y'$$

in $\mathcal{C}$ such that $f \rightsquigarrow g$, we have $ufv \rightsquigarrow ugv$.

A precongruence $(\sim)$ on $\mathcal{C}$ is called a *congruence* if it is, in addition, an equivalence relation of $\mathrm{Mor}(\mathcal{C})$.

The equivalence class of $f \in \mathrm{Mor}(\mathcal{C})$ with respect to $(\sim)$ is then written $[f]_\sim$ or just $[f]$.

Lemma 3. Suppose given a congruence $(\sim)$ on $\mathcal{C}$.

We aim to form the category $\mathcal{C}/(\sim)$ as follows.

Let $\mathrm{Ob}(\mathcal{C}/(\sim)) := \mathrm{Ob}(\mathcal{C})$.

Let $\mathrm{Mor}(\mathcal{C}/(\sim)) := \mathrm{Mor}(\mathcal{C})/(\sim) = \{[f] \,:\, f \in \mathrm{Mor}(\mathcal{C})\}$.

Let Source($[f]$) := Source(f) and Target($[f]$) := Target(f) for $f \in \mathrm{Mor}(\mathcal{C})$.

Given

$$X \xrightarrow{[f]} Y \xrightarrow{[g]} Z$$

in $\mathcal{C}/(\sim)$, we let their composite be defined by $[f][g] := [fg]$.

The following assertions (1–3) hold.

(1) We have the category $\mathcal{C}/(\sim)$, called the *factor category* of $\mathcal{C}$ modulo $(\sim)$.

(2) We have the *residue class functor* $R_{\mathcal{C},(\sim)} : \mathcal{C} \to \mathcal{C}/(\sim)$ that maps $X \xrightarrow{f} Y$ to $X \xrightarrow{[f]} Y$.

(3) Given a functor $F : \mathcal{C} \to \mathcal{D}$ such that $Ff = Ff'$ for $f,\ f' \in \mathrm{Mor}(\mathcal{C})$ with $f \sim f'$, we have a unique functor $\bar{F} : \mathcal{C}/(\sim) \to \mathcal{D}$ such that $F = \bar{F} \circ R_{\mathcal{C},(\sim)}$.

The functor $\bar{F}$ maps $X \xrightarrow{[f]} Y$ to $(\bar{F}X \xrightarrow{\bar{F}[f]} \bar{F}Y) = (FX \xrightarrow{Ff} FY)$.

Proof. Ad (1). We show that composition is well-defined. Suppose given $X \xrightarrow{f} Y$ and $X \xrightarrow{f'} Y$ such that $[f] = [f']$ and $Y \xrightarrow{g} Z$ and $Y \xrightarrow{g'} Z$ such that $[g] = [g']$. We have $fg \sim fg' \sim f'g'$, by (Con 2). So it follows that $[fg] = [f'g']$. This shows that composition is well-defined.

Furthermore, given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} N$ in $\mathcal{C}$, we obtain $([f][g])[h] = [fg][h] = [(fg)h] = [f(gh)] = [f][gh] = [f]([g][h])$. So associativity of the composition holds.

Moreover, for $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$ we have $[f][\mathrm{id}_Y] = [f\,\mathrm{id}_Y] = [f]$ and $[\mathrm{id}_Y][g] = [\mathrm{id}_Y\,g] = [g]$. So we have $\mathrm{id}_X^{\mathcal{C}/(\sim)} = [\mathrm{id}_X^{\mathcal{C}}]$ for $X \in \mathrm{Ob}(\mathcal{C})$.

Ad (3). *Uniqueness.* Suppose given a functor $\tilde{F} : \mathcal{C} \to \mathcal{D}$ with $F = \tilde{F} \circ R_{\mathcal{C},(\sim)}$. We have $\tilde{F}[f] = (\tilde{F} \circ R_{\mathcal{C},(\sim)})(f) = Ff$ for $f \in \mathrm{Mor}(\mathcal{C})$.

Existence. We define $\bar{F}(X \xrightarrow{[f]} Y) := (FX \xrightarrow{Ff} FY)$ for $(X \xrightarrow{[f]} Y)$ in $\mathcal{C}/(\sim)$. To show that this is well-defined, we note that $[f] = [f']$ implies $Ff = Ff'$. To prove functoriality, we note that $\bar{F}([f] \cdot [g]) = \bar{F}([f \cdot g]) = F(f \cdot g) = Ff \cdot Fg = \bar{F}[f] \cdot \bar{F}[g]$ for $(\,X \xrightarrow{f} Y \xrightarrow{g} Z\,)$ in $\mathcal{C}$ and $\bar{F}[\mathrm{id}_Y] = F\,\mathrm{id}_Y = \mathrm{id}_{FY} = \mathrm{id}_{\bar{F}Y}$ for $Y \in \mathrm{Ob}(\mathcal{C}/(\sim))$. □

Lemma 4. Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$. Then for $f, g \in \mathrm{Mor}(\mathcal{C})$ the relation

$$f \underset{F}{\sim} g :\Leftrightarrow (Ff = Fg \text{ and } \mathrm{Source}(f) = \mathrm{Source}(g) \text{ and } \mathrm{Target}(f) = \mathrm{Target}(g))$$

defines a congruence $(\underset{F}{\sim})$ on $\mathcal{C}$. We call this congruence the *congruence induced by* F.

Proof. The relation $(\underset{F}{\sim})$ is an equivalence relation on $\mathrm{Mor}(\mathcal{C})$.

Ad (Con 1). This follows from the construction.

Ad (Con 2). Suppose given $X' \xrightarrow{u} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{v} Y'$ in $\mathcal{C}$ such that $f \underset{F}{\sim} g$. Then Source(ufv) $= X' =$ Source(ugv) and Target(ufv) $= Y' =$ Target(ugv) and $F(ufv) = Fu \cdot Ff \cdot Fv = Fu \cdot Fg \cdot Fv = F(ugv)$. So $ufv \underset{F}{\sim} ugv$. □

Lemma 5. Suppose given a precongruence $(\rightsquigarrow)$ on $\mathcal{C}$.

Let $(\sim)$ be the equivalence relation on $\mathrm{Mor}(\mathcal{C})$ generated by $(\rightsquigarrow)$. Let $(\underset{X,Y}{\sim})$ be the equivalence relation on ${}_{\mathcal{C}}(X,Y)$ generated by $(\rightsquigarrow) \cap {}_{\mathcal{C}}(X,Y)^{\times 2}$.

Let $(\underset{\sqcup}{\sim}) := \dot{\bigcup}_{(X,Y)\in \mathrm{Ob}(\mathcal{C})^{\times 2}} (\underset{X,Y}{\sim})$.

Then $(\underset{\sqcup}{\sim}) = (\sim)$.

Proof. Ad $(\subseteq)$. It suffices to show that $(\underset{X,Y}{\sim}) \overset{!}{\subseteq} (\sim)$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$, i.e. $(\underset{X,Y}{\sim}) \overset{!}{\subseteq} (\sim) \cap {}_{\mathcal{C}}(X,Y)^{\times 2}$. We have $(\rightsquigarrow) \subseteq (\sim)$ and hence $(\rightsquigarrow) \cap {}_{\mathcal{C}}(X,Y)^{\times 2} \subseteq (\sim) \cap {}_{\mathcal{C}}(X,Y)^{\times 2}$. Since $(\sim) \cap {}_{\mathcal{C}}(X,Y)^{\times 2}$ is an equivalence relation on ${}_{\mathcal{C}}(X,Y)$ it follows that $(\underset{X,Y}{\sim}) \subseteq (\sim) \cap {}_{\mathcal{C}}(X,Y)^{\times 2}$.

Ad $(\supseteq)$. We show that $(\rightsquigarrow) \overset{!}{\subseteq} (\underset{\sqcup}{\sim})$. Suppose given $f, g \in \mathrm{Mor}(\mathcal{C})$ such that $f \rightsquigarrow g$. Then $\mathrm{Source}(f) = \mathrm{Source}(g) =: X$ and $\mathrm{Target}(f) = \mathrm{Target}(g) =: Y$ by (Con 1). Hence $f \underset{X,Y}{\sim} g$ and thus $f \underset{\sqcup}{\sim} g$. We show that $(\underset{\sqcup}{\sim})$ is an equivalence relation on $\mathrm{Mor}(\mathcal{C})$. To show reflexivity we note that for $(X \xrightarrow{f} Y) \in \mathrm{Mor}(\mathcal{C})$ we have $f \in {}_{\mathcal{C}}(X,Y)$ and therefore $f \underset{X,Y}{\sim} f$ and therefore $f \underset{\sqcup}{\sim} f$. To show symmetry we note that for $f, g \in \mathrm{Mor}(\mathcal{C})$ satisfying $f \underset{\sqcup}{\sim} g$ there exist $X, Y \in \mathrm{Ob}(\mathcal{C})$ such that $f \underset{X,Y}{\sim} g$, whence $g \underset{X,Y}{\sim} f$ and therefore $g \underset{\sqcup}{\sim} f$. To show transitivity we consider $f, g, h \in \mathrm{Mor}(\mathcal{C})$ satisfying $f \underset{\sqcup}{\sim} g$ and $g \underset{\sqcup}{\sim} h$. So there exist $X, Y \in \mathrm{Ob}(\mathcal{C})$ such that $f \underset{X,Y}{\sim} g$ and $X', Y' \in \mathrm{Ob}(\mathcal{C})$ such that $g \underset{X',Y'}{\sim} h$. We have $X = \mathrm{Source}(g) = X'$ and $Y = \mathrm{Target}(g) = Y'$. Hence $g \underset{X,Y}{\sim} h$. So $f \underset{X,Y}{\sim} h$ and thus $f \underset{\sqcup}{\sim} h$. So it follows that $(\sim) \subseteq (\underset{\sqcup}{\sim})$.

□

Lemma 6. Suppose given a precongruence $(\rightsquigarrow)$ on $\mathcal{C}$. Let $(\sim)$ be the equivalence relation on $\mathrm{Mor}(\mathcal{C})$ generated by $(\rightsquigarrow)$. Then $(\sim)$ is a congruence.

Proof. Let $(\underset{X,Y}{\sim})$ be the equivalence relation on ${}_{\mathcal{C}}(X,Y)$ generated by $(\rightsquigarrow) \cap {}_{\mathcal{C}}(X,Y)^{\times 2}$. Let $(\underset{\sqcup}{\sim}) := \dot{\bigcup}_{(X,Y)\in \mathrm{Ob}(\mathcal{C})^{\times 2}} (\underset{X,Y}{\sim})$. By Lemma 5 it holds that $(\underset{\sqcup}{\sim}) = (\sim)$.

Ad (Con 1). Suppose given $f, g \in \mathrm{Mor}(\mathcal{C})$ such that $f \sim g$, i.e. $f \underset{\sqcup}{\sim} g$. Then there exist $X, Y \in \mathrm{Ob}(\mathcal{C})$ such that $f \underset{X,Y}{\sim} g$. Hence $\mathrm{Target}(f) = Y = \mathrm{Target}(g)$ and $\mathrm{Source}(f) = X = \mathrm{Source}(g)$.

Ad (Con 2). Suppose given $X' \xrightarrow{u} X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{v} Y'$ in $\mathcal{C}$ such that $f \sim g$. We have to show that $ufv \overset{!}{\sim} ugv$.

Consider the map $t : {}_{\mathcal{C}}(X,Y) \to {}_{\mathcal{C}}(X',Y')/(\sim)$, $h \mapsto [uhv]$. For $h_1, h_2 \in {}_{\mathcal{C}}(X,Y)$ such that $h_1 \rightsquigarrow h_2$ we have $uh_1v \rightsquigarrow uh_2v$ by (Con 2). Hence $uh_1v \sim uh_2v$, i.e. $[uh_1v] = [uh_2v]$. By Remark 1 the map t has the property that for $h_1 \underset{X,Y}{\sim} h_2$, we have $(h_1)t = (h_2)t$.

Since it follows from $f \sim g$ that$f \underset{\sqcup}{\sim} g$ and thus $f \underset{X,Y}{\sim} g$, one gets $(f)t = (g)t$, i.e. $ufv \sim ugv$. □

Definition 7. Let $(\rightsquigarrow)$ be a precongruence on $\mathcal{C}$. Let $(\sim)$ be the equivalence relation on $\mathrm{Mor}(\mathcal{C})$

generated by $(\rightsquigarrow)$. By Lemma 6, $(\sim)$ is a congruence on $\mathcal{C}$. We say that $(\sim)$ is the *congruence generated by* $(\rightsquigarrow)$.

Lemma 8. Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$. Let $(\rightsquigarrow)$ be a precongruence on $\mathcal{C}$. Suppose that F has the property that for $f, g \in \mathrm{Mor}(\mathcal{C})$ with $f \rightsquigarrow g$, we have $Ff = Fg$. Let $(\sim)$ be the congruence generated by $(\rightsquigarrow)$. Recall the functor $R_{\mathcal{C},(\sim)} : \mathcal{C} \to \mathcal{C}/(\sim)$, cf. Lemma 3.

There exists a unique functor $\bar{F} : \mathcal{C}/(\sim) \to \mathcal{D}$ such that $\bar{F} \circ R_{\mathcal{C},(\sim)} = F$.

The functor $\bar{F}$ maps $X \xrightarrow{[f]} Y$ to $(\bar{F}X \xrightarrow{\bar{F}[f]} \bar{F}Y) = (FX \xrightarrow{Ff} FY)$.

Proof. By Lemma 3 it suffices to show that for $f, g \in \mathrm{Mor}(\mathcal{C})$ such that $f \sim g$ we have $Ff = Fg$.

But this follows from Remark 1, applied to $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, $f \mapsto Ff$. □

Lemma 9. Let $(\sim)$ be a congruence on $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors such that for $f, g \in \mathrm{Mor}(\mathcal{C})$ satisfying $f \sim g$ we have $Ff = Fg$ and $Gf = Gg$. Let $\alpha = (\alpha_X)_{X \in \mathrm{Ob}(\mathcal{C})} : F \to G$ be a transformation.

By Lemma 3 we have the functors $\bar{F} : \mathcal{C}/(\sim) \to \mathcal{D}$ and $\bar{G} : \mathcal{C}/(\sim) \to \mathcal{D}$ satisfying $\bar{F} \circ R_{\mathcal{C},(\sim)} = F$ and $\bar{G} \circ R_{\mathcal{C},(\sim)} = G$. Recall that $\mathrm{Ob}(\mathcal{C}/(\sim)) = \mathrm{Ob}(\mathcal{C})$. The tuple $\bar{\alpha} := (\alpha_X)_{X \in \mathrm{Ob}\,\mathcal{C}/(\sim)}$ is the unique transformation from $\bar{F}$ to $\bar{G}$ satisfying $\bar{\alpha} R_{\mathcal{C},(\sim)} = \alpha$.

Proof. Existence. We show that $\bar{\alpha}$ is a transformation.

Let $(X \xrightarrow{[f]} Y) \in \mathrm{Mor}(\mathcal{C}/(\sim))$. We have $\bar{\alpha}_X \cdot \bar{G}[f] = \alpha_X \cdot \bar{G}[f] = \alpha_X \cdot (\bar{G} \circ R_{\mathcal{C},\sim})(f) = \alpha_X \cdot Gf = Ff \cdot \alpha_Y = (\bar{F} \circ R_{\mathcal{C},(\sim)})(f) \cdot \bar{\alpha}_Y = \bar{F}[f] \cdot \bar{\alpha}_Y$.

Furthermore for $X \in \mathrm{Ob}(\mathcal{C})$ we have $(\bar{\alpha} R_{\mathcal{C},\sim})_X = \alpha_{R_{\mathcal{C},\sim} X} = \alpha_X$.

Uniqueness. Suppose that a tranformation $\beta : \bar{F} \to \bar{G}$ satisfies $\beta \bar{\alpha} R_{\mathcal{C},(\sim)} = \alpha$.

For $X \in \mathrm{Ob}(\mathcal{C})$ we have $\beta_X = \beta_{R_{\mathcal{C},(\sim)} X} = (\beta R_{\mathcal{C},(\sim)})_X = \alpha_X$. □

Chapter 3

Semisimplicial and simplicial homotopy

3.1 Two lemmas on the simplex category Δ

Lemma 10. Let $0 \leqslant k \leqslant n$. Let $[n] \xrightarrow{f} [k]$ be a surjective monotone map.

Let $D_f := \{i \in [n-1] : (i)f = (i+1)f\} \subseteq [n-1]$.

(i) We write $D_f = \{i_1, \ldots, i_{n-k}\}$, where $0 \leqslant i_1 < \cdots < i_{n-k} \leqslant n-1$. We have $f = {}^{[n]}\prod_{j \in \lfloor n-k\,,\,1 \rfloor}^{[k]} \sigma_{i_j}^{k+j-1}$.

(ii) Suppose given $0 \leqslant i_1 < \cdots < i_{n-k} \leqslant n-1$ such that $f = {}^{[n]}\prod_{j \in \lfloor n-k\,,\,1 \rfloor}^{[k]} \sigma_{i_j}^{k+j-1}$.
Then $D_f = \{i_1, \ldots, i_{n-k}\}$.

So every surjective monotone map $[n] \xrightarrow{f} [k]$ has a unique factorisation $f = \sigma_{i_{n-k}}^{n-1} \cdots \sigma_{i_1}^{k}$ such that $i_1 < \cdots < i_{n-k}$, where existence follows from (i) and uniqueness follows from (ii).

Proof.

Ad (i). We prove this by induction on $n-k$.

Let $n-k=0$. Then we have $f = \mathrm{id}_{[n]} = {}^{[n]}\prod_{j \in \lfloor 0\,,\,1 \rfloor}^{[n]} \sigma_{i_j}^{n+j-1}$. Since $f = \mathrm{id}_{[n]}$ is injective, we have $D_{\mathrm{id}_{[n]}} = \emptyset$.

Let $n-k \geqslant 1$. Suppose that (i) is proven for all surjective maps $[n-1] \to [k]$.

Since $0 \leqslant k < n$, the map f is not injective. So $D_f \neq \emptyset$. Let $e := \max D_f = i_{n-k}$.

First claim. We have $\sigma_e^{n-1} \cdot \partial_{e+1}^n \cdot f = f$. We prove this by evaluating at each element $x \in [n]$.

Case $x \leqslant e$.

We have $(x)(\sigma_e^{n-1} \cdot \partial_{e+1}^n \cdot f) = (x)(\partial_{e+1}^n \cdot f) = (x)f$.

Case $x = e+1$.

We have $(e+1)(\sigma_e^{n-1} \cdot \partial_{e+1}^n \cdot f) = (e)(\partial_{e+1}^n \cdot f) = (e)f = (e+1)f$, since $e \in D_f$.

Case $x \geqslant e+2$.

We have $(x)(\sigma_e^{n-1} \cdot \partial_{e+1}^n \cdot f) = (x-1)(\partial_{e+1}^n \cdot f) = (x)f$.

This proves the *first claim.*

Second claim. We have $D_f = D_{(\partial_{e+1}^n \cdot f)} \,\dot\cup\, \{e\}$.

Ad $(\subseteq)$.

Let $x \in D_f \setminus \{e\}$. We have $(x)(\partial^n_{e+1} \cdot f) \overset{x \leqq e}{=} (x)f = (x+1)f \overset{x+1 \leqslant e}{=} (x+1)(\partial^n_{e+1} \cdot f)$, since $x, x+1 \in [0, e]$. Hence $x \in D_{(\partial^n_{e+1} \cdot f)}$

Ad $(\supseteq)$ and ad $e \notin D_{(\partial^n_{e+1} \cdot f)}$.

Let $x \in D_{(\partial^n_{e+1} \cdot f)}$.

Assume that $x \geqslant e+1$. We have $(x+1)f = (x)(\partial^n_{e+1} \cdot f) = (x+1)(\partial^n_{e+1} \cdot f) = (x+2)f$. Hence $x+1 \in D_f$ and $x+1 > e$, which *contradicts* the maximality of e.

Assume that $x = e$. We have $(e)f = (e)(\partial^n_{e+1} \cdot f) = (e+1)(\partial^n_{e+1} \cdot f) = (e+2)f$. So $(e)f = (e+1)f = (e+2)f$, whence $e+1 \in D_f f$, which *contradicts* the maximality of e.

So we have $x \leqslant e-1$.

We have $(x)f = (x)(\partial^n_{e+1} \cdot f) = (x+1)(\partial^n_{e+1} \cdot f) = (x+1)f$. Hence $x \in D_f$. Moreover, $D_{(\partial^n_{e+1} \cdot f)} \subseteq [0, e-1]$, and thus $e \notin D_{(\partial^n_{e+1} \cdot f)}$.

So $D_f = D_{(\partial^n_{e+1} \cdot f)} \dot{\cup} \{e\}$.

This proves the *second claim.*

Since $e \in D_f$, the map $\partial^n_{e+1} \cdot f$ is surjective. So we may use the induction hypothesis to write $\partial^n_{e+1} \cdot f = {}^{[n-1]}\prod^{[k]}_{j \in \lfloor n-k-1\,,\,1 \rfloor} \sigma^{k+j-1}_{i_j}$, where $D_{(\partial^n_{e+1} \cdot f)} = \{i_1, \ldots, i_{n-k-1}\}$ and $i_1 < \cdots < i_{n-k-1}$.

Hence we have $f = \sigma^{n-1}_e \cdot \partial^n_{e+1} \cdot f = \sigma^{n-1}_e \cdot {}^{[n-1]}\prod^{[k]}_{j \in \lfloor n-k-1\,,\,1 \rfloor} \sigma^{k+j-1}_{i_j} = {}^{[n]}\prod^{[k]}_{j \in \lfloor n-k\,,\,1 \rfloor} \sigma^{k+j-1}_{i_j}$ and $D_f = \{i_1, \ldots, i_{n-k-1}\} \cup \{e\}$ and $e > i_{n-k-1} > \cdots > i_1 \geqslant 0$.

Ad (ii).

We prove this by induction on $n-k$.

Suppose that $n-k = 0$. Then we have $f = \mathrm{id}_{[n]} = {}^{[n]}\prod^{[n]}_{j \in \lfloor 0\,,\,1 \rfloor} \sigma^{n+j-1}_{i_j}$. Since $f = \mathrm{id}_{[n]}$ is injective, we have $D_{\mathrm{id}_{[n]}} = \emptyset$.

Suppose that $n-k \geqslant 1$. Suppose that (ii) is proven for all surjective maps $[n-1] \to [k]$.

Write $g := {}^{[n-k-1]}\prod^{[k]}_{j \in \lfloor n-1\,,\,1 \rfloor} \sigma^{k+j-1}_{i_j}$. By induction hypothesis we have $D_g = \{i_1, \ldots, i_{n-k-1}\}$.

We *claim* that $D_f = D_g \cup \{i_{n-k}\}$.

Ad $(\subseteq)$.

Let $x \in D_f$.

Assume that $x > i_{n-k}$.

We have $(x)f = (x)(\sigma^{n-1}_{i_{n-k}} \cdot g) = (x-1)g$ and $(x+1)f = (x+1)(\sigma^{n-1}_{i_{n-k}} \cdot g) = (x)g$.

Since $x-1 \geqslant i_{n-k} > i_{n-k-1} > \cdots > i_1$ and $D_g = \{i_1, \ldots, i_{n-k-1}\}$, we have $x-1 \notin D_g$. Hence $(x-1)g \neq (x)g$. It follows that $(x)f = (x-1)g \neq (x)g = (x+1)f$, so $x \notin D_f$. This is a *contradiction.*

So $x \leqslant i_{n-k}$.

We may assume that $x \leqslant i_{n-k} - 1$.

We have $(x)g \overset{x \leqslant i_{n-k}}{=} (x)(\sigma^{n-1}_{i_{n-k}} \cdot g) = (x)f = (x+1)f = (x+1)(\sigma^{n-1}_{i_{n-k}} \cdot g) \overset{x+1 \leqslant i_{n-k}}{=} (x+1)g$. So $x \in D_g$.

Ad $(\supseteq)$.

Let $x \in D_g$. We have $(x)f = (x)(\sigma^{n-1}_{i_{n-k}} \cdot g) \overset{x \leqslant i_{n-k}}{=} (x)g = (x+1)g \overset{x+1 \leqslant i_{n-k}}{=} (x+1)(\sigma^{n-1}_{i_{n-k}} \cdot g) = (x+1)f$. So $x \in D_f$.

We have $(i_{n-k})f = (i_{n-k})(\sigma^{n-1}_{i_{n-k}} \cdot g) = (i_{n-k}+1)(\sigma^{n-1}_{i_{n-k}} \cdot g) = (i_{n-k}+1)f$. So $i_{n-k} \in D_f$.

This proves the *claim*. □

Lemma 11. Let $\ell \geqslant 0$. Let $k \in [0, \ell+1]$.

Let $0 \leqslant i_1 < i_2 < \cdots < i_{\ell-k+1} \leqslant \ell$. Write $i_{\ell-k+2} := \ell+2$. Suppose given $j \in [0, \ell+1]$.

Let $q := \min(\{m \in [1, \ell-k+1] : i_m > j\} \cup \{\ell-k+2\}) = \min(\{m \in [1, \ell-k+2] : i_m > j\})$.

Then

$$\partial^{\ell+1}_j \cdot {}^{[\ell+1]}\prod^{[k]}_{m \in \lfloor \ell-k+1\,,\,1 \rfloor} \sigma^{k+m-1}_{i_m}$$

$$= \begin{cases} {}^{[\ell]}\prod^{[k+q-2]}_{m \in \lfloor \ell-k+1\,,\,q \rfloor} \sigma^{k+m-2}_{i_m-1} \cdot {}^{[k+q-2]}\prod^{[k-1]}_{m \in \lfloor q-1\,,\,1 \rfloor} \sigma^{k+m-2}_{i_m} \cdot \partial^k_{j-q+1} & \text{if } q = 1 \text{ or } (q \in [2, \ell-k+2] \text{ and } j \in [i_{q-1}+2, i_q-1]) \\ {}^{[\ell]}\prod^{[k+q-2]}_{m \in \lfloor \ell-k+1\,,\,q \rfloor} \sigma^{k+m-2}_{i_m-1} \cdot {}^{[k+q-2]}\prod^{[k]}_{m \in \lfloor q-2\,,\,1 \rfloor} \sigma^{k+m-1}_{i_m} & \text{if } q \in [2, \ell-k+2] \text{ and } j \in [i_{q-1}, i_{q-1}+1]\,. \end{cases}$$

We remark that if $k = 0$, we have $i_s = s-1$ for $s \in [1, \ell+1]$. So $j \geqslant i_1$. Hence $q \geqslant 2$ and $j = i_{q-1}$. So we are in the second case.

We remark that if $q \in [2, \ell-k+2]$ and $j \in [i_{q-1}+2, i_q-1]$, then $i_q - 1 \geqslant j \geqslant i_{q-1}+2 > i_{q-1}$. Moreover, if $q \in [3, \ell-k+2]$ and $j \in [i_{q-1}, i_{q-1}+1]$, then $i_q - 1 \geqslant i_{q-1} > i_{q-2}$. So in both cases we get a strictly decreasing sequence of the indices of the surjective maps.

Proof. Note that for $n \geqslant 0$, $u \in [0, n]$ and $v \in [0, n+1]$, we have

$$\partial^{n+1}_v \cdot \sigma^n_u = \begin{cases} \sigma^{n-1}_{u-1} \cdot \partial^n_v & \text{if } v \in [0, u-1] & \text{(Case 1)} \\ \mathrm{id}_{[n]} & \text{if } v \in [u, u+1] & \text{(Case 2)} \\ \sigma^{n-1}_u \cdot \partial^n_{v-1} & \text{if } v \in [u+2, n+1] & \text{(Case 3).} \end{cases}$$

If $q = 1$ or $(q \in [2, \ell-k+2]$ and $j \in [i_{q-1}+2, i_q-1])$, then for the factors $\sigma^{k+m-1}_{i_m}$ indexed by $m \in \lfloor \ell-k+1\,,\,q \rfloor$, Case 1 applies, yielding

$${}^{[\ell]}\prod^{[k+q-2]}_{m \in \lfloor \ell-k+1\,,\,q \rfloor} \sigma^{k+m-2}_{i_m-1} \cdot \partial^{k+q-1}_j \cdot {}^{[k+q-1]}\prod^{[k]}_{m \in \lfloor q-1\,,\,1 \rfloor} \sigma^{k+m-1}_{i_m}\,.$$

For the factors $\sigma^{k+m-1}_{i_m}$ indexed by $m \in \lfloor q-1\,,\,1 \rfloor$, Case 3 applies since firstly, $j \geqslant i_{q-1}+2$, then $j-1 \geqslant i_{q-2}+2$, then $j-2 \geqslant i_{q-3}+2$, and so on. This then yields

$${}^{[\ell]}\prod^{[k+q-2]}_{m \in \lfloor \ell-k+1\,,\,q \rfloor} \sigma^{k+m-2}_{i_m-1} \cdot {}^{[k+q-2]}\prod^{[k-1]}_{m \in \lfloor q-1\,,\,1 \rfloor} \sigma^{k+m-2}_{i_m} \cdot \partial^k_{j-q+1}$$

as claimed.

If $q \in [2, \ell-k+2]$ and $j \in [i_{q-1}, i_{q-1}+1]$, then for the factors $\sigma^{k+m-1}_{i_m}$ indexed by $m \in \lfloor \ell-k+1\,,\,q \rfloor$, Case 1 applies, yielding

$${}^{[\ell]}\prod^{[k+q-2]}_{m \in \lfloor \ell-k+1\,,\,q \rfloor} \sigma^{k+m-2}_{i_m-1} \cdot \partial^{k+q-1}_j \cdot {}^{[k+q-1]}\prod^{[k]}_{m \in \lfloor q-1\,,\,1 \rfloor} \sigma^{k+m-1}_{i_m}\,.$$

By Case 2, we get

$$\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[\ell]\;\;[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot \prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k+q-2]\;\;[k]} \sigma_{i_m}^{k+m-1}$$

as claimed. $\square$

3.2 Cartesian products of simplicial sets

Remark 12. Let $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ and $Y : \Delta^{\mathrm{op}} \to \mathrm{Set}$ be simplicial sets. Then $X \times Y : \Delta^{\mathrm{op}} \to \mathrm{Set}$ is the functor defined by

$$\begin{array}{rcll} [m] & \mapsto & X_m \times Y_m & \text{for } [m] \in \mathrm{Ob}\,\Delta \\ f & \mapsto & X_f \times Y_f & \text{for } f \in \mathrm{Mor}\,\Delta\,. \end{array}$$

Remark 13. We have a functor

$$\begin{array}{rlcll} (\times): & \mathrm{Simp(Set)} \times \mathrm{Simp(Set)} & \to & \mathrm{Simp(Set)} & \\ & (X,Y) & \mapsto & X \times Y & \text{for } (X,Y) \in \mathrm{Ob}\,\mathrm{Simp(Set)} \times \mathrm{Simp(Set)} \\ & ((f_n)_{n\geqslant 0}, (g_n)_{n\geqslant 0}) & \mapsto & (f_n \times g_n)_{n\geqslant 0} & \text{for } (f,g) \in \mathrm{Mor}\,\mathrm{Simp(Set)} \times \mathrm{Simp(Set)}\,. \end{array}$$

3.3 Homotopy over the category Set

3.3.1 Simplicial homotopy for simplicial sets

Definition 14. Let $n \geqslant 0$. We define the functor $\triangle^n := {}_{\Delta}(-,[n]) : \Delta^{\mathrm{op}} \to \mathrm{Set}$. So

$$\begin{array}{lcll} [m] & \mapsto & {}_{\Delta}([m],[n]) & \text{for } [m] \in \mathrm{Ob}\,\Delta \\ ([m_1] \xrightarrow{f^{\mathrm{op}}} [m_2]) & \mapsto & (([m_2] \xrightarrow{g} [n]) \mapsto ([m_1] \xrightarrow{fg} [n])) & \text{for } ([m_1] \xleftarrow{f} [m_2]) \in \mathrm{Mor}\,\Delta\,. \end{array}$$

Remark 15. The functor $\triangle^n = {}_{\Delta}(-,[n])$ is a simplicial set, namely the image of $[n]$ under the Yoneda embedding from Δ to Simp(Set).

Example 16.

(i) For $[n] \in \mathrm{Ob}\,\Delta$ there is exactly one map $O_n : [n] \to [0]$, so $\triangle^0_n = \{O_n\}$ is a set with exactly one element. Moreover, for $([m] \xrightarrow{f} [n]) \in \mathrm{Mor}\,\Delta$, we have $(O_n)\triangle^0_f = O_m$.

(ii) For $[n] \in \mathrm{Ob}\,\Delta$ and $i \in [0, n+1]$ we define a map

$$\begin{array}{rrcl} a_{n,i}: & [n] & \to & [1] \\ & j & \mapsto & \begin{cases} 0 & \text{if } j < i \\ 1 & \text{if } j \geqslant i\,. \end{cases} \end{array}$$

We find that $\triangle^1_n = \{a_{n,i} : i \in [0, n+1]\}$.

Remark 17. We have simplicial maps $\triangle^0 \xrightarrow{\iota_0} \triangle^1$ and $\triangle^0_n \xrightarrow{\iota_1} \triangle^1$ defined by

$$\begin{array}{rrcl} \iota_{0,n}: & \triangle^0_n & \to & \triangle^1_n \\ & O_n & \mapsto & a_{n,0} \end{array} \quad \text{and} \quad \begin{array}{rrcl} \iota_{1,n}: & \triangle^0_n & \to & \triangle^1_n \\ & O_n & \mapsto & a_{n,n+1}\,. \end{array}$$

These are the images of $\partial^1_0 : [0] \to [1]$ and $\partial^1_1 : [0] \to [1]$ under the Yoneda embedding.

Definition 18 (simplicial homotopy)**.** Let X, Y be simplicial sets. Let $\varphi : X \to X \times \triangle^0$ be the simplicial map defined by

$$\begin{array}{rccl} \varphi_n : & X_n & \to & (X \times \triangle^0)_n \\ & x & \mapsto & (x, O_n)\,. \end{array}$$

We remark that φ is an isomorphism.

Let $f : X \to Y$ and $g : X \to Y$ be simplicial maps. A simplicial map $H : X \times \triangle^1 \to Y$ is called an *elementary simplicial homotopy* from f to g if the following diagram commutes.

$$\begin{array}{ccc} X \times \triangle^0 & \xleftarrow{\ \varphi\ } & X \\ {\scriptstyle \mathrm{id} \times \iota_1}\downarrow & & \downarrow{\scriptstyle f} \\ X \times \triangle^1 & \xrightarrow{\ H\ } & Y \\ {\scriptstyle \mathrm{id} \times \iota_0}\uparrow & & \uparrow{\scriptstyle g} \\ X \times \triangle^0 & \xleftarrow[\ \varphi\ ]{} & X \end{array}$$

So we require $(x, a_{n,0})H_n = (x)g_n$ and $(x, a_{n,n+1})H_n = (x)f_n$ for $n \geqslant 0$ and $x \in X_n$.

We call f *elementary simplicially homotopic* to g if there exists an elementary simplicial homotopy from f to g. We then write $f \rightsquigarrow g$.

Lemma 19. We have the following equalities for $n \geqslant 0$.

(i) $\quad \partial_j^{n+1} \cdot a_{n+1,i} = a_{n,i} \quad$ for $0 \leqslant i \leqslant j \leqslant n+1$

(ii) $\quad \partial_j^{n+1} \cdot a_{n+1,i} = a_{n,i-1} \quad$ for $0 \leqslant j < i \leqslant n+2$

(iii) $\quad \sigma_j^n \cdot a_{n,i} = a_{n+1,i} \quad$ for $0 \leqslant i \leqslant j \leqslant n$

(iv) $\quad \sigma_j^n \cdot a_{n,i} = a_{n+1,i+1} \quad$ for $0 \leqslant j < i \leqslant n+1$

Proof. By Definition we have $(j)a_{n,i} = \lfloor j \geqslant i \rfloor$.

Ad (i). Let $\ell \in [n]$. We have $\ell(\partial_j^{n+1} \cdot a_{n+1,i}) = \lfloor (\ell)\partial_j^{n+1} \geqslant i \rfloor = \lfloor \ell + \lfloor \ell \geqslant j \rfloor \geqslant i \rfloor = \lfloor \ell \geqslant i \rfloor = (\ell)a_{n,i}$.

Ad (ii). Let $\ell \in [n]$. We have $\ell(\partial_j^{n+1} \cdot a_{n+1,i}) = \lfloor (\ell)\partial_j^{n+1} \geqslant i \rfloor = \lfloor \ell + \lfloor \ell \geqslant j \rfloor \geqslant i \rfloor = \lfloor \ell \geqslant i-1 \rfloor = (\ell)a_{n,i-1}$.

Ad (iii). Let $\ell \in [n+1]$. We have $\ell(\sigma_j^n \cdot a_{n,i}) = \lfloor (\ell)\sigma_j^n \geqslant i \rfloor = \lfloor \ell - \lfloor \ell > j \rfloor \geqslant i \rfloor = \lfloor \ell \geqslant i \rfloor = (\ell)a_{n+1,i}$.

Ad (iv). Let $\ell \in [n+1]$. We have $\ell(\sigma_j^n \cdot a_{n,i}) = \lfloor (\ell)\sigma_j^n \geqslant i \rfloor = \lfloor \ell - \lfloor \ell > j \rfloor \geqslant i \rfloor = \lfloor \ell \geqslant i+1 \rfloor = (\ell)a_{n+1,i+1}$.

□

Lemma 20 (Characterization of being simplicially homotopic)**.** Suppose given simplicial sets X and Y and simplicial maps $f, g : X \to Y$. Then f is elementary simplicially homotopic to g if and only if there exist maps $h_i^\ell : X_\ell \to Y_{\ell+1}$ for $\ell \geqslant 0$ and $i \in [0, \ell]$ such that the following conditions (i - viii) hold for $\ell \geqslant 0$.

(i) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_{j-1}^{X,\ell+1} \cdot h_i^\ell \quad$ for $i \in [0, \ell]$ and $j \in [i+2, \ell+2]$

(ii) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i+1}^{\ell+1} \cdot d_j^{Y,\ell+2} \quad$ for $i \in [0, \ell]$ and $j = i+1$

(iii) $h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i-1}^{\ell+1} \cdot d_j^{Y,\ell+2}$ for $i \in [1, \ell+1]$ and $j = i$

(iv) $h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_j^{X,\ell+1} \cdot h_{i-1}^{\ell}$ for $i \in [1, \ell+1]$ and $j \in [0, i-1]$

(v) $h_i^{\ell} \cdot s_j^{Y,\ell+1} = s_{j-1}^{X,\ell} \cdot h_i^{\ell+1}$ for $i \in [0, \ell]$ and $j \in [i+1, \ell+1]$

(vi) $h_i^{\ell} \cdot s_j^{Y,\ell+1} = s_j^{X,\ell} \cdot h_{i+1}^{\ell+1}$ for $i \in [0, \ell]$ and $j \in [0, i]$

(vii) $h_\ell^{\ell} \cdot d_{\ell+1}^{Y,\ell+1} = f_\ell$

(viii) $h_0^{\ell} \cdot d_0^{Y,\ell+1} = g_\ell$

Note that (ii) and (iii) are equivalent.

The tuple $((h_i^\ell)_{i\in[0,\ell]})_{\ell \geqslant 0}$ is also called an *elementary simplicial homotopy* from f to g.

Proof. Ad $\Rightarrow$.

Suppose given an elementary simplicial homotopy $H : X \times \triangle^1 \to Y$ between $f : X \to Y$ and $g : X \to Y$. Let H be given by a tuple $H = (H_n : X_n \times \triangle_n^1 \to Y_n)_{n \geqslant 0}$. Recall that $\triangle_n^1$ is given by the set of all monotone maps from $[n]$ to $[1]$, viz. $\triangle_n^1 = \{a_{n,i} : i \in [0, n+1]\}$, cf. Example 16.

For $\ell \geqslant 0$ and $i \in [0, \ell]$ we define the map

$$\begin{array}{rccl} h_i^\ell : & X_\ell & \to & Y_{\ell+1} \\ & x & \mapsto & (x s_i^{X,\ell}, a_{\ell+1,i+1}) H_{\ell+1} \,. \end{array}$$

We have to show that the tuple of tuples $((h_i^\ell)_{i\in[0,\ell]})_{\ell \geqslant 0}$ satisfies the conditions (i) - (viii).

Ad (i).

For $\ell \geqslant 0$ and $i \in [0, \ell]$ we verify equality by evaluating the maps at each element $x \in X_{\ell+1}$. We have

$$\begin{aligned} & x(h_i^{\ell+1} \cdot d_j^{Y,\ell+2}) = (x s_i^{X,\ell+1}, a_{\ell+2,i+1})(H_{\ell+2} \cdot d_j^{Y,\ell+2}) = (x s_i^{X,\ell+1}, a_{\ell+2,i+1})(d_j^{X\times\triangle^1,\ell+2} \cdot H_{\ell+1}) \\ & = (x(s_i^{X,\ell+1} \cdot d_j^{X,\ell+2}), \partial_j^{\ell+2} \cdot a_{\ell+2,i+1}) H_{\ell+1} \,. \end{aligned}$$

Since $j \in [i+2, \ell+1]$ we have $j > i+1$, so $s_i^{X,\ell+1} \cdot d_j^{X,\ell+2} = d_{j-1}^{X,\ell+1} \cdot s_i^{X,\ell}$ and $\partial_j^{\ell+2} \cdot a_{\ell+2,i+1} = a_{\ell+1,i+1}$, cf. Lemma 19 (i).

So continuing the chain of equalities above we get

$$x(h_i^{\ell+1} \cdot d_j^{Y,\ell+2}) = (x(d_{j-1}^{X,\ell+1} \cdot s_i^{X,\ell}), a_{\ell+1,i+1}) H_{\ell+1} = x(d_{j-1}^{X,\ell+1} \cdot h_i^\ell) \,.$$

Ad (ii).

For $\ell \geqslant 0$ and $i \in [0, \ell]$ we verify equality by evaluating the maps at each element $x \in X_\ell$.

We have

$$\begin{aligned} & x(h_i^{\ell} \cdot d_j^{Y,\ell+1}) = (x s_i^{X,\ell}, a_{\ell+1,i+1})(H_{\ell+1} \cdot d_j^{Y,\ell+1}) = (x s_i^{X,\ell}, a_{\ell+1,i+1})(d_j^{X\times\triangle^1,\ell+1} \cdot H_{\ell}) \\ & = (x(s_i^{X,\ell} \cdot d_j^{X,\ell+1}), \partial_j^{\ell+1} \cdot a_{\ell+1,i+1}) H_{\ell} \,. \end{aligned}$$

Since $j = i+1$ we have $s_i^{X,\ell} \cdot d_j^{X,\ell+1} = \mathrm{id}_{X_\ell}$ and $\partial_j^{\ell+1} \cdot a_{\ell+1,i+1} = a_{\ell,i+1}$, cf. Lemma 19 (i).

So continuing the chain of equalities above we get

$$x(h_i^{\ell+1} \cdot d_j^{Y,\ell+2}) = (x, a_{\ell,i+1}) H_\ell \,.$$

On the other hand we get

$$x(h^{\ell}_{i+1}\cdot d^{Y,\ell+1}_j) = (xs^{X,\ell}_{i+1}, a_{\ell+1,i+2})(H_{\ell+1}\cdot d^{Y,\ell+1}_j) = (xs^{X,\ell}_{i+1}, a_{\ell+1,i+2})(d^{X\times\triangle^1,\ell+1}_j \cdot H_\ell)$$
$$= (x(s^{X,\ell}_{i+1}\cdot d^{X,\ell+1}_j), \partial^{\ell+1}_j \cdot a_{\ell+1,i+2})H_\ell = (x, a_{\ell,i+1})H_\ell\,,$$

since we have $j = i+1$ and therefore $\partial^{\ell+1}_j \cdot a_{\ell+1,i+2} = a_{\ell+1,i+1}$, cf. Lemma 19 (ii).

Ad (iv).

For $\ell \geqslant 0$ and $i \in [0,\ell]$ we verify equality by evaluating the maps at each element $x \in X_\ell$.

