A calculus of fractions for the homotopy category of a Brown cofibration category

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Introduction

1 How to calculate in homotopy categories?

Homotopy categories

Homotopical algebra may be thought of as the study of homotopy categories in the following sense. We consider a category $\mathcal{C}$ that is equipped with a set ([1]) of morphisms that we want to call *weak equivalences*. We would like to consider the objects in $\mathcal{C}$ that are connected by weak equivalences as essentially equal, although a given weak equivalence in $\mathcal{C}$ is not an isomorphism in general. To make this mathematically precise, we have to pass to the *homotopy category* $\operatorname{Ho}\mathcal{C}$ of $\mathcal{C}$, which is defined to be the localisation of $\mathcal{C}$ with respect to the weak equivalences. Here localisation is a purely category theoretical device that produces the universal category in which the weak equivalences become isomorphisms – the idea being borrowed from localisation of rings.
The archetypical example is given by the category of topological spaces, with the weak equivalences being continuous maps that induce isomorphisms on all homotopy groups. Similarly, we may consider the category of simplicial sets, with the weak equivalences being simplicial maps that induce, after topological realisation, isomorphisms on all homotopy groups. An additive example is given by the category of complexes with entries in an abelian category, with weak equivalences being the quasi-isomorphisms, that is, the complex morphisms that induce isomorphisms on all (co)homology objects. A further example, which is somehow degenerate from our point of view, is given by an abelian category, with the weak equivalences being those morphisms having kernel and cokernel in a chosen thick subcategory.
The homotopy category of topological spaces is then equivalent to the homotopy category of simplicial sets, and also equivalent to the category consisting of CW-spaces and homotopy classes of continuous maps.

Gabriel-Zisman localisation

By a theorem of Gabriel and Zisman [12, sec. 1.1], a localisation of a category with respect to an arbitrary set of weak equivalences exists, the *Gabriel-Zisman localisation* ([2]); and as a localisation is defined via a universal property, it is unique up to a unique isomorphism of categories. While the objects in the Gabriel-Zisman localisation are the same as in $\mathcal{C}$, the morphisms are equivalence classes of zigzags

$$X \longrightarrow \ \underset{\approx}{\longleftarrow} \ \longrightarrow \dots \underset{\approx}{\longleftarrow} \ \longrightarrow Y$$

of finite but arbitrary length, where the "backward" arrows (labeled by "$\approx$") are supposed to be weak equivalences. So roughly said, the morphisms in the Gabriel-Zisman localisation consist of arbitrarily many numerators and denominators. To decide whether two such zigzags represent the same morphism, the definition provides an equivalence relation generated by certain elementary relations – which leads to a word problem.

Brown's homotopy 2-arrow calculus

Since our four examples share more structure, we can do better in our situation. A *Brown cofibration category* is a category $\mathcal{C}$ that is not only equipped with a set of weak equivalences, leading via localisation to its homotopy category, but moreover with a set of morphisms called *cofibrations*, fulfilling a short list of axioms, see section 3

[1] For the purpose of this introduction, we ignore set-theoretical difficulties.
[2] To the author's knowledge, this general construction first explicitly appeared in the monograph of Gabriel and Zisman [12, sec. 1.1]. One can find earlier mentions, for example in [15, ch. I, §3, rem., p. 29] and in [37, ch. I, §2, n. 3, p. 17]. In the latter source, one finds moreover a citation "[C.G.G.]", which might be the unpublished manuscript *Catégories et foncteurs* of Chevalley, Gabriel and Grothendieck occurring in the bibliography of [32].

below or definition (3.52)(a). For example, the category of simplicial sets or the category of complexes in an abelian category, with weak equivalences as described above, become Brown cofibration categories if we equip these categories with monomorphisms as cofibrations. The category of topological spaces together with the weak equivalences as above and with the Serre fibrations becomes a *Brown fibration category*, that is, it fulfills axioms dual to that of a Brown cofibration category. Finally, in the example of an abelian category we may add all morphisms as cofibrations to the data to obtain a Brown cofibration category.
By Browns homotopy 2-arrow calculus [7, dual of th. 1 and proof], the morphisms in the homotopy category of a Brown cofibration category $\mathcal{C}$ may be described as follows. Every morphism in $\operatorname{Ho}\mathcal{C}$ is represented by a diagram

$$X \longrightarrow \tilde{Y} \xleftarrow[\approx]{} Y \ ,$$

called a 2-*arrow* [3]. Two such 2-arrows represent the same morphism in $\operatorname{Ho}\mathcal{C}$ if and only if they can be embedded as the top and the bottom row in a diagram of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & \tilde{Y}_1 \arrow[d, "\approx"'] & Y \arrow[l, "\approx"'] \arrow[d, equal] \\
X \arrow[r] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, "\approx"'] \arrow[d, equal] \\
X \arrow[r] & \tilde{Y}_2 \arrow[u, "\approx"] & Y \arrow[l, "\approx"']
\end{tikzcd}$$

that is commutative up to a suitable notion of *homotopy*.

Z-2-arrow calculus

In the example of an abelian category, one even has a *strict* 2-arrow calculus, that is, one gets a strictly commutative 2-by-2 diagram as above for two 2-arrows representing the same morphism. This, however, does not hold in an arbitrary Brown cofibration category $\mathcal{C}$. If we want to work with a strictly commutative diagram, we have to pick certain 2-arrows: Every morphism in $\operatorname{Ho}\mathcal{C}$ is represented by a so-called *Z-2-arrow*, that is, a 2-arrow

$$X \longrightarrow \tilde{Y} \xleftarrow[\approx]{} Y$$

such that the induced morphism $X \amalg Y \to \tilde{Y}$ is a cofibration. Two such Z-2-arrows represent the same morphism in $\operatorname{Ho}\mathcal{C}$ if and only if they can be embedded as the top and the bottom row in a strictly commutative 2-by-2 diagram of the above form. Since in the example of an abelian category all morphisms are cofibrations, all 2-arrows are Z-2-arrows; and so the calculus with Z-2-arrows may be seen as a generalisation of the strict calculus in the example of an abelian category to arbitrary Brown cofibration categories.
It is not hard to derive Brown's homotopy 2-arrow calculus from the Z-2-arrow calculus, see theorem (3.132). It is possible, but more complicated, to derive the Z-2-arrow calculus from Brown's homotopy 2-arrow calculus. We will, however, develop the Z-2-arrow calculus ab ovo.
The Z-2-arrow calculus will be applied to construct an unstable variant of a higher triangulated structure on the homotopy category of a Brown cofibration category; cf. section 2 for details.

A comparison: How to calculate in derived categories

We reconsider our example of the category of complexes $\mathrm{C}(\mathcal{A})$ with entries in an abelian category $\mathcal{A}$, equipped with the quasi-isomorphisms as weak equivalences and with the monomorphisms as cofibrations. In this case, the homotopy category $\operatorname{Ho}\mathrm{C}(\mathcal{A})$ is the derived category $\mathrm{D}(\mathcal{A})$. Beside this, there is also the so-called homotopy

[3] In the main text, this will be called an *S-2-arrow* to distinguish it from the dual situation. We will omit the "S" for the purpose of this introduction.

category of complexes $\mathrm{K}(\mathcal{A})$ ([4]) that has the same objects as $\mathrm{C}(\mathcal{A})$, but as morphisms the homotopy classes of complex morphisms.
Verdier has shown that the derived category $\mathrm{D}(\mathcal{A})$ may be constructed as the Verdier quotient of $\mathrm{K}(\mathcal{A})$ modulo the thick subcategory of acyclic complexes. As a strict 2-arrow calculus is valid for every Verdier quotient and as every morphism in $\mathrm{K}(\mathcal{A})$ is a homotopy class of morphisms in $\mathrm{C}(\mathcal{A})$, this leads to a homotopy 2-arrow calculus for the derived category in the following sense. Every morphism in $\mathrm{D}(\mathcal{A})$ is represented by a 2-arrow, and two 2-arrows represent the same morphism in $\mathrm{D}(\mathcal{A})$ if and only if they can be embedded as the top and the bottom row in a 2-by-2 diagram of the above form that is commutative up to complex homotopy.
Brown's homotopy 2-arrow calculus is a generalisation of this example to arbitrary Brown cofibration categories. In contrast, the Z-2-arrow calculus yields a possible way to calculate in the derived category $\mathrm{D}(\mathcal{A})$ as a localisation of $\mathrm{C}(\mathcal{A})$, which is more handy than Gabriel-Zisman and which circumvents the homotopy category of complexes $\mathrm{K}(\mathcal{A})$.

Related concepts

There are several concepts related to that of a Brown cofibration category, the most popular one being that of a *Quillen model category* [28, ch. I, sec. 1, def. 1]. An overview can be found in [30, ch. 2]. Every Quillen model category has a Brown cofibration category as a subcategory, namely the *full subcategory of cofibrant objects*. The homotopy categories of a Quillen model category and its full subcategory of cofibrant objects are equivalent by Quillen's homotopy category theorem [28, ch. I, sec. 1, th. 1].
In many examples of Quillen model categories, all objects are either cofibrant or fibrant, that is, they are either Brown cofibration categories or Brown fibration categories, whence Brown's homotopy 2-arrow calculus as well as the Z-2-arrow calculus (resp. its dual) apply. In the general case, one obtains a (strict) 3-arrow calculus, as was proven by Dwyer, Hirschhorn, Kan and Smith [11, sec. 10, sec. 36], provided the Quillen model category at hand admits functorial factorisations in the sense of [11, sec. 9.1, ax. MC5]. The requirement of functorial factorisations was shown to be redundant by the author [36].
In that work, a 3-arrow calculus has been developed in the context of *uni-fractionable categories*, which is applicable to Quillen model categories as well as to their subcategories of cofibrant, fibrant and bifibrant objects, see [36, def. 3.1, th. 5.13, ex. 6.1]. Although it has been announced that the results developed in loc. cit. would play a role in this thesis, the author decided not to use them, as the Z-2-arrow calculus seems to be more practicable. However, some of the methods survived and are used in chapter II, see in particular section 6.

2 An unstable higher triangulated structure on the homotopy category

In the following, we will illustrate what we mean by a higher triangulated structure. Although we work unstably in the main text, we begin our explanation with the stable situation (in the sense below) as this is the classical case.

The shift on the homotopy category

We suppose given a zero-pointed Brown cofibration category $\mathcal{C}$ ([5]), that is, a Brown cofibration category together with a distinguished zero object. As K. Brown has shown [7, dual of th. 3], the homotopy category $\operatorname{Ho}\mathcal{C}$ has a canonical endofunctor $\mathrm{T}\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$, called the *shift* of $\operatorname{Ho}\mathcal{C}$.
For example, on the derived category we get the usual shift of complexes. On the homotopy category of pointed topological spaces, using the dual notion of a zero-pointed Brown fibration category, we get the loop space functor.
A *stable Brown cofibration category* is a zero-pointed Brown cofibration category $\mathcal{C}$ such that the shift on $\operatorname{Ho}\mathcal{C}$ is invertible. Schwede [33, th. A.12] has shown that the homotopy category of a stable Brown cofibration category carries the structure of a triangulated category in the sense of Verdier [37, ch. I, §1, n° 1, sec. 1-1]. Precursors and variants of this result are reported in [33, rem. A.13].

[4] The clash of notation "homotopy category of complexes" vs. "homotopy category in the sense of homotopical algebra" may be explained as follows. There is another Brown cofibration structure on $\mathrm{C}(\mathcal{A})$ where the cofibrations are given by the pointwise split monomorphisms and where the weak equivalences are given by the homotopy equivalences of complexes. With respect to this structure, the homotopy category $\operatorname{Ho}\mathrm{C}(\mathcal{A})$ is $\mathrm{K}(\mathcal{A})$.

[5] In the literature, a zero-pointed category is often just called a *pointed* category.

Higher triangles

A *Verdier triangulated category* consists of an additive category $\mathcal{T}$, equipped with an autofunctor $\mathrm{T}\colon \mathcal{T} \to \mathcal{T}$, called *shift*, and a set of diagrams in $\mathcal{T}$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} C \xrightarrow{w} \mathrm{T}X$$

called *Verdier triangles*, such that certain axioms are fulfilled. Such a Verdier triangle in $\mathcal{T}$ is sometimes depicted as

$$\begin{tikzcd} & C \arrow[dl, Rightarrow, "w"'] & \\ X \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd} ,$$

where the double-arrow notation indicates that w is in fact a morphism $C \to \mathrm{T}X$.
One of the axioms of a Verdier triangulated category $\mathcal{T}$ is the so-called *octahedral axiom*, which states the following. For all morphisms $u_1\colon X \to Y$, $u_2\colon Y \to Z$ in $\mathcal{T}$ there exists a diagram in $\mathcal{T}$ of the form

$$\begin{tikzcd} C \arrow[dr, Rightarrow] \arrow[ddrr, bend right] & & Y \arrow[ll] \arrow[dr, "u_2"] & & A \arrow[ll, Rightarrow] \arrow[llll, Rightarrow, bend right] \\ & X \arrow[ur, "u_1"] \arrow[rr] & & Z \arrow[ur] \arrow[dl] & \\ & & B \arrow[ul, Rightarrow] \arrow[uurr, bend right] & & \end{tikzcd}$$

such that $(X, Y, C, \mathrm{T}X)$, $(Y, Z, A, \mathrm{T}Y)$, $(X, Z, B, \mathrm{T}X)$, $(C, B, A, \mathrm{T}C)$ are Verdier triangles, and such that the triangles (X, Y, Z), $(C, B, \mathrm{T}X)$, $(A, \mathrm{T}Y, \mathrm{T}C)$, (Z, B, A) and the quadrangles (Y, Z, C, B), $(B, \mathrm{T}X, A, \mathrm{T}Y)$ commute. Such a diagram is called a *Verdier octahedron.*
In every Verdier triangle

$$\begin{tikzcd} & C \arrow[dl, Rightarrow, "w"'] & \\ X \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd} ,$$

the composites $X \to C$, $Y \to \mathrm{T}X$, $C \to \mathrm{T}Y$ are zero morphisms. So, a bit redundantly, this Verdier triangle may be also depicted as a commutative diagram of the form

$$\begin{tikzcd} & & & 0 \\ & & 0 \arrow[r] & \mathrm{T}Y \arrow[u] \\ & 0 \arrow[r] & C \arrow[u] \arrow[r, "w"] & \mathrm{T}X \arrow[u, "\mathrm{T}u"'] \\ 0 \arrow[r] & X \arrow[u] \arrow[r, "u"] & Y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u] \end{tikzcd} .$$

On the other hand, in a Verdier octahedron

$$\begin{tikzcd} C \arrow[dr, Rightarrow] \arrow[ddrr, bend right] & & Y \arrow[ll] \arrow[dr, "u_2"] & & A \arrow[ll, Rightarrow] \arrow[llll, Rightarrow, bend right] \\ & X \arrow[ur, "u_1"] \arrow[rr] & & Z \arrow[ur] \arrow[dl] & \\ & & B \arrow[ul, Rightarrow] \arrow[uurr, bend right] & & \end{tikzcd}$$

the morphisms $X \to Z$, $C \to \mathrm{T}X$, $A \to \mathrm{T}C$, $Z \to A$ are uniquely determined as composites of two other morphisms. Moreover, as such a Verdier octahedron consists of Verdier triangles, several composites of morphisms in it are zero morphisms. So this Verdier octahedron may be depicted as a commutative diagram of the form

$$\begin{tikzcd}
 & & & & 0 \\
 & & & 0 \arrow[r] & \mathrm{T}Z \arrow[u] \\
 & & 0 \arrow[r] & A \arrow[r] \arrow[u] & \mathrm{T}Y \arrow[u, "\mathrm{T}u_2"'] \\
 & 0 \arrow[r] & C \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & \mathrm{T}X \arrow[u, "\mathrm{T}u_1"'] \\
0 \arrow[r] & X \arrow[r, "u_1"] \arrow[u] & Y \arrow[r, "u_2"] \arrow[u] & Z \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd} .$$

If we prolongate this diagram periodically (up to shift), we may read off the four contained Verdier triangles (also periodically prolongated), cf. figure 1.
In fact, writing Verdier triangles and Verdier octahedra in this way corresponds to their usual construction when $\mathcal{T} = \operatorname{Ho}\mathcal{C}$, the homotopy category of a stable Brown cofibration category: Verdier triangles arise from certain diagrams of the form

$$\begin{tikzcd}
 & & & M_2 \\
 & & N_2 \arrow[r] & T_Y \arrow[u] \\
 & N_1 \arrow[r] & C \arrow[r] \arrow[u] & T_X \arrow[u] \\
M_0 \arrow[r] & X \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & M_1 \arrow[u]
\end{tikzcd}$$

in $\mathcal{C}$ such that $M_0 \cong M_1 \cong M_2 \cong N_1 \cong N_2 \cong 0$ and $T_X \cong \mathrm{T}X$, $T_Y \cong \mathrm{T}Y$ in $\operatorname{Ho}\mathcal{C}$. Likewise, the Verdier octahedra that are usually constructed to verify the octahedral axiom arise from certain diagrams of the form

$$\begin{tikzcd}
 & & & & M_2 \\
 & & & N_3 \arrow[r] & T_Z \arrow[u] \\
 & & N_2 \arrow[r] & A \arrow[r] \arrow[u] & T_Y \arrow[u] \\
 & N_1 \arrow[r] & C \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & T_X \arrow[u] \\
M_0 \arrow[r] & X \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u] & M_1 \arrow[u]
\end{tikzcd}$$

in $\mathcal{C}$ such that $M_0 \cong M_1 \cong M_2 \cong N_1 \cong N_2 \cong N_3 \cong 0$ and $T_X \cong \mathrm{T}X$, $T_Y \cong \mathrm{T}Y$, $T_Z \cong \mathrm{T}Z$ in $\operatorname{Ho}\mathcal{C}$.

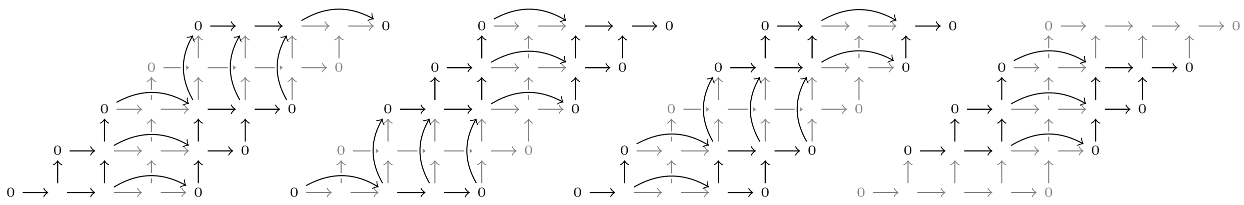


Figure 1: The four Verdier triangles in a Verdier octahedron.

In the same style, one may construct certain diagrams of the form

$$\begin{tikzcd}
 & & & & & M_2 \\
 & & & & N_n \arrow[r] & T_{X_n} \arrow[u] \\
 & & & \iddots & \vdots \arrow[u] & \vdots \arrow[u] \\
 & & N_2 \arrow[r] & \ldots \arrow[r] & C_{2,n} \arrow[r] \arrow[u] & T_{X_2} \arrow[u] \\
 & N_1 \arrow[r] & C_{1,2} \arrow[r] \arrow[u] & \ldots \arrow[r] & C_{1,n} \arrow[r] \arrow[u] & T_{X_1} \arrow[u] \\
M_0 \arrow[r] & X_1 \arrow[r] \arrow[u] & X_2 \arrow[r] \arrow[u] & \ldots \arrow[r] & X_n \arrow[r] \arrow[u] & M_1 \arrow[u]
\end{tikzcd}$$

in $\mathcal{C}$ that yield diagrams of the form

$$\begin{tikzcd}
 & & & & & 0 \\
 & & & & 0 \arrow[r] & \mathrm{T}X_n \arrow[u] \\
 & & & \iddots & \vdots \arrow[u] & \vdots \arrow[u] \\
 & & 0 \arrow[r] & \ldots \arrow[r] & C_{2,n} \arrow[r] \arrow[u] & \mathrm{T}X_2 \arrow[u] \\
 & 0 \arrow[r] & C_{1,2} \arrow[r] \arrow[u] & \ldots \arrow[r] & C_{1,n} \arrow[r] \arrow[u] & \mathrm{T}X_1 \arrow[u] \\
0 \arrow[r] & X_1 \arrow[r] \arrow[u] & X_2 \arrow[r] \arrow[u] & \ldots \arrow[r] & X_n \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

in $\operatorname{Ho}\mathcal{C}$. These diagrams in $\operatorname{Ho}\mathcal{C}$ (periodically prolonged) are called *n-triangles*. For $m \leq n$, an n-triangle contains several m-triangles, cf. figure 2.

By definition, a Verdier triangle in $\operatorname{Ho}\mathcal{C}$ is obtained from a diagram in the Brown cofibration category $\mathcal{C}$ as indicated above, so the Verdier triangles in $\operatorname{Ho}\mathcal{C}$ are precisely the 2-triangles. In contrast, the definition of a Verdier octahedron is only requiring a diagram (of the form as described above) that contains four Verdier

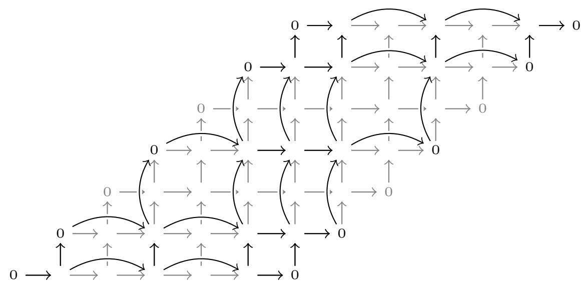


Figure 2: A 3-triangle in a 5-triangle.

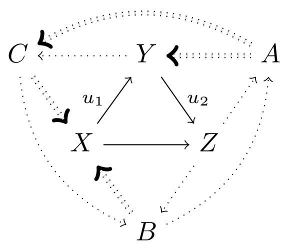
triangles, as stated in the octahedral axiom. As 3-triangles fulfill this property, they are particular Verdier octahedra. In general, there are Verdier octahedra in $\operatorname{Ho}\mathcal{C}$ that are not isomorphic to a 3-triangle in $\operatorname{Ho}\mathcal{C}$ [24, lem. 3, lem. 7].
Moreover, since an n-triangle for $n \geq 2$ contains several Verdier triangles, a kind of a higher octahedral axiom is fulfilled, cf. [5, rem. 1.1.14(d)].

Basic properties of n-triangles

In this thesis, we show that some of the properties of Verdier triangles in the homotopy category $\operatorname{Ho}\mathcal{C}$ of a stable Brown cofibration category $\mathcal{C}$ generalise to n-triangles (and therefore may be asked as axioms in a suitable notion of triangulated category with n-triangles at disposal, see Künzer [22, def. 2.1.2] and, independently, Maltsiniotis [25, sec. 1.4]). We will explain these basic properties of n-triangles and describe their relationship to the corresponding axioms of a Verdier triangulated category in the following. In doing so, by a *morphism of n-triangles* we mean a diagram morphism that is periodic up to shift.
Closed under isomorphisms. Like Verdier triangles, general n-triangles are closed under isomorphisms already by definition.
Prolongation on the objects. In every Verdier triangulated category, and therefore in particular in $\operatorname{Ho}\mathcal{C}$, one has the following two properties. First, one has *prolongation of morphisms to Verdier triangles*: Every morphism $u\colon X \to Y$ may be prolonged to a Verdier triangle.

$$X \xrightarrow{u} Y \xrightarrow{v} C \xrightarrow{w} \mathrm{T}X$$

Second, one has the *octahedral axiom*, that is, *prolongation of pairs of composable morphisms to Verdier octahedra*: All morphisms $u_1\colon X \to Y$, $u_2\colon Y \to Z$ may be prolonged to a Verdier octahedron. ([6])



[6]This is equivalent to (TR4) in [37, ch. I, §1, n° 1, sec. 1-1] in view of (TR3) in loc. cit.

So summarised, these two properties state that every sequence of 1 resp. 2 composable morphisms may be prolonged to a Verdier triangle resp. to a Verdier octahedron.

$$
\begin{tikzcd}
 & & & 0 \\
 & & 0 \arrow[r] & \mathrm{T}Y \arrow[u] \\
 & 0 \arrow[r] & C \arrow[u] \arrow[r, dotted] & \mathrm{T}X \arrow[u] \\
0 \arrow[r] & X \arrow[u] \arrow[r] & Y \arrow[u, dotted] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & & 0 \\
 & & & 0 \arrow[r] & \mathrm{T}Z \arrow[u] \\
 & & 0 \arrow[r] & A \arrow[u] \arrow[r, dotted] & \mathrm{T}Y \arrow[u] \\
 & 0 \arrow[r] & C \arrow[u] \arrow[r, dotted] & B \arrow[u, dotted] \arrow[r, dotted] & \mathrm{T}X \arrow[u] \\
0 \arrow[r] & X \arrow[u] \arrow[r] & Y \arrow[u, dotted] \arrow[r] & Z \arrow[u, dotted] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

As explained above, the Verdier octahedra constructed in the verification of the octahedral axiom arise from certain diagrams in $\mathcal{C}$, and so they are in fact 3-triangles. We show that an analogous prolongation property holds for n-triangles in $\operatorname{Ho}\mathcal{C}$, see theorem (5.55)(a): Every sequence of $n-1$ composable morphisms in $\operatorname{Ho}\mathcal{C}$ may be prolonged to an n-triangle.

$$
\begin{tikzcd}
 & & & & & 0 \\
 & & & & 0 \arrow[r] & \mathrm{T}X_n \arrow[u] \\
 & & & \iddots & \vdots \arrow[u] & \vdots \arrow[u] \\
 & & 0 \arrow[r] & \ldots \arrow[r, dotted] & C_{2,n} \arrow[u, dotted] \arrow[r, dotted] & \mathrm{T}X_2 \arrow[u] \\
 & 0 \arrow[r] & C_{1,2} \arrow[u] \arrow[r, dotted] & \ldots \arrow[r, dotted] & C_{1,n} \arrow[u, dotted] \arrow[r, dotted] & \mathrm{T}X_1 \arrow[u] \\
0 \arrow[r] & X_1 \arrow[u] \arrow[r] & X_2 \arrow[u, dotted] \arrow[r] & \ldots \arrow[r] & X_n \arrow[u, dotted] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

We call the lowest row of an n-triangle its *base*. With this terminology, the stated property may be reformulated as follows: The restriction functor that assigns to an n-triangle its base (from the category of n-triangles in $\operatorname{Ho}\mathcal{C}$ to the diagram category whose objects are $n-1$ composable morphisms in $\operatorname{Ho}\mathcal{C}$) is (strictly) surjective on the objects.

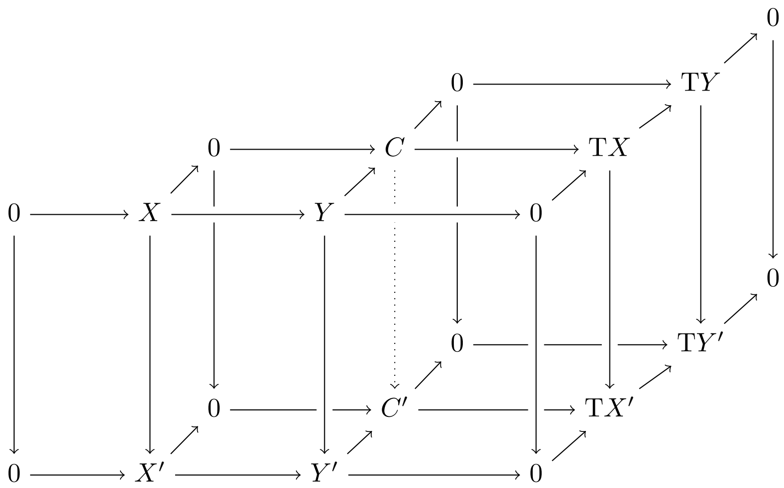
Prolongation on the morphisms. In every Verdier triangulated category, and therefore in particular in $\operatorname{Ho}\mathcal{C}$, one has *prolongation of morphisms of morphisms to morphisms of Verdier triangles*: Given a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\alpha"] & Y \arrow[r, "v"] \arrow[d, "\beta"] & C \arrow[r, "w"] & \mathrm{T}X \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \mathrm{T}X'
\end{tikzcd}
$$

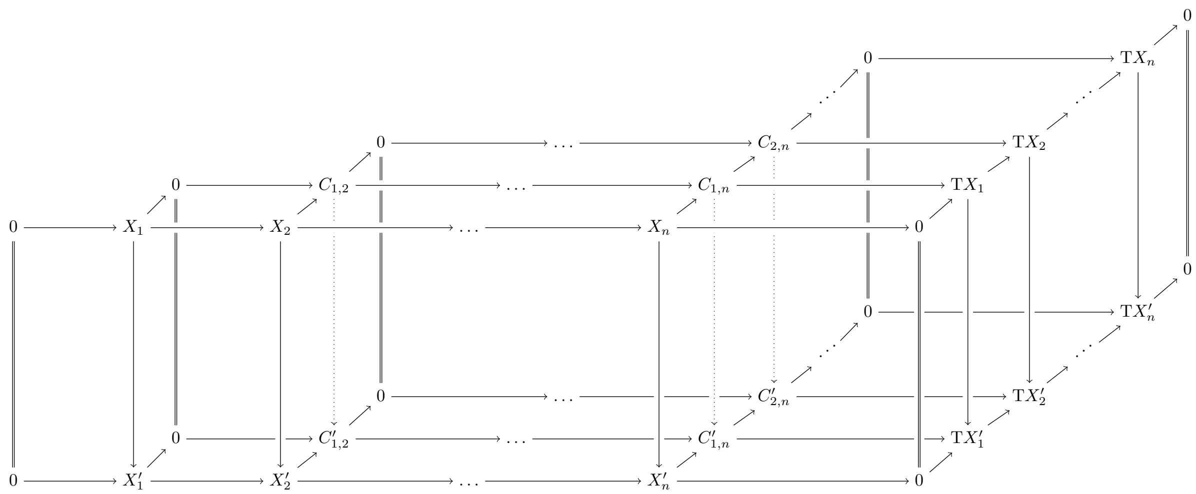
whose rows are supposed to be Verdier triangles, there exists a morphism $\gamma\colon C \to C'$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\alpha"] & Y \arrow[r, "v"] \arrow[d, "\beta"] & C \arrow[r, "w"] \arrow[d, dotted, "\gamma"] & \mathrm{T}X \arrow[d, "\mathrm{T}\alpha"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \mathrm{T}X'
\end{tikzcd}$$

So with the notion of a base as just introduced, this property states that every morphism of bases of Verdier triangles may be prolonged to a morphism of Verdier triangles.



We show that an analogous prolongation property holds for n-triangles in $\operatorname{Ho}\mathcal{C}$, see theorem (5.55)(b): Every morphism of bases of n-triangles in $\operatorname{Ho}\mathcal{C}$ may be prolonged to a morphism of n-triangles.



In other words: The restriction functor that assigns to an n-triangle its base is full.

Stability under generalised simplicial operations. Every 3-triangle in $\operatorname{Ho}\mathcal{C}$, being a Verdier octahedron, contains four 2-triangles (in the notation above, they have the bases $X \to Y$ resp. $Y \to Z$ resp. $X \to Z$ resp. $C \to B$). Every 2-triangle contains three 1-triangles (in the notation above, they have the bases X resp. Y resp. C). Conversely, every 1-triangle may be considered as a degenerate 2-triangle in two ways (the existence of one of these 2-triangles is an axiom of a Verdier triangulated category).

$$\begin{tikzcd}
& & 0 \\
& 0 \arrow[r] & \mathrm{T}X \arrow[u] \\
0 \arrow[r] & X \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & 0 \\
& & 0 \arrow[r] & \mathrm{T}X \arrow[u] \\
& 0 \arrow[r, equal] & 0 \arrow[r] \arrow[u, equal] & \mathrm{T}X \arrow[u, equal] \\
0 \arrow[r] & X \arrow[r, equal] \arrow[u] & X \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & 0 \\
& & 0 \arrow[r] & \mathrm{T}X \arrow[u] \\
& 0 \arrow[r] & X \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r, equal] & 0 \arrow[r] \arrow[u, equal] & X \arrow[r] \arrow[u, equal] & 0 \arrow[u, equal]
\end{tikzcd}$$

These relationships between n-triangles can be shortly expressed by the statement that n-triangles are stable under simplicial operations. In other words, n-triangles may be organised in a simplicial set that has as n-simplices precisely the n-triangles.
Moreover, given a Verdier triangle

$$\begin{tikzcd}
& & 0 \\
& 0 \arrow[r] & \mathrm{T}Y \arrow[u] \\
0 \arrow[r] & C \arrow[r, "w"] \arrow[u] & \mathrm{T}X \arrow[u, "\mathrm{T}u"'] \\
0 \arrow[r] & X \arrow[r, "u"] \arrow[u] & Y \arrow[r] \arrow[u, "v"'] & 0 \arrow[u]
\end{tikzcd}$$

in $\operatorname{Ho}\mathcal{C}$, applying the *rotation axiom* of a Verdier triangulated category twice shows that

$$\begin{tikzcd}
& & & 0 \\
& & 0 \arrow[r] & \mathrm{T}^2X \arrow[u] \\
& 0 \arrow[r] & \mathrm{T}Y \arrow[r, "\mathrm{T}v"] \arrow[u] & \mathrm{T}C \arrow[u, "\mathrm{T}w"'] \\
0 \arrow[r] & C \arrow[r, "w"] \arrow[u] & \mathrm{T}X \arrow[r] \arrow[u, "\mathrm{T}u"'] & 0 \arrow[u]
\end{tikzcd}$$

is also a Verdier triangle in $\operatorname{Ho}\mathcal{C}$. The stability under such an operation can also be generalised to arbitrary n-triangles: Given an n-triangle, the diagram obtained by taking as new base the second lowest row (in the periodic prolongation) is again an n-triangle. One says that n-triangles are *stable under translation.*

The unstable case

To state and prove the properties of n-triangles described above, one never uses the invertibility of the shift. In other words, "unstable n-triangles" may be defined in the homotopy category of *every* zero-pointed Brown cofibration category and then have the asserted properties.

However, there are some differences to the stable case: As the homotopy category of a stable Brown cofibration category is a Verdier triangulated category, it is in particular an additive category [33, prop. A.8(iii)]. This additivity does no longer hold in the general unstable case. Moreover, in the stable case, Verdier triangles may be periodically prolonged in two directions, using also the negative powers of the shift functor, and as a consequence of the rotation axiom, they are also "stable under translation in the negative direction". In the unstable case, the considered diagrams, which we then call *n-cosemitriangles*, are only stable under periodic prolongations in one direction, for lack of negative powers of the shift.

Combinatorics

Since n-cosemitriangles are quite large diagrams, the bookkeeping of the occurring data is a non-trivial task. To manage this, an underlying combinatorics for cosemitriangles is developed, as an unstable analogon to the combinatorics for Heller triangulated categories [23, sec. 1.1].
This combinatorics consists of two parts: First, we obtain for every $n \in \mathbb{N}_0$ a diagram category in which our n-cosemitriangles live, the *category of n-cosemistrips*. Second, these diagram categories in turn may be organised using a combinatorics that is a generalisation of the well-known combinatorics for simplicial sets: they form a so-called *semiquasicyclic category*. The stability of cosemitriangles under simplicial operations and translation may be shortly expressed as the fact that cosemitriangles form a semiquasicyclic subcategory of the semiquasicyclic category of n-cosemistrips, cf. proposition (5.50).

3 The main results

In this section, we state our main results, partly in informal terms and not necessarily in full generality.

Z-fractionable categories and the Z-2-arrow calculus

To prove the Z-2-arrow calculus, we work axiomatically and introduce the following notion.

Definition (Z-fractionable category, see (2.81)(a), (2.80)(a), (2.1)(a), (1.1)(a), (2.10), (1.35), (1.37), (2.65), (2.62), (2.68), (2.56), (2.70), (2.72), (2.75))**.** A *Z-fractionable category* consists of a category $\mathcal{C}$ together with the following data that is subject to the axioms listed below.

- Distinguished morphisms in $\mathcal{C}$, called *denominators*, which will in diagrams be depicted as

$$X \xrightarrow[\approx]{d} Y \ .$$

- Distinguished denominators in $\mathcal{C}$, called *S-denominators*, which will in diagrams be depicted as

$$X \overset{i}{\multimap} Y \ .$$

- Distinguished diagrams of the form

$$X \xrightarrow{f} \tilde{Y} \xleftarrow{i} Y$$

  in $\mathcal{C}$ ([7]), called *Z-2-arrows*.

A general diagram of the form

$$X \xrightarrow{f} \tilde{Y} \xleftarrow[\approx]{a} Y$$

in $\mathcal{C}$ will be called an *S-2-arrow* in $\mathcal{C}$, often denoted by (f, a).
The following axioms are supposed to hold.

(Cat) *Multiplicativity.* The denominators and the S-denominators are closed under composition in $\mathcal{C}$ and contain all identities in $\mathcal{C}$.

[7] In general, we do *not* take all diagrams of this form.

($2\,\mathrm{of}\,3_{\mathrm{S}}$) *S-part of 2 out of 3 axiom.* For all morphisms f and g in $\mathcal{C}$ such that f and fg are denominators, it follows that g is also a denominator.

($\mathrm{Ore}_{\mathrm{S}}^{\mathrm{wu}}$) *Weakly universal S-Ore completion axiom.* For every morphism f and every S-denominator i in $\mathcal{C}$ with $\operatorname{Source} f = \operatorname{Source} i$ there exists an S-2-arrow (f', i') in $\mathcal{C}$ such that i' is an S-denominator with $fi' = if'$, and such that for every S-2-arrow (g, a) in $\mathcal{C}$ with $fa = ig$ there exists a morphism c in $\mathcal{C}$ with $a = i'c$ and $g = f'c$.

$$\begin{tikzcd}
 & & {} \\
{} \arrow[r, "f'"] \arrow[urr, bend left, "g"] & {} \arrow[ur, dotted, "c"] & \\
{} \arrow[u, "i"] \arrow[r, "f"] & {} \arrow[u, "i'"] \arrow[uur, bend right, "a"', "\approx"] &
\end{tikzcd}$$

($\mathrm{Rpl}_{\mathrm{Z}}$) *Z-replacement axiom.* For every S-2-arrow (f, a) in $\mathcal{C}$ there exists a Z-2-arrow $(\dot{f}, \dot{a})$ and a morphism s in $\mathcal{C}$ with $f = \dot{f}s$ and $a = \dot{a}s$.

$$\begin{tikzcd}
{} \arrow[r, dotted, "\dot{f}"] \arrow[d, equal] & {} \arrow[d, dotted, "s"] & {} \arrow[l, dotted, "\dot{a}"'] \arrow[d, equal] \\
{} \arrow[r, "f"] & {} & {} \arrow[l, "a"', "\approx"]
\end{tikzcd}$$

($\mathrm{Rpl}_{\mathrm{Z}}^{\mathrm{den}}$) *Z-replacement axiom for denominators.* For every S-2-arrow (d, a) in $\mathcal{C}$ with denominator d there exists a Z-2-arrow $(\dot{d}, \dot{a})$ in $\mathcal{C}$ with denominator $\dot{d}$ and a morphism s in $\mathcal{C}$ with $d = \dot{d}s$ and $a = \dot{a}s$.

$$\begin{tikzcd}
{} \arrow[r, dotted, "\dot{d}", "\approx"'] \arrow[d, equal] & {} \arrow[d, dotted, "s"] & {} \arrow[l, dotted, "\dot{a}"'] \arrow[d, equal] \\
{} \arrow[r, "d", "\approx"'] & {} & {} \arrow[l, "a"', "\approx"]
\end{tikzcd}$$

($\mathrm{Rpl}_{\mathrm{Z}}^{\mathrm{rel}}$) *Relative Z-replacement axiom.* We suppose given a Z-2-arrow (f_1, i_1), an S-2-arrow (f_2, a_2) and morphisms $g_1, g_2, \tilde{g}_2$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, "\tilde{g}_2"] & {} \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
{} \arrow[r, "f_2"] & {} & {} \arrow[l, "a_2"', "\approx"]
\end{tikzcd}$$

commutes. Then there exist a Z-2-arrow $(\dot{f}_2, \dot{a}_2)$ and morphisms s, g in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & {} \arrow[rr, "f_1"] \arrow[dl, equal] \arrow[dd, "g_1"] & & {} \arrow[dl, equal] \arrow[dd, "\tilde{g}_2"] & & {} \arrow[ll, "i_1"'] \arrow[dl, equal] \arrow[dd, "g_2"] \\
{} \arrow[rr, "f_1"] \arrow[dd, "g_1"] & & {} \arrow[dd, dotted, "g"] & & {} \arrow[ll, "i_1"'] \arrow[dd, "g_2"] & \\
 & {} \arrow[rr, "f_2"] \arrow[dl, equal] & & {} & & {} \arrow[ll, "a_2"', "\approx"] \arrow[dl, equal] \\
{} \arrow[rr, dotted, "\dot{f}_2"] & & {} \arrow[ur, dotted, "s"] & & {} \arrow[ll, dotted, "\dot{a}_2"'] &
\end{tikzcd}$$

Moreover, we suppose to have the following additional assertions, respectively.

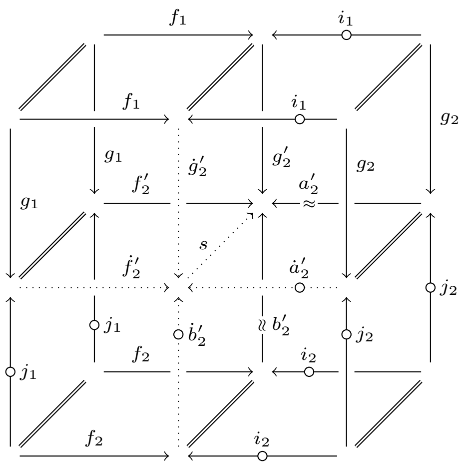
If g_1 and g_2 are denominators, then we suppose that g may be chosen to be a denominator.

If g_1 and g_2 are S-denominators, then we suppose that g may be chosen to be an S-denominator.

($\mathrm{Rpl}_{\mathrm{Z}}^{\mathrm{rel,Z}}$) *Relative Z-replacement axiom for Z-2-arrows.* We suppose given Z-2-arrows (f_1, i_1), (f_2, i_2), (g_1, j_1), (g_2, j_2) and S-2-arrows (f_2', a_2'), (g_2', b_2') in $\mathcal{C}$ such that the diagram

$$
\begin{tikzcd}
\cdot \arrow[r, "f_1"] \arrow[d, "g_1"'] & \cdot \arrow[d, "g_2'"'] & \cdot \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
\cdot \arrow[r, "f_2'"] & \cdot & \cdot \arrow[l, "a_2'"', "\approx"] \\
\cdot \arrow[u, "j_1"] \arrow[r, "f_2"'] & \cdot \arrow[u, "b_2'", "\approx"'] & \cdot \arrow[u, "j_2"'] \arrow[l, "i_2"]
\end{tikzcd}
$$

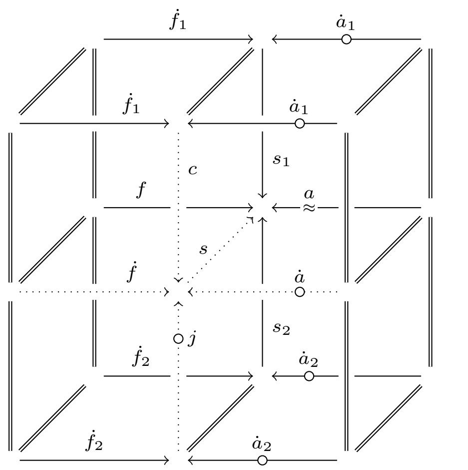
commutes. Then there exist Z-2-arrows $(\dot{f}_2', \dot{a}_2')$, $(\dot{g}_2', \dot{b}_2')$ and a morphism s in $\mathcal{C}$ such that the following diagram commutes.



($\mathrm{Cpr}_{\mathrm{Z}}$) *Z-comparison axiom.* We suppose given an S-2-arrow (f, a), Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{f}_2, \dot{a}_2)$ and morphisms s_1, s_2 in $\mathcal{C}$ such that the diagram

$$
\begin{tikzcd}
\cdot \arrow[r, "\dot{f}_1"] \arrow[d, equal] & \cdot \arrow[d, "s_1"] & \cdot \arrow[l, "\dot{a}_1"'] \arrow[d, equal] \\
\cdot \arrow[r, "f"] & \cdot & \cdot \arrow[l, "a"', "\approx"] \\
\cdot \arrow[u, equal] \arrow[r, "\dot{f}_2"'] & \cdot \arrow[u, "s_2"'] & \cdot \arrow[u, equal] \arrow[l, "\dot{a}_2"]
\end{tikzcd}
$$

commutes. Then there exist a Z-2-arrow $(\dot{f}, \dot{a})$, a normal S-2-arrow (c, j) and a morphism s in $\mathcal{C}$ such that the following diagram commutes.



(Cct$_\mathrm{Z}$) *Z-concatenation axiom.* For all Z-2-arrows (f_1, i_1), (f_2, i_2) in $\mathcal{C}$ with $\operatorname{Target}(f_1, i_1) = \operatorname{Source}(f_2, i_2)$ there exists a weakly universal S-Ore completion (f_2', i_1') for f_2 and i_1 such that $(f_1 f_2', i_2 i_1')$ is a Z-2-arrow in $\mathcal{C}$.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] & {} \arrow[rr, dotted, "f_2'"] & & {} \\
& & {} \arrow[ul, "i_1"] \arrow[rr, "f_2"] & & {} \arrow[ul, dotted, "i_1'"] \\
& & & & & {} \arrow[ul, "i_2"]
\end{tikzcd}$$

(Inv$_\mathrm{Z}$) *Z-inversion axiom.* Given a Z-2-arrow (f, i) in $\mathcal{C}$ such that f is a denominator, then (i, f) is a Z-2-arrow in $\mathcal{C}$.

(Num$_\mathrm{Z}$) *Z-numerator axiom.* For every Z-2-arrow (f, i) and every denominator d in $\mathcal{C}$ with $\operatorname{Source}(f, i) = \operatorname{Source} d$ there exists an S-2-arrow (f', d') in $\mathcal{C}$ with $fd' = df'$.

$$\begin{tikzcd}
{} \arrow[r, dotted, "f'"] & {} \\
{} \arrow[u, "d", "\wr"'] \arrow[r, "f"] & {} \arrow[u, dotted, "d'"', "\wr"] & {} \arrow[l, "i"']
\end{tikzcd}$$

(Exp$_\mathrm{Z}$) *Z-expansion axiom* Given a Z-2-arrow (f, i) and an S-denominator j in $\mathcal{C}$ with $\operatorname{Target} f = \operatorname{Target} i = \operatorname{Source} j$, then (fj, ij) is a Z-2-arrow in $\mathcal{C}$.

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} \arrow[d, "j"] & {} \arrow[l, "i"'] \arrow[d, equal] \\
{} \arrow[r, "fj"] & {} & {} \arrow[l, "ij"']
\end{tikzcd}$$

Theorem (construction of the S-Ore localisation, Z-2-arrow calculus, see (2.85), (2.93))**.** We suppose given a Z-fractionable category $\mathcal{C}$.

(a) There is a localisation $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$ of $\mathcal{C}$, called the *S-Ore localisation* of $\mathcal{C}$, whose objects are the same as the objects in $\mathcal{C}$ and whose morphisms are represented by S-2-arrows in $\mathcal{C}$.

(b) Every morphism in $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$ is actually represented by a Z-2-arrow in $\mathcal{C}$.

(c) Z-2-arrows (f, i), (f', i') in $\mathcal{C}$ represent the same morphism in $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$ if and only if they fit in a commutative diagram in $\mathcal{C}$ as follows.

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} \arrow[d, dotted, "\wr"] & {} \arrow[l, "i"'] \arrow[d, equal] \\
{} \arrow[r, dotted] & {} & {} \arrow[l, dotted] \\
{} \arrow[r, "f'"] \arrow[u, equal] & {} \arrow[u, dotted, "\wr"] & {} \arrow[l, "i'"'] \arrow[u, equal]
\end{tikzcd}$$

(d) We suppose given morphisms φ_1, φ_2, ψ_1, ψ_2 in $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$. Moreover, we suppose given Z-2-arrows (f_1, i_1), (f_2, i_2) and S-2-arrows (g_1, b_1), (g_2, b_2) in $\mathcal{C}$, representing φ_1, φ_2, ψ_1, ψ_2, respectively. We have $\varphi_1 \psi_2 = \psi_1 \varphi_2$ in $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$ if and only if the given S-2-arrows fit in a commutative diagram in $\mathcal{C}$ as follows.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, dotted, "\tilde{g}_2"] & {} \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
{} \arrow[r, dotted, "\tilde{f}_2"] & {} & {} \arrow[l, dotted, "\tilde{i}_2"'] \\
{} \arrow[r, "f_2"] \arrow[u, "\wr", "b_1"'] & {} \arrow[u, dotted, "\wr", "\tilde{b}_2"'] & {} \arrow[l, "i_2"'] \arrow[u, "\wr", "b_2"']
\end{tikzcd}$$

There is also the notion of a *Z-prefractionable category*, see definition (2.80)(a). Such a Z-prefractionable category has the same data as a Z-fractionable category, but only the axioms (Cat), ($2\,\mathrm{of}\,3_{\mathrm{S}}$), ($\mathrm{Ore}_{\mathrm{S}}^{\mathrm{wu}}$), ($\mathrm{Rpl}_{\mathrm{Z}}$), ($\mathrm{Rpl}_{\mathrm{Z}}^{\mathrm{rel}}$), ($\mathrm{Cpr}_{\mathrm{Z}}$) from the definition above are supposed to hold. Much of the theory for Z-fractionable categories developed in this thesis already holds for Z-prefractionable categories, for example, parts (a) to (c) and a weaker form of part (d) of the preceding theorem, see theorem (2.93)(c).

Cylinders in Brown cofibration categories

To make the results obtained for Z-fractionable categories available in the context of Brown cofibration categories, we have to show that a Brown cofibration category gives rise to a Z-fractionable category. For convenience, we recall the definition of a Brown cofibration category. The axioms listed here are equivalent to the dual axioms in [7, sec. 1, p. 421].

Definition (Brown cofibration category, see (3.52)(a), (3.51)(a), (3.30)(a), (3.1)(a), (3.14)(a), (1.35), (1.37), (1.36), (3.29), (3.40), (3.43)(c))**.** A *Brown cofibration category* consists of a category $\mathcal{C}$ together with the following data that is subject to the axioms listed below.

- Distinguished morphisms in $\mathcal{C}$, called *weak equivalences*, which will in diagrams be depicted as

$$X \xrightarrow[\approx]{w} Y\ .$$

- Distinguished morphisms in $\mathcal{C}$, called *cofibrations*, which will in diagrams be depicted as

$$X \overset{i}{\rightarrowtail} Y\ .$$

The following axioms are supposed to hold.

(Cat) *Multiplicativity.* The weak equivalences and the cofibrations are closed under composition in $\mathcal{C}$ and contain all identities in $\mathcal{C}$.

(2 of 3) *2 out of 3 axiom.* If two out of the three morphisms in a commutative triangle are weak equivalences, then so is the third.

(Iso) *Isosaturatedness.* Every isomorphism in $\mathcal{C}$ is a weak equivalence and a cofibration.

($\mathrm{Ini_c}$) *Existence of an initial object.* There exists an initial object in $\mathcal{C}$.

($\mathrm{Push_c}$) *Pushout axiom for cofibrations.* Given a morphism $f\colon X \to Y$ and a cofibration $i\colon X \to X'$ in $\mathcal{C}$, there exists a pushout rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i", tail] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

in $\mathcal{C}$ such that i' is a cofibration.

(Cof) *Cofibrancy axiom.* For every object X in $\mathcal{C}$ there exists an initial object I in $\mathcal{C}$ such that the unique morphism $I \to X$ is a cofibration.

($\mathrm{Fac_c}$) *Factorisation axiom for cofibrations.* For every morphism $f\colon X \to Y$ in $\mathcal{C}$ there exist a cofibration $i\colon X \to \tilde{Y}$ and a weak equivalence $w\colon \tilde{Y} \to Y$ in $\mathcal{C}$ such that $f = iw$.

$$\begin{tikzcd} & \tilde{Y} \arrow[dr, dotted, "w", "\approx"'] & \\ X \arrow[ur, dotted, tail, "i"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

(Inc$_\mathrm{c}$) *Incision axiom.* Given a pushout rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i", "\approx"'] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

in $\mathcal{C}$ such that i is a cofibration and a weak equivalence, then i' is a weak equivalence.

From the existence of an initial object, the cofibrancy axiom and the pushout axiom for cofibrations, it follows that every Brown cofibration category has finite coproducts.

Theorem (Brown cofibration categories as Z-fractionable categories, see (3.127), (3.124), (3.39), (3.7))**.** Every Brown cofibration category becomes a Z-fractionable category, where the denominators are the weak equivalences, the S-denominators are the weak equivalences that are cofibrations, and the Z-2-arrows are the S-2-arrows

$$X \xrightarrow{f} \tilde{Y} \xleftarrow[\approx]{i} Y$$

such that the induced morphism $X \amalg Y \to \tilde{Y}$ is a cofibration.

We conclude that the homotopy category of every Brown cofibration category admits a Z-2-arrow calculus as explained above.
The structure of a Z-fractionable category on a Brown cofibration category is based on the properties of cylinders in the following sense, which is a generalisation of Quillen's cylinder notion in [28, ch. I, sec. 1, def. 4].

Definition (cylinder, see (3.108)(a))**.** We suppose given a Brown cofibration category $\mathcal{C}$ and an S-2-arrow

$$X \xrightarrow{f} \tilde{Y} \xleftarrow[\approx]{u} Y$$

in $\mathcal{C}$. A *cylinder* of (f, u) consists of

- an object Z,
- a morphism $\mathrm{ins}_0 \colon X \to Z$, called *start insertion*,
- a weak equivalence $\mathrm{ins}_1 \colon Y \to Z$, called *end insertion*, and
- a weak equivalence $\mathrm{s} \colon Z \to \tilde{Y}$, called *cylinder equivalence*,

such that $i_0 s = f$, $i_1 s = u$, and such that the induced morphism $X \amalg Y \to Z$ is a cofibration.

$$\begin{tikzcd} X \arrow[r, "\mathrm{ins}_0"] \arrow[d, equal] & Z \arrow[d, "\mathrm{s}", "\approx"'] & Y \arrow[l, "\mathrm{ins}_1"', "\approx"] \arrow[d, equal] \\ X \arrow[r, "f"] & \tilde{Y} & Y \arrow[l, "u"', "\approx"] \end{tikzcd}$$

The absolute version in part (a) of the following lemma, which is central to our approach via Z-fractionable categories, is a generalisation of K. Brown's factorisation lemma in [7, sec. 1, p. 421] to S-2-arrows.

Lemma (Brown factorisation lemma, see (3.113))**.** We suppose given a Brown cofibration category $\mathcal{C}$.

(a) There exists a cylinder of every S-2-arrow in $\mathcal{C}$.

(b) We suppose given a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & \tilde{Y}_1 \arrow[d, "\tilde{g}_2"] & Y_1 \arrow[l, "u_1"', "\approx"] \arrow[d, "g_2"] \\ X'_2 \arrow[r, "f'_2"] & \tilde{Y}'_2 & Y'_2 \arrow[l, "u'_2"', "\approx"] \\ X_2 \arrow[u, "v_1"', "\approx"] \arrow[r, "f_2"] & \tilde{Y}_2 \arrow[u, "\tilde{v}_2"', "\approx"] & Y_2 \arrow[l, "u_2"', "\approx"] \arrow[u, "v_2"', "\approx"] \end{tikzcd}$$

in $\mathcal{C}$. For every cylinder Z_1 of (f_1, u_1) and every cylinder Z_2 of (f_2, u_2) there exists a cylinder Z'_2 of (f'_2, u'_2), fitting into a commutative diagram as follows.

$$\begin{tikzcd}
& X_1 \arrow[rr, "f_1"] \arrow[dl, equal] \arrow[dd, "g_1" near start] & & \tilde{Y}_1 \arrow[dd, "\tilde{g}_2" near start] & & Y_1 \arrow[ll, "u_1"', "\approx"] \arrow[dl, equal] \arrow[dd, "g_2"] \\
X_1 \arrow[rr, "\mathrm{ins}_0^{Z_1}"] \arrow[dd, "g_1"] & & Z_1 \arrow[ur, "\mathrm{s}^{Z_1}", "\sim"'] \arrow[dd, dotted, "g" near start] & & Y_1 \arrow[ll, "\mathrm{ins}_1^{Z_1}"', "\approx"] \arrow[dd, "g_2" near start] & \\
& X'_2 \arrow[rr, "f'_2"] \arrow[dl, equal] & & \tilde{Y}'_2 & & Y'_2 \arrow[ll, "u'_2"', "\approx"] \arrow[dl, equal] \\
X'_2 \arrow[rr, dotted, "\mathrm{ins}_0^{Z'_2}"] & & Z'_2 \arrow[ur, dotted, "\mathrm{s}^{Z'_2}", "\sim"'] & & Y'_2 \arrow[ll, dotted, "\mathrm{ins}_1^{Z'_2}"', "\approx"] & \\
& X_2 \arrow[rr, "f_2"] \arrow[uu, "v_1", "\sim"' near start] \arrow[dl, equal] & & \tilde{Y}_2 \arrow[uu, "\tilde{v}_2", "\sim"' near start] & & Y_2 \arrow[ll, "u_2"', "\approx"] \arrow[uu, "v_2", "\sim"'] \arrow[dl, equal] \\
X_2 \arrow[uu, "v_1", "\sim"'] \arrow[rr, "\mathrm{ins}_0^{Z_2}"] & & Z_2 \arrow[uu, dotted, "v", "\sim"'] \arrow[ur, "\mathrm{s}^{Z_2}", "\sim"'] & & Y_2 \arrow[uu, "v_2", "\sim"'] \arrow[ll, "\mathrm{ins}_1^{Z_2}"', "\approx"] &
\end{tikzcd}$$

Cosemitriangles on the homotopy category

From now on, we suppose given a zero-pointed Brown cofibration category, that is, a Brown cofibration category that is equipped with a (distinguished) zero object. The homotopy category of a Brown cofibration category carries a *shift functor*, as shown by K. Brown [7, dual of th. 3]. We give an isomorphic construction of this shift functor in chapter V, section 2, suitable to our needs.
In the following, we deal with diagrams on the semistrip type $\#_{+}^{n}$ for some $n \in \mathbb{N}_0$, a combinatorial construct introduced in definition (4.42). It may be depicted as follows.

$$\begin{tikzcd}[column sep=small]
& & & & & & & & \iddots & \vdots & \vdots & \\
& & & & & & & 2^{[1]}/2^{[1]} \arrow[r] & \dots \arrow[r] & n^{[1]}/2^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/2^{[1]} \arrow[r] \arrow[u] & \dots \\
& & & & & & 1^{[1]}/1^{[1]} \arrow[r] & 2^{[1]}/1^{[1]} \arrow[r] \arrow[u] & \dots \arrow[r] & n^{[1]}/1^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/1^{[1]} \arrow[r] \arrow[u] & \dots \\
& & & & & 0^{[1]}/0^{[1]} \arrow[r] & 1^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 2^{[1]}/0^{[1]} \arrow[r] \arrow[u] & \dots \arrow[r] & n^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/0^{[1]} \arrow[u] & \\
& & & & n/n \arrow[r] & 0^{[1]}/n \arrow[r] \arrow[u] & 1^{[1]}/n \arrow[r] \arrow[u] & 2^{[1]}/n \arrow[r] \arrow[u] & \dots \arrow[r] & n^{[1]}/n \arrow[u] & & \\
& & & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \iddots & & & \\
& & 2/2 \arrow[r] & \dots \arrow[r] & n/2 \arrow[r] \arrow[u] & 0^{[1]}/2 \arrow[r] \arrow[u] & 1^{[1]}/2 \arrow[r] \arrow[u] & 2^{[1]}/2 \arrow[u] & & & & \\
& 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & \dots \arrow[r] & n/1 \arrow[r] \arrow[u] & 0^{[1]}/1 \arrow[r] \arrow[u] & 1^{[1]}/1 \arrow[u] & & & & & \\
0/0 \arrow[r] & 1/0 \arrow[r] \arrow[u] & 2/0 \arrow[r] \arrow[u] & \dots \arrow[r] & n/0 \arrow[r] \arrow[u] & 0^{[1]}/0 \arrow[u] & & & & & &
\end{tikzcd}$$

The n-cosemitriangles in $\operatorname{Ho}\mathcal{C}$ are defined in three steps as follows.

Definition (Heller n-cosemistrip, see (5.33))**.** A *Heller n-cosemistrip* is a $\#_{+}^{n}$-commutative diagram X in $\mathcal{C}$ such that the entries on the "boundaries" are *coacyclic*, that is, the morphism from 0 to such an entry is a weak equivalence, and such that the "visible" quadrangles as depicted above are pushout rectangles with "vertical" cofibrations.

Definition (standard n-cosemitriangle, see (5.45))**.** A *standard n-cosemitriangle* in $\operatorname{Ho}\mathcal{C}$ is a diagram Y in $\operatorname{Ho}\mathcal{C}$ that is obtained from a Heller n-cosemistrip by "canonical isomorphic replacements".

Standard n-cosemitriangles have zeros at the "boundaries" by construction. Moreover, they turn out to be *periodic* diagrams in the sense of definition (4.55)(b).

Definition (n-cosemitriangle, see (5.51))**.** An *n-cosemitriangle* in $\operatorname{Ho}\mathcal{C}$ is a diagram Y in $\operatorname{Ho}\mathcal{C}$ that is isomorphic (in the category of periodic diagrams with zeros at the "boundaries") to a standard n-cosemitriangle.

The following theorem should be seen in analogy to some of the axioms of a triangulated category in the sense of Verdier, as explained in section 2.

Theorem (prolongation theorem, see (5.55))**.**

(a) Every diagram of $n-1$ composable morphisms in $\operatorname{Ho}\mathcal{C}$ may be prolonged to an n-cosemitriangle that has these $n-1$ composable morphisms in its lowest row, its *base*.

(b) Given n-cosemitriangles X and Y, then every morphism between its bases may be prolonged to a morphism in the category of such periodic diagrams.

Outline

We give a brief chapter-wise summary of the contents of this thesis. More details can be found in the introductions to each chapter.
In chapter I, we define localisations of categories, fix notations and terminology and recall some basic results. Then in chapter II, we develop our localisation theory leading to the Z-2-arrow calculus: We postulate the axioms of a Z-fractionable category, construct the S-Ore localisation of such a structure and show that this localisation admits a Z-2-arrow calculus in the sense of theorem (2.93). The results are applied to Brown cofibration categories in chapter III, where we show that the latter fit into the framework of Z-fractionable categories. In chapter IV, we study the combinatorics for an unstable higher triangulation on the homotopy category of a Brown cofibration category, which is finally introduced in chapter V by means of the Z-2-arrow calculus.

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Aachen, August 28, 2012
Sebastian Thomas

Conventions and notations

We use the following conventions and notations.

- The composite of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is usually denoted by $fg\colon X \to Z$. The composite of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is usually denoted by $G \circ F\colon \mathcal{C} \to \mathcal{E}$.

- Given objects X and Y in a category $\mathcal{C}$, we denote the set of morphisms from X to Y by ${}_{\mathcal{C}}(X, Y)$.

- Given a category $\mathcal{C}$, we denote by $\operatorname{Iso}\mathcal{C}$ the set of isomorphisms in $\mathcal{C}$.

- If X is isomorphic to Y, we write $X \cong Y$.

- We suppose given categories $\mathcal{C}$ and $\mathcal{D}$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be an isofunctor if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$. The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be isomorphic, written $\mathcal{C} \cong \mathcal{D}$, if an isofunctor $F\colon \mathcal{C} \to \mathcal{D}$ exists.

  A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be an equivalence (of categories) if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$. Such a functor G is then called an isomorphism inverse of F. The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent, written $\mathcal{C} \simeq \mathcal{D}$, if an equivalence of categories $F\colon \mathcal{C} \to \mathcal{D}$ exists.

- Given a category $\mathcal{C}$ and a graph S, a diagram in $\mathcal{C}$ over S is a graph morphism $X\colon S \to \mathcal{C}$. The category of diagrams in $\mathcal{C}$ over S is denoted by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{Grph}}$. Given a diagram X in $\mathcal{C}$ over S, we usually denote the image of a morphism $a\colon i \to j$ in S by $X_a\colon X_i \to X_j$.

- Given categories $\mathcal{C}$ and S, an S-commutative diagram in $\mathcal{C}$ is a functor $X\colon S \to \mathcal{C}$. The category of S-commutative diagrams in $\mathcal{C}$ is denoted by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{Cat}}$. Given an S-commutative diagram X in $\mathcal{C}$, we usually denote the image of a morphism $a\colon i \to j$ in S by $X_a\colon X_i \to X_j$. In particular contexts, we also use the notation $X^a\colon X^i \to X^j$.

- The opposite category of a category $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$.

- We usually identify a poset X and its associated category that has as set of objects the underlying set of X and precisely one morphism $x \to y$ for $x, y \in \operatorname{Ob} P = P$ if and only if $x \leq y$. A full subposet is a subposet that is full as a subcategory.

- Given a subobject U of an object X, we denote by $\mathrm{inc} = \mathrm{inc}^U\colon U \to X$ the inclusion. Dually, given a quotient object Q of an object X, we denote by $\mathrm{quo} = \mathrm{quo}^Q\colon X \to Q$ the quotient morphism.

- Given a coproduct C of X_1 and X_2, the embedding $X_k \to C$ is denoted by $\mathrm{emb}_k = \mathrm{emb}^C_k$ for $k \in \{1, 2\}$. Given morphisms $f_k\colon X_k \to Y$ for $k \in \{1, 2\}$, the induced morphism $C \to Y$ is denoted by $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^C$.

- Given an initial object I, the unique morphism $I \to X$ to an object X will be denoted by $\mathrm{ini} = \mathrm{ini}_X = \mathrm{ini}^I_X$. Dually, given a terminal object T, the unique morphism $X \to T$ from an object T will be denoted by $\mathrm{ter} = \mathrm{ter}_X = \mathrm{ter}^T_X$. Given a zero object N, the unique morphism $X \to Y$ that factors over N will be denoted by 0.

- Given a category that has an initial object, we denote by ¡ a chosen initial object. Given a category that has binary coproducts and objects X_1, X_2, we denote by $X_1 \amalg X_2$ a chosen coproduct. Analogously, given morphisms $f_k\colon X_k \to Y_k$ for $k \in \{1, 2\}$, the coproduct of f_1 and f_2 is denoted by $f_1 \amalg f_2$.

- Given a category that has a zero object, we denote by 0 a chosen zero object.
- A zero-pointed category is a category together with a (distinguished) zero object. A morphism of zero-pointed categories is a functor that preserves the zero-objects.
- Arrows a and b in an (oriented) graph are called parallel if $\operatorname{Source} a = \operatorname{Source} b$ and $\operatorname{Target} a = \operatorname{Target} b$.
- We use the notations $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- Given integers $a, b \in \mathbb{Z}$, we write $[a, b] := \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ for the set of integers lying between a and b. Sometimes (for example in composites), we need some specified orientation, then we write $\lceil a, b \rceil := (z \in \mathbb{Z} \mid a \leq z \leq b)$ for the *ascending interval* and $\lfloor a, b \rfloor = (z \in \mathbb{Z} \mid a \geq z \geq b)$ for the *descending interval.*
- Given a map $f \colon X \to Y$ and subsets $X' \subseteq X$, $Y' \subseteq Y$ with $X'f \subseteq Y'$, we denote by $f|_{X'}^{Y'}$ the map $X' \to Y'$, $x' \mapsto x'f$. In the special cases, where $Y' = Y$ resp. $X' = X$, we also write $f|_{X'} := f|_{X'}^{Y}$ resp. $f|^{Y'} := f|_{X}^{Y'}$. Likewise for functors.
- When defining a category via its hom-sets, these are considered to be formally disjoint. In other words, a morphism between two given objects is formally seen a triple consisting of an underlying morphism and its source and target object. Cf. appendix A, section 1.
- In a poset, an expression like "$i \leq k$, $j \leq l$" has to be read as $i \leq k$ and $j \leq l$ (and not as $i \leq k \leq l$ and $i \leq j \leq l$).
- If unambigous, we denote a twoangle, a triangle, a quadrangle occurring in a diagram as the tuple of its corners.
- Given a quadrangle X in a category $\mathcal{C}$, that is, a $\square$-commutative diagram in $\mathcal{C}$, where $\square = \Delta^1 \times \Delta^1$, we write $X_{(0,0),(1,0),(0,1),(1,1)} = (X_{0,0}, X_{1,0}, X_{0,1}, X_{1,1}) = X$.
- For $n \in \mathbb{N}_0$, we denote by $\Delta^n = \Delta^n_{\mathbf{Cat}}$ the n-th simplex type, that is, the poset given by the underlying set $[0, n]$ together with the natural order.

A remark on Grothendieck universes To avoid set-theoretical difficulties, we work with Grothendieck universes [1, exp. I, sec. 0] in this thesis. In particular, every category has a *set* of objects and a *set* of morphisms. Given a Grothendieck universe $\mathfrak{U}$, we say that a set X is a $\mathfrak{U}$-set if it is an element of $\mathfrak{U}$. We say that a category $\mathcal{C}$ is a $\mathfrak{U}$-*category* if $\operatorname{Ob} \mathcal{C}$ and $\operatorname{Mor} \mathcal{C}$ are elements of $\mathfrak{U}$. The *category of* $\mathfrak{U}$-*categories*, whose set of objects consists of all $\mathfrak{U}$-categories and whose set of morphisms consists of all functors between $\mathfrak{U}$-categories (and source, target, composition and identities given by ordinary source, target, composition of functors and the identity functors, respectively), will be denoted $\mathbf{Cat} = \mathbf{Cat}_{(\mathfrak{U})}$.

Chapter I

Localisations of categories

A localisation of a category $\mathcal{C}$ with respect to a subset D of its set of morphisms $\operatorname{Mor}\mathcal{C}$ is the universal category where the morphisms in D become invertible. Such a localisation always exists by a theorem of Gabriel and Zisman [12, sec. 1.1], cf. theorem (1.24). We will not make use of this result in this and the following chapter. In this chapter, we will recall the precise definition of a localisation, see definition (1.11)(a), and deduce some standard properties. The obtained results are not very difficult to prove and are folklore. In particular, the author does not claim any originality for the content of this chapter.
The main purpose of this chapter is to fix notation and to prepare the language for chapter II, where a localisation for a so-called Z-fractionable category, see definition (2.81)(a), and so in particular for a Brown cofibration category, cf. theorem (3.127), is constructed and several properties in that context, in particular the Z-2-arrow calculus (2.93), are proven.
The chapter is organised as follows. In section 1, we introduce the structure of a category with denominators, which allows us to define localisation as a categorical concept. The notion of a localisation is studied in section 2, together with some general consequences that can be deduced from the universal property. At the end of section 2, we briefly recall the Gabriel-Zisman localisation. Finally, we consider the saturation and some notions of saturatedness in section 3, that is, various closure properties of the set of denominators, the strongest one demanding that all morphisms that become isomorphisms in the localisation are already denominators.

1 Categories with denominators

A localisation of a given category $\mathcal{C}$ can be defined with respect to every subset D of $\operatorname{Mor}\mathcal{C}$, see definition (1.11)(a). We may consider $\mathcal{C}$ together with such a distinguished subset as a structure, called a category with denominators, see definition (1.1)(a). This allows us to embed localisation theory of categories in a categorical setup. For example, as localisations are defined via a universal property, we will get some general properties of localisations from the general theory on couniversal objects, cf. appendix B, remark (1.13) and corollary (1.14). In particular, we may construct a functor that maps categories with denominators to (a choice of) respective localisations, see corollary (1.14)(d).

Definition of a category with denominators

(1.1) Definition (category with denominators)**.**

(a) A *category with denominators* consists of a category $\mathcal{C}$ together with a subset $D \subseteq \operatorname{Mor}\mathcal{C}$. By abuse of notation, we refer to the said category with denominators as well as to its underlying category just by $\mathcal{C}$. The elements of D are called *denominators* in $\mathcal{C}$.

Given a category with denominators $\mathcal{C}$ with set of denominators D, we write $\operatorname{Den}\mathcal{C} := D$. In diagrams, a denominator $d\colon X \to Y$ in $\mathcal{C}$ will usually be depicted as

$$X \xrightarrow[\approx]{d} Y\ .$$

(b) We suppose given categories with denominators $\mathcal{C}$ and $\mathcal{D}$. A *morphism of categories with denominators* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ that *preserves denominators*, that is, such that Fd is a denominator in $\mathcal{D}$ for every denominator d in $\mathcal{C}$.

(1.2) Example.

(a) Every category $\mathcal{C}$ carries the structure of a category with denominators having

$$\operatorname{Den}\mathcal{C} = \{1_X \mid X \in \operatorname{Ob}\mathcal{C}\}.$$

(b) Every category $\mathcal{C}$ carries the structure of a category with denominators having

$$\operatorname{Den}\mathcal{C} = \operatorname{Iso}\mathcal{C}.$$

The notion of a category with denominators is self-dual:

(1.3) Remark. Given a category with denominators $\mathcal{C}$, its opposite category $\mathcal{C}^{\mathrm{op}}$ becomes a category with denominators with $\operatorname{Den}(\mathcal{C}^{\mathrm{op}}) = \operatorname{Den}\mathcal{C}$.

The category of categories with denominators

(1.4) Definition (category with denominators with respect to a Grothendieck universe)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. A category with denominators $\mathcal{C}$ is called a *category with denominators with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$-*category with denominators*) if its underlying category is a $\mathfrak{U}$-category.

(1.5) Remark.

(a) We suppose given a Grothendieck universe $\mathfrak{U}$. A category with denominators $\mathcal{C}$ is a $\mathfrak{U}$-category with denominators if and only if it is an element of $\mathfrak{U}$.

(b) For every category with denominators $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a $\mathfrak{U}$-category with denominators.

(1.6) Remark. For every Grothendieck universe $\mathfrak{U}$ we have a category $\mathbf{CatD}_{(\mathfrak{U})}$, given as follows. The set of objects of $\mathbf{CatD}_{(\mathfrak{U})}$ is given by

$$\operatorname{Ob}\mathbf{CatD}_{(\mathfrak{U})} = \{\mathcal{C} \mid \mathcal{C} \text{ is a } \mathfrak{U}\text{-category with denominators}\}.$$

For objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathbf{CatD}_{(\mathfrak{U})}$, we have the hom-set

$${}_{\mathbf{CatD}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D}) = \{F \mid F \text{ is a morphism of categories with denominators from } \mathcal{C} \text{ to } \mathcal{D}\}.$$

For morphisms $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ in $\mathbf{CatD}_{(\mathfrak{U})}$, the composite of F and G in $\mathbf{CatD}_{(\mathfrak{U})}$ is given by the composite of the underlying functors $G \circ F\colon \mathcal{C} \to \mathcal{E}$. For an object $\mathcal{C}$ in $\mathbf{CatD}_{(\mathfrak{U})}$, the identity morphism on $\mathcal{C}$ in $\mathbf{CatD}_{(\mathfrak{U})}$ is given by the underlying identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.

(1.7) Definition (category of categories with denominators)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. The category $\mathbf{CatD} = \mathbf{CatD}_{(\mathfrak{U})}$ as considered in remark (1.6) is called the *category of categories with denominators* (more precisely, the *category of* $\mathfrak{U}$-*categories with denominators*).

The discrete structure and the isomorphism structure

In example (1.2)(b), we have seen that every category can be equipped with the structure of a category with denominators, where the set of denominators is given by the subset of its isomorphisms. Since we will need this canonical structure in section 2 to embed localisation theory of categories in a categorical setup, we assign a name to it.

(1.8) Definition (discrete structure, isomorphism structure)**.** We suppose given a category $\mathcal{C}$.

(a) We denote by $\mathcal{C}_{\mathrm{disc}}$ the category with denominators whose underlying category is $\mathcal{C}$ and whose set of denominators is

$$\operatorname{Den}\mathcal{C}_{\mathrm{disc}} = \{1_X \mid X \in \operatorname{Ob}\mathcal{C}\}.$$

The structure of a category with denominators of $\mathcal{C}_{\mathrm{disc}}$ is called the *discrete structure* (*of a category with denominators*) on $\mathcal{C}$.

(b) We denote by $\mathcal{C}_{\mathrm{iso}}$ the category with denominators whose underlying category is $\mathcal{C}$ and whose set of denominators is

$$\operatorname{Den}\mathcal{C}_{\mathrm{iso}} = \operatorname{Iso}\mathcal{C}.$$

The structure of a category with denominators of $\mathcal{C}_{\mathrm{disc}}$ is called the *isomorphism structure* (*of a category with denominators*) on $\mathcal{C}$.

(1.9) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$.

(a) We have a functor

$$-_{\mathrm{disc}} \colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{CatD}_{(\mathfrak{U})},$$

given on the morphisms by $F_{\mathrm{disc}} = F$ for $F \in \operatorname{Mor}\mathbf{Cat}_{(\mathfrak{U})}$, which is full, faithful and injective on the objects.

(b) We have a functor

$$-_{\mathrm{iso}} \colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{CatD}_{(\mathfrak{U})},$$

given on the morphisms by $F_{\mathrm{iso}} = F$ for $F \in \operatorname{Mor}\mathbf{Cat}_{(\mathfrak{U})}$, which is full, faithful and injective on the objects.

Diagram categories

Given a category with denominators $\mathcal{C}$ and a category S, we denote by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{Cat}}$ the category of S-commutative diagrams in $\mathcal{C}$ (that is, the category of functors from S to $\mathcal{C}$). If unambiguous, we will consider $\mathcal{C}^S$ as a category with denominators in the following way, without further comment.

(1.10) Remark. Given a category with denominators $\mathcal{C}$ and a category S, then $\mathcal{C}^S$ becomes a category with denominators having

$$\operatorname{Den}\mathcal{C}^S = \{d \in \operatorname{Mor}\mathcal{C}^S \mid d_i \text{ is a denominator in } \mathcal{C} \text{ for every } i \in \operatorname{Ob}S\}.$$

2 Localisations

In this section, we introduce localisations of categories with denominators, see definition (1.11), and deduce some general properties. The developed facts are direct consequences of the universal property that defines a localisation, see in particular remark (1.13) and corollary (1.14). At the end of the section, we will briefly recall the Gabriel-Zisman localisation, but we will not make use of it in the rest of this chapter I and in chapter II. We will use the Gabriel-Zisman localisation in the definition of the homotopy category for an arbitrary category with weak equivalences, see definition (3.8).

The existence of a localisation for a so-called Z-prefractionable category, see definition (2.80)(a), will be shown in theorem (2.85).

Definition of a localisation

For the definition of a category with denominators, see definition (1.1).

(1.11) Definition (localisation)**.**

(a) We suppose given a category $\mathcal{C}$ and a subset D of $\operatorname{Mor}\mathcal{C}$. A *localisation* of $\mathcal{C}$ with respect to D consists of a category $\mathcal{L}$ and a functor $L\colon \mathcal{C} \to \mathcal{L}$ with Ld invertible in $\mathcal{L}$ for every $d \in D$, and such that for every category $\mathcal{D}$ and every functor $F\colon \mathcal{C} \to \mathcal{D}$ with Fd invertible in $\mathcal{D}$ for every $d \in D$, there exists a unique functor $\hat{F}\colon \mathcal{L} \to \mathcal{D}$ with $F = \hat{F} \circ L$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \\ \mathcal{L} \arrow[ur, dotted, "\hat{F}"'] & \end{tikzcd}$$

By abuse of notation, we refer to the said localisation as well as to its underlying category by $\mathcal{L}$. The functor L is said to be the *localisation functor* of $\mathcal{L}$.

Given a localisation $\mathcal{L}$ of $\mathcal{C}$ with localisation functor $L\colon \mathcal{C} \to \mathcal{L}$, we write $\mathrm{loc} = \mathrm{loc}^{\mathcal{L}} := L$.

(b) Given a category with denominators $\mathcal{C}$, a *localisation* of $\mathcal{C}$ is a localisation of the underlying category of $\mathcal{C}$ with respect to its set of denominators $\operatorname{Den}\mathcal{C}$.

The definition (1.11)(a) of a localisation of a category with respect to a given subset of morphisms and the definition (1.11)(b) of a localisation of a category with denominators describe almost the same issue, but from different point of views. While it is more convenient to speak of localisations with respect to a subset when such a subset is varied, we very often deal with fixed subsets and therefore prefer to work with categories with denominators.

Gabriel and Zisman have shown in [12, sec. 1.1] that there exists a localisation of every category $\mathcal{C}$ with respect to an arbitrary subset D of $\operatorname{Mor}\mathcal{C}$, see theorem (1.24). We will not make use of this result in the construction in chapter II. Rather, given a Z-prefractionable category, see definition (2.80)(a), we construct a localisation directly, see theorem (2.85), generalising the construction of the well-known Ore localisation, cf. chapter II, section 3.

(1.12) Example. We suppose given a category $\mathcal{C}$.

(a) The category $\mathcal{C}$ becomes a localisation of the discrete structure $\mathcal{C}_{\mathrm{disc}}$, where the localisation functor $\mathrm{loc}\colon \mathcal{C}_{\mathrm{disc}} \to \mathcal{C}$ is given by $\mathrm{loc} = \mathrm{id}_{\mathcal{C}}$.

(b) The category $\mathcal{C}$ becomes a localisation of the isomorphism structure $\mathcal{C}_{\mathrm{iso}}$, where the localisation functor $\mathrm{loc}\colon \mathcal{C}_{\mathrm{iso}} \to \mathcal{C}$ is given by $\mathrm{loc} = \mathrm{id}_{\mathcal{C}}$.

Proof.

(a) The identity 1_X of every object X in $\mathcal{C}$ is invertible, that is, $\mathrm{id}_{\mathcal{C}}(d) = d$ is invertible in $\mathcal{C}$ for every denominator d in $\mathcal{C}_{\mathrm{disc}}$. To show that $\mathcal{C}$ becomes a localisation of $\mathcal{C}_{\mathrm{disc}}$ with $\mathrm{loc} = \mathrm{id}_{\mathcal{C}}$, we suppose given a category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible for every denominator d in $\mathcal{C}$. Since the image of an identity under an arbitrary functor is an identity and hence invertible, this just means that F is an arbitrary functor. Now $\hat{F} = F\colon \mathcal{C} \to \mathcal{D}$ is the unique functor with $F = \hat{F} \circ \mathrm{id}_{\mathcal{C}}$.

(b) Similarly to (a). □

Consequences of the universal property

Localisations are defined by a universal property and can therefore be interpreted as couniversal objects, see definition (B.2):

(1.13) Remark. We suppose given a category with denominators $\mathcal{C}$. Given a localisation $\mathcal{L}$ of $\mathcal{C}$, then $\mathcal{L}$ becomes a couniversal object under $\mathcal{C}$ along the functor $-_{\mathrm{iso}}\colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{CatD}_{(\mathfrak{U})}$ for every Grothendieck universe $\mathfrak{U}$ with $\mathcal{C} \in \operatorname{Ob}\mathbf{CatD}_{(\mathfrak{U})}$, $\mathcal{L} \in \operatorname{Ob}\mathbf{Cat}_{(\mathfrak{U})}$, where the universal morphism is given by $\mathrm{uni}^{\mathcal{L}} = \mathrm{loc}^{\mathcal{L}}$. Conversely, given a category $\mathcal{L}$ and a functor $L\colon \mathcal{C} \to \mathcal{L}$ such that $\mathcal{L}$ becomes a couniversal object under $\mathcal{C}$ along the functor $-_{\mathrm{iso}}\colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{CatD}_{(\mathfrak{U})}$ for every Grothendieck universe $\mathfrak{U}$ with $\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})}$, $\mathcal{L} \in \operatorname{Ob} \mathbf{Cat}_{(\mathfrak{U})}$ and such that $\mathrm{uni}^{\mathcal{L}} = L$, then $\mathcal{L}$ becomes a localisation of $\mathcal{C}$ with localisation functor $\mathrm{loc}^{\mathcal{L}} = \mathrm{uni}^{\mathcal{L}}$.

(1.14) Corollary.

(a) We suppose given a category with denominators $\mathcal{C}$ and localisations $\mathcal{L}$, $\mathcal{L}'$ of $\mathcal{C}$. We let $\hat{L}\colon \mathcal{L}' \to \mathcal{L}$ denote the unique functor with $\mathrm{loc}^{\mathcal{L}} = \hat{L} \circ \mathrm{loc}^{\mathcal{L}'}$, and we let $\hat{L}'\colon \mathcal{L} \to \mathcal{L}'$ denote the unique functor with $\mathrm{loc}^{\mathcal{L}'} = \hat{L}' \circ \mathrm{loc}^{\mathcal{L}}$. Then $\hat{L}$ and $\hat{L}'$ are mutually inverse isofunctors.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{loc}^{\mathcal{L}'}"] \arrow[d, "\mathrm{loc}^{\mathcal{L}}"'] & \mathcal{L}' \arrow[dl, bend right, "\hat{L}"'] \arrow[dl, phantom, "\cong"] \\ \mathcal{L} \arrow[ur, bend right, "\hat{L}'"'] & \end{tikzcd}$$

(b) We suppose given a morphism of categories with denominators $F\colon \mathcal{C}_1 \to \mathcal{C}_2$, a localisation $\mathcal{L}_1$ of $\mathcal{C}_1$ and a localisation $\mathcal{L}_2$ of $\mathcal{C}_2$. There exists a unique functor $\hat{F}\colon \mathcal{L}_1 \to \mathcal{L}_2$ with $\mathrm{loc}^{\mathcal{L}_2} \circ F = \hat{F} \circ \mathrm{loc}^{\mathcal{L}_1}$.

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[d, "\mathrm{loc}^{\mathcal{L}_1}"'] & \mathcal{C}_2 \arrow[d, "\mathrm{loc}^{\mathcal{L}_2}"] \\ \mathcal{L}_1 \arrow[r, dotted, "\hat{F}"] & \mathcal{L}_2 \end{tikzcd}$$

(c) We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. Moreover, we suppose given an isomorphism of categories with denominators $F\colon \mathcal{C} \to \mathcal{C}'$ and an isofunctor $G\colon \mathcal{L} \to \mathcal{L}'$. Then $\mathcal{L}'$ becomes a localisation of $\mathcal{C}'$ with localisation functor $\mathrm{loc}^{\mathcal{L}'} = G \circ \mathrm{loc}^{\mathcal{L}} \circ F^{-1}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F", "\cong"'] \arrow[d, "\mathrm{loc}^{\mathcal{L}}"'] & \mathcal{C}' \arrow[d, "\mathrm{loc}^{\mathcal{L}'}"] \\ \mathcal{L} \arrow[r, "G", "\cong"'] & \mathcal{L}' \end{tikzcd}$$

Given a category $\mathcal{D}$ and a functor $H'\colon \mathcal{C}' \to \mathcal{D}$ such that $H'd'$ is invertible for every denominator d' in $\mathcal{C}'$, the unique functor $\hat{H}'\colon \mathcal{L}' \to \mathcal{D}$ with $H' = \hat{H}' \circ \mathrm{loc}^{\mathcal{L}'}$ is given by

$$\hat{H}' = \hat{H} \circ \hat{F}^{-1},$$

where $\hat{H}\colon \mathcal{L} \to \mathcal{D}$ is the unique functor with $H' \circ F = \hat{H} \circ \mathrm{loc}^{\mathcal{L}}$.

(d) We suppose given a Grothendieck universe $\mathfrak{U}$ and a subcategory $\mathcal{U}$ of $\mathbf{CatD}_{(\mathfrak{U})}$. Moreover, we suppose given a family $(\mathcal{L}_{\mathcal{C}})_{\mathcal{C} \in \operatorname{Ob} \mathcal{U}}$ such that $\mathcal{L}_{\mathcal{C}}$ is a localisation of $\mathcal{C}$ and such that the underlying category of $\mathcal{L}_{\mathcal{C}}$ is a $\mathfrak{U}$-category for every $\mathcal{C} \in \operatorname{Ob} \mathcal{U}$. Then we have a functor $\mathcal{L}\colon \mathcal{U} \to \mathbf{Cat}_{(\mathfrak{U})}$, given as follows. For $\mathcal{C} \in \operatorname{Ob} \mathcal{U}$, we have

$$\mathcal{L}\mathcal{C} = \mathcal{L}_{\mathcal{C}}.$$

For every morphism $F\colon \mathcal{C} \to \mathcal{C}'$ in $\mathcal{U}$, the morphism $\mathcal{L}F\colon \mathcal{L}\mathcal{C} \to \mathcal{L}\mathcal{C}'$ in $\mathbf{Cat}_{(\mathfrak{U})}$ is the unique morphism in $\mathbf{Cat}_{(\mathfrak{U})}$ with $\mathrm{loc}^{\mathcal{L}\mathcal{C}'} \circ F = \mathcal{L}F \circ \mathrm{loc}^{\mathcal{L}\mathcal{C}}$.

(e) We suppose given a Grothendieck universe $\mathfrak{U}$, a $\mathfrak{U}$-category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$ such that the underlying category of $\mathcal{L}$ is a $\mathfrak{U}$-category. The maps

$${}_{\mathbf{Cat}_{(\mathfrak{U})}}(\mathcal{L}, \mathcal{D}) \to {}_{\mathbf{CatD}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D}_{\mathrm{iso}}),\ G \mapsto G \circ \mathrm{loc}^{\mathcal{L}}$$

for $\mathcal{D} \in \operatorname{Ob} \mathbf{Cat}_{(\mathfrak{U})}$ define an isotransformation

$${}_{\mathbf{Cat}_{(\mathfrak{U})}}(\mathcal{L}, -) \to {}_{\mathbf{CatD}_{(\mathfrak{U})}}(\mathcal{C}, -_{\mathrm{iso}}).$$

(f) We suppose given a Grothendieck universe $\mathfrak{U}$, a family $(\mathcal{L}_{\mathcal{C}})_{\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})}}$ such that $\mathcal{L}_{\mathcal{C}}$ is a localisation of $\mathcal{C}$ and such that the underlying category of $\mathcal{L}_{\mathcal{C}}$ is a $\mathfrak{U}$-category for every $\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})}$. Moreover we let $\mathcal{L} \colon \mathbf{CatD}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})}$ be the functor with $\mathcal{L}\mathcal{C} = \mathcal{L}_{\mathcal{C}}$ for $\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})}$ and where $\mathcal{L}F \colon \mathcal{L}\mathcal{C} \to \mathcal{L}\mathcal{C}'$ for a morphism $F \colon \mathcal{C} \to \mathcal{C}'$ in $\mathbf{CatD}_{(\mathfrak{U})}$ is the unique morphism in $\mathbf{Cat}_{(\mathfrak{U})}$ with $\operatorname{loc}^{\mathcal{L}_{\mathcal{C}'}} \circ F = \mathcal{L}F \circ \operatorname{loc}^{\mathcal{L}_{\mathcal{C}}}$. Then $\mathcal{L}$ is left adjoint to the functor $-_{\text{iso}} \colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{CatD}_{(\mathfrak{U})}$. An adjunction $\Phi \colon \mathcal{L} \dashv -_{\text{iso}}$ is given by

$$\Phi_{\mathcal{C},\mathcal{D}} \colon {}_{\mathbf{Cat}_{(\mathfrak{U})}}(\mathcal{L}\mathcal{C}, \mathcal{D}) \to {}_{\mathbf{CatD}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D}),\ G \mapsto G \circ \operatorname{loc}^{\mathcal{L}\mathcal{C}}$$

for $\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})}$, $\mathcal{D} \in \operatorname{Ob} \mathbf{Cat}_{(\mathfrak{U})}$.

(g) Given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$, the following assertions are equivalent.

  (i) Every denominator in $\mathcal{C}$ is invertible in $\mathcal{C}$.

  (ii) The localisation functor $\operatorname{loc} \colon \mathcal{C} \to \mathcal{L}$ is an isofunctor.

Proof.

(a) This follows from remark (1.13) and remark (B.4).

(b) This follows from remark (1.13) and remark (B.5).

(c) This follows from remark (1.13) and corollary (B.7).

(d) This follows from remark (1.13) and remark (B.19).

(e) This follows from remark (1.13) and remark (B.3)(a).

(f) This follows from remark (1.13) and theorem (B.21).

(g) This follows from remark (1.13) and proposition (B.8). □

(1.15) Proposition. We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. The localisation functor $\operatorname{loc} \colon \mathcal{C} \to \mathcal{L}$ is surjective on the objects.

Proof. We let $\mathcal{U}$ be the full subcategory of $\mathcal{L}$ with $\operatorname{Ob} \mathcal{U} = \operatorname{Im}(\operatorname{Ob} \operatorname{loc})$. By the universal property of $\mathcal{L}$, there exists a unique functor $\hat{L} \colon \mathcal{L} \to \mathcal{U}$ with $\operatorname{loc}|^{\mathcal{U}} = \hat{L} \circ \operatorname{loc}$. Thus we have $\operatorname{loc} = \operatorname{inc}^{\mathcal{U}} \circ \operatorname{loc}|^{\mathcal{U}} = \operatorname{inc}^{\mathcal{U}} \circ \hat{L} \circ \operatorname{loc}$ and therefore $\operatorname{id}_{\mathcal{L}} = \operatorname{inc}^{\mathcal{U}} \circ \hat{L}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\operatorname{loc}|^{\mathcal{U}}"] \arrow[d, "\operatorname{loc}"'] & \mathcal{U} \arrow[r, "\operatorname{inc}"] & \mathcal{L} \\
\mathcal{L} \arrow[ur, dotted, "\hat{L}"] \arrow[urr, equal] & &
\end{tikzcd}$$

Hence $\operatorname{Ob} \operatorname{inc}^{\mathcal{U}}$ is surjective, and so we have $\mathcal{U} = \mathcal{L}$. In particular, we have $\operatorname{Im}(\operatorname{Ob} \operatorname{loc}) = \operatorname{Ob} \mathcal{U} = \operatorname{Ob} \mathcal{L}$, that is, loc is surjective on the objects. □

The following proposition states that localisations also fulfill a universal property with respect to transformations. This 2-universality is a consequence of the 1-universality that holds by definition. The trick is to rewrite transformations as functors. The author learned this trick from Denis-Charles Cisinski.

(1.16) Proposition. We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. For every category $\mathcal{D}$, all functors $G, G' \colon \mathcal{L} \to \mathcal{D}$ and every transformation $\alpha \colon G \circ \operatorname{loc} \to G' \circ \operatorname{loc}$ there exists a unique transformation $\hat{\alpha} \colon G \to G'$ with $\alpha = \hat{\alpha} * \operatorname{loc}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "G \circ \operatorname{loc}", ""{name=A, below}] \arrow[r, bend right, "G' \circ \operatorname{loc}"', ""{name=B, above}] \arrow[Rightarrow, from=A, to=B, "\alpha"] \arrow[d, "\operatorname{loc}"'] & \mathcal{D} \\
\mathcal{L} \arrow[ur, bend left, "G", ""{name=C, below}] \arrow[ur, bend right, "G'"', ""{name=E, above}] \arrow[Rightarrow, dotted, from=C, to=E, "\hat{\alpha}"] &
\end{tikzcd}$$

Proof. We suppose given a category $\mathcal{D}$, functors $G, G' \colon \mathcal{L} \to \mathcal{D}$ and a transformation $\alpha \colon G \circ \mathrm{loc} \to G' \circ \mathrm{loc}$. We obtain a functor $H \colon \mathcal{C} \to \mathcal{D}^{\Delta^1}$ with $\mathrm{Source} \circ H = G \circ \mathrm{loc}$ and $\mathrm{Target} \circ H = G' \circ \mathrm{loc}$, given as follows. For $X \in \mathrm{Ob}\,\mathcal{C}$, we have $(HX)_0 = G(\mathrm{loc}(X))$, $(HX)_1 = G'(\mathrm{loc}(X))$ and $(HX)_{0,1} = \alpha_X$. For a morphism $f \colon X \to Y$ in $\mathcal{C}$, the morphism $Hf \colon HX \to HY$ in $\mathcal{D}^{\Delta^1}$ is given by $(Hf)_0 = G(\mathrm{loc}(f))$ and $(Hf)_1 = G'(\mathrm{loc}(f))$.

$$\begin{tikzcd} G(\mathrm{loc}(X)) \arrow[r, "\alpha_X"] \arrow[d, "G(\mathrm{loc}(f))"'] & G'(\mathrm{loc}(X)) \arrow[d, "G'(\mathrm{loc}(f))"] \\ G(\mathrm{loc}(Y)) \arrow[r, "\alpha_Y"] & G'(\mathrm{loc}(Y)) \end{tikzcd}$$

Since $G(\mathrm{loc}(d))$ and $G'(\mathrm{loc}(d))$ are isomorphisms in $\mathcal{D}$ for every denominator d in $\mathcal{C}$, it follows that Hd is an isomorphism in $\mathcal{D}^{\Delta^1}$. So by the universal property of $\mathcal{L}$, there exists a unique functor $\hat{H} \colon \mathcal{L} \to \mathcal{D}^{\Delta^1}$ with $H = \hat{H} \circ \mathrm{loc}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "\mathrm{loc}"'] & \mathcal{D}^{\Delta^1} \\ \mathcal{L} \arrow[ur, dotted, "\hat{H}"'] & \end{tikzcd}$$

As

$$\begin{aligned} \mathrm{Source} \circ \hat{H} \circ \mathrm{loc} &= \mathrm{Source} \circ H = G \circ \mathrm{loc}, \\ \mathrm{Target} \circ \hat{H} \circ \mathrm{loc} &= \mathrm{Target} \circ H = G' \circ \mathrm{loc}, \end{aligned}$$

the universal property of $\mathcal{L}$ yields $\mathrm{Source} \circ \hat{H} = G$ and $\mathrm{Target} \circ \hat{H} = G'$. We obtain a transformation $\hat{\alpha} \colon G \to G'$, given by $\hat{\alpha}_{\hat{X}} = (\hat{H}\hat{X})_{0,1}$ for $\hat{X} \in \mathrm{Ob}\,\mathcal{L}$. In particular, we have

$$\hat{\alpha}_{\mathrm{loc}(X)} = (\hat{H}(\mathrm{loc}(X)))_{0,1} = (HX)_{0,1} = \alpha_X$$

for $X \in \mathrm{Ob}\,\mathcal{C}$, that is, $\hat{\alpha} * \mathrm{loc} = \alpha$.
Conversely, given an arbitrary transformation $\beta \colon G \to G'$ with $\alpha = \beta * \mathrm{loc}$, we have $\beta_{\mathrm{loc}(X)} = \alpha_X = \hat{\alpha}_{\mathrm{loc}(X)}$ for $X \in \mathrm{Ob}\,\mathcal{C}$. But this already implies that $\beta = \hat{\alpha}$ as loc is surjective on the objects by proposition (1.15). □

(1.17) Corollary. We suppose given a category with denominators $\mathcal{C}$ and a functor $L \colon \mathcal{C} \to \mathcal{L}$. The following conditions are equivalent.

(a) The category $\mathcal{L}$ becomes a localisation of $\mathcal{C}$ with localisation functor $\mathrm{loc}^{\mathcal{L}} = L$.

(b) For every category $\mathcal{D}$, the induced map

$$\mathbf{Cat}(\mathcal{L}, \mathcal{D}) \to \mathbf{CatD}(\mathcal{C}, \mathcal{D}_{\mathrm{iso}}),\ G \mapsto G \circ L$$

is invertible.

(c) For every category $\mathcal{D}$, the induced functor

$$\mathbf{Cat}(\mathcal{L}, \mathcal{D}) \to \mathbf{Cat}(\mathcal{C}, \mathcal{D}),\ G \mapsto G \circ L,\ \beta \mapsto \beta * L$$

is full, faithful and injective on the objects, and its image is the full subcategory of $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ with set of objects $\mathbf{CatD}(\mathcal{C}, \mathcal{D}_{\mathrm{iso}})$.

(1.18) Corollary. We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. Moreover, we suppose given a category $\mathcal{D}$, functors $G, G' \colon \mathcal{L} \to \mathcal{D}$. A transformation $\beta \colon G \to G'$ is an isotransformation if and only if $\beta * \mathrm{loc}$ is an isotransformation.

Adjunctions on localisation level

(1.19) Proposition. We suppose given morphisms of categories with denominators $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$, a localisation $\mathcal{L}$ of $\mathcal{C}$ and a localisation $\mathcal{M}$ of $\mathcal{D}$. Moreover, we let $\hat{F}\colon \mathcal{L} \to \mathcal{M}$ be the unique functor with $\mathrm{loc}^{\mathcal{M}} \circ F = \hat{F} \circ \mathrm{loc}^{\mathcal{L}}$, and we let $\hat{G}\colon \mathcal{M} \to \mathcal{L}$ be the unique functor with $\mathrm{loc}^{\mathcal{L}} \circ G = \hat{G} \circ \mathrm{loc}^{\mathcal{M}}$.
For every adjunction $\Phi\colon F \dashv G$, we obtain an adjunction $\hat{\Phi}\colon \hat{F} \dashv \hat{G}$, whose unit $\eta^{\hat{\Phi}}\colon \mathrm{id}_{\mathcal{L}} \to \hat{G} \circ \hat{F}$ is the unique transformation with $\mathrm{loc}^{\mathcal{L}} * \eta^{\Phi} = \eta^{\hat{\Phi}} * \mathrm{loc}^{\mathcal{L}}$ and whose counit $\varepsilon^{\hat{\Phi}}\colon \hat{F} \circ \hat{G} \to \mathrm{id}_{\mathcal{M}}$ is the unique transformation with $\mathrm{loc}^{\mathcal{M}} * \varepsilon^{\Phi} = \varepsilon^{\hat{\Phi}} * \mathrm{loc}^{\mathcal{M}}$.

Proof. We suppose given an adjunction $\Phi\colon F \dashv G$. Moreover, we let $\hat{\eta}\colon \mathrm{id}_{\mathcal{L}} \to \hat{G} \circ \hat{F}$ be the unique transformation with $\mathrm{loc}^{\mathcal{L}} * \eta^{\Phi} = \hat{\eta} * \mathrm{loc}^{\mathcal{L}}$ and we let $\hat{\varepsilon}\colon \hat{F} \circ \hat{G} \to \mathrm{id}_{\mathcal{M}}$ be the unique transformation with $\mathrm{loc}^{\mathcal{M}} * \varepsilon^{\Phi} = \hat{\varepsilon} * \mathrm{loc}^{\mathcal{M}}$. Then we have

$$\begin{aligned}((\hat{F} * \hat{\eta})(\hat{\varepsilon} * \hat{F})) * \mathrm{loc}^{\mathcal{L}} &= (\hat{F} * \hat{\eta} * \mathrm{loc}^{\mathcal{L}})(\hat{\varepsilon} * \hat{F} * \mathrm{loc}^{\mathcal{L}}) = (\hat{F} * \mathrm{loc}^{\mathcal{L}} * \eta^{\Phi})(\hat{\varepsilon} * \mathrm{loc}^{\mathcal{M}} * F) \\ &= (\mathrm{loc}^{\mathcal{M}} * F * \eta^{\Phi})(\mathrm{loc}^{\mathcal{M}} * \varepsilon^{\Phi} * F) = \mathrm{loc}^{\mathcal{M}} * ((F * \eta^{\Phi})(\varepsilon^{\Phi} * F)) = \mathrm{loc}^{\mathcal{M}} * 1_F \\ &= 1_{\mathrm{loc}^{\mathcal{M}} \circ F} = 1_{\hat{F} \circ \mathrm{loc}^{\mathcal{L}}} = 1_{\hat{F}} * \mathrm{loc}^{\mathcal{L}}\end{aligned}$$

and therefore $(\hat{F} * \hat{\eta})(\hat{\varepsilon} * \hat{F}) = 1_{\hat{F}}$ by proposition (1.16). Dually, we have $(\hat{\eta} * \hat{G})(\hat{G} * \hat{\varepsilon}) = 1_{\hat{G}}$. Thus we obtain an adjunction $\hat{\Phi}\colon \hat{F} \dashv \hat{G}$ with unit $\eta^{\hat{\Phi}} = \hat{\eta}$ and counit $\varepsilon^{\hat{\Phi}} = \hat{\varepsilon}$. □

(1.20) Corollary. We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$.

(a) Given an initial object I in $\mathcal{C}$, then $\mathrm{loc}(I)$ is an initial object in $\mathcal{L}$.

(b) Given a terminal object T in $\mathcal{C}$, then $\mathrm{loc}(T)$ is a terminal object in $\mathcal{L}$.

Proof.

(a) The 0-th simplex type Δ^0, considered as a category having $\mathrm{Ob}\,\Delta^0 = \{0\}$ and $\mathrm{Mor}\,\Delta^0 = \{1_0\}$, is terminal. The defining universal property of an initial object says that I becomes a couniversal object under 0 along the unique functor $\mathrm{ter}_{\mathcal{C}}\colon \mathcal{C} \to \Delta^0$, where the universal morphism $\mathrm{uni}\colon 0 \to \mathrm{ter}_{\mathcal{C}}(I)$ is given by $\mathrm{uni} = 1_0$.

$$\begin{tikzcd} 0 \arrow[r, "1_0"] \arrow[d, "1_0"'] & \mathrm{ter}_{\mathcal{C}}(X) \\ \mathrm{ter}_{\mathcal{C}}(I) \arrow[ur, dotted, "\mathrm{ter}_{\mathcal{C}}(\mathrm{ini}^I_X)"'] & \end{tikzcd}$$

So by remark (B.19), we get a functor $F\colon \Delta^0 \to \mathcal{C}$, given on the objects by $F0 = I$, and this functor is left adjoint to $\mathrm{ter}_{\mathcal{C}}$ by theorem (B.21).

We consider Δ^0 as a category with denominators having $\mathrm{Den}\,\Delta^0 = \mathrm{Mor}\,\Delta^0 = \{1_0\}$, so that $F\colon \Delta^0 \to \mathcal{C}$ and $\mathrm{ter}_{\mathcal{C}}\colon \mathcal{C} \to \Delta^0$ becomes a morphism of categories with denominators. As this structure of a category with denominators is the isomorphism structure, Δ^0 becomes a localisation of Δ^0 with $\mathrm{loc}^{\Delta^0} = \mathrm{id}_{\Delta^0}$.

$$\begin{tikzcd} \Delta^0 \arrow[r, "F"] \arrow[d, equal] & \mathcal{C} \arrow[r, "\mathrm{ter}_{\mathcal{C}}"] \arrow[d, "\mathrm{loc}"] & \Delta^0 \arrow[d, equal] \\ \Delta^0 \arrow[r, "\mathrm{loc} \circ F"] & \mathcal{L} \arrow[r, "\mathrm{ter}_{\mathcal{L}}"] & \Delta^0 \end{tikzcd}$$

By proposition (1.19), it follows that $\mathrm{loc} \circ F\colon \Delta^0$ is left adjoint to the unique functor $\mathrm{ter}_{\mathcal{L}}\colon \mathcal{L} \to \Delta^0$, so that $\mathrm{loc}(F(0)) = \mathrm{loc}(I)$ is an initial object in $\mathcal{L}$. □

(Co)retractions in the localisation

(1.21) Remark. We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. Moreover, we suppose given a morphism f in $\mathcal{C}$.

(a) If there exists a morphism h in $\mathcal{C}$ such that fh is a denominator in $\mathcal{C}$, then $\operatorname{loc}(f)$ is a coretraction in $\mathcal{L}$, and a retraction of $\operatorname{loc}(f)$ is given by $\operatorname{loc}(h)\operatorname{loc}(fh)^{-1}$.

(b) If there exists a morphism g in $\mathcal{C}$ such that gf is a denominator in $\mathcal{C}$, then $\operatorname{loc}(f)$ is a retraction in $\mathcal{L}$, and a coretraction of $\operatorname{loc}(f)$ is given by $\operatorname{loc}(gf)^{-1}\operatorname{loc}(g)$.

Proof.

(a) We suppose that there exists a morphism h in $\mathcal{C}$ such that fh is a denominator. As $\mathcal{L}$ is a localisation of $\mathcal{C}$, the morphism $\operatorname{loc}(fh)$ is invertible in $\mathcal{L}$, and we have

$$\operatorname{loc}(f)\operatorname{loc}(h)\operatorname{loc}(fh)^{-1} = \operatorname{loc}(fh)\operatorname{loc}(fh)^{-1} = 1.$$

Thus $\operatorname{loc}(f)$ is a coretraction in $\mathcal{L}$, and a retraction of $\operatorname{loc}(f)$ is given by $\operatorname{loc}(h)\operatorname{loc}(fh)^{-1}$. □

(1.22) Corollary. We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. Moreover, we suppose given a morphism f in $\mathcal{C}$.

(a) If there exist morphisms h and h' in $\mathcal{C}$ such that fh and hh' are denominators in $\mathcal{C}$, then $\operatorname{loc}(f)$ is an isomorphism in $\mathcal{L}$ with

$$\operatorname{loc}(f)^{-1} = \operatorname{loc}(h)\operatorname{loc}(fh)^{-1}.$$

(b) If there exist morphisms g and g' in $\mathcal{C}$ such that gf and $g'g$ are denominators in $\mathcal{C}$, then $\operatorname{loc}(f)$ is an isomorphism in $\mathcal{L}$ with

$$\operatorname{loc}(f)^{-1} = \operatorname{loc}(gf)^{-1}\operatorname{loc}(g).$$

(c) If there exist morphisms g and h in $\mathcal{C}$ such that gf and fh are denominators in $\mathcal{C}$, then $\operatorname{loc}(f)$ is an isomorphism in $\mathcal{L}$ with

$$\operatorname{loc}(f)^{-1} = \operatorname{loc}(gf)^{-1}\operatorname{loc}(g) = \operatorname{loc}(h)\operatorname{loc}(fh)^{-1}.$$

Proof.

(c) We suppose that there exist morphisms g and h in $\mathcal{C}$ such that gf and fh are denominators in $\mathcal{C}$. By remark (1.21)(a), the morphism $\operatorname{loc}(f)$ is a coretraction in $\mathcal{L}$, and a retraction of $\operatorname{loc}(f)$ is given by $\operatorname{loc}(h)\operatorname{loc}(fh)^{-1}$. Moreover, by remark (1.21)(b), the morphism $\operatorname{loc}(f)$ is also a retraction in $\mathcal{L}$, and a coretraction of $\operatorname{loc}(f)$ is given by $\operatorname{loc}(gf)^{-1}\operatorname{loc}(g)$. But then $\operatorname{loc}(f)$ is an isomorphism in $\mathcal{L}$ with inverse

$$\operatorname{loc}(f)^{-1} = \operatorname{loc}(gf)^{-1}\operatorname{loc}(g) = \operatorname{loc}(h)\operatorname{loc}(fh)^{-1}.$$

(a) We suppose that there exist morphisms h and h' in $\mathcal{C}$ such that fh and hh' are denominators in $\mathcal{C}$. By remark (1.21)(a), the morphism $\operatorname{loc}(f)$ is a coretraction in $\mathcal{L}$, and a retraction of $\operatorname{loc}(f)$ is given by $\operatorname{loc}(h)\operatorname{loc}(fh)^{-1}$. Moreover, $\operatorname{loc}(h)$ is an isomorphism in $\mathcal{L}$ with

$$\operatorname{loc}(h)^{-1} = \operatorname{loc}(fh)^{-1}\operatorname{loc}(f)$$

by (c). But this implies that $\operatorname{loc}(h)\operatorname{loc}(fh)^{-1}\operatorname{loc}(f) = 1$, and so $\operatorname{loc}(f)$ is an isomorphism in $\mathcal{L}$ with

$$\operatorname{loc}(f)^{-1} = \operatorname{loc}(h)\operatorname{loc}(fh)^{-1}.$$

(b) This is dual to (a). □

The Gabriel-Zisman localisation

Gabriel and Zisman showed in [12, sec. 1.1] that there exists a localisation of a category $\mathcal{C}$ with respect to an arbitrary subset of $\operatorname{Mor}\mathcal{C}$. They gave a concrete construction of such a localisation using presentations of categories, which we restate in theorem (1.24).
The Gabriel-Zisman localisation will only be used in the definition of the homotopy category of an arbitrary category with weak equivalences, see definition (3.8). It will not be used in section 3 of the current chapter I or in the following chapter II.
Given sets X and Y, we denote by $X \sqcup Y$ their disjoint union and by $\mathrm{emb}_1 \colon X \to X \sqcup Y$, $\mathrm{emb}_2 \colon Y \to X \sqcup Y$ the embeddings.

(1.23) Definition (Gabriel-Zisman graph)**.** We suppose given a category with denominators $\mathcal{C}$. The *Gabriel-Zisman graph* is defined to be the graph $\mathrm{GZ}(\mathcal{C})$ with set of objects $\operatorname{Ob}\mathrm{GZ}(\mathcal{C}) := \operatorname{Ob}\mathcal{C}$ and set of arrows $\operatorname{Arr}\mathrm{GZ}(\mathcal{C}) := \operatorname{Mor}\mathcal{C} \sqcup \operatorname{Den}\mathcal{C}$, and where source and target are given by

$$\mathrm{Source}^{\mathrm{GZ}(\mathcal{C})} a := \begin{cases} \mathrm{Source}^{\mathcal{C}} f & \text{if } a = \mathrm{emb}_1(f) \text{ for some } f \in \operatorname{Mor}\mathcal{C}, \\ \mathrm{Target}^{\mathcal{C}} d & \text{if } a = \mathrm{emb}_2(d) \text{ for some } d \in \operatorname{Den}\mathcal{C}, \end{cases}$$

$$\mathrm{Target}^{\mathrm{GZ}(\mathcal{C})} a := \begin{cases} \mathrm{Target}^{\mathcal{C}} f & \text{if } a = \mathrm{emb}_1(f) \text{ for some } f \in \operatorname{Mor}\mathcal{C}, \\ \mathrm{Source}^{\mathcal{C}} d & \text{if } a = \mathrm{emb}_2(d) \text{ for some } d \in \operatorname{Den}\mathcal{C} \end{cases}$$

for $a \in \operatorname{Arr}\mathrm{GZ}(\mathcal{C})$.

(1.24) Theorem (Gabriel, Zisman [12, sec. 1.1, lem. 1.2])**.** We suppose given a category with denominators $\mathcal{C}$. Moreover, we let $\mathsf{GZ}(\mathcal{C})$ be the category that is given by the following presentation. The Gabriel-Zisman graph $\mathrm{GZ}(\mathcal{C})$ generates $\mathsf{GZ}(\mathcal{C})$, and the generators are subject to the following relations. For $f, g \in \operatorname{Mor}\mathcal{C}$ with $\mathrm{Target}\, f = \mathrm{Source}\, g$, we have $\mathrm{emb}_1(f)\,\mathrm{emb}_1(g) = \mathrm{emb}_1(fg)$; for $X \in \operatorname{Ob}\mathcal{C}$, we have $\mathrm{emb}_1(1_X) = 1_X$; and for $d \in \operatorname{Den}\mathcal{C}$, we have $\mathrm{emb}_1(d)\,\mathrm{emb}_2(d) = 1_{\mathrm{Source}\, d}$ and $\mathrm{emb}_2(d)\,\mathrm{emb}_1(d) = 1_{\mathrm{Target}\, d}$.
Then $\mathsf{GZ}(\mathcal{C})$ becomes a localisation of $\mathcal{C}$, where the localisation functor $\mathrm{loc}\colon \mathcal{C} \to \mathsf{GZ}(\mathcal{C})$ is given on the objects by

$$\mathrm{loc}(X) = X$$

for $X \in \operatorname{Ob}\mathcal{C}$ and on the morphisms by

$$\mathrm{loc}(f) = \mathrm{emb}_1(f)$$

for $f \in \operatorname{Mor}\mathcal{C}$.
For every denominator d in $\mathcal{C}$, the inverse of $\mathrm{loc}(d)$ is given by

$$\mathrm{loc}(d)^{-1} = \mathrm{emb}_2(d).$$

Without proof. □

(1.25) Definition (Gabriel-Zisman localisation)**.** We suppose given a category with denominators $\mathcal{C}$. The localisation $\mathsf{GZ}(\mathcal{C})$ as constructed in theorem (1.24) is called the *Gabriel-Zisman localisation* of $\mathcal{C}$.