We have

$$x(h^{\ell}_{i}\cdot d^{Y,\ell+1}_j) = (xs^{X,\ell}_{i}, a_{\ell+1,i+1})(H_{\ell+1}\cdot d^{Y,\ell+1}_j) = (xs^{X,\ell}_{i}, a_{\ell+1,i+1})(d^{X\times\triangle^1,\ell+1}_j \cdot H_\ell)$$
$$= (x(s^{X,\ell}_{i}\cdot d^{X,\ell+1}_j), \partial^{\ell+1}_j \cdot a_{\ell+1,i+1})H_\ell\,.$$

Since $j < i$ we have $s^{X,\ell}_i \cdot d^{X,\ell+1}_j = d^{X,\ell}_j \cdot s^{X,\ell-1}_{i-1}$ and $\partial^{\ell+1}_j \cdot a_{\ell+1,i+1} = a_{\ell,i}$, cf. Lemma 19 (ii).

So continuing the chain of equalities above we get

$$x(h^{\ell+1}_{i}\cdot d^{Y,\ell+2}_j) = (x(d^{X,\ell}_j \cdot s^{X,\ell-1}_{i-1}), a_{\ell,i})H_\ell = x(d^{X,\ell}_j \cdot h^{\ell-1}_{i-1})\,.$$

Ad (v).

For $\ell \geqslant 0$ and $i \in [0,\ell]$ we verify equality by evaluating the maps at each element $x \in X_\ell$.

We have

$$x(h^{\ell}_{i}\cdot s^{Y,\ell+1}_j) = (xs^{X,\ell}_{i}, a_{\ell+1,i+1})(H_{\ell+1}\cdot s^{Y,\ell+1}_j) = (xs^{X,\ell}_{i}, a_{\ell+1,i+1})(s^{X\times\triangle^1,\ell+1}_j \cdot H_{\ell+2})$$
$$= (x(s^{X,\ell}_{i}\cdot s^{X,\ell+1}_j), \sigma^{\ell+1}_j \cdot a_{\ell+1,i+1})H_{\ell+2}$$

Since $j > i$ we have $s^{X,\ell}_i \cdot s^{X,\ell+1}_j = s^{X,\ell}_{j-1} \cdot s^{X,\ell+1}_{i}$ and $\sigma^{\ell+1}_j \cdot a_{\ell+1,i+1} = a_{\ell+2,i+1}$ cf. Lemma 19 (iii).

So continuing the chain of equalities above we get

$$x(h^{\ell}_{i}\cdot s^{Y,\ell+1}_j) = (x(s^{X,\ell}_{j-1} \cdot s^{X,\ell+1}_{i}), a_{\ell+2,i+1})H_{\ell+2} = x(s^{X,\ell}_{j-1} \cdot h^{\ell+1}_{i})\,.$$

Ad (vi).

For $\ell \geqslant 0$ and $i \in [0,\ell]$ we verify equality by evaluating the maps at each element $x \in X_\ell$.

We have

$$x(h^{\ell}_{i}\cdot s^{Y,\ell+1}_j) = (xs^{X,\ell}_{i}, a_{\ell+1,i+1})(H_{\ell+1}\cdot s^{Y,\ell+1}_j) = (xs^{X,\ell}_{i}, a_{\ell+1,i+1})(s^{X\times\triangle^1,\ell+1}_j \cdot H_{\ell+2})$$
$$= (x(s^{X,\ell}_{i}\cdot s^{X,\ell+1}_j), \sigma^{\ell+1}_j \cdot a_{\ell+1,i+1})H_{\ell+2}$$

Since $j \leqslant i$ we have $s^{X,\ell}_i \cdot s^{X,\ell+1}_j = s^{X,\ell}_{j} \cdot s^{X,\ell+1}_{i+1}$ and $\sigma^{\ell+1}_j \cdot a_{\ell+1,i+1} = a_{\ell+2,i+2}$ cf. Lemma 19 (iv).

So continuing the chain of equalities above we get

$$x(h^{\ell}_{i}\cdot s^{Y,\ell+1}_j) = (x(s^{X,\ell}_{j} \cdot s^{X,\ell+1}_{i+1}), a_{\ell+2,i+2})H_{\ell+2} = x(s^{X,\ell}_{j} \cdot h^{\ell+1}_{i+1})\,.$$

Ad (vii).

We verify equality by evaluating the maps at each element $x \in X_\ell$.

We have

$$\begin{aligned}&x(h_\ell^\ell \cdot d_{\ell+1}^{Y,\ell+1}) = (xs_\ell^{X,\ell}, a_{\ell+1,\ell+1})(H_{\ell+1} \cdot d_{\ell+1}^{Y,\ell+1}) = (xs_\ell^{X,\ell}, a_{\ell+1,\ell+1})(d_{\ell+1}^{X\times\triangle^1,\ell+1} \cdot H_\ell)\\ &= (x(s_\ell^{X,\ell} \cdot d_{\ell+1}^{X,\ell+1}), \partial_{\ell+1}^{\ell+1} \cdot a_{\ell+1,\ell+1})H_\ell = (x, a_{\ell,\ell+1})H_\ell\,,\end{aligned}$$

cf. Lemma 19 (i).

Since H is an elementary simplicial homotopy between f and g, we have $(x, a_{\ell,\ell+1})H_\ell = (x)f_\ell$.

Ad (viii).

We verify equality by evaluating the maps at each element $x \in X_\ell$.

We have

$$\begin{aligned}&x(h_0^\ell \cdot d_0^{Y,\ell+1}) = (xs_0^{X,\ell}, a_{\ell+1,1})(H_{\ell+1} \cdot d_0^{Y,\ell+1}) = (xs_0^{X,\ell}, a_{\ell+1,1})(d_0^{X\times\triangle^1,\ell+1} \cdot H_\ell)\\ &= (x(s_0^{X,\ell} \cdot d_0^{X,\ell+1}), \partial_0^{\ell+1} \cdot a_{\ell+1,1})H_\ell = (x, a_{\ell,0})H_\ell\,,\end{aligned}$$

cf. Lemma 19 (ii).

Since H is an elementary simplicial homotopy between f and g, we have $(x, a_{\ell,0})H_\ell = (x)g_\ell$.

Ad $\Leftarrow$.

Suppose given a tuple $((X_n \xrightarrow{h_i^n} Y_{n+1})_{i\in[0,n]})_{n\geqslant 0}$ satisfying (i) - (viii).

For $n \geqslant 0$, we define

$$\begin{array}{rcl} H_n: \quad X_n \times \triangle_n^1 & \to & Y_n \\ (x, a_{n,i}) & \mapsto & \begin{cases} x(h_i^n \cdot d_i^{Y,n+1}) & \text{for } i \in [1,n] \\ xg_n & \text{for } i = 0 \\ xf_n & \text{for } i = n+1 \end{cases} \end{array}$$

It remains to verify that $(H_n)_{n\geqslant 0}$ is a simplicial map.

I. Let $n \geqslant 0$ and $i \in [0, n+1]$. We verify $d_i^{X\times\triangle^1,n+1} \cdot H_n = H_{n+1} \cdot d_i^{Y,n+1}$ by evaluating at each element $(x, a_{n+1,j}) \in X_{n+1} \times \triangle_{n+1}^1$, where $j \in [0, n+2]$.

Case $0 < j < i \leqslant n+1$.

We have $(x, a_{n+1,j})(d_i^{X\times\triangle^1,n+1} \cdot H_n) = (xd_i^{X,n+1}, \partial_i^{n+1} \cdot a_{n+1,j})H_n \overset{\text{Lm 19 (i)}}{=} (xd_i^{X,n+1}, a_{n,j})H_n = x(d_i^{X,n+1} \cdot h_j^n \cdot d_j^{Y,n+1}) \overset{\text{(i)}}{=} x(h_j^{n+1} \cdot d_{i+1}^{Y,n+2} \cdot d_j^{Y,n+1}) = x(h_j^{n+1} \cdot d_j^{Y,n+2} \cdot d_i^{Y,n+1}) = (x, a_{n+1,j})(H_{n+1} \cdot d_i^{Y,n+1})$.

Case $0 = j < i \leqslant n+1$.

We have $(x, a_{n+1,0})(d_i^{X\times\triangle^1,n+1} \cdot H_n) = (xd_i^{X,n+1}, \partial_i^{n+1} \cdot a_{n+1,0})H_n \overset{\text{Lm 19 (i)}}{=} (xd_i^{X,n+1}, a_{n,0})H_n = x(d_i^{X,n+1} \cdot g_n) = x(g_{n+1} \cdot d_i^{Y,n+1}) = (x, a_{n+1,0})(H_{n+1} \cdot d_i^{Y,n+1})$.

Case $0 < i = j \leqslant n$.

We have $(x, a_{n+1,i})(d_i^{X\times\triangle^1,n+1} \cdot H_n) = (xd_i^{X,n+1}, \partial_i^{n+1} \cdot a_{n+1,i})H_n \overset{\text{Lm 19 (i)}}{=} (xd_i^{X,n+1}, a_{n,i})H_n = x(d_i^{X,n+1} \cdot h_i^n \cdot d_i^{Y,n+1}) \overset{\text{(iv)}}{=} x(h_{i+1}^{n+1} \cdot d_i^{Y,n+2} \cdot d_i^{Y,n+1}) = x(h_{i+1}^{n+1} \cdot d_{i+1}^{Y,n+2} \cdot d_i^{Y,n+1}) \overset{\text{(iii)}}{=} (h_i^{n+1} \cdot d_{i+1}^{Y,n+2} \cdot d_i^{Y,n+1}) = x(h_i^{n+1} \cdot d_i^{Y,n+2} \cdot d_i^{Y,n+1}) = (x, a_{n+1,i})(H_{n+1} \cdot d_i^{Y,n+1})$.

Case $0 = i = j \leqslant n$.

We have $(x, a_{n+1,0})(d_0^{X\times\triangle^1,n+1} \cdot H_n) = (xd_0^{X,n+1}, \partial_0^{n+1} \cdot a_{n+1,0})H_n \overset{\text{Lm 19 (i)}}{=} (xd_0^{X,n+1}, a_{n,0})H_n = x(d_0^{X,n+1} \cdot g_n) = x(g_{n+1} \cdot d_0^{Y,n+1}) = (x, a_{n+1,0})(H_{n+1} \cdot d_0^{Y,n+1})$.

Case $0 \leqslant i = j = n+1$.

We have $(x, a_{n+1,n+1})(d_{n+1}^{X\times\triangle^1,n+1} \cdot H_n) = (xd_{n+1}^{X,n+1}, \partial_{n+1}^{n+1} \cdot a_{n+1,n+1})H_n \overset{\text{Lm 19 (i)}}{=} (xd_{n+1}^{X,n+1}, a_{n,n+1})H_n = x(d_{n+1}^{X,n+1} \cdot f_n) = x(f_{n+1} \cdot d_{n+1}^{Y,n+1}) \overset{\text{(vii)}}{=} x(h_{n+1}^{n+1} \cdot d_{n+2}^{Y,n+2} \cdot d_{n+1}^{Y,n+1}) = x(h_{n+1}^{n+1} \cdot d_{n+1}^{Y,n+2} \cdot d_{n+1}^{Y,n+1}) = (x, a_{n+1,n+1})(H_{n+1} \cdot d_{n+1}^{Y,n+1})$.

Case $0 < i < j = i+1 < n+2$.

We have $(x, a_{n+1,i+1})(d_i^{X\times\triangle^1,n+1} \cdot H_n) = (xd_i^{X,n+1}, \partial_i^{n+1} \cdot a_{n+1,i+1})H_n \overset{\text{Lm 19 (ii)}}{=} (xd_i^{X,n+1}, a_{n,i})H_n = x(d_i^{X,n+1} \cdot h_i^n \cdot d_i^{Y,n+1}) \overset{\text{(iv)}}{=} x(h_{i+1}^{n+1} \cdot d_i^{Y,n+2} \cdot d_i^{Y,n+1}) = x(h_{i+1}^{n+1} \cdot d_{i+1}^{Y,n+2} \cdot d_i^{Y,n+1}) = (x, a_{n+1,i+1})(H_{n+1} \cdot d_i^{Y,n+1})$.

Case $0 < i < j = i+1 = n+2$.

We have $(x, a_{n+1,n+2})(d_{n+1}^{X\times\triangle^1,n+1} \cdot H_n) = (xd_{n+1}^{X,n+1}, \partial_{n+1}^{n+1} \cdot a_{n+1,n+2})H_n \overset{\text{Lm 19 (ii)}}{=} (xd_{n+1}^{X,n+1}, a_{n,n+1})H_n = x(d_{n+1}^{X,n+1} \cdot f_n) = x(f_{n+1} \cdot d_{n+1}^{Y,n+1}) = (x, a_{n+1,n+2})(H_{n+1} \cdot d_{n+1}^{Y,n+1})$.

Case $0 = i < j = i+1 < n+2$.

We have $(x, a_{n+1,1})(d_0^{X\times\triangle^1,n+1} \cdot H_n) = (xd_0^{X,n+1}, \partial_0^{n+1} \cdot a_{n+1,1})H_n \overset{\text{Lm 19 (ii)}}{=} (xd_0^{X,n+1}, a_{n,0})H_n = x(d_0^{X,n+1} \cdot g_n) = x(g_{n+1} \cdot d_0^{Y,n+1}) \overset{\text{(viii)}}{=} x(h_0^{n+1} \cdot d_0^{Y,n+2} \cdot d_0^{Y,n+1}) = x(h_0^{n+1} \cdot d_1^{Y,n+2} \cdot d_0^{Y,n+1}) \overset{\text{(ii)}}{=} x(h_1^{n+1} \cdot d_1^{Y,n+2} \cdot d_0^{Y,n+1}) = (x, a_{n+1,1})(H_{n+1} \cdot d_0^{Y,n+1})$.

Case $0 \leqslant i < j \leqslant n+1$ and $j \geqslant i+2$.

We have $(x, a_{n+1,j})(d_i^{X\times\triangle^1,n+1} \cdot H_n) = (xd_i^{X,n+1}, \partial_i^{n+1} \cdot a_{n+1,j})H_n \overset{\text{Lm 19 (ii)}}{=} (xd_i^{X,n+1}, a_{n,j-1})H_n = x(d_i^{X,n+1} \cdot h_{j-1}^n \cdot d_{j-1}^{Y,n+1}) \overset{\text{(iv)}}{=} x(h_j^{n+1} \cdot d_i^{Y,n+2} \cdot d_{j-1}^{Y,n+1}) = x(h_j^{n+1} \cdot d_j^{Y,n+2} \cdot d_i^{Y,n+1}) = (x, a_{n+1,j})(H_{n+1} \cdot d_i^{Y,n+1})$.

Case $0 \leqslant i < j = n+2$ and $j \geqslant i+2$.

We have $(x, a_{n+1,n+2})(d_i^{X\times\triangle^1,n+1} \cdot H_n) = (xd_i^{X,n+1}, \partial_i^{n+1} \cdot a_{n+1,n+2})H_n \overset{\text{Lm 19 (ii)}}{=} (xd_i^{X,n+1}, a_{n,n+1})H_n = x(d_i^{X,n+1} \cdot f_n) = x(f_{n+1} \cdot d_i^{Y,n+1}) = (x, a_{n+1,n+2})(H_{n+1} \cdot d_i^{Y,n+1})$.

II. Let $n \geqslant 0$ and $i \in [0, n]$. We verify $s_i^{X\times\triangle^1,n} \cdot H_{n+1} = H_n \cdot s_i^{Y,n}$ by evaluating at each element $(x, a_{n,j}) \in X_n \times \triangle_n^1$, where $j \in [0, n+1]$.

Case $0 \leqslant i < j < n+1$.

We have $(x, a_{n,j})(s_i^{X\times\triangle^1,n} \cdot H_{n+1}) = (xs_i^{X,n}, \sigma_i^n \cdot a_{n,j})H_{n+1} \overset{\text{Lm 19 (iv)}}{=} (xs_i^{X,n}, a_{n+1,j+1})H_{n+1} = x(s_i^{X,n} \cdot h_{j+1}^{n+1} \cdot d_{j+1}^{Y,n+2}) \overset{\text{(vi)}}{=} x(h_j^n \cdot s_i^{Y,n+1} \cdot d_{j+1}^{Y,n+1}) = x(h_j^n \cdot d_j^{Y,n+1} \cdot s_i^{Y,n}) = (x, a_{n,j})(H_n \cdot s_i^{Y,n})$.

Case $0 \leqslant i < j = n+1$.

We have $(x, a_{n,n+1})(s_i^{X\times\triangle^1,n} \cdot H_{n+1}) = (xs_i^{X,n}, \sigma_i^n \cdot a_{n,n+1})H_{n+1} \overset{\text{Lm 19 (iv)}}{=} (xs_i^{X,n}, a_{n+1,n+2})H_{n+1} = x(s_i^{X,n} \cdot f_{n+1}) = x(f_n \cdot s_i^{Y,n}) = (x, a_{n,n+1})(H_n \cdot s_i^{Y,n})$.

Case $0 < j \leqslant i \leqslant n$.

We have $(x, a_{n,j})(s_i^{X\times\triangle^1,n} \cdot H_{n+1}) = (xs_i^{X,n}, \sigma_i^n \cdot a_{n,j})H_{n+1} \overset{\text{Lm 19 (iii)}}{=} (xs_i^{X,n}, a_{n+1,j})H_{n+1} = x(s_i^{X,n} \cdot h_j^{n+1} \cdot d_j^{Y,n+2}) \overset{\text{(v)}}{=} x(h_j^n \cdot s_{i+1}^{Y,n+1} \cdot d_j^{Y,n+2}) = x(h_j^n \cdot d_j^{Y,n+1} \cdot s_i^{Y,n}) = (x, a_{n,j})(H_n \cdot s_i^{Y,n})$.

Case $0 = j \leqslant i \leqslant n$.

We have $(x, a_{n,0})(s_i^{X\times\triangle^1,n} \cdot H_{n+1}) = (xs_i^{X,n}, \sigma_i^n \cdot a_{n,0})H_{n+1} \overset{\text{Lm 19 (iii)}}{=} (xs_i^{X,n}, a_{n+1,0})H_{n+1} = x(s_i^{X,n} \cdot g_{n+1}) = x(g_n \cdot s_i^{Y,n}) = (x, a_{n,0})(H_n \cdot s_i^{Y,n})$.

□

Remark 21. The conditions (ii) and (iii) in Lemma 20 are equivalent to each other.

Remark 22. The constructions in the proof of Lemma 20 invert each other.

Suppose given an elementary simplicial homotopy $\tilde{H}$ from f to g. By $\Rightarrow$ we attach to $\tilde{H}$ a tuple

$$((X_n \xrightarrow{h_i^n} Y_{n+1})_{i\in[0,n]})_{n\geqslant 0}$$

satisfying (i) - (viii) by setting

$$\begin{array}{rccl} h_i^n: & X_n & \to & Y_{n+1} \\ & x & \mapsto & (xs_i^{X,n}, a_{n+1,i+1})\tilde{H}_{n+1}\,. \end{array}$$

To this tuple, we attach an elementary simplicial homotopy H as in $\Leftarrow$ by setting

$$\begin{array}{rccl} H_n: & X_n \times \triangle_n^1 & \to & Y_n \\ & (x, a_{n,i}) & \mapsto & \begin{cases} x(h_i^n \cdot d_i^{Y,n+1}) & \text{for } i \in [1,n] \\ xg_n & \text{for } i = 0 \\ xf_n & \text{for } i = n+1 \end{cases} \end{array}$$

Then
$(x, a_{n,i})H_n = x(h_i^n \cdot d_i^{Y,n+1}) = (xs_i^{X,n}, a_{n+1,i+1})(\tilde{H}_{n+1} \cdot d_i^{Y,n+1}) = (xs_i^{X,n}, a_{n+1,i+1})(d_i^{X\times\triangle^1,n+1} \cdot \tilde{H}_n) = (x(s_i^{X,n} \cdot d_i^{X,n+1}), \partial_i^{n+1} \cdot a_{n+1,i+1})\tilde{H}_n = (x, a_{n,i})\tilde{H}_n$,
for $i \in [1,n]$ and $x \in X_n$.
Moreover, $(x, a_{n,0})H_n = g_n = (x, a_{n,0})\tilde{H}_n$ and $(x, a_{n,n+1})H_n = f_n = (x, a_{n,n+1})\tilde{H}_n$ for $x \in X_n$.
So we get $H = \tilde{H}$.

Conversely suppose given a tuple $((X_n \xrightarrow{\tilde{h}_i^n} Y_{n+1})_{i\in[0,n]})_{n\geqslant 0}$ satisfying (i) - (viii) of Lemma 20.
By $\Leftarrow$ we attach to this tuple an elementary simplicial homotopy H from f to g by setting

$$\begin{array}{rccl} H_n: & X_n \times \triangle_n^1 & \to & Y_n \\ & (x, a_{n,i}) & \mapsto & \begin{cases} x(\tilde{h}_i^n \cdot d_i^{Y,n+1}) & \text{for } i \in [1,n] \\ xg_n & \text{for } i = 0 \\ xf_n & \text{for } i = n+1 \end{cases} \end{array}$$

To H, we attach a tuple $((X_n \xrightarrow{h_i^n} Y_{n+1})_{i\in[0,n]})_{n\geqslant 0}$ satisfying (i) - (viii) of Lemma 20 as in $\Rightarrow$ by setting

$$\begin{array}{rccl} h_i^n: & X_n & \to & Y_{n+1} \\ & x & \mapsto & (xs_i^{X,n}, a_{n+1,i+1})H_{n+1}\,. \end{array}$$

Then $xh_i^n = (xs_i^{X,n}, a_{n+1,i+1})H_{n+1} = (xs_i^{X,n})(\tilde{h}_{i+1}^{n+1} \cdot d_{i+1}^{Y,n+2}) \overset{\text{(iv)}}{=} x(\tilde{h}_i^n \cdot s_i^{Y,n+1} \cdot d_{i+1}^{Y,n+2}) = x\tilde{h}_i^n$ for $i \in [0,n]$ and $x \in X_n$. So we find $h_i^n = \tilde{h}_i^n$ for $i \in [0,n]$.

3.3.2 Semisimplicial homotopy for semisimplicial sets

Definition 23.

Let $f : X \to Y$ and $g : X \to Y$ be morphisms in SemiSimp(Set).

An *elementary semisimplicial homotopy* from f to g is a tuple

$$((h_i^\ell : X_\ell \to Y_{\ell+1})_{i\in[0,\ell]})_{\ell\geqslant 0}$$

of maps such that the following conditions (i - vi) hold for $\ell \geqslant 0$.

(i) $h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_{j-1}^{X,\ell+1} \cdot h_i^\ell$ for $i \in [0,\ell]$ and $j \in [i+2, \ell+2]$

(ii) $h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i+1}^{\ell+1} \cdot d_j^{Y,\ell+2}$ for $i \in [0,\ell]$ and $j = i+1$

(iii) $h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i-1}^{\ell+1} \cdot d_j^{Y,\ell+2}$ for $i \in [1,\ell+1]$ and $j = i$

(iv) $h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_j^{X,\ell+1} \cdot h_{i-1}^\ell$ for $i \in [1,\ell+1]$ and $j \in [0, i-1]$

(v) $h_\ell^\ell \cdot d_{\ell+1}^{Y,\ell+1} = f_\ell$

(vi) $h_0^\ell \cdot d_0^{Y,\ell+1} = g_\ell$

Note that (ii) and (iii) are equivalent.

We call f *elementary semisimplicially homotopic* to g if there exists an elementary semisimplicial homotopy from f to g. We then write $f \rightsquigarrow g$.

3.3.3 From semisimplicial to simplicial homotopy

We recall the functor $\mathcal{F}_{\mathrm{Set}} : \mathrm{SemiSimp(Set)} \to \mathrm{Simp(Set)}$ from **[1,** Lemma 53].

Lemma 24. Let $X \in \operatorname{Ob}\mathrm{SemiSimp(Set)}$ be a semisimplicial set.

Write $\tilde{X} := \mathcal{F}_{\mathrm{Set}}(X) \in \operatorname{Ob}\mathrm{Simp(Set)}$. Write $\tilde{d}_j^\ell := \tilde{X}_{\partial_j^\ell}$ for $\ell \geqslant 1$ and $j \in [0,\ell]$.

Let $\ell \geqslant 0$. Suppose given

$$(x, \;{}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \in \tilde{X}_{\ell+1}\,,$$

where $k \in [0,\ell+1]$, where $\ell \geqslant i_{\ell-k+1} > \cdots > i_1 \geqslant 0$ and where $x \in X_k$.

Write $i_{\ell-k+2} := \ell+2$.

Suppose given $j \in [0,\ell+1]$. Let $q := \min(\{m \in [1,\ell-k+1] : i_m > j\} \cup \{\ell-k+2\})$.

If $q = 1$ or ($q \in [2,\ell-k+2]$ and $j \in [i_{q-1}+2, i_q-1]$) then we have

$$(x, \;{}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{d}_j^{\ell+1} \;=\; (x d_{j-q+1}^k\,,\; {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})\,.$$

If $q \in [2,\ell-k+2]$ and $j \in [i_{q-1}, i_{q-1}+1]$ then we have

$$(x, \;{}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{d}_j^{\ell+1} \;=\; (x\;,\; {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\,.$$

Proof. By Lemma 11, we have

$$\partial_j^{\ell+1}\cdot \;{}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}$$

$$= \begin{cases} {}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\cdot \partial_{j-q+1}^k & \text{if } q=1 \text{ or } (q\in[2,\ell-k+2] \text{ and } j\in[i_{q-1}+2,i_q-1]) \\ & \text{(Case 1)} \\ {}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1} & \text{if } q\in[2,\ell-k+2] \text{ and } j\in[i_{q-1},i_{q-1}+1] \\ & \text{(Case 2)}\,. \end{cases}$$

In particular, we may read off the factorisation into a surjective monotone map, followed by an injective monotone map. The latter is ∂_{j-q+1}^k in Case 1, and $\mathrm{id}_{[k]}$ in Case 2.

So by [**1,** Lemma 50.(ii)], we have

$$\begin{aligned}
&(x,\; {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{d}_j^{\ell+1} \\
&= \Big(xX_{\big(\partial_j^{\ell+1}\cdot\, {}^{[\ell+1]}\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)^{\bullet}}\;,\; \overline{\partial_j^{\ell+1}\cdot\, {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}}\Big) \\
&= \begin{cases} \Big(xX_{\partial_{j-q+1}^k}\,,\; {}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) & \text{in Case 1} \\ \Big(xX_{\mathrm{id}_{[k]}}\,,\; {}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) & \text{in Case 2} \end{cases} \\
&= \begin{cases} \Big(xd_{j-q+1}^k\,,\; {}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) & \text{in Case 1} \\ \Big(x\,,\; {}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) & \text{in Case 2}\,. \end{cases}
\end{aligned}$$

□

Proposition 25. Suppose given semisimplicial maps $f : X \to Y$ and $g : X \to Y$ and an elementary semisimplicial homotopy $((h_i^\ell)_{i\in[0,\ell]})_{\ell\geqslant 0}$ from f to g.

Then there exists an elementary simplicial homotopy from $\mathcal{F}_{\mathrm{Set}}(f)$ to $\mathcal{F}_{\mathrm{Set}}(g)$, cf. [**1,** Lemma 53].

Proof. We write $\tilde{X} := \mathcal{F}_{\mathrm{Set}}(X)$, $\tilde{Y} := \mathcal{F}_{\mathrm{Set}}(Y)$, $\tilde{f} := \mathcal{F}_{\mathrm{Set}}(f)$, $\tilde{g} := \mathcal{F}_{\mathrm{Set}}(g)$.

We write $\tilde{s}_i^\ell := \tilde{X}_{\sigma_i^\ell}$ for $\ell \geqslant 0$ and $i \in [0,\ell]$. We write $\tilde{d}_i^\ell := \tilde{X}_{\partial_i^\ell}$ for $\ell \geqslant 1$ and $i \in [0,\ell]$.

We write $\tilde{s}_i^\ell := \tilde{Y}_{\sigma_i^\ell}$ for $\ell \geqslant 0$ and $i \in [0,\ell]$. We write $\tilde{d}_i^\ell := \tilde{Y}_{\partial_i^\ell}$ for $\ell \geqslant 1$ and $i \in [0,\ell]$.

We recall that

$$\begin{aligned} \tilde{X}_\ell &= \{(x,h) : h : [\ell] \to [k] \text{ surjective monotone, } x \in X_k\} \\ \tilde{Y}_\ell &= \{(y,h) : h : [\ell] \to [k] \text{ surjective monotone, } y \in Y_k\}\,. \end{aligned}$$

We recall that for a monotone map $h : [\ell] \to [k]$, there exists a unique surjective monotone map $\bar{h} : [\ell] \to [n]$ and a unique injective montone map $\dot{h} : [n] \to [k]$ such that $h = \bar{h} \cdot \dot{h}$.

It holds that $\tilde{s}_i^\ell : \tilde{X}_\ell \to \tilde{X}_{\ell+1}, (x,h) \mapsto (xX_{(\sigma_i^\ell \cdot h)^\bullet}, \sigma_i^\ell \cdot h)$ and $\tilde{d}_i^\ell : \tilde{X}_\ell \to \tilde{X}_{\ell-1}, (x,h) \mapsto (xd_i^{X,\ell}, \overline{\partial_i^\ell \cdot h})$. In particular, for $h = \mathrm{id}_{[\ell]}$ we have $(x, \mathrm{id}_{[\ell]})\tilde{s}_i^\ell = (x, \sigma_i^\ell)$ and $(x, \mathrm{id}_{[\ell]})\tilde{d}_i^\ell) = (xd_i^\ell, \mathrm{id}_{[\ell]})$.

Furthermore, we have $\tilde{f} : \tilde{X} \to \tilde{Y}, (x, h : [\ell] \to [k]) \mapsto ((x)f_k, h : [\ell] \to [k])$.

We have to construct a tuple $((\tilde{X}_\ell \xrightarrow{\tilde{h}_i^\ell} \tilde{Y}_\ell)_{i\in[0,\ell]})_{\ell\geqslant 0}$ of maps satisfying conditions (i) - (viii) in Lemma 20.

We define

$$\begin{array}{rlcl} \tilde{h}_0^0 : & \tilde{X}_0 & \to & \tilde{Y}_1 \\ & (x, \mathrm{id}_{[0]}) & \mapsto & (xh_0^0, \mathrm{id}_{[1]})\,. \end{array}$$

Let $\ell \geqslant 1$. Suppose $\tilde{h}_i^{\ell-1}$ already has been constructed for $i \in [0, \ell-1]$. For a monotone surjective map $v : [\ell] \to [k]$ we use Lemma 10 to write $v = {}^{[\ell]}\prod^{[k]}_{m\in\lfloor \ell-k\,,\,1\rfloor} \sigma_{i_m}^{k+m-1}$, where $\ell - 1 \geqslant i_{\ell-k} > \cdots > i_1 \geqslant 0$.

Given $i \in [0, \ell]$, we define

$$\begin{array}{rlcll} \tilde{h}_i^\ell : & \tilde{X}_\ell & \to & \tilde{Y}_{\ell+1} & \\ & (x, {}^{[\ell]}\prod^{[k]}_{m\in\lfloor\ell-k\,,\,1\rfloor} \sigma_{i_m}^{k+m-1}) & \mapsto & \begin{cases} (xh_i^\ell, \mathrm{id}_{[\ell+1]}) & \text{for } \ell = k \\ (x, {}^{[\ell-1]}\prod^{[k]}_{m\in\lfloor\ell-k-1\,,\,1\rfloor} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k}+1}^\ell) & \text{for } \ell > k \text{ and } i_{\ell-k} \geqslant i \\ (x, {}^{[\ell-1]}\prod^{[k]}_{m\in\lfloor\ell-k-1\,,\,1\rfloor} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k}}^\ell) & \text{for } \ell > k \text{ and } i_{\ell-k} < i \end{cases} \end{array}$$

We have to verify that the tuple $((\tilde{X}_\ell \xrightarrow{\tilde{h}_i^\ell} \tilde{Y}_{\ell+1})_{i\in[0,\ell]})_{\ell\geqslant 0}$ satisfies (i) - (viii) in Lemma 20.

Ad (i).

We prove $\tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2} = \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{h}_i^\ell$ for $i \in [0, \ell]$ and $j \in [i+2, \ell+2]$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by evaluating at each element $(x, v) \in \tilde{X}_1$.

Case $v = \mathrm{id}_{[1]}$ and $x \in X_1$.

We have $(x, \mathrm{id}_{[1]})(\tilde{h}_0^1 \cdot \tilde{d}_2^2) = (xh_0^1, \mathrm{id}_{[2]})\tilde{d}_2^2 = (x(h_0^1 \cdot d_2^2), \mathrm{id}_{[1]}) \overset{\text{Def 27 (i)}}{=} (x(d_1^1 \cdot h_0^0), \mathrm{id}_{[1]}) = (xd_1^1, \mathrm{id}_{[0]})\tilde{h}_0^0 = (x, \mathrm{id}_{[1]})(\tilde{d}_1^1 \cdot \tilde{h}_0^0)$.

Case $v = \sigma_0^0$ and $x \in X_0$.

We have $(x, \sigma_0^0)(\tilde{h}_0^1 \cdot \tilde{d}_2^2) = (x, \mathrm{id}_{[0]})(\tilde{h}_0^0 \cdot \tilde{s}_1^1 \cdot \tilde{d}_2^2) = (x, \mathrm{id}_{[0]})\tilde{h}_0^0 = (xh_0^0, \mathrm{id}_{[1]})$.

We have $(x, \sigma_0^0)(\tilde{d}_1^1 \cdot \tilde{h}_0^0) = (xX_{(\partial_1^1 \cdot \sigma_0^0)^\bullet}, \overline{\partial_1^1 \cdot \sigma_0^0})\tilde{h}_0^0 = (x, \mathrm{id}_{[0]})\tilde{h}_0^0 = (xh_0^0, \mathrm{id}_{[1]})$.

Let $\ell \geqslant 1$. Suppose we have $\tilde{h}_i^\ell \cdot \tilde{d}_j^{\ell+1} = \tilde{d}_{j-1}^\ell \cdot \tilde{h}_i^{\ell-1}$.

We verify equality by evaluating at each element $(x, {}^{[\ell+1]}\prod^{[k]}_{m\in\lfloor\ell-k+1\,,\,1\rfloor} \sigma_{i_m}^{k+m-1}) \in \tilde{X}_{\ell+1}$, where $k \in [0, \ell+1]$ and $x \in X_k$ and $\ell \geqslant i_{\ell-k+1} > \cdots > i_1 \geqslant 0$. Write $i_{\ell-k+2} := \ell + 2$.

Let $q := \min(\{m \in [1, \ell-k+1] : i_m > j-1\} \cup \{\ell-k+2\})$.

Suppose that $k = \ell + 1$.

We have

$$\begin{aligned}
(x,\mathrm{id}_{[\ell+1]})(\tilde h_i^{\ell+1}\cdot\tilde d_j^{\ell+2}) &= (xh_i^{\ell+1},\mathrm{id}_{[\ell+2]})\tilde d_j^{\ell+2}\\
&= (x(h_i^{\ell+1}\cdot d_j^{\ell+2}),\mathrm{id}_{[\ell+1]})\\
&\overset{\text{Def 27 (i)}}{=} (x(d_{j-1}^{\ell+1}\cdot h_i^{\ell}),\mathrm{id}_{[\ell+1]})\\
&= (xd_{j-1}^{\ell+1},\mathrm{id}_{[\ell]})\tilde h_i^{\ell}\\
&= (x,\mathrm{id}_{[\ell+1]})(\tilde d_{j-1}^{\ell+1}\cdot\tilde h_i^{\ell})\,.
\end{aligned}$$

Suppose that $k\leqslant\ell$.

Case $q\in[2,\ell-k+1]$ and $j-1\in[i_{q-1},i_{q-1}+1]$. Note that both in case $j-1<i_{\ell-k}$ and in case $j-1\in[i_{\ell-k},i_{\ell-k}+1]$, we have $q=\min(\{m\in[1,\ell-k] : i_m>j-1\}\cup\{\ell-k+1\})$. Note that $q\in[2,\ell-k+1]$ implies $\ell-k+1\geqslant 2$, which means $\ell-k\geqslant 1$.

We have

$$\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor\ell-k+1\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde d_{j-1}^{\ell+1}\cdot\tilde h_i^{\ell})\\
\overset{\text{Lm 24}}{=}\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor\ell-k+1\,,\,q\rfloor}^{[k+q-2]}\sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})\tilde h_i^{\ell}\\
=\ & (x,\ {}^{[\ell-1]}\prod_{m\in\lfloor\ell-k\,,\,q\rfloor}^{[k+q-2]}\sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde h_i^{\ell-1}\cdot\tilde s_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}$$

We have

$$\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor\ell-k+1\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde h_i^{\ell+1}\cdot\tilde d_j^{\ell+2})\\
=\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor\ell-k\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde h_i^{\ell}\cdot\tilde s_{i_{\ell-k+1}+1}^{\ell+1}\cdot\tilde d_j^{\ell+2})\\
=\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor\ell-k\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde h_i^{\ell}\cdot\tilde d_j^{\ell+1}\cdot\tilde s_{i_{\ell-k+1}}^{\ell})\\
\overset{\text{ind. hyp.}}{=}\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor\ell-k\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde d_{j-1}^{\ell}\cdot\tilde h_i^{\ell-1}\cdot\tilde s_{i_{\ell-k+1}}^{\ell})\\
\overset{\text{Lm 24}}{=}\ & (x,\ {}^{[\ell-1]}\prod_{m\in\lfloor\ell-k\,,\,q\rfloor}^{[k+q-2]}\sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde h_i^{\ell-1}\cdot\tilde s_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}$$

Case $q=\ell-k+2$ and $j-1\in[i_{\ell-k+1},i_{\ell-k+1}+1]$. Note that $i_{\ell-k+1}\geqslant i$, since $i_{\ell-k+1}\leqslant i-1$ together with $j\geqslant i+2$ would imply $j-1\geqslant i_{\ell-k+1}+2$, which is not the case.

We have

$$(x,\ {}^{[\ell+1]}\prod_{m\in\lfloor\ell-k+1\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde d_{j-1}^{\ell+1}\cdot\tilde h_i^{\ell}) \overset{\text{Lm 24}}{=} (x,\ {}^{[\ell]}\prod_{m\in\lfloor\ell-k\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})\tilde h_i^{\ell}\,.$$

We have

$$\begin{aligned}
(x,\ {}^{[\ell+1]}\prod_{m\in\lfloor\ell-k+1\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde h_i^{\ell+1}\cdot\tilde d_j^{\ell+2}) &= (x,\ {}^{[\ell]}\prod_{m\in\lfloor\ell-k\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})(\tilde h_i^{\ell}\cdot\tilde s_{i_{\ell-k+1}+1}^{\ell+1}\cdot\tilde d_j^{\ell+2})\\
&= (x,\ {}^{[\ell]}\prod_{m\in\lfloor\ell-k\,,\,1\rfloor}^{[k]}\sigma_{i_m}^{k+m-1})\tilde h_i^{\ell}\,.
\end{aligned}$$

Case $q=1$. This implies $k\geqslant 1$, since otherwise we would have a factor $\sigma_{i_1}^{0+1-1}$ in the product, hence $i_1=0$, but $i_1>j-1\geqslant 1$.

We have

$$
\begin{aligned}
(x,\ {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-1}^{\ell+1}\cdot \tilde{h}_i^{\ell}) &\overset{\text{Lm } 24}{=} (xd_{j-1}^{k}\,,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2})\tilde{h}_i^{\ell} \\
&= (xd_{j-1}^{k}\,,\ {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2})(\tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
(x,\ {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) &= (x,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
&= (x,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
&\overset{\text{ind. hyp.}}{=} (x,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-1}^{\ell}\cdot \tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
&\overset{\text{Lm } 24}{=} (xd_{j-1}^{k}\,,\ {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2})(\tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}
$$

Case $q\in[2,\ell-k+1]$ and $j-1\in[i_{q-1}+2,i_q-1]$. This implies $k\geqslant 1$, because otherwise $k=0$ and therefore $i_q=q-1$ and $i_{q-1}=q-2$ because they must fit into the strictly decreasing sequence $\ell\geqslant i_{\ell+1}>\cdots>i_q>i_{q-1}>\cdots>i_1\geqslant 0$, leading to $q-1=i_q>j-1\geqslant i_{q-1}+2=q-2+2=q$, which is impossible, cf. also the Remark in Lemma 11.

Note that both in case $j-1<i_{\ell-k}$ and in case $j-1\in[i_{\ell-k}+2,i_{\ell-k+1}-1]$, we still have $q=\min(\{m\in[1,\ell-k]:i_m>j-1\}\cup\{\ell-k+1\})$.

We have

$$
\begin{aligned}
&(x,\ {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-1}^{\ell+1}\cdot \tilde{h}_i^{\ell}) \\
\overset{\text{Lm } 24}{=}\ &(xd_{j-q}^{k}\,,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})\tilde{h}_i^{\ell} \\
\overset{i_{\ell-k+1}-1\,\geqslant\, i}{=}\ &(xd_{j-q}^{k}\,,\ {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
&(x,\ {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
=\ &(x,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
=\ &(x,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
\overset{\text{ind. hyp.}}{=}\ &(x,\ {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-1}^{\ell}\cdot \tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
\overset{\text{Lm } 24}{=}\ &(xd_{j-q}^{k}\,,\ {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\!\!\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}
$$

Case $q = \ell-k+2$ and $j-1 \in [i_{\ell-k+1}+1, \ell+1]$ and $i_{\ell-k+1} \geqslant i$. Note that we have $i_{\ell-k}+1 < i_{\ell-k+1}+1 < j-1$ and therefore $j-2 \geqslant i_{\ell-k}+2$. So $\min(\{m \in [1, \ell-k] : i_m > j-2\} \cup \{\ell-k+1\}) = \ell-k+1$.

We have

$$\begin{aligned}(x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-1}^{\ell+1}\cdot\tilde{h}_i^{\ell}) &\overset{\text{Lm }24}{=} (xd^k_{j-\ell+k-2}\,, {}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})\tilde{h}_i^{\ell} \\ &= (xd^k_{j-\ell+k-2}\,, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_i^{\ell-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}+1})\,.\end{aligned}$$

We have

$$\begin{aligned}(x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot\tilde{d}_j^{\ell+2}) &= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot\tilde{s}^{\ell+1}_{i_{\ell-k+1}+1}\cdot\tilde{d}_j^{\ell+2}) \\ &= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot\tilde{d}_{j-1}^{\ell+1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}+1}) \\ &\overset{\text{ind. hyp.}}{=} (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-2}^{\ell}\cdot\tilde{h}_i^{\ell-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}+1}) \\ &\overset{\text{Lm }24}{=} (xd^k_{j-\ell+k-2}\,, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_i^{\ell-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}+1})\,.\end{aligned}$$

Case $q = \ell-k+2$ and $j-1 \in [i_{\ell-k+1}+1, \ell+1]$ and $i_{\ell-k+1} < i$. Note that we have $i_{\ell-k}+1 < i_{\ell-k+1}+1 < j-1$ and therefore $j-2 > i_{\ell-k}+1$. So $\min(\{m \in [1, \ell-k] : i_m > j-2\} \cup \{\ell-k+1\}) = \ell-k+1$.

We have

$$\begin{aligned}(x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-1}^{\ell+1}\cdot\tilde{h}_i^{\ell}) &\overset{\text{Lm }24}{=} (xd^k_{j-\ell+k-2}\,, {}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})\tilde{h}_i^{\ell} \\ &= (xd^k_{j-\ell+k-2}\,, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_{i-1}^{\ell-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}})\,.\end{aligned}$$

We have

$$\begin{aligned}(x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot\tilde{d}_j^{\ell+2}) &= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{s}^{\ell+1}_{i_{\ell-k+1}}\cdot\tilde{d}_j^{\ell+2}) \\ &= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{d}_{j-1}^{\ell+1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}}) \\ &\overset{\text{ind. hyp.}}{=} (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-2}^{\ell}\cdot\tilde{h}_{i-1}^{\ell-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}}) \\ &\overset{\text{Lm }24}{=} (xd^k_{j-\ell+k-2}\,, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_{i-1}^{\ell-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}})\,.\end{aligned}$$

Ad (ii).

We prove $\tilde{h}_i^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2} = \tilde{h}_{i+1}^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2}$ for $i \in [0,\ell]$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by evaluating at each element $(x, v) \in \tilde{X}_1$.