Next, we turn the Gabriel-Zisman localisation into a functor.

(1.26) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. If $\mathcal{C}$ is a $\mathfrak{U}$-category with denominators, then $\mathsf{GZ}(\mathcal{C})$ is a $\mathfrak{U}$-category.

(1.27) Corollary. We suppose given a Grothendieck universe $\mathfrak{U}$.

(a) We have a functor

$$\mathsf{GZ}\colon \mathbf{CatD}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})},$$

given on the morphisms as follows. For every morphism $F\colon \mathcal{C} \to \mathcal{C}'$ in $\mathbf{CatD}_{(\mathfrak{U})}$, the morphism $\mathsf{GZ}(F)\colon \mathsf{GZ}(\mathcal{C}) \to \mathsf{GZ}(\mathcal{C}')$ in $\mathbf{Cat}_{(\mathfrak{U})}$ is the unique morphism in $\mathbf{Cat}_{(\mathfrak{U})}$ with $\mathrm{loc}^{\mathsf{GZ}(\mathcal{C}')} \circ F = \mathsf{GZ}(F) \circ \mathrm{loc}^{\mathsf{GZ}(\mathcal{C})}$.

(b) The functor GZ is left adjoint to the functor $-_{\mathrm{iso}}\colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{CatD}_{(\mathfrak{U})}$. An adjunction $\Phi\colon \mathsf{GZ} \dashv -_{\mathrm{iso}}$ is given by

$$\Phi_{\mathcal{C},\mathcal{D}}\colon {}_{\mathbf{Cat}_{(\mathfrak{U})}}(\mathsf{GZ}(\mathcal{C}),\mathcal{D}) \to {}_{\mathbf{CatD}_{(\mathfrak{U})}}(\mathcal{C},\mathcal{D}),\ G \mapsto G \circ \mathrm{loc}^{\mathsf{GZ}(\mathcal{C})}$$

for $\mathcal{C} \in \mathrm{Ob}\,\mathbf{CatD}_{(\mathfrak{U})}$, $\mathcal{D} \in \mathrm{Ob}\,\mathbf{Cat}_{(\mathfrak{U})}$.

Proof. This follows from remark (1.26) and corollary (1.14)(d), (f). □

3 Saturatedness

We suppose given a category $\mathcal{C}$, a subset D of $\mathrm{Mor}\,\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$ with respect to D. By definition of a localisation, see definition (1.11)(a), every element of D is made invertible in $\mathcal{L}$, or, said more precisely, $\mathrm{loc}(d)$ is invertible for every $d \in D$. But in general, not every morphism f in $\mathcal{C}$ that is invertible in $\mathcal{L}$ has to be an element of D. This gives rise to the definition of the saturation, see definition (1.28)(a), that is, the subset of precisely those morphisms that become invertible in the localisation $\mathcal{L}$.
If every morphism that becomes invertible in $\mathcal{L}$ actually lies in D, this subset is called saturated, see definition (1.39)(a). This property has several weaker variants, which we introduce from definition (1.35) on and which we relate to each other in proposition (1.43).

The saturation

(1.28) Definition (saturation)**.**

(a) We suppose given a subset D of $\mathrm{Mor}\,\mathcal{C}$ such that a localisation of $\mathcal{C}$ with respect to D exists ([1]). The *saturation* of D in $\mathcal{C}$ is defined to be the set

$$\mathrm{Sat}\,D = \mathrm{Sat}_{\mathcal{C}}\,D := \{f \in \mathrm{Mor}\,\mathcal{C} \mid \mathrm{loc}(f) \text{ is invertible in } \mathcal{L}\}$$

for a (and hence any) localisation $\mathcal{L}$ of $\mathcal{C}$ with respect to D.

(b) We suppose given a category with denominators $\mathcal{C}$ such that a localisation of $\mathcal{C}$ exists. The *saturation* of $\mathcal{C}$ is defined to be the category with denominators $\mathrm{Sat}\,\mathcal{C}$ whose underlying category is given by $\mathcal{C}$ and whose set of denominators is given by

$$\mathrm{Den}\,\mathrm{Sat}\,\mathcal{C} := \mathrm{Sat}_{\mathcal{C}}(\mathrm{Den}\,\mathcal{C}).$$

(1.29) Example. We suppose given a category $\mathcal{C}$.

(a) The saturation of the discrete structure $\mathcal{C}_{\mathrm{disc}}$ is the isomorphism structure $\mathcal{C}_{\mathrm{iso}}$.

(b) The saturation of the isomorphism structure $\mathcal{C}_{\mathrm{iso}}$ is the isomorphism structure $\mathcal{C}_{\mathrm{iso}}$.

Proof.

(a) By example (1.12)(a), the category $\mathcal{C}$ becomes a localisation of $\mathcal{C}_{\mathrm{disc}}$, where the localisation functor is given by $\mathrm{loc} = \mathrm{id}_{\mathcal{C}}$. Hence we have

$$\mathrm{Den}\,\mathrm{Sat}\,\mathcal{C}_{\mathrm{disc}} = \{f \in \mathrm{Mor}\,\mathcal{C} \mid f \text{ is invertible in } \mathcal{C}\} = \mathrm{Den}\,\mathcal{C}_{\mathrm{iso}}$$

and therefore $\mathrm{Sat}\,\mathcal{C}_{\mathrm{disc}} = \mathcal{C}_{\mathrm{iso}}$.

(b) This is proven analogously to (a). □

(1.30) Remark. Given a category $\mathcal{C}$ and a subset D of $\mathrm{Mor}\,\mathcal{C}$ such that a localisation of $\mathcal{C}$ with respect to D exists, then we have $D \subseteq \mathrm{Sat}\,D$.

(1.31) Proposition. We suppose given a category $\mathcal{C}$ and subsets D, D' of $\mathrm{Mor}\,\mathcal{C}$ such that a localisation of $\mathcal{C}$ with respect to D and with respect to D' exists. If $D \subseteq D'$, then we also have

$$\mathrm{Sat}\,D \subseteq \mathrm{Sat}\,D'.$$

[1] We do not want to use the Gabriel-Zisman localisation in the following.

Proof. We let $\mathcal{L}$ be a localisation of $\mathcal{C}$ with respect to D and we let $\mathcal{L}'$ be a localisation of $\mathcal{C}$ with respect to D'. As $D \subseteq D'$, the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ maps elements of D to elements of D'. So by corollary (1.14)(b), there exists a unique functor $\hat{E} \colon \mathcal{L} \to \mathcal{L}'$ with $\mathrm{loc}^{\mathcal{L}'} = \hat{E} \circ \mathrm{loc}^{\mathcal{L}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, equal] \arrow[d, "\mathrm{loc}^{\mathcal{L}}"'] & \mathcal{C} \arrow[d, "\mathrm{loc}^{\mathcal{L}'}"] \\ \mathcal{L} \arrow[r, dotted, "\hat{E}"] & \mathcal{L}' \end{tikzcd}$$

As every functor preserves isomorphisms, we obtain

$$\begin{aligned} \mathrm{Sat}\, D &= \{ f \in \mathrm{Mor}\, \mathcal{C} \mid \mathrm{loc}^{\mathcal{L}}(f) \text{ is invertible in } \mathcal{L} \} = \{ f \in \mathrm{Mor}\, \mathcal{C} \mid \hat{E}(\mathrm{loc}^{\mathcal{L}}(f)) \text{ is invertible in } \mathcal{L}' \} \\ &= \{ f \in \mathrm{Mor}\, \mathcal{C} \mid \mathrm{loc}^{\mathcal{L}'}(f) \text{ is invertible in } \mathcal{L}' \} = \mathrm{Sat}\, D'. \end{aligned}$$ □

(1.32) Proposition. We suppose given a category $\mathcal{C}$ and a subset D of $\mathrm{Mor}\, \mathcal{C}$ such that a localisation of $\mathcal{C}$ with respect to D exists. Moreover, we suppose given a subset D' of $\mathrm{Mor}\, \mathcal{C}$ with $D \subseteq D' \subseteq \mathrm{Sat}\, D$. Every localisation of $\mathcal{C}$ with respect to D is also a localisation of $\mathcal{C}$ with respect to D' and every localisation of $\mathcal{C}$ with respect to D' is also a localisation of $\mathcal{C}$ with respect to D. In particular, there exists a localisation of $\mathcal{C}$ with respect to D'.

Proof. We let $\mathcal{L}$ be a localisation of $\mathcal{C}$ with respect to D. Then $\mathrm{loc}(d')$ is invertible in $\mathcal{L}$ for every $d' \in D'$ since $D' \subseteq \mathrm{Sat}\, D$. Moreover, given a category $\mathcal{D}$ and a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that Fd' is invertible in $\mathcal{D}$ for every $d' \in D'$, then in particular Fd is invertible in $\mathcal{D}$ for every $d \in D$ and hence there exists a unique functor $\hat{F} \colon \mathcal{L} \to \mathcal{D}$ with $F = \hat{F} \circ \mathrm{loc}$. So $\mathcal{L}$ is also a localisation of $\mathcal{C}$ with respect to D'.
Conversely, we suppose given a localisation $\mathcal{L}'$ of $\mathcal{C}$ with respect to D'. Then $\mathrm{loc}(d')$ is invertible in $\mathcal{L}$ for every $d' \in D'$, and so in particular $\mathrm{loc}^{\mathcal{L}'}(d)$ is invertible in $\mathcal{L}$ for every $d \in D$. To show that $\mathcal{L}'$ is a localisation of $\mathcal{C}$ with respect to D, we suppose given a category $\mathcal{D}$ and a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible in $\mathcal{D}$ for every $d \in D$. Since $\mathcal{L}$ is a localisation of $\mathcal{C}$ with respect to D, there exists a unique functor $\hat{F} \colon \mathcal{L} \to \mathcal{D}$ with $F = \hat{F} \circ \mathrm{loc}^{\mathcal{L}}$. Moreover, $D' \subseteq \mathrm{Sat}\, D$ implies that $\mathrm{loc}^{\mathcal{L}}(d')$ is invertible in $\mathcal{D}$ and hence that $Fd' = \hat{F}\, \mathrm{loc}^{\mathcal{L}}(d')$ is invertible for every $d' \in D'$. Thus there exists a unique functor $\hat{F}' \colon \mathcal{L}' \to \mathcal{D}$ with $F = \hat{F}' \circ \mathrm{loc}^{\mathcal{L}'}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{loc}^{\mathcal{L}}"'] & \mathcal{D} \\ \mathcal{L} \arrow[ur, dotted, "\hat{F}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{loc}^{\mathcal{L}'}"'] & \mathcal{D} \\ \mathcal{L}' \arrow[ur, dotted, "\hat{F}'"'] & \end{tikzcd}$$

Thus $\mathcal{L}'$ is also a localisation of $\mathcal{C}$ with respect to D. □

(1.33) Corollary. We suppose given a category $\mathcal{C}$, a subset D of $\mathrm{Mor}\, \mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$ with respect to D. Then $\mathcal{L}$ is also a localisation of $\mathcal{C}$ with respect to $\mathrm{Sat}\, D$, and we have

$$\mathrm{Sat}\, \mathrm{Sat}\, D = \mathrm{Sat}\, D.$$

Proof. By proposition (1.32), we know that $\mathcal{L}$ is a localisation of $\mathcal{C}$ with respect to $\mathrm{Sat}\, D$. In particular, we have

$$\mathrm{Sat}\, \mathrm{Sat}\, D = \{ f \in \mathrm{Mor}\, \mathcal{C} \mid \mathrm{loc}^{\mathcal{L}}(f) \text{ is invertible in } \mathcal{L} \} = \mathrm{Sat}\, D$$

by definition of the saturation. □

The preceding proposition states that when we study localisations of a category $\mathcal{C}$ with respect to a subset D of $\mathrm{Mor}\, \mathcal{C}$, we can replace D without loss of generality by a subset D' of $\mathrm{Mor}\, \mathcal{C}$ with $D \subseteq D' \subseteq \mathrm{Sat}\, D$ and study localisations of $\mathcal{C}$ with respect to D' instead. We will study some examples for such a denominator set D' in proposition (1.45) below.

(1.34) Proposition. We suppose given an isomorphism of categories with denominators $F \colon \mathcal{C} \to \mathcal{C}'$. If a localisation of $\mathcal{C}$ or $\mathcal{C}'$ exists, then there exists a localisation of both $\mathcal{C}$ and $\mathcal{C}'$, and we have

$$\mathrm{Den}\, \mathrm{Sat}\, \mathcal{C}' = F(\mathrm{Den}\, \mathrm{Sat}\, \mathcal{C}).$$

Proof. By corollary (1.14)(c), there exists a localisation of $\mathcal{C}$ if and only if there exists a localisation of $\mathcal{C}'$. We suppose that a localisation $\mathcal{L}$ of $\mathcal{C}$ exists. By corollary (1.14)(c), the underlying category of $\mathcal{L}$ becomes a localisation $\mathcal{L}'$ of $\mathcal{C}'$ with localisation functor $\mathrm{loc}^{\mathcal{L}'} = \mathrm{loc}^{\mathcal{L}} \circ F^{-1}$, and we get

$$\begin{aligned}\operatorname{Den}\operatorname{Sat}\mathcal{C}' &= \{f' \in \operatorname{Mor}\mathcal{C}' \mid \mathrm{loc}^{\mathcal{L}'}(f') \text{ is invertible in } \mathcal{L}'\} \\ &= \{f' \in \operatorname{Mor}\mathcal{C}' \mid \mathrm{loc}^{\mathcal{L}}(F^{-1}f') \text{ is invertible in } \mathcal{L}\} \\ &= \{Ff \mid f \in \operatorname{Mor}\mathcal{C} \text{ and } \mathrm{loc}^{\mathcal{L}}(f) \text{ is invertible in } \mathcal{L}\} \\ &= F(\{f \in \operatorname{Mor}\mathcal{C} \mid \mathrm{loc}^{\mathcal{L}}(f) \text{ is invertible in } \mathcal{L}\}) = F(\operatorname{Den}\operatorname{Sat}\mathcal{C}).\end{aligned}$$ □

Levels of saturatedness

(1.35) Definition (multiplicativity)**.**

(a) We suppose given a category $\mathcal{C}$. A subset D of $\operatorname{Mor}\mathcal{C}$ is said to be *multiplicative* (in $\mathcal{C}$) if it fulfills:

(Cat) *Multiplicativity.* For all $d, e \in D$ with $\operatorname{Target} d = \operatorname{Source} e$, we have $de \in D$, and for every object X in $\mathcal{C}$, we have $1_X \in D$.

(b) (i) A category with denominators $\mathcal{C}$ is said to be *multiplicative* if its set of denominators $\operatorname{Den}\mathcal{C}$ is a multiplicative subset of $\mathcal{C}$.

(ii) We suppose given a Grothendieck universe $\mathfrak{U}$. The full subcategory $\mathbf{CatD}_{\mathrm{mul}} = \mathbf{CatD}_{\mathrm{mul},(\mathfrak{U})}$ of $\mathbf{CatD}_{(\mathfrak{U})}$ with

$$\operatorname{Ob}\mathbf{CatD}_{\mathrm{mul},(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob}\mathbf{CatD}_{(\mathfrak{U})} \mid \mathcal{C} \text{ is multiplicative}\}$$

is called the *category of multiplicative categories with denominators* (more precisely, the *category of multiplicative* $\mathfrak{U}$*-categories with denominators*).

(1.36) Definition (isosaturatedness)**.**

(a) We suppose given a category $\mathcal{C}$. A subset D of $\operatorname{Mor}\mathcal{C}$ is said to *contain all isomorphisms* (or to be *isosaturated*) in $\mathcal{C}$ if it fulfills:

(Iso) *Isosaturatedness.* For all isomorphisms f in $\mathcal{C}$, we have $f \in D$.

(b) A category with denominators $\mathcal{C}$ is said to be *isosaturated* if its set of denominators $\operatorname{Den}\mathcal{C}$ is an isosaturated subset of $\mathcal{C}$.

(1.37) Definition (semisaturatedness)**.**

(a) We suppose given a category $\mathcal{C}$.

(i) A subset D of $\operatorname{Mor}\mathcal{C}$ is said to be *S-semisaturated* (in $\mathcal{C}$) if it is multiplicative and fulfills:

($2\,\mathrm{of}\,3_{\mathrm{S}}$) *S-part of 2 out of 3 axiom.* For all morphisms f and g in $\mathcal{C}$ with $f, fg \in D$, we have also $g \in D$.

(ii) A subset D of $\operatorname{Mor}\mathcal{C}$ is said to be *T-semisaturated* (in $\mathcal{C}$) if it is multiplicative and fulfills:

($2\,\mathrm{of}\,3_{\mathrm{T}}$) *T-part of 2 out of 3 axiom.* For all morphisms f and g in $\mathcal{C}$ with $g, fg \in D$, we have also $f \in D$.

(iii) A subset D of $\operatorname{Mor}\mathcal{C}$ is said to be *semisaturated* (in $\mathcal{C}$) ([2]) if it is S-semisaturated and T-semisaturated.

(b) A category with denominators $\mathcal{C}$ is said to be *S-semisaturated* resp. *T-semisaturated* resp. *semisaturated* if its set of denominators $\operatorname{Den}\mathcal{C}$ is an S-semisaturated resp. a T-semisaturated resp. a semisaturated subset of $\mathcal{C}$.

(1.38) Definition (weak saturatedness)**.**

(a) We suppose given a category $\mathcal{C}$. A subset D of $\operatorname{Mor}\mathcal{C}$ is said to be *weakly saturated* (in $\mathcal{C}$) if it is multiplicative and fulfills:

[2] In the literature, semisaturatedness is sometimes called *saturatedness*; and saturatedness in our sense, see definition (1.39), is sometimes called *strong saturatedness*.

(2 of 6) *2 out of 6 axiom.* For all morphisms f, g, h in $\mathcal{C}$ with $fg, gh \in D$, we have also $f, g, h, fgh \in D$.

(b) A category with denominators $\mathcal{C}$ is said to be *weakly saturated* if its set of denominators $\operatorname{Den} \mathcal{C}$ is a weakly saturated subset of $\mathcal{C}$.

(1.39) Definition (saturatedness)**.**

(a) We suppose given a category $\mathcal{C}$. A subset D of $\operatorname{Mor} \mathcal{C}$ is said to be *saturated* (in $\mathcal{C}$) if it fulfills:

(Sat) *Saturatedness.* There exists a localisation of $\mathcal{C}$ with respect to D and we have $\operatorname{Sat} D = D$.

(b) A category with denominators $\mathcal{C}$ is said to be *saturated* if its set of denominators $\operatorname{Den} \mathcal{C}$ is a saturated subset of $\mathcal{C}$.

(1.40) Example. We suppose given a category $\mathcal{C}$.

(a) The discrete structure $\mathcal{C}_{\text{disc}}$ is semisaturated.

(b) The isomorphism structure $\mathcal{C}_{\text{iso}}$ is saturated.

Proof.

(a) We suppose given morphisms f and g in $\mathcal{C}$ with $\operatorname{Target} f = \operatorname{Source} g$ such that two out of the three morphisms f, g, fg are denominators in $\mathcal{C}_{\text{disc}}$. Then these two are equal to an identity morphism and therefore all three are equal to an identity morphism in $\mathcal{C}$. But this means that all three are denominators in $\mathcal{C}_{\text{disc}}$. Moreover, 1_X is a denominator in $\mathcal{C}_{\text{disc}}$ for every object X in $\mathcal{C}$, and so $\mathcal{C}_{\text{disc}}$ is semisaturated.

(b) This follows from example (1.29)(b). □

(1.41) Proposition. We suppose given an isomorphism of categories with denominators $F \colon \mathcal{C} \to \mathcal{D}$.

(a) The category with denominators $\mathcal{C}$ is multiplicative if and only if $\mathcal{D}$ is multiplicative.

(b) The category with denominators $\mathcal{C}$ is isosaturated if and only if $\mathcal{D}$ is isosaturated.

(c) The category with denominators $\mathcal{C}$ is S-semisaturated if and only if $\mathcal{D}$ is S-semisaturated. The category with denominators $\mathcal{C}$ is T-semisaturated if and only if $\mathcal{D}$ is T-semisaturated.

(d) The category with denominators $\mathcal{C}$ is weakly saturated if and only if $\mathcal{D}$ is weakly saturated.

(e) The category with denominators $\mathcal{C}$ is saturated if and only if $\mathcal{D}$ is saturated.

Proof.

(a) We suppose that $\mathcal{C}$ is multiplicative, and we suppose given denominators e and e' in $\mathcal{D}$ with $\operatorname{Target} e = \operatorname{Source} e'$. Then $F^{-1}e$ and $F^{-1}e'$ are denominators in $\mathcal{C}$, and since $\mathcal{C}$ is multiplicative, it follows that $(F^{-1}e)(F^{-1}e')$ is a denominator in $\mathcal{C}$. But this implies that $ee' = F((F^{-1}e)(F^{-1}e'))$ is a denominator in $\mathcal{D}$. Moreover, given an object Y in $\mathcal{D}$, we have $1_Y = F(1_{F^{-1}Y})$, and as $1_{F^{-1}Y}$ is a denominator in $\mathcal{C}$, it follows that 1_Y is a denominator in $\mathcal{D}$. Altogether, $\mathcal{D}$ is multiplicative.

The other implication follows by symmetry.

(b) We suppose that $\mathcal{C}$ is isosaturated, and we suppose given an isomorphism g in $\mathcal{D}$. Then $F^{-1}g$ is an isomorphism in $\mathcal{C}$, and since $\mathcal{C}$ is isosaturated, it follows that $F^{-1}g$ is an isomorphism in $\mathcal{C}$. But this implies that $g = FF^{-1}g$ is a denominator in $\mathcal{D}$.

The other implication follows by symmetry.

(c) We suppose that $\mathcal{C}$ is S-semisaturated, and we suppose given morphisms g and g' in $\mathcal{D}$ with $\operatorname{Target} g = \operatorname{Source} g'$ such that g and gg' are denominators in $\mathcal{D}$. Then $F^{-1}g$ and $(F^{-1}g)(F^{-1}g') = F^{-1}(gg')$ are denominators in $\mathcal{C}$. Since $\mathcal{C}$ is S-semisaturated, it follows that $(F^{-1}g')$ is a denominator in $\mathcal{C}$. But then $g' = F(F^{-1}g')$ is a denominator in $\mathcal{D}$. Thus $\mathcal{D}$ fulfills the S-part of the 2 out of 3 axiom. As $\mathcal{D}$ is multiplicative by (a), we conclude that $\mathcal{D}$ is semisaturated.

By duality, we obtain: If $\mathcal{C}$ is T-semisaturated, then $\mathcal{D}$ is T-semisaturated.

The other implications follow by symmetry.

(d) We suppose that $\mathcal{C}$ is weakly saturated, and we suppose given morphisms g, g', g'' in $\mathcal{D}$ such that gg' and $g'g''$ are denominators in $\mathcal{D}$. Then $(F^{-1}g)(F^{-1}g') = F^{-1}(gg')$ and $(F^{-1}g')(F^{-1}g'') = F^{-1}(g'g'')$ are denominators in $\mathcal{C}$, and since $\mathcal{C}$ is weakly saturated, it follows that $F^{-1}g$, $F^{-1}g'$, $F^{-1}g''$, $(F^{-1}g)(F^{-1}g')(F^{-1}g'')$ are denominators in $\mathcal{C}$. But this implies that $g = F(F^{-1}g)$, $g' = F(F^{-1}g')$, $g'' = F(F^{-1}g'')$, $gg'g'' = F((F^{-1}g)(F^{-1}g')(F^{-1}g''))$ are denominators in $\mathcal{D}$. As $\mathcal{D}$ is multiplicative by (a), we conclude that $\mathcal{D}$ is weakly saturated.

The other implication follows by symmetry.

(e) If $\mathcal{C}$ is saturated, that is, if there exists a localisation of $\mathcal{C}$ and we have $\operatorname{Sat}\mathcal{C} = \mathcal{C}$, then by proposition (1.34) there exists a localisation of $\mathcal{D}$ and we have

$$\operatorname{Den}\operatorname{Sat}\mathcal{D} = F(\operatorname{Den}\operatorname{Sat}\mathcal{C}) = F(\operatorname{Den}\mathcal{C}) = \operatorname{Den}\mathcal{D},$$

that is, $\mathcal{D}$ is saturated.

The other implication follows by symmetry. □

We suppose given a category with denominators $\mathcal{C}$ and a category S. By remark (1.10), we may consider $\mathcal{C}^S$, the category of S-commutative diagrams in $\mathcal{C}$, as a category with denominators, having pointwise denominators. The following proposition states that various notions of saturatedness are enherited to the diagram category.

(1.42) Proposition. We suppose given a category with denominators $\mathcal{C}$ and a category S.

(a) If $\mathcal{C}$ is multiplicative, then $\mathcal{C}^S$ is multiplicative.

(b) If $\mathcal{C}$ is isosaturated, then $\mathcal{C}^S$ is isosaturated.

(c) If $\mathcal{C}$ is S-semisaturated, then $\mathcal{C}^S$ is S-semisaturated. If $\mathcal{C}$ is T-semisaturated, then $\mathcal{C}^S$ is T-semisaturated.

(d) If $\mathcal{C}$ is weakly saturated, then $\mathcal{C}^S$ is weakly saturated.

Proof.

(a) We suppose that $\mathcal{C}$ is multiplicative. Moreover, we suppose given denominators d, e in $\mathcal{C}^S$ with $\operatorname{Target} d = \operatorname{Source} e$. Then d_i and e_i are denominators in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$. It follows that $(de)_i = d_i e_i$ is a denominator in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$, that is, de is a denominator in $\mathcal{C}^S$. Moreover, given an object X in $\mathcal{C}^S$, then $(1_X)_i = 1_{X_i}$ is a denominator in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$, whence 1_X is a denominator in $\mathcal{C}^S$. Altogether, $\mathcal{C}^S$ is multiplicative.

(b) We suppose that $\mathcal{C}$ is isosaturated. Moreover, we suppose given an isomorphism f in $\mathcal{C}^S$. Then f_i is an isomorphism in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$. The isosaturatedness of $\mathcal{C}$ implies that f_i is a denominator in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$, that is, f is a denominator in $\mathcal{C}^S$. Thus $\mathcal{C}^S$ is isosaturated.

(c) We suppose that $\mathcal{C}$ is S-semisaturated. Moreover, we suppose given morphisms f, g in $\mathcal{C}^S$ such that f and fg are denominators in $\mathcal{C}^S$. Then f_i and $f_i g_i = (fg)_i$ are denominators in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$. It follows that g_i is a denominator in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$, that is, g is a denominator in $\mathcal{C}^S$. Thus $\mathcal{C}^S$ fulfills the S-part of the 2 out of 3 axiom. As $\mathcal{C}^S$ is multiplicative by (a), we conclude that $\mathcal{C}^S$ is S-semisaturated.

The other implication follows by duality.

(d) We suppose that $\mathcal{C}$ is weakly saturated. Moreover, we suppose given morphisms f, g, h in $\mathcal{C}^S$ such that fg and gh are denominators in $\mathcal{C}^S$. Then $f_i g_i = (fg)_i$ and $g_i h_i = (gh)_i$ are denominators in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$. It follows that f_i, g_i, h_i, $(fgh)_i = f_i g_i h_i$ are denominators in $\mathcal{C}$ for every $i \in \operatorname{Ob} S$, that is, f, g, h, fgh are denominators in $\mathcal{C}^S$. Thus $\mathcal{C}^S$ fulfills the 2 out of 6 axiom. As $\mathcal{C}^S$ is multiplicative by (a), we conclude that $\mathcal{C}^S$ is weakly saturated. □

The following proposition states how the different variations of the notion of saturatedness introduced in definition (1.35) to definition (1.39) are related. Cf. figure 1.

(1.43) Proposition. We suppose given a category with denominators $\mathcal{C}$.

(a) If $\mathcal{C}$ is saturated, then $\mathcal{C}$ is weakly saturated.

$$\begin{tikzcd}
\text{(Cat)} & \text{(Cat), (2 of 3}_{\mathrm{S}}\text{), (2 of 3}_{\mathrm{T}}\text{)} \arrow[l, Rightarrow] & \text{(Cat), (2 of 6)} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \text{(Sat)} \arrow[l, Rightarrow] \\
 & & \text{(Iso)} &
\end{tikzcd}$$

Figure 1: Levels of saturatedness.

(b) If $\mathcal{C}$ is weakly saturated, then $\mathcal{C}$ is semisaturated and isosaturated.

(c) If $\mathcal{C}$ is semisaturated, then $\mathcal{C}$ is multiplicative.

Proof.

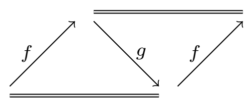
(a) We suppose that $\mathcal{C}$ is saturated and we let $\mathcal{L}$ be a localisation of $\mathcal{C}$. Moreover, we suppose given morphisms f, g, h in $\mathcal{C}$ such that fg and gh are denominators in $\mathcal{C}$. By corollary (1.22), it follows that $\mathrm{loc}(f)$, $\mathrm{loc}(g)$, $\mathrm{loc}(h)$ are invertible in $\mathcal{L}$, that is, f, g, h are denominators in the saturation $\mathrm{Sat}\,\mathcal{C}$. Moreover, $\mathrm{loc}(fgh) = \mathrm{loc}(f)\,\mathrm{loc}(g)\,\mathrm{loc}(h)$ is invertible in $\mathcal{L}$ as a composite of invertible morphisms, that is, fgh is a denominator in the saturation $\mathrm{Sat}\,\mathcal{C}$. But as $\mathcal{C}$ is saturated, we have $\mathrm{Sat}\,\mathcal{C} = \mathcal{C}$, and so f, g, h, fgh are in fact denominators in $\mathcal{C}$. So we have shown that $\mathcal{C}$ fulfills the 2 out of 6 axiom.

In particular, given denominators f, g in $\mathcal{C}$, then $f\,1$ and $1\,g$ are denominators in $\mathcal{C}$ and hence $fg = f\,1\,g$ is a denominator in $\mathcal{C}$. Moreover, given an object X in $\mathcal{C}$, the morphism $\mathrm{loc}(1_X) = 1_{\mathrm{loc}(X)}$ is invertible in $\mathcal{L}$ and hence 1_X is a denominator in $\mathrm{Sat}\,\mathcal{C} = \mathcal{C}$. Hence $\mathcal{C}$ is also multiplicative and therefore is weakly saturated.

(b) We suppose that $\mathcal{C}$ is weakly saturated. Then $\mathcal{C}$ is in particular multiplicative.

To show that $\mathcal{C}$ is semisaturated, we suppose given morphisms f and g in $\mathcal{C}$ with $\mathrm{Target}\,f = \mathrm{Source}\,g$. If f and fg are denominators in $\mathcal{C}$, then $1\,f$ and fg are denominators in $\mathcal{C}$ and hence g is a denominator in $\mathcal{C}$ by the 2 out of 6 axiom. Dually, if g and fg are denominators in $\mathcal{C}$, then fg and $g\,1$ are denominators in $\mathcal{C}$ and hence f is a denominator in $\mathcal{C}$ by the 2 out of 6 axiom. Thus $\mathcal{C}$ is semisaturated.

To show that $\mathcal{C}$ is isosaturated, we suppose given an isomorphism f in $\mathcal{C}$, so that there exists a morphism g in $\mathcal{C}$ with $fg = 1$ and $gf = 1$. Since in particular identities are denominators in $\mathcal{C}$, it follows that f is a denominator in $\mathcal{C}$ by the 2 out of 6 axiom.



(c) This holds by definition. $\square$

Now we may give an example of a semisaturated category with denominators that is not weakly saturated.

(1.44) Example. We suppose given a category $\mathcal{C}$ that contains a non-identical isomorphism. Then the discrete structure $\mathcal{C}_{\mathrm{disc}}$ is semisaturated, but not weakly saturated.

Proof. The discrete structure $\mathcal{C}_{\mathrm{disc}}$ is always semisaturated by example (1.40)(a), but if there exists a non-identical isomorphism f in $\mathcal{C}$, then f is not a denominator in $\mathcal{C}_{\mathrm{disc}}$, and so $\mathcal{C}_{\mathrm{disc}}$ is not weakly saturated by proposition (1.43)(b). $\square$

(1.45) Proposition. We suppose given a category $\mathcal{C}$, a subset D of $\mathrm{Mor}\,\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$ with respect to D. Moreover, we let

$$\begin{aligned}
D_{\mathrm{mul}} &:= \bigcap \{U \subseteq \mathrm{Mor}\,\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is multiplicative}\},\\
D_{\mathrm{ssat}} &:= \bigcap \{U \subseteq \mathrm{Mor}\,\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is semisaturated}\},\\
D_{\mathrm{wsat}} &:= \bigcap \{U \subseteq \mathrm{Mor}\,\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is weakly saturated}\},\\
D_{\mathrm{sat}} &:= \bigcap \{U \subseteq \mathrm{Mor}\,\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is saturated}\}.
\end{aligned}$$

(a) (i) The subset D_{mul} is the smallest multiplicative subset of $\operatorname{Mor}\mathcal{C}$ that contains D.

(ii) The subset D_{ssat} is the smallest semisaturated subset of $\operatorname{Mor}\mathcal{C}$ that contains D.

(iii) The subset D_{wsat} is the smallest weakly saturated subset of $\operatorname{Mor}\mathcal{C}$ that contains D.

(iv) The subset D_{sat} is the smallest saturated subset of $\operatorname{Mor}\mathcal{C}$ that contains D.

(b) We have $D \subseteq D_{\mathrm{mul}} \subseteq D_{\mathrm{ssat}} \subseteq D_{\mathrm{wsat}} \subseteq D_{\mathrm{sat}} = \operatorname{Sat} D$.

(c) The category $\mathcal{L}$ is a localisation of $\mathcal{C}$ with respect to D, to D_{mul}, to D_{ssat}, to D_{wsat} and to $D_{\mathrm{sat}} = \operatorname{Sat} D$.

Proof. We set

$$\begin{aligned}
\mathcal{U} &:= \{U \subseteq \operatorname{Mor}\mathcal{C} \mid D \subseteq U\},\\
\mathcal{U}_{\mathrm{mul}} &:= \{U \subseteq \operatorname{Mor}\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is multiplicative}\},\\
\mathcal{U}_{\mathrm{ssat}} &:= \{U \subseteq \operatorname{Mor}\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is semisaturated}\},\\
\mathcal{U}_{\mathrm{wsat}} &:= \{U \subseteq \operatorname{Mor}\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is weakly saturated}\},\\
\mathcal{U}_{\mathrm{sat}} &:= \{U \subseteq \operatorname{Mor}\mathcal{C} \mid D \subseteq U \text{ and } U \text{ is saturated}\},
\end{aligned}$$

so that $D = \bigcap \mathcal{U}$, $D_{\mathrm{mul}} = \bigcap \mathcal{U}_{\mathrm{mul}}$, $D_{\mathrm{ssat}} = \bigcap \mathcal{U}_{\mathrm{ssat}}$, $D_{\mathrm{wsat}} = \bigcap \mathcal{U}_{\mathrm{wsat}}$, $D_{\mathrm{sat}} = \bigcap \mathcal{U}_{\mathrm{sat}}$.

(a) (i) We suppose given $d, e \in D_{\mathrm{mul}}$ with $\operatorname{Target} d = \operatorname{Source} e$. For $U \in \mathcal{U}_{\mathrm{mul}}$, we have $D_{\mathrm{mul}} = \bigcap \mathcal{U}_{\mathrm{mul}} \subseteq U$, so it follows that $d, e \in U$ and therefore $de \in U$ by the multiplicativity of U. Thus we have $de \in \bigcap \mathcal{U}_{\mathrm{mul}} = D_{\mathrm{mul}}$.

Moreover, we suppose given $X \in \operatorname{Ob}\mathcal{C}$. Then for all $U \in \mathcal{U}_{\mathrm{mul}}$, we have $1_X \in U$ by the multiplicativity of U, and therefore $1_X \in \bigcap \mathcal{U}_{\mathrm{mul}} = D_{\mathrm{mul}}$.

Altogether, D_{mul} is a multiplicative subset of $\operatorname{Mor}\mathcal{C}$.

Moreover, given an arbitrary multiplicative subset U of $\operatorname{Mor}\mathcal{C}$, we have $D_{\mathrm{mul}} \subseteq U$ by definition of D_{mul}, so D_{mul} is in fact the smallest multiplicative subset of $\operatorname{Mor}\mathcal{C}$.

(ii) This is proven analogously to (i).

(iii) This is proven analogously to (i).

(iv) As $\operatorname{Sat} D \in \mathcal{U}_{\mathrm{sat}}$, we have $D_{\mathrm{sat}} = \bigcap \mathcal{U}_{\mathrm{sat}} \subseteq \operatorname{Sat} D$. Moreover, for all $U \in \mathcal{U}_{\mathrm{sat}}$, we have $D \subseteq U$ and therefore $\operatorname{Sat} D \subseteq \operatorname{Sat} U = U$ by proposition (1.31) and the saturatedness of U. Thus we also have $\operatorname{Sat} D \subseteq \bigcap \mathcal{U}_{\mathrm{sat}} = D_{\mathrm{sat}}$.

Altogether, we have $D_{\mathrm{sat}} = \operatorname{Sat} D$. In particular, D_{sat} is a saturated subset of $\operatorname{Mor}\mathcal{C}$ by corollary (1.33).

Moreover, given an arbitrary saturated subset U of $\operatorname{Mor}\mathcal{C}$, we have $D_{\mathrm{sat}} \subseteq U$ by definition of D_{sat}, so D_{sat} is in fact the smallest saturated subset of $\operatorname{Mor}\mathcal{C}$.

(b) By proposition (1.43), we have

$$\mathcal{U} \supseteq \mathcal{U}_{\mathrm{mul}} \supseteq \mathcal{U}_{\mathrm{ssat}} \supseteq \mathcal{U}_{\mathrm{wsat}} \supseteq \mathcal{U}_{\mathrm{sat}}$$

and therefore

$$\bigcap \mathcal{U} \subseteq \bigcap \mathcal{U}_{\mathrm{mul}} \subseteq \bigcap \mathcal{U}_{\mathrm{ssat}} \subseteq \bigcap \mathcal{U}_{\mathrm{wsat}} \subseteq \bigcap \mathcal{U}_{\mathrm{sat}}$$

that is,

$$D \subseteq D_{\mathrm{mul}} \subseteq D_{\mathrm{ssat}} \subseteq D_{\mathrm{wsat}} \subseteq D_{\mathrm{sat}} = \operatorname{Sat} D.$$

(c) This follows from (b) and proposition (1.32). □

Chapter II

Z-2-arrow calculus

By a theorem of Gabriel and Zisman [12, sec. 1.1], we know that there exists a localisation of every category $\mathcal{C}$ with respect to every subset D of its set of morphisms $\operatorname{Mor}\mathcal{C}$, cf. theorem (1.24). The objects in the Gabriel-Zisman localisation of $\mathcal{C}$ are precisely the objects in $\mathcal{C}$; and the morphisms are equivalence classes of zigzags

$$\longrightarrow \xleftarrow{\approx} \longrightarrow \dots \xleftarrow{\approx} \longrightarrow$$

of finite but arbitrary length, where the "backward" arrows are in D and where the defining equivalence relation is generated by certain elementary relations. As a consequence, the question of equality of representatives leads to a word problem. In this generality, however, there does not seem to exist a more convenient calculus.

There are other constructions for localisations in particular cases. For example, the classical construction of the derived category of an abelian category by Verdier [37, ch. II, §1, not. 1.1] is done in two steps: First, one starts with the category of complexes in the given abelian category and passes to the homotopy category of complexes, a quotient of additive categories. Second, one localises this homotopy category of complexes at the (homotopy classes of) quasi-isomorphisms using a procedure called Ore localisation (more precisely, S-Ore localisation in our terminology), which has its historical origins in ring theory, cf. the works of Ore [27, sec. 2] and Asano [2, Satz 1]. We recall this classical construction briefly in section 3.

For the S-Ore localisation, one has more convenient results that answer the question about representatives and equality of representatives: Every morphism in this localisation is represented by a diagram

$$\longrightarrow \xleftarrow{\approx} \ ,$$

which we call an S-2-arrow ([1]). The first arrow we consider as its numerator, the second as its denominator – like in rational numbers, but with a directed numerator and a directed denominator. Moreover, already from the construction of the S-Ore localisation it follows that two of these diagrams represent the same morphism if and only if they can be embedded as the top and the bottom row in a commutative diagram of the following form.

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, equal] & {} \arrow[d] & {} \arrow[l, "\approx"'] \arrow[d, equal] \\
{} \arrow[r] & {} & {} \arrow[l, "\approx"'] \\
{} \arrow[r] \arrow[u, equal] & {} \arrow[u, "\approx"] & {} \arrow[l, "\approx"'] \arrow[u, equal]
\end{tikzcd}$$

So roughly said, two numerator-denominator pairs represent the same morphism if and only if they have a common expansion, again like in rational numbers. We say that the S-Ore localisation admits a (strict) S-2-arrow calculus.

In our example of the derived category, one has S-2-arrows as representatives, and two such S-2-arrows represent the same morphism in the derived category if and only if they can be embedded in a 2-by-2 diagram as above that is commutative in the homotopy category of complexes. In other words: The equality of S-2-arrows is characterised by such a 2-by-2 diagram in the category of complexes that is commutative up to homotopy. We say

[1]The prefix "S-" is used to distinguish our situation from the dual case, here and in several other notions below.

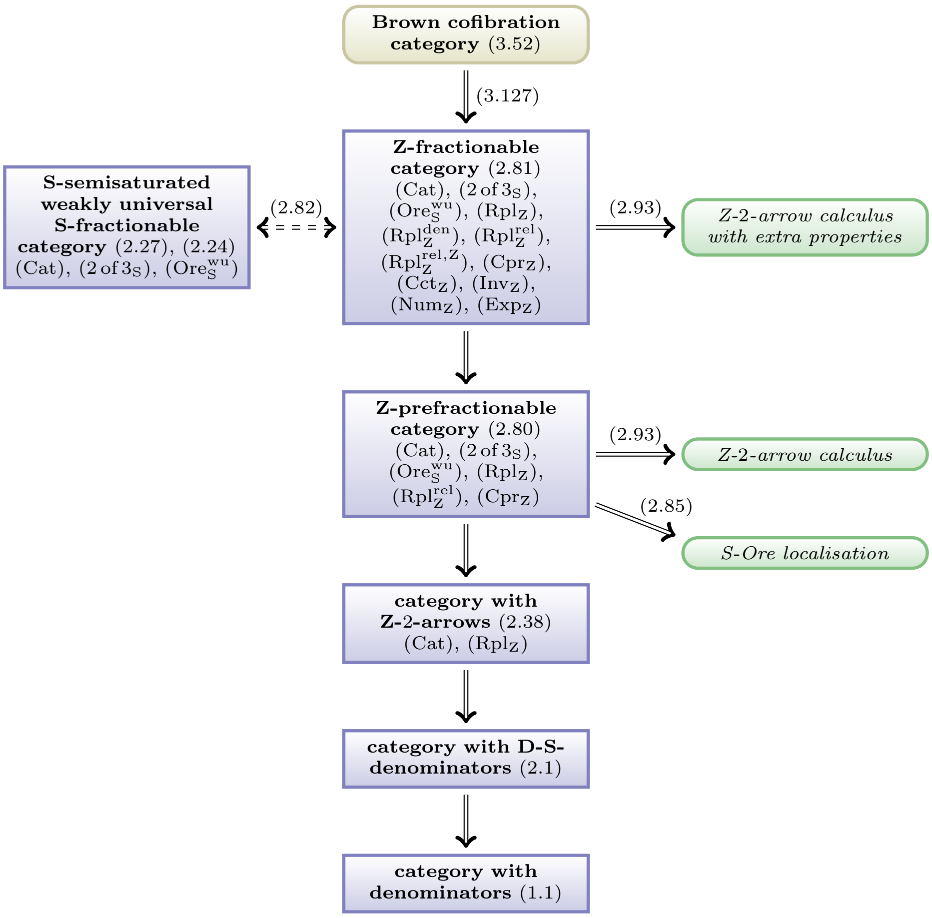


Figure 1: Z-fractionable categories: a localisation theory for Brown cofibration categories.

that the derived category admits a homotopy S-2-arrow calculus. The idea of such a two-step construction was taken up by Brown in the more general framework of a Brown cofibration category ([2]), see definition (3.52)(a); he developed a 2-arrow calculus up to homotopy in this context [7, dual of th. 1 and proof], cf. theorem (3.132).
In this chapter, we develop an axiomatic approach for a kind of strict 2-arrow calculus for so-called Z-fractionable categories, see definition (2.81)(a): Instead of working with all S-2-arrows as representatives, we restrict our attention to particular S-2-arrows, the so-called Z-2-arrows, which still represent all morphisms in the localisation, see theorem (2.93)(a). The question of the equality of morphisms represented by given Z-2-arrows is then answered by a strict 2-by-2 diagram, see theorem (2.93)(b). The axioms of a Z-fractionable category are fulfilled by a Brown cofibration category, see theorem (3.127).
In fact, most results developed in this chapter still hold if we forget about half of the axioms of a Z-fractionable category, and so we often work with so-called Z-prefractionable categories, see definition (2.80)(a). For Z-fractionable categories, the Z-2-arrow calculus is more flexible, see theorem (2.93)(d), (e), and the composition rule is simpler, see remark (2.103)(a). The author does not know of a Z-prefractionable category that is not a Z-fractionable category.
The chapter is organised as follows. In section 1, we introduce categories with denominators and S-denominators, which is an expansion of the notion of a category with denominators, see definition (1.1)(a), where several denominators are distinguished. Thereafter, we study the S-2-arrow graph of a category with denominators in section 2, a graph having a quotient that becomes a localisation of the category with denominators we started with. In section 3, we generalise the classical notion of an S-Ore completion to S-denominators, and, moreover, we briefly recall the classical S-Ore construction. Then in section 4, Z-2-arrows are introduced and first properties are collected that follow from the fact that S-2-arrows may be replaced by Z-2-arrows in the sense of definition (2.38)(a). After that, we introduce the axioms of a Z-fractionable category in section 5 and deduce

[2] In fact, he studied the dual notion of a Brown fibration category and used the terminology *category of fibrant objects* [7, sec. 1].

some facts from these axioms. Moreover, we compare the classical approach of an S-fractionable category with that of a Z-fractionable category. The construction of the S-Ore localisation is then generalised to the framework of a Z-prefractionable category in section 6. In particular, theorem (2.93) yields a generalisation for the classical S-2-arrow calculus. Although Z-2-arrows play a prominent role, we still also work with arbitrary S-2-arrows in section 6 to gain more flexibility. However, it is possible to work only with Z-2-arrows as representatives, and this approach will be indicated in section 7. Finally, in section 8, we compare our approach to the 3-arrow approach for Brown cofibration categories of Maltsiniotis [26].

1 Categories with denominators and S-denominators

Categories with denominators, see definition (1.1)(a), provide the categorical concept for localisation, see chapter I, sections 1 and 2, in particular definition (1.11)(b) and corollary (1.14)(d). In this section, we introduce the concept of a category with denominators and S-denominators, that is, a structure where particular denominators are distinguished. These so-called S-denominators may fulfill certain properties that need not necessarily hold for all denominators, cf. for example definition (2.23).

Definition of a category with D-S-denominators

For the definition of a category with denominators and of a morphism of categories with denominators, see definition (1.1).

(2.1) Definition (category with D-S-denominators)**.**

(a) A *category with denominators and S-denominators* (or *category with D-S-denominators*, for short) consists of a category with denominators $\mathcal{C}$ together with a multiplicative subset S of $\operatorname{Den} \mathcal{C}$. By abuse of notation, we refer to the said category with D-S-denominators as well as to its underlying category with denominators just by $\mathcal{C}$. The elements of S are called *S-denominators* in $\mathcal{C}$.

Given a category with D-S-denominators $\mathcal{C}$ with set of S-denominators S, we write $\operatorname{SDen} \mathcal{C} := S$. In diagrams, an S-denominator $i \colon X \to Y$ in $\mathcal{C}$ will usually be depicted as

$$X \overset{i}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;\;} Y \,.$$

(b) We suppose given categories with D-S-denominators $\mathcal{C}$ and $\mathcal{D}$. A *morphism of categories with denominators and S-denominators* (or *morphism of categories with D-S-denominators*, for short) from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of categories with denominators $F \colon \mathcal{C} \to \mathcal{D}$ that *preserves S-denominators*, that is, such that Fi is an S-denominator in $\mathcal{D}$ for every S-denominator i in $\mathcal{C}$.

Although the following example is quite obvious, it will give us a canonical connection between categories with denominators and categories with D-S-denominators.

(2.2) Example. Every multiplicative category with denominators $\mathcal{C}$ carries the structure of a category with D-S-denominators having

$$\operatorname{SDen} \mathcal{C} = \operatorname{Den} \mathcal{C}.$$

The category of categories with D-S-denominators

(2.3) Definition (category with D-S-denominators with respect to a Grothendieck universe)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. A category with D-S-denominators $\mathcal{C}$ is called a *category with D-S-denominators with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$-*category with D-S-denominators*) if its underlying category with denominators is a $\mathfrak{U}$-category with denominators.

(2.4) Remark.

(a) We suppose given a Grothendieck universe $\mathfrak{U}$. A category with D-S-denominators $\mathcal{C}$ is a $\mathfrak{U}$-category with D-S-denominators if and only if it is an element of $\mathfrak{U}$.

(b) For every category with D-S-denominators $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a $\mathfrak{U}$-category with D-S-denominators.

(2.5) Remark. For every Grothendieck universe $\mathfrak{U}$ we have a category $\mathbf{CatDS}_{(\mathfrak{U})}$, given as follows. The set of objects of $\mathbf{CatDS}_{(\mathfrak{U})}$ is given by

$$\operatorname{Ob}\mathbf{CatDS}_{(\mathfrak{U})} = \{\mathcal{C} \mid \mathcal{C} \text{ is a } \mathfrak{U}\text{-category with D-S-denominators}\}.$$

For objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathbf{CatDS}_{(\mathfrak{U})}$, we have the hom-set

$${}_{\mathbf{CatDS}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D}) = \{F \mid F \text{ is a morphism of categories with D-S-denominators from } \mathcal{C} \text{ to } \mathcal{D}\}.$$

For morphisms $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{D} \to \mathcal{E}$ in $\mathbf{CatDS}_{(\mathfrak{U})}$, the composite of F and G in $\mathbf{CatDS}_{(\mathfrak{U})}$ is given by the composite of the underlying morphisms of categories with denominators $G \circ F \colon \mathcal{C} \to \mathcal{E}$. For an object $\mathcal{C}$ in $\mathbf{CatDS}_{(\mathfrak{U})}$, the identity morphism on $\mathcal{C}$ in $\mathbf{CatDS}_{(\mathfrak{U})}$ is given by the underlying identity morphism of categories with denominators $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$.

(2.6) Definition (category of categories with D-S-denominators)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. The category $\mathbf{CatDS} = \mathbf{CatDS}_{(\mathfrak{U})}$ as considered in remark (2.5) is called the *category of categories with D-S-denominators* (more precisely, the *category of* $\mathfrak{U}$*-categories with D-S-denominators*).

The S-structure

In example (2.2), we have seen that there can be defined a structure of a category with D-S-denominators on every multiplicative category with denominators. Since we will need this structure later, we assign a name to it.

(2.7) Definition (S-structure)**.** Given a multiplicative category with denominators $\mathcal{C}$, we denote by $\mathcal{C}_{\mathrm{S}}$ the category with D-S-denominators whose underlying category with denominators is $\mathcal{C}$ and whose set of S-denominators is given by

$$\operatorname{SDen}\mathcal{C}_{\mathrm{S}} = \operatorname{Den}\mathcal{C}.$$

The structure of a category with D-S-denominators of $\mathcal{C}_{\mathrm{S}}$ is called the *S-structure* (*of a category with D-S-denominators*) on $\mathcal{C}$.

(2.8) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. We have a functor

$$-_{\mathrm{S}} \colon \mathbf{CatD}_{\mathrm{mul},(\mathfrak{U})} \to \mathbf{CatDS}_{(\mathfrak{U})},$$

given on the morphisms by $F_{\mathrm{S}} = F$ for $F \in \operatorname{Mor}\mathbf{CatD}_{\mathrm{mul},(\mathfrak{U})}$, which is full, faithful and injective on the objects.

2 S-2-arrows

Like the ordinary S-Ore localisation, the S-Ore localisation of a Z-prefractionable category fulfills some kind of 2-arrow calculus, cf. theorem (2.35) and theorem (2.93). In particular, the morphisms in the localisation are represented by so-called S-2-arrows, that is, diagrams consisting of two arrows, where one of them is formally inverted. Moreover, in both cases, the S-Ore localisation is constructed ab ovo using S-2-arrows, see definition (2.30) and definition (2.86).
In this section, we introduce the S-2-arrow graph for a given a category with denominators, whose objects are the same objects as in our given category and whose arrows are precisely the S-2-arrows. When the category with denominators at hand is moreover equipped with a subset of S-denominators, one has in addition a variant of the S-2-arrow graph involving only those S-2-arrows whose denominator is actually an S-denominator. This variant allows to generalise the classical notion of an S-Ore completion, see definition (2.23)(a). Finally, we consider a congruence called S-fraction equality on the S-2-arrow graph. Later, the S-Ore localisation will have as underlying graph precisely the quotient graph obtained from the S-2-arrow graph modulo S-fraction equality, see definition (2.30) and definition (2.86).

The (normal) S-2-arrow graph

(2.9) Definition (S-2-arrow shape)**.** The *S-2-arrow shape* is defined to be the graph $\mathbf{\Theta}_{\mathrm{S}}$ given by

$$\operatorname{Ob} \mathbf{\Theta}_{\mathrm{S}} = \{1,2,3\},$$
$$\operatorname{Arr} \mathbf{\Theta}_{\mathrm{S}} = \{(1,2),(3,2)\},$$

and where $\operatorname{Source}(1,2) = 1$, $\operatorname{Target}(1,2) = 3$, $\operatorname{Source}(3,2) = 3$, $\operatorname{Target}(3,2) = 2$.

$$1 \longrightarrow 2 \longleftarrow 3$$

A diagram of shape $\mathbf{\Theta}_{\mathrm{S}}$ in $\mathcal{C}$ is just a graph morphism $X\colon \mathbf{\Theta}_{\mathrm{S}} \to \mathcal{C}$. Given a diagram X of shape $\mathbf{\Theta}_{\mathrm{S}}$ in $\mathcal{C}$, we write $X_i = X(i)$ for $i \in \operatorname{Ob} \mathbf{\Theta}_{\mathrm{S}}$ and $X_a = X(a)$ for $a \in \operatorname{Arr} \mathbf{\Theta}_{\mathrm{S}}$. Given diagrams X and Y, a diagram morphism from X to Y is a family $f = (f_i)_{i \in \operatorname{Ob} \mathbf{\Theta}_{\mathrm{S}}}$ in $\operatorname{Mor} \mathcal{C}$ with $X_a f_j = f_i Y_a$ for all arrows $a\colon i \to j$ in $\mathbf{\Theta}_{\mathrm{S}}$. The category consisting of diagrams of shape $\mathbf{\Theta}_{\mathrm{S}}$ in $\mathcal{C}$ as objects and diagram morphisms between those diagrams as morphisms will be denoted by $\mathcal{C}^{\mathbf{\Theta}_{\mathrm{S}}}$. ([3])
Given a graph $\mathcal{G}$, a subgraph $\mathcal{U}$ of $\mathcal{G}$ is said to be wide if $\operatorname{Ob} \mathcal{U} = \operatorname{Ob} \mathcal{G}$.
For the definition of a category with denominators, see definition (1.1)(a). For the definition of a category with D-S-denominators, see definition (2.1)(a).

(2.10) Definition ((normal) S-2-arrow graph)**.**

(a) We suppose given a category with denominators $\mathcal{C}$. The *S-2-arrow graph* of $\mathcal{C}$ is defined to be the graph $\operatorname{AG}_{\mathrm{S}} \mathcal{C}$ given by

$$\operatorname{Ob} \operatorname{AG}_{\mathrm{S}} \mathcal{C} = \operatorname{Ob} \mathcal{C},$$
$$\operatorname{Arr} \operatorname{AG}_{\mathrm{S}} \mathcal{C} = \{A \in \operatorname{Ob} \mathcal{C}^{\mathbf{\Theta}}_{\mathrm{S}} \mid A_{3,2} \text{ is a denominator in } \mathcal{C}\},$$

and where $\operatorname{Source} A = A_1$ and $\operatorname{Target} A = A_3$ for $A \in \operatorname{Arr} \operatorname{AG}_{\mathrm{S}} \mathcal{C}$.

An arrow A in $\operatorname{AG}_{\mathrm{S}} \mathcal{C}$ is called an *S-2-arrow* in $\mathcal{C}$. Given a morphism $f\colon X \to \tilde{Y}$ and a denominator $a\colon Y \to \tilde{Y}$ in $\mathcal{C}$, we abuse notation and denote the unique S-2-arrow A with $A_{1,2} = f$ and $A_{3,2} = a$ by $(f, a) := A$. Moreover, we use the notation $(f, a)\colon X \to \tilde{Y} \leftarrow Y$.

$$X \xrightarrow{f} \tilde{Y} \overset{a}{\underset{\approx}{\longleftarrow}} Y$$

(b) We suppose given a category with D-S-denominators $\mathcal{C}$. The wide subgraph $\operatorname{AG}_{\mathrm{S,n}} \mathcal{C}$ of $\operatorname{AG}_{\mathrm{S}} \mathcal{C}$ with

$$\operatorname{Arr} \operatorname{AG}_{\mathrm{S,n}} \mathcal{C} = \{(f, i) \in \operatorname{Arr} \operatorname{AG}_{\mathrm{S}} \mathcal{C} \mid i \text{ is an S-denominator}\}$$

is called the *normal S-2-arrow graph* of $\mathcal{C}$. An S-2-arrow in $\mathcal{C}$ that is an arrow in $\operatorname{AG}_{\mathrm{S,n}} \mathcal{C}$ is said to be *normal*.

So if we consider in an S-2-arrow (f, a) the first morphism f as the "numerator part" and the second morphism a as the "denominator part" of (f, a), then an S-2-arrow may have an arbitrary denominator as the denominator part, whereas the denominator part of a normal S-2-arrow is an S-denominator.
The next remark shows that the S-2-arrow graph may be seen as a particular case of the normal S-2-arrow graph.

(2.11) Remark. For every category with denominators $\mathcal{C}$, we have

$$\operatorname{AG}_{\mathrm{S,n}} \mathcal{C}_{\mathrm{S}} = \operatorname{AG}_{\mathrm{S}} \mathcal{C}.$$

[3] By the adjunction "free category on a graph – underlying graph of a category", diagrams of shape $\mathbf{\Theta}_{\mathrm{S}}$ in $\mathcal{C}$ correspond in a unique way to functors from the free category on $\mathbf{\Theta}_{\mathrm{S}}$ to $\mathcal{C}$, and diagram morphisms correspond to transformations.

(2.12) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\mathbf{\Theta}_{\mathrm{S}}$ is in $\mathfrak{U}$.

(a) We suppose given a category with denominators $\mathcal{C}$. If $\mathcal{C}$ is a $\mathfrak{U}$-category with denominators, then $\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$ is a $\mathfrak{U}$-graph.

(b) We suppose given a category with D-S-denominators $\mathcal{C}$. If $\mathcal{C}$ is a $\mathfrak{U}$-category with D-S-denominators, then $\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$ is a $\mathfrak{U}$-graph.

(2.13) Proposition. We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\mathbf{\Theta}_{\mathrm{S}}$ is in $\mathfrak{U}$.

(a) We have a functor

$$\mathrm{AG}_{\mathrm{S}} \colon \mathbf{CatD}_{(\mathfrak{U})} \to \mathbf{Grph}_{(\mathfrak{U})},$$

given on the morphisms as follows. For every morphism $F \colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{CatD}_{(\mathfrak{U})}$, the morphism $\mathrm{AG}_{\mathrm{S}}\,F \colon \mathrm{AG}_{\mathrm{S}}\,\mathcal{C} \to \mathrm{AG}_{\mathrm{S}}\,\mathcal{D}$ is given on the objects by

$$(\mathrm{AG}_{\mathrm{S}}\,F)X = FX$$

for $X \in \operatorname{Ob}\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$ and on the arrows by

$$(\mathrm{AG}_{\mathrm{S}}\,F)(f, a) = (Ff, Fa)$$

for $(f, a) \in \operatorname{Arr}\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$.

(b) We have a functor

$$\mathrm{AG}_{\mathrm{S,n}} \colon \mathbf{CatDS}_{(\mathfrak{U})} \to \mathbf{Grph}_{(\mathfrak{U})},$$

given on the morphisms as follows. For every morphism $F \colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{CatDS}_{(\mathfrak{U})}$, the morphism $\mathrm{AG}_{\mathrm{S,n}}\,F \colon \mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C} \to \mathrm{AG}_{\mathrm{S,n}}\,\mathcal{D}$ is given on the objects by

$$(\mathrm{AG}_{\mathrm{S,n}}\,F)X = FX$$

for $X \in \operatorname{Ob}\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$ and on the arrows by

$$(\mathrm{AG}_{\mathrm{S,n}}\,F)(f, i) = (Ff, Fi)$$

for $(f, i) \in \operatorname{Arr}\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$.

Proof.

(a) This follows from remark (2.11) and (b).

(b) We suppose given $\mathcal{C}, \mathcal{D} \in \operatorname{Ob}\mathbf{CatDS}$. For every morphism $F \colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{CatDS}$ and for $(f, i) \in \operatorname{Arr}\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$, we have $(Ff, Fi) \in \operatorname{Arr}\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{D}$ as F preserves S-denominators and

$$\begin{aligned}\operatorname{Source}(Ff, Fi) &= \operatorname{Source}Ff = F(\operatorname{Source}f) = F(\operatorname{Source}(f, i)),\\ \operatorname{Target}(Ff, Fi) &= \operatorname{Source}Fi = F(\operatorname{Source}i) = F(\operatorname{Target}(f, i)).\end{aligned}$$

Hence we obtain a well-defined map

$$A_{\mathcal{C},\mathcal{D}} \colon {}_{\mathbf{CatDS}}(\mathcal{C}, \mathcal{D}) \to {}_{\mathbf{Grph}}(\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}, \mathrm{AG}_{\mathrm{S,n}}\,\mathcal{D}),$$

where $A_{\mathcal{C},\mathcal{D}}(F)$ for $F \in {}_{\mathbf{CatDS}}(\mathcal{C}, \mathcal{D})$ is given by $A_{\mathcal{C},\mathcal{D}}(F)X = FX$ for $X \in \operatorname{Ob}\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$ and $A_{\mathcal{C},\mathcal{D}}(F)(f, i) = (Ff, Fi)$ for $(f, i) \in \operatorname{Arr}\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$.

For morphisms $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ in $\mathbf{CatDS}$, we get

$$A_{\mathcal{C},\mathcal{E}}(G \circ F)X = (G \circ F)X = A_{\mathcal{D},\mathcal{E}}(G)A_{\mathcal{C},\mathcal{D}}(F)X$$

for $X \in \operatorname{Ob}\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$ and

$$A_{\mathcal{C},\mathcal{E}}(G \circ F)(f, i) = ((G \circ F)f, (G \circ F)i) = (GFf, GFi) = A_{\mathcal{D},\mathcal{E}}(G)A_{\mathcal{C},\mathcal{D}}(F)(f, i)$$

for $(f, i) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$, that is, $A_{\mathcal{C},\mathcal{E}}(G \circ F) = A_{\mathcal{D},\mathcal{E}}(G) \circ A_{\mathcal{C},\mathcal{D}}(F)$. Moreover, for every object $\mathcal{C}$ in **CatDS**, we get

$$A_{\mathcal{C},\mathcal{C}}(\mathrm{id}_{\mathcal{C}})X = \mathrm{id}_{\mathcal{C}} X = X = \mathrm{id}_{\mathrm{AG}_{\mathrm{S,n}} \mathcal{C}} X$$

for $X \in \operatorname{Ob} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$ and

$$A_{\mathcal{C},\mathcal{C}}(\mathrm{id}_{\mathcal{C}})(f, i) = (\mathrm{id}_{\mathcal{C}} f, \mathrm{id}_{\mathcal{C}} i) = (f, i) = \mathrm{id}_{\mathrm{AG}_{\mathrm{S,n}} \mathcal{C}}(f, i)$$

for $(f, i) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$, that is, $A_{\mathcal{C},\mathcal{C}}(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\mathrm{AG}_{\mathrm{S,n}} \mathcal{C}}$. Thus we have a functor

$$\mathrm{AG}_{\mathrm{S,n}} \colon \mathbf{CatDS} \to \mathbf{Grph}$$

that is given on the morphisms by $\mathrm{AG}_{\mathrm{S,n}} F = A_{\mathcal{C},\mathcal{D}}(F)$ for $F \in {}_{\mathbf{CatDS}}(\mathcal{C}, \mathcal{D})$. □

(Normal) S-fraction equality

Our next step will be the introduction of equivalence relations on the sets of arrows of the S-2-arrow graph resp. the normal S-2-arrow graph.

(2.14) Definition ((normal) S-fraction equality)**.**

(a) We suppose given a category with denominators $\mathcal{C}$. The equivalence relation $\equiv_{\mathrm{S}}$ on $\operatorname{Arr} \mathrm{AG}_{\mathrm{S}} \mathcal{C}$ is defined to be generated by the following relation on $\operatorname{Arr} \mathrm{AG}_{\mathrm{S}} \mathcal{C}$: Given $(f, a) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{S}} \mathcal{C}$ and a morphism c in $\mathcal{C}$ such that ac is a denominator in $\mathcal{C}$, then (f, a) is in relation to (fc, ac).

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, equal] & \cdot \arrow[d, "c"] & \cdot \arrow[l, "a"', "\approx"] \arrow[d, equal] \\ \cdot \arrow[r, "fc"'] & \cdot & \cdot \arrow[l, "ac"', "\approx"] \end{tikzcd}$$

Given $(f, a), (\tilde{f}, \tilde{a}) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{S}} \mathcal{C}$ with $(f, a) \equiv_{\mathrm{S}} (\tilde{f}, \tilde{a})$, we say that (f, a) and $(\tilde{f}, \tilde{a})$ are *S-fraction equal.*

(b) We suppose given a category with D-S-denominators $\mathcal{C}$. The equivalence relation $\equiv_{\mathrm{S,n}}$ on $\operatorname{Arr} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$ is defined to be generated by the following relation on $\operatorname{Arr} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$: Given $(f, i) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$ and a morphism c in $\mathcal{C}$ such that ic is an S-denominator in $\mathcal{C}$, then (f, i) is in relation to (fc, ic).

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, equal] & \cdot \arrow[d, "c"] & \cdot \arrow[l, "i"'] \arrow[d, equal] \\ \cdot \arrow[r, "fc"'] & \cdot & \cdot \arrow[l, "ic"'] \end{tikzcd}$$

Given $(f, i), (\tilde{f}, \tilde{i}) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$ with $(f, i) \equiv_{\mathrm{S,n}} (\tilde{f}, \tilde{i})$, we say that (f, i) and $(\tilde{f}, \tilde{i})$ are *normally S-fraction equal.*

If the category with denominators $\mathcal{C}$ in definition (2.14)(a) resp. the category with D-S-denominators $\mathcal{C}$ in definition (2.14)(b) is S-semisaturated, then the morphism c in loc. cit. is automatically a denominator, respectively.

(2.15) Remark. We suppose given a category with denominators $\mathcal{C}$. For S-2-arrows (f, a), $(\tilde{f}, \tilde{a})$ in $\mathcal{C}$, we have $(f, a) \equiv_{\mathrm{S,n}} (\tilde{f}, \tilde{a})$ in $\operatorname{Arr} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}_{\mathrm{S}}$ if and only if $(f, a) \equiv_{\mathrm{S}} (\tilde{f}, \tilde{a})$ in $\operatorname{Arr} \mathrm{AG}_{\mathrm{S}} \mathcal{C}$.

(2.16) Remark. We suppose given a category with D-S-denominators $\mathcal{C}$ and normal S-2-arrows (f, i), $(\tilde{f}, \tilde{i})$ in $\mathcal{C}$. If $(f, i) \equiv_{\mathrm{S,n}} (\tilde{f}, \tilde{i})$, then $(f, i) \equiv_{\mathrm{S}} (\tilde{f}, \tilde{i})$.

Proof. This holds as every normal S-2-arrow is in particular an S-2-arrow. □

(2.17) Remark. We suppose given a category with denominators $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. Moreover, we suppose given S-2-arrows (f, a), $(\tilde{f}, \tilde{a})$ in $\mathcal{C}$. If $(f, a) \equiv_{\mathrm{S}} (\tilde{f}, \tilde{a})$, then

$$\operatorname{loc}(f) \operatorname{loc}(a)^{-1} = \operatorname{loc}(\tilde{f}) \operatorname{loc}(\tilde{a})^{-1}$$

in $\mathcal{L}$.

Proof. For every morphism c in $\mathcal{C}$ such that ac is a denominators in $\mathcal{C}$, we have

$$\mathrm{loc}(f)\,\mathrm{loc}(a)^{-1}\,\mathrm{loc}(ac) = \mathrm{loc}(f)\,\mathrm{loc}(a)^{-1}\,\mathrm{loc}(a)\,\mathrm{loc}(c) = \mathrm{loc}(f)\,\mathrm{loc}(c) = \mathrm{loc}(fc)$$

and therefore

$$\mathrm{loc}(f)\,\mathrm{loc}(a)^{-1} = \mathrm{loc}(fc)\,\mathrm{loc}(ac)^{-1}.$$ □

(2.18) Remark. We suppose given a semisaturated category with denominators $\mathcal{C}$ and S-2-arrows (f, a), $(\tilde{f}, \tilde{a})$ in $\mathcal{C}$. If $(f, a) \equiv_{\mathrm{S}} (\tilde{f}, \tilde{a})$, then f is a denominator in $\mathcal{C}$ if and only if $\tilde{f}$ is a denominator in $\mathcal{C}$.

Proof. This follows by the definition of S-fraction equality (2.14)(a) and by the semisaturatedness of $\mathcal{C}$. □

(2.19) Remark.

(a) We suppose given a multiplicative category with denominators $\mathcal{C}$ and S-2-arrows (f, a), $(\tilde{f}, \tilde{a})$ in $\mathcal{C}$. If $(f, a) \equiv_{\mathrm{S}} (\tilde{f}, \tilde{a})$, then $(gf, da) \equiv_{\mathrm{S}} (g\tilde{f}, d\tilde{a})$ for every morphism g in $\mathcal{C}$ with $\mathrm{Target}\, g = \mathrm{Source}\,(f, a) = \mathrm{Source}\,(\tilde{f}, \tilde{a})$ and for every denominator d in $\mathcal{C}$ with $\mathrm{Target}\, d = \mathrm{Target}\,(f, a) = \mathrm{Target}\,(\tilde{f}, \tilde{a})$.

(b) We suppose given a multiplicative category with D-S-denominators $\mathcal{C}$ and normal S-2-arrows (f, i), $(\tilde{f}, \tilde{i})$ in $\mathcal{C}$. If $(f, i) \equiv_{\mathrm{S,n}} (\tilde{f}, \tilde{i})$, then $(gf, ji) \equiv_{\mathrm{S,n}} (g\tilde{f}, j\tilde{i})$ for every morphism g in $\mathcal{C}$ with $\mathrm{Target}\, g = \mathrm{Source}\,(f, i) = \mathrm{Source}\,(\tilde{f}, \tilde{i})$ and for every S-denominator j in $\mathcal{C}$ with $\mathrm{Target}\, j = \mathrm{Target}\,(f, i) = \mathrm{Target}\,(\tilde{f}, \tilde{i})$.

Proof.

(a) This follows from remark (2.15) and (b).

(b) This follows by the definition of normal S-fraction equality (2.14)(b). □

In the next remark, we will show that (normal) S-fraction equality respects the graph structure on the (normal) S-2-arrow graph, and so we may pass to quotient graphs.

(2.20) Remark.

(a) We suppose given a category with denominators $\mathcal{C}$. The S-fraction equality relation $\equiv_{\mathrm{S}}$ is a graph congruence on $\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$.

(b) We suppose given a category with D-S-denominators $\mathcal{C}$. The normal S-fraction equality relation $\equiv_{\mathrm{S,n}}$ is a graph congruence on $\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$.

Proof.

(a) This follows from remark (2.15) and (b).

(b) For $(f, i) \in \mathrm{Arr}\,\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$, $c \in \mathrm{Mor}\,\mathcal{C}$ with $ic \in \mathrm{SDen}\,\mathcal{C}$, we have

$$\mathrm{Source}\,(fc, ic) = \mathrm{Source}(fc) = \mathrm{Source}\, f = \mathrm{Source}\,(f, i),$$
$$\mathrm{Target}\,(fc, ic) = \mathrm{Source}(ic) = \mathrm{Source}\, i = \mathrm{Target}\,(f, i).$$ □

(2.21) Definition ((normal) S-fraction)**.**

(a) We suppose given a category with denominators $\mathcal{C}$. Given an S-2-arrow (f, a) in $\mathcal{C}$, its equivalence class in the quotient graph $(\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ is denoted by $f/a := [(f, a)]_{\equiv_{\mathrm{S}}}$ and is said to be the *S-fraction* of (f, a).

(b) We suppose given a category with D-S-denominators $\mathcal{C}$. Given a normal S-2-arrow (f, i) in $\mathcal{C}$, its equivalence class in the quotient graph $(\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C})/{\equiv_{\mathrm{S,n}}}$ is said to be the *normal S-fraction* of (f, i). If no confusion arises, we abuse notation and also write $f/i := [(f, i)]_{\equiv_{\mathrm{S,n}}}$ ([4]).

(2.22) Remark. We suppose given a category with D-S-denominators $\mathcal{C}$. The inclusion $\mathrm{inc}\colon \mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C} \to \mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$ induces a well-defined graph morphism

$$(\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C})/{\equiv_{\mathrm{S,n}}} \to (\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/{\equiv_{\mathrm{S}}},$$

which is identical on the objects and maps the normal S-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{S,n}}}$ of some $(f, i) \in \mathrm{Arr}\,\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$ to the S-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{S}}}$.

Proof. This follows from remark (2.16). □

[4] This abuse will be justified for the case where $\mathcal{C}$ is a Z-prefractionable category in corollary (2.61).

3 S-Ore completions and the classical S-Ore localisation

This section has two aims: First, we will introduce S-Ore completions, that is, certain S-2-arrows that make two given morphisms with the same source object into a commutative quadrangle, see definition (2.23)(a) and definition (2.24). Second, we will recall the ordinary S-Ore completion for so-called S-fractionable categories, see definition (2.30), and the S-2-arrow calculus, see theorem (2.35). S-fractionable categories are categories with denominators that admit S-Ore completions and fulfill an extra condition, see definition (2.27)(a). This second part is well-known – except possibly theorem (2.37), which states that S-fractionable categories are the only multiplicative categories with denominators that admit an S-2-arrow calculus. We include it to be able to conveniently compare our approach for Z-(pre)fractionable categories, see section 6, and the classical one. In contrast to the S-Ore completions, the second part of this section will not be used elsewhere in this thesis.
The basic ideas of the classical S-Ore localisation have their historical origin in ring theory, in particular in the works of Ore [27, sec. 2] and Asano [2, Satz 1]. The categorical version comes from the Grothendieck school, see Verdier [37, ch. I, §2, sec. 3.2] and Grothendieck and Hartshorne [15, ch. I, §3, prop. 3.1], inspired by the work of Serre [34, ch. I, sec. 2].

S-Ore completions

We start with the definition of S-Ore completions. As already mentioned above, we think of S-2-arrows as representatives for fractions, like the rational numbers, but with directed numerator and directed denominator. Having this image in mind, an S-Ore completion is then, roughly said, a method to replace a diagram, where numerator and denominator are in a wrong order, by an actual S-2-arrow.
While classical Ore completions are defined via arbitrary denominators, they will be introduced here using S-denominators, as this is the form in which we use them later. However, the classical definition is reobtained if we interpret a category with denominators canonically as a category with D-S-denominators, see definition (2.7) and definition (2.24)(b).
For the structure of a category with D-S-denominators, see definition (2.1)(a).

(2.23) Definition (S-Ore completion)**.** We suppose given a category with D-S-denominators $\mathcal{C}$, a morphism f and an S-denominator i in $\mathcal{C}$ with $\operatorname{Source} f = \operatorname{Source} i$.

(a) An *S-Ore completion* for f and i is a normal S-2-arrow (f', i') in $\mathcal{C}$ with $fi' = if'$.

$$\begin{tikzcd} \cdot \arrow[r, "f'"] & \cdot \\ \cdot \arrow[u, "i"] \arrow[r, "f"'] & \cdot \arrow[u, "i'"'] \end{tikzcd}$$

(b) An S-Ore completion (f', i') for f and i is said to be *weakly universal* if for every S-2-arrow (g, a) in $\mathcal{C}$ with $fa = ig$ there exists a morphism c in $\mathcal{C}$ with $a = i'c$ and $g = f'c$.

$$\begin{tikzcd} & & \cdot \\ \cdot \arrow[r, "f'"] \arrow[urr, bend left, "g"] & \cdot \arrow[ur, dotted, "c"] & \\ \cdot \arrow[u, "i"] \arrow[r, "f"'] & \cdot \arrow[u, "i'"'] \arrow[uur, bend right, "a"'] & \end{tikzcd}$$

(2.24) Definition (S-Ore completion axiom)**.**

(a) (i) A category with D-S-denominators $\mathcal{C}$ is said to fulfill the *S-Ore completion axiom* if the following holds.

($\mathrm{Ore}_{\mathrm{S}}$) *S-Ore completion axiom.* There exists an S-Ore completion for every morphism f and every S-denominator i in $\mathcal{C}$ with $\operatorname{Source} f = \operatorname{Source} i$.

(ii) A category with D-S-denominators $\mathcal{C}$ is said to fulfill the *weakly universal S-Ore completion axiom* if the following holds.

($\mathrm{Ore}_{\mathrm{S}}^{\mathrm{wu}}$) *Weakly universal S-Ore completion axiom.* There exists a weakly universal S-Ore completion for every morphism f and every S-denominator i in $\mathcal{C}$ with $\operatorname{Source} f = \operatorname{Source} i$.

(b) A category with denominators $\mathcal{C}$ is said to fulfill the *S-Ore completion axiom* resp. the *weakly universal S-Ore completion axiom* if the S-structure $\mathcal{C}_{\mathrm{S}}$ fulfills the S-Ore completion axiom resp. the weakly universal S-Ore completion axiom.

The S-Ore completion axiom yields the following technical lemma, which will be used several times throughout this chapter.

(2.25) Lemma (flipping lemma for S-2-arrows)**.** We suppose given a category with D-S-denominators $\mathcal{C}$ that fulfills the S-Ore completion axiom, and we suppose given a commutative diagram

$$\begin{tikzcd}
\cdot \arrow[r, "f_1"] & \cdot & \cdot \arrow[l, "a_1"', "\approx"] \\
\cdot \arrow[u, "b_1"', "\approx"] \arrow[r, "\tilde{f}_1"] \arrow[d, "g_1"] & \cdot \arrow[u, "j"', circ] \arrow[d, "\tilde{g}_2"] & \cdot \arrow[l, "\tilde{a}_1"', "\approx"] \arrow[u, "b_2"', "\approx"] \arrow[d, "g_2"] \\
\cdot \arrow[r, "f_2"] & \cdot & \cdot \arrow[l, "a_2"', "\approx"]
\end{tikzcd}$$

in $\mathcal{C}$ with S-2-arrows (f_1, a_1), (f_2, a_2), $(\tilde{f}_1, \tilde{a}_1)$, denominators b_1, b_2 and S-denominator j. For $k \in \{1, 2\}$, we suppose that $g_k = 1$ or $b_k = 1$, and we set

$$(g'_k, b'_k) := \begin{cases} (g_k, 1) & \text{if } b_k = 1, \\ (1, b_k) & \text{if } g_k = 1. \end{cases}$$

Then there exist morphisms $\tilde{f}_2$, $\tilde{a}_2$ and a normal S-2-arrow $(\tilde{g}'_2, j')$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "f_1"] \arrow[d, "g'_1"] & \cdot \arrow[d, dotted, "\tilde{g}'_2"] & \cdot \arrow[l, "a_1"', "\approx"] \arrow[d, "g'_2"] \\
\cdot \arrow[r, dotted, "\tilde{f}_2"] & \cdot & \cdot \arrow[l, dotted, "\tilde{a}_2"'] \\
\cdot \arrow[u, "b'_1"', "\approx"] \arrow[r, "f_2"] & \cdot \arrow[u, dotted, "j'"', circ] & \cdot \arrow[l, "a_2"', "\approx"] \arrow[u, "b'_2"', "\approx"]
\end{tikzcd}$$

commutes.

Proof. By the S-Ore completion axiom, there exists an S-Ore completion $(\tilde{g}'_2, j')$ for $\tilde{g}_2$ and j.

$$\begin{tikzcd}
\cdot \arrow[r, "j", circ] \arrow[d, "\tilde{g}_2"'] & \cdot \arrow[d, "\tilde{g}'_2"] \\
\cdot \arrow[r, "j'", circ] & \cdot
\end{tikzcd}$$

Moreover, as

$$\begin{tikzcd}
\cdot \arrow[r, "b_1", "\approx"'] \arrow[d, "g_1"'] & \cdot \arrow[d, "g'_1"] \\
\cdot \arrow[r, "b'_1", "\approx"'] & \cdot
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\cdot \arrow[r, "b_1", "\approx"'] \arrow[d, "g_1"'] & \cdot \arrow[d, "g'_2"] \\
\cdot \arrow[r, "b'_2", "\approx"'] & \cdot
\end{tikzcd}$$

are pushout rectangles in $\mathcal{C}$ by definition of (g'_1, b'_1) and (g'_2, b'_2), we get induced morphisms $\tilde{f}_2$ and $\tilde{a}_2$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
& \cdot \arrow[rr, "f_1"] \arrow[dd, "g'_1"] & & \cdot \arrow[dd, dotted, "\tilde{g}'_2"] & & \cdot \arrow[ll, "a_1"', "\approx"] \arrow[dd, "g'_2"] \\
\cdot \arrow[ur, "b_1", "\approx"'] \arrow[rr, "\tilde{f}_1"] \arrow[dd, "g_1"] & & \cdot \arrow[ur, "j", circ] \arrow[dd, "\tilde{g}_2"] & & \cdot \arrow[ll, "\tilde{a}_1"', "\approx"] \arrow[ur, "b_2", "\approx"'] \arrow[dd, "g_2"] & \\
& \cdot \arrow[rr, dotted, "\tilde{f}_2"] & & \cdot & & \cdot \arrow[ll, dotted, "\tilde{a}_2"'] \\
\cdot \arrow[ur, "b'_1", "\approx"'] \arrow[rr, "f_2"] & & \cdot \arrow[ur, "j'", circ] & & \cdot \arrow[ll, "a_2"', "\approx"] \arrow[ur, "b'_2", "\approx"'] &
\end{tikzcd}$$

□

If the category with D-S-denominators $\mathcal{C}$ in the flipping lemma (2.25) is S-semisaturated, then the morphism $\tilde{a}_2$ in loc. cit. is automatically a denominator, so we have an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$.

S-fractionable categories

Next, we will introduce S-fractionable categories: categories with denominators that fulfill the S-Ore completion axiom and the so-called S-Ore expansibility axiom.

(2.26) Definition (S-Ore expansibility axiom)**.**

(a) A category with D-S-denominators $\mathcal{C}$ is said to fulfill the *S-Ore expansibility axiom* if the following holds.

($\mathrm{Ore}_{\mathrm{S,e}}$) *S-Ore expansibility axiom.* We suppose given parallel morphisms f_1, f_2 in $\mathcal{C}$. If there exists an S-denominator i in $\mathcal{C}$ with $if_1 = if_2$, then there exists an S-denominator i' in $\mathcal{C}$ such that $f_1 i' = f_2 i'$.

$$\begin{tikzcd} {} \arrow[r, "i", "\circ" marking] & {} \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & {} \arrow[r, dotted, "i'", "\circ" marking] & {} \end{tikzcd}$$

(b) A category with denominators $\mathcal{C}$ is said to fulfill the *S-Ore expansibility axiom* if the S-structure $\mathcal{C}_{\mathrm{S}}$ fulfills the S-Ore expansibility axiom.

(2.27) Definition (S-fractionable category)**.**

(a) An *S-fractionable category* is a multiplicative category with denominators $\mathcal{C}$ that fulfills the S-Ore completion axiom and the S-Ore expansibility axiom.

(b) We suppose given S-fractionable categories $\mathcal{C}$ and $\mathcal{D}$. A *morphism of S-fractionable categories* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of categories with denominators from $\mathcal{C}$ to $\mathcal{D}$.

(c) We suppose given a Grothendieck universe $\mathfrak{U}$. The full subcategory $\mathbf{SFrCat} = \mathbf{SFrCat}_{(\mathfrak{U})}$ of $\mathbf{CatD}_{(\mathfrak{U})}$ with

$$\operatorname{Ob}\mathbf{SFrCat}_{(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob}\mathbf{CatD}_{(\mathfrak{U})} \mid \mathcal{C} \text{ is an S-fractionable category}\}$$

is called the *category of S-fractionable categories* (more precisely, the *category of S-fractionable $\mathfrak{U}$-categories*). An object in $\mathbf{SFrCat}_{(\mathfrak{U})}$ is called an *S-fractionable $\mathfrak{U}$-category*, and a morphism in $\mathbf{SFrCat}_{(\mathfrak{U})}$ is called a *$\mathfrak{U}$-morphism of S-fractionable categories.*

If the S-Ore completions that an S-fractionable category admits may be chosen weakly universally and the S-fractionable category is S-semisaturated, see definition (1.37)(b), then the S-Ore expansibility axiom turns out to be redundant:

(2.28) Proposition. We suppose given an S-semisaturated category with denominators $\mathcal{C}$. If $\mathcal{C}$ fulfills the weakly universal S-Ore completion axiom, then $\mathcal{C}$ is an S-fractionable category.

Proof. We suppose that $\mathcal{C}$ fulfills the weakly universal S-Ore completion axiom. To show that $\mathcal{C}$ is an S-fractionable category, it suffices to show that it fulfills the S-Ore expansibility axiom. To this end, we suppose given parallel morphisms f_1, f_2 and a denominator d in $\mathcal{C}$ with $df_1 = df_2$. We choose a weakly universal S-Ore completion $(f', \tilde{d}')$ for $f := df_1 = df_2$ and d, so that there exist induced morphisms d_1, d_2 in $\mathcal{C}$ with $f_1 = f'd_1$, $1 = \tilde{d}'d_1$, $f_2 = f'd_2$, $1 = \tilde{d}'d_2$.

$$\begin{tikzcd}
& & & {} \\
{} \arrow[rr, dotted, "f'"] \arrow[urrr, bend left, "f_2"] \arrow[dr, bend right, "f_1"'] & & {} \arrow[ur, dotted, "d_2"] \arrow[dl, dotted, "d_1"'] & \\
& {} & & \\
{} \arrow[uu, "d", "\wr"'] \arrow[rr, "f"'] & & {} \arrow[uu, dotted, "\tilde{d}'"', "\wr"] \arrow[ul, equal] \arrow[uuur, equal] &
\end{tikzcd}$$

By S-semisaturatedness, d_1 is a denominator in $\mathcal{C}$. We choose an S-Ore completion (d_2', d_1') for d_2 and d_1.

$$\begin{tikzcd}
{} \arrow[r, dotted, "d_2'"] & {} \\
{} \arrow[u, "d_1", "\wr"'] \arrow[r, "d_2"'] & {} \arrow[u, dotted, "\wr", "d_1'"']
\end{tikzcd}$$

Then we get

$$
\begin{aligned}
f_1 d_2' &= f' d_1 d_2' = f' d_2 d_1' = f_2 d_1', \\
d_2' &= \tilde{d}' d_1 d_2' = \tilde{d}' d_2 d_1' = d_1',
\end{aligned}
$$

and so we have $f_1 d' = f_2 d'$ for $d' := d_1' = d_2'$. □

The classical S-Ore localisation

We briefly recall the classical S-Ore localisation. Cf. theorem (2.85).

(2.29) Theorem. We suppose given an S-fractionable category $\mathcal{C}$.

(a) There is a category structure on $(\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$, where the composition and the identities are given as follows. Given $(f_1, a_1), (f_2, a_2) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$ with $\operatorname{Target}(f_1, a_1) = \operatorname{Source}(f_2, a_2)$, we choose a morphism f_2' and a denominator a_1' with $a_1 f_2' = f_2 a_1'$. Then $(f_1/a_1)(f_2/a_2) = f_1 f_2'/a_2 a_1'$.

$$
\begin{tikzcd}
\cdot \arrow[rr, "f_1"] & & \cdot \arrow[rr, "f_2'"] & & \cdot & & \\
& & & \cdot \arrow[ul, "a_1"', "\approx"] \arrow[rr, "f_2"] & & \cdot \arrow[ul, "a_1'"', "\approx"] & \\
& & & & & & \cdot \arrow[ul, "a_2"', "\approx"]
\end{tikzcd}
$$

The identity of $X \in \operatorname{Ob}(\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ is given by $1_X = 1_X/1_X$.

(b) The quotient graph $(\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ together with the category structure from (a) becomes a localisation of $\mathcal{C}$, where the localisation functor $\operatorname{loc}\colon \mathcal{C} \to (\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ is given on the objects by $\operatorname{loc}(X) = X$ for $X \in \operatorname{Ob}\mathcal{C}$ and on the morphisms by $\operatorname{loc}(f) = f/1$ for $f \in \operatorname{Mor}\mathcal{C}$.

For every denominator d in $\mathcal{C}$, the inverse of $\operatorname{loc}(d)$ is given by $\operatorname{loc}(d)^{-1} = 1/d$.

Proof. Cf. [13, sec. III.2, lem. 8]. □

(2.30) Definition (S-Ore localisation)**.** We suppose given an S-fractionable category $\mathcal{C}$. The *S-Ore localisation* of $\mathcal{C}$ is defined to be the localisation $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ of $\mathcal{C}$, whose underlying category is the quotient graph $(\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ together with composition and identities as in theorem (2.29)(a), and whose localisation functor is given as in theorem (2.29)(b).

The S-2-arrow calculus

Next, we recall the S-2-arrow calculus of an S-fractionable category. Cf. theorem (2.93).

(2.31) Definition (S-2-arrow conditions)**.** We suppose given a multiplicative category with denominators $\mathcal{C}$, a category $\mathcal{L}$ and a functor $L\colon \mathcal{C} \to \mathcal{L}$ such that Ld is invertible in $\mathcal{L}$ for every denominator d in $\mathcal{C}$.

(a) We say that $(\mathcal{L}, L)$ fulfills the *S-2-arrow representative condition* if the following holds.

($2\mathrm{ac}_{\mathrm{S,r}}$) *S-2-arrow representative condition.* We have

$$\operatorname{Mor}\mathcal{L} = \{(Lf)(La)^{-1} \mid (f, a) \text{ is an S-2-arrow in } \mathcal{C}\}.$$

(b) We say that $(\mathcal{L}, L)$ fulfills the *S-2-arrow equality condition* if the following holds.

($2\mathrm{ac}_{\mathrm{S,e}}$) *S-2-arrow equality condition.* Given S-2-arrows (f, a), (f', a') in $\mathcal{C}$ with

$$(Lf)(La)^{-1} = (Lf')(La')^{-1}$$

in $\mathcal{L}$, there exist S-2-arrows $(\tilde{f}', \tilde{a}')$, (c, d) in $\mathcal{C}$ such that the following diagram commutes.

$$
\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, equal] & \cdot \arrow[d, dotted, "c"] & \cdot \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[r, dotted, "\tilde{f}'"] & \cdot & \cdot \arrow[l, dotted, "\tilde{a}'"', "\approx"] \\
\cdot \arrow[r, "f'"] \arrow[u, equal] & \cdot \arrow[u, dotted, "d", "\approx"'] & \cdot \arrow[l, "a'"', "\approx"] \arrow[u, equal]
\end{tikzcd}
$$

(c) We say that $(\mathcal{L}, L)$ fulfills the *S-2-arrow composition condition* if the following holds.

($2\mathrm{acs}_{\mathrm{S,c}}$) *S-2-arrow composition condition.* Given S-2-arrows (f_1, a_1), (f_2, a_2), (g_1, b_1), (g_2, b_2) in $\mathcal{C}$ with

$$(Lf_1)(La_1)^{-1}(Lg_2)(Lb_2)^{-1} = (Lg_1)(Lb_1)^{-1}(Lf_2)(La_2)^{-1}$$

in $\mathcal{L}$, there exist an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$ and morphisms $\tilde{g}_2$, $\tilde{b}_2$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, dotted, "\tilde{g}_2"] & {} \arrow[l, "a_1", "\approx"'] \arrow[d, "g_2"] \\
{} \arrow[r, dotted, "\tilde{f}_2"] & {} & {} \arrow[l, dotted, "\tilde{a}_2", "\approx"'] \\
{} \arrow[u, "b_1", "\approx"'] \arrow[r, "f_2"] & {} \arrow[u, dotted, "\tilde{b}_2"] & {} \arrow[l, "a_2", "\approx"'] \arrow[u, "b_2", "\approx"']
\end{tikzcd}$$

If the category with denominators $\mathcal{C}$ in definition (2.31) is S-semisaturated, then the morphism $\tilde{b}_2$ in part (c) of loc. cit. is automatically a denominator, so we have an S-2-arrow $(\tilde{g}_2, \tilde{b}_2)$.

(2.32) Remark. We suppose given a multiplicative category with denominators $\mathcal{C}$, a category $\mathcal{L}$ and a functor $L \colon \mathcal{C} \to \mathcal{L}$ such that Ld is invertible in $\mathcal{L}$ for every denominator d in $\mathcal{C}$.

(a) If $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition, then L is surjective on the objects.

(b) If $(\mathcal{L}, L)$ fulfills the S-2-arrow equality condition, then L is injective on the objects.

Proof.

(a) We suppose that $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition. To show that $\mathcal{L}$ is surjective on the objects, we suppose given an object $\hat{X}$ in $\mathcal{L}$. By the S-2-arrow representative condition, there exists an S-2-arrow $(f, a) \colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $1_{\hat{X}} = (Lf)(La)^{-1}$. We get

$$\hat{X} = \operatorname{Source} 1_{\hat{X}} = \operatorname{Source}((Lf)(La)^{-1}) = \operatorname{Source} Lf = L(\operatorname{Source} f) = LX.$$

Thus L is surjective on the objects.

(b) We suppose that $(\mathcal{L}, L)$ fulfills the S-2-arrow equality condition. To show that $\mathcal{L}$ is injective on the objects, we suppose given objects X, Y in $\mathcal{C}$ such that $LX = LY$ in $\mathcal{L}$. Then we have

$$L1_X = 1_{LX} = 1_{LY} = L1_Y,$$

and so by the S-2-arrow equality condition we in particular have

$$X = \operatorname{Source} 1_X = \operatorname{Source} 1_Y = Y.$$

Thus L is injective on the objects. $\square$

(2.33) Proposition. We suppose given a multiplicative category with denominators $\mathcal{C}$, a category $\mathcal{L}$ and a functor $L \colon \mathcal{C} \to \mathcal{L}$ such that Ld is invertible in $\mathcal{L}$ for every denominator d in $\mathcal{C}$.

(a) If $(\mathcal{L}, L)$ fulfills the S-2-arrow equality condition, then $\mathcal{C}$ fulfills the S-Ore expansibility axiom.

(b) If $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition and the S-2-arrow equality condition, then $\mathcal{C}$ fulfills the S-Ore completion axiom.

Proof.

(a) We suppose that $(\mathcal{L}, L)$ fulfills the S-2-arrow equality condition. To show that $\mathcal{C}$ fulfills the S-Ore expansibility axiom, we suppose given parallel morphisms f_1, f_2 and a denominator d in $\mathcal{C}$ with $df_1 = df_2$. Then we have

$$(Ld)(Lf_1) = L(df_1) = L(df_2) = (Ld)(Lf_2)$$

and hence $Lf_1 = Lf_2$ since Ld is invertible in $\mathcal{L}$. As $(\mathcal{L}, L)$ fulfills the S-2-arrow equality condition, there exists a denominator d' in $\mathcal{C}$ such that $f_1 d' = f_2 d'$. Thus $\mathcal{C}$ fulfills the S-Ore expansibility axiom.

(b) We suppose that $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition and the S-2-arrow equality condition. To show that $\mathcal{C}$ fulfills the S-Ore completion axiom, we suppose given a morphism f and a denominator d in $\mathcal{C}$ with $\operatorname{Source} f = \operatorname{Source} d$. As $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition, there exists an S-2-arrow (g, a) in $\mathcal{C}$ with $(Ld)^{-1}(Lf) = (Lg)(La)^{-1}$. We get

$$L(fa) = (Lf)(La) = (Ld)(Lg) = L(dg),$$

and so as $(\mathcal{L}, L)$ fulfills the S-2-arrow equality condition, there exists a denominator e in $\mathcal{C}$ such that $fae = dge$.

$$\begin{tikzcd}
 & & {} \\
{} \arrow[r, dotted, "g"] & {} \arrow[ur, dotted, "e"] & \\
{} \arrow[u, "d \wr"] \arrow[r, "f"] & {} \arrow[u, dotted, "\wr a"'] &
\end{tikzcd}$$

We set $f' := ge$ and $d' := ae$, so that $fd' = df'$. Moreover, d' is a denominator in $\mathcal{C}$ by multiplicativity.

$$\begin{tikzcd}
 & & {} \\
{} \arrow[r, "g"] \arrow[urr, bend left, "f'"] & {} \arrow[ur, "e"] & \\
{} \arrow[u, "d \wr"] \arrow[r, "f"] & {} \arrow[u, "\wr a"'] \arrow[uur, bend right, "d'"'] &
\end{tikzcd}$$

Thus $\mathcal{C}$ fulfills the S-Ore completion axiom. □

(2.34) Proposition. We suppose given a multiplicative category with denominators $\mathcal{C}$, a category $\mathcal{L}$ and a functor $\mathcal{L}\colon \mathcal{C} \to \mathcal{L}$ such that Ld is invertible in $\mathcal{L}$ for every denominator d in $\mathcal{C}$.

(a) If $(\mathcal{L}, L)$ fulfills the S-2-arrow composite condition, then it also fulfills the S-2-arrow equality condition.

(b) If $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition and the S-2-arrow equality condition, then it also fulfills the S-2-arrow composition condition.

Proof.

(b) We suppose that $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition and the S-2-arrow equality condition. To show that $(\mathcal{L}, L)$ fulfills the S-2-arrow composition condition, we suppose given S-2-arrows (f_1, a_1), (f_2, a_2), (g_1, b_1), (g_2, b_2) in $\mathcal{C}$ with

$$(Lf_1)(La_1)^{-1}(Lg_2)(Lb_2)^{-1} = (Lg_1)(Lb_1)^{-1}(Lf_2)(La_2)^{-1}$$

in $\mathcal{L}$. By proposition (2.33)(b), we know that $\mathcal{C}$ fulfills the S-Ore completion axiom. In particular, there exist an S-Ore completion (a_2', b_2') for a_2 and b_2, an S-Ore completion (f_2', b_1') for $f_2 b_2'$ and b_1, and an S-Ore completion (g_2', a_1') for $g_2 a_2' b_1'$ and a_1.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, dotted, "g_2'"] & {} \arrow[l, "a_1", "\approx"'] \arrow[d, "g_2"] \\
{} \arrow[dr, dotted, "f_2'"] & {} & {} \arrow[ddl, dotted, "a_2'"] \\
 & {} \arrow[u, dotted, "\wr a_1'"'] & \\
 & {} \arrow[u, dotted, "b_1' \wr"] & \\
{} \arrow[uuuu, "\wr b_1"'] \arrow[r, "f_2"] & {} \arrow[u, dotted, "b_2'"] & {} \arrow[l, "a_2", "\approx"'] \arrow[uuuu, "\wr b_2"']
\end{tikzcd}$$

We obtain

$$\begin{aligned}
L(f_1 g_2') &= (Lf_1)(Lg_2') = (Lf_1)(La_1)^{-1}(Lg_2)(La_2')(Lb_1')(La_1') \\
&= (Lf_1)(La_1)^{-1}(Lg_2)(Lb_2)^{-1}(La_2)(Lb_2')(Lb_1')(La_1') \\
&= (Lg_1)(Lf_2')(La_1') = L(g_1 f_2' a_1').
\end{aligned}$$

So as $(\mathcal{L}, L)$ fulfills the S-2-arrow expansibility axiom, there exists a denominator d in $\mathcal{C}$ with $f_1 g_2' d = g_1 f_2' a_1' d$.

$$\begin{tikzcd}
{} \arrow[rr, "f_1"] \arrow[dd, "g_1"] & & {} \arrow[dd, "g_2'"] & & {} \arrow[ll, "a_1"', "\approx" description] \arrow[dd, "g_2"] \\
& & & & \\
{} \arrow[rr, "f_2' a_1'"] & & {} & & {} \arrow[ll, "a_2' b_1' a_1'"', "\approx" description] \\
& {} \arrow[ur, dotted, "d"] & & & \\
{} \arrow[rr, "f_2"] \arrow[uu, "b_1", "\approx"'] & & {} \arrow[uu, "b_2' b_1' a_1'"] & & {} \arrow[ll, "a_2"', "\approx" description] \arrow[uu, "b_2", "\approx"']
\end{tikzcd}$$

Setting $\tilde{f}_2 := f_2' a_1' d$, $\tilde{a}_2 := a_2' b_1' a_1' d$, $\tilde{g}_2 := g_2' d$, $\tilde{b}_2 := b_2' b_1' a_1' d$ yields

$$\begin{aligned}
&f_1 \tilde{g}_2 = f_1 g_2' d = g_1 f_2' a_1' d = g_1 \tilde{f}_2, \\
&a_1 \tilde{g}_2 = a_1 g_2' d = g_2 a_2' b_1' a_1' d = g_2 \tilde{a}_2, \\
&f_2 \tilde{b}_2 = f_2 b_2' b_1' a_1' d = b_1 f_2' a_1' d = b_1 \tilde{f}_2, \\
&a_2 \tilde{b}_2 = a_2 b_2' b_1' a_1' d = b_2 a_2' b_1' a_1' d = b_2 \tilde{a}_2.
\end{aligned}$$

Moreover, $\tilde{a}_2$ is a denominator in $\mathcal{C}$ by multiplicativity.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, "\tilde{g}_2"] & {} \arrow[l, "a_1"', "\approx" description] \arrow[d, "g_2"] \\
{} \arrow[r, "\tilde{f}_2"] & {} & {} \arrow[l, "\tilde{a}_2"', "\approx" description] \\
{} \arrow[r, "f_2"] \arrow[u, "b_1", "\approx"'] & {} \arrow[u, "\tilde{b}_2"] & {} \arrow[l, "a_2"', "\approx" description] \arrow[u, "b_2", "\approx"']
\end{tikzcd}$$

Thus $(\mathcal{L}, L)$ fulfills the S-2-arrow composition condition. □

(2.35) Theorem (S-2-arrow calculus)**.** Given an S-fractionable category $\mathcal{C}$, then $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ fulfills the S-2-arrow representative condition and the S-2-arrow equality condition.

Proof. Cf. [13, sec. III.2, lem. 8]. □

(2.36) Proposition. We suppose given a multiplicative category with denominators $\mathcal{C}$, a category $\mathcal{L}$ and a functor $L\colon \mathcal{C} \to \mathcal{L}$ such that Ld is invertible in $\mathcal{L}$ for every denominator d in $\mathcal{C}$. If $(\mathcal{L}, L)$ fulfills the S-2-arrow representative condition and the S-2-arrow equality condition, then $\mathcal{L}$ becomes a localisation of $\mathcal{C}$ with localisation functor $\mathrm{loc}^{\mathcal{L}} = L$.

Proof. By proposition (2.33), we know that $\mathcal{C}$ fulfills the S-Ore expansibility axiom and the S-Ore completion axiom, that is, $\mathcal{C}$ is an S-fractionable category. In particular, the S-Ore localisation $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ of $\mathcal{C}$ is defined. By the universal property of $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, there exists a unique functor $\hat{L}\colon \mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) \to \mathcal{L}$ with $L = \hat{L} \circ \mathrm{loc}^{\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "L"] \arrow[d, "\mathrm{loc}^{\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})}"'] & \mathcal{L} \\
\mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) \arrow[ur, dotted, "\hat{L}"'] &
\end{tikzcd}$$

The S-Ore localisation $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ fulfills the S-2-arrow representative condition and the S-2-arrow equality conditionby theorem (2.35), so in particular, $\hat{L}$ is given by

$$\hat{L}X = LX$$

for every object X in $\mathcal{C}$ and by

$$\hat{L}(\mathrm{loc}^{\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})}(f)\,\mathrm{loc}^{\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})}(a)^{-1}) = (Lf)(La)^{-1}$$

for every S-2-arrow (f, a) in $\mathcal{C}$. We want to show that $\hat{L}$ is an isofunctor. Indeed, $\mathrm{Mor}\,\hat{L}$ is surjective as $(\mathcal{L}, L)$ and $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ fulfill the S-2-arrow representative condition, and $\mathrm{Mor}\,\hat{L}$ is injective as $(\mathcal{L}, L)$ and $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ fulfill the S-2-arrow equality condition. Altogether, $\mathrm{Mor}\,\hat{L}$ is a bijection. But this already implies that $\hat{L}$ is an isofunctor. Thus $\mathcal{L}$ becomes a localisation of $\mathcal{C}$ with $\mathrm{loc}^{\mathcal{L}} = L$. □

The next theorem states that the axiomatics of an S-fractionable category is, in some precise sense, the best to obtain an S-2-arrow calculus in the sense of theorem (2.35).

(2.37) Theorem. We suppose given a multiplicative category with denominators $\mathcal{C}$. The following conditions are equivalent.

(a) The category with denominators $\mathcal{C}$ is an S-fractionable category.

(b) There exists a localisation of $\mathcal{C}$ that fulfills the S-2-arrow representative condition and the S-2-arrow equality condition.

(c) There exists a localisation of $\mathcal{C}$ that fulfills the S-2-arrow composition condition.

Proof. If condition (a) holds, that is, if $\mathcal{C}$ is an S-fractionable category, then by theorem (2.35), the S-Ore localisation $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ fulfills the S-2-arrow representative condition and the S-2-arrow equality condition, and so condition (b) holds.
Moreover, if condition (b) holds, that is, if there exists a localisation $\mathcal{L}$ of $\mathcal{C}$ that fulfills the S-2-arrow representative condition and the S-2-arrow equality condition, then this localisation also fulfills the S-2-arrow composite condition by proposition (2.34)(b).
Finally, we suppose that condition (c) holds, that is, we suppose that there exists a localisation $\mathcal{L}$ of $\mathcal{C}$ that fulfills the S-2-arrow composition condition. Then $\mathcal{L}$ fulfills in particular the S-2-arrow equality condition and therefore $\mathcal{C}$ fulfills the S-Ore expansibility axiom by proposition (2.33)(a). To show that $\mathcal{C}$ fulfills the S-Ore completion axiom, we suppose given a morphism f and a denominator d in $\mathcal{C}$ with $\mathrm{Source}\,f = \mathrm{Source}\,d$. Then we have $\mathrm{loc}(d)^{-1}\,\mathrm{loc}(f) = \mathrm{loc}(d)^{-1}\,\mathrm{loc}(f)$ in $\mathcal{L}$, and so the S-2-arrow composition condition in particular yields an S-Ore completion (f', d') for f and d.

$$\begin{tikzcd}
{} \arrow[r, equal] \arrow[d, equal] & {} \arrow[d, dotted, "f'"] & {} \arrow[l, "d"', "\approx"] \arrow[d, "f"] \\
{} \arrow[r, dotted, "f'"] & {} & {} \arrow[l, dotted, "d'"', "\approx"] \arrow[d, equal] \\
{} \arrow[u, "d"', "\approx"] \arrow[r, "f"] & {} \arrow[u, dotted, "d'"', "\approx"] \arrow[r, equal] & {}
\end{tikzcd}$$

Hence $\mathcal{C}$ is an S-fractionable category, that is, condition (a) holds.
Altogether, we have shown that condition (a), condition (b) and condition (c) are equivalent. □

4 Z-2-arrows

As just shown in theorem (2.37), S-fractionable categories, as introduced in definition (2.27)(a), characterise those multiplicative categories with denominators that admit an S-2-arrow calculus in the sense of theorem (2.35). So by contraposition, if a multiplicative category with denominators does not fulfill the axioms of an S-fractionable category, it cannot admit such a pure S-2-arrow calculus, even if we know that every morphism in the localisation is represented by an S-2-arrow, see definition (2.31)(a). So if we still want to work with strictly commutative diagrams as in the S-2-arrow equality condition, see definition (2.31)(b), we have to restrict our attention to a subset of S-2-arrows that fulfills the following two requirements simultaneously. First, it must be small enough such that two S-2-arrows that are contained in the subset represent the same morphism in the localisation if and only if they may be embedded in a 2-by-2 diagram as in definition (2.31)(b). Second, it must still be large enough such that every morphism in the localisation is represented by an S-2-arrow that lies in the subset.

In this section, we are going to introduce the notion of a category with Z-2-arrows, see definition (2.38)(a), that is, a category with denominators and S-denominators equipped with a distinguished subset of normal S-2-arrows, see definition (2.1)(a) and definition (2.10). Such a category with Z-2-arrows is the basic structure for our axiomatic localisation approach, but it does not yet necessarily fulfill enough axioms to construct a generalisation of the S-Ore localisation, cf. definition (2.30). Those axioms will be introduced in section 5.

After the definition of categories with Z-2-arrows, we develop some basic properties that follow from the Z-replacement axiom. Thereafter, we introduce the Z-fraction equality, see definition (2.50), a congruence on the Z-2-arrow graph that is analogously defined to the S-fraction equality on the S-2-arrow graph resp. the normal S-fraction equality on the normal S-2-arrow graph, cf. definition (2.14).

Categories with Z-2-arrows

For the definition of a category with D-S-denominators and of a morphism of categories with D-S-denominators, see definition (2.1).

(2.38) Definition (category with Z-2-arrows)**.**

(a) A *category with Z-2-arrows* consists of a multiplicative category with D-S-denominators $\mathcal{C}$ together with a subgraph $\mathcal{Z}$ of $\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$ such that the following axiom holds.

($\mathrm{Rpl}_{\mathrm{Z}}$) *Z-replacement axiom.* For every S-2-arrow (f, a) in $\mathcal{C}$ there exists an arrow $(\dot{f}, \dot{a})$ in $\mathcal{Z}$ and a morphism s in $\mathcal{C}$ with $(f, a) = (\dot{f}s, \dot{a}s)$.

$$\begin{tikzcd} {} \arrow[r, dotted, "\dot{f}"] \arrow[d, equal] & {} \arrow[d, dotted, "s"] & {} \arrow[l, dotted, "\dot{a}"'] \arrow[d, equal] \\ {} \arrow[r, "f"] & {} & {} \arrow[l, "a"', "\approx"] \end{tikzcd}$$

By abuse of notation, we refer to the said category with Z-2-arrows as well as to its underlying category with D-S-denominators just by $\mathcal{C}$. The subgraph $\mathcal{Z}$ is called the *Z-2-arrow graph* of $\mathcal{C}$, the arrows in $\mathcal{Z}$ are called *Z-2-arrows* in $\mathcal{C}$.

Given a category with Z-2-arrows $\mathcal{C}$ with Z-2-arrow graph $\mathcal{Z}$, we write $\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C} := \mathcal{Z}$.

(b) We suppose given categories with Z-2-arrows $\mathcal{C}$ and $\mathcal{D}$. A *morphism of categories with Z-2-arrows* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of categories with D-S-denominators $F\colon \mathcal{C} \to \mathcal{D}$ that *preserves Z-2-arrows*, that is, such that (Ff, Fi) is a Z-2-arrow in $\mathcal{D}$ for every Z-2-arrow (f, i) in $\mathcal{C}$.

If a category with Z-2-arrows $\mathcal{C}$ is S-semisaturated, then the morphism s in the Z-replacement axiom in definition (2.38)(a) is automatically a denominator.

While S-2-arrows and normal S-2-arrows are defined via a property, see definition (2.10), the Z-2-arrows of a category with Z-2-arrows are a distinguished part of the structure.

(2.39) Example. The S-structure of every multiplicative category with denominators $\mathcal{C}$ carries the structure of a category with Z-2-arrows having

$$\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C}_{\mathrm{S}} = \mathrm{AG}_{\mathrm{S}}\,\mathcal{C}.$$

Proof. The Z-replacement axiom is fulfilled as every identity morphism in $\mathcal{C}$ is a denominator in $\mathcal{C}$ by multiplicativity.

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, equal] & {} \arrow[d, equal] & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\ {} \arrow[r, "f"] & {} & {} \arrow[l, "a"', "\approx"] \end{tikzcd}$$

□

The category of categories with Z-2-arrows

(2.40) Definition (category with Z-2-arrows with respect to a Grothendieck universe)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. A category with Z-2-arrows $\mathcal{C}$ is called a *category with Z-2-arrows with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$-*category with Z-2-arrows*) if its underlying category with D-S-denominators is a $\mathfrak{U}$-category with D-S-denominators.

(2.41) Remark.

(a) We suppose given a Grothendieck universe $\mathfrak{U}$. A category with Z-2-arrows $\mathcal{C}$ is a $\mathfrak{U}$-category with Z-2-arrows if and only if it is an element of $\mathfrak{U}$.

(b) For every category with Z-2-arrows $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a $\mathfrak{U}$-category with Z-2-arrows.

(2.42) Remark. For every Grothendieck universe $\mathfrak{U}$ we have a category $\mathbf{CatZ}_{(\mathfrak{U})}$, given as follows. The set of objects of $\mathbf{CatZ}_{(\mathfrak{U})}$ is given by

$$\operatorname{Ob}\mathbf{CatZ}_{(\mathfrak{U})} = \{\mathcal{C} \mid \mathcal{C} \text{ is a } \mathfrak{U}\text{-category with Z-2-arrows}\}.$$

For objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathbf{CatZ}_{(\mathfrak{U})}$, we have the hom-set

$${}_{\mathbf{CatZ}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D}) = \{F \mid F \text{ is a morphism of categories with Z-2-arrows from } \mathcal{C} \text{ to } \mathcal{D}\}.$$

For morphisms $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ in $\mathbf{CatZ}_{(\mathfrak{U})}$, the composite of F and G in $\mathbf{CatZ}_{(\mathfrak{U})}$ is given by the composite of the underlying morphisms of categories with D-S-denominators $G \circ F\colon \mathcal{C} \to \mathcal{E}$. For an object $\mathcal{C}$ in $\mathbf{CatZ}_{(\mathfrak{U})}$, the identity morphism on $\mathcal{C}$ in $\mathbf{CatZ}_{(\mathfrak{U})}$ is given by the underlying identity morphism of categories with D-S-denominators $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.

(2.43) Definition (category of categories with Z-2-arrows)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. The category $\mathbf{CatZ} = \mathbf{CatZ}_{(\mathfrak{U})}$ as considered in remark (2.42) is called the *category of categories with Z-2-arrows* (more precisely, the *category of* $\mathfrak{U}$*-categories with Z-2-arrows*).

The Z-structure

In example (2.39), we have seen that there can be defined a structure of a category with Z-2-arrows on every multiplicative category with denominators. Since we will need this structure to compare our approach to the classical one introduced in section 3, we assign a name to it.

(2.44) Definition (Z-structure)**.** We suppose given a multiplicative category with denominators $\mathcal{C}$. The category with Z-2-arrows $\mathcal{C}_{\mathrm{Z}}$ whose underlying category with D-S-denominators is $\mathcal{C}_{\mathrm{S}}$ and whose Z-2-arrow graph is given by

$$\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C}_{\mathrm{Z}} = \mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$$

is called the *Z-structure* of $\mathcal{C}$.

(2.45) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. We have a functor

$$-_{\mathrm{Z}}\colon \mathbf{CatD}_{\mathrm{mul},(\mathfrak{U})} \to \mathbf{CatZ}_{(\mathfrak{U})},$$

given on the morphisms by $F_{\mathrm{Z}} = F$ for $F \in \operatorname{Mor}\mathbf{CatD}_{\mathrm{mul},(\mathfrak{U})}$, which is full, faithful and injective on the objects.

Properties of Z-2-arrows

We show some simple properties of Z-2-arrows.

(2.46) Remark. We suppose given a category with Z-2-arrows $\mathcal{C}$. The Z-2-arrow graph $\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C}$ is a wide subgraph of $\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$.

Proof. By the Z-replacement axiom, for every $X \in \operatorname{Ob} \mathrm{AG}_{\mathrm{S,n}} \mathcal{C} = \operatorname{Ob} \mathcal{C}$ there exists a Z-2-arrow (e, i) and a morphism s in $\mathcal{C}$ with $es = is = 1_X$, so that $X = \operatorname{Source}(e, i) = \operatorname{Target}(e, i) \in \operatorname{Ob} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, equal] & Z \arrow[d, "s"] & X \arrow[l, "i"', circle] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

□

(2.47) Remark. We suppose given a category with Z-2-arrows $\mathcal{C}$. For every S-2-arrow (f, a) in $\mathcal{C}$ there exists a Z-2-arrow $(\dot{f}, \dot{a})$ in $\mathcal{C}$ with

$$(f, a) \equiv_{\mathrm{S}} (\dot{f}, \dot{a}).$$

Proof. This follows from the Z-replacement axiom. □

For the formulation of the S-2-arrow representative condition, see definition (2.31)(a).

(2.48) Corollary. We suppose given a category with Z-2-arrows $\mathcal{C}$ and a localisation $\mathcal{L}$ of $\mathcal{C}$. If $\mathcal{L}$ fulfills the S-2-arrow representative condition, then

$$\operatorname{Mor} \mathcal{L} = \{\operatorname{loc}(f) \operatorname{loc}(i)^{-1} \mid (f, i) \text{ is a Z-2-arrow in } \mathcal{C}\}.$$

Proof. This follows from remark (2.47) and remark (2.17). □

(2.49) Corollary. We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the S-Ore completion axiom, and we suppose given S-2-arrows (f_1, a_1) and (f_2, a_2) in $\mathcal{C}$ with $\operatorname{Target}(f_1, a_1) = \operatorname{Target}(f_2, a_2)$. Then there exist normal S-2-arrows $(\tilde{f}_1, i)$ and $(\tilde{f}_2, i)$ in $\mathcal{C}$ with

$$(f_1, a_1) \equiv_{\mathrm{S}} (\tilde{f}_1, i),$$
$$(f_2, a_2) \equiv_{\mathrm{S}} (\tilde{f}_2, i).$$

Proof. By remark (2.47), there exist Z-2-arrows $(\dot{f}_k, \dot{a}_k)$ in $\mathcal{C}$ with $(f_k, a_k) \equiv_{\mathrm{S}} (\dot{f}_k, \dot{a}_k)$ for $k \in \{1, 2\}$. In particular, we have $\operatorname{Source}(\dot{f}_k, \dot{a}_k) = \operatorname{Source}(f_k, a_k)$ and $\operatorname{Target}(\dot{f}_k, \dot{a}_k) = \operatorname{Target}(f_k, a_k)$ for $k \in \{1, 2\}$. Hence $\operatorname{Target}(f_1, a_1) = \operatorname{Target}(f_2, a_2)$ implies that $\operatorname{Target}(\dot{f}_1, \dot{a}_1) = \operatorname{Target}(\dot{f}_2, \dot{a}_2)$. We let $(\dot{a}_2', \dot{a}_1')$ be an S-Ore completion of $\dot{a}_2$ and $\dot{a}_1$ and set $\tilde{f}_1 := \dot{f}_1 \dot{a}_2'$, $\tilde{f}_2 := \dot{f}_2 \dot{a}_1'$ and $i := \dot{a}_1 \dot{a}_2' = \dot{a}_2 \dot{a}_1'$.

$$\begin{tikzcd}
{} \arrow[r, "\dot{f}_1"] \arrow[d, equal] & {} \arrow[d, "\dot{a}_2'"] & {} \arrow[l, "\dot{a}_1"', circle] \arrow[d, equal] \\
{} \arrow[r, "\tilde{f}_1"] \arrow[r, shift right, "\tilde{f}_2"'] & {} & {} \arrow[l, "i"', circle] \\
{} \arrow[r, "\dot{f}_2"] \arrow[u, equal] & {} \arrow[u, "\dot{a}_1'", circle] & {} \arrow[l, "\dot{a}_2"', circle] \arrow[u, equal]
\end{tikzcd}$$

By multiplicativity, $i = \dot{a}_2 \dot{a}_1'$ is an S-denominator in $\mathcal{C}$, and we have

$$(\tilde{f}_1, i) = (\dot{f}_1 \dot{a}_2', \dot{a}_1 \dot{a}_2') \equiv_{\mathrm{S}} (\dot{f}_1, \dot{a}_1) \equiv_{\mathrm{S}} (f_1, a_1),$$
$$(\tilde{f}_2, i) = (\dot{f}_2 \dot{a}_1', \dot{a}_2 \dot{a}_1') \equiv_{\mathrm{S}} (\dot{f}_2, \dot{a}_2) \equiv_{\mathrm{S}} (f_2, a_2).$$

□

Z-fraction equality

In analogy to the S-fraction equality relation $\equiv_{\mathrm{S}}$ on the S-2-arrow graph resp. to the normal S-fraction equality relation $\equiv_{\mathrm{S,n}}$ on the normal S-2-arrow graph, see definition (2.14), we may introduce an equivalence relation on the set of arrows of the Z-2-arrow graph:

(2.50) Definition (Z-fraction equality)**.** We suppose given a category with Z-2-arrows $\mathcal{C}$. The equivalence relation $\equiv_{\mathrm{Z}}$ on $\operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$ is defined to be generated by the following relation on $\operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$: Given $(f, i) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$ and a morphism c in $\mathcal{C}$ such that $(fc, ic) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$, then (f, i) is in relation to (fc, ic).

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, equal] & \cdot \arrow[d, "c"] & \cdot \arrow[l, "i"'] \arrow[d, equal] \\ \cdot \arrow[r, "fc"'] & \cdot & \cdot \arrow[l, "ic"] \end{tikzcd}$$

Given $(f, i), (\tilde{f}, \tilde{i}) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$ with $(f, i) \equiv_{\mathrm{Z}} (\tilde{f}, \tilde{i})$, we say that (f, i) and $(\tilde{f}, \tilde{i})$ are *Z-fraction equal.*

(2.51) Remark. We suppose given a multiplicative category with denominators $\mathcal{C}$. For S-2-arrows (f, a), $(\tilde{f}, \tilde{a})$ in $\mathcal{C}$, we have $(f, a) \equiv_{\mathrm{Z}} (\tilde{f}, \tilde{a})$ in $\operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}_{\mathrm{Z}}$ if and only if $(f, a) \equiv_{\mathrm{S}} (\tilde{f}, \tilde{a})$ in $\operatorname{Arr} \mathrm{AG}_{\mathrm{S}} \mathcal{C}$.

(2.52) Remark. We suppose given a category with Z-2-arrows $\mathcal{C}$ and Z-2-arrows (f, i), $(\tilde{f}, \tilde{i})$ in $\mathcal{C}$. If $(f, i) \equiv_{\mathrm{Z}} (\tilde{f}, \tilde{i})$, then $(f, i) \equiv_{\mathrm{S,n}} (\tilde{f}, \tilde{i})$.

Proof. This holds as every Z-2-arrow is in particular a normal S-2-arrow. □

(2.53) Remark. We suppose given a category with Z-2-arrows $\mathcal{C}$. The Z-fraction equality relation $\equiv_{\mathrm{Z}}$ on $\operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$ defines a graph congruence on $\mathrm{AG}_{\mathrm{Z}} \mathcal{C}$.

Proof. For $(f, i) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$, $c \in \operatorname{Mor} \mathcal{C}$ with $(fc, ic) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$, we have

$$\operatorname{Source}(fc, ic) = \operatorname{Source}(fc) = \operatorname{Source} f = \operatorname{Source}(f, i),$$
$$\operatorname{Target}(fc, ic) = \operatorname{Source}(ic) = \operatorname{Source} i = \operatorname{Target}(f, i).$$

□

(2.54) Definition (Z-fraction)**.** We suppose given a category with Z-2-arrows $\mathcal{C}$. Given a Z-2-arrow (f, i) in $\mathcal{C}$, its equivalence class in the quotient graph $(\mathrm{AG}_{\mathrm{Z}} \mathcal{C})/{\equiv_{\mathrm{Z}}}$ is said to be the *Z-fraction* of (f, i). If no confusion arises, we abuse notation and also write $f/i := [(f, i)]_{\equiv_{\mathrm{Z}}}$ ([5]).

(2.55) Remark. We suppose given a category with Z-2-arrows $\mathcal{C}$. The inclusion $\mathrm{inc}\colon \mathrm{AG}_{\mathrm{Z}} \mathcal{C} \to \mathrm{AG}_{\mathrm{S,n}} \mathcal{C}$ induces a well-defined graph morphism

$$(\mathrm{AG}_{\mathrm{Z}} \mathcal{C})/{\equiv_{\mathrm{Z}}} \to (\mathrm{AG}_{\mathrm{S,n}} \mathcal{C})/{\equiv_{\mathrm{S,n}}},$$

which is identical on the objects and maps the Z-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{Z}}}$ of some $(f, i) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{Z}} \mathcal{C}$ to the normal S-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{S,n}}}$.

Proof. This follows from remark (2.52). □

5 Z-fractionable categories

In this section, we consider several axioms that a category with Z-2-arrows, see definition (2.38)(a), may fulfill, deduce some consequences, and define the concepts of a Z-prefractionable category and of a Z-fractionable category, see definition (2.80)(a) and definition (2.81)(a). All these axioms are fulfilled by a Brown cofibration category, that is, every Brown cofibration category may be seen as a Z-fractionable category, see theorem (3.127). Moreover, we relate the concepts of a Z-(pre)fractionable category to the classical concept of an S-fractionable category, see definition (2.27)(a).

The axioms of a Z-prefractionable category are sufficient to construct the S-Ore localisation of a category with Z-2-arrows in analogy to the S-Ore localisation of an S-fractionable category, see definition (2.30) and definition (2.101). However, the additional axioms of a Z-fractionable category enable us for example to calculate composites and inverses of morphisms in the localisation in a nice way, cf. remark (2.103), and they moreover yield some additional nice properties, such as for example (2.93)(d). While the axioms of a Z-prefractionable category pervade (at least implicitly) the rest of this chapter, some of the additional axioms of a Z-fractionable category will be used precisely once outside this section.

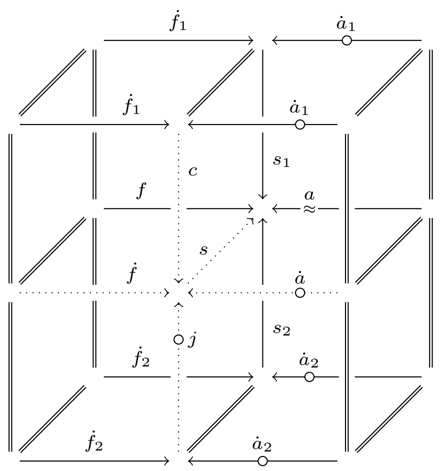
[5] This abuse of notation will be justified in corollary (2.61).

The axioms of a Z-prefractionable category

We begin with the essential axioms of a Z-fractionable category, that is, the axioms of a Z-prefractionable category, see definition (2.80)(a).

(2.56) Definition (Z-comparison axiom)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *Z-comparison axiom* if the following holds.

(Cpr$_{\mathrm{Z}}$) *Z-comparison axiom.* We suppose given an S-2-arrow (f, a), Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{f}_2, \dot{a}_2)$ and morphisms s_1, s_2 in $\mathcal{C}$ such that $(f, a) = (\dot{f}_1 s_1, \dot{a}_1 s_1) = (\dot{f}_2 s_2, \dot{a}_2 s_2)$. Then there exist a Z-2-arrow $(\dot{f}, \dot{a})$, a normal S-2-arrow (c, j) and a morphism s in $\mathcal{C}$ such that the following diagram commutes.



If a category with Z-2-arrows $\mathcal{C}$ is S-semisaturated, then the morphisms s_1, s_2, s in the Z-comparison axiom are automatically denominators.

(2.57) Remark. Given a multiplicative category with denominators $\mathcal{C}$, the Z-structure $\mathcal{C}_{\mathrm{Z}}$ fulfills the Z-comparison axiom.

(2.58) Proposition. We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the Z-comparison axiom. Given Z-2-arrows (f, i), (f', i') in $\mathcal{C}$, the following conditions are equivalent.

(a) We have $(f, i) \equiv_{\mathrm{S}} (f', i')$ in $\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$.

(b) We have $(f, i) \equiv_{\mathrm{S,n}} (f', i')$ in $\mathrm{AG}_{\mathrm{S,n}}\,\mathcal{C}$.

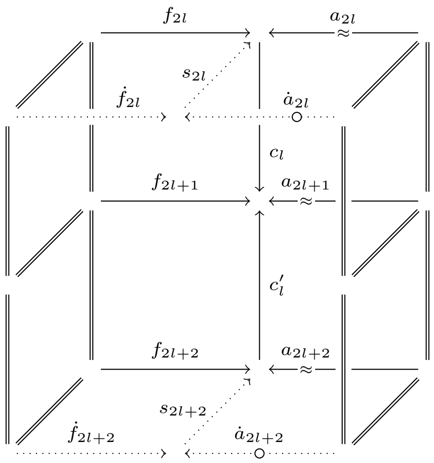
(c) We have $(f, i) \equiv_{\mathrm{Z}} (f', i')$ in $\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C}$.

Proof. If $(f, i) \equiv_{\mathrm{Z}} (f', i')$, then in particular $(f, i) \equiv_{\mathrm{S,n}} (f', i')$, and if $(f, i) \equiv_{\mathrm{S,n}} (f', i')$, then in particular $(f, i) \equiv_{\mathrm{S}} (f', i')$. So condition (c) implies condition (b), and condition (b) implies condition (a).
Let us finally suppose that condition (a) holds, that is, we suppose that $(f, i) \equiv_{\mathrm{S}} (f', i')$ in $\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$. Then there exist $n \in \mathbb{N}_0$, S-2-arrows (f_l, a_l) for $l \in [0, 2n]$ and morphisms c_l, c'_l in $\mathcal{C}$ for $l \in [0, n-1]$ with $(f_0, a_0) = (f, i)$, $(f_{2n}, a_{2n}) = (f', i')$ and $(f_{2l} c_l, a_{2l} c_l) = (f_{2l+1}, a_{2l+1}) = (f_{2l+2} c'_l, a_{2l+2} c'_l)$ for $l \in [0, n-1]$.

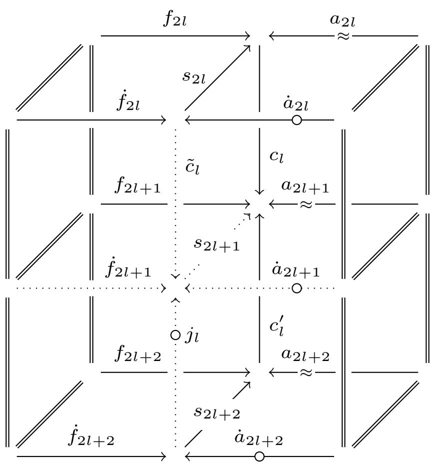
$$\begin{tikzcd}
\cdot \arrow[r, "f_{2l}"] \arrow[d, equal] & \cdot \arrow[d, "c_l"] & \cdot \arrow[l, "a_{2l}"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[r, "f_{2l+1}"] & \cdot & \cdot \arrow[l, "a_{2l+1}"', "\approx"] \\
\cdot \arrow[r, "f_{2l+2}"] \arrow[u, equal] & \cdot \arrow[u, "c'_l"'] & \cdot \arrow[l, "a_{2l+2}"', "\approx"] \arrow[u, equal]
\end{tikzcd}$$

For $l \in [1, n-1]$, we choose Z-2-arrows $(\dot{f}_{2l}, \dot{a}_{2l})$ and morphisms s_{2l} in $\mathcal{C}$ with $(f_{2l}, a_{2l}) = (\dot{f}_{2l} s_{2l}, \dot{a}_{2l} s_{2l})$.

Moreover, we choose $(\dot{f}_0, \dot{a}_0) := (f_0, a_0) = (f, i)$, $s_0 := 1$ and $(\dot{f}_{2n}, \dot{a}_{2n}) := (f_{2n}, a_{2n}) = (f', i')$, $s_{2n} := 1$.



Then for $l \in [0, n-1]$, the Z-comparison axiom yields a Z-2-arrow $(\dot{f}_{2l+1}, \dot{a}_{2l+1})$, a morphism s_{2l+1} and a normal S-2-arrow $(\tilde{c}_l, j_l)$ in $\mathcal{C}$ such that the following diagram commutes.



We have $(f, i) = (\dot{f}_0, \dot{a}_0) \equiv_{\mathrm{Z}} (\dot{f}_{2l}, \dot{a}_{2l}) = (f', i')$, that is, condition (c) holds.
Altogether, the conditions (a), (b) and (c) are equivalent. $\square$

(2.59) Remark (flipping lemma for Z-2-arrows)**.** We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the S-Ore completion axiom and the Z-comparison axiom, and we suppose given a commutative diagram

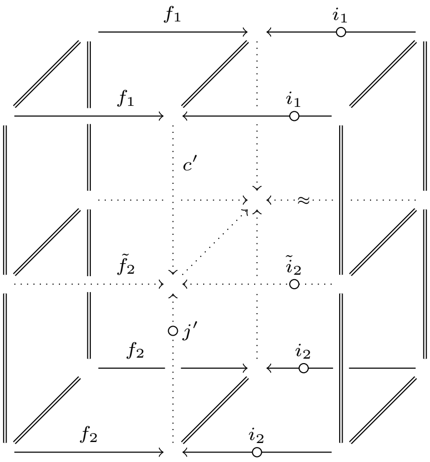
$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "i_1"'] \arrow[d, equal] \\
\bullet \arrow[r, "\tilde{f}_1"] \arrow[d, equal] & \bullet \arrow[u, "j"] \arrow[d, "c"] & \bullet \arrow[l, "\tilde{a}_1"', "\approx"] \arrow[d, equal] \\
\bullet \arrow[r, "f_2"] & \bullet & \bullet \arrow[l, "i_2"']
\end{tikzcd}$$

in $\mathcal{C}$ with Z-2-arrows (f_1, i_1), (f_2, i_2), S-2-arrow $(\tilde{f}_1, \tilde{a}_1)$ and S-denominator j. Then there exist a Z-2-arrow $(\tilde{f}_2, \tilde{i}_2)$ and a normal S-2-arrow (c', j') in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, equal] & \bullet \arrow[d, dotted, "c'"] & \bullet \arrow[l, "i_1"'] \arrow[d, equal] \\
\bullet \arrow[r, dotted, "\tilde{f}_2"] \arrow[d, equal] & \bullet & \bullet \arrow[l, dotted, "\tilde{i}_2"'] \arrow[d, equal] \\
\bullet \arrow[r, "f_2"] & \bullet \arrow[u, dotted, "j'"] & \bullet \arrow[l, "i_2"']
\end{tikzcd}$$

commutes.

Proof. This follows from the flipping lemma for S-2-arrows (2.25) and the Z-comparison axiom.



□

The following theorem gives a more concrete description of the S-fraction equality relation $\equiv_{\mathrm{S}}$ in a category with Z-2-arrows that fulfills the S-Ore completion axiom and the Z-comparison axiom. It is one of the main ingredients for the Z-2-arrow calculus (2.93) and corollary (2.94).

(2.60) Theorem. We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the S-Ore completion axiom and the Z-comparison axiom.

(a) Given S-2-arrows (f, a), (f', a') in $\mathcal{C}$, the following conditions are equivalent.

(i) We have $(f, a) \equiv_{\mathrm{S}} (f', a')$ in $\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$.

(ii) For every Z-2-arrow $(\dot{f}, \dot{a})$ and every morphism s in $\mathcal{C}$ with $(f, a) = (\dot{f}s, \dot{a}s)$ there exist an S-2-arrow $(\tilde{f}', \tilde{a}')$ and a normal S-2-arrow (c, j) in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} & {} \arrow[l, "a", "\approx"'] \arrow[d, equal] \\
{} \arrow[r, "\dot{f}"] \arrow[d, equal] & {} \arrow[u, "s"'] \arrow[d, dotted, "c"'] & {} \arrow[l, "\dot{a}"] \arrow[d, equal] \\
{} \arrow[r, dotted, "\tilde{f}'"] \arrow[d, equal] & {} & {} \arrow[l, dotted, "\tilde{a}'", "\approx"'] \arrow[d, equal] \\
{} \arrow[r, "f'"] & {} \arrow[u, dotted, "j"'] & {} \arrow[l, "a'", "\approx"']
\end{tikzcd}$$

(iii) There exist a Z-2-arrow $(\dot{f}, \dot{a})$, an S-2-arrow $(\tilde{f}', \tilde{a}')$, a normal S-2-arrow (c, j) and a morphism s in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} & {} \arrow[l, "a", "\approx"'] \arrow[d, equal] \\
{} \arrow[r, dotted, "\dot{f}"] \arrow[d, equal] & {} \arrow[u, dotted, "s"'] \arrow[d, dotted, "c"'] & {} \arrow[l, dotted, "\dot{a}"] \arrow[d, equal] \\
{} \arrow[r, dotted, "\tilde{f}'"] \arrow[d, equal] & {} & {} \arrow[l, dotted, "\tilde{a}'", "\approx"'] \arrow[d, equal] \\
{} \arrow[r, "f'"] & {} \arrow[u, dotted, "j"'] & {} \arrow[l, "a'", "\approx"']
\end{tikzcd}$$

(b) Given normal S-2-arrows (f, i), (f', i') in $\mathcal{C}$, the following conditions are equivalent.

  (i) We have $(f, i) \equiv_{\mathrm{S}} (f', i')$ in $\mathrm{AG}_{\mathrm{S}}\, \mathcal{C}$.

  (ii) We have $(f, i) \equiv_{\mathrm{S,n}} (f', i')$ in $\mathrm{AG}_{\mathrm{S,n}}\, \mathcal{C}$.

  (iii) For every Z-2-arrow $(\dot{f}, \dot{i})$ and every morphism s in $\mathcal{C}$ with $(f, i) = (\dot{f}s, \dot{i}s)$ there exist a normal S-2-arrow $(\tilde{f}', \tilde{i}')$ and a normal S-2-arrow (c, j) in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} & {} \arrow[l, "i"'] \arrow[d, equal] \\
{} \arrow[r, "\dot{f}"] \arrow[d, equal] & {} \arrow[u, "s"'] \arrow[d, dotted, "c"] & {} \arrow[l, "\dot{i}"'] \arrow[d, equal] \\
{} \arrow[r, dotted, "\tilde{f}'"] \arrow[d, equal] & {} & {} \arrow[l, dotted, "\tilde{i}'"'] \arrow[d, equal] \\
{} \arrow[r, "f'"] & {} \arrow[u, dotted, "j"'] & {} \arrow[l, "i'"']
\end{tikzcd}$$

  (iv) There exist a Z-2-arrow $(\dot{f}, \dot{i})$, normal S-2-arrows $(\tilde{f}', \tilde{i}')$, (c, j) and a morphism s in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} & {} \arrow[l, "i"'] \arrow[d, equal] \\
{} \arrow[r, dotted, "\dot{f}"] \arrow[d, equal] & {} \arrow[u, dotted, "s"'] \arrow[d, dotted, "c"] & {} \arrow[l, dotted, "\dot{i}"'] \arrow[d, equal] \\
{} \arrow[r, dotted, "\tilde{f}'"] \arrow[d, equal] & {} & {} \arrow[l, dotted, "\tilde{i}'"'] \arrow[d, equal] \\
{} \arrow[r, "f'"] & {} \arrow[u, dotted, "j"'] & {} \arrow[l, "i'"']
\end{tikzcd}$$

(c) Given Z-2-arrows (f, i), (f', i') in $\mathcal{C}$, the following conditions are equivalent.

  (i) We have $(f, i) \equiv_{\mathrm{S}} (f', i')$ in $\mathrm{AG}_{\mathrm{S}}\, \mathcal{C}$.

  (ii) We have $(f, i) \equiv_{\mathrm{S,n}} (f', i')$ in $\mathrm{AG}_{\mathrm{S,n}}\, \mathcal{C}$.

  (iii) We have $(f, i) \equiv_{\mathrm{Z}} (f', i')$ in $\mathrm{AG}_{\mathrm{Z}}\, \mathcal{C}$.

  (iv) There exist a Z-2-arrow $(\tilde{f}', \tilde{i}')$ and a normal S-2-arrow (c, j) in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} \arrow[d, dotted, "c"] & {} \arrow[l, "i"'] \arrow[d, equal] \\
{} \arrow[r, dotted, "\tilde{f}'"] \arrow[d, equal] & {} & {} \arrow[l, dotted, "\tilde{i}'"'] \arrow[d, equal] \\
{} \arrow[r, "f'"] & {} \arrow[u, dotted, "j"'] & {} \arrow[l, "i'"']
\end{tikzcd}$$

  commutes.

Proof.

(c) The equivalence of condition (i), condition (ii) and condition (iii) follows from (2.58). The equivalence of condition (iii) and condition (iv) follows from the flipping lemma for Z-2-arrows (2.59).

(a) First, we suppose that condition (i) holds, that is, we suppose that $(f, a) \equiv_{\mathrm{S}} (f', a')$, and we suppose given a Z-2-arrow $(\dot{f}, \dot{a})$ and a morphism s in $\mathcal{C}$ with $(f, a) = (\dot{f}s, \dot{a}s)$. Moreover, we choose a Z-2-arrow $(\dot{f}', \dot{a}')$ and a morphism s' in $\mathcal{C}$ with $(f', a') = (\dot{f}'s', \dot{a}'s')$. Then we have

$$(\dot{f}, \dot{a}) \equiv_{\mathrm{S}} (f, a) \equiv_{\mathrm{S}} (f', a') \equiv_{\mathrm{S}} (\dot{f}', \dot{a}')$$

By (c), there exist a Z-2-arrow $(\bar{f}', \bar{a}')$ and a normal S-2-arrow $(\tilde{c}, \tilde{j})$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, equal] & {} & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
{} \arrow[r, "\dot{f}"] \arrow[d, equal] & {} \arrow[u, "s"'] \arrow[d, dotted, "\tilde{c}"'] & {} \arrow[l, "\dot{a}"', "\circ"] \arrow[d, equal] \\
{} \arrow[r, dotted, "\bar{f}'"] \arrow[d, equal] & {} & {} \arrow[l, dotted, "\bar{a}'"', "\circ"] \arrow[d, equal] \\
{} \arrow[r, "\dot{f}'"] \arrow[d, equal] & {} \arrow[u, dotted, "\tilde{j}"', "\circ"] \arrow[d, "s'"'] & {} \arrow[l, "\dot{a}'"', "\circ"] \arrow[d, equal] \\
{} \arrow[r, "f'"] & {} & {} \arrow[l, "a'"', "\approx"]
\end{tikzcd}$$

Applying the flipping lemma for S-2-arrows (2.25) to the rectangle

$$\begin{tikzcd}
{} \arrow[r, "\bar{f}'"] \arrow[d, equal] & {} & {} \arrow[l, "\bar{a}'"', "\circ"] \arrow[d, equal] \\
{} \arrow[r, "\dot{f}'"] \arrow[d, equal] & {} \arrow[u, "\tilde{j}"', "\circ"] \arrow[d, "s'"'] & {} \arrow[l, "\dot{a}'"', "\circ"] \arrow[d, equal] \\
{} \arrow[r, "f'"] & {} & {} \arrow[l, "a'"', "\approx"]
\end{tikzcd}$$

and composing yields the asserted diagram of condition (ii).

Condition (ii) and the Z-replacement axiom imply condition (iii).

Finally, if condition (iii) holds, then we have in particular

$$(f, a) \equiv_{\mathrm{S}} (\dot{f}, \dot{a}) \equiv_{\mathrm{S}} (\tilde{f}', \tilde{a}') \equiv_{\mathrm{S}} (f', a'),$$

and so condition (i) holds.

Altogether, the three conditions (i), (ii) and (iii) are equivalent.

(b) By (a), condition (i), condition (iii) and condition (iv) are equivalent.

Moreover, if condition (iv) hold, then we have in particular

$$(f, i) \equiv_{\mathrm{S,n}} (\dot{f}, \dot{i}) \equiv_{\mathrm{S,n}} (\tilde{f}', \tilde{i}') \equiv_{\mathrm{S,n}} (f', i'),$$

and so condition (ii) holds.

Finally, condition (ii) implies condition (i) by remark (2.16).

Altogether, the four conditions (i), (ii), (iii) and (iv) are equivalent. □

(2.61) Corollary. We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the S-Ore completion axiom and the Z-comparison axiom. The inclusions $\mathrm{inc}\colon \mathrm{AG_Z}\,\mathcal{C} \to \mathrm{AG_{S,n}}\,\mathcal{C}$ and $\mathrm{inc}\colon \mathrm{AG_{S,n}}\,\mathcal{C} \to \mathrm{AG_S}\,\mathcal{C}$ induce graph isomorphisms

$$(\mathrm{AG_Z}\,\mathcal{C})/{\equiv_{\mathrm{Z}}} \to (\mathrm{AG_{S,n}}\,\mathcal{C})/{\equiv_{\mathrm{S,n}}} \to (\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}.$$

Proof. The induced graph morphisms are identical on the objects and map the Z-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{Z}}}$ of $(f, i) \in \mathrm{Arr}\,\mathrm{AG_Z}\,\mathcal{C}$ to the normal S-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{S,n}}}$, cf. remark (2.55), resp. the normal S-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{S,n}}}$ of $(f, i) \in \mathrm{Arr}\,\mathrm{AG_{S,n}}\,\mathcal{C}$ to the S-fraction $f/i = [(f, i)]_{\equiv_{\mathrm{S}}}$, cf. remark (2.22). The injectivity of the maps on the sets of arrows follows from theorem (2.60)(c), (b), the surjectivity from remark (2.47). □

(2.62) Definition (relative Z-replacement axiom)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *relative Z-replacement axiom* if the following holds.

($\mathrm{Rpl}_{\mathrm{Z}}^{\mathrm{rel}}$) *Relative Z-replacement axiom.* We suppose given a Z-2-arrow (f_1, i_1), an S-2-arrow (f_2, a_2) and morphisms $g_1, g_2, \tilde{g}_2$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, "g_1"] & \bullet \arrow[d, "\tilde{g}_2"] & \bullet \arrow[l, "i_1"', "\circ" marking] \arrow[d, "g_2"] \\
\bullet \arrow[r, "f_2"] & \bullet & \bullet \arrow[l, "a_2"', "\approx" marking]
\end{tikzcd}$$

commutes. Then there exist a Z-2-arrow $(\dot{f}_2, \dot{a}_2)$ and morphisms s, g in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
& \bullet \arrow[rr, "f_1"] \arrow[dl, equal] \arrow[dd, "g_1"] & & \bullet \arrow[dl, equal] \arrow[dd, "\tilde{g}_2"] & & \bullet \arrow[ll, "i_1"', "\circ" marking] \arrow[dl, equal] \arrow[dd, "g_2"] \\
\bullet \arrow[rr, "f_1"] \arrow[dd, "g_1"] & & \bullet \arrow[dd, dotted, "g"] & & \bullet \arrow[ll, "i_1"', "\circ" marking] \arrow[dd, "g_2"] & \\
& \bullet \arrow[rr, "f_2"] \arrow[dl, equal] & & \bullet & & \bullet \arrow[ll, "a_2"', "\approx" marking] \arrow[dl, equal] \\
\bullet \arrow[rr, dotted, "\dot{f}_2"] & & \bullet \arrow[ur, dotted, "s"] & & \bullet \arrow[ll, dotted, "\dot{a}_2"', "\circ" marking] &
\end{tikzcd}$$

Moreover, we suppose to have the following additional assertions, respectively.

If g_1 and g_2 are denominators, then we suppose that g may be chosen to be a denominator.

If g_1 and g_2 are S-denominators, then we suppose that g may be chosen to be an S-denominator.

If a category with Z-2-arrows $\mathcal{C}$ is S-semisaturated, then the morphism s in the relative Z-replacement axiom is automatically a denominator.

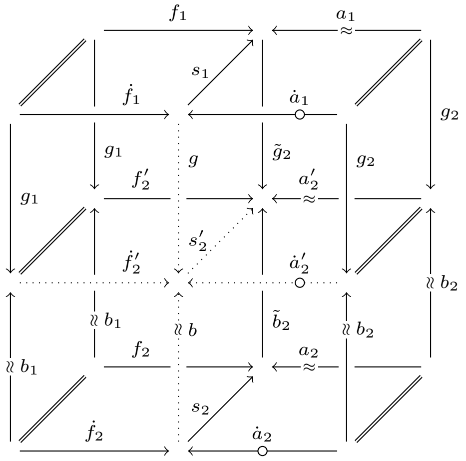
(2.63) Remark. Given an S-semisaturated category with denominators $\mathcal{C}$, the Z-structure $\mathcal{C}_{\mathrm{Z}}$ fulfills the relative Z-replacement axiom.

We deduce a variant of the relative Z-replacement axiom for S-2-arrows:

(2.64) Lemma (Z-replacement lemma)**.** We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the relative Z-replacement axiom and the Z-comparison axiom. Moreover, we suppose given S-2-arrows (f_1, a_1), (f_2, a_2), (f'_2, a'_2), (g_1, b_1), (g_2, b_2) and morphisms $\tilde{g}_2$, $\tilde{b}_2$ in $\mathcal{C}$ such that the diagram

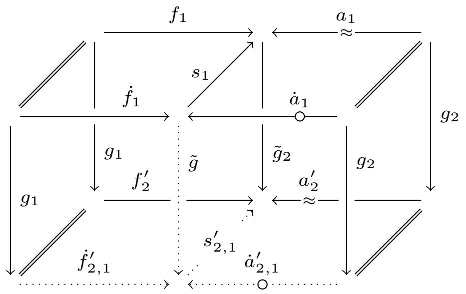
$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, "g_1"] & \bullet \arrow[d, "\tilde{g}_2"] & \bullet \arrow[l, "a_1"', "\approx" marking] \arrow[d, "g_2"] \\
\bullet \arrow[r, "f'_2"] & \bullet & \bullet \arrow[l, "a'_2"', "\approx" marking] \\
\bullet \arrow[r, "f_2"] \arrow[u, "b_1", "\approx"'] & \bullet \arrow[u, "\tilde{b}_2"] & \bullet \arrow[l, "a_2"', "\approx" marking] \arrow[u, "b_2", "\approx"']
\end{tikzcd}$$

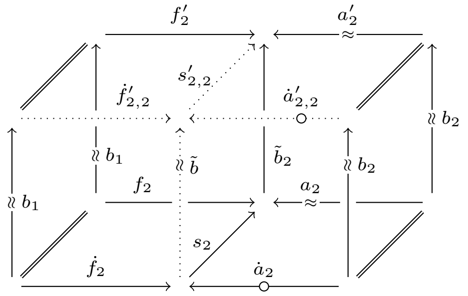
commutes. For all Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{f}_2, \dot{a}_2)$ and all morphisms s_1, s_2 in $\mathcal{C}$ with $(f_1, a_1) = (\dot{f}_1 s_1, \dot{a}_1 s_1)$, $(f_2, a_2) = (\dot{f}_2 s_2, \dot{a}_2 s_2)$ there exist a Z-2-arrow $(\dot{f}'_2, \dot{a}'_2)$, an S-2-arrow (g, b) and a morphism s'_2 in $\mathcal{C}$ such that the following diagram commutes.



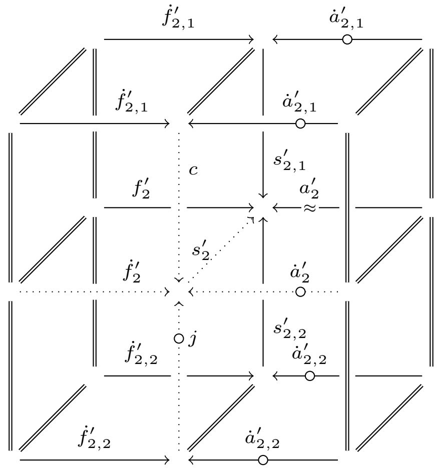
If, in addition, (g_1, b_1) and (g_2, b_2) are normal S-2-arrows, then (g, b) may be chosen to be a normal S-2-arrow.

Proof. We suppose given Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{f}_2, \dot{a}_2)$ and morphisms s_1, s_2 in $\mathcal{C}$ with $(f_1, a_1) = (\dot{f}_1 s_1, \dot{a}_1 s_1)$, $(f_2, a_2) = (\dot{f}_2 s_2, \dot{a}_2 s_2)$. By the relative Z-replacement axiom, there exist Z-2-arrows $(\dot{f}'_{2,1}, \dot{a}'_{2,1})$, $(\dot{f}'_{2,2}, \dot{a}'_{2,2})$, morphisms $s'_{2,1}$, $s'_{2,2}$, $\tilde{g}$ and a denominator $\tilde{b}$ in $\mathcal{C}$ such that the following diagrams commute.



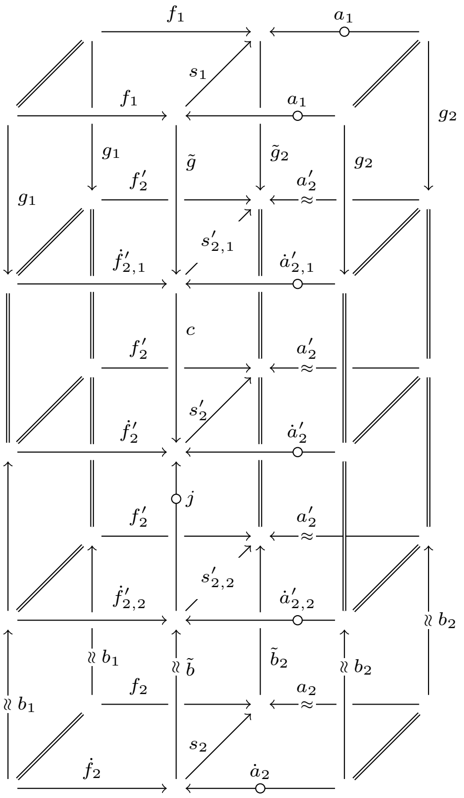


The Z-comparison axiom yields a Z-2-arrow $(\dot{f}'_2, \dot{a}'_2)$, a normal S-2-arrow (c, j) and a morphism s'_2 in $\mathcal{C}$ such that



commutes.

Altogether, we obtain the commutative diagram



in $\mathcal{C}$, so that setting $g := \tilde{g}c$ and $b := \tilde{b}j$ yields the asserted commutative diagram, where b is a denominator by multiplicativity.
Moreover, if b_1, b_2 are S-denominators, then $\tilde{b}$ may be chosen to be an S-denominator, and so b will be an S-denominator by multiplicativity. □

The additional axioms of a Z-fractionable category

Next, we introduce some minor supplemental axioms that turn a Z-prefractionable category into a Z-fractionable category, see definition (2.81)(a).

(2.65) Definition (Z-replacement axiom for denominators)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *Z-replacement axiom for denominators* if the following holds.

($\mathrm{Rpl}_{\mathrm{Z}}^{\mathrm{den}}$) *Z-replacement axiom for denominators.* For every S-2-arrow (d, a) in $\mathcal{C}$ with denominator d there exists a Z-2-arrow $(\dot{d}, \dot{a})$ in $\mathcal{C}$ with denominator $\dot{d}$ and a morphism s in $\mathcal{C}$ with $(d, a) = (\dot{d}s, \dot{a}s)$.

$$\begin{tikzcd}
\cdot \arrow[r, dotted, "\dot{d}", "\approx"'] \arrow[d, equal] & \cdot \arrow[d, dotted, "s"] & \cdot \arrow[l, dotted, "\dot{a}"'] \arrow[d, equal] \\
\cdot \arrow[r, "d", "\approx"'] & \cdot & \cdot \arrow[l, "a"', "\approx"]
\end{tikzcd}$$

(2.66) Remark. Given a multiplicative category with denominators $\mathcal{C}$, the Z-structure $\mathcal{C}_{\mathrm{Z}}$ fulfills the Z-replacement axiom for denominators.

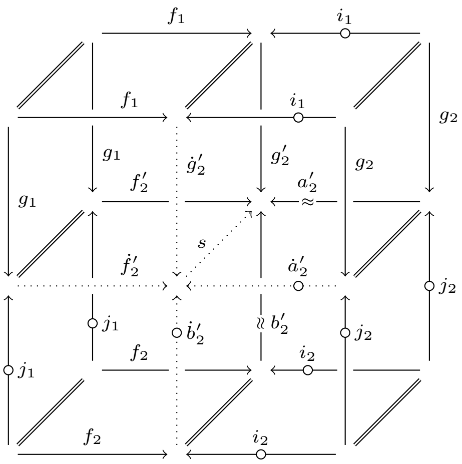
(2.67) Remark. Every T-semisaturated category with Z-2-arrows fulfills the Z-replacement axiom for denominators.

(2.68) Definition (relative Z-replacement axiom for Z-2-arrows)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *relative Z-replacement axiom for Z-2-arrows* if the following holds.

($\mathrm{Rpl}_{\mathrm{Z}}^{\mathrm{rel,Z}}$) *Relative Z-replacement axiom for Z-2-arrows.* We suppose given Z-2-arrows (f_1, i_1), (f_2, i_2), (g_1, j_1), (g_2, j_2) and S-2-arrows (f_2', a_2'), (g_2', b_2') in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"'] & {} \arrow[d, "g_2'"'] & {} \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
{} \arrow[r, "f_2'"] & {} & {} \arrow[l, "a_2'"', "\approx"] \\
{} \arrow[r, "f_2"] \arrow[u, "j_1"] & {} \arrow[u, "b_2'", "\approx"'] & {} \arrow[l, "i_2"'] \arrow[u, "j_2"']
\end{tikzcd}$$

commutes. Then there exist Z-2-arrows $(\dot{f}_2', \dot{a}_2')$, $(\dot{g}_2', \dot{b}_2')$ and a morphism s in $\mathcal{C}$ such that the following diagram commutes.



(2.69) Remark. Given a multiplicative category with denominators $\mathcal{C}$, the Z-structure $\mathcal{C}_{\mathrm{Z}}$ fulfills the relative Z-replacement axiom for Z-2-arrows.

(2.70) Definition (Z-concatenation axiom)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *Z-concatenation axiom* if the following holds.

($\mathrm{Cct}_{\mathrm{Z}}$) *Z-concatenation axiom.* For all Z-2-arrows (f_1, i_1), (f_2, i_2) in $\mathcal{C}$ with $\mathrm{Target}\,(f_1, i_1) = \mathrm{Source}\,(f_2, i_2)$ there exists a weakly universal S-Ore completion (f_2', i_1') for f_2 and i_1 such that $(f_1 f_2', i_2 i_1')$ is a Z-2-arrow in $\mathcal{C}$.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] & {} \arrow[rr, dotted, "f_2'"] & & {} & & \\
& & {} \arrow[ul, "i_1"] \arrow[rr, "f_2"] & & {} \arrow[ul, dotted, "i_1'"] & \\
& & & & & {} \arrow[ul, "i_2"]
\end{tikzcd}$$

(2.71) Remark. We suppose given a multiplicative category with denominators $\mathcal{C}$. If $\mathcal{C}$ fulfills the weakly universal S-Ore completion axiom, then the Z-structure $\mathcal{C}_{\mathrm{Z}}$ fulfills the Z-concatenation axiom.

(2.72) Definition (Z-inversion axiom)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *Z-inversion axiom* if the following holds.

($\mathrm{Inv}_{\mathrm{Z}}$) *Z-inversion axiom.* Given a Z-2-arrow (f, i) in $\mathcal{C}$ such that f is a denominator, then (i, f) is a Z-2-arrow in $\mathcal{C}$.

(2.73) Remark. Given a multiplicative category with denominators $\mathcal{C}$, the Z-structure $\mathcal{C}_{\mathrm{Z}}$ fulfills the Z-inversion axiom.

(2.74) Remark. We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the Z-inversion axiom and a Z-2-arrow (f, i) in $\mathcal{C}$. If f is a denominator in $\mathcal{C}$, then f is an S-denominator in $\mathcal{C}$.

Proof. If f is a denominator in $\mathcal{C}$, then (i, f) is a Z-2-arrow in $\mathcal{C}$ by the Z-inversion axiom. So f is an S-denominator in $\mathcal{C}$ as every Z-2-arrow is a normal S-2-arrow. □

(2.75) Definition (Z-numerator axiom)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *Z-numerator axiom* if the following holds.

(Num$_\mathrm{Z}$) *Z-numerator axiom.* For every Z-2-arrow (f, i) and every denominator d in $\mathcal{C}$ with $\operatorname{Source}(f, i) = \operatorname{Source} d$ there exists an S-2-arrow (f', d') in $\mathcal{C}$ with $fd' = df'$.

$$\begin{tikzcd} {} \arrow[r, dotted, "f'"] & {} & \\ {} \arrow[u, "d", "\wr"'] \arrow[r, "f"'] & {} \arrow[u, dotted, "\wr", "d'"'] & {} \arrow[l, "i"'] \end{tikzcd}$$

(2.76) Remark. We suppose given a multiplicative category with denominators $\mathcal{C}$. If $\mathcal{C}$ fulfills the S-Ore completion axiom, then the Z-structure $\mathcal{C}_\mathrm{Z}$ fulfills the Z-numerator axiom.

(2.77) Definition (Z-expansion axiom)**.** A category with Z-2-arrows $\mathcal{C}$ is said to fulfill the *Z-expansion axiom* if the following holds.

(Exp$_\mathrm{Z}$) *Z-expansion axiom* Given a Z-2-arrow (f, i) and an S-denominator j in $\mathcal{C}$ with $\operatorname{Target} f = \operatorname{Target} i = \operatorname{Source} j$, then (fj, ij) is a Z-2-arrow in $\mathcal{C}$.

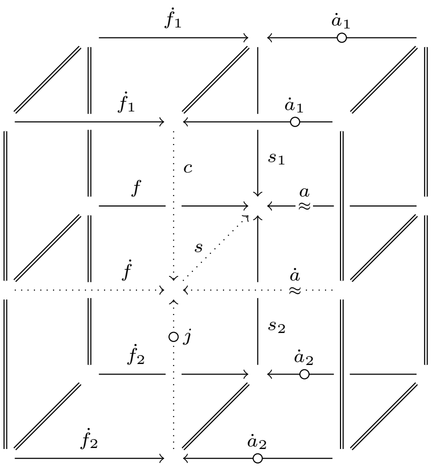
$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, equal] & {} \arrow[d, "j"] & {} \arrow[l, "i"'] \arrow[d, equal] \\ {} \arrow[r, "fj"] & {} & {} \arrow[l, "ij"'] \end{tikzcd}$$

(2.78) Remark. Given a multiplicative category with denominators $\mathcal{C}$, the Z-structure $\mathcal{C}_\mathrm{Z}$ fulfills the Z-expansion axiom.

(2.79) Remark. We suppose given a category with Z-2-arrows $\mathcal{C}$. If $\mathcal{C}$ fulfills the Z-expansion axiom, then the following conditions are equivalent.

(a) The category with Z-2-arrows $\mathcal{C}$ fulfills the Z-comparison axiom.

(b) We suppose given an S-2-arrow (f, a), Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{f}_2, \dot{a}_2)$ and morphisms s_1, s_2 in $\mathcal{C}$ such that $(f, a) = (\dot{f}_1 s_1, \dot{a}_1 s_1) = (\dot{f}_2 s_2, \dot{a}_2 s_2)$. Then there exist an S-2-arrow $(\dot{f}, \dot{a})$, a normal S-2-arrow (c, j) and a morphism s in $\mathcal{C}$ such that the following diagram commutes.



Definition of a Z-(pre)fractionable category

Finally, after collecting all the axioms and some consequences, we are able to define Z-prefractionable categories and Z-fractionable categories.

(2.80) Definition (Z-prefractionable category)**.**

(a) A *Z-prefractionable category* is an S-semisaturated category with Z-2-arrows that fulfills the weakly universal S-Ore completion axiom, the relative Z-replacement axiom and the Z-comparison axiom.

(b) We suppose given Z-prefractionable categories $\mathcal{C}$ and $\mathcal{D}$. A *morphism of Z-prefractionable categories* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of categories with Z-2-arrows from $\mathcal{C}$ to $\mathcal{D}$.

(c) We suppose given a Grothendieck universe $\mathfrak{U}$. The full subcategory $\mathbf{ZPFrCat} = \mathbf{ZPFrCat}_{(\mathfrak{U})}$ of $\mathbf{CatZ}_{(\mathfrak{U})}$ with

$$\operatorname{Ob} \mathbf{ZPFrCat}_{(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob} \mathbf{CatZ}_{(\mathfrak{U})} \mid \mathcal{C} \text{ is a Z-prefractionable category}\}$$

is called the *category of Z-prefractionable categories* (more precisely, the *category of Z-prefractionable $\mathfrak{U}$-categories*). An object in $\mathbf{ZPFrCat}_{(\mathfrak{U})}$ is called a *Z-prefractionable $\mathfrak{U}$-category*, and a morphism in $\mathbf{ZPFrCat}_{(\mathfrak{U})}$ is called a *$\mathfrak{U}$-morphism of Z-prefractionable categories*.

(d) The full subcategory $\mathbf{CatD}_{\mathrm{ZPFr}} = \mathbf{CatD}_{\mathrm{ZPFr},(\mathfrak{U})}$ of $\mathbf{CatD}_{(\mathfrak{U})}$ with

$$\operatorname{Ob} \mathbf{CatD}_{\mathrm{ZPFr},(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})} \mid \text{there exist } S \subseteq \operatorname{Den} \mathcal{C} \text{ and } \mathcal{Z} \leq \mathrm{AG_S}\, \mathcal{C} \text{ such that } \mathcal{C} \text{ becomes a Z-prefractionable category with } \operatorname{SDen} \mathcal{C} = S \text{ and } \mathrm{AG_Z}\, \mathcal{C} = \mathcal{Z}\},$$

is called the *category of categories with denominators admitting the structure of a Z-prefractionable category* (more precisely, the *category of $\mathfrak{U}$-categories with denominators admitting the structure of a Z-prefractionable category*).

(2.81) Definition (Z-fractionable category)**.**

(a) A *Z-fractionable category* is a Z-prefractionable category $\mathcal{C}$ that fulfills the Z-replacement axiom for denominators, the relative Z-replacement axiom for Z-2-arrows, the Z-concatenation axiom, the Z-inversion axiom, the Z-numerator axiom and the Z-expansion axiom.

(b) We suppose given Z-fractionable categories $\mathcal{C}$ and $\mathcal{D}$. A *morphism of Z-fractionable categories* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of Z-prefractionable categories from $\mathcal{C}$ to $\mathcal{D}$.

(c) We suppose given a Grothendieck universe $\mathfrak{U}$. The full subcategory $\mathbf{ZFrCat} = \mathbf{ZFrCat}_{(\mathfrak{U})}$ of $\mathbf{ZPFrCat}_{(\mathfrak{U})}$ with

$$\operatorname{Ob} \mathbf{ZFrCat}_{(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob} \mathbf{ZPFrCat}_{(\mathfrak{U})} \mid \mathcal{C} \text{ is a Z-fractionable category}\}$$

is called the *category of Z-fractionable categories* (more precisely, the *category of Z-fractionable $\mathfrak{U}$-categories*). An object in $\mathbf{ZFrCat}_{(\mathfrak{U})}$ is called a *Z-fractionable $\mathfrak{U}$-category*, and a morphism in $\mathbf{ZFrCat}_{(\mathfrak{U})}$ is called a *$\mathfrak{U}$-morphism of Z-fractionable categories*.

(d) The full subcategory $\mathbf{CatD}_{\mathrm{ZFr}} = \mathbf{CatD}_{\mathrm{ZFr},(\mathfrak{U})}$ of $\mathbf{CatD}_{(\mathfrak{U})}$ with

$$\operatorname{Ob} \mathbf{CatD}_{\mathrm{ZFr},(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})} \mid \text{there exist } S \subseteq \operatorname{Den} \mathcal{C} \text{ and } \mathcal{Z} \leq \mathrm{AG_S}\, \mathcal{C} \text{ such that } \mathcal{C} \text{ becomes a Z-fractionable category with } \operatorname{SDen} \mathcal{C} = S \text{ and } \mathrm{AG_Z}\, \mathcal{C} = \mathcal{Z}\},$$

is called the *category of categories with denominators admitting the structure of a Z-fractionable category* (more precisely, the *category of $\mathfrak{U}$-categories with denominators admitting the structure of a Z-fractionable category*).

The connection between S-fractionable categories and Z-fractionable categories is as follows.

(2.82) Remark. Given an S-semisaturated category with denominators $\mathcal{C}$, the following conditions are equivalent:

(a) The category with denominators $\mathcal{C}$ fulfills the weakly universal S-Ore completion axiom. ([6])

(b) The Z-structure $\mathcal{C}_{\mathrm{Z}}$ is a Z-prefractionable category.

[6] So in particular, $\mathcal{C}$ is an S-fractionable category by proposition (2.28).

(c) The Z-structure $\mathcal{C}_{\mathrm{Z}}$ is a Z-fractionable category.

Proof. If condition (c) holds, then in particular condition (b) holds, and if condition (b) holds, then in particular condition (a) holds. So to show that the three conditions are equivalent, it remains to show that condition (a) implies condition (c). Indeed, $\mathcal{C}_{\mathrm{Z}}$ always fulfills the Z-comparison axiom, the Z-replacement axiom for denominators, the relative Z-replacement axiom for Z-2-arrows, the Z-inversion axiom. Moreover, the S-semisaturatedness implies implies the relative Z-replacement axiom and the weakly universal S-Ore completion axiom implies the Z-concatenation axiom and the Z-numerator axiom. Altogether, $\mathcal{C}_{\mathrm{Z}}$ is a Z-fractionable category. □

6 The S-Ore localisation of a Z-prefractionable category

In this section, we develop the two main results of this chapter. First, we show that the quotient graph $(\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/\!\equiv_{\mathrm{S}}$ of a Z-prefractionable category $\mathcal{C}$ carries the structure of a localisation of $\mathcal{C}$, as it is well-known in the particular case of the S-Ore localisation of an S-fractionable category, see definition (2.30) and remark (2.82). Here, the author has been guided by the interpretation of an S-2-arrow (f, a) in $\mathcal{C}$ as a 3-arrow $(1, f, a)$ in the sense of definition (2.110)(a) ([7]), see also [36, def. 4.2], and then to apply similar methods as in [36, sec. 5]. Second, we show that the so defined localisation admits an S-2-arrow calculus type criterion for equality of S-fractions, but restricted to Z-2-arrows, see theorem (2.93). As a corollary, we also get a criterion for the equality of arbitrary S-2-arrows, see corollary (2.94)(b).

The completion lemma and the comparison lemma

We begin with two technical lemmata, which will be used several times in the construction of the category structure on $(\mathrm{AG}_{\mathrm{S}}\,\mathcal{C})/\!\equiv_{\mathrm{S}}$, see theorem (2.85).
For the definition of a category with Z-2-arrows, see definition (2.38)(a); and for the S-Ore completion axiom, see definition (2.23)(a).

(2.83) Lemma (completion lemma)**.** We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the S-Ore completion axiom. Given morphisms f, g and a denominator d in $\mathcal{C}$ with $\operatorname{Target} f = \operatorname{Target} d$ and $\operatorname{Source} g = \operatorname{Source} d$, there exist morphisms f', g', a morphism s and S-denominators i, i' in $\mathcal{C}$ with $d = is$, $f's = f$, $ig' = gi'$.

$$\begin{tikzcd}
 & \cdot \arrow[rr, dotted, "f'"] \arrow[dr, dotted, "s"'] & & \cdot \arrow[rr, dotted, "g'"] & & \cdot & \\
\cdot \arrow[ur, equal] \arrow[rr, "f"] & & \cdot & & \cdot \arrow[ll, "d", "\approx"'] \arrow[ul, dotted, "i"] \arrow[rr, "g"] & & \cdot \arrow[ul, dotted, "i'"]
\end{tikzcd}$$

Proof. This follows from the Z-replacement axiom and the S-Ore completion axiom. □

For the definition of a Z-prefractionable category, see definition (2.80)(a).

(2.84) Lemma (comparison lemma)**.** We suppose given a Z-prefractionable category $\mathcal{C}$. Given a commutative diagram

$$\begin{tikzcd}
 & \cdot \arrow[rr, "f_1'"] \arrow[dr, "b_1", "\approx"'] & & \cdot \arrow[rr, "g_1'"] & & \cdot & & & \\
\cdot \arrow[ur, equal] \arrow[rr, "f_1"] \arrow[dd, equal] & & \cdot \arrow[dd, "h"] & & \cdot \arrow[ll, "d_1", "\approx"'] \arrow[ul, "a_1", "\approx"'] \arrow[rr, "g_1"] \arrow[dd, "h'"'] & & \cdot \arrow[ul, "a_1'", "\approx"'] \arrow[dd, "c"] & & \cdot \arrow[ll, "e_1", "\approx"'] \arrow[dd, equal] \\
 & \cdot \arrow[rr, "f_2'"] \arrow[dr, "b_2", "\approx"'] & & \cdot \arrow[rr, "g_2'"] & & \cdot & & & \\
\cdot \arrow[ur, equal] \arrow[rr, "f_2"] & & \cdot & & \cdot \arrow[ll, "d_2", "\approx"'] \arrow[ul, "a_2", "\approx"'] \arrow[rr, "g_2"] & & \cdot \arrow[ul, "a_2'", "\approx"'] & & \cdot \arrow[ll, "e_2", "\approx"']
\end{tikzcd}$$

in $\mathcal{C}$ with denominators d_1, d_2, e_1, e_2, a_1, a_1', a_2, a_2', b_1, b_2, we have

$$(f_1'g_1', e_1a_1') \equiv_{\mathrm{S}} (f_2'g_2', e_2a_2').$$

[7] In fact, this interpretation is the author's reason for the terminology "S-2-arrow" – such an S-2-arrow may be seen as a 3-arrow where the "T-part" is trivial.

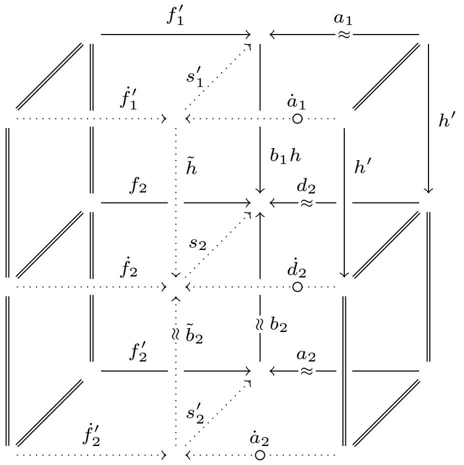
Moreover, if e_1, e_2, a'_1, a'_2 are S-denominators, then

$$(f'_1 g'_1, e_1 a'_1) \equiv_{\mathrm{S,n}} (f'_2 g'_2, e_2 a'_2).$$

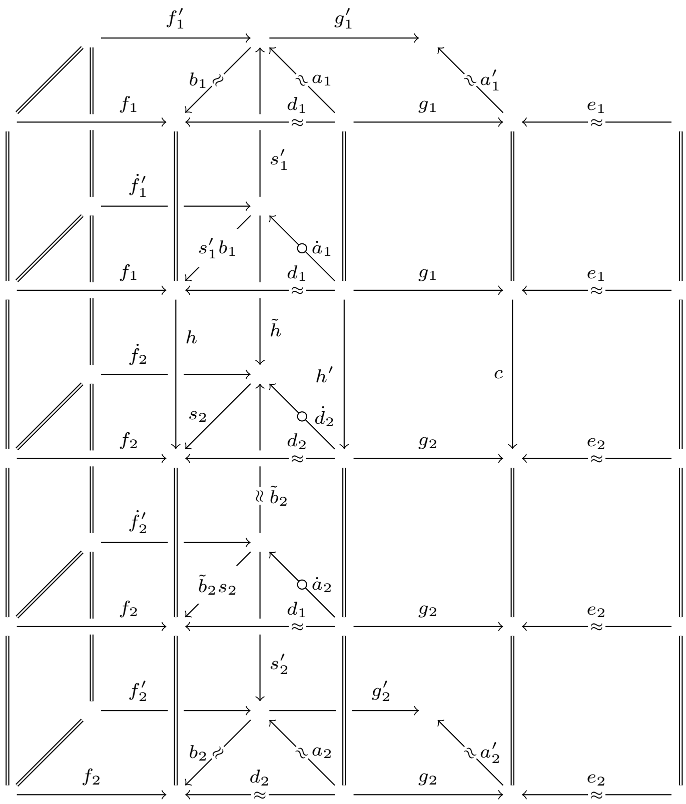
Proof. The diagram

$$\begin{tikzcd}
\cdot \arrow[r, "f'_1"] \arrow[d, equal] & \cdot \arrow[d, "b_1 h"] & \cdot \arrow[l, "a_1"', "\approx"] \arrow[d, "h'"] \\
\cdot \arrow[r, "f_2"] \arrow[d, equal] & \cdot & \cdot \arrow[l, "d_2"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[r, "f'_2"] & \cdot \arrow[u, "\approx", "b_2"'] & \cdot \arrow[l, "a_2"', "\approx"]
\end{tikzcd}$$

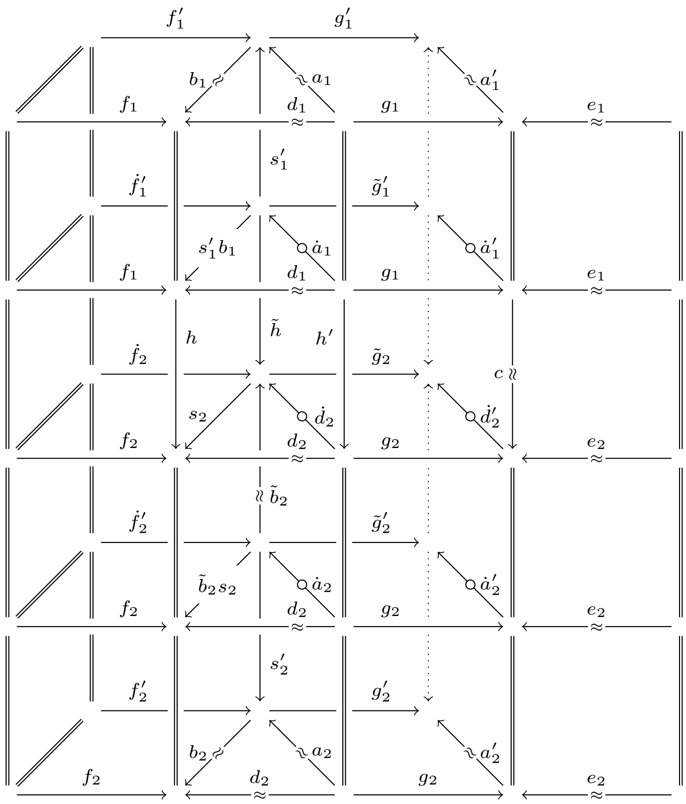
in $\mathcal{C}$ commutes, and thus the Z-replacement axiom and the Z-replacement lemma (2.64) imply that there exist Z-2-arrows $(\dot{f}'_1, \dot{a}_1)$, $(\dot{f}_2, \dot{d}_2)$, $(\dot{f}'_2, \dot{a}_2)$, an S-2-arrow $(\tilde{h}, \tilde{b}_2)$ and morphisms s'_1, s_2, s'_2 such that



commutes. This yields the following commutative diagram in $\mathcal{C}$.



By the weakly universal S-Ore completion axiom, there exist a weakly universal S-Ore completion $(\tilde{g}'_1, \dot{a}'_1)$ for g_1 and $\dot{a}_1$, a weakly universal S-Ore completion $(\tilde{g}_2, \dot{d}'_2)$ for g_2 and $\dot{d}_2$, and a weakly universal S-Ore completion $(\tilde{g}'_2, \dot{a}'_2)$ for g_2 and $\dot{a}_2$. Moreover, the morphism c is a denominator in $\mathcal{C}$ by S-semisaturatedness. So the weakly universal S-Ore completions induce morphisms, yielding a commutative diagram as follows.



In particular, we have

$$(f'_1 g'_1, e_1 a'_1) \equiv_{\mathrm{S}} (\dot{f}'_1 \tilde{g}'_1, e_1 \dot{a}'_1) \equiv_{\mathrm{S}} (\dot{f}_2 \tilde{g}_2, e_2 \dot{d}'_2) \equiv_{\mathrm{S}} (\dot{f}'_2 \tilde{g}'_2, e_2 \dot{a}'_2) \equiv_{\mathrm{S}} (f'_2 g'_2, e_2 a'_2).$$

Moreover, if e_1, e_2, a'_1, a'_2 are S-denominators in $\mathcal{C}$, then

$$(f'_1 g'_1, e_1 a'_1) \equiv_{\mathrm{S,n}} (\dot{f}'_1 \tilde{g}'_1, e_1 \dot{a}'_1) \equiv_{\mathrm{S,n}} (\dot{f}_2 \tilde{g}_2, e_2 \dot{d}'_2) \equiv_{\mathrm{S,n}} (\dot{f}'_2 \tilde{g}'_2, e_2 \dot{a}'_2) \equiv_{\mathrm{S,n}} (f'_2 g'_2, e_2 a'_2)$$

as the occurring S-2-arrows are normal. □

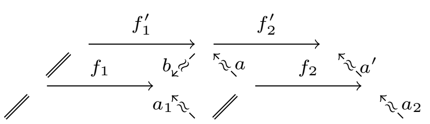
Construction of the S-Ore localisation

With the two previous lemmata at hand, we may construct a localisation structure on the quotient graph of the S-2-arrow graph modulo S-fraction equality, see definition (2.10)(a) and definition (2.14)(a).

(2.85) Theorem. We suppose given a Z-prefractionable category $\mathcal{C}$.

(a) There is a category structure on $(\mathrm{AG_S}\,\mathcal{C})/\equiv_{\mathrm{S}}$, where the composition and the identities are given as follows.

Given $(f_1, a_1), (f_2, a_2) \in \mathrm{Arr}\,\mathrm{AG_S}\,\mathcal{C}$ with $\mathrm{Target}\,(f_1, a_1) = \mathrm{Source}\,(f_2, a_2)$, we choose morphisms f'_1, f'_2 and denominators a, a', b with $a_1 = ab$, $f'_1 b = f_1$, $a f'_2 = f_2 a'$.



Then, for any such choice,

$$(f_1/a_1)(f_2/a_2) = f'_1 f'_2/a_2 a'.$$

The identity of $X \in \mathrm{Ob}\,(\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ is given by

$$1_X = 1_X/1_X.$$

(b) The quotient graph $(\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ together with the category structure from (a) becomes a localisation of $\mathcal{C}$, where the localisation functor $\mathrm{loc}\colon \mathcal{C} \to (\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ is given on the objects by

$$\mathrm{loc}(X) = X$$

for $X \in \mathrm{Ob}\,\mathcal{C}$ and on the morphisms by

$$\mathrm{loc}(f) = f/1$$

for $f \in \mathrm{Mor}\,\mathcal{C}$.

For every denominator d in $\mathcal{C}$, the inverse of $\mathrm{loc}(d)$ is given by

$$\mathrm{loc}(d)^{-1} = 1/d.$$

Given a category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible for every denominator d in $\mathcal{C}$, the unique functor $\hat{F}\colon (\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}} \to \mathcal{D}$ with $F = \hat{F} \circ \mathrm{loc}$ is given on the objects by

$$\hat{F}X = FX$$

for $X \in \mathrm{Ob}(\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ and on the morphisms by

$$\hat{F}(f/a) = (Ff)(Fa)^{-1}$$

for $(f, a) \in \mathrm{Arr}\,\mathrm{AG_S}\,\mathcal{C}$.

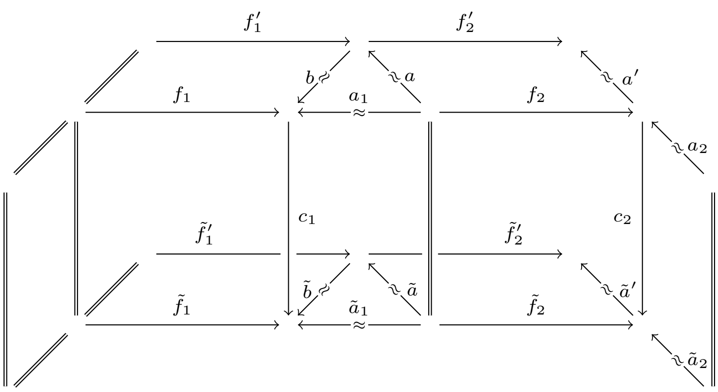
Proof.

(a) The completion lemma (2.83) and the S-semisaturatedness of $\mathcal{C}$ show that the construction of the composites described above is feasible in $\mathcal{C}$.

It is our first aim to show that this construction is independent of all choices. To this end, we suppose given $(f_l, a_l), (\tilde{f}_l, \tilde{a}_l) \in \mathrm{Arr}\,\mathrm{AG_S}\,\mathcal{C}$ and $c_l \in \mathrm{Mor}\,\mathcal{C}$ with $(\tilde{f}_l, \tilde{a}_l) = (f_l c_l, a_l c_l)$ for $l \in \{1, 2\}$, and such that $\mathrm{Target}\,(f_1, a_1) = \mathrm{Source}\,(f_2, a_2)$.

$$\begin{tikzcd}
\cdot \arrow[r, "f_1"] \arrow[d, equal] & \cdot \arrow[d, "c_1"] & \cdot \arrow[l, "a_1"', "\approx"] \arrow[r, "f_2"] \arrow[d, equal] & \cdot \arrow[d, "c_2"] & \cdot \arrow[l, "a_2"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[r, "\tilde{f}_1"] & \cdot & \cdot \arrow[l, "\tilde{a}_1"', "\approx"] \arrow[r, "\tilde{f}_2"] & \cdot & \cdot \arrow[l, "\tilde{a}_2"', "\approx"]
\end{tikzcd}$$

Moreover, we suppose given morphisms f'_1, f'_2 and denominators a, a', b with $a_1 = ab$, $f'_1 b = f_1$, $a f'_2 = f_2 a'$, and we suppose given morphisms $\tilde{f}'_1$, $\tilde{f}'_2$ and denominators $\tilde{a}$, $\tilde{a}'$, $\tilde{b}$ with $\tilde{a}_1 = \tilde{a}\tilde{b}$, $\tilde{f}'_1 \tilde{b} = \tilde{f}_1$, $\tilde{a}\tilde{f}'_2 = \tilde{f}_2 \tilde{a}'$.



Then the comparison lemma (2.84) yields $(f_1'f_2', a_2a') \equiv_{\mathrm{S}} (\tilde{f}_1'\tilde{f}_2', \tilde{a}_2\tilde{a}')$ in $\operatorname{AG}_{\mathrm{S}}\mathcal{C}$ and therefore $f_1'f_2'/a_2a' = \tilde{f}_1'\tilde{f}_2'/\tilde{a}_2\tilde{a}'$ in $(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}}$.

In the special case where $c_1 = 1$ and $c_2 = 1$, we see that different choices made in the construction lead to the same S-fraction $f_1'f_2'/a_2a' = \tilde{f}_1'\tilde{f}_2'/\tilde{a}_2\tilde{a}'$. Hence we obtain a well-defined map

$$c\colon \operatorname{Arr}\operatorname{AG}_{\mathrm{S}}\mathcal{C} \;{}_{\mathrm{Target}}\!\times_{\mathrm{Source}} \operatorname{Arr}\operatorname{AG}_{\mathrm{S}}\mathcal{C} \to \operatorname{Arr}(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}}, ((f_1, a_1), (f_2, a_2)) \mapsto f_1'f_2'/a_2a',$$

where f_1', f_2', a' are chosen as described above. Moreover, the general case shows that c is independent of the choice of the representatives in the equivalence classes with respect to $\equiv_{\mathrm{S}}$, whence we obtain an induced map

$$\bar{c}\colon \operatorname{Arr}(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}} \;{}_{\mathrm{Target}}\!\times_{\mathrm{Source}} \operatorname{Arr}(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}} \to \operatorname{Arr}(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}}$$

given by

$$\bar{c}(f_1/a_1, f_2/a_2) = c((f_1, a_1), (f_2, a_2)) = f_1'f_2'/a_2a'$$

for $(f_1, a_1), (f_2, a_2) \in \operatorname{Arr}\operatorname{AG}_{\mathrm{S}}\mathcal{C}$ with $\operatorname{Target}(f_1, a_1) = \operatorname{Source}(f_2, a_2)$.

In addition to $\bar{c}$, we define the map

$$e\colon \operatorname{Ob}(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}} \to \operatorname{Arr}(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}}, X \mapsto 1_X/1_X.$$

To show that $(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}}$ is a category with composition $\bar{c}$ and identity map e, it remains to verify the category axioms. We suppose given $(f_1, a_1), (f_2, a_2) \in \operatorname{Arr}\operatorname{AG}_{\mathrm{S}}\mathcal{C}$ with $\operatorname{Target} f_1/a_1 = \operatorname{Source} f_2/a_2$, and we choose morphisms f_1', f_2' and denominators a, a', b with $a_1 = ab$, $f_1'b = f_1$, $af_2' = f_2a'$. Then we obtain

$$\begin{aligned}
\operatorname{Source}\bar{c}(f_1/a_1, f_2/a_2) &= \operatorname{Source} f_1'f_2'/a_2a' = \operatorname{Source}(f_1'f_2') = \operatorname{Source} f_1' = \operatorname{Source} f_1 = \operatorname{Source} f_1/a_1,\\
\operatorname{Target}\bar{c}(f_1/a_1, f_2/a_2) &= \operatorname{Target} f_1'f_2'/a_2a' = \operatorname{Source}(a_2a') = \operatorname{Source} a_2 = \operatorname{Target} f_2/a_2.
\end{aligned}$$

Moreover, for $X \in \operatorname{Ob}(\operatorname{AG}_{\mathrm{S}}\mathcal{C})/{\equiv_{\mathrm{S}}}$, we get

$$\begin{aligned}
\operatorname{Source} e(X) &= \operatorname{Source} 1_X/1_X = \operatorname{Source} 1_X = X,\\
\operatorname{Target} e(X) &= \operatorname{Target} 1_X/1_X = \operatorname{Source} 1_X = X.
\end{aligned}$$

For the associativity of $\bar{c}$, we suppose given $(f_1, a_1), (f_2, a_2), (f_3, a_3) \in \operatorname{Arr}\operatorname{AG}_{\mathrm{S}}\mathcal{C}$ such that $\operatorname{Target} f_1/a_1 = \operatorname{Source} f_2/a_2$ and $\operatorname{Target} f_2/a_2 = \operatorname{Source} f_3/a_3$. We choose morphisms f_1', f_2' and denominators a, a', b with $a_1 = ab$, $f_1'b = f_1$, $af_2' = f_2a'$, and we choose morphisms $\tilde{f}_2'$, $\tilde{f}_3'$ and denominators $\tilde{a}$, $\tilde{a}'$, $\tilde{b}$ with $a_2 = \tilde{a}\tilde{b}$, $\tilde{f}_2'\tilde{b} = f_2$, $\tilde{a}\tilde{f}_3' = f_3\tilde{a}'$.

$$\begin{tikzcd}
& & \arrow[rr, "f_1'"] \arrow[dl, equal] & & \arrow[rrr, "f_2'"] \arrow[dl, "b"'] & & \arrow[rrr, "\tilde{f}_2'"] \arrow[dl, equal] & & & \arrow[rr, "\tilde{f}_3'"] \arrow[dl, "\tilde{b}"'] & & & & \\
& \arrow[rr, "f_1"] \arrow[dl, equal] & & & & \arrow[rrr, "f_2"] \arrow[ul, "a"'] \arrow[dl, equal] & & & \arrow[ul, "a'"'] & & \arrow[rr, "f_3"] \arrow[ul, "\tilde{a}"'] \arrow[dl, equal] & & \arrow[ul, "\tilde{a}'"'] & \\
& & & & \arrow[ul, "a_1"'] & & & & & \arrow[ul, "a_2"'] & & & & \arrow[ul, "a_3"']
\end{tikzcd}$$

By definition of $\bar{c}$, we obtain

$$\begin{aligned}
\bar{c}(f_1/a_1, f_2/a_2) &= f_1'f_2'/a_2a',\\
\bar{c}(f_2/a_2, f_3/a_3) &= \tilde{f}_2'\tilde{f}_3'/a_3\tilde{a}'.
\end{aligned}$$

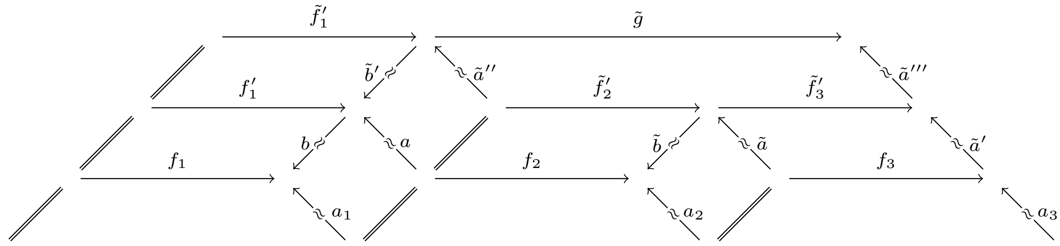
Now $\tilde{b}a'$ is a denominator in $\mathcal{C}$ by multiplicativity. By the completion lemma (2.83) and S-semisaturatedness, there exist morphisms g, f_3', and denominators a'', a''', b' with $\tilde{b}a' = a''b'$, $gb' = f_1'f_2'$, $a''f_3' = \tilde{f}_3'a'''$.

$$\begin{tikzcd}
& & & \arrow[rrrrr, "g"] & & & & & \arrow[rrr, "f_3'"] \arrow[dl, "b'"'] & & & & & & \\
& & \arrow[rrr, "f_1'"] \arrow[ur, equal] \arrow[dl, equal] & & & \arrow[rr, "f_2'"] \arrow[dl, "b"'] & & & & \arrow[rrr, "\tilde{f}_3'"] \arrow[ul, "a''"'] \arrow[dl, "\tilde{b}"'] & & & \arrow[ul, "a'''"'] & & \\
& \arrow[rrr, "f_1"] \arrow[dl, equal] & & & & & \arrow[rr, "f_2"] \arrow[ul, "a"'] \arrow[dl, equal] & & \arrow[ul, "a'"'] & & \arrow[rrr, "f_3"] \arrow[ul, "\tilde{a}"'] \arrow[dl, equal] & & & \arrow[ul, "\tilde{a}'"'] & \\
& & & & & \arrow[ul, "a_1"'] & & & & \arrow[ul, "a_2"'] & & & & & \arrow[ul, "a_3"']
\end{tikzcd}$$

Then we have $a_2 a' = \tilde{a}\tilde{b}a' = \tilde{a}a''b'$ and $\tilde{a}a'' f_3' = \tilde{a}\tilde{f}_3' a''' = f_3 \tilde{a}' a'''$, whence

$$\bar{c}(\bar{c}(f_1/a_1, f_2/a_2), f_3/a_3) = \bar{c}(f_1' f_2'/a_2 a', f_3/a_3) = g f_3'/a_3 \tilde{a}' a'''.$$

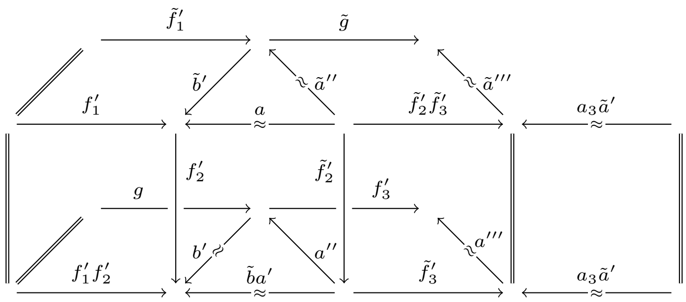
Moreover, the completion lemma (2.83) and S-semisaturatedness yield morphisms $\tilde{f}_1'$, $\tilde{g}$ and denominators $\tilde{a}''$, $\tilde{a}'''$, $\tilde{b}'$ with $a = \tilde{a}''\tilde{b}'$, $\tilde{f}_1'\tilde{b}' = f_1'$, $\tilde{a}''\tilde{g} = \tilde{f}_2'\tilde{f}_3'\tilde{a}'''$.



Then we have $a_1 = ab = \tilde{a}''\tilde{b}'b$ and $\tilde{f}_1'\tilde{b}'b = f_1' b = f_1$, whence

$$\bar{c}(f_1/a_1, \bar{c}(f_2/a_2, f_3/a_3)) = \bar{c}(f_1/a_1, \tilde{f}_2'\tilde{f}_3'/a_3\tilde{a}') = \tilde{f}_1'\tilde{g}/a_3\tilde{a}'\tilde{a}'''.$$

But as $af_2' = f_2 a' = \tilde{f}_2'\tilde{b}a'$, the diagram



in $\mathcal{C}$ commutes. Thus we have $(\tilde{f}_1'\tilde{g}, a_3\tilde{a}'\tilde{a}''') \equiv_{\mathrm{S}} (gf_3', a_3\tilde{a}'a''')$ by the comparison lemma (2.84) and therefore

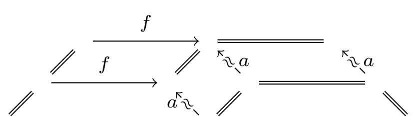
$$\bar{c}(\bar{c}(f_1/a_1, f_2/a_2), f_3/a_3) = gf_3'/a_3\tilde{a}'a''' = \tilde{f}_1'\tilde{g}/a_3\tilde{a}'\tilde{a}''' = \bar{c}(f_1/a_1, \bar{c}(f_2/a_2, f_3/a_3))$$

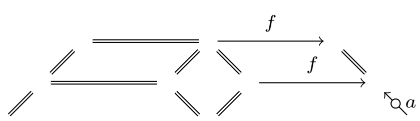
in $(\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$. Hence $\bar{c}$ is associative.

Finally, we have

$$\bar{c}(f/a, e(\mathrm{Target}\, f/a)) = \bar{c}(f/a, 1/1) = f1/1a = f/a,$$
$$\bar{c}(e(\mathrm{Source}\, f/a), f/a) = \bar{c}(1/1, f/a) = 1f/a1 = f/a.$$

for $(f, a) \in \mathrm{Arr}\,\mathrm{AG_S}\,\mathcal{C}$.





Altogether, $(\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ becomes a category with $(f_1/a_1)(f_2/a_2) = \bar{c}(f_1/a_1, f_2/a_2)$ for $(f_1, a_1), (f_2, a_2) \in \mathrm{Arr}\,\mathrm{AG_S}\,\mathcal{C}$ with $\mathrm{Target}\, f_1/a_1 = \mathrm{Source}\, f_2/a_2$ and $1_X = e(X)$ for $X \in \mathrm{Ob}\,(\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$.

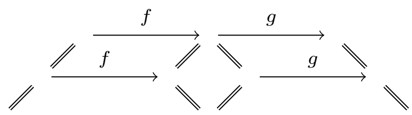
(b) We define a graph morphism $L\colon \mathcal{C} \to (\mathrm{AG_S}\,\mathcal{C})/{\equiv_{\mathrm{S}}}$ on the objects by $LX := X$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and on the arrows by $Lf := f/1$ for $f \in \mathrm{Mor}\,\mathcal{C}$. Then we get

$$L(fg) = fg/1 = (f/1)(g/1) = (Lf)(Lg)$$

for $f, g \in \operatorname{Mor} \mathcal{C}$ with $\operatorname{Target} f = \operatorname{Source} g$ and

$$L1_X = 1_X/1_X = 1_{LX}$$

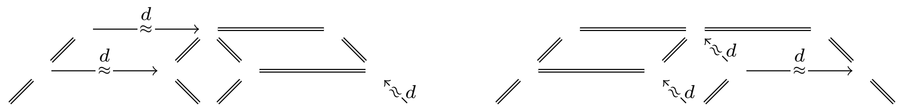
for $X \in \operatorname{Ob} \mathcal{C}$, that is, L is a functor.



For every denominator d in $\mathcal{C}$, we have

$$\begin{aligned}(Ld)(1/d) &= (d/1)(1/d) = d/d = 1/1 = 1,\\ (1/d)(Ld) &= (1/d)(d/1) = 1/1 = 1,\end{aligned}$$

that is, Ld is invertible in $(\operatorname{AG_S} \mathcal{C})/{\equiv_S}$ with $(Ld)^{-1} = 1/d$.



To show that $(\operatorname{AG_S} \mathcal{C})/{\equiv_S}$ becomes a localisation of $\mathcal{C}$ with localisation functor L, we suppose given a category $\mathcal{D}$ and a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible in $\mathcal{D}$ for every denominator d in $\mathcal{C}$. Since

$$\begin{aligned}&\operatorname{Source}((Ff)(Fa)^{-1}) = \operatorname{Source} Ff = F(\operatorname{Source} f) = F(\operatorname{Source}(f, a)),\\ &\operatorname{Target}((Ff)(Fa)^{-1}) = \operatorname{Target}(Fa)^{-1} = \operatorname{Source} Fa = F(\operatorname{Source} a) = F(\operatorname{Target}(f, a))\end{aligned}$$

for $(f, a) \in \operatorname{Arr} \operatorname{AG_S} \mathcal{C}$, there is a graph morphism $F' \colon \operatorname{AG_S} \mathcal{C} \to \mathcal{D}$ given on the objects by $F'X = FX$ for $X \in \operatorname{Ob} \operatorname{AG_S} \mathcal{C}$ and on the arrows by $F'(f, a) = (Ff)(Fa)^{-1}$ for $(f, a) \in \operatorname{Arr} \operatorname{AG_S} \mathcal{C}$. Moreover, for $(f, a) \in \operatorname{Arr} \operatorname{AG_S} \mathcal{C}$ and $c \in \operatorname{Mor} \mathcal{C}$ with $ac \in \operatorname{Den} \mathcal{C}$, we obtain

$$\begin{aligned}F'(fc, ac) &= F(fc)F(ac)^{-1} = (Ff)(Fc)F(ac)^{-1} = (Ff)(Fa)^{-1}(Fa)(Fc)F(ac)^{-1} = (Ff)(Fa)^{-1}\\ &= F'(f, a).\end{aligned}$$

Hence F' maps S-fraction equal S-2-arrows to the same morphism and so we obtain an induced graph morphism $\hat{F} \colon (\operatorname{AG_S} \mathcal{C})/{\equiv_S} \to \mathcal{D}$ with $F' = \hat{F} \circ \operatorname{quo}$, given by

$$\hat{F}X = F'X = FX$$

for $X \in \operatorname{Ob} (\operatorname{AG_S} \mathcal{C})/{\equiv_S}$ and by

$$\hat{F}(f/a) = F'(f, a) = (Ff)(Fa)^{-1}$$

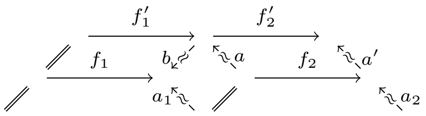
for $(f, a) \in \operatorname{Arr} \operatorname{AG_S} \mathcal{C}$.

$$\begin{tikzcd}
\operatorname{AG_S} \mathcal{C} \arrow[r, "F'"] \arrow[d, "\operatorname{quo}"'] & \mathcal{D} \\
(\operatorname{AG_S} \mathcal{C})/{\equiv_S} \arrow[ur, dotted, "\hat{F}"']
\end{tikzcd}$$

For $(f_1, a_1), (f_2, a_2) \in \operatorname{Arr} \operatorname{AG_S} \mathcal{C}$ with $\operatorname{Target}(f_1, a_1) = \operatorname{Source}(f_2, a_2)$, we have

$$\begin{aligned}\hat{F}((f_1/a_1)(f_2/a_2)) &= \hat{F}(f_1' f_2'/a_2 a') = F(f_1' f_2')\, F(a_2 a')^{-1} = (Ff_1')(Ff_2')(Fa')^{-1}(Fa_2)^{-1}\\ &= (Ff_1)(Fb)^{-1}(Fa)^{-1}(Ff_2)(Fa_2)^{-1} = (Ff_1)(Fa_1)^{-1}(Ff_2)(Fa_2)^{-1}\\ &= \hat{F}(f_1/a_1)\, \hat{F}(f_2/a_2),\end{aligned}$$

where f_1', f_2', a, a', b are supposed to be chosen as in (a).



Moreover, we have

$$\hat{F}(1_X) = \hat{F}(1_X/1_X) = (F1_X)(F1_X)^{-1} = 1_{FX}1_{FX}^{-1} = 1_{FX} = 1_{\hat{F}X}$$

for $X \in \mathrm{Ob}\,(\mathrm{AG_S}\,\mathcal{C})/{\equiv_\mathrm{S}}$. So $\hat{F}\colon (\mathrm{AG_S}\,\mathcal{C})/{\equiv_\mathrm{S}} \to \mathcal{D}$ is a functor. As

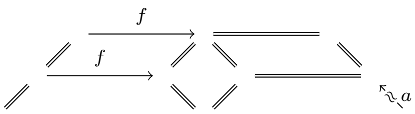
$$\hat{F}Lf = \hat{F}(f/1) = (Ff)(F1)^{-1} = (Ff)\,1^{-1} = Ff$$

for $f \in \mathrm{Mor}\,\mathcal{C}$, we have $\hat{F} \circ L = F$.

Conversely, given an arbitrary functor $G\colon (\mathrm{AG_S}\,\mathcal{C})/{\equiv_\mathrm{S}} \to \mathcal{D}$ with $F = G \circ L$, we conclude that

$$G(f/a) = G((f/1)(1/a)) = G((Lf)(La)^{-1}) = (GLf)(GLa)^{-1} = (Ff)(Fa)^{-1}$$

for $(f, a) \in \mathrm{Arr}\,\mathrm{AG_S}\,\mathcal{C}$.



Altogether, $(\mathrm{AG_S}\,\mathcal{C})/{\equiv_\mathrm{S}}$ becomes a localisation of $\mathcal{C}$ with localisation functor $\mathrm{loc}^{(\mathrm{AG_S}\,\mathcal{C})/\equiv_\mathrm{S}} = L$. □

(2.86) Definition (S-Ore localisation)**.** We suppose given a Z-prefractionable category $\mathcal{C}$. The *S-Ore localisation* of $\mathcal{C}$ is defined to be the localisation $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$ of $\mathcal{C}$, whose underlying category is the quotient graph $(\mathrm{AG_S}\,\mathcal{C})/{\equiv_\mathrm{S}}$ together with composition and identities as in theorem (2.85)(a), and whose localisation functor is given as in theorem (2.85)(b).

(2.87) Remark. Given Z-prefractionable categories $\mathcal{C}$ and $\mathcal{C}'$ such that their underlying categories with denominators coincide, we have $\mathsf{Ore}_\mathrm{S}(\mathcal{C}) = \mathsf{Ore}_\mathrm{S}(\mathcal{C}')$.

Proof. The definition of the category structure of $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$ is independent of $\mathrm{SDen}\,\mathcal{C}$ and $\mathrm{AG_Z}\,\mathcal{C}$, see theorem (2.85)(a). Analogously for $\mathcal{C}'$, and so we have $\mathsf{Ore}_\mathrm{S}(\mathcal{C}) = \mathsf{Ore}_\mathrm{S}(\mathcal{C}')$. □

(2.88) Definition (S-Ore localisation)**.** We suppose given a category with denominators $\mathcal{C}$ that admits the structure of a Z-prefractionable category. The *S-Ore localisation* of $\mathcal{C}$ is defined to be the S-Ore localisation of $\mathcal{C}$ equipped with an arbitrary choice of a structure of a Z-prefractionable category on $\mathcal{C}$.

Next, we turn the S-Ore localisation into a functor.

(2.89) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\boldsymbol{\Theta}_\mathrm{S}$ is in $\mathfrak{U}$ and a category with denominators $\mathcal{C}$ that admits the structure of a Z-prefractionable category. If $\mathcal{C}$ is a $\mathfrak{U}$-category with denominators, then $\mathsf{Ore}_\mathrm{S}(\mathcal{C})$ is a $\mathfrak{U}$-category.

(2.90) Corollary. We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\boldsymbol{\Theta}_\mathrm{S}$ is in $\mathfrak{U}$. Then we have a functor

$$\mathsf{Ore}_\mathrm{S}\colon \mathbf{CatD}_{\mathrm{ZPFr},(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})},$$

given on the morphisms as follows. For every morphism $F\colon \mathcal{C} \to \mathcal{C}'$ in $\mathbf{CatD}_{\mathrm{ZPFr},(\mathfrak{U})}$, the morphism $\mathsf{Ore}_\mathrm{S}(F)\colon \mathsf{Ore}_\mathrm{S}(\mathcal{C}) \to \mathsf{Ore}_\mathrm{S}(\mathcal{C}')$ in $\mathbf{Cat}_{(\mathfrak{U})}$ is the unique morphism in $\mathbf{Cat}_{(\mathfrak{U})}$ with $\mathrm{loc}^{\mathsf{Ore}_\mathrm{S}(\mathcal{C}')} \circ F = \mathsf{Ore}_\mathrm{S}(F) \circ \mathrm{loc}^{\mathsf{Ore}_\mathrm{S}(\mathcal{C})}$.

Proof. This follows from remark (2.89) and corollary (1.14)(d). □

The following remark allows us to rewrite the concrete realisation of the morphisms in the S-Ore localisation in terms of the localisation functor.

(2.91) Remark (splitting S-fractions)**.** We suppose given a Z-prefractionable category $\mathcal{C}$. For every S-2-arrow (f, a) in $\mathcal{C}$, we have

$$f/a = \mathrm{loc}(f)\,\mathrm{loc}(a)^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

Proof. As $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ is a localisation of $\mathcal{C}$, it follows that $\mathrm{loc}(d)$ is invertible for every denominator d in $\mathcal{C}$. By theorem (2.85)(b), the unique functor $\hat{L}\colon \mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) \to \mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ with $\mathrm{loc} = \hat{L} \circ \mathrm{loc}$ is given by $\hat{L}(f/a) = \mathrm{loc}(f)\,\mathrm{loc}(a)^{-1}$ for $(f, a) \in \mathrm{Arr}\,\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$. But since $\mathrm{loc} = \mathrm{id}_{\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})} \circ \mathrm{loc}$, we necessarily have $\hat{L} = \mathrm{id}_{\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})}$ and therefore $f/a = \mathrm{loc}(f)\,\mathrm{loc}(a)^{-1}$ for $(f, a) \in \mathrm{Arr}\,\mathrm{AG}_{\mathrm{S}}\,\mathcal{C}$. □

For the definition of the S-2-arrow representative condition, see definition (2.31)(a).

(2.92) Corollary. Given a Z-prefractionable category $\mathcal{C}$, the S-Ore localisation $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ fulfills the S-2-arrow representative condition.

The Z-2-arrow calculus

Next, we will deduce an S-2-arrow calculus type criterion for the morphisms in the S-Ore localisation $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, cf. theorem (2.35), but restricted to Z-2-arrows.
For the definition of a Z-(pre)fractionable category and of the various axioms needed, see section 5.

(2.93) Theorem (Z-2-arrow calculus)**.** We suppose given a Z-prefractionable category $\mathcal{C}$.

(a) We have

$$\mathrm{Mor}\,\mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) = \{\mathrm{loc}(f)\,\mathrm{loc}(i)^{-1} \mid (f, i) \text{ is a Z-2-arrow in } \mathcal{C}\}.$$

(b) Given Z-2-arrows (f, i), (f', i') in $\mathcal{C}$, we have

$$\mathrm{loc}(f)\,\mathrm{loc}(i)^{-1} = \mathrm{loc}(f')\,\mathrm{loc}(i')^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ if and only if there exist a Z-2-arrow $(\tilde{f}', \tilde{i}')$, a denominator c and an S-denominator j in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, equal] & \bullet \arrow[d, dotted, "\wr\, c"] & \bullet \arrow[l, "i"'] \arrow[d, equal] \\
\bullet \arrow[r, dotted, "\tilde{f}'"] & \bullet & \bullet \arrow[l, dotted, "\tilde{i}'"'] \\
\bullet \arrow[u, equal] \arrow[r, "f'"] & \bullet \arrow[u, dotted, "j"'] & \bullet \arrow[l, "i'"'] \arrow[u, equal]
\end{tikzcd}$$

(c) Given Z-2-arrows (f_1, i_1), (f_2, i_2), a normal S-2-arrow (g_1, j_1) and an S-2-arrow (g_2, b_2) in $\mathcal{C}$, we have

$$\mathrm{loc}(f_1)\,\mathrm{loc}(i_1)^{-1}\,\mathrm{loc}(g_2)\,\mathrm{loc}(b_2)^{-1} = \mathrm{loc}(g_1)\,\mathrm{loc}(j_1)^{-1}\,\mathrm{loc}(f_2)\,\mathrm{loc}(i_2)^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ if and only if there exist a Z-2-arrow $(\tilde{f}_2, \tilde{i}_2)$ and an S-2-arrow $(\tilde{g}_2, \tilde{b}_2)$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, "g_1"'] & \bullet \arrow[d, dotted, "\tilde{g}_2"] & \bullet \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
\bullet \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{i}_2"'] \\
\bullet \arrow[u, "j_1"] \arrow[r, "f_2"] & \bullet \arrow[u, dotted, "\wr\, \tilde{b}_2"] & \bullet \arrow[l, "i_2"'] \arrow[u, "\wr\, b_2"']
\end{tikzcd}$$

If, in addition, (g_2, b_2) is a normal S-2-arrow, then $(\tilde{g}_2, \tilde{b}_2)$ may be chosen to be a normal S-2-arrow.

(d) We suppose that $\mathcal{C}$ fulfills the relative Z-replacement axiom for Z-2-arrows. Given Z-2-arrows (f_1, i_1), (f_2, i_2), (g_1, j_1), (g_2, j_2) in $\mathcal{C}$, we have

$$\operatorname{loc}(f_1) \operatorname{loc}(i_1)^{-1} \operatorname{loc}(g_2) \operatorname{loc}(j_2)^{-1} = \operatorname{loc}(g_1) \operatorname{loc}(j_1)^{-1} \operatorname{loc}(f_2) \operatorname{loc}(i_2)^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ if and only if there exist Z-2-arrows $(\tilde{f}_2, \tilde{i}_2)$, $(\tilde{g}_2, \tilde{j}_2)$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, "g_1"] & \bullet \arrow[d, dotted, "\tilde{g}_2"] & \bullet \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
\bullet \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{i}_2"'] \\
\bullet \arrow[u, "j_1"'] \arrow[r, "f_2"] & \bullet \arrow[u, dotted, "\tilde{j}_2"'] & \bullet \arrow[l, "i_2"'] \arrow[u, "j_2"']
\end{tikzcd}$$

(e) We suppose that $\mathcal{C}$ fulfills the Z-numerator axiom. Given Z-2-arrows (f_1, i_1), (f_2, i_2) and S-2-arrows (g_1, b_1), (g_2, b_2) in $\mathcal{C}$, we have

$$\operatorname{loc}(f_1) \operatorname{loc}(i_1)^{-1} \operatorname{loc}(g_2) \operatorname{loc}(b_2)^{-1} = \operatorname{loc}(g_1) \operatorname{loc}(b_1)^{-1} \operatorname{loc}(f_2) \operatorname{loc}(i_2)^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ if and only if there exist a Z-2-arrow $(\tilde{f}_2, \tilde{i}_2)$ and an S-2-arrow $(\tilde{g}_2, \tilde{b}_2)$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, "g_1"] & \bullet \arrow[d, dotted, "\tilde{g}_2"] & \bullet \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
\bullet \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{i}_2"'] \\
\bullet \arrow[u, "b_1"'] \arrow[r, "f_2"] & \bullet \arrow[u, dotted, "\tilde{b}_2"'] & \bullet \arrow[l, "i_2"'] \arrow[u, "b_2"']
\end{tikzcd}$$

(f) Given a category $\mathcal{D}$ and a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible in $\mathcal{D}$ for every denominator d in $\mathcal{C}$, the unique functor $\hat{F} \colon \mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) \to \mathcal{D}$ with $F = \hat{F} \circ \operatorname{loc}$ is given on the objects by

$$\hat{F}X = FX$$

for $X \in \operatorname{Ob} \mathcal{C}$ and on the morphisms by

$$\hat{F}(\operatorname{loc}(f) \operatorname{loc}(i)^{-1}) = (Ff)(Fi)^{-1}$$

for every Z-2-arrow (f, i) in $\mathcal{C}$.

Proof.

(a) This follows from corollary (2.92) and corollary (2.48).

(b) By remark (2.91), we have $\operatorname{loc}(f) \operatorname{loc}(i)^{-1} = \operatorname{loc}(f') \operatorname{loc}(i')^{-1}$ if and only if $f/i = f'/i'$, that is, if and only if $(f, i) \equiv_{\mathrm{S}} (f', i')$. So the assertion follows from theorem (2.60)(c) and S-semisaturatedness.

(c) By remark (2.91), we have

$$\operatorname{loc}(f_1) \operatorname{loc}(i_1)^{-1} \operatorname{loc}(g_2) \operatorname{loc}(b_2)^{-1} = \operatorname{loc}(g_1) \operatorname{loc}(j_1)^{-1} \operatorname{loc}(f_2) \operatorname{loc}(i_2)^{-1}$$

if and only if $(f_1/i_1)(g_2/b_2) = (g_1/j_1)(f_2/i_2)$.

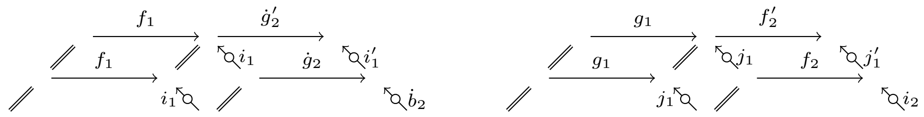
If we have a commutative diagram as stated, then we have

$$(f_1/i_1)(g_2/b_2) = f_1\tilde{g}_2/b_2\tilde{i}_2 = g_1\tilde{f}_2/i_2\tilde{b}_2 = (g_1/j_1)(f_2/i_2).$$

Conversely, we suppose that $(f_1/i_1)(g_2/b_2) = (g_1/j_1)(f_2/i_2)$. We choose a Z-2-arrow $(\dot{g}_2, \dot{b}_2)$ and a morphism t_2 in $\mathcal{C}$ with $(g_2, b_2) = (\dot{g}_2 t_2, \dot{b}_2 t_2)$, so that $(g_2, b_2) \equiv_{\mathrm{S}} (\dot{g}_2, \dot{b}_2)$ and therefore $g_2/b_2 = \dot{g}_2/\dot{b}_2$. Moreover, we choose an S-Ore completion $(\dot{g}_2', i_1')$ for $\dot{g}_2$ and i_1 and an S-Ore completion (f_2', j_1') for f_2 and j_1, so that

$$f_1\dot{g}_2'/\dot{b}_2 i_1' = (f_1/i_1)(\dot{g}_2/\dot{b}_2) = (f_1/i_1)(g_2/b_2) = (g_1/j_1)(f_2/i_2) = g_1 f_2'/i_2 j_1'$$

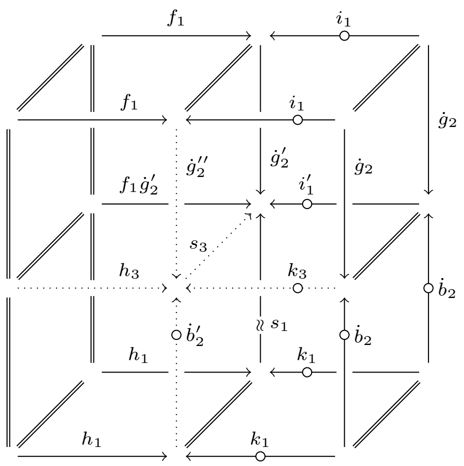
by theorem (2.85)(a).



By theorem (2.60)(b), there exist a Z-2-arrow (h_1, k_1), normal S-2-arrows (h_2, k_2), (c, l) and a denominator s_1 in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdot \arrow[r, "f_1\dot{g}_2'"] \arrow[d, equal] & \cdot & \cdot \arrow[l, "\dot{b}_2 i_1'"'] \arrow[d, equal] \\
\cdot \arrow[r, dotted, "h_1"] \arrow[d, equal] & \cdot \arrow[u, dotted, "\approx s_1"] \arrow[d, dotted, "c"] & \cdot \arrow[l, dotted, "k_1"'] \arrow[d, equal] \\
\cdot \arrow[r, dotted, "h_2"] \arrow[d, equal] & \cdot & \cdot \arrow[l, dotted, "k_2"'] \arrow[d, equal] \\
\cdot \arrow[r, "g_1 f_2'"] & \cdot \arrow[u, dotted, "l"] & \cdot \arrow[l, "i_2 j_1'"']
\end{tikzcd}$$

By the Z-replacement lemma (2.64), there exist a Z-2-arrow (h_3, k_3), a normal S-2-arrow $(\dot{g}_2'', \dot{b}_2')$ and a morphism s_3 in $\mathcal{C}$ such that the following diagram commutes.



Since $\mathcal{C}$ is S-semisaturated, an application of the flipping lemma for S-2-arrows (2.25) to the commutative diagram

$$\begin{tikzcd}
\cdot \arrow[r, "h_3"] \arrow[d, equal] & \cdot & \cdot \arrow[l, "k_3"'] \\
\cdot \arrow[r, "h_1"] \arrow[d, equal] & \cdot \arrow[u, "\dot{b}_2'"] \arrow[d, "c"] & \cdot \arrow[l, "k_1"'] \arrow[u, "\dot{b}_2"'] \arrow[d, equal] \\
\cdot \arrow[r, "h_2"] & \cdot & \cdot \arrow[l, "k_2"']
\end{tikzcd}$$

yields an S-2-arrow (h_4, a) and a normal S-2-arrow $(e, \dot{b}_2'')$ such that the diagram

$$\begin{tikzcd}
{} \arrow[r, "h_3"] \arrow[d, equal] & {} \arrow[d, dotted, "e"] & {} \arrow[l, "k_3"'] \arrow[d, equal] \\
{} \arrow[r, dotted, "h_4"] & {} & {} \arrow[l, dotted, "a" "\approx"'] \\
{} \arrow[r, "h_2"] \arrow[u, equal] & {} \arrow[u, dotted, "\dot{b}_2''"'] & {} \arrow[l, "k_2"'] \arrow[u, "\dot{b}_2"']
\end{tikzcd}$$

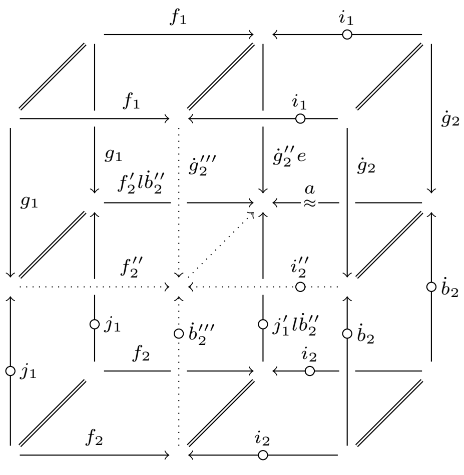
commutes. Now we have

$$\begin{aligned}
&f_1\dot{g}_2''e = h_3e = h_4 = h_2\dot{b}_2'' = g_1f_2'l\dot{b}_2'',\\
&i_1\dot{g}_2''e = \dot{g}_2k_3e = \dot{g}_2a,\\
&f_2j_1'l\dot{b}_2'' = j_1f_2'l\dot{b}_2'',\\
&i_2j_1'l\dot{b}_2'' = k_2\dot{b}_2'' = \dot{b}_2a.
\end{aligned}$$

Moreover, $j_1'l\dot{b}_2''$ is an S-denominator by multiplicativity.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, "\dot{g}_2''e"] & {} \arrow[l, "i_1"'] \arrow[d, "\dot{g}_2"] \\
{} \arrow[r, "f_2'l\dot{b}_2''"] & {} & {} \arrow[l, "a" "\approx"'] \\
{} \arrow[r, "f_2"] \arrow[u, "j_1"] & {} \arrow[u, "j_1'l\dot{b}_2''"'] & {} \arrow[l, "i_2"'] \arrow[u, "\dot{b}_2"']
\end{tikzcd}$$

By the Z-replacement lemma (2.64), there exist a Z-2-arrow (f_2'', i_2'') and a normal S-2-arrow $(\dot{g}_2''', \dot{b}_2''')$ such that the following diagram commutes.



An application of the flipping lemma for S-2-arrows (2.25) to the commutative diagram

$$\begin{tikzcd}
{} \arrow[d, "\dot{g}_2'''"] & {} \arrow[l, "i_1"'] \arrow[r, equal] \arrow[d, "\dot{g}_2"] & {} \arrow[d, "g_2"] \\
{} & {} \arrow[l, "i_2''"'] \arrow[r, "t_2"] & {} \\
{} \arrow[u, "\dot{b}_2'''"] & {} \arrow[l, "i_2"'] \arrow[r, equal] \arrow[u, "\dot{b}_2"] & {} \arrow[u, "b_2" "\approx"']
\end{tikzcd}$$

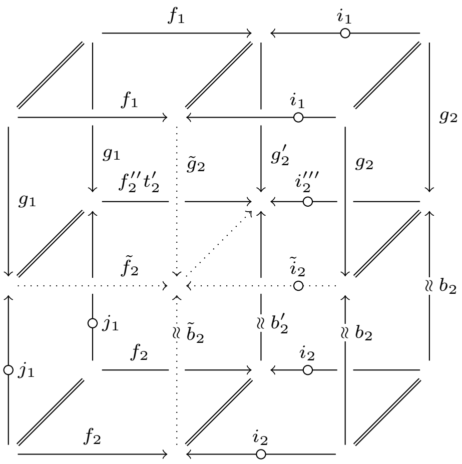
yields an S-2-arrow (g_2', b_2') and a normal S-2-arrow (t_2', i_2''') such that the diagram

$$\begin{tikzcd}
{} \arrow[r, equal] \arrow[d, "\dot{g}_2'''"] & {} \arrow[d, dotted, "g_2'"] & {} \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
{} \arrow[r, dotted, "t_2'"] & {} & {} \arrow[l, dotted, "i_2'''"'] \\
{} \arrow[u, "\dot{b}_2'''"'] \arrow[r, equal] & {} \arrow[u, dotted, "\wr\, b_2'"'] & {} \arrow[l, "i_2"'] \arrow[u, "\wr\, b_2"']
\end{tikzcd}$$

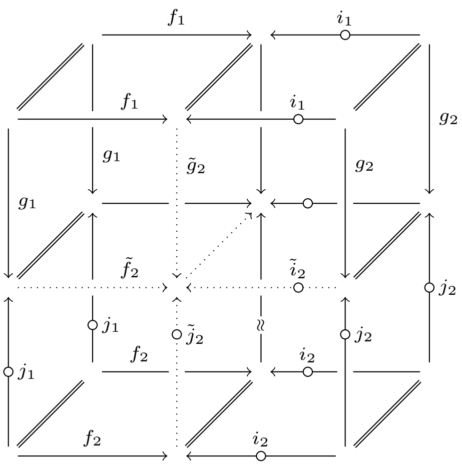
commutes. So we have

$$f_1 g_2' = f_1 \dot{g}_2''' t_2' = g_1 f_2'' t_2',$$
$$f_2 b_2' = f_2 \dot{b}_2''' t_2' = j_1 f_2'' t_2'.$$

Finally, the assertion follows by another application of the Z-replacement lemma (2.64).



(d) This follows from (c) and the relative Z-replacement axiom for Z-2-arrows.



(e) We suppose that $\operatorname{loc}(f_1)\operatorname{loc}(i_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(b_2)^{-1} = \operatorname{loc}(g_1)\operatorname{loc}(b_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(i_2)^{-1}$ in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Moreover, we choose a Z-2-arrow $(\dot{g}_1, \dot{b}_1)$ and a denominator t_1 in $\mathcal{C}$ with $(g_1, b_1) = (\dot{g}_1 t_1, \dot{b}_1 t_1)$. By (c), there exist a Z-2-arrow (f_2', i_2') and an S-2-arrow (g_2', b_2') in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[r, equal] \arrow[d, "g_1"] & {} \arrow[r, "f_1"] \arrow[d, "\dot{g}_1"] & {} \arrow[d, dotted, "g_2'"] & {} \arrow[l, "i_1"'] \arrow[d, "g_2"] \\
{} & {} \arrow[l, "t_1"', "\approx"] \arrow[r, dotted, "f_2'"] & {} & {} \arrow[l, dotted, "i_2'"'] \\
{} \arrow[u, "\wr\, b_1"'] \arrow[r, equal] & {} \arrow[u, "\dot{b}_1"'] \arrow[r, "f_2"'] & {} \arrow[u, dotted, "\wr\, b_2'"'] & {} \arrow[l, "i_2"] \arrow[u, "\wr\, b_2"']
\end{tikzcd}$$

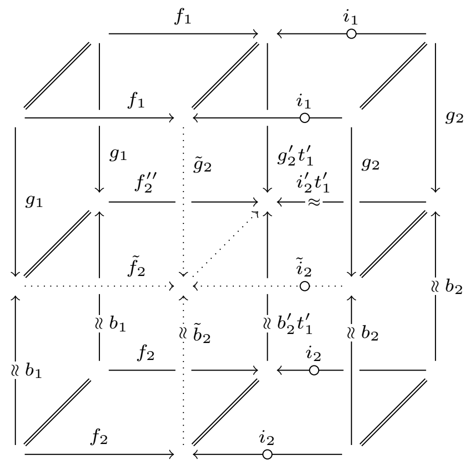
By the Z-numerator axiom, there exists an S-2-arrow (f_2'', t_1') in $\mathcal{C}$ with $f_2' t_1' = t_1 f_2''$.

$$\begin{tikzcd}
\cdot \arrow[r, dotted, "f_2''"] & \cdot & \\
\cdot \arrow[u, "t_1", "\approx"'] \arrow[r, "f_2'"'] & \cdot \arrow[u, dotted, "\approx"', "t_1'"] & \cdot \arrow[l, "i_2'"']
\end{tikzcd}$$

So we get

$$\begin{aligned}
&f_1 g_2' t_1' = \dot{g}_1 f_2' t_1' = \dot{g}_1 t_1 f_2'' = g_1 f_2'', \\
&i_1 g_2' t_1' = g_2 i_2' t_1', \\
&f_2 b_2' t_1' = \dot{b}_1 f_2' t_1' = \dot{b}_1 t_1 f_2'' = b_1 f_2'', \\
&i_2 b_2' t_1' = b_2 i_2' t_1'.
\end{aligned}$$

The assertion follows by an application of the Z-replacement lemma (2.64).