Case $v = \mathrm{id}_{[1]}$ and $x \in X_1$.

We have $(x, \mathrm{id}_{[1]})(\tilde{h}_0^1\cdot\tilde{d}_1^2) = (xh_0^1, \mathrm{id}_{[2]})\tilde{d}_1^2 = (x(h_0^1\cdot d_1^2), \mathrm{id}_{[1]}) = (x(h_1^1\cdot d_1^2), \mathrm{id}_{[1]}) = (xh_1^1, \mathrm{id}_{[2]})\tilde{d}_1^2 = (x, \mathrm{id}_{[1]})(\tilde{h}_1^1\cdot\tilde{d}_1^2)\,.$

Case $v = \sigma_0^0$ and $x \in X_0$.

We have $(x, \sigma_0^0)(\tilde{h}_0^1 \cdot \tilde{d}_1^2) = (x, \mathrm{id}_{[0]})(\tilde{h}_0^0 \cdot \tilde{s}_1^1 \cdot \tilde{d}_1^2) = (x, \mathrm{id}_{[0]})\tilde{h}_0^0$.

We have $(x, \sigma_0^0)(\tilde{h}_1^1 \cdot \tilde{d}_1^2) = (x, \mathrm{id}_{[0]})(\tilde{h}_0^0 \cdot \tilde{s}_0^1 \cdot \tilde{d}_1^2) = (x, \mathrm{id}_{[0]})\tilde{h}_0^0$.

Let $\ell \geqslant 1$. Suppose we have $\tilde{h}_i^\ell \cdot \tilde{d}_{i+1}^{\ell+1} = \tilde{h}_{i+1}^\ell \cdot \tilde{d}_{i+1}^{\ell+1}$.

We verify equality by evaluating at each element $(x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \in \tilde{X}_{\ell+1}$, where $k \in [0,\ell]$ and $\ell \geqslant i_{\ell-k+1} > \cdots > i_1 \geqslant 0$.

Suppose that $\ell + 1 = k$.

We have

$$\begin{aligned}
(x, \mathrm{id}_{[\ell+1]})(\tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2}) &= (xh_i^{\ell+1}, \mathrm{id}_{[\ell+2]})\tilde{d}_{i+1}^{\ell+2} \\
&= (x(h_i^{\ell+1} \cdot d_{i+1}^{\ell+2}), \mathrm{id}_{[\ell+1]}) \\
&= (x(h_{i+1}^{\ell+1} \cdot d_{i+1}^{\ell+2}), \mathrm{id}_{[\ell+1]}) \\
&= (xh_{i+1}^{\ell+1}, \mathrm{id}_{[\ell+2]})\tilde{d}_{i+1}^{\ell+2} \\
&= (x, \mathrm{id}_{[\ell+1]})(\tilde{h}_{i+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2})\,.
\end{aligned}$$

Suppose that $k \leqslant \ell$.

Case $i_{\ell-k+1} \geqslant i + 1$. It follows that $i < i_{\ell-k+1} \leqslant \ell$.

We have

$$\begin{aligned}
(x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2}) &= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^\ell \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2}) \\
&= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^\ell \cdot \tilde{d}_{i+1}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^\ell) \\
&\overset{\text{ind. hyp.}}{=} (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i+1}^\ell \cdot \tilde{d}_{i+1}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^\ell) \\
&= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i+1}^\ell \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2}) \\
&= (x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2})\,.
\end{aligned}$$

Case $i_{\ell-k+1} = i$.

We have

$$\begin{aligned}
(x, {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2}) &= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^\ell \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2}) \\
&= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_i^\ell\,.
\end{aligned}$$

We have

$$
\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i+1}^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2}) &= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i}^{\ell}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2}) \\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\quad[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i}^{\ell}\,.
\end{aligned}
$$

Case $i_{\ell-k+1} < i$.

We have

$$
\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i}^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2}) &= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2}) \\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{d}_{i}^{\ell+1}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
&\overset{\text{ind. hyp.}}{=} (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i}^{\ell}\cdot\tilde{d}_{i}^{\ell+1}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i}^{\ell}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2}) \\
&= (x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1]\quad[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i+1}^{\ell+1}\cdot\tilde{d}_{i+1}^{\ell+2})\,.
\end{aligned}
$$

Ad (iv).

We prove $\tilde{h}_i^{\ell+1}\cdot\tilde{d}_j^{\ell+2} = \tilde{d}_j^{\ell+1}\cdot\tilde{h}_{i-1}^{\ell}$ for $i\in[1,\ell+1]$ and $j\in[0,i-1]$ by induction on $\ell\geqslant 0$.

Let $\ell = 0$. We verify equality by evaluating at each element $(x,v)\in\tilde{X}_1$.

Case $v = \mathrm{id}_{[1]}$ and $x\in X_1$.

We have $(x,\mathrm{id}_{[1]})(\tilde{h}_1^1\cdot\tilde{d}_0^2) = (xh_1^1,\mathrm{id}_{[2]})\tilde{d}_0^2 = (x(h_1^1\cdot d_0^2),\mathrm{id}_{[1]}) = (x(d_0^1\cdot h_0^0),\mathrm{id}_{[1]}) = (xd_0^1,\mathrm{id}_{[0]})\tilde{h}_0^0 = (x,\mathrm{id}_{[1]})(\tilde{d}_0^1\cdot\tilde{h}_0^0)$.

Case $v = \sigma_0^0$ and $x\in X_0$.

We have $(x,\sigma_0^0)(\tilde{h}_1^1\cdot\tilde{d}_0^2) = (x,\mathrm{id}_{[0]})(\tilde{h}_0^0\cdot\tilde{s}_0^1\cdot\tilde{d}_0^2) = (x,\mathrm{id}_{[0]})\tilde{h}_0^0$.

We have $(x,\sigma_0^0)(\tilde{d}_0^1\cdot\tilde{h}_0^0) = (xX_{(\partial_0^1\cdot\sigma_0^0)^\bullet},\overline{\partial_0^1\cdot\sigma_0^0})\tilde{h}_0^0 = (x,\mathrm{id}_{[0]})\tilde{h}_0^0$.

Let $\ell\geqslant 1$. Suppose we have $\tilde{h}_i^{\ell}\cdot\tilde{d}_j^{\ell+1} = \tilde{d}_j^{\ell}\cdot\tilde{h}_{i-1}^{\ell-1}$.

We verify equality by evaluating at each element $(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1]\quad[k]} \sigma_{i_m}^{k+m-1})\in\tilde{X}_{\ell+1}$, where $k\in[0,\ell]$ and $\ell\geqslant i_{\ell-k+1} > \cdots > i_1\geqslant 0$.

Let $q := \min(\{m\in[1,\ell-k+1] : i_m > j\}\cup\{\ell-k+2\})$.

Suppose that $k = \ell+1$.

We have

$$
\begin{aligned}
(x,\mathrm{id}_{[\ell+1]})(\tilde{h}_i^{\ell+1}\cdot\tilde{d}_j^{\ell+2}) &= (xh_i^{\ell+1},\mathrm{id}_{[\ell+2]})\tilde{d}_j^{\ell+2} \\
&= (x(h_i^{\ell+1}\cdot d_j^{\ell+2}),\mathrm{id}_{[\ell+1]}) \\
&= (x(d_j^{\ell+1}\cdot h_{i-1}^{\ell}),\mathrm{id}_{[\ell+1]}) \\
&= (xd_j^{\ell+1},\mathrm{id}_{[\ell]})\tilde{h}_{i-1}^{\ell} \\
&= (x,\mathrm{id}_{[\ell+1]})(\tilde{d}_j^{\ell+1}\cdot\tilde{h}_{i-1}^{\ell})\,.
\end{aligned}
$$

Suppose $k \leqslant \ell$.

Case $q \geqslant 2$ and $j \in [i_{q-1}, i_{q-1}+1]$ and $i_{\ell-k+1} \geqslant i$. It follows that $\ell-k+1 \geqslant q$ and $j < i \leqslant i_{\ell-k+1} \leqslant \ell$. Therefore we have $q = \min(\{m \in [1, \ell-k] : i_m > j\} \cup \{\ell-k+1\})$.

We have

$$\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1} \cdot \tilde{h}_{i-1}^{\ell}) \\
\overset{\text{Lm 24}}{=}\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i-1}^{\ell} \\
=\ & (x,\ {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}$$

We have

$$\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2}) \\
=\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_j^{\ell+2}) \\
=\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell} \cdot \tilde{d}_j^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
\overset{\text{ind. hyp.}}{=}\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell} \cdot \tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
\overset{\text{Lm 24}}{=}\ & (x,\ {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}$$

Case $\ell-k+1 \geqslant q \geqslant 2$ and $j \in [i_{q-1}, i_{q-1}+1]$ and $i_{\ell-k+1} < i$. Note that $j < i_q \leqslant i_{\ell-k+1} < i$ since $\ell-k+1 \geqslant q$. Therefore we have $q = \min(\{m \in [1, \ell-k] : i_m > j\} \cup \{\ell-k+1\})$.

We have

$$\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1} \cdot \tilde{h}_{i-1}^{\ell}) \\
\overset{\text{Lm 24}}{=}\ & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i-1}^{\ell} \\
=\ & (x,\ {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-2}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell})\,.
\end{aligned}$$

We have

$$
\begin{aligned}
& (x, \; {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
= \quad & (x, \; {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
= \quad & (x, \; {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell}) \\
\overset{\text{ind. hyp.}}{=} \quad & (x, \; {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell}\cdot \tilde{h}_{i-2}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell}) \\
\overset{\text{Lm 24}}{=} \quad & (x, \; {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-2}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell})\,.
\end{aligned}
$$

Case $q = \ell - k + 2$ and $j \in [i_{\ell-k+1}, i_{\ell-k+1}+1]$. It follows that $i_{\ell-k+1} \leqslant j < i$.

We have

$$
(x, \; {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1}\cdot \tilde{h}_{i-1}^{\ell}) \overset{\text{Lm 24}}{=} (x, \; {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i-1}^{\ell}\,.
$$

We have

$$
\begin{aligned}
(x, \; {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \quad &= \quad (x, \; {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
&= \quad (x, \; {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i-1}^{\ell}\,.
\end{aligned}
$$

Case $q \in [2, \ell-k+2]$ and $j \in [i_{q-1}+2, i_q - 1]$ and $i_{\ell-k+1} \geqslant i$. It follows that $j < i \leqslant i_{\ell-k+1} \leqslant \ell$ and thus $\ell - k + 1 \geqslant q$. Therefore we have $q = \min(\{m \in [1, \ell-k] : i_m > j\} \cup \{\ell-k+1\})$.

We have

$$
\begin{aligned}
& (x, \; {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1}\cdot \tilde{h}_{i-1}^{\ell}) \\
\overset{\text{Lm 24}}{=} \quad & (xd_{j-q+1}^{k}, \; {}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})\tilde{h}_{i-1}^{\ell} \\
= \quad & (xd_{j-q+1}^{k}, \; {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_{i-1}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}).
\end{aligned}
$$

We have

$$
\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
= \quad & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{s}^{\ell+1}_{i_{\ell-k+1}+1}\cdot \tilde{d}_j^{\ell+2}) \\
= \quad & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}} \\
\stackrel{\text{ind. hyp.}}{=} \quad & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell}\cdot \tilde{h}_{i-1}^{\ell-1}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}}) \\
\stackrel{\text{Lm 24}}{=} \quad & (xd^k_{j-q+1},\ {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]}\sigma^{k+m-2}_{i_{m-1}}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]}\sigma^{k+m-2}_{i_m})(\tilde{h}^{\ell-1}_{i-1}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}}).
\end{aligned}
$$

Case $q\in[2,\ell-k+1]$ and $j\in[i_{q-1}+2,i_q-1]$ and $i_{\ell-k+1}<i$. Note that $j<i_q\leqslant i_{\ell-k+1}\leqslant \ell$ and $j<i_{\ell-k+1}<i$ and therefore $i-1>j$. Again we have $q=\min(\{m\in[1,\ell-k]:i_m>j\}\cup\{\ell-k+1\})$. We have

$$
\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1}\cdot \tilde{h}_{i-1}^{\ell}) \\
\stackrel{\text{Lm 24}}{=} \quad & (xd^k_{j-q+1},\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]}\sigma^{k+m-2}_{i_{m-1}}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]}\sigma^{k+m-2}_{i_m})\tilde{h}^{\ell}_{i-1} \\
= \quad & (xd^k_{j-q+1},\ {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]}\sigma^{k+m-2}_{i_{m-1}}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]}\sigma^{k+m-2}_{i_m})\tilde{h}^{\ell-1}_{i-2}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}-1}\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
& (x,\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) \\
= \quad & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot \tilde{s}^{\ell+1}_{i_{\ell-k+1}}\cdot \tilde{d}_j^{\ell+2}) \\
= \quad & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}-1}) \\
\stackrel{\text{ind. hyp.}}{=} \quad & (x,\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell}\cdot \tilde{h}_{i-2}^{\ell-1}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}-1}) \\
\stackrel{\text{Lm 24}}{=} \quad & (xd^k_{j-q+1},\ {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]}\sigma^{k+m-2}_{i_{m-1}}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]}\sigma^{k+m-2}_{i_m})(\tilde{h}^{\ell-1}_{i-2}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}-1})\,.
\end{aligned}
$$

Case $q=\ell-k+2$ and $j\in[i_{\ell-k+1}+2,\ell+1]$. It follows that $i_{\ell-k+1}+2\leqslant j<i$ and thus $i_{\ell-k+1}<i-1$. Note that $j-1\geqslant i_{\ell-k+1}+1\geqslant i_{\ell-k}+2$. We have

$$
\begin{aligned}
(x,\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1}\cdot \tilde{h}_{i-1}^{\ell}) \quad \stackrel{\text{Lm 24}}{=} \quad & (xd^k_{j-\ell+k-1},\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})\tilde{h}^{\ell}_{i-1} \\
= \quad & (xd^k_{j-\ell+k-1},\ {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}^{\ell-1}_{i-2}\cdot \tilde{s}^{\ell}_{i_{\ell-k+1}})\,.
\end{aligned}
$$

We have

$$\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1][k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot\tilde{d}_j^{\ell+2}) &= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell][k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot\tilde{d}_j^{\ell+2}) \\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell][k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{d}_{j-1}^{\ell+1}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
&\overset{\text{ind. hyp.}}{=} (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell][k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j-1}^{\ell}\cdot\tilde{h}_{i-2}^{\ell-1}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell}) \\
&\overset{\text{Lm 24}}{=} (xd_{j-\ell+k-1}^{k}, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell-1][k-1]} \sigma_{i_m}^{k+m-2})(\tilde{h}_{i-2}^{\ell-1}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}$$

Case $q=1$ and $i_{\ell-k+1}<i$. Note that $j<i_1\leqslant i_{\ell-k+1}<i$.

Therefore we have $q=\min(\{m\in[1,\ell-k]: i_m>j\}\cup\{\ell-k+1\})$.

We have

$$\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1][k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1}\cdot\tilde{h}_{i-1}^{\ell}) &\overset{\text{Lm 24}}{=} (xd_j^{k}, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell][k-1]} \sigma_{i_m-1}^{k+m-2})\tilde{h}_{i-1}^{\ell} \\
&= (xd_j^{k}, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell-1][k-1]} \sigma_{i_m-1}^{k+m-2})(\tilde{h}_{i-2}^{\ell-1}\cdot\tilde{s}_{i_{\ell-k+1}-1}^{\ell})\,.
\end{aligned}$$

We have

$$\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1][k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot\tilde{d}_j^{\ell+2}) &= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell][k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot\tilde{d}_j^{\ell+2}) \\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell][k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-1}^{\ell}\cdot\tilde{d}_{j}^{\ell+1}\cdot\tilde{s}_{i_{\ell-k+1}-1}^{\ell}) \\
&\overset{\text{ind. hyp.}}{=} (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell][k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_{j}^{\ell}\cdot\tilde{h}_{i-2}^{\ell-1}\cdot\tilde{s}_{i_{\ell-k+1}-1}^{\ell}) \\
&\overset{\text{Lm 24}}{=} (xd_j^{k}, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell-1][k-1]} \sigma_{i_m-1}^{k+m-2})(\tilde{h}_{i-2}^{\ell-1}\cdot\tilde{s}_{i_{\ell-k+1}-1}^{\ell})\,.
\end{aligned}$$

Case $q=1$ and $i_{\ell-k+1}\geqslant i$. Note that $j<i\leqslant i_{\ell-k+1}\leqslant \ell$.

Therefore we have $q=\min(\{m\in[1,\ell-k]: i_m>j\}\cup\{\ell-k+1\})$.

We have

$$\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1][k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell+1}\cdot\tilde{h}_{i-1}^{\ell}) &\overset{\text{Lm 24}}{=} (xd_j^{k}, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell][k-1]} \sigma_{i_m-1}^{k+m-2})\tilde{h}_{i-1}^{\ell} \\
&= (xd_j^{k}, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell-1][k-1]} \sigma_{i_m-1}^{k+m-2})(\tilde{h}_{i-1}^{\ell-1}\cdot\tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}$$

We have

$$
\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[\ell+1]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2}) &= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde{d}_j^{\ell+2})\\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\\
&\overset{\text{ind. hyp.}}{=} (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{d}_j^{\ell}\cdot \tilde{h}_{i-1}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\\
&\overset{\text{Lm 24}}{=} (xd_j^k, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell-1]\ \ [k-1]} \sigma_{i_m-1}^{k+m-2})(\tilde{h}_{i-1}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell})\,.
\end{aligned}
$$

Ad (v).

We prove $\tilde{s}_{j-1}^{\ell}\cdot \tilde{h}_i^{\ell+1} = \tilde{h}_i^{\ell}\cdot \tilde{s}_j^{\ell+1}$ for $i\in[0,\ell]$ and $j\in[i+1,\ell+1]$ by evaluating at each element $(x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})\in \tilde{X}_\ell$, where $k\in[0,\ell]$ and $\ell-1\geqslant i_{\ell-k}>\cdots>i_1\geqslant 0$.

Let $e := \max(\{m\in[1,\ell-k] : j-1>i_m\}\cup\{0\})$.

We have

$$
\begin{aligned}
(x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{s}_{j-1}^{\ell}\cdot \tilde{h}_i^{\ell+1}) &= (x, \sigma_{j-1}^{\ell}\cdot \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})\tilde{h}_i^{\ell+1}\\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,e+1\rfloor}^{[\ell+1]\ \ [k+e+1]} \sigma_{i_m+1}^{k+m}\cdot \sigma_{j-1}^{k+e}\cdot \prod_{m\in\lfloor e\,,\,1\rfloor}^{[k+e]\ \ [k]} \sigma_{i_m}^{k+m-1})\tilde{h}_i^{\ell+1}\\
&= (x, \sigma_{j-1}^{k+e}\cdot \prod_{m\in\lfloor e\,,\,1\rfloor}^{[k+e]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{k+e+1}\cdot \prod_{m\in\lceil e+1\,,\,\ell-k\rceil}^{\tilde{Y}_{k+e+2}\ \ \tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1})\\
&= (x, \prod_{m\in\lfloor e\,,\,1\rfloor}^{[k+e]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{k+e}\cdot \tilde{s}_j^{k+e+1}\cdot \prod_{m\in\lceil e+1\,,\,\ell-k\rceil}^{\tilde{Y}_{k+e+2}\ \ \tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1})\\
&= (x, \prod_{m\in\lfloor e\,,\,1\rfloor}^{[k+e]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{k+e}\cdot \prod_{m\in\lceil e+1\,,\,\ell-k\rceil}^{\tilde{Y}_{k+e+1}\ \ \tilde{Y}_{\ell+1}} \tilde{s}_{i_m+1}^{k+m}\cdot \tilde{s}_j^{\ell+1})\\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,e+1\rfloor}^{[\ell]\ \ [k+e]} \sigma_{i_m}^{k+m-1}\cdot \prod_{m\in\lfloor e\,,\,1\rfloor}^{[k+e]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{s}_j^{\ell+1})\\
&= (x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell}\cdot \tilde{s}_j^{\ell+1})\,.
\end{aligned}
$$

Ad (vi).

We prove $\tilde{s}_j^{\ell}\cdot \tilde{h}_{i+1}^{\ell+1} = \tilde{h}_i^{\ell}\cdot \tilde{s}_j^{\ell+1}$ for $i\in[0,\ell]$ and $j\in[0,i]$ by evaluating at each element $(x, \prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[\ell]\ \ [k]} \sigma_{i_m}^{k+m-1})\in \tilde{X}_\ell$, where $k\in[0,\ell]$ and $\ell-1\geqslant i_{\ell-k}>\cdots>i_1\geqslant 0$.

Let $e := \max(\{m\in[1,\ell-k] : j>i_m\}\cup\{0\})$.

Let $c := \max(\{m\in[1,\ell-k] : i>i_m\}\cup\{0\})$.

We have $c\geqslant e$, since $i\geqslant j$.

We have

$$
\begin{aligned}
& (x, \, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{s}_j^{\ell} \cdot \tilde{h}_{i+1}^{\ell+1}) \\
= \;& (x, \sigma_j^{\ell} \cdot {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i+1}^{\ell+1} \\
= \;& (x, \, {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,e+1\rfloor}^{[k+e+1]} \sigma_{i_m+1}^{k+m} \cdot \sigma_j^{k+e} \cdot {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i+1}^{\ell+1} \\
= \;& (x, \, {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,c+1\rfloor}^{[k+c+1]} \sigma_{i_m+1}^{k+m} \cdot {}^{[k+c+1]}\!\!\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e+1]} \sigma_{i_m+1}^{k+m} \cdot \sigma_j^{k+e} \cdot {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{h}_{i+1}^{\ell+1} \\
= \;& (x, \, {}^{[k+c+1]}\!\!\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e+1]} \sigma_{i_m+1}^{k+m} \cdot \sigma_j^{k+e} \cdot {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i+1}^{k+c+1} \cdot {}^{\tilde{Y}_{k+c+2}}\!\!\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1}) \\
= \;& (x, \sigma_j^{k+e} \cdot {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i+1-(c-e)}^{k+e+1} \cdot {}^{\tilde{Y}_{k+e+2}}\!\!\prod_{m\in\lceil e+1\,,\,c\rceil}^{\tilde{Y}_{k+c+2}} \tilde{s}_{i_m+1}^{k+m+1} \cdot {}^{\tilde{Y}_{k+c+2}}\!\!\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1}) \\
= \;& (x, \, {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-(c-e)}^{k+e} \cdot \tilde{s}_j^{k+e+1} \cdot {}^{\tilde{Y}_{k+e+2}}\!\!\prod_{m\in\lceil e+1\,,\,c\rceil}^{\tilde{Y}_{k+c+2}} \tilde{s}_{i_m+1}^{k+m+1} \cdot {}^{\tilde{Y}_{k+c+2}}\!\!\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1}) \\
= \;& (x, \, {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{i-(c-e)}^{k+e} \cdot {}^{\tilde{Y}_{k+e+1}}\!\!\prod_{m\in\lceil e+1\,,\,c\rceil}^{\tilde{Y}_{k+c+1}} \tilde{s}_{i_m}^{k+m} \cdot {}^{\tilde{Y}_{k+c+1}}\!\!\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+1}} \tilde{s}_{i_m+1}^{k+m} \cdot \tilde{s}_j^{\ell+1}) \\
= \;& (x, \, {}^{[k+c]}\!\!\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e]} \sigma_{i_m}^{k+m-1} \cdot {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{k+c} \cdot {}^{\tilde{Y}_{k+c+1}}\!\!\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+1}} \tilde{s}_{i_m+1}^{k+m} \cdot \tilde{s}_j^{\ell+1}) \\
= \;& (x, \, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,c+1\rfloor}^{[k+c]} \sigma_{i_m}^{k+m-1} \cdot {}^{[k+c]}\!\!\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e]} \sigma_{i_m}^{k+m-1} \cdot {}^{[k+e]}\!\!\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell} \cdot \tilde{s}_j^{\ell+1}) \\
= \;& (x, \, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_i^{\ell} \cdot \tilde{s}_j^{\ell+1})\,.
\end{aligned}
$$

Ad (vii).

We prove $\tilde{h}_\ell^\ell \cdot \tilde{d}_{\ell+1}^{\ell+1} = \tilde{f}_\ell$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by evaluating at each element $(x, \mathrm{id}_{[0]}) \in \tilde{X}_0$.

We have $(x, \mathrm{id}_{[0]})(\tilde{h}_0^0 \cdot \tilde{d}_1^1) = (xh_0^0, \mathrm{id}_{[1]})\tilde{d}_1^1 = (x(h_0^0 \cdot d_1^1), \mathrm{id}_{[0]}) = (xf_0, \mathrm{id}_{[0]}) = (x, \mathrm{id}_{[0]})\tilde{f}_0\,.$

Let $\ell \geqslant 1$. Suppose we have $\tilde{h}_{\ell-1}^{\ell-1} \cdot \tilde{d}_\ell^\ell = \tilde{f}_{\ell-1}$.

We verify equality by evaluating at each element $(x, \, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \in \tilde{X}_\ell\,,$ where $k \in [0, \ell]$ and $\ell - 1 \geqslant i_{\ell-k} > \cdots > i_1 \geqslant 0$.

Case $\ell = k$.

We have $(x, \mathrm{id}_{[\ell]})(\tilde{h}_\ell^\ell \cdot \tilde{d}_{\ell+1}^{\ell+1}) = (xh_\ell^\ell, \mathrm{id}_{[\ell+1]})\tilde{d}_{\ell+1}^{\ell+1} = (x(h_\ell^\ell \cdot d_{\ell+1}^{\ell+1}), \mathrm{id}_{[\ell]}) = (xf_\ell, \mathrm{id}_{[\ell]}) = (x, \mathrm{id}_{[\ell]})\tilde{f}_\ell\,.$

Case $\ell > k$.

We have

$$\begin{aligned}
(x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_\ell^\ell \cdot \tilde{d}_{\ell+1}^{\ell+1}) &= (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{\ell-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k}}^{\ell} \cdot \tilde{d}_{\ell+1}^{\ell+1}) \\
&= (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_{\ell-1}^{\ell-1} \cdot \tilde{d}_\ell^\ell \cdot \tilde{s}_{i_{\ell-k}}^{\ell-1}) \\
&\overset{\text{ind. hyp.}}{=} (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{f}_{\ell-1} \cdot \tilde{s}_{i_{\ell-k}}^{\ell-1}) \\
&= (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{s}_{i_{\ell-k}}^{\ell-1} \cdot \tilde{f}_\ell) \\
&= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{f}_\ell\,.
\end{aligned}$$

Ad (viii).

We prove $\tilde{h}_0^\ell \cdot \tilde{d}_0^{\ell+1} = \tilde{g}_\ell$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by evaluating at each element $(x, \mathrm{id}_{[0]}) \in \tilde{X}_0$.

We have $(x, \mathrm{id}_{[0]})(\tilde{h}_0^0 \cdot \tilde{d}_0^1) = (xh_0^0, \mathrm{id}_{[1]})\tilde{d}_0^1 = (x(h_0^0 \cdot d_0^1), \mathrm{id}_{[0]}) = (xg_0, \mathrm{id}_{[0]}) = (x, \mathrm{id}_{[0]})\tilde{g}_0\,.$

Let $\ell \geqslant 1$. Suppose property (viii) is proven for $\ell - 1$.

We verify equality by evaluating at each element $(x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \in \tilde{X}_\ell$, where $k \in [0, \ell]$ and $\ell - 1 \geqslant i_{\ell-k} > \cdots > i_1 \geqslant 0$.

Case $\ell = k$.

We have $(x, \mathrm{id}_{[\ell]})(\tilde{h}_0^\ell \cdot \tilde{d}_0^{\ell+1}) = (xh_0^\ell, \mathrm{id}_{[\ell+1]})\tilde{d}_0^{\ell+1} = (x(h_0^\ell \cdot d_0^{\ell+1}), \mathrm{id}_{[\ell]}) = (xg_\ell, \mathrm{id}_{[\ell]}) = (x, \mathrm{id}_{[\ell]})\tilde{g}_\ell\,.$

Case $\ell > k$.

We have

$$\begin{aligned}
(x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_0^\ell \cdot \tilde{d}_0^{\ell+1}) &= (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_0^{\ell-1} \cdot \tilde{s}_{i_{\ell-k}+1}^{\ell} \cdot \tilde{d}_0^{\ell+1}) \\
&= (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{h}_0^{\ell-1} \cdot \tilde{d}_0^\ell \cdot \tilde{s}_{i_{\ell-k}}^{\ell-1}) \\
&\overset{\text{ind. hyp.}}{=} (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{g}_{\ell-1} \cdot \tilde{s}_{i_{\ell-k}}^{\ell-1}) \\
&= (x, {}^{[\ell-1]}\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})(\tilde{s}_{i_{\ell-k}}^{\ell-1} \cdot \tilde{g}_\ell) \\
&= (x, {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\tilde{g}_\ell\,.
\end{aligned}$$

□

3.4 Homotopy over a general category

Let $\mathcal{C}$ be a category.

3.4.1 Homotopy for simplicial and semisimplicial objects

Definition 26 (simplicial homotopy)**.**

Let $f : X \to Y$ and $g : X \to Y$ be morphisms in $\mathrm{Simp}(\mathcal{C})$.

An *elementary simplicial homotopy* from f to g is a tuple

$$((h_i^\ell : X_\ell \to Y_{\ell+1})_{i\in[0,\ell]})_{\ell\geqslant 0}$$

of morphism in $\mathcal{C}$ such that the following conditions (i - viii) hold for $\ell \geqslant 0$. Cf. Lemma 20.

(i) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_{j-1}^{X,\ell+1} \cdot h_i^\ell \quad$ for $i \in [0,\ell]$ and $j \in [i+2, \ell+2]$

(ii) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i+1}^{\ell+1} \cdot d_j^{Y,\ell+2} \quad$ for $i \in [0,\ell]$ and $j = i+1$

(iii) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i-1}^{\ell+1} \cdot d_j^{Y,\ell+2} \quad$ for $i \in [1,\ell+1]$ and $j = i$

(iv) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_j^{X,\ell+1} \cdot h_{i-1}^\ell \quad$ for $i \in [1,\ell+1]$ and $j \in [0, i-1]$

(v) $\quad h_i^\ell \cdot s_j^{Y,\ell+1} = s_{j-1}^{X,\ell} \cdot h_i^{\ell+1} \quad$ for $i \in [0,\ell]$ and $j \in [i+1, \ell+1]$

(vi) $\quad h_i^\ell \cdot s_j^{Y,\ell+1} = s_j^{X,\ell} \cdot h_{i+1}^{\ell+1} \quad$ for $i \in [0,\ell]$ and $j \in [0, i]$

(vii) $\quad h_\ell^\ell \cdot d_{\ell+1}^{Y,\ell+1} = f_\ell$

(viii) $\quad h_0^\ell \cdot d_0^{Y,\ell+1} = g_\ell$

Note that (ii) and (iii) are equivalent.

We call f *elementary simplicially homotopic* to g if there exists an elementary simplicial homotopy from f to g. This defines a relation $(\rightsquigarrow)$ of elementary simplicial homotopy on $\mathrm{Mor}(\mathrm{Simp}(\mathcal{C}))$. We write $(\underset{X,Y}{\rightsquigarrow}) := (\rightsquigarrow) \cap {}_{\mathrm{Simp}(\mathcal{C})}(X,Y)^{\times 2}$.

Definition 27 (semisimplicial homotopy)**.**

Let $f : X \to Y$ and $g : X \to Y$ be morphisms in $\mathrm{SemiSimp}(\mathcal{C})$.

An *elementary semisimplicial homotopy* from f to g is a tuple

$$((h_i^\ell : X_\ell \to Y_{\ell+1})_{i\in[0,\ell]})_{\ell\geqslant 0}$$

of morphism in $\mathcal{C}$ such that the following conditions (i - vi) hold for $\ell \geqslant 0$.

(i) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_{j-1}^{X,\ell+1} \cdot h_i^\ell \quad$ for $i \in [0,\ell]$ and $j \in [i+2, \ell+2]$

(ii) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i+1}^{\ell+1} \cdot d_j^{Y,\ell+2} \quad$ for $i \in [0,\ell]$ and $j = i+1$

(iii) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = h_{i-1}^{\ell+1} \cdot d_j^{Y,\ell+2} \quad$ for $i \in [1,\ell+1]$ and $j = i$

(iv) $\quad h_i^{\ell+1} \cdot d_j^{Y,\ell+2} = d_j^{X,\ell+1} \cdot h_{i-1}^\ell \quad$ for $i \in [1,\ell+1]$ and $j \in [0, i-1]$

(v) $\quad h_\ell^\ell \cdot d_{\ell+1}^{Y,\ell+1} = f_\ell$

(vi) $\quad h_0^\ell \cdot d_0^{Y,\ell+1} = g_\ell$

Note that (ii) and (iii) are equivalent.

We call f *elementary semisimplicially homotopic* to g if there exists an elementary semisimplicial homotopy from f to g. This defines a relation $(\rightsquigarrow)$ of elementary semisimplicial homotopy on $\operatorname{Mor}(\operatorname{SemiSimp}(\mathcal{C}))$. We write $(\underset{X,Y}{\rightsquigarrow}) := (\rightsquigarrow) \cap {}_{\operatorname{Simp}(\mathcal{C})}(X,Y)^{\times 2}$.

3.4.2 From semisimplicial to simplicial homotopy

Suppose that $\mathcal{C}$ has finite coproducts.

Reminder 28. Let $X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathcal{C})$ be a semisimplicial object.

Write $\tilde{X} := \mathcal{F}_{\mathcal{C}}(X) \in \operatorname{Ob}\operatorname{Simp}(\mathcal{C})$, cf. [**1,** Lemmas 58, 60]. Write $\tilde{d}_j^\ell := \tilde{X}_{\partial_j^\ell}$ for $\ell \geqslant 1$ and $j \in [0,\ell]$.

Write $\tilde{s}_i^\ell := \tilde{X}_{\sigma_i^\ell}$ for $\ell \geqslant 0$ and $i \in [0,\ell]$.

We write $\tilde{g} := \mathcal{F}_{\mathcal{C}}(g)$ for a semisimplicial morphism g.

We recall that for a monotone map $v : [\ell] \to [k]$, there exists a unique surjective monotone map $\bar{v} : [\ell] \to [n]$ and a unique injective montone map $\dot{v} : [n] \to [k]$ such that $v = \bar{v} \cdot \dot{v}$.

We recall from [**1,** Lemma 58 (i)] that

$$\tilde{X}_\ell = \coprod_{\substack{(v:[\ell]\to[k])\in\operatorname{surj} \\ \text{for some } k\in[0,\ell]}} X_k$$

for $\ell \geqslant 0$, where

$$\Big(X_m \xrightarrow{\iota(f)} \coprod_{\substack{(v:[\ell]\to[k])\in\operatorname{surj} \\ \text{for some } k\in[0,\ell]}} X_k\Big)_{\substack{(f:[\ell]\to[m])\in\operatorname{surj} \\ \text{for some } m\in[0,\ell]}}$$

is a chosen coproduct of the tuple $(X_k)_{\substack{(f:[\ell]\to[k])\in\operatorname{surj} \\ \text{for some } k\in[0,\ell]}}$.

For $m \in [0,\ell]$ and $(f : [\ell] \to [m]) \in \operatorname{surj}$ we call the morphism $\iota(f) : X_m \to \coprod_{\substack{(v:[\ell]\to[k])\in\operatorname{surj} \\ \text{for some } k\in[0,\ell]}} X_k$ *inclusion morphism.*

Lemma 29. Let $X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathcal{C})$ be a semisimplicial object.

Let $\ell \geqslant 0$. Suppose given

$$^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1} \in \operatorname{surj}\,,$$

where $k \in [0,\ell+1]$ and where $\ell \geqslant i_{\ell-k+1} > \cdots > i_1 \geqslant 0$.

Write $i_{\ell-k+2} := \ell+2$.

Suppose given $j \in [0,\ell+1]$. Let $q := \min(\{m \in [1,\ell-k+1] : i_m > j\} \cup \{\ell-k+2\})$.

If $q = 1$ or ($q \in [2,\ell-k+2]$ and $j \in [i_{q-1}+2, i_q-1]$) then we have

$$\iota\Big(\, ^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_j^{\ell+1} \;=\; d_{j-q+1}^k \cdot \iota\Big(\, ^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot \, ^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big)\,.$$

If $q \in [2,\ell-k+2]$ and $j \in [i_{q-1}, i_{q-1}+1]$ then we have

$$\iota\Big(\, ^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_j^{\ell+1} \;=\; \iota\Big(\, ^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot \, ^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\,.$$

Proof. By Lemma 11, we have

$$
\partial_j^{\ell+1} \cdot \ {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}
$$

$$
= \begin{cases}
{}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2} \cdot \partial_{j-q+1}^{k} & \text{if } q=1 \text{ or } (q\in[2,\ell-k+2] \text{ and } j\in[i_{q-1}+2, i_q-1]) \text{ (Case 1)} \\[2ex]
{}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1} & \text{if } q\in[2,\ell-k+2] \text{ and } j\in[i_{q-1}, i_{q-1}+1] \text{ (Case 2)}\,.
\end{cases}
$$

In particular, we may read off the factorisation into a surjective monotone map, followed by an injective monotone map. The latter is ∂_{j-q+1}^{k} in Case 1, and $\mathrm{id}_{[k]}$ in Case 2.

So by **[1,** Lemma 58.(ii)], we have

$$
\begin{aligned}
& \iota\Big({}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1} \Big) \cdot \tilde{d}_j^{\ell+1} \\
= \ & X_{\Big(\partial_j^{\ell+1}\cdot \ {}^{[\ell+1]}\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)^{\bullet}} \cdot \iota\Big(\overline{\partial_j^{\ell+1} \cdot \ {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}} \Big) \\
= \ & \begin{cases}
X_{\partial_{j-q+1}^{k}} \cdot \iota\Big({}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2} \Big) & \text{in Case 1} \\
X_{\mathrm{id}_{[k]}} \cdot \iota\Big({}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1} \Big) & \text{in Case 2}
\end{cases} \\
= \ & \begin{cases}
d_{j-q+1}^{k} \cdot \iota\Big({}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2} \Big) & \text{in Case 1} \\
\iota\Big({}^{[\ell]}\!\!\prod\limits_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\!\!\prod\limits_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1} \Big) & \text{in Case 2}\,.
\end{cases}
\end{aligned}
$$

□

Proposition 30. Suppose given semisimplicial morphisms $f : X \to Y$ and $g : X \to Y$ and an elementary semisimplicial homotopy $((h_i^\ell)_{i\in[0,\ell]})_{\ell\geqslant 0}$ from f to g.

Then there exists an elementary simplicial homotopy from $\mathcal{F}_{\mathcal{C}}(f)$ to $\mathcal{F}_{\mathcal{C}}(g)$. Cf. **[1,** Lemma 60].

Proof. Let $v : [\ell] \to [k]$ be a surjective, monotone map.

It holds that $\iota(v)\cdot \tilde{s}_i^\ell = \iota(\sigma_i^\ell \cdot v)$, for $i\in[0,\ell]$. In particular, for $v=\mathrm{id}_{[\ell]}$ we have $\iota(\mathrm{id}_{[\ell]})\cdot \tilde{s}_i^\ell = \iota(\sigma_i^\ell)$.

It holds that $\iota(v)\cdot \tilde{d}_i^\ell = X_{(\partial_i^\ell\cdot v)^\bullet}\cdot \iota(\overline{\partial_i^\ell\cdot v})$ for $\ell\geqslant 1$ and $i\in[0,\ell]$. In particular, for $v=\mathrm{id}_{[\ell]}$ we have $\iota(\mathrm{id}_{[\ell]})\cdot \tilde{d}_i^\ell = d_i^\ell \cdot \iota(\mathrm{id}_{[\ell-1]})$.

Furthermore, we have $\iota(v)\cdot \tilde{f}_\ell = f_k \cdot \iota(v)$.

We have to construct a tuple $((\tilde{X}_\ell \xrightarrow{\tilde{h}_i^\ell} \tilde{Y}_\ell)_{i\in[0,\ell]})_{\ell\geqslant 0}$ of morphisms satisfying conditions (i – viii) in Definition 26.

We define $\tilde{h}_0^0$ to be the unique morphism making the following diagram commute.

$$\begin{array}{ccc}
\tilde{X}_0 & \xrightarrow{\tilde{h}_0^0} & \tilde{Y}_1 \\
{\scriptstyle \iota(\mathrm{id}_{[0]})}\uparrow & & \uparrow{\scriptstyle \iota(\mathrm{id}_{[1]})} \\
X_0 & \xrightarrow[h_0^0]{} & Y_1
\end{array}$$

Let $\ell \geqslant 1$. Suppose that $\tilde{h}_i^{\ell-1}$ has already been constructed for $i \in [0, \ell-1]$.

Given $i \in [0, \ell]$, we define the morphism $\tilde{h}_i^\ell$ to be the unique morphism making a diagram as follows commute for each $k \in [0, l]$ and each surjective monotone map $v : [\ell] \to [k]$.

We use Lemma 10 to write $v = {}^{[\ell]}\prod^{[k]}_{m \in \lfloor \ell-k\,,\,1 \rfloor} \sigma_{i_m}^{k+m-1}$, where $\ell - 1 \geqslant i_{\ell-k} > \cdots > i_1 \geqslant 0$.

Case $k = \ell$. Note that $v = \mathrm{id}_{[\ell]}$.

$$\begin{array}{ccc}
\tilde{X}_\ell & \xrightarrow{\tilde{h}_i^\ell} & \tilde{Y}_{\ell+1} \\
{\scriptstyle \iota(v)}\uparrow & & \uparrow{\scriptstyle \iota(\mathrm{id}_{[\ell+1]})} \\
X_\ell & \xrightarrow[h_i^\ell]{} & Y_{\ell+1}
\end{array}$$

Case $k \in [0, \ell-1]$ and $i_{\ell-k} \geqslant i$.

$$\begin{array}{ccc}
\tilde{X}_\ell & \xrightarrow{\tilde{h}_i^\ell} & \tilde{Y}_{\ell+1} \\
{\scriptstyle \iota(v)}\uparrow & & \uparrow{\scriptstyle \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k}+1}^\ell} \\
X_k & \xrightarrow[\iota\left({}^{[\ell-1]}\prod^{[k]}_{m \in \lfloor \ell-k-1\,,\,1 \rfloor} \sigma_{i_m}^{k+m-1}\right)]{} & \tilde{X}_{\ell-1}
\end{array}$$

Case $k \in [0, \ell-1]$ and $i_{\ell-k} < i$.

$$\begin{array}{ccc}
\tilde{X}_\ell & \xrightarrow{\tilde{h}_i^\ell} & \tilde{Y}_{\ell+1} \\
{\scriptstyle \iota(v)}\uparrow & & \uparrow{\scriptstyle \tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k}}^\ell} \\
X_k & \xrightarrow[\iota\left({}^{[\ell-1]}\prod^{[k]}_{m \in \lfloor \ell-k-1\,,\,1 \rfloor} \sigma_{i_m}^{k+m-1}\right)]{} & \tilde{X}_{\ell-1}
\end{array}$$

We have to verify that the tuple $((\tilde{X}_\ell \xrightarrow{\tilde{h}_i^\ell} \tilde{Y}_{\ell+1})_{i \in [0,\ell]})_{\ell \geqslant 0}$ satisfies (i) - (viii) in Definition 26.

Ad (i).

We prove $\tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2} = \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{h}_i^{\ell}$ for $i \in [0,\ell]$ and $j \in [i+2,\ell+2]$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by precomposing with each inclusion morphism $\iota(v)$ for all surjective maps $v : [1] \to [k]$, where $k \in \{0,1\}$.

Case $v = \mathrm{id}_{[1]}$.

We have $\iota(\mathrm{id}_{[1]}) \cdot \tilde{h}_0^1 \cdot \tilde{d}_2^2 = h_0^1 \cdot \iota(\mathrm{id}_{[2]}) \cdot \tilde{d}_2^2 = h_0^1 \cdot d_2^2 \cdot \iota(\mathrm{id}_{[1]}) \overset{\text{Def 27 (i)}}{=} d_1^1 \cdot h_0^0 \cdot \iota(\mathrm{id}_{[1]}) = d_1^1 \cdot \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0 = \iota(\mathrm{id}_{[1]}) \cdot \tilde{d}_1^1 \cdot \tilde{h}_0^0$.

Case $v = \sigma_0^0$.

We have $\iota(\sigma_0^0) \cdot \tilde{h}_0^1 \cdot \tilde{d}_2^2 = \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0 \cdot \tilde{s}_1^1 \cdot \tilde{d}_2^2 = \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0$.

We have $\iota(\sigma_0^0) \cdot \tilde{d}_1^1 \cdot \tilde{h}_0^0 = X_{(\partial_1^1 \cdot \sigma_0^0)\bullet} \cdot \iota(\overline{\partial_1^1 \cdot \sigma_0^0}) \cdot \tilde{h}_0^0 = \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0$.

Let $\ell \geqslant 1$.

We verify equality by precomposing with each inclusion morphism $\iota\Big({}^{[\ell+1]}\prod^{[k]}_{m \in \lfloor \ell-k+1\,,\,1\rfloor} \sigma_{i_m}^{k+m-1} \Big)$, where $k \in [0,\ell+1]$ and $\ell \geqslant i_{\ell-k+1} > \dots > i_1 \geqslant 0$, cf. Lemma 10.

Write $i_{\ell-k+2} := \ell+2$.

Let $q := \min(\{m \in [1,\ell-k+1] : i_m > j-1\} \cup \{\ell-k+2\})$.

Suppose that $k = \ell+1$.

We have

$$
\begin{aligned}
\iota(\mathrm{id}_{[\ell+1]}) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2} &= h_i^{\ell+1} \cdot \iota(\mathrm{id}_{[\ell+2]}) \cdot \tilde{d}_j^{\ell+2} \\
&= h_i^{\ell+1} \cdot d_j^{\ell+2} \cdot \iota(\mathrm{id}_{[\ell+1]}) \\
&= d_{j-1}^{\ell+1} \cdot h_i^{\ell} \cdot \iota(\mathrm{id}_{[\ell+1]}) \\
&= d_{j-1}^{\ell+1} \cdot \iota(\mathrm{id}_{[\ell]}) \cdot \tilde{h}_i^{\ell} \\
&= \iota(\mathrm{id}_{[\ell+1]}) \cdot \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{h}_i^{\ell} .
\end{aligned}
$$

Suppose that $k \leqslant \ell$.

Case $q \in [2,\ell-k+1]$ and $j-1 \in [i_{q-1}, i_{q-1}+1]$. Note that both in case $j-1 < i_{\ell-k}$ and in case $j-1 \in [i_{\ell-k}, i_{\ell-k}+1]$, we have $q = \min(\{m \in [1,\ell-k] : i_m > j-1\} \cup \{\ell-k+1\})$. Note that $q \in [2,\ell-k+1]$ implies $\ell-k+1 \geqslant 2$, which means $\ell-k \geqslant 1$.

We have

$$
\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod^{[k]}_{m \in \lfloor \ell-k+1\,,\,1\rfloor} \sigma_{i_m}^{k+m-1} \Big) \cdot \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{h}_i^{\ell} \\
\overset{\text{Lm 29}}{=} \;& \iota\Big({}^{[\ell]}\prod^{[k+q-2]}_{m \in \lfloor \ell-k+1\,,\,q\rfloor} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod^{[k]}_{m \in \lfloor q-2\,,\,1\rfloor} \sigma_{i_m}^{k+m-1} \Big) \cdot \tilde{h}_i^{\ell} \\
= \;& \iota\Big({}^{[\ell-1]}\prod^{[k+q-2]}_{m \in \lfloor \ell-k\,,\,q\rfloor} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod^{[k]}_{m \in \lfloor q-2\,,\,1\rfloor} \sigma_{i_m}^{k+m-1} \Big) \cdot \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} .
\end{aligned}
$$

We have

$$
\begin{aligned}
& \iota\big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
= \;& \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
= \;& \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
\overset{\text{ind. hyp.}}{=} \;& \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{d}_{j-1}^{\ell}\cdot \tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
\overset{\text{Lm 29}}{=} \;& \iota\big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-2\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}
$$

Case $q = \ell-k+2$ and $j-1 \in [i_{\ell-k+1}, i_{\ell-k+1}+1]$. Note that $i_{\ell-k+1} \geqslant i$, since $i_{\ell-k+1} \leqslant i-1$ together with $j \geqslant i+2$ would imply $j-1 \geqslant i_{\ell-k+1}+2$, which is not the case.

We have

$$
\iota\big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{d}_{j-1}^{\ell+1}\cdot \tilde{h}_i^{\ell} \overset{\text{Lm 29}}{=} \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell}\,.
$$

We have

$$
\begin{aligned}
\iota\big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2} &= \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
&= \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell}\,.
\end{aligned}
$$

Case $q = 1$. This implies $k \geqslant 1$, since otherwise we would have a factor $\sigma_{i_1}^{0+1-1}$ in the product, hence $i_1 = 0$, but $i_1 > j-1 \geqslant 1$.

We have

$$
\begin{aligned}
\iota\big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{d}_{j-1}^{\ell+1}\cdot \tilde{h}_i^{\ell} &\overset{\text{Lm 29}}{=} d_{j-1}^{k}\cdot \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\big)\cdot \tilde{h}_i^{\ell} \\
&= d_{j-1}^{k}\cdot \iota\big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\big)\cdot \tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
\iota\big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2} &= \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
&= \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{h}_i^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
&\overset{\text{ind. hyp.}}{=} \iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde{d}_{j-1}^{\ell}\cdot \tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
&\overset{\text{Lm 29}}{=} d_{j-1}^{k}\cdot \iota\big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\big)\cdot \tilde{h}_i^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}
$$

Case $q \in [2, \ell-k+1]$ and $j-1 \in [i_{q-1}+2, i_q-1]$. This implies $k \geqslant 1$, because otherwise $k = 0$ and therefore $i_q = q-1$ and $i_{q-1} = q-2$ because they must fit into the strictly decreasing sequence

$\ell \geqslant i_{\ell+1} > \cdots > i_q > i_{q-1} > \cdots > i_1 \geqslant 0$, leading to $q-1 = i_q > j-1 \geqslant i_{q-1}+2 = q-2+2 = q$, which is impossible, cf. also the Remark in Lemma 11.

Note that both in case $j-1 < i_{\ell-k}$ and in case $j-1 \in [i_{\ell-k}+2, i_{\ell-k+1}-1]$, we still have $q = \min(\{m \in [1,\ell-k] : i_m > j-1\} \cup \{\ell-k+1\})$.

We have

$$\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{h}_i^{\ell} \\
\overset{\text{Lm 29}}{=}\; & d_{j-q}^{k} \cdot \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_i^{\ell} \\
\overset{i_{\ell-k+1}-1 \geqslant i}{=}\; & d_{j-q}^{k} \cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}$$

We have

$$\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \\
=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \\
=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell} \cdot \tilde{d}_j^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
\overset{\text{ind. hyp.}}{=}\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_{j-1}^{\ell} \cdot \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
\overset{\text{Lm 29}}{=}\; & d_{j-q}^{k} \cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}$$

Case $q = \ell-k+2$ and $j-1 \in [i_{\ell-k+1}+2, \ell+1]$ and $i_{\ell-k+1} \geqslant i$. Note that we have $i_{\ell-k}+1 < i_{\ell-k+1}+1 < j-1$ and therefore $j-2 \geqslant i_{\ell-k}+2$. So $\min(\{m \in [1,\ell-k] : i_m > j-2\} \cup \{\ell-k+1\}) = \ell-k+1$.

We have

$$\begin{aligned}
\iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{h}_i^{\ell} \;\overset{\text{Lm 29}}{=}\; & d_{j-\ell+k-2}^{k} \cdot \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_i^{\ell} \\
=\; & d_{j-\ell+k-2}^{k} \cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell}\,.
\end{aligned}$$

We have

$$\begin{aligned}
\iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \;=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \\
=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell} \cdot \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell} \\
\overset{\text{ind. hyp.}}{=}\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_{j-2}^{\ell} \cdot \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell} \\
\overset{\text{Lm 29}}{=}\; & d_{j-\ell+k-2}^{k} \cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_i^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell}\,.
\end{aligned}$$

Case $q = \ell-k+2$ and $j-1 \in [i_{\ell-k+1}+2, \ell+1]$ and $i_{\ell-k+1} < i$. Note that we have $i_{\ell-k}+1 < i_{\ell-k+1}+1 < j-1$ and therefore $j-2 > i_{\ell-k}+1$. So $\min(\{m \in [1, \ell-k] : i_m > j-2\} \cup \{\ell-k+1\}) = \ell-k+1$.

We have

$$\begin{aligned} \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{h}_i^{\ell} &\overset{\text{Lm 29}}{=} d_{j-\ell+k-2}^{k} \cdot \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_i^{\ell} \\ &= d_{j-\ell+k-2}^{k} \cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} . \end{aligned}$$

We have

$$\begin{aligned} \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2} &= \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \\ &= \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} \cdot \tilde{d}_{j-1}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\ &\overset{\text{ind. hyp.}}{=} \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_{j-2}^{\ell} \cdot \tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\ &\overset{\text{Lm 29}}{=} d_{j-\ell+k-2}^{k} \cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big) \cdot \tilde{h}_{i-1}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} . \end{aligned}$$

Ad (ii).

We prove $\tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} = \tilde{h}_{i+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2}$ for $i \in [0, \ell]$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by precomposing with each inclusion morphism $\iota(v)$ for all surjective maps $v : [1] \to [k]$, where $k \in \{0, 1\}$.

Case $v = \mathrm{id}_{[1]}$.

We have $\iota(\mathrm{id}_{[1]})\cdot\tilde{h}_0^1\cdot\tilde{d}_1^2 = h_0^1\cdot\iota(\mathrm{id}_{[2]})\cdot\tilde{d}_1^2 = h_0^1\cdot d_1^2\cdot\iota(\mathrm{id}_{[1]}) = h_1^1\cdot d_1^2\cdot\iota(\mathrm{id}_{[1]}) = h_1^1\cdot\iota(\mathrm{id}_{[2]})\cdot\tilde{d}_1^2 = \iota(\mathrm{id}_{[1]})\cdot\tilde{h}_1^1\cdot\tilde{d}_1^2$.

Case $v = \sigma_0^0$.

We have $\iota(\sigma_0^0) \cdot \tilde{h}_0^1 \cdot \tilde{d}_1^2 = \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0 \cdot \tilde{s}_1^1 \cdot \tilde{d}_1^2 = \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0$.

We have $\iota(\sigma_0^0) \cdot \tilde{h}_1^1 \cdot \tilde{d}_1^2 = \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0 \cdot \tilde{s}_0^1 \cdot \tilde{d}_1^2 = \iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0$.

Let $\ell \geqslant 1$.

We verify equality by precomposing with each inclusion morphism $\iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)$, where $k \in [0, \ell+1]$ and $\ell \geqslant i_{\ell-k+1} > \cdots > i_1 \geqslant 0$, cf. Lemma 10.

Suppose that $\ell + 1 = k$.

We have

$$\begin{aligned} \iota(\mathrm{id}_{[\ell+1]}) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} &= h_i^{\ell+1} \cdot \iota(\mathrm{id}_{[\ell+2]}) \cdot \tilde{d}_{i+1}^{\ell+2} \\ &= h_i^{\ell+1} \cdot d_{i+1}^{\ell+2} \cdot \iota(\mathrm{id}_{[\ell+1]}) \\ &= h_{i+1}^{\ell+1} \cdot d_{i+1}^{\ell+2} \cdot \iota(\mathrm{id}_{[\ell+1]}) \\ &= h_{i+1}^{\ell+1} \cdot \iota(\mathrm{id}_{[\ell+2]}) \cdot \tilde{d}_{i+1}^{\ell+2} \\ &= \iota(\mathrm{id}_{[\ell+1]}) \cdot \tilde{h}_{i+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} . \end{aligned}$$

Suppose that $k \leqslant \ell$.

Case $i_{\ell-k+1} \geqslant i+1$. It follows that $i < i_{\ell-k+1} \leqslant \ell$.

We have

$$\begin{aligned}
\iota(\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} &= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} \\
&= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell} \cdot \tilde{d}_{i+1}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
&\overset{\text{ind. hyp.}}{=} \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i+1}^{\ell} \cdot \tilde{d}_{i+1}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
&= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i+1}^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} \\
&= \iota(\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} .
\end{aligned}$$

Case $i_{\ell-k+1} = i$.

We have

$$\begin{aligned}
\iota(\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} &= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} \\
&= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell} .
\end{aligned}$$

We have

$$\begin{aligned}
\iota(\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} &= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} \\
&= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell} .
\end{aligned}$$

Case $i_{\ell-k+1} < i$.

We have

$$\begin{aligned}
\iota(\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_i^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} &= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i-1}^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} \\
&= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i-1}^{\ell} \cdot \tilde{d}_{i}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
&\overset{\text{ind. hyp.}}{=} \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i}^{\ell} \cdot \tilde{d}_{i}^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
&= \iota(\ {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i}^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} \\
&= \iota(\ {}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}) \cdot \tilde{h}_{i+1}^{\ell+1} \cdot \tilde{d}_{i+1}^{\ell+2} .
\end{aligned}$$

Ad (iv).

We prove $\tilde{h}_i^{\ell+1} \cdot \tilde{d}_j^{\ell+2} = \tilde{d}_j^{\ell+1} \cdot \tilde{h}_{i-1}^{\ell}$ for $i \in [1, \ell+1]$ and $j \in [0, i-1]$ by induction on $\ell \geqslant 0$.

Suppose that $\ell = 0$. We verify equality by precomposing with each inclusion morphism $\iota(v)$ for all surjective maps $v : [1] \to [k]$, where $k \in \{0, 1\}$.

Case $v = \mathrm{id}_{[1]}$.

We have $\iota(\mathrm{id}_{[1]})\cdot\tilde{h}^1_1\cdot\tilde{d}^2_0 = h^1_1\cdot\iota(\mathrm{id}_{[2]})\cdot\tilde{d}^2_0 = h^1_1\cdot d^2_0\cdot\iota(\mathrm{id}_{[1]}) = d^1_0\cdot h^0_0\cdot\iota(\mathrm{id}_{[1]}) = d^1_0\cdot\iota(\mathrm{id}_{[0]})\cdot\tilde{h}^0_0 = \iota(\mathrm{id}_{[1]})\cdot\tilde{d}^1_0\cdot\tilde{h}^0_0$.

Case $v = \sigma^0_0$.

We have $\iota(\sigma^0_0)\cdot\tilde{h}^1_1\cdot\tilde{d}^2_0 = \iota(\mathrm{id}_{[0]})\cdot\tilde{h}^0_0\cdot\tilde{s}^1_0\cdot\tilde{d}^2_0 = \iota(\mathrm{id}_{[0]})\cdot\tilde{h}^0_0$.

We have $\iota(\sigma^0_0)\cdot\tilde{d}^1_0\cdot\tilde{h}^0_0 = X_{(\partial^1_0\cdot\sigma^0_0)^\bullet}\cdot\iota(\overline{\partial^1_0\cdot\sigma^0_0})\cdot\tilde{h}^0_0 = \iota(\mathrm{id}_{[0]})\cdot\tilde{h}^0_0$.

Suppose that $\ell \geqslant 1$.

We verify equality by precomposing with each inclusion morphism $\iota\Big({}^{[\ell+1]}\prod^{[k]}_{m\in\lfloor \ell-k+1\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)$, where $k\in[0,\ell+1]$ and $\ell\geqslant i_{\ell-k+1}>\cdots>i_1\geqslant 0$, cf. Lemma 10.

Let $q := \min(\{m\in[1,\ell-k+1] : i_m > j\}\cup\{\ell-k+2\})$.

Suppose that $k = \ell+1$.

We have

$$\begin{aligned}
\iota(\mathrm{id}_{[\ell+1]})\cdot\tilde{h}^{\ell+1}_i\cdot\tilde{d}^{\ell+2}_j &= h^{\ell+1}_i\cdot\iota(\mathrm{id}_{[\ell+2]})\cdot\tilde{d}^{\ell+2}_j\\
&= h^{\ell+1}_i\cdot d^{\ell+2}_j\cdot\iota(\mathrm{id}_{[\ell+1]})\\
&= d^{\ell+1}_j\cdot h^{\ell}_{i-1}\cdot\iota(\mathrm{id}_{[\ell+1]})\\
&= d^{\ell+1}_j\cdot\iota(\mathrm{id}_{[\ell]})\cdot\tilde{h}^{\ell}_{i-1}\\
&= \iota(\mathrm{id}_{[\ell+1]})\cdot\tilde{d}^{\ell+1}_j\cdot\tilde{h}^{\ell}_{i-1}\,.
\end{aligned}$$

Suppose that $k\leqslant\ell$.

Case $q\in[2,\ell-k+2]$ and $j\in[i_{q-1},i_{q-1}+1]$ and $i_{\ell-k+1}\geqslant i$. It follows that $j<i\leqslant i_{\ell-k+1}\leqslant\ell$ and $\ell-k+1\geqslant q$. Therefore we have $q=\min(\{m\in[1,\ell-k] : i_m>j\}\cup\{\ell-k+1\})$.

We have

$$\begin{aligned}
&\iota\Big({}^{[\ell+1]}\prod^{[k]}_{m\in\lfloor \ell-k+1\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{d}^{\ell+1}_j\cdot\tilde{h}^{\ell}_{i-1}\\
\overset{\text{Lm 29}}{=}\;&\iota\Big({}^{[\ell]}\prod^{[k+q-2]}_{m\in\lfloor \ell-k+1\,,\,q\rfloor}\sigma^{k+m-2}_{i_m-1}\cdot{}^{[k+q-2]}\prod^{[k]}_{m\in\lfloor q-2\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{h}^{\ell}_{i-1}\\
=\;&\iota\Big({}^{[\ell-1]}\prod^{[k+q-2]}_{m\in\lfloor \ell-k\,,\,q\rfloor}\sigma^{k+m-2}_{i_m-1}\cdot{}^{[k+q-2]}\prod^{[k]}_{m\in\lfloor q-2\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{h}^{\ell-1}_{i-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}}\,.
\end{aligned}$$

We have

$$\begin{aligned}
&\iota\Big({}^{[\ell+1]}\prod^{[k]}_{m\in\lfloor \ell-k+1\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{h}^{\ell+1}_i\cdot\tilde{d}^{\ell+2}_j\\
=\;&\iota\Big({}^{[\ell]}\prod^{[k]}_{m\in\lfloor \ell-k\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{h}^{\ell}_i\cdot\tilde{s}^{\ell+1}_{i_{\ell-k+1}+1}\cdot\tilde{d}^{\ell+2}_j\\
=\;&\iota\Big({}^{[\ell]}\prod^{[k]}_{m\in\lfloor \ell-k\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{h}^{\ell}_i\cdot\tilde{d}^{\ell+1}_j\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}}\\
\overset{\text{ind. hyp.}}{=}\;&\iota\Big({}^{[\ell]}\prod^{[k]}_{m\in\lfloor \ell-k\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{d}^{\ell}_j\cdot\tilde{h}^{\ell-1}_{i-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}}\\
\overset{\text{Lm 29}}{=}\;&\iota\Big({}^{[\ell-1]}\prod^{[k+q-2]}_{m\in\lfloor \ell-k\,,\,q\rfloor}\sigma^{k+m-2}_{i_m-1}\cdot{}^{[k+q-2]}\prod^{[k]}_{m\in\lfloor q-2\,,\,1\rfloor}\sigma^{k+m-1}_{i_m}\Big)\cdot\tilde{h}^{\ell-1}_{i-1}\cdot\tilde{s}^{\ell}_{i_{\ell-k+1}}\,.
\end{aligned}$$

Case $q \in [2, \ell-k+1]$ and $j \in [i_{q-1}, i_{q-1}+1]$ and $i_{\ell-k+1} < i$. Note that $j < i_q \leqslant i_{\ell-k+1} < i$ since $\ell - k + 1 \geqslant q$. Therefore we have $q = \min(\{m \in [1, \ell-k] : i_m > j\} \cup \{\ell-k+1\})$.

We have

$$
\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m \in \lfloor \ell-k+1\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_j^{\ell+1} \cdot \tilde{h}_{i-1}^{\ell} \\
\overset{\text{Lm 29}}{=} \; & \iota\Big({}^{[\ell]}\prod_{m \in \lfloor \ell-k+1\,,\,q \rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m \in \lfloor q-2\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} \\
= \; & \iota\Big({}^{[\ell-1]}\prod_{m \in \lfloor \ell-k\,,\,q \rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m \in \lfloor q-2\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-2}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell} .
\end{aligned}
$$

We have

$$
\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m \in \lfloor \ell-k+1\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i}^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \\
= \; & \iota\Big({}^{[\ell]}\prod_{m \in \lfloor \ell-k\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \\
= \; & \iota\Big({}^{[\ell]}\prod_{m \in \lfloor \ell-k\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} \cdot \tilde{d}_j^{\ell+1} \cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell} \\
\overset{\text{ind. hyp.}}{=} \; & \iota\Big({}^{[\ell]}\prod_{m \in \lfloor \ell-k\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_j^{\ell} \cdot \tilde{h}_{i-2}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell} \\
\overset{\text{Lm 29}}{=} \; & \iota\Big({}^{[\ell-1]}\prod_{m \in \lfloor \ell-k\,,\,q \rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2} \cdot {}^{[k+q-2]}\prod_{m \in \lfloor q-2\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-2}^{\ell-1} \cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell} .
\end{aligned}
$$

Case $q = \ell - k + 2$ and $j \in [i_{\ell-k+1}, i_{\ell-k+1}+1]$. It follows that $i_{\ell-k+1} < j < i$.

We have

$$
\iota\Big({}^{[\ell+1]}\prod_{m \in \lfloor \ell-k+1\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{d}_j^{\ell+1} \cdot \tilde{h}_{i-1}^{\ell} \overset{\text{Lm 29}}{=} \iota\Big({}^{[\ell]}\prod_{m \in \lfloor \ell-k\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} .
$$

We have

$$
\begin{aligned}
\iota\Big({}^{[\ell+1]}\prod_{m \in \lfloor \ell-k+1\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i}^{\ell+1} \cdot \tilde{d}_j^{\ell+2} & = \iota\Big({}^{[\ell]}\prod_{m \in \lfloor \ell-k\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} \cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1} \cdot \tilde{d}_j^{\ell+2} \\
& = \iota\Big({}^{[\ell]}\prod_{m \in \lfloor \ell-k\,,\,1 \rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-1}^{\ell} .
\end{aligned}
$$

Case $q \in [2, \ell-k+2]$ and $j \in [i_{q-1}+2, i_q-1]$ and $i_{\ell-k+1} \geqslant i$. It follows that $j < i \leqslant i_{\ell-k+1} \leqslant \ell$ and thus $\ell - k + 1 \geqslant q$. Therefore we have $q = \min(\{m \in [1, \ell-k] : i_m > j\} \cup \{\ell-k+1\})$.

We have

$$\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{h}_{i-1}^{\ell} \\
\overset{\text{Lm 29}}{=}\; & d_{j-q+1}^{k}\cdot \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big)\cdot \tilde{h}_{i-1}^{\ell} \\
=\; & d_{j-q+1}^{k}\cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big)\cdot \tilde{h}_{i-1}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}$$

We have

$$\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{h}_i^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{h}_i^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
\overset{\text{ind. hyp.}}{=}\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{d}_j^{\ell}\cdot \tilde{h}_{i-1}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell} \\
\overset{\text{Lm 29}}{=}\; & d_{j-q+1}^{k}\cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big)\cdot \tilde{h}_{i-1}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}$$

Case $q\in[2,\ell-k+1]$ and $j\in[i_{q-1}+2,i_q-1]$ and $i_{\ell-k+1}<i$. Note that $j<i_q\leqslant i_{\ell-k+1}\leqslant \ell$ and $j<i_{\ell-k+1}<i$ and therefore $i-1>j$. Again we have $q=\min(\{m\in[1,\ell-k]:i_m>j\}\cup\{\ell-k+1\})$.

We have

$$\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{h}_{i-1}^{\ell} \\
\overset{\text{Lm 29}}{=}\; & d_{j-q+1}^{k}\cdot \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big)\cdot \tilde{h}_{i-1}^{\ell} \\
=\; & d_{j-q+1}^{k}\cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big)\cdot \tilde{h}_{i-2}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell}\,.
\end{aligned}$$

We have

$$\begin{aligned}
& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{h}_i^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{h}_{i-1}^{\ell}\cdot \tilde{s}_{i_{\ell-k+1}}^{\ell+1}\cdot \tilde{d}_j^{\ell+2} \\
=\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{h}_{i-1}^{\ell}\cdot \tilde{d}_j^{\ell+1}\cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell} \\
\overset{\text{ind. hyp.}}{=}\; & \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde{d}_j^{\ell}\cdot \tilde{h}_{i-2}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell} \\
\overset{\text{Lm 29}}{=}\; & d_{j-q+1}^{k}\cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,q\rfloor}^{[k+q-2]} \sigma_{i_m-1}^{k+m-2}\cdot {}^{[k+q-2]}\prod_{m\in\lfloor q-1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\Big)\cdot \tilde{h}_{i-2}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k+1}-1}^{\ell}\,.
\end{aligned}$$

Case $q = \ell-k+2$ and $j \in [i_{\ell-k+1}+2, \ell+1]$. It follows that $i_{\ell-k+1}+2 \leqslant j < i$ and thus $i_{\ell-k+1} < i-1$. Note that $j-1 \geqslant i_{\ell-k+1}+1 \geqslant i_{\ell-k}+2$. We have

$$\begin{aligned}
\iota\big(\, {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde d_j^{\ell+1}\cdot \tilde h_{i-1}^{\ell} &\overset{\text{Lm 29}}{=} d^k_{j-\ell+k-1}\cdot \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\big)\cdot \tilde h_{i-1}^{\ell}\\
&= d^k_{j-\ell+k-1}\cdot \iota\big(\, {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\big)\cdot \tilde h_{i-2}^{\ell-1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}}\,.
\end{aligned}$$

We have

$$\begin{aligned}
\iota\big(\, {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde h_i^{\ell+1}\cdot \tilde d_j^{\ell+2} &= \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde h_{i-1}^{\ell}\cdot \tilde s^{\ell+1}_{i_{\ell-k+1}}\cdot \tilde d_j^{\ell+2}\\
&= \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde h_{i-1}^{\ell}\cdot \tilde d_{j-1}^{\ell+1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}}\\
&\overset{\text{ind. hyp.}}{=} \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde d_{j-1}^{\ell}\cdot \tilde h_{i-2}^{\ell-1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}}\\
&\overset{\text{Lm 29}}{=} d^k_{j-\ell+k-1}\cdot \big(\, {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m}^{k+m-2}\big)\cdot \tilde h_{i-2}^{\ell-1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}}\,.
\end{aligned}$$

Case $q = 1$ and $i_{\ell-k+1} < i$. Note that $j < i_1 \leqslant i_{\ell-k+1} < i$.

Therefore we have $q = \min(\{m \in [1, \ell-k] : i_m > j\} \cup \{\ell-k+1\})$.

We have

$$\begin{aligned}
\iota\big(\, {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde d_j^{\ell+1}\cdot \tilde h_{i-1}^{\ell} &\overset{\text{Lm 29}}{=} d^k_{j}\cdot \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\big)\cdot \tilde h_{i-1}^{\ell}\\
&= d^k_{j}\cdot \iota\big(\, {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\big)\cdot \tilde h_{i-2}^{\ell-1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}-1}\,.
\end{aligned}$$

We have

$$\begin{aligned}
\iota\big(\, {}^{[\ell+1]}\!\!\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde h_i^{\ell+1}\cdot \tilde d_j^{\ell+2} &= \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde h_{i-1}^{\ell}\cdot \tilde s^{\ell+1}_{i_{\ell-k+1}}\cdot \tilde d_j^{\ell+2}\\
&= \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde h_{i-1}^{\ell}\cdot \tilde d_{j}^{\ell+1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}-1}\\
&\overset{\text{ind. hyp.}}{=} \iota\big(\, {}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)\cdot \tilde d_{j}^{\ell}\cdot \tilde h_{i-2}^{\ell-1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}-1}\\
&\overset{\text{Lm 29}}{=} d^k_{j}\cdot \iota\big(\, {}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\big)\cdot \tilde h_{i-2}^{\ell-1}\cdot \tilde s^{\ell}_{i_{\ell-k+1}-1}\,.
\end{aligned}$$

Case $q = 1$ and $i_{\ell-k+1} \geqslant i$. Note that $j < i \leqslant i_{\ell-k+1} \leqslant \ell$.

Therefore we have $q = \min(\{m \in [1, \ell-k] : i_m > j\} \cup \{\ell-k+1\})$.

We have

$$
\begin{aligned}
\iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde d_j^{\ell+1}\cdot \tilde h_{i-1}^{\ell} &\overset{\text{Lm 29}}{=} d_j^k\cdot \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\Big)\cdot \tilde h_{i-1}^{\ell} \\
&= d_j^k\cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\Big)\cdot \tilde h_{i-1}^{\ell-1}\cdot \tilde s_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
\iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k+1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_{i}^{\ell+1}\cdot \tilde d_{j}^{\ell+2} &= \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_{i}^{\ell}\cdot \tilde s_{i_{\ell-k+1}+1}^{\ell+1}\cdot \tilde d_j^{\ell+2} \\
&= \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_{i}^{\ell}\cdot \tilde d_j^{\ell+1}\cdot \tilde s_{i_{\ell-k+1}}^{\ell} \\
&\overset{\text{ind. hyp.}}{=} \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde d_j^{\ell}\cdot \tilde h_{i-1}^{\ell-1}\cdot \tilde s_{i_{\ell-k+1}}^{\ell} \\
&\overset{\text{Lm 29}}{=} d_j^k\cdot \iota\Big({}^{[\ell-1]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k-1]} \sigma_{i_m-1}^{k+m-2}\Big)\cdot \tilde h_{i-1}^{\ell-1}\cdot \tilde s_{i_{\ell-k+1}}^{\ell}\,.
\end{aligned}
$$

Ad (v).

We prove $\tilde s_{j-1}^{\ell}\cdot \tilde h_i^{\ell+1} = \tilde h_i^{\ell}\cdot \tilde s_j^{\ell+1}$ for $i\in[0,\ell]$ and $j\in[i+1,\ell+1]$ by precomposing with each inclusion morphism $\iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)$, where $k\in[0,\ell]$ and $\ell-1\geqslant i_{\ell-k}>\cdots>i_1\geqslant 0$, cf. Lemma 10.

Let $e := \max(\{m\in[1,\ell-k] : j-1>i_m\}\cup\{0\})$.

We have

$$
\begin{aligned}
\iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde s_{j-1}^{\ell}\cdot \tilde h_i^{\ell+1} &= \iota\Big(\sigma_{j-1}^{\ell}\cdot {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_i^{\ell+1} \\
&= \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k\,,\,e+1\rfloor}^{[k+e+1]} \sigma_{i_m+1}^{k+m}\cdot \sigma_{j-1}^{k+e}\cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_i^{\ell+1} \\
&= \iota\Big(\sigma_{j-1}^{k+e}\cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_i^{k+e+1}\cdot {}^{\tilde Y_{k+e+2}}\prod_{m\in\lceil e+1\,,\,\ell-k\rfloor}^{\tilde Y_{\ell+2}} \tilde s_{i_m+2}^{k+m+1} \\
&= \iota\Big({}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_i^{k+e}\cdot \tilde s_j^{k+e+1}\cdot {}^{\tilde Y_{k+e+2}}\prod_{m\in\lceil e+1\,,\,\ell-k\rfloor}^{\tilde Y_{\ell+2}} \tilde s_{i_m+2}^{k+m+1} \\
&= \iota\Big({}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_i^{k+e}\cdot {}^{\tilde Y_{k+e+1}}\prod_{m\in\lceil e+1\,,\,\ell-k\rfloor}^{\tilde Y_{\ell+1}} \tilde s_{i_m+1}^{k+m}\cdot \tilde s_j^{\ell+1} \\
&= \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,e+1\rfloor}^{[k+e]} \sigma_{i_m}^{k+m-1}\cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_i^{\ell}\cdot \tilde s_j^{\ell+1} \\
&= \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big)\cdot \tilde h_i^{\ell}\cdot \tilde s_j^{\ell+1}\,.
\end{aligned}
$$

Ad (vi).

We prove $\tilde s_j^{\ell}\cdot \tilde h_{i+1}^{\ell+1} = \tilde h_i^{\ell}\cdot \tilde s_j^{\ell+1}$ for $i\in[0,\ell]$ and $j\in[0,i]$ by precomposing with each inclusion morphism $\iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)$, where $k\in[0,\ell]$ and $\ell-1\geqslant i_{\ell-k}>\cdots>i_1\geqslant 0$, cf. Lemma 10.

Let $e := \max(\{m \in [1, \ell - k] : j > i_m\} \cup \{0\})$.

Let $c := \max(\{m \in [1, \ell - k] : i > i_m\} \cup \{0\})$.

We have $c \geqslant e$, since $i \geqslant j$.

We have

$$
\begin{aligned}
& \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{s}_j^{\ell} \cdot \tilde{h}_{i+1}^{\ell+1} \\
= \;& \iota\Big(\sigma_j^{\ell} \cdot {}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i+1}^{\ell+1} \\
= \;& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k\,,\,e+1\rfloor}^{[k+e+1]} \sigma_{i_m+1}^{k+m} \cdot \sigma_j^{k+e} \cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i+1}^{\ell+1} \\
= \;& \iota\Big({}^{[\ell+1]}\prod_{m\in\lfloor \ell-k\,,\,c+1\rfloor}^{[k+c+1]} \sigma_{i_m+1}^{k+m} \cdot {}^{[k+c+1]}\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e+1]} \sigma_{i_m+1}^{k+m} \cdot \sigma_j^{k+e} \cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i+1}^{\ell+1} \\
= \;& \iota\Big({}^{[k+c+1]}\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e+1]} \sigma_{i_m+1}^{k+m} \cdot \sigma_j^{k+e} \cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i+1}^{k+c+1} \cdot {}^{\tilde{Y}_{k+c+2}}\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1} \\
= \;& \iota\Big(\sigma_j^{k+e} \cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i+1-(c-e)}^{k+e+1} \cdot {}^{\tilde{Y}_{k+e+2}}\prod_{m\in\lceil e+1\,,\,c\rceil}^{\tilde{Y}_{k+c+2}} \tilde{s}_{i_m+1}^{k+m+1} \cdot {}^{\tilde{Y}_{k+c+2}}\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1} \\
= \;& \iota\Big({}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-(c-e)}^{k+e} \cdot \tilde{s}_j^{k+e+1} \cdot {}^{\tilde{Y}_{k+e+2}}\prod_{m\in\lceil e+1\,,\,c\rceil}^{\tilde{Y}_{k+c+2}} \tilde{s}_{i_m+1}^{k+m+1} \cdot {}^{\tilde{Y}_{k+c+2}}\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+2}} \tilde{s}_{i_m+2}^{k+m+1} \\
= \;& \iota\Big({}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_{i-(c-e)}^{k+e} \cdot {}^{\tilde{Y}_{k+e+1}}\prod_{m\in\lceil e+1\,,\,c\rceil}^{\tilde{Y}_{k+c+1}} \tilde{s}_{i_m}^{k+m} \cdot {}^{\tilde{Y}_{k+c+1}}\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+1}} \tilde{s}_{i_m+1}^{k+m} \cdot \tilde{s}_j^{\ell+1} \\
= \;& \iota\Big({}^{[k+c]}\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e]} \sigma_{i_m}^{k+m-1} \cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{k+c} \cdot {}^{\tilde{Y}_{k+c+1}}\prod_{m\in\lceil c+1\,,\,\ell-k\rceil}^{\tilde{Y}_{\ell+1}} \tilde{s}_{i_m+1}^{k+m} \cdot \tilde{s}_j^{\ell+1} \\
= \;& \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,c+1\rfloor}^{[k+c]} \sigma_{i_m}^{k+m-1} \cdot {}^{[k+c]}\prod_{m\in\lfloor c\,,\,e+1\rfloor}^{[k+e]} \sigma_{i_m}^{k+m-1} \cdot {}^{[k+e]}\prod_{m\in\lfloor e\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell} \cdot \tilde{s}_j^{\ell+1} \\
= \;& \iota\Big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\Big) \cdot \tilde{h}_i^{\ell} \cdot \tilde{s}_j^{\ell+1}\,.
\end{aligned}
$$

Ad (vii).

We prove $\tilde{h}_\ell^\ell \cdot \tilde{d}_{\ell+1}^{\ell+1} = \tilde{f}_\ell$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by precomposing with the inclusion morphism $\iota(\mathrm{id}_{[0]})$.

We have $\iota(\mathrm{id}_{[0]}) \cdot \tilde{h}_0^0 \cdot \tilde{d}_1^1 = h_0^0 \cdot \iota(\mathrm{id}_{[1]}) \cdot \tilde{d}_1^1 = h_0^0 \cdot d_1^1 \cdot \iota(\mathrm{id}_{[0]}) = f_0 \cdot \iota(\mathrm{id}_{[0]}) = \iota(\mathrm{id}_{[0]}) \cdot \tilde{f}_0\,$.

Let $\ell \geqslant 1$.

We verify equality by precomposing with each inclusion morphism $\iota\big({}^{[\ell]}\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1}\big)$, where $k \in [0, \ell]$ and $\ell - 1 \geqslant i_{\ell-k} > \cdots > i_1 \geqslant 0$, cf. Lemma 10.

Case $\ell = k$.

We have $\iota(\mathrm{id}_{[\ell]}) \cdot \tilde{h}_\ell^\ell \cdot \tilde{d}_{\ell+1}^{\ell+1} = h_\ell^\ell \cdot \iota(\mathrm{id}_{[\ell+1]}) \cdot \tilde{d}_{\ell+1}^{\ell+1} = h_\ell^\ell \cdot d_{\ell+1}^{\ell+1} \cdot \iota(\mathrm{id}_{[\ell]}) = f_\ell \cdot \iota(\mathrm{id}_{[\ell]}) = \iota(\mathrm{id}_{[\ell]}) \cdot \tilde{f}_\ell\,$.

Case $\ell > k$.

We have

$$
\begin{aligned}
\iota(\,{}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{h}_\ell^\ell \cdot \tilde{d}_{\ell+1}^{\ell+1} &= \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{h}_{\ell-1}^{\ell-1}\cdot \tilde{s}_{i_{\ell-k}}^{\ell}\cdot \tilde{d}_{\ell+1}^{\ell+1} \\
&= \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{h}_{\ell-1}^{\ell-1}\cdot \tilde{d}_{\ell}^{\ell}\cdot \tilde{s}_{i_{\ell-k}}^{\ell-1} \\
&\overset{\text{ind. hyp.}}{=} \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{f}_{\ell-1}\cdot \tilde{s}_{i_{\ell-k}}^{\ell-1} \\
&= \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{s}_{i_{\ell-k}}^{\ell-1}\cdot \tilde{f}_{\ell} \\
&= \iota(\,{}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{f}_{\ell}\,.
\end{aligned}
$$

Ad (viii).

We prove $\tilde{h}_0^\ell \cdot \tilde{d}_0^{\ell+1} = \tilde{g}_\ell$ by induction on $\ell \geqslant 0$.

Let $\ell = 0$. We verify equality by precomposing with the inclusion morphism $\iota(\mathrm{id}_{[0]})$.

We have $\iota(\mathrm{id}_{[0]})\cdot \tilde{h}_0^0\cdot \tilde{d}_0^1 = h_0^0\cdot \iota(\mathrm{id}_{[1]})\cdot \tilde{d}_0^1 = h_0^0\cdot d_0^1\cdot \iota(\mathrm{id}_{[0]}) = g_0\cdot \iota(\mathrm{id}_{[0]}) = \iota(\mathrm{id}_{[0]})\cdot \tilde{g}_0\,.$

Let $\ell \geqslant 1$.

We verify equality by precomposing with each inclusion morphism $\iota(\,{}^{[\ell]}\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})$, where $k\in[0,\ell]$ and $\ell-1\geqslant i_{\ell-k} > \cdots > i_1 \geqslant 0$, cf. Lemma 10.

Case $\ell = k$.

We have $\iota(\mathrm{id}_{[\ell]})\cdot \tilde{h}_0^\ell\cdot \tilde{d}_0^{\ell+1} = h_0^\ell\cdot \iota(\mathrm{id}_{[\ell+1]})\cdot \tilde{d}_0^{\ell+1} = h_0^\ell\cdot d_0^{\ell+1}\cdot \iota(\mathrm{id}_{[\ell]}) = g_\ell\cdot \iota(\mathrm{id}_{[\ell]}) = \iota(\mathrm{id}_{[\ell]})\cdot \tilde{g}_\ell\,.$

Case $\ell > k$.

We have

$$
\begin{aligned}
\iota(\,{}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{h}_0^\ell \cdot \tilde{d}_0^{\ell+1} &= \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{h}_0^{\ell-1}\cdot \tilde{s}_{i_{\ell-k}+1}^{\ell}\cdot \tilde{d}_0^{\ell+1} \\
&= \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{h}_0^{\ell-1}\cdot \tilde{d}_0^{\ell}\cdot \tilde{s}_{i_{\ell-k}}^{\ell-1} \\
&\overset{\text{ind. hyp.}}{=} \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{g}_{\ell-1}\cdot \tilde{s}_{i_{\ell-k}}^{\ell-1} \\
&= \iota(\,{}^{[\ell-1]}\!\!\prod_{m\in\lfloor \ell-k-1\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{s}_{i_{\ell-k}}^{\ell-1}\cdot \tilde{g}_{\ell} \\
&= \iota(\,{}^{[\ell]}\!\!\prod_{m\in\lfloor \ell-k\,,\,1\rfloor}^{[k]} \sigma_{i_m}^{k+m-1})\cdot \tilde{g}_{\ell}\,.
\end{aligned}
$$

□

3.4.3 The homotopy categories of simplicial and of semisimplicial objects in $\mathcal{C}$

Let $\mathcal{C}$ be a category.

Definition 31 (Homotopy in $\mathrm{Simp}(\mathcal{C})$).

Suppose given $X,\,Y \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))$. Let $(\underset{X,Y}{\sim})$ be the equivalence relation on ${}_{\mathrm{Simp}(\mathcal{C})}(X,Y)$ generated by the relation of elementary simplicial homotopy $(\underset{X,Y}{\rightsquigarrow})$; cf. Definition 26.

Let

$$(\sim) \;:=\; \dot{\bigcup_{(X,Y)\,\in\,\mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))\times\mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))}} (\underset{X,Y}{\sim})$$

be the relation of *simplicial homotopy* or, short, *homotopy* on $\mathrm{Mor}(\mathrm{Simp}(\mathcal{C}))$.

So given $f,\,g \in \mathrm{Mor}(\mathrm{Simp}(\mathcal{C}))$, we have that f is homotopic to g, written $f \sim g$, if $\mathrm{Source}(f) = \mathrm{Source}(g) =: X$ and $\mathrm{Target}(f) = \mathrm{Target}(g) =: Y$ and $f \underset{X,Y}{\sim} g$.

Lemma 32. The relation of elementary simplicial homotopy $(\rightsquigarrow)$, cf. Definition 26, is a precongruence on $\mathrm{Simp}(\mathcal{C})$; cf. Definition 2.

The homotopy relation $(\sim)$ is the equivalence relation generated by $(\rightsquigarrow)$.

The homotopy relation $(\sim)$ is a congruence on $\mathrm{Simp}(\mathcal{C})$; cf. Definition 2.

Proof. We first show that $(\rightsquigarrow)$ is a precongruence.

Ad (Con 1). Given two morphisms that are elementary simplicially homotopic, they have the same source and the same target by Definition 26.

Ad (Con 2).

Suppose given $X,\,Y \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))$. Suppose given $X' \xrightarrow{u} X$ and $Y \xrightarrow{v} Y'$ in $\mathrm{Simp}(\mathcal{C})$. Given $f,\,g \in {}_{\mathrm{Simp}(\mathcal{C})}(X,Y)$ such that $f \rightsquigarrow g$, we have to show that $ufv \overset{!}{\rightsquigarrow} ugv$.