(f) This follows from theorem (2.85)(b) and remark (2.91). □

(2.94) Corollary. We suppose given a Z-prefractionable category $\mathcal{C}$.

(a) We have

$$\operatorname{Mor} \mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) = \{\operatorname{loc}(f)\operatorname{loc}(a)^{-1} \mid (f, a) \text{ is an S-2-arrow in } \mathcal{C}\}.$$

(b) We suppose given S-2-arrows (f, a), (f', a') in $\mathcal{C}$. The following conditions are equivalent.

(i) We have

$$\operatorname{loc}(f)\operatorname{loc}(a)^{-1} = \operatorname{loc}(f')\operatorname{loc}(a')^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

(ii) For every Z-2-arrow $(\dot{f}, \dot{a})$ and every morphism s in $\mathcal{C}$ with $(f, a) = (\dot{f}s, \dot{a}s)$ there exist an S-2-arrow $(\tilde{f}', \tilde{a}')$, a denominator c and an S-denominator j in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, equal] & \cdot & \cdot \arrow[l, "a", "\approx"'] \arrow[d, equal] \\
\cdot \arrow[r, "\dot{f}"] \arrow[d, equal] & \cdot \arrow[u, "s"'] \arrow[d, dotted, "\approx"', "c"] & \cdot \arrow[l, "\dot{a}"'] \arrow[d, equal] \\
\cdot \arrow[r, dotted, "\tilde{f}'"] \arrow[d, equal] & \cdot & \cdot \arrow[l, dotted, "\tilde{a}'", "\approx"'] \arrow[d, equal] \\
\cdot \arrow[r, "f'"] & \cdot \arrow[u, dotted, "j"'] & \cdot \arrow[l, "a'", "\approx"']
\end{tikzcd}$$

(iii) There exist a Z-2-arrow $(\dot{f}, \dot{a})$, an S-2-arrow $(\tilde{f}', \tilde{a}')$, denominators c, s and an S-denominator j in $\mathcal{C}$ such that the following diagram commutes.

$$
\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a"', "\approx" description] \arrow[d, equal] \\
\bullet \arrow[r, dotted, "\dot{f}"] \arrow[d, equal] & \bullet \arrow[u, dotted, "\approx" description, "s"'] \arrow[d, dotted, "\approx" description, "c"'] & \bullet \arrow[l, dotted, "\dot{a}"', "\circ" description] \arrow[d, equal] \\
\bullet \arrow[r, dotted, "\tilde{f}'"] \arrow[d, equal] & \bullet & \bullet \arrow[l, dotted, "\tilde{a}'"', "\approx" description] \arrow[d, equal] \\
\bullet \arrow[r, "f'"] & \bullet \arrow[u, dotted, "\circ" description, "j"'] & \bullet \arrow[l, "a'"', "\approx" description]
\end{tikzcd}
$$

(c) We suppose given S-2-arrows (f_1, a_1), (f_2, a_2), (g_1, b_1) and a normal S-2-arrow (g_2, j_2) in $\mathcal{C}$. The following conditions are equivalent.

(i) We have

$$\operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} = \operatorname{loc}(g_1)\operatorname{loc}(b_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

(ii) For every Z-2-arrow $(\dot{f}_1, \dot{a}_1)$, every normal S-2-arrow $(\dot{g}_1, \dot{b}_1)$ and all morphisms s_1, t_1 in $\mathcal{C}$ with $(f_1, a_1) = (\dot{f}_1 s_1, \dot{a}_1 s_1)$, $(g_1, b_1) = (\dot{g}_1 t_1, \dot{b}_1 t_1)$ there exist an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$ and a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)$ in $\mathcal{C}$ such that the following diagram commutes.

$$
\begin{tikzcd}
 & \bullet \arrow[r, "f_1"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a_1"', "\approx" description] \arrow[d, equal] \\
\bullet \arrow[r, equal] \arrow[d, "g_1"'] & \bullet \arrow[r, "\dot{f}_1"] \arrow[d, "\dot{g}_1"'] & \bullet \arrow[u, "s_1"'] \arrow[d, dotted, "\tilde{g}_2"'] & \bullet \arrow[l, "\dot{a}_1"', "\circ" description] \arrow[d, "g_2"] \\
\bullet & \bullet \arrow[l, "t_1"'] \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{a}_2"', "\approx" description] \\
\bullet \arrow[r, equal] \arrow[u, "\approx" description, "b_1"] & \bullet \arrow[u, "\circ" description, "\dot{b}_1"'] \arrow[r, "f_2"] & \bullet \arrow[u, dotted, "\circ" description, "\tilde{j}_2"'] & \bullet \arrow[l, "a_2"', "\approx" description] \arrow[u, "\circ" description, "j_2"']
\end{tikzcd}
$$

(iii) There exist Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{g}_1, \dot{b}_1)$, an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$, a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)$ and denominators s_1, t_1 in $\mathcal{C}$ such that the following diagram commutes.

$$
\begin{tikzcd}
 & \bullet \arrow[r, "f_1"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a_1"', "\approx" description] \arrow[d, equal] \\
\bullet \arrow[r, equal] \arrow[d, "g_1"'] & \bullet \arrow[r, dotted, "\dot{f}_1"] \arrow[d, dotted, "\dot{g}_1"'] & \bullet \arrow[u, dotted, "\approx" description, "s_1"'] \arrow[d, dotted, "\tilde{g}_2"'] & \bullet \arrow[l, dotted, "\dot{a}_1"', "\circ" description] \arrow[d, "g_2"] \\
\bullet & \bullet \arrow[l, dotted, "t_1"', "\approx" description] \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{a}_2"', "\approx" description] \\
\bullet \arrow[r, equal] \arrow[u, "\approx" description, "b_1"] & \bullet \arrow[u, dotted, "\circ" description, "\dot{b}_1"'] \arrow[r, "f_2"] & \bullet \arrow[u, dotted, "\circ" description, "\tilde{j}_2"'] & \bullet \arrow[l, "a_2"', "\approx" description] \arrow[u, "\circ" description, "j_2"']
\end{tikzcd}
$$

(d) Given a category $\mathcal{D}$ and a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible in $\mathcal{D}$ for every denominator d in $\mathcal{C}$, the unique functor $\hat{F} \colon \mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) \to \mathcal{D}$ with $F = \hat{F} \circ \operatorname{loc}$ is given on the objects by

$$\hat{F}X = FX$$

for $X \in \operatorname{Ob}\mathcal{C}$ and on the morphisms by

$$\hat{F}(\operatorname{loc}(f)\operatorname{loc}(a)^{-1}) = (Ff)(Fa)^{-1}$$

for every S-2-arrow (f, a) in $\mathcal{C}$.

Proof.

(a) This follows from theorem (2.93)(a).

(b) This follows from remark (2.91), theorem (2.60)(a) and S-semisaturatedness.

(c) First, we suppose that condition (i) holds, that is, we suppose that

$$\operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} = \operatorname{loc}(g_1)\operatorname{loc}(b_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, and we suppose given a Z-2-arrow $(\dot{f}_1, \dot{a}_1)$, a normal S-2-arrow $(\dot{g}_1, \dot{b}_1)$ and morphisms s_1, t_1 in $\mathcal{C}$ with $(f_1, a_1) = (\dot{f}_1 s_1, \dot{a}_1 s_1)$, $(g_1, b_1) = (\dot{g}_1 t_1, \dot{b}_1 t_1)$. Moreover, we choose a Z-2-arrow $(\dot{f}_2, \dot{a}_2)$ and a morphism s_2 in $\mathcal{C}$ with $(f_2, a_2) = (\dot{f}_2 s_2, \dot{a}_2 s_2)$. By remark (2.17), we have

$$\begin{aligned}\operatorname{loc}(\dot{f}_1)\operatorname{loc}(\dot{a}_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} &= \operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} \\ &= \operatorname{loc}(g_1)\operatorname{loc}(b_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1} \\ &= \operatorname{loc}(\dot{g}_1)\operatorname{loc}(\dot{b}_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1}.\end{aligned}$$

As (g_2, j_2) is a normal S-2-arrow, by theorem (2.93)(c) there exist a Z-2-arrow $(\bar{f}_2, \bar{a}_2)$ and a normal S-2-arrow $(\bar{g}_2, \bar{j}_2)$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a_1"', "\approx"] \arrow[d, equal] \\
\bullet \arrow[r, "\dot{f}_1"] \arrow[d, "\dot{g}_1"'] & \bullet \arrow[u, "s_1"'] \arrow[d, dotted, "\bar{g}_2"] & \bullet \arrow[l, "\dot{a}_1"', "\circ"] \arrow[d, "g_2"] \\
\bullet \arrow[r, dotted, "\bar{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\bar{a}_2"', "\circ"] \\
\bullet \arrow[r, "\dot{f}_2"] \arrow[u, "\circ", "\dot{b}_1"'] \arrow[d, equal] & \bullet \arrow[u, dotted, "\circ", "\bar{j}_2"'] \arrow[d, "s_2"'] & \bullet \arrow[l, "\dot{a}_2"', "\circ"] \arrow[u, "\circ", "j_2"'] \arrow[d, equal] \\
\bullet \arrow[r, "f_2"] & \bullet & \bullet \arrow[l, "a_2"', "\approx"]
\end{tikzcd}$$

Applying the flipping lemma for S-2-arrows (2.25) to the rectangle

$$\begin{tikzcd}
\bullet \arrow[r, "\bar{f}_2"] & \bullet & \bullet \arrow[l, "\bar{a}_2"', "\circ"] \\
\bullet \arrow[r, "\dot{f}_2"] \arrow[u, "\circ", "\dot{b}_1"'] \arrow[d, equal] & \bullet \arrow[u, "\circ", "\bar{j}_2"'] \arrow[d, "s_2"'] & \bullet \arrow[l, "\dot{a}_2"', "\circ"] \arrow[u, "\circ", "j_2"'] \arrow[d, equal] \\
\bullet \arrow[r, "f_2"] & \bullet & \bullet \arrow[l, "a_2"', "\approx"]
\end{tikzcd}$$

and composing yields the asserted diagram of condition (ii).

$$\begin{tikzcd}
 & \bullet \arrow[r, "f_1"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a_1"', "\approx"] \arrow[d, equal] \\
\bullet \arrow[r, equal] \arrow[d, "g_1"'] & \bullet \arrow[r, "\dot{f}_1"] \arrow[d, "\dot{g}_1"'] & \bullet \arrow[u, "s_1"'] \arrow[d, dotted, "\tilde{g}_2"] & \bullet \arrow[l, "\dot{a}_1"', "\circ"] \arrow[d, "g_2"] \\
\bullet & \bullet \arrow[l, "t_1"'] \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{a}_2"', "\approx"] \\
\bullet \arrow[r, equal] \arrow[u, "\approx", "b_1"'] & \bullet \arrow[r, "f_2"] \arrow[u, "\circ", "\dot{b}_1"'] & \bullet \arrow[u, dotted, "\circ", "\tilde{j}_2"'] & \bullet \arrow[l, "a_2"', "\approx"] \arrow[u, "\circ", "j_2"']
\end{tikzcd}$$

Condition (ii) and the Z-replacement axiom imply condition (iii).

Finally, if condition (iii) holds, then we have

$$\begin{aligned}
\operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} &= \operatorname{loc}(\dot{f}_1)\operatorname{loc}(\dot{a}_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} \\
&= \operatorname{loc}(\dot{f}_1)\operatorname{loc}(\tilde{g}_2)\operatorname{loc}(\tilde{a}_2)^{-1}\operatorname{loc}(j_2)^{-1} \\
&= \operatorname{loc}(\dot{g}_1)\operatorname{loc}(\tilde{f}_2)\operatorname{loc}(\tilde{j}_2)^{-1}\operatorname{loc}(a_2)^{-1} \\
&= \operatorname{loc}(\dot{g}_1)\operatorname{loc}(\dot{b}_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1} \\
&= \operatorname{loc}(g_1)\operatorname{loc}(b_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1},
\end{aligned}$$

that is, condition (i) holds.

Altogether, the three conditions (i), (ii) and (iii) are equivalent.

(d) This follows from theorem (2.85)(b) and remark (2.91). □

(2.95) Corollary. We suppose given a semisaturated Z-prefractionable category $\mathcal{C}$.

(a) We suppose given S-2-arrows (f_1, a_1), (f_2, a_2), (f, a) in $\mathcal{C}$ such that

$$\operatorname{loc}(f)\operatorname{loc}(a)^{-1} = \operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. If two out of the morphisms f_1, f_2, f are denominators in $\mathcal{C}$, then so is the third.

(b) We suppose given S-2-arrows (f, a), (f', a') in $\mathcal{C}$ such that

$$(\operatorname{loc}(f)\operatorname{loc}(a)^{-1})^{-1} = \operatorname{loc}(f')\operatorname{loc}(a')^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Then f is a denominator in $\mathcal{C}$ if and only if f' is a denominator in $\mathcal{C}$.

Proof.

(a) By corollary (2.94)(c), there exist S-2-arrows $(\tilde{f}_1, \tilde{a}_1)$, $(\tilde{f}_2, \tilde{a}_2)$, $(\tilde{f}, \tilde{a})$, $(\tilde{e}, \tilde{b})$ and denominators s, s_1 in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & \cdot \arrow[r, "f"] \arrow[d, equal] & \cdot & \cdot \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[r, equal] \arrow[d, "f_1"] & \cdot \arrow[r, dotted, "\tilde{f}"] \arrow[d, dotted, "\tilde{f}_1"] & \cdot \arrow[u, dotted, "s", "\approx"'] \arrow[d, dotted, "\tilde{e}"] & \cdot \arrow[l, dotted, "\tilde{a}"', "\approx"] \arrow[d, equal] \\
\cdot & \cdot \arrow[l, dotted, "s_1"', "\approx"] \arrow[r, dotted, "\tilde{f}_2"] & \cdot & \cdot \arrow[l, dotted, "\tilde{a}_2"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[u, "a_1", "\approx"'] \arrow[r, equal] & \cdot \arrow[u, dotted, "\tilde{a}_1", "\approx"'] \arrow[r, "f_2"] & \cdot \arrow[u, dotted, "\tilde{b}", "\approx"'] & \cdot \arrow[l, "a_2"', "\approx"]
\end{tikzcd}$$

The semisaturatedness of $\mathcal{C}$ implies that f_1 resp. f_2 resp. f is a denominator if and only if $\tilde{f}_1$ resp. $\tilde{f}_2$ resp. $\tilde{f}$ is a denominator. So, if two out of the morphisms f_1, f_2, f are denominators, then two out of the morphisms $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}$ are denominators. But as $\tilde{f}_1 \tilde{f}_2 = \tilde{f}\tilde{e}$ and $\tilde{e}$ is a denominator by semisaturatedness, if two out of the morphisms $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}$ are denominators, then so is the third.

(b) This follows from (a) and

$$\operatorname{loc}(f)\operatorname{loc}(a)^{-1}\operatorname{loc}(f')\operatorname{loc}(a')^{-1} = \operatorname{loc}(1)\operatorname{loc}(1)^{-1}.$$ □

(2.96) Remark. We suppose given a Z-prefractionable category $\mathcal{C}$. For all morphisms φ_1 and φ_2 in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ with $\operatorname{Target}\varphi_1 = \operatorname{Target}\varphi_2$ there exist normal S-2-arrows (f_1, i) and (f_2, i) in $\mathcal{C}$ with

$$\begin{aligned}
\varphi_1 &= \operatorname{loc}(f_1)\operatorname{loc}(i)^{-1}, \\
\varphi_2 &= \operatorname{loc}(f_2)\operatorname{loc}(i)^{-1}.
\end{aligned}$$

Proof. By corollary (2.94)(a), there exist S-2-arrows $(\tilde{f}_1, a_1)$, $(\tilde{f}_2, a_2)$ in $\mathcal{C}$ with $\varphi_1 = \operatorname{loc}(\tilde{f}_1)\operatorname{loc}(a_1)^{-1}$ and $\varphi_2 = \operatorname{loc}(\tilde{f}_2)\operatorname{loc}(a_2)^{-1}$. Moreover, by corollary (2.49), there exist normal S-2-arrows (f_1, i), (f_2, i) in $\mathcal{C}$ with $(\tilde{f}_1, a_1) \equiv_{\mathrm{S}} (f_1, i)$ and $(\tilde{f}_2, a_2) \equiv_{\mathrm{S}} (f_2, i)$. Thus remark (2.17) implies that

$$\begin{aligned}
\varphi_1 &= \operatorname{loc}(\tilde{f}_1)\operatorname{loc}(a_1)^{-1} = \operatorname{loc}(f_1)\operatorname{loc}(i)^{-1}, \\
\varphi_2 &= \operatorname{loc}(\tilde{f}_2)\operatorname{loc}(a_2)^{-1} = \operatorname{loc}(f_2)\operatorname{loc}(i)^{-1}.
\end{aligned}$$ □

A saturatedness criterion

Our next aim is to give a sufficient (and necessary) criterion for saturatedness.

(2.97) Proposition. We suppose given a Z-prefractionable category $\mathcal{C}$ and a morphism f in $\mathcal{C}$.

(a) We suppose that $\mathcal{C}$ fulfills the Z-replacement axiom for denominators. The following two conditions are equivalent.

(i) The morphism $\operatorname{loc}(f)$ is a coretraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

(ii) There exists a morphism h in $\mathcal{C}$ such that fh is a denominator in $\mathcal{C}$.

If $\mathcal{C}$ is T-semisaturated or fulfills the Z-expansion axiom, then these conditions are also equivalent to the following condition.

(iii) There exists a Z-2-arrow (h, k) in $\mathcal{C}$ such that fh is a denominator in $\mathcal{C}$.

(b) The following conditions are equivalent.

(i) The morphism $\operatorname{loc}(f)$ is a retraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

(ii) There exist a morphism $\tilde f$ and a Z-2-arrow (g, j) in $\mathcal{C}$ with $fg\tilde f = j\tilde f$ and such that $g\tilde f$ is an S-denominator in $\mathcal{C}$.

(iii) There exist morphisms $\tilde f$, g and denominators a, b in $\mathcal{C}$ with $fb = a\tilde f$ and such that $g\tilde f$ is a denominator in $\mathcal{C}$.

(c) The following three conditions are equivalent.

(i) The morphism $\operatorname{loc}(f)$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

(ii) There exist morphisms $\tilde f$, $\tilde g$ and Z-2-arrows (g, j), (g', j') in $\mathcal{C}$ with $fg\tilde f = j\tilde f$, $gg'\tilde g = j'\tilde g$, and such that $g\tilde f$ and $g'\tilde g$ are S-denominators in $\mathcal{C}$.

(iii) There exist morphisms $\tilde f$, g, $\tilde g$, g' and denominators a, b, a', b' in $\mathcal{C}$ with $fb = a\tilde f$, $gb' = a'\tilde g$, and such that $g\tilde f$ and $g'\tilde g$ are denominators in $\mathcal{C}$.

If $\mathcal{C}$ fulfills the Z-replacement axiom for denominators, then these conditions are also equivalent to each of the following three conditions.

(iv) There exist morphisms h and h' in $\mathcal{C}$ such that fh and hh' are denominators in $\mathcal{C}$.

(v) There exist morphisms $\tilde f$, h and a Z-2-arrow (g, j) in $\mathcal{C}$ with $fg\tilde f = j\tilde f$ and such that $g\tilde f$ is an S-denominator and fh is a denominator in $\mathcal{C}$.

(vi) There exist morphisms $\tilde f$, g, h and denominators a, b in $\mathcal{C}$ with $fb = a\tilde f$ and such that $g\tilde f$ and fh are denominators in $\mathcal{C}$.

If $\mathcal{C}$ is T-semisaturated or fulfills the Z-replacement axiom for denominators and the Z-expansion axiom, then these conditions are also equivalent to each of the following two conditions.

(vii) There exist Z-2-arrows (h, k) and (h', k') in $\mathcal{C}$ such that fh and hh' are denominators in $\mathcal{C}$.

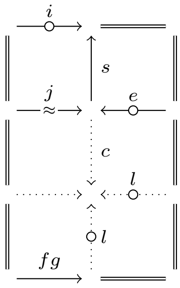
(viii) There exist a morphism $\tilde f$ and Z-2-arrows (g, j), (h, k) in $\mathcal{C}$ with $fg\tilde f = j\tilde f$ and such that $g\tilde f$ is an S-denominator and fh is a denominator in $\mathcal{C}$.

Proof.

(a) Condition (ii) implies condition (i) by remark (1.21)(a). Moreover, condition (iii) always implies condition (ii). To show the asserted equivalence, we show that condition (i) implies condition (ii), as well as condition (iii) under one of the additional assumptions.

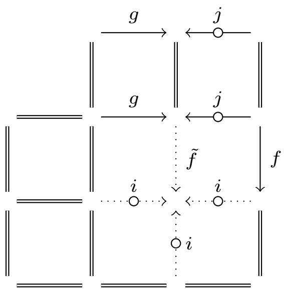
So we suppose that condition (i) holds, that is, we suppose that $\operatorname{loc}(f)$ is a coretraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. By theorem (2.93)(a), there exist a Z-2-arrow (g, i) in $\mathcal{C}$ such that $\operatorname{loc}(f)\operatorname{loc}(g)\operatorname{loc}(i)^{-1} = 1$. We obtain $\operatorname{loc}(i) = \operatorname{loc}(f)\operatorname{loc}(g) = \operatorname{loc}(fg)$. By the Z-replacement axiom for denominators, there exist a Z-2-arrow

(j, e) with denominator j and a morphism s in $\mathcal{C}$ with $(i, 1) = (js, es)$. Now corollary (2.94)(b) yields a normal S-2-arrows (c, l) in $\mathcal{C}$ such that $(jc, ec) = (fgl, l)$.



By S-semisaturatedness, c is a denominator in $\mathcal{C}$. But then $h := gl$ yields $fh = fgl = jc$, and so fh is a denominator in $\mathcal{C}$ by multiplicativity. Hence condition (ii) holds. If $\mathcal{C}$ fulfills the Z-expansion axiom, then setting $k := il$ yields a Z-2-arrow $(h, k) = (gl, il)$, so even condition (iii) holds. Finally, if $\mathcal{C}$ is T-semisaturated, then fg is a denominator in $\mathcal{C}$, so condition (iii) is also valid in this case.

(b) First, we suppose that condition (i) holds, that is, we suppose that $\operatorname{loc}(f)$ is a retraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. By theorem (2.93)(a), there exist a Z-2-arrow (g, j) in $\mathcal{C}$ such that $\operatorname{loc}(g) \operatorname{loc}(j)^{-1} \operatorname{loc}(f) = 1$. Corollary (2.94)(c) yields a normal S-2-arrow $(\tilde{f}, i)$ in $\mathcal{C}$ such that $g\tilde{f} = i$ and $j\tilde{f} = fi$.



We obtain $fg\tilde{f} = fi = j\tilde{f}$, and $g\tilde{f} = i$ is an S-denominator in $\mathcal{C}$. Thus condition (ii) holds.

If condition (ii) holds, then in particular condition (iii) holds.

Finally, we suppose that condition (iii) holds, that is, we suppose that there exist morphisms $\tilde{f}$, g and denominators a, b in $\mathcal{C}$ with $fb = a\tilde{f}$ and such that $g\tilde{f}$ is a denominator in $\mathcal{C}$. Then $\operatorname{loc}(\tilde{f})$ is a retraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ by remark (1.21)(b), and therefore $\operatorname{loc}(f) = \operatorname{loc}(a) \operatorname{loc}(\tilde{f}) \operatorname{loc}(b)^{-1}$ is also a retraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Hence condition (i) holds.

(c) First, we show that condition (i), condition (ii) and condition (iii) are equivalent.

We suppose that condition (i) holds, that is, we suppose that $\operatorname{loc}(f)$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Then $\operatorname{loc}(f)$ is in particular a retraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, and so (b) implies that there exist a morphism $\tilde{f}$ and a Z-2-arrow (g, j) in $\mathcal{C}$ with $fg\tilde{f} = j\tilde{f}$ and such that $g\tilde{f}$ is an S-denominator in $\mathcal{C}$. We obtain

$$\operatorname{loc}(f) \operatorname{loc}(g\tilde{f}) = \operatorname{loc}(fg\tilde{f}) = \operatorname{loc}(j\tilde{f}) = \operatorname{loc}(j) \operatorname{loc}(\tilde{f}).$$

As $\operatorname{loc}(f)$, $\operatorname{loc}(g\tilde{f})$ and $\operatorname{loc}(j)$ are isomorphisms in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, it follows that $\operatorname{loc}(\tilde{f})$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. But then $\operatorname{loc}(g) = \operatorname{loc}(g\tilde{f}) \operatorname{loc}(\tilde{f})^{-1}$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, and therefore in particular a retraction. By (b), there exist a morphism $\tilde{g}$ and a Z-2-arrow (g', j') in $\mathcal{C}$ with $gg'\tilde{g} = j'\tilde{g}$ and such that $g'\tilde{g}$ is an S-denominator in $\mathcal{C}$. Thus condition (ii) holds.

If condition (ii) holds, then in particular condition (iii) holds.

We suppose that condition (iii) holds, that is, we suppose that there exist morphisms $\tilde{f}$, g, $\tilde{g}$, g' and denominators a, b, a', b' in $\mathcal{C}$ with $fb = a\tilde{f}$, $gb' = a'\tilde{g}$, and such that $g\tilde{f}$ and $g'\tilde{g}$ are denominators in $\mathcal{C}$. Then $\operatorname{loc}(g)$ is a retraction by (b). Moreover, $\operatorname{loc}(g)$ is a coretraction by remark (1.21)(a), whence an isomorphism. But then $\operatorname{loc}(\tilde{f}) = \operatorname{loc}(g)^{-1} \operatorname{loc}(g\tilde{f})$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, and therefore also $\operatorname{loc}(f) = \operatorname{loc}(a) \operatorname{loc}(\tilde{f}) \operatorname{loc}(b)^{-1}$. Thus condition (i) holds.

Second, we show that condition (i) is equivalent to condition (iv), to condition (v), and to condition (vi). So from now on, we suppose that $\mathcal{C}$ fulfills the Z-replacement axiom for denominators.

As condition (i) means that $\mathrm{loc}(f)$ is a coretraction and a retraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, the equivalence of condition (i), condition (v) and condition (vi) follows from (a) and (b). Moreover, condition (iv) implies condition (i) by corollary (1.22)(a).

It remains to show that condition (i) implies condition (iv). So we suppose that condition (i) holds, that is, we suppose that $\mathrm{loc}(f)$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Then $\mathrm{loc}(f)$ is in particular a coretraction in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, and so (a) implies that there exists a morphism h in $\mathcal{C}$ such that fh is a denominator in $\mathcal{C}$. But then $\mathrm{loc}(h) = \mathrm{loc}(f)^{-1} \mathrm{loc}(fh)$ is also an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$, and therefore in particular a coretraction. By (a), there exist a morphism h' in $\mathcal{C}$ such that hh' is a denominator in $\mathcal{C}$. Thus condition (iv) holds.

Third, we suppose, in addition, that $\mathcal{C}$ is T-semisaturated (in this case, $\mathcal{C}$ automatically fulfills the Z-replacement axiom for denominators) or that $\mathcal{C}$ fulfills the Z-expansion axiom. Then condition (iv) is equivalent to condition (vii) by (a), and condition (i) is equivalent to condition (viii) by (a) and (b). □

(2.98) Corollary (cf. [11, sec. 36.4], [36, prop. 5.10])**.** A Z-prefractionable category is saturated if and only if it is weakly saturated.

Proof. We suppose given a Z-prefractionable category $\mathcal{C}$. Since saturatedness always implies weak saturatedness, see proposition (1.43)(a), it suffices to show that if $\mathcal{C}$ is weakly saturated, then it is already saturated. So we suppose that $\mathcal{C}$ is weakly saturated and we suppose given a morphism f in $\mathcal{C}$ such that $\mathrm{loc}(f)$ is invertible in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Then $\mathcal{C}$ is semisaturated by proposition (1.43)(b), and so it fulfills the Z-replacement axiom for denominators. Hence proposition (2.97)(c) implies that there exist morphisms h and h' in $\mathcal{C}$ such that fh and hh' are denominators in $\mathcal{C}$. But then the 2 out of 6 axiom implies that f is also a denominator in $\mathcal{C}$. Thus $\mathcal{C}$ is saturated. □

(2.99) Corollary. We suppose given a weakly saturated Z-prefractionable category $\mathcal{C}$. The set of isomorphisms in the S-Ore localisation of $\mathcal{C}$ is given by

$$\begin{aligned}\mathrm{Iso}\,\mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) &= \{\mathrm{loc}(f)\,\mathrm{loc}(a)^{-1} \mid (f,a) \text{ is an S-2-arrow in } \mathcal{C} \text{ with denominator } f\} \\ &= \{\mathrm{loc}(f)\,\mathrm{loc}(i)^{-1} \mid (f,i) \text{ is a Z-2-arrow in } \mathcal{C} \text{ with denominator } f\}.\end{aligned}$$

Proof. Given an S-2-arrow (f,a) in $\mathcal{C}$ with denominator f, then $\mathrm{loc}(f)$ and $\mathrm{loc}(a)$ are isomorphisms in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ and hence $\mathrm{loc}(f)\,\mathrm{loc}(a)^{-1}$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Conversely, we suppose given an isomorphism φ in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. We choose an S-2-arrow (f,a) in $\mathcal{C}$ with $\varphi = \mathrm{loc}(f)\,\mathrm{loc}(a)^{-1}$. Since a is a denominator in $\mathcal{C}$, the morphism $\mathrm{loc}(a)$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ and thus $\mathrm{loc}(f) = \varphi\,\mathrm{loc}(a)$ is an isomorphism in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. But $\mathcal{C}$ is saturated by corollary (2.98), whence f is a denominator in $\mathcal{C}$.
Thus we have

$$\mathrm{Iso}\,\mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) = \{\mathrm{loc}(f)\,\mathrm{loc}(a)^{-1} \mid (f,a) \text{ is an S-2-arrow in } \mathcal{C} \text{ with denominator } f\}.$$

As $\mathcal{C}$ is weakly saturated, it is T-semisaturated by proposition (1.43)(b). In particular, we also have

$$\mathrm{Iso}\,\mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) = \{\mathrm{loc}(f)\,\mathrm{loc}(i)^{-1} \mid (f,i) \text{ is a Z-2-arrow in } \mathcal{C} \text{ with denominator } f\}$$

by remark (2.67). □

7 The Z-Ore localisation

Theorem (2.93) helps us to understand the morphisms of the S-Ore localisation $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$ of a Z-prefractionable category $\mathcal{C}$ if we work with Z-2-arrows. Nonetheless, the S-Ore localisation is defined using arbitrary S-2-arrows as representatives. So it seems to be a natural question whether it is possible to work *solely* with Z-2-arrows. If one is willing to get S-2-arrows as intermediate steps and to replace them by Z-2-arrows, for example in the computation of a composite, then the following proposition gives a positive answer to this. However, if $\mathcal{C}$ fulfills the additional axioms of a Z-fractionable category, see definition (2.81), we can even avoid replacements and compose Z-2-arrows directly to Z-2-arrows, cf. remark (2.103).

Construction of the Z-Ore localisation

(2.100) Proposition. We suppose given a Z-prefractionable category $\mathcal{C}$.

(a) There is a category structure on $(\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C})/{\equiv_{\mathrm{Z}}}$, where the composition and the identity morphisms are constructed as follows.

We suppose given $(f_1, i_1), (f_2, i_2) \in \operatorname{Arr} \mathrm{AG}_{\mathrm{Z}}\,\mathcal{C}$ with $\operatorname{Target}(f_1, i_1) = \operatorname{Source}(f_2, i_2)$. First, we choose a morphism f_2' and an S-denominator i_1' with $i_1 f_2' = f_2 i_1'$. Second, we choose a Z-2-arrow (f, i) and a morphism s in $\mathcal{C}$ with $f_1 f_2' = fs$ and $i_2 i_1' = is$.

$$\begin{tikzcd}
{} \arrow[rrr, "f"] \arrow[d, equal] & & & {} \arrow[d, "s"] & & \\
{} \arrow[r, "f_1"] & {} \arrow[rr, "f_2'"] & & {} & & \\
& & {} \arrow[ul, "i_1"] \arrow[rr, "f_2"] & & {} \arrow[ul, "i_1'"] & {} \arrow[uull, "i"] \arrow[d, equal] \\
& & & & & {} \arrow[ul, "i_2"]
\end{tikzcd}$$

Then

$$(f_1/i_1)(f_2/i_2) = f/i.$$

Given $X \in \operatorname{Ob}(\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C})/{\equiv_{\mathrm{Z}}}$, we choose a Z-2-arrow (e, i) and a morphism s in $\mathcal{C}$ with $(1_X, 1_X) = (es, is)$.

$$\begin{tikzcd}
{} \arrow[r, "e"] \arrow[d, equal] & {} \arrow[d, "s"] & {} \arrow[l, "i"'] \arrow[d, equal] \\
{} \arrow[r, equal] & {} & {} \arrow[l, equal]
\end{tikzcd}$$

Then

$$1_X = e/i.$$

(b) The quotient graph $(\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C})/{\equiv_{\mathrm{Z}}}$ together with the category structure from (a) becomes a localisation of $\mathcal{C}$, where the localisation functor $\operatorname{loc}\colon \mathcal{C} \to (\mathrm{AG}_{\mathrm{Z}}\,\mathcal{C})/{\equiv_{\mathrm{Z}}}$ is given on the objects by

$$\operatorname{loc}(X) = X$$

for $X \in \operatorname{Ob}\mathcal{C}$ and is constructed on the morphisms as follows. Given $f \in \operatorname{Mor}\mathcal{C}$, we choose a Z-2-arrow $(\dot{f}, \dot{e})$ and a morphism s in $\mathcal{C}$ with $(f, 1) = (\dot{f}s, \dot{e}s)$.

$$\begin{tikzcd}
{} \arrow[r, "\dot{f}"] \arrow[d, equal] & {} \arrow[d, "s"] & {} \arrow[l, "\dot{e}"'] \arrow[d, equal] \\
{} \arrow[r, "f"] & {} & {} \arrow[l, equal]
\end{tikzcd}$$

Then

$$\operatorname{loc}(f) = \dot{f}/\dot{e}.$$

For every denominator d in $\mathcal{C}$, the inverse of $\operatorname{loc}(d)$ is constructed as follows. We choose a Z-2-arrow $(\dot{e}, \dot{d})$ and a morphism s in $\mathcal{C}$ with $(1, d) = (\dot{e}s, \dot{d}s)$.

$$\begin{tikzcd}
{} \arrow[r, "\dot{e}"] \arrow[d, equal] & {} \arrow[d, "s"] & {} \arrow[l, "\dot{d}"'] \arrow[d, equal] \\
{} \arrow[r, equal] & {} & {} \arrow[l, "d"', "\approx"]
\end{tikzcd}$$

Then

$$\operatorname{loc}(d)^{-1} = \dot{e}/\dot{d}.$$

Given a category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible for every denominator d in $\mathcal{C}$, the unique functor $\hat{F}\colon (\mathrm{AG_Z}\,\mathcal{C})/{\equiv_\mathrm{Z}} \to \mathcal{D}$ with $F = \hat{F} \circ \operatorname{loc}$ is given by

$$\hat{F}(f/i) = (Ff)(Fi)^{-1}$$

for $(f, i) \in \operatorname{Arr} \mathrm{AG_Z}\,\mathcal{C}$.

Proof. By corollary (2.61), the inclusion morphism $\operatorname{inc}\colon \mathrm{AG_Z}\,\mathcal{C} \to \mathrm{AG_S}\,\mathcal{C}$ induces a graph isomorphism

$$(\mathrm{AG_Z}\,\mathcal{C})/{\equiv_\mathrm{Z}} \to (\mathrm{AG_S}\,\mathcal{C})/{\equiv_\mathrm{S}}.$$

Thus the assertion follows from theorem (2.85) by transport of structure, cf. corollary (1.14)(c). □

(2.101) Definition (Z-Ore localisation)**.** We suppose given a Z-prefractionable category $\mathcal{C}$. The *Z-Ore localisation* of $\mathcal{C}$ is defined to be the localisation $\mathsf{Ore}_\mathrm{Z}(\mathcal{C})$ of $\mathcal{C}$, whose underlying category is the quotient graph $(\mathrm{AG_Z}\,\mathcal{C})/{\equiv_\mathrm{Z}}$ together with composition and identities as in proposition (2.100)(a), and whose localisation functor is given as in proposition (2.100)(b).

So by construction of the Z-Ore localisation, we get:

(2.102) Remark. We suppose given a Z-prefractionable category $\mathcal{C}$. The unique isofunctor

$$I\colon \mathsf{Ore}_\mathrm{Z}(\mathcal{C}) \to \mathsf{Ore}_\mathrm{S}(\mathcal{C})$$

with $\operatorname{loc}^{\mathsf{Ore}_\mathrm{S}(\mathcal{C})} = I \circ \operatorname{loc}^{\mathsf{Ore}_\mathrm{Z}(\mathcal{C})}$ is given on the objects by

$$IX = X$$

for $X \in \operatorname{Ob} \mathsf{Ore}_\mathrm{Z}(\mathcal{C})$ and on the morphisms by

$$I(f/i) = f/i$$

for $(f, i) \in \operatorname{Arr} \mathrm{AG_Z}\,\mathcal{C}$.

Composites and inverses in the S-Ore localisation of a Z-fractionable category

The additional axioms of a Z-fractionable category yield the following simplified constructions for composites and inverses in the Z-Ore localisation.

(2.103) Remark. We suppose given a Z-prefractionable category $\mathcal{C}$.

(a) We suppose that $\mathcal{C}$ fulfills the Z-concatenation axiom. Given Z-2-arrows (f_1, i_1), (f_2, i_2) in $\mathcal{C}$ with $\operatorname{Target}(f_1, i_1) = \operatorname{Source}(f_2, i_2)$, the composite $(f_1/i_1)(f_2/i_2)$ in $\mathsf{Ore}_\mathrm{Z}(\mathcal{C})$ can be constructed as follows.

We choose an S-Ore completion (f_2', i_1') for f_2 and i_1 such that $(f_1 f_2', i_2 i_1')$ is a Z-2-arrow in $\mathcal{C}$.

$$\begin{tikzcd}
{} \arrow[r, "f_1"] & {} \arrow[r, "f_2'"] & {} & & \\
& & {} \arrow[ul, "i_1"] \arrow[r, "f_2"] & {} \arrow[ul, "i_1'"] & \\
& & & & {} \arrow[ul, "i_2"]
\end{tikzcd}$$

Then

$$(f_1/i_1)(f_2/i_2) = f_1 f_2'/i_2 i_1'.$$

(b) We suppose that $\mathcal{C}$ fulfills the Z-inversion axiom. Given a Z-2-arrow (f, i) in $\mathcal{C}$ such that f is a denominator in $\mathcal{C}$, the Z-fraction f/i is invertible in $\mathsf{Ore}_\mathrm{Z}(\mathcal{C})$ with

$$(f/i)^{-1} = i/f.$$

8 Maltsiniotis' 3-arrow calculus

Inspired by the 3-arrow calculus of Dwyer, Hirschhorn, Kan and Smith for so-called *homotopical categories admitting a 3-arrow calculus* [11, sec. 36.1, sec. 36.3], which may be seen as a generalisation of Quillen model categories (that admit functorial factorisations), Georges Maltsiniotis developed a 3-arrow calculus for Brown fibration categories. The key technique in his proof was Brown's homotopy 2-arrow calculus [7, th. 1 and proof], cf. the dual of theorem (3.132). We discuss (the duals of) his results using the Z-2-arrow calculus for Z-fractionable categories, see theorem (2.93).

Categories with Z-2-arrows as categories with D-S-T-denominators

To obtain a 3-arrow calculus, we need a notion that is dual to that of an S-denominator.

(2.104) Definition (category with D-S-T-denominators)**.** A *category with denominators, S-denominators and T-denominators* (or *category with D-S-T-denominators*, for short) consists of a category with D-S-denominators $\mathcal{C}$ together with a multiplicative subset T of $\operatorname{Den}\mathcal{C}$. By abuse of notation, we refer to the said category with D-S-T-denominators as well as to its underlying category with D-S-denominators just by $\mathcal{C}$. The elements of T are called *T-denominators* in $\mathcal{C}$.
Given a category with D-S-T-denominators $\mathcal{C}$ with set of T-denominators T, we write $\operatorname{TDen}\mathcal{C} := T$. In diagrams, a T-denominator p in $\mathcal{C}$ will usually be depicted as

$$\xrightarrow{\;p\;} \, .$$

Throughout this section, we will consider the underlying category with D-S-denominators of a Z-prefractionable category as a category with D-S-T-denominators as in the following remark, without further comment.

(2.105) Remark. Given a multiplicative category with D-S-denominators $\mathcal{C}$, then $\mathcal{C}$ becomes a category with D-S-T-denominators, where the set of T-denominators is given by

$$\operatorname{TDen}\mathcal{C} = \{p \in \operatorname{Den}\mathcal{C} \mid \text{there exists an } S\text{-denominator } i \text{ in } \mathcal{C} \text{ with } ip = 1\}.$$

Proof. We set $T := \{p \in \operatorname{Den}\mathcal{C} \mid \text{there exists an } S\text{-denominator } i \text{ in } \mathcal{C} \text{ with } ip = 1\}$. To show that T is multiplicative, we suppose given $p, q \in T$ with $\operatorname{Target} p = \operatorname{Source} q$. Then p and q are denominators in $\mathcal{C}$, and so pq is a denominator in $\mathcal{C}$ by the multiplicativity of $\operatorname{Den}\mathcal{C}$. Moreover, there exist S-denominators i and j in $\mathcal{C}$ with $ip = 1$ and $jq = 1$. But then we also have $jipq = jq = 1$, and as ji is an S-denominator by the multiplicativity of $\operatorname{SDen}\mathcal{C}$, it follows that $pq \in T$. Finally, given an object X in $\mathcal{C}$, we have $1_X \in T$ since $1_X \, 1_X = 1_X$ and since 1_X is an S-denominator in $\mathcal{C}$. Altogether, T is a multiplicative subset of $\operatorname{Mor}\mathcal{C}$, and so $\mathcal{C}$ becomes a category with D-S-T-denominators having $\operatorname{TDen}\mathcal{C} = T$. □

(2.106) Remark. We suppose given a category with D-S-denominators $\mathcal{C}$. Moreover, we suppose given an S-2-arrow (f, p) with T-denominator p, a normal S-2-arrow (g, j) and a denominator s in $\mathcal{C}$ with $(f, p) = (gs, js)$.

$$\begin{tikzcd} {} \arrow[r, "g"] \arrow[d, equal] & {} \arrow[d, "s"] & {} \arrow[l, "j"'] \arrow[d, equal] \\ {} \arrow[r, "f"] & {} & {} \arrow[l, "p"'] \end{tikzcd}$$

Then s is a T-denominator in $\mathcal{C}$.

Proof. As p is a T-denominator in $\mathcal{C}$, there exists an S-denominator i in $\mathcal{C}$ with $ip = 1$. But then we also have $ijs = ip = 1$, and since ij is an S-denominator by multiplicativity, it follows that s is a T-denominator in $\mathcal{C}$.

$$\begin{tikzcd} {} \arrow[r, "g"] \arrow[d, equal] & {} \arrow[d, "s"] & {} \arrow[l, "j"'] \arrow[d, equal] & {} \arrow[l, "i"'] \arrow[d, equal] \\ {} \arrow[r, "f"] & {} & {} \arrow[l, "p"'] & {} \arrow[l, "i"'] \end{tikzcd}$$

□

(2.107) Lemma (factorisation lemma, cf. [36, (Fac) in def. (3.1)(a), lem. (5.1)])**.** We suppose given a category with Z-2-arrows $\mathcal{C}$ that fulfills the Z-replacement axiom for denominators and the Z-inversion axiom.

(a) For every denominator d in $\mathcal{C}$ there exist an S-denominator i and a T-denominator p in $\mathcal{C}$ with $d = ip$.

$$\begin{tikzcd} & \cdot \arrow[dr, dotted, "p"] & \\ \cdot \arrow[ur, dotted, "i"] \arrow[rr, "d"', "\approx"] & & \cdot \end{tikzcd}$$

(b) We suppose that $\mathcal{C}$ is S-semisaturated and that $\mathcal{C}$ fulfills the weakly universal S-Ore completion axiom. We suppose given S-2-arrows (f_1, a_2), (f_2, a_2) and denominators d_1, d_2, $\tilde{d}_2$ in $\mathcal{C}$ such that $f_1 \tilde{d}_2 = d_1 f_2$ and $a_1 \tilde{d}_2 = d_2 a_2$. Moreover, we suppose given S-denominators i_1, i_2 and T-denominators p_1, p_2 in $\mathcal{C}$ with $d_1 = i_1 p_1$, $d_2 = i_2 p_2$. Then there exist an S-denominator $\tilde{i}_2$, a T-denominator $\tilde{p}_2$ and an S-2-arrow (f, a) in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd} & \cdot \arrow[dr, "p_1"] \arrow[dd, dotted, "f"] & \\ \cdot \arrow[ur, "i_1"] \arrow[rr, "d_1", "\approx"'] \arrow[dd, "f_1"'] & & \cdot \arrow[dd, "f_2"] \\ & \cdot \arrow[dr, dotted, "\tilde{p}_2"] & \\ \cdot \arrow[ur, dotted, "\tilde{i}_2"] \arrow[rr, "\tilde{d}_2", "\approx"'] & & \cdot \\ & \cdot \arrow[dr, "p_2"] \arrow[uu, dotted, "\approx" , "a"'] & \\ \cdot \arrow[ur, "i_2"] \arrow[rr, "d_2", "\approx"'] \arrow[uu, "a_1", "\approx"'] & & \cdot \arrow[uu, "\approx", "a_2"'] \end{tikzcd}$$

Proof.

(a) We suppose given a denominator d in $\mathcal{C}$. By the Z-replacement axiom for denominators, there exist a Z-2-arrow (i, e) with denominator i and a morphism p in $\mathcal{C}$ with $(d, 1) = (ip, ep)$. By S-semisaturatedness, p is a denominator, and since every Z-2-arrow is a normal S-2-arrow, it follows that p is in fact a T-denominator. Moreover, since $\mathcal{C}$ fulfills the Z-inversion axiom, it follows that i is an S-denominator by remark (2.74).

$$\begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, equal] & \cdot \arrow[d, "p"] & \cdot \arrow[l, "e"'] \arrow[d, equal] \\ \cdot \arrow[r, "d"', "\approx"] & \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

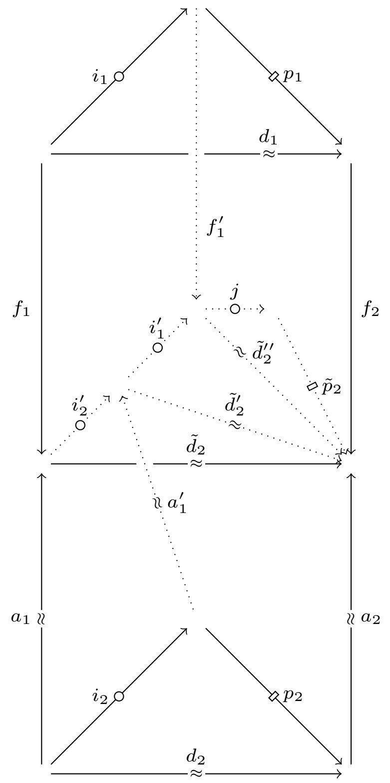
(b) By the weakly universal S-Ore completion axiom, there exist a weakly universal S-Ore completion (a'_1, i'_2) for a_1 and i_2, and there exist a weakly universal S-Ore completion (f'_1, i'_1) for $f_1 i'_2$ and i_1. As

$$a_1 \tilde{d}_2 = d_2 a_2 = i_2 p_2 a_2,$$

there exists a morphism $\tilde{d}'_2$ with $p_2 a_2 = a'_1 \tilde{d}'_2$ and $\tilde{d}_2 = i'_2 \tilde{d}'_2$. By S-semisaturatedness, a'_1 and $\tilde{d}'_2$ are denominators in $\mathcal{C}$. We obtain

$$f_1 i'_2 \tilde{d}'_2 = f_1 \tilde{d}_2 = d_1 f_2 = i_1 p_1 f_2,$$

and so there exists a morphism $\tilde{d}''_2$ with $p_1 f_2 = f'_1 \tilde{d}''_2$ and $\tilde{d}'_2 = i_1 \tilde{d}''_2$. By S-semisaturatedness, $\tilde{d}''_2$ is a denominator in $\mathcal{C}$. Finally, by (a) there exist an S-denominator j and a T-denominator $\tilde{p}_2$ in $\mathcal{C}$ such that $\tilde{d}''_2 = j \tilde{p}_2$.

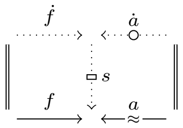


We set $f := f_1' j$, $a := a_1' i_1' j$, $\tilde{\tilde{i}}_2 = i_2' i_1' j$, and get

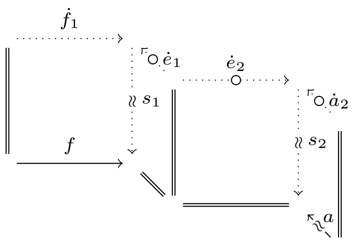
$$f_1 \tilde{\tilde{i}}_2 = f_1 i_2' i_1' j = i_1 f_1' j = i_1 f,$$
$$f \tilde{p}_2 = f_1' j \tilde{p}_2 = f_1' \tilde{d}_2'' = p_1 f_2,$$
$$a_1 \tilde{\tilde{i}}_2 = a_1 i_2' i_1' j = i_2 a_1' i_1' j = i_2 a,$$
$$a \tilde{p}_2 = a_1' i_1' j \tilde{p}_2 = a_1' i_1' \tilde{d}_2'' = a_1' \tilde{d}_2' = p_2 a_2.$$

Moreover, $a = a_1' i_1' j$ is a denominator and $\tilde{\tilde{i}}_2 = i_2' i_1' j$ is an S-denominator in $\mathcal{C}$ by multiplicativity. □

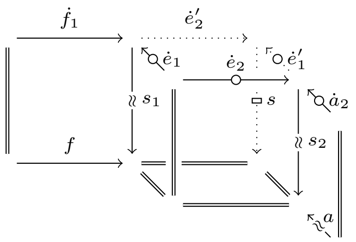
(2.108) Proposition. We suppose given an S-semisaturated category with Z-2-arrows $\mathcal{C}$ that fulfills the Z-concatenation axiom and the Z-inversion axiom. For every S-2-arrow (f, a) in $\mathcal{C}$ there exist a Z-2-arrow $(\dot{f}, \dot{a})$ and a T-denominator s in $\mathcal{C}$ with $(f, a) = (\dot{f} s, \dot{a} s)$.



Proof. We suppose given an S-2-arrow (f, a) in $\mathcal{C}$. By the Z-replacement axiom, there exist a Z-2-arrow $(\dot{f}_1, \dot{e}_1)$ and a morphism s_1 in $\mathcal{C}$ with $(f, 1) = (\dot{f}_1 s_1, \dot{e}_1 s_1)$, and by the Z-replacement axiom for denominators, there exist a Z-2-arrow $(\dot{e}_2, \dot{a}_2)$ with denominator $\dot{e}_2$ and a morphism s_2 in $\mathcal{C}$ with $(1, a) = (\dot{e}_2 s_2, \dot{a}_2 s_2)$. Then $\dot{e}_2$ is in fact an S-denominator by remark (2.74), and s_1, s_2 are denominators by S-semisaturatedness.



Moreover, by the Z-concatenation axiom, there exist a weakly universal S-Ore completion $(\dot{e}'_2, \dot{e}'_1)$ for $\dot{e}_2$ and $\dot{e}_1$ such that $(\dot{f}, \dot{a}) := (\dot{f}_1 \dot{e}'_2, \dot{a}_2 \dot{e}'_1)$ is a Z-2-arrow in $\mathcal{C}$. As $\dot{e}_2 s_2 = 1 = \dot{e}_1 s_1$ and s_2 is a denominator by S-semisaturatedness, there exist a morphism s with $s_1 = \dot{e}'_2 s$ and $s_2 = \dot{e}'_1 s$. We obtain $\dot{f}_1 \dot{e}'_2 s = \dot{f}_1 s_1 = f$ as well as $\dot{a}_2 \dot{e}'_1 s = \dot{a}_2 s_2 = a$. Finally, s is a denominator in $\mathcal{C}$ by S-semisaturatedness. So since $\dot{e}_2 \dot{e}'_1 s = \dot{e}_2 s_2 = 1$ and $\dot{e}_2 \dot{e}'_1$ is an S-denominator by multiplicativity, it follows that s is a T-denominator in $\mathcal{C}$.



□

(Normal) 3-arrows

Analogously to the S-2-arrow graph and the normal S-2-arrow graph, see definition (2.10), we will now define the 3-arrow graph and the normal 3-arrow graph. In contrast to the former, where we used a quotient of the S-2-arrow graph to construct the S-Ore localisation, see theorem (2.85), we will not make explicit use of the graph structure on the 3-arrow graph here – we will just use (the language for) its arrows. An analogous construction applied to an analogous quotient of the 3-arrow graph in a somewhat different context can be found in [36, prop. 5.2, prop. 5.5].

(2.109) Definition (3-arrow shape)**.** The 3-*arrow shape* is defined to be the graph $\mathbf{\Theta}$ given by

$$\operatorname{Ob}\mathbf{\Theta} = \{0, 1, 2, 3\},$$
$$\operatorname{Arr}\mathbf{\Theta} = \{(1, 0), (1, 2), (3, 2)\},$$

and where Source $(1, 0) = 1$, Target $(1, 0) = 0$, Source $(1, 2) = 1$, Target $(1, 2) = 3$, Source $(3, 2) = 3$, Target $(3, 2) = 2$.

$$0 \longleftarrow 1 \longrightarrow 2 \longleftarrow 3$$

(2.110) Definition ((normal) 3-arrow graph)**.**

(a) We suppose given a category with denominators $\mathcal{C}$. The 3-*arrow graph* of $\mathcal{C}$ is defined to be the graph $\operatorname{AG}\mathcal{C}$ given by

$$\operatorname{Ob}\operatorname{AG}\mathcal{C} = \operatorname{Ob}\mathcal{C},$$
$$\operatorname{Arr}\operatorname{AG}\mathcal{C} = \{A \in \operatorname{Ob}\mathcal{C}^{\mathbf{\Theta}} \mid A_{1,0} \text{ and } A_{3,2} \text{ are denominators in } \mathcal{C}\},$$

and where Source $A = A_0$ resp. Target $A = A_3$ for $A \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$.

An arrow A in $\operatorname{AG}\mathcal{C}$ is called a 3-*arrow* in $\mathcal{C}$. Given a denominator $b\colon \tilde{X} \to X$, a morphism $f\colon \tilde{X} \to \tilde{Y}$ and a denominator $a\colon Y \to \tilde{Y}$ in $\mathcal{C}$, we abuse notation and denote the unique 3-arrow A with $A_{1,0} = b$, $A_{1,2} = f$, $A_{3,2} = a$ by $(b, f, a) := A$. Moreover, we use the notation $(b, f, a)\colon X \leftarrow \tilde{X} \to \tilde{Y} \leftarrow Y$.

$$X \xleftarrow[\approx]{b} \tilde{X} \xrightarrow{f} \tilde{Y} \xleftarrow[\approx]{a} Y$$

(b) We suppose given a category with D-S-T-denominators $\mathcal{C}$. A 3-arrow $(p, f, i)\colon X \leftarrow \tilde{X} \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ is said to be *normal* if p is a T-denominator and i is an S-denominator in $\mathcal{C}$.

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} \tilde{Y} \xleftarrow{i} Y$$

The *normal* 3-*arrow graph* of $\mathcal{C}$ is defined to be the wide subgraph $\operatorname{AG}_{\mathrm{n}}\mathcal{C}$ of $\operatorname{AG}\mathcal{C}$ with

$$\operatorname{Arr}\operatorname{AG}_{\mathrm{n}}\mathcal{C} = \{A \in \operatorname{Arr}\operatorname{AG}\mathcal{C} \mid A \text{ is normal}\}.$$

A 3-arrow calculus for Z-(pre)fractionable categories

In the framework of Brown fibration categories, Georges Maltsiniotis found a 3-arrow calculus in the sense of the validity of a "3-arrow representative condition" and a "3-arrow equality condition", cf. [11, sec. 36.2–3] (cf. definition (2.31) for the respective notions for S-2-arrows). In his proof, he used Brown's homotopy S-2-arrow calculus [7, th. 1 and proof]. We obtain (the dual of) his 3-arrow calculus in the slightly more general framework of Z-fractionable categories, cf. theorem (3.127), using the Z-2-arrow calculus instead of Brown's homotopy S-2-arrow calculus.

(2.111) Theorem (Maltsiniotis' 3-arrow calculus [26, p. 32])**.** We suppose given a Z-prefractionable category $\mathcal{C}$.

(a) We have

$$\operatorname{Mor} \mathsf{Ore}_{\mathrm{S}}(\mathcal{C}) = \{\operatorname{loc}(b)^{-1} \operatorname{loc}(f) \operatorname{loc}(a)^{-1} \mid (b, f, a) \text{ is a 3-arrow in } \mathcal{C}\}.$$

(b) We suppose given 3-arrows (b, f, a), (b', f', a') in $\mathcal{C}$. The following conditions are equivalent.

(i) We have

$$\operatorname{loc}(b)^{-1} \operatorname{loc}(f) \operatorname{loc}(a)^{-1} = \operatorname{loc}(b')^{-1} \operatorname{loc}(f') \operatorname{loc}(a')^{-1}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

(ii) For every Z-2-arrow $(\dot{f}, \dot{a})$, every normal S-2-arrow $(\dot{b}, \dot{b}')$ and all morphisms s, t with $(f, a) = (\dot{f}s, \dot{a}s)$, $(b, b') = (\dot{b}t, \dot{b}'t)$, there exist an S-2-arrow $(\tilde{f}', \tilde{a}')$ and a normal S-2-arrow (c, j) such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, "f"] \arrow[d, equal] & {} & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, "\dot{f}"] \arrow[d, "\dot{b}"] & {} \arrow[u, "s"'] \arrow[d, dotted, "c"] & {} \arrow[l, "\dot{a}"', "\circ"] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "t"'] \arrow[r, dotted, "\tilde{f}'"] & {} & {} \arrow[l, dotted, "\tilde{a}'"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "b'"', "\approx"] \arrow[r, "f'"] \arrow[u, "\dot{b}'", "\circ"'] & {} \arrow[u, dotted, "j"', "\circ"] & {} \arrow[l, "a'"', "\approx"]
\end{tikzcd}$$

(iii) There exist Z-2-arrows $(\dot{f}, \dot{a})$, $(\dot{b}, \dot{b}')$, an S-2-arrow $(\tilde{f}', \tilde{a}')$, a normal S-2-arrow (c, j) and denominators s, t in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, "f"] \arrow[d, equal] & {} & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, dotted, "\dot{f}"] \arrow[d, dotted, "\dot{b}"] & {} \arrow[u, dotted, "s"', "\approx"] \arrow[d, dotted, "c"] & {} \arrow[l, dotted, "\dot{a}"', "\circ"] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, dotted, "t"', "\approx"] \arrow[r, dotted, "\tilde{f}'"] & {} & {} \arrow[l, dotted, "\tilde{a}'"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "b'"', "\approx"] \arrow[r, "f'"] \arrow[u, dotted, "\dot{b}'", "\circ"'] & {} \arrow[u, dotted, "j"', "\circ"] & {} \arrow[l, "a'"', "\approx"]
\end{tikzcd}$$

If $\mathcal{C}$ fulfills the Z-concatenation axiom and the Z-inversion axiom, then these three conditions are furthermore equivalent to the following condition.

(iv) There exist Z-2-arrows $(\dot{f}, \dot{a})$, $(\dot{b}, \dot{b}')$, an S-2-arrow $(\tilde{f}', \tilde{a}')$, a normal S-2-arrow (c, j) and T-denominators s, t in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bullet \arrow[d, equal] & \bullet \arrow[l, "b"', "\approx" description] \arrow[r, "f"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a"', "\approx" description] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "b"', "\approx" description] \arrow[r, dotted, "\dot{f}"] \arrow[d, dotted, "\dot{b}"] & \bullet \arrow[u, dotted, "s"'] \arrow[d, dotted, "c"'] & \bullet \arrow[l, dotted, "\dot{a}"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, dotted, "t"'] \arrow[r, dotted, "\tilde{f}'"] & \bullet & \bullet \arrow[l, dotted, "\tilde{a}'"', "\approx" description] \arrow[d, equal] \\
\bullet & \bullet \arrow[l, "b'"', "\approx" description] \arrow[r, "f'"] \arrow[u, dotted, "\dot{b}'"'] & \bullet \arrow[u, dotted, "j"'] & \bullet \arrow[l, "a'"', "\approx" description]
\end{tikzcd}$$

(c) We suppose given 3-arrows (b_1, f_1, a_1), (b_2, f_2, a_2), (p_1, g_1, d_1), (e_2, g_2, j_2) in $\mathcal{C}$ such that p_1 is a T-denominator and j_2 is an S-denominator. The following conditions are equivalent.

(i) We have

$$\begin{aligned} & \operatorname{loc}(b_1)^{-1} \operatorname{loc}(f_1) \operatorname{loc}(a_1)^{-1} \operatorname{loc}(e_1)^{-1} \operatorname{loc}(g_2) \operatorname{loc}(j_2)^{-1} \\ & = \operatorname{loc}(p_1)^{-1} \operatorname{loc}(g_1) \operatorname{loc}(d_1)^{-1} \operatorname{loc}(b_2)^{-1} \operatorname{loc}(f_2) \operatorname{loc}(a_2)^{-1} \end{aligned}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$.

(ii) For every Z-2-arrow $(\dot{f}_1, \dot{a}_1)$, every normal S-2-arrow $(\dot{g}_1, \dot{d}_1)$ and all morphisms s_1, t_1, c in $\mathcal{C}$ with $f_1 = \dot{f}_1 s_1$, $e_1 a_1 = \dot{a}_1 s_1$, $c g_1 = \dot{g}_1 t_1$, $b_2 d_1 = \dot{d}_1 t_1$, $c p_1 = b_1$, there exist an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$ and a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)$ such that the following diagram commutes.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "b_1"', "\approx" description] \arrow[r, "f_1"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a_1"', "\approx" description] \\
\bullet \arrow[u, "p_1"] \arrow[d, "g_1"'] & \bullet \arrow[l, "c"'] \arrow[r, "\dot{f}_1"] \arrow[d, "\dot{g}_1"'] & \bullet \arrow[u, "s_1"'] \arrow[d, dotted, "\tilde{g}_2"'] & \bullet \arrow[l, "\dot{a}_1"'] \arrow[u, "e_1"', "\approx" description] \arrow[d, "g_2"] \\
\bullet & \bullet \arrow[l, "t_1"'] \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{a}_2"', "\approx" description] \\
\bullet \arrow[u, "d_1", "\approx" description] & \bullet \arrow[l, "b_2"', "\approx" description] \arrow[r, "f_2"] \arrow[u, "\dot{d}_1"'] & \bullet \arrow[u, dotted, "\tilde{j}_2"'] & \bullet \arrow[l, "a_2"', "\approx" description] \arrow[u, "j_2"']
\end{tikzcd}$$

(iii) There exist Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{g}_1, \dot{d}_1)$, an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$, a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)$, denominators s_1, t_1 and an S-denominator i_1 in $\mathcal{C}$ such that $i_1 p_1 = 1$ and such that the following diagram commutes.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "b_1"', "\approx" description] \arrow[r, "f_1"] \arrow[d, equal] & \bullet & \bullet \arrow[l, "a_1"', "\approx" description] \\
\bullet \arrow[u, "p_1"] \arrow[d, "g_1"'] & \bullet \arrow[l, dotted, "b_1 i_1"', "\approx" description] \arrow[r, dotted, "\dot{f}_1"] \arrow[d, dotted, "\dot{g}_1"'] & \bullet \arrow[u, dotted, "s_1"', "\approx" description] \arrow[d, dotted, "\tilde{g}_2"'] & \bullet \arrow[l, dotted, "\dot{a}_1"'] \arrow[u, "e_1"', "\approx" description] \arrow[d, "g_2"] \\
\bullet & \bullet \arrow[l, dotted, "t_1"', "\approx" description] \arrow[r, dotted, "\tilde{f}_2"] & \bullet & \bullet \arrow[l, dotted, "\tilde{a}_2"', "\approx" description] \\
\bullet \arrow[u, "d_1", "\approx" description] & \bullet \arrow[l, "b_2"', "\approx" description] \arrow[r, "f_2"] \arrow[u, dotted, "\dot{d}_1"'] & \bullet \arrow[u, dotted, "\tilde{j}_2"'] & \bullet \arrow[l, "a_2"', "\approx" description] \arrow[u, "j_2"']
\end{tikzcd}$$

If $\mathcal{C}$ fulfills the Z-concatenation axiom and the Z-inversion axiom, then these three conditions are furthermore equivalent to the following condition.

(iv) There exist Z-2-arrows $(\dot{f}_1, \dot{a}_1)$, $(\dot{g}_1, \dot{d}_1)$, an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$, a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)$, T-denominators s_1, t_1 and an S-denominator i_1 in $\mathcal{C}$ such that $i_1 p_1 = 1$ and such that the following diagram

commutes.

$$\begin{tikzcd}
{} & {} \arrow[l, "b_1"', "\approx"] \arrow[r, "f_1"] & {} & {} \arrow[l, "a_1"', "\approx"] \\
{} \arrow[u, "p_1"] \arrow[d, "g_1"] & {} \arrow[l, dotted, "b_1 i_1"', "\approx"] \arrow[u, equal] \arrow[r, dotted, "\dot{f}_1"] \arrow[d, dotted, "\dot{g}_1"] & {} \arrow[u, dotted, "s_1"] \arrow[d, dotted, "\tilde{g}_2"] & {} \arrow[u, "e_1", "\approx"'] \arrow[l, dotted, "\dot{a}_1"'] \arrow[d, "g_2"] \\
{} & {} \arrow[l, dotted, "t_1"'] \arrow[r, dotted, "\tilde{f}_2"] & {} & {} \arrow[l, dotted, "\tilde{a}_2"', "\approx"] \\
{} \arrow[u, "d_1", "\approx"'] & {} \arrow[l, "b_2"', "\approx"] \arrow[u, dotted, "\dot{d}_1"] \arrow[r, "f_2"] & {} \arrow[u, dotted, "\tilde{j}_2"] & {} \arrow[l, "a_2"', "\approx"] \arrow[u, "j_2"]
\end{tikzcd}$$

Proof.

(a) This follows from corollary (2.94)(a).

(c) First, we suppose that condition (i) holds, that is, we suppose that

$$\begin{aligned}
&\operatorname{loc}(b_1)^{-1}\operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(e_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} \\
&= \operatorname{loc}(p_1)^{-1}\operatorname{loc}(g_1)\operatorname{loc}(d_1)^{-1}\operatorname{loc}(b_2)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1}
\end{aligned}$$

in $\mathsf{Ore}_{\mathrm{S}}(\mathcal{C})$. Moreover, we suppose given a Z-2-arrow $(\dot{f}_1, \dot{a}_1)$, a normal S-2-arrow $(\dot{g}_1, \dot{d}_1)$ and morphisms s_1, t_1, c in $\mathcal{C}$ with $f_1 = \dot{f}_1 s_1$, $e_1 a_1 = \dot{a}_1 s_1$, $c g_1 = \dot{g}_1 t_1$, $b_2 d_1 = \dot{d}_1 t_1$, $c p_1 = b_1$. Then we get

$$\begin{aligned}
&\operatorname{loc}(f_1)\operatorname{loc}(e_1 a_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} = \operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(e_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} \\
&= \operatorname{loc}(b_1)\operatorname{loc}(p_1)^{-1}\operatorname{loc}(g_1)\operatorname{loc}(d_1)^{-1}\operatorname{loc}(b_2)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1} \\
&= \operatorname{loc}(c)\operatorname{loc}(g_1)\operatorname{loc}(d_1)^{-1}\operatorname{loc}(b_2)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1} = \operatorname{loc}(c g_1)\operatorname{loc}(b_2 d_1)^{-1}\operatorname{loc}(f_2)\operatorname{loc}(a_2)^{-1},
\end{aligned}$$

and so by corollary (2.94)(c) there exist an S-2-arrow $(\tilde{f}_2, \tilde{a}_2)$ and a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & {} \arrow[r, "f_1"] & {} & {} \arrow[l, "e_1 a_1"', "\approx"] \\
{} \arrow[r, equal] \arrow[d, "c g_1"] & {} \arrow[u, equal] \arrow[r, "\dot{f}_1"] \arrow[d, "\dot{g}_1"] & {} \arrow[u, "s_1"] \arrow[d, dotted, "\tilde{g}_2"] & {} \arrow[u, equal] \arrow[l, "\dot{a}_1"'] \arrow[d, "g_2"] \\
{} & {} \arrow[l, "t_1"'] \arrow[r, dotted, "\tilde{f}_2"] & {} & {} \arrow[l, dotted, "\tilde{a}_2"', "\approx"] \\
{} \arrow[u, "b_2 d_1", "\approx"'] \arrow[r, equal] & {} \arrow[u, "\dot{d}_1"] \arrow[r, "f_2"] & {} \arrow[u, dotted, "\tilde{j}_2"] & {} \arrow[l, "a_2"', "\approx"] \arrow[u, "j_2"]
\end{tikzcd}$$

But then the following diagram also commutes, that is, condition (ii) holds.

$$\begin{tikzcd}
{} & {} \arrow[l, "b_1"', "\approx"] \arrow[r, "f_1"] & {} & {} \arrow[l, "a_1"', "\approx"] \\
{} \arrow[u, "p_1"] \arrow[d, "g_1"] & {} \arrow[l, "c"'] \arrow[u, equal] \arrow[r, "\dot{f}_1"] \arrow[d, "\dot{g}_1"] & {} \arrow[u, "s"] \arrow[d, "\tilde{g}_2"] & {} \arrow[u, "e_1", "\approx"'] \arrow[l, "\dot{a}_1"'] \arrow[d, "g_2"] \\
{} & {} \arrow[l, "t"'] \arrow[r, "\tilde{f}_2"] & {} & {} \arrow[l, "\tilde{a}_2"', "\approx"] \\
{} \arrow[u, "d_1", "\approx"'] & {} \arrow[l, "b_2"', "\approx"] \arrow[u, "\dot{d}_1"] \arrow[r, "f_2"] & {} \arrow[u, "\tilde{j}_2"] & {} \arrow[l, "a_2"', "\approx"] \arrow[u, "j_2"]
\end{tikzcd}$$

Condition (ii) and the Z-replacement axiom imply condition (iii).

Finally, if condition (iii) holds, then we have

$$\operatorname{loc}(b_1)^{-1}\operatorname{loc}(f_1)\operatorname{loc}(a_1)^{-1}\operatorname{loc}(e_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1} = \operatorname{loc}(b_1)^{-1}\operatorname{loc}(\dot{f}_1)\operatorname{loc}(\dot{a}_1)^{-1}\operatorname{loc}(g_2)\operatorname{loc}(j_2)^{-1}$$

$$= \operatorname{loc}(b_1)^{-1} \operatorname{loc}(\dot{f}_1) \operatorname{loc}(\tilde{g}_2) \operatorname{loc}(\tilde{a}_2)^{-1} \operatorname{loc}(j_2)^{-1}$$
$$= \operatorname{loc}(p_1)^{-1} \operatorname{loc}(b_1 i_1)^{-1} \operatorname{loc}(\dot{g}_1) \operatorname{loc}(\tilde{f}_2) \operatorname{loc}(\tilde{j}_2)^{-1} \operatorname{loc}(a_2)^{-1}$$
$$= \operatorname{loc}(p_1)^{-1} \operatorname{loc}(g_1) \operatorname{loc}(t)^{-1} \operatorname{loc}(\dot{d}_1)^{-1} \operatorname{loc}(f_2) \operatorname{loc}(a_2)^{-1}$$
$$= \operatorname{loc}(p_1)^{-1} \operatorname{loc}(g_1) \operatorname{loc}(d_1)^{-1} \operatorname{loc}(b_2)^{-1} \operatorname{loc}(f_2) \operatorname{loc}(a_2)^{-1},$$

that is, condition (i) holds.

Altogether, the three conditions (i), (ii) and (iii) are equivalent.

If $\mathcal{C}$ fulfills the Z-concatenation axiom and the Z-inversion axiom, then condition (ii) implies condition (iv) by proposition (2.108), and condition (iv) is a particular case of condition (iii). So in this case, the four conditions (i), (ii), (iii) and (iv) are equivalent.

(b) This follows from (c). □

Chapter III

Cofibration categories

In homotopical algebra, we study categories with weak equivalences and their homotopy categories in the following sense: A category with weak equivalences consists of a category that is equipped with a sort of distinguished morphisms called weak equivalences, cf. definition (3.1)(a). We would like to consider objects that are connected by a weak equivalence as essentially equal. So as weak equivalences are not isomorphisms in general, we define the homotopy category of a category with weak equivalences as its localisation with respect to the subset of weak equivalences, cf. definition (1.11) and definition (1.25), that is, as the universal category where the weak equivalences become invertible.

To study homotopy categories, it seems hardly possible to work with weak equivalences alone. However, the naturally occurring examples of categories with weak equivalences share more structure; for example, they are equipped with cofibrations (in the sense of definition (3.14)(a)) or fibrations (in a sense dual to definition (3.14)(a)) or both. Whereas the weak equivalences form the important part of a category with cofibrations and weak equivalences in the sense that they suffice to *define* and *construct* the homotopy category, the (co)fibrations are usually seen as auxiliary tools to provide constructions and, in consequence, to *understand* the homotopy category. While we restrict our attention to cofibrations in this thesis, every notion has a dual notion and every assertion has a dual assertion.

In this chapter, we study the basic homotopical algebra of Brown cofibration categories and the slightly more general Cisinski cofibration categories. Both notions of cofibration categories are particular well behaved categories with cofibrations and weak equivalences. In particular, we show in theorem (3.127) that every Brown cofibration category admits the structure of a Z-fractionable category in the sense of definition (2.81)(a), so that we may apply our results from chapter II to obtain a description of the hom-sets of the homotopy category of a Brown cofibration category, see theorem (3.128). As the homotopy category of every Cisinski cofibration category is equivalent to the homotopy category of its full subcategory of cofibrant objects [9, prop. 1.8], which is a Brown cofibration category by remark (3.53), this also gives us a convenient calculus for the morphisms in the homotopy category of every Cisinski cofibration category (and so, in particular, of every Quillen model category).

Some of the facts and proofs presented in this chapter are folklore or known in the (more particular) context of Quillen model categories. The author's guide was the extensive manuscript of Rădulescu-Banu [30]. Many assertions are applicable to other contexts as well, such as Waldhausen (cofibration) categories or exact categories. In order not to exclude these possible applications, we shall point out which axioms are actually needed at each point. The main innovation is the relativisation of Quillen's cylinder notion of an object to S-2-arrows.

The chapter is organised as follows. In section 1, we define categories with weak equivalences and their homotopy categories. The notion of a category with cofibrations, which is an auxiliary tool from our point of view, is defined in section 2, and we combine both structures to the notion of a category with cofibrations and weak equivalences in section 3. Then in section 4, we discuss the interplay between cofibrations and weak equivalences and define Cisinski and Brown cofibration categories, our main objects of study. After that, we study the somewhat technical notion of a coreedian rectangle in section 5, which is used to define some cofibration category structures on diagram categories in section 6 and occurs furthermore at some other places in this and the following chapter. In section 7, we generalise the well-known notion of a cylinder of an object to the notion of a cylinder of an S-2-arrow and study their main properties. Cylinders are used to give a proof of the so-called gluing lemma in section 8. Moreover, in section 9, we will see that cylinders yield a concept to turn a Brown

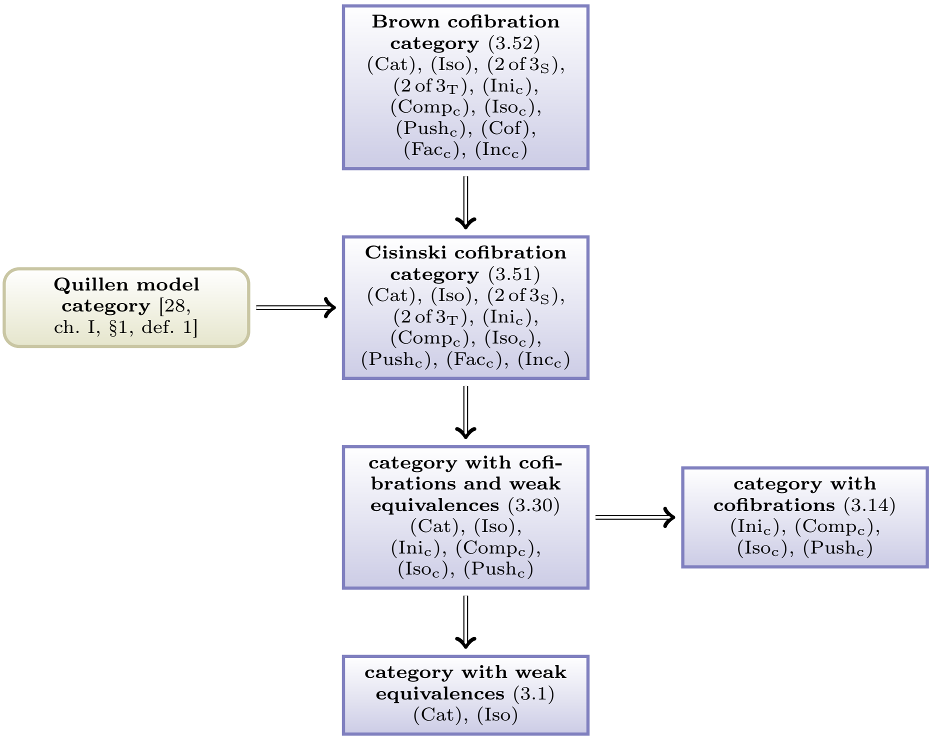


Figure 1: Hierarchy of some structures in homotopical algebra.

cofibration category into a Z-fractionable category, see theorem (3.127). In particular, the homotopy category of every Brown cofibration category admits a Z-2-arrow calculus as in theorem (3.128). This Z-2-arrow calculus is used to give a new proof for Brown's homotopy S-2-arrow calculus, see theorem (3.132).

1 Categories with weak equivalences

In this section, we define categories with weak equivalences, that is, categories equipped with a distinguished subset of morphisms called weak equivalences that fulfill some closure properties, as well as their homotopy categories.

Definition of a category with weak equivalences

Given a category $\mathcal{C}$, a subset of the set of morphisms $\operatorname{Mor}\mathcal{C}$ is called multiplicative in $\mathcal{C}$ if it is closed under composition and contains all identity morphisms in $\mathcal{C}$, see definition (1.35)(a).

(3.1) Definition (category with weak equivalences)**.**

(a) A *category with weak equivalences* consists of a category $\mathcal{C}$ together with a multiplicative subset $W \subseteq \operatorname{Mor}\mathcal{C}$ that contains all isomorphisms in $\mathcal{C}$. By abuse of notation, we refer to the said category with weak equivalences as well as to its underlying category just by $\mathcal{C}$. The elements of W are called *weak equivalences* in $\mathcal{C}$.

Given a category with weak equivalences $\mathcal{C}$ with set of weak equivalences W, we write $\operatorname{We}\mathcal{C} := W$. In diagrams, a weak equivalence $w\colon X \to Y$ in $\mathcal{C}$ will usually be depicted as

$$X \xrightarrow[\approx]{w} Y \, .$$

(b) We suppose given categories with weak equivalences $\mathcal{C}$ and $\mathcal{D}$. A *morphism of categories with weak equivalences* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ that *preserves weak equivalences*, that is, such that Fw is a weak equivalence in $\mathcal{D}$ for every weak equivalence w in $\mathcal{C}$.

The notion of a category with weak equivalences is closely related to that of a *relative category* by Barwick and Kan [3, sec. 3.1] and to that of a *category pair* by Rădulescu-Banu [30, def. 1.8.2].
Formally seen, a category with weak equivalences is the same as a multiplicative category with denominators where the denominators contain all isomorphisms. Indeed, we will often take this point of view, see remark (3.7) and definition (3.8). However, the notion of a weak equivalence, originally introduced by Quillen [28, ch. I, §1, def. 1, ex.] as an abstraction of the notion of a weak homotopy equivalence from topology, is historically established.
We change our point of view now: The localisation *construction* steps in the background, whereas *properties* of the localisation step in the foreground. As we study the localisation and their models with the methods of homotopical algebra, denominators will be called weak equivalences, and the localisation will be called homotopy category, see definition (3.8).

(3.2) Remark. Given a category with weak equivalences $\mathcal{C}$, its opposite category $\mathcal{C}^{\mathrm{op}}$ becomes a category with weak equivalences having $\mathrm{We}(\mathcal{C}^{\mathrm{op}}) = \mathrm{We}\,\mathcal{C}$.

The category of categories with weak equivalences

(3.3) Definition (category with weak equivalences with respect to a Grothendieck universe)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. A category with weak equivalences $\mathcal{C}$ is called a *category with weak equivalences with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$-*category with weak equivalences*) if its underlying category is a $\mathfrak{U}$-category.

(3.4) Remark.

(a) We suppose given a Grothendieck universe $\mathfrak{U}$. A category with weak equivalences $\mathcal{C}$ is a $\mathfrak{U}$-category with weak equivalences if and only if it is an element of $\mathfrak{U}$.

(b) For every category with weak equivalences $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a $\mathfrak{U}$-category with weak equivalences.

(3.5) Remark. For every Grothendieck universe $\mathfrak{U}$ we have a category $\mathbf{CatW}_{(\mathfrak{U})}$, given as follows. The set of objects of $\mathbf{CatW}_{(\mathfrak{U})}$ is given by

$$\mathrm{Ob}\,\mathbf{CatW}_{(\mathfrak{U})} = \{\mathcal{C} \mid \mathcal{C} \text{ is a } \mathfrak{U}\text{-category with weak equivalences}\}.$$

For objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathbf{CatW}_{(\mathfrak{U})}$, we have the hom-set

$${}_{\mathbf{CatW}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D}) = \{F \mid F \text{ is a morphism of categories with weak equivalences from } \mathcal{C} \text{ to } \mathcal{D}\}.$$

For morphisms $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ in $\mathbf{CatW}_{(\mathfrak{U})}$, the composite of F and G in $\mathbf{CatW}_{(\mathfrak{U})}$ is given by the composite of the underlying functors $G \circ F\colon \mathcal{C} \to \mathcal{E}$. For an object $\mathcal{C}$ in $\mathbf{CatW}_{(\mathfrak{U})}$, the identity morphism on $\mathcal{C}$ in $\mathbf{CatW}_{(\mathfrak{U})}$ is given by the underlying identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.

(3.6) Definition (category of categories with weak equivalences)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. The category $\mathbf{CatW} = \mathbf{CatW}_{(\mathfrak{U})}$ as considered in remark (3.5) is called the *category of categories with weak equivalences* (more precisely, the *category of* $\mathfrak{U}$-*categories with weak equivalences*).

The homotopy category

If unambiguous, we will consider a category with weak equivalences as a category with denominators, see definition (1.1)(a), in the following way, without further comment.

(3.7) Remark.

(a) Given a category with weak equivalences $\mathcal{C}$, the underlying category of $\mathcal{C}$ becomes a multiplicative and isosaturated category with denominators having

$$\mathrm{Den}\,\mathcal{C} = \mathrm{We}\,\mathcal{C}.$$

(b) Given a morphism of categories with weak equivalences $F\colon \mathcal{C} \to \mathcal{D}$, then F becomes a morphism of categories with denominators from $\mathcal{C}$ to $\mathcal{D}$.

Categories with denominators have been introduced to be localised. The reason why we consider a category with weak equivalences as a category with denominators is that we want to study its localisation, see definition (1.11)(b), which is unique up to isomorphism of categories.

(3.8) Definition (homotopy category)**.** We suppose given a category with weak equivalences $\mathcal{C}$. The *homotopy category* of $\mathcal{C}$ is the Gabriel-Zisman localisation $\operatorname{Ho}\mathcal{C} := \mathsf{GZ}(\mathcal{C})$.

(3.9) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. We have a functor

$$\operatorname{Ho}\colon \mathbf{CatW}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})},$$

given on the morphisms as follows. For every morphism $F\colon \mathcal{C} \to \mathcal{C}'$ in $\mathbf{CatW}_{(\mathfrak{U})}$, the morphism $\operatorname{Ho} F\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}'$ in $\mathbf{Cat}_{(\mathfrak{U})}$ is the unique morphism in $\mathbf{Cat}_{(\mathfrak{U})}$ with $\operatorname{loc}^{\operatorname{Ho}\mathcal{C}'} \circ F = (\operatorname{Ho} F) \circ \operatorname{loc}^{\operatorname{Ho}\mathcal{C}}$.

Proof. This follows from corollary (1.27)(a). □

The zero-pointed case

A zero-pointed category is a category together with a (distinguished) zero object. A morphism of zero-pointed categories is a functor that preserves the zero-objects.

(3.10) Definition (zero-pointed category with weak equivalences)**.** A *zero-pointed category with weak equivalences* consists of a category with weak equivalences $\mathcal{C}$ together with a (distinguished) zero object N in $\mathcal{C}$. By abuse of notation, we refer to the said zero-pointed category with weak equivalences as well as to its underlying category with weak equivalences just by $\mathcal{C}$. The zero object N is called the *zero object* (or the *distinguished zero object*) in $\mathcal{C}$.
Given a zero-pointed category with weak equivalences $\mathcal{C}$ with distinguished zero object N, we write $0 = 0^{\mathcal{C}} := N$.

(3.11) Remark. Given a zero-pointed category with weak equivalences $\mathcal{C}$, the homotopy category $\operatorname{Ho}\mathcal{C}$ becomes a zero-pointed category having

$$0^{\operatorname{Ho}\mathcal{C}} = 0^{\mathcal{C}}.$$

In particular, the localisation functor $\operatorname{loc}\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ becomes a morphism of zero-pointed categories.

Proof. This follows from corollary (1.20). □

The saturation of a category with weak equivalences

The interpretation of a category with weak equivalences as a category with denominators gives us the notion of the saturation of a category with weak equivalences:

(3.12) Remark. Given a category with weak equivalences $\mathcal{C}$, the saturation $\operatorname{Sat}\mathcal{C}$ becomes a category with weak equivalences having

$$\operatorname{We}\operatorname{Sat}\mathcal{C} = \operatorname{Den}\operatorname{Sat}\mathcal{C}.$$

Proof. This follows from proposition (1.43). □

2 Categories with cofibrations

Is is a hard task to study the homotopy categories of arbitrary categories with weak equivalences. To remedy this, one studies categories with weak equivalences that have additional structure such as cofibrations or, dually, fibrations. While the weak equivalences suffice to define the homotopy category, cofibrations allow us to do constructions that give us more information and additional structure on the homotopy category. For example, the description of the hom-sets of the homotopy category of a Brown cofibration category developed in section 9, see theorem (3.128), remark (3.129) and theorem (3.132), strongly relies on the notion of a cofibration. Moreover, the construction of the triangulated structure in chapter V involves Coquillen rectangles, see definition (3.101), and therefore the cofibrations.

We will introduce categories with cofibrations and weak equivalences in definition (3.30)(a) in section 3. However, there are some facts that may be deduced solely from the presence of cofibrations, and so they may be also used in examples where one has no natural notion of a weak equivalence at hand. For example, every exact category in the sense of QUILLEN [29, §2, pp. 99–100], cf. also [20, app. A], [8, def. 2.1], becomes a category with cofibrations, where the cofibrations are precisely those monomorphisms that occur as kernels in distinguished short exact sequences.

Definition of a category with cofibrations

To state the axioms of a category with cofibrations, the notion of a cofibrant object with respect to a given multiplicative subset will be defined first:

(3.13) Definition (C-cofibrant object)**.** We suppose given a category $\mathcal{C}$ and a subset $C \subseteq \operatorname{Mor}\mathcal{C}$. The full subcategory $\mathcal{C}_{C\text{-cof}}$ of $\mathcal{C}$ with

$$\operatorname{Ob}\mathcal{C}_{C\text{-cof}} = \{X \in \operatorname{Ob}\mathcal{C} \mid \text{there exists an initial object } I \text{ in } \mathcal{C} \text{ such that } \operatorname{ini}_X^I \in C\}$$

is called the *full subcategory of C-cofibrant objects* in $\mathcal{C}$. An object in $\mathcal{C}$ that lies in $\mathcal{C}_{C\text{-cof}}$ is said to be *cofibrant with respect to C* (or *C-cofibrant*).

(3.14) Definition (category with cofibrations)**.**

(a) A *category with cofibrations* ([1]) consists of a category $\mathcal{C}$ together with a subset $C \subseteq \operatorname{Mor}\mathcal{C}$ such that the following axioms hold.

(Ini_c) *Existence of a cofibrant initial object.* There exists an initial object in $\mathcal{C}$ that is C-cofibrant.

(Comp_c) *Composition axiom for cofibrations.* The subset C is closed under composition in $\mathcal{C}$.

(Iso_c) *Isomorphism axiom for cofibrations.* Every isomorphism with C-cofibrant source object is in C.

(Push_c) *Pushout axiom for cofibrations.* Given morphisms $f\colon X \to Y$ and $i\colon X \to X'$ in $\mathcal{C}_{C\text{-cof}}$ with $i \in C$, there exists a pushout rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

in $\mathcal{C}$ such that $i' \in C$.

By abuse of notation, we refer to the said category with cofibrations as well as to its underlying category just by $\mathcal{C}$. The elements of C are called *cofibrations* in $\mathcal{C}$.

Given a category with cofibrations $\mathcal{C}$ with set of cofibrations C, we write $\operatorname{Cof}\mathcal{C} := C$. In diagrams, a cofibration $i\colon X \to Y$ in $\mathcal{C}$ will usually be depicted as

$$X \overset{i}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} Y\ .$$

(b) We suppose given categories with cofibrations $\mathcal{C}$ and $\mathcal{D}$. A *morphism of categories with cofibrations* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ that *preserves cofibrations*, that is, such that Fi is a cofibration in $\mathcal{D}$ for every cofibration i in $\mathcal{C}$.

[1] In the particular case where $\mathcal{C}$ has a (distinguished) zero object and $\mathcal{C}_{C\text{-cof}} = \mathcal{C}$, cf. definition (3.29), this is called a *c-category* by HELLER [17, sec. 3] and a *category with cofibrations* by WALDHAUSEN [38, sec. 1.1].

The category of categories with cofibrations

(3.15) Definition (category with cofibrations with respect to a Grothendieck universe)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. A category with cofibrations $\mathcal{C}$ is called a *category with cofibrations with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$-*category with cofibrations*) if its underlying category is a $\mathfrak{U}$-category.

(3.16) Remark.

(a) We suppose given a Grothendieck universe $\mathfrak{U}$. A category with cofibrations $\mathcal{C}$ is a $\mathfrak{U}$-category with cofibrations if and only if it is an element of $\mathfrak{U}$.

(b) For every category with cofibrations $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a $\mathfrak{U}$-category with cofibrations.

(3.17) Remark. For every Grothendieck universe $\mathfrak{U}$ we have a category $\mathbf{CatC}_{(\mathfrak{U})}$, given as follows. The set of objects of $\mathbf{CatC}_{(\mathfrak{U})}$ is given by

$$\operatorname{Ob}\mathbf{CatC}_{(\mathfrak{U})} = \{\mathcal{C} \mid \mathcal{C} \text{ is a } \mathfrak{U}\text{-category with cofibrations}\}.$$

For objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathbf{CatC}_{(\mathfrak{U})}$, we have the hom-set

$${}_{\mathbf{CatC}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D}) = \{F \mid F \text{ is a morphism of categories with cofibrations from } \mathcal{C} \text{ to } \mathcal{D}\}.$$

For morphisms $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ in $\mathbf{CatC}_{(\mathfrak{U})}$, the composite of F and G in $\mathbf{CatC}_{(\mathfrak{U})}$ is given by the composite of the underlying functors $G \circ F\colon \mathcal{C} \to \mathcal{E}$. For an object $\mathcal{C}$ in $\mathbf{CatC}_{(\mathfrak{U})}$, the identity morphism on $\mathcal{C}$ in $\mathbf{CatC}_{(\mathfrak{U})}$ is given by the underlying identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.

(3.18) Definition (category of categories with cofibrations)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. The category $\mathbf{CatC} = \mathbf{CatC}_{(\mathfrak{U})}$ as considered in remark (3.17) is called the *category of categories with cofibrations* (more precisely, the *category of* $\mathfrak{U}$-*categories with cofibrations*).

Cofibrant objects

In a category with cofibrations $\mathcal{C}$, one has the notion of a cofibrant object with respect to the set of cofibrations $\operatorname{Cof}\mathcal{C}$, see definition (3.13). We abbreviate the terminology:

(3.19) Definition (cofibrant object)**.** We suppose given a category with cofibrations $\mathcal{C}$. The full subcategory $\mathcal{C}_{\mathrm{cof}} := \mathcal{C}_{\operatorname{Cof}(\mathcal{C})\text{-cof}}$ of $\mathcal{C}$ is called the *full subcategory of cofibrant objects* in $\mathcal{C}$. An object in $\mathcal{C}$ that lies in $\mathcal{C}_{\mathrm{cof}}$ is said to be *cofibrant*, and a morphism in $\mathcal{C}_{\mathrm{cof}}$ is called a *morphism of cofibrant objects* in $\mathcal{C}$.

Given a category with cofibrations $\mathcal{C}$, there *exists* an initial object in $\mathcal{C}$ that is cofibrant. Moreover, an object X in $\mathcal{C}$ is cofibrant if there *exists* an initial object I such that the unique morphism $\mathrm{ini}^I_X\colon I \to X$ is a cofibration in $\mathcal{C}$. The following two remarks show the independence of the notion of cofibrancy from the considered initial object. Likewise, remark (3.25) shows the independence of the considered pushout in the pushout axiom for cofibrations.

(3.20) Remark. Every initial object in a category with cofibrations is cofibrant.

Proof. We suppose given a category with cofibrations $\mathcal{C}$ and an initial object I in $\mathcal{C}$. There exists an initial object I' in $\mathcal{C}$ that is cofibrant, and so as $\mathrm{ini}^{I'}_I\colon I' \to I$ is an isomorphism in $\mathcal{C}$, it is a cofibration in $\mathcal{C}$. But this means that I is cofibrant. □

(3.21) Remark. We suppose given a category with cofibrations $\mathcal{C}$. An object X in $\mathcal{C}$ is cofibrant if and only if for every initial object I in $\mathcal{C}$, the unique morphism $\mathrm{ini}^I_X\colon I \to X$ is a cofibration in $\mathcal{C}$.

Proof. We suppose given an object X in $\mathcal{C}$. First, we suppose that X is cofibrant, that is, we suppose that there exists an initial object I' in $\mathcal{C}$ such that $\mathrm{ini}^{I'}_X\colon I' \to X$ is a cofibration. Moreover, we let I be an arbitrary initial object in $\mathcal{C}$. Then the unique morphism $\mathrm{ini}^I_{I'}\colon I \to I'$ is an isomorphism and therefore a cofibration as I is cofibrant by remark (3.20). But then also $\mathrm{ini}^I_X = \mathrm{ini}^I_{I'} \mathrm{ini}^{I'}_X$ is a cofibration as cofibrations are closed under composition. Conversely, if $\mathrm{ini}^I_X\colon I \to X$ is a cofibration for every initial object I in $\mathcal{C}$, then X is cofibrant since there exists an initial object in $\mathcal{C}$. □

If unambiguous, we will consider the full subcategory of cofibrant objects in a category with cofibrations, see definition (3.19) and definition (3.14)(a), as a category with cofibrations in the following way, without further comment.

(3.22) Remark. Given a category with cofibrations $\mathcal{C}$, the full subcategory of cofibrant objects $\mathcal{C}_{\mathrm{cof}}$ becomes a category with cofibrations having

$$\operatorname{Cof} \mathcal{C}_{\mathrm{cof}} = \operatorname{Cof} \mathcal{C} \cap \operatorname{Mor} \mathcal{C}_{\mathrm{cof}}.$$

Moreover, $\operatorname{Cof} \mathcal{C}_{\mathrm{cof}}$ is a multiplicative subset of $\operatorname{Mor} \mathcal{C}_{\mathrm{cof}}$.

(3.23) Remark. Given a category with cofibrations $\mathcal{C}$, then every cofibration in $\mathcal{C}$ with cofibrant source object has a cofibrant target object.

Proof. We suppose given a cofibration $i \colon X \to Y$ in $\mathcal{C}$ such that X is cofibrant. Then ini_X is a cofibration, and hence $\mathrm{ini}_Y = \mathrm{ini}_X \, i$ is a cofibration as cofibrations are closed under composition. Thus Y is cofibrant. □

(3.24) Corollary. Given a category with cofibrations $\mathcal{C}$, the full subcategory of cofibrant objects $\mathcal{C}_{\mathrm{cof}}$ is closed under isomorphisms in $\mathcal{C}$.

Proof. We suppose given an isomorphism $f \colon X \to Y$ in $\mathcal{C}$ such that X is cofibrant. Then f is a cofibration by the isomorphism axiom for cofibrations, and hence Y is cofibrant by remark (3.23). □

(3.25) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and a pushout rectangle

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"']
\end{tikzcd}$$

in $\mathcal{C}$ such that $f \colon X \to Y$ is a morphism and $i \colon X \to X'$ is a cofibration in $\mathcal{C}_{\mathrm{cof}}$. Then i' is a cofibration and Y' is cofibrant in $\mathcal{C}$.

Proof. As $\mathcal{C}$ is a category with cofibrations, there exists a pushout rectangle

$$\begin{tikzcd}
X' \arrow[r, "\tilde{f}'"] & \tilde{Y}' \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "\tilde{i}'"']
\end{tikzcd}$$

in $\mathcal{C}$ such that $\tilde{i}' \colon Y \to \tilde{Y}'$ is a cofibration. Then $\tilde{Y}'$ is cofibrant since Y is cofibrant and $\tilde{i}'$ is a cofibration. Moreover, since $(X, Y, X', \tilde{Y}')$ and (X, Y, X', Y') are pushout rectangles in $\mathcal{C}$, the unique morphism $g \colon \tilde{Y}' \to Y'$ with $f' = \tilde{f}' g$ and $i' = \tilde{i}' g$ is an isomorphism. By the isomorphism axiom for cofibrations, it follows that g is a cofibration as $\tilde{Y}'$ is cofibrant. In particular, $i' = \tilde{i}' g$ is a cofibration as cofibrations are closed under composition, and Y' is cofibrant by remark (3.23).

$$\begin{tikzcd}
 & & Y' \\
X' \arrow[r, "\tilde{f}'"] \arrow[urr, bend left, "f'"] & \tilde{Y}' \arrow[ur, "g", "\cong"'] & \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "\tilde{i}'"'] \arrow[uur, bend right, "i'"'] &
\end{tikzcd}$$

□

(3.26) Proposition (cf. [30, lem. 1.2.1])**.** We suppose given a category with cofibrations $\mathcal{C}$.

(a) The full subcategory of cofibrant objects $\mathcal{C}_{\mathrm{cof}}$ has finite coproducts. Given $n \in \mathbb{N}_0$ and objects X_k in $\mathcal{C}_{\mathrm{cof}}$ for $k \in [1, n]$, the coproduct $\coprod_{i \in [1,n]} X_i$ in $\mathcal{C}_{\mathrm{cof}}$ is a coproduct in $\mathcal{C}$ and the embedding $\mathrm{emb}_k \colon X_k \to \coprod_{i \in [1,n]} X_i$ is a cofibration in $\mathcal{C}$ for every $k \in [1, n]$.

(b) Given $n \in \mathbb{N}_0$ and cofibrations $i_k \colon X_k \to Y_k$ in $\mathcal{C}_{\mathrm{cof}}$ for $k \in [1, n]$, the coproduct $\coprod_{k \in [1,n]} i_k \colon \coprod_{k \in [1,n]} X_k \to \coprod_{k \in [1,n]} Y_k$ is a cofibration.

Proof.

(a) As ¡ is cofibrant, for cofibrant objects X_1, X_2 in $\mathcal{C}$, there exists a pushout C of ini_{X_1} and ini_{X_2} by the pushout axiom for cofibrations. The embeddings emb_1 and emb_2 are cofibrations and C is cofibrant by remark (3.25). Moreover, C is a coproduct of X_1 and X_2 in $\mathcal{C}$.

$$\begin{tikzcd}
X_2 \arrow[r, "\mathrm{emb}_2"] & C \\
¡ \arrow[u, "\mathrm{ini}_{X_2}"] \arrow[r, "\mathrm{ini}_{X_1}"] & X_1 \arrow[u, "\mathrm{emb}_1"']
\end{tikzcd}$$

The assertion follows by induction, using the closedness of $\mathrm{Cof}\,\mathcal{C}_{\mathrm{cof}}$ under composition and the isomorphism axiom for cofibrations.

(b) As ¡ is cofibrant, the identity morphism $1_{¡} = \mathrm{ini}_{¡} \colon ¡ \to ¡$ is a cofibration. Given cofibrations $i_1 \colon X_1 \to Y_1$, $i_2 \colon X_2 \to Y_2$ in $\mathcal{C}_{\mathrm{cof}}$, we have the following pushout rectangles, in which $i_1 \amalg 1_{X_2}$ and $1_{Y_1} \amalg i_2$ are cofibrations by remark (3.25).

$$\begin{tikzcd}
Y_1 \arrow[r, "\mathrm{emb}_1"] & Y_1 \amalg X_2 & Y_2 \arrow[r, "\mathrm{emb}_2"] & Y_1 \amalg Y_2 \\
X_1 \arrow[u, "i_1"] \arrow[r, "\mathrm{emb}_1"] & X_1 \amalg X_2 \arrow[u, "i_1 \amalg 1_{X_2}"'] & X_2 \arrow[u, "i_2"] \arrow[r, "\mathrm{emb}_2"] & Y_1 \amalg X_2 \arrow[u, "1_{Y_1} \amalg i_2"']
\end{tikzcd}$$

Thus $i_1 \amalg i_2 = (i_1 \amalg 1_{X_2})(1_{Y_1} \amalg i_2)$ is a cofibration.

For $n \in \mathbb{N}_0$ arbitrary, the assertion follows by induction. □

(3.27) Corollary. We suppose given a category with cofibrations $\mathcal{C}$, an $n \in \mathbb{N}_0$ and morphisms $i_k \colon X_k \to Y$ in $\mathcal{C}_{\mathrm{cof}}$ for $k \in [1, n]$. If $(\, i_k \,)_{k \in [1,n]} \colon \coprod_{k \in [1,n]} X_k \to Y$ is a cofibration in $\mathcal{C}$, then $i_k \colon X_k \to Y$ is a cofibration in $\mathcal{C}$ for every $k \in [1, n]$.

Proof. As X_k for $k \in [1, n]$ is cofibrant, the embedding $\mathrm{emb}_k \colon X_k \to \coprod_{k \in [1,n]} X_k$ is a cofibration by proposition (3.26)(a). So if $(\, i_k \,)_{k \in [1,n]}$ is a cofibration, then $i_l = \mathrm{emb}_l \,(\, i_k \,)_{k \in [1,n]}$ is a cofibration for every $l \in [1, n]$ by closedness under composition. □

(3.28) Proposition. We suppose given a category with cofibrations $\mathcal{C}$ and morphisms $i_1 \colon X_1 \to X$, $i_2 \colon X_2 \to X$, $f \colon X_2 \to Y$ in $\mathcal{C}_{\mathrm{cof}}$. If $\binom{i_1}{i_2} \colon X_1 \amalg X_2 \to X$ is a cofibration in $\mathcal{C}$, then

$$\begin{pmatrix} i_1 \mathrm{emb}_1^{X \amalg^{X_2} Y} \\ \mathrm{emb}_2^{X \amalg^{X_2} Y} \end{pmatrix} \colon X_1 \amalg Y \to X \amalg^{X_2} Y$$

is a cofibration in $\mathcal{C}$.

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"] & X \arrow[r, "\mathrm{emb}_1"] & X \amalg^{X_2} Y \\
& X_2 \arrow[u, "i_2"'] \arrow[r, "f"] & Y \arrow[u, "\mathrm{emb}_2"']
\end{tikzcd}$$

Proof. We have

$$\mathrm{emb}_2^{X_1 \amalg X_2}(1_{X_1} \amalg f) = f \mathrm{emb}_2^{X_1 \amalg Y},$$
$$\left(\begin{smallmatrix} i_1 \\ i_2 \end{smallmatrix}\right) \mathrm{emb}_1^{X \amalg^{X_2} Y} = \begin{pmatrix} i_1 \mathrm{emb}_1^{X \amalg^{X_2} Y} \\ i_2 \mathrm{emb}_1^{X \amalg^{X_2} Y} \end{pmatrix} = \begin{pmatrix} i_1 \mathrm{emb}_1^{X \amalg^{X_2} Y} \\ f \mathrm{emb}_2^{X \amalg^{X_2} Y} \end{pmatrix} = (1_{X_1} \amalg f) \begin{pmatrix} i_1 \mathrm{emb}_1^{X \amalg^{X_2} Y} \\ \mathrm{emb}_2^{X \amalg^{X_2} Y} \end{pmatrix},$$

that is, the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{emb}_1"] & X \amalg^{X_2} Y \\
X_1 \amalg X_2 \arrow[u, "\left(\begin{smallmatrix} i_1 \\ i_2 \end{smallmatrix}\right)"] \arrow[r, "1_{X_1} \amalg f"] & X_1 \amalg Y \arrow[u, "\left(\begin{smallmatrix} i_1 \mathrm{emb}_1 \\ \mathrm{emb}_2 \end{smallmatrix}\right)"'] \\
X_2 \arrow[u, "\mathrm{emb}_2"] \arrow[r, "f"] & Y \arrow[u, "\mathrm{emb}_2"']
\end{tikzcd}$$

commutes. So since $(X_2, Y, X_1 \amalg X_2, X_1 \amalg Y)$ and $(X_2, Y, X, X \amalg^{X_2} Y)$ are pushout rectangles, the quadrangle $(X_1 \amalg X_2, X_1 \amalg Y, X, X \amalg^{X_2} Y)$ is also a pushout rectangle, whence $\begin{pmatrix} i_1 \mathrm{emb}_1^{X \amalg^{X_2} Y} \\ \mathrm{emb}_2^{X \amalg^{X_2} Y} \end{pmatrix}$ is a cofibration by remark (3.25). □

Cofibrancy axiom

(3.29) Definition (cofibrancy axiom)**.** A category with cofibrations $\mathcal{C}$ is said to fulfill the *cofibrancy axiom* if the following holds.

(Cof) *Cofibrancy axiom.* Every object in $\mathcal{C}$ is cofibrant.

3 Categories with cofibrations and weak equivalences

In this section, we combine the notion of a category with weak equivalences from section 1 with that of a category with cofibrations from section 2 and introduce the notion of category with cofibrations and weak equivalences, see definition (3.30)(a). The two underlying structures given by the cofibrations on the one hand, and by the weak equivalences on the other hand, are completely independent so far; there are no axioms that describe the interplay between cofibrations and weak equivalences. This will be done in the next section 4, where we present some properties such a category with cofibrations and weak equivalences may fulfill.

Definition of a category with cofibrations and weak equivalences

For the definition of a category with cofibrations and of a morphism of categories with cofibrations, see definition (3.14). For the definition of a category with weak equivalences and of a morphism of categories with weak equivalences, see definition (3.1).

(3.30) Definition (category with cofibrations and weak equivalences)**.**

(a) A *category with cofibrations and weak equivalences* consists of a category $\mathcal{C}$ together with subsets $C, W \subseteq \operatorname{Mor} \mathcal{C}$ such that $\mathcal{C}$ becomes a category with cofibrations having $\operatorname{Cof} \mathcal{C} = C$ and a category with weak equivalences having $\operatorname{We} \mathcal{C} = W$.

(b) We suppose given categories with cofibrations and weak equivalences $\mathcal{C}$ and $\mathcal{D}$. A *morphism of categories with cofibrations and weak equivalences* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F \colon \mathcal{C} \to \mathcal{D}$ that is a morphism of categories with cofibrations and a morphism of categories with weak equivalences.

As for categories with weak equivalences, cf. definition (3.10), we can define a zero-pointed variant, which will become important in chapter V.

(3.31) Definition (zero-pointed category with cofibrations and weak equivalences)**.** A *zero-pointed category with cofibrations and weak equivalences* consists of a category with cofibrations and weak equivalences $\mathcal{C}$ together with a (distinguished) zero object N in $\mathcal{C}$. By abuse of notation, we refer to the said zero-pointed category with cofibrations and weak equivalences as well as to its underlying category with cofibrations and weak equivalences just by $\mathcal{C}$. The zero object N is called the *zero object* (or the *distinguished zero object*) in $\mathcal{C}$.

The category of categories with cofibrations and weak equivalences

(3.32) Definition (category with cofibrations and weak equivalences with respect to a Grothendieck universe)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. A category with cofibrations and weak equivalences $\mathcal{C}$ is called a *category with cofibrations and weak equivalences with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$-*category with cofibrations and weak equivalences*) if its underlying category is a $\mathfrak{U}$-category.

(3.33) Remark.

(a) We suppose given a Grothendieck universe $\mathfrak{U}$. A category with cofibrations and weak equivalences $\mathcal{C}$ is a $\mathfrak{U}$-category with cofibrations and weak equivalences if and only if it is an element of $\mathfrak{U}$.

(b) For every category with cofibrations and weak equivalences $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a $\mathfrak{U}$-category with cofibrations and weak equivalences.

(3.34) Remark. For every Grothendieck universe $\mathfrak{U}$ we have a category $\mathbf{CatCW}_{(\mathfrak{U})}$, given as follows. The set of objects of $\mathbf{CatCW}_{(\mathfrak{U})}$ is given by

$$\operatorname{Ob}\mathbf{CatCW}_{(\mathfrak{U})} = \{\mathcal{C} \mid \mathcal{C} \text{ is a } \mathfrak{U}\text{-category with cofibrations and weak equivalences}\}.$$

For objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathbf{CatCW}_{(\mathfrak{U})}$, we have the hom-set

$$\begin{aligned}&{}_{\mathbf{CatCW}_{(\mathfrak{U})}}(\mathcal{C}, \mathcal{D})\\ &= \{F \mid F \text{ is a morphism of categories with cofibrations and weak equivalences from } \mathcal{C} \text{ to } \mathcal{D}\}.\end{aligned}$$

For morphisms $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ in $\mathbf{CatCW}_{(\mathfrak{U})}$, the composite of F and G in $\mathbf{CatCW}_{(\mathfrak{U})}$ is given by the composite of the underlying functors $G \circ F\colon \mathcal{C} \to \mathcal{E}$. For an object $\mathcal{C}$ in $\mathbf{CatCW}_{(\mathfrak{U})}$, the identity morphism on $\mathcal{C}$ in $\mathbf{CatCW}_{(\mathfrak{U})}$ is given by the underlying identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.

(3.35) Definition (category of categories with cofibrations and weak equivalences)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. The category $\mathbf{CatCW} = \mathbf{CatCW}_{(\mathfrak{U})}$ as considered in remark (3.34) is called the *category of categories with cofibrations and weak equivalences* (more precisely, the *category of* $\mathfrak{U}$*-categories with cofibrations and weak equivalences*).

Cofibrant objects in a category with cofibrations and weak equivalences

By remark (3.22), we may consider the full subcategory of cofibrant objects in a category with cofibrations, see definition (3.19) and definition (3.14)(a), as a category with cofibrations. Likewise, if unambiguous, we will consider the full subcategory of cofibrant objects in a category with cofibrations and weak equivalences as a category with cofibrations and weak equivalences in the following way, without further comment.

(3.36) Remark. Given a category with cofibrations and weak equivalences $\mathcal{C}$, the full subcategory of cofibrant objects $\mathcal{C}_{\mathrm{cof}}$ becomes a category with cofibrations and weak equivalences having

$$\begin{aligned}&\operatorname{Cof}\mathcal{C}_{\mathrm{cof}} = \operatorname{Cof}\mathcal{C} \cap \operatorname{Mor}\mathcal{C}_{\mathrm{cof}},\\ &\operatorname{We}\mathcal{C}_{\mathrm{cof}} = \operatorname{We}\mathcal{C} \cap \operatorname{Mor}\mathcal{C}_{\mathrm{cof}}.\end{aligned}$$

The saturation of a category with cofibrations and weak equivalences

(3.37) Remark. Given a category with cofibrations and weak equivalences $\mathcal{C}$, the saturation $\operatorname{Sat}\mathcal{C}$ becomes a category with cofibrations and weak equivalences having

$$\begin{aligned}&\operatorname{Cof}\operatorname{Sat}\mathcal{C} = \operatorname{Cof}\mathcal{C},\\ &\operatorname{We}\operatorname{Sat}\mathcal{C} = \operatorname{Den}\operatorname{Sat}\mathcal{C}.\end{aligned}$$

Proof. This follows from remark (3.12). □

Acyclic cofibrations

(3.38) Definition (acyclic cofibrations)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$. A cofibration i in $\mathcal{C}$ is said to be *acyclic* if it is a weak equivalence. The set of acyclic cofibrations in $\mathcal{C}$ is denoted by $\mathrm{aCof}\,\mathcal{C} := \mathrm{Cof}\,\mathcal{C} \cap \mathrm{We}\,\mathcal{C}$.

By remark (3.7)(a), we may consider a category with weak equivalences, see definition (3.1)(a), as a category with denominators, see definition (1.1)(a). Likewise, if unambiguous, we will consider a category with cofibrations and weak equivalences that fulfills the cofibrancy axiom, see definition (3.29), as a category with D-S-denominators, see definition (2.1)(a), in the following way, without further comment.

(3.39) Remark.