We choose an elementary simplicial homotopy $((h_i^\ell)_{i\in[0,\ell]})_{\ell\geqslant 0}$ from f to g. Note that $h_i^\ell : X_\ell \to Y_{\ell+1}$.

We *claim* that $((u_\ell h_i^\ell v_{\ell+1})_{i\in[0,\ell]})_{\ell\geqslant 0}$ is an elementary simplicial homotopy from ufv to ugv. We have to verify the properties from Definition 26.

Ad (i). Suppose given $i \in [0,\ell]$ and $j \in [i+2,\ell+2]$. We obtain

$$(u_{\ell+1}h_i^{\ell+1}v_{\ell+2})\cdot d_j^{Y',\,\ell+2} \;=\; u_{\ell+1}\cdot h_i^{\ell+1}\cdot d_j^{Y,\,\ell+2}\cdot v_{\ell+1} \;=\; u_{\ell+1}\cdot d_{j-1}^{X,\,\ell+1}\cdot h_i^\ell\cdot v_{\ell+1} \;=\; d_{j-1}^{X',\,\ell+1}\cdot(u_\ell h_i^\ell v_{\ell+1})\,.$$

Ad (ii). Suppose given $i \in [0,\ell]$. We obtain

$$(u_{\ell+1}h_i^{\ell+1}v_{\ell+2})\cdot d_{i+1}^{Y',\,\ell+2} \;=\; u_{\ell+1}\cdot h_i^{\ell+1}\cdot d_{i+1}^{Y,\,\ell+2}\cdot v_{\ell+1} \;=\; u_{\ell+1}\cdot h_{i+1}^{\ell+1}\cdot d_{i+1}^{Y,\,\ell+2}\cdot v_{\ell+1} \;=\; (u_{\ell+1}h_{i+1}^{\ell+1}v_{\ell+2})\cdot d_{i+1}^{Y',\,\ell+2}\,.$$

Ad (iv). Suppose given $i \in [1,\ell+1]$ and $j \in [0,i-1]$. We obtain

$$(u_{\ell+1}h_i^{\ell+1}v_{\ell+2})\cdot d_j^{Y',\,\ell+2} \;=\; u_{\ell+1}\cdot h_i^{\ell+1}\cdot d_j^{Y,\,\ell+2}\cdot v_{\ell+1} \;=\; u_{\ell+1}\cdot d_j^{X,\,\ell+1}\cdot h_{i-1}^\ell\cdot v_{\ell+1} \;=\; d_j^{X',\,\ell+1}\cdot(u_\ell h_{i-1}^\ell v_{\ell+1})\,.$$

Ad (v). Suppose given $i \in [0,\ell]$ and $j \in [i+1,l+1]$. We obtain

$$(u_\ell h_i^\ell v_{\ell+1})\cdot s_j^{Y',\,\ell+1} \;=\; u_\ell\cdot h_i^\ell\cdot s_j^{Y,\,\ell+1}\cdot v_{\ell+2} \;=\; u_\ell\cdot s_{j-1}^{X,\,\ell}\cdot h_i^{\ell+1}\cdot v_{\ell+2} \;=\; s_{j-1}^{X',\ell}\cdot(u_{\ell+1}h_i^{\ell+1}v_{\ell+2})\,.$$

Ad (vi). Suppose given $i \in [0,\ell]$ and $j \in [0,i]$. We obtain

$$(u_\ell h_i^\ell v_{\ell+1})\cdot s_j^{Y',\,\ell+1} \;=\; u_\ell\cdot h_i^\ell\cdot s_j^{Y,\,\ell+1}\cdot v_{\ell+2} \;=\; u_\ell\cdot s_j^{X,\,\ell}\cdot h_{i+1}^{\ell+1}\cdot v_{\ell+2} \;=\; s_j^{X',\,\ell}\cdot(u_{\ell+1}h_{i+1}^{\ell+1}v_{\ell+2})\,.$$

Ad (vii). We have

$$(u_\ell h_i^\ell v_{\ell+1})\cdot d_{\ell+1}^{Y',\,l+1} \;=\; u_\ell\cdot h_i^\ell\cdot d_{\ell+1}^{Y,\,l+1}\cdot v_\ell \;=\; u_\ell f_\ell v_\ell\,.$$

Ad (viii). We have

$$(u_\ell h_i^\ell v_{\ell+1}) \cdot d_0^{Y',l+1} \; = \; u_\ell \cdot h_i^\ell \cdot d_0^{Y,l+1} \cdot v_\ell \; = \; u_\ell g_\ell v_\ell \; .$$

This proves the *claim.*

By Lemma 5, $(\sim)$ is the equivalence relation on Mor(Simp($\mathcal{C}$)) generated by $(\rightsquigarrow)$, which is a precongruence. So by Lemma 6, $(\sim)$ is a congruence.

□

Definition 33. Consider the category Simp($\mathcal{C}$) of simplicial objects in $\mathcal{C}$.

Its *homotopy category* is defined to be

$$\mathrm{HoSimp}(\mathcal{C}) := \mathrm{Simp}(\mathcal{C})/(\sim)\,,$$

cf. Definition 31 and Lemmas 32 and 3.

For $f \in \mathrm{Mor}(\mathrm{Simp}(\mathcal{C}))$, we write $[f] \in \mathrm{Mor}(\mathrm{HoSimp}(\mathcal{C}))$ for the equivalence class of f with respect to homotopy.

Definition 34 (Homotopy in SemiSimp($\mathcal{C}$))**.**

Suppose given $X,\, Y \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{C}))$. Let $(\underset{X,Y}{\sim})$ be the equivalence relation on ${}_{\mathrm{SemiSimp}(\mathcal{C})}(X,Y)$ generated by the relation of elementary semisimplicial homotopy $(\underset{X,Y}{\rightsquigarrow})$; cf. Definition 27.

Let

$$(\sim) \; := \; \dot{\bigcup_{(X,Y)\,\in\,\mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{C}))\times\mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{C}))}} (\underset{X,Y}{\sim})$$

be the relation of *semisimplicial homotopy*, or short, *homotopy* on Mor(SemiSimp($\mathcal{C}$)).

So given $f,\, g \in \mathrm{Mor}(\mathrm{SemiSimp}(\mathcal{C}))$, we have that f is homotopic to g, written $f \sim g$, if $\mathrm{Source}(f) = \mathrm{Source}(g) =: X$ and $\mathrm{Target}(f) = \mathrm{Target}(g) =: Y$ and $f \underset{X,Y}{\sim} g$.

Lemma 35. The relation of elementary semisimplicial homotopy $(\rightsquigarrow)$, cf. Definition 27, is a precongruence on SemiSimp($\mathcal{C}$); cf. Definition 2.

The homotopy relation $(\sim)$ is the equivalence relation generated by $(\rightsquigarrow)$.

The homotopy relation $(\sim)$ is a congruence on SemiSimp($\mathcal{C}$); cf. Definition 2.

Proof. We first show that $(\rightsquigarrow)$ is a precongruence.

Ad (Con 1). Given two morphisms that are elementary simplicially homotopic, they have the same source and the same target by Definition 27.

Ad (Con 2).

Suppose given $X,\, Y \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{C}))$. Suppose given $X' \xrightarrow{u} X$ and $Y \xrightarrow{v} Y'$ in SemiSimp($\mathcal{C}$). Given $f,\, g \in {}_{\mathrm{SemiSimp}(\mathcal{C})}(X,Y)$ such that f is elementary semisimplicially homotopic to g, we have to show that $ufv \overset{!}{\sim} ugv$.

We choose an elementary semisimplicial homotopy $((h_i^\ell)_{i\in[0,\ell]})_{\ell\geqslant 0}$ from f to g. Note that $h_i^\ell : X_\ell \to Y_{\ell+1}$.

We *claim* that $((u_\ell h_i^\ell v_{\ell+1})_{i\in[0,\ell]})_{\ell\geqslant 0}$ is an elementary semisimplicial homotopy from ufv to ugv. We have to verify the properties from Definition 27.

Ad (i). Suppose given $i \in [0, \ell]$ and $j \in [i+2, \ell+2]$. We obtain

$$(u_{\ell+1} h_i^{\ell+1} v_{\ell+2}) \cdot d_j^{Y',\ell+2} = u_{\ell+1} \cdot h_i^{\ell+1} \cdot d_j^{Y,\ell+2} \cdot v_{\ell+1} = u_{\ell+1} \cdot d_{j-1}^{X,\ell+1} \cdot h_i^{\ell} \cdot v_{\ell+1} = d_{j-1}^{X',\ell+1} \cdot (u_\ell h_i^\ell v_{\ell+1}) .$$

Ad (ii). Suppose given $i \in [0, \ell]$. We obtain

$$(u_{\ell+1} h_i^{\ell+1} v_{\ell+2}) \cdot d_{i+1}^{Y',\ell+2} = u_{\ell+1} \cdot h_i^{\ell+1} \cdot d_{i+1}^{Y,\ell+2} \cdot v_{\ell+1} = u_{\ell+1} \cdot h_{i+1}^{\ell+1} \cdot d_{i+1}^{Y,\ell+2} \cdot v_{\ell+1} = (u_{\ell+1} h_{i+1}^{\ell+1} v_{\ell+2}) \cdot d_{i+1}^{Y',\ell+2} .$$

Ad (iv). Suppose given $i \in [1, \ell+1]$ and $j \in [0, i-1]$. We obtain

$$(u_{\ell+1} h_i^{\ell+1} v_{\ell+2}) \cdot d_j^{Y',\ell+2} = u_{\ell+1} \cdot h_i^{\ell+1} \cdot d_j^{Y,\ell+2} \cdot v_{\ell+1} = u_{\ell+1} \cdot d_j^{X,\ell+1} \cdot h_{i-1}^{\ell} \cdot v_{\ell+1} = d_j^{X',\ell+1} \cdot (u_\ell h_{i-1}^\ell v_{\ell+1}) .$$

Ad (v). We have

$$(u_\ell h_i^\ell v_{\ell+1}) \cdot d_{\ell+1}^{Y',l+1} = u_\ell \cdot h_i^\ell \cdot d_{\ell+1}^{Y,l+1} \cdot v_\ell = u_\ell f_\ell v_\ell .$$

Ad (vi). We have

$$(u_\ell h_i^\ell v_{\ell+1}) \cdot d_0^{Y',l+1} = u_\ell \cdot h_i^\ell \cdot d_0^{Y,l+1} \cdot v_\ell = u_\ell g_\ell v_\ell .$$

This proves the *claim*.

By Lemma 5, $(\sim)$ is the equivalence relation on $\mathrm{Mor}(\mathrm{SemiSimp}(\mathcal{C}))$ generated by $(\rightsquigarrow)$, which is a precongruence. So by Lemma 6, $(\sim)$ is a congruence.

□

Definition 36. Consider the category $\mathrm{SemiSimp}(\mathcal{C})$ of semisimplicial objects in $\mathcal{C}$.

Its *homotopy category* is defined to be

$$\mathrm{HoSemiSimp}(\mathcal{C}) := \mathrm{SemiSimp}(\mathcal{C})/(\sim) ,$$

cf. Definition 34 and Lemmas 35 and 3.

For $f \in \mathrm{Mor}(\mathrm{SemiSimp}(\mathcal{C}))$, we write $[f] \in \mathrm{Mor}(\mathrm{HoSemiSimp}(\mathcal{C}))$ for the equivalence class of f with respect to homotopy.

Proposition 37. Suppose that $\mathcal{C}$ has finite coproducts.

(1) We have the functor

$$\begin{array}{rlcll} \bar{\mathcal{F}}_{\mathcal{C}} : & \mathrm{HoSemiSimp}(\mathcal{C}) & \to & \mathrm{HoSimp}(\mathcal{C}) & \\ & X & \mapsto & \bar{\mathcal{F}}_{\mathcal{C}}(X) := \mathcal{F}_{\mathcal{C}}(X) & \text{for } X \in \mathrm{Ob}(\mathrm{HoSemiSimp}(\mathcal{C})) \\ & [f] & \mapsto & \bar{\mathcal{F}}_{\mathcal{C}}([f]) := [\mathcal{F}_{\mathcal{C}}(f)] & \text{for } [f] \in \mathrm{Mor}(\mathrm{HoSemiSimp}(\mathcal{C})) . \end{array}$$

(2) We have the functor

$$\begin{array}{rlcll} \bar{\mathcal{V}}_{\mathcal{C}} : & \mathrm{HoSimp}(\mathcal{C}) & \to & \mathrm{HoSemiSimp}(\mathcal{C}) & \\ & X & \mapsto & \bar{\mathcal{V}}_{\mathcal{C}}(X) := \mathcal{V}_{\mathcal{C}}(X) & \text{for } X \in \mathrm{Ob}(\mathrm{HoSimp}(\mathcal{C})) \\ & [f] & \mapsto & \bar{\mathcal{V}}_{\mathcal{C}}([f]) := [\mathcal{V}_{\mathcal{C}}(f)] & \text{for } [f] \in \mathrm{Mor}(\mathrm{HoSimp}(\mathcal{C})) . \end{array}$$

Proof. Ad (1). By Proposition 30, we have that for $f, g \in \mathrm{Mor}(\mathrm{SemiSimp}(\mathcal{C}))$ satisfying $f \rightsquigarrow g$, it holds that $\mathcal{F}_{\mathcal{C}} f \rightsquigarrow \mathcal{F}_{\mathcal{C}} g$, so in particular $\mathcal{F}_{\mathcal{C}} f \sim \mathcal{F}_{\mathcal{C}} g$. In the case $\mathcal{C} = \mathrm{Set}$, one can use Proposition 25 instead.

Consider the functor $R_{\mathrm{Simp}(\mathcal{C}),(\sim)} \circ \mathcal{F}_{\mathcal{C}} : \mathrm{SemiSimp}(\mathcal{C}) \to \mathrm{HoSimp}(\mathcal{C})$. For $f, g \in \mathrm{Mor}(\mathcal{C})$ satisfying $f \rightsquigarrow g$, we then have $(R_{\mathrm{Simp}(\mathcal{C}),(\sim)} \circ \mathcal{F}_{\mathcal{C}})f = [\mathcal{F}_{\mathcal{C}} f] = [\mathcal{F}_{\mathcal{C}} g] = (R_{\mathrm{Simp}(\mathcal{C}),(\sim)} \circ \mathcal{F}_{\mathcal{C}})g$.

So by Lemma 8, there exists a functor

$$\bar{\mathcal{F}}_{\mathcal{C}} := \overline{R_{\mathrm{Simp}(\mathcal{C}),(\sim)} \circ \mathcal{F}_{\mathcal{C}}} : \mathrm{SemiSimp}(\mathcal{C})/(\sim) = \mathrm{HoSemiSimp}(\mathcal{C}) \to \mathrm{HoSimp}(\mathcal{C})$$

mapping $X \xrightarrow{[f]} Y$ to $\mathcal{F}_{\mathcal{C}} X \xrightarrow{[\mathcal{F}_{\mathcal{C}} f]} \mathcal{F}_{\mathcal{C}} Y$.

Ad (2). We *claim* that $f \rightsquigarrow g$ implies $\mathcal{V}_{\mathcal{C}}(f) \rightsquigarrow \mathcal{V}_{\mathcal{C}}(g)$ for $f, g \in \mathrm{Mor}(\mathrm{Simp}(\mathcal{C}))$.

So suppose that $X \xrightarrow{f} Y$ is elementary simplicially homotopic to $X \xrightarrow{g} Y$. This means that there exists a tuple of morphisms $((X_\ell \xrightarrow{h_i^\ell} Y_{\ell+1})_{i \in [0,\ell]})_{\ell \geqslant 0}$ in $\mathcal{C}$ satisfying conditions (i - viii) of Definition 26. Thus we can show that there exists an elementary semisimplicial homotopy from $\mathcal{V}(f) = (f_\ell)_{\ell \geqslant 0}$ to $\mathcal{V}(g) = (g_\ell)_{\ell \geqslant 0}$ as follows. We may choose as an elementary semisimplicial homotopy the tuple $((h_i^\ell)_{i \in [0,\ell]})_{\ell \geqslant 0}$, since it satisfies conditions (i - vi) of Definition 27.

This proves the *claim*.

Consider the functor $R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)} \circ \mathcal{V}_{\mathcal{C}} : \mathrm{Simp}(\mathcal{C}) \to \mathrm{HoSemiSimp}(\mathcal{C})$. For $f, g \in \mathrm{Mor}(\mathrm{Simp}(\mathcal{C}))$ satisfying $f \rightsquigarrow g$, we then have $(R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)} \circ \mathcal{V}_{\mathcal{C}})f = [\mathcal{V}_{\mathcal{C}} f] = [\mathcal{V}_{\mathcal{C}} g] = (R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)} \circ \mathcal{V}_{\mathcal{C}})g$.

So by Lemma 8, there exists a functor

$$\bar{\mathcal{V}}_{\mathcal{C}} := \overline{R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)} \circ \mathcal{V}_{\mathcal{C}}} : \mathrm{Simp}(\mathcal{C})/(\sim) = \mathrm{HoSimp}(\mathcal{C}) \to \mathrm{HoSemiSimp}(\mathcal{C})$$

mapping $X \xrightarrow{[f]} Y$ to $\mathcal{V}_{\mathcal{C}} X \xrightarrow{[\mathcal{V}_{\mathcal{C}} f]} \mathcal{V}_{\mathcal{C}} Y$. □

Reminder 38.

(1) We have the transformation $\iota : \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{C})} \to \mathcal{V}_{\mathcal{C}} \circ \mathcal{F}_{\mathcal{C}}$ given by the inclusion morphisms of X_n into $\coprod_{\substack{(f:[n]\to[k])\in \mathrm{surj} \\ \text{for some } k \in [0,n]}} X_k$ for $n \geqslant 0$ and $X \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{C}))$, cf. [**1,** Remark 61, 62].

(2) For $n \geqslant 0$ and $X \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))$ we have the morphism $\eta_{X,n}$ uniquely determined by making the diagrams

$$\begin{array}{ccc} \coprod\limits_{\substack{(f:[n]\to[k])\in \mathrm{surj} \\ \text{for some } k \in [0,n]}} X_k & \xrightarrow{\eta_{X,n}} & X_n \\ {\scriptstyle \iota(v)}\uparrow & \nearrow_{X_v} & \\ X_\ell & & \end{array}$$

commutative for $\ell \in [0,n]$ and $(v : [n] \to [\ell]) \in \mathrm{surj}$. We have the transformation $\eta : \mathcal{F}_{\mathcal{C}} \circ \mathcal{V}_{\mathcal{C}} \to \mathrm{id}_{\mathrm{Simp}(\mathcal{C})}$ given by the tuple $((\eta_{X,n})_{n \geqslant 0})_{X \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))}$, cf. [**1,** Remark 63, 64].

(3) We have $\mathcal{F}_{\mathcal{C}} \dashv \mathcal{V}_{\mathcal{C}}$, cf. [**1,** Proposition 65]. More precisely, we have the commutativity of the following diagrams.

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{C}} & \xrightarrow{\mathcal{F}_{\mathcal{C}}\iota} & \mathcal{F}_{\mathcal{C}} \circ \mathcal{V}_{\mathcal{C}} \circ \mathcal{F}_{\mathcal{C}} \\ & {\scriptstyle \mathrm{id}_{\mathcal{F}_{\mathcal{C}}}} \searrow & \downarrow {\scriptstyle \eta\mathcal{F}_{\mathcal{C}}} \\ & & \mathcal{F}_{\mathcal{C}} \end{array} \qquad \begin{array}{ccc} \mathcal{V}_{\mathcal{C}} & \xrightarrow{\iota\mathcal{V}_{\mathcal{C}}} & \mathcal{V}_{\mathcal{C}} \circ \mathcal{F}_{\mathcal{C}} \circ \mathcal{V}_{\mathcal{C}} \\ & {\scriptstyle \mathrm{id}_{\mathcal{V}_{\mathcal{C}}}} \searrow & \downarrow {\scriptstyle \mathcal{V}_{\mathcal{C}}\eta} \\ & & \mathcal{V}_{\mathcal{C}} \end{array}$$

Theorem 39. The functor

$$\bar{\mathcal{F}}_{\mathcal{C}} : \mathrm{HoSemiSimp}(\mathcal{C}) \to \mathrm{HoSimp}(\mathcal{C})$$

is left adjoint to the functor

$$\bar{\mathcal{V}}_{\mathcal{C}} : \mathrm{HoSimp}(\mathcal{C}) \to \mathrm{HoSemiSimp}(\mathcal{C})\,,$$

i.e. $\bar{\mathcal{F}}_{\mathcal{C}} \dashv \bar{\mathcal{V}}_{\mathcal{C}}$.

More precisely we have a unit

$$\bar{\iota} = (\bar{\iota}_X)_{X\in\mathrm{Ob}(\mathrm{HoSemiSimp}(\mathcal{C}))} := ([\iota_X])_{X\in\mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{C}))} : \mathrm{id}_{\mathrm{HoSemiSimp}(\mathcal{C})} \to \bar{\mathcal{V}}_{\mathcal{C}} \circ \bar{\mathcal{F}}_{\mathcal{C}}$$

and a counit

$$\bar{\eta} = (\bar{\eta}_X)_{X\in\mathrm{Ob}(\mathrm{HoSimp}(\mathcal{C}))} := ([\eta_X])_{X\in\mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))} : \bar{\mathcal{F}}_{\mathcal{C}} \circ \bar{\mathcal{V}}_{\mathcal{C}} \to \mathrm{id}_{\mathrm{HoSimp}(\mathcal{C})}\,;$$

cf. Reminder 38.

Proof. We write $\mathcal{F} := \mathcal{F}_{\mathcal{C}}$, $\bar{\mathcal{F}} := \bar{\mathcal{F}}_{\mathcal{C}}$, $\mathcal{V} := \mathcal{V}_{\mathcal{C}}$, $\bar{\mathcal{V}} := \bar{\mathcal{V}}_{\mathcal{C}}$, $R_{\mathrm{SemiSimp}(\mathcal{C})} := R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)}$, $R_{\mathrm{Simp}(\mathcal{C})} := R_{\mathrm{Simp}(\mathcal{C}),(\sim)}$.

Note that we have $(R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathcal{V}_{\mathcal{C}} \circ \mathcal{F}) = (\bar{\mathcal{V}} \circ R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F}) = (\bar{\mathcal{V}} \circ \bar{\mathcal{F}} \circ R_{\mathrm{SemiSimp}(\mathcal{C})})$ and therefore $\overline{R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathcal{V} \circ \mathcal{F}} = \bar{\mathcal{V}} \circ \bar{\mathcal{F}}$ in the sense of Lemma 8.

Note that we have $(R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F} \circ \mathcal{V}) = (\bar{\mathcal{F}} \circ R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathcal{V}) = (\bar{\mathcal{F}} \circ \bar{\mathcal{V}} \circ R_{\mathrm{Simp}(\mathcal{C})})$ and therefore $\overline{R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F} \circ \mathcal{V}} = \bar{\mathcal{F}} \circ \bar{\mathcal{V}}$ in the sense of Lemma 8.

Given $f, g \in \mathrm{Mor}(\mathrm{SemiSimp}(\mathcal{C})$ such that $f \sim g$, we have

$$(R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{C})})(f) = [f] = [g] = (R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{C})})(g)$$

and

$$(R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathcal{V} \circ \mathcal{F})(f) = (\bar{\mathcal{V}} \circ \bar{\mathcal{F}})([f]) = (\bar{\mathcal{V}} \circ \bar{\mathcal{F}})([g]) = (R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathcal{V} \circ \mathcal{F})(g)\,.$$

We have the transformation $R_{\mathrm{SemiSimp}(\mathcal{C})}\iota : R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{C})} \to R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathcal{V} \circ \mathcal{F}$; cf. Reminder 38 (1).

So by Lemma 9, there exists the unique transformation

$$\bar{\iota} := \overline{R_{\mathrm{SemiSimp}(\mathcal{C})}\iota} : \overline{R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{C})}} = \mathrm{id}_{\mathrm{HoSemiSimp}(\mathcal{C})} \to \overline{R_{\mathrm{SemiSimp}(\mathcal{C})} \circ \mathcal{V} \circ \mathcal{F}} = \bar{\mathcal{V}} \circ \bar{\mathcal{F}}$$

satisfying $\bar{\iota}R_{\mathrm{SemiSimp}(\mathcal{C})} = R_{\mathrm{SemiSimp}(\mathcal{C})}\iota$ and being given by $\bar{\iota} = ([\iota_X])_{X\in\mathrm{Ob}(\mathrm{HoSemiSimp}(\mathcal{C}))}$.

Given $f, g \in \mathrm{Mor}(\mathrm{Simp}(\mathcal{C})$ such that $f \sim g$ we have

$$(R_{\mathrm{Simp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{Simp}(\mathcal{C})})(f) = [f] = [g] = (R_{\mathrm{Simp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{Simp}(\mathcal{C})})(g)$$

and

$$(R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F} \circ \mathcal{V})(f) = (\bar{\mathcal{F}} \circ \bar{\mathcal{V}})([f]) = (\bar{\mathcal{F}} \circ \bar{\mathcal{V}})([g]) = (R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F} \circ \mathcal{V})(g)\,.$$

We have the transformation $R_{\mathrm{Simp}(\mathcal{C})}\eta : R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F} \circ \mathcal{V} \to R_{\mathrm{Simp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{Simp}(\mathcal{C})}$; cf. Reminder 38 (2).

So by Lemma 9, there exists the unique transformation

$$\bar{\eta} := \overline{R_{\mathrm{Simp}(\mathcal{C})}\eta} : \overline{R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F} \circ \mathcal{V}} = \bar{\mathcal{F}} \circ \bar{\mathcal{V}} \to \overline{R_{\mathrm{Simp}(\mathcal{C})} \circ \mathrm{id}_{\mathrm{Simp}(\mathcal{C})}} = \mathrm{id}_{\mathrm{HoSimp}(\mathcal{C})}$$

satisfying $\bar{\eta}R_{\mathrm{Simp}(\mathcal{C})} = R_{\mathrm{Simp}(\mathcal{C})}\eta$ and being given by $\bar{\eta} = ([\eta_X])_{X\in\mathrm{Ob}(\mathrm{HoSimp}(\mathcal{C}))}$.

Let $X \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{C}))$. We have

$$(\bar{\mathcal{F}}\bar{\iota})_X = \bar{\mathcal{F}}(\bar{\iota}_X) = \bar{\mathcal{F}}([\iota_X]) = \bar{\mathcal{F}} \circ R_{\mathrm{SemiSimp}(\mathcal{C})}(\iota_X) = R_{\mathrm{Simp}(\mathcal{C})} \circ \mathcal{F}(\iota_X) = R_{\mathrm{Simp}(\mathcal{C})}((\mathcal{F}\iota)_X)$$

and

$$(\bar{\eta}\bar{\mathcal{F}})_X = \bar{\eta}_{\bar{\mathcal{F}}X} = [\eta_{\mathcal{F}X}] = R_{\mathrm{Simp}(\mathcal{C})}(\eta_{\mathcal{F}X})\,.$$

So we have

$$\begin{aligned}(\bar{\mathcal{F}}\bar{\iota}\cdot\bar{\eta}\bar{\mathcal{F}})_X &= (\bar{\mathcal{F}}\bar{\iota})_X\cdot(\bar{\eta}\bar{\mathcal{F}})_X = R_{\mathrm{Simp}(\mathcal{C})}((\mathcal{F}\iota)_X)\cdot R_{\mathrm{Simp}(\mathcal{C})}(\eta_{\mathcal{F}X}) = R_{\mathrm{Simp}(\mathcal{C})}((\mathcal{F}\iota)_X\cdot\eta_{\mathcal{F}X})\\ &\overset{\text{R 38 (3)}}{=} R_{\mathrm{Simp}(\mathcal{C})}(\mathrm{id}_{\mathcal{F}X}) = \mathrm{id}_{\bar{\mathcal{F}}X}\end{aligned}$$

Hence the diagram

$$\begin{array}{ccc} \bar{\mathcal{F}} & \xrightarrow{\bar{\mathcal{F}}\bar{\iota}} & \bar{\mathcal{F}}\circ\bar{\mathcal{V}}\circ\bar{\mathcal{F}} \\ & {}_{\mathrm{id}_{\bar{\mathcal{F}}}}\searrow & \downarrow{\scriptstyle \bar{\eta}\bar{\mathcal{F}}} \\ & & \bar{\mathcal{F}} \end{array}$$

commutes.

Let $X \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))$. We have

$$(\bar{\iota}\bar{\mathcal{V}})_X = \bar{\iota}_{\bar{\mathcal{V}}X} = [\iota_{\mathcal{V}X}] = R_{\mathrm{SemiSimp}(\mathcal{C})}(\iota_{\mathcal{V}X})$$

and

$$(\bar{\mathcal{V}}\bar{\eta})_X = \bar{\mathcal{V}}(\bar{\eta}_X) = \bar{\mathcal{V}}([\eta_X]) = \bar{\mathcal{V}}\circ R_{\mathrm{Simp}(\mathcal{C})}(\eta_X) = R_{\mathrm{SemiSimp}(\mathcal{C})}\circ\mathcal{V}(\eta_X) = R_{\mathrm{SemiSimp}(\mathcal{C})}((\mathcal{V}\eta)_X)\,.$$

So we have

$$\begin{aligned}(\bar{\iota}\bar{\mathcal{V}}\cdot\bar{\mathcal{V}}\bar{\eta})_X &= (\bar{\iota}\bar{\mathcal{V}})_X\cdot(\bar{\mathcal{V}}\bar{\eta})_X = R_{\mathrm{SemiSimp}(\mathcal{C})}(\iota_{\mathcal{V}X})\cdot R_{\mathrm{SemiSimp}(\mathcal{C})}((\mathcal{V}\eta)_X) = R_{\mathrm{SemiSimp}(\mathcal{C})}(\iota_{\mathcal{V}X}\cdot(\mathcal{V}\eta)_X)\\ &\overset{\text{R 38 (3)}}{=} R_{\mathrm{SemiSimp}(\mathcal{C})}(\mathrm{id}_{\mathcal{V}X}) = \mathrm{id}_{\bar{\mathcal{V}}X}\;.\end{aligned}$$

Hence the diagram

$$\begin{array}{ccc} \bar{\mathcal{V}} & \xrightarrow{\bar{\iota}\bar{\mathcal{V}}} & \bar{\mathcal{V}}\circ\bar{\mathcal{F}}\circ\bar{\mathcal{V}} \\ & {}_{\mathrm{id}_{\bar{\mathcal{V}}}}\searrow & \downarrow{\scriptstyle \bar{\mathcal{V}}\bar{\eta}} \\ & & \bar{\mathcal{V}} \end{array}$$

commutes. □

Remark 40. In [**3**], Rourke and Sanderson consider the forgetful functor from Simp(Set) to SemiSimp(Set), which they call F. They construct its left adjoint $G \dashv F$. They define a homotopy category of Kan semisimplicial sets, with a suitable definition of being Kan. Then the functor F induces a functor $\bar{F}$ on the homotopy categories. Conversely, they first construct a horn functor H from Simp(Set) to its full subcategory of Kan simplicial sets [**3,** p. 334, l. -4]. They show that $H\circ G$ induces a functor $\overline{H\circ G}$ on the homotopy categories and that $\bar{F}$ and $\overline{H\circ G}$ are mutually inverse equivalences. Their proof uses topological methods [**3,** Th 6.8].

I do not know for which categories, the functor $\bar{\mathcal{F}}$ is an equivalence. Cf. Corollary 75.

Chapter 4

The resolution functor

4.1 Augmented semisimplicial resolutions and construction of the functor $\mathcal{E}$

Let $\mathcal{C}$ be a category having finite limits. Suppose given a full subcategory $\mathcal{P} \subseteq \mathcal{C}$ having finite coproducts. Recall that a morphism $g \in \mathrm{Mor}(\mathcal{C})$ is called $\mathcal{P}$-epic if ${}_{\mathcal{C}}(P, g)$ is a surjective map for $P \in \mathrm{Ob}(\mathcal{P})$; cf. [**1,** Definition 25]. We require $\mathcal{P} \subseteq \mathcal{C}$ to be a resolving subcategory, which means that for each object X in $\mathcal{C}$ we require the existence of a $\mathcal{P}$-epimorphism $P \to X$ for some $P \in \mathrm{Ob}(\mathcal{P})$; cf. [**1,** Definition 25, 27].

Definition 41. We define the category $\Delta_{\mathrm{inj,aug}}$ as subcategory of Set as follows. Let

$$\mathrm{Ob}(\Delta_{\mathrm{inj,aug}}) := \{[i] : i \in \mathbb{Z}_{\geqslant -1}\}\,.$$

For $[a], [b] \in \mathrm{Ob}(\Delta_{\mathrm{inj,aug}})$ let

$${}_{\Delta_{\mathrm{inj,aug}}}([a],[b]) := \{f \in {}_{\mathrm{Set}}([a],[b]) : (x > y \Rightarrow xf > yf) \text{ for } x, y \in [a]\}$$

be the set of injective monotone maps.

Note that $\Delta_{\mathrm{inj}} \subseteq \Delta_{\mathrm{inj,aug}}$ is a full subcategory; cf. [**1,** Definition 40 (ii)].

Note that $\mathrm{Mor}(\Delta_{\mathrm{inj,aug}}) = \mathrm{Mor}(\Delta_{\mathrm{inj}}) \cup \{\emptyset \to [i] : i \in \mathbb{Z}_{\geqslant -1}\}$.

Let $\partial_0^0 := (\emptyset \to [0]) \in \mathrm{Mor}(\Delta_{\mathrm{inj,aug}})$. We have $\partial_i^{n-1} \cdot \partial_j^n = \partial_{j-1}^{n-1} \cdot \partial_i^n$ for $n \geqslant 1$ and $0 \leqslant i < j \leqslant n$.

Definition 42.

Let $\mathrm{AugSemiSimp}(\mathcal{C}) := \mathcal{C}^{\Delta^{\mathrm{op}}_{\mathrm{inj,aug}}}$ be the category of *augmented semisimplicial objects in* $\mathcal{C}$.

For $X \in \mathrm{Ob}(\mathrm{AugSemiSimp}(\mathcal{C}))$ we often write $X_n := X[n]$ for $n \in \mathbb{Z}_{\geqslant -1}$.

We write $d_i^{X,n} := X(\partial_i^n)^{\mathrm{op}}$ for $n \geqslant 0$ and $i \in [0, n]$. We often abbreviate $d_i^n := d_i^{X,n}$.

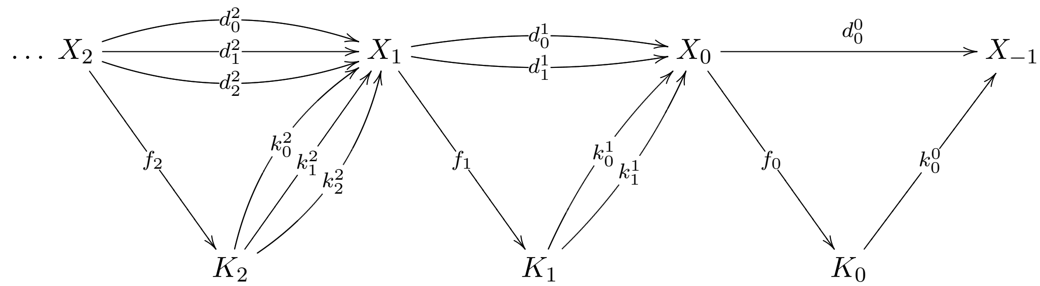
Note that we have $d_j^n \cdot d_i^{n-1} = X(\partial_i^{n-1} \cdot \partial_j^n)^{\mathrm{op}} = X(\partial_{j-1}^{n-1} \cdot \partial_i^n)^{\mathrm{op}} = d_i^n \cdot d_{j-1}^{n-1}$ for $n \geqslant 1$ and $0 \leqslant i < j \leqslant n$.

Remark 43. We can extend our notion of a simplicial kernel of a finite tuple of morphisms defined in [**1,** Definition 22] to the empty tuple.

Suppose given $X \in \mathrm{Ob}(\mathcal{C})$ and an empty tuple of morphisms starting in X. Then a simplicial kernel of this empty tuple is defined to be tuple $(K, (K \xrightarrow{k} X))$ satisfying the empty condition having the following universal property. Given $(K' \xrightarrow{k'} X)$ satisfying the empty condition, there exists a unique morphism $\alpha : K' \to K$ such that $\alpha \cdot k = k'$.

The tuple (X, id_X) is a simplicial kernel of X, together with the empty tuple of morphisms. Hence $(K, (K \xrightarrow{k} X))$ is simplicial kernel of X, together with the empty tuple of morphisms if and only if k is an isomorphism, cf. [**1,** Remark 3].

Definition 44. Suppose given $X \in \mathrm{Ob}(\mathrm{AugSemiSimp}(\mathcal{C}))$. Choose simplicial kernels $(K_n, (k_i^n)_{i\in[0,n]})$ of $(d_i^{n-1})_{i\in[0,n-1]}$ for $n \geqslant 1$, cf. [**1,** Definition 22, Proposition 23] and Remark 43. By the universal property of the simplicial kernel, there exists a unique morphism $f_n : X_n \to K_n$ such that $f_n \cdot k_i^n = d_i^n$ for $i \in [0, n]$ for $n \geqslant 0$, note that $d_j^n \cdot d_i^{n-1} = d_i^n \cdot d_{j-1}^{n-1}$ for $n \geqslant 0$ and $0 \leqslant i < j \leqslant n$, cf. [**1,** Definition 22] and Remark 43.



If the morphisms f_n and are $\mathcal{P}$-epic and $X_n \in \mathrm{Ob}(\mathcal{P})$ for $n \geqslant 0$, then we call X an *augmented semisimplicial resolution in $\mathcal{C}$ with respect to $\mathcal{P}$*. This definition is independent from the choice of the simplicial kernels. Note that k_0^0 can be chosen to be $\mathrm{id}_{X_{-1}}$ and hence the morphism d_0^0 is $\mathcal{P}$-epic. Sometimes, we call X an augmented semisimplicial resolution of X_{-1}.

Proof. We show the claimed independence. Suppose given simplicial kernels $(\tilde{K}, (\tilde{k}_i^n)_{i\in[0,n]})$ of $(d_i^{n-1})_{i\in[0,n-1]}$ for $n \geqslant 0$. Then we have unique morphisms $\tilde{f}_n : X_n \to K_n$ such that $\tilde{f}_n \cdot \tilde{k}_i^n = d_i^n$ for $n \geqslant 1$ and $i \in [0, n]$. There are isomorphisms $\varphi_n : K^n \to \tilde{K}^n$ such that $\varphi_n \cdot \tilde{k}_i^n = k_i^n$ for $n \geqslant 0$ and $i \in [0, n]$, cf. [**1,** Remark 3]. It follows that $\tilde{f}_n = f_n \cdot \varphi_n$ since $(f_n \cdot \varphi_n) \cdot \tilde{k}_i^n = f_n \cdot k_i^n = d_i^n$ for $i \in [0, n]$. So for $P \in \mathrm{Ob}(\mathcal{P})$ we have ${}_{\mathcal{C}}(P, \tilde{f}_n) = {}_{\mathcal{C}}(P, f_n \cdot \varphi_n) = {}_{\mathcal{C}}(P, f_n) \cdot {}_{\mathcal{C}}(P, \varphi_n)$. Since ${}_{\mathcal{C}}(P, \varphi_n)$ is a bijection, the map ${}_{\mathcal{C}}(P, \tilde{f}_n)$ is surjective if and only if the map ${}_{\mathcal{C}}(P, f_n)$ is surjective for $n \geqslant 0$. Hence f_n is $\mathcal{P}$-epic if and only if $\tilde{f}_n$ is $\mathcal{P}$-epic for $n \geqslant 0$. $\square$

Definition 45. We define the category $\mathrm{AugSemiSimpRes}(\mathcal{C}, \mathcal{P})$ to be the full subcategory of $\mathrm{AugSemiSimp}(\mathcal{C})$ with

$$\begin{aligned} &\mathrm{Ob}(\mathrm{AugSemiSimpRes}(\mathcal{C}, \mathcal{P})) := \\ &\{X \in \mathrm{Ob}(\mathrm{AugSemiSimp}(\mathcal{C})) : X \text{ is an augmented semisimplicial resolution in } \mathcal{C} \text{ with respect to } \mathcal{P}\}\,. \end{aligned}$$

Definition 46. (i) Let $I_1 : \Delta_{\mathrm{inj}}^{\mathrm{op}} \hookrightarrow \Delta_{\mathrm{inj,aug}}^{\mathrm{op}}$ denote the inclusion functor.

We have the functor

$$\begin{array}{rccc} I_{\mathrm{aug},1} : & \mathrm{AugSemiSimp}(\mathcal{C}) & \to & \mathrm{SemiSimp}(\mathcal{C}) \\ & (X \xrightarrow{f} Y) & \mapsto & (X \circ I_1 \xrightarrow{fI_1} Y \circ I_1)\,. \end{array}$$

(ii) Let $\mathcal{O}$ denote the full subcategory of $\Delta_{\mathrm{inj,aug}}^{\mathrm{op}}$ with $\mathrm{Ob}(\mathcal{O}) = \{[-1]\} = \{\emptyset\}$. Let $I_2 : \mathcal{O} \hookrightarrow \Delta_{\mathrm{inj,aug}}^{\mathrm{op}}$ denote the inclusion functor.

We identify $\mathcal{C}$ with $\mathcal{C}^{\mathcal{O}}$ via

$$\begin{array}{ccc} \mathcal{C}^{\mathcal{O}} & \to & \mathcal{C} \\ (X \xrightarrow{f} Y) & \mapsto & (X(\emptyset) \xrightarrow{f_\emptyset} Y(\emptyset))\,. \end{array}$$

Then we have the functor

$$\begin{array}{rccc} I_{\mathrm{aug},2}: & \mathrm{AugSemiSimp}(\mathcal{C}) & \to & \mathcal{C} \\ & (X \xrightarrow{f} Y) & \mapsto & (X \circ I_2 \xrightarrow{fI_2} Y \circ I_2) = (X_{-1} \xrightarrow{f_{-1}} Y_{-1}) \,. \end{array}$$

Definition 47 (homotopy on AugSemiSimp($\mathcal{C}$))**.** Consider the functor

$$\begin{array}{rccc} H: & \mathrm{AugSemiSimp}(\mathcal{C}) & \to & \mathrm{HoSemiSimp}(\mathcal{C}) \times \mathcal{C} \\ & (X \xrightarrow{f} Y) & \mapsto & \big((R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)} \circ I_{\mathrm{aug},1})(X \xrightarrow{f} Y), I_{\mathrm{aug},2}(X \xrightarrow{f} Y)\big) \,. \end{array}$$

We define the category HoAugSemiSimp($\mathcal{C}$) := AugSemiSimp($\mathcal{C}$)/($\underset{H}{\sim}$) to be the factor category of AugSemiSimp($\mathcal{C}$) with respect to the congruence induced by H, cf. Lemmas 4, 3.

We call this category the *homotopy category of augmented semisimplicial objects in* $\mathcal{C}$.

Note that for $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in AugSemiSimp($\mathcal{C}$), we have $f \underset{H}{\sim} g$ if and only if $fI_1 \sim gI_1$ and $f_{-1} = g_{-1}$.

Definition 48. We define the category HoAugSemiSimpRes($\mathcal{C}, \mathcal{P}$) as the full subcategory of HoAugSemiSimp($\mathcal{C}$) with

$$\mathrm{Ob}(\mathrm{HoAugSemiSimpRes}(\mathcal{C},\mathcal{P})) := \{X \in \mathrm{Ob}(\mathrm{HoAugSemiSimp}(\mathcal{C})) : X \in \mathrm{Ob}(\mathrm{AugSemiSimpRes}(\mathcal{C},\mathcal{P}))\} \,.$$

Write $H' := H|_{\mathrm{AugSemiSimpRes}(\mathcal{C},\mathcal{P})} : \mathrm{AugSemiSimpRes}(\mathcal{C},\mathcal{P}) \to \mathrm{HoAugSemiSimpRes}(\mathcal{C},\mathcal{P})$. Note that we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{AugSemiSimpRes}(\mathcal{C},\mathcal{P}) & \hookrightarrow & \mathrm{AugSemiSimp}(\mathcal{C}) \\ \downarrow{\scriptstyle R_{\mathrm{AugSemiSimpRes}(\mathcal{C},\mathcal{P}),(\underset{H'}{\sim})}} & & \downarrow{\scriptstyle R_{\mathrm{AugSemiSimp}(\mathcal{C}),(\underset{H}{\sim})}} \\ \mathrm{HoAugSemiSimpRes}(\mathcal{C},\mathcal{P}) & \hookrightarrow & \mathrm{HoAugSemiSimp}(\mathcal{C}) \end{array}$$

By construction of ($\underset{H}{\sim}$), we have that $f \underset{H}{\sim} g$ implies

$$(R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)} \circ I_{\mathrm{aug},1})f = (R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)} \circ I_{\mathrm{aug},1})g$$

for $f, g \in \mathrm{Mor}(\mathrm{AugSemiSimp}(\mathcal{C}))$. By Lemma 3 (3), we have a unique functor

$$\bar{I}_{\mathrm{aug},1} : \mathrm{HoAugSemiSimp}(\mathcal{C}) \to \mathrm{HoSemiSimp}(\mathcal{C})$$

making the following diagram commutative.

$$\begin{array}{ccc} \mathrm{AugSemiSimp}(\mathcal{C}) & \xrightarrow{I_{\mathrm{aug},1}} & \mathrm{SemiSimp}(\mathcal{C}) \\ \downarrow{\scriptstyle R_{\mathrm{AugSemiSimp}(\mathcal{C}),(\underset{H}{\sim})}} & & \downarrow{\scriptstyle R_{\mathrm{SemiSimp}(\mathcal{C}),(\sim)}} \\ \mathrm{HoAugSemiSimp}(\mathcal{C}) & \xrightarrow[\bar{I}_{\mathrm{aug},1}]{} & \mathrm{HoSemiSimp}(\mathcal{C}) \end{array}$$

By construction of ($\underset{H}{\sim}$), we have that $f \underset{H}{\sim} g$ implies

$$I_{\mathrm{aug},2}(f) = I_{\mathrm{aug},2}(g)$$

for $f, g \in \mathrm{Mor}(\mathrm{AugSemiSimp}(\mathcal{C}))$. By Lemma 3 (3), we have a unique functor

$$\bar{I}_{\mathrm{aug},2} : \mathrm{HoAugSemiSimp}(\mathcal{C}) \to \mathcal{C}$$

making the following diagram commutative.

$$\begin{array}{ccc} \mathrm{AugSemiSimp}(\mathcal{C}) & \xrightarrow{I_{\mathrm{aug},2}} & \mathcal{C} \\ {\scriptstyle R_{\mathrm{AugSemiSimp}(\mathcal{C}),(\underset{H}{\sim})}}\downarrow & \nearrow{\scriptstyle \bar{I}_{\mathrm{aug},2}} & \\ \mathrm{HoAugSemiSimp}(\mathcal{C}) & & \end{array}$$

In the remainder of this §4 we consider the functor

$$\mathcal{E} := \bar{I}_{\mathrm{aug},2}|_{\mathrm{HoAugSemiSimpRes}(\mathcal{C},\mathcal{P})} : \mathrm{HoAugSemiSimpRes}(\mathcal{C},\mathcal{P}) \to \mathcal{C}$$

We aim to show that $\mathcal{E}$ is an equivalence of categories; cf. Theorem 54 below.

4.2 Construction of an augmented semisimplicial resolution: $\mathcal{E}$ is dense

Remark 49. Let $([n] \xrightarrow{f} [m]) \in \mathrm{Mor}(\Delta_{\mathrm{inj,aug}})$. Let $0 \leqslant k_1 < \cdots < k_{m-n} \leqslant m$ denote the elements of $[m]$ that do not appear in the image of f. So $[m] = [n]f \,\dot{\cup}\, \{k_1, \ldots, k_{m-n}\}$. Then $f = {}^{[n]}\prod_{i\in\lceil 1\,,\,m-n\rceil}^{[m]} \partial^{n+i}_{k_i}$.

Proof. Case $n \geqslant 0$. In this case we have $f \in \mathrm{Mor}(\Delta_{\mathrm{inj}})$. Then $f = {}^{[n]}\prod_{i\in\lceil 1\,,\,m-n\rceil}^{[m]} \partial^{n+i}_{k_i}$ by **[1,** Lemma 42].

Case $n = -1$. In this case f maps from $[-1] = \emptyset$. Hence $f = {}^{[-1]}\prod_{i\in\lceil 1\,,\,m+1\rceil}^{[m]} \partial^{i-1}_{i-1}$. □

Lemma 50. Suppose given a tuple $((X_n)_{n\geqslant -1}, ((d_i^n)_{i\in[0,n]})_{n\geqslant 0})$, where $X_n \in \mathrm{Ob}(\mathcal{C})$ for $n \geqslant -1$ and $d_i^n \in {}_{\mathcal{C}}(X_n, X_{n-1})$ for $n \geqslant 0$ and $i \in [0,n]$ such that $d_j^{n+1} \cdot d_i^n = d_i^{n+1} \cdot d_{j-1}^n$ for $0 \leqslant i < j \leqslant n+1$ and $n \geqslant 0$.

There exists a unique functor $X : \Delta^{\mathrm{op}}_{\mathrm{inj,aug}} \to \mathcal{C}$, such that $X[n] = X_n$ for $n \geqslant -1$ and $X\partial_i^n = d_i^n$ for $0 \leqslant i \leqslant n$.

Proof.

Existence. By **[1,** Proposition 43], there exists a unique functor $Y : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{C}$, such that $Y[n] = X_n$ for $n \geqslant 0$ and $Y(\partial_i^n)^{\mathrm{op}} = d_i^n$ for $i \in [0,n]$ and $n \geqslant 1$. We define

$$\begin{array}{rccl} X: & \Delta^{\mathrm{op}}_{\mathrm{inj,aug}} & \to & \mathcal{C} \\ & ([n] \xrightarrow{f} [m]) & \mapsto & \begin{cases} Yf & \text{if } f \in \mathrm{Mor}(\Delta^{\mathrm{op}}_{\mathrm{inj}}) \\ {}^{X_n}\prod_{i\in\lfloor n\,,\,0\rfloor}^{X_{-1}} d_0^i & \text{if } m = -1 \end{cases} \end{array}$$

We claim that X is a functor. Note that X maps identities to identities.

Suppose given $([n] \xrightarrow{f} [m] \xrightarrow{g} [k])$ in $\Delta^{\mathrm{op}}_{\mathrm{inj,aug}}$.

Case $f, g \in \mathrm{Mor}(\Delta^{\mathrm{op}}_{\mathrm{inj}})$. We have $X(f \cdot g) = Y(f \cdot g) = Yf \cdot Yg = Xf \cdot Xg$.

Case $f \in \mathrm{Mor}(\Delta^{\mathrm{op}}_{\mathrm{inj}})$ and $k = -1$. We have $f = \prod^{[n]\quad[m]}_{i\in\lfloor n\,,\,m+1\rfloor} \partial^{i\ \mathrm{op}}_{j_i}$, for some $j_i \in [0, i]$ for $i \in [m+1, n]$, cf. Remark 49 or [**1,** Lemma 42]. Then

$$
\begin{aligned}
Xf \cdot Xg &= Y\Big(\prod^{[n]\quad[m]}_{i\in\lfloor n\,,\,m+1\rfloor} \partial^{i\ \mathrm{op}}_{j_i}\Big) \cdot \Big(\prod^{X_m\quad X_{-1}}_{i\in\lfloor m\,,\,0\rfloor} d^i_0\Big) = \Big(\prod^{X_n\quad X_m}_{i\in\lfloor n\,,\,m+1\rfloor} Y\partial^{i\ \mathrm{op}}_{j_i}\Big) \cdot \Big(\prod^{X_m\quad X_{-1}}_{i\in\lfloor m\,,\,0\rfloor} d^i_0\Big) \\
&= \Big(\prod^{X_n\quad X_m}_{i\in\lfloor n\,,\,m+1\rfloor} d^i_{j_i}\Big) \cdot \Big(\prod^{X_m\quad X_{-1}}_{i\in\lfloor m\,,\,0\rfloor} d^i_0\Big) \overset{(*)}{=} \prod^{X_n\quad X_{-1}}_{i\in\lfloor n\,,\,0\rfloor} d^i_0 = X(f \cdot g)\,.
\end{aligned}
$$

To show the equality $(*)$ note that one can use relation

$$d^{n+1}_j \cdot d^n_0 = d^{n+1}_0 \cdot d^n_{j-1}$$

for $0 < j \leqslant n+1$ to decrease the sum of the indices as long as this sum is bigger than zero.

Case $m = -1$ and $k = -1$. We have $Xf \cdot Xg = Xf \cdot X\,\mathrm{id}_{[-1]} = Xf \cdot \mathrm{id}_{X_{-1}} = Xf = X(f \cdot \mathrm{id}_{[-1]}) = X(f \cdot g)$.

Uniqueness. Suppose given a functor $\tilde{X}$ satisfying $\tilde{X}[n] = X_n$ for $n \geqslant -1$ and $\tilde{X}\partial^n_i = d^n_i$ for $0 \leqslant i \leqslant n$. Then $\tilde{X} \circ I_1$ is functor satisfying $(\tilde{X} \circ I_1)[n] = X_n$ for $n \geqslant 0$ and $(\tilde{X} \circ I_1)\partial^n_i = d^n_i$ for $i \in [0, n]$ and $n \geqslant 1$. Hence $\tilde{X} \circ I_1 = Y$. Hence for $[n] \xrightarrow{f} [m]) \in \mathrm{Mor}(\Delta^{\mathrm{op}}_{\mathrm{inj}})$ we have $\tilde{X}f = (\tilde{X} \circ I_1)f = Yf = Xf$.

For $([n] \xrightarrow{f} [m]) \in \mathrm{Mor}(\Delta^{\mathrm{op}}_{\mathrm{inj,aug}})$ such that $[m] = [-1]$ we have

$$\tilde{X}f = \tilde{X}(\emptyset \hookrightarrow [n])^{\mathrm{op}} = \tilde{X}\Big(\prod^{[n]\quad\emptyset}_{i\in\lfloor n\,,\,0\rfloor} (\partial^i_0)^{\mathrm{op}}\Big) = \prod^{X_n\quad X_{-1}}_{i\in\lfloor n\,,\,0\rfloor} \tilde{X}(\partial^i_0)^{\mathrm{op}} = \prod^{X_n\quad X_{-1}}_{i\in\lfloor n\,,\,0\rfloor} d^i_0 = Xf$$

This shows uniqueness. □

The construction in the following lemma goes back to Tierney and Vogel [**2,** (2.3)] .

Lemma 51 (density of $\mathcal{E}$)**.** Given $X \in \mathrm{Ob}(\mathcal{C})$, there exists $Y \in \mathrm{Ob}(\mathrm{HoAugSemiSimpRes}(\mathcal{C}, \mathcal{P}))$ such that $\mathcal{E}(Y) = X$. In particular $\mathcal{E}$ is dense.

Proof. By the construction in [**1,** Definition 36] we can construct a tuple $((Y_n)_{n\geqslant -1}, ((d^n_i)_{i\in[0,n]})_{n\geqslant 0})$, such that the following conditions hold.

- We have $Y_n \in \mathrm{Ob}(\mathcal{P})$ for $n \geqslant 0$.
- We have $Y_{-1} = X$.
- We have $(Y_n \xrightarrow{d^n_i} Y_{n-1}) \in \mathrm{Mor}(\mathcal{C})$ and $d^{n+1}_j \cdot d^n_i = d^{n+1}_i \cdot d^n_{j-1}$ for $0 \leqslant i < j \leqslant n+1$.
- For $n \geqslant 0$ the morphism $f_n : Y_n \to K_n$ defined by $d^n_i = f_n \cdot k^n_i$ for $i \in [0, n]$ and for a chosen simplicial kernel $(K_n, (k^n_i)_{i\in[0,n]})$ of $(d^{n-1}_i)_{i\in[0,n-1]}$ is $\mathcal{P}$-epic. In particular, we choose $k^0_0 := \mathrm{id}_X$ and obtain $d^0_0 = f_0 : Y_0 \to X$.

Lemma 50 shows that there exists a unique functor $Y : \Delta^{\mathrm{op}}_{\mathrm{inj,aug}} \to \mathcal{C}$ such that $Y[n] = Y_n$ for $n \geqslant -1$ and $Y(\partial^n_i)^{\mathrm{op}} = d^n_i$ for $0 \leqslant i \leqslant n$. Then $Y \in \mathrm{Ob}(\mathrm{AugSemiSimpRes}(\mathcal{C}, \mathcal{P}))$, cf. Definitions 44 and 45, and $\mathcal{E}(Y) = Y_{-1} = X$. □

4.3 Semisimplicial resolution of morphisms: $\mathcal{E}$ is full

Suppose given $X, \tilde{X} \in \mathrm{Ob}(\mathcal{C})$. Suppose given augmented semisimplicial resolutions of X and $\tilde{X}$, i.e. $Y, \tilde{Y} \in \mathrm{Ob}(\mathrm{AugSemiSimpRes}(\mathcal{C}, \mathcal{P}))$ such that $\mathcal{E}(Y) = Y_{-1} = X$ and $\mathcal{E}(\tilde{Y}) = \tilde{Y}_{-1} = \tilde{X}$.

We write $d_i^n := d_i^{n,Y}$ and $\tilde{d}_i^n := d_i^{n,\tilde{Y}}$ for $0 \leqslant i \leqslant n$. For $n \geqslant 0$ we may choose simplicial kernels $(K_n, (k_i^n)_{i\in[0,n]})$ of $(d_i^{n-1})_{i\in[0,n-1]}$ and $(\tilde{K}_n, (\tilde{k}_i^n)_{i\in[0,n]})$ of $(\tilde{d}_i^{n-1})_{i\in[0,n-1]}$. For $n \geqslant 0$ let $f_n : Y_n \to K_n$ and $\tilde{f}_n : \tilde{Y}_n \to \tilde{K}_n$ be the unique morphisms satisfying $f_n \cdot k_i^n = d_i^n$ and $\tilde{f}_n \cdot \tilde{k}_i^n = \tilde{d}_i^n$ for $i \in [0, n]$.

Lemma 52. Suppose given a morphism $g : X \to \tilde{X}$ in $\mathcal{C}$. Then there exists $\hat{g} : Y \to \tilde{Y}$ in $\mathrm{AugSemiSimpRes}(\mathcal{C}, \mathcal{P})$ such that $\mathcal{E}([\hat{g}]) = g$.

In particular the functor $\mathcal{E}$ is full.

Proof. We construct a tuple $(Y_n \xrightarrow{g_n} \tilde{Y}_n)_{n\geqslant -1}$ of morphisms in $\mathcal{C}$ satisfying $g_n \cdot \tilde{d}_i^n = d_i^n \cdot g_{n-1}$ for $0 \geqslant i \geqslant n$.

Construction of g_{-1}. Let $g_{-1} = g$.

Construction of g_0. Since $\tilde{d}_0^0 : \tilde{Y}_0 \to \tilde{X}$ is $\mathcal{P}$-epic and $Y_0 \in \mathrm{Ob}(\mathcal{P})$, there exists a morphism $g_0 : Y_0 \to \tilde{Y}_0$ satisfying $g_0 \cdot \tilde{d}_0^0 = d_0 \cdot g$.

Construction of g_n for $n \geqslant 1$. Suppose that $n \geqslant 1$ and that the morphism g_{n-1} has already been constructed and satisfies $g_{n-1} \cdot \tilde{d}_i^{n-1} = d_i^{n-1} \cdot g_{n-2}$ for $i \in [0, n-1]$. Consider the tuple $(k_i^n \cdot g_{n-1})_{i\in[0,n]}$. For $0 \leqslant i < j \leqslant n$ we have $k_j^n \cdot g_{n-1} \cdot \tilde{d}_i^{n-1} = k_j^n \cdot d_i^{n-1} \cdot g_{n-2} = k_i^n \cdot d_{j-1}^{n-1} \cdot g_{n-2} = k_i^n \cdot g_{n-1} \cdot \tilde{d}_{j-1}^{n-1}$. Hence by the universal property of the simplicial kernel of $(\tilde{d}_i^n)_{i\in[0,n]}$ there exists a unique morphism $\gamma_n : K_n \to \tilde{K}_n$ satisfying $\gamma_n \cdot \tilde{k}_i^n = k_i^n \cdot g_{n-1}$ for $i \in [0, n]$. Since $\tilde{f}_n$ is $\mathcal{P}$-epic and $Y_n \in \mathrm{Ob}(\mathcal{P})$, there exists a morphism $g_n : Y_n \to \tilde{Y}_n$ satisfying $g_n \cdot \tilde{f}_n = f_n \cdot \gamma_n$. It follows that $g_n \cdot \tilde{d}_i^n = g_n \cdot \tilde{f}_n \cdot \tilde{k}_i^n = f_n \cdot \gamma_n \cdot \tilde{k}_i^n = f_n \cdot k_i^n \cdot g_{n-1} = d_i^n \cdot g_{n-1}$ for $i \in [0, n]$.

For $([n] \xrightarrow{h} [m]) \in \mathrm{Mor}(\Delta_{\mathrm{inj,aug}})$ we have $h = {}^{[n]}\prod^{[m]}_{i\in\lceil 1\,,\,m-n\rceil} \partial_{k_i}^{n+i}$ for certain $0 \leqslant k_1 < \cdots < k_{m-n} \leqslant m$, cf. Remark 49.

Then

$$\begin{aligned} Y h^{op} \cdot g_n &= Y\Big({}^{[m]}\prod^{[n]}_{i\in\lfloor m-n\,,\,1\rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}}\Big) \cdot g_n = {}^{Y_m}\prod^{Y_n}_{i\in\lfloor m-n\,,\,1\rfloor} d_{k_i}^{n+i} \cdot g_n = g_m \cdot {}^{\tilde{Y}_m}\prod^{\tilde{Y}_n}_{i\in\lfloor m-n\,,\,1\rfloor} \tilde{d}_{k_i}^{n+i} \\ &= g_m \cdot \tilde{Y}\Big({}^{[m]}\prod^{[n]}_{i\in\lfloor m-n\,,\,1\rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}}\Big) = g_m \cdot \tilde{Y} h^{\mathrm{op}} . \end{aligned}$$

□

4.4 Construction of a semisimplicial homotopy: $\mathcal{E}$ is faithful

The following proposition is due to Tierney and Vogel [**2,** Theorem 2.4].

Proposition 53 ($\mathcal{E}$ is faithful)**.** Suppose given morphisms

$$(W \xrightarrow{f} Z), (W \xrightarrow{g} Z) \in \mathrm{Mor}(\mathrm{AugSemiSimpRes}(\mathcal{C}, \mathcal{P}))$$

such that $\mathcal{E}([f]) = \mathcal{E}([g])$. Then we have $[f] = [g]$.

In particular, the functor $\mathcal{E}$ is faithful.

Proof. Let $f = (W_n \xrightarrow{f_n} Z_n)_{n\geqslant -1}$ and $g = (W_n \xrightarrow{g_n} Z_n)_{n\geqslant -1}$.

For $n \geqslant 1$ we introduce the following notation.

Let $(K_n, (k_i^n)_{i\in[0,n]})$ be a simplicial kernel of the tuple $(d_i^{Z,n-1})_{i\in[0,n-1]}$ for $n \geqslant 1$.

Let $p_n : Z_n \to K_n$ be the unique morphism satisfying $p_n \cdot k_i^n = d_i^{Z,n}$ for $i \in [0,n]$.

Note that p_n is $\mathcal{P}$-epic, cf. Definition 44.

We show the existence of a tuple $((W_n \xrightarrow{h_i^n} Z_{n+1})_{i\in[0,n]})_{n\geqslant 0}$ of morphisms in $\mathcal{C}$ satisfying the following conditions (i - vi) for $n \geqslant 0$, cf. Definition 27.

(i) $h_i^{n+1} \cdot d_j^{Z,n+2} = d_{j-1}^{W,n+1} \cdot h_i^n$ for $i \in [0,n]$ and $j \in [i+2, n+2]$

(ii) $h_i^{n+1} \cdot d_j^{Z,n+2} = h_{i+1}^{n+1} \cdot d_j^{Z,n+2}$ for $i \in [0,n]$ and $j = i+1$

(iii) $h_i^{n+1} \cdot d_j^{Z,n+2} = h_{i-1}^{n+1} \cdot d_j^{Z,n+2}$ for $i \in [1,n+1]$ and $j = i$

(iv) $h_i^{n+1} \cdot d_j^{Z,n+2} = d_j^{W,n+1} \cdot h_{i-1}^n$ for $i \in [1,n+1]$ and $j \in [0,i-1]$

(v) $h_n^n \cdot d_{n+1}^{Z,n+1} = f_n$

(vi) $h_0^n \cdot d_0^{Z,n+1} = g_n$.

Then by Definition 27 we have $I_{\mathrm{aug},1}f \sim I_{\mathrm{aug},1}g$, which together with $I_{\mathrm{aug},2}f = \mathcal{E}[f] = f_{-1} = g_{-1} = \mathcal{E}[g] = I_{\mathrm{aug},2}g$ implies $f \sim g$; cf. Definition 47.

Construction of h_0^0.

Since $f_0 \cdot d_0^0 = d_0^0 \cdot f_{-1} = d_0^0 \cdot g_{-1} = g_0 \cdot d_0^0$ by the universal property of the simplicial kernel there exists a unique morphism $\delta_0^0 : W_0 \to L_1$ such that $g_0 = \delta_0^0 \cdot l_0^1$ and $f_0 = \delta_0^0 \cdot l_1^1$. Since p_1 is $\mathcal{P}$-epic and $W_0 \in \mathrm{Ob}(\mathcal{P})$ there exists a unique morphism $h_0^0 : W_0 \to Z_1$ satisfying $h_0^0 \cdot p_1 = \delta$.

Ad (v). We have $h_0^0 \cdot d_1^1 = h_0^0 \cdot p_1 \cdot l_1^1 = \delta_0^0 \cdot l_1^1 = f_0$.

Ad (vi). We have $h_0^0 \cdot d_0^1 = h_0^0 \cdot p_1 \cdot l_0^1 = \delta_0^0 \cdot l_0^1 = g_0$.

Construction of h_i^n for $n \geqslant 1$ and $i \in [0,n]$.

Suppose that we already have constructed morphisms $W_k \xrightarrow{h_i^k} Z_{k+1}$ for $k \in [0, n-1]$ and $i \in [0,k]$ satisfying conditions (i - vi).

> We shall roughly proceed as follows. We aim to construct the tuple $(h_i^n)_{i\in[0,n]}$ using the universal property of the simplicial kernel $(K_{n+1}, (K_{n+1} \xrightarrow{k_i^{n+1}} Z_n)_{i\in[0,n+1]})$, followed by a lift along p_{n+1}. To induce h_i^n in this manner, we need for each $i \in [0,n]$ a tuple $(W_n \xrightarrow{\alpha_{i,j}^n} Z_n)_{j\in[0,n+1]}$ satisfying
>
> $$\alpha_{i,j}^n \cdot d_\ell^n = \alpha_{i,\ell}^n \cdot d_{j-1}^n$$
>
> for $0 \leqslant \ell < j \leqslant n+1$. The morphisms $\alpha_{i,j}^n$ may be chosen to be of the form $d_*^n \cdot h_*^{n-1}$, as we are led to by the conditions (i - vi), except for $\alpha_{i-1,i}^n$ and $\alpha_{i,i}^n$, where we require equality and search for $\beta_i = \alpha_{i,i}^n = \alpha_{i-1,i}^n$, for $i \in [1,n]$. To construct these morphisms β_i we use the universal property of the simplicial kernel K_n, followed by a lift along p_n, to induce β_i. So for each $i \in [1,n]$ we need a tuple $(W_n \xrightarrow{\gamma_{i,j}^n} Z_{n-1})_{j\in[0,n]}$ of morphisms satisfying
>
> $$\gamma_{i,j}^n \cdot d_\ell^{n-1} = \gamma_{i,\ell}^n \cdot d_{j-1}^{n-1}$$
>
> for $0 \leqslant \ell < j \leqslant n$.

For $i \in [1,n]$ and $j \in [0,n]$ we define the morphism

$$\gamma_{i,j}^n := \left\{ \begin{array}{ll} d_j^n \cdot h_{i-1}^{n-1} \cdot d_i^n & \text{for } j \geqslant i \\ d_j^n \cdot h_{i-1}^{n-1} \cdot d_{i-1}^n & \text{for } j < i. \end{array} \right\} : W_n \to Z_{n-1}$$

We show that $\gamma^n_{i,j} \cdot d^{n-1}_\ell \overset{!}{=} \gamma^n_{i,\ell} \cdot d^{n-1}_{j-1}$ *for* $0 \leqslant \ell < j \leqslant n$ *and* $i \in [1,n]$

Case $j \geqslant i > \ell$ *and* $i < n$ *and* $j > i$. *Note that* $n \geqslant 2$.

We have

$$\gamma^n_{i,j} \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_{i-1} \cdot d^n_i \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_i \cdot d^n_i \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_i \cdot d^n_\ell \cdot d^{n-1}_{i-1} = d^n_j \cdot d^{n-1}_\ell \cdot h^{n-2}_{i-1} \cdot d^{n-1}_{i-1}$$
$$= d^n_\ell \cdot d^{n-1}_{j-1} \cdot h^{n-2}_{i-1} \cdot d^{n-1}_{i-1} = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_j \cdot d^{n-1}_{i-1} = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_{i-1} \cdot d^{n-1}_{j-1} = \gamma^n_{i,\ell} \cdot d^{n-1}_{j-1}\,.$$

Case $j \geqslant i > \ell$ *and* $i < n$ *and* $j = i$. *Note that* $n \geqslant 2$.

We have

$$\gamma^n_{i,j} \cdot d^{n-1}_\ell = \gamma^n_{j,j} \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_{j-1} \cdot d^n_j \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_j \cdot d^n_j \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_j \cdot d^n_\ell \cdot d^{n-1}_{j-1}$$
$$= d^n_j \cdot d^{n-1}_\ell \cdot h^{n-2}_{j-1} \cdot d^{n-1}_{j-1} = d^n_\ell \cdot d^{n-1}_{j-1} \cdot h^{n-2}_{j-1} \cdot d^{n-1}_{j-1} = d^n_\ell \cdot h^{n-1}_j \cdot d^n_{j-1} \cdot d^{n-1}_{j-1} = d^n_\ell \cdot h^{n-1}_j \cdot d^n_j \cdot d^{n-1}_{j-1}$$
$$= d^n_\ell \cdot h^{n-1}_{j-1} \cdot d^n_j \cdot d^{n-1}_{j-1} = d^n_\ell \cdot h^{n-1}_{j-1} \cdot d^n_{j-1} \cdot d^{n-1}_{j-1} = \gamma^n_{j,\ell} \cdot d^{n-1}_{j-1} = \gamma^n_{i,\ell} \cdot d^{n-1}_{j-1}\,.$$

Case $j \geqslant i > \ell$ *and* $i = n$. *It follows that* $j = i = n$.

We have

$$\gamma^n_{i,j} \cdot d^{n-1}_\ell = d^n_n \cdot h^{n-1}_{n-1} \cdot d^n_n \cdot d^{n-1}_\ell = d^n_n \cdot f_{n-1} \cdot d^{n-1}_\ell = d^n_n \cdot d^{n-1}_\ell \cdot f_{n-2} = d^n_\ell \cdot d^{n-1}_{n-1} \cdot f_{n-2} = d^n_\ell \cdot f_{n-1} \cdot d^{n-1}_{n-1}$$
$$= d^n_\ell \cdot h^{n-1}_{n-1} \cdot d^n_n \cdot d^{n-1}_{n-1} = d^n_\ell \cdot h^{n-1}_{n-1} \cdot d^n_{n-1} \cdot d^{n-1}_{n-1} = \gamma^n_{n,\ell} \cdot d^{n-1}_{n-1} = \gamma^n_{i,\ell} \cdot d^{n-1}_{j-1}\,.$$

Case $j \geqslant i$ *and* $\ell \geqslant i$. *It follows that* $j > i$. *Note that* $n \geqslant 2$.

We have

$$\gamma^n_{i,j} \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_{i-1} \cdot d^n_i \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_{i-1} \cdot d^n_{\ell+1} \cdot d^{n-1}_i = d^n_j \cdot d^{n-1}_\ell \cdot h^{n-2}_{i-1} \cdot d^{n-1}_i = d^n_\ell \cdot d^{n-1}_{j-1} \cdot h^{n-2}_{i-1} \cdot d^{n-1}_i$$
$$= d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_j \cdot d^{n-1}_i = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_i \cdot d^{n-1}_{j-1} = \gamma^n_{i,\ell} \cdot d^{n-1}_{j-1}\,.$$

Case $i > j$. *It follows that* $i > j > \ell$. *Note that* $n \geqslant 2$.

We have

$$\gamma^n_{i,j} \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_{i-1} \cdot d^n_{i-1} \cdot d^{n-1}_\ell = d^n_j \cdot h^{n-1}_{i-1} \cdot d^n_\ell \cdot d^{n-1}_{i-2} = d^n_j \cdot d^{n-1}_\ell \cdot h^{n-2}_{i-2} \cdot d^{n-1}_{i-2} = d^n_\ell \cdot d^{n-1}_{j-1} \cdot h^{n-2}_{i-2} \cdot d^{n-1}_{i-2}$$
$$= d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_{j-1} \cdot d^{n-1}_{i-2} = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_{i-1} \cdot d^{n-1}_{j-1} = \gamma^n_{i,\ell} \cdot d^{n-1}_{j-1}\,.$$

So for $i \in [1,n]$ by the universal property of the simplicial kernel $(K_n, (k^n_j)_{j\in[0,n]})$ there exists a unique morphism $\epsilon^n_i : W_n \to K_n$ satisfying $\epsilon^n_i \cdot k^n_j = \gamma^n_{i,j}$ for $j \in [0,n]$.

Since p_n is $\mathcal{P}$-epic and $W_n \in \mathrm{Ob}(\mathcal{P})$, there exists a morphism $\beta^n_i : W_n \to Z_n$ such that $\beta^n_i \cdot p_n = \epsilon^n_i$ for $i \in [1,n]$. Let $\beta^n_0 := g_n$ and $\beta^n_{n+1} := f_n$.

Suppose given $i \in [0,n]$. For $j \in [0,n+1]$, we define the morphisms

$$\alpha^n_{i,j} := \left\{ \begin{array}{ll} d^n_{j-1} \cdot h^{n-1}_i & \text{if } j > i+1 \\ \beta^n_j & \text{if } j \in [i,i+1] \\ d^n_j \cdot h^{n-1}_{i-1} & \text{if } j < i \end{array} \right\} : W_n \to Z_n$$

We show that $\alpha^n_{i,j} \cdot d^n_\ell \overset{!}{=} \alpha^n_{i,\ell} \cdot d^n_{j-1}$ *for* $0 \leqslant \ell < j \leqslant n+1$ *and* $i \in [0,n]$.

Case $j > i+1$ *and* $i > \ell$. *Note that* $n \geqslant 2$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = d^n_{j-1} \cdot h^{n-1}_i \cdot d^n_\ell = d^n_{j-1} \cdot d^{n-1}_\ell \cdot h^{n-2}_{i-1} = d^n_\ell \cdot d^{n-1}_{j-2} \cdot h^{n-2}_{i-1} = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_{j-1} = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j > i+1$ and $i \leqslant \ell$ and $\ell = i = 0$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = d^n_{j-1} \cdot h^{n-1}_0 \cdot d^n_0 = d^n_{j-1} \cdot g_{n-1} = g_n \cdot d^n_{j-1} = \beta^n_0 \cdot d^n_{j-1} = \alpha^n_{0,0} \cdot d^n_{j-1} = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j > i+1$ and $i \leqslant \ell$ and $\ell \in [i, i+1]$ and $0 < \ell < n$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = d^n_{j-1} \cdot h^{n-1}_i \cdot d^n_\ell\,.$$

We have

$$\alpha^n_{i,\ell} \cdot d^n_{j-1} = \beta^n_\ell \cdot d^n_{j-1} = \beta^n_\ell \cdot p_n \cdot k^n_{j-1} = \epsilon^n_\ell \cdot k^n_{j-1} = \gamma^n_{\ell,j-1} = d^n_{j-1} \cdot h^{n-1}_{\ell-1} \cdot d^n_\ell = d^n_{j-1} \cdot h^{n-1}_i \cdot d^n_\ell\,.$$

Case $j > i+1$ and $i \leqslant \ell$ and $\ell \in [i, i+1]$ and $\ell = n$. It follows that $i = n-1$ and $j = n+1$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = \alpha^n_{n-1,n+1} \cdot d^n_n = d^n_n \cdot h^{n-1}_{n-1} \cdot d^n_n\,.$$

We have

$$\alpha^n_{i,\ell} \cdot d^n_{j-1} = \alpha^n_{n-1,n} \cdot d^n_n = \beta^n_n \cdot d^n_n = \beta^n_n \cdot p_n \cdot k^n_n = \epsilon^n_n \cdot k^n_n = \gamma^n_{n,n} = d^n_n \cdot h^{n-1}_{n-1} \cdot d^n_n\,.$$

Case $j > i+1$ and $i \leqslant \ell$ and $\ell > i+1$. It follows that $j > i+2$. Note that $n \geqslant 2$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = d^n_{j-1} \cdot h^{n-1}_i \cdot d^n_\ell = d^n_{j-1} \cdot d^{n-1}_{\ell-1} \cdot h^{n-2}_i = d^n_{\ell-1} \cdot d^{n-1}_{j-2} \cdot h^{n-2}_i = d^n_{\ell-1} \cdot h^{n-1}_i \cdot d^n_{j-1} = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j \in [i, i+1]$ and $j = i$. Note that $j \in [1, n]$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = \beta^n_j \cdot d^n_\ell = \beta^n_j \cdot p_n \cdot k^n_\ell = \epsilon^n_j \cdot k^n_\ell = \gamma^n_{j,\ell} = d^n_\ell \cdot h^{n-1}_{j-1} \cdot d^n_{j-1} = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_{j-1} = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j \in [i, i+1]$ and $j = i+1$ and $\ell < i < n$. Note that $j \in [1, n]$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = \beta^n_j \cdot d^n_\ell = \beta^n_j \cdot p_n \cdot k^n_\ell = \epsilon^n_j \cdot k^n_\ell = \gamma^n_{j,\ell} = d^n_\ell \cdot h^{n-1}_{j-1} \cdot d^n_{j-1} = d^n_\ell \cdot h^{n-1}_i \cdot d^n_i = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_i$$
$$= \alpha^n_{i,\ell} \cdot d^n_i = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j \in [i, i+1]$ and $j = i+1$ and $\ell < i = n$. Note that $j = n+1$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = \alpha^n_{n,n+1} \cdot d^n_\ell = \beta^n_{n+1} \cdot d^n_\ell = f_n \cdot d^n_\ell = d^n_\ell \cdot f_{n-1} = d^n_\ell \cdot h^{n-1}_{n-1} \cdot d^n_n = \alpha^n_{n,\ell} \cdot d^n_n = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j \in [i, i+1]$ and $j = i+1$ and $i \leqslant \ell$ and $j = 1$. It follows that $0 = i = \ell = j-1$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = \alpha^n_{0,1} \cdot d^n_0 = \beta^n_1 \cdot d^n_0 = \beta^n_1 \cdot p_n \cdot k^n_0 = \epsilon^n_1 \cdot k^n_0 = \gamma^n_{1,0} = d^n_0 \cdot k^{n-1}_0 \cdot d^n_0 = d^n_0 \cdot g_{n-1} = g_n \cdot d^n_0$$
$$= \beta^n_0 \cdot d^n_0 = \alpha^n_{0,0} \cdot d^n_0 = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j \in [i, i+1]$ and $j = i+1$ and $i \leqslant \ell$ and $2 \leqslant j \leqslant n$. It follows that $i = \ell = j-1$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = \beta^n_j \cdot d^n_\ell = \beta^n_j \cdot p_n \cdot k^n_\ell = \epsilon^n_j \cdot k^n_\ell = \gamma^n_{j,\ell} = d^n_\ell \cdot h^{n-1}_{j-1} \cdot d^n_{j-1} = d^n_\ell \cdot h^{n-1}_i \cdot d^n_i = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_i$$
$$= \gamma^n_{i,\ell} = \epsilon^n_i \cdot k^n_\ell = \beta^n_i \cdot p_n \cdot k^n_\ell = \alpha^n_{i,i} \cdot d^n_\ell = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j \in [i, i+1]$ and $j = i+1$ and $i \leqslant \ell$ and $j = n+1$. It follows that $i = \ell = j-1 = n$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = \alpha^n_{n,n+1} \cdot d^n_n = \beta^n_{n+1} \cdot d^n_n = f_n \cdot d^n_n = d^n_n \cdot f_{n-1} = d^n_n \cdot h^{n-1}_{n-1} \cdot d^n_n = \gamma^n_{n,n} = \epsilon^n_n \cdot k^n_n = \beta^n_n \cdot p_n \cdot k^n_n$$
$$= \beta^n_n \cdot d^n_n = \alpha^n_{n,n} \cdot d^n_n = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

Case $j < i$. It follows that $i > \ell + 1$. Note that $n \geqslant 2$.

We have

$$\alpha^n_{i,j} \cdot d^n_\ell = d^n_j \cdot h^{n-1}_{i-1} \cdot d^n_\ell = d^n_j \cdot d^{n-1}_\ell \cdot h^{n-2}_{i-2} = d^n_\ell \cdot d^{n-1}_{j-1} \cdot h^{n-2}_{i-2} = d^n_\ell \cdot h^{n-1}_{i-1} \cdot d^n_{j-1} = \alpha^n_{i,\ell} \cdot d^n_{j-1}\,.$$

So for $i \in [0, n]$ by the universal property of the simplicial kernel $(K_{n+1}, (k^{n+1}_j)_{j\in[0,n+1]})$ there exists a unique morphism $\delta^n_i : W_n \to K_{n+1}$ satisfying $\delta^n_i \cdot k^{n+1}_j = \alpha^n_{i,j}$ for $j \in [0, n+1]$.

Since p_{n+1} is $\mathcal{P}$-epic and $W_n \in \mathrm{Ob}(\mathcal{P})$, there exists a morphism $h^n_i : W_n \to Z_{n+1}$ such that $h^n_i \cdot p_{n+1} = \delta^n_i$ for $i \in [0, n]$.

Ad (i). Let $i \in [0, n-1]$ and $j \in [i+2, n+1]$.

We have
$$h^n_i \cdot d^{n+1}_j = h^n_i \cdot p_{n+1} \cdot k^{n+1}_j = \delta^n_i \cdot k^{n+1}_j = \alpha^n_{i,j} = d^n_{j-1} \cdot h^{n-1}_i\,.$$

Ad (ii). Let $i \in [0, n-1]$ and $j = i+1$.

We have

$$h^n_i \cdot d^{n+1}_j = h^n_i \cdot p_{n+1} \cdot k^{n+1}_j = \delta^n_i \cdot k^{n+1}_j = \alpha^n_{i,j} = \beta^n_j = \alpha^n_{i+1,j} = \delta^n_{i+1} \cdot k^{n+1}_j = h^n_{i+1} \cdot p_{n+1} \cdot k^{n+1}_j = h^n_{i+1} \cdot d^{n+1}_j\,.$$

Ad (iii). Cf. Definition 27.

Ad (iv). Let $i \in [1, n]$ and $j \in [0, i-1]$.

We have
$$h^n_i \cdot d^{n+1}_j = h^n_i \cdot p_{n+1} \cdot k^{n+1}_j = \delta^n_i \cdot k^{n+1}_j = \alpha^n_{i,j} = d^n_j \cdot h^{n-1}_{i-1}\,.$$

Ad (v).

We have
$$h^n_n \cdot d^{n+1}_{n+1} = h^n_n \cdot p_{n+1} \cdot k^{n+1}_{n+1} = \delta^n_n \cdot k^{n+1}_{n+1} = \alpha^n_{n,n+1} = \beta^n_{n+1} = f_n\,.$$

Ad (vi).

We have
$$h^n_0 \cdot d^{n+1}_0 = h^n_0 \cdot p_{n+1} \cdot k^{n+1}_0 = \delta^n_0 \cdot k^{n+1}_0 = \alpha^n_{0,0} = \beta^n_0 = g_n\,.$$

□

4.5 The resolution functor

Recall that $\mathcal{C}$ is a category that has finite limits and that $\mathcal{P} \subseteq \mathcal{C}$ is a resolving subcategory with finite coproducts.

Recall that we have the functor

$$\mathcal{E} : \text{HoAugSemiSimpRes}(\mathcal{C}, \mathcal{P}) \to \mathcal{C}\,;$$

cf. Definition 48.

It maps an augmented semisimplicial object, with objects in $\mathcal{P}$ at positions $k \geqslant 0$ and with all induced morphisms into the occurring simplicial kernels being $\mathcal{P}$-epic, to its object at position $k = -1$.

Theorem 54. The functor $\mathcal{E}$ is an equivalence of categories.

Proof. It suffices to show that $\mathcal{E}$ is faithful, full and dense.

Faithful: Cf. Proposition 53.

Full: Cf. Lemma 52.

Dense: Cf. Lemma 51. □

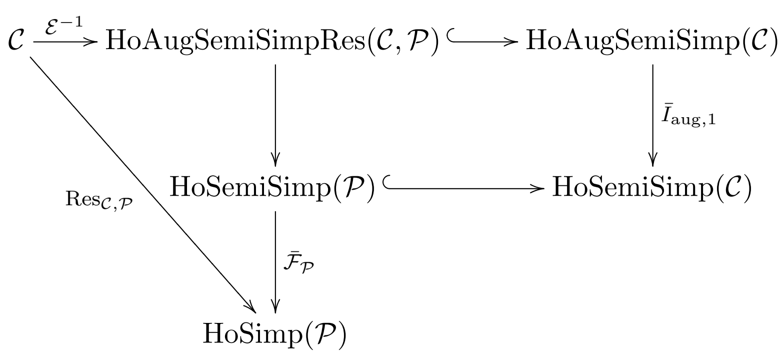
Definition 55 (resolution functor)**.**

We choose an inverse equivalence $\mathcal{E}^{-1} : \mathcal{C} \to \text{HoAugSemiSimpRes}(\mathcal{C}, \mathcal{P})$. We define the *resolution functor* to be

$$\begin{array}{rlcl} \text{Res}_{\mathcal{C},\mathcal{P}} : & \mathcal{C} & \to & \text{HoSimp}(\mathcal{P}) \\ & (X \xrightarrow{f} Y) & \mapsto & \text{Res}_{\mathcal{C},\mathcal{P}}(X \xrightarrow{f} Y) := \bar{\mathcal{F}}_{\mathcal{P}}(\bar{I}_{\text{aug},1}(\mathcal{E}^{-1}(X \xrightarrow{f} Y)))\,. \end{array}$$

Concerning $\bar{I}_{\text{aug},1}$, see Definition 48. Concerning $\bar{\mathcal{F}}_{\mathcal{P}}$, see Proposition 37.

We have the following commutative diagram.



Lemma 56. Suppose given a category $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

Given $X \xrightarrow{f} Y$ in $\text{Simp}(\mathcal{C})$, we recall that $X, Y : \Delta^{\text{op}} \to \mathcal{C}$ are functors and that f is a transformation from X to Y.

Hence we may define a functor $\text{Simp}(F) : \text{Simp}(\mathcal{C}) \to \text{Simp}(\mathcal{D})$ by

$$\text{Simp}(F)(X \xrightarrow{f} Y) := ((F \circ X) \xrightarrow{Ff} (F \circ Y))\,.$$

Note that $d_i^{\text{Simp}(F)(X),\ell+1} = F(d_i^{X,\ell+1})$ and $s_i^{\text{Simp}(F)(X),\ell} = F(s_i^{X,\ell})$ for $\ell \geqslant 0$ and $i \in [0, \ell]$.

Note that $(\mathrm{Simp}(F)(f))_\ell = F(f_\ell)$ for $\ell \geqslant 0$.