(a) Given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the cofibrancy axiom, the category with denominators $\mathcal{C}$ becomes a category with D-S-denominators having

$$\mathrm{SDen}\,\mathcal{C} = \mathrm{aCof}\,\mathcal{C}.$$

(b) Given a morphism of categories with cofibrations and weak equivalences $F\colon \mathcal{C} \to \mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ fulfill the cofibrancy axiom, then F becomes a morphism of categories with D-S-denominators from $\mathcal{C}$ to $\mathcal{D}$.

4 Cofibration categories

We consider some axioms that a category with cofibrations and weak equivalences as introduced in definition (3.30)(a) may fulfill. Moreover, we discuss the relationship between some of the axioms and deduce some simple consequences of them. At the end, we define the concepts of a Cisinski cofibration category and of a Brown cofibration category, see definition (3.51)(a) and definition (3.52)(a).

The factorisation axiom for cofibrations

The factorisation axiom for cofibrations roughly states that every morphism with cofibrant source object in a given category with cofibrations and weak equivalences is a *cofibration up to an approximation by a weak equivalence.*

(3.40) Definition (factorisation axiom for cofibrations)**.** A category with cofibrations and weak equivalences $\mathcal{C}$ is said to fulfill the *factorisation axiom for cofibrations* if the following holds.

($\mathrm{Fac}_{\mathrm{c}}$) *Factorisation axiom for cofibrations.* For every morphism $f\colon X \to Y$ with X cofibrant there exist a cofibration $i\colon X \to \tilde{Y}$ and a weak equivalence $w\colon \tilde{Y} \to Y$ such that $f = iw$.

$$\begin{tikzcd}
 & \tilde{Y} \arrow[dr, dotted, "w", "\sim"'] & \\
X \arrow[ur, dotted, "i \bullet"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

(3.41) Remark. Given a category with cofibrations and weak equivalences $\mathcal{C}$, if $\mathcal{C}$ fulfills the factorisation axiom for cofibrations, then $\mathcal{C}_{\mathrm{cof}}$ fulfills the factorisation axiom for cofibrations.

(3.42) Remark. The saturation of a category with cofibrations and weak equivalences that fulfills the factorisation axiom for cofibrations also fulfills the factorisation axiom for cofibrations.

Proof. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations. Moreover, we suppose given a morphism $f\colon X \to Y$ with X cofibrant in $\mathrm{Sat}\,\mathcal{C}$. Since $\mathrm{Cof}\,\mathrm{Sat}\,\mathcal{C} = \mathrm{Cof}\,\mathcal{C}$, it follows that X is cofibrant in $\mathcal{C}$. So as $\mathcal{C}$ fulfills the factorisation axiom for cofibrations, there exists a cofibration $i\colon X \to \tilde{Y}$ and a weak equivalence $w\colon \tilde{Y} \to Y$ in $\mathcal{C}$ with $f = iw$. But since $\mathrm{Cof}\,\mathrm{Sat}\,\mathcal{C} = \mathrm{Cof}\,\mathcal{C}$ and $\mathrm{We}\,\mathcal{C} = \mathrm{Den}\,\mathcal{C} \subseteq \mathrm{Den}\,\mathrm{Sat}\,\mathcal{C} = \mathrm{We}\,\mathrm{Sat}\,\mathcal{C}$, the morphism i is also a cofibration in $\mathrm{Sat}\,\mathcal{C}$ and w is also a weak equivalence in $\mathrm{Sat}\,\mathcal{C}$. Thus $\mathrm{Sat}\,\mathcal{C}$ fulfills the factorisation axiom for cofibrations. $\square$

Gluing, excision and incision

Next, we will introduce the gluing axiom, the excision axiom and the incision axiom for categories with cofibrations and weak equivalences. In general, the gluing axiom is stronger than the excision axiom and the incision axiom, as shown in proposition (3.46). However, if a given category with cofibrations and weak equivalences is semisaturated and fulfills the factorisation axiom for cofibrations introduced in definition (3.40), then these three axioms are equivalent, see Rădulescu-Banu's criterion (3.123).

(3.43) Definition (gluing axiom, excision axiom, incision axiom)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$.

(a) We say that $\mathcal{C}$ fulfills the *gluing axiom* if the following holds.

(Glu$_\mathrm{c}$) *Gluing axiom.* We suppose given a commutative cuboid

$$\begin{tikzcd}
 & X_1' \arrow[rr, "f_1'"] \arrow[dd, "g_1'" near end] & & Y_1' \arrow[dd, "g_2'"] \\
X_1 \arrow[ur, "i_1", "\bullet"' description] \arrow[rr, "f_1" near end] \arrow[dd, "g_1"] & & Y_1 \arrow[ur, "i_1'", "\bullet"' description] \arrow[dd, "g_2" near start] & \\
 & X_2' \arrow[rr, "f_2'" near start] & & Y_2' \\
X_2 \arrow[ur, "i_2", "\bullet"' description] \arrow[rr, "f_2"] & & Y_2 \arrow[ur, "i_2'", "\bullet"' description] &
\end{tikzcd}$$

in $\mathcal{C}_\mathrm{cof}$ such that i_1 and i_2 are cofibrations and such that (X_1, Y_1, X_1', Y_1') and (X_2, Y_2, X_2', Y_2') are pushout rectangles in $\mathcal{C}$. If g_1, g_2, g_1' are weak equivalences, then so is g_2'.

(b) We say that $\mathcal{C}$ fulfills the *excision axiom* if the following holds.

(Exc$_\mathrm{c}$) *Excision axiom.* Given a pushout rectangle

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i", "\bullet"' description] \arrow[r, "f", "\approx"'] & Y \arrow[u, "i'"', "\bullet" description]
\end{tikzcd}$$

in $\mathcal{C}$ such that f is a weak equivalence and i is a cofibration in $\mathcal{C}_\mathrm{cof}$, then f' is a weak equivalence.

(c) We say that $\mathcal{C}$ fulfills the *incision axiom* if the following holds.

(Inc$_\mathrm{c}$) *Incision axiom.* Given a pushout rectangle

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i", "\approx"'] \arrow[r, "f"] & Y \arrow[u, "i'"', "\bullet" description]
\end{tikzcd}$$

in $\mathcal{C}$ such that f is a morphism in $\mathcal{C}_\mathrm{cof}$ and i is an acyclic cofibration, then i' is an acyclic cofibration.

(3.44) Remark. Given a category with cofibrations and weak equivalences $\mathcal{C}$, if $\mathcal{C}$ fulfills the gluing axiom resp. the excision axiom resp. the incision axiom, then $\mathcal{C}_\mathrm{cof}$ fulfills the gluing axiom resp. the excision axiom resp. the incision axiom.

$$
\begin{tikzcd}
(\mathrm{Exc_c}) & (\mathrm{Glu_c}) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (\mathrm{Inc_c})
\end{tikzcd}
$$

Figure 2: Gluing implies excision and incision.

By remark (3.39)(a), we may consider every category with cofibrations and weak equivalences that fulfills the cofibrancy axiom as a category with denominators and S-denominators, see definition (2.1)(a) – the denominators being the weak equivalences and the S-denominators being the acyclic cofibrations.

(3.45) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the cofibrancy axiom. If $\mathcal{C}$ fulfills the incision axiom, then the category with D-S-denominators $\mathcal{C}$ fulfills the weakly universal S-Ore completion axiom.

Proof. This follows from the pushout axiom for cofibrations and the incision axiom. □

If a semisaturated category with cofibrations and weak equivalences fulfills the factorisation axiom for cofibrations, then the gluing axiom, the excision axiom and the incision axiom are equivalent, see Rădulescu-Banu's criterion (3.123). The more elementary parts of this theorem, namely that the gluing axiom implies the excision and the incision axiom, will be proven in the following proposition. These two implications hold even if we do not require the factorisation axiom for cofibrations, so for example in any *Waldhausen cofibration category* ([2]). The statement that the excision axiom implies the gluing axiom will be proven in proposition (3.66), after we have proven the factorisation lemma (3.65). Finally, the implication that the gluing axiom can be derived from the incision axiom is known as the gluing lemma in the literature, which will be shown in section 8, see corollary (3.121). Its proof implicitly involves cylinders, which will be introduced in section 7.

(3.46) Proposition. If a category with cofibrations and weak equivalences fulfills the gluing axiom, then it fulfills the excision axiom and the incision axiom.

Proof. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the gluing axiom, and we let

$$
\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"']
\end{tikzcd}
$$

be a pushout rectangle in $\mathcal{C}$ such that f is a morphism and i is a cofibration in $\mathcal{C}_{\mathrm{cof}}$.
Since in the commutative cuboid

$$
\begin{tikzcd}
& X' \arrow[rr, equal] \arrow[dd, equal] & & X' \arrow[dd, "f'"] \\
X \arrow[ur, "i"] \arrow[rr, equal] \arrow[dd, equal] & & X \arrow[ur, "i"] \arrow[dd, "f"] & \\
& X' \arrow[rr, "f'"] & & Y' \\
X \arrow[ur, "i"] \arrow[rr, "f"] & & Y \arrow[ur, "i'"] &
\end{tikzcd}
$$

the quadrangles (X, X, X', X') and (X, Y, X', Y') are pushout rectangles, it follows that if f is a weak equivalence, then f' is a weak equivalence. Thus $\mathcal{C}$ fulfills the excision axiom.

[2] Waldhausen uses the terminology *category with cofibrations and weak equivalences* [38, sec. 1.2]. Many authors call this just a *Waldhausen category*.

Moreover, since in the commutative cuboid

$$\begin{tikzcd}
& X \arrow[rr, "f"] \arrow[dl, equal] \arrow[dd, "i"] & & X' \arrow[dl, equal] \arrow[dd, "i'"] \\
X \arrow[rr, "f"] \arrow[dd, equal] & & X' \arrow[dd, equal] & \\
& X' \arrow[rr, "f'"] & & Y' \\
X \arrow[rr, "f"] \arrow[ur, "i"] & & Y \arrow[ur, "i'"] &
\end{tikzcd}$$

the quadrangles (X, X', X, X') and (X, Y, X', Y') are pushout rectangles, it follows that if i is a weak equivalence, then i' is a weak equivalence. Thus $\mathcal{C}$ fulfills the incision axiom. □

(3.47) Proposition. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the excision axiom. Given $n \in \mathbb{N}_0$ and weak equivalences $w_k \colon X_k \to Y_k$ in $\mathcal{C}_{\mathrm{cof}}$ for $k \in [1, n]$, the coproduct $\coprod_{k\in[1,n]} w_k \colon \coprod_{k\in[1,n]} X_k \to \coprod_{k\in[1,n]} Y_k$ is a weak equivalence.

Proof. This is proven analogously to proposition (3.26)(b). □

(3.48) Proposition (cf. [30, lem. 1.2.1])**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the incision axiom or the excision axiom. Given $n \in \mathbb{N}_0$ and acyclic cofibrations $i_k \colon X_k \to Y_k$ in $\mathcal{C}_{\mathrm{cof}}$ for $k \in [1, n]$, the coproduct $\coprod_{k\in[1,n]} i_k \colon \coprod_{k\in[1,n]} X_k \to \coprod_{k\in[1,n]} Y_k$ is an acyclic cofibration.

Proof. If $\mathcal{C}$ fulfills the incision axiom, this is proven analogously to proposition (3.26)(b). If $\mathcal{C}$ fulfills the excision axiom, this follows from proposition (3.26)(b) and proposition (3.47). □

Gunnarsson's cuboid lemma

The following lemma is purely category theoretic – we do not need the specific structure of a category with cofibrations and weak equivalences. It will be used in proposition (3.50) and proposition (3.60).

(3.49) Lemma (Gunnarsson's cuboid lemma, cf. [14, proof of lem. 7.4])**.** We suppose given a category $\mathcal{C}$ and a commutative cuboid

$$\begin{tikzcd}
& X_1' \arrow[rr, "f_1'"] \arrow[dd, "h_1'"] & & Y_1' \arrow[dd, "h_2'"] \\
X_1 \arrow[rr, "f_1"] \arrow[ur, "g_1"] \arrow[dd, "h_1"] & & Y_1 \arrow[ur, "g_1'"] \arrow[dd, "h_2"] & \\
& X_2' \arrow[rr, "f_2'"] & & Y_2' \\
X_2 \arrow[rr, "f_2"] \arrow[ur, "g_2"] & & Y_2 \arrow[ur, "g_2'"] &
\end{tikzcd}$$

in $\mathcal{C}$ such that (X_1, Y_1, X_1', Y_1') and (X_2, Y_2, X_2', Y_2') are pushout rectangles. Given a pushout C of h_1 and g_1 and a pushout D of h_2 and g_1' in $\mathcal{C}$, there exist unique morphisms $C \to X_2'$, $D \to Y_2'$, $C \to D$ such that the following diagram commutes. Moreover, the quadrangles (X_2, Y_2, C, D) and (C, D, X_2', Y_2') in this diagram are pushout rectangles.

$$
\begin{tikzcd}
 & X'_1 \arrow[rr, "f'_1"] \arrow[dd] \arrow[ddd, "h'_1"] & & Y'_1 \arrow[dd] \arrow[ddd, "h'_2"] \\
X_1 \arrow[ur, "g_1"] \arrow[rr, "f_1"] \arrow[ddd, "h_1"] & & Y_1 \arrow[ur, "g'_1"] \arrow[ddd, "h_2"] & \\
 & C \arrow[rr] \arrow[d] & & D \arrow[d] \\
 & X'_2 \arrow[rr, "f'_2"] & & Y'_2 \\
X_2 \arrow[uur] \arrow[ur, "g_2"] \arrow[rr, "f_2"] & & Y_2 \arrow[uur] \arrow[ur, "g'_2"] &
\end{tikzcd}
$$

Proof. We let

$$
\begin{tikzcd}
X_1 \arrow[r, "g_1"] \arrow[d, "h_1"'] & X'_1 \arrow[d, "\tilde{h}'_1"] \\
X_2 \arrow[r, "\tilde{g}_2"] & C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Y_1 \arrow[r, "g'_1"] \arrow[d, "h_2"'] & Y'_1 \arrow[d, "\tilde{h}'_2"] \\
Y_2 \arrow[r, "\tilde{g}'_2"] & D
\end{tikzcd}
$$

be pushout rectangles. Moreover, we let $k_1 \colon C \to X'_2$ be the unique morphism with $h'_1 = \tilde{h}'_1 k_1$ and $g_2 = \tilde{g}_2 k_1$, we let $k_2 \colon D \to Y'_2$ be the unique morphism with $h'_2 = \tilde{h}'_2 k_2$ and $g'_2 = \tilde{g}'_2 k_2$, and we let $f \colon C \to D$ be the unique morphism with $f_2 \tilde{g}'_2 = \tilde{g}_2 f$ and $f'_1 \tilde{h}'_2 = \tilde{h}'_1 f$. Then we have

$$
\begin{aligned}
&\tilde{g}_2 f k_2 = f_2 \tilde{g}'_2 k_2 = f_2 g'_2 = g_2 f'_2 = \tilde{g}_2 k_1 f'_2, \\
&\tilde{h}'_1 f k_2 = f'_1 \tilde{h}_2 k_2 = f'_1 h'_2 = h'_1 f'_2 = \tilde{h}'_1 k_1 f'_2,
\end{aligned}
$$

and therefore $f k_2 = k_1 f'_2$.

$$
\begin{tikzcd}
 & X'_1 \arrow[rr, "f'_1"] \arrow[dd, "\tilde{h}'_1"'] \arrow[ddd, "h'_1"] & & Y'_1 \arrow[dd, "\tilde{h}'_2"'] \arrow[ddd, "h'_2"] \\
X_1 \arrow[ur, "g_1"] \arrow[rr, "f_1"] \arrow[ddd, "h_1"] & & Y_1 \arrow[ur, "g'_1"] \arrow[ddd, "h_2"] & \\
 & C \arrow[rr, "f"] \arrow[d, "k_1"'] & & D \arrow[d, "k_2"'] \\
 & X'_2 \arrow[rr, "f'_2"] & & Y'_2 \\
X_2 \arrow[uur, "\tilde{g}_2"] \arrow[ur, "g_2"] \arrow[rr, "f_2"] & & Y_2 \arrow[uur, "\tilde{g}'_2"] \arrow[ur, "g'_2"] &
\end{tikzcd}
$$

In the diagram above, (X_1, Y_1, X'_1, Y'_1) and (Y_1, Y_2, Y'_1, D) are pushout rectangles, whence (X_1, Y_2, X'_1, D) is a pushout rectangle. Further, since (X_1, X_2, X'_1, C) and (X_1, Y_2, X'_1, D) are pushout rectangles, it follows that (X_2, Y_2, C, D) is a pushout rectangle. Finally, as (X_2, Y_2, C, D) and (X_2, Y_2, X'_2, Y'_2) are pushout rectangles, we conclude that (C, D, X'_2, Y'_2) is a pushout rectangle. □

(3.50) Proposition. We suppose given an S-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the excision axiom, and we suppose given a commutative cuboid

$$\begin{tikzcd}
& X'_1 \arrow[rr, "f'_1"] \arrow[dd, "g'_1"] & & Y'_1 \arrow[dd, "g'_2"] \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1"] \arrow[dd, "g_1"] & & Y_1 \arrow[ur, "i'_1"] \arrow[dd, "g_2"] & \\
& X'_2 \arrow[rr, "j'_2"] & & Y'_2 \\
X_2 \arrow[ur, "i_2"] \arrow[rr, "j_2"] & & Y_2 \arrow[ur, "i'_2"] &
\end{tikzcd}$$

in $\mathcal{C}_{\mathrm{cof}}$ such that i_1, i_2 and j_2 are cofibrations and such that (X_1, Y_1, X'_1, Y'_1) and (X_2, Y_2, X'_2, Y'_2) are pushout rectangles in $\mathcal{C}$. If g_1, g_2, g'_1 are weak equivalences, then so is g'_2.

Proof. We let

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"] \arrow[d, "g_1"'] & X'_1 \arrow[d, "\tilde{g}'_1"] \\
X_2 \arrow[r, "\tilde{i}_2"] & C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Y_1 \arrow[r, "i'_1"] \arrow[d, "g_2"'] & Y'_1 \arrow[d, "\tilde{g}'_2"] \\
Y_2 \arrow[r, "\tilde{i}'_2"] & D
\end{tikzcd}$$

be pushout rectangles. We let $h_1 \colon C \to X'_2$ and $h_2 \colon D \to Y'_2$ and $j \colon C \to D$ be the unique morphisms such that the following diagram commutes.

$$\begin{tikzcd}
& X'_1 \arrow[rr, "f'_1"] \arrow[dd, "\tilde{g}'_1"'] \arrow[ddd, "g'_1"] & & Y'_1 \arrow[dd, "\tilde{g}'_2"'] \arrow[ddd, "g'_2"] \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1"] \arrow[ddd, "g_1"] & & Y_1 \arrow[ur, "i'_1"] \arrow[ddd, "g_2"] & \\
& C \arrow[rr, "j"] \arrow[d, "h_1"'] & & D \arrow[d, "h_2"'] \\
& X'_2 \arrow[rr, "j'_2"] & & Y'_2 \\
X_2 \arrow[uur, "\tilde{i}_2"] \arrow[ur, "i_2"'] \arrow[rr, "j_2"] & & Y_2 \arrow[uur, "\tilde{i}'_2"] \arrow[ur, "i'_2"'] &
\end{tikzcd}$$

By Gunnarsson's cuboid lemma (3.49), the quadrangles (X_2, Y_2, C, D) and (C, D, X'_2, Y'_2) are pushout rectangles. In particular, j is a cofibration. As $\mathcal{C}$ fulfills the excision axiom, if g_1 resp. g_2 resp. h_1 is a weak equivalence, then $\tilde{g}'_1$ resp. $\tilde{g}'_2$ resp. h_2 is a weak equivalence. Thus if g_1, g_2 and g'_1 are weak equivalences, then $\tilde{g}'_1$ and $\tilde{g}'_2$ are weak equivalences, therefore h_1 is a weak equivalence by S-semisaturatedness, hence h_2 is a weak equivalence, and finally $g'_2 = \tilde{g}'_2 h_2$ is a weak equivalence by multiplicativity. □

Cisinski cofibration categories and Brown cofibration categories

Now we can give the definitions of a Cisinski cofibration category and of a Brown cofibration category. For the definition of a category with cofibrations and weak equivalences, see definition (3.30)(a), and for the definition of semisaturatedness, see definition (1.37)(b), cf. also remark (3.7)(a).

(3.51) Definition (Cisinski cofibration category)**.**

(a) A *Cisinski cofibration category* ([3]) is a semisaturated category with cofibrations and weak equivalences that fulfills the factorisation axiom for cofibrations and the incision axiom.

(b) We suppose given Cisinski cofibration categories $\mathcal{C}$ and $\mathcal{D}$. A *morphism of Cisinski cofibration categories* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of categories with cofibrations and weak equivalences from $\mathcal{C}$ to $\mathcal{D}$.

(c) We suppose given a Grothendieck universe $\mathfrak{U}$. The full subcategory $\mathbf{CisCofCat} = \mathbf{CisCofCat}_{(\mathfrak{U})}$ of $\mathbf{CatCW}_{(\mathfrak{U})}$ with

$$\operatorname{Ob}\mathbf{CisCofCat}_{(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob}\mathbf{CatCW}_{(\mathfrak{U})} \mid \mathcal{C} \text{ is a Cisinski cofibration category}\}$$

is called the *category of Cisinski cofibration categories* (more precisely, the *category of* $\mathfrak{U}$*-Cisinski cofibration categories*). An object in $\mathbf{CisCofCat}_{(\mathfrak{U})}$ is called a $\mathfrak{U}$*-Cisinski cofibration category*, and a morphism in $\mathbf{CisCofCat}_{(\mathfrak{U})}$ is called a $\mathfrak{U}$*-morphism of Cisinski cofibration categories*.

(d) The full subcategory $\mathbf{CatW}^{\mathrm{co}}_{\mathrm{Cis}} = \mathbf{CatW}^{\mathrm{co}}_{\mathrm{Cis},(\mathfrak{U})}$ of $\mathbf{CatW}_{(\mathfrak{U})}$ with

$$\operatorname{Ob}\mathbf{CatW}^{\mathrm{co}}_{\mathrm{Cis},(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob}\mathbf{CatW}_{(\mathfrak{U})} \mid \text{there exists } C \subseteq \operatorname{Mor}\mathcal{C} \text{ such that } \mathcal{C} \text{ becomes a Cisinski cofibration category with } \operatorname{Cof}\mathcal{C} = C\},$$

is called the *category of categories with weak equivalences admitting the structure of a Cisinski cofibration category* (more precisely, the *category of* $\mathfrak{U}$*-categories with weak equivalences admitting the structure of a Cisinski cofibration category*).

A category with cofibrations is said to fulfill the cofibrancy axiom, see definition (3.29), if all of its objects are cofibrant.

(3.52) Definition (Brown cofibration category)**.**

(a) A *Brown cofibration category* ([4]) is a Cisinski cofibration category $\mathcal{C}$ that fulfills the cofibrancy axiom.

(b) We suppose given Brown cofibration categories $\mathcal{C}$ and $\mathcal{D}$. A *morphism of Brown cofibration categories* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of categories with cofibrations and weak equivalences from $\mathcal{C}$ to $\mathcal{D}$.

(c) We suppose given a Grothendieck universe $\mathfrak{U}$. The full subcategory $\mathbf{BrCofCat} = \mathbf{BrCofCat}_{(\mathfrak{U})}$ of $\mathbf{CatCW}_{(\mathfrak{U})}$ with

$$\operatorname{Ob}\mathbf{BrCofCat}_{(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob}\mathbf{CatCW}_{(\mathfrak{U})} \mid \mathcal{C} \text{ is a Brown cofibration category}\}$$

is called the *category of Brown cofibration categories* (more precisely, the *category of* $\mathfrak{U}$*-Brown cofibration categories*). An object in $\mathbf{BrCofCat}_{(\mathfrak{U})}$ is called a $\mathfrak{U}$*-Brown cofibration category*, and a morphism in $\mathbf{BrCofCat}_{(\mathfrak{U})}$ is called a $\mathfrak{U}$*-morphism of Brown cofibration categories*.

(d) The full subcategory $\mathbf{CatW}^{\mathrm{co}}_{\mathrm{Br}} = \mathbf{CatW}^{\mathrm{co}}_{\mathrm{Br},(\mathfrak{U})}$ of $\mathbf{CatW}_{(\mathfrak{U})}$ with

$$\operatorname{Ob}\mathbf{CatW}^{\mathrm{co}}_{\mathrm{Br},(\mathfrak{U})} = \{\mathcal{C} \in \operatorname{Ob}\mathbf{CatW}_{(\mathfrak{U})} \mid \text{there exists } C \subseteq \operatorname{Mor}\mathcal{C} \text{ such that } \mathcal{C} \text{ becomes a Brown cofibration category with } \operatorname{Cof}\mathcal{C} = C\},$$

is called the *category of categories with weak equivalences admitting the structure of a Brown cofibration category* (more precisely, the *category of* $\mathfrak{U}$*-categories with weak equivalences admitting the structure of a Brown cofibration category*).

[3] Cisinski uses the terminology *catégorie dérivable à droite* (*right derivable category*) [9, sec. 2.22]. Rădulescu-Banu uses the terminology (*Anderson-Brown-Cisinski*) *precofibration category* [30, def. 1.1.1].

[4] In the dual situation, K. Brown uses the terminology *category of fibrant objects* [7, sec. 1, p. 420].

There are other definitions of cofibration categories by several authors, which are more or less similar to those of Cisinski and Brown. A precise comparison between some of them can be found in Rădulescu-Banu's manuscript [30, ch. 2].

Every Brown cofibration category is a Cisinski cofibration category by definition. On the other hand, we have:

(3.53) Remark. Given a Cisinski cofibration category $\mathcal{C}$, the full subcategory of cofibrant objects $\mathcal{C}_{\mathrm{cof}}$ is a Brown cofibration category.

By a theorem of Cisinski [9, prop. 1.8], which is a variant of Quillen's homotopy category theorem [28, ch. I, sec. 1, th. 1], the homotopy category of a Cisinski cofibration category $\mathcal{C}$ and its full subcategory of cofibrant objects $\mathcal{C}_{\mathrm{cof}}$ are equivalent.

A zero-pointed Cisinski cofibration category is a zero-pointed category with cofibrations and weak equivalences whose underlying category with cofibrations and weak equivalences is a Cisinski cofibration category, that is, a Cisinski cofibration category together with a (distinguished) zero object. Likewise, a zero-pointed Brown cofibration category is a zero-pointed category with cofibrations and weak equivalences whose underlying category with cofibrations and weak equivalences is a Brown cofibration category, that is, a Brown cofibration category together with a (distinguished) zero object.

5 Coreedian rectangles

In this section, we introduce coreedian rectangles and study their properties. They will occur in the construction of a structure of a category with cofibrations on a diagram category, see definition (3.82)(a) and definition (3.88), in the gluing lemma for cofibrations and acyclic cofibrations (3.61), as well as in several factorisation lemmata, see for example the Brown factorisation lemma (3.113).

Definition of a coreedian rectangle

(3.54) Definition ((acyclicly) coreedian rectangle)**.**

(a) We suppose given a category with cofibrations $\mathcal{C}$. A *Coreedy rectangle* (or *coreedian rectangle* or *coreedian quadrangle*) in $\mathcal{C}$ is a commutative quadrangle X in $\mathcal{C}_{\mathrm{cof}}$ such that $X_{(0,0),(0,1)}$ is a cofibration and such that there exists a pushout C of $X|_{\llcorner}$ in $\mathcal{C}$ such that the induced morphism

$$\begin{pmatrix} X_{(1,0),(1,1)} \\ X_{(0,1),(1,1)} \end{pmatrix}^{C} : C \to X_{(1,1)}$$

is a cofibration.

$$\begin{tikzcd}
X_{0,1} \arrow[rr] \arrow[dr, dotted] & & X_{1,1} \\
& C \arrow[ur, dotted, "\bullet" description] & \\
X_{0,0} \arrow[rr] \arrow[uu, "\bullet" description] & & X_{1,0} \arrow[uu, "\bullet" description] \arrow[ul, dotted, "\bullet" description]
\end{tikzcd}$$

(b) We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$. An *acyclic Coreedy rectangle* (or *acyclicly coreedian rectangle* or *acyclicly coreedian quadrangle*) in $\mathcal{C}$ is a commutative quadrangle X in $\mathcal{C}_{\mathrm{cof}}$ such that $X_{(0,0),(0,1)}$ is a cofibration and such that there exists a pushout C of $X|_{\llcorner}$ in $\mathcal{C}$ such that the induced morphism

$$\begin{pmatrix} X_{(1,0),(1,1)} \\ X_{(0,1),(1,1)} \end{pmatrix}^{C} : C \to X_{(1,1)}$$

is an acyclic cofibration.

$$\begin{tikzcd}
X_{0,1} \arrow[rr] \arrow[dr, dotted] & & X_{1,1} \\
& C \arrow[ur, dotted, "\bullet" marking, "\sim"'] & \\
X_{0,0} \arrow[rr] \arrow[uu, "\bullet" marking] & & X_{1,0} \arrow[uu, "\bullet" marking] \arrow[ul, dotted, "\bullet" marking]
\end{tikzcd}$$

The definition of a coreedian rectangle is *not* symmetric. So if we say that a quadrangle (X_1, Y_1, X_2, Y_2) is coreedian, then the morphism $X_1 \to X_2$ is meant to be a cofibration. However, we will sometimes be slightly unprecise when we draw a quadrangle (which might occur from the data of another diagram) and say that this quadrangle is coreedian. In this case, we will see which of the morphisms are cofibrations from the respective situation.

(3.55) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and a commutative quadrangle X in $\mathcal{C}_{\mathrm{cof}}$ such that $X_{(0,0),(0,1)}$ is a cofibration. Moreover, we suppose given a pushout C of $X|_{\llcorner}$ in $\mathcal{C}$.

(a) If X is coreedian, then C is cofibrant and $\begin{pmatrix} X_{(1,0),(1,1)} \\ X_{(0,1),(1,1)} \end{pmatrix}^C : C \to X_{(1,1)}$ is a cofibration.

(b) We suppose that $\mathcal{C}$ carries the structure of a category with cofibrations and weak equivalences. If X is acyclicly coreedian, then C is cofibrant and $\begin{pmatrix} X_{(1,0),(1,1)} \\ X_{(0,1),(1,1)} \end{pmatrix}^C : C \to X_{(1,1)}$ is an acyclic cofibration.

Proof.

(a) We suppose that X is coreedian, so that there exists a pushout $\tilde{C}$ of $X|_{\llcorner}$ in $\mathcal{C}$ such that $\tilde{C}$ is cofibrant and $\begin{pmatrix} X_{(1,0),(1,1)} \\ X_{(0,1),(1,1)} \end{pmatrix}^{\tilde{C}} : \tilde{C} \to X_{(1,1)}$ is a cofibration. Then $\begin{pmatrix} \mathrm{emb}_1^C \\ \mathrm{emb}_2^C \end{pmatrix}^{\tilde{C}} : \tilde{C} \to C$ is an isomorphism and therefore a cofibration by the isomorphism axiom for cofibrations. In particular, C is cofibrant. Moreover, $\begin{pmatrix} \mathrm{emb}_1^{\tilde{C}} \\ \mathrm{emb}_2^{\tilde{C}} \end{pmatrix}^C : C \to \tilde{C}$ is a cofibration, and so

$$\begin{pmatrix} X_{(1,0),(1,1)} \\ X_{(0,1),(1,1)} \end{pmatrix}^C = \begin{pmatrix} \mathrm{emb}_1^{\tilde{C}} \\ \mathrm{emb}_2^{\tilde{C}} \end{pmatrix}^C \begin{pmatrix} X_{(1,0),(1,1)} \\ X_{(0,1),(1,1)} \end{pmatrix}^{\tilde{C}}$$

is a cofibration by closedness under composition.

(b) This is proven analogously to (a). □

(3.56) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and a commutative quadrangle X in $\mathcal{C}_{\mathrm{cof}}$.

(a) We suppose that X is coreedian. Then $X_{(1,0),(1,1)}$ is a cofibration. If, in addition, $X_{(0,0),(1,0)}$ is a cofibration, then $X_{(0,1),(1,1)}$ is a cofibration.

(b) We suppose that $\mathcal{C}$ carries the structure of a category with cofibrations and weak equivalences that fulfills the incision axiom. Moreover, we suppose that X is acyclicly coreedian. Then $X_{(1,0),(1,1)}$ is an acyclic cofibration. If, in addition, $X_{(0,0),(1,0)}$ is an acyclic cofibration, then $X_{(0,1),(1,1)}$ is an acyclic cofibration.

Proof.

(a) This follows from remark (3.25) and closedness under composition.

(b) This follows from the incision axiom and closedness under composition. □

(3.57) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and a commutative quadrangle X in $\mathcal{C}_{\mathrm{cof}}$ such that $X_{(0,0),(0,1)}$ is an isomorphism.

(a) The quadrangle X is coreedian if and only if $X_{(1,0),(1,1)}$ is a cofibration in $\mathcal{C}$.

(b) We suppose that $\mathcal{C}$ carries the structure of a category with cofibrations and weak equivalences. The quadrangle X is acyclicly coreedian if and only if $X_{(1,0),(1,1)}$ is an acyclic cofibration in $\mathcal{C}$.

Proof. This follows from remark (3.55) as

$$\begin{tikzcd}
X_{(0,1)} \arrow[r, "X^{-1}_{(0,0),(0,1)} X_{(0,0),(1,0)}"] & X_{(1,0)} \\
X_{(0,0)} \arrow[u, "\cong"] \arrow[r] & X_{(1,0)} \arrow[u, equal]
\end{tikzcd}$$

is a pushout rectangle in $\mathcal{C}$. □

(3.58) Corollary.

(a) We suppose given a category with cofibrations $\mathcal{C}$. A morphism $i\colon X \to Y$ in $\mathcal{C}_{\text{cof}}$ is a cofibration if and only if

$$\begin{tikzcd}
¡ \arrow[r, "\bullet" marking] & Y \\
¡ \arrow[u, equal] \arrow[r, "\bullet" marking] & X \arrow[u, "i"']
\end{tikzcd}$$

is a Coreedy rectangle.

(b) We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$. A morphism $i\colon X \to Y$ in $\mathcal{C}_{\text{cof}}$ is an acyclic cofibration if and only if

$$\begin{tikzcd}
¡ \arrow[r, "\bullet" marking] & Y \\
¡ \arrow[u, equal] \arrow[r, "\bullet" marking] & X \arrow[u, "i"']
\end{tikzcd}$$

is an acyclic Coreedy rectangle.

(3.59) Proposition. We suppose given a category with cofibrations $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd}
X_2 \arrow[r, "f_2"] & Y_2 \arrow[r, "g_2"] & Z_2 \\
X_1 \arrow[u, "h_1"'] \arrow[r, "f_1"] & Y_1 \arrow[u, "h_2"'] \arrow[r, "g_1"] & Z_1 \arrow[u, "h_3"']
\end{tikzcd}$$

in $\mathcal{C}_{\text{cof}}$.

(a) If (X_1, Y_1, X_2, Y_2) and (Y_1, Z_1, Y_2, Z_2) are coreedian rectangles, then (X_1, Z_1, X_2, Z_2) is also a coreedian rectangle. If (X_1, X_2, Y_1, Y_2) and (Y_1, Y_2, Z_1, Z_2) are coreedian rectangles, then (X_1, X_2, Z_1, Z_2) is also a coreedian rectangle.

(b) We suppose that $\mathcal{C}$ carries the structure of a category with cofibrations and weak equivalences that fulfills the incision axiom. If (X_1, Y_1, X_2, Y_2) and (Y_1, Z_1, Y_2, Z_2) are acyclicly coreedian rectangles, then (X_1, Z_1, X_2, Z_2) is also an acyclicly coreedian rectangle. If (X_1, X_2, Y_1, Y_2) and (Y_1, Y_2, Z_1, Z_2) are acyclicly coreedian rectangle, then (X_1, X_2, Z_1, Z_2) is also an acyclicly coreedian rectangle.

Proof.

(a) We suppose that (X_1, Y_1, X_2, Y_2) and (Y_1, Z_1, Y_2, Z_2) or that (X_1, X_2, Y_1, Y_2) and (Y_1, Y_2, Z_1, Z_2) are Coreedy rectangles, so that there exist pushout rectangles

$$\begin{tikzcd}
X_2 \arrow[r, "f'_1"] & C_1 \\
X_1 \arrow[u, "h_1"] \arrow[r, "f_1"] & Y_1 \arrow[u, "h'_1"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
Y_2 \arrow[r, "g'_1"] & C_2 \\
Y_1 \arrow[u, "h_2"] \arrow[r, "g_1"] & Z_1 \arrow[u, "h'_2"']
\end{tikzcd}$$

in $\mathcal{C}$ such that C_1 and C_2 are cofibrant and such that the unique morphism $i_1\colon C_1 \to Y_2$ with $h_2 = h'_1 i_1$ and $f_2 = f'_1 i_1$ and the unique morphism $i_2\colon C_2 \to Z_2$ with $h_3 = h'_2 i_2$ and $g_2 = g'_1 i_2$ are cofibrations.

$$\begin{tikzcd}
X_2 \arrow[rr, "f_2"] \arrow[dr, "f'_1"'] & & Y_2 \arrow[rr, "g_2"] \arrow[dr, "g'_1"'] & & Z_2 \\
& C_1 \arrow[ur, "i_1"] & & C_2 \arrow[ur, "i_2"] & \\
X_1 \arrow[uu, "h_1"'] \arrow[rr, "f_1"] & & Y_1 \arrow[uu, "h_2"'] \arrow[ul, "h'_1"] \arrow[rr, "g_1"] & & Z_1 \arrow[uu, "h_3"'] \arrow[ul, "h'_2"]
\end{tikzcd}$$

Moreover, as $f_1 g_1$ or h_1 is a cofibration, there exists a pushout rectangle

$$\begin{tikzcd}
X_2 \arrow[r, "k"] & Z_1 \\
X_1 \arrow[u, "h_1"] \arrow[r, "f_1 g_1"] & C \arrow[u, "h''_1"']
\end{tikzcd}$$

in $\mathcal{C}$ such that C is cofibrant. Since $h_1 k = f_1 g_1 h''_1$, there exists a unique morphism $g''_1\colon C_1 \to C$ with $k = f'_1 g''_1$ and $g_1 h''_1 = h'_1 g''_1$.

$$\begin{tikzcd}
X_2 \arrow[r, "f'_1"'] \arrow[rr, bend left, "k"] & C_1 \arrow[r, "g''_1"'] & C \\
X_1 \arrow[u, "h_1"'] \arrow[r, "f_1"] & Y_1 \arrow[u, "h'_1"'] \arrow[r, dotted, "g_1"] & Z_1 \arrow[u, "h''_1"']
\end{tikzcd}$$

As (X_1, Y_1, X_2, C_1) and (X_1, Z_1, X_2, C) are pushout rectangles, we conclude that (Y_1, Z_1, C_1, C) is a pushout rectangle. So since $h'_1 i_1 g'_1 = h_2 g'_1 = g_1 h'_2$, there exists a unique morphism $i\colon C \to C_2$ with $i_1 g'_1 = g''_1 i$ and $h'_2 = h''_1 i$.

$$\begin{tikzcd}
X_2 \arrow[rrrr, "f_2"] \arrow[drrr, "f'_1"'] & & & & Y_2 \arrow[rrrr, "g_2"] \arrow[drrr, "g'_1"'] & & & & Z_2 \\
& & & C_1 \arrow[ur, "i_1"] \arrow[drrr, "g''_1"'] & & & & C_2 \arrow[ur, "i_2"] & \\
& & & & & & C \arrow[ur, dotted, "i"] & & \\
X_1 \arrow[uuu, "h_1"'] \arrow[rrrr, "f_1"] & & & & Y_1 \arrow[uuu, "h_2"'] \arrow[uul, "h'_1"] \arrow[rrrr, "g_1"] & & & & Z_1 \arrow[uuu, "h_3"'] \arrow[uul, "h'_2"] \arrow[ull, "h''_1"]
\end{tikzcd}$$

Since (Y_1, Z_1, C_1, C) and (Y_1, Z_1, Y_2, C_2) are pushout rectangles, it follows that (C_1, C, Y_2, C_2) is a pushout rectangle. But then i is a cofibration since i_1 is a cofibration. Finally, we have

$$\begin{aligned} &kii_2 = f_1' g_1'' ii_2 = f_1' i_1 g_1' i_2 = f_2 g_2, \\ &h_1'' ii_2 = h_2' i_2 = h_3, \end{aligned}$$

and ii_2 is a cofibration as cofibrations are closed under composition. Thus (X_1, Z_1, X_2, Z_2) resp. (X_1, X_2, Z_1, Z_2) is coreedian.

(b) This is proven analogously to (a). □

The gluing lemma for cofibrations and acyclic cofibrations

Next, we are going to prove the gluing lemma for cofibrations (3.61)(a) and the gluing lemma for acyclic cofibrations (3.61)(b), which may be seen as the building blocks of the (ordinary) gluing lemma, see corollary (3.121) and its proof.

(3.60) Proposition. We suppose given a category with cofibrations $\mathcal{C}$ and a commutative cuboid

$$\begin{tikzcd}
& X_1' \arrow[rr, "f_1'"] \arrow[dd, "g_1'"] & & Y_1' \arrow[dd, "g_2'"] \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1"] \arrow[dd, "g_1"] & & Y_1 \arrow[ur, "i_1'"] \arrow[dd, "g_2"] & \\
& X_2' \arrow[rr, "f_2'"] & & Y_2' \\
X_2 \arrow[ur, "i_2"] \arrow[rr, "f_2"] & & Y_2 \arrow[ur, "i_2'"] &
\end{tikzcd}$$

in $\mathcal{C}_{\mathrm{cof}}$ such that i_1 and i_2 are cofibrations and such that (X_1, Y_1, X_1', Y_1') and (X_2, Y_2, X_2', Y_2') are pushout rectangles.

(a) If (X_1, X_2, X_1', X_2') is coreedian, then (Y_1, Y_2, Y_1', Y_2') is also coreedian.

(b) We suppose that $\mathcal{C}$ carries the structure of a category with cofibrations and weak equivalences that fulfills the incision axiom. If (X_1, X_2, X_1', X_2') is acyclicly coreedian, then (Y_1, Y_2, Y_1', Y_2') is also acyclicly coreedian.

Proof.

(a) We suppose that (X_1, X_2, X_1', X_2') is a Coreedy rectangle, so that there exists in particular a pushout C of g_1 and i_1 that is cofibrant. Moreover, as i_1' is a cofibration, there exists a pushout D of g_2 and i_1' that is cofibrant. By Gunnarson's cuboid lemma (3.49), the following quadrangle, where $f\colon C \to D$ is the unique morphism on the pushouts induced by f_1, f_2, f_1', is a pushout rectangle in $\mathcal{C}$.

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\binom{i_2}{g_1'}^C"'] & D \arrow[d, "\binom{i_2'}{g_2'}^D"] \\
X_2' \arrow[r, "f_2'"] & Y_2'
\end{tikzcd}$$

So as $\binom{i_2}{g_1'}^C$ is a cofibration, it follows that $\binom{i_2'}{g_2'}^D$ is also a cofibration by remark (3.25). Thus the quadrangle (Y_1, Y_2, Y_1', Y_2') is coreedian.

(b) This is proven analogously to (a). □

(3.61) Corollary (gluing lemma for (acyclic) cofibrations)**.** We suppose given a category with cofibrations $\mathcal{C}$ and a commutative cuboid

$$\begin{tikzcd}
& X'_1 \arrow[rr, "f'_1"] \arrow[dd, "g'_1" near start] & & Y'_1 \arrow[dd, "g'_2"] \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1" near start] \arrow[dd, "g_1"] & & Y_1 \arrow[ur, "i'_1"] \arrow[dd, "g_2" near start] & \\
& X'_2 \arrow[rr, "f'_2" near start] & & Y'_2 \\
X_2 \arrow[ur, "i_2"] \arrow[rr, "f_2"] & & Y_2 \arrow[ur, "i'_2"] &
\end{tikzcd}$$

in $\mathcal{C}_{\mathrm{cof}}$ such that i_1 and i_2 are cofibrations and such that (X_1, Y_1, X'_1, Y'_1) and (X_2, Y_2, X'_2, Y'_2) are pushout rectangles.

(a) If g_1 and g_2 are cofibrations and (X_1, X_2, X'_1, X'_2) is coreedian, then g'_1 and g'_2 are cofibrations and (Y_1, Y_2, Y'_1, Y'_2) is coreedian.

(b) We suppose that $\mathcal{C}$ carries the structure of a category with cofibrations and weak equivalences that fulfills the incision axiom. If g_1 and g_2 are acyclic cofibrations and (X_1, X_2, X'_1, X'_2) is acyclicly coreedian, then g'_1 and g'_2 are acyclic cofibrations and (Y_1, Y_2, Y'_1, Y'_2) is acyclicly coreedian.

Proof.

(a) This follows from proposition (3.60)(a) and remark (3.56)(a).

(b) This follows from proposition (3.60)(b) and remark (3.56)(b). □

Alternative proof for proposition (3.26)(b). For $n = 2$, the assertion follows from corollary (3.58)(a) and the gluing lemma for cofibrations (3.61)(a).

$$\begin{tikzcd}
& X_2 \arrow[rr, "\mathrm{emb}_2"] \arrow[dd, "i_2" near start] & & X_1 \amalg X_2 \arrow[dd, "i_1 \amalg i_2"] \\
¡ \arrow[ur, "\mathrm{ini}_{X_2}"] \arrow[rr, "\mathrm{ini}_{X_1}" near end] \arrow[dd, equal] & & X_1 \arrow[ur, "\mathrm{emb}_1"] \arrow[dd, "i_1" near start] & \\
& Y_2 \arrow[rr, "\mathrm{emb}_2" near start] & & Y_1 \amalg Y_2 \\
¡ \arrow[ur, "\mathrm{ini}_{Y_2}"] \arrow[rr, "\mathrm{ini}_{Y_1}"] & & Y_1 \arrow[ur, "\mathrm{emb}_1"] &
\end{tikzcd}$$

For $n \in \mathbb{N}_0$ arbitrary, the assertion follows by induction. □

Analogously, the gluing lemma for acyclic cofibrations (3.61)(b) yields an alternative proof for proposition (3.48).

Coreedian rectangles and coproducts

(3.62) Proposition. We suppose given a category with cofibrations $\mathcal{C}$ and a commutative quadrangle

$$\begin{tikzcd}
X_1 \amalg X_2 \arrow[r, "\binom{f_1}{f_2}"] \arrow[d, "h_1 \amalg h_2"'] & X \arrow[d, "h"] \\
Y_1 \amalg Y_2 \arrow[r, "\binom{g_1}{g_2}"] & Y
\end{tikzcd}$$

in $\mathcal{C}_{\mathrm{cof}}$ with X_1, X_2, Y_1, Y_2 cofibrant.

(a) If h_2 is a cofibration and $(X_1 \amalg X_2, Y_1 \amalg Y_2, X, Y)$ is a coreedian rectangle, then (X_1, Y_1, X, Y) is a coreedian rectangle. If h_1 and h_2 are cofibrations and $(X_1 \amalg X_2, X, Y_1 \amalg Y_2, Y)$ is a coreedian rectangle, then (X_1, X, Y_1, Y) is a coreedian rectangle.

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & X \arrow[d, "h"] \\
Y_1 \arrow[r, "g_1"] & Y
\end{tikzcd}$$

(b) We suppose that $\mathcal{C}$ carries the structure of a category with cofibrations and weak equivalences that fulfills the incision axiom. If h_2 is an acyclic cofibration and $(X_1 \amalg X_2, Y_1 \amalg Y_2, X, Y)$ is an acyclicly coreedian rectangle, then (X_1, Y_1, X, Y) is an acyclicly coreedian rectangle. If h_1 is a cofibration, h_2 is an acyclic cofibration and $(X_1 \amalg X_2, X, Y_1 \amalg Y_2, Y)$ is an acyclicly coreedian rectangle, then (X_1, X, Y_1, Y) is an acyclic coreedian rectangle.

Proof.

(a) We suppose that h_2 is a cofibration, so that

$$\begin{tikzcd}
\text{¡} \arrow[r, "\mathrm{ini}_X"] \arrow[d, equal] & X_2 \arrow[d, "h_2"] \\
\text{¡} \arrow[r, "\mathrm{ini}_Y"] & Y_2
\end{tikzcd}$$

is a coreedian rectangle. As the cuboid

$$\begin{tikzcd}
& X_2 \arrow[rr, "\mathrm{emb}_2"] \arrow[dd, "h_2"] & & X_1 \amalg X_2 \arrow[dd, "h_1 \amalg h_2"] \\
\text{¡} \arrow[ur, "\mathrm{ini}_{X_2}"] \arrow[rr, "\mathrm{ini}_{X_1}"] \arrow[dd, equal] & & X_1 \arrow[ur, "\mathrm{emb}_1"] \arrow[dd, "h_1"] & \\
& Y_2 \arrow[rr, "\mathrm{emb}_2"] & & Y_1 \amalg Y_2 \\
\text{¡} \arrow[ur, "\mathrm{ini}_{Y_2}"] \arrow[rr, "\mathrm{ini}_{Y_1}"] & & Y_1 \arrow[ur, "\mathrm{emb}_1"] &
\end{tikzcd}$$

commutes, $(X_1, Y_1, X_1 \amalg X_2, Y_1 \amalg Y_2)$ is coreedian by proposition (3.60)(a). This coreedian quadrangle fits into the following commutative diagram.

$$\begin{tikzcd}
X_1 \arrow[r, "\mathrm{emb}_1"] \arrow[d, "h_1"] & X_1 \amalg X_2 \arrow[r, "\binom{f_1}{f_2}"] \arrow[d, "h_1 \amalg h_2"] & X \arrow[d, "h"] \\
Y_1 \arrow[r, "\mathrm{emb}_1"] & Y_1 \amalg Y_2 \arrow[r, "\binom{g_1}{g_2}"] & Y
\end{tikzcd}$$

So if $(X_1 \amalg X_2, Y_1 \amalg Y_2, X, Y)$ is coreedian, then (X_1, Y_1, X, Y) is coreedian by proposition (3.59)(a). Moreover, if h_1 is a cofibration and $(X_1 \amalg X_2, X, Y_1 \amalg Y_2, Y)$ is coreedian, then $(X_1, X_1 \amalg X_2, Y_1, Y_1 \amalg Y_2)$ is coreedian, whence (X_1, X, Y_1, Y) is coreedian by proposition (3.59)(a).

(b) This is proven analogously to (a). $\square$

(3.63) Corollary. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & \tilde{Y}_1 \arrow[d, "\tilde{g}_2"] & Y_1 \arrow[l, "u_1"'] \arrow[d, "g_2"] \\
X'_2 \arrow[r, "f'_2"] & \tilde{Y}'_2 & Y'_2 \arrow[l, "u'_2"'] \\
X_2 \arrow[r, "f_2"] \arrow[u, "v_1"] & \tilde{Y}_2 \arrow[u, "\tilde{v}_2"] & Y_2 \arrow[l, "u_2"'] \arrow[u, "v_2"]
\end{tikzcd}$$

in $\mathcal{C}_{\mathrm{cof}}$ such that $\left(\begin{smallmatrix} f_1 \\ u_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} f_2 \\ u_2 \end{smallmatrix}\right)$ are cofibrations and such that

$$\begin{tikzcd}
(X_1 \amalg Y_1) \amalg (X_2 \amalg Y_2) \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f_1 \\ u_1 \end{smallmatrix}\right) \amalg \left(\begin{smallmatrix} f_2 \\ u_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} g_1 \amalg g_2 \\ v_1 \amalg v_2 \end{smallmatrix}\right)"'] & \tilde{Y}_1 \amalg \tilde{Y}_2 \arrow[d, "\left(\begin{smallmatrix} \tilde{g}_2 \\ \tilde{v}_2 \end{smallmatrix}\right)"] \\
X'_2 \amalg Y'_2 \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f'_2 \\ u'_2 \end{smallmatrix}\right)"] & \tilde{Y}'_2
\end{tikzcd}$$

is a coreedian rectangle.

(a) The following quadrangle is coreedian.

$$\begin{tikzcd}
(X_1 \amalg X_2) \amalg (Y_1 \amalg Y_2) \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f_1 \amalg f_2 \\ u_1 \amalg u_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} g_1 \\ v_1 \end{smallmatrix}\right) \amalg \left(\begin{smallmatrix} g_2 \\ v_2 \end{smallmatrix}\right)"'] & \tilde{Y}_1 \amalg \tilde{Y}_2 \arrow[d, "\left(\begin{smallmatrix} \tilde{g}_2 \\ \tilde{v}_2 \end{smallmatrix}\right)"] \\
X'_2 \amalg Y'_2 \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f'_2 \\ u'_2 \end{smallmatrix}\right)"] & \tilde{Y}'_2
\end{tikzcd}$$

(b) The following quadrangles are coreedian.

$$\begin{tikzcd}
X_1 \amalg Y_1 \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f_1 \\ u_1 \end{smallmatrix}\right)"] \arrow[d, "g_1 \amalg g_2"'] & \tilde{Y}_1 \arrow[d, "\tilde{g}_2"] \\
X'_2 \amalg Y'_2 \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f'_2 \\ u'_2 \end{smallmatrix}\right)"] & \tilde{Y}'_2
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \amalg Y_2 \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f_2 \\ u_2 \end{smallmatrix}\right)"] \arrow[d, "v_1 \amalg v_2"'] & \tilde{Y}_2 \arrow[d, "\tilde{v}_2"] \\
X'_2 \amalg Y'_2 \arrow[r, "\bullet" marking, "\left(\begin{smallmatrix} f'_2 \\ u'_2 \end{smallmatrix}\right)"] & \tilde{Y}'_2
\end{tikzcd}$$

(c) If g_2 resp. g_1 resp. v_2 resp. v_1 is a cofibration, then $(X_1, X'_2, \tilde{Y}_1, \tilde{Y}'_2)$ resp. $(Y_1, Y'_2, \tilde{Y}_1, \tilde{Y}'_2)$ resp. $(X_2, X'_2, \tilde{Y}_2, \tilde{Y}'_2)$ resp. $(Y_2, Y'_2, \tilde{Y}_2, \tilde{Y}'_2)$ is coreedian.

(d) If g_1 and g_2 are cofibrations, then $\tilde{g}_2$ is a cofibration. If v_1 and v_2 are cofibrations, then $\tilde{v}_2$ is a cofibration. If $\left(\begin{smallmatrix} g_1 \\ v_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} g_2 \\ v_2 \end{smallmatrix}\right)$ are cofibrations, then $\left(\begin{smallmatrix} \tilde{g}_2 \\ \tilde{v}_2 \end{smallmatrix}\right)$ is a cofibration.

Proof.

(a) This holds as the quadrangles $((X_1 \amalg Y_1) \amalg (X_2 \amalg Y_2), \tilde{Y}_1 \amalg \tilde{Y}_2, X'_2 \amalg Y'_2, \tilde{Y}'_2)$ and $((X_1 \amalg X_2) \amalg (Y_1 \amalg Y_2), \tilde{Y}_1 \amalg \tilde{Y}_2, X'_2 \amalg Y'_2, \tilde{Y}'_2)$ are isomorphic.

(b) This follows from proposition (3.62).

(c) This follows from (b) and proposition (3.62).

(d) If g_1 and g_2 are cofibrations, then $g_1 \amalg g_2$ is a cofibration by proposition (3.26)(b), and so $\tilde{g}_2$ is a cofibration by remark (3.56)(a) as the rectangle $(X_1 \amalg Y_1, \tilde{Y}_1, X'_2 \amalg Y'_2, \tilde{Y}'_2)$ is coreedian by (b).

Analogously, if v_1 and v_2 are cofibrations, then $v_1 \amalg v_2$ is a cofibration, and so $\tilde{v}_2$ is a cofibration by (b) and remark (3.56)(a).

Finally, is $\left(\begin{smallmatrix} g_1 \\ v_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} g_2 \\ v_2 \end{smallmatrix}\right)$ are cofibrations, then $\left(\begin{smallmatrix} g_1 \\ v_1 \end{smallmatrix}\right) \amalg \left(\begin{smallmatrix} g_2 \\ v_2 \end{smallmatrix}\right)$ is a cofibration, and so $\left(\begin{smallmatrix} \tilde{g}_2 \\ \tilde{v}_2 \end{smallmatrix}\right)$ is a cofibration by (a) and remark (3.56)(a). □

The Coreedy approximation lemma and the factorisation lemma

Given a category with cofibrations and weak equivalences that fulfills the factorisation axiom for cofibrations, every morphism is a cofibration up to an approximation by a weak equivalence, see definition (3.40). This generalises to commutative quadrangles in the following sense:

(3.64) Lemma (Coreedy approximation lemma)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations and a commutative quadrangle

$$\begin{tikzcd}
X_1 \arrow[r, "f_1", "\bullet" marking] \arrow[d, "g_1"'] & Y_1 \arrow[d, "g_2"] \\
X_2 \arrow[r, "f_2"] & Y_2
\end{tikzcd}$$

in $\mathcal{C}$ such that X_1, X_2, Y_1 are cofibrant and such that f_1 is a cofibration. Then there exist a cofibration $\tilde{f}_2\colon X_2 \to \tilde{Y}_2$, a morphism $\tilde{g}_2\colon Y_1 \to \tilde{Y}_2$ and a weak equivalence $w\colon \tilde{Y}_2 \to Y_2$ such that the diagram

$$\begin{tikzcd}
X_1 \arrow[rr, "f_1", "\bullet" marking] \arrow[dd, "g_1"'] & & Y_1 \arrow[dd, "g_2"] \arrow[dl, dotted, "\tilde{g}_2"'] \\
& \tilde{Y}_2 \arrow[dr, dotted, "w", "\approx"'] & \\
X_2 \arrow[ur, dotted, "\tilde{f}_2", "\bullet" marking] \arrow[rr, "f_2"] & & Y_2
\end{tikzcd}$$

in $\mathcal{C}$ commutes and such that $(X_1, X_2, Y_1, \tilde{Y}_2)$ is coreedian.

Proof. By the pushout axiom for cofibrations, there exists a pushout rectangle

$$\begin{tikzcd}
X_1 \arrow[r, "f_1", "\bullet" marking] \arrow[d, "g_1"'] & Y_1 \arrow[d, "g_1'"] \\
X_2 \arrow[r, "f_1'", "\bullet" marking] & C
\end{tikzcd}$$

in $\mathcal{C}$ such that C is cofibrant. Since the quadrangle (X_1, Y_1, X_2, Y_2) commutes, there exists a unique morphism $h\colon C \to Y_2$ such that $f_2 = f_1' h$ and $g_2 = g_1' h$. Moreover, as $\mathcal{C}$ fulfills the factorisation axiom for cofibrations and C is cofibrant, there exist a cofibration $i\colon C \to \tilde{Y}_2$ and a weak equivalence $w\colon \tilde{Y}_2 \to Y_2$ with $h = iw$.

$$\begin{tikzcd}
X_1 \arrow[rrr, "f_1", "\bullet" marking] \arrow[dd, "g_1"'] & & & Y_1 \arrow[dd, "g_2"] \arrow[dl, "g_1'"'] \\
& & C \arrow[r, dotted, "i", "\bullet" marking] \arrow[dr, dotted, "h"'] & \tilde{Y}_2 \arrow[d, dotted, "w", "\approx"'] \\
X_2 \arrow[urr, "f_1'", "\bullet" marking] \arrow[rrr, "f_2"] & & & Y_2
\end{tikzcd}$$

Setting $\tilde{f}_2 := f_1' i$ and $\tilde{g}_2 := g_1' i$ yields the assertion. $\square$

(3.65) Lemma (factorisation lemma, cf. [30, lem. 1.3.3])**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations.

(a) We suppose given morphisms $f_1\colon X_1 \to Y_1$, $f_2\colon X_2 \to Y_2$, $g_1\colon X_1 \to X_2$, $g_2\colon Y_1 \to Y_2$ in $\mathcal{C}$ with X_1, X_2 cofibrant and such that $f_1 g_2 = g_1 f_2'$.

Given a cofibration $i_1\colon X_1 \to \tilde{Y}_1$ and a weak equivalence $w_1\colon \tilde{Y}_1 \to Y_1$ with $f_1 = i_1 w_1$, there exist a cofibration $i_2\colon X_2 \to \tilde{Y}_2$, a weak equivalence $w_2\colon \tilde{Y}_2 \to Y_2$ and a morphism $\tilde{g}_2\colon \tilde{Y}_1 \to \tilde{Y}_2$ such that the diagram

$$\begin{tikzcd}
 & \tilde{Y}_1 \arrow[dr, "w_1", "\sim"'] \arrow[dd, dotted, "\tilde{g}_2"] & \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1"] \arrow[ddd, "g_1"'] & & Y_1 \arrow[ddd, "g_2"] \\
 & \tilde{Y}_2 \arrow[ddr, dotted, "w_2", "\sim"'] & \\
 & & \\
X_2 \arrow[uur, dotted, "i_2"] \arrow[rr, "f_2"] & & Y_2
\end{tikzcd}$$

commutes and such that $(X_1, X_2, \tilde{Y}_1, \tilde{Y}_2)$ is a coreedian rectangle.

(b) We suppose that $\mathcal{C}$ is T-semisaturated. Moreover, we suppose given morphisms $f_1\colon X_1 \to Y_1$, $f_2\colon X_2 \to Y_2$, $f_2'\colon X_2' \to Y_2'$ and S-2-arrows $(g_1, u_1)\colon X_1 \to X_2' \leftarrow X_2$, $(g_2, u_2)\colon Y_1 \to Y_2' \leftarrow Y_2$ in $\mathcal{C}$ with X_1, X_2, X_2' cofibrant and such that $f_1 g_2 = g_1 f_2'$ and $f_2 u_2 = u_1 f_2'$.

Given cofibrations $i_1\colon X_1 \to \tilde{Y}_1$, $i_2\colon X_2 \to \tilde{Y}_2$ and weak equivalences $w_1\colon \tilde{Y}_1 \to Y_1$, $w_2\colon \tilde{Y}_2 \to Y_2$ with $f_1 = i_1 w_1$ and $f_2 = i_2 w_2$, there exist a cofibration $i_2'\colon X_2' \to \tilde{Y}_2'$, a weak equivalence $w_2'\colon \tilde{Y}_2' \to Y_2$ and an S-2-arrow $(\tilde{g}_2, \tilde{u}_2)\colon \tilde{Y}_1 \to \tilde{Y}_2' \leftarrow \tilde{Y}_2$ such that the diagram

$$\begin{tikzcd}
 & \tilde{Y}_1 \arrow[dr, "w_1", "\sim"'] \arrow[dd, dotted, "\tilde{g}_2"] & \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1"] \arrow[dd, "g_1"'] & & Y_1 \arrow[dd, "g_2"] \\
 & \tilde{Y}_2' \arrow[dr, dotted, "w_2'", "\sim"'] & \\
X_2' \arrow[ur, dotted, "i_2'"] \arrow[rr, "f_2'"] & & Y_2' \\
 & \tilde{Y}_2 \arrow[dr, "w_2", "\sim"'] \arrow[uuu, dotted, "\tilde{u}_2", "\sim"'] & \\
X_2 \arrow[ur, "i_2"] \arrow[rr, "f_2"] \arrow[uu, "u_1", "\sim"'] & & Y_2 \arrow[uu, "u_2"', "\sim"]
\end{tikzcd}$$

commutes and such that the following quadrangle is coreedian.

$$\begin{tikzcd}
X_1 \amalg X_2 \arrow[r, "i_1 \amalg i_2"] \arrow[d, "\left(\begin{smallmatrix} g_1 \\ u_1 \end{smallmatrix}\right)"'] & \tilde{Y}_1 \amalg \tilde{Y}_2 \arrow[d, "\left(\begin{smallmatrix} \tilde{g}_2 \\ \tilde{u}_2 \end{smallmatrix}\right)"] \\
X_2' \arrow[r, "i_2'"] & \tilde{Y}_2'
\end{tikzcd}$$

Proof.

(a) Since $i_1 w_1 g_2 = f_1 g_2 = g_1 f_2$, the Coreedy approximation lemma (3.64) yields a cofibration $i_2 \colon X_2 \to \tilde{Y}_2$, a morphism $\tilde{g}_2 \colon \tilde{Y}_1 \to \tilde{Y}_2$ and a weak equivalence $w_2 \colon \tilde{Y}_2 \to Y_2$ such that the diagram

$$\begin{tikzcd}
X_1 \arrow[rr, "i_1"] \arrow[dd, "g_1"'] & & \tilde{Y}_1 \arrow[dl, dotted, "\tilde{g}_2"'] \arrow[dd, "w_1 g_2"] \\
& \tilde{Y}_2 \arrow[dr, dotted, "w_2", "\sim"'] & \\
X_2 \arrow[ur, dotted, "i_2"] \arrow[rr, "f_2"'] & & Y_2
\end{tikzcd}$$

commutes and such that the quadrangle $(X_1, X_2, \tilde{Y}_1, \tilde{Y}_2)$ is coreedian.

(b) By proposition (3.26)(b), the coproduct $i_1 \amalg i_2 \colon X_1 \amalg X_2 \to \tilde{Y}_1 \amalg \tilde{Y}_2$ is a cofibration. Since

$$(i_1 \amalg i_2)\left(\begin{smallmatrix} w_1 g_2 \\ w_2 u_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} i_1 w_1 g_2 \\ i_2 w_2 u_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_1 g_2 \\ f_2 u_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} g_1 f_2' \\ u_1 f_2' \end{smallmatrix}\right) = \left(\begin{smallmatrix} g_1 \\ u_1 \end{smallmatrix}\right) f_2',$$

the Coreedy approximation lemma (3.64) yields a cofibration $i_2' \colon X_2' \to \tilde{Y}_2'$, a morphism $\left(\begin{smallmatrix} \tilde{g}_2 \\ \tilde{u}_2 \end{smallmatrix}\right) \colon \tilde{Y}_1 \amalg \tilde{Y}_2 \to \tilde{Y}_2'$ and a weak equivalence $w_2' \colon \tilde{Y}_2' \to Y_2'$ such that the diagram

$$\begin{tikzcd}
X_1 \amalg X_2 \arrow[rr, "i_1 \amalg i_2"] \arrow[dd, "\left(\begin{smallmatrix} g_1 \\ u_1 \end{smallmatrix}\right)"'] & & \tilde{Y}_1 \amalg \tilde{Y}_2 \arrow[dl, dotted, "\left(\begin{smallmatrix} \tilde{g}_2 \\ \tilde{u}_2 \end{smallmatrix}\right)"'] \arrow[dd, "\left(\begin{smallmatrix} w_1 g_2 \\ w_2 u_2 \end{smallmatrix}\right)"] \\
& \tilde{Y}_2' \arrow[dr, dotted, "w_2'", "\sim"'] & \\
X_2' \arrow[ur, dotted, "i_2'"] \arrow[rr, "f_2'"'] & & Y_2'
\end{tikzcd}$$

commutes and such that $(X_1 \amalg X_2, \tilde{Y}_1 \amalg \tilde{Y}_2, X_2', \tilde{Y}_2')$ is a coreedian rectangle. The morphism $\tilde{u}_2$ is a weak equivalence in $\mathcal{C}$ by T-semisaturatedness.

$$\begin{tikzcd}
& \tilde{Y}_1 \arrow[dr, "w_1", "\sim"'] \arrow[dd, "\tilde{g}_2"] & \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1"] \arrow[dd, "g_1"'] & & Y_1 \arrow[dd, "g_2"] \\
& \tilde{Y}_2' \arrow[dr, "w_2'", "\sim"'] & \\
X_2' \arrow[ur, "i_2'"] \arrow[rr, "f_2'"] & & Y_2' \\
& \tilde{Y}_2 \arrow[uu, "\tilde{u}_2", "\sim"'] \arrow[dr, "w_2", "\sim"'] & \\
X_2 \arrow[uu, "u_1", "\sim"'] \arrow[ur, "i_2"] \arrow[rr, "f_2"] & & Y_2 \arrow[uu, "u_2"', "\sim"]
\end{tikzcd}$$

□

As an application of the factorisation lemma, we show that the gluing axiom and the excision axiom are equivalent under certain additional conditions.

(3.66) Proposition. We suppose given an S-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations. Then $\mathcal{C}$ fulfills the gluing axiom if and only if it fulfills the excision axiom. In particular, if $\mathcal{C}$ fulfills the excision axiom, then it fulfills the incision axiom.

Proof. If $\mathcal{C}$ fulfills the gluing axiom, then it fulfills the excision axiom by proposition (3.46). So we suppose conversely that $\mathcal{C}$ fulfills the excision axiom, and we suppose given a commutative cuboid

$$\begin{tikzcd}
 & X'_1 \arrow[rr, "f'_1"] \arrow[dd, "g'_1", "\wr"'] & & Y'_1 \arrow[dd, "g'_2"] \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "f_1"] \arrow[dd, "g_1", "\wr"'] & & Y_1 \arrow[ur, "i'_1"] \arrow[dd, "g_2", "\wr"'] & \\
 & X'_2 \arrow[rr, "f'_2"] & & Y'_2 \\
X_2 \arrow[ur, "i_2"] \arrow[rr, "f_2"] & & Y_2 \arrow[ur, "i'_2"] &
\end{tikzcd}$$

in $\mathcal{C}_{\mathrm{cof}}$ such that i_1, i_2 are cofibrations, g_1, g_2, g'_1 are weak equivalences, and such that (X_1, Y_1, X'_1, Y'_1) and (X_2, Y_2, X'_2, Y'_2) are pushout rectangles. By the factorisation axiom for cofibrations and the factorisation lemma (3.65)(a), there exist cofibrations $j_k \colon X_k \to \tilde{Y}_k$ and weak equivalences $w_k \colon \tilde{Y}_k \to Y_k$ for $k \in \{1, 2\}$ and a morphism $\tilde{g}_2 \colon \tilde{Y}_1 \to \tilde{Y}_2$ with $f_1 = j_1 w_1$, $f_2 = j_2 w_2$, $j_1 \tilde{g}_2 = g_1 j_2$, $w_1 g_2 = \tilde{g}_2 w_2$.

$$\begin{tikzcd}
 & \tilde{Y}_1 \arrow[dr, dotted, "w_1", "\wr"'] \arrow[dd, dotted, "\tilde{g}_2", "\wr"'] & \\
X_1 \arrow[ur, dotted, "j_1"] \arrow[rr, "f_1"] \arrow[dd, "g_1"', "\wr"] & & Y_1 \arrow[dd, "g_2", "\wr"'] \\
 & \tilde{Y}_2 \arrow[dr, dotted, "w_2", "\wr"'] & \\
X_2 \arrow[ur, dotted, "j_2"] \arrow[rr, "f_2"] & & Y_2
\end{tikzcd}$$

Next, we let

$$\begin{tikzcd}
X'_1 \arrow[r, "j'_1"] & \tilde{Y}'_1 \\
X_1 \arrow[u, "i_1"] \arrow[r, "j_1"] & \tilde{Y}_1 \arrow[u, "\tilde{i}'_1"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X'_2 \arrow[r, "j'_2"] & \tilde{Y}'_2 \\
X_2 \arrow[u, "i_2"] \arrow[r, "j_2"] & \tilde{Y}_2 \arrow[u, "\tilde{i}'_2"']
\end{tikzcd}$$

be pushout rectangles. Since $f_k i'_k = i_k f'_k$ for $k \in \{1, 2\}$, there exists a unique morphism $w'_k \colon \tilde{Y}'_k \to Y'_k$ with $f'_k = j'_k w'_k$ and $w_k i'_k = \tilde{i}'_k w'_k$ for $k \in \{1, 2\}$.

$$\begin{tikzcd}
X'_1 \arrow[r, "j'_1"] \arrow[rr, bend left, "f'_1"] & \tilde{Y}'_1 \arrow[r, dotted, "w'_1"] & Y'_1 \\
X_1 \arrow[u, "i_1"'] \arrow[r, "j_1"] & \tilde{Y}_1 \arrow[u, "\tilde{i}'_1"'] \arrow[r, "w_1", "\approx"'] & Y_1 \arrow[u, "i'_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X'_2 \arrow[r, "j'_2"] \arrow[rr, bend left, "f'_2"] & \tilde{Y}'_2 \arrow[r, dotted, "w'_2"] & Y'_2 \\
X_2 \arrow[u, "i_2"'] \arrow[r, "j_2"] & \tilde{Y}_2 \arrow[u, "\tilde{i}'_2"'] \arrow[r, "w_2", "\approx"'] & Y_2 \arrow[u, "i'_2"']
\end{tikzcd}$$

As $(X_k, \tilde{Y}_k, X'_k, \tilde{Y}'_k)$ and (X_k, Y_k, X'_k, Y'_k) are pushout rectangles for $k \in \{1, 2\}$, it follows that $(\tilde{Y}_k, Y_k, \tilde{Y}'_k, Y'_k)$ is a pushout rectangle for $k \in \{1, 2\}$, and hence w'_1 and w'_2 are weak equivalences by the excision axiom.
Moreover, since $(X_1, \tilde{Y}_1, X'_1, \tilde{Y}'_1)$ is a pushout rectangle, there exists a unique morphism $\tilde{g}'_2 \colon \tilde{Y}'_1 \to \tilde{Y}'_2$ with

$g_1' j_2' = j_1' \tilde{g}_2'$, $\tilde{g}_2 \tilde{i}_2' = \tilde{i}_1' \tilde{g}_2'$ and $\tilde{g}_2' w_2' = w_1' g_2'$.

$$\begin{tikzcd}
& X_1' \arrow[rr, "j_1'", "\bullet"{marking}] \arrow[dd, "g_1'", "\approx"'] & & \tilde{Y}_1' \arrow[rr, "w_1'", "\approx"'] \arrow[dd, dotted, "\tilde{g}_2'"] & & Y_1' \arrow[dd, "g_2'"] \\
X_1 \arrow[ur, "i_1", "\bullet"{marking}] \arrow[rr, "j_1", "\bullet"{marking}] \arrow[dd, "g_1", "\approx"'] & & \tilde{Y}_1 \arrow[ur, "\tilde{i}_1'", "\bullet"{marking}] \arrow[rr, "w_1", "\approx"'] \arrow[dd, "\tilde{g}_2", "\approx"'] & & Y_1 \arrow[ur, "i_1'", "\bullet"{marking}] \arrow[dd, "g_2", "\approx"'] & \\
& X_2' \arrow[rr, "j_2'", "\bullet"{marking}] & & \tilde{Y}_2' \arrow[rr, "w_2'", "\approx"'] & & Y_2' \\
X_2 \arrow[ur, "i_2", "\bullet"{marking}] \arrow[rr, "j_2", "\bullet"{marking}] & & \tilde{Y}_2 \arrow[ur, "\tilde{i}_2'", "\bullet"{marking}] \arrow[rr, "w_2", "\approx"'] & & Y_2 \arrow[ur, "i_2'"] &
\end{tikzcd}$$

By proposition (3.50) it follows that $\tilde{g}_2'$ is a weak equivalence and therefore that g_2' is a weak equivalence by S-semisaturatedness. Thus $\mathcal{C}$ fulfills the gluing axiom.
Altogether, $\mathcal{C}$ fulfills the gluing axiom if and only if $\mathcal{C}$ fulfills the excision axiom. In particular, if $\mathcal{C}$ fulfills the excision axiom, then $\mathcal{C}$ fulfills the incision axiom by proposition (3.46). □

6 Some structures on diagram categories

In this section, we show how structures of categories with weak equivalences and categories with cofibrations on a diagram category can be inherited from such a respective structure on the base category. We essentially follow Rădulescu-Banu [30, sec. 9.2], but we use a slightly more general definition of a Reedy cofibration in our particular context, see definition (3.82).
Given categories $\mathcal{C}$ and S, we denote by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{Cat}}$ the category of S-commutative diagrams in $\mathcal{C}$ (that is, the category of functors from S to $\mathcal{C}$).

Pointwise weak equivalences

First, we introduce the pointwise structure of a category with weak equivalences, see definition (3.1)(a), on a diagram category.

(3.67) Definition (pointwise weak equivalence)**.** We suppose given a category with weak equivalences $\mathcal{C}$ and a category S. A morphism of S-commutative diagrams $w\colon X \to Y$ in $\mathcal{C}$ is called a *pointwise weak equivalence* if $w_k\colon X_k \to Y_k$ is a weak equivalence in $\mathcal{C}$ for every $k \in \operatorname{Ob} S$.

(3.68) Remark. Given a category with weak equivalences $\mathcal{C}$ and a category S, then $\mathcal{C}^S$ becomes a category with weak equivalences having

$$\operatorname{We} \mathcal{C}^S = \{w \in \operatorname{Mor} \mathcal{C}^S \mid w \text{ is a pointwise weak equivalence}\}.$$

Proof. This follows from proposition (1.42)(a), (b). □

(3.69) Definition (pointwise structure)**.** Given a category with weak equivalences $\mathcal{C}$ and a category S, we denote by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{ptw}}$ the category with weak equivalences whose underlying category is $\mathcal{C}^S$ and whose set of weak equivalences is

$$\operatorname{We} \mathcal{C}^S_{\mathrm{ptw}} = \{w \in \operatorname{Mor} \mathcal{C}^S \mid w \text{ is a pointwise weak equivalence}\}.$$

The structure of a category with weak equivalences of $\mathcal{C}^S_{\mathrm{ptw}}$ is called the *pointwise structure* (*of a category with weak equivalences*) on $\mathcal{C}^S$.

(3.70) Remark. We suppose given a category with weak equivalences $\mathcal{C}$ and a category S. If $\mathcal{C}$ is semisaturated, then so is $\mathcal{C}^S$.

Proof. This follows from proposition (1.42)(c). □

Restriction functors and the diagram functor

(3.71) Remark. We suppose given a category with weak equivalences $\mathcal{C}$ and a category S.

(a) For every subcategory U of S, the restriction functor $(-)|_U \colon \mathcal{C}^S \to \mathcal{C}^U$ is a morphism of categories with weak equivalences. In particular, there exists a unique functor $\mathrm{Ho}\,(-)|_U \colon \mathrm{Ho}\,\mathcal{C}^S \to \mathrm{Ho}\,\mathcal{C}^U$ with $\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^U} \circ (-)|_U = \mathrm{Ho}\,(-)|_U \circ \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^S}$.

(b) For every object k in S, the evaluation functor $-_k \colon \mathcal{C}^S \to \mathcal{C}$ is a morphism of categories with weak equivalences. In particular, there exists a unique functor $\mathrm{Ho}\,-_k \colon \mathrm{Ho}\,\mathcal{C}^S \to \mathrm{Ho}\,\mathcal{C}$ with $\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}} \circ -_k = \mathrm{Ho}\,-_k \circ \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^S}$.

(3.72) Notation. We suppose given a category with weak equivalences $\mathcal{C}$ and a category S.

(a) Given a subcategory U of S, we abuse notation and write

$$(-)|_U := \mathrm{Ho}\,(-)|_U \colon \mathrm{Ho}\,\mathcal{C}^S \to \mathrm{Ho}\,\mathcal{C}^U.$$

In particular, we write $\varphi|_U = (\mathrm{Ho}\,(-)|_U)\varphi$ for every morphism $\varphi \colon X \to Y$ in $\mathrm{Ho}\,\mathcal{C}^S$.

(b) Given an object k in S, we abuse notation and write

$$-_k := \mathrm{Ho}\,-_k \colon \mathrm{Ho}\,\mathcal{C}^S \to \mathrm{Ho}\,\mathcal{C}.$$

In particular, we write $\varphi_k = (\mathrm{Ho}\,-|_k)\varphi$ for every morphism $\varphi \colon X \to Y$ in $\mathrm{Ho}\,\mathcal{C}^S$.

Given a category with weak equivalences $\mathcal{C}$ and a category S, the homotopy category of the diagram category $\mathrm{Ho}\,\mathcal{C}^S$ and the diagram category on the homotopy category $(\mathrm{Ho}\,\mathcal{C})^S$ are, in general, not equivalent. However, every object in $\mathrm{Ho}\,\mathcal{C}^S$, that is, every S-commutative diagram in $\mathcal{C}$, yields an S-commutative diagram in $\mathrm{Ho}\,\mathcal{C}$, that is, an object in $(\mathrm{Ho}\,\mathcal{C})^S$, by pointwise application of the localisation functor $\mathrm{loc} \colon \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$. More precisely, we obtain a canonical functor between both categories as described in the following remark.

(3.73) Remark. We suppose given a category with weak equivalences $\mathcal{C}$ and a category S. There exists a unique functor $\mathrm{dia} \colon \mathrm{Ho}\,\mathcal{C}^S \to (\mathrm{Ho}\,\mathcal{C})^S$ with $(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}})^S = \mathrm{dia} \circ \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^S}$.

$$\begin{tikzcd}
\mathcal{C}^S \arrow[r, "(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}})^S"] \arrow[d, "\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^S}"'] & (\mathrm{Ho}\,\mathcal{C})^S \\
\mathrm{Ho}\,\mathcal{C}^S \arrow[ur, "\mathrm{dia}"'] &
\end{tikzcd}$$

Proof. We suppose given a pointwise weak equivalence of S-commutative diagrams $w \colon X \to Y$ in $\mathcal{C}$. Then $w_k \colon X_k \to Y_k$ is a weak equivalence in $\mathcal{C}$ and therefore $\mathrm{loc}(w_k) \colon X_k \to Y_k$ is an isomorphism in $\mathrm{Ho}\,\mathcal{C}$ for every $k \in \mathrm{Ob}\,S$. But this means that $(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}})^S w$ is an isomorphism in $(\mathrm{Ho}\,\mathcal{C})^S$.
So $(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}})^S \colon \mathcal{C}^S \to (\mathrm{Ho}\,\mathcal{C})^S$ maps weak equivalences in $\mathcal{C}^S$ to isomorphisms in $(\mathrm{Ho}\,\mathcal{C})^S$, and the assertion follows from the universal property of $\mathrm{Ho}\,\mathcal{C}^S$. □

(3.74) Definition (diagram functor)**.** We suppose given a category with weak equivalences $\mathcal{C}$ and a category S. The unique functor $\mathrm{dia} = \mathrm{dia}_S \colon \mathrm{Ho}\,\mathcal{C}^S \to (\mathrm{Ho}\,\mathcal{C})^S$ with $(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}})^S = \mathrm{dia} \circ \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^S}$ is called the *diagram functor* with respect to S.

(3.75) Remark. We suppose given a category with weak equivalences $\mathcal{C}$ and a category S. For an S-commutative diagram X in $\mathcal{C}$, we have

$$\mathrm{dia}(X)_k = X_k$$

for every $k \in \mathrm{Ob}\,S$ and

$$\mathrm{dia}(X)_a = \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}}(X_a)$$

for every $a \in \mathrm{Mor}\,S$. For a morphism of S-commutative diagrams f in $\mathcal{C}$, we have

$$\mathrm{dia}(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^S}(f))_k = \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}}(f_k)$$

for every $k \in \mathrm{Ob}\,S$.

We will see examples of full and dense diagram functors in proposition (5.53).

Pointwise cofibrations

Like we did with pointwise weak equivalences, we can turn a diagram category into a category with cofibrations, see definition (3.14)(a), using pointwise cofibrations.

(3.76) Definition (pointwise cofibration)**.** We suppose given a category with cofibrations $\mathcal{C}$ and a category S.

(a) A morphism of S-commutative diagrams $i\colon X \to Y$ in $\mathcal{C}$ is called a *pointwise cofibration* if $i_k\colon X_k \to Y_k$ is a cofibration in $\mathcal{C}$ for every $k \in \operatorname{Ob} S$.

(b) An S-commutative diagram X in $\mathcal{C}$ is said to be *pointwise cofibrant* if it is cofibrant with respect to $\{i \in \operatorname{Mor} \mathcal{C}^S \mid i \text{ is a pointwise cofibration}\}$.

(3.77) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and a category S. An S-commutative diagram X in $\mathcal{C}$ is pointwise cofibrant if and only if X_k is cofibrant for every $k \in \operatorname{Ob} S$.

(3.78) Proposition. Given a category with cofibrations $\mathcal{C}$ and a category S, then $\mathcal{C}^S$ becomes a category with cofibrations having

$$\operatorname{Cof} \mathcal{C}^S = \{i \in \operatorname{Mor} \mathcal{C}^S \mid i \text{ is a pointwise cofibration}\}.$$

Proof. We set $C := \{i \in \operatorname{Mor} \mathcal{C}^S \mid i \text{ is a pointwise cofibration}\}$. In the following, we verify the axioms of a category with cofibrations.
The category $\mathcal{C}^S$ has an initial object I given by $I_k = \text{¡}^{\mathcal{C}}$ for $k \in \operatorname{Ob} S$ and by $I_a = 1_{\text{¡}^{\mathcal{C}}}$ for every morphism $a\colon k \to l$ in S. Moreover, I is pointwise cofibrant as $1_I = \operatorname{ini}^I_I\colon I \to I$ is a pointwise cofibration.
The closedness under composition of C in $\mathcal{C}^S$ is proven analogously to proposition (1.42)(a).
To show the isomorphism axiom for cofibrations, we suppose given an isomorphism $f\colon X \to Y$ in $\mathcal{C}^S$ such that X is pointwise cofibrant. Then $f_k\colon X_k \to Y_k$ is an isomorphism and X_k is cofibrant in $\mathcal{C}$ for every $k \in \operatorname{Ob} S$. So since $\mathcal{C}$ fulfills the isomorphism axiom for cofibrations, it follows that $f_k\colon X_k \to Y_k$ is a cofibration in $\mathcal{C}$ for every $k \in \operatorname{Ob} S$, that is, $f\colon X \to Y$ is a pointwise cofibration.
To show the pushout axiom for cofibrations, we suppose given a morphism $f\colon X \to Y$ and a pointwise cofibration $i\colon X \to X'$ of S-commutative diagrams in $\mathcal{C}$ such that X, Y, X' are pointwise cofibrant. As $\mathcal{C}$ fulfills the pushout axiom for cofibrations, there exists a pushout of f_k and i_k for every $k \in \operatorname{Ob} S$. We obtain an object Y' and morphisms $f'\colon X' \to Y'$, $i'\colon Y \to Y'$ in $\mathcal{C}^S$, where

$$\begin{tikzcd}
X'_k \arrow[r, "f'_k"] & Y'_k \\
X_k \arrow[u, "i_k"] \arrow[r, "f_k"] & Y_k \arrow[u, "i'_k"']
\end{tikzcd}$$

is a pushout rectangle in $\mathcal{C}$ for every $k \in \operatorname{Ob} S$, and where $Y'_a\colon Y'_k \to Y'_l$ for a morphism $a\colon k \to l$ in S is the unique morphism in $\mathcal{C}$ such that $X'_a f'_l = f'_k Y'_a$ and $Y_a i'_l = i'_k Y'_a$. But then the quadrangle

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"']
\end{tikzcd}$$

is a pushout rectangle in $\mathcal{C}^S$ and i' is a pointwise cofibration.
Altogether, $\mathcal{C}^S$ becomes a category with cofibrations having $\operatorname{Cof} \mathcal{C}^S = C$. □

(3.79) Definition (pointwise structure)**.** We suppose given a category S.

(a) Given a category with cofibrations $\mathcal{C}$, we denote by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{ptw}}$ the category with cofibrations whose underlying category is $\mathcal{C}^S$ and whose set of cofibrations is

$$\operatorname{Cof} \mathcal{C}^S_{\mathrm{ptw}} = \{i \in \operatorname{Mor} \mathcal{C}^S \mid i \text{ is a pointwise cofibration}\}.$$

The structure of a category with cofibrations of $\mathcal{C}^S_{\mathrm{ptw}}$ is called the *pointwise structure* (*of a category with cofibrations*) on $\mathcal{C}^S$.

(b) Given a category with cofibrations and weak equivalences $\mathcal{C}$, we denote by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{ptw}}$ the category with cofibrations and weak equivalences whose underlying category is $\mathcal{C}^S$, whose underlying structure of a category with cofibrations is the pointwise structure of a category with cofibrations on $\mathcal{C}^S$, and whose underlying structure of a category with weak equivalences is the pointwise structure of a category with weak equivalences on $\mathcal{C}^S$. The structure of a category with cofibrations and weak equivalences of $\mathcal{C}^S_{\mathrm{ptw}}$ is called the *pointwise structure* (*of a category with cofibrations and weak equivalences*) on $\mathcal{C}^S$.

(3.80) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and a category S. If $\mathcal{C}$ fulfills the cofibrancy axiom, then so does $\mathcal{C}^S$.

(3.81) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and a category S. If $\mathcal{C}$ fulfills the incision axiom, then so does $\mathcal{C}^S$.

Proof. This is proven similarly to the verification of the pushout axiom for cofibrations in proposition (3.78). □

Reedy cofibrations

Next, we introduce a sort of cofibrations on a diagram category that are a bit more complicated to define. For the purpose of this thesis, it suffices to consider the particular case where the shape category is given by $S = \Delta^n$ for some $n \in \mathbb{N}_0$, and so we restrict our attention to this case. A more general Reedy theory for Cisinski cofibration categories, where S may be a so-called *finite directed category*, can be found in the work of Rădulescu-Banu [30, ch. 9]. However, the Reedy cofibrations defined here are slightly more general than those of [30, def. 9.2.2(1)(b)] as we do *not* require a Reedy cofibration to have a Reedy cofibrant source object.

(3.82) Definition (Reedy cofibration)**.** We suppose given a category with cofibrations $\mathcal{C}$ and an $n \in \mathbb{N}_0$.

(a) A morphism of Δ^n-commutative diagrams $i\colon X \to Y$ in $\mathcal{C}$ is called a *Reedy cofibration* if X and Y are pointwise cofibrant, if $i_0\colon X_0 \to Y_0$ is a cofibration in $\mathcal{C}$, and if $(X_{k-1}, X_k, Y_{k-1}, Y_k)$ is a coreedian rectangle in $\mathcal{C}$ for $k \in \Delta^n \setminus \{0\}$.

(b) A Δ^n-commutative diagram X in $\mathcal{C}$ is said to be *Reedy cofibrant* if it is cofibrant with respect to $\{i \in \operatorname{Mor} \mathcal{C}^{\Delta^n} \mid i \text{ is a Reedy cofibration}\}$.

(3.83) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and an $n \in \mathbb{N}_0$. An S-commutative diagram X in $\mathcal{C}$ is Reedy cofibrant if and only if X_0 is cofibrant and $X_{k-1,k}$ is a cofibration in $\mathcal{C}$ for $k \in \Delta^n \setminus \{0\}$.

Proof. First, we suppose that X is Reedy cofibrant, that is, there exists an initial object I in $\mathcal{C}^{\Delta^n}$ such that $\mathrm{ini}^I_X\colon I \to X$ is a Reedy cofibration. In particular, X is pointwise cofibrant, and so X_0 is cofibrant in $\mathcal{C}$. Moreover, for $k \in \Delta^n \setminus \{0\}$, the morphism $I_{k-1,k} = \mathrm{ini}^{I_{k-1}}_{I_k}$ is an isomorphism, and so $X_{k-1,k}$ is a cofibration as $(I_{k-1}, I_k, X_{k-1}, X_k)$ is coreedian.

$$\begin{tikzcd}
I_{k-1} \arrow[rr, "\cong"] \arrow[dd, "\mathrm{ini}^{I_{k-1}}_{X_{k-1}}"', tail] & & I_k \arrow[dl, "\mathrm{ini}^{I_k}_{X_{k-1}}"', tail] \arrow[dd, "\mathrm{ini}^{I_k}_{X_k}", tail] \\
& X_{k-1} \arrow[dr, tail] & \\
X_{k-1} \arrow[ur, equal] \arrow[rr, tail] & & X_k
\end{tikzcd}$$

Conversely, we suppose that X_0 is cofibrant and that $X_{k-1,k}$ is a cofibration for $k \in \Delta^n \setminus \{0\}$. Then X is pointwise cofibrant by induction. Moreover, it is Reedy cofibrant as we have an initial object I in $\mathcal{C}^{\Delta^n}$ given by $I_k = ¡^{\mathcal{C}}$ for $k \in \Delta^n$ and by $I_{k,l} = 1_{¡^{\mathcal{C}}}$ for all $k, l \in \Delta^n$ with $k \leq l$. □

(3.84) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and an $n \in \mathbb{N}_0$. Every Reedy cofibration of Δ^n-commutative diagrams in $\mathcal{C}$ is a pointwise cofibration. In particular, every Reedy cofibrant Δ^n-commutative diagram in $\mathcal{C}$ is pointwise cofibrant.

Proof. This follows from remark (3.56)(a) by an induction on n. □

(3.85) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and an $n \in \mathbb{N}_0$. Every isomorphism of Δ^n-commutative diagrams in $\mathcal{C}$ with pointwise cofibrant source object is a Reedy cofibration.

Proof. We suppose given an isomorphism of Δ^n-commutative diagrams $f\colon X \to Y$ in $\mathcal{C}$ such that X is pointwise cofibrant. Then f is a pointwise cofibration by the isomorphism axiom for cofibrations for $\mathcal{C}^{\Delta^n}_{\mathrm{ptw}}$. So in particular, Y is pointwise cofibrant and $f_0\colon X_0 \to Y_0$ is a cofibration. Moreover, $(X_{k-1}, X_k, Y_{k-1}, Y_k)$ is coreedian for $k \in \Delta^n \setminus \{0\}$ by remark (3.57)(a). $\square$

(3.86) Proposition. We suppose given a category with cofibrations $\mathcal{C}$ and an $n \in \mathbb{N}_0$.

(a) We suppose given a morphism $f\colon X \to Y$ and a Reedy cofibration of Δ^n-commutative diagrams $i\colon X \to X'$ in $\mathcal{C}$ such that X, Y, X' are pointwise cofibrant. Then there exists a pushout rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

in $\mathcal{C}^{\Delta^n}$.

(b) We suppose given a pushout rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

in $\mathcal{C}^{\Delta^n}$ such that X, Y, X' are pointwise cofibrant and such that $i\colon X \to X'$ is a Reedy cofibration of Δ^n-commutative diagrams in $\mathcal{C}$. Then $i'\colon Y \to Y'$ is a Reedy cofibration of Δ^n-commutative diagrams in $\mathcal{C}$.

Proof.

(a) Every Reedy cofibration is a pointwise cofibration by remark (3.84), so a pushout rectangle exists as $\mathcal{C}^S_{\mathrm{ptw}}$ fulfills the pushout axiom for cofibrations.

(b) By remark (3.84), i is a pointwise cofibration, and so i' is a pointwise cofibration by remark (3.25). So in particular, Y' is pointwise cofibrant and $i'_0\colon Y_0 \to Y'_0$ is a cofibration. Moreover, for $k \in \Delta^n \setminus \{0\}$, the coreedianess of $(X_{k-1}, X_k, X'_{k-1}, X'_k)$ implies the coreedianess of $(Y_{k-1}, Y_k, Y'_{k-1}, Y'_k)$ by proposition (3.60), whence $i'\colon Y \to Y'$ is a Reedy cofibration.

$$\begin{tikzcd}
& X'_{k-1} \arrow[rr, "f'_{k-1}"] \arrow[dd] & & Y'_{k-1} \arrow[dd] \\
X_{k-1} \arrow[ur, "i_{k-1}"] \arrow[rr, "f_{k-1}", near end] \arrow[dd] & & Y_{k-1} \arrow[ur, "i'_{k-1}"] \arrow[dd] & \\
& X'_k \arrow[rr, "f'_k", near end] & & Y'_k \\
X_k \arrow[ur, "i_k"] \arrow[rr, "f_k"] & & Y_k \arrow[ur, "i'_k"] &
\end{tikzcd}$$

$\square$

(3.87) Proposition. Given a category with cofibrations $\mathcal{C}$ and an $n \in \mathbb{N}_0$, then $\mathcal{C}^{\Delta^n}$ becomes a category with cofibrations having

$$\operatorname{Cof} \mathcal{C}^{\Delta^n} = \{ i \in \operatorname{Mor} \mathcal{C}^{\Delta^n} \mid i \text{ is a Reedy cofibration} \}.$$

Proof. We set $C := \{i \in \operatorname{Mor} \mathcal{C}^{\Delta^n} \mid i \text{ is a Reedy cofibration}\}$. In the following, we verify the axioms of a category with cofibrations.
The category $\mathcal{C}^{\Delta^n}$ has an initial object I given by $I_k = \text{¡}^{\mathcal{C}}$ for $k \in \Delta^n$ and by $I_{k,l} = 1_{\text{¡}^{\mathcal{C}}}$ for all $k, l \in \Delta^n$ with $k \leq l$. Moreover, I is Reedy cofibrant as $I_0 = \text{¡}^{\mathcal{C}}$ is cofibrant and $I_{k-1,k} = 1_{\text{¡}^{\mathcal{C}}}$ is a cofibration in $\mathcal{C}$ for $k \in \Delta \setminus \{0\}$.
To show that C is closed under composition, we suppose given Reedy cofibrations of Δ^n-commutative diagrams $i\colon X \to Y$, $j\colon Y \to Z$ in $\mathcal{C}$, so that X, Y, Z are pointwise cofibrant, $i_0\colon X_0 \to Y_0$, $j_0\colon Y_0 \to Z_0$ are cofibrations in $\mathcal{C}$, and $(X_{k-1}, X_k, Y_{k-1}, Y_k)$, $(Y_{k-1}, Y_k, Z_{k-1}, Z_k)$ are coreedian rectangles in $\mathcal{C}$ for $k \in \Delta^n \setminus \{0\}$. But then $i_0 j_0\colon X_0 \to Z_0$ is also a cofibration in $\mathcal{C}$ by the multiplicativity of $\operatorname{Cof} \mathcal{C}$, and $(X_{k-1}, X_k, Z_{k-1}, Z_k)$ is a coreedian rectangle in $\mathcal{C}$ for $k \in \Delta^n \setminus \{0\}$ by proposition (3.59)(a). Hence $ij\colon X \to Z$ is a Reedy cofibration.

$$\begin{tikzcd}
X_{k-1} \arrow[r] \arrow[d, "i_{k-1}"' , "\bullet" marking] & X_k \arrow[d, "i_k", "\bullet" marking] \\
Y_{k-1} \arrow[r] \arrow[d, "j_{k-1}"' , "\bullet" marking] & Y_k \arrow[d, "j_k", "\bullet" marking] \\
Z_{k-1} \arrow[r] & Z_k
\end{tikzcd}$$

Finally, the isomorphism axiom for cofibrations follows from remark (3.85), and the pushout axiom for cofibrations follows from proposition (3.86). □

(3.88) Definition (Reedy structure)**.** We suppose given an $n \in \mathbb{N}_0$.

(a) Given a category with cofibrations $\mathcal{C}$, we denote by $\mathcal{C}^{\Delta^n}_{\text{Reedy}}$ the category with cofibrations whose underlying category is $\mathcal{C}^{\Delta^n}$ and whose set of cofibrations is

$$\operatorname{Cof} \mathcal{C}^{\Delta^n}_{\text{Reedy}} = \{i \in \operatorname{Mor} \mathcal{C}^{\Delta^n} \mid i \text{ is a Reedy cofibration}\}.$$

The structure of a category with cofibrations of $\mathcal{C}^{\Delta^n}_{\text{Reedy}}$ is called the *Reedy structure* (*of a category with cofibrations*) on $\mathcal{C}^{\Delta^n}$.

(b) Given a category with cofibrations and weak equivalences $\mathcal{C}$, we denote by $\mathcal{C}^{\Delta^n}_{\text{Reedy}}$ the category with cofibrations and weak equivalences whose underlying category is $\mathcal{C}^{\Delta^n}$, whose underlying structure of a category with cofibrations is the Reedy structure of a category with cofibrations on $\mathcal{C}^{\Delta^n}$, and whose underlying structure of a category with weak equivalences is the pointwise structure of a category with weak equivalences on $\mathcal{C}^{\Delta^n}$. The structure of a category with cofibrations and weak equivalences of $\mathcal{C}^{\Delta^n}_{\text{Reedy}}$ is called the *Reedy structure* (*of a category with cofibrations and weak equivalences*) on $\mathcal{C}^{\Delta^n}$.

(3.89) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and an $n \in \mathbb{N}_0$. If $\mathcal{C}$ fulfills the incision axiom, then so does $\mathcal{C}^{\Delta^n}_{\text{Reedy}}$.

Proof. This follows from proposition (3.86)(b) and remark (3.81). □

(3.90) Proposition. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations, and we suppose given a morphism of Δ^n-commutative diagrams $f\colon X \to Y$ for some $n \in \mathbb{N}_0$. Moreover, we suppose given a Reedy cofibration of Δ^m-commutative diagrams $i_{\text{res}}\colon X|_{\Delta^m} \to \tilde{Y}_{\text{res}}$ and a pointwise weak equivalence of Δ^m-commutative diagrams $w_{\text{res}}\colon \tilde{Y}_{\text{res}} \to Y|_{\Delta^m}$ in $\mathcal{C}$ for some $m \in \mathbb{N}_0$ with $m \leq n$ such that $f|_{\Delta^m} = i_{\text{res}} w_{\text{res}}$. Then there exist a Reedy cofibration of Δ^n-commutative diagrams $i\colon X \to \tilde{Y}$ and a pointwise weak equivalence of Δ^n-commutative diagrams $w\colon \tilde{Y} \to Y$ in $\mathcal{C}$ such that $i_{\text{res}} = i|_{\Delta^m}$, $w_{\text{res}} = w|_{\Delta^m}$ and $f = iw$.

$$\begin{tikzcd}
& X_0 \arrow[r] \arrow[dl, "i_{\text{res},0}"' , "\bullet" marking] \arrow[dd, "f_0"] & \cdots \arrow[rr] & & X_m \arrow[r] \arrow[dl, "i_{\text{res},m}"' , "\bullet" marking] \arrow[dd, "f_m"] & \cdots \arrow[rr] & & X_n \arrow[dl, dotted, "i_n"' , "\bullet" marking] \arrow[dd, "f_n"] \\
\tilde{Y}_{\text{res},0} \arrow[rr] \arrow[dr, "w_{\text{res},0}"' , "\sim"] & & \cdots \arrow[r] & \tilde{Y}_{\text{res},m} \arrow[rr, dotted] \arrow[dr, "w_{\text{res},m}"' , "\sim"] & & \cdots \arrow[r, dotted] & \tilde{Y}_n \arrow[dr, dotted, "w_n"' , "\sim"] & \\
& Y_0 \arrow[r] & \cdots \arrow[rr] & & Y_m \arrow[r] & \cdots \arrow[rr] & & Y_n
\end{tikzcd}$$

Proof. For $m = n$, there is nothing to show. For $m = 0$, $n = 1$, the assertion follows from the factorisation lemma (3.65)(a). For $m, n \in \mathbb{N}_0$ with $m < n$ arbitrary, the assertion follows by an induction on $n - m$. $\square$

(3.91) Corollary. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and an $n \in \mathbb{N}_0$. If $\mathcal{C}$ fulfills the factorisation axiom for cofibrations, then so does $\mathcal{C}^{\Delta^n}_{\mathrm{Reedy}}$ and $\mathcal{C}^{\Delta^n}_{\mathrm{ptw}}$.

Proof. This follows from proposition (3.90) and remark (3.84). $\square$

For the definition of a Cisinski cofibration category and of a Brown cofibration category, see definition (3.51)(a) and definition (3.52)(a).

(3.92) Theorem. We suppose given a Cisinski cofibration category $\mathcal{C}$ and an $n \in \mathbb{N}_0$. Then $\mathcal{C}^{\Delta^n}_{\mathrm{Reedy}}$ and $\mathcal{C}^{\Delta^n}_{\mathrm{ptw}}$ are Cisinski cofibration categories.

Proof. This follows from remark (3.70), remark (3.81), remark (3.89) and corollary (3.91). $\square$

(3.93) Corollary. We suppose given a Brown cofibration category $\mathcal{C}$ and an $n \in \mathbb{N}_0$. Then $\mathcal{C}^{\Delta^n}_{\mathrm{ptw}}$ is a Brown cofibration category.

Proof. This follows from theorem (3.92) and remark (3.80). $\square$

The Quillen structure on the category of spans

Now we consider the shape category $S = \llcorner$, that is, the full subposet of $\square = \Delta^1 \times \Delta^1$ with underlying set $\{(0,0),(1,0),(0,1)\}$. Given a category $\mathcal{C}$, the diagram category $\mathcal{C}^{\llcorner}$ is called the *category of spans* in $\mathcal{C}$. We define a sort of cofibrations for the category of spans in a category with cofibrations that is a mixture of a pointwise cofibration (on the restriction to $\{(0,0),(1,0)\}$) and of a Reedy cofibration (on the restriction to $\{(0,0),(0,1)\}$).

(3.94) Definition (Quillen cofibration)**.** We suppose given a category with cofibrations $\mathcal{C}$.

(a) A morphism of spans $i \colon X \to Y$ in $\mathcal{C}$ is called a *Quillen cofibration* if it is a pointwise cofibration such that $i|_{\{(0,0),(0,1)\}}$ is a Reedy cofibration (via the poset isomorphism $\Delta^1 \cong \{(0,0),(0,1)\}$).

(b) A span X in $\mathcal{C}$ is said to be *Quillen cofibrant* if it is cofibrant with respect to $\{i \in \operatorname{Mor} \mathcal{C}^{\llcorner} \mid i \text{ is a Quillen cofibration}\}$.

(3.95) Remark. We suppose given a category with cofibrations $\mathcal{C}$. A span X in $\mathcal{C}$ is Quillen cofibrant if and only if it is pointwise cofibrant and $X_{(0,0),(0,1)}$ is a cofibration in $\mathcal{C}$.

(3.96) Remark. Given a category with cofibrations $\mathcal{C}$, then $\mathcal{C}^{\llcorner}$ becomes a category with cofibrations having

$$\operatorname{Cof} \mathcal{C}^{\llcorner} = \{i \in \operatorname{Mor} \mathcal{C}^{\llcorner} \mid i \text{ is a Quillen cofibration}\}.$$

Proof. We set $\Delta := \{(0,0),(0,1)\}$ and $C := \{i \in \operatorname{Mor} \mathcal{C}^{\llcorner} \mid i \text{ is a Quillen cofibration}\}$. In the following, we verify the axioms of a category with cofibrations.

The category $\mathcal{C}^{\llcorner}$ has an initial object I given by $I_k = ¡^{\mathcal{C}}$ for $k \in \llcorner$ and by $I_{k,l} = 1_{¡^{\mathcal{C}}}$ for all $k, l \in \llcorner$ with $k \leq l$. Moreover, I is Quillen cofibrant as it is pointwise cofibrant and $I_{(0,0),(0,1)} = 1_{¡^{\mathcal{C}}}$ is a cofibration in $\mathcal{C}$.

To show that C is closed under composition, we suppose given Quillen cofibrations of spans $i \colon X \to Y$, $j \colon Y \to Z$ in $\mathcal{C}$. Then i and j are pointwise cofibrations, and so ij is a pointwise cofibration. Moreover, $i|_{\Delta}$ and $j|_{\Delta}$ are Reedy cofibrations, and so $(ij)|_{\Delta} = i|_{\Delta}\, j|_{\Delta}$ is a Reedy cofibration. Thus $ij \colon X \to Z$ is a Quillen cofibration.

To show the isomorphism axiom for cofibrations, we suppose given an isomorphism $f \colon X \to Y$ in $\mathcal{C}^{\llcorner}$ such that X is Quillen cofibrant. Then X is pointwise cofibrant and therefore f is a pointwise cofibration. Moreover, $X|_{\Delta}$ is Reedy cofibrant and therefore $f|_{\Delta}$ is a Reedy cofibration. Thus f is a Quillen cofibration.

To show the pushout axiom for cofibrations, we suppose given morphisms of spans $f \colon X \to Y$, $i \colon X \to X'$ in $\mathcal{C}$ such that X, Y, X' are Quillen cofibrant and i is a Quillen cofibration. Then X, Y, X' are pointwise cofibrant and i is a pointwise cofibration, and so there exists a pushout rectangle

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"']
\end{tikzcd}$$

in $\mathcal{C}^{\llcorner}$ such that i' is a pointwise cofibration. But then in particular

$$\begin{tikzcd} X'|_{\triangle} \arrow[r, "f'|_{\triangle}"] & Y'|_{\triangle} \\ X|_{\triangle} \arrow[u, "i|_{\triangle}"] \arrow[r, "f|_{\triangle}"] & Y|_{\triangle} \arrow[u, "i'|_{\triangle}"'] \end{tikzcd}$$

is a pushout rectangle in $\mathcal{C}^{\triangle}$, and so as $X|_{\triangle}$ and $Y|_{\triangle}$ are Reedy cofibrant and $i|_{\triangle}$ is a Reedy cofibration, it follows that $i'|_{\triangle}$ is a Reedy cofibration. Thus i' is a Quillen cofibration.
Altogether, $\mathcal{C}^{\llcorner}$ becomes a category with cofibrations having $\operatorname{Cof} \mathcal{C}^{\llcorner} = C$. □

(3.97) Definition (Quillen structure)**.**

(a) Given a category with cofibrations $\mathcal{C}$, we denote by $\mathcal{C}^{\llcorner}_{\text{Quillen}}$ the category with cofibrations whose underlying category is $\mathcal{C}^{\llcorner}$ and whose set of cofibrations is

$$\operatorname{Cof} \mathcal{C}^{\llcorner}_{\text{Quillen}} = \{i \in \operatorname{Mor} \mathcal{C}^{\llcorner} \mid i \text{ is a Quillen cofibration}\}.$$

The structure of a category with cofibrations of $\mathcal{C}^{\llcorner}_{\text{Quillen}}$ is called the *Quillen structure* (*of a category with cofibrations*) on $\mathcal{C}^{\llcorner}$.

(b) Given a category with cofibrations and weak equivalences $\mathcal{C}$, we denote by $\mathcal{C}^{\llcorner}_{\text{Quillen}}$ the category with cofibrations and weak equivalences whose underlying category is $\mathcal{C}^{\llcorner}$, whose underlying structure of a category with cofibrations is the Quillen structure of a category with cofibrations on $\mathcal{C}^{\llcorner}$, and whose underlying structure of a category with weak equivalences is the pointwise structure of a category with weak equivalences on $\mathcal{C}^{\llcorner}$. The structure of a category with cofibrations and weak equivalences of $\mathcal{C}^{\llcorner}_{\text{Quillen}}$ is called the *Quillen structure* (*of a category with cofibrations and weak equivalences*) on $\mathcal{C}^{\llcorner}$.

(3.98) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$. If $\mathcal{C}$ fulfills the incision axiom, then so does $\mathcal{C}^{\llcorner}_{\text{Quillen}}$.

Proof. This is proven similarly to the verification of the pushout axiom for cofibrations in remark (3.96). □

(3.99) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$. If $\mathcal{C}$ fulfills the factorisation axiom for cofibrations, then so does $\mathcal{C}^{\llcorner}_{\text{Quillen}}$.

Proof. This follows from proposition (3.90). □

(3.100) Theorem. Given a Cisinski cofibration category $\mathcal{C}$, then $\mathcal{C}^{\llcorner}_{\text{Quillen}}$ is a Cisinski cofibration category.

Proof. This follows from remark (3.70), remark (3.98) and remark (3.99). □

The category of Coquillen rectangles

The Quillen cofibrant spans in a category with cofibrations $\mathcal{C}$ are precisely those spans in $\mathcal{C}$ that may be, by the pushout axiom for cofibrations, by all means completed to a pushout rectangle. From the structure of a category with cofibrations we now deduce a structure of a category with cofibrations on the category of these particular pushout rectangles.

(3.101) Definition (category of Coquillen rectangles)**.** We suppose given a category with cofibrations $\mathcal{C}$. The full subcategory $\mathcal{C}^{\square}_{\text{coqu}}$ of $\mathcal{C}^{\square}$ with

$$\operatorname{Ob} \mathcal{C}^{\square}_{\text{coqu}} = \{X \in \operatorname{Ob} \mathcal{C}^{\square} \mid X \text{ is a pushout rectangle in } \mathcal{C} \text{ and } X|_{\llcorner} \text{ is a Quillen cofibrant span in } \mathcal{C}\}$$

is called the *category of Coquillen rectangles* (or the *category of coquillenian rectangles*) in $\mathcal{C}$. An object in $\mathcal{C}^{\square}_{\text{coqu}}$ is called a *Coquillen rectangle* (or *coquillenian rectangle* or *coquillenian quadrangle*) in $\mathcal{C}$, and a morphism in $\mathcal{C}^{\square}_{\text{coqu}}$ is called a *morphism of Coquillen rectangles* (or a *morphism of coquillenian rectangles*).

(3.102) Definition (Quillen cofibration)**.** We suppose given a category with cofibrations $\mathcal{C}$. A morphism of Coquillen rectangles $i \colon X \to Y$ in $\mathcal{C}$ is called a *Quillen cofibration* if it is a pointwise cofibration such that $i|_{\llcorner}$ is a Quillen cofibration.

(3.103) Remark. We suppose given a category with cofibrations $\mathcal{C}$. Every Coquillen rectangle in $\mathcal{C}$ is cofibrant with respect to $\{i \in \operatorname{Mor} \mathcal{C}^{\square}_{\mathrm{coqu}} \mid i \text{ is a Quillen cofibration}\}$.

(3.104) Remark. We suppose given a category with cofibrations $\mathcal{C}$ and a Quillen cofibration of Coquillen rectangles $i\colon X \to Y$ in $\mathcal{C}$. Then $i|_{\{(0,0),(0,1)\}}$ and $i|_{\{(1,0),(1,1)\}}$ are Reedy cofibrations.