There exists a unique functor $\mathrm{HoSimp}(F) : \mathrm{HoSimp}(\mathcal{C}) \to \mathrm{HoSimp}(\mathcal{D})$ making the following diagram commutative.

$$\begin{array}{ccc} \mathrm{Simp}(\mathcal{C}) & \xrightarrow{\mathrm{Simp}(F)} & \mathrm{Simp}(\mathcal{D}) \\ {\scriptstyle R_{\mathrm{Simp}(\mathcal{C}),(\sim)}}\downarrow & & \downarrow{\scriptstyle R_{\mathrm{Simp}(\mathcal{D}),(\sim)}} \\ \mathrm{HoSimp}(\mathcal{C}) & \xrightarrow[\mathrm{HoSimp}(F)]{} & \mathrm{HoSimp}(\mathcal{D}) \end{array}$$

Proof. Suppose given morphisms $f, g : X \to Y$. Suppose given a simpicial homotopy $((h_i^n)_{i\in[0,n]})_{n\geqslant 0}$ from f to g, i. e. a tuple of morphisms satisfying the relations (i - viii) in Definition 26. We *claim* that the tuple $((Fh_i^n)_{i\in[0,n]})_{n\geqslant 0}$ is an elementary simplicial homotopy from $\mathrm{Simp}(F)f$ to $\mathrm{Simp}(F)g$.

Let $n \geqslant 0$.

Ad (i).

Let $i \in [0, n]$ and $j \in [i+2, n+2]$.

We have

$$Fh_i^{n+1} \cdot d_j^{\mathrm{Simp}(F)Y,n+2} = Fh_i^{n+1} \cdot Fd_j^{Y,n+2} = F(h_i^{n+1} \cdot d_j^{Y,n+2}) = F(d_{j-1}^{X,n+1} \cdot h_i^n) = Fd_{j-1}^{X,n+1} \cdot Fh_i^n$$
$$= d_{j-1}^{\mathrm{Simp}(F)X,n+1} \cdot Fh_i^n .$$

Ad (ii).

Let $i \in [0, n]$ and $j = i+1$.

We have

$$Fh_i^{n+1} \cdot d_j^{\mathrm{Simp}(F)Y,n+2} = Fh_i^{n+1} \cdot Fd_j^{Y,n+2} = F(h_i^{n+1} \cdot d_j^{Y,n+2}) = F(h_{i+1}^{n+1} \cdot d_j^{Y,n+2}) = Fh_{i+1}^{n+1} \cdot Fd_j^{Y,n+2}$$
$$= Fh_{i+1}^{n+1} \cdot d_j^{\mathrm{Simp}(F)Y,n+2} .$$

Ad (iv).

Let $i \in [1, n+1]$ and $j \in [0, i-1]$.

We have

$$Fh_i^{n+1} \cdot d_j^{\mathrm{Simp}(F)Y,n+2} = Fh_i^{n+1} \cdot Fd_j^{Y,n+2} = F(h_i^{n+1} \cdot d_j^{Y,n+2}) = F(d_j^{X,n+1} \cdot h_{i-1}^n) = Fd_j^{X,n+1} \cdot Fh_{i-1}^n$$
$$= d_j^{\mathrm{Simp}(F)X,n+1} \cdot Fh_{i-1}^n .$$

Ad (v).

Let $i \in [0, n]$ and $j \in [i+1, n+1]$.

We have

$$Fh_i^n \cdot s_j^{\mathrm{Simp}(F)Y,n+1} = Fh_i^n \cdot Fs_j^{Y,n+1} = F(h_i^n \cdot s_j^{Y,n+1}) = F(s_{j-1}^{X,n} \cdot h_i^{n+1}) = Fs_{j-1}^{X,n} \cdot Fh_i^{n+1}$$
$$= s_{j-1}^{\mathrm{Simp}(F)X,n} \cdot Fh_i^{n+1} .$$

Ad (vi).

Let $i \in [0, n]$ and $j \in [0, i]$.

We have

$$Fh_i^n \cdot s_j^{\mathrm{Simp}(F)Y,n+1} = Fh_i^n \cdot Fs_j^{Y,n+1} = F(h_i^n \cdot s_j^{Y,n+1}) = F(s_j^{X,n} \cdot h_{i+1}^{n+1}) = Fs_j^{X,n} \cdot Fh_{i+1}^{n+1}$$
$$= s_j^{\mathrm{Simp}(F)X,n} \cdot Fh_{i+1}^{n+1} .$$

Ad (vii).

We have

$$Fh_n^n \cdot d_{n+1}^{\mathrm{Simp}(F)Y,n+1} = Fh_n^n \cdot Fd_{n+1}^{Y,n+1} = F(h_n^n \cdot d_{n+1}^{Y,n+1}) = F(f_n) = (\mathrm{Simp}(F)f)_n .$$

Ad (viii).

We have

$$Fh_0^n \cdot d_0^{\mathrm{Simp}(F)Y,n+1} = Fh_0^n \cdot Fd_0^{Y,n+1} = F(h_0^n \cdot d_0^{Y,n+1}) = F(g_n) = (\mathrm{Simp}(F)g)_n .$$

So for f elementary simplicially homotopic to g, we have

$$(R_{\mathrm{Simp}(\mathcal{D}),\sim} \circ \mathrm{Simp}(F))f = (R_{\mathrm{Simp}(\mathcal{D}),\sim} \circ \mathrm{Simp}(F))g .$$

By Lemma 8 and Lemma 32 there exists a unique functor $\mathrm{HoSimp}(F) : \mathrm{HoSimp}(\mathcal{C}) \to \mathrm{HoSimp}(\mathcal{D})$ satisfying

$$\mathrm{HoSimp}(F) \circ R_{\mathrm{Simp}(\mathcal{C}),\sim} = R_{\mathrm{Simp}(\mathcal{D}),\sim} \circ \mathrm{Simp}(F) .$$

□

We end with a provisorial definition.

Definition 57. Suppose given a category $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

We define the functor by

$$\mathrm{L}F := \mathrm{HoSimp}(F) \circ \mathrm{Res}_{\mathcal{C},\mathcal{P}} : \mathcal{C} \to \mathrm{Simp}(\mathcal{D})$$

and call it the *left derived functor of* F; cf. [**2,** §2, p. 6].

Chapter 5

Dold-Puppe-Kan correspondence

Let R be a ring.

In this chapter $\mathcal{A} = R$-Mod will be the category of left R-modules and R-linear maps.

We aim to give a proof of the classical Dold-Puppe-Kan theorem, cf. [**5,** Theorems 8.1, 8.2], [**6,** Theorem 1.5], [**7,** Satz 3.6], with the help of the functors $\mathcal{F}_{\mathcal{C}}$ and $\bar{\mathcal{F}}_{\mathcal{C}}$.

5.1 From semisimplicial modules to $\geqslant$0-complexes and back

Recall that a complex C over $\mathcal{A}$ consists of a tuple of morphisms $(C_{n+1} \xrightarrow{\delta_n^C} C_n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$ such that $\delta_{n+1}^C \cdot \delta_n^C = 0$ for $n \in \mathbb{Z}$.

$$C = (\ \ldots \xrightarrow{\delta_{n+1}^C} C_{n+1} \xrightarrow{\delta_n^C} C_n \xrightarrow{\delta_{n-1}^C} \ldots\ )$$

A morphism between complexes C and D is a tuple $(C_n \xrightarrow{\alpha_n} D_n)_{n\in\mathbb{Z}}$ with $\delta_{n+1}^C \cdot \alpha_n = \alpha_{n+1} \cdot d_{n+1}^D$ for $n \in \mathbb{Z}$.

We denote the category of complexes over $\mathcal{A}$ by $\mathrm{C}(\mathcal{A})$.

Definition 58. A $\geqslant$0-*complex* is a complex C such that $C_i = 0$ for $-1 \geqslant i$. We have the full subcategory $\mathrm{C}(\mathcal{A})_{\geqslant 0}$ of $\geqslant$0-complexes in $\mathrm{C}(\mathcal{A})$.

Remark 59. In the following section we write a semisimplicial object in $\mathcal{A}$ as $X = (X_n, (d_i^{X,n})_{i\in[0,n+1]})_{n\geqslant 0}$ with $X_n \in \mathrm{Ob}\,\mathcal{A}$ and $d_i^{X,n+1} {\in}_{\mathcal{A}}(X_{n+1}, X_n)$ such that $d_i^{X,n+1} \cdot d_j^{X,n} = d_{j+1}^{X,n+1} \cdot d_i^{X,n}$ for $0 \leqslant i \leqslant j \leqslant n$, cf. [**1,** Proposition 43].

Consider a tuple $(X_n \xrightarrow{\alpha_n} Y_n)_{n\geqslant 0}$, where $X, Y \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{A}))$. For $(X_n \xrightarrow{\alpha_n} Y_n)_{n\geqslant 0}$ to be a semisimplicial morphism it suffices to have $d_i^{X,n+1} \cdot \alpha_n = \alpha_{n+1} \cdot d_i^{Y,n+1}$ for $n \geqslant 0$ and $i \in [0, n+1]$.

Definition 60. Suppose given $X \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{A}))$.

Write $\bigcap_{i\in[1,0]} \ker d_i^{X,0} := X_0$.

We define a $\geqslant$0-complex $\tilde{X} = (\tilde{X}_n, \delta_{n+1}^{\tilde{X}})_{n\geqslant 0}$ over $\mathcal{A}$ by $\tilde{X}_n := \bigcap_{i\in[1,n]} \ker d_i^{X,n}$ and $\delta_{n+1}^{\tilde{X}} := d_0^{X,n+1}|_{\tilde{X}_{n+1}}^{\tilde{X}_n}$ for $n \geqslant 0$.

For $n \geqslant 1$ and for $x \in \tilde{X}_{n+1}$ and $i \in [1, n]$ we have $x(d_0^{X,n+1} \cdot d_i^{X,n}) = x(d_{i+1}^{X,n+1} \cdot d_0^{X,n}) = 0 d_0^{X,n} = 0$, so that $\delta_{n+1}^{\tilde{X}}$ is a well-defined R-linear map.

For $n \geqslant 1$ and $x \in \tilde{X}_{n+1}$ we have $x(\delta^{\tilde{X}}_{n+1} \cdot \delta^{\tilde{X}}_{n}) = x(d_0^{X,n+1} \cdot d_0^{X,n}) = x(d_1^{X,n+1} \cdot d_0^{X,n}) = 0 d_0^{X,n} = 0$, so that $\tilde{X}$ is indeed a complex.

We define the *Moore complex* functor

$$\begin{array}{rlcll} \mathcal{M}: & \mathrm{SemiSimp}(\mathcal{A}) & \to & \mathrm{C}(\mathcal{A})_{\geqslant 0} & \\ & X & \mapsto & \tilde{X} & \text{for } X \in \mathrm{Ob}\,\mathrm{SemiSimp}(\mathcal{A}) \\ & (X_n \xrightarrow{\alpha_n} Y_n)_{n\geqslant 0} & \mapsto & (\tilde{X}_n \xrightarrow{\alpha_n|^{\tilde{Y}_n}_{\tilde{X}_n}} \tilde{Y}_n)_{n\geqslant 0} & \text{for } \alpha \in \mathrm{Mor}\,\mathrm{SemiSimp}(\mathcal{A}) \end{array}$$

This is well-defined since for $n \geqslant 0$ and $x \in \tilde{X}_{n+1}$ we have $x(\alpha_{n+1} \cdot d_i^{Y,n+1}) = x(d_i^{X,n+1} \cdot \alpha_n) = 0\alpha_n = 0$ for $i \in [1, n+1]$. We have $\alpha_n \cdot \delta^{\tilde{Y}}_n = \alpha_n \cdot d_0^{Y,n} = d_0^{X,n} \cdot \alpha_{n-1} = \delta^{\tilde{X}}_n \cdot \alpha_{n-1}$, so α is indeed a complex morphism. Note that in fact $\mathcal{M}(\mathrm{id}_X) = \mathrm{id}_{\tilde{X}}$ and $\mathcal{M}(\alpha \cdot \beta) = \mathcal{M}(\alpha) \cdot \mathcal{M}(\beta)$ by construction.

Definition 61. For $C \in \mathrm{Ob}(\mathrm{C}(\mathcal{A})_{\geqslant 0})$ we define a semisimplicial object $\hat{C}$ in $\mathcal{A}$ by $\hat{C}_n := C_n$, by $d_0^{\hat{C},n+1} := \delta^C_{n+1}$ and by $d_i^{\hat{C},n+1} = 0$ for $n \geqslant 0$ and $i \in [1, n+1]$.

We have $d_i^{\hat{C},n+2} \cdot d_j^{\hat{C},n+1} = 0 = d_{j+1}^{\hat{C},n+2} \cdot d_i^{\hat{C},n+1}$ for $0 \leqslant i \leqslant j \leqslant n+1$, so that $(\hat{C}, ((d_i^{\hat{C},n})_{i\in[0,n]})_{n\geqslant 0})$ is indeed a semisimplicial object in $\mathcal{A}$; cf. [**1,** Proposition 43].

We define a functor

$$\begin{array}{rlcll} \mathcal{N}: & \mathrm{C}(\mathcal{A})_{\geqslant 0} & \to & \mathrm{SemiSimp}(\mathcal{A}) & \\ & C & \mapsto & \hat{C} & \text{for } C \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})_{\geqslant 0} \\ & (\alpha_n)_{n\geqslant 0} & \mapsto & (\alpha_n)_{n\geqslant 0} & \text{for } (\alpha_n)_{n\geqslant 0} \in \mathrm{Mor}\,\mathrm{C}(\mathcal{A})_{\geqslant 0} \end{array}$$

This is well-defined since we have $\alpha_{n+1} \cdot d_i^{\hat{D},n+1} = 0 = d_i^{\hat{C},n+1} \cdot \alpha_n$ for $i \in [1, n+1]$ and $\alpha_{n+1} \cdot d_0^{\hat{D},n+1} = \alpha_{n+1} \cdot \delta^D_{n+1} = \delta^C_{n+1} \cdot \alpha_n = d_0^{\hat{C},n+1} \cdot \alpha_n$ for $n \geqslant 0$ and for $C \xrightarrow{\alpha} D$ in $\mathrm{C}(\mathcal{A})_{\geqslant 0}$.

Remark 62. The functor $\mathcal{M} \circ \mathcal{N} : \mathrm{C}(\mathcal{A})_{\geqslant 0} \to \mathrm{C}(\mathcal{A})_{\geqslant 0}$ is the identity functor.

Proof. For $C \in \mathrm{C}(\mathcal{A})_{\geqslant 0}$ we have $((\mathcal{M} \circ \mathcal{N})C)_n = \bigcap_{i\in[1,n]} \ker d_i^{\mathcal{N}C,n} = \bigcap_{i\in[1,n]} \ker 0 = \bigcap_{i\in[1,n]} C_n = C_n$ and $\delta_n^{(\mathcal{M}\circ\mathcal{N})C} = d_0^{\mathcal{N}C,n} = \delta^C_n$.

Suppose given $n \geqslant 0$ and a morphism $C \xrightarrow{\alpha} D$. We find that $((\mathcal{M} \circ \mathcal{N})\alpha)_n = (\mathcal{N}\alpha)_n|^{\mathcal{M}(\mathcal{N}(C))_n}_{\mathcal{M}(\mathcal{N}(C))_n} = \alpha_n|^{D_n}_{C_n} = \alpha_n$. So indeed $(\mathcal{M} \circ \mathcal{N})\alpha = \alpha$. □

Lemma 63. For $X \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{A}))$ and $n \geqslant 0$ we define

$$\begin{array}{rlcl} \mu_{X,n}: & ((\mathcal{N} \circ \mathcal{M})X)_n & \to & X_n \\ & x & \mapsto & x \end{array}$$

to be the inclusion map, where $((\mathcal{N} \circ \mathcal{M})X)_n = \bigcap_{i\in[1,n]} \ker d_i^{X,n}$.

Then $\mu_X := (\mu_{X,n})_{n\geqslant 0} : (\mathcal{N} \circ \mathcal{M})X \to X$ is a semisimplicial morphism.

Moreover, the tuple $\mu := (\mu_X)_{X\in\mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{A}))}$ is a transformation from $\mathcal{N} \circ \mathcal{M}$ to $\mathrm{id}_{\mathrm{SemiSimp}(\mathcal{A})}$.

Proof. We show that μ_X is a semisimplicial morphism.

Let $n \geqslant 1$. For $x \in (\mathcal{N} \circ \mathcal{M}(X))_n = \bigcap_{i\in[1,n]} \ker d_i^{X,n}$ we have

$x(\mu_{X,n} \cdot d_i^{X,n}) = x d_i^{X,n} = 0 = x(0 \cdot \mu_{X,n-1}) = x(d_i^{\mathcal{N} \circ \mathcal{M}(X),n} \cdot \mu_{X,n-1})$, if $i \in [1,n]$, and we have $x(\mu_{X,n} \cdot d_0^{X,n}) = x d_0^{X,n} = x(d_0^{\mathcal{N} \circ \mathcal{M}(X),n} \cdot \mu_{X,n-1})$.

So μ_X is a semisimplicial morphim for $X \in \mathrm{Ob}(\mathrm{SemiSimp}(\mathcal{A}))$.

Let $(X_n \xrightarrow{\alpha_n} Y_n)_{n \geqslant 0}$ be a semisimplicial morphism.

Let $n \geqslant 1$.

Let $x \in ((\mathcal{N} \circ \mathcal{M})X)_n = \bigcap\limits_{i \in [1,n]} \ker d^n_{X,i}$.

We have $x((\mathcal{N} \circ \mathcal{M})\alpha)_n \cdot \mu_{Y,n}) = x((\alpha_n|^{\bigcap_{i\in[1,n]} \ker d_i^{Y,n}}_{\bigcap_{i\in[1,n]} \ker d_i^{X,n}}) \cdot \mu_{Y,n}) = (x\alpha_n)\mu_{Y,n} = x\alpha_n = x(\mu_{X,n} \cdot \alpha_n)$.

$$
\begin{array}{ccc}
\bigcap\limits_{i\in[1,n]} \ker d_i^{X,n} & \xhookrightarrow{\mu_{X,n}} & X_n \\
\Big\downarrow {\scriptstyle \alpha_n|^{\bigcap_{i\in[1,n]} \ker d_i^{Y,n}}_{\bigcap_{i\in[1,n]} \ker d_i^{X,n}}} & & \Big\downarrow {\scriptstyle \alpha_n} \\
\bigcap\limits_{i\in[1,n]} \ker d_i^{Y,n} & \xhookrightarrow[\mu_{Y,n}]{} & Y_n
\end{array}
$$

□

Proposition 64. The functor $\mathcal{N}$ is left adjoint to the functor $\mathcal{M}$.

Moreover $\nu := \mathrm{id}_{\mathrm{id}_{\mathrm{C}(\mathcal{A})_{\geqslant 0}}} : \mathrm{id}_{\mathrm{C}(\mathcal{A})_{\geqslant 0}} \to \mathcal{M} \circ \mathcal{N}$ is a unit and $\mu : \mathcal{N} \circ \mathcal{M} \to \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{A})}$ is a counit of this adjunction.

Proof. We have to show commutativity of the following diagram.

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\mathcal{N}\nu} & \mathcal{N} \circ \mathcal{M} \circ \mathcal{N} \\
& {\scriptstyle \mathrm{id}_{\mathcal{N}}} \searrow & \Big\downarrow {\scriptstyle \mu\mathcal{N}} \\
& & \mathcal{N}
\end{array}
$$

This means we have to show commutativity of the diagram

$$
\begin{array}{ccc}
(\mathcal{N}C)_n & \xrightarrow{(\mathcal{N}\,\mathrm{id}_C)_n} & ((\mathcal{N} \circ \mathcal{M} \circ \mathcal{N})C)_n \\
& {\scriptstyle \mathrm{id}_{(\mathcal{N}C)_n}} \searrow & \Big\downarrow {\scriptstyle \mu_{\mathcal{N}C,n}} \\
& & (\mathcal{N}C)_n
\end{array}
$$

for $C \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})_{\geqslant 0}$ and $n \geqslant 0$.

But $\mu_{\mathcal{N}C,n}$ is the inclusion map $((\mathcal{N} \circ \mathcal{M} \circ \mathcal{N})C)_n \hookrightarrow (\mathcal{N}C)_n$ and according to Remark 62 we have $((\mathcal{N} \circ \mathcal{M} \circ \mathcal{N})C)_n = (\mathcal{N}C)_n$. So $\mu_{\mathcal{N}C,n} = \mathrm{id}_{(\mathcal{N}C)_n}$.

So we have $(\mathcal{N}\,\mathrm{id}_C)_n \cdot \mu_{\mathcal{N}C,n} = \mathrm{id}_{(\mathcal{N}C)_n} \cdot \mathrm{id}_{(\mathcal{N}C)_n} = \mathrm{id}_{(\mathcal{N}C)_n}$.

We also have to show commutativity of the following diagram.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\nu\mathcal{M}} & \mathcal{M}\circ\mathcal{N}\circ\mathcal{M} \\ & \searrow^{\mathrm{id}_{\mathcal{M}}} & \downarrow{\scriptstyle \mathcal{M}\mu} \\ & & \mathcal{M} \end{array}$$

This means we have to show commutativity of the diagram

$$\begin{array}{ccc} (\mathcal{M}X)_n & \xrightarrow{\mathrm{id}_{(\mathcal{M}X)_n}} & ((\mathcal{M}\circ\mathcal{N}\circ\mathcal{M})X)_n \\ & \searrow^{\mathrm{id}_{(\mathcal{M}X)_n}} & \downarrow{\scriptstyle (\mathcal{M}\mu_X)_n} \\ & & (\mathcal{M}X)_n \end{array}$$

for $X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathcal{A})$ and $n \geqslant 0$.

But we have $(\mathcal{M}\mu_X)_n = \mu_{X,n}|^{(\mathcal{M}X)_n}_{((\mathcal{M}\circ\mathcal{N}\circ\mathcal{M})X)_n} \overset{\text{Rm 62}}{=} \mu_{X,n}|^{(\mathcal{M}X)_n}_{(\mathcal{M}X)_n} = \mathrm{id}_{(\mathcal{M}X)_n}$.

So we have $\mathrm{id}_{(\mathcal{M}X)_n} \cdot \mathcal{M}(\mu_X)_n = \mathrm{id}_{(\mathcal{M}X)_n} \cdot \mathrm{id}_{(\mathcal{M}X)_n} = \mathrm{id}_{(\mathcal{M}X)_n}$.

□

5.2 From simplicial modules to $\geqslant 0$-complexes and back

Recall the functors $\mathcal{F}_\mathcal{A} : \operatorname{SemiSimp}(\mathcal{A}) \to \operatorname{Simp}(\mathcal{A})$ and $\mathcal{V}_\mathcal{A} : \operatorname{Simp}(\mathcal{A}) \to \operatorname{SemiSimp}(\mathcal{A})$, cf. [**1,** Lemma 60, Definition 48] . We write $\mathcal{F} := \mathcal{F}_\mathcal{A}$ and $\mathcal{V} := \mathcal{V}_\mathcal{A}$.

For $X \in \operatorname{Ob}(\operatorname{SemiSimp}(\mathcal{A}))$ we have $(\mathcal{F}X)_n = \bigoplus_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k\in[0,n]}} X_k$.

For $n \geqslant 0$ and $k \in [0, n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$ we have the inclusion morphisms

$$\iota_{X,f} : X_k \to \bigoplus_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n]}} X_\ell \,.$$

For $f = \mathrm{id}_{[n]}$ we denote $\iota_{X,n} := \iota_{X,\mathrm{id}_{[n]}}$.

For $n \geqslant 0$ and $k \in [0, n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$ we also have the projection morphisms

$$\pi_{X,f} : \bigoplus_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n]}} X_\ell \to X_k \,.$$

Then $\mathrm{id}_{(\mathcal{F}X)_n} = \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k\in[0,n]}} \pi_{X,f} \cdot \iota_{X,f}$ and $\mathrm{id}_{X_k} = \iota_{X,f} \cdot \pi_{X,f}$ for $k \in [0, n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$.

Furthermore $0 = \iota_{X,f} \cdot \pi_{X,g}$ for $k, \ell \in [0, n]$ and $(f : [n] \to [k]), (g : [n] \to [\ell]) \in \mathrm{surj}$ such that $f \neq g$.

Definition 65. We have the transformation $\iota := ((\iota_{X,n})_{n\geqslant 0})_{X\in\operatorname{Ob}(\operatorname{SemiSimp}(\mathcal{A}))} : \mathrm{id}_{\operatorname{SemiSimp}(\mathcal{A})} \to \mathcal{V}\circ\mathcal{F}$; cf. [**1,** Remark 62].

For $X \in \operatorname{Ob}(\operatorname{Simp}(\mathcal{A}))$ and $n \geqslant 0$ we have, by the universal property of the coproduct, a unique morphism $\eta_{X,n} : ((\mathcal{F}\circ\mathcal{V})X)_n \to X_n$ satisfying $\iota_{\mathcal{V}X,f} \cdot \eta_{X,n} = X_f$ for $k \in [0, n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$.

We have the transformation $\eta := ((\eta_{X,n})_{n\geqslant 0})_{X\in\operatorname{Ob}(\operatorname{Simp}(\mathcal{A}))} : \mathcal{F}\circ\mathcal{V} \to \mathrm{id}_{\operatorname{Simp}(\mathcal{A})}$; cf. [**1,** Remark 64].

We have $\mathcal{F} \dashv \mathcal{V}$ and ι is a unit and η is a counit of this adjunction; cf. [**1,** Proposition 65].

Recall that $\mathcal{N} \dashv \mathcal{M}$ and that the transformation $\nu = \mathrm{id}_{\mathrm{id}_{\mathrm{C}(\mathcal{A})_{\geqslant 0}}}$ is a unit and $\mu : \mathcal{N} \circ \mathcal{M} \to \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{A})}$ is a counit of this adjunction; cf. Lemma 63, Proposition 64.

It follows that $\mathcal{F} \circ \mathcal{N} \dashv \mathcal{M} \circ \mathcal{V}$ and that

$$\tau := \nu \cdot (\mathcal{M}\iota\mathcal{N}) : \mathrm{id}_{\mathrm{C}(\mathcal{A})_{\geqslant 0}} \to (\mathcal{M} \circ \mathcal{V}) \circ (\mathcal{F} \circ \mathcal{N})$$

is a unit and that

$$\epsilon := (\mathcal{F}\mu\mathcal{V}) \cdot \eta : (\mathcal{F} \circ \mathcal{N}) \circ (\mathcal{M} \circ \mathcal{V}) \to \mathrm{id}_{\mathrm{Simp}(\mathcal{A})}$$

is a counit of this adjunction.

Our aim in this section is to show that $\mathcal{F} \circ \mathcal{N}$ and $\mathcal{M} \circ \mathcal{V}$ are in fact equivalences of categories. So we have to show that τ and ϵ are isotransformations.

Lemma 66. The transformation $\tau : \mathrm{id}_{\mathrm{C}(\mathcal{A})_{\geqslant 0}} \to (\mathcal{M} \circ \mathcal{V}) \circ (\mathcal{F} \circ \mathcal{N})$ is an isotransformation.

Proof. We show that for $C \in \mathrm{Ob}(\mathrm{C}(\mathcal{A})_{\geqslant 0})$ and $n \geqslant 0$ the R-linear map $\tau_{C,n} := (\tau_C)_n$ is an isomorphism of R-modules.

For $n = 0$, we have

$$\tau_{C,0} = \nu_{C,0} \cdot (\mathcal{M}\iota_{\mathcal{N}C})_0 = \iota_{\mathcal{N}C,0} = \iota_{\mathcal{N}C,\mathrm{id}_{[0]}} : (\mathcal{N}C)_0 \to \bigoplus_{\substack{(g:[0]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,0]}} (\mathcal{N}C)_\ell \,.$$

So $\tau_{C,0} = \mathrm{id}_{C_0}$.

For $n \geqslant 1$, we have

$$((\mathcal{M} \circ \mathcal{V} \circ \mathcal{F} \circ \mathcal{N})C)_n = \bigcap_{i\in[1,n]} \ker d_i^{(\mathcal{F}\circ\mathcal{N})C,n} \subseteq \bigoplus_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n]}} (\mathcal{N}C)_\ell$$

and thus

$$\tau_{C,n} = \nu_{C,n} \cdot (\mathcal{M}\iota_{\mathcal{N}C,n}) = \iota_{\mathcal{N}C,n}\Big|_{\bigcap_{i\in[1,n]} \ker d_i^{\mathcal{N}C,n}}^{\bigcap_{i\in[1,n]} \ker d_i^{(\mathcal{V}\circ\mathcal{F}\circ\mathcal{N})C,n}} = \iota_{\mathcal{N}C,n}\Big|^{\bigcap_{i\in[1,n]} \ker d_i^{(\mathcal{F}\circ\mathcal{N})C,n}} .$$

Ad injectivity. The map $\iota_{\mathcal{N}C,n}$ is injective, since it is the inclusion map of a direct summand. Hence any restriction of it is injective.

Ad surjectivity. Let $x \in ((\mathcal{M} \circ \mathcal{V} \circ \mathcal{F} \circ \mathcal{N})C)_n = \bigcap_{i\in[1,n]} \ker d_i^{(\mathcal{F}\circ\mathcal{N})C,n}$.

We have $x = x\,\mathrm{id}_{((\mathcal{F}\circ\mathcal{N})C)_n} = \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k\in[0,n]}} x(\pi_{\mathcal{N}C,f} \cdot \iota_{\mathcal{N}C,f})$.

So in order to show that $x \in \mathrm{Im}\,\iota_{\mathcal{N}C,n} = \mathrm{Im}\,\iota_{\mathcal{N}C,\mathrm{id}_{[n]}}$, it suffices to show that $x\pi_{\mathcal{N}C,f} \overset{!}{=} 0$ for $k \in [0,n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$ such that $f \neq \mathrm{id}_{[n]}$.

For $f : [n] \to [k] \in \mathrm{surj}$ such that $f \neq \mathrm{id}_{[n]}$ the map f is not injective. Hence we may define $e_f := \max\{i \in [0, n-1] : if = (i+1)f\}$.

Let $k \in [0,n]$ and $f : [n] \to [k] \in \mathrm{surj}$ such that $f \neq \mathrm{id}_{[n]}$.

We now prove $x\pi_{\mathcal{N}C,f} \overset{!}{=} 0$ by induction over $n - 1 - e_f$.

We assume by induction hypothesis that for $\ell \in [0, n]$ and $g : [n] \to [\ell]$ in surj such that $g \neq \{\mathrm{id}_{[n]}\}$ and such that $n - 1 - e_g < n - 1 - e_f$, we have $x\pi_{\mathcal{N}C,g} = 0$.

We now consider the following equation.

$$0 = xd^{(\mathcal{F}\circ\mathcal{N})C,n}_{e_f+1} = \sum_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n]}} x(\pi_{\mathcal{N}C,g} \cdot \iota_{\mathcal{N}C,g} \cdot d^{(\mathcal{F}\circ\mathcal{N})C,n}_{e_f+1})$$

Recall that for $\ell \in [0, n]$ and $g : [n] \to [\ell]$ in surj and $i \in [0, n]$, we have

$$\iota_{\mathcal{N}C,g} \cdot d^{(\mathcal{F}\circ\mathcal{N})C,n}_{i} = (\mathcal{N}C)_{(\partial^n_i \cdot g)^\bullet} \cdot \iota_{\mathcal{N}C,\overline{\partial^n_i \cdot g}}\,, \text{ cf. } [\mathbf{1}, \text{ Lemma 58}],$$

where we regard $\mathcal{N}C$ as a functor $\mathcal{N}C : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{A}$.

For $g = \mathrm{id}_{[n]}$ we get $\iota_{\mathcal{N}C,g} \cdot d^{(\mathcal{F}\circ\mathcal{N})C,n}_{e_f+1} = d^{\mathcal{N}C,n}_{e_f+1} \cdot \iota_{\mathcal{N}C,\mathrm{id}_{[n-1]}} = 0$ and hence

$$0 = \sum_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n-1]}} x(\pi_{\mathcal{N}C,g} \cdot \iota_{\mathcal{N}C,g} \cdot d^{(\mathcal{F}\circ\mathcal{N})C,n}_{e_f+1})\,.$$

The induction hypothesis leads further to

$$0 = \sum_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n-1] \\ e_g\leqslant e_f}} x(\pi_{\mathcal{N}C,g} \cdot \iota_{\mathcal{N}C,g} \cdot d^{(\mathcal{F}\circ\mathcal{N})C,n}_{e_f+1})\,.$$

We have $\iota_{\mathcal{N}C,g} \cdot d^{(\mathcal{F}\circ\mathcal{N})C,n}_{e_f+1} = (\mathcal{N}C)_{(\partial^n_{e_f+1} \cdot g)^\bullet} \cdot \iota_{\mathcal{N}C,\overline{\partial^n_{e_f+1} \cdot g}}\,.$

Now for $\ell \in [0, n-1]$ and $(g : [n] \to [\ell]) \in$ surj such that $e_g \leqslant e_f$ there are two cases.

Case 1. We have $e_g = e_f$. Then $\partial^n_{e_f+1} \cdot g$ is surjective. We conclude that $(\partial^n_{e_f+1} \cdot g)^\bullet = \mathrm{id}_{[\ell]}$ and $(\mathcal{N}C)_{(\partial^n_{e_f+1} \cdot g)^\bullet} = \mathrm{id}_{C_\ell}$.

Case 2. We have $e_g < e_f$. Then $\partial^n_{e_f+1} \cdot g$ is not surjective, since $i := (e_f + 1)g$ is not in the image of $\partial^n_{e_f+1} \cdot g$, since otherwise either $(e_f)g = (e_f + 1)g$ or $(e_f + 1)g = (e_f + 2)g$, both contradicting the maximality of e_g . So $(\partial^n_{e_f+1} \cdot g)^\bullet = \partial^\ell_i$. We have $i > 0$ since otherwise $0g = 0(\partial^n_{e_f+1} \cdot g) = 0\partial^{ell}_i = 0\partial^\ell_0 > 0$ and g would not be surjective. We conclude that $(\mathcal{N}C)_{(\partial^n_{e_f+1} \cdot g)^\bullet} = d^{\mathcal{N}C,\ell}_i = 0$ by construction of $\mathcal{N}C$.

So we further reduce to

$$0 = \sum_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n-1] \\ e_g\leqslant e_f}} x(\pi_{\mathcal{N}C,g} \cdot \iota_{\mathcal{N}C,g} \cdot d^{(\mathcal{F}\circ\mathcal{N})C,n}_{e_f+1}) = \sum_{\substack{(g:[n]\to[\ell])\in\mathrm{surj} \\ \text{for some } \ell\in[0,n-1] \\ e_g\leqslant e_f}} x(\pi_{\mathcal{N}C,g} \cdot \iota_{\mathcal{N}C,\overline{\partial^n_{e_f+1} \cdot g}})\,.$$

For $g \in$ surj starting from $[n]$ and such that $g \neq \mathrm{id}_{[n]}$ and $e_g = e_f$ we have

$$g \neq f \quad\Rightarrow\quad \overline{\partial^n_{e_f+1} \cdot g} \neq \overline{\partial^n_{e_f+1} \cdot f}\quad .$$

Indeed, if $\overline{\partial^n_{e_f+1} \cdot g} = \overline{\partial^n_{e_f+1} \cdot f}$ we get $\partial^n_{e_f+1} \cdot g = \overline{\partial^n_{e_f+1} \cdot g} = \overline{\partial^n_{e_f+1} \cdot f} = \partial^n_n \cdot f$, so $g|_{[n]\setminus\{e_f+1\}} = f|_{[n-1]\setminus\{e_f+1\}}$, which together with $(e_f + 1)g = (e_f)g = (e_f)f = (e_f + 1)f$ implies $g = f$.

Applying $\pi_{\mathcal{N}C,\overline{\partial^n_{e_f+1}\cdot f}}$, we can conclude that

$$0 = 0\pi_{\mathcal{N}C,\overline{\partial^n_{e_f+1}\cdot f}} = \sum_{\substack{(g:[n]\to[\ell])\in\text{surj} \\ \text{for some } \ell\in[0,n-1] \\ e_g=e_f}} x(\pi_{\mathcal{N}C,g}\cdot\iota_{\mathcal{N}C,\overline{\partial^n_{e_f+1}\cdot g}}\cdot\pi_{\mathcal{N}C,\overline{\partial^n_{e_f+1}\cdot f}}) = x\pi_{\mathcal{N}C,f}\,.$$

□

For the proof that ϵ is an isotransformation we follow an approach I have learned from Marc Stephan [**4,** I Prop. 3.21, p. 29 and II Cor. 3.4, p. 51]. As a preperation, we need the folllowing Lemma.

Lemma 67. The functor $\mathcal{M}\circ\mathcal{V}$ reflects isomorphisms. I.e. given $(f : X \to Y) \in \mathrm{Mor}(\mathrm{Simp}(\mathcal{A}))$ such that $(\mathcal{M}\circ\mathcal{V})f$ is an isomorphism, it follows that f is an isomorphism.

Proof. Suppose given $(f : X \to Y) \in \mathrm{Mor}(\mathrm{Simp}(\mathcal{A}))$ such that $(\mathcal{M}\circ\mathcal{V})f$ is an isomorphism. This means that for $n \geqslant 0$ the R-linear map $((\mathcal{M}\circ\mathcal{V})f)_n : ((\mathcal{M}\circ\mathcal{V})X)_n \to ((\mathcal{M}\circ\mathcal{V})Y)_n$ is bijective. It suffices to show that the R-linear map $f_n = (\mathcal{V}f)_n$ is bijective for $n \geqslant 0$.

We introduce the following notation. Suppose given $a \in [0, n]$. For $b \in [1, a+1]$ we define

$$X_{a,b} := \bigcap_{i\in[b,a]} \ker d_i^{X,a} \subseteq X_a\,.$$

In particular, $X_{a,a+1} := X_a$.

For $b \in [1, a]$, we have the inclusion map

$$e_{X,a,b} : \begin{array}{ccc} X_{a,b} & \to & X_{a,b+1} \\ x & \mapsto & x \end{array}.$$

For $a \in [0, n]$ and $b \in [1, a]$ and $x \in X_{a,b+1}$ and $i \in [b, a-1]$ we have

$$x(d_b^{X,a}\cdot d_i^{X,a-1}) = x(d_{i+1}^{X,a}\cdot d_b^{X,a-1}) = 0d_b^{X,a-1} = 0\,.$$

Hence there exist the maps $d_b^{X,a}|_{X_{a,b+1}}^{X_{a-1,b}}$.

For $a \in [0, n]$ and $b \in [1, a]$ we have the following short exact sequence.

$$0 \longrightarrow X_{a,b} \xrightarrow{e_{X,a,b}} X_{a,b+1} \xrightarrow{d_b^{X,a}|_{X_{a,b+1}}^{X_{a-1,b}}} X_{a-1,b} \longrightarrow 0$$

At $X_{a,b}$ it is exact since the inclusion map $e_{X,a,b}$ is injective. At $X_{a,b+1}$ it is exact since for $x \in X_{a,b+1}$ we have

$$xd_b^{X,a} = 0 \Leftrightarrow x \in X_{a,b}\,.$$

For $x \in X_{a-1,b}$ and $i \in [b+1, a]$ we have

$$x(s_{b-1}^{X,a-1}\cdot d_i^{X,a}) = x(d_{i-1}^{X,a-1}\cdot s_{b-1}^{X,a-2}) = 0s_{b-1}^{X,a-2} = 0$$

and so $xs_{b-1}^{X,a-1} \in X_{a,b+1}$. Since $x(s_{b-1}^{X,a-1}\cdot d_b^{X,a}) = x\,\mathrm{id}_{X_{a-1}} = x$, it follows that $d_b^{X,a}|_{X_{a,b+1}}^{X_{a-1,b}}$ is surjective and hence we have exactness at $X_{a-1,b}$.

If $x \in X_{a,b}$ we have $x(f_a \cdot d_i^{Y,a}) = x(d_i^{X,a} \cdot f_{a-1}) = 0f_{a-1} = 0$ for $i \in [b,a]$. Hence we may define $f_{a,b} := f_a|_{X_{a,b}}^{Y_{a,b}}$.

We have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_{a,b} & \xrightarrow{e_{X,a,b}} & X_{a,b+1} & \xrightarrow{d_b^{X,a}|_{X_{a,b+1}}^{X_{a-1,b}}} & X_{a-1,b} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{a,b}} & & \downarrow{\scriptstyle f_{a,b+1}} & & \downarrow{\scriptstyle f_{a-1,b}} & & \\
0 & \longrightarrow & Y_{a,b} & \xrightarrow[e_{Y,a,b}]{} & Y_{a,b+1} & \xrightarrow[d_b^{Y,a}|_{Y_{a,b+1}}^{Y_{a-1,b}}]{} & Y_{a-1,b} & \longrightarrow & 0
\end{array}$$

So if $f_{a,b}$ and $f_{a-1,b}$ are bijective, then $f_{a,b+1}$ is bijective.

We prove that $f_{a,b}$ is bijective for $a \in [0,n]$ and $b \in [1, a+1]$ by induction on b.

Induction base. Note that $f_{a,1} = (\mathcal{M} \circ \mathcal{V} f)_a$ for $a \in [0,n]$, which is bijective.

Induction step. Suppose that $b \geqslant 2$. Since $f_{a,b-1}$ and $f_{a-1,b-1}$ are bijective, $f_{a,b}$ is bijective for $a \in [0,n]$.

So in particular $f_{n,n+1} = f_n$ is bijective. □

Lemma 68. The transformation $\epsilon : (\mathcal{F} \circ \mathcal{N}) \circ (\mathcal{M} \circ \mathcal{V}) \to \mathrm{id}_{\mathrm{Simp}(\mathcal{A})}$ is an isotransformation.

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{A}))$. We have $\tau_{(\mathcal{M}\circ\mathcal{V})X} \cdot (\mathcal{M} \circ \mathcal{V})\epsilon_X = \mathrm{id}_{(\mathcal{M}\circ\mathcal{V})X}$ by adjunction. By Lemma 66 we know that $\tau_{(\mathcal{M}\circ\mathcal{V})X}$ is an isomorphism. Hence $(\mathcal{M} \circ \mathcal{V})\epsilon_X = (\tau_{(\mathcal{M}\circ\mathcal{V})X})^{-1}$ is an isomorphism. Hence by Lemma 67 we have that ϵ_X is an isomorphism. □

So we already proved the first part of Dold-Puppe-Kan, i.e. of Proposition 72 below. Now we treat the question of what the inverse transformation of ϵ explicitly is.

Definition 69. Let $X \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{A}))$ and $n \geqslant 0$.

Suppose given $k \in [0,n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$. Suppose given a subset $A \subseteq [1,k]$.

We define a monotone map

$$\begin{array}{rccl}
g_{f,A} : & [k] & \to & [n] \\
 & i & \mapsto & \begin{cases} \min f^{-1}(i) & \text{if } i \notin A \\ \big(\min f^{-1}(i)\big) - 1 & \text{if } i \in A \end{cases}
\end{array}$$

Note that $(0)g_{f,A} = 0$, since $0 \notin A$. Note that for $0 \leqslant i < j \leqslant k$ we have $\min f^{-1}(i) < \min f^{-1}(j)$, so $(i)g_{f,A} \leqslant (j)g_{f,A}$.

We define an R-linear map $\tilde{\delta}_{X,f} : X_n \to X_k$ by

$$\tilde{\delta}_{X,f} := \sum_{A \subseteq [1,k]} (-1)^{|A|} X_{g_{f,A}} .$$

Suppose given $i \in [1,k]$.

Note that for $A \subseteq [1,k]$ such that $i \notin A$, we have $\partial_i^k \cdot g_{f,A \dot\cup \{i\}} = \partial_i^k \cdot g_{f,A}$.

Thus we have

$$\begin{aligned}\tilde{\delta}_{X,f} \cdot d_i^{X,k} &= \Big(\sum_{A\subseteq[1,k]} (-1)^{|A|} X_{g_{f,A}}\Big) \cdot X_{\partial_i^k} = \sum_{A\subseteq[1,k]} (-1)^{|A|} X_{\partial_i^k \cdot g_{f,A}} \\ &= \sum_{\substack{A\subseteq[1,k]\\ i\notin A}} (-1)^{|A|} X_{\partial_i^k \cdot g_{f,A}} + \sum_{\substack{A\subseteq[1,k]\\ i\in A}} (-1)^{|A|} X_{\partial_i^k \cdot g_{f,A}} \\ &= \sum_{\substack{A\subseteq[1,k]\\ i\notin A}} (-1)^{|A|} X_{\partial_i^k \cdot g_{f,A}} + \sum_{\substack{A\subseteq[1,k]\\ i\notin A}} (-1)^{|A\dot\cup\{i\}|} X_{\partial_i^k \cdot g_{f,A\dot\cup\{i\}}} \\ &= \sum_{\substack{A\subseteq[1,k]\\ i\notin A}} (-1)^{|A|} X_{\partial_i^k \cdot g_{f,A}} - \sum_{\substack{A\subseteq[1,k]\\ i\notin A}} (-1)^{|A|} X_{\partial_i^k \cdot g_{f,A}} = 0\,.\end{aligned}$$

So $\operatorname{Im}\tilde{\delta}_{X,f} \subseteq \bigcap_{i\in[1,k]} \ker d_i^{X,k}$, hence we may define

$$\delta_{X,f} := \tilde{\delta}_{X,f}|^{\bigcap_{i\in[1,k]} \ker d_i^{X,k}}\,.$$

So $\delta_{X,f} \cdot \mu_{\mathcal{V}X,k} = \tilde{\delta}_{X,f}$. In particular, if $k = 0$ then $\delta_{X,f} = \tilde{\delta}_{X,f}$.

We define the R-linear map $\delta_{X,n} : X_n \to ((\mathcal{F}\circ\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X)_n$ by requiring

$$\delta_{X,n} \cdot \pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} = \delta_{X,f}$$

for $k \in [0,n]$ and $(f : [n] \to [k]) \in \text{surj}$ using the universal property of the product.

Lemma 70. For $X \in \text{Ob}(\text{Simp}(\mathcal{A}))$ and $n \geqslant 0$, we have $(\epsilon_X)_n \cdot \delta_{X,n} = \text{id}_{((\mathcal{F}\circ\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X)_n}$.

Proof. Recall that $((\mathcal{F}\circ\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X)_n = \bigoplus_{\substack{(f:[n]\to[k])\in\text{surj}\\ \text{for some } k\in[0,n]}} \bigcap_{i\in[1,k]} \ker d_i^{X,k}$.

Recall that

$$\bigoplus_{\substack{(f:[n]\to[k])\in\text{surj}\\ \text{for some } k\in[0,n]}} \bigcap_{i\in[1,k]} \ker d_i^{X,k} \xrightarrow{(\epsilon_X)_n} X_n \xrightarrow{\delta_{X,n}} \bigoplus_{\substack{(f:[n]\to[k])\in\text{surj}\\ \text{for some } k\in[0,n]}} \bigcap_{i\in[1,k]} \ker d_i^{X,k}\,.$$

To show that $(\epsilon_X)_n \cdot \delta_{X,n} = \text{id}_{((\mathcal{F}\circ\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X)_n}$ we have to show that for $k, \tilde{k} \in [0,n]$, $(f : [n] \to [k]) \in \text{surj}$ and $(\tilde{f} : [n] \to [\tilde{k}]) \in \text{surj}$ we have

$$\iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \cdot (\epsilon_X)_n \cdot \delta_{X,n} \cdot \pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,\tilde{f}} \overset{!}{=} \begin{cases} \text{id}_{((\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X)_k} & \text{if } f = \tilde{f} \\ 0 & \text{if } f \neq \tilde{f}\,. \end{cases}$$

Hence by the universal properties of the product and the coproduct it follows that $(\epsilon_X)_n \cdot \delta_{X,n} = \text{id}_{((\mathcal{F}\circ\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X)_n}$.

We observe that for $k \in [0,n]$ and $(f : [n] \to [k]) \in \text{surj}$ we have, using the construction of $\mathcal{F}$ from [**1,** Lemmas 59, 60],

$$\iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \cdot (\epsilon_X)_n = \iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \cdot \mathcal{F}(\mu_{\mathcal{V}X})_n \cdot \eta_{X,n} = \mu_{\mathcal{V}X,k} \cdot \iota_{\mathcal{V}X,f} \cdot \eta_{X,n} = \mu_{\mathcal{V}X,k} \cdot X_f\,.$$

We have $\mu_{\mathcal{V}X,k} \cdot X_f = X_f|_{\bigcap_{i\in[1,k]} \ker d_i^{X,k}}$; cf. Lemma 63.

So we have

$$\iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f}\cdot(\epsilon_X)_n\cdot\delta_{X,n}\cdot\pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,\tilde f} = X_f|_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}\cdot\tilde\delta_{X,\tilde f}|^{\bigcap\limits_{i\in[1,\tilde k]}\ker d_i^{X,\tilde k}}$$
$$= \Big(\sum_{A\subseteq[1,\tilde k]}(-1)^{|A|}X_f\cdot X_{g_{\tilde f,A}}\Big)|^{\bigcap\limits_{i\in[1,\tilde k]}\ker d_i^{X,\tilde k}}_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}} = \Big(\sum_{A\subseteq[1,\tilde k]}(-1)^{|A|}X_{g_{\tilde f,A}\cdot f}\Big)|^{\bigcap\limits_{i\in[1,\tilde k]}\ker d_i^{X,\tilde k}}_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}\,.$$

Case $f=\tilde f$. In particular, $k=\tilde k$.

For $\emptyset\neq A\subseteq[1,k]$ and $j=\max A$ we have $j(g_{f,A}\cdot f)=((\min f^{-1}(j))-1)f=j-1$ and $i(g_{f,A}\cdot f)=(\min f^{-1}(i))f=i$ for $i\in[j+1,k]$. So $j\notin\operatorname{Im}(g_{f,A}\cdot f)$. Hence there exists a monotone map $u:[k]\to[k-1]$ such that $g_{f,A}\cdot f=u\cdot\partial_j^k$.

For $x\in\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}$ we have

$$xX_{g_{f,A}\cdot f}=xX_{u\cdot\partial_j^k}=x(d_j^{X,k}\cdot X_u)=0X_u=0\,.$$

So $X_{g_{f,A}\cdot f}|_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}=0$.

We further have $i(g_{f,\emptyset}\cdot f)=(\min f^{-1}(i))f=i$ for $i\in[0,k]$. So $X_{g_{f,\emptyset}\cdot f}=X_{\mathrm{id}_{[k]}}=\mathrm{id}_{X_k}$.

Combining these results we get

$$\iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f}\cdot(\epsilon_X)_n\cdot\delta_{X,n}\cdot\pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,\tilde f} = \sum_{A\subseteq[1,k]}(-1)^{|A|}X_{g_{f,A}\cdot f}|^{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}$$
$$= \mathrm{id}_{X_k}|^{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}} = \mathrm{id}_{((\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X)_k}\,.$$

Case $\tilde k<k$.

Suppose given $A\subseteq[1,\tilde k]$. The map $g_{\tilde f,A}\cdot f:[\tilde k]\to[k]$ is not surjective, since $\tilde k<k$. We have $0(g_{\tilde f,A}\cdot f)=0$. Thus there exists $j\in[1,k]$ such that $j\notin\operatorname{Im}g_{\tilde f,A}\cdot f$. Hence there exists a monotone map $u:[\tilde k]\to[k-1]$ such that $g_{\tilde f,A}\cdot f=u\cdot\partial_j^k$.

For $x\in\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}$ we have

$$xX_{g_{\tilde f,A}\cdot f}=xX_{u\cdot\partial_j^k}=x(d_j^{X,k}\cdot X_u)=0X_u=0\,.$$

So $X_{g_{\tilde f,A}\cdot f}|_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}=0$ for $A\subseteq[1,\tilde k]$.

So we have

$$\iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f}\cdot(\epsilon_X)_n\cdot\delta_{X,n}\cdot\pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,\tilde f} = \Big(\sum_{A\subseteq[1,\tilde k]}(-1)^{|A|}X_{g_{\tilde f,A}\cdot f}\Big)|^{\bigcap\limits_{i\in[1,\tilde k]}\ker d_i^{X,\tilde k}}_{\bigcap\limits_{i\in[1,k]}\ker d_i^{X,k}}=0\,.$$

Case $\tilde k\geqslant k$ and $\tilde f\neq f$.

Note that for $h : [n] \to [\ell]$ surjective and monotone, we have $\ell = |\operatorname{Im} g_{\tilde{f},\emptyset}| - 1$ and $(i)h = |\operatorname{Im} g_{h,\emptyset} \cap [1, i]|$ for $i \in [0, n]$. Hence h is determined by $\operatorname{Im} g_{h,\emptyset}$.

We *assume* that $\operatorname{Im} g_{\tilde{f},\emptyset} \subseteq \operatorname{Im} g_{f,\emptyset}$. So $\tilde{k} = |\operatorname{Im} g_{\tilde{f},\emptyset}| - 1 \leqslant |\operatorname{Im} g_{f,\emptyset}| - 1 = k \leqslant \tilde{k}$. Thus $|\operatorname{Im} g_{\tilde{f},\emptyset}| = |\operatorname{Im} g_{f,\emptyset}|$. Hence $f = \tilde{f}$, which *contradicts* the assumption.

Hence there exists $j \in \operatorname{Im} g_{\tilde{f},\emptyset}$ such that $j \notin \operatorname{Im} g_{f,\emptyset}$. We have $j > 0$ and $(j)f = (j-1)f$, since otherwise $j = \min f^{-1}(jf) = j(f \cdot g_{f,\emptyset}) \in \operatorname{Im} g_{f,\emptyset}$.

Let $e \in [1, \tilde{k}]$ be the unique element in $g^{-1}_{\tilde{f},\emptyset}(j)$.

Suppose given $A \subseteq [1, \tilde{k}]$ such that $e \notin A$. Then $e(g_{\tilde{f},A} \cdot f) = (j)f = (j-1)f = e(g_{\tilde{f},A \dot\cup \{e\}} \cdot f)$. For $i \in [0, \tilde{k}] \setminus \{e\}$ we have $(i)g_{\tilde{f},A} = (i)g_{\tilde{f},A\dot\cup\{e\}}$. Hence we have $g_{\tilde{f},A} \cdot f = g_{\tilde{f},A\dot\cup\{e\}} \cdot f$.

So we get

$$
\begin{aligned}
\iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \cdot (\epsilon_X)_n \cdot \delta_{X,n} \cdot \pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,\tilde{f}} &= \Big(\sum_{A \subseteq [1,\tilde{k}]} (-1)^{|A|} X_{g_{\tilde{f},A} \cdot f} \Big)\Big|^{\bigcap_{i\in[1,\tilde{k}]} \ker d_i^{X,\tilde{k}}}_{\bigcap_{i\in[1,k]} \ker d_i^{X,k}} \\
&= \Big(\sum_{\substack{A \subseteq [1,\tilde{k}] \\ e \notin A}} (-1)^{|A|} X_{g_{\tilde{f},A} \cdot f} + \sum_{\substack{A \subseteq [1,\tilde{k}] \\ e \in A}} (-1)^{|A|} X_{g_{\tilde{f},A} \cdot f} \Big)\Big|^{\bigcap_{i\in[1,\tilde{k}]} \ker d_i^{X,\tilde{k}}}_{\bigcap_{i\in[1,k]} \ker d_i^{X,k}} \\
&= \Big(\sum_{\substack{A \subseteq [1,\tilde{k}] \\ e \notin A}} (-1)^{|A|} X_{g_{\tilde{f},A} \cdot f} + \sum_{\substack{A \subseteq [1,\tilde{k}] \\ e \notin A}} (-1)^{|A \dot\cup \{e\}|} X_{g_{\tilde{f},A\dot\cup\{e\}} \cdot f} \Big)\Big|^{\bigcap_{i\in[1,\tilde{k}]} \ker d_i^{X,\tilde{k}}}_{\bigcap_{i\in[1,k]} \ker d_i^{X,k}} \\
&= \Big(\sum_{\substack{A \subseteq [1,\tilde{k}] \\ e \notin A}} (-1)^{|A|} X_{g_{\tilde{f},A} \cdot f} - \sum_{\substack{A \subseteq [1,\tilde{k}] \\ e \notin A}} (-1)^{|A|} X_{g_{\tilde{f},A} \cdot f} \Big)\Big|^{\bigcap_{i\in[1,\tilde{k}]} \ker d_i^{X,\tilde{k}}}_{\bigcap_{i\in[1,k]} \ker d_i^{X,k}} = 0 \,.
\end{aligned}
$$

□

Lemma 71. For $X \in \operatorname{Ob}(\operatorname{Simp}(\mathcal{A}))$ the tuple $(\delta_{X,n})_{n \geqslant 0}$ is an isomorphism from X to $(\mathcal{F} \circ \mathcal{N} \circ \mathcal{M} \circ \mathcal{V})X$ in $\operatorname{Simp}(\mathcal{A})$.

The tuple $\delta := ((\delta_{X,n})_{n \geqslant 0})_{X \in \operatorname{Ob}(\operatorname{Simp}(\mathcal{A}))}$, is an isotransfomation from $\operatorname{id}_{\operatorname{Simp}(\mathcal{A})}$ to $(\mathcal{F} \circ \mathcal{N}) \circ (\mathcal{M} \circ \mathcal{V})$.

The transformations ϵ and δ are mutually inverse isotransformations.

Proof. The transformation ϵ is an isotransformation; cf. Lemma 68. Suppose given $X \in \operatorname{Ob}(\operatorname{Simp}(\mathcal{A}))$.

By Lemma 70, we have $(\epsilon_X)_n \cdot \delta_{X,n} = \operatorname{id}_{((\mathcal{F}\circ\mathcal{N}\circ\mathcal{M}\mathcal{V})X)_n}$ for $n \geqslant 0$. Since ϵ_X is an isomorphism in $\operatorname{Simp}(\mathcal{A})$, the tuple $(\delta_{X,n})_{n \geqslant 0}$ is its inverse isomorphism.

Since ϵ is an isotransformation, it follows that ϵ and δ are mutually inverse isotransformations. □

The following proposition originates from [**5,** Theorems 8.1, 8.2], [**6,** Theorem 1.5] and for abelian categories from [**7,** Satz 3.6].

Proposition 72 (Dold-Puppe-Kan)**.**

We have equivalences of categories

$$
\operatorname{Simp}(\mathcal{A}) \underset{\mathcal{F}\circ\mathcal{N}}{\overset{\mathcal{M}\circ\mathcal{V}}{\rightleftarrows}} \mathrm{C}(\mathcal{A})_{\geqslant 0} \qquad (\sim)
$$

mutually inverse up to the isotransformations

$$\tau : \mathrm{id}_{\mathrm{C}(\mathcal{A})_{\geqslant 0}} \to (\mathcal{M} \circ \mathcal{V}) \circ (\mathcal{F} \circ \mathcal{N})$$

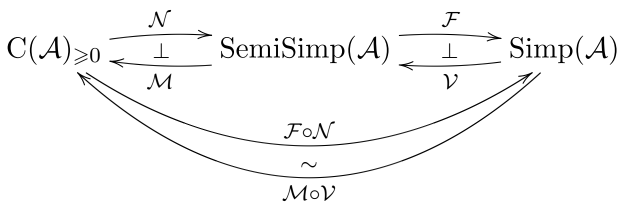
and

$$\epsilon : (\mathcal{F} \circ \mathcal{N}) \circ (\mathcal{M} \circ \mathcal{V}) \to \mathrm{id}_{\mathrm{Simp}(\mathcal{A})} \, .$$

The inverse isotransformation of ϵ is given by δ.

Proof. Cf. Definition 65, Lemma 66, Definition 69 and Lemma 71. □

So altogether we have the following diagram.



Remark 73. We can also show by a direct calculation that $\delta \cdot \epsilon = \mathrm{id}_{\mathrm{id}_{\mathrm{Simp}(\mathcal{A})}}$.

Let $X \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{A}))$ and $n \geqslant 0$. We have to show that $\delta_{X,n} \cdot (\epsilon_X)_n \overset{!}{=} \mathrm{id}_{X_n}$.

We have

$$\begin{aligned}
\delta_{X,n} \cdot (\epsilon_X)_n &= \delta_{X,n} \cdot \Big(\sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k \in [0,n]}} \pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \cdot \iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \Big) \cdot (\epsilon_X)_n \\
&= \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k \in [0,n]}} \delta_{X,n} \cdot \pi_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \cdot \iota_{(\mathcal{N}\circ\mathcal{M}\circ\mathcal{V})X,f} \cdot (\epsilon_X)_n \overset{\text{proof of L 70}}{=} \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k \in [0,n]}} \delta_{X,f} \cdot \mu_{\mathcal{V}X,k} \cdot X_f \\
&= \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k \in [0,n]}} \tilde{\delta}_{X,f} \cdot X_f = \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k \in [0,n]}} \sum_{A\subseteq[1,k]} (-1)^{|A|} X_{g_{f,A}} \cdot X_f \\
&= \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k \in [0,n]}} \sum_{A\subseteq[1,k]} (-1)^{|A|} X_{f \cdot g_{f,A}} \, .
\end{aligned}$$

In order to show that this sum equals id_{X_n}, we show that all summands except for $(-1)^{|\emptyset|} X_{\mathrm{id}_{[n]} \cdot g_{\mathrm{id}_{[n]},\emptyset}} = \mathrm{id}_{X_n}$ cancel out. We therefore aim to define a map through which we can form pairs of summands that cancel out each other.

Suppose given $k \in [0,n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$ and $A \subseteq [1,k]$ such that $(f,A) \neq (\mathrm{id}_{[n]}, \emptyset)$.

We call the map $g_{f,A}$ *ascending* if we have $(i)g_{f,A} \geqslant i$ for $i \in [0,k]$.

For instance, $g_{f,\emptyset}$ is ascending.

Let

$$M_1 := \{(f,A) : (f : [n] \to [k]) \in \mathrm{surj} \text{ for some } k \in [0,n-1], A \subseteq [1,k],\ g_{f,A} \text{ ascending}\} \, .$$

Let

$$M_2 := \{(f,A) : (f : [n] \to [k]) \in \mathrm{surj} \text{ for some } k \in [1,n], A \subseteq [1,k],\ g_{f,A} \text{ not ascending}\} \, .$$

We will see that

$$\{(f,A) : (f : [n] \to [k]) \in \text{surj for some } k \in [1,n], A \subseteq [1,k], (f,A) \neq (\mathrm{id}_{[n]}, \emptyset)\} \stackrel{!}{=} M_1 \,\dot{\cup}\, M_2\,.$$

We aim to construct maps

$$\begin{array}{rlll} \varphi : & M_1 & \to & M_2 \\ & (f,A) & \mapsto & ((f,A)\varphi_1, (f,A)\varphi_2) \end{array}$$

and

$$\begin{array}{rlll} \psi : & M_2 & \to & M_1 \\ & (f,A) & \mapsto & ((f,A)\psi_1, (f,A)\psi_2)\,. \end{array}$$

Suppose given $k \in [0,n]$ and $(f : [n] \to [k]) \in$ surj and $A \subseteq [1,k]$ such that $(f,A) \neq (\mathrm{id}_{[n]}, \emptyset)$.

Suppose that $g_{f,A}$ is ascending.

We *assume* that $f = \mathrm{id}_{[n]}$. Then we have $(i)g_{f,A} \leqslant (i)g_{f,\emptyset} = i \leqslant (i)g_{f,A}$ for $i \in [0,n]$. Hence $g_{f,A} = \mathrm{id}_{[n]}$ and $A = \emptyset$. This is a *contradiction*. Hence $f \neq \mathrm{id}_{[n]}$. In particular, $k \in [0, n-1]$.

So there exists $e_f := \min\{i \in [0,k] : |f^{-1}(i)| > 1\}$. For $i \in [0, e_f]$ we have $(i)f = i$ and $\min f^{-1}(i) = i$. So for $i \in [0, e_f]$ we have $i \notin A$ since otherwise $(i)g_{f,A} = (\min f^{-1}(i)) - 1 = i - 1 < i$. In other words, $[0, e_f] \cap A = \emptyset$.

We define the surjective monotone map

$$\begin{array}{rlll} (f,A)\varphi_1 : & [n] & \to & [k+1] \\ & i & \mapsto & \begin{cases} (i)f = i & \text{if } i \leqslant e_f \\ (i)f + 1 & \text{if } i > e_f\,. \end{cases} \end{array}$$

We define $(f,A)\varphi_2 := \{a+1 : a \in A \,\dot{\cup}\, \{e_f\}\} \subseteq [1, k+1]$.

We show that $f \cdot g_{f,A} \stackrel{!}{=} (f,A)\varphi_1 \cdot g_{(f,A)\varphi_1,(f,A)\varphi_2}$.

Indeed, if $i \in [0, e_f]$, we have

$$i((f,A)\varphi_1 \cdot g_{(f,A)\varphi_1,(f,A)\varphi_2}) = (i)g_{(f,A)\varphi_1,(f,A)\varphi_2} = \min((f,A)\varphi_1)^{-1}(i) = i = (i)g_{f,A} = i(f \cdot g_{f,A})\,.$$

If $i \in [e_f + 1, n]$ and $(i)f = e_f$, we have

$$\begin{aligned} & i((f,A)\varphi_1 \cdot g_{(f,A)\varphi_1,(f,A)\varphi_2}) = (e_f+1)g_{(f,A)\varphi_1,(f,A)\varphi_2} = (\min((f,A)\varphi_1)^{-1}(e_f+1)) - 1 \\ & = e_f + 1 - 1 = e_f = \min f^{-1}(e_f) = i(f \cdot g_{f,A})\,. \end{aligned}$$

If $i \in [e_f + 1, n]$ and $(i)f \in [e_f + 1, k] \cap A$, we have

$$\begin{aligned} & i((f,A)\varphi_1 \cdot g_{(f,A)\varphi_1,(f,A)\varphi_2}) = ((i)f+1)g_{(f,A)\varphi_1,(f,A)\varphi_2} = (\min((f,A)\varphi_1)^{-1}((i)f+1)) - 1 \\ & = \min f^{-1}((i)f) - 1 = i(f \cdot g_{f,A})\,. \end{aligned}$$

If $i \in [e_f + 1, n]$ and $(i)f \in [e_f + 1, k] \setminus A$, we have

$$\begin{aligned} & i((f,A)\varphi_1 \cdot g_{(f,A)\varphi_1,(f,A)\varphi_2}) = ((i)f+1)g_{(f,A)\varphi_1,(f,A)\varphi_2} = \min(((f,A)\varphi_1)^{-1}((i)f+1)) \\ & = \min f^{-1}((i)f) = i(f \cdot g_{f,A})\,. \end{aligned}$$

Note that $(e_f+1)g_{(f,A)\varphi_1,(f,A)\varphi_2} = (\min((f,A)\varphi_1)^{-1}(e_f+1)) - 1 = e_f + 1 - 1 = e_f < e_f + 1$. So $g_{(f,A)\varphi_1,(f,A)\varphi_2}$ is not ascending.

Now suppose that $g_{f,A}$ is not ascending.

Note that $g_{f,\emptyset}$ is ascending. Hence $A \neq \emptyset$ and $k \geqslant 1$. Let $u_A := \min A$.

We show that $(i)g_{f,\emptyset} \overset{!}{=} i$ and $(i)f \overset{!}{=} i$ for $i \in [0, u_A]$.

Assume that there exists $i \in [0, u_A]$ such that $(i)g_{f,\emptyset} > i$. Then for $j \in [i, k]$ we have

$$(j)g_{f,A} \geqslant (j)g_{f,\emptyset} - 1 \geqslant (i)g_{f,\emptyset} - 1 + (j - i) \geqslant i + 1 - 1 + (j - i) = j.$$

Furthermore, since $i \leqslant u_A$ we have for $j \in [0, i-1]$ that $(j)g_{f,A} = (j)g_{f,\emptyset} \geqslant j$. Hence $g_{f,A}$ is ascending, which is a *contradiction*. Hence $(i)g_{f,\emptyset} = i$ for $i \in [0, u_A]$.

In particular, we have $(u_A)g_{f,A} = (u_A)g_{f,\emptyset} - 1 = u_A - 1$. For $i \in [0, u_A - 1]$ we have $(i)g_{f,A} = (i)g_{f,\emptyset} = i$. In particular, $(u_A - 1)g_{f,A} = u_A - 1$.

We have $\min f^{-1}(u_A) = (u_A)g_{f,\emptyset} = u_A$. Hence $u_A \in f^{-1}(u_A)$ and hence $(u_A)f = u_A$. Since f is surjective it follows that $(i)f = i$ for $i \in [0, u_A]$.

We define a surjective monotone map

$$\begin{array}{rccl} (f,A)\psi_1 : & [n] & \to & [k-1] \\ & i & \mapsto & \begin{cases} (i)f & \text{if } (i)f < u_A \\ (i)f - 1 & \text{if } (i)f \geqslant u_A \,. \end{cases} \end{array}$$

Comparing $(f, A)\psi_1$ to f, we get the following equalities.

For $i \in [0, u_A - 2]$, we have

$$((f, A)\psi_1)^{-1}(i) = f^{-1}(i) = \{i\}\,.$$

We have

$$((f, A)\psi_1)^{-1}(u_A - 1) = f^{-1}(u_A - 1) \cup f^{-1}(u_A) = \{u_A - 1\} \cup f^{-1}(u_A)\,.$$

For $i \in [u_A, k - 1]$, we have

$$((f, A)\psi_1)^{-1}(i) = f^{-1}(i + 1)\,.$$

We define $(f, A)\psi_2 := \{a - 1 : a \in A \setminus \{u_A\}\} \subseteq [1, k - 1]$.

Note that $(u_A)((f, A)\psi_1) = u_A - 1$. Hence, if $u_A \leqslant k - 1$ we have

$$(u_A)g_{(f,A)\psi_1,(f,A)\psi_2} \geqslant \min(((f, A)\psi_1)^{-1}(u_A)) - 1 \geqslant u_A + 1 - 1 = u_A\,.$$

Note that $(i)g_{(f,A)\psi_1,(f,A)\psi_2} = (i)g_{f,A} = i$ for $i \in [0, u_A - 1]$. Furthermore, for $i \in [u_A + 1, k - 1]$, we have

$$(i)g_{(f,A)\psi_1,(f,A)\psi_2} \geqslant (i)g_{(f,A)\psi_1,\emptyset} - 1 \geqslant \min(((f, A)\psi_1)^{-1}(u_A)) + (i - u_A) - 1 \geqslant u_A + 1 + (i - u_A) - 1 = i\,.$$

So it follows that $g_{(f,A)\psi_1,(f,A)\psi_2}$ is ascending.

This concludes the construction of the maps φ and ψ.

Now suppose given $k \in [0, n]$ and $(f : [n] \to [k]) \in \text{surj}$ and $A \subseteq [1, k]$ such that $(f, A) \neq (\text{id}_{[n]}, \emptyset)$ and such that $g_{f,A}$ is ascending.

We have $u_{(f,A)\varphi_2} = \min(f, A)\varphi_2 = e_f + 1$.

We show that $((f, A)\varphi_1, (f, A)\varphi_2)\psi_1 \overset{!}{=} f$. Let $i \in [0, k]$.

If $i \leqslant e_f$, we have

$$(i)(f, A)\varphi_1 = (i)f < e_f + 1 = u_{(f,A)\varphi_2}\,.$$

So we have
$$(i)(((f,A)\varphi_1,(f,A)\varphi_2)\psi_1) = (i)((f,A)\varphi_1) = (i)f\,.$$

If $i > e_f$, we have
$$(i)(f,A)\varphi_1 = (i)f + 1 \geqslant (e_f)f + 1 = e_f + 1 = u_{(f,A)\varphi_2}\,.$$

So we have
$$(i)(((f,A)\varphi_1,(f,A)\varphi_2)\psi_1) = (i)((f,A)\varphi_1) - 1 = (i)f + 1 - 1 = (i)f\,.$$

We show that $((f,A)\varphi_1,(f,A)\varphi_2)\psi_2 \overset{!}{=} A$.

Indeed, we have
$$\begin{aligned} & a \in ((f,A)\varphi_1,(f,A)\varphi_2)\psi_2 \\ \Leftrightarrow\ & a+1 \in (f,A)\varphi_2 \setminus \{e_f+1\} \\ \Leftrightarrow\ & a+1 \in \{\tilde{a}+1 : \tilde{a} \in (A \,\dot{\cup}\, \{e_f\})\} \setminus \{e_f+1\} \\ \Leftrightarrow\ & a \in A\,. \end{aligned}$$

Now suppose given $k \in [0,n]$ and $(f : [n] \to [k]) \in \mathrm{surj}$ and $A \subseteq [1,k]$ such that $(f,A) \neq (\mathrm{id}_{[n]}, \emptyset)$ and such that $(i)g_{f,A}$ is not ascending.

We have $e_{(f,A)\psi_1} = u_A - 1$.

We show that $((f,A)\psi_1,(f,A)\psi_2)\varphi_1 = f$. Let $i \in [0,k]$.

If $(i)f < u_A$, we have $i \leqslant u_A - 1 = e_{(f,A)\psi_1}$. So we have
$$(i)(((f,A)\psi_1,(f,A)\psi_2)\varphi_1) = (i)((f,A)\psi_1) = (i)f\,.$$

If $(i)f \geqslant u_A$, we have $i > u_A - 1 = e_{(f,A)\psi_1}$. So we have
$$(i)(((f,A)\psi_1,(f,A)\psi_2)\varphi_1) = (i)(f,A)\psi_1 + 1 = (i)f - 1 + 1 = (i)f\,.$$

We show that $((f,A)\psi_1,(f,A)\psi_2)\varphi_2 \overset{!}{=} A$.

Indeed, we have
$$\begin{aligned} & a((f,A)\psi_1,(f,A)\psi_2)\varphi_2 \\ \Leftrightarrow\ & a-1 \in (f,A)\psi_2 \,\dot{\cup}\, \{u_A - 1\} \\ \Leftrightarrow\ & a-1 \in \{\tilde{a}-1 : \tilde{a} \in (A \setminus \{u_A\})\} \,\dot{\cup}\, \{u_A - 1\} \\ \Leftrightarrow\ & a \in A\,. \end{aligned}$$

So we have
$$\begin{aligned} \delta_{X,n} \cdot (\epsilon_X)_n &= \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k\in[0,n] \\ A\subseteq[1,k]}} (-1)^{|A|} X_{f\cdot g_{f,A}} = \mathrm{id}_{X_n} + \sum_{\substack{(f:[n]\to[k])\in\mathrm{surj} \\ \text{for some } k\in[0,n] \\ A\subseteq[1,k] \\ (f,A)\neq(\mathrm{id}_{[n]},\emptyset)}} (-1)^{|A|} X_{f\cdot g_{f,A}} \\ &= \mathrm{id}_{X_n} + \sum_{(f,A)\in M_1} (-1)^{|A|} X_{f\cdot g_{f,A}} + \sum_{(f,A)\in M_2} (-1)^{|A|} X_{f\cdot g_{f,A}} \\ &= \mathrm{id}_{X_n} + \sum_{(f,A)\in M_1} (-1)^{|A|} X_{f\cdot g_{f,A}} + \sum_{(f,A)\in M_1} (-1)^{|(f,A)\varphi_2|} X_{(f,A)\varphi_1\cdot g_{(f,A)\varphi_1,(f,A)\varphi_2}} \\ &= \mathrm{id}_{X_n} + \sum_{(f,A)\in M_1} (-1)^{|A|} X_{f\cdot g_{f,A}} - \sum_{(f,A)\in M_1} (-1)^{|A|} X_{f\cdot g_{f,A}} = \mathrm{id}_{X_n}\ . \end{aligned}$$

5.3 The functor $\mathcal{F}$ is not an equivalence

We assume throughout this section that R is not the zero ring. So there exists $0 \neq M \in \mathrm{Ob}(\mathcal{A})$.

Lemma 74. The functor $\mathcal{N} : \mathrm{C}(\mathcal{A})_{\geqslant 0} \to \mathrm{SemiSimp}(\mathcal{A})$ from Definition 61 is not an equivalence of categories.

Proof. We show that $\mathcal{N}$ is not dense. We choose $0 \neq M \in \mathrm{Ob}(\mathcal{A})$. Let

$$\begin{array}{rccc} A: & \Delta^{\mathrm{op}}_{\mathrm{inj}} & \to & \mathcal{A} \\ & f & \mapsto & \mathrm{id}_M \end{array}$$

be a constant functor, which is a semisimplicial object. Suppose given $B \in \mathrm{C}(\mathcal{A})_{\geqslant 0}$. We show that $\mathcal{N}B$ is not isomorphic to A. *Assume* the contrary. Then there exists an isomorphism φ given by a tuple of bijective R-linear maps $((\mathcal{N}B)_n \xrightarrow{\varphi_n} A_n)_{n \geqslant 0}$. Then, as maps from $(\mathcal{N}B)_1$ to $A_0 = M$, we have

$$0 = 0 \cdot \varphi_0 = d_1^{\mathcal{N}B,1} \cdot \varphi_0 = \varphi_1 \cdot d_1^{A,1} = \varphi_1$$

and hence $\mathrm{id}_M = \varphi_1^{-1} \cdot \varphi_1 = 0$. This is a *contradiction*. Hence $\mathcal{N}$ is not dense. □

Corollary 75. The functor $\mathcal{F} : \mathrm{SemiSimp}(\mathcal{A}) \to \mathrm{Simp}(\mathcal{A})$ from [**1,** Lemma 60] is not an equivalence.

Proof. *Assume* that $\mathcal{F}$ is an equivalence. Since $\mathcal{F} \circ \mathcal{N}$ is an equivalence, cf. Proposition 72, it follows that $\mathcal{N}$ is an equivalence. This *contradicts* the previous Lemma 74. □

Corollary 76. The functor $\mathcal{M} : \mathrm{SemiSimp}(\mathcal{A}) \to \mathrm{C}(\mathcal{A})_{\geqslant 0}$ from Definition 60 is not an equivalence.

Proof. *Assume* that $\mathcal{M}$ is an equivalence. Let $\mathcal{M}^{-1}$ be an inverse equivalence. Then $\mathcal{M}^{-1} \dashv \mathcal{M}$, cf. e.g. [**8,** Lemma 36]. Since also $\mathcal{N} \dashv \mathcal{M}$ there exists an isotransformation from $\mathcal{M}^{-1}$ to $\mathcal{N}$. Hence $\mathcal{N}$ is an equivalence. This is a *contradiction* to Lemma 74. □

Corollary 77. The functor $\mathcal{V} : \mathrm{Simp}(\mathcal{A}) \to \mathrm{SemiSimp}(\mathcal{A})$ from [**1,** Definition 48] is not an equivalence.

Proof. *Assume* that $\mathcal{V}$ is an equivalence. Since $\mathcal{V} \circ \mathcal{M}$ is an equivalence, cf. Proposition 72, it follows that $\mathcal{M}$ is an equivalence. This *contradicts* the previous Corollary 76. □

5.4 The functor $\mathcal{N}$ is not compatible with homotopy

We assume throughout this section that R is not the zero ring.

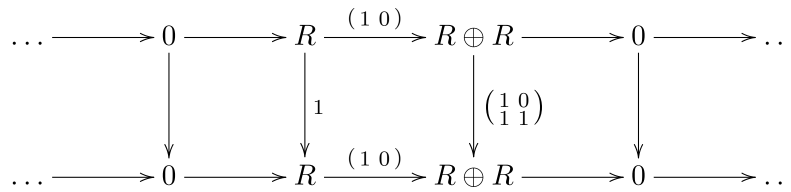
Recall that on $\mathrm{C}(\mathcal{A})$ we have the homotopy congruence, having the homotopy category $\mathrm{K}(\mathcal{A})$ as factor category. Recall that $\mathcal{N}$ maps from $\mathrm{C}(\mathcal{A})_{\geqslant 0}$ to $\mathrm{SemiSimp}(\mathcal{A})$; cf. Definition 61. We write $\mathrm{K}(\mathcal{A})_{\geqslant 0}$ for the image of $\mathrm{C}(\mathcal{A})_{\geqslant 0}$ in $\mathrm{K}(\mathcal{A})$.

Remark 78. In $\mathrm{C}(\mathcal{A})_{\geqslant 0}$, there exist morphisms φ and ψ such that φ and ψ are homotopic, but such that $\mathcal{N}\varphi$ and $\mathcal{N}\psi$ are not semisimplicially homotopic, cf. Definition 34.

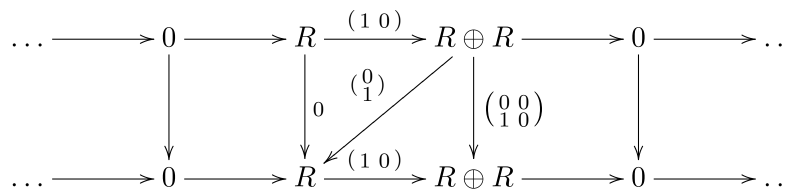
Proof. We use matrix calculus to describe R-linear maps between direct sums. Let $C \in \mathrm{Ob}(\mathrm{C}(\mathcal{A})_{\geqslant 0})$ defined by $C_0 := R \oplus R$, $C_1 := R$, $C_i = 0$ for $i \notin \{0, 1\}$ and $d_0^C := (R \xrightarrow{(1\ 0)} R \oplus R)$.

$$C = (\ldots \longrightarrow 0 \longrightarrow R \xrightarrow{(1\ 0)} R \oplus R \longrightarrow 0 \longrightarrow \ldots)$$

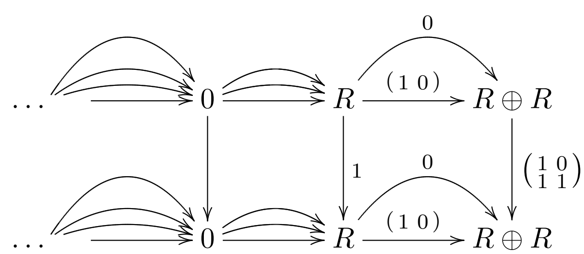
Let $\varphi : C \to C$ be defined by $\varphi_0 := (R \oplus R \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)} R \oplus R)$ and $\varphi_1 := \mathrm{id}_R$.



We define $\psi := \mathrm{id}_C$. The morphism φ is homotopic to ψ, i.e. $\varphi - \psi$ is homotopic to zero, as can be seen through the following diagram.



Now it remains to show that $\mathcal{N}\varphi \overset{!}{\not\rightsquigarrow} \mathcal{N}\psi$. The morphism $\mathcal{N}\varphi$ is pictured in the following diagram.



Suppose given morphisms $a, b : \mathcal{N}C \to \mathcal{N}C$ such that $a \rightsquigarrow b$. Then there exists an R-linear map

$$R \oplus R \xrightarrow{h_0^0 =: \left(\begin{smallmatrix} h_{0,1}^0 \\ h_{0,2}^0 \end{smallmatrix}\right)} R$$

such that $h_0^0 \cdot d_1^{\mathcal{N}C,1} = a_0$, cf. Definition 27 (v). Since $d_1^{\mathcal{N}C,1} = 0$, we have $a_0 = 0 \neq (\mathcal{N}\varphi)_0$. Hence

$\mathcal{N}\varphi \neq a$. We further have $\left(\begin{smallmatrix} h^0_{0,1} \\ h^0_{0,2} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) = h^0_0 \cdot d_0^{\mathcal{N}C,1} \cdot b_0$, cf. Definition 27 (vi). Hence $b_0 = \left(\begin{smallmatrix} h^0_{0,1} & 0 \\ h^0_{0,2} & 0 \end{smallmatrix}\right) \neq (\mathcal{N}\varphi)_0$. Thus $\mathcal{N}\varphi \neq b$. So we obtain the congruence class $[\mathcal{N}\varphi] = \{\mathcal{N}\varphi\}$, which consists only of $\mathcal{N}\varphi$. In particular, $\mathcal{N}\varphi \not\sim \mathcal{N}\psi$. □

Corollary 79. There does not exist a functor from $\mathrm{K}(\mathcal{A})_{\geqslant 0}$ to $\mathrm{HoSemiSimp}(\mathcal{A})$ that completes the diagram

$$\begin{array}{ccc} \mathrm{C}(\mathcal{A})_{\geqslant 0} & \xrightarrow{\mathcal{N}} & \mathrm{SemiSimp}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{A})_{\geqslant 0} & & \mathrm{HoSemiSimp}(\mathcal{A}) \,, \end{array}$$

with vertical residue class functors, to a commutative quadrangle.

Proof. By Remark 78, there exist morphisms $\varphi, \psi \in \mathrm{Mor}(\mathrm{C}(\mathcal{A})_{\geqslant 0})$ such that φ is homotopic to ψ but such that $\mathcal{N}\varphi \not\sim \mathcal{N}\psi$. If there existed a functor $\bar{\mathcal{N}} : \mathrm{K}(\mathcal{A})_{\geqslant 0} \to \mathrm{HoSemiSimp}(\mathcal{A})$ making the diagram

$$\begin{array}{ccc} \mathrm{C}(\mathcal{A})_{\geqslant 0} & \xrightarrow{\mathcal{N}} & \mathrm{SemiSimp}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{A})_{\geqslant 0} & \xrightarrow{\bar{\mathcal{N}}} & \mathrm{HoSemiSimp}(\mathcal{A}) \end{array}$$

commutative, we would have $[\mathcal{N}\varphi] = \bar{\mathcal{N}}[\varphi] = \bar{\mathcal{N}}[\psi] = [\mathcal{N}\psi]$, which is not the case. □

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Zusammenfassung

Sei Δ die Simplexkategorie, die aus den geordneten Mengen $\{0, 1, \ldots, n\}$ für $n \geqslant 0$ und monotonen Abbildungen besteht. Darin sei Δ_{inj} die Teilkategorie der injektiven Abbildungen.

Sei $\mathcal{C}$ eine Kategorie. Die Kategorie $\text{Simp}(\mathcal{C})$ ist die Funktorkategorie $\mathcal{C}^{\Delta^{\text{op}}}$ der kontravarianten Funktoren von Δ nach $\mathcal{C}$. Ihre Objekte heißen simpliziale Objekte über $\mathcal{C}$. Die Kategorie $\text{SemiSimp}(\mathcal{C})$ ist die Funktorkategorie $\mathcal{C}^{\Delta_{\text{inj}}^{\text{op}}}$ der kontravarianten Funktoren von Δ_{inj} nach $\mathcal{C}$. Ihre Objekte heißen semisimpliziale Objekte über $\mathcal{C}$.

Wenn $\mathcal{C}$ endliche Koprodukte hat, gibt es einen zum Vergissfunktor $\mathcal{V} : \text{Simp}(\mathcal{C}) \to \text{SemiSimp}(\mathcal{C})$ linksadjungierten Funktor $\mathcal{F} : \text{SemiSimp}(\mathcal{C}) \to \text{Simp}(\mathcal{C})$.

Nach Tierney und Vogel kann man in einer Kategorie $\mathcal{C}$ mit endlichen Limiten und mit einer Teilkategorie $\mathcal{P}$, die gewisse Eigenschaften erfüllt, simplizial auflösen. Das heißt, es existiert eine Konstruktion, die Morphismen aus $\mathcal{C}$ auf Morphismen in $\text{Simp}(\mathcal{P})$ abbildet, wobei der Funktor $\mathcal{F}$ in die Konstruktion mit eingeht.

Um diese Konstruktion funktoriell werden zu lassen, beschäftigen wir uns in dieser Arbeit mit Homotopie auf den Kategorien $\text{Simp}(\mathcal{P})$ und $\text{SemiSimp}(\mathcal{P})$ und zeigen, dass $\mathcal{F}$ mit Homotopie verträglich ist. Wir erhalten einen induzierten Funktor $\bar{\mathcal{F}} : \text{HoSemiSimp}(\mathcal{P}) \to \text{HoSimp}(\mathcal{P})$ zwischen den jeweiligen Homotopiekategorien. Diesen komponieren wir mit dem von Tierney und Vogel konstruierten Funktor $\mathcal{C} \to \text{HoSemiSimp}(\mathcal{P})$ zum gewünschten Auflösungsfunktor $\mathcal{C} \to \text{HoSimp}(\mathcal{P})$.

Zudem bauen wir den Funktor $\mathcal{F}$ in einen Beweis des klassischen Satzes von Dold-Puppe-Kan ein. Aus diesem folgt dann, dass die Begriffe des semisimplizialen und simplizialen Objekts in einer Modulkategorie wesentlich verschieden sind, wohingegen man nach unten beschränkte Komplexe und simpliziale Objekte als äquivalent ansehen kann.

Hiermit versichere ich,

(1) dass ich meine Arbeit selbstständig verfasst habe,

(2) dass ich keine anderen als die angegebenen Quellenangaben benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,

(3) dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und

(4) dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, Dezember 2019

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