Proof. This follows from definition (3.94)(a) and the gluing lemma for cofibrations (3.61)(a). $\square$

(3.105) Remark. Given a category with cofibrations $\mathcal{C}$, then $\mathcal{C}^{\square}_{\mathrm{coqu}}$ becomes a category with cofibrations having

$$\operatorname{Cof} \mathcal{C}^{\square}_{\mathrm{coqu}} = \{i \in \operatorname{Mor} \mathcal{C}^{\square}_{\mathrm{coqu}} \mid i \text{ is a Quillen cofibration}\}.$$

Proof. We set $C := \{i \in \operatorname{Mor} \mathcal{C}^{\square}_{\mathrm{coqu}} \mid i \text{ is a Quillen cofibration}\}$. In the following, we verify the axioms of a category with cofibrations.
The category $\mathcal{C}^{\square}_{\mathrm{coqu}}$ has an initial object I given by $I_k = \text{¡}^{\mathcal{C}}$ for $k \in \square$ and by $I_{k,l} = 1_{\text{¡}^{\mathcal{C}}}$ for $k, l \in \square$ with $k \leq l$. Moreover, I is C-cofibrant by remark (3.103).
To show that C is closed under composition, we suppose given Quillen cofibrations of Coquillen rectangles $i\colon X \to Y$, $j\colon Y \to Z$ in $\mathcal{C}$. Then i and j are pointwise cofibrations, and so ij is a pointwise cofibration. Moreover, $i|_{\llcorner}$ and $j|_{\llcorner}$ are Quillen cofibrations, and so $(ij)|_{\llcorner} = i|_{\llcorner}\, j|_{\llcorner}$ is a Quillen cofibration. Thus $ij\colon X \to Z$ is a Quillen cofibration.
To show the isomorphism axiom for cofibrations, we suppose given an isomorphism $f\colon X \to Y$ in $\mathcal{C}^{\square}_{\mathrm{coqu}}$. Then X is pointwise cofibrant and therefore f is a pointwise cofibration. Moreover, $X|_{\llcorner}$ is Quillen cofibrant and therefore $f|_{\llcorner}$ is a Quillen cofibration. Thus f is a Quillen cofibration.
To show the pushout axiom for cofibrations, we suppose given morphisms of Coquillen rectangles $f\colon X \to Y$, $i\colon X \to X'$ in $\mathcal{C}$ such that i is a Quillen cofibration. Then X, Y, X' are pointwise cofibrant and i is a pointwise cofibration, and so there exists a pushout rectangle

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"']
\end{tikzcd}$$

in $\mathcal{C}^{\square}$ such that i' is a pointwise cofibration. But then

$$\begin{tikzcd}
X'|_{\llcorner} \arrow[r, "f'|_{\llcorner}"] & Y'|_{\llcorner} \\
X|_{\llcorner} \arrow[u, "i|_{\llcorner}"] \arrow[r, "f|_{\llcorner}"] & Y|_{\llcorner} \arrow[u, "i'|_{\llcorner}"']
\end{tikzcd}$$

is a pushout rectangle in $\mathcal{C}^{\llcorner}$, and so as $X|_{\llcorner}$ and $Y|_{\llcorner}$ are Quillen cofibrant and $i|_{\llcorner}$ is a Quillen cofibration, it follows that $i'|_{\llcorner}$ is a Quillen cofibration and that $Y'|_{\llcorner}$ is Quillen cofibrant span. Moreover, Y' is a pushout rectangle as X, Y and X' are pushout rectangles. Thus Y' is a Coquillen rectangle in $\mathcal{C}$ and i' is a Quillen cofibration.
Altogether, $\mathcal{C}^{\square}_{\mathrm{coqu}}$ becomes a category with cofibrations having $\operatorname{Cof} \mathcal{C}^{\square}_{\mathrm{coqu}} = C$. $\square$

(3.106) Definition (Quillen structure)**.**

(a) Given a category with cofibrations $\mathcal{C}$, we denote by $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ the category with cofibrations whose underlying category is $\mathcal{C}^{\square}_{\mathrm{coqu}}$ and whose set of cofibrations is

$$\operatorname{Cof} (\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}} = \{i \in \operatorname{Mor} \mathcal{C}^{\square}_{\mathrm{coqu}} \mid i \text{ is a Quillen cofibration}\}.$$

The structure of a category with cofibrations of $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ is called the *Quillen structure* (*of a category with cofibrations*) on $\mathcal{C}^{\square}_{\mathrm{coqu}}$.

(b) Given a category with cofibrations and weak equivalences $\mathcal{C}$, we denote by $(\mathcal{C}^{\square}_{\text{coqu}})_{\text{Quillen}}$ the category with cofibrations and weak equivalences whose underlying category is $\mathcal{C}^{\square}_{\text{coqu}}$, whose underlying structure of a category with cofibrations is the Quillen structure of a category with cofibrations on $\mathcal{C}^{\square}_{\text{coqu}}$, and whose underlying structure of a category with weak equivalences is the pointwise structure of a category with weak equivalences on $\mathcal{C}^{\square}_{\text{coqu}}$. The structure of a category with cofibrations and weak equivalences of $(\mathcal{C}^{\square}_{\text{coqu}})_{\text{Quillen}}$ is called the *Quillen structure* (*of a category with cofibrations and weak equivalences*) on $\mathcal{C}^{\square}_{\text{coqu}}$.

(3.107) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$. If $\mathcal{C}$ fulfills the incision axiom, then so does $(\mathcal{C}^{\square}_{\text{coqu}})_{\text{Quillen}}$.

Proof. This follows from remark (3.81) and remark (3.98). $\square$

Given a Cisinski cofibration category $\mathcal{C}$, we will show in corollary (3.122) that $(\mathcal{C}^{\square}_{\text{coqu}})_{\text{Quillen}}$ is a Brown cofibration category. To this end, we implicitly use the notion of a cylinder of an S-2-arrow, which we will introduce in the next section.

7 Cylinders

In this section, we introduce the notion of a cylinder of an S-2-arrow in a category with cofibrations and weak equivalences, cf. definition (3.30)(a). This is a relative version of the common notion of a cylinder of an object as occurring for example in the works of Quillen [28, ch. I, sec. 1, def. 4] and Brown [7, dual of sec. 1]. We will see that the cylinder of an S-2-arrow is an appropriate notion for a convenient "factorisation" of an S-2-arrow, see the Brown factorisation lemma (3.113)(a), and therefore yields a convenient representative of a morphism in the homotopy category, cf. section 9, in particular, theorem (3.128).

Definition of a cylinder

(3.108) Definition (cylinder)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$.

(a) We suppose given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$. A *cylinder* (or *cylinder object*) of (f, u) consists of an object Z together with a morphism $i_0\colon X \to Z$, a weak equivalence $i_1\colon Y \to Z$ and a weak equivalence $s\colon Z \to \tilde{Y}$ in $\mathcal{C}$ such that $i_0 s = f$ and $i_1 s = u$, and such that there exists a coproduct C of X and Y in $\mathcal{C}$ such that $\binom{\text{ins}_0}{\text{ins}_1}\colon C \to Z$ is a cofibration. By abuse of notation, we refer to the said cylinder as well as to its underlying object by Z. The morphism i_0 is called the *start insertion* (or the *insertion at* 0) of Z, and the morphism i_1 is called the *end insertion* (or the *insertion at* 1) of Z. The morphism s is called the *cylinder equivalence* of Z.

Given a cylinder Z of (f, u) with start insertion i_0, end insertion i_1 and cylinder equivalence s, we write $\text{ins}_0 = \text{ins}_0^Z := i_0$, $\text{ins}_1 = \text{ins}_1^Z := i_1$ and $\text{s} = \text{s}^Z := s$.

$$\begin{tikzcd} X \arrow[r, "\text{ins}_0"] \arrow[d, equal] & Z \arrow[d, "\text{s}", "\wr"'] & Y \arrow[l, "\text{ins}_1"', "\approx"] \arrow[d, equal] \\ X \arrow[r, "f"] & \tilde{Y} & Y \arrow[l, "u"', "\approx"] \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dr, "\text{s}", "\wr"'] & \\ C \arrow[ur, tail, "\binom{\text{ins}_0}{\text{ins}_1}"] \arrow[rr, "\binom{f}{u}"] & & \tilde{Y} \end{tikzcd}$$

(b) A *cylinder* of a morphism $f\colon X \to Y$ in $\mathcal{C}$ is a cylinder of $(f, 1_Y)$.

$$\begin{tikzcd} X \arrow[r, "\text{ins}_0"] \arrow[d, equal] & Z \arrow[d, "\text{s}", "\wr"'] & Y \arrow[l, "\text{ins}_1"', "\approx"] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, equal] & Y \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dr, "\text{s}", "\wr"'] & \\ C \arrow[ur, tail, "\binom{\text{ins}_0}{\text{ins}_1}"] \arrow[rr, "\binom{f}{1}"] & & Y \end{tikzcd}$$

(c) A *cylinder* of an object X in $\mathcal{C}$ is a cylinder of 1_X.

$$\begin{tikzcd} X \arrow[r, "\mathrm{ins}_0"] \arrow[d, equal] & Z \arrow[d, "\mathrm{s}", "\wr"'] & X \arrow[l, "\mathrm{ins}_1"', "\approx"] \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dr, "\mathrm{s}", "\wr"'] & \\ C \arrow[ur, "\left(\begin{smallmatrix}\mathrm{ins}_0 \\ \mathrm{ins}_1\end{smallmatrix}\right)", tail] \arrow[rr, "\left(\begin{smallmatrix}1 \\ 1\end{smallmatrix}\right)"] & & X \end{tikzcd}$$

(3.109) Notation. In the context of cylinders, we use a different notation for the embeddings into a binary coproduct. Given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$, a cylinder Z of (f, u) and a coproduct C of X and Y, we write $\mathrm{emb}_0 = \mathrm{emb}_0^C\colon X \to C$ and $\mathrm{emb}_1 = \mathrm{emb}_1^C\colon Y \to C$ (instead of emb_1 and emb_2) and $\mathrm{ins} = \mathrm{ins}^Z = \left(\begin{smallmatrix}\mathrm{ins}_0 \\ \mathrm{ins}_1\end{smallmatrix}\right)^C : C \to Z$, so that we have $\mathrm{ins}_0 = \mathrm{emb}_0 \,\mathrm{ins}$ and $\mathrm{ins}_1 = \mathrm{emb}_1 \,\mathrm{ins}$.

(3.110) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$, an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with X and Y cofibrant and a cylinder Z of (f, u). For every coproduct C of X and Y, the induced morphism $\mathrm{ins}^C : C \to Z$ is a cofibration in $\mathcal{C}$.

Proof. We suppose given an arbitrary coproduct C of X and Y. As Z is a cylinder, there exists a coproduct $\tilde{C}$ of X and Y such that $\left(\begin{smallmatrix}\mathrm{ins}_0 \\ \mathrm{ins}_1\end{smallmatrix}\right)^{\tilde{C}} : \tilde{C} \to Z$ is a cofibration. By proposition (3.26)(a), the object C is cofibrant, whence the canonical isomorphism

$$\begin{pmatrix}\mathrm{emb}_0^{\tilde{C}} \\ \mathrm{emb}_1^{\tilde{C}}\end{pmatrix}^C : C \to \tilde{C}$$

is a cofibration. But then

$$\left(\begin{smallmatrix}\mathrm{ins}_0 \\ \mathrm{ins}_1\end{smallmatrix}\right)^C = \begin{pmatrix}\mathrm{emb}_0^{\tilde{C}} \\ \mathrm{emb}_1^{\tilde{C}}\end{pmatrix}^C \left(\begin{smallmatrix}\mathrm{ins}_0 \\ \mathrm{ins}_1\end{smallmatrix}\right)^{\tilde{C}}$$

is a cofibration by closedness under composition. □

(3.111) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with X and Y cofibrant. Given a cylinder Z of (f, u) in $\mathcal{C}$, the start insertion ins_0 is a cofibration and the end insertion ins_1 is an acyclic cofibration in $\mathcal{C}$.

Proof. This follows from corollary (3.27). □

For the definition of T-semisaturatedness, see definition (1.37) and remark (3.7)(a).

(3.112) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$. Moreover, we suppose given a cofibration $i\colon C \to Z$ for some coproduct C of X and Y in $\mathcal{C}$ and a weak equivalence $s\colon Z \to \tilde{Y}$ in $\mathcal{C}$ such that $\left(\begin{smallmatrix}f \\ u\end{smallmatrix}\right)^C = is$. If $\mathcal{C}$ is T-semisaturated, then Z becomes a cylinder of (f, u) with $\mathrm{ins}_0^Z = \mathrm{emb}_0^C i$, $\mathrm{ins}_1^Z = \mathrm{emb}_1^C i$ and $\mathrm{s}^Z = s$.

Proof. As $\left(\begin{smallmatrix}f \\ u\end{smallmatrix}\right)^C = is$, we have $\mathrm{emb}_0^C is = \mathrm{emb}_0^C \left(\begin{smallmatrix}f \\ u\end{smallmatrix}\right)^C = f$ and $\mathrm{emb}_1^C is = \mathrm{emb}_1^C \left(\begin{smallmatrix}f \\ u\end{smallmatrix}\right)^C = u$. Moreover, since s and u are weak equivalences in $\mathcal{C}$, it follows that $\mathrm{emb}_1 i$ is a weak equivalence in $\mathcal{C}$ by T-semisaturatedness. Thus Z becomes a cylinder of (f, u) with $\mathrm{ins}_0^Z = \mathrm{emb}_0^C i$, $\mathrm{ins}_1^Z = \mathrm{emb}_1^C i$, $\mathrm{s}^Z = s$. □

The Brown factorisation lemma

The following lemma gives a sufficient criterion for the existence of cylinders in a category with cofibrations and weak equivalences. A category with cofibrations and weak equivalences fulfills the factorisation axiom for cofibrations if each of its morphisms with cofibrant source factors into a cofibration followed by a weak equivalence, see definition (3.40).

(3.113) Lemma (Brown factorisation lemma, cf. [7, factorisation lemma, p. 421])**.** We suppose given a T-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations.

(a) We suppose given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$. If X and Y are cofibrant, then there exists a cylinder Z of (f, u).

(b) We suppose given a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & \tilde{Y}_1 \arrow[d, "\tilde{g}_2"] & Y_1 \arrow[l, "u_1"', "\approx"] \arrow[d, "g_2"] \\
X'_2 \arrow[r, "f'_2"] & \tilde{Y}'_2 & Y'_2 \arrow[l, "u'_2"', "\approx"] \\
X_2 \arrow[r, "f_2"] \arrow[u, "v_1", "\approx"'] & \tilde{Y}_2 \arrow[u, "\tilde{v}_2", "\approx"'] & Y_2 \arrow[l, "u_2"', "\approx"] \arrow[u, "v_2", "\approx"']
\end{tikzcd}$$

in $\mathcal{C}$ with X_1, X_2, X'_2, Y_1, Y_2, Y'_2 cofibrant and with weak equivalences u_1, u_2, u'_2, v_1, v_2, $\tilde{v}_2$. For every cylinder Z_1 of (f_1, u_1) and every cylinder Z_2 of (f_2, u_2) there exists a cylinder Z'_2 of (f'_2, u'_2) and an S-2-arrow $(g, v)\colon Z_1 \to Z'_2 \leftarrow Z_2$ such that the diagram

$$\begin{tikzcd}
& X_1 \arrow[rr, "f_1"] \arrow[dd, "g_1"] & & \tilde{Y}_1 \arrow[dd, "\tilde{g}_2"] & & Y_1 \arrow[ll, "u_1"', "\approx"] \arrow[dd, "g_2"] \\
X_1 \arrow[ur, equal] \arrow[rr, "\mathrm{ins}_0^{Z_1}"] \arrow[dd, "g_1"] & & Z_1 \arrow[ur, "\mathrm{s}^{Z_1}", "\approx"'] \arrow[dd, dotted, "g"] & & Y_1 \arrow[ur, equal] \arrow[ll, "\mathrm{ins}_1^{Z_1}"', "\approx"] \arrow[dd, "g_2"] & \\
& X'_2 \arrow[rr, "f'_2"] & & \tilde{Y}'_2 & & Y'_2 \arrow[ll, "u'_2"', "\approx"] \\
X'_2 \arrow[ur, equal] \arrow[rr, dotted, "\mathrm{ins}_0^{Z'_2}"] & & Z'_2 \arrow[ur, dotted, "\mathrm{s}^{Z'_2}", "\approx"'] & & Y'_2 \arrow[ur, equal] \arrow[ll, dotted, "\mathrm{ins}_1^{Z'_2}"', "\approx"] & \\
& X_2 \arrow[rr, "f_2"] \arrow[uu, "v_1", "\approx"'] & & \tilde{Y}_2 \arrow[uu, "\tilde{v}_2", "\approx"'] & & Y_2 \arrow[ll, "u_2"', "\approx"] \arrow[uu, "v_2", "\approx"'] \\
X_2 \arrow[ur, equal] \arrow[rr, "\mathrm{ins}_0^{Z_2}"] \arrow[uu, "v_1", "\approx"'] & & Z_2 \arrow[ur, "\mathrm{s}^{Z_2}", "\approx"'] \arrow[uu, dotted, "v", "\approx"'] & & Y_2 \arrow[ur, equal] \arrow[ll, "\mathrm{ins}_1^{Z_2}"', "\approx"] \arrow[uu, "v_2", "\approx"'] &
\end{tikzcd}$$

commutes and such that the following quadrangle is coreedian.

$$\begin{tikzcd}
(X_1 \amalg Y_1) \amalg (X_2 \amalg Y_2) \arrow[r, "\mathrm{ins}^{Z_1} \amalg \mathrm{ins}^{Z_2}"] \arrow[d, "\left(\begin{smallmatrix} g_1 \amalg g_2 \\ v_1 \amalg v_2 \end{smallmatrix}\right)"'] & Z_1 \amalg Z_2 \arrow[d, "\left(\begin{smallmatrix} g \\ v \end{smallmatrix}\right)"] \\
X'_2 \amalg Y'_2 \arrow[r, "\mathrm{ins}^{Z'_2}"] & Z'_2
\end{tikzcd}$$

In any such completion such that this quadrangle is coreedian, we have the following additional assertions.

(i) The following quadrangles are coreedian.

$$\begin{tikzcd}
X_1 \amalg Y_1 \arrow[r, "\mathrm{ins}^{Z_1}"] \arrow[d, "g_1 \amalg g_2"'] & Z_1 \arrow[d, "g"] \\
X'_2 \amalg Y'_2 \arrow[r, "\mathrm{ins}^{Z'_2}"] & Z'_2
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \amalg Y_2 \arrow[r, "\mathrm{ins}^{Z_2}"] \arrow[d, "v_1 \amalg v_2"', "\approx"] & Z_2 \arrow[d, "v", "\approx"'] \\
X'_2 \amalg Y'_2 \arrow[r, "\mathrm{ins}^{Z'_2}"] & Z'_2
\end{tikzcd}$$

(ii) If g_2 resp. g_1 resp. v_2 resp. v_1 is a cofibration, then (X_1, X'_2, Z_1, Z'_2) resp. (Y_1, Y'_2, Z_1, Z'_2) resp. (X_2, X'_2, Z_2, Z'_2) resp. (Y_2, Y'_2, Z_2, Z'_2) is a coreedian rectangle.

(iii) If g_1 and g_2 are cofibrations, then g is a cofibration.

(iv) If v_1 and v_2 are acyclic cofibrations, then v is an acyclic cofibration.

(v) If $\left(\begin{smallmatrix} g_1 \\ v_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} g_2 \\ v_2 \end{smallmatrix}\right)$ are cofibrations, then $\left(\begin{smallmatrix} g \\ v \end{smallmatrix}\right)$ is a cofibration.

Proof.

(a) By proposition (3.26)(a), there exist finite coproducts of cofibrant objects in $\mathcal{C}$ and these finite coproducts are again cofibrant. So if X and Y are cofibrant, then the factorisation axiom for cofibrations implies that there exists a cofibration $i\colon X \amalg Y \to Z$ and a weak equivalence $s\colon Z \to \tilde{Y}$ such that $\left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right) = is$.

$$\begin{tikzcd}
& Z \arrow[dr, dotted, "s", "\sim"'] & \\
X \amalg Y \arrow[ur, dotted, "i\,\bullet"] \arrow[rr, "\left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right)"] & & \tilde{Y}
\end{tikzcd}$$

By remark (3.112), Z becomes a cylinder of (f, u) with $\mathrm{ins}^Z = i$ and $\mathrm{s}^Z = s$.

(b) We suppose given a cylinder Z_1 of (f_1, u_1) and a cylinder Z_2 of (f_2, u_2). Since

$$\begin{aligned}
\left(\begin{smallmatrix} f_1 \\ u_1 \end{smallmatrix}\right) \tilde{g}_2 &= \begin{pmatrix} f_1 \tilde{g}_2 \\ u_1 \tilde{g}_2 \end{pmatrix} = \begin{pmatrix} g_1 f_2' \\ g_2 u_2' \end{pmatrix} = (g_1 \amalg g_2) \begin{pmatrix} f_2' \\ u_2' \end{pmatrix}, \\
\left(\begin{smallmatrix} f_2 \\ u_2 \end{smallmatrix}\right) \tilde{v}_2 &= \begin{pmatrix} f_2 \tilde{v}_2 \\ u_2 \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} v_1 f_2' \\ v_2 u_2' \end{pmatrix} = (v_1 \amalg v_2) \begin{pmatrix} f_2' \\ u_2' \end{pmatrix},
\end{aligned}$$

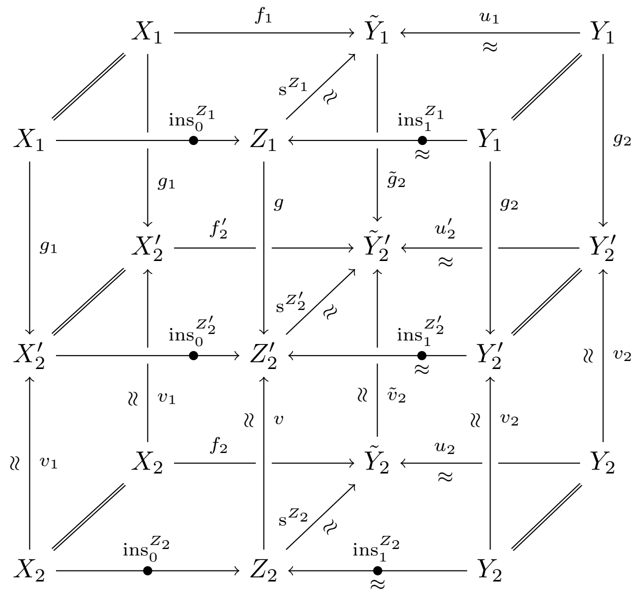
and since $X_1 \amalg Y_1$, $X_2 \amalg Y_2$, $X_2' \amalg Y_2'$ are cofibrant by proposition (3.26)(a), the factorisation lemma (3.65)(b) implies that there exist a cofibration $i_2'\colon X_2' \amalg Y_2' \to Z_2'$, a weak equivalence $s_2'\colon Z_2' \to \tilde{Y}_2'$ and an S-2-arrow $(g, v)\colon Z_1 \to Z_2' \leftarrow Z_2$ such that the diagram

$$\begin{tikzcd}
& Z_1 \arrow[dd, dotted, "g"] \arrow[dr, "\mathrm{s}^{Z_1}", "\sim"'] & \\
X_1 \amalg Y_1 \arrow[ur, "\mathrm{ins}^{Z_1}\,\bullet"] \arrow[rr, "\left(\begin{smallmatrix} f_1 \\ u_1 \end{smallmatrix}\right)"] \arrow[dd, "g_1 \amalg g_2"'] & & \tilde{Y}_1 \arrow[dd, "\tilde{g}_2"] \\
& Z_2' \arrow[dr, dotted, "s_2'", "\sim"'] & \\
X_2' \amalg Y_2' \arrow[ur, dotted, "i_2'\,\bullet"] \arrow[rr, "\left(\begin{smallmatrix} f_2' \\ u_2' \end{smallmatrix}\right)"] & & \tilde{Y}_2' \\
& Z_2 \arrow[uu, dotted, "v", "\sim"'] \arrow[dr, "\mathrm{s}^{Z_2}", "\sim"'] & \\
X_2 \amalg Y_2 \arrow[ur, "\mathrm{ins}^{Z_2}\,\bullet"] \arrow[rr, "\left(\begin{smallmatrix} f_2 \\ u_2 \end{smallmatrix}\right)"] \arrow[uu, "v_1 \amalg v_2", "\sim"'] & & \tilde{Y}_2 \arrow[uu, "\tilde{v}_2"', "\sim"]
\end{tikzcd}$$

commutes and such that

$$\begin{tikzcd}
(X_1 \amalg Y_1) \amalg (X_2 \amalg Y_2) \arrow[r, "\mathrm{ins}^{Z_1} \amalg \mathrm{ins}^{Z_2}", "\bullet"'] \arrow[d, "\left(\begin{smallmatrix} g_1 \amalg g_2 \\ v_1 \amalg v_2 \end{smallmatrix}\right)"'] & Z_1 \amalg Z_2 \arrow[d, "\left(\begin{smallmatrix} g \\ v \end{smallmatrix}\right)"] \\
X_2' \amalg Y_2' \arrow[r, "i_2'", "\bullet"'] & Z_2'
\end{tikzcd}$$

is a coreedian rectangle. By remark (3.112), Z'_2 becomes a cylinder of (f'_2, u'_2) with $\mathrm{ins}^{Z'_2} = i'_2$ and $\mathrm{s}^{Z'_2} = s'_2$.



We verify the additional assertions.

(i) This follows from corollary (3.63)(b).

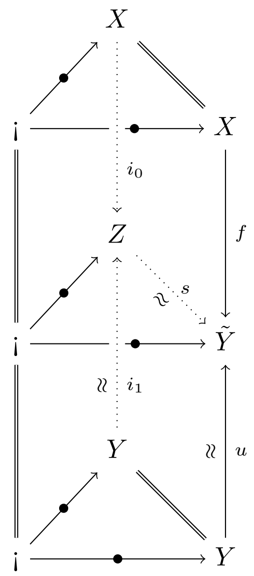
(ii) This follows from corollary (3.63)(c).

(iii) This follows from corollary (3.63)(d).

(iv) This follows from corollary (3.63)(d).

(v) This follows from corollary (3.63)(d). □

Alternative proof of the Brown factorisation lemma (3.113)(a). We suppose that X and Y are cofibrant. By the factorisation lemma (3.65)(b), there exist a weak equivalence $s\colon Z \to \tilde{Y}$ and an S-2-arrow $(i_0, i_1)\colon X \to Z \leftarrow Y$ such that the diagram



commutes and such that

$$\begin{tikzcd}
¡ \amalg ¡ \arrow[r, "\mathrm{ini}_X \amalg \mathrm{ini}_Y", "\bullet" marking] \arrow[d, "\begin{pmatrix} 1_¡ \\ 1_¡ \end{pmatrix}"'] & X \amalg Y \arrow[d, "\begin{pmatrix} i_0 \\ i_1 \end{pmatrix}"] \\
¡ \arrow[r, "\bullet" marking] & Z
\end{tikzcd}$$

is a coreedian rectangle. As $\mathrm{ini}_{¡ \amalg ¡} = \mathrm{emb}_0^{¡ \amalg ¡} = \mathrm{emb}_1^{¡ \amalg ¡}$ is an isomorphism, the quadrangle

$$\begin{tikzcd}
¡ \arrow[r, "\bullet" marking] \arrow[d, equal] & X \amalg Y \arrow[d, "\begin{pmatrix} i_0 \\ i_1 \end{pmatrix}"] \\
¡ \arrow[r, "\bullet" marking] & Z
\end{tikzcd}$$

is a coreedian rectangle, that is, $\begin{pmatrix} i_0 \\ i_1 \end{pmatrix}$ is a cofibration. Thus Z becomes a cylinder of (f, u) with $\mathrm{ins}_0^Z = i_0$, $\mathrm{ins}_1^Z = i_1$, $\mathrm{s}^Z = s$. □

Concatenations and inversions

In analogy to the case of cylinders of objects (cf. [28, ch. I, sec. 1, lem. 3, proof of lem. 4]), one may define concatenations and inversions of cylinders of S-2-arrows.

(3.114) Definition (concatenation, inversion)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$.

(a) We suppose given S-2-arrows $(f_0, u_0)\colon X_0 \to \tilde{X}_1 \leftarrow X_1$, $(f_1, u_1)\colon X_1 \to \tilde{X}_2 \leftarrow X_2$, $(f_1', u_0')\colon \tilde{X}_1 \to \bar{X}_2 \leftarrow \tilde{X}_2$ in $\mathcal{C}$ with $u_0 f_1' = f_1 u_0'$. Given a cylinder Z_0 of (f_0, u_0) and a cylinder Z_1 of (f_1, u_1), a *concatenation* of Z_0 and Z_1 with respect to (f_0, u_0), (f_1, u_1), (f_1', u_0') is a cylinder Z of $(f_0 f_1', u_1 u_0')\colon X_0 \to \bar{X}_2 \leftarrow X_2$ such that the underlying object of Z is a pushout of $\mathrm{ins}_1^{Z_0}$ and $\mathrm{ins}_0^{Z_1}$ in $\mathcal{C}$ and such that $\mathrm{ins}_0^Z = \mathrm{ins}_0^{Z_0} \mathrm{emb}_0^Z$, $\mathrm{ins}_1^Z = \mathrm{ins}_1^{Z_1} \mathrm{emb}_1^Z$, $\mathrm{s}^Z = \begin{pmatrix} \mathrm{s}^{Z_0} f_1' \\ \mathrm{s}^{Z_1} u_0' \end{pmatrix}^Z$.

$$\begin{tikzcd}
& & & Z \arrow[ddd, "\mathrm{s}^Z", "\sim"'] & & \\
& Z_0 \arrow[urr, "\mathrm{emb}_0^Z"] \arrow[ddd, "\mathrm{s}^{Z_0}", "\sim"'] & & & Z_1 \arrow[ul, "\mathrm{emb}_1^Z", "\sim"'] \arrow[ddd, "\mathrm{s}^{Z_1}", "\sim"'] & \\
X_0 \arrow[ur, "\mathrm{ins}_0^{Z_0}"] \arrow[ddd, equal] & & X_1 \arrow[ul, "\mathrm{ins}_1^{Z_0}", "\sim"'] \arrow[urr, "\mathrm{ins}_0^{Z_1}"] \arrow[ddd, equal] & & & X_2 \arrow[ul, "\mathrm{ins}_1^{Z_1}", "\sim"'] \arrow[ddd, equal] \\
& & & \bar{X}_2 & & \\
& \tilde{X}_1 \arrow[urr, "f_1'"] & & & \tilde{X}_2 \arrow[ul, "u_0'", "\sim"'] & \\
X_0 \arrow[ur, "f_0"] & & X_1 \arrow[ul, "u_0", "\sim"'] \arrow[urr, "f_1"] & & & X_2 \arrow[ul, "u_1", "\sim"']
\end{tikzcd}$$

(b) We suppose given an S-2-arrow $(w, u)\colon X \to \bar{X} \leftarrow \tilde{X}$ in $\mathcal{C}$ such that w is a weak equivalence. Given a cylinder Z of (w, u), an *inversion* of Z is a cylinder Z' of $(u, w)\colon \tilde{X} \to \bar{X} \leftarrow X$ with underlying object Z and with $\mathrm{ins}_0^{Z'} = \mathrm{ins}_1^Z$, $\mathrm{ins}_1^{Z'} = \mathrm{ins}_0^Z$, $\mathrm{s}^{Z'} = \mathrm{s}^Z$.

The proof of the next proposition follows the arguments of Brown [7, sec. 1, proof of factorisation lemma].

(3.115) Proposition. We suppose given an S-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$.

(a) We suppose that $\mathcal{C}$ fulfills the incision axiom. Moreover, we suppose given S-2-arrows $(f_0, u_0)\colon X_0 \to \tilde{X}_1 \leftarrow X_1$, $(f_1, u_1)\colon X_1 \to \tilde{X}_2 \leftarrow X_2$, $(f_1', u_0')\colon \tilde{X}_1 \to \bar{X}_2 \leftarrow \tilde{X}_2$ in $\mathcal{C}$ with X_0, X_1, X_2 cofibrant and such that $u_0 f_1' = f_1 u_0'$. There exists a concatenation of every cylinder of (f_0, u_0) and every cylinder of (f_1, u_1) with respect to (f_0, u_0), (f_1, u_1), (f_1', u_0').

(b) We suppose given an S-2-arrow $(w, u)\colon X \to \bar{X} \leftarrow \tilde{X}$ in $\mathcal{C}$ with X, $\tilde{X}$ cofibrant and such that w a weak equivalence. There exists a (unique) inversion of every cylinder of (w, u).

Proof.

(a) We suppose given a cylinder Z_0 of (f_0, u_0) and a cylinder Z_1 of (f_1, u_1). By remark (3.111), the insertions $\mathrm{ins}_0^{Z_0}$, $\mathrm{ins}_1^{Z_0}$, $\mathrm{ins}_0^{Z_1}$, $\mathrm{ins}_1^{Z_1}$ are cofibrations as X_0, X_1, X_2 are cofibrant. So by the pushout axiom for cofibrations, there exists a pushout Z of $\mathrm{ins}_1^{Z_0}$ and $\mathrm{ins}_0^{Z_1}$. Moreover, $\mathrm{emb}_0^Z\colon Z_0 \to Z$ is a cofibration by remark (3.25) and $\mathrm{emb}_1^Z\colon Z_1 \to Z$ is an acyclic cofibration by the incision axiom. As

$$\mathrm{ins}_1^{Z_0} \mathrm{s}^{Z_0} f_1' = u_0 f_1' = f_1 u_0' = \mathrm{ins}_0^{Z_1} \mathrm{s}^{Z_1} u_0',$$

the induced morphism $\begin{pmatrix} \mathrm{s}^{Z_0} f_1' \\ \mathrm{s}^{Z_1} u_0' \end{pmatrix}^Z \colon Z \to \bar{X}_2$ exists. The morphism $\mathrm{ins}_0^{Z_0} \mathrm{emb}_0^Z$ is a cofibration and the morphism $\mathrm{ins}_1^{Z_1} \mathrm{emb}_1^Z$ is an acyclic cofibration by closedness under composition, and $\begin{pmatrix} \mathrm{s}^{Z_0} f_1' \\ \mathrm{s}^{Z_1} u_0' \end{pmatrix}^Z$ is a weak equivalence by S-semisaturatedness.

$$\begin{tikzcd}
 & & & Z \arrow[ddd, "\begin{pmatrix} \mathrm{s}^{Z_0} f_1' \\ \mathrm{s}^{Z_1} u_0' \end{pmatrix}^Z", "\wr"'] & & \\
 & Z_0 \arrow[urr, "\mathrm{emb}_0^Z"] \arrow[ddd, "\mathrm{s}^{Z_0}", "\wr"'] & & & Z_1 \arrow[ul, "\mathrm{emb}_1^Z", "\wr"'] \arrow[ddd, "\mathrm{s}^{Z_1}", "\wr"'] & \\
X_0 \arrow[ur, "\mathrm{ins}_0^{Z_0}"] \arrow[ddd, equal] & & X_1 \arrow[ul, "\mathrm{ins}_1^{Z_0}", "\wr"'] \arrow[urr, "\mathrm{ins}_0^{Z_1}"] \arrow[ddd, equal] & & & X_2 \arrow[ul, "\mathrm{ins}_1^{Z_1}", "\wr"'] \arrow[ddd, equal] \\
 & & & \bar{X}_2 & & \\
 & \tilde{X}_1 \arrow[urr, "f_1'"] & & & \tilde{X}_2 \arrow[ul, "u_0'", "\wr"'] & \\
X_0 \arrow[ur, "f_0"] & & X_1 \arrow[ul, "u_0", "\wr"'] \arrow[urr, "f_1"] & & & X_2 \arrow[ul, "u_1", "\wr"']
\end{tikzcd}$$

As $\begin{pmatrix} \mathrm{ins}_0^{Z_0} \\ \mathrm{ins}_1^{Z_0} \end{pmatrix}\colon X_0 \amalg X_1 \to Z_0$ is a cofibration, the morphism $\begin{pmatrix} \mathrm{ins}_0^{Z_0} \mathrm{emb}_0^Z \\ \mathrm{emb}_1^Z \end{pmatrix}\colon X_0 \amalg Z_1 \to Z$ is also a cofibration by proposition (3.28). Moreover, as $\mathrm{ins}_1^{Z_1}$ is a cofibration, the coproduct $1_{X_0} \amalg \mathrm{ins}_1^{Z_1}$ is a cofibration by proposition (3.26)(b), and so $\begin{pmatrix} \mathrm{ins}_0^{Z_0} \mathrm{emb}_0^Z \\ \mathrm{ins}_1^{Z_1} \mathrm{emb}_1^Z \end{pmatrix} = (1_{X_0} \amalg \mathrm{ins}_1^{Z_1}) \begin{pmatrix} \mathrm{ins}_0^{Z_0} \mathrm{emb}_0^Z \\ \mathrm{emb}_1^Z \end{pmatrix}$ is a cofibration by closedness under composition. Altogether, Z becomes a cylinder of $(f_0 f_1', u_1 u_0')$ with $\mathrm{ins}_0^Z = \mathrm{ins}_0^{Z_0} \mathrm{emb}_0^Z$, $\mathrm{ins}_1^Z = \mathrm{ins}_1^{Z_1} \mathrm{emb}_1^Z$, $\mathrm{s}^Z = \begin{pmatrix} \mathrm{s}^{Z_0} f_1' \\ \mathrm{s}^{Z_1} u_0' \end{pmatrix}^Z$, that is, a concatenation of Z_0 and Z_1 with respect to (f_0, u_0), (f_1, u_1), (f_1', u_0').

(b) We suppose given a cylinder Z of (w, u). Then $\begin{pmatrix} \mathrm{ins}_1^Z \\ \mathrm{ins}_0^Z \end{pmatrix}$ is a cofibration since the quadrangle

$$\begin{tikzcd}
Z \arrow[r, equal] & Z' \\
X \amalg \tilde{X} \arrow[u, "\mathrm{ins}^Z"] \arrow[r, "\begin{pmatrix} \mathrm{emb}_1^{\tilde{X} \amalg X} \\ \mathrm{emb}_0^{\tilde{X} \amalg X} \end{pmatrix}", "\cong"'] & \tilde{X} \amalg X \arrow[u, "\begin{pmatrix} \mathrm{ins}_1^Z \\ \mathrm{ins}_0^Z \end{pmatrix}"']
\end{tikzcd}$$

is cocartesian and ins^Z is a cofibration. Thus we have a cylinder Z' of (u, w) with underlying object Z and such that $\mathrm{ins}_0^{Z'} = \mathrm{ins}_1^Z$, $\mathrm{ins}_1^{Z'} = \mathrm{ins}_0^Z$, $\mathrm{s}^{Z'} = \mathrm{s}^Z$, that is, an inversion of Z. $\square$

Refinements

Refinements of cylinders of objects have been considered by Rădulescu-Banu [30, sec. 6.3, pp. 69–70]. We generalise to S-2-arrows:

(3.116) Definition (refinement)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$. Given cylinders Z and Z' of (f, u), we say that Z' is a *refinement* of Z (or that Z is a *coarsening* of Z') if there exists a cofibration $i\colon Z \to Z'$ such that $\mathrm{ins}^{Z'} = \mathrm{ins}^{Z} i$ and $\mathrm{s}^{Z} = i\mathrm{s}^{Z'}$.

$$\begin{tikzcd}
 & X \arrow[rr, "f"] \arrow[dl, equal] \arrow[dd, equal] & & \tilde{Y} \arrow[dd, equal] & & Y \arrow[ll, "u"', "\approx"] \arrow[dl, equal] \arrow[dd, equal] \\
X \arrow[rr, "\mathrm{ins}_0^Z"] \arrow[dd, equal] & & Z \arrow[ur, "\mathrm{s}^Z", "\approx"'] \arrow[dd, tail, "i", "\wr"'] & & Y_1 \arrow[ll, "\mathrm{ins}_1^Z"', "\approx"] \arrow[dd, equal] & \\
 & X \arrow[rr, "f"] \arrow[dl, equal] & & \tilde{Y} & & Y \arrow[ll, "u"', "\approx"] \arrow[dl, equal] \\
X \arrow[rr, "\mathrm{ins}_0^{Z'}"] & & Z' \arrow[ur, "\mathrm{s}^{Z'}", "\approx"'] & & Y \arrow[ll, "\mathrm{ins}_1^{Z'}"', "\approx"] &
\end{tikzcd}$$

(3.117) Remark. We suppose given a T-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations, and we suppose given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with X and Y cofibrant. For all cylinders Z and $\tilde{Z}$ of (f, u), there exists a cylinder Z' of (f, u) that is a refinement of Z and $\tilde{Z}$.

Proof. This is a particular case of the Brown factorisation lemma (3.113)(b). □

8 The gluing lemma

As an application of cylinders, we show in this section that every Cisinski cofibration category fulfills the gluing axiom. This fact, known as *gluing lemma* in the literature, was proven in this axiomatic approach in a particular case by Brown [7, sec. 4, lem. 2], cf. proposition (3.46) and theorem (3.123), and in full generality by Gunnarsson in his thesis [14, lem. 7.4]. The idea behind the proof presented here is due to Cisinski.

A characterisation of morphisms of categories with weak equivalences

A morphism of categories with weak equivalences is a functor that preserves weak equivalences, see definition (3.1)(b). Similarly, a morphism of categories with cofibrations and weak equivalences is a functor that preserves cofibrations and weak equivalences, see definition (3.30)(b).

(3.118) Lemma (cf. [19, lem. 1.1.12])**.** We suppose given a T-semisaturated category with cofibrations and weak equivances $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations and an S-semisaturated category with weak equivalences $\mathcal{D}$. A functor $F\colon \mathcal{C}_{\mathrm{cof}} \to \mathcal{D}$ is a morphism of categories with weak equivalences if and only if it maps acyclic cofibrations in $\mathcal{C}_{\mathrm{cof}}$ to weak equivalences in $\mathcal{D}$.

Proof. We suppose given a functor $F\colon \mathcal{C}_{\mathrm{cof}} \to \mathcal{D}$. If F is a morphism of categories with weak equivalences, that is, if it maps weak equivalences in $\mathcal{C}_{\mathrm{cof}}$ to weak equivalences in $\mathcal{D}$, then it maps in particular acyclic cofibrations in $\mathcal{C}_{\mathrm{cof}}$ to weak equivalences in $\mathcal{D}$. Conversely, we suppose that F maps acyclic cofibrations in $\mathcal{C}_{\mathrm{cof}}$ to weak equivalences in $\mathcal{D}$ and we suppose given an arbitrary weak equivalence $w\colon X \to Y$ in $\mathcal{C}_{\mathrm{cof}}$. By the Brown factorisation lemma (3.113)(a), there exists a cylinder Z of w. The insertions $\mathrm{ins}_0\colon X \to Z$ and $\mathrm{ins}_1\colon Y \to Z$ are acyclic cofibrations by remark (3.111) and T-semisaturatedness, and thus $F\mathrm{ins}_0$ and $F\mathrm{ins}_1$ are weak equivalences in $\mathcal{D}$. Since

$$(F\mathrm{ins}_1)(F\mathrm{s}) = F(\mathrm{ins}_1\mathrm{s}) = F1_Y = 1_{FY}$$

is a weak equivalence, we conclude that $F\mathrm{s}$ is a weak equivalence by S-semisaturatedness. But then

$$Fw = F(\mathrm{ins}_0\mathrm{s}) = (F\mathrm{ins}_0)(F\mathrm{s})$$

is a weak equivalence by multiplicativity. □

(3.119) Corollary. We suppose given a T-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations, and we suppose given an S-semisaturated category with cofibrations and weak equivalences $\mathcal{D}$. A functor $F\colon \mathcal{C}_{\mathrm{cof}} \to \mathcal{D}$ is a morphism of categories with cofibrations and weak equivalences if and only if F preserves cofibrations and acyclic cofibrations.

The pushout functor for Cisinski cofibration categories

For the following proposition, we recall some notations and definitions. The poset $\llcorner$ is the full subposet of $\square = \Delta^1 \times \Delta^1$ with underlying set $\{(0,0),(1,0),(0,1)\}$. Given a Cisinski cofibration category $\mathcal{C}$, cf. definition (3.51)(a), the category of spans $\mathcal{C}^{\llcorner}$ together with the Quillen structure of a category with cofibrations and weak equivalences becomes a Cisinski cofibration category $\mathcal{C}^{\llcorner}_{\mathrm{Quillen}}$ by theorem (3.100). A span X in $\mathcal{C}$ is Quillen cofibrant, that is, cofibrant in $\mathcal{C}^{\llcorner}_{\mathrm{Quillen}}$, if and only if it is pointwise cofibrant and $X_{(0,0),(0,1)}$ is a cofibration. By the pushout axiom for cofibrations, every Quillen cofibrant span in $\mathcal{C}$ can be prolonged to a Coquillen rectangle as introduced in definition (3.101). The category of Coquillen rectangles then becomes a Brown cofibration category, where cofibrations and weak equivalences are defined via restriction to $(\mathcal{C}^{\llcorner}_{\mathrm{Quillen}})_{\mathrm{cof}}$.

(3.120) Proposition. We suppose given a Cisinski cofibration category $\mathcal{C}$. The pushout functor ([5])

$$(\mathcal{C}^{\llcorner}_{\mathrm{Quillen}})_{\mathrm{cof}} \to (\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$$

is a morphism of Brown cofibration categories.

Proof. The pushout functor $(\mathcal{C}^{\llcorner}_{\mathrm{Quillen}})_{\mathrm{cof}} \to (\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ preserves cofibrations resp. acyclic cofibrations by the gluing lemma for cofibrations resp. acyclic cofibrations (3.61). Thus it is a morphism of categories with cofibrations and weak equivalences by corollary (3.119), that is, a morphism of Brown cofibration categories. □

(3.121) Corollary (gluing lemma, cf. [14, lem. 7.4], [30, lem. 1.4.1])**.** Every Cisinski cofibration category fulfills the gluing axiom.

Proof. We suppose given a Cisinski cofibration category $\mathcal{C}$. Then $\mathcal{C}^{\llcorner}_{\mathrm{Quillen}}$ is also a Cisinski cofibration category by theorem (3.92). By proposition (3.120), the pushout functor $(\mathcal{C}^{\llcorner}_{\mathrm{Quillen}})_{\mathrm{cof}} \to (\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ is a morphism of Brown cofibration categories. In particular, it maps weak equivalences in $(\mathcal{C}^{\llcorner}_{\mathrm{Quillen}})_{\mathrm{cof}}$ to weak equivalences in $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$, that is, $\mathcal{C}$ fulfills the gluing axiom, cf. definition (3.43)(a). □

(3.122) Corollary. Given a Cisinski cofibration category $\mathcal{C}$, the Quillen structure on the category of Coquillen rectangles $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ is a Brown cofibration category.

Proof. The category with cofibrations and weak equivalences $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ is semisaturated by remark (3.70) and fulfills the incision axiom by remark (3.107). Moreover, every object in $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ is cofibrant by remark (3.103). So to show that $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ is a Brown cofibration category, it remains to verify the factorisation axiom for cofibrations. To this end, we suppose given a morphism of Coquillen rectangles $f\colon X \to Y$ in $\mathcal{C}$. As $X|_{\llcorner}$ is Quillen cofibrant and $\mathcal{C}^{\llcorner}_{\mathrm{Quillen}}$ fulfills the factorisation axiom for cofibrations by remark (3.99), there exists a Quillen cofibration of spans $i_{\mathrm{res}}\colon X|_{\llcorner} \to \tilde{Y}_{\mathrm{res}}$ and a pointwise weak equivalence of spans $w_{\mathrm{res}}\colon \tilde{Y}_{\mathrm{res}} \to Y|_{\llcorner}$ in $\mathcal{C}$ with $f|_{\llcorner} = i_{\mathrm{res}} w_{\mathrm{res}}$. By the pushout axiom for cofibrations, there exist a Coquillen rectangle $\tilde{Y}$ in $\mathcal{C}$ with $\tilde{Y}|_{\llcorner} = \tilde{Y}_{\mathrm{res}}$. Moreover, as X and Y are pushout rectangles, there exist morphisms of Coquillen rectangles $i\colon X \to \tilde{Y}$ and $w\colon \tilde{Y} \to Y$ in $\mathcal{C}$ such that $i|_{\llcorner} = i_{\mathrm{res}}$ and $w|_{\llcorner} = w_{\mathrm{res}}$, and we have $f_{1,1} = i_{1,1} w_{1,1}$ and therefore $f = iw$. Finally, i is a Quillen cofibration by the gluing lemma for cofibrations (3.61)(a) and w is a pointwise weak equivalence by the gluing lemma (3.121). Thus $(\mathcal{C}^{\square}_{\mathrm{coqu}})_{\mathrm{Quillen}}$ fulfills the factorisation axiom for cofibrations. □

A characterisation of Cisinski cofibration categories

For the formulation of the factorisation axiom for cofibrations, see definition (3.40); for the gluing axiom, the excision axiom and the incision axiom, see definition (3.43).

(3.123) Theorem (Rădulescu-Banu's criterion [30, lem. 1.4.3])**.** We suppose given a semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the factorisation axiom for cofibrations. The following conditions are equivalent.

[5] Defined via a *choice* of pushout rectangles, cf. appendix A, section 1.

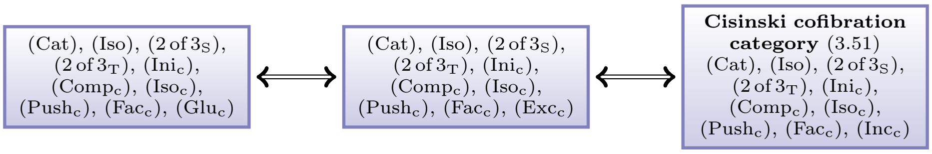


Figure 3: Rădulescu-Banu's criterion.

(a) The gluing axiom holds in $\mathcal{C}$.

(b) The excision axiom holds in $\mathcal{C}$.

(c) The incision axiom holds in $\mathcal{C}$, that is, the category with cofibrations and weak equivalences $\mathcal{C}$ is a Cisinski cofibration category.

Proof. By proposition (3.66), $\mathcal{C}$ fulfills the gluing axiom if and only if $\mathcal{C}$ fulfills the excision axiom. Moreover, if $\mathcal{C}$ fulfills the gluing axiom, then $\mathcal{C}$ fulfills the incision axiom by proposition (3.46). Finally, if $\mathcal{C}$ fulfills the incision axiom, that is, if $\mathcal{C}$ is a Cisinski cofibration category, then $\mathcal{C}$ fulfills the gluing axiom by the gluing lemma (3.121). □

9 The homotopy category of a Brown cofibration category

In this section, we apply our results on cylinders of S-2-arrows to give a description of the hom-sets of a Brown cofibration category as in definition (3.52)(a). More precisely, we show that every Brown cofibration category fits into our axiomatic framework of a Z-fractionable category introduced in chapter II, sections 4 to 5, see theorem (3.127). As localisations are defined via a universal property, it follows that the homotopy category of every Brown cofibration category is isomorphic to the S-Ore localisation constructed in chapter II, section 6, cf. corollary (1.14)(a). We conclude that the Z-2-arrow calculus developed in theorem (2.93) holds for any Brown cofibration category, cf. theorem (3.128). Finally, we apply the Z-2-arrow calculus to give a new proof for the classical homotopy S-2-arrow calculus of Brown [7, dual of th. 1 and proof], see theorem (3.132).

Z-2-arrows in Brown cofibration categories

We consider a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the cofibrancy axiom, see definition (3.30)(a), as a category with D-S-denominators, see definition (2.1)(a), as in remark (3.39). Under certain additional assumptions, we even consider $\mathcal{C}$ as a category with Z-2-arrows, see definition (2.38)(a), in the following way, without further comment.

(3.124) Remark. We suppose given a T-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the cofibrancy axiom and the factorisation axiom for cofibrations. The category with D-S-denominators $\mathcal{C}$ becomes a category with Z-2-arrows, having a Z-2-arrow graph given on the arrows by

$$\operatorname{Arr}\operatorname{AG}_{\mathrm{Z}}\mathcal{C} = \{(i_0, i_1) \in \operatorname{Arr}\operatorname{AG}_{\mathrm{S}}\mathcal{C} \mid \binom{i_0}{i_1} \text{ is a cofibration}\}.$$

Proof. We let $\mathcal{Z}$ be the wide subgraph of $\operatorname{AG}_{\mathrm{S}}\mathcal{C}$ with $\operatorname{Arr}\mathcal{Z} = \{(i_0, i_1) \in \operatorname{Arr}\operatorname{AG}_{\mathrm{S}}\mathcal{C} \mid \binom{i_0}{i_1} \text{ is a cofibration}\}$.
Given an arbitrary arrow in $\mathcal{Z}$, that is, an S-2-arrow $(i_0, i_1)\colon X \to Z \leftarrow Y$ in $\mathcal{C}$ such that $\binom{i_0}{i_1}\colon X \amalg Y \to Z$ is a cofibration, then Z becomes a cylinder of (i_0, i_1) with $\mathrm{ins}^Z = \binom{i_0}{i_1}$ and $\mathrm{s}^Z = 1_Z$. In particular, $\mathrm{ins}_1^Z = i_1$ is an acyclic cofibration by remark (3.111), that is, $\mathcal{Z}$ is a subgraph of $\operatorname{AG}_{\mathrm{S,n}}\mathcal{C}$.
To show the Z-replacement axiom, we suppose given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$. By the Brown factorisation lemma (3.113)(a), there exists a cylinder Z of (f, u). So $\binom{\mathrm{ins}_0}{\mathrm{ins}_1}\colon X \amalg Y \to \tilde{Y}$ is a cofibration, that is, $(\mathrm{ins}_0, \mathrm{ins}_1)$ is an arrow in $\mathcal{Z}$. Moreover, we have $(f, u) = (\mathrm{ins}_0\,\mathrm{s}, \mathrm{ins}_1\,\mathrm{s})$.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ins}_0"] \arrow[d, equal] & Z \arrow[d, "\mathrm{s}", "\wr"'] & Y \arrow[l, "\mathrm{ins}_1"', "\approx"] \arrow[d, equal] \\
X \arrow[r, "f"] & \tilde{Y} & Y \arrow[l, "u"', "\approx"]
\end{tikzcd}$$

Altogether, $\mathcal{C}$ becomes a category with Z-2-arrows having $\operatorname{AG}_{\mathrm{Z}}\mathcal{C} = \mathcal{Z}$. □

In the proof of the last remark, we have seen that the insertions of every cylinder (of S-2-arrows) yield a Z-2-arrow – roughly spoken, cylinders are replacements of S-2-arrows by Z-2-arrows. The following remark states that every Z-2-arrow can be seen as some kind of trivial cylinder.

(3.125) Remark. We suppose given a semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the cofibrancy axiom and the factorisation axiom for cofibrations. Given a Z-2-arrow $(i_0, i_1)\colon X \to Z \leftarrow Y$ in $\mathcal{C}$, then Z becomes a cylinder of (i_0, i_1) with $\mathrm{ins}^Z = \binom{i_0}{i_1}$ and $\mathrm{s}^Z = 1_Z$.

(3.126) Remark. We suppose given a T-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the cofibrancy axiom and the factorisation axiom for cofibrations. The category with Z-2-arrows $\mathcal{C}$ fulfills the Z-replacement axiom for denominators, the relative Z-replacement axiom, the relative Z-replacement axiom for Z-2-arrows, the Z-comparison axiom, the Z-inversion axiom and the Z-expansion axiom.

Proof. The Z-replacement axiom for denominators follows from the Z-replacement axiom and the T-semisaturatedness. The relative Z-replacement axiom and the relative Z-replacement axiom for Z-2-arrows and the Z-comparison axiom follow from the Brown factorisation lemma (3.113)(b). The Z-inversion axiom follows from proposition (3.115)(b). The Z-expansion axiom holds as cofibrations are closed under composition. □

For the definition of a Brown cofibration category, see definition (3.52)(a).

(3.127) Theorem. We suppose given a Brown cofibration category $\mathcal{C}$. The category with Z-2-arrows $\mathcal{C}$ is a Z-fractionable category.

Proof. By remark (3.45) and remark (3.126), the category with Z-2-arrows $\mathcal{C}$ fulfills the weakly universal S-Ore completion axiom, the Z-replacement axiom for denominators, the relative Z-replacement axiom, the relative Z-replacement axiom for Z-2-arrows, the Z-comparison axiom, the Z-inversion axiom and the Z-expansion axiom. The Z-concatenation axiom follows from proposition (3.115)(a). The Z-numerator axiom follows from the pushout axiom for cofibrations and the excision axiom, which holds by Rădulescu-Banu's criterion (3.123). □

The Z-2-arrow calculus for Brown cofibration categories

As every Brown cofibration category carries the structure of a Z-fractionable category by theorem (3.127), we can apply our results on Z-fractionable categories from chapter II to Brown cofibration categories. In particular, we obtain the following descriptions of the hom-sets.

(3.128) Theorem (Z-2-arrow calculus)**.** We suppose given a Brown cofibration category $\mathcal{C}$.

(a) We have

$$\operatorname{Mor}\operatorname{Ho}\mathcal{C} = \{\operatorname{loc}(f)\operatorname{loc}(i)^{-1} \mid (f, i) \text{ is a Z-2-arrow in } \mathcal{C}\}.$$

(b) Given Z-2-arrows $(f_1, i_1)\colon X \to \tilde{Y}_1 \leftarrow Y$, $(f_2, i_2)\colon X \to \tilde{Y}_2 \leftarrow Y$ in $\mathcal{C}$, we have

$$\operatorname{loc}(f_1)\operatorname{loc}(i_1)^{-1} = \operatorname{loc}(f_2)\operatorname{loc}(i_2)^{-1}$$

in $\operatorname{Ho}\mathcal{C}$ if and only if there exist a Z-2-arrow $(f, i)\colon X \to \tilde{Y} \leftarrow Y$ and acyclic cofibrations $j_1\colon \tilde{Y}_1 \to Y$ and $j_2\colon \tilde{Y}_2 \to Y$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & \tilde{Y}_1 \arrow[d, dotted, "j_1", "\wr"'] & Y \arrow[l, "i_1"', "\approx"] \arrow[d, equal] \\
X \arrow[r, dotted, "f"] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, dotted, "i"', "\approx"] \arrow[d, equal] \\
X \arrow[r, "f_2"] & \tilde{Y}_2 \arrow[u, dotted, "j_2"', "\wr"] & Y \arrow[l, "i_2"', "\approx"]
\end{tikzcd}$$

commutes and such that the following quadrangle is coreedian.

$$\begin{tikzcd}
(X \amalg Y) \amalg (X \amalg Y) \arrow[r, "\binom{f_1}{i_1} \amalg \binom{f_2}{i_2}"] \arrow[d, "\binom{1}{1}"'] & \tilde{Y}_1 \amalg \tilde{Y}_2 \arrow[d, "\binom{j_1}{j_2}"] \\
X \amalg Y \arrow[r, "\binom{f}{i}"] & \tilde{Y}
\end{tikzcd}$$

In any such completion such that this quadrangle is coreedian, we have the following additional assertions.

(i) The following quadrangles are coreedian.

$$\begin{tikzcd}
X \amalg Y \arrow[r, "\binom{f_1}{i_1}"] \arrow[d, equal] & \tilde{Y}_1 \arrow[d, "j_1", "\approx"'] \\
X \amalg Y \arrow[r, "\binom{f}{i}"] & \tilde{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \amalg Y \arrow[r, "\binom{f_2}{i_2}"] \arrow[d, equal] & \tilde{Y}_2 \arrow[d, "j_2", "\approx"'] \\
X \amalg Y \arrow[r, "\binom{f}{i}"] & \tilde{Y}
\end{tikzcd}$$

(ii) The quadrangles $(X, X, \tilde{Y}_1, \tilde{Y})$, $(Y, Y, \tilde{Y}_1, \tilde{Y})$, $(X, X, \tilde{Y}_2, \tilde{Y})$, $(Y, Y, \tilde{Y}_2, \tilde{Y})$ are coreedian.

(c) Given Z-2-arrows $(f_1, i_1)\colon X_1 \to \tilde{Y}_1 \leftarrow Y_1$, $(f_2, i_2)\colon X_2 \to \tilde{Y}_2 \leftarrow Y_2$ and S-2-arrows $(g_1, v_1)\colon X_1 \to X_2' \leftarrow X_2$, $(g_2, v_2)\colon Y_1 \to Y_2' \leftarrow Y_2$ in $\mathcal{C}$, we have

$$\operatorname{loc}(f_1) \operatorname{loc}(i_1)^{-1} \operatorname{loc}(g_2) \operatorname{loc}(v_2)^{-1} = \operatorname{loc}(g_1) \operatorname{loc}(v_1)^{-1} \operatorname{loc}(f_2) \operatorname{loc}(i_2)^{-1}$$

in $\operatorname{Ho}\mathcal{C}$ if and only if there exist a Z-2-arrow $(f_2', i_2')\colon X_2' \to \tilde{Y}_2' \leftarrow Y_2'$ and an S-2-arrow $(\tilde{g}_2, \tilde{v}_2)\colon \tilde{Y}_1 \to \tilde{Y}_2' \leftarrow \tilde{Y}_2$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & \tilde{Y}_1 \arrow[d, dotted, "\tilde{g}_2"] & Y_1 \arrow[l, "i_1"', "\approx"] \arrow[d, "g_2"] \\
X_2' \arrow[r, dotted, "f_2'"] & \tilde{Y}_2' & Y_2' \arrow[l, dotted, "i_2'"', "\approx"] \\
X_2 \arrow[r, "f_2"] \arrow[u, "v_1", "\approx"'] & \tilde{Y}_2 \arrow[u, dotted, "\tilde{v}_2", "\approx"'] & Y_2 \arrow[l, "i_2"', "\approx"] \arrow[u, "v_2", "\approx"']
\end{tikzcd}$$

commutes and such that such that the following quadrangle is coreedian.

$$\begin{tikzcd}
(X_1 \amalg Y_1) \amalg (X_2 \amalg Y_2) \arrow[r, "\binom{f_1}{i_1} \amalg \binom{f_2}{i_2}"] \arrow[d, "\binom{g_1 \amalg g_2}{v_1 \amalg v_2}"'] & \tilde{Y}_1 \amalg \tilde{Y}_2 \arrow[d, "\binom{\tilde{g}_2}{\tilde{v}_2}"] \\
X_2' \amalg Y_2' \arrow[r, "\binom{f_2'}{i_2'}"] & \tilde{Y}_2'
\end{tikzcd}$$

In any such completion such that this quadrangle is coreedian, we have the following additional assertions.

(i) The following quadrangles are coreedian.

$$\begin{tikzcd}
X_1 \amalg Y_1 \arrow[r, "\binom{f_1}{i_1}"] \arrow[d, "g_1 \amalg g_2"'] & \tilde{Y}_1 \arrow[d, "\tilde{g}_2"] \\
X_2' \amalg Y_2' \arrow[r, "\binom{f_2'}{i_2'}"] & \tilde{Y}_2'
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \amalg Y_2 \arrow[r, "\binom{f_2}{i_2}"] \arrow[d, "v_1 \amalg v_2"', "\approx"] & \tilde{Y}_2 \arrow[d, "\tilde{v}_2", "\approx"'] \\
X_2' \amalg Y_2' \arrow[r, "\binom{f_2'}{i_2'}"] & \tilde{Y}_2'
\end{tikzcd}$$

(ii) If g_2 resp. g_1 resp. v_2 resp. v_1 is a cofibration, then $(X_1, X'_2, \tilde{Y}_1, \tilde{Y}'_2)$ resp. $(Y_1, Y'_2, \tilde{Y}_1, \tilde{Y}'_2)$ resp. $(X_2, X'_2, \tilde{Y}_2, Z'_2)$ resp. $(Y_2, Y'_2, Z_2, \tilde{Y}'_2)$ is a coreedian rectangle.

(iii) If g_1 and g_2 are cofibrations, then $\tilde{g}_2$ is a cofibration.

(iv) If v_1 and v_2 are acyclic cofibrations, then $\tilde{v}_2$ is an acyclic cofibration.

(v) If (g_1, v_1) and (g_2, v_2) are Z-2-arrows, then $(\tilde{g}_2, \tilde{v}_2)$ is a Z-2-arrow.

(d) Given a category $\mathcal{D}$ and a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that Fw is invertible in $\mathcal{D}$ for every weak equivalence w in $\mathcal{C}$, the unique functor $\hat{F} \colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ with $F = \hat{F} \circ \operatorname{loc}$ is given on the objects by

$$\hat{F}X = FX$$

for $X \in \operatorname{Ob}\mathcal{C}$ and on the morphisms by

$$\hat{F}(\operatorname{loc}(f)\operatorname{loc}(i)^{-1}) = (Ff)(Fi)^{-1}$$

for every Z-2-arrow (f, i) in $\mathcal{C}$.

Proof.

(a) This follows from theorem (3.127) and theorem (2.93)(a).

(c) This follows from theorem (3.127) and theorem (2.93)(c), remark (3.125) and the Brown factorisation lemma (3.113)(b).

(b) This follows from (c).

(d) This follows from theorem (3.127) and theorem (2.93)(f). □

(3.129) Remark. We suppose given a Brown cofibration category $\mathcal{C}$.

(a) We have

$$\operatorname{Mor}\operatorname{Ho}\mathcal{C} = \{\operatorname{loc}(f)\operatorname{loc}(u)^{-1} \mid (f, u) \text{ is an S-2-arrow in } \mathcal{C}\}.$$

(b) We suppose given S-2-arrows $(f_1, u_1) \colon X \to \tilde{Y}_1 \leftarrow Y$, $(f_2, u_2) \colon X \to \tilde{Y}_2 \leftarrow Y$ in $\mathcal{C}$. The following conditions are equivalent.

(i) We have

$$\operatorname{loc}(f_1)\operatorname{loc}(u_1)^{-1} = \operatorname{loc}(f_2)\operatorname{loc}(u_2)^{-1}$$

in $\operatorname{Ho}\mathcal{C}$.

(ii) For every cylinder Z_1 of (f_1, u_1) there exist an S-2-arrow $(f, u) \colon X \to \tilde{Y} \leftarrow Y$ and a normal S-2-arrow $(c, j) \colon Z_1 \to \tilde{Y} \leftarrow \tilde{Y}_2$ in $\mathcal{C}$ with weak equivalence c such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & \tilde{Y}_1 & Y \arrow[l, "u_1", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "\mathrm{ins}_0", "\bullet" marking] \arrow[d, equal] & Z_1 \arrow[u, "s", "\wr"'] \arrow[d, dotted, "c", "\wr"'] & Y \arrow[l, "\mathrm{ins}_1", "\approx"', "\bullet" marking] \arrow[d, equal] \\
X \arrow[r, dotted, "f"] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, dotted, "u", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "f_2"] & \tilde{Y}_2 \arrow[u, dotted, "j", "\wr"', "\bullet" marking] & Y \arrow[l, "u_2", "\approx"']
\end{tikzcd}$$

(iii) There exist a cylinder Z_1 of (f_1, u_1), an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ and a normal S-2-arrow $(c, j)\colon Z_1 \to \tilde{Y} \leftarrow \tilde{Y}_2$ in $\mathcal{C}$ with weak equivalence c such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & \tilde{Y}_1 & Y \arrow[l, "u_1", "\approx"'] \arrow[d, equal] \\
X \arrow[r, dotted, "\mathrm{ins}_0"] \arrow[d, equal] & Z_1 \arrow[u, dotted, "\mathrm{s}"', "\wr"] \arrow[d, dotted, "c"', "\wr"] & Y \arrow[l, dotted, "\mathrm{ins}_1", "\approx"'] \arrow[d, equal] \\
X \arrow[r, dotted, "f"] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, dotted, "u", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "f_2"] & \tilde{Y}_2 \arrow[u, dotted, "j"', "\wr"] & Y \arrow[l, "u_2", "\approx"']
\end{tikzcd}$$

(c) We suppose given S-2-arrows $(f_1, u_1)\colon X_1 \to \tilde{Y}_1 \leftarrow Y_1$, $(f_2, u_2)\colon X_2 \to \tilde{Y}_2 \leftarrow Y_2$, $(g_1, b_1)\colon X_1 \to X'_2 \leftarrow X_2$ and a normal S-2-arrow $(g_2, j_2)\colon Y_1 \to Y'_2 \leftarrow Y_2$ in $\mathcal{C}$. The following conditions are equivalent.

(i) We have

$$\mathrm{loc}(f_1)\,\mathrm{loc}(u_1)^{-1}\,\mathrm{loc}(g_2)\,\mathrm{loc}(j_2)^{-1} = \mathrm{loc}(g_1)\,\mathrm{loc}(v_1)^{-1}\,\mathrm{loc}(f_2)\,\mathrm{loc}(u_2)^{-1}$$

in $\mathrm{Ho}\,\mathcal{C}$.

(ii) For every cylinder Z_1 of (f_1, u_1), every normal S-2-arrow $(\dot{g}_1, \dot{v}_1)\colon X_1 \to \tilde{X}'_2 \leftarrow X_2$ and every morphism $t\colon \tilde{X}'_2 \to X'_2$ in $\mathcal{C}$ with $(g_1, v_1) = (\dot{g}_1 t, \dot{v}_1 t)$ there exist an S-2-arrow $(\tilde{f}_2, \tilde{u}_2)\colon \tilde{X}'_2 \to \tilde{Y}'_2 \leftarrow Y'_2$ and a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)\colon Z_1 \to \tilde{Y}'_2 \leftarrow \tilde{Y}_2$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & X_1 \arrow[r, "f_1"] \arrow[d, equal] & \tilde{Y}_1 & Y_1 \arrow[l, "u_1", "\approx"'] \arrow[d, equal] \\
X_1 \arrow[r, equal] \arrow[d, "g_1"] & X_1 \arrow[r, "\mathrm{ins}_0"] \arrow[d, "\dot{g}_1"] & Z_1 \arrow[u, "\mathrm{s}", "\wr"'] \arrow[d, dotted, "\tilde{g}_2"] & Y_1 \arrow[l, "\mathrm{ins}_1", "\approx"'] \arrow[d, "g_2"] \\
X'_2 & \tilde{X}'_2 \arrow[l, "t"'] \arrow[r, dotted, "\tilde{f}_2"] & \tilde{Y}'_2 & Y'_2 \arrow[l, dotted, "\tilde{u}_2", "\approx"'] \\
X_2 \arrow[r, equal] \arrow[u, "v_1", "\wr"'] & X_2 \arrow[r, "f_2"] \arrow[u, "\dot{v}_1", "\wr"'] & \tilde{Y}_2 \arrow[u, dotted, "\tilde{j}_2", "\wr"'] & Y_2 \arrow[l, "u_2", "\approx"'] \arrow[u, "j_2", "\wr"']
\end{tikzcd}$$

(iii) There exist a cylinder Z_1 of (f_1, u_1), a cylinder A_1 of (g_1, v_1), an S-2-arrow $(\tilde{f}_2, \tilde{u}_2)\colon A_1 \to \tilde{Y}'_2 \leftarrow Y'_2$ and a normal S-2-arrow $(\tilde{g}_2, \tilde{j}_2)\colon Z_1 \to \tilde{Y}'_2 \leftarrow \tilde{Y}_2$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & X_1 \arrow[r, "f_1"] \arrow[d, equal] & \tilde{Y}_1 & Y_1 \arrow[l, "u_1", "\approx"'] \arrow[d, equal] \\
X_1 \arrow[r, equal] \arrow[d, "g_1"] & X_1 \arrow[r, dotted, "\mathrm{ins}_0^{Z_1}"] \arrow[d, dotted, "\mathrm{ins}_0^{A_1}"] & Z_1 \arrow[u, dotted, "\mathrm{s}^{Z_1}", "\wr"'] \arrow[d, dotted, "\tilde{g}_2"] & Y_1 \arrow[l, dotted, "\mathrm{ins}_1^{Z_1}", "\approx"'] \arrow[d, "g_2"] \\
X'_2 & A_1 \arrow[l, dotted, "\mathrm{s}^{A_1}"', "\approx"] \arrow[r, dotted, "\tilde{f}_2"] & \tilde{Y}'_2 & Y'_2 \arrow[l, dotted, "\tilde{u}_2", "\approx"'] \\
X_2 \arrow[r, equal] \arrow[u, "v_1", "\wr"'] & X_2 \arrow[r, "f_2"] \arrow[u, dotted, "\mathrm{ins}_1^{A_1}", "\wr"'] & \tilde{Y}_2 \arrow[u, dotted, "\tilde{j}_2", "\wr"'] & Y_2 \arrow[l, "u_2", "\approx"'] \arrow[u, "j_2", "\wr"']
\end{tikzcd}$$

(d) Given a category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fw is invertible in $\mathcal{D}$ for every weak equivalence w in $\mathcal{C}$, the unique functor $\hat{F}\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ with $F = \hat{F} \circ \operatorname{loc}$ is given on the objects by

$$\hat{F}X = FX$$

for $X \in \operatorname{Ob}\mathcal{C}$ and on the morphisms by

$$\hat{F}(\operatorname{loc}(f)\operatorname{loc}(u)^{-1}) = (Ff)(Fu)^{-1}$$

for every S-2-arrow (f, u) in $\mathcal{C}$.

Proof. This follows from corollary (2.94). □

In [36, ex. (7.1)], we have shown that the full subcategory of cofibrant objects $\mathcal{M}_{\mathrm{cof}}$ in a Quillen model category $\mathcal{M}$ admits a 3-arrow calculus in the sense of [36, th. (5.18)], where we have used the lifting axiom in the proof. Now theorem (3.128) and remark (3.129) apply to the full subcategory of cofibrant objects in a Quillen model category $\mathcal{M}$ [7, sec. 1, p. 421], yielding a better description, without using the lifting axiom in the proof.

Brown's homotopy S-2-arrow calculus

To calculate in homotopy categories of a Brown fibration category, Brown developed a homotopy 2-arrow calculus in analogy to the homotopy 2-arrow calculus that one obtains by the construction of the derived category as a localisation of the homotopy category of complexes, cf. the introduction, section 1, p. vi. We sketch his approach in the dual situation of Brown cofibration categories: First, he introduced a weak cylinder homotopy relation $\overset{\mathrm{c}}{\approx}$, which is weaker than the cylinder homotopy relation $\overset{\mathrm{c}}{\sim}$ from definition (3.130)(a)(ii) below, and showed that $\overset{\mathrm{c}}{\approx}$ is a congruence of categories and that the homotopy category can be obtained as a localisation of the corresponding quotient category $\mathcal{C}/\overset{\mathrm{c}}{\approx}$. This quotient $\mathcal{C}/\overset{\mathrm{c}}{\approx}$, equipped with the images of the weak equivalences as denominators, turns out to become an S-fractionable category [7, prop. 2], cf. definition (2.27)(a). But this implies that $\mathcal{C}/\overset{\mathrm{c}}{\approx}$ admits an S-2-arrow calculus by theorem (2.35). In other words, $\mathcal{C}$ admits an S-2-arrow calculus up to the congruence $\overset{\mathrm{c}}{\approx}$. However, this turns out to be equivalent to the fact that $\mathcal{C}$ admits an S-2-arrow calculus up to the stronger relation $\overset{\mathrm{c}}{\sim}$, that is, $\mathcal{C}$ admits a homotopy S-2-arrow calculus.
We give an alternative proof of Brown's result, using the Z-2-arrow calculus (3.128). To formulate the theorem, we have to recall Quillen's definition of a cylinder homotopy [28, ch. I, §1, def. 3, def. 4, lem. 1].

(3.130) Definition (cylinder homotopy)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$.

(a) We suppose given morphisms $f_0, f_1\colon X \to Y$ in $\mathcal{C}$.

(i) Given a cylinder $\dot{X}$ of X, a *cylinder homotopy* ([6]) from f_0 to f_1 with respect to $\dot{X}$ is a morphism $f\colon \dot{X} \to Y$ in $\mathcal{C}$ with $f_0 = \operatorname{ins}_0 f$ and $f_1 = \operatorname{ins}_1 f$.

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, "\operatorname{ins}_0", "\wr"'] & Y \arrow[d, equal] \\
\dot{X} \arrow[r, "f"] & Y \\
X \arrow[r, "f_1"] \arrow[u, "\operatorname{ins}_1"', "\wr"] & Y \arrow[u, equal]
\end{tikzcd}$$

For a cylinder homotopy f from f_0 to f_1 with respect to $\dot{X}$, we usually write $f\colon f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f_1$.

(ii) Given a cylinder $\dot{X}$ of X, we say that f_0 is *cylinder homotopic* to f_1 *with respect to* $\dot{X}$, written $f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f_1$, if there exists a cylinder homotopy from f_0 to f_1 with respect to $\dot{X}$.
We say that f_0 is *cylinder homotopic* to f_1, written $f_0 \overset{\mathrm{c}}{\sim} f_1$, if we have $f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f_1$ for some cylinder $\dot{X}$ of X.

[6] Quillen uses the terminology *left homotopy* [28, ch. I, §1, def. 4].

(b) We suppose given S-2-arrows $(f_0, u_0), (f_1, u_1)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$.

(i) Given a cylinder $\dot{X}$ of X and a cylinder $\dot{Y}$ of Y, a *cylinder homotopy* from (f_0, u_0) to (f_1, u_1) with respect to $(\dot{X}, \dot{Y})$ is an S-2-arrow $(f, u)\colon \dot{X} \to \tilde{Y} \leftarrow \dot{Y}$ in $\mathcal{C}$ with $f\colon f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f_1$ and $u\colon u_0 \overset{\mathrm{c}}{\sim}_{\dot{Y}} u_1$.

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, "\mathrm{ins}_0", "\wr"'] & \tilde{Y} \arrow[d, equal] & Y \arrow[l, "u_0"', "\approx"] \arrow[d, "\mathrm{ins}_0", "\wr"'] \\
\dot{X} \arrow[r, "f"] & \tilde{Y} & \dot{Y} \arrow[l, "u"', "\approx"] \\
X \arrow[r, "f_1"] \arrow[u, "\mathrm{ins}_1"', "\wr"] & \tilde{Y} \arrow[u, equal] & Y \arrow[l, "u_1"', "\approx"] \arrow[u, "\mathrm{ins}_1"', "\wr"]
\end{tikzcd}$$

For a cylinder homotopy (f, u) from (f_0, u_0) to (f_1, u_1) with respect to $(\dot{X}, \dot{Y})$, we usually write $(f, u)\colon (f_0, u_0) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (f_1, u_1)$.

(ii) Given a cylinder $\dot{X}$ of X and a cylinder $\dot{Y}$ of Y, we say that (f_0, u_0) is *cylinder homotopic* to (f_1, u_1) *with respect to* $(\dot{X}, \dot{Y})$, written $(f_0, u_0) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (f_1, u_1)$, if there exists a cylinder homotopy from (f_0, u_0) to (f_1, u_1) with respect to $(\dot{X}, \dot{Y})$.

We say that (f_0, u_0) is *cylinder homotopic* to (f_1, u_1), written $(f_0, u_0) \overset{\mathrm{c}}{\sim} (f_1, u_1)$, if we have $(f_0, u_0) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (f_1, u_1)$ for some cylinder $\dot{X}$ of X and some cylinder $\dot{Y}$ of Y.

(3.131) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$.

(a) Given morphisms $f_0, f_1\colon X \to Y$ in $\mathcal{C}$ with $f_0 \overset{\mathrm{c}}{\sim} f_1$, we have

$$\mathrm{loc}(f_0) = \mathrm{loc}(f_1)$$

in $\mathrm{Ho}\,\mathcal{C}$.

(b) Given S-2-arrows $(f_0, u_0), (f_1, u_1)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $(f_0, u_0) \overset{\mathrm{c}}{\sim} (f_1, u_1)$, we have

$$\mathrm{loc}(f_0)\,\mathrm{loc}(u_0)^{-1} = \mathrm{loc}(f_1)\,\mathrm{loc}(u_1)^{-1}$$

in $\mathrm{Ho}\,\mathcal{C}$.

Proof.

(a) We suppose that $f_0 \overset{\mathrm{c}}{\sim} f_1$, that is, there exists a cylinder $\dot{X}$ of X and a cylinder homotopy $f\colon f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f_1$. As the cylinder equivalence $\mathrm{s}\colon \dot{X} \to X$ is a weak equivalence in $\mathcal{C}$, the morphism $\mathrm{loc}(\mathrm{s})$ is invertible in $\mathrm{Ho}\,\mathcal{C}$. So $\mathrm{ins}_0\,\mathrm{s} = \mathrm{ins}_1\,\mathrm{s} = 1_X$ implies that $\mathrm{loc}(\mathrm{ins}_0) = \mathrm{loc}(\mathrm{s})^{-1} = \mathrm{loc}(\mathrm{ins}_1)$, and so we obtain

$$\mathrm{loc}(f_0) = \mathrm{loc}(\mathrm{ins}_0)\,\mathrm{loc}(f) = \mathrm{loc}(\mathrm{ins}_1)\,\mathrm{loc}(f) = \mathrm{loc}(f_1).$$

(b) This follows from (a). □

(3.132) Theorem (Brown's homotopy S-2-arrow calculus [7, dual of th. 1 and proof])**.** We suppose given a Brown cofibration category $\mathcal{C}$.

(a) We have

$$\mathrm{Mor}\,\mathrm{Ho}\,\mathcal{C} = \{\mathrm{loc}(f)\,\mathrm{loc}(u)^{-1} \mid (f, u) \text{ is an S-2-arrow in } \mathcal{C}\}.$$

(b) We suppose given S-2-arrows $(f_0, u_0)\colon X \to \tilde{Y}_0 \leftarrow Y$, $(f_1, u_1)\colon X \to \tilde{Y}_1 \leftarrow Y$ in $\mathcal{C}$. The following conditions are equivalent.

(i) We have

$$\mathrm{loc}(f_0)\,\mathrm{loc}(u_0)^{-1} = \mathrm{loc}(f_1)\,\mathrm{loc}(u_1)^{-1}$$

in $\mathrm{Ho}\,\mathcal{C}$.

(ii) For every cylinder $\dot{X}$ of X and every cylinder $\dot{Y}$ of Y there exist a Z-2-arrow $(j_0, j_1)\colon \tilde{Y}_0 \to \tilde{Y} \leftarrow \tilde{Y}_1$ in $\mathcal{C}$ with acyclic cofibration j_0 such that $(f_0 j_0, u_0 j_0) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (f_1 j_1, u_1 j_1)$.

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}_{\dot{X}}"] & \tilde{Y}_0 \arrow[d, dotted, "\wr\bullet\, j_0"] & Y \arrow[l, "u_0"', "\approx"] \arrow[d, equal] \arrow[dl, phantom, "\overset{\mathrm{c}}{\sim}_{\dot{Y}}"] \\
X \arrow[r, dotted, "f_1 j_1"] & \tilde{Y} & Y \arrow[l, dotted, "u_1 j_1"', "\approx"] \\
X \arrow[r, "f_1"] \arrow[u, equal] & \tilde{Y}_1 \arrow[u, dotted, "\wr\bullet\, j_1"] & Y \arrow[l, "u_1"', "\approx"] \arrow[u, equal]
\end{tikzcd}$$

(iii) There exist a Z-2-arrow $(j_0, j_1)\colon \tilde{Y}_0 \to \tilde{Y} \leftarrow \tilde{Y}_1$ in $\mathcal{C}$ with acyclic cofibration j_0 such that $(f_0 j_0, u_0 j_0) \overset{\mathrm{c}}{\sim} (f_1 j_1, u_1 j_1)$.

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}"] & \tilde{Y}_0 \arrow[d, dotted, "\wr\bullet\, j_0"] & Y \arrow[l, "u_0"', "\approx"] \arrow[d, equal] \arrow[dl, phantom, "\overset{\mathrm{c}}{\sim}"] \\
X \arrow[r, dotted, "f_1 j_1"] & \tilde{Y} & Y \arrow[l, dotted, "u_1 j_1"', "\approx"] \\
X \arrow[r, "f_1"] \arrow[u, equal] & \tilde{Y}_1 \arrow[u, dotted, "\wr\bullet\, j_1"] & Y \arrow[l, "u_1"', "\approx"] \arrow[u, equal]
\end{tikzcd}$$

Proof.

(a) This follows from remark (3.129)(a).

(b) First, we suppose that condition (i) holds, that is, we suppose that $\operatorname{loc}(f_0) \operatorname{loc}(u_0)^{-1} = \operatorname{loc}(f_1) \operatorname{loc}(u_1)^{-1}$ in $\operatorname{Ho}\mathcal{C}$, and we suppose given a cylinder $\dot{X}$ of X and a cylinder $\dot{Y}$ of Y. Then we have $(1_X, 1_X) \equiv_{\mathrm{S}} (\operatorname{ins}_0^{\dot{X}}, \operatorname{ins}_1^{\dot{X}})$ and $(1_Y, 1_Y) \equiv_{\mathrm{S}} (\operatorname{ins}_0^{\dot{Y}}, \operatorname{ins}_1^{\dot{Y}})$. By remark (2.17), we obtain

$$\begin{aligned}
\operatorname{loc}(\operatorname{ins}_0^{\dot{X}}) \operatorname{loc}(\operatorname{ins}_1^{\dot{X}})^{-1} \operatorname{loc}(f_1) \operatorname{loc}(u_1)^{-1} &= \operatorname{loc}(f_1) \operatorname{loc}(u_1)^{-1} = \operatorname{loc}(f_0) \operatorname{loc}(u_0)^{-1} \\
&= \operatorname{loc}(f_0) \operatorname{loc}(u_0)^{-1} \operatorname{loc}(\operatorname{ins}_0^{\dot{Y}}) \operatorname{loc}(\operatorname{ins}_1^{\dot{Y}})^{-1}.
\end{aligned}$$

As $(\operatorname{ins}_0^{\dot{X}}, \operatorname{ins}_1^{\dot{X}})$ and $(\operatorname{ins}_0^{\dot{Y}}, \operatorname{ins}_1^{\dot{Y}})$ are Z-2-arrows in $\mathcal{C}$, by theorem (3.128)(c) there exist a Z-2-arrow $(j_0, j_1)\colon \tilde{Y}_0 \to \tilde{Y} \leftarrow \tilde{Y}_1$ and an S-2-arrow $(g, v)\colon \dot{X} \to \tilde{Y} \leftarrow \dot{Y}$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, "\wr\bullet\, \operatorname{ins}_0^{\dot{X}}"] & \tilde{Y}_0 \arrow[d, dotted, "\wr\bullet\, j_0"] & Y \arrow[l, "u_0"', "\approx"] \arrow[d, "\bullet\, \operatorname{ins}_0^{\dot{Y}}"] \\
\dot{X} \arrow[r, dotted, "g"] & \tilde{Y} & \dot{Y} \arrow[l, dotted, "v"', "\approx"] \\
X \arrow[r, "f_1"] \arrow[u, "\wr\bullet\, \operatorname{ins}_1^{\dot{X}}"] & \tilde{Y}_1 \arrow[u, dotted, "\wr\bullet\, j_1"] & Y \arrow[l, "u_1"', "\approx"] \arrow[u, "\wr\bullet\, \operatorname{ins}_1^{\dot{Y}}"]
\end{tikzcd}$$

But then (g, v) is a cylinder homotopy from $(f_0 j_0, u_0 j_0)$ to $(f_1 j_1, u_1 j_1)$ with respect to $(\dot{X}, \dot{Y})$. Thus we have $(f_0 j_0, u_0 j_0) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (f_1 j_1, u_1 j_1)$, that is, condition (ii) holds.

Condition (ii) and the Brown factorisation lemma (3.113)(a) imply condition (iii).

Finally, if condition (iii) holds, then we have

$$\operatorname{loc}(f_0) \operatorname{loc}(u_0)^{-1} = \operatorname{loc}(f_0 j_0) \operatorname{loc}(u_0 j_0)^{-1} = \operatorname{loc}(f_1 j_1) \operatorname{loc}(u_1 j_1)^{-1} = \operatorname{loc}(f_1) \operatorname{loc}(u_1)^{-1}$$

by remark (2.17) and remark (3.131)(b), that is, condition (i) holds.

Altogether, the three conditions (i), (ii) and (iii) are equivalent. $\square$

We have only formulated and proven an S-2-arrow equality condition up to cylinder homotopy, cf. definition (2.31)(b). An S-2-arrow composition condition up to cylinder homotopy, cf. definition (2.31)(c), is also valid, cf. theorem (2.37).

Chapter IV

Combinatorics for unstable triangulations

In a Verdier triangulated category $\mathcal{T}$, one deals with a so-called shift functor $\mathrm{T}\colon \mathcal{T} \to \mathcal{T}$ and certain diagrams, called Verdier triangles and Verdier octahedra. As indicated in the introduction, section 2, we may think of a Verdier triangle as a diagram of the form

$$\begin{tikzcd}
& & & 0 \\
& & 0 \arrow[r] & \mathrm{T}Y \arrow[u] \\
& 0 \arrow[r] & C \arrow[r] \arrow[u] & \mathrm{T}X \arrow[u] \\
0 \arrow[r] & X \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd} ,$$

and of a Verdier octahedron as a diagram of the form

$$\begin{tikzcd}
& & & & 0 \\
& & & 0 \arrow[r] & \mathrm{T}Z \arrow[u] \\
& & 0 \arrow[r] & A \arrow[r] \arrow[u] & \mathrm{T}Y \arrow[u] \\
& 0 \arrow[r] & C \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & \mathrm{T}X \arrow[u] \\
0 \arrow[r] & X \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd} .$$

Likewise, for $n \in \mathbb{N}_0$, we may think of an n-cosemitriangle in the homotopy category of a zero-pointed Brown cofibration category as an analogous diagram where the lowest row is supposed to have $n+1$ objects.
To work conveniently with these diagrams, it is desirable to write them as objects of a diagram category $\mathcal{T}^{S_0}$ for a suitable category S_0. One advantage of such an approach is the possibility to easily address composites, as one has already a corresponding composite in the shape category S_0. For instance, in the example of the Verdier octahedron drawn above, if this Verdier octahedron was an object V in $\mathcal{T}^{S_0}$ for a suitable category S_0, then the composite of the morphisms $C \to B$ and $B \to \mathrm{T}X$ could be written in the form V_a for a suitable $a \in \operatorname{Mor} S_0$. One can find such a category S_0; in fact, S_0 will be a poset ([1]), and so the described composite is of the form $V_{i,j}$

[1] We have $S_0 = \#_0^3$, cf. definition (4.45).

for $i, j \in \operatorname{Ob} S_0$ such that $C = V_i$ and $\mathrm{T}X = V_j$. This makes formal manipulations of the morphisms in such a diagram more convenient.
However, in the particular situation, such a poset S_0 is still insufficient as one would also like to manage composites that are not actually visible in the drawn picture. For example, in the case of the Verdier octahedron drawn above, we would like to address the composite of $A \to \mathrm{T}Y$ and the shift of $Y \to C$, which is a morphism $A \to \mathrm{T}C$. This composite can be seen in the picture if we *prolongate* the diagram *periodically*:

$$\begin{tikzcd}
 & & & & 0 \arrow[r] & \mathrm{T}^2X \arrow[r] & \mathrm{T}^2Y \arrow[r] & \mathrm{T}^2Z \arrow[r] & 0 \\
 & & & 0 \arrow[r] & \mathrm{T}Z \arrow[r] \arrow[u] & \mathrm{T}B \arrow[r] \arrow[u] & \mathrm{T}A \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & & 0 \arrow[r] & A \arrow[r] \arrow[u] & \mathrm{T}Y \arrow[r] \arrow[u] & \mathrm{T}C \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
 & 0 \arrow[r] & C \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & \mathrm{T}X \arrow[r] \arrow[u] & 0 \arrow[u] & & & \\
0 \arrow[r] & X \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u] & 0 \arrow[u] & & & &
\end{tikzcd}$$

We can manage the task of addressing composites with shifted morphisms by considering a category S that is equipped with a shift functor $\mathrm{T}\colon S \to S$ and that has S_0 as a subcategory, and to work with a suitable S-commutative diagram W in $\mathcal{T}$ that is compatible with the shifts in the sense of definition (4.17)(a). In fact, S will again be a poset ([2]). The diagram W then carries a lot of redundant information; however, for formal manipulations it is often easier to work with W instead of V.
In this chapter, we will develop the combinatorics for the triangulated structure studied in chapter V. We will introduce the semistrip types $\#^n_+$ for $n \in \mathbb{N}_0$, see definition (4.42), which are posets equipped with a shift functor that will play the role of "shape posets" for cosemitriangles: An n-cosemitriangle in the homotopy category of a zero-pointed Brown cofibration category as introduced in definition (5.33) will be a particular $\#^n_+$-commutative diagram.
To define the semistrip types, we first study the semiquasicyclic types Θ^n_+ for $n \in \mathbb{N}_0$, see definition (4.24). These posets may be thought of analoga of the ordinary simplex types Δ^n for $n \in \mathbb{N}_0$, but suitably modified such that they carry a shift. The semiquasicyclic types Θ^n_+ for $n \in \mathbb{N}_0$ have two roles: First, they appear in the definition of the semistrip types $\#^n_+$ for $n \in \mathbb{N}_0$, and as an n-cosemitriangle will be a diagram over $\#^n_+$, the semiquasicyclic types will be used to *define* the cosemitriangles. On the other hand, they will be used to *organise* the cosemitriangles: We will see in proposition (5.50) that cosemitriangles are stable under semiquasicyclic operations, which are, roughly said, simplicial operations plus a translation operation. In other words, the sets of cosemitriangles form a structure that is a variant of a simplicial set, a so-called semiquasicyclic set, see definition (4.38)(b).
The combinatorics for an unstable triangulated structure defined in this chapter is an unstable analogon to Künzer's combinatorics for Heller triangulated categories, see [23, sec. 1.1].
The chapter is organised as follows. First, we define objects with shift as a structure in an arbitrary category in section 1, as we will need this in several ways: The homotopy category of a zero-pointed Brown cofibration category becomes a zero-pointed category with shift, see convention (5.44), and the combinatorics for cosemitriangles is also pervaded by shifts. Then in section 2, we show how a shift functor can be induced on a diagram category, and define shift compatible diagrams. In section 3 to 5, the described combinatorics around the semiquasicyclic types and the semistrip types is developed. Finally, we introduce cosemistrips and the more restrictively defined cosemicomplexes in section 6.

[2] We have $S = \#^3_+$, cf. definition (4.42).

1 Objects with shift

A triangulated category in the sense of Verdier [37, ch. I, §1, n° 1, sec. 1-1] consists of an additive category $\mathcal{T}$ that is equipped with an autofunctor $\mathrm{T}\colon \mathcal{T} \to \mathcal{T}$, usually called the *shift* of $\mathcal{T}$, and additional structure. Variants of this notion, where the autofunctor is replaced by a not necessarily invertible endofunctor, have been studied by Keller and Vossieck [21, sec. 1.1] and, independently, Beligiannis and Marmaridis [6, def. 2.2, def. 2.3]. So all these structures have an underlying category that is equipped with an endomorphism. Such a construct will be called a *category with shift*, see definition (4.5)(c).
In this section, we introduce the notion of an object with shift in an arbitrary category, see definition (4.1)(a). In particular, we obtain the notion of a poset with shift, which is central in the combinatorics for higher unstable triangulations developed in section 3 to 6.

Definition of an object with shift

(4.1) Definition (object with shift)**.** We suppose given a category Ω.

(a) An *object with shift* in Ω consists of an object X together with a morphism $T\colon X \to X$ in Ω. By abuse of notation, we refer to the said object with shift as well as to its underlying object just by X. The endomorphism T is called the *shift morphism* (or just the *shift*) of X.

Given an object with shift X in Ω with shift morphism T, we write $\mathrm{T} = \mathrm{T}^X := T$.

(b) We suppose given objects with shift X and Y in Ω. A *morphism of objects with shift* in Ω from X to Y is a morphism $f\colon X \to Y$ in Ω that *preserves the shifts*, that is, such that $\mathrm{T}^X f = f \mathrm{T}^Y$.

(4.2) Remark. We suppose given a category Ω and an object with shift X in Ω. The shift morphism $\mathrm{T}\colon X \to X$ is a morphism of objects with shift.

The category of objects with shift

(4.3) Remark. For every category Ω, we have a category $\mathbf{T}\Omega$, given as follows. The set of objects of $\mathbf{T}\Omega$ is given by

$$\operatorname{Ob} \mathbf{T}\Omega = \{X \mid X \text{ is an object with shift in } \Omega\}.$$

For objects X and Y in $\mathbf{T}\Omega$, we have the hom-set

$$_{\mathbf{T}\Omega}(X, Y) = \{f \mid f \text{ is a morphism of objects with shift from } X \text{ to } Y\}.$$

For morphisms $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathbf{T}\Omega$, the composite of f and g in $\mathbf{T}\Omega$ is given by the composite of the underlying morphisms $fg\colon X \to Z$ in Ω. For an object X in $\mathbf{T}\Omega$, the identity morphism on X in $\mathbf{T}\Omega$ is given by the underlying identity morphisms $1_X\colon X \to X$ in Ω.

(4.4) Definition (category of objects with shift)**.** We suppose given a category Ω. The category $\mathbf{T}\Omega$ as considered in remark (4.3) is called the *category of objects with shift* in Ω.

Some instances of objects with shift

So far, we have introduced the categorical concept of an object with shift in an arbitrary category Ω. Now we particularise this notion for concretely given categories like $\mathbf{Set}_{(\mathfrak{U})}$ for some Grothendieck universe $\mathfrak{U}$. Moreover, we introduce universe-free variants of the notions obtained in this way.

(4.5) Definition (set with shift, poset with shift, category with shift)**.**

(a) Given a Grothendieck universe $\mathfrak{U}$, the category $\mathbf{TSet}_{(\mathfrak{U})}$ is called the *category of sets with shift* (more precisely, the *category of* $\mathfrak{U}$*-sets with shift*), an object in $\mathbf{TSet}_{(\mathfrak{U})}$ is called a *set with shift with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$*-set with shift*), and a morphism in $\mathbf{TSet}_{(\mathfrak{U})}$ is called a *morphism of sets with shift with respect to* $\mathfrak{U}$ (or a $\mathfrak{U}$*-morphism of sets with shift*).

A *set with shift* is a $\mathfrak{U}$-set with shift for some Grothendieck universe $\mathfrak{U}$. Given a set with shift X, the shift morphism of X is also called the *shift map* of X. A *morphism of sets with shift* (or a *shift preserving map*) is a $\mathfrak{U}$-morphism of sets with shift for some Grothendieck universe $\mathfrak{U}$.

(b) Given a Grothendieck universe $\mathfrak{U}$, the category $\mathbf{TPoset}_{(\mathfrak{U})}$ is called the *category of posets with shift* (more precisely, the *category of $\mathfrak{U}$-posets with shift*), an object in $\mathbf{TPoset}_{(\mathfrak{U})}$ is called a *poset with shift with respect to* $\mathfrak{U}$ (or a *$\mathfrak{U}$-poset with shift*), and a morphism in $\mathbf{TPoset}_{(\mathfrak{U})}$ is called a *morphism of posets with shift with respect to* $\mathfrak{U}$ (or a *$\mathfrak{U}$-morphism of posets with shift*).

A *poset with shift* is a $\mathfrak{U}$-poset with shift for some Grothendieck universe $\mathfrak{U}$. A *morphism of posets with shift* (or a *shift preserving poset morphism*) is a $\mathfrak{U}$-morphism of posets with shift for some Grothendieck universe $\mathfrak{U}$.

(c) Given a Grothendieck universe $\mathfrak{U}$. The category $\mathbf{TCat}_{(\mathfrak{U})}$ is called the *category of categories with shift* (more precisely, the *category of $\mathfrak{U}$-categories with shift*), an object in $\mathbf{TCat}_{(\mathfrak{U})}$ is called a *category with shift with respect to* $\mathfrak{U}$ (or a *$\mathfrak{U}$-category with shift*), and a morphism in $\mathbf{TCat}_{(\mathfrak{U})}$ is called a *morphism of categories with shift with respect to* $\mathfrak{U}$ (or a *$\mathfrak{U}$-morphism of categories with shift*).

A *category with shift* is a $\mathfrak{U}$-category with shift for some Grothendieck universe $\mathfrak{U}$. Given a category with shift $\mathcal{C}$, the shift morphism of $\mathcal{C}$ is also called the *shift functor* of $\mathcal{C}$. A *morphism of categories with shift* (or a *shift preserving functor*) is a $\mathfrak{U}$-morphism of categories with shift for some Grothendieck universe $\mathfrak{U}$.

(d) Given a Grothendieck universe $\mathfrak{U}$, the category $\mathbf{TCat}_{0,(\mathfrak{U})}$ is called the *category of zero-pointed categories with shift* (more precisely, the *category of zero-pointed $\mathfrak{U}$-categories with shift*), an object in $\mathbf{TCat}_{0,(\mathfrak{U})}$ is called a *zero-pointed category with shift with respect to* $\mathfrak{U}$ (or a *zero-pointed $\mathfrak{U}$-category with shift*), and a morphism in $\mathbf{TCat}_{(\mathfrak{U})}$ is called a *morphism of zero-pointed categories with shift with respect to* $\mathfrak{U}$ (or a *$\mathfrak{U}$-morphism of zero-pointed categories with shift*).

A *zero-pointed category with shift* is a zero-pointed $\mathfrak{U}$-category with shift for some Grothendieck universe $\mathfrak{U}$. A *morphism of zero-pointed categories with shift* (or a *shift preserving morphism of zero-pointed categories*) is a $\mathfrak{U}$-morphism of zero-pointed categories with shift for some Grothendieck universe $\mathfrak{U}$.

(4.6) Remark.

(a) (i) We suppose given a Grothendieck universe $\mathfrak{U}$. A set with shift X is a $\mathfrak{U}$-set with shift if and only if it is an element of $\mathfrak{U}$.

(ii) For every set with shift X there exists a Grothendieck universe $\mathfrak{U}$ such that X is a $\mathfrak{U}$-set with shift.

(b) (i) We suppose given a Grothendieck universe $\mathfrak{U}$. A poset with shift X is a $\mathfrak{U}$-poset with shift if and only if it is an element of $\mathfrak{U}$.

(ii) For every poset with shift X there exists a Grothendieck universe $\mathfrak{U}$ such that X is a $\mathfrak{U}$-poset with shift.

(c) (i) We suppose given a Grothendieck universe $\mathfrak{U}$. A category with shift $\mathcal{C}$ is a $\mathfrak{U}$-category with shift if and only if it is an element of $\mathfrak{U}$.

(ii) For every category with shift $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a $\mathfrak{U}$-category with shift.

(d) (i) We suppose given a Grothendieck universe $\mathfrak{U}$. A zero-pointed category with shift $\mathcal{C}$ is a zero-pointed $\mathfrak{U}$-category with shift if and only if it is an element of $\mathfrak{U}$.

(ii) For every zero-pointed category with shift $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is a zero-pointed $\mathfrak{U}$-category with shift.

(4.7) Definition (subobject with shift)**.**

(a) We suppose given a set with shift X. A *subset with shift* of X is a set with shift U whose underlying set is a subset of X and whose shift is given by $\mathrm{T}^U = \mathrm{T}^X|_U^U$.

(b) We suppose given a poset with shift X. A *subposet with shift* of X is a poset with shift U whose underlying poset is a subposet of X and whose shift is given by $\mathrm{T}^U = \mathrm{T}^X|_U^U$.

(c) We suppose given a category with shift $\mathcal{C}$. A *subcategory with shift* of $\mathcal{C}$ is a category with shift $\mathcal{U}$ whose underlying category is a subcategory of $\mathcal{C}$ and whose shift is given by $\mathrm{T}^{\mathcal{U}} = \mathrm{T}^{\mathcal{C}}|_{\mathcal{U}}^{\mathcal{U}}$.

(d) We suppose given a zero-pointed category with shift $\mathcal{C}$. A *zero-pointed subcategory with shift* of $\mathcal{C}$ is a zero-pointed category with shift $\mathcal{U}$ whose underlying zero-pointed category is a zero-pointed subcategory of $\mathcal{C}$ and whose shift is given by $\mathrm{T}^{\mathcal{U}} = \mathrm{T}^{\mathcal{C}}|_{\mathcal{U}}^{\mathcal{U}}$.

(4.8) Remark.

(a) Given sets with shift X and U such that the underlying set of U is a subset of X, then U is a subset with shift of X if and only if the inclusion $\mathrm{inc}\colon U \to X$ is a morphism of sets with shift.

(b) Given posets with shift X and U such that the underlying poset of U is a subposet of X, then U is a subposet with shift of X if and only if the inclusion $\mathrm{inc}\colon U \to X$ is a morphism of posets with shift.

(c) Given categories with shift $\mathcal{C}$ and $\mathcal{U}$ such that the underlying category of $\mathcal{U}$ is a subcategory of $\mathcal{C}$, then $\mathcal{U}$ is a subcategory with shift of $\mathcal{C}$ if and only if the inclusion $\mathrm{inc}\colon \mathcal{U} \to \mathcal{C}$ is a morphism of categories with shift.

(d) Given zero-pointed categories with shift $\mathcal{C}$ and $\mathcal{U}$ such that the underlying zero-pointed category of $\mathcal{U}$ is a zero-pointed subcategory of $\mathcal{C}$, then $\mathcal{U}$ is a zero-pointed subcategory with shift of $\mathcal{C}$ if and only if the inclusion $\mathrm{inc}\colon \mathcal{U} \to \mathcal{C}$ is a morphism of zero-pointed categories with shift.

(4.9) Definition (full subobject with shift)**.**

(a) We suppose given a poset with shift X. A subposet with shift U of X is said to be *full* if its underlying poset is a full subposet of the underlying poset of X.

(b) We suppose given a category with shift $\mathcal{C}$. A subcategory with shift $\mathcal{U}$ of $\mathcal{C}$ is said to be *full* if its underlying category is a full subcategory of the underlying category of $\mathcal{C}$.

(c) We suppose given a zero-pointed category with shift $\mathcal{C}$. A zero-pointed subcategory with shift $\mathcal{U}$ of $\mathcal{C}$ is said to be *full* if its underlying zero-pointed category is a full zero-pointed subcategory of the underlying zero-pointed category of $\mathcal{C}$.

To abbreviate, we use the following notation for the shift map of a set with shift.

(4.10) Notation (element notation for the shift)**.** Given a set with shift X, we write

$$x^{[m]} := x\mathrm{T}^m$$

for $x \in X$, $m \in \mathbb{N}_0$. ([3])

If unambiguous, we will consider the set of objects and the set of morphisms of a category with shift as sets with shift in the following way, without further comment.

(4.11) Remark. Given a category with shift $\mathcal{C}$, the set of objects $\mathrm{Ob}\,\mathcal{C}$ becomes a set with shift having

$$\mathrm{T}^{\mathrm{Ob}\,\mathcal{C}} = \mathrm{Ob}\,\mathrm{T}^{\mathcal{C}},$$

and the set of morphisms $\mathrm{Mor}\,\mathcal{C}$ becomes a set with shift having

$$\mathrm{T}^{\mathrm{Mor}\,\mathcal{C}} = \mathrm{Mor}\,\mathrm{T}^{\mathcal{C}}.$$

2 Diagram categories on categories with shift

We suppose given categories $\mathcal{C}$ and S. In this section, we will show that a shift functor on $\mathcal{C}$ induces a shift functor on the diagram category $\mathcal{C}^S$ (see below for details). Moreover, we will introduce the notion of a shift compatible diagram, which is defined when both categories $\mathcal{C}$ and S are equipped with a shift functor. Cosemitriangles as introduced in chapter V, section 4 will be particular shift compatible diagrams.

[3] In the literature, the notation $x[m] := x\mathrm{T}^m$ is often used.

The inner shift functor

We suppose given categories $\mathcal{C}$ and S. An S-commutative diagram in $\mathcal{C}$ is a functor X from S to $\mathcal{C}$. The category $\mathcal{C}$ is called the base category, and the category S is called the shape category. We denote the category of S-commutative diagrams in $\mathcal{C}$ by $\mathcal{C}^S = \mathcal{C}^S_{\mathbf{Cat}}$. Given an object i in S, we usually write X_i for the image of i under X, and given a morphism $a\colon i \to j$ in S, we usually write $X_a\colon X_i \to X_j$ for the image a under X.
Given a Grothendieck universe $\mathfrak{U}$, we have a diagram functor

$$(=)^{(-)} = (=)^{(-)}_{\mathbf{Cat}}\colon \mathbf{Cat}^{\mathrm{op}}_{(\mathfrak{U})} \times \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})},$$

given on the objects by $(=)^{(-)}(S, \mathcal{C}) = \mathcal{C}^S$ for $\mathcal{C}, S \in \operatorname{Ob}\mathbf{Cat}_{(\mathfrak{U})}$, and given on the morphisms by $(=)^{(-)}(G, F) = F^G_{\mathrm{Cat}} = F^G\colon \mathcal{C}^S \to \mathcal{D}^R$ for morphisms $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon R \to S$ in $\mathbf{Cat}_{(\mathfrak{U})}$, where $F^G(X) = F \circ X \circ G$ for $X \in \operatorname{Ob}\mathcal{C}^S$ and $F^G(f) = F * f * G$ for $f \in \operatorname{Mor}\mathcal{C}^S$. We abbreviate $F^S = F^{\mathrm{id}_S}$ for a morphism $F\colon \mathcal{C} \to \mathcal{D}$ and an object S in $\mathbf{Cat}_{(\mathfrak{U})}$.

(4.12) Definition (inner structure)**.** Given a category with shift $\mathcal{C}$ and a category S, we denote by $\mathcal{C}^S = \mathcal{C}^S_{\mathrm{in}} = \mathcal{C}^S_{\mathbf{Cat},\mathrm{in}}$ the category with shift whose underlying category is $\mathcal{C}^S$ and whose shift functor is given by

$$\mathrm{T}^{\mathcal{C}^S_{\mathrm{in}}} = (\mathrm{T}^{\mathcal{C}})^S\colon \mathcal{C}^S \to \mathcal{C}^S.$$

The structure of a category with shift on $\mathcal{C}^S_{\mathrm{in}}$ is called the *inner structure* (*of a category with shift*) on $\mathcal{C}^S$, and the shift functor $\mathrm{T}_{\mathrm{in}} = \mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}} = \mathrm{T}^{\mathcal{C}^S_{\mathbf{Cat}}}_{\mathrm{in}} := \mathrm{T}^{\mathcal{C}^S_{\mathbf{Cat},\mathrm{in}}}$ is called the *inner shift functor* (or just the *inner shift*) on $\mathcal{C}^S$.

There is also a notion of an *outer shift* on the diagram category $\mathcal{C}^S$ for a category $\mathcal{C}$ and a category with shift S, cf. [23, sec. 1.2.1.2, p. 246].
For later use, we deduce explicit formulas for the inner shift functor:

(4.13) Remark. We suppose given a category with shift $\mathcal{C}$ and a category S. The inner shift functor T_{in} on $\mathcal{C}^S$ is given on the objects by

$$(\mathrm{T}_{\mathrm{in}}X)_i = X^{[1]}_i$$

for $i \in \operatorname{Ob} S$ and by

$$(\mathrm{T}_{\mathrm{in}}X)_a = X^{[1]}_a$$

for $a \in \operatorname{Mor} S$, $X \in \operatorname{Ob}\mathcal{C}^S$, and on the morphisms by

$$(\mathrm{T}_{\mathrm{in}}f)_i = f^{[1]}_i$$

for $i \in \operatorname{Ob} S$, $f \in \operatorname{Mor}\mathcal{C}^S$.

Proof. We have

$$(\mathrm{T}_{\mathrm{in}}X)_i = ((\mathrm{T}^{\mathcal{C}})^S X)_i = (\mathrm{T}^{\mathcal{C}} \circ X)_i = \mathrm{T}^{\mathcal{C}} X_i = X^{[1]}_i$$

for $i \in \operatorname{Ob} S$ and

$$(\mathrm{T}_{\mathrm{in}}X)_a = ((\mathrm{T}^{\mathcal{C}})^S X)_a = (\mathrm{T}^{\mathcal{C}} \circ X)_a = \mathrm{T}^{\mathcal{C}} X_a = X^{[1]}_a$$

for $a \in \operatorname{Mor} S$, $X \in \operatorname{Ob}\mathcal{C}^S$. Moreover, we have

$$(\mathrm{T}_{\mathrm{in}}f)_i = ((\mathrm{T}^{\mathcal{C}})^S f)_i = (\mathrm{T}^{\mathcal{C}} * f)_i = \mathrm{T}^{\mathcal{C}} f_i = f^{[1]}_i$$

for $i \in \operatorname{Ob} S$, $f \in \operatorname{Mor}\mathcal{C}^S$. □

(4.14) Notation. Given a category with shift $\mathcal{C}$, we usually abbreviate $X^{[1]} = \mathrm{T}X$ and $f^{[1]} = \mathrm{T}f$; cf. notation (4.10). When dealing with a diagram category $\mathcal{C}^S$ for a category S, we also use this notation for the inner shift $\mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}} = (\mathrm{T}^{\mathcal{C}})^S$ as this shift is obtained by object- and morphismwise application of $\mathrm{T}^{\mathcal{C}}$. So given an S-commutative diagram X in $\mathcal{C}$, we have $(X^{[1]})_i = X^{[1]}_i$ for $i \in \operatorname{Ob} S$ and $(X^{[1]})_a = X^{[1]}_a$ for $a \in \operatorname{Mor} S$. Likewise for morphisms of S-commutative diagrams.

The diagram functor for the inner structure

We turn the inner structure of a category with shift into a functor.

(4.15) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. The diagram functor

$$(=)^{(-)}\colon \mathbf{Cat}^{\mathrm{op}}_{(\mathfrak{U})} \times \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})}$$

induces a functor

$$(=)^{(-)}_{\mathrm{in}} = (=)^{(-)}_{\mathbf{Cat},\mathrm{in}}\colon \mathbf{Cat}^{\mathrm{op}}_{(\mathfrak{U})} \times \mathbf{TCat}_{(\mathfrak{U})} \to \mathbf{TCat}_{(\mathfrak{U})},$$

given on the morphisms by

$$F^G_{\mathrm{in}} = F^G\colon \mathcal{C}^S_{\mathrm{in}} \to \mathcal{D}^R_{\mathrm{in}}$$

for morphisms $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}_{(\mathfrak{U})}$ and $G\colon R \to S$ in $\mathbf{Cat}_{(\mathfrak{U})}$.

Proof. Given morphisms $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}$ and $G\colon R \to S$ in $\mathbf{Cat}$, we have

$$\mathrm{T}^{\mathcal{D}^R_{\mathrm{in}}} \circ F^G = (\mathrm{T}^{\mathcal{D}})^{\mathrm{id}_R} \circ F^G = (\mathrm{T}^{\mathcal{D}} \circ F)^{G \circ \mathrm{id}_R} = (F \circ \mathrm{T}^{\mathcal{C}})^{\mathrm{id}_S \circ G} = F^G \circ (\mathrm{T}^{\mathcal{C}})^{\mathrm{id}_S} = F^G \circ \mathrm{T}^{\mathcal{C}^S_{\mathrm{in}}},$$

that is, $F^G\colon \mathcal{C}^S_{\mathrm{in}} \to \mathcal{D}^R_{\mathrm{in}}$ is a morphism of categories with shift.

$$\begin{tikzcd}
\mathcal{C}^S_{\mathrm{in}} \arrow[r, "\mathrm{T}^{\mathcal{C}^S_{\mathrm{in}}}"] \arrow[d, "F^G"'] & \mathcal{C}^S_{\mathrm{in}} \arrow[d, "F^G"] \\
\mathcal{D}^R_{\mathrm{in}} \arrow[r, "\mathrm{T}^{\mathcal{D}^R_{\mathrm{in}}}"] & \mathcal{D}^R_{\mathrm{in}}
\end{tikzcd}$$

The functoriality of the induced graph morphism $(=)^{(-)}_{\mathrm{in}}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{TCat} \to \mathbf{TCat}$, which is given on the morphisms by $F^G_{\mathrm{in}} = F^G$ for $G \in \operatorname{Mor} \mathbf{Cat}$, $F \in \operatorname{Mor} \mathbf{TCat}$, follows from the functoriality of $(=)^{(-)}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Cat}$. □

We suppose given a Grothendieck universe $\mathfrak{U}$. The diagram functor $(=)^{(-)} = (=)^{(-)}_{\mathbf{Cat}}\colon \mathbf{Cat}^{\mathrm{op}}_{(\mathfrak{U})} \times \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})}$ induces a diagram functor $(=)^{(-)} = (=)^{(-)}_{\mathbf{Cat}}\colon \mathbf{Cat}^{\mathrm{op}}_{(\mathfrak{U})} \times \mathbf{Cat}_{0,(\mathfrak{U})} \to \mathbf{Cat}_{0,(\mathfrak{U})}$, where $0^{\mathcal{C}^S}$ for $\mathcal{C} \in \operatorname{Ob} \mathbf{Cat}_{0,(\mathfrak{U})}$, $S \in \operatorname{Ob} \mathbf{Cat}_{(\mathfrak{U})}$ is given by $(0^{\mathcal{C}^S})_i = 0^{\mathcal{C}}$ for $i \in \operatorname{Ob} S$.

(4.16) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. The diagram functor

$$(=)^{(-)}\colon \mathbf{Cat}^{\mathrm{op}}_{(\mathfrak{U})} \times \mathbf{Cat}_{0,(\mathfrak{U})} \to \mathbf{Cat}_{0,(\mathfrak{U})}$$

induces a functor

$$(=)^{(-)}_{\mathrm{in}} = (=)^{(-)}_{\mathbf{Cat},\mathrm{in}}\colon \mathbf{Cat}^{\mathrm{op}}_{(\mathfrak{U})} \times \mathbf{TCat}_{0,(\mathfrak{U})} \to \mathbf{TCat}_{0,(\mathfrak{U})},$$

given on the morphisms by

$$F^G_{\mathrm{in}} = F^G\colon \mathcal{C}^S_{\mathrm{in}} \to \mathcal{D}^R_{\mathrm{in}}$$

for morphisms $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}_{0,(\mathfrak{U})}$ and $G\colon R \to S$ in $\mathbf{Cat}_{(\mathfrak{U})}$.

Proof. By remark (4.15), the diagram functor $(=)^{(-)}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Cat}$ induces a functor $(=)^{(-)}_{\mathrm{in}}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{TCat} \to \mathbf{TCat}$. In particular, given $\mathcal{C} \in \operatorname{Ob} \mathbf{TCat}_0$, $S \in \operatorname{Ob} \mathbf{Cat}$, the diagram category $\mathcal{C}^S$ carries the structure of a category with shift. Moreover, $\mathcal{C}^S$ is a zero-pointed category. To show that $\mathcal{C}^S$ carries the structure of a zero-pointed category with shift, it remains to show that $\mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}}\colon \mathcal{C}^S \to \mathcal{C}^S$ is a morphism of zero-pointed categories. Indeed, $\mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}} 0^{\mathcal{C}^S}$ is given by

$$(\mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}} 0^{\mathcal{C}^S})_i = \mathrm{T}^{\mathcal{C}}(0^{\mathcal{C}^S})_i = \mathrm{T}^{\mathcal{C}} 0^{\mathcal{C}} = 0^{\mathcal{C}}$$

for $i \in \operatorname{Ob} S$, and so we have $\mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}} 0^{\mathcal{C}^S} = 0^{\mathcal{C}^S}$.

Moreover, given morphisms $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}_0$ and $G\colon R \to S$ in $\mathbf{Cat}$, then $F^G\colon \mathcal{C}^S \to \mathcal{D}^R$ is a morphism of categories with shift by remark (4.15) and a morphism of zero-pointed categories, so it is a morphism of zero-pointed categories with shift.

The functoriality of the induced graph morphism $(=)^{(-)}_{\mathrm{in}}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{TCat}_0 \to \mathbf{TCat}_0$, which is given on the morphisms by $F^G_{\mathrm{in}} = F^G$ for $G \in \operatorname{Mor} \mathbf{Cat}$, $F \in \operatorname{Mor} \mathbf{TCat}_0$, follows from the functoriality of $(=)^{(-)}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat}_0 \to \mathbf{Cat}_0$. □

Shift compatible diagrams

So far, we have only considered diagram categories where the base category carries the structure of a category with shift. If the shape category is also equipped with a shift functor, as it will be in the case $S = \#_+^n$ for some $n \in \mathbb{N}_0$ in section 6, we can study those diagrams that preserves the shifts, so-called shift compatible diagrams:

(4.17) Definition (shift compatible diagram)**.** We suppose given categories with shift $\mathcal{C}$ and S.

(a) An S-commutative diagram X in $\mathcal{C}$ is said to be *shift compatible* if $X \circ \mathrm{T}^S = \mathrm{T}^{\mathcal{C}} \circ X$.

(b) A morphism of S-commutative diagrams f in $\mathcal{C}$ is said to be *shift compatible* if $f * \mathrm{T}^S = \mathrm{T}^{\mathcal{C}} * f$.

(4.18) Remark. We suppose given categories with shift $\mathcal{C}$ and S. An S-commutative diagram X in $\mathcal{C}$ is shift compatible if and only if the functor $X\colon S \to \mathcal{C}$ is a morphism of categories with shift.

(4.19) Remark. We suppose given categories with shift $\mathcal{C}$ and S.

(a) An S-commutative diagram X in $\mathcal{C}$ is shift compatible if and only if $X_{i^{[1]}} = X_i^{[1]}$ for all $i \in \operatorname{Ob} S$ and $X_{a^{[1]}} = X_a^{[1]}$ for all $a \in \operatorname{Mor} S$.

(b) A morphism of S-commutative diagrams $f\colon X \to Y$ in $\mathcal{C}$ is shift compatible if and only if X, Y are shift compatible and $f_{i^{[1]}} = f_i^{[1]}$ for all $i \in \operatorname{Ob} S$.

(4.20) Remark. For all categories with shift $\mathcal{C}$ and S, we have a subcategory with shift $\mathcal{C}^S_{\mathbf{TCat}}$ of $\mathcal{C}^S$, given by

$$\operatorname{Ob} \mathcal{C}^S_{\mathbf{TCat}} = \{X \in \operatorname{Ob} \mathcal{C}^S \mid X \text{ is shift compatible}\},$$
$$\operatorname{Mor} \mathcal{C}^S_{\mathbf{TCat}} = \{f \in \operatorname{Mor} \mathcal{C}^S \mid f \text{ is shift compatible}\}.$$

Proof. Given a shift compatible diagram morphism f in $\mathcal{C}$ over S, then $\operatorname{Source} f$ and $\operatorname{Target} f$ are shift compatible diagrams in $\mathcal{C}$ over S. Given shift compatible diagram morphisms f, g in $\mathcal{C}$ over S with $\operatorname{Target} f = \operatorname{Source} g$, we have

$$(fg) * \mathrm{T}^S = (f * \mathrm{T}^S)(g * \mathrm{T}^S) = (\mathrm{T}^{\mathcal{C}} * f)(\mathrm{T}^{\mathcal{C}} * g) = \mathrm{T}^{\mathcal{C}} * (fg),$$

that is, the composite fg is shift compatible. Finally, given a shift compatible diagram X in $\mathcal{C}$ over S, we have

$$1_X * \mathrm{T}^S = 1_{X \circ \mathrm{T}^S} = 1_{\mathrm{T}^{\mathcal{C}} \circ X} = \mathrm{T}^{\mathcal{C}} * 1_X,$$

so the identity morphism 1_X is shift compatible.
Altogether, the set of shift compatible diagrams in $\mathcal{C}$ over S resp. the set of shift compatible diagram morphisms in $\mathcal{C}$ over S form the set of objects resp. the set of morphisms of a subcategory $\mathcal{C}^S_{\mathbf{TCat}}$ of $\mathcal{C}^S$.
Given $X \in \operatorname{Ob} \mathcal{C}^S_{\mathbf{TCat}}$, we have

$$\mathrm{T}_{\mathrm{in}} X \circ \mathrm{T}^S = \mathrm{T}^{\mathcal{C}} \circ X \circ \mathrm{T}^S = \mathrm{T}^{\mathcal{C}} \circ \mathrm{T}^{\mathcal{C}} \circ X = \mathrm{T}^{\mathcal{C}} \circ \mathrm{T}_{\mathrm{in}} X,$$

that is, $\mathrm{T}_{\mathrm{in}} X \in \operatorname{Ob} \mathcal{C}^S_{\mathbf{TCat}}$. Moreover, given $f \in \operatorname{Mor} \mathcal{C}^S_{\mathbf{TCat}}$, we have

$$\mathrm{T}_{\mathrm{in}} f * \mathrm{T}^S = \mathrm{T}^{\mathcal{C}} * f * \mathrm{T}^S = \mathrm{T}^{\mathcal{C}} * \mathrm{T}^{\mathcal{C}} * f = \mathrm{T}^{\mathcal{C}} * \mathrm{T}_{\mathrm{in}} f,$$

that is, $\mathrm{T}_{\mathrm{in}} f \in \operatorname{Ob} \mathcal{C}^S_{\mathbf{TCat}}$. Hence T_{in} maps shift compatible diagrams resp. shift compatible diagram morphisms to shift compatible diagrams resp. shift compatible diagram morphisms, and $\mathcal{C}^S_{\mathbf{TCat}}$ becomes a category with shift having $\mathrm{T}^{\mathcal{C}^S_{\mathbf{TCat}}} = \mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}}|^{\mathcal{C}^S_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}$.

$$\begin{tikzcd}
\mathcal{C}^S_{\mathbf{TCat}} \arrow[r, "\mathrm{T}_{\mathrm{in}}|^{\mathcal{C}^S_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}"] \arrow[d, "\mathrm{inc}"'] & \mathcal{C}^S_{\mathbf{TCat}} \arrow[d, "\mathrm{inc}"] \\
\mathcal{C}^S \arrow[r, "\mathrm{T}_{\mathrm{in}}"] & \mathcal{C}^S
\end{tikzcd}$$

□

(4.21) Definition (category of shift compatible diagrams)**.** We suppose given categories with shift $\mathcal{C}$ and S. The category with shift $\mathcal{C}^S_{\mathbf{TCat}}$ as considered in remark (4.20) is called the *category of shift compatible S-commutative diagrams* in $\mathcal{C}$.

The diagram functor for shift compatible diagrams

We show that the diagram functors for the inner structure, see remark (4.15) and remark (4.16), induce respective diagram functors for the categories of shift compatible diagrams.

(4.22) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. The diagram functor

$$(=)^{(-)}\colon \mathbf{Cat}_{(\mathfrak{U})}^{\mathrm{op}} \times \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})}$$

induces a functor

$$(=)^{(-)}_{\mathbf{TCat}}\colon (\mathbf{TCat}_{(\mathfrak{U})})^{\mathrm{op}} \times \mathbf{TCat}_{(\mathfrak{U})} \to \mathbf{TCat}_{(\mathfrak{U})},$$

given on the morphisms by

$$F^G_{\mathbf{TCat}} = F^G_{\mathbf{Cat}}|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}\colon \mathcal{C}^S_{\mathbf{TCat}} \to \mathcal{D}^R_{\mathbf{TCat}}$$

for morphisms $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon R \to S$ in $\mathbf{TCat}_{(\mathfrak{U})}$.

Proof. We suppose given morphisms $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon R \to S$ in $\mathbf{TCat}$. Then for $X \in \operatorname{Ob} \mathcal{C}^S_{\mathbf{TCat}}$, we have

$$(F^G X) \circ \mathrm{T}^R = F \circ X \circ G \circ \mathrm{T}^R = F \circ X \circ \mathrm{T}^S \circ G = F \circ \mathrm{T}^{\mathcal{C}} \circ X \circ G = \mathrm{T}^{\mathcal{D}} \circ F \circ X \circ G = \mathrm{T}^{\mathcal{D}} \circ (F^G X),$$

that is, $F^G X \in \operatorname{Ob} \mathcal{D}^R_{\mathbf{TCat}}$. Moreover, for $f \in \operatorname{Mor} \mathcal{D}^R_{\mathbf{TCat}}$, we have

$$(F^G f) * \mathrm{T}^R = F * f * G * \mathrm{T}^R = F * f * \mathrm{T}^S * G = F * \mathrm{T}^{\mathcal{C}} * f * G = \mathrm{T}^{\mathcal{D}} * F * f \circ G = \mathrm{T}^{\mathcal{D}} * (F^G f),$$

that is, $F^G f \in \operatorname{Mor} \mathcal{D}^R_{\mathbf{TCat}}$. Hence $F^G\colon \mathcal{C}^S \to \mathcal{D}^R$ maps shift compatible S-commutative diagrams resp. shift compatible morphisms of S-commutative diagrams in $\mathcal{C}$ to shift compatible R-commutative diagrams resp. shift compatible morphisms of R-commutative diagrams in $\mathcal{D}$.

$$\begin{tikzcd}
\mathcal{C}^S_{\mathbf{TCat}} \arrow[r, "F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}"] \arrow[d, "\mathrm{inc}"'] & \mathcal{D}^R_{\mathbf{TCat}} \arrow[d, "\mathrm{inc}"] \\
\mathcal{C}^S \arrow[r, "F^G"] & \mathcal{D}^R
\end{tikzcd}$$

Moreover, as the shift on $\mathcal{C}^S_{\mathbf{TCat}}$ resp. $\mathcal{D}^R_{\mathbf{TCat}}$ is induced from the inner shift on $\mathcal{C}^S$ resp. $\mathcal{D}^R$, remark (4.15) implies that

$$\begin{aligned}
\mathrm{inc}^{\mathcal{D}^R_{\mathbf{TCat}}} \circ \mathrm{T}^{\mathcal{D}^R_{\mathbf{TCat}}} \circ F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}} &= \mathrm{T}^{\mathcal{D}^R}_{\mathrm{in}} \circ \mathrm{inc}^{\mathcal{D}^R_{\mathbf{TCat}}} \circ F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}} = \mathrm{T}^{\mathcal{D}^R}_{\mathrm{in}} \circ F^G \circ \mathrm{inc}^{\mathcal{C}^S_{\mathbf{TCat}}} \\
&= F^G \circ \mathrm{T}^{\mathcal{C}^S}_{\mathrm{in}} \circ \mathrm{inc}^{\mathcal{C}^S_{\mathbf{TCat}}} = F^G \circ \mathrm{inc}^{\mathcal{C}^S_{\mathbf{TCat}}} \circ \mathrm{T}^{\mathcal{C}^S_{\mathbf{TCat}}} \\
&= \mathrm{inc}^{\mathcal{D}^R_{\mathbf{TCat}}} \circ F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}} \circ \mathrm{T}^{\mathcal{C}^S_{\mathbf{TCat}}}.
\end{aligned}$$

and therefore $\mathrm{T}^{\mathcal{D}^R_{\mathbf{TCat}}} \circ F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}} = F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}} \circ \mathrm{T}^{\mathcal{C}^S_{\mathbf{TCat}}}$. Hence $F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}$ is a morphism of categories with shift.

$$\begin{tikzcd}
\mathcal{C}^S_{\mathbf{TCat}} \arrow[r, "\mathrm{T}^{\mathcal{C}^S_{\mathbf{TCat}}}"] \arrow[d, "F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}"'] & \mathcal{C}^S_{\mathbf{TCat}} \arrow[d, "F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}"] \\
\mathcal{D}^R_{\mathbf{TCat}} \arrow[r, "\mathrm{T}^{\mathcal{D}^R_{\mathbf{TCat}}}"] & \mathcal{D}^R_{\mathbf{TCat}}
\end{tikzcd}$$

The functoriality of the induced graph morphism $(=)^{(-)}_{\mathbf{TCat}}\colon \mathbf{TCat}^{\mathrm{op}} \times \mathbf{TCat} \to \mathbf{TCat}$, which is given on the morphisms by $F^G_{\mathbf{TCat}} = F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}$ for morphisms $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}$ and $G\colon R \to S$ in $\mathbf{TCat}$, follows from the functoriality of $(=)^{(-)}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Cat}$. $\square$

(4.23) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$. The diagram functor

$$(=)^{(-)}\colon \mathbf{Cat}_{(\mathfrak{U})}^{\mathrm{op}} \times \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})}$$

induces a functor

$$(=)^{(-)}_{\mathbf{TCat}}\colon (\mathbf{TCat}_{(\mathfrak{U})})^{\mathrm{op}} \times \mathbf{TCat}_{0,(\mathfrak{U})} \to \mathbf{TCat}_{0,(\mathfrak{U})},$$

given on the morphisms by

$$F^G_{\mathbf{TCat}} = F^G|^{\mathcal{D}^R_{\mathbf{TCat}}}_{\mathcal{C}^S_{\mathbf{TCat}}}\colon \mathcal{C}^S_{\mathbf{TCat}} \to \mathcal{D}^R_{\mathbf{TCat}}$$

for morphisms $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}_{0,(\mathfrak{U})}$ and $G\colon R \to S$ in $\mathbf{TCat}_{(\mathfrak{U})}$.

Proof. Given $S \in \operatorname{Ob}\mathbf{TCat}$, $\mathcal{C} \in \operatorname{Ob}\mathbf{TCat}_0$, then $\mathcal{C}^S_{\mathbf{TCat}}$ is a zero-pointed subcategory with shift of $\mathcal{C}^S_{\mathrm{in}}$ as $(0^{\mathcal{C}^S})_{i^{[1]}} = 0^{\mathcal{C}} = (0^{\mathcal{C}})^{[1]} = (0^{\mathcal{C}^S})^{[1]}_i$ for $i \in \operatorname{Ob}S$, that is, $0^{\mathcal{C}^S} \in \operatorname{Ob}\mathcal{C}^S_{\mathbf{TCat}}$. The assertion follows from remark (4.16) and remark (4.22). □

3 Semiquasicyclic types

The cosemitriangles in the homotopy category of a zero-pointed Brown cofibration category, see definition (5.51), as well as already their models, see definition (5.33), will be organised in a so-called semiquasicyclic category, that is, a semiquasicyclic object in the category of categories **Cat**, see definition (4.38). A semiquasicyclic object in turn may be seen as a variant of a simplicial object.

Simplicial objects in a category Ω are presheaves with values in Ω over the category of simplex types $\mathbf{\Delta}$, that is, $\mathbf{\Delta}^{\mathrm{op}}$-commutative diagrams in Ω, where the category $\mathbf{\Delta}$ consists of the simplex types Δ^p for $p \in \mathbb{N}_0$ and morphisms of posets. In contrast, semiquasicyclic objects in Ω are presheaves with values in Ω over the category of semiquasicyclic types $\mathbf{\Theta}_+$, consisting of the semiquasicyclic types Θ^p_+ for $p \in \mathbb{N}_0$ and morphisms of posets with shift, see definition (4.24). The stable analogon, so-called *quasicyclic objects*, has been introduced by Künzer [23, sec. 5.2].

In this section, we define the category of semiquasicyclic types and study some of their properties. As a poset, the p-th semiquasicyclic type Θ^p_+ for some $p \in \mathbb{N}_0$ will be just the poset of non-negative integers $\mathbb{N}_0$, see definition (4.24)(a). However, Θ^p_+ will be a poset with shift, and the shift morphism on Θ^p_+ will influence the way we think of Θ^p_+, namely as result of a gluing of cells Θ^p_m for $m \in \mathbb{N}_0$ in the sense of definition (4.27), see corollary (4.34). Moreover, we give a presentation of the poset structure of Θ^p_+ by means of shift values of the elements of the cell Θ^p_0, see proposition (4.32), and show that the inclusion $\mathrm{inc}\colon \Theta^p_0 \to \Theta^p_+$ fulfills a universal property, see proposition (4.35).

Definition of semiquasicyclic types

(4.24) Definition (semiquasicyclic type)**.**

(a) We suppose given $p \in \mathbb{N}_0$. The *p-th semiquasicyclic type* is defined to be the poset with shift Θ^p_+ with underlying poset $\mathbb{N}_0$ and whose shift is given by

$$\mathrm{T}^{\Theta^p_+}\colon \Theta^p_+ \to \Theta^p_+,\ i \mapsto i + (p+1).$$

(b) We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\mathbb{N}_0$ is in $\mathfrak{U}$. The full subcategory $\mathbf{\Theta}_+$ in $\mathbf{TPoset}_{(\mathfrak{U})}$ with

$$\operatorname{Ob}\mathbf{\Theta}_+ = \{\Theta^p_+ \mid p \in \mathbb{N}_0\}$$

is called the *category of semiquasicyclic types*. A morphism in $\mathbf{\Theta}_+$ is called a *morphism of semiquasicyclic types*.

(4.25) Example. In Θ^3_+, we have $0^{[0]} = 0$, $1^{[0]} = 1$, $2^{[0]} = 2$, $3^{[0]} = 3$, $0^{[1]} = 4$, $1^{[1]} = 5$, $2^{[1]} = 6$, $3^{[1]} = 7$, $0^{[2]} = 8$, $1^{[2]} = 9$, etc.

In [23, sec. 1.1, p. 243], Künzer's stable analogon to the semiquasicyclic type Θ^p_+ is the *periodic repetition* of Δ_p (this is Δ^p in our notation), denoted by $\bar{\Delta}_p$ in loc. cit.

(4.26) Remark. For every $p \in \mathbb{N}_0$, the shift $\mathrm{T}^{\Theta^p_+}\colon \Theta^p_+ \to \Theta^p_+$ is an injective morphism of posets with shift that reflects the order of Θ^p_+.

The cell decomposition

One can think of Θ^p_+ for some $p \in \mathbb{N}_0$ as an $|\mathbb{N}_0|$-fold copy of Δ^p, see example (4.25) and corollary (4.33), cf. also the definition of Künzer's stable analogon [23, sec. 1.1, p. 243]. For some technical purposes, as in the proof of proposition (4.35), this is very convenient. Sometimes, however, we prefer to think of Θ^p_+ as the consequence of a gluing of cells in the following sense, see corollary (4.32).

(4.27) Definition (cells of Θ^p_+)**.** For $p, m \in \mathbb{N}_0$, we let Θ^p_m be the full subposet in Θ^p_+ with underlying set given by

$$\Theta^p_m = \{i^{[m]} \mid i \in [0, p+1]\}.$$

(4.28) Example. We have

$$\begin{aligned}\Theta^3_0 &= \{0^{[0]}, 1^{[0]}, 2^{[0]}, 3^{[0]}, 4^{[0]}\} = \{0, 1, 2, 3, 4\},\\ \Theta^3_1 &= \{0^{[1]}, 1^{[1]}, 2^{[1]}, 3^{[1]}, 4^{[1]}\} = \{4, 5, 6, 7, 8\},\\ \Theta^3_2 &= \{0^{[2]}, 1^{[2]}, 2^{[2]}, 3^{[2]}, 4^{[2]}\} = \{8, 9, 10, 11, 12\}.\end{aligned}$$

(4.29) Notation. As usual, we interpret the posets Θ^p_+ and Θ^p_0 for some $p \in \mathbb{N}_0$ as categories. In particular, the least element in Θ^p_0 is a unique initial object and will be denoted by $¡ = ¡^{\Theta^p_0} = 0$, and the greatest element in Θ^p_0 is a unique terminal object and will be denoted by $! = !^{\Theta^p_0} = p+1$.

(4.30) Notation. For $p \in \mathbb{N}_0$, we denote by Δ^p the full subposet of Θ^p_0 with underlying subset $\Theta^p_0 \setminus \{!\} = [0, p]$, and we denote by $\dot{\Delta}^p$ the full subposet of Θ^p_0 with underlying subset $\Theta^p_0 \setminus \{¡, !\} = [1, p]$.

(4.31) Example. We have

$$\begin{aligned}\Theta^3_0 &= \{0, 1, 2, 3, 4\} = \{0^{[0]}, 1^{[0]}, 2^{[0]}, 3^{[0]}, 0^{[1]}\} = \{¡^{[0]}\} \mathbin{\dot\cup} \{1^{[0]}, 2^{[0]}, 3^{[0]}\} \mathbin{\dot\cup} \{!^{[0]}\},\\ \Theta^3_1 &= \{4, 5, 6, 7, 8\} = \{0^{[1]}, 1^{[1]}, 2^{[1]}, 3^{[1]}, 0^{[2]}\} = \{¡^{[1]}\} \mathbin{\dot\cup} \{1^{[1]}, 2^{[1]}, 3^{[1]}\} \mathbin{\dot\cup} \{!^{[1]}\},\\ \Theta^3_2 &= \{8, 9, 10, 11, 12\} = \{0^{[2]}, 1^{[2]}, 2^{[2]}, 3^{[2]}, 0^{[3]}\} = \{¡^{[2]}\} \mathbin{\dot\cup} \{1^{[2]}, 2^{[2]}, 3^{[2]}\} \mathbin{\dot\cup} \{!^{[2]}\}\end{aligned}$$

and $!^{[0]} = 0^{[1]} = ¡^{[1]}$, $!^{[1]} = 0^{[2]} = ¡^{[2]}$.

(4.32) Proposition. We suppose given $p \in \mathbb{N}_0$.

(a) We have

$$\begin{aligned}\Theta^p_0 &= \{¡\} \mathbin{\dot\cup} \dot{\Delta}^p \mathbin{\dot\cup} \{!\},\\ \Theta^p_+ &= \{i^{[m]} \mid i \in \Theta^p_0,\ m \in \mathbb{N}_0\}.\end{aligned}$$

(b) Given $i, j \in \Theta^p_0$, $m, n \in \mathbb{N}_0$, we have

$$i^{[m]} = j^{[n]}$$

in Θ^p_+ if and only if

$$\begin{aligned}&m = n,\ i = j \text{ or}\\ &m + 1 = n,\ i = !,\ j = ¡ \text{ or}\\ &m = n + 1,\ i = ¡,\ j = !.\end{aligned}$$

(c) Given $i, j \in \Theta^p_0$, $m, n \in \mathbb{N}_0$, we have

$$i^{[m]} \leq j^{[n]}$$

in Θ^p_+ if and only if

$$\begin{aligned}&m < n \text{ or}\\ &m = n,\ i \leq j \text{ or}\\ &m = n + 1,\ i = ¡,\ j = !.\end{aligned}$$

Proof.

(a) Given $k \in \Theta^p_+$, there exist $i, m \in \mathbb{N}_0$ with

$$k = i + m(p+1) = i^{[m]}$$

and $i \in [0, p] = \Delta^p \subseteq \Theta^p_0$ by division with remainders.

(c) The condition $i^{[m]} \leq j^{[n]}$ is equivalent to $i + m(p+1) \leq j + n(p+1)$. As $i, j \in \Theta^p_0 = [0, p+1]$, we have $j - i \in [-(p+1), (p+1)]$. Thus $i + m(p+1) \leq j + n(p+1)$ implies that

$$m(p+1) \leq (j-i) + n(p+1) \leq (n+1)(p+1),$$

so we necessarily have $m \leq n+1$.

If $m = n+1$, then $i + m(p+1) \leq j + n(p+1)$ is equivalent to $i + (p+1) \leq j$, that is, to $i = 0 = ¡$ and $j = p+1 = !$.

If $m = n$, then $i + m(p+1) \leq j + n(p+1)$ is equivalent to $i \leq j$.

If $m < n$, then $i + m(p+1) \leq j + n(p+1)$ is equivalent to $i \leq j + (n-m)(p+1)$, and this holds without restriction as $i \in \Theta^p_0 = [0, p+1]$ implies that $i \leq p+1 \leq j + (n-m)(p+1)$.

(b) We have $i^{[m]} = j^{[n]}$ if and only if $i^{[m]} \leq j^{[n]}$ and $j^{[n]} \leq i^{[m]}$. By (c), we have $i^{[m]} \leq j^{[n]}$ if and only if $m < n$ or $m = n$, $i \leq j$ or $m = n+1$, $i = ¡$, $j = !$, and we have $j^{[m]} \leq i^{[n]}$ if and only if $n < m$ or $n = m$, $j \leq i$ or $n = m+1$, $j = ¡$, $i = !$. In particular, we have the three cases $m = n$ or $m+1 = n$ or $m = n+1$.

If $m = n$, then $i^{[m]} = j^{[n]}$ is equivalent to $i \leq j$ and $j \leq i$, that is, to $i = j$. If $m+1 = n$, then $i^{[m]} = j^{[n]}$ is equivalent to $i = !$, $j = ¡$. If $m = n+1$, then $i^{[m]} = j^{[n]}$ is equivalent to $i = ¡$, $j = !$. □

(4.33) Corollary. We suppose given $p \in \mathbb{N}_0$.

(a) We have

$$\Delta^p = \{¡\} \,\dot\cup\, \dot\Delta^p,$$
$$\Theta^p_+ = \{i^{[m]} \mid i \in \Delta^p,\ m \in \mathbb{N}_0\}.$$

(b) Given $i, j \in \Delta^p$, $m, n \in \mathbb{N}_0$, we have $i^{[m]} = j^{[n]}$ in Θ^p_+ if and only if $m = n$ and $i = j$.

(c) Given $i, j \in \Delta^p$, $m, n \in \mathbb{N}_0$, we have $i^{[m]} \leq j^{[n]}$ in Θ^p_+ if and only if $m < n$ or $m = n$, $i \leq j$.

Proof.

(a) By proposition (4.32)(a), we have

$$\begin{aligned}\Theta^p_+ &= \{i^{[m]} \mid i \in \Theta^p_0,\ m \in \mathbb{N}_0\} = \{i^{[m]} \mid i \in \Delta^p,\ m \in \mathbb{N}_0\} \cup \{!^{[m]} \mid m \in \mathbb{N}_0\} \\ &= \{i^{[m]} \mid i \in \Delta^p,\ m \in \mathbb{N}_0\} \cup \{¡^{[m+1]} \mid m \in \mathbb{N}_0\} = \{i^{[m]} \mid i \in \Delta^p,\ m \in \mathbb{N}_0\}.\end{aligned}$$

(b) This follows from proposition (4.32)(b).

(c) This follows from proposition (4.32)(c). □

The semiquasicyclic types decompose into the cells introduced in definition (4.27):

(4.34) Corollary. We suppose given $p \in \mathbb{N}_0$.

(a) For $m \in \mathbb{N}_0$, we have

$$\Theta^p_m = \{i^{[m]} \mid i \in \Theta^p_0\} = \{k \in \Theta^p_+ \mid ¡^{[m]} \leq k \leq !^{[m]}\}.$$

(b) We have

$$\Theta^p_+ = \bigcup_{m \in \mathbb{N}_0} \Theta^p_m.$$

(c) For $m \in \mathbb{N}_0$, we have

$$\Theta^p_m \cap \Theta^p_{m+1} = \{!^{[m]}\} = \{¡^{[m+1]}\},$$
$$\Theta^p_m \cap \Theta^p_{m+k} = \emptyset \text{ for } k \in \mathbb{N} \text{ with } k \geq 2.$$

(d) For $m \in \mathbb{N}_0$, $i, j \in \Theta^p_0$, we have $i^{[m]} \leq j^{[m]}$ in Θ^p_m if and only if $i \leq j$ in Θ^p_0.

(e) For $m, n \in \mathbb{N}_0$ with $m < n$, we have $k \leq l$ in Θ^p_+ for all $k \in \Theta^p_m$, $l \in \Theta^p_n$, where $k = l$ holds if and only if $m + 1 = n$, $k = !^{[m]}$, $l = ¡^{[m+1]}$.

Proof.

(a) This follows from definition (4.27).

(b) By proposition (4.32)(a), we have

$$\Theta^p_+ = \{i^{[m]} \mid i \in \Theta^p_0,\, m \in \mathbb{N}_0\} = \bigcup_{m \in \mathbb{N}_0} \{i^{[m]} \mid i \in \Theta^p_0\} = \bigcup_{m \in \mathbb{N}_0} \Theta^p_m.$$

(c) This follows from proposition (4.32)(a), (b).

(d) This follows from proposition (4.32)(c).

(e) This follows from proposition (4.32)(c), (b). □

The universal property

The semiquasicyclic types, considered as categories with shift, admit the following universal property.

(4.35) Proposition. We suppose given $p \in \mathbb{N}_0$. Then we have $! = ¡^{[1]}$ in Θ^p_+.
Moreover, for every category with shift $\mathcal{C}$ and every functor $F \colon \Theta^p_0 \to \mathcal{C}$ with $F! = (F¡)^{[1]}$, there exists a unique morphism of categories with shift $\hat{F} \colon \Theta^p_+ \to \mathcal{C}$ with $F = \hat{F}|_{\Theta^p_0}$, given on the objects by

$$\hat{F}(i^{[m]}) = (Fi)^{[m]}$$

for $i \in \Theta^p_0$, $m \in \mathbb{N}_0$, and on the morphisms by

$$\hat{F}(i^{[m]}, j^{[n]}) = \begin{cases} 1_{(F¡)^{[m]}} & \text{if } m = n + 1, \\ F(i, j)^{[m]} & \text{if } m = n, \\ F(i, !)^{[m]} \, (\bullet_{r \in \lceil m+1, n-1 \rceil} F(¡, !)^{[r]}) F(¡, j)^{[n]} & \text{if } m < n, \end{cases}$$

for $i, j \in \Theta^p_0$, $m, n \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]}$.

Proof. We suppose given a category with shift $\mathcal{C}$ and a functor $F \colon \Theta^p_0 \to \mathcal{C}$ with $F! = (F¡)^{[1]}$. To construct a functor $\hat{F} \colon \Theta^p_+ \to \mathcal{C}$ with $F = \hat{F}|_{\Theta^p_0}$, we will use the asymmetric description of Θ^p_+ as given in corollary (4.33). We define a map $\hat{F}_0 \colon \operatorname{Ob} \Theta^p_+ \to \operatorname{Mor} \mathcal{C}$ by

$$\hat{F}_0(i^{[m]}) := (Fi)^{[m]}$$

for $i \in \Delta^p$, $m \in \mathbb{N}_0$, and we define a map $\hat{F}_1 \colon \operatorname{Mor} \Theta^p_+ \to \operatorname{Mor} \mathcal{C}$ by

$$\hat{F}_1(i^{[m]}, j^{[n]}) := \begin{cases} F(i, j)^{[m]} & \text{if } m = n, \\ F(i, !)^{[m]} \, (\bullet_{r \in \lceil m+1, n-1 \rceil} F(¡, !)^{[r]}) F(¡, j)^{[n]} & \text{if } m < n \end{cases}$$

for $i, j \in \Delta^p$, $m, n \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]}$. Then we have

$$\operatorname{Source}^{\mathcal{C}} \hat{F}_1(i^{[m]}, j^{[n]}) = \left\{ \begin{matrix} \operatorname{Source}^{\mathcal{C}} F(i, j)^{[m]} & \text{if } m = n, \\ \operatorname{Source}^{\mathcal{C}} (F(i, !)^{[m]} \, (\bullet_{r \in \lceil m+1, n-1 \rceil} F(¡, !)^{[r]}) F(¡, j)^{[n]}) & \text{if } m < n \end{matrix} \right\}$$

$$
\begin{aligned}
&= \begin{cases} \operatorname{Source}^{\mathcal{C}} F(i,j)^{[m]} & \text{if } m = n, \\ \operatorname{Source}^{\mathcal{C}} F(i,!)^{[m]} & \text{if } m < n \end{cases} = \begin{cases} F(\operatorname{Source}^{\Theta^p_+}(i,j))^{[m]} & \text{if } m = n, \\ F(\operatorname{Source}^{\Theta^p_+}(i,!))^{[m]} & \text{if } m < n \end{cases} \\
&= (Fi)^{[m]} = \hat{F}_0(i^{[m]}) = \hat{F}_0(\operatorname{Source}^{\Theta^p_+}(i^{[m]}, j^{[n]}))
\end{aligned}
$$

and, analogously,

$$\operatorname{Target}^{\mathcal{C}} \hat{F}_1(i^{[m]}, j^{[n]}) = \hat{F}_0(\operatorname{Target}^{\Theta^p_+}(i^{[m]}, j^{[n]}))$$

for $i, j \in \Delta^p$, $m, n \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]}$. Moreover, we get

$$
\begin{aligned}
\hat{F}_1(i^{[m]}, j^{[n]})\, \hat{F}_1(j^{[n]}, k^{[q]}) &= \begin{cases} F(i,j)^{[m]} F(j,k)^{[n]} & \text{if } m = n,\ n = q, \\ F(i,j)^{[m]} F(j,!)^{[n]}\, (\bullet_{r \in [n+1,q-1]} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m = n,\ n < q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,n-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},j)^{[n]} F(j,k)^{[n]} & \text{if } m < n,\ n = q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,n-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},j)^{[n]} & \\ \qquad \cdot F(j,!)^{[n]}\, (\bullet_{r \in [n+1,q-1]} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m < n,\ n < q \end{cases} \\
&= \begin{cases} F(i,j)^{[m]} F(j,k)^{[m]} & \text{if } m = n,\ n = q, \\ F(i,j)^{[m]} F(j,!)^{[m]}\, (\bullet_{r \in \lceil m+1,q-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m = n,\ n < q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,q-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},j)^{[q]} F(j,k)^{[q]} & \text{if } m < n,\ n = q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,n-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},j)^{[n]} & \\ \qquad \cdot F(j,!)^{[n]}\, (\bullet_{r \in [n+1,q-1]} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m < n,\ n < q \end{cases} \\
&= \begin{cases} F(i,k)^{[m]} & \text{if } m = n,\ n = q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,q-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m = n,\ n < q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,q-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m < n,\ n = q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,q-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m < n,\ n < q \end{cases} \\
&= \begin{cases} F(i,k)^{[m]} & \text{if } m = q, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,q-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},k)^{[q]} & \text{if } m < q \end{cases} = \hat{F}_1(i^{[m]}, k^{[q]}) \\
&= \hat{F}_1((i^{[m]}, j^{[n]})(j^{[n]}, k^{[q]}))
\end{aligned}
$$

for $i, j, k \in \dot{\Delta}^p$, $m, n, q \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]} \leq k^{[q]}$, and

$$\hat{F}_1(1_{i^{[m]}}) = \hat{F}_1(i^{[m]}, i^{[m]}) = F(i,i)^{[m]} = F(1_i)^{[m]} = 1_{(Fi)^{[m]}} = 1_{\hat{F}_0(i^{[m]})}$$

for $i \in \dot{\Delta}^p$, $m \in \mathbb{N}_0$. Thus we have a functor $\hat{F}\colon \Theta^p_+ \to \mathcal{C}$ with $\operatorname{Ob} \hat{F} = \hat{F}_0$ and $\operatorname{Mor} \hat{F} = \hat{F}_1$, that is, such that

$$\hat{F}(i^{[m]}) = (Fi)^{[m]}$$

for $i \in \dot{\Delta}^p$, $m \in \mathbb{N}_0$, and

$$\hat{F}(i^{[m]}, j^{[n]}) = \begin{cases} F(i,j)^{[m]} & \text{if } m = n, \\ F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,n-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},j)^{[n]} & \text{if } m < n, \end{cases}$$

for $i, j \in \dot{\Delta}^p$, $m, n \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]}$.
As

$$
\begin{aligned}
\hat{F}((i^{[m]}, j^{[n]})^{[1]}) = \hat{F}(i^{[m+1]}, j^{[n+1]}) &= \begin{cases} F(i,j)^{[m+1]} & \text{if } m+1 = n+1, \\ F(i,!)^{[m+1]}\, (\bullet_{r \in \lceil m+2,n \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},j)^{[n+1]} & \text{if } m+1 < n+1 \end{cases} \\
&= \begin{cases} F(i,j)^{[m+1]} & \text{if } m = n, \\ F(i,!)^{[m+1]}\, (\bullet_{r \in \lceil m+1,n-1 \rceil} F(\text{¡},!)^{[r+1]}) F(\text{¡},j)^{[n+1]} & \text{if } m < n \end{cases} \\
&= \begin{cases} (F(i,j)^{[m]})^{[1]} & \text{if } m = n, \\ (F(i,!)^{[m]}\, (\bullet_{r \in \lceil m+1,n-1 \rceil} F(\text{¡},!)^{[r]}) F(\text{¡},j)^{[n]})^{[1]} & \text{if } m < n \end{cases} = \hat{F}(i^{[m]}, j^{[n]})^{[1]}
\end{aligned}
$$

for $i, j \in \dot{\Delta}^p$, $m, n \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]}$, we have $\hat{F} \circ \mathrm{T}^{\Theta^p_+} = \mathrm{T}^{\mathcal{C}} \circ \hat{F}$. Thus $\hat{F}$ is in fact a morphism of categories with shift.
Moreover, as $F! = (F¡)^{[1]}$, we have

$$F(¡, ¡)^{[1]} = F(1_¡)^{[1]} = 1_{(F¡)^{[1]}} = 1_{F!} = F1_! = F(!, !)$$

and hence

$$\begin{aligned}
\hat{F}(i, j) &= \begin{Bmatrix} \hat{F}(i, j) & \text{if } i \in \dot{\Delta}^p,\ j \in \dot{\Delta}^p, \\ \hat{F}(i, ¡^{[1]}) & \text{if } i \in \dot{\Delta}^p,\ j = !, \\ \hat{F}(¡^{[1]}, ¡^{[1]}) & \text{if } i = !,\ j = ! \end{Bmatrix} = \begin{Bmatrix} F(i, j) & \text{if } i \in \dot{\Delta}^p,\ j \in \dot{\Delta}^p, \\ F(i, !)F(¡, ¡)^{[1]} & \text{if } i \in \dot{\Delta}^p,\ j = !, \\ F(¡, ¡)^{[1]} & \text{if } i = !,\ j = ! \end{Bmatrix} \\
&= \begin{Bmatrix} F(i, j) & \text{if } i \in \dot{\Delta}^p,\ j \in \dot{\Delta}^p, \\ F(i, !) & \text{if } i \in \dot{\Delta}^p,\ j = !, \\ F(!, !) & \text{if } i = !,\ j = ! \end{Bmatrix} = F(i, j)
\end{aligned}$$

for $i, j \in \Theta^p_0$ with $i \leq j$. Thus we have $F = \hat{F}|_{\Theta^p_0}$. In particular, $\hat{F}$ is necessarily given by

$$\hat{F}(i^{[m]}) = (\hat{F}i)^{[m]} = (Fi)^{[m]}$$

for $i \in \Theta^p_0$, $m \in \mathbb{N}_0$, and by

$$\begin{aligned}
\hat{F}(i^{[m]}, j^{[n]}) &= \begin{Bmatrix} \hat{F}(¡^{[m]}, !^{[m-1]}) & \text{if } m = n + 1, \\ \hat{F}(i^{[m]}, j^{[m]}) & \text{if } m = n, \\ \hat{F}((i^{[m]}, !^{[m]}) \, (\bullet_{r \in [m+1, n-1]} (¡^{[r]}, !^{[r]})) \, (¡^{[n]}, j^{[n]})) & \text{if } m < n, \end{Bmatrix} \\
&= \begin{Bmatrix} \hat{F}(¡^{[m]}, ¡^{[m]}) & \text{if } m = n + 1, \\ \hat{F}(i^{[m]}, j^{[m]}) & \text{if } m = n, \\ \hat{F}(i^{[m]}, !^{[m]}) \, (\bullet_{r \in [m+1, n-1]} \hat{F}(¡^{[r]}, !^{[r]})) \, \hat{F}(¡^{[n]}, j^{[n]}) & \text{if } m < n, \end{Bmatrix} \\
&= \begin{Bmatrix} \hat{F}1_{¡^{[m]}} & \text{if } m = n + 1, \\ \hat{F}(i, j)^{[m]} & \text{if } m = n, \\ \hat{F}(i, !)^{[m]} \, (\bullet_{r \in [m+1, n-1]} \hat{F}(¡, !)^{[r]}) \, \hat{F}(¡, j)^{[n]} & \text{if } m < n, \end{Bmatrix} \\
&= \begin{Bmatrix} 1_{\hat{F}(¡)^{[m]}} & \text{if } m = n + 1, \\ \hat{F}(i, j)^{[m]} & \text{if } m = n, \\ \hat{F}(i, !)^{[m]} \, (\bullet_{r \in [m+1, n-1]} \hat{F}(¡, !)^{[r]}) \, \hat{F}(¡, j)^{[n]} & \text{if } m < n, \end{Bmatrix} \\
&= \begin{Bmatrix} 1_{(F¡)^{[m]}} & \text{if } m = n + 1, \\ F(i, j)^{[m]} & \text{if } m = n, \\ F(i, !)^{[m]} \, (\bullet_{r \in [m+1, n-1]} F(¡, !)^{[r]}) \, F(¡, j)^{[n]} & \text{if } m < n, \end{Bmatrix}
\end{aligned}$$

for $i, j \in \Theta^p_0$, $m, n \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]}$.
Conversely, given an arbitrary morphism of categories with shift $G\colon \Theta^p_+ \to \mathcal{C}$ with $F = G|_{\Theta^p_0}$, then we necessarily have

$$\begin{aligned}
G(i^{[m]}, j^{[n]}) &= \begin{Bmatrix} G(i^{[m]}, j^{[m]}) & \text{if } m = n, \\ G((i^{[m]}, ¡^{[m+1]}) \, (\bullet_{r \in [m+1, n-1]} (¡^{[r]}, ¡^{[r+1]})) \, (¡^{[n]}, j^{[n]})) & \text{if } m < n \end{Bmatrix} \\
&= \begin{Bmatrix} G(i^{[m]}, j^{[m]}) & \text{if } m = n, \\ G(i^{[m]}, ¡^{[m+1]}) \, (\bullet_{r \in [m+1, n-1]} G(¡^{[r]}, ¡^{[r+1]})) \, G(¡^{[n]}, j^{[n]}) & \text{if } m < n \end{Bmatrix} \\
&= \begin{Bmatrix} G(i^{[m]}, j^{[m]}) & \text{if } m = n, \\ G(i^{[m]}, !^{[m]}) \, (\bullet_{r \in [m+1, n-1]} G(¡^{[r]}, !^{[r]})) \, G(¡^{[n]}, j^{[n]}) & \text{if } m < n \end{Bmatrix} \\
&= \begin{Bmatrix} G((i, j)^{[m]}) & \text{if } m = n, \\ G((i, !)^{[m]}) \, (\bullet_{r \in [m+1, n-1]} G((¡, !)^{[r]})) \, G((¡, j)^{[n]}) & \text{if } m < n \end{Bmatrix}
\end{aligned}$$

$$= \left\{\begin{array}{ll} G(i,j)^{[m]} & \text{if } m = n, \\ G(i,!)^{[m]}\,(\bullet_{r\in\lceil m+1,n-1\rceil} G(¡,!)^{[r]})\,G(¡,j)^{[n]} & \text{if } m < n \end{array}\right\}$$
$$= \begin{cases} F(i,j)^{[m]} & \text{if } m = n, \\ F(i,!)^{[m]}\,(\bullet_{r\in\lceil m+1,n-1\rceil} F(¡,!)^{[r]})\,F(¡,j)^{[n]} & \text{if } m < n \end{cases}$$

for $i, j \in \dot{\Delta}^p$, $m, n \in \mathbb{N}_0$ with $i^{[m]} \leq j^{[n]}$, that is, $G = \hat{F}$. □

(4.36) Corollary. For every poset morphism $\alpha_0 \colon \Theta_0^p \to \Theta_+^q$ for $p, q \in \mathbb{N}_0$ with $!\alpha_0 = (¡\alpha_0)^{[1]}$ there exists a unique morphism of semiquasicyclic types $\alpha \colon \Theta_+^p \to \Theta_+^q$ with $\alpha_0 = \alpha|_{\Theta_0^p}$, given by

$$i^{[m]}\alpha = (i\alpha_0)^{[m]}$$

for $i \in \Theta_0^p$, $m \in \mathbb{N}_0$.

4 (Co)semiquasicyclic objects

In this section, we define semiquasicyclic objects in a category Ω as presheaves with values in Ω over the category of semiquasicyclic types $\mathbf{\Theta}_+$ as introduced in definition (4.24)(b), that is, as objects in the diagram category $\Omega^{\mathbf{\Theta}_+^{\mathrm{op}}}$. Likewise, we introduce the dual notion of a cosemiquasicyclic object as an object in $\Omega^{\mathbf{\Theta}_+}$.
We have a faithful functor $I \colon \mathbf{\Delta} \to \mathbf{\Theta}_+$, given on the objects by $I\Delta^p = \Theta_+^p$ for $p \in \mathbb{N}_0$, and on the morphism as follows. Given a morphism of simplex types $\alpha \colon \Delta^p \to \Delta^q$ for some $p, q \in \mathbb{N}_0$, then $I\alpha$ is the unique morphism of semiquasicyclic types with $(I\alpha)|_{\Delta^p}^{\Delta^q} = \alpha$ and $!(I\alpha) = (0\alpha)^{[1]}$, cf. corollary (4.36). In particular, every semiquasicyclic object has an underlying simplicial object. So we adapt the usual terminology and notations from simplicial algebraic topology.

The category of (co)semiquasicyclic objects

(4.37) Definition ((co)semiquasicyclic object)**.** We suppose given a category Ω.

(a) The *category of cosemiquasicyclic objects* in Ω is defined to be the category $\mathbf{cq}_+\Omega := \Omega^{\mathbf{\Theta}_+}$. An object in $\mathbf{cq}_+\Omega$ is called a *cosemiquasicyclic object* in Ω, a morphism in $\mathbf{cq}_+\Omega$ is called a *morphism of cosemiquasicyclic objects* in Ω.

We suppose given a cosemiquasicyclic object X in Ω. For $p \in \mathbb{N}_0$, we write X^p for the image of Θ_+^p under X. Given a morphism of semiquasicyclic types $\alpha \colon \Theta_+^p \to \Theta_+^q$ for $p, q \in \mathbb{N}_0$, the image $X^\alpha \colon X^p \to X^q$ of α under X is called the *cosemiquasicyclic operation* induced by α.

(b) The *category of semiquasicyclic objects* in Ω is defined to be the category $\mathbf{q}_+\Omega := \Omega^{\mathbf{\Theta}_+^{\mathrm{op}}}$. An object in $\mathbf{q}_+\Omega$ is called a *semiquasicyclic object* in Ω, a morphism in $\mathbf{q}_+\Omega$ is called a *morphism of semiquasicyclic objects* in Ω.

We suppose given a semiquasicyclic object X in Ω. For $p \in \mathbb{N}_0$, we write X_p for the image of Θ_+^p under X. Given a morphism of semiquasicyclic types $\alpha \colon \Theta_+^p \to \Theta_+^q$ for $p, q \in \mathbb{N}_0$, the image $X_\alpha \colon X_q \to X_p$ of α under X is called the *semiquasicyclic operation* induced by α.

Some instances of (co)semiquasicyclic objects

(4.38) Definition ((co)semiquasicyclic set)**.**

(a) Given a Grothendieck universe $\mathfrak{U}$, the category $\mathbf{cq}_+\mathbf{Set}_{(\mathfrak{U})}$ is called the *category of cosemiquasicyclic sets* (more precisely, the *category of cosemiquasicyclic* $\mathfrak{U}$*-sets*), an object in $\mathbf{cq}_+\mathbf{Set}_{(\mathfrak{U})}$ is called a *cosemiquasicyclic set with respect to* $\mathfrak{U}$ (or *cosemiquasicyclic* $\mathfrak{U}$*-set*), and a morphism in $\mathbf{cq}_+\mathbf{Set}_{(\mathfrak{U})}$ is called a *cosemiquasicyclic map with respect to* $\mathfrak{U}$ (or *cosemiquasicyclic* $\mathfrak{U}$*-map*).

A *cosemiquasicyclic set* is a cosemiquasicyclic $\mathfrak{U}$-set for some Grothendieck universe $\mathfrak{U}$, and a *cosemiquasicyclic map* is a cosemiquasicyclic $\mathfrak{U}$-map for some Grothendieck universe $\mathfrak{U}$.

(b) Given a Grothendieck universe $\mathfrak{U}$, the category $\mathbf{q}_+\mathbf{Set}_{(\mathfrak{U})}$ is called the *category of semiquasicyclic sets* (more precisely, the *category of semiquasicyclic $\mathfrak{U}$-sets*), an object in $\mathbf{q}_+\mathbf{Set}_{(\mathfrak{U})}$ is called a *semiquasicyclic set with respect to* $\mathfrak{U}$ (or *semiquasicyclic $\mathfrak{U}$-set*), and a morphism in $\mathbf{q}_+\mathbf{Set}_{(\mathfrak{U})}$ is called a *semiquasicyclic map with respect to* $\mathfrak{U}$ (or *semiquasicyclic $\mathfrak{U}$-map*).

A *semiquasicyclic set* is a semiquasicyclic $\mathfrak{U}$-set for some Grothendieck universe $\mathfrak{U}$, and a *semiquasicyclic map* is a semiquasicyclic $\mathfrak{U}$-map for some Grothendieck universe $\mathfrak{U}$.

Analogously, one defines *semiquasicyclic posets* and *semiquasicyclic poset morphisms*, *semiquasicyclic categories* and *semiquasicyclic functors*, *semiquasicyclic zero-pointed categories* and *semiquasicyclic morphisms of zero-pointed categories*, *semiquasicyclic sets with shift* and *semiquasicyclic morphisms of sets with shift*, *semiquasicyclic posets with shift* and *semiquasicyclic morphisms of posets with shift*, *semiquasicyclic categories with shift* and *semiquasicyclic morphisms of categories with shift*, *semiquasicyclic zero-pointed categories with shift* and *semiquasicyclic morphisms of zero-pointed categories with shift*; and their *cosemiquasicyclic* variants.

(4.39) Definition ((co)semiquasicyclic subset)**.**

(a) We suppose given a cosemiquasicyclic set X. A *cosemiquasicyclic subset* of X is a cosemiquasicyclic set U such that U^p is a subset of X^p for all $p \in \mathbb{N}_0$ and such that $U^\alpha = X^\alpha|_{U^p}^{U^q}$ for every morphism of semiquasicyclic types $\alpha\colon \Theta_+^p \to \Theta_+^q$, where $p, q \in \mathbb{N}_0$.

(b) We suppose given a semiquasicyclic set X. A *semiquasicyclic subset* of X is a semiquasicyclic set U such that U_p is a subset of X_p for all $p \in \mathbb{N}_0$ and such that $U_\alpha = X_\alpha|_{U_q}^{U_p}$ for every morphism of semiquasicyclic types $\alpha\colon \Theta_+^p \to \Theta_+^q$, where $p, q \in \mathbb{N}_0$.

Analogously, one defines *semiquasicyclic subposets*, *semiquasicyclic subcategories*, *semiquasicyclic zero-pointed subcategories*, *semiquasicyclic subsets with shift*, *semiquasicyclic subposets with shift*, *semiquasicyclic subcategories with shift*, *semiquasicyclic zero-pointed subcategories with shift*; and their *cosemiquasicyclic* variants.

(4.40) Definition (full (co)semiquasicyclic subcategory)**.**

(a) We suppose given a cosemiquasicyclic category $\mathcal{C}$. A cosemiquasicyclic subcategory $\mathcal{U}$ of $\mathcal{C}$ is said to be *full* if $\mathcal{U}^p$ is a full subcategory of $\mathcal{C}^p$ for all $p \in \mathbb{N}_0$.

(b) We suppose given a semiquasicyclic category $\mathcal{C}$. A semiquasicyclic subcategory $\mathcal{U}$ of $\mathcal{C}$ is said to be *full* if $\mathcal{U}_p$ is a full subcategory of $\mathcal{C}_p$ for all $p \in \mathbb{N}_0$.

Analogously, one defines *full semiquasicyclic subposets*, *full semiquasicyclic zero-pointed subcategories*, *full semiquasicyclic subposets with shift*, *full semiquasicyclic subcategories with shift*, *full semiquasicyclic zero-pointed subcategories with shift*; and their *cosemiquasicyclic* variants.

5 Semistrip types

We suppose given $p \in \mathbb{N}_0$. In this section, we define the semistrip type $\#_+^p$ as a certain extension of the semiquasicyclic type Θ_+^p as introduced in definition (4.24)(a), cf. corollary (4.50). A p-cosemitriangle as introduced in definition (5.51) will be in particular a p-cosemistrip, see definition (4.55)(a), that is, a $\#_+^p$-commutative diagram.
While the semiquasicyclic types are organised in a category, see definition (4.24)(b), the semistrip types will be organised in a cosemiquasicyclic poset, see definition (4.38), for their use in the next section 6. Similarly to the description of Θ_+^p as a gluing of cells in corollary (4.34), we will deduce a cell decomposition of $\#_+^p$, see definition (4.45) and corollary (4.49). Moreover, we will show that $\#_+^p$ fulfills a universal property, see proposition (4.53), in analogy to proposition (4.35). As a consequence of this universal property, we will see that p-cosemitriangles and morphisms of p-cosemitriangles are uniquely determined by their values on the finite subposet $\#_0^p$ of $\#_+^p$, cf. proposition (4.60).

The cosemiquasicyclic poset of semistrip types

(4.41) Remark. There is a cosemiquasicyclic poset with shift $\#_+$, given as follows. For $p \in \mathbb{N}_0$, the underlying poset of $\#_+^p$ is the full subposet of $\Theta_+^p \times \Theta_+^p$ (with the componentwise order) given by

$$\#_+^p = \{(k, i) \in \Theta_+^p \times \Theta_+^p \mid i \leq k \leq i^{[1]}\}.$$

The shift of $\#_+^p$ is given by

$$\mathrm{T}^{\#_+^p} \colon \#_+^p \to \#_+^p,\ (k,i) \mapsto (i^{[1]}, k).$$

For a morphism of semiquasicyclic types $\alpha \colon \Theta_+^p \to \Theta_+^q$ for $p, q \in \mathbb{N}_0$, the cosemiquasicyclic operation induced by α is given by

$$\#_+^\alpha \colon \#_+^p \to \#_+^q,\ (k,i) \mapsto (k\alpha, i\alpha).$$

Proof. For $p \in \mathbb{N}_0$, we let X^p be the subposet of $\Theta_+^p \times \Theta_+^p$ given by $X^p = \{(k,i) \in \Theta_+^p \times \Theta_+^p \mid i \leq k \leq i^{[1]}\}$.
We suppose given $p \in \mathbb{N}_0$. For $(k,i) \in X^p$, we have $i \leq k \leq i^{[1]}$ and therefore $k \leq i^{[1]} \leq k^{[1]}$ as the shift $\mathrm{T}^{\Theta_+^p} \colon \Theta_+^p \to \Theta_+^p$ is a monotone map, whence $(i^{[1]}, k) \in X^p$. So the underlying set of X^p becomes a set with shift where $(k,i)^{[1]} = (i^{[1]}, k)$ for $(k,i) \in X^p$. To show that the poset X^p becomes a poset with shift, we have to show that T^{X^p} is a monotone map. Indeed, given $(k,i), (l,j) \in X^p$ with $(k,i) \leq (l,j)$, that is, such that $i \leq j$ and $k \leq l$, we also have $i^{[1]} \leq j^{[1]}$ as $\mathrm{T}^{\Theta_+^p} \colon \Theta_+^p \to \Theta_+^p$ is monotone, and therefore

$$(k,i)^{[1]} = (i^{[1]}, k) \leq (j^{[1]}, l) = (l,j)^{[1]}.$$

Hence T^{X^p} is indeed a monotone map, that is, X^p is a poset with shift.
Next, we suppose given $p, q \in \mathbb{N}_0$ and a morphism of semiquasicyclic types $\alpha \colon \Theta_+^p \to \Theta_+^q$ for $p, q \in \mathbb{N}_0$. For $k/i \in X^p$, we have $k \leq i^{[1]}$ and therefore $k\alpha \leq i^{[1]}\alpha = (i\alpha)^{[1]}$ as α preserves the shifts, so $(k\alpha, i\alpha) \in X^q$. Thus we obtain a well-defined map

$$X_{p,q}^\alpha \colon X^p \to X^q,\ (k,i) \mapsto (k\alpha, i\alpha).$$

For $(k,i), (l,j) \in X^p$ with $(k,i) \leq (l,j)$, that is, such that $i \leq j$ and $k \leq l$, we also have $i\alpha \leq j\alpha$ and $k\alpha \leq l\alpha$ as α is monotone, and therefore

$$(k,i)X_{p,q}^\alpha = (k\alpha, i\alpha) \leq (l\alpha, j\alpha) = (l,j)X_{p,q}^\alpha.$$

Hence $X_{p,q}^\alpha$ is a monotone map. Moreover, since

$$((k,i)X_{p,q}^\alpha)^{[1]} = (k\alpha, i\alpha)^{[1]} = ((i\alpha)^{[1]}, k\alpha) = (i^{[1]}\alpha, k\alpha) = (i^{[1]}, k)X_{p,q}^\alpha = (k,i)^{[1]}X_{p,q}^\alpha$$

for $(k,i) \in \#_+^p$, the poset morphism $X_{p,q}^\alpha \colon X^p \to X^q$ is a morphism of posets with shift.
Given morphisms of semiquasicyclic types $\alpha \colon \Theta_+^p \to \Theta_+^q$, $\beta \colon \Theta_+^q \to \Theta_+^r$ for $p, q, r \in \mathbb{N}_0$, we have

$$(k,i)\#_+^\alpha \#_+^\beta = (k\alpha, i\alpha)\#_+^\beta = (k\alpha\beta, i\alpha\beta) = (k,i)\#_+^{\alpha\beta}.$$

Moreover, for $p \in \mathbb{N}_0$, we have

$$(k,i)\#_+^{\mathrm{id}_{\Theta_+^p}} = (k\,\mathrm{id}_{\Theta_+^p}, i\,\mathrm{id}_{\Theta_+^p}) = (k,i) = (k,i)\,\mathrm{id}_{\#_+^p}.$$

Altogether, we obtain a cosemiquasicyclic poset with shift $\#_+$, given by $\#_+^p = X^p$ for $p \in \mathbb{N}_0$ and by $\#_+^\alpha = X_{p,q}^\alpha$ for a morphism of semiquasicyclic types $\alpha \colon \#_+^p \to \#_+^q$, where $p, q \in \mathbb{N}_0$. □

(4.42) Definition (cosemiquasicyclic poset of semistrip types)**.** The cosemiquasicyclic poset with shift $\#_+ = \#_+^\bullet$ as in remark (4.41) is called the *cosemiquasicyclic poset of semistrip types*. For $p \in \mathbb{N}_0$, the poset with shift $\#_+^p$ is called the *p-th semistrip type*. The elements of $\#_+^p$ will be denoted by $k/i := (k,i)$.

We suppose given $p \in \mathbb{N}_0$. For $i, j \in \Theta_+^p$, we have $i \leq j$ if and only if there exists a morphism from i to j in Θ_+^p. In particular, $\#_+^p$ as in definition (4.42) is isomorphic to a full subposet of the diagram category $(\Theta_+^p)^{\Delta^1}$.
In [23, sec. 1.1, p. 243], KÜNZER's stable analogon to the semistrip type $\#_+^p$ is the *strip of the periodic repetition* of Δ_p, denoted by $\bar{\Delta}_p^\#$ in loc. cit.

(4.43) Example. The shape of $\#^3_+$ may be displayed as follows.

$$
\begin{tikzcd}
&&&&&&& \vdots & \vdots \\
&&&&&& 2^{[1]}/2^{[1]} \arrow[r] & 3^{[1]}/2^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/2^{[1]} \arrow[r] \arrow[u] & \ldots \\
&&&&& 1^{[1]}/1^{[1]} \arrow[r] & 2^{[1]}/1^{[1]} \arrow[r] \arrow[u] & 3^{[1]}/1^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/1^{[1]} \arrow[r] \arrow[u] & \ldots \\
&&&& 0^{[1]}/0^{[1]} \arrow[r] & 1^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 2^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 3^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/0^{[1]} \arrow[u] \\
&&& 3/3 \arrow[r] & 0^{[1]}/3 \arrow[r] \arrow[u] & 1^{[1]}/3 \arrow[r] \arrow[u] & 2^{[1]}/3 \arrow[r] \arrow[u] & 3^{[1]}/3 \arrow[u] \\
&& 2/2 \arrow[r] & 3/2 \arrow[r] \arrow[u] & 0^{[1]}/2 \arrow[r] \arrow[u] & 1^{[1]}/2 \arrow[r] \arrow[u] & 2^{[1]}/2 \arrow[u] \\
& 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & 3/1 \arrow[r] \arrow[u] & 0^{[1]}/1 \arrow[r] \arrow[u] & 1^{[1]}/1 \arrow[u] \\
0/0 \arrow[r] & 1/0 \arrow[r] \arrow[u] & 2/0 \arrow[r] \arrow[u] & 3/0 \arrow[r] \arrow[u] & 0^{[1]}/0 \arrow[u]
\end{tikzcd}
$$

(4.44) Remark. For every $p \in \mathbb{N}_0$, the shift $\mathrm{T}^{\#^p_+} \colon \#^p_+ \to \#^p_+$ is an injective morphism of posets with shift that reflects the order of $\#^p_+$.

The cell decomposition

(4.45) Definition (cells of $\#^p_+$)**.** For $p, m \in \mathbb{N}_0$, we let $\#^p_m$ be the full subposet in $\#^p_+$ with underlying set given by

$$\#^p_m = \{(k/i)^{[m]} \mid i, k \in \Theta^p_0,\, i \leq k\}.$$

(4.46) Example. The cells $\#^3_0$, $\#^3_1$, $\#^3_2$ of $\#^3_+$ may be displayed as follows.

$$
\begin{tikzcd}
&&&& 0^{[2]}/0^{[2]} \\
&&& 3^{[1]}/3^{[1]} \arrow[r] & 0^{[2]}/3^{[1]} \arrow[u] \\
&& 2^{[1]}/2^{[1]} \arrow[r] & 3^{[1]}/2^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/2^{[1]} \arrow[u] \\
& 1^{[1]}/1^{[1]} \arrow[r] & 2^{[1]}/1^{[1]} \arrow[r] \arrow[u] & 3^{[1]}/1^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/1^{[1]} \arrow[u] \\
0^{[1]}/0^{[1]} \arrow[r] & 1^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 2^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 3^{[1]}/0^{[1]} \arrow[r] \arrow[u] & 0^{[2]}/0^{[1]} \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
&&&& 0^{[1]}/0^{[1]} \\
&&& 3/3 \arrow[r] & 0^{[1]}/3 \arrow[u] \\
&& 2/2 \arrow[r] & 3/2 \arrow[r] \arrow[u] & 0^{[1]}/2 \arrow[u] \\
& 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & 3/1 \arrow[r] \arrow[u] & 0^{[1]}/1 \arrow[u] \\
0/0 \arrow[r] & 1/0 \arrow[r] \arrow[u] & 2/0 \arrow[r] \arrow[u] & 3/0 \arrow[r] \arrow[u] & 0^{[1]}/0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
0^{[1]}/0^{[1]} \arrow[r] & 1^{[1]}/0^{[1]} \arrow[r] & 2^{[1]}/0^{[1]} \arrow[r] & 3^{[1]}/0^{[1]} \arrow[r] & 0^{[2]}/0^{[1]} \\
0^{[1]}/3 \arrow[r] \arrow[u] & 1^{[1]}/3 \arrow[r] \arrow[u] & 2^{[1]}/3 \arrow[r] \arrow[u] & 3^{[1]}/3 \arrow[u] \\
0^{[1]}/2 \arrow[r] \arrow[u] & 1^{[1]}/2 \arrow[r] \arrow[u] & 2^{[1]}/2 \arrow[u] \\
0^{[1]}/1 \arrow[r] \arrow[u] & 1^{[1]}/1 \arrow[u] \\
0^{[1]}/0 \arrow[u]
\end{tikzcd}
$$

We would like to emphasise that an expression as "$i \leq j,\ k \leq l$" as occurring in part (c) of the following proposition has to be read as "$i \leq k$ and $j \leq l$" (and not as "$i \leq k \leq l$ and $i \leq j \leq l$").

(4.47) Proposition. We suppose given $p \in \mathbb{N}_0$.

(a) We have

$$\#_0^p = \{k/i \mid i, k \in \Theta_0^p,\ i \leq k\},$$
$$\#_+^p = \{(k/i)^{[m]} \mid k/i \in \#_0^p,\ m \in \mathbb{N}_0\}.$$

(b) Given $k/i, l/j \in \#_0^p$, $m, n \in \mathbb{N}_0$, we have

$$(k/i)^{[m]} = (l/j)^{[n]}$$

in $\#_+^p$ if and only if

$$m = n,\ i = j,\ k = l \text{ or}$$
$$m + 1 = n,\ i = l,\ k = \text{!},\ j = \text{¡} \text{ or}$$
$$m = n + 1,\ j = k,\ i = \text{¡},\ l = \text{!} \text{ or}$$
$$m + 2 = n,\ i = k = \text{!},\ j = l = \text{¡} \text{ or}$$
$$m = n + 2,\ i = k = \text{¡},\ j = l = \text{!}.$$

(c) Given $k/i, l/j \in \#_0^p$, $m, n \in \mathbb{N}_0$, we have

$$(k/i)^{[m]} \leq (l/j)^{[n]}$$

in $\#_+^p$ if and only if

$$m + 1 < n \text{ or}$$
$$m + 1 = n,\ i \leq l \text{ or}$$
$$m = n,\ i \leq j,\ k \leq l \text{ or}$$
$$m = n + 1,\ k \leq j,\ i = \text{¡},\ l = \text{!} \text{ or}$$
$$m = n + 2,\ i = k = \text{¡},\ j = l = \text{!}.$$

Proof.

(a) We suppose given an arbitrary element $l/j \in \#_+^p$, that is, we suppose given $j, l \in \Theta_+^p$ with $j \leq l \leq j^{[1]}$. By corollary (4.33)(a), there exist $i, k \in \Delta^p$, $m, n \in \mathbb{N}_0$, with $j = i^{[m]}$, $l = k^{[n]}$. As

$$i^{[m]} = j \leq l = k^{[n]},$$

we obtain $m < n$ or $m = n$, $i \leq k$ by corollary (4.33)(c), and as

$$k^{[n]} = l \leq j^{[1]} = i^{[m+1]},$$

we obtain $n < m + 1$ or $n = m + 1$, $k \leq i$. So we have $n = m$ and $i \leq k$ or we have $n = m + 1$ and $k \leq i$. If $n = m$ and $i \leq k$, then

$$l/j = k^{[m]}/i^{[m]} = (k/i)^{[2m]},$$

and if $n = m + 1$ and $k \leq i$, then

$$l/j = k^{[m+1]}/i^{[m]} = (i^{[m]}/k^{[m]})^{[1]} = (i/k)^{[2m+1]}.$$

Conversely, we suppose given $i, k \in \Theta_0^p$ with $i \leq k$. Then we have $i \leq k \leq i^{[1]}$ by corollary (4.34)(e), hence $k/i \in \#_+^p$ and therefore $(k/i)^{[m]} \in \#_+^p$ for all $m \in \mathbb{N}_0$.

(c) First, we suppose that $m \leq n$, so that

$$(l/j)^{[n]} = \begin{cases} (l^{[q]}/j^{[q]})^{[m]} & \text{if } n = m + 2q \text{ for some } q \in \mathbb{N}_0, \\ (j^{[q+1]}/l^{[q]})^{[m]} & \text{if } n = m + 2q + 1 \text{ for some } q \in \mathbb{N}_0. \end{cases}$$

As $\mathrm{T}^{\#_+^p}$ reflects the order of $\#_+^p$ by remark (4.44), the condition $(k/i)^{[m]} \leq (l/j)^{[n]}$ is equivalent to

$$k/i \leq \begin{cases} l^{[q]}/j^{[q]} & \text{if } n = m + 2q \text{ for some } q \in \mathbb{N}_0, \\ j^{[q+1]}/l^{[q]} & \text{if } n = m + 2q + 1 \text{ for some } q \in \mathbb{N}_0. \end{cases}$$

By proposition (4.32)(c), this holds if and only if $n > m + 1$ or $n = m + 1$, $i \leq l$ or $n = m$, $k \leq l$, $i \leq j$.

Next, we suppose that $m > n$. Analogously, we see that $(k/i)^{[m]} \leq (l/j)^{[n]}$ is equivalent to

$$l/j \geq \begin{cases} k^{[q]}/i^{[q]} & \text{if } m = n + 2q \text{ for some } q \in \mathbb{N}, \\ i^{[q+1]}/k^{[q]} & \text{if } m = n + 2q + 1 \text{ for some } q \in \mathbb{N}_0. \end{cases}$$

By proposition (4.32)(c), this holds if and only if $m = n + 2$, $l = j = !$, $k = i = \text{¡}$ or $m = n + 1$, $l = !$, $i = \text{¡}$, $k \leq j$.

(b) We have $(k/i)^{[m]} = (l/j)^{[n]}$ if and only if $(k/i)^{[m]} \leq (l/j)^{[n]}$ and $(l/j)^{[n]} \leq (k/i)^{[m]}$. By (c), we have $(k/i)^{[m]} \leq (l/j)^{[n]}$ if and only if $m + 1 < n$ or $m + 1 = n$, $i \leq l$ or $m = n$, $i \leq j$, $k \leq l$ or $m = n + 1$, $k \leq j$, $i = \text{¡}$, $l = !$ or $m = n + 2$, $i = k = \text{¡}$, $j = l = !$, and we have $(l/j)^{[n]} \leq (k/i)^{[m]}$ if and only if $n + 1 < m$ or $n + 1 = m$, $j \leq k$ or $n = m$, $j \leq i$, $l \leq k$ or $n = m + 1$, $l \leq i$, $j = \text{¡}$, $k = !$ or $n = m + 2$, $j = l = \text{¡}$, $i = k = !$. In particular, we have the five cases $m = n$ or $m + 1 = n$ or $m = n + 1$ or $m + 1 < n$ or $m < n + 1$.

If $m = n$, then $(k/i)^{[m]} = (l/j)^{[n]}$ is equivalent to $i \leq j$, $k \leq l$ and $j \leq i$, $l \leq k$, that is, to $i = j$, $k = l$. If $m + 1 = n$, then $(k/i)^{[m]} = (l/j)^{[n]}$ is equivalent to $i \leq l$ and $l \leq i$, $j = \text{¡}$, $k = !$, that is, to $i = l$, $j = \text{¡}$, $k = !$. If $m = n + 1$, then $(k/i)^{[m]} = (l/j)^{[n]}$ is equivalent to $k \leq j$, $i = \text{¡}$, $l = !$ and $j \leq k$, that is, to $j = k$, $i = \text{¡}$, $l = !$. If $m + 1 < n$, then $(k/i)^{[m]} = (l/j)^{[n]}$ is equivalent to $n = m + 2$, $j = l = \text{¡}$, $i = k = !$. If $n + 1 < m$, then $(k/i)^{[m]} = (l/j)^{[n]}$ is equivalent to $m = n + 2$, $i = k = \text{¡}$, $j = l = !$. Altogether, $(k/i)^{[m]} = (l/j)^{[n]}$ is equivalent to

$m = n$, $i = j$, $k = l$ or
$m + 1 = n$, $i = l$, $k = !$, $j = \text{¡}$ or
$m = n + 1$, $j = k$, $i = \text{¡}$, $l = !$ or
$m + 2 = n$, $i = k = !$, $j = l = \text{¡}$ or
$m = n + 2$, $i = k = \text{¡}$, $j = l = !$. □

(4.48) Corollary. We suppose given $p \in \mathbb{N}_0$.

(a) We have

$$\#_+^p = \{(k/i)^{[m]} \mid i, k \in \Delta^p,\ i \leq k,\ m \in \mathbb{N}_0\}.$$

(b) Given $i, j, k, l \in \Delta^p$ with $i \leq k$, $j \leq l$ and $m, n \in \mathbb{N}_0$, we have

$$(k/i)^{[m]} = (l/j)^{[n]}$$

in $\#_+^p$ if and only if

$$m = n,\ i = j,\ k = l.$$

(c) Given $i, j, k, l \in \Delta^p$ with $i \leq k$, $j \leq l$ and $m, n \in \mathbb{N}_0$, we have

$$(k/i)^{[m]} \leq (l/j)^{[n]}$$

in $\#_+^p$ if and only if

$m + 1 < n$ or
$m + 1 = n$, $i \leq l$ or
$m = n$, $i \leq j$, $k \leq l$.

Proof.

(a) By proposition (4.47)(a), we have

$$\begin{aligned}\#_+^p &= \{(k/i)^{[m]} \mid i, k \in \Theta_0^p,\, i \leq k,\, m \in \mathbb{N}_0\} \\ &= \{(k/i)^{[m]} \mid i, k \in \Delta^p,\, i \leq k,\, m \in \mathbb{N}_0\} \cup \{(!/i)^{[m]} \mid i \in \Delta^p,\, m \in \mathbb{N}_0\} \cup \{(!/!)^{[m]} \mid m \in \mathbb{N}_0\} \\ &= \{(k/i)^{[m]} \mid i, k \in \Delta^p,\, i \leq k,\, m \in \mathbb{N}_0\} \cup \{(i/\text{¡})^{[m+1]} \mid i \in \Delta^p,\, m \in \mathbb{N}_0\} \cup \{(\text{¡}/\text{¡})^{[m+2]} \mid m \in \mathbb{N}_0\} \\ &= \{(k/i)^{[m]} \mid i, k \in \Delta^p,\, i \leq k,\, m \in \mathbb{N}_0\}.\end{aligned}$$

(b) This follows from proposition (4.47)(b).

(c) This follows from proposition (4.47)(c). □

In analogy to corollary (4.34), we obtain the following cell decomposition for the semiquasicyclic types:

(4.49) Corollary. We suppose given $p \in \mathbb{N}_0$.

(a) For $m \in \mathbb{N}_0$, we have

$$\#_m^p = \{(k/i)^{[m]} \mid k/i \in \#_0^p\}.$$

(b) We have

$$\#_+^p = \bigcup_{m \in \mathbb{N}_0} \#_m^p.$$

(c) For $m \in \mathbb{N}_0$, we have

$$\begin{aligned}&\#_m^p \cap \#_{m+1}^p = \Theta_{m+1}^p, \\ &\#_m^p \cap \#_{m+2}^p = \#_m^p \cap \#_{m+1}^p \cap \#_{m+2}^p = \{(!/\text{¡})^{[m+1]}\} = \{(\text{¡}/\text{¡})^{[m+2]}\}, \\ &\#_m^p \cap \#_{m+k}^p = \emptyset \text{ for } k \in \mathbb{N} \text{ with } k \geq 3.\end{aligned}$$

(d) For $m \in \mathbb{N}_0$, $k/i, l/j \in \#_0^p$, we have $(k/i)^{[m]} \leq (l/j)^{[m]}$ in $\#_+^p$ if and only if $k/i \leq l/j$ in $\#_0^p$, and we have $(k/i)^{[m]} \leq (l/j)^{[m+1]}$ in $\#_+^p$ if and only if $k/i \leq j^{[1]}/l$ in $\#_0^p \cup \#_1^p$.

(e) For $m, n \in \mathbb{N}_0$ with $m + 1 < n$, we have $k/i \leq l/j$ in $\#_+^p$ for all $k/i \in \#_m^p$, $l/j \in \#_n^p$, where $k/i = l/j$ holds if and only if $n = m + 2$, $k/i = (!/!)^{[m]}$, $l/j = (\text{¡}/\text{¡})^{[m+2]}$.

(f) For $m \in \mathbb{N}_0$, we have

$$\#_m^p = \{k/i \in \#_+^p \mid (\text{¡}/\text{¡})^{[m]} \leq k/i \leq (!/!)^{[m]}\}.$$

Proof.

(a) This follows from definition (4.45).

(b) This follows from proposition (4.47)(a).

(c) This follows from proposition (4.47)(b) and (b).

(d) This follows from proposition (4.47)(c).

(e) This follows from proposition (4.47)(c), (b).

(f) We suppose given $m \in \mathbb{N}_0$ and $l/j \in \#_+^p$. First, we suppose that $l/j \in \#_m^p$, so that there exists $k/i \in \#_0^p$ with $l/j = (k/i)^{[m]}$. Since ¡ is the least element and ! is the greatest element of Θ_0^p and since the shift morphism is monotone, we have

$$(\text{¡}/\text{¡})^{[m]} \leq (k/i)^{[m]} \leq (!/!)^{[m]}$$

and so $(\text{¡}/\text{¡})^{[m]} \leq l/j \leq (!/!)^{[m]}$. Conversely, we suppose that l/j fulfills $(\text{¡}/\text{¡})^{[m]} \leq l/j \leq (!/!)^{[m]}$. By proposition (4.47)(a), there exist $k/i \in \#_0^p$, $n \in \mathbb{N}_0$ with $l/j = (k/i)^{[n]}$. As $(\text{¡}/\text{¡})^{[m]} \leq l/j = (k/i)^{[n]}$, we have $n \geq m$ or $n = m - 1$, $l/j = (!/i)^{[m-1]} = (i/\text{¡})^{[m]}$ or $n = m - 2$, $l/j = (!/!)^{[m-2]} = (\text{¡}/\text{¡})^{[m]}$ by proposition (4.47)(c). As $(k/i)^{[n]} = l/j \leq (!/!)^{[m]}$, we have $n \leq m$ or $n = m+1$, $l/j = (k/\text{¡})^{[m+1]} = (!/k)^{[m]}$ or $n = m + 2$, $l/j = (\text{¡}/\text{¡})^{[m+2]} = (!/!)^{[m]}$ by proposition (4.47)(c). Thus we have $l/j \in \#_m^p$. □

(4.50) Corollary. We have an injective morphism of posets with shift $b\colon \Theta_+^P \to \#_+^P$ that reflects the orders, given by

$$i^{[m]}b = (i/¡)^{[m]}$$

for $i \in \Theta_0^p$, $m \in \mathbb{N}_0$, with $\Theta_0^p b \subseteq \#_0^p$.

$$\begin{tikzcd}
\Theta_0^p \arrow[r, "b|_{\Theta_0^p}^{\#_0^p}"] \arrow[d, "\mathrm{inc}"'] & \#_0^p \arrow[d, "\mathrm{inc}"] \\
\Theta_+^p \arrow[r, "b"] & \#_+^p
\end{tikzcd}$$

Proof. We let $b_0\colon \Theta_0^p \to \#_0^p$ be given by $ib_0 := i/¡$ for $i \in \Theta_0^p$. Then b_0 is a poset morphism and we have

$$!b_0 = !/¡ = ¡^{[1]}/¡ = (¡/¡)^{[1]} = (¡b_0)^{[1]}$$

in $\#_+^p$. By proposition (4.35), there exists a unique morphism of posets with shift $b\colon \Theta_+^p \to \#_+^p$ with $b_0 = b|_{\Theta_0^p}^{\#_0^p}$, given by $i^{[m]}b = (ib_0)^{[m]} = (i/¡)^{[m]}$ for $i \in \Theta_0^p$, $m \in \mathbb{N}_0$. To show that b reflects the orders, we suppose given $i, j \in \Theta_0^p$, $m, n \in \mathbb{Z}$ with $i^{[m]}b \le j^{[n]}b$, that is, with $(i/¡)^{[m]} \le (j/¡)^{[n]}$. By proposition (4.47)(c), it follows that

$m + 1 < n$ or
$m + 1 = n$, $¡ \le j$ or
$m = n$, $¡ \le ¡$, $i \le j$ or
$m = n + 1$, $i \le ¡$, $¡ = ¡$, $j = !$ or
$m = n + 2$, $¡ = i = ¡$, $¡ = j = !$,

that is, $m < n$ or $m = n$, $i \le j$ or $m = n{+}1$, $i = ¡$, $j = !$. In each case we have $i^{[m]} \le j^{[n]}$ by proposition (4.32)(c). Thus b reflects the orders, and so it is in particular injective. □

(4.51) Convention. We suppose given $p \in \mathbb{N}_0$. From now on, we identify Θ_+^p with the image of the injective morphism of posets with shift $b\colon \Theta_+^p \to \#_+^p$ from corollary (4.50). That is, by abuse of notation, we write Θ_+^p instead of $\operatorname{Im} b$, and, given $i \in \Theta_0^p$, $m \in \mathbb{N}_0$, the image $i^{[m]}b = (i/¡)^{[m]} \in \#_+^p$ of $i^{[m]} \in \Theta_+^p$ will also be denoted by $i^{[m]}$. Accordingly, although the objects $¡ = ¡^{\Theta_0^p} \in \Theta_0^p \subseteq \Theta_+^p$ resp. $! = !^{\Theta_0^p} \in \Theta_0^p \subseteq \Theta_+^p$ are no longer initial resp. terminal in $\#_+^p$, we will still use this notation for these elements in $\#_+^p$.

With this convention, the semiquasicyclic type Θ_+^p for $p \in \mathbb{N}_0$ lies like a snake in the semistrip type $\#_+^p$:

(4.52) Example.

(a) The shape of $\Theta_+^3 \subseteq \#_+^3$ may be displayed as follows.

$$\begin{tikzcd}
 & & & & & & \vdots & \\
 & & & & & 1^{[1]}/1^{[1]} \arrow[r] & 2^{[1]}/1^{[1]} \arrow[r] \arrow[u] & \ldots \\
 & & & & !^{[1]} \arrow[r] & 1^{[2]} \arrow[r] \arrow[u] & 2^{[2]} \arrow[r] \arrow[u] & \ldots \\
 & & & 3/3 \arrow[r] & 3^{[1]} \arrow[r] \arrow[u] & 1^{[1]}/3 \arrow[r] \arrow[u] & 2^{[1]}/3 \arrow[r] \arrow[u] & \ldots \\
 & & 2/2 \arrow[r] & 3/2 \arrow[r] \arrow[u] & 2^{[1]} \arrow[r] \arrow[u] & 1^{[1]}/2 \arrow[r] \arrow[u] & 2^{[1]}/2 \arrow[u] & \\
 & 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & 3/1 \arrow[r] \arrow[u] & 1^{[1]} \arrow[r] \arrow[u] & 1^{[1]}/1 \arrow[u] & & \\
¡ \arrow[r] & 1 \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & 3 \arrow[r] \arrow[u] & ! \arrow[u] & & &
\end{tikzcd}$$

(b) The cells $\#_0^3$, $\#_1^3$, $\#_2^3$ of $\#_+^3$ may be displayed as follows.

$$\begin{tikzcd}
 & & & & {!}^{[4]} \\
 & & & (3/3)^{[2]} \arrow[r] & 3^{[3]} \arrow[u] \\
 & & (2/2)^{[2]} \arrow[r] & (3/2)^{[2]} \arrow[r] \arrow[u] & 2^{[3]} \arrow[u] \\
 & (1/1)^{[2]} \arrow[r] & (2/1)^{[2]} \arrow[r] \arrow[u] & (3/1)^{[2]} \arrow[r] \arrow[u] & 1^{[3]} \arrow[u] \\
\text{¡}^{[2]} \arrow[r] & 1^{[2]} \arrow[r] \arrow[u] & 2^{[2]} \arrow[r] \arrow[u] & 3^{[2]} \arrow[r] \arrow[u] & {!}^{[2]} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & {!}^{[1]} & {!}^{[1]} \arrow[r] & 1^{[2]} \arrow[r] & 2^{[2]} \arrow[r] & 3^{[2]} \arrow[r] & {!}^{[2]} \\
 & & & 3/3 \arrow[r] & 3^{[1]} \arrow[u] & 3^{[1]} \arrow[r] \arrow[u] & (3/1)^{[1]} \arrow[r] \arrow[u] & (3/2)^{[1]} \arrow[r] \arrow[u] & (3/3)^{[1]} \arrow[u] & \\
 & & 2/2 \arrow[r] & 3/2 \arrow[r] \arrow[u] & 2^{[1]} \arrow[u] & 2^{[1]} \arrow[r] \arrow[u] & (2/1)^{[1]} \arrow[r] \arrow[u] & (2/2)^{[1]} \arrow[u] & & \\
 & 1/1 \arrow[r] & 2/1 \arrow[r] \arrow[u] & 3/1 \arrow[r] \arrow[u] & 1^{[1]} \arrow[u] & 1^{[1]} \arrow[r] \arrow[u] & (1/1)^{[1]} \arrow[u] & & & \\
\text{¡} \arrow[r] & 1 \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & 3 \arrow[r] \arrow[u] & {!} \arrow[u] & \text{¡}^{[1]} \arrow[u] & & & &
\end{tikzcd}$$

The universal property

Analogously to proposition (4.35), we will now prove a universal property for the semistrip types.

(4.53) Proposition. We suppose given $p \in \mathbb{N}_0$. For every category with shift $\mathcal{C}$ and every functor $F\colon \#_0^p \to \mathcal{C}$ with $F \circ \mathrm{T}^{\#_+^p}|_{\Theta_0^p}^{\#_0^p} = \mathrm{T}^{\mathcal{C}} \circ F|_{\Theta_0^p}$, there exists a unique morphism of categories with shift $\hat{F}\colon \#_+^p \to \mathcal{C}$ with $F = \hat{F}|_{\#_0^p}$, given on the objects by

$$\hat{F}((k/i)^{[m]}) = F(k/i)^{[m]}$$

for $k/i \in \#_0^p$, $m \in \mathbb{N}_0$, and on the morphisms by

$$\hat{F}((k/i)^{[m]}, (l/j)^{[n]}) = \begin{cases} 1_{(F\text{¡})^{[m]}} & \text{if } m = n+2, \\ F(k,j)^{[m]} & \text{if } m = n+1, \\ F(k/i, l/j)^{[m]} & \text{if } m = n, \\ F(k/i, i_{m+1}^{[1]})^{[m]}\, (\bullet_{r \in \lceil m+1, n-1 \rceil} F(i_r, i_{r+1}^{[1]})^{[r]})\, F(i_n, l/j)^{[n]} & \text{if } m < n \end{cases}$$

for $k/i, l/j \in \#_0^p$, $m, n \in \mathbb{N}_0$ with $(k/i)^{[m]} \leq (l/j)^{[n]}$, and for arbitrarily chosen $i_r \in \Theta_0^p$, $r \in [m+1, n]$, with $k/i \leq i_{m+1}^{[1]}$ and $i_n \leq l/j$ in the case $m < n$.

Proof. We suppose given a category with shift $\mathcal{C}$ and a functor $F\colon \#_0^p \to \mathcal{C}$ with $F \circ \mathrm{T}^{\#_+^p}|_{\Theta_0^p}^{\#_0^p} = \mathrm{T}^{\mathcal{C}} \circ F|_{\Theta_0^p}$. To construct a functor $\hat{F}\colon \#_+^p \to \mathcal{C}$ with $F = \hat{F}|_{\#_0^p}$, we will use the asymmetric description of $\#_+^p$ as given in corollary (4.48). We define a map $\hat{F}_0\colon \operatorname{Ob} \#_+^p \to \operatorname{Ob} \mathcal{C}$ by

$$\hat{F}_0((k/i)^{[m]}) := F(k/i)^{[m]}$$

for $i, k \in \Delta^p$ with $i \leq k$, $m \in \mathbb{N}_0$, and we define a map $\hat{F}_1\colon \operatorname{Mor} \#_+^p \to \operatorname{Mor} \mathcal{C}$ by

$$\hat{F}_1((k/i)^{[m]}, (l/j)^{[n]}) := \begin{cases} F(k/i, l/j)^{[m]} & \text{if } m = n, \\ F(k/i, i^{[1]})^{[m]}\, F(i, l/j)^{[m+1]} & \text{if } m+1 = n, \\ F(k/i, {!}^{[1]})^{[m]}\, (\bullet_{r \in \lceil m+2, n-1 \rceil} F(\text{¡}, {!})^{[r]})\, F(\text{¡}, l/j)^{[n]} & \text{if } m+1 < n \end{cases}$$

for $i, j, k, l \in \Delta^p$ with $i \leq k$, $j \leq l$ and $m, n \in \mathbb{N}_0$ such that $(k/i)^{[m]} \leq (l/j)^{[n]}$.
Then we have

$$
\begin{aligned}
&\mathrm{Source}^{\mathcal{C}}\, \hat{F}_1((k/i)^{[m]}, (l/j)^{[n]}) \\
&= \begin{cases} \mathrm{Source}^{\mathcal{C}}\, F(k/i, l/j)^{[m]} & \text{if } m = n, \\ \mathrm{Source}^{\mathcal{C}}(F(k/i, i^{[1]})^{[m]}\, F(i, l/j)^{[m+1]}) & \text{if } m+1 = n, \\ \mathrm{Source}^{\mathcal{C}}(F(k/i, !^{[1]})^{[m]}\, (\bullet_{r \in \lceil m+2, n-1 \rceil} F(¡, !)^{[r]})\, F(¡, l/j)^{[n]}) & \text{if } m+1 < n \end{cases} \\
&= \begin{cases} \mathrm{Source}^{\mathcal{C}}\, F(k/i, l/j)^{[m]} & \text{if } m = n, \\ \mathrm{Source}^{\mathcal{C}}\, F(k/i, i^{[1]})^{[m]} & \text{if } m+1 = n, \\ \mathrm{Source}^{\mathcal{C}}\, F(k/i, !^{[1]})^{[m]} & \text{if } m+1 < n \end{cases} = \begin{cases} F(\mathrm{Source}^{\#_0^p}(k/i, l/j))^{[m]} & \text{if } m = n, \\ F(\mathrm{Source}^{\#_0^p}(k/i, i^{[1]}))^{[m]} & \text{if } m+1 = n, \\ F(\mathrm{Source}^{\#_0^p}(k/i, !^{[1]}))^{[m]} & \text{if } m+1 < n \end{cases} \\
&= F(k/i)^{[m]} = \hat{F}_0((k/i)^{[m]}) = \hat{F}_0(\mathrm{Source}^{\#_+^p}((k/i)^{[m]}, (l/j)^{[n]}))
\end{aligned}
$$

and, analogously,

$$\mathrm{Target}^{\mathcal{C}}\, \hat{F}_1((k/i)^{[m]}, (l/j)^{[n]}) = \hat{F}_0(\mathrm{Target}^{\#_+^p}((k/i)^{[m]}, (l/j)^{[n]}))$$

for $i, j, k, l \in \Delta^p$ with $i \leq k$, $j \leq l$ and $m, n \in \mathbb{N}_0$ such that $(k/i)^{[m]} \leq (l/j)^{[n]}$. Moreover, we get

$$
\begin{aligned}
&\hat{F}_1((k_1/i_1)^{[m_1]}, (k_2/i_2)^{[m_2]})\, \hat{F}_1((k_2/i_2)^{[m_2]}, (k_3/i_3)^{[m_3]}) \\
&= \begin{cases}
F(k_1/i_1, k_2/i_2)^{[m_1]}\, F(k_2/i_2, k_3/i_3)^{[m_2]} & \text{if } m_1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, k_2/i_2)^{[m_1]}\, F(k_2/i_2, i_2^{[1]})^{[m_2]}\, F(i_2, k_3/i_3)^{[m_2+1]} & \text{if } m_1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, k_2/i_2)^{[m_1]}\, F(k_2/i_2, !^{[1]})^{[m_2]} & \\
\quad \cdot (\bullet_{r \in \lceil m_2+2, m_3-1 \rceil} F(¡, !)^{[r]})\, F(¡, k_3/i_3)^{[m_3]} & \text{if } m_1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_2/i_2)^{[m_1+1]}\, F(k_2/i_2, k_3/i_3)^{[m_2]} & \text{if } m_1 + 1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_2/i_2)^{[m_1+1]}\, F(k_2/i_2, i_2^{[1]})^{[m_2]} & \\
\quad \cdot F(i_2, k_3/i_3)^{[m_2+1]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_2/i_2)^{[m_1+1]}\, F(k_2/i_2, !^{[1]})^{[m_2]} & \\
\quad \cdot (\bullet_{r \in \lceil m_2+2, m_3-1 \rceil} F(¡, !)^{[r]})\, F(¡, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(¡, !)^{[r]})\, F(¡, k_2/i_2)^{[m_2]} & \\
\quad \cdot F(k_2/i_2, k_3/i_3)^{[m_2]} & \text{if } m_1 + 1 < m_2,\ m_2 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(¡, !)^{[r]})\, F(¡, k_2/i_2)^{[m_2]} & \\
\quad \cdot F(k_2/i_2, i_2^{[1]})^{[m_2]}\, F(i_2, k_3/i_3)^{[m_2+1]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(¡, !)^{[r]}) & \\
\quad \cdot F(¡, k_2/i_2)^{[m_2]}\, F(k_2/i_2, !^{[1]})^{[m_2]} & \\
\quad \cdot (\bullet_{r \in \lceil m_2+2, m_3-1 \rceil} F(¡, !)^{[r]})\, F(¡, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 < m_3
\end{cases}
\end{aligned}
$$

$$
= \left\{ \begin{array}{ll}
F(k_1/i_1, k_2/i_2)^{[m_1]}\, F(k_2/i_2, k_3/i_3)^{[m_1]} & \text{if } m_1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, k_2/i_2)^{[m_1]}\, F(k_2/i_2, i_2^{[1]})^{[m_1]}\, F(i_2, k_3/i_3)^{[m_1+1]} & \text{if } m_1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, k_2/i_2)^{[m_1]}\, F(k_2/i_2, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]}) & \\
\quad \cdot F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_2/i_2)^{[m_1+1]}\, F(k_2/i_2, k_3/i_3)^{[m_1+1]} & \text{if } m_1 + 1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_2/i_2)^{[m_1+1]}\, F(k_2/i_2, i_2^{[1]})^{[m_1+1]} & \\
\quad \cdot F(i_2, k_3/i_3)^{[m_1+2]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_2/i_2)^{[m_1+1]}\, F(k_2/i_2, !^{[1]})^{[m_1+1]} & \\
\quad \cdot (\bullet_{r \in \lceil m_1+3, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_2/i_2)^{[m_2]} & \\
\quad \cdot F(k_2/i_2, k_3/i_3)^{[m_2]} & \text{if } m_1 + 1 < m_2,\ m_2 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_2/i_2)^{[m_2]} & \\
\quad \cdot F(k_2/i_2, i_2^{[1]})^{[m_2]}\, F(i_2, k_3/i_3)^{[m_2+1]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]}) & \\
\quad \cdot F(\text{¡}, k_2/i_2)^{[m_2]}\, F(k_2/i_2, !^{[1]})^{[m_2]} & \\
\quad \cdot (\bullet_{r \in \lceil m_2+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 < m_3
\end{array} \right\}
$$

$$
= \left\{ \begin{array}{ll}
F(k_1/i_1, k_3/i_3)^{[m_1]} & \text{if } m_1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1^{[1]}, i_2^{[1]})^{[m_1]}\, F(i_2, k_3/i_3)^{[m_1+1]} & \text{if } m_1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_3/i_3)^{[m_1+1]} & \text{if } m_1 + 1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, !)^{[m_1+1]}\, F(!, i_2^{[1]})^{[m_1+1]}\, F(i_2, k_3/i_3)^{[m_1+2]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, !)^{[m_1+1]}\, F(!, !^{[1]})^{[m_1+1]} & \\
\quad \cdot (\bullet_{r \in \lceil m_1+3, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_2]} & \text{if } m_1 + 1 < m_2,\ m_2 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, !)^{[m_2]}\, F(!, i_2^{[1]})^{[m_2]} & \\
\quad \cdot F(i_2, k_3/i_3)^{[m_2+1]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, !)^{[m_2]}\, F(!, !^{[1]})^{[m_2]} & \\
\quad \cdot (\bullet_{r \in \lceil m_2+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 < m_3
\end{array} \right\}
$$

$$
= \left\{ \begin{array}{ll}
F(k_1/i_1, k_3/i_3)^{[m_1]} & \text{if } m_1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, i_2)^{[m_1+1]}\, F(i_2, k_3/i_3)^{[m_1+1]} & \text{if } m_1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_3/i_3)^{[m_1+1]} & \text{if } m_1 + 1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1^{[1]}, !^{[1]})^{[m_1]}\, F(\text{¡}, i_2)^{[m_1+2]}\, F(i_2, k_3/i_3)^{[m_1+2]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1^{[1]}, !^{[1]})^{[m_1]}\, F(\text{¡}, !)^{[m_1+2]} & \\
\quad \cdot (\bullet_{r \in \lceil m_1+3, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_2]} & \text{if } m_1 + 1 < m_2,\ m_2 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, i_2)^{[m_2+1]} & \\
\quad \cdot F(i_2, k_3/i_3)^{[m_2+1]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, !)^{[m_2+1]} & \\
\quad \cdot (\bullet_{r \in \lceil m_2+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 < m_3
\end{array} \right\}
$$

$$
= \begin{cases}
F(k_1/i_1, k_3/i_3)^{[m_1]} & \text{if } m_1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_3/i_3)^{[m_1+1]} & \text{if } m_1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]}\, F(i_1, k_3/i_3)^{[m_1+1]} & \text{if } m_1 + 1 = m_2,\ m_2 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 = m_2,\ m_2 + 1 < m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_2]} & \text{if } m_1 + 1 < m_2,\ m_2 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_2 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_2+1]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]})\, F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 < m_2,\ m_2 + 1 < m_3
\end{cases}
$$

$$
= \begin{cases}
F(k_1/i_1, k_3/i_3)^{[m_1]} & \text{if } m_1 = m_3, \\
F(k_1/i_1, i_1^{[1]})^{[m_1]} F(i_1, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 = m_3, \\
F(k_1/i_1, !^{[1]})^{[m_1]}\, (\bullet_{r \in \lceil m_1+2, m_3-1 \rceil} F(\text{¡}, !)^{[r]}) F(\text{¡}, k_3/i_3)^{[m_3]} & \text{if } m_1 + 1 < m_3
\end{cases}
$$

$$
= \hat{F}_1((k_1/i_1)^{[m_1]}, (k_3/i_3)^{[m_3]}).
$$

for $i_1, i_2, i_3, k_1, k_2, k_3 \in \Delta^p$ with $i_1 \leq k_1$, $i_2 \leq k_2$, $i_3 \leq k_3$ and $m_1, m_2, m_3 \in \mathbb{N}_0$ such that $(k_1/i_1)^{[m_1]} \leq (k_2/i_2)^{[m_2]} \leq (k_3/i_3)^{[m_3]}$, and

$$
\hat{F}_1(1_{(k/i)^{[m]}}) = \hat{F}_1((k/i)^{[m]}, (k/i)^{[m]}) = F(k/i, k/i)^{[m]} = F(1_{k/i})^{[m]} = 1_{F(k/i)^{[m]}} = 1_{\hat{F}_0((k/i)^{[m]})}
$$

for $i, k \in \Delta^p$ with $i \leq k$ and $m \in \mathbb{N}_0$.
Thus we have a functor $\hat{F} \colon \#_+^p \to \mathcal{C}$ with $\operatorname{Ob} \hat{F} = \hat{F}_0$ and $\operatorname{Mor} \hat{F} = \hat{F}_1$, that is, such that

$$
\hat{F}((k/i)^{[m]}) = F(k/i)^{[m]}
$$

for $i, k \in \Delta^p$ with $i \leq k$, $m \in \mathbb{N}_0$, and

$$
\hat{F}((k/i)^{[m]}, (l/j)^{[n]}) = \begin{cases}
F(k/i, l/j)^{[m]} & \text{if } m = n, \\
F(k/i, i^{[1]})^{[m]} F(i, l/j)^{[n]} & \text{if } m + 1 = n, \\
F(k/i, !^{[1]})^{[m]}\, (\bullet_{r \in \lceil m+2, n-1 \rceil} F(\text{¡}, !)^{[r]}) F(\text{¡}, l/j)^{[n]} & \text{if } m + 1 < n
\end{cases}
$$

for $i, j, k, l \in \Delta^p$ with $i \leq k$, $j \leq l$ and $m, n \in \mathbb{N}_0$ such that $(k/i)^{[m]} \leq (l/j)^{[n]}$.
Moreover, as

$$
\hat{F}\big(((k/i)^{[m]}, (l/j)^{[n]})^{[1]}\big) = \hat{F}\big((k/i)^{[m+1]}, (l/j)^{[n+1]}\big)
$$

$$
= \begin{cases}
F(k/i, l/j)^{[m+1]} & \text{if } m + 1 = n + 1, \\
F(k/i, i^{[1]})^{[m+1]} F(i, l/j)^{[n+1]} & \text{if } m + 2 = n + 1, \\
F(k/i, !^{[1]})^{[m+1]}\, (\bullet_{r \in \lceil m+3, n \rceil} F(\text{¡}, !)^{[r]}) F(\text{¡}, l/j)^{[n+1]} & \text{if } m + 2 < n + 1
\end{cases}
$$

$$
= \begin{cases}
(F(k/i, l/j)^{[m]})^{[1]} & \text{if } m = n, \\
(F(k/i, i^{[1]})^{[m]} F(i, l/j)^{[n]})^{[1]} & \text{if } m + 1 = n, \\
(F(k/i, !/!)^{[m]}\, (\bullet_{r \in \lceil m+2, n-1 \rceil} F(\text{¡}, !)^{[r]}) F(\text{¡}, l/j)^{[n]})^{[1]} & \text{if } m + 1 < n
\end{cases} = \hat{F}((k/i)^{[m]}, (l/j)^{[n]})^{[1]}
$$

for $i, j, k, l \in \Delta^p$ with $i \leq k$, $j \leq l$ and $m, n \in \mathbb{N}_0$ such that $(k/i)^{[m]} \leq (l/j)^{[n]}$, we have $\operatorname{Mor} \hat{F} \circ \operatorname{Mor} \mathrm{T}^{\#_+^p} = \operatorname{Mor} \mathrm{T}^{\mathcal{C}} \circ \operatorname{Mor} \hat{F}$. Hence $\hat{F} \circ \mathrm{T}^{\#_+^p} = \mathrm{T}^{\mathcal{C}} \circ \hat{F}$, that is, $\hat{F}$ is in fact a morphism of categories with shift.
Finally, we get

$$
\hat{F}(k/i, l/j) = \begin{cases}
\hat{F}(k/i, l/j) & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j \in \Delta^p,\ l \in \Delta^p, \\
\hat{F}(k/i, (j/\text{¡})^{[1]}) & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j \in \Delta^p,\ l = !, \\
\hat{F}(k/i, (\text{¡}/\text{¡})^{[2]}) & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j = !,\ l = !, \\
\hat{F}((i/\text{¡})^{[1]}, (j/\text{¡})^{[1]}) & \text{if } i \in \Delta^p,\ k = !,\ j \in \Delta^p,\ l = !, \\
\hat{F}((i/\text{¡})^{[1]}, (\text{¡}/\text{¡})^{[2]}) & \text{if } i \in \Delta^p,\ k = !,\ j = !,\ l = !, \\
\hat{F}((\text{¡}/\text{¡})^{[2]}, (\text{¡}/\text{¡})^{[2]}) & \text{if } i = !,\ k = !,\ j = !,\ l = !
\end{cases}
$$

$$= \begin{cases} F(k/i, l/j) & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j \in \Delta^p,\ l \in \Delta^p, \\ F(k/i, i^{[1]})F(i,j)^{[1]} & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j \in \Delta^p,\ l = !, \\ F(k/i, !^{[1]})F(\text{¡},\text{¡})^{[2]} & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j = !,\ l = !, \\ F(i,j)^{[1]} & \text{if } i \in \Delta^p,\ k = !,\ j \in \Delta^p,\ l = !, \\ F(i,!)^{[1]}F(\text{¡},\text{¡})^{[2]} & \text{if } i \in \Delta^p,\ k = !,\ j = !,\ l = !, \\ F(\text{¡},\text{¡})^{[2]} & \text{if } i = !,\ k = !,\ j = !,\ l = ! \end{cases}$$

$$= \begin{cases} F(k/i, l/j) & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j \in \Delta^p,\ l \in \Delta^p, \\ F(k/i, i^{[1]})F(i^{[1]},j^{[1]}) & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j \in \Delta^p,\ l = !, \\ F(k/i, !^{[1]}) & \text{if } i \in \Delta^p,\ k \in \Delta^p,\ j = !,\ l = !, \\ F(i^{[1]},j^{[1]}) & \text{if } i \in \Delta^p,\ k = !,\ j \in \Delta^p,\ l = !, \\ F(i^{[1]},!^{[1]}) & \text{if } i \in \Delta^p,\ k = !,\ j = !,\ l = !, \\ F(!^{[1]},!^{[1]}) & \text{if } i = !,\ k = !,\ j = !,\ l = ! \end{cases} = F(k/i, l/j)$$

for $k/i, l/j \in \#_0^p$ with $k/i \leq l/j$, that is, we have $\operatorname{Mor} F = \operatorname{Mor} \hat{F}|_{\#_0^p}$ and therefore $F = \hat{F}|_{\#_0^p}$. In particular, $\hat{F}$ is necessarily given on the objects by

$$\hat{F}((k/i)^{[m]}) = \hat{F}(k/i)^{[m]} = F(k/i)^{[m]}$$

for $k/i \in \#_0^p$, $m \in \mathbb{N}_0$. To derive a formula for the values of $\hat{F}$ on the morphisms, we suppose given $k/i, l/j \in \#_0^p$, $m, n \in \mathbb{N}_0$ with $(k/i)^{[m]} \leq (l/j)^{[n]}$. By proposition (4.47)(c), we have $m+1 < n$ or $m+1 = n$, $i \leq l$ or $m = n$, $i \leq j$, $k \leq l$ or $m = n+1$, $k \leq j$, $i = \text{¡}$, $l = !$ or $m = n+2$, $i = k = \text{¡}$, $j = l = !$. In particular, we have the four cases $m = n$ or $m = n+1$ or $m = n+2$ or $m < n$. If $m = n$, then we necessarily have $i \leq j$, $k \leq l$ and

$$\hat{F}((k/i)^{[m]}, (l/j)^{[n]}) = \hat{F}((k/i)^{[m]}, (l/j)^{[m]}) = \hat{F}((k/i, l/j)^{[m]}) = \hat{F}(k/i, l/j)^{[m]} = F(k/i, l/j)^{[m]}.$$

If $m = n+1$, then we necessarily have $k \leq j$, $i = \text{¡}$, $l = !$ and

$$\hat{F}((k/i)^{[m]}, (l/j)^{[n]}) = \hat{F}((k/\text{¡})^{[m]}, (!/j)^{[m-1]}) = \hat{F}(k^{[m]}, j^{[m]}) = \hat{F}(k,j)^{[m]} = F(k,j)^{[m]}.$$

If $m = n+2$, then we necessarily have $i = k = \text{¡}$, $j = l = !$ and

$$\begin{aligned} \hat{F}((k/i)^{[m]}, (l/j)^{[n]}) &= \hat{F}((\text{¡}/\text{¡})^{[m]}, (!/!)^{[m-2]}) = \hat{F}(\text{¡}^{[m]}, \text{¡}^{[m]}) = \hat{F}(\text{¡},\text{¡})^{[m]} = F(\text{¡},\text{¡})^{[m]} \\ &= F(1_{\text{¡}})^{[m]} = 1_{(F\text{¡})^{[m]}}. \end{aligned}$$

Finally, we suppose that $m < n$. We choose $i_r \in \Theta_0^p$ for $r \in [m+1, n]$ with $k/i \leq i_{m+1}^{[1]}$ and $i_n \leq l/j$. (If $m = n-1$, it is possible to choose i_{m+1} with $k/i \leq i_{m+1}^{[1]}$ and $i_{m+1} \leq l/j$, that is, with $k/i \leq i_{m+1}^{[1]} \leq (l/j)^{[1]}$, since in this case we have $i \leq l$ by proposition (4.47)(c).) Then we necessarily have

$$\begin{aligned} \hat{F}((k/i)^{[m]}, (l/j)^{[n]}) &= \hat{F}\big(((k/i)^{[m]}, i_{m+1}^{[m+1]})\, (\bullet_{r \in \lceil m+1, n-1 \rceil} (i_r^{[r]}, i_{r+1}^{[r+1]}))\, (i_n^{[n]}, l/j^{[n]})\big) \\ &= \hat{F}((k/i)^{[m]}, (i_{m+1}^{[1]})^{[m]})\, (\bullet_{r \in \lceil m+1, n-1 \rceil} \hat{F}(i_r^{[r]}, (i_{r+1}^{[1]})^{[r]}))\, \hat{F}(i_n^{[n]}, (l/j)^{[n]}) \\ &= \hat{F}(k/i, i_{m+1}^{[1]})^{[m]}\, (\bullet_{r \in \lceil m+1, n-1 \rceil} \hat{F}(i_r, i_{r+1}^{[1]})^{[r]})\, \hat{F}(i_n, l/j)^{[n]} \\ &= F(k/i, i_{m+1}^{[1]})^{[m]}\, (\bullet_{r \in \lceil m+1, n-1 \rceil} F(i_r, i_{r+1}^{[1]})^{[r]})\, F(i_n, l/j)^{[n]}. \end{aligned}$$

Conversely, given an arbitrary morphism of categories with shift $G\colon \#_+^p \to \mathcal{C}$ with $F = G|_{\#_0^p}$, we necessarily have

$$\begin{aligned} G((k/i)^{[m]}, (l/j)^{[n]}) &= \begin{cases} G((k/i)^{[m]}, (l/j)^{[m]}) & \text{if } m = n, \\ G\big(((k/i)^{[m]}, (i^{[1]})^{[m]})\, (i^{[m+1]}, (l/j)^{[m+1]})\big) & \text{if } m+1 = n, \\ G\big(((k/i)^{[m]}, (!^{[1]})^{[m]})\, (\bullet_{r \in \lceil m+2, n-1 \rceil} (\text{¡}^{[r]}, !^{[r]}))\, (\text{¡}^{[n]}, (l/j)^{[n]})\big) & \text{if } m+1 < n \end{cases} \\ &= \begin{cases} G((k/i)^{[m]}, (l/j)^{[m]}) & \text{if } m = n, \\ G((k/i)^{[m]}, (i^{[1]})^{[m]})\, G(i^{[m+1]}, (l/j)^{[m+1]}) & \text{if } m+1 = n, \\ G((k/i)^{[m]}, (!^{[1]})^{[m]})\, (\bullet_{r \in \lceil m+2, n-1 \rceil} G(\text{¡}^{[r]}, !^{[r]}))\, G(\text{¡}^{[n]}, (l/j)^{[n]}) & \text{if } m+1 < n \end{cases} \end{aligned}$$

$$= \left\{\begin{array}{ll} G(k/i,l/j)^{[m]} & \text{if } m=n, \\ G(k/i,i^{[1]})^{[m]}\, G(i,l/j)^{[m+1]} & \text{if } m+1=n, \\ G(k/i,!^{[1]})^{[m]}\, (\bullet_{r\in\lceil m+2,n-1\rceil} G(¡,!)^{[r]})\, G(¡,l/j)^{[n]} & \text{if } m+1<n \end{array}\right\}$$

$$= \begin{cases} F(k/i,l/j)^{[m]} & \text{if } m=n, \\ F(k/i,i^{[1]})^{[m]}\, F(i,l/j)^{[m+1]} & \text{if } m+1=n, \\ F(k/i,!^{[1]})^{[m]}\, (\bullet_{r\in\lceil m+2,n-1\rceil} F(¡,!)^{[r]})\, F(¡,l/j)^{[n]} & \text{if } m+1<n \end{cases}$$

for $i,j,k,l \in \Delta^p$ with $i \leq k$, $j \leq l$ and $m,n \in \mathbb{N}_0$ such that $(k/i)^{[m]} \leq (l/j)^{[n]}$, whence $G = \hat{F}$. □

6 Cosemistrips and cosemicomplexes

We suppose given $p \in \mathbb{N}_0$. In this section, we will introduce p-cosemistrips in a given category $\mathcal{C}$ as $\#_+^p$-commutative diagrams therein, see definition (4.55)(a). The semistrip type $\#_+^p$ is a poset with shift, cf. definition (4.42). If $\mathcal{C}$ is also equipped with a shift functor, we may consider the particular shift compatible $\#_+^p$-commutative diagrams as introduced in definition (4.17)(a): In such a diagram, the morphism on $(k/i)^{[1]} \to (l/j)^{[1]}$ for $k/i, l/j \in \#_+^p$ with $k/i \leq l/j$ is obtained by an application of $\mathrm{T}^{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ on the morphism on $k/i \to l/j$. Because of the injectivity (on the objects and the morphisms) of $\mathrm{T}^{\#_+^p}\colon \#_+^p \to \#_+^p$, cf. remark (4.44), these particular diagrams are called periodic p-cosemistrips, see definition (4.55)(b). Likewise, periodic morphisms of p-cosemistrips are defined. The universal property of $\#_+^p$ of proposition (4.53) will show that periodic p-cosemistrips and periodic morphisms of p-cosemistrips are uniquely determined by their values on the (finite) subposet $\#_0^p$ of $\#_+^p$, cf. definition (4.27) and proposition (4.60).
Examples of cosemistrips will be the Heller cosemistrips in a zero-pointed Brown cofibration category, see definition (5.33), which will be the models for the cosemitriangles in the corresponding homotopy category, see definition (5.45) and definition (5.51). The cosemitriangles will be in fact particular periodic cosemistrips with zeros at the "boundaries", so-called cosemicomplexes as in definition (4.62).

The semiquasicyclic categories of cosemistrips and periodic cosemistrips

(4.54) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$ that contains $\mathbb{N}_0$.

(a) We have a functor

$$\mathrm{Strips}^{\mathrm{co},+}\colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{q}_+\mathbf{Cat}_{(\mathfrak{U})},$$

given by

$$\mathrm{Strips}^{\mathrm{co},+}(-) = \mathrm{Strips}^{\mathrm{co},+}_\bullet(-) := -^{\#_+^\bullet}.$$

For every morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}_{(\mathfrak{U})}$, the morphism $\mathrm{Strips}^{\mathrm{co},+}(F)\colon \mathrm{Strips}^{\mathrm{co},+}(\mathcal{C}) \to \mathrm{Strips}^{\mathrm{co},+}(\mathcal{D})$ in $\mathbf{q}_+\mathbf{Cat}_{(\mathfrak{U})}$ is given by

$$\mathrm{Strips}^{\mathrm{co},+}_n(F)X = FX$$

for $X \in \operatorname{Ob}\mathrm{Strips}^{\mathrm{co},+}_n(\mathcal{C})$ and

$$\mathrm{Strips}^{\mathrm{co},+}_n(F)f = Ff$$

for $f \in \operatorname{Mor}\mathrm{Strips}^{\mathrm{co},+}_n(\mathcal{C})$, $n \in \mathbb{N}_0$.

(b) We have a functor

$$\mathrm{Strips}^{\mathrm{co},+}_{\mathrm{per}}\colon \mathbf{TCat}_{(\mathfrak{U})} \to \mathbf{q}_+\mathbf{TCat}_{(\mathfrak{U})},$$

given by

$$\mathrm{Strips}^{\mathrm{co},+}_{\mathrm{per}}(-) = \mathrm{Strips}^{\mathrm{co},+}_{\mathrm{per},\bullet}(-) = -^{\#_+^\bullet}_{\mathbf{TCat}}.$$

For every morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}_{(\mathfrak{U})}$, the morphism $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per}}(F)\colon \operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{C}) \to \operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{D})$ in $\mathbf{q}_+\mathbf{TCat}_{(\mathfrak{U})}$ is given by

$$\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per},n}(F) = \operatorname{Strips}^{\mathrm{co},+}_{n}(F)|^{\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per},n}(\mathcal{D})}_{\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per},n}(\mathcal{C})}$$

for $n \in \mathbb{N}_0$.

Proof.

(b) This follows from remark (4.20) and remark (4.22). □

(4.55) Definition (semiquasicyclic category of (periodic) cosemistrips)**.**

(a) We suppose given a category $\mathcal{C}$. The semiquasicyclic category $\operatorname{Strips}^{\mathrm{co},+}(\mathcal{C}) = \operatorname{Strips}^{\mathrm{co},+}_{\bullet}(\mathcal{C}) = \mathcal{C}^{\#^{\bullet}_{+}}$ as considered in remark (4.54)(a) is called the *semiquasicyclic category of cosemistrips* in $\mathcal{C}$. For $p \in \mathbb{N}_0$, the category with shift $\operatorname{Strips}^{\mathrm{co},+}_{p}(\mathcal{C})$ is called the *category of p-cosemistrips* in $\mathcal{C}$, an object in $\operatorname{Strips}^{\mathrm{co},+}_{p}(\mathcal{C})$ is called a *p-cosemistrip* in $\mathcal{C}$, and a morphism in $\operatorname{Strips}^{\mathrm{co},+}_{p}(\mathcal{C})$ is called a *morphism of p-cosemistrips* in $\mathcal{C}$.

We suppose given a p-cosemistrip X in $\mathcal{C}$ for some $p \in \mathbb{N}_0$. Given $k/i \in \#^{p}_{+}$, we write $X^{k/i}$ for the image of k/i under X. Given $k/i, l/j \in \#^{p}_{+}$ with $k/i \leq l/j$, we write $X^{k/i,l/j}\colon X^{k/i} \to X^{l/j}$ for the image of $(k/i, l/j)$ under X.

(b) We suppose given a category with shift $\mathcal{C}$. The semiquasicyclic category with shift $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{C}) = \operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per},\bullet}(\mathcal{C}) = \mathcal{C}^{\#^{\bullet}_{+}}_{\mathbf{TCat}}$ as considered in remark (4.54)(b) is called the *semiquasicyclic category of periodic cosemistrips* in $\mathcal{C}$. For $p \in \operatorname{Ob}\mathbb{N}_0$, the category with shift $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C})$ is called the *category of periodic p-cosemistrips* in $\mathcal{C}$, an object in $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C})$ is called a *periodic p-cosemistrip* in $\mathcal{C}$, and a morphism in $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C})$ is called a *periodic morphism of p-cosemistrips* in $\mathcal{C}$.

(4.56) Example.

(a) A 3-cosemistrip X in a category $\mathcal{C}$ may be displayed as follows.

$$\begin{tikzcd}
 & & & & & & & \vdots & \vdots & \\
 & & & & & & X^{2^{[1]}/2^{[1]}} \arrow[r] & X^{3^{[1]}/2^{[1]}} \arrow[r] \arrow[u] & X^{0^{[2]}/2^{[1]}} \arrow[r] \arrow[u] & \dots \\
 & & & & & X^{1^{[1]}/1^{[1]}} \arrow[r] & X^{2^{[1]}/1^{[1]}} \arrow[r] \arrow[u] & X^{3^{[1]}/1^{[1]}} \arrow[r] \arrow[u] & X^{0^{[2]}/1^{[1]}} \arrow[r] \arrow[u] & \dots \\
 & & & & X^{0^{[1]}/0^{[1]}} \arrow[r] & X^{1^{[1]}/0^{[1]}} \arrow[r] \arrow[u] & X^{2^{[1]}/0^{[1]}} \arrow[r] \arrow[u] & X^{3^{[1]}/0^{[1]}} \arrow[r] \arrow[u] & X^{0^{[2]}/0^{[1]}} \arrow[u] & \\
 & & & X^{3/3} \arrow[r] & X^{0^{[1]}/3} \arrow[r] \arrow[u] & X^{1^{[1]}/3} \arrow[r] \arrow[u] & X^{2^{[1]}/3} \arrow[r] \arrow[u] & X^{3^{[1]}/3} \arrow[u] & & \\
 & & X^{2/2} \arrow[r] & X^{3/2} \arrow[r] \arrow[u] & X^{0^{[1]}/2} \arrow[r] \arrow[u] & X^{1^{[1]}/2} \arrow[r] \arrow[u] & X^{2^{[1]}/2} \arrow[u] & & & \\
 & X^{1/1} \arrow[r] & X^{2/1} \arrow[r] \arrow[u] & X^{3/1} \arrow[r] \arrow[u] & X^{0^{[1]}/1} \arrow[r] \arrow[u] & X^{1^{[1]}/1} \arrow[u] & & & & \\
X^{0/0} \arrow[r] & X^{1/0} \arrow[r] \arrow[u] & X^{2/0} \arrow[r] \arrow[u] & X^{3/0} \arrow[r] \arrow[u] & X^{0^{[1]}/0} \arrow[u] & & & & &
\end{tikzcd}$$

(b) A periodic 3-cosemistrip X in a category with shift $\mathcal{C}$ may be displayed as follows.

$$\begin{tikzcd}[column sep=small]
& & & & & & & \vdots & \vdots & \\
& & & & & & (X^{2/2})^{[2]} \arrow[r] & (X^{3/2})^{[2]} \arrow[r] \arrow[u] & (X^{2})^{[3]} \arrow[r] \arrow[u] & \ldots \\
& & & & & (X^{1/1})^{[2]} \arrow[r] & (X^{2/1})^{[2]} \arrow[r] \arrow[u] & (X^{3/1})^{[2]} \arrow[r] \arrow[u] & (X^{1})^{[3]} \arrow[r] \arrow[u] & \ldots \\
& & & & (X^{\mathrm{i}})^{[2]} \arrow[r] & (X^{1})^{[2]} \arrow[r] \arrow[u] & (X^{2})^{[2]} \arrow[r] \arrow[u] & (X^{3})^{[2]} \arrow[r] \arrow[u] & (X^{\mathrm{i}})^{[3]} \arrow[u] & \\
& & & X^{3/3} \arrow[r] & (X^{3})^{[1]} \arrow[r] \arrow[u] & (X^{3/1})^{[1]} \arrow[r] \arrow[u] & (X^{3/2})^{[1]} \arrow[r] \arrow[u] & (X^{3/3})^{[1]} \arrow[u] & & \\
& & X^{2/2} \arrow[r] & X^{3/2} \arrow[r] \arrow[u] & (X^{2})^{[1]} \arrow[r] \arrow[u] & (X^{2/1})^{[1]} \arrow[r] \arrow[u] & (X^{2/2})^{[1]} \arrow[u] & & & \\
& X^{1/1} \arrow[r] & X^{2/1} \arrow[r] \arrow[u] & X^{3/1} \arrow[r] \arrow[u] & (X^{1})^{[1]} \arrow[r] \arrow[u] & (X^{1/1})^{[1]} \arrow[u] & & & & \\
X^{\mathrm{i}} \arrow[r] & X^{1} \arrow[r] \arrow[u] & X^{2} \arrow[r] \arrow[u] & X^{3} \arrow[r] \arrow[u] & (X^{\mathrm{i}})^{[1]} \arrow[u] & & & & &
\end{tikzcd}$$

(4.57) Remark. We suppose given a category $\mathcal{C}$. For every morphism of semiquasicyclic types $\alpha\colon \Theta_+^m \to \Theta_+^n$ for $m, n \in \mathbb{N}_0$, the semiquasicyclic operation $\operatorname{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})\colon \operatorname{Strips}_n^{\mathrm{co},+}(\mathcal{C}) \to \operatorname{Strips}_m^{\mathrm{co},+}(\mathcal{C})$ is given on the objects by

$$(\operatorname{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})X)^{k/i} = X^{k\alpha/i\alpha}$$

for $k/i \in \#_+^m$ and

$$(\operatorname{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})X)^{k/i,l/j} = X^{k\alpha/i\alpha,l\alpha/j\alpha}$$

for $k/i, l/j \in \#_+^m$ with $k/i \leq l/j$, $X \in \operatorname{Ob}\operatorname{Strips}_n^{\mathrm{co},+}(\mathcal{C})$, and on the morphisms by

$$(\operatorname{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})f)^{k/i} = f^{k\alpha/i\alpha}$$

for $k/i \in \#_+^m$, $f \in \operatorname{Mor}\operatorname{Strips}_n^{\mathrm{co},+}(\mathcal{C})$.

Proof.

(a) We have

$$\begin{aligned}(\operatorname{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})X)^{k/i,l/j} &= (\mathcal{C}^{\#_+^\alpha}(X))^{k/i,l/j} = (X \circ \#_+^\alpha)^{k/i,l/j} = X^{(k/i,l/j)\#_+^\alpha} = X^{(k/i)\#_+^\alpha,(l/j)\#_+^\alpha} \\ &= X^{k\alpha/i\alpha,l\alpha/j\alpha}\end{aligned}$$

for $k/i, l/j \in \#_+^m$ with $k/i \leq l/j$, $X \in \operatorname{Ob}\operatorname{Strips}_n^{\mathrm{co},+}(\mathcal{C})$, and we have

$$(\operatorname{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})f)^{k/i} = (\mathcal{C}^{\#_+^\alpha}(f))^{k/i} = (f * \#_+^\alpha)^{k/i} = f^{(k/i)\#_+^\alpha} = f^{k\alpha/i\alpha}$$

for $k/i \in \#_+^m$, $f \in \operatorname{Mor}\operatorname{Strips}_n^{\mathrm{co},+}(\mathcal{C})$. □

(4.58) Remark. We suppose given a category with shift $\mathcal{C}$ and a $p \in \mathbb{N}_0$.

(a) A p-cosemistrip X in $\mathcal{C}$ is periodic if and only if

$$X^{i^{[1]}/k,j^{[1]}/l} = (X^{k/i,l/j})^{[1]}$$

for $k/i, l/j \in \#_+^p$ with $k/i \leq l/j$.

(b) A morphism of p-cosemistrips $f\colon X \to Y$ in $\mathcal{C}$ is periodic if and only if X, Y are periodic and

$$f^{i^{[1]}/k} = (f^{k/i})^{[1]}$$

for $k/i \in \#_+^p$.

Proof. This follows from remark (4.19). □

(4.59) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$ that contains $\mathbb{N}_0$.

(a) The functor $\mathrm{Strips}^{\mathrm{co},+}\colon \mathbf{Cat}_{(\mathfrak{U})} \to \mathbf{q}_+\mathbf{Cat}_{(\mathfrak{U})}$ induces a functor

$$\mathrm{Strips}^{\mathrm{co},+}\colon \mathbf{Cat}_{0,(\mathfrak{U})} \to \mathbf{q}_+\mathbf{Cat}_{0,(\mathfrak{U})},$$

where $0^{\mathrm{Strips}_n^{\mathrm{co},+}(\mathcal{C})}$ for $p \in \mathbb{N}_0$ is given by $(0^{\mathrm{Strips}_p^{\mathrm{co},+}(\mathcal{C})})^{k/i} = 0^{\mathcal{C}}$ for $k/i \in \#_+^p$.

(b) The functor $\mathrm{Strips}_{\mathrm{per}}^{\mathrm{co},+}\colon \mathbf{TCat}_{(\mathfrak{U})} \to \mathbf{q}_+\mathbf{TCat}_{(\mathfrak{U})}$ induces a functor

$$\mathrm{Strips}_{\mathrm{per}}^{\mathrm{co},+}\colon \mathbf{TCat}_{0,(\mathfrak{U})} \to \mathbf{q}_+\mathbf{TCat}_{0,(\mathfrak{U})},$$

where $0^{\mathrm{Strips}_{\mathrm{per},p}^{\mathrm{co},+}(\mathcal{C})} = 0^{\mathrm{Strips}_p^{\mathrm{co},+}(\mathcal{C})}$ for $p \in \mathbb{N}_0$.

Proof.

(b) This follows from remark (4.23). □

The following proposition explains in which sense an p-cosemistrip resp. a morphism of p-cosemistrips for some $p \in \mathbb{N}_0$ is periodic:

(4.60) Proposition. We suppose given a category with shift $\mathcal{C}$ and a $p \in \mathbb{N}_0$.

(a) For every $\#_0^p$-commutative diagram X_0 in $\mathcal{C}$ with $X_0^{i^{[1]},j^{[1]}} = (X_0^{i,j})^{[1]}$ for $i, j \in \Theta_0^p$ with $i \leq j$ there exists a unique periodic p-cosemistrip X in $\mathcal{C}$ with $X_0 = X|_{\#_0^p}$, given by

$$X^{(k/i)^{[m]}} = (X_0^{k/i})^{[m]}$$

for $k/i \in \#_0^p$, $m \in \mathbb{N}_0$, and by

$$X^{(k/i)^{[m]},(l/j)^{[n]}} = \begin{cases} 1_{(X_0^{\mathrm{i}})^{[m]}} & \text{if } m = n+2, \\ (X_0^{k,j})^{[m]} & \text{if } m = n+1, \\ (X_0^{k/i,l/j})^{[m]} & \text{if } m = n, \\ (X_0^{k/i,i_{m+1}^{[1]}})^{[m]} \, (\bullet_{r \in \lceil m+1,n-1 \rceil} (X_0^{i_r,i_{r+1}^{[1]}})^{[r]}) \, (X_0^{i_n,l/j})^{[n]} & \text{if } m < n, \end{cases}$$

for $k/i, l/j \in \#_0^p$, $m, n \in \mathbb{N}_0$ with $(k/i)^{[m]} \leq (l/j)^{[n]}$, and for arbitrarily chosen $i_r \in \Theta_0^p$, $r \in [m+1, n]$, with $k/i \leq i_{m+1}^{[1]}$ and $i_n \leq l/j$ in the case $m < n$.

(b) We suppose given periodic p-cosemistrips X, Y in $\mathcal{C}$. For every morphism of $\#_0^p$-commutative diagrams $f_0\colon X|_{\#_0^p} \to Y|_{\#_0^p}$ with $f_0^{i^{[1]}} = (f_0^i)^{[1]}$ for $i \in \Theta_0^p$ there exists a unique periodic morphism of p-cosemistrips $f\colon X \to Y$ in $\mathcal{C}$ with $f_0 = f|_{\#_0^p}$, given by

$$f^{(k/i)^{[m]}} = (f_0^{k/i})^{[m]}$$

for $k/i \in \#_0^p$, $m \in \mathbb{N}_0$.

Proof.

(a) This is a reformulation of proposition (4.35).

(b) We suppose given a morphism of $\#_0^p$-commutative diagrams $f_0\colon X|_{\#_0^p} \to Y|_{\#_0^p}$ such that $f_0^{i^{[1]}} = (f_0^i)^{[1]}$ for all $i \in \Theta_0^p$. Then we obtain a $\#_0^p$-commutative diagram H_0 in $\mathcal{C}^{\Delta^1}$ with $\mathrm{Source} \circ H_0 = X|_{\#_0^p}$ and $\mathrm{Target} \circ H_0 = Y|_{\#_0^p}$, given by $(H_0^{k/i})_{0,1} = f_0^{k/i}$ for $k/i \in \#_0^p$ and by $H_0^{k/i,l/j} = ((X|_{\#_0^p})^{k/i,l/j}, (Y|_{\#_0^p})^{k/i,l/j}) = (X^{k/i,l/j}, Y^{k/i,l/j})$ for $k/i, l/j \in \#_0^p$ with $k/i \leq l/j$.

$$\begin{array}{ccc} X^{k/i} & \longrightarrow & X^{l/j} \\ {\scriptstyle f_0^{k/i}} \big\downarrow & & \big\downarrow {\scriptstyle f_0^{l/j}} \\ Y^{k/i} & \longrightarrow & Y^{l/j} \end{array}$$

Since $f_0^{i^{[1]}} = (f_0^i)^{[1]}$ for $i \in \Theta_0^p$, we have

$$H_0^{i^{[1]},j^{[1]}} = (X^{i^{[1]},j^{[1]}}, Y^{i^{[1]},j^{[1]}}) = ((X^{i,j})^{[1]}, (Y^{i,j})^{[1]}) = (X^{i,j}, Y^{i,j})^{[1]} = (H_0^{i,j})^{[1]}$$

for $i, j \in \Theta_0^p$ with $i \leq j$. So by (a) there exists a unique periodic p-cosemistrip H in $\mathcal{C}^{\Delta^1}$ with $H_0 = H|_{\#_0^p}$. As

$$(\mathrm{Source} \circ H)|_{\#_0^p} = \mathrm{Source} \circ H|_{\#_0^p} = \mathrm{Source} \circ H_0 = X|_{\#_0^p},$$
$$(\mathrm{Target} \circ H)|_{\#_0^p} = \mathrm{Target} \circ H|_{\#_0^p} = \mathrm{Target} \circ H_0 = Y|_{\#_0^p},$$

it follows that $\mathrm{Source} \circ H = X$ and $\mathrm{Target} \circ H = Y$ by (a). So we obtain a morphism of p-cosemistrips $f \colon X \to Y$, given by

$$f^{(k/i)^{[m]}} = (H^{(k/i)^{[m]}})_{0,1} = (H_0^{k/i})_{0,1}^{[m]} = (f_0^{k/i})^{[m]}$$

for $k/i \in \#_0^p$, $m \in \mathbb{N}_0$. Moreover, we have

$$f^{(k/i)^{[1]}} = (H^{(k/i)^{[1]}})_{0,1} = (H^{k/i})_{0,1}^{[1]} = (f^{k/i})^{[1]}$$

for $k/i \in \#_+^p$, that is, f is periodic, and we have $(\mathrm{Source} \circ H)|_{\#_0^p} = X|_{\#_0^p}$, $(\mathrm{Target} \circ H)|_{\#_0^p} = Y|_{\#_0^p}$ and

$$f^{k/i} = (H^{k/i})_{0,1} = (H_0^{k/i})_{0,1} = f_0^{k/i}$$

for $k/i \in \#_0^p$, that is, $f|_{\#_0^p} = f_0$.

Conversely, we suppose given an arbitrary periodic morphism of p-cosemistrips $g \colon X \to Y$ in $\mathcal{C}$ with $f_0 = g|_{\#_0^p}$. We obtain a p-cosemistrip K in $\mathcal{C}^{\Delta^1}$ with $\mathrm{Source} \circ K = X$ and $\mathrm{Target} \circ K = Y$, given by $(K^{k/i})_{0,1} = g^{k/i}$ for $k/i \in \#_+^p$ and by $K^{k/i,l/j} = (X^{k/i,l/j}, Y^{k/i,l/j})$ for $k/i, l/j \in \#_+^p$ with $k/i \leq l/j$.

$$\begin{tikzcd} X^{k/i} \arrow[r] \arrow[d, "g^{k/i}"'] & X^{l/j} \arrow[d, "g^{l/j}"] \\ Y^{k/i} \arrow[r] & Y^{l/j} \end{tikzcd}$$

The n-cosemistrip K is periodic since

$$\begin{aligned} K^{(k/i)^{[1]},(l/j)^{[1]}} &= (X^{(k/i)^{[1]},(l/j)^{[1]}}, Y^{(k/i)^{[1]},(l/j)^{[1]}}) = ((X^{k/i,l/j})^{[1]}, (Y^{k/i,l/j})^{[1]}) \\ &= (X^{k/i,l/j}, Y^{k/i,l/j})^{[1]} = (K^{k/i,l/j})^{[1]} \end{aligned}$$

for $k/i, l/j \in \#_+^p$ with $k/i \leq l/j$. Moreover, we have

$$K^{k/i,l/j} = (X^{k/i,l/j}, Y^{k/i,l/j}) = H_0^{k/i,l/j}$$

for $k/i, l/j \in \#_0^p$ with $k/i \leq l/j$, that is, $K|_{\#_0^p} = H_0$. Thus we have $K = H$, and therefore $g = f$. □

The semiquasicyclic categories of cosemicomplexes and periodic cosemicomplexes

Finally, we will introduce cosemicomplexes, that is, cosemistrips with zeros at the "boundaries".

(4.61) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$ that contains $\mathbb{N}_0$.

(a) We have a functor

$$\mathrm{Com}^{\mathrm{co},+} \colon \mathbf{Cat}_{0,(\mathfrak{U})} \to \mathbf{q}_+\mathbf{Cat}_{0,(\mathfrak{U})},$$

given as follows. For $\mathcal{C} \in \mathrm{Ob}\,\mathbf{Cat}_{0,(\mathfrak{U})}$, the semiquasicyclic zero-pointed category $\mathrm{Com}^{\mathrm{co},+}(\mathcal{C})$ is the full semiquasicyclic zero-pointed subcategory of $\mathrm{Strips}^{\mathrm{co},+}(\mathcal{C})$ given by

$$\mathrm{Ob}\,\mathrm{Com}_p^{\mathrm{co},+}(\mathcal{C}) = \{X \in \mathrm{Ob}\,\mathrm{Strips}_p^{\mathrm{co},+}(\mathcal{C}) \mid X^{(i/i)^{[m]}} = 0 \text{ for } i \in \Theta_0^p,\, m \in \mathbb{N}_0\}$$

for $p \in \mathbb{N}_0$. For every morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}_{0,(\mathfrak{U})}$, the morphism $\mathrm{Com}^{\mathrm{co},+}(F)\colon \mathrm{Com}^{\mathrm{co},+}(\mathcal{C}) \to \mathrm{Com}^{\mathrm{co},+}(\mathcal{D})$ in $\mathbf{q}_+\mathbf{Cat}_{0,(\mathfrak{U})}$ is given by

$$\mathrm{Com}^{\mathrm{co},+}_p(F) = \mathrm{Strips}^{\mathrm{co},+}_p(F)|^{\mathrm{Com}^{\mathrm{co},+}_p(\mathcal{D})}_{\mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C})}$$

for $p \in \mathbb{N}_0$.

(b) We have a functor

$$\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per}}\colon \mathbf{TCat}_{0,(\mathfrak{U})} \to \mathbf{q}_+\mathbf{TCat}_{0,(\mathfrak{U})},$$

given as follows. For $\mathcal{C} \in \operatorname{Ob} \mathbf{TCat}_{0,(\mathfrak{U})}$, the semiquasicyclic zero-pointed category with shift $\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{C})$ is the full semiquasicyclic zero-pointed subcategory with shift of $\mathrm{Strips}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{C})$ given by

$$\operatorname{Ob} \mathrm{Com}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C}) = \operatorname{Ob} \mathrm{Strips}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C}) \cap \operatorname{Ob} \mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C})$$

for $p \in \mathbb{N}_0$. For every morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{TCat}_{0,(\mathfrak{U})}$, the morphism $\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per}}(F)\colon \mathrm{Com}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{C}) \to \mathrm{Com}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{D})$ in $\mathbf{q}_+\mathbf{TCat}_{0,(\mathfrak{U})}$ is given by

$$\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per},p}(F) = \mathrm{Strips}^{\mathrm{co},+}_{\mathrm{per},p}(F)|^{\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{D})}_{\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per},n}(\mathcal{C})}$$

for $p \in \mathbb{N}_0$.

Proof.

(a) First, we suppose given an object $\mathcal{C}$ in $\mathbf{Cat}_0$. Given a morphism of semiquasicyclic types $\alpha\colon \Theta^p_+ \to \Theta^q_+$ for $p, q \in \mathbb{N}_0$ and $X \in \operatorname{Ob} \mathrm{Strips}^{\mathrm{co},+}_q(\mathcal{C})$ such that $X^{(j/j)^{[n]}} = 0$ for $j \in \Theta^q_0$, $n \in \mathbb{N}_0$, we also have

$$(\mathrm{Strips}^{\mathrm{co},+}_\alpha(\mathcal{C})X)^{(i/i)^{[m]}} = X^{(i\alpha/i\alpha)^{[m]}} = 0$$

for $i \in \Theta^p_0$, $m \in \mathbb{N}_0$. Thus we have a full semiquasicyclic zero-pointed subcategory $\mathrm{Com}^{\mathrm{co},+}(\mathcal{C})$ of $\mathrm{Strips}^{\mathrm{co},+}(\mathcal{C})$ given by

$$\operatorname{Ob} \mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C}) = \{X \in \operatorname{Ob} \mathrm{Strips}^{\mathrm{co},+}_P(\mathcal{C}) \mid X^{(i/i)^{[m]}} = 0 \text{ for } i \in \Theta^p_0,\ m \in \mathbb{N}_0\}$$

for $p \in \mathbb{N}_0$.

Next, we suppose given a morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}_0$. We have to show that the morphisms of zero-pointed categories with shift $\mathrm{Strips}^{\mathrm{co},+}_p(F)\colon \mathrm{Strips}^{\mathrm{co},+}_p(\mathcal{C}) \to \mathrm{Strips}^{\mathrm{co},+}_p(\mathcal{D})$ for $p \in \mathbb{N}_0$ map objects in $\mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C})$ to objects in $\mathrm{Com}^{\mathrm{co},+}_p(\mathcal{D})$. So we suppose given $p \in \mathbb{N}_0$ and $X \in \operatorname{Ob} \mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C})$, so that $X^{(i/i)^{[m]}} = 0^{\mathcal{C}}$ for $i \in \Theta^p_0$, $m \in \mathbb{N}_0$. As F is a morphism of zero-pointed categories, it follows that

$$(\mathrm{Strips}^{\mathrm{co},+}_P(F)X)^{(i/i)^{[m]}} = FX^{(i/i)^{[m]}} = F0^{\mathcal{C}} = 0^{\mathcal{D}}$$

for $i \in \Theta^p_0$, $m \in \mathbb{N}_0$. Thus $\mathrm{Strips}^{\mathrm{co},+}_P(F)X \in \operatorname{Ob} \mathrm{Com}^{\mathrm{co},+}_P(\mathcal{D})$. As $X \in \operatorname{Ob} \mathrm{Com}^{\mathrm{co},+}_P(\mathcal{C})$ was arbitrary, it follows that the restriction $\mathrm{Strips}^{\mathrm{co},+}_P(F)|^{\mathrm{Com}^{\mathrm{co},+}_P(\mathcal{D})}_{\mathrm{Com}^{\mathrm{co},+}_P(\mathcal{C})}$ exists.

The functoriality of $\mathrm{Com}^{\mathrm{co},+}$ follows from the functoriality of $\mathrm{Strips}^{\mathrm{co},+}\colon \mathbf{Cat}_0 \to \mathbf{q}_+\mathbf{Cat}_0$, see remark (4.54)(a).

(b) This follows from (a) and remark (4.54)(b). □

(4.62) Definition (semiquasicyclic category of (periodic) cosemicomplexes)**.**

(a) We suppose given a zero-pointed category $\mathcal{C}$. The full semiquasicyclic zero-pointed subcategory $\mathrm{Com}^{\mathrm{co},+}(\mathcal{C})$ of $\mathrm{Strips}^{\mathrm{co},+}(\mathcal{C})$ as considered in remark (4.61)(a) is called the *semiquasicyclic category of cosemicomplexes* in $\mathcal{C}$. For $p \in \mathbb{N}_0$, the zero-pointed category $\mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C})$ is called the *category of p-cosemicomplexes* in $\mathcal{C}$, an object in $\mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C})$ is called a *p-cosemicomplex* in $\mathcal{C}$, and a morphism in $\mathrm{Com}^{\mathrm{co},+}_p(\mathcal{C})$ is called a *morphism of p-cosemicomplexes* in $\mathcal{C}$.

(b) We suppose given a zero-pointed category with shift $\mathcal{C}$. The full semiquasicyclic subcategory with shift $\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{C})$ of $\mathrm{Strips}^{\mathrm{co},+}_{\mathrm{per}}(\mathcal{C})$ as considered in remark (4.61)(b) is called the *semiquasicyclic category of periodic cosemicomplexes* in $\mathcal{C}$. For $p \in \mathbb{N}_0$, the category with shift $\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C})$ is called the *category of periodic p-cosemicomplexes* in $\mathcal{C}$, an object in $\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C})$ is called a *periodic p-cosemicomplex* in $\mathcal{C}$, and a morphism in $\mathrm{Com}^{\mathrm{co},+}_{\mathrm{per},p}(\mathcal{C})$ is called a *periodic morphism of p-cosemicomplexes* in $\mathcal{C}$.

(4.63) Example.

(a) A 3-cosemicomplex X in a zero-pointed category $\mathcal{C}$ may be displayed as follows.

$$\begin{tikzcd}
&&&&&&& \vdots & \vdots \\
&&&&&& 0 \arrow[r] & X^{3^{[1]}/2^{[1]}} \arrow[r] \arrow[u] & X^{0^{[2]}/2^{[1]}} \arrow[r] \arrow[u] & \dots \\
&&&&& 0 \arrow[r] & X^{2^{[1]}/1^{[1]}} \arrow[r] \arrow[u] & X^{3^{[1]}/1^{[1]}} \arrow[r] \arrow[u] & X^{0^{[2]}/1^{[1]}} \arrow[r] \arrow[u] & \dots \\
&&&& 0 \arrow[r] & X^{1^{[1]}/0^{[1]}} \arrow[r] \arrow[u] & X^{2^{[1]}/0^{[1]}} \arrow[r] \arrow[u] & X^{3^{[1]}/0^{[1]}} \arrow[r] \arrow[u] & 0 \arrow[u] \\
&&& 0 \arrow[r] & X^{0^{[1]}/3} \arrow[r] \arrow[u] & X^{1^{[1]}/3} \arrow[r] \arrow[u] & X^{2^{[1]}/3} \arrow[r] \arrow[u] & 0 \arrow[u] \\
&& 0 \arrow[r] & X^{3/2} \arrow[r] \arrow[u] & X^{0^{[1]}/2} \arrow[r] \arrow[u] & X^{1^{[1]}/2} \arrow[r] \arrow[u] & 0 \arrow[u] \\
& 0 \arrow[r] & X^{2/1} \arrow[r] \arrow[u] & X^{3/1} \arrow[r] \arrow[u] & X^{0^{[1]}/1} \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] & X^{1/0} \arrow[r] \arrow[u] & X^{2/0} \arrow[r] \arrow[u] & X^{3/0} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

(b) A periodic 3-cosemicomplex X in a zero-pointed category with shift $\mathcal{C}$ may be displayed as follows.

$$\begin{tikzcd}
&&&&&&& \vdots & \vdots \\
&&&&&& 0 \arrow[r] & (X^{3/2})^{[2]} \arrow[r] \arrow[u] & (X^{2})^{[3]} \arrow[r] \arrow[u] & \dots \\
&&&&& 0 \arrow[r] & (X^{2/1})^{[2]} \arrow[r] \arrow[u] & (X^{3/1})^{[2]} \arrow[r] \arrow[u] & (X^{1})^{[3]} \arrow[r] \arrow[u] & \dots \\
&&&& 0 \arrow[r] & (X^{1})^{[2]} \arrow[r] \arrow[u] & (X^{2})^{[2]} \arrow[r] \arrow[u] & (X^{3})^{[2]} \arrow[r] \arrow[u] & 0 \arrow[u] \\
&&& 0 \arrow[r] & (X^{3})^{[1]} \arrow[r] \arrow[u] & (X^{3/1})^{[1]} \arrow[r] \arrow[u] & (X^{3/2})^{[1]} \arrow[r] \arrow[u] & 0 \arrow[u] \\
&& 0 \arrow[r] & X^{3/2} \arrow[r] \arrow[u] & (X^{2})^{[1]} \arrow[r] \arrow[u] & (X^{2/1})^{[1]} \arrow[r] \arrow[u] & 0 \arrow[u] \\
& 0 \arrow[r] & X^{2/1} \arrow[r] \arrow[u] & X^{3/1} \arrow[r] \arrow[u] & (X^{1})^{[1]} \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] & X^{1} \arrow[r] \arrow[u] & X^{2} \arrow[r] \arrow[u] & X^{3} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

By proposition (4.60), a periodic p-cosemistrip for some $p \in \mathbb{N}_0$ is uniquely determined by its values on $\#^p_0$.

(4.64) Remark. We suppose given a zero-pointed category with shift $\mathcal{C}$ and a $p \in \mathbb{N}_0$. A periodic p-cosemistrip X is a p-cosemicomplex if and only if $X^{i/i} = 0$ for $i \in \Theta^p_0$.

Proof. If X is a p-cosemicomplex, then in particular $X^{i/i} = 0$ for $i \in \Theta^p_0$. Conversely, if we have $X^{i/i} = 0$ for $i \in \Theta^p_0$, then we also have

$$X^{(i/i)^{[m]}} = (X^{i/i})^{[m]} = 0^{[m]} = 0$$

for $i \in \Theta^p_0$, $m \in \mathbb{N}_0$ as X is periodic and $\mathrm{T}^{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is a morphism of zero-pointed categories, and so X is a p-cosemicomplex in $\mathcal{C}$. $\square$

Chapter V

The triangulated structure

We suppose given a zero-pointed Brown cofibration category $\mathcal{C}$, that is, a Brown cofibration category as in definition (3.52)(a) that is equipped with a (distinguished) zero object. Brown has shown in [7, dual of th. 3] that the homotopy category $\operatorname{Ho}\mathcal{C}$, see definition (3.8), carries the structure of a category with shift as introduced in definition (4.5)(c). If $\mathcal{C}$ is *stable*, that is, if the shift on $\operatorname{Ho}\mathcal{C}$ is invertible, then $\operatorname{Ho}\mathcal{C}$ becomes a triangulated category in the sense of Verdier [37, ch. I, §1, n° 1, sec. 1-1], as proven in this generality by Schwede [33, th. A.12]. In the case where the shift is not necessarily invertible, a variant of this structure involving *homotopy cofibre sequences* was already studied by Brown [7, dual of sec. 4, pp. 430–434].

In this chapter, we construct an unstable analogon of higher triangles in the homotopy category, called *cosemitriangles*, in the spirit of Künzer [22, def. 2.1.2] and Maltsiniotis [25, sec. 1.4]. We show that these cosemitriangles may be organised in a semiquasicyclic category in the sense of definition (4.38) and that prolongation properties analogous to those for the Verdier triangles in a Verdier triangulated category hold, see theorem (5.55). For more detailed explanations, see the introduction, section 2. A key tool on our way is proposition (5.53) due to Cisinski [9, prop. 2.15], which roughly states that the objects resp. morphisms in $(\operatorname{Ho}\mathcal{C})^{\dot{\Delta}^n}$ for some $n \in \mathbb{N}_0$, cf. notation (4.30), may be strictified to objects resp. S-2-arrows in $\mathcal{C}^{\dot{\Delta}^n}$. We give a new proof for this result using the Z-2-arrow calculus (3.128). We do *not* show an analogon to the rotation axiom in a Verdier triangulated category.

A comment on the terminology: While Verdier triangulated categories are self-dual, the "higher unstable triangles" on $\operatorname{Ho}\mathcal{C}$, which we call cosemitriangles, are of course not. The "semi" in cosemitriangles should indicate that they are only defined in a "positive area of the plane", cf. example (4.63)(b), using only non-negative powers of the shift on $\operatorname{Ho}\mathcal{C}$. The "co" refers to the direction of the arrows in the cosemitriangle; the "higher unstable triangles" in the homotopy category of a zero-pointed Brown fibration category would be called *semitriangles*. For "unstable Verdier triangles" in an additive framework, see also the work of Keller and Vossieck [21] and, independently, Beligiannis and Marmaridis [6].

The chapter is organised as follows. In section 1, we study cones, which are convenient models for a morphism to the zero object in the homotopy category. Using this, we introduce Coheller rectangles in section 2 and construct the shift on the homotopy category by a choice of such Coheller rectangles. Moreover, we define the Coheller construction, see definition (5.22), which is a choice-free variant of the shift. Finally, we study the models for cosemitriangles in section 3 and the cosemitriangles in section 4. The prolongation results for cosemitriangles can be found in theorem (5.55).

1 Cones

Throughout this section, we suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that has a zero object, cf. definition (3.30)(a).

In the construction of the Coheller shift and the cosemitriangles on the homotopy category $\operatorname{Ho}\mathcal{C}$, cf. definition (5.28) and definition (5.51), cones play a prominent role. The main property of cones is that they are suitable replacements in $\mathcal{C}$ for a morphism to the zero object in $\operatorname{Ho}\mathcal{C}$, cf. remark (5.7).

Coacyclic objects

(5.1) Definition (coacyclic object)**.** An object A in $\mathcal{C}$ is said to be *coacyclic* if it is cofibrant and if there exists a zero object N in $\mathcal{C}$ such that $\mathrm{ini}_A^N \colon N \to A$ is a weak equivalence.

(5.2) Remark. We suppose given an coacyclic object A in $\mathcal{C}$. For every zero object N in $\mathcal{C}$, the unique morphism $\mathrm{ini}_A^N \colon N \to A$ is an acyclic cofibration.

Proof. We suppose given a zero object N in $\mathcal{C}$. As A is acyclic, there exists a zero object $\tilde{N}$ such that $\mathrm{ini}_A^{\tilde{N}} \colon \tilde{N} \to A$ is a weak equivalence. Moreover, $\mathrm{ini}_A^{\tilde{N}}$ is a cofibration by remark (3.21). But N is cofibrant, whence the canonical isomorphism $\mathrm{ini}_{\tilde{N}}^N \colon N \to \tilde{N}$ is an acyclic cofibration. But then $\mathrm{ini}_A^N = \mathrm{ini}_{\tilde{N}}^N \mathrm{ini}_A^{\tilde{N}}$ is an acyclic cofibration by closedness under composition. □

(5.3) Remark. Every coacyclic object in $\mathcal{C}$ is a zero object in $\mathrm{Ho}\,\mathcal{C}$.

(5.4) Remark. We suppose given a coacyclic object A in $\mathcal{C}$. If $\mathcal{C}$ fulfills the incision axiom or the excision axiom, then $\mathrm{emb}_1 \colon X \to X \amalg A$ is an acyclic cofibration for every cofibrant object X in $\mathcal{C}$.

Proof. This holds as

$$\begin{tikzcd}
A \arrow[r, "\mathrm{emb}_2"] & X \amalg A \\
0 \arrow[u, "\wr"] \arrow[r] & X \arrow[u, "\mathrm{emb}_1"']
\end{tikzcd}$$

is a pushout rectangle in $\mathcal{C}$. □

(5.5) Remark. We suppose given a morphism $f \colon X \to Y$ in $\mathcal{C}_{\mathrm{cof}}$.

(a) If X is coacyclic and f is a weak equivalence, then Y is coacyclic.

(b) We suppose that $\mathcal{C}$ is T-semisaturated. If Y is coacyclic and f is a weak equivalence, then X.

(c) We suppose that $\mathcal{C}$ is S-semisaturated. If X and Y are coacyclic, then f is a weak equivalence.

Proof. We have $\mathrm{ini}_X f = \mathrm{ini}_Y$, that is, the following diagram commutes.

$$\begin{tikzcd}
& 0 \arrow[dl, "\mathrm{ini}_X"'] \arrow[dr, "\mathrm{ini}_Y"] & \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

(a) If ini_X and f are weak equivalences, then ini_Y is a weak equivalence by multiplicativity. That is, if X is coacyclic and f is a weak equivalence, then Y is coacyclic.

(b) If ini_Y and f are weak equivalences, then ini_X is a weak equivalence by T-semisaturatedness. That is, if Y is coacyclic and f is a weak equivalence, then X is coacyclic.

(c) If X and Y are coacyclic, that is, if ini_X and ini_Y are weak equivalences, then f is a weak equivalence by S-semisaturatedness. □

Definition of cones

With the notion of a coacyclic object at hand, we are able to define cones:

(5.6) Definition (cone)**.** We suppose given a cofibrant object X in $\mathcal{C}$. A *cone* of X consists of a coacyclic object C in $\mathcal{C}$ together with a cofibration $i \colon X \to C$. By abuse of notation, we refer to the said cone as well as to its underlying object by C. The cofibration i is called the *insertion* of C. Given a cone C of X with insertion i, we write $\mathrm{ins} = \mathrm{ins}^C := i$.

Cones in $\mathcal{C}$ may be seen as cofibrant models (in the sense of the Reedy structure (3.88)(b)) for morphisms to the zero object in the homotopy category:

(5.7) Remark. Given a cofibrant object X and a cone C of X in $\mathcal{C}$, then $(\mathrm{ins}, \mathrm{ini}_C)$ is a Z-2-arrow in $\mathcal{C}$ and we have

$$\mathrm{ter}_X = \mathrm{loc}(\mathrm{ins})\,\mathrm{loc}(\mathrm{ini}_C)^{-1}$$

in $\mathrm{Ho}\,\mathcal{C}$.

Proof. This holds as the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ins}", tail] \arrow[d, equal] & C \arrow[d] & 0 \arrow[l, tail, "\approx"'] \arrow[d, equal] \\
X \arrow[r, "\mathrm{ter}_X"] & 0 \arrow[r, equal] & 0
\end{tikzcd}$$

□

(5.8) Remark. We suppose given a cofibrant object X in $\mathcal{C}$. Moreover, we suppose given a cofibration $i \colon X \to C$ in $\mathcal{C}$ such that $\mathrm{ter}_C \colon C \to 0$ is a weak equivalence in $\mathcal{C}$. If $\mathcal{C}$ is T-semisaturated, then C becomes a cone of X with $\mathrm{ins}^C = i$.

Proof. As $\mathrm{ter}_C \colon C \to 0$ is a weak equivalence, it follows that $\mathrm{ini}_C \colon 0 \to C$ is a weak equivalence by T-semisaturatedness, that is, C is coacyclic. Thus C becomes a cone of X with $\mathrm{ins}^C = i$. □

Cones behave somehow like injective objects. More precisely, we have the following lemma. For the formulation of the factorisation axiom for cofibrations, see definition (3.40).

(5.9) Lemma (cf. [17, lem. 5.2])**.** We suppose that $\mathcal{C}$ is T-semisaturated and fulfills the factorisation axiom for cofibrations. Moreover, we suppose given an S-2-arrow $(f, u) \colon X \to \tilde{Y} \leftarrow Y$ and a cofibration $i \colon X \to X'$ in $\mathcal{C}_{\mathrm{cof}}$. For every cone C of Y there exist a cone $\tilde{C}$ of $\tilde{Y}$ and an S-2-arrow $(g, v) \colon X' \to \tilde{C} \leftarrow C$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "i", tail] \arrow[d, "f"] & X' \arrow[d, dotted, "g"] \\
\tilde{Y} \arrow[r, dotted, tail, "\mathrm{ins}^{\tilde{C}}"] & \tilde{C} \\
Y \arrow[u, "u", "\approx"'] \arrow[r, tail, "\mathrm{ins}^C"] & C \arrow[u, dotted, "v", "\approx"']
\end{tikzcd}$$

commutes and such that the following quadrangle is coreedian.

$$\begin{tikzcd}
X \amalg Y \arrow[r, tail, "i \amalg \mathrm{ins}^C"] \arrow[d, "\binom{f}{u}"'] & X' \amalg C \arrow[d, "\binom{g}{v}"] \\
\tilde{Y} \arrow[r, tail, "\mathrm{ins}^{\tilde{C}}"] & \tilde{C}
\end{tikzcd}$$

Proof. By the factorisation lemma (3.65)(b), there exist a cofibration $\tilde{j} \colon \tilde{Y} \to \tilde{C}$ and an S-2-arrow $(g, v)\colon X' \to \tilde{C} \leftarrow C$ in $\mathcal{C}$ such that $\mathrm{ter}_{\tilde{C}} \colon \tilde{C} \to 0$ is a weak equivalence, such that

$$\begin{tikzcd}
X \amalg Y \arrow[r, tail, "i \amalg \mathrm{ins}^C"] \arrow[d, "\binom{f}{u}"'] & X' \amalg C \arrow[d, "\binom{g}{v}"] \\
\tilde{Y} \arrow[r, tail, "\tilde{j}"] & \tilde{C}
\end{tikzcd}$$

is coreedian and such that the diagram

$$\begin{tikzcd}
 & X' \arrow[dd, dotted, "g"] \arrow[dr, equal] & \\
X \arrow[ur, "i"] \arrow[rr, "i"] \arrow[dd, "f"'] & & X' \arrow[dd] \\
 & \tilde{C} \arrow[dr, dotted, "\wr"] & \\
\tilde{Y} \arrow[ur, dotted, "\tilde{j}"] \arrow[rr] & & 0 \arrow[dd, equal] \\
 & C \arrow[uu, dotted, "\wr", "v"'] \arrow[dr, "\wr"] & \\
Y \arrow[uu, "u", "\wr"'] \arrow[ur, "\mathrm{ins}^C"] \arrow[rr] & & 0
\end{tikzcd}$$

commutes. But then $\tilde{C}$ becomes a formal cone of $\tilde{Y}$ with $\mathrm{ins}^{\tilde{C}} = \tilde{j}$ by remark (5.8). □

(5.10) Corollary (Heller factorisation lemma)**.** We suppose that $\mathcal{C}$ is T-semisaturated and fulfills the factorisation axiom for cofibrations.

(a) There exists a cone of every cofibrant object in $\mathcal{C}$.

(b) We suppose given an S-2-arrow $(f, u)\colon X_1 \to \tilde{X}_2 \leftarrow X_2$ in $\mathcal{C}_{\mathrm{cof}}$. For all cones C_1 of X_1 and C_2 of X_2, there exist a cone $\tilde{C}_2$ of $\tilde{X}_2$ and an S-2-arrow $(g, v)\colon C_1 \to \tilde{C}_2 \leftarrow C_2$ such that the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\mathrm{ins}^{C_1}"] \arrow[d, "f"'] & C_1 \arrow[d, dotted, "g", "\wr"'] \\
\tilde{X}_2 \arrow[r, dotted, "\mathrm{ins}^{\tilde{C}_2}"] & \tilde{C}_2 \\
X_2 \arrow[u, "u", "\wr"'] \arrow[r, "\mathrm{ins}^{C_2}"] & C_2 \arrow[u, dotted, "v"', "\wr"]
\end{tikzcd}$$

commutes and such that the following quadrangle is coreedian.

$$\begin{tikzcd}
X_1 \amalg X_2 \arrow[r, "\mathrm{ins}^{C_1} \amalg \mathrm{ins}^{C_2}"] \arrow[d, "\binom{f}{u}"'] & C_1 \amalg C_2 \arrow[d, "\binom{g}{v}"] \\
\tilde{X}_2 \arrow[r, "\mathrm{ins}^{\tilde{C}_2}"] & \tilde{C}_2
\end{tikzcd}$$

Proof.

(a) This follows from the factorisation axiom for cofibrations and remark (5.8).

$$\begin{tikzcd}
 & C \arrow[dr, dotted, "p", "\wr"'] & \\
X \arrow[ur, dotted, "i"] \arrow[rr, "0"] & & 0
\end{tikzcd}$$

(b) This follows from lemma (5.9). □

2 The Coheller shift

We suppose given a Brown cofibration category $\mathcal{C}$ that has a zero object, cf. definition (3.52)(a). In this section, we will turn the homotopy category $\operatorname{Ho}\mathcal{C}$ into a category with shift as in definition (4.5)(c), that is, we define a suitable endofunctor on $\operatorname{Ho}\mathcal{C}$. Such an endofunctor $\Sigma\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$, called *suspension*, was constructed by Brown [7, dual of th. 3] on the objects as follows. For every object X in $\mathcal{C}$, he chose a Coquillen rectangle, cf. definition (3.101), of the form

$$\begin{tikzcd} Z_X \arrow[r, "\mathrm{quo}"] & \Sigma_X \\ X \amalg X \arrow[u, "\mathrm{ins}"] \arrow[r] & 0 \arrow[u] \end{tikzcd} ,$$

where Z_X is a (chosen) cylinder of X, cf. definition (3.108)(c), and set $\Sigma X := \Sigma_X$. To construct higher triangles, as we will do in section 3 and section 4, it is more convenient to have a construction of a shift via cones instead of cylinders, cf. definition (5.6); more precisely, via a choice of Coquillen rectangles of the form

$$\begin{tikzcd} C_X \arrow[r, "\mathrm{quo}"] & T_X \\ X \arrow[u, "\mathrm{ins}"] \arrow[r] & 0 \arrow[u] \end{tikzcd} ,$$

as Heller did in his framework of h-c-categories in [17, prop. 5.3] (or in an additive case already in [16, sec. 3]). Brown's construction may be seen as a particular case of Heller's one as every cylinder Z_X of an object X gives rise to a commutative diagram of the form

$$\begin{tikzcd} Z_X \arrow[r, "\mathrm{quo}"] & C_X \arrow[r, "\mathrm{quo}"] & \Sigma_X \\ X \amalg X \arrow[u, "\mathrm{ins}^{Z_X}"] \arrow[r, "\mathrm{quo}"] & X \arrow[u, "\mathrm{ins}^{C_X}"] \arrow[r] & 0 \arrow[u] \\ X \arrow[u, "\mathrm{emb}_0"] \arrow[r] & 0 \arrow[u] & \end{tikzcd} ,$$

in which $(X, 0, X \amalg X, X)$, $(X \amalg X, X, Z_X, C_X)$, $(X, 0, C_X, \Sigma_X)$ are Coquillen rectangles. As $\mathrm{ins}_0^{Z_X} = \mathrm{emb}_0 \, \mathrm{ins}^{Z_X}$ is an acyclic cofibration, the unique morphism $\mathrm{ini}_{C_X}\colon 0 \to C_X$ is also an acyclic cofibration, whence C_X is a cone of X.

However, different choices of cones lead to isomorphic shift functors, see remark (5.31), so from a philosophical point of view, Brown's shift is as good as Heller's. Finally, it is not necessary to have a zero object in the chosen Coquillen rectangles; we actually construct a shift via an arbitrary choice of Coheller rectangles as introduced in definition (5.11) for the objects in $\mathcal{C}$, see definition (5.28).

A comment on the terminology: In additive frameworks, the dual of our shift construction is often called the *Heller operator* in honour of Heller's work [16, sec. 3]. As we adopt Heller's ideas to the framework of Brown cofibration categories, the author chose the word *Coheller shift* for the shift constructed in this thesis, cf. definition (5.28), leaving the notion of a *suspension* for those particular Coheller shifts constructed via (cones that arise from) cylinders. In an arbitrary Brown cofibration category that has a zero object, it seems unlikely that every cone as in definition (5.6) appears as a quotient of a cylinder as in definition (3.108)(c).

From now on, throughout the rest of this section, we suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that has a zero object.

Coheller rectangles

A Coquillen rectangle in $\mathcal{C}$ is a pushout rectangle X in $\mathcal{C}_{\mathrm{cof}}$ such that $X_{(0,0),(0,1)}$ is a cofibration, cf. definition (3.101). The category of Coquillen rectangles in $\mathcal{C}$ is denoted by $\mathcal{C}^{\square}_{\mathrm{coqu}}$.

(5.11) Definition (category of Coheller rectangles)**.** The full subcategory $\mathcal{C}^{\square}_{\mathrm{cohel}}$ of $\mathcal{C}^{\square}_{\mathrm{coqu}}$ with

$$\operatorname{Ob}\mathcal{C}^{\square}_{\mathrm{cohel}} = \{X \in \operatorname{Ob}\mathcal{C}^{\square}_{\mathrm{coqu}} \mid X_{1,0} \text{ and } X_{0,1} \text{ are coacyclic objects in } \mathcal{C}\}$$

is called the *category of Coheller rectangles* (or the *category of cohellerian rectangles*) in $\mathcal{C}$. An object in $\mathcal{C}^{\square}_{\mathrm{cohel}}$ is called a *Coheller rectangle* (or *cohellerian rectangle* or *cohellerian quadrangle*) in $\mathcal{C}$, and a morphism in $\mathcal{C}^{\square}_{\mathrm{cohel}}$ is called a *morphism of Coheller rectangles* (or a *morphism of cohellerian rectangles*).

(5.12) Remark. If $\mathcal{C}$ is equipped with the structure of a zero-pointed category with cofibrations and weak equivalences, then the category of Coheller rectangles $\mathcal{C}^{\square}_{\mathrm{cohel}}$ becomes a zero-pointed category, where the zero object $0^{\mathcal{C}^{\square}_{\mathrm{cohel}}}$ is given by $0^{\mathcal{C}^{\square}_{\mathrm{cohel}}}_k = 0^{\mathcal{C}}$ for $k \in \operatorname{Ob}\square$.

(5.13) Remark. We suppose given a pointwise weak equivalence of Coquillen rectangles $f\colon X \to Y$ in $\mathcal{C}$.

(a) If X is a Coheller rectangle in $\mathcal{C}$, then Y is a Coheller rectangle in $\mathcal{C}$.

(b) We suppose that $\mathcal{C}$ is T-semisaturated. If Y is a Coheller rectangle in $\mathcal{C}$, then X is a Coheller rectangle in $\mathcal{C}$.

Proof.

(a) This follows from remark (5.5)(a).

(b) This follows from remark (5.5)(b). □

The following remark gives a connection between the concept of a Coheller rectangle and that of a cone as introduced definition (5.6).

(5.14) Remark.

(a) Given a Coheller rectangle Y in $\mathcal{C}$, then $Y_{0,1}$ becomes a cone of $Y_{0,0}$ with $\mathrm{ins}^{Y_{0,1}} = Y_{(0,0),(0,1)}$.

(b) For every object X and every cone C of X there exists a Coheller rectangle Y in $\mathcal{C}$ with $Y_{(0,0),(0,1)} = \mathrm{ins}^C$ and $Y_{1,0} = 0$.

Proof.

(b) This follows from the pushout axiom for cofibrations. □

For the definition of a Cisinski cofibration category, see definition (3.51)(a).

(5.15) Remark. We suppose that $\mathcal{C}$ is a Cisinski cofibration category. A morphism of Coheller rectangles $f\colon X \to Y$ in $\mathcal{C}$ is a pointwise weak equivalence if and only if $f_{0,0}$ is a weak equivalence in $\mathcal{C}$.

Proof. We suppose given a morphism of Coheller rectangles $f\colon X \to Y$. As $X_{1,0}$, $X_{0,1}$, $Y_{1,0}$, $Y_{0,1}$ are coacyclic objects, the components $f_{1,0}\colon X_{1,0} \to Y_{1,0}$ and $f_{0,1}\colon X_{0,1} \to Y_{0,1}$ are weak equivalences by remark (5.5)(c). So if f is a pointwise weak equivalence, then in particular $f_{0,0}$ is a weak equivalence, and conversely, if $f_{0,0}$ is a weak equivalence, then f is a pointwise weak equivalence by the gluing lemma (3.121).

$$\begin{tikzcd}
 & X_{0,1} \arrow[rr] \arrow[dd, "\wr"', "f_{0,1}"] & & X_{1,1} \arrow[dd, "f_{1,1}"] \\
X_{0,0} \arrow[ur, "\bullet" description] \arrow[rr] \arrow[dd, "f_{0,0}"] & & X_{1,0} \arrow[ur, "\bullet" description] \arrow[dd, "\wr"', "f_{1,0}"] & \\
 & Y_{0,1} \arrow[rr] & & Y_{1,1} \\
Y_{0,0} \arrow[ur, "\bullet" description] \arrow[rr] & & Y_{1,0} \arrow[ur, "\bullet" description] &
\end{tikzcd}$$

□

The following lemma is our main tool for the construction of the Coheller construction and the Coheller shift, see definition (5.22) and definition (5.28). We make use of notation (3.72)(b).

(5.16) Lemma (Heller lemma, cf. [17, lem. 5.2])**.** We suppose that $\mathcal{C}$ is a Cisinski cofibration category. For every Coquillen rectangle X and every Coheller rectangle Y in $\mathcal{C}$, the evaluation functor $-_{0,0}\colon \mathcal{C}^{\square}_{\mathrm{coqu}} \to \mathcal{C}_{\mathrm{cof}}$ induces a bijection

$${}_{\mathrm{Ho}\,\mathcal{C}^{\square}_{\mathrm{coqu}}}(X, Y) \to {}_{\mathrm{Ho}\,\mathcal{C}_{\mathrm{cof}}}(X_{0,0}, Y_{0,0}),\ \varphi \mapsto \varphi_{0,0}.$$

Proof. We suppose given a Coquillen rectangle X and a Coheller rectangle Y in $\mathcal{C}$. As $\mathcal{C}^{\square}_{\mathrm{coqu}}$ is a Brown cofibration category by corollary (3.122), the induced map ${}_{\mathrm{Ho}\,\mathcal{C}^{\square}_{\mathrm{coqu}}}(X, Y) \to {}_{\mathrm{Ho}\,\mathcal{C}_{\mathrm{cof}}}(X_{0,0}, Y_{0,0})$ is given by

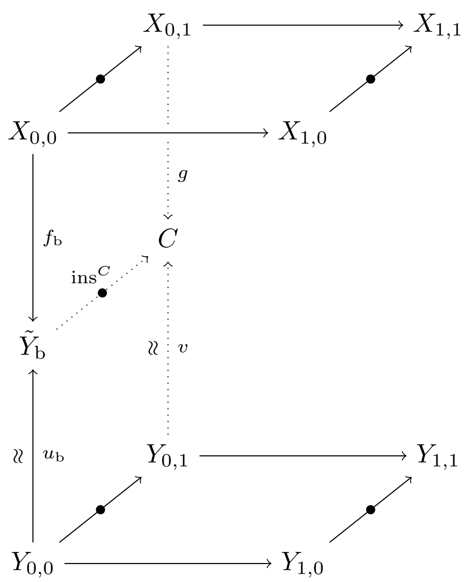
$$(\mathrm{loc}(f)\,\mathrm{loc}(u)^{-1})_{0,0} = \mathrm{loc}(f_{0,0})\,\mathrm{loc}(u_{0,0})^{-1}$$

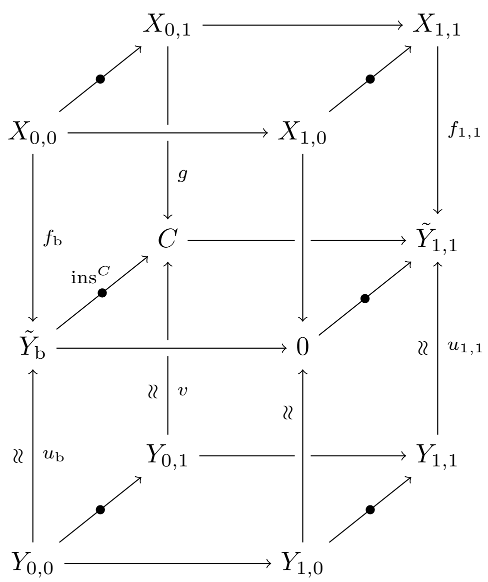
for every S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}^{\square}_{\mathrm{coqu}}$, see corollary (2.94)(d).

To show the surjectivity of ${}_{\mathrm{Ho}\,\mathcal{C}^{\square}_{\mathrm{coqu}}}(X, Y) \to {}_{\mathrm{Ho}\,\mathcal{C}_{\mathrm{cof}}}(X_{0,0}, Y_{0,0})$, $\varphi \mapsto \varphi_{0,0}$, we suppose given a morphism $\varphi_{\mathrm{b}}\colon X_{0,0} \to Y_{0,0}$ in $\mathrm{Ho}\,\mathcal{C}_{\mathrm{cof}}$. By corollary (2.94)(a), there exist an S-2-arrow $(f_{\mathrm{b}}, u_{\mathrm{b}})\colon X_{0,0} \to \tilde{Y}_{\mathrm{b}} \leftarrow Y_{0,0}$ in $\mathcal{C}_{\mathrm{cof}}$ with $\varphi_{\mathrm{b}} = \mathrm{loc}(f_{\mathrm{b}})\,\mathrm{loc}(u_{\mathrm{b}})^{-1}$. As Y is a Coheller rectangle, $Y_{0,1}$ becomes a cone of $Y_{0,0}$ with insertion $\mathrm{ins}^{Y_{0,1}} = Y_{(0,0),(0,1)}$ by remark (5.14)(a). So by lemma (5.9), there exist a cone C of $\tilde{Y}_{\mathrm{b}}$ and an S-2-arrow $(g, v)\colon X_{0,1} \to C \leftarrow Y_{0,1}$ in $\mathcal{C}$ such that $X_{(0,0),(0,1)}g = f_{\mathrm{b}}\mathrm{ins}^{C}$ and $Y_{(0,0),(0,1)}v = u_{\mathrm{b}}\mathrm{ins}^{C}$. By remark (5.14)(b), there exists a Coheller rectangle $\tilde{Y}$ in $\mathcal{C}$ with $\tilde{Y}_{(0,0),(0,1)} = \mathrm{ins}^{C}$ and $\tilde{Y}_{1,0} = 0$. Moreover, as X and Y are pushout rectangles, there exist morphisms of Coquillen rectangles $f\colon X \to \tilde{Y}$ and $u\colon Y \to \tilde{Y}$ in $\mathcal{C}$ such that $f_{0,0} = f_{\mathrm{b}}$, $f_{0,1} = g$, $f_{1,0} = \mathrm{ter}_{X_{1,0}}$, $u_{0,0} = u_{\mathrm{b}}$, $u_{0,1} = v$, $u_{1,0} = \mathrm{ter}_{Y_{1,0}}$, and u is a pointwise weak equivalence by remark (5.15). So we have

$$(\mathrm{loc}(f)\,\mathrm{loc}(u)^{-1})_{0,0} = \mathrm{loc}(f_{0,0})\,\mathrm{loc}(u_{0,0})^{-1} = \mathrm{loc}(f_{\mathrm{b}})\,\mathrm{loc}(u_{\mathrm{b}})^{-1} = \varphi_{\mathrm{b}}.$$

Thus the induced map is surjective.





To show injectivity, we suppose given morphisms $\varphi, \varphi'\colon X \to Y$ in $\mathrm{Ho}\,\mathcal{C}^{\square}_{\mathrm{coqu}}$ such that $\varphi_{0,0} = \varphi'_{0,0}$. By theorem (3.128)(a), there exist Z-2-arrows of Coquillen rectangles $(f, i)\colon X \to \tilde{Y} \leftarrow Y$ and $(f', i')\colon X \to \tilde{Y}' \leftarrow Y$ in $\mathcal{C}$ with $\varphi = \mathrm{loc}(f)\,\mathrm{loc}(i)^{-1}$ and $\varphi' = \mathrm{loc}(f')\,\mathrm{loc}(i')^{-1}$. Hence we have

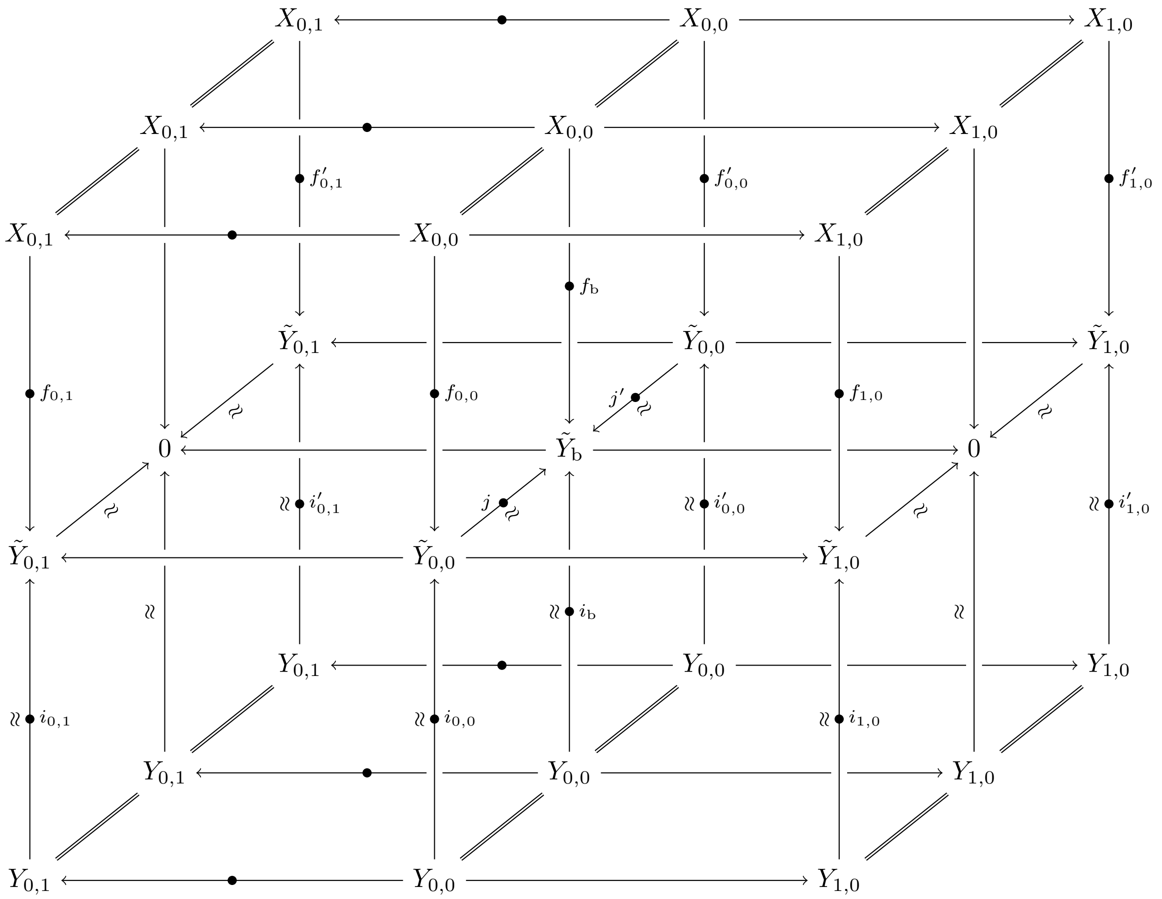
$$\mathrm{loc}(f_{0,0})\,\mathrm{loc}(i_{0,0})^{-1} = \varphi_{0,0} = \varphi'_{0,0} = \mathrm{loc}(f'_{0,0})\,\mathrm{loc}(i'_{0,0})^{-1},$$

and so by theorem (3.128)(b) there exist a Z-2-arrow $(f_{\mathrm{b}}, i_{\mathrm{b}})\colon X_{0,0} \to \tilde{Y}_{\mathrm{b}} \leftarrow Y_{0,0}$ and acyclic cofibrations

$j\colon \tilde{Y}_{0,0} \to \tilde{Y}_{\mathrm{b}}$, $j'\colon \tilde{Y}'_{0,0} \to \tilde{Y}_{\mathrm{b}}$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
X_{0,0} \arrow[r, equal] \arrow[d, "f_{0,0}"] & X_{0,0} \arrow[r, equal] \arrow[d, dotted, "f_{\mathrm{b}}"] & X_{0,0} \arrow[d, "f'_{0,0}"] \\
\tilde{Y}_{0,0} \arrow[r, dotted, "j", "\approx"'] & \tilde{Y}_{\mathrm{b}} & \tilde{Y}'_{0,0} \arrow[l, dotted, "j'"', "\approx"] \\
Y_{0,0} \arrow[r, equal] \arrow[u, "i_{0,0}", "\approx"'] & Y_{0,0} \arrow[r, equal] \arrow[u, dotted, "i_{\mathrm{b}}", "\approx"'] & Y_{0,0} \arrow[u, "i'_{0,0}", "\approx"']
\end{tikzcd}$$

Moreover, as $Y_{1,0}$ and $Y_{0,1}$ are coacyclic, the unique morphisms $\mathrm{ter}_{Y_{1,0}}\colon Y_{1,0} \to 0$ and $\mathrm{ter}_{Y_{0,1}}\colon Y_{0,1} \to 0$ are weak equivalences. So since the diagram



commutes, we have $\mathrm{loc}(f|_{\llcorner}) \, \mathrm{loc}(i|_{\llcorner})^{-1} = \mathrm{loc}(f'|_{\llcorner}) \, \mathrm{loc}(i'|_{\llcorner})^{-1}$ in $\mathrm{Ho}\, \mathcal{C}^{\llcorner}$. As $X|_{\llcorner}$ and $Y|_{\llcorner}$ are Quillen cofibrant, we even have $\mathrm{loc}(f|_{\llcorner}) \, \mathrm{loc}(i|_{\llcorner})^{-1} = \mathrm{loc}(f'|_{\llcorner}) \, \mathrm{loc}(i'|_{\llcorner})^{-1}$ in $\mathrm{Ho}\,(\mathcal{C}^{\llcorner}_{\mathrm{Quillen}})_{\mathrm{cof}}$ by [9, prop. 1.8]. But then we also have

$$\varphi = \mathrm{loc}(f) \, \mathrm{loc}(i)^{-1} = \mathrm{loc}(f') \, \mathrm{loc}(i')^{-1} = \varphi'$$

in $\mathrm{Ho}\, \mathcal{C}^{\square}_{\mathrm{coqu}}$. □

The Coheller construction

As already indicated at the beginning of this section, the Coheller shift will be defined via a *choice* of a Coheller rectangle for each cofibrant object in $\mathcal{C}$. Before we do so, we present a uniform variant of the shift construction, which does not necessitate choices, using the theory developed in appendix B, section 1. This Coheller construction as introduced in definition (5.22) will be useful in our treatment of cosemitriangles in section 4; in particular, it will be used in definition (5.45)(a) of a standard cosemitriangle.

(5.17) Definition (Coheller rectangle of an object)**.** Given a cofibrant object X and a Coheller rectangle R in $\mathcal{C}$ such that $R_{0,0} = X$, we say that R is a *Coheller rectangle of* X.

(5.18) Remark. We suppose that $\mathcal{C}$ is T-semisaturated and fulfills the factorisation axiom for cofibrations. Then there exists a Coheller rectangle of every cofibrant object in $\mathcal{C}$.

Proof. This follows from the Heller factorisation lemma (5.10)(a) and remark (5.14)(b). □

For the concept of the structure category, see definition (A.2).

(5.19) Definition (Coheller category)**.** For $X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}_{\mathrm{cof}} = \operatorname{Ob}\mathcal{C}_{\mathrm{cof}}$, we let $\mathfrak{R}_X$ be the set of Coheller rectangles of X. The structure category

$$\mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) := (\operatorname{Ho}\mathcal{C}_{\mathrm{cof}})_{\mathfrak{R}}$$

is called the *Coheller category* of $\mathcal{C}$.

(5.20) Remark. We have

$$\operatorname{Ob}\mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) = \{(X, R) \mid X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}_{\mathrm{cof}} \text{ and } R \text{ is a Coheller rectangle of } X\}.$$

For objects (X, R) and (Y, S) in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, we have the hom-set

$${}_{\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})}((X, R), (Y, S)) = {}_{\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}}(X, Y).$$

For morphisms $\varphi\colon (X, R) \to (Y, S)$, $\psi\colon (Y, S) \to (Z, T)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, the composite $\varphi\psi\colon (X, R) \to (Z, T)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$ has the underlying morphism $\varphi\psi\colon X \to Z$ in $\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$. For an object (X, R) in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, the identity morphism $1_{(X,R)}\colon (X, R) \to (X, R)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$ has the underlying morphism $1_X\colon X \to X$ in $\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$. The forgetful functor $\mathrm{U}\colon \mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) \to \operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$ is given on the objects by

$$\mathrm{U}_R X = X$$

for $(X, R) \in \operatorname{Ob}\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, and on the morphisms by

$$\mathrm{U}_{R,S}\varphi = \varphi$$

for every morphism $\varphi\colon (X, R) \to (Y, S)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$.

If $\mathcal{C}$ is a Cisinski cofibration category, then the evaluation functor $-_{0,0}\colon \mathcal{C}^{\square}_{\mathrm{coqu}} \to \mathcal{C}_{\mathrm{cof}}$ induces a bijection

$${}_{\operatorname{Ho}\mathcal{C}^{\square}_{\mathrm{coqu}}}(R, S) \to {}_{\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}}(R_{0,0}, S_{0,0}),\ \psi \mapsto \psi_{0,0}$$

for all Coheller rectangles R and S in $\mathcal{C}$, see the Heller lemma (5.16). This gives rise to the following construction.

(5.21) Proposition. We suppose that $\mathcal{C}$ is a Cisinski cofibration category. Then we have a functor

$$\mathsf{H}\colon \mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) \to \operatorname{Ho}\mathcal{C}_{\mathrm{cof}},$$

given on the objects by

$$\mathsf{H}_R(X) := R_{1,1}$$

for $(X, R) \in \operatorname{Ob}\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, and on the morphisms as follows. We suppose given a morphism $\varphi\colon (X, R) \to (Y, S)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$. Moreover, we let $\psi\colon R \to S$ be the unique morphism in $\operatorname{Ho}\mathcal{C}^{\square}_{\mathrm{coqu}}$ with $\varphi = \psi_{0,0}$ in $\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$. Then

$$\mathsf{H}_{R,S}(\varphi) = \psi_{1,1}.$$

Proof. We define a map

$$H_0\colon \operatorname{Ob}\mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) \to \operatorname{Ob}\operatorname{Ho}\mathcal{C}_{\mathrm{cof}},\ (X, R) \mapsto R_{1,1}.$$

We suppose given $(X, R), (Y, S) \in \operatorname{Ob}\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$. As the evaluation functor $-_{0,0}\colon \mathcal{C}^{\square}_{\mathrm{coqu}} \to \mathcal{C}_{\mathrm{cof}}$ induces a bijection

$$\Phi_{R,S}\colon {}_{\operatorname{Ho}\mathcal{C}^{\square}_{\mathrm{coqu}}}(R, S) \to {}_{\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}}(R_{0,0}, S_{0,0}),\ \psi \mapsto \psi_{0,0}$$

by the Heller lemma (5.16), we obtain a well-defined map

$$H_{(X,R),(Y,S)} \colon {}_{\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})}((X,R),(Y,S)) \to {}_{\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}}(R_{1,1}, S_{1,1}),\ \varphi \mapsto (\varphi \Phi_{R,S}^{-1})_{1,1}.$$

Given morphisms $\varphi\colon (X,R) \to (Y,S)$ and $\rho\colon (Y,S) \to (Z,T)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, we have

$$((\varphi \Phi_{R,S}^{-1})(\rho \Phi_{S,T}^{-1}))_{0,0} = (\varphi \Phi_{R,S}^{-1})_{0,0}\,(\rho \Phi_{S,T}^{-1})_{0,0} = \varphi\rho$$

and therefore

$$\begin{aligned} H_{(X,R),(Z,T)}(\varphi\varphi') &= ((\varphi\rho)\Phi_{R,T}^{-1})_{1,1} = ((\varphi \Phi_{R,S}^{-1})(\rho \Phi_{S,T}^{-1}))_{1,1} = (\varphi \Phi_{R,S}^{-1})_{1,1}\,(\rho \Phi_{S,T}^{-1})_{1,1} \\ &= H_{(X,R),(Y,S)}(\varphi)\, H_{(Y,S),(Z,T)}(\rho). \end{aligned}$$

Given an object (X,R) in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, the identity $1_R\colon R \to R$ in $\operatorname{Ho}\mathcal{C}^{\square}_{\mathrm{coqu}}$ fulfills $(1_R)_{0,0} = 1_{R_{0,0}} = 1_X$, and so we have

$$H_{(X,R),(X,R)}(1_X) = (1_R)_{1,1} = 1_{R_{1,1}} = 1_{H_0(X,R)}.$$

Thus we have a functor $\mathsf{H}\colon \mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) \to \operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$ given by $\mathsf{H}_R X = H_0(X,R)$ for $(X,R) \in \operatorname{Ob}\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$ and by $\mathsf{H}_{R,S}\varphi = H_{(X,R),(Y,S)}\varphi$ for every morphism $\varphi\colon (X,R) \to (Y,S)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$. □

(5.22) Definition (Coheller construction)**.** We suppose that $\mathcal{C}$ is a Cisinski cofibration category. The functor $\mathsf{H}\colon \mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) \to \operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$ from proposition (5.21) is called the *total Coheller construction functor*. For an object (X,R) in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, the object $\mathsf{H}_R(X)$ in $\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$ is called the *Coheller construction* of X with respect to R. For a morphism $\varphi\colon (X,R) \to (Y,S)$ in $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$, the morphism $\mathsf{H}_{R,S}(\varphi)\colon \mathsf{H}_R(X) \to \mathsf{H}_S(Y)$ in $\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$ is called the *Coheller construction* of $\varphi\colon X \to Y$ with respect to R and S.

(5.23) Remark. We suppose that $\mathcal{C}$ is a equipped with the structure of a zero-pointed Cisinski cofibration category. The Coheller category $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$ becomes a zero-pointed category having the zero object $0^{\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})} = (0^{\operatorname{Ho}\mathcal{C}}, 0^{\mathcal{C}^{\square}_{\mathrm{cohel}}})$. Moreover, the Coheller construction

$$\mathsf{H}\colon \mathbf{Hel}^{\mathrm{co}}(\mathcal{C}) \to \operatorname{Ho}\mathcal{C}_{\mathrm{cof}}$$

is a morphism of zero-pointed categories with respect to this structure on $\mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$.

Proof. We have

$$\mathsf{H}_{0^{\mathcal{C}^{\square}_{\mathrm{cohel}}}}(0^{\operatorname{Ho}\mathcal{C}}) = 0^{\mathcal{C}^{\square}_{\mathrm{cohel}}}_{1,1} = 0^{\mathcal{C}} = 0^{\operatorname{Ho}\mathcal{C}}.$$ □

Construction of the Coheller shift

Now we are ready to define the Coheller shift as the Coheller construction via a choice of a Coheller rectangle for each cofibrant object in $\mathcal{C}$.

(5.24) Definition (choice of Coheller rectangles)**.**

(a) A *choice of Coheller rectangles* in $\mathcal{C}$ is a family $(R_X)_{X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}}$ such that R_X is a Coheller rectangle of X for each $X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}_{\mathrm{cof}} = \operatorname{Ob}\mathcal{C}_{\mathrm{cof}}$.

(b) We suppose that $\mathcal{C}$ is equipped with the structure of a zero-pointed category with cofibrations and weak equivalences. A choice of Coheller rectangles $(R_X)_{X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}_{\mathrm{cof}}}$ in $\mathcal{C}$ is said to be *zero-pointed* if $R_{0^{\mathcal{C}}} = 0^{\mathcal{C}^{\square}_{\mathrm{cohel}}}$.

(5.25) Remark. We suppose that $\mathcal{C}$ is T-semisaturated and fulfills the factorisation axiom for cofibrations.

(a) There exists a choice of Coheller rectangles in $\mathcal{C}$.

(b) If $\mathcal{C}$ is equipped with the structure of a zero-pointed category with cofibrations and weak equivalences, there exists a zero-pointed choice of Coheller rectangles in $\mathcal{C}$.

Proof. This follows from remark (5.18). □

(5.26) Remark. For $X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}} = \operatorname{Ob} \mathcal{C}_{\mathrm{cof}}$, we let $\mathfrak{R}_X$ be the set of Coheller rectangles of X. A choice of Coheller rectangles of $\mathcal{C}$ is precisely a choice of structures with respect to $\mathfrak{R} = (\mathfrak{R}_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}}$.

For the definition of the structure choice functor with respect to a choice of structures, see definition (A.8). In the case of a choice of Coheller rectangles, the structure choice functor is given as follows.

(5.27) Remark. We suppose given a choice of Coheller rectangles $R = (R_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}}$ of $\mathcal{C}$. The structure choice functor $\mathrm{I}_R \colon \operatorname{Ho} \mathcal{C}_{\mathrm{cof}} \to \mathbf{Hel}^{\mathrm{co}}(\mathcal{C})$ is given on the objects by

$$\mathrm{I}_R X = (X, R_X)$$

for $X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$, and on the morphisms by

$$\mathrm{I}_R \varphi = \varphi \colon (X, R_X) \to (Y, R_Y)$$

for every morphism $\varphi \colon X \to Y$ in $\operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$.

(5.28) Definition (Coheller shift)**.** We suppose that $\mathcal{C}$ is a Cisinski cofibration category. Moreover, we suppose given a choice of Coheller rectangles $R = (R_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}}$. The composite

$$\mathrm{T}^{\mathrm{co}}_{\mathrm{Heller}} = \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} := \mathsf{H} \circ \mathrm{I}_R \colon \operatorname{Ho} \mathcal{C}_{\mathrm{cof}} \to \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$$

is called the *Coheller shift* (or *Coheller shift functor*) on $\operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$ with respect to R.

(5.29) Remark. We suppose that $\mathcal{C}$ is a Cisinski cofibration category, and we suppose given a choice of Coheller rectangles $R = (R_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}}$. For $X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$, we have

$$\mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} X = \mathsf{H}_{R_X}(X).$$

For a morphism $\varphi \colon X \to Y$ in $\operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$, we have

$$\mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} \varphi = \mathsf{H}_{R_X, R_Y}(\varphi).$$

Proof. This follows from remark (5.27) and proposition (5.21). □

A zero-pointed Cisinski cofibration category is a Cisinski cofibration category as in definition (3.52)(a), equipped with a (distinguished) zero object. Then $\mathcal{C}_{\mathrm{cof}}$ becomes a zero-pointed Brown cofibration category, the homotopy category $\operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$ becomes a zero-pointed category and the localisation functor becomes a morphism of zero-pointed categories, cf. remark (3.53) and remark (3.11).

(5.30) Remark. We suppose that $\mathcal{C}$ is a zero-pointed Cisinski cofibration category. The Coheller shift $\mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} \colon \operatorname{Ho} \mathcal{C}_{\mathrm{cof}} \to \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}$ is a morphism of zero-pointed categories for every zero-pointed choice of Coheller rectangles $R = (R_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}}$ in $\mathcal{C}$.

Proof. By remark (5.23), we have

$$\mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} 0^{\operatorname{Ho} \mathcal{C}} = \mathsf{H}_{R_{0^{\operatorname{Ho} \mathcal{C}}}}(0^{\operatorname{Ho} \mathcal{C}}) = \mathsf{H}_{0^{\mathcal{C}^{\square}_{\mathrm{cohel}}}}(0^{\operatorname{Ho} \mathcal{C}}) = 0^{\operatorname{Ho} \mathcal{C}}.$$ □

(5.31) Remark. We suppose that $\mathcal{C}$ is a Cisinski cofibration category. Moreover, we suppose given choices of Coheller rectangles $R = (R_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}}$ and $R' = (R'_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}_{\mathrm{cof}}}$. Then we have

$$\mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} \cong \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R'}.$$

An isotransformation $\alpha_{R,R'} \colon \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} \to \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R'}$ is given by

$$(\alpha_{R,R'})_X = \mathsf{H}_{R_X, R'_X}(1_X) \colon \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} X \to \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R'} X$$

for $X \in \operatorname{Ob} \mathcal{C}$. The inverse of $\alpha_{R,R'}$ is given by $\alpha_{R,R'}^{-1} = \alpha_{R',R}$.

Proof. This follows from corollary (A.12). □

3 Heller cosemistrips

We suppose given a zero-pointed Brown cofibration category $\mathcal{C}$, that is, a Brown cofibration category as in definition (3.52)(a) equipped with a (distinguished) zero object. Cosemitriangles in $\operatorname{Ho}\mathcal{C}$ will be diagrams that arise, up to isomorphism, in a suitable manner from a diagram in $\mathcal{C}$. This section is dedicated to the study of these models, called Heller cosemistrips.
Throughout this section, we suppose given a category with cofibrations and weak equivalences $\mathcal{C}$, see definition (3.30)(a), that has a zero object.

The semiquasicyclic category of Heller cosemistrips

For $n \in \mathbb{N}_0$, an n-cosemistrip in $\mathcal{C}$ is just a $\#^n_+$-commutative diagram in $\mathcal{C}$, see definition (4.55)(a) and definition (4.42). The cosemistrips in $\mathcal{C}$ are organised in a semiquasicyclic category $\operatorname{Strips}^{\mathrm{co},+}(\mathcal{C})$, cf. definition (4.38). In the following remark, we construct a semiquasicyclic subcategory of $\operatorname{Strips}^{\mathrm{co},+}(\mathcal{C})$.
Given a cosemistrip X in $\mathcal{C}$, we denote by $X^{k/i,l/i,k/j,l/j} = (X^{k/i}, X^{l/i}, X^{k/j}, X^{l/j})$ the unique commutative quadrangle in X that is determined by the four indicated vertices.

(5.32) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that has a zero object. We have a full semiquasicyclic subcategory $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller}}(\mathcal{C})$ of $\operatorname{Strips}^{\mathrm{co},+}(\mathcal{C})$ given by

$$\begin{aligned}\operatorname{Ob}\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C}) = \{X \in \operatorname{Ob}\operatorname{Strips}^{\mathrm{co},+}_n(\mathcal{C}) \mid\ & X^{i/i},\ X^{i^{[1]}/i} \text{ are coacyclic for } i \in \Theta^n_+ \text{ and } X^{k/i,l/i,k/j,l/j} \\ & \text{is a Coquillen rectangle for } k/i, l/j \in \#^n_+ \text{ with } k/i \leq l/j \leq (k/i)^{[1]}\}\end{aligned}$$

for $n \in \mathbb{N}_0$.

Proof. We suppose given a morphism of semiquasicyclic types $\alpha\colon \Theta^m_+ \to \Theta^n_+$ for $m, n \in \mathbb{N}_0$ and an object X in $\operatorname{Strips}^{\mathrm{co},+}_n(\mathcal{C})$ such that $X^{j/j}$ and $X^{j^{[1]}/j}$ are coacyclic for $j \in \Theta^n_+$, and such that $X^{l/j,l'/j,l/j',l'/j'}$ is a Coquillen rectangle for $l/j, l'/j' \in \#^n_+$ with $l/j \leq l'/j' \leq (l/j)^{[1]}$. But then also $(\operatorname{Strips}^{\mathrm{co},+}_\alpha(\mathcal{C})X)^{i/i} = X^{i\alpha/i\alpha}$ and $(\operatorname{Strips}^{\mathrm{co},+}_\alpha(\mathcal{C})X)^{i^{[1]}/i} = X^{i^{[1]}\alpha/i\alpha} = X^{(i\alpha)^{[1]}/i\alpha}$ are coacyclic for $i \in \Theta^m_+$, and $(\operatorname{Strips}^{\mathrm{co},+}_\alpha(\mathcal{C})X)^{k/i,k'/i,k/i',k'/i'} = X^{k\alpha/i\alpha,k'\alpha/i\alpha,k\alpha/i'\alpha,k'\alpha/i'\alpha}$ is a Coquillen rectangle for $k/i, k'/i' \in \#^m_+$ with $k/i \leq k'/i' \leq (k/i)^{[1]}$. $\square$

(5.33) Definition (semiquasicyclic category of Heller cosemistrips)**.** The full semiquasicyclic subcategory $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller}}(\mathcal{C})$ of $\operatorname{Strips}^{\mathrm{co},+}(\mathcal{C})$ as in remark (5.32) is called the *semiquasicyclic category of Heller cosemistrips* in $\mathcal{C}$. For $n \in \mathbb{N}_0$, the category $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C})$ is called the *category of Heller n-cosemistrips* in $\mathcal{C}$, an object in $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C})$ is called a *Heller n-cosemistrip* in $\mathcal{C}$, and a morphism in $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C})$ is called a *morphism of Heller n-cosemistrips* in $\mathcal{C}$.

If unambiguous, we will consider the category of Heller n-cosemistrips for $n \in \mathbb{N}_0$ as a category with weak equivalences in the following way, without further comment.

(5.34) Remark. For every $n \in \mathbb{N}_0$, the category of Heller n-cosemistrips $\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C})$ becomes a category with weak equivalences having

$$\operatorname{We}\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C}) = \operatorname{We}(\operatorname{Strips}^{\mathrm{co},+}_n(\mathcal{C}))_{\mathrm{ptw}} \cap \operatorname{Mor}\operatorname{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C}).$$

Coheller rectangles in Heller cosemistrips

As the “boundaries” of every Heller cosemistrip consist of coacyclic objects, some of the Coquillen rectangles occurring in such a Heller cosemistrip are actually Coheller rectangles as introduced in definition (5.11).

(5.35) Remark. Given a Heller n-cosemistrip X in $\mathcal{C}$ for some $n \in \mathbb{N}_0$, then $X^{k/i,i^{[1]}/i,k/k,i^{[1]}/k}$ is a Coheller rectangle in $\mathcal{C}$ for every $k/i \in \#^n_+$.

(5.36) Definition (Coheller rectangles in Heller n-cosemistrip)**.** We suppose given a Heller n-cosemistrip X in $\mathcal{C}$ for some $n \in \mathbb{N}_0$. For $k/i \in \#^n_+$, the quadrangle

$$\mathrm{R}^{k/i}(X) := X^{k/i,i^{[1]}/i,k/k,i^{[1]}/k}$$

is called the *Coheller rectangle at position k/i* in X.

(5.37) Remark. Given a morphism of semiquasicyclic types $\alpha\colon \Theta_+^m \to \Theta_+^n$ for $m, n \in \mathbb{N}_0$ and a Heller n-cosemistrip X in $\mathcal{C}$, we have

$$\mathrm{R}^{k/i}(\mathrm{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})X) = \mathrm{R}^{k\alpha/i\alpha}(X)$$

for $k/i \in \#_+^n$.

Proof. For $k/i \in \#_+^n$, we have

$$\begin{aligned}\mathrm{R}^{k/i}(\mathrm{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})X) &= (\mathrm{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})X)^{k/i,i^{[1]}/i,k/k,i^{[1]}/k} = X^{k\alpha/i\alpha,i^{[1]}\alpha/i\alpha,k\alpha/k\alpha,i^{[1]}\alpha/k\alpha} \\ &= X^{k\alpha/i\alpha,(i\alpha)^{[1]}/i\alpha,k\alpha/k\alpha,(i\alpha)^{[1]}/k\alpha} = \mathrm{R}^{k\alpha/i\alpha}(X).\end{aligned}$$

□

For the definition of a Cisinski cofibration category, see definition (3.51)(a). For the definition of the Coheller construction, see definition (5.22).

(5.38) Lemma. We suppose that $\mathcal{C}$ is a Cisinski cofibration category, and we suppose given a commutative diagram

$$\begin{tikzcd} & C \arrow[r, "b''"] & T_Y \\ A \arrow[r, "f'"] & Z \arrow[u, "c"] \arrow[r, "b'"] & T_X \arrow[u, "c'"'] \\ X \arrow[u, "a"] \arrow[r, "f"] & Y \arrow[u, "a'"] \arrow[r, "b"] & B \arrow[u, "a''"'] \end{tikzcd}$$

in $\mathcal{C}$.

(a) If (X, B, A, T_X) is a Coheller rectangle of X and (Y, B, C, T_Y) is a Coheller rectangle of Y, then we have

$$\mathrm{loc}(c') = \mathsf{H}_{(X,B,A,T_X),(Y,B,C,T_Y)}(\mathrm{loc}(f)).$$

(b) If (X, A, B, T_X) is a Coheller rectangle of X and (Y, C, B, T_Y) is a Coheller rectangle of Y, then we have

$$\mathrm{loc}(c') = \mathsf{H}_{(X,A,B,T_X),(Y,C,B,T_Y)}(\mathrm{loc}(f)).$$

Proof. This holds by definition of the Coheller construction as the cuboid

$$\begin{tikzcd} & A \arrow[rr, "f'b'"] \arrow[dd, "f'c"] & & T_X \arrow[dd, "c'"] \\ X \arrow[ur, "a"] \arrow[rr, "fb"] \arrow[dd, "f"] & & B \arrow[ur, "a''"] \arrow[dd, equal] & \\ & C \arrow[rr, "b''"] & & T_Y \\ Y \arrow[ur, "a'c"] \arrow[rr, "b"] & & B \arrow[ur, "a''c'"] & \end{tikzcd}$$

in $\mathcal{C}$ commutes. □

(5.39) Proposition. We suppose that $\mathcal{C}$ is a Cisinski cofibration category, and we suppose given a Heller n-cosemistrip X in $\mathcal{C}$ for some $n \in \mathbb{N}_0$.

(a) For $k/i \in \#_+^n$, we have

$$X^{(k/i)^{[1]}} = \mathsf{H}_{\mathrm{R}^{k/i}(X)}(X^{k/i}).$$

(b) For $k/i, l/j \in \#_+^n$ with $k/i \leq l/j \leq (k/i)^{[1]}$, we have

$$\operatorname{loc}(X^{(k/i)^{[1]},(l/j)^{[1]}}) = \mathsf{H}_{\mathrm{R}^{k/i}(X),\mathrm{R}^{l/j}(X)}(\operatorname{loc}(X^{k/i,l/j})).$$

Proof.

(a) For $k/i \in \#_+^n$, the Coheller rectangle at position k/i in X is given by $\mathrm{R}^{k/i}(X) = X^{k/i,i^{[1]}/i,k/k,i^{[1]}/k}$, and so we have

$$X^{(k/i)^{[1]}} = X^{i^{[1]}/k} = \mathsf{H}_{\mathrm{R}^{k/i}(X)}(X^{k/i}).$$

(b) Given $k/i, l/j \in \#_+^n$ with $k/i \leq l/j \leq (k/i)^{[1]}$, the diagram

$$\begin{tikzcd}
 & X^{l/l} \arrow[r] & X^{i^{[1]}/l} \arrow[r] & X^{j^{[1]}/l} \\
X^{k/k} \arrow[r] & X^{l/k} \arrow[r] \arrow[u] & X^{i^{[1]}/k} \arrow[r] \arrow[u] & X^{j^{[1]}/k} \arrow[u] \\
X^{k/j} \arrow[r] \arrow[u] & X^{l/j} \arrow[r] \arrow[u] & X^{i^{[1]}/j} \arrow[r] \arrow[u] & X^{j^{[1]}/j} \arrow[u] \\
X^{k/i} \arrow[r] \arrow[u] & X^{l/i} \arrow[r] \arrow[u] & X^{i^{[1]}/i} \arrow[u] &
\end{tikzcd}$$

commutes as X is a Heller n-cosemistrip. Since $\mathrm{R}^{k/i}(X) = X^{k/i,i^{[1]}/i,k/k,i^{[1]}/k}$ is a Coheller rectangle of $X^{k/i}$ and $\mathrm{R}^{l/i}(X) = X^{l/i,i^{[1]}/i,l/l,i^{[1]}/l}$ is a Coheller rectangle of $X^{l/i}$, we have

$$\operatorname{loc}(X^{i^{[1]}/k,i^{[1]}/l}) = \mathsf{H}_{\mathrm{R}^{k/i}(X),\mathrm{R}^{l/i}(X)}(\operatorname{loc}(X^{k/i,l/i}))$$

by lemma (5.38)(a). Analogously, since $\mathrm{R}^{l/i}(X) = X^{l/i,i^{[1]}/i,l/l,i^{[1]}/l}$ is a Coheller rectangle of $X^{l/i}$ and $\mathrm{R}^{l/j}(X) = X^{l/j,j^{[1]}/j,l/l,j^{[1]}/l}$ is a Coheller rectangle of $X^{l/j}$, we have

$$\operatorname{loc}(X^{i^{[1]}/l,j^{[1]}/l}) = \mathsf{H}_{\mathrm{R}^{l/i}(X),\mathrm{R}^{l/j}(X)}(\operatorname{loc}(X^{l/i,l/j}))$$

by lemma (5.38)(b). Altogether, we have

$$\begin{aligned}
\operatorname{loc}(X^{(k/i)^{[1]},(l/j)^{[1]}}) &= \operatorname{loc}(X^{(k/i)^{[1]},(l/i)^{[1]}}) \operatorname{loc}(X^{(l/i)^{[1]},(l/j)^{[1]}}) = \operatorname{loc}(X^{i^{[1]}/k,i^{[1]}/l}) \operatorname{loc}(X^{i^{[1]}/l,j^{[1]}/l}) \\
&= \mathsf{H}_{\mathrm{R}^{k/i}(X),\mathrm{R}^{l/i}(X)}(\operatorname{loc}(X^{k/i,l/i}))\, \mathsf{H}_{\mathrm{R}^{l/i}(X),\mathrm{R}^{l/j}(X)}(\operatorname{loc}(X^{l/i,l/j})) \\
&= \mathsf{H}_{\mathrm{R}^{k/i}(X),\mathrm{R}^{l/j}(X)}(\operatorname{loc}(X^{k/i,l/i}) \operatorname{loc}(X^{l/i,l/j})) \\
&= \mathsf{H}_{\mathrm{R}^{k/i}(X),\mathrm{R}^{l/j}(X)}(\operatorname{loc}(X^{k/i,l/j})).
\end{aligned}$$

□

The prolongation lemma

(5.40) Remark. We suppose that $\mathcal{C}$ is a Cisinski cofibration category. A morphism of Heller n-cosemistrips $f\colon X \to Y$ for some $n \in \mathbb{N}_0$ is a pointwise weak equivalence if and only if $f|_{\dot{\Delta}^n}$ is a pointwise weak equivalence.

Proof. This follows from the gluing lemma (3.121) and remark (5.5)(c). □

(5.41) Remark. We suppose that $\mathcal{C}$ is a Cisinski cofibration category, and we suppose given $n \in \mathbb{N}_0$. Moreover, we let

$$s := \begin{cases} 0^{[1]} & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{cases}$$

and we let $\Xi^n := \{k/i \in \#^n_+ \mid i \in \{0, s\}\}$. For the purpose of this remark, an *n-layer* in $\mathcal{C}$ is a Ξ^n-commutative diagram X in $\mathcal{C}_{\mathrm{cof}}$ such that $X^{0/0}$, $X^{0^{[1]}/0}$, $X^{s/s}$, $X^{s^{[1]}/s}$ are coacyclic and such that $X^{k/0,l/0,k/s,l/s}$ is a Coquillen rectangle for $k, l \in \Theta^n_0$ with $0 < k \leq l$.

$$\begin{tikzcd}
 & X^{s/s} \arrow[r] & X^{2/s} \arrow[r] & \dots \arrow[r] & X^{n/s} \arrow[r] & X^{0^{[1]}/s} \arrow[r] & X^{s^{[1]}/s} \\
X^{0/0} \arrow[r] & X^{1/0} \arrow[r] \arrow[u] & X^{2/0} \arrow[r] \arrow[u] & \dots \arrow[r] & X^{n/0} \arrow[r] \arrow[u] & X^{0^{[1]}/0} \arrow[u] &
\end{tikzcd}$$

(a) For every Θ^n_0-commutative diagram X_{res} in $\mathcal{C}_{\mathrm{cof}}$ with $X^{¡}_{\mathrm{res}}$ and $X^{!}_{\mathrm{res}}$ coacyclic there exists an n-layer X in $\mathcal{C}$ with $X_{\mathrm{res}} = X|_{\Theta^n_0}$.

(b) For all n-layers X, Y in $\mathcal{C}$ and every S-2-arrow of Θ^n_0-commutative diagrams $(f_{\mathrm{res}}, u_{\mathrm{res}})\colon X|_{\Theta^n_0} \to \tilde{Y}_{\mathrm{res}} \leftarrow Y|_{\Theta^n_0}$ in $\mathcal{C}_{\mathrm{cof}}$ there exists an S-2-arrow of n-layers $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $(f_{\mathrm{res}}, u_{\mathrm{res}}) = (f|_{\Theta^n_0}, u|_{\Theta^n_0})$.

Proof.

(a) This follows from the Heller factorisation lemma (5.10)(a) and the pushout axiom for cofibrations.

(b) This follows from the Heller factorisation lemma (5.10)(b), the pushout axiom for cofibrations and the gluing lemma (3.121). □

(5.42) Lemma (prolongation lemma)**.** We suppose that $\mathcal{C}$ is a Cisinski cofibration category, and we suppose given $n \in \mathbb{N}_0$.

(a) For every $\dot{\Delta}^n$-commutative diagram X_{b} in $\mathcal{C}_{\mathrm{cof}}$ there exists a Heller n-cosemistrip X in $\mathcal{C}$ with $X_{\mathrm{b}} = X|_{\dot{\Delta}^n}$.

(b) For all Heller n-cosemistrips X and Y in $\mathcal{C}$ and every S-2-arrow $(f_{\mathrm{b}}, u_{\mathrm{b}})\colon X|_{\dot{\Delta}^n} \to \tilde{Y}_{\mathrm{b}} \leftarrow Y|_{\dot{\Delta}^n}$ in $\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}}$ there exists an S-2-arrow of Heller n-cosemistrips $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $(f_{\mathrm{b}}, u_{\mathrm{b}}) = (f|_{\dot{\Delta}^n}, u|_{\dot{\Delta}^n})$.

Proof.

(a) Given a $\dot{\Delta}^n$-commutative diagram X_{b} in $\mathcal{C}_{\mathrm{cof}}$, then by remark (5.41)(a) and an induction there exists a Heller n-cosemistrip X in $\mathcal{C}$ with $X^{¡} = 0$, $X|_{\dot{\Delta}^n} = X_{\mathrm{b}}$, $X^{!} = 0$.

(b) We suppose given Heller n-cosemistrips X and Y in $\mathcal{C}$ and an S-2-arrow $(f_{\mathrm{b}}, u_{\mathrm{b}})\colon X|_{\dot{\Delta}^n} \to \tilde{Y}_{\mathrm{b}} \leftarrow Y|_{\dot{\Delta}^n}$ in $\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}}$. Remark (5.41)(b) and an induction show that there exists an S-2-arrow of Heller n-cosemistrips $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $(f^{¡}, u^{¡}) = (\mathrm{ter}_{X^{¡}}, \mathrm{ini}_{Y^{¡}})$, $(f|_{\dot{\Delta}^n}, u|_{\dot{\Delta}^n}) = (f_{\mathrm{b}}, u_{\mathrm{b}})$, $(f^{!}, u^{!}) = (\mathrm{ter}_{X^{!}}, \mathrm{ini}_{Y^{!}})$. □

(5.43) Corollary. We suppose that $\mathcal{C}$ is a Cisinski cofibration category. For every $n \in \mathbb{N}_0$, the restriction functor

$$(-)|_{\dot{\Delta}^n}\colon \operatorname{Ho}\mathrm{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C}) \to \operatorname{Ho}\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}}$$

is surjective on the objects and full.

Proof. We suppose given $n \in \mathbb{N}_0$. The prolongation lemma (5.42)(a) implies the surjectivity on the objects. To show fullness, we suppose given Heller n-cosemistrips X and Y in $\mathcal{C}$ and a morphism $\varphi_{\mathrm{b}}\colon X|_{\dot{\Delta}^n} \to Y|_{\dot{\Delta}^n}$ in $\operatorname{Ho}\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}}$. As $\mathcal{C}$ is a Cisinski cofibration category, the diagram category $(\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}})_{\mathrm{ptw}}$ is a Brown cofibration category by corollary (3.93). So by remark (3.129)(a), there exists an S-2-arrow $(f_{\mathrm{b}}, u_{\mathrm{b}})\colon X|_{\dot{\Delta}^n} \to \tilde{Y}_{\mathrm{b}} \leftarrow Y|_{\dot{\Delta}^n}$ in $(\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}})_{\mathrm{ptw}}$ with $\varphi_{\mathrm{b}} = \mathrm{loc}(f_{\mathrm{b}}) \, \mathrm{loc}(u_{\mathrm{b}})^{-1}$ in $\operatorname{Ho}\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}}$. The prolongation lemma (5.42)(b) shows that there exists an S-2-arrow of Heller n-cosemistrips $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $(f|_{\dot{\Delta}^n}, u|_{\dot{\Delta}^n}) = (f_{\mathrm{b}}, u_{\mathrm{b}})$. We obtain

$$\varphi_{\mathrm{b}} = \mathrm{loc}(f_{\mathrm{b}}) \, \mathrm{loc}(u_{\mathrm{b}})^{-1} = \mathrm{loc}(f|_{\dot{\Delta}^n}) \, \mathrm{loc}(u|_{\dot{\Delta}^n})^{-1} = (\mathrm{loc}(f) \, \mathrm{loc}(u)^{-1})|_{\dot{\Delta}^n}.$$

Thus $(-)|_{\dot{\Delta}^n}\colon \operatorname{Ho}\mathrm{Strips}^{\mathrm{co},+}_{\mathrm{Heller},n}(\mathcal{C}) \to \operatorname{Ho}\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{cof}}$ is full. □

4 Cosemitriangles

Throughout this section, we suppose given a zero-pointed Brown cofibration category $\mathcal{C}$, that is, a Brown cofibration category as in definition (3.52)(a) equipped with a (distinguished) zero object. Then the homotopy category $\operatorname{Ho}\mathcal{C}$ becomes a zero-pointed category and the localisation functor becomes a morphism of zero-pointed categories, cf. remark (3.11). Moreover, we suppose given a zero-pointed choice of Coheller rectangles $R = (R_X)_{X \in \operatorname{Ob}\mathcal{C}}$ as introduced in definition (5.24). (The zero-pointedness of $\mathcal{C}$ is not needed in the proof of proposition (5.53).)

We move on from Heller cosemistrips in $\mathcal{C}$ as introduced in definition (5.33) to cosemitriangles in $\operatorname{Ho}\mathcal{C}$, see definition (5.45) and definition (5.51), and show that they fulfill prolongation properties similar to those of ordinary triangles in a Verdier triangulated category, see theorem (5.55).

(5.44) Convention. From now on, we consider $\operatorname{Ho}\mathcal{C}$ as a category with shift having $\mathrm{T}^{\operatorname{Ho}\mathcal{C}} = \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R}$. In particular, for every object X in $\operatorname{Ho}\mathcal{C}$ we write $X^{[1]} = \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} X = \mathsf{H}_{R_X}(X)$, and for every morphism $\varphi\colon X \to Y$ in $\operatorname{Ho}\mathcal{C}$ we write $\varphi^{[1]} = \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R}\varphi = \mathsf{H}_{R_X,R_Y}(\varphi)$.

From Heller cosemistrips to standard cosemitriangles

We suppose given a Heller n-cosemistrip X in $\mathcal{C}$ for some $n \in \mathbb{N}_0$. Then the "boundary entries" $X^{i/i}$ and $X^{i^{[1]}/i}$ for $i \in \Theta^n_+$ are coacyclic objects in $\mathcal{C}$, whence

$$X^{i/i} \cong X^{i^{[1]}/i} \cong 0^{\operatorname{Ho}\mathcal{C}}$$

in $\operatorname{Ho}\mathcal{C}$. So $\operatorname{loc}(X)$ is almost an n-cosemicomplex in $\operatorname{Ho}\mathcal{C}$, we only might have some "wrong" zero objects at the boundaries. Moreover, we have

$$X^{(k/i)^{[1]}} = \mathsf{H}_{\mathrm{R}^{k/i}(X)}(X^{k/i}) \cong \mathsf{H}_{R_{X^{k/i}}}(X^{k/i}) = \mathrm{T}^{\mathrm{co}}_{\mathrm{Heller},R} X^{k/i} = (X^{k/i})^{[1]}$$

for $k/i \in \#^n_+$ by proposition (5.39)(a) and remark (5.31). So $\operatorname{loc}(X)$ is almost periodic (at least on the objects), we only might have some "wrong" shift objects on the respectively shifted indices. The standard n-cosemitriangle obtained from X will be defined by an isomorphic replacement of the respective entries, so that we obtain a periodic n-cosemicomplex:

(5.45) Definition (standard n-cosemitriangle)**.** We suppose given $n \in \mathbb{N}_0$.

(a) Given a Heller n-cosemistrip X in $\mathcal{C}$, we define an n-cosemicomplex X^{per} and an isomorphism of n-cosemistrips $\kappa_X\colon \operatorname{loc}(X) \to X^{\mathrm{per}}$ in $\operatorname{Ho}\mathcal{C}$ as follows.

For $i, k \in \Delta^n = (\dot{\Delta}^n)_{\mathsf{i}}$, $m \in \mathbb{N}_0$, we define $\kappa_X^{(k/i)^{[m]}}$ recursively by

$$\kappa_X^{(k/i)^{[m]}} := \begin{cases} \mathrm{ter}_{X^{i/i}} & \text{for } m = 0,\ k = i, \\ 1_{X^{k/i}} & \text{for } m = 0,\ k > i, \\ \mathsf{H}_{\mathrm{R}^{(k/i)^{[m-1]}}(X), R_0}(\kappa_X^{(k/i)^{[m-1]}}) & \text{for } m > 0,\ k = i, \\ \mathsf{H}_{\mathrm{R}^{(k/i)^{[m-1]}}(X), R_{(X^{k/i})^{[m-1]}}}(\kappa_X^{(k/i)^{[m-1]}}) & \text{for } m > 0,\ k > i. \end{cases}$$

The n-cosemistrip X^{per} is called the *standard n-cosemitriangle* obtained from X, and the isomorphism $\kappa_X\colon \operatorname{loc}(X) \to X^{\mathrm{per}}$ is called the *compatibility isomorphism* of X.

A *standard n-cosemitriangle* in $\operatorname{Ho}\mathcal{C}$ is a standard n-cosemitriangle obtained from some Heller n-cosemistrip X in $\mathcal{C}$.

(b) Given a morphism of Heller n-cosemistrips $f\colon X \to Y$, we define a morphism of n-cosemicomplexes $f^{\mathrm{per}}\colon X^{\mathrm{per}} \to Y^{\mathrm{per}}$ by $f^{\mathrm{per}} := \kappa_X^{-1} \operatorname{loc}(f)\, \kappa_Y$.

$$\begin{tikzcd} \operatorname{loc}(X) \arrow[r, "\kappa_X", "\cong"'] \arrow[d, "\operatorname{loc}(f)"'] & X^{\mathrm{per}} \arrow[d, "f^{\mathrm{per}}"] \\ \operatorname{loc}(Y) \arrow[r, "\kappa_Y", "\cong"'] & Y^{\mathrm{per}} \end{tikzcd}$$

The morphism of n-cosemistrips f^{per} is called the *morphism of standard n-cosemitriangles* obtained from f.

Periodicity

Our next aim is to show that every standard n-cosemitriangle in $\operatorname{Ho}\mathcal{C}$ for $n \in \mathbb{N}_0$ is periodic in the sense of definition (b).

(5.46) Remark. For $n \in \mathbb{N}_0$, $k/i \in \#_0^n$ with $i, k \in \Delta^n = (\dot{\Delta}^n)_\mathrm{i}$, $m \in \mathbb{N}_0$, we have

$$(X^{\mathrm{per}})^{(k/i)^{[m]}} = \begin{Bmatrix} 0 & \text{if } k = i, \\ (X^{k/i})^{[m]} & \text{if } k > i \end{Bmatrix} = ((X^{\mathrm{per}})^{k/i})^{[m]}.$$

(5.47) Proposition. For $n \in \mathbb{N}_0$, $k/i \in \#_+^n$, $m \in \mathbb{N}$, we have

$$\kappa_X^{(k/i)^{[m]}} = \mathsf{H}_{\mathrm{R}^{(k/i)^{[m-1]}}(X), R_{((X^{\mathrm{per}})^{k/i})^{[m-1]}}}(\kappa_X^{(k/i)^{[m-1]}}).$$

Proof. There exist $i_0, k_0 \in \Delta^n$, $r \in \mathbb{N}_0$, with $k/i = (k_0/i_0)^{[r]}$. By remark (5.46), we obtain

$$\begin{aligned} \kappa_X^{(k/i)^{[m]}} &= \kappa_X^{(k_0/i_0)^{[r+m]}} = \begin{Bmatrix} \mathsf{H}_{\mathrm{R}^{(k_0/i_0)^{[r+m-1]}}(X), R_0}(\kappa_X^{(k_0/i_0)^{[r+m-1]}}) & \text{if } k_0 = i_0, \\ \mathsf{H}_{\mathrm{R}^{(k_0/i_0)^{[r+m-1]}}(X), R_{(X^{k_0/i_0})^{[r+m-1]}}}(\kappa_X^{(k_0/i_0)^{[r+m-1]}}) & \text{if } k_0 > i_0 \end{Bmatrix} \\ &= \mathsf{H}_{\mathrm{R}^{(k_0/i_0)^{[r+m-1]}}(X), R_{(X^{\mathrm{per}})^{(k_0/i_0)^{[r+m-1]}}}}(\kappa_X^{(k_0/i_0)^{[r+m-1]}}) \\ &= \mathsf{H}_{\mathrm{R}^{(k/i)^{[m-1]}}(X), R_{(X^{\mathrm{per}})^{(k/i)^{[m-1]}}}}(\kappa_X^{(k/i)^{[m-1]}}). \end{aligned}$$

□

(5.48) Corollary. We suppose given $n \in \mathbb{N}_0$.

(a) Given a Heller n-cosemistrip X in $\mathcal{C}$, the standard n-triangle X^{per} obtained from X is a periodic n-cosemicomplex.

(b) Given a morphism of Heller n-cosemistrips $f \colon X \to Y$ in $\mathcal{C}$, the morphism of standard n-triangles $f^{\mathrm{per}} \colon X^{\mathrm{per}} \to Y^{\mathrm{per}}$ obtained from f is a periodic morphism of n-cosemicomplexes.

Proof.

(a) For $i \in \Theta_0^n$, $m \in \mathbb{N}_0$, we have $(X^{\mathrm{per}})^{(i/i)^{[m]}} = 0$, that is, X is an n-cosemicomplex. Moreover, for $k/i, l/j \in \#_+^n$ with $k/i \leq l/j \leq (k/i)^{[1]}$, we have

$$\begin{aligned} &(X^{\mathrm{per}})^{(k/i)^{[1]}, (l/j)^{[1]}} = (\kappa_X^{(k/i)^{[1]}})^{-1} \operatorname{loc}(X^{(k/i)^{[1]}, (l/j)^{[1]}}) \, \kappa_X^{(l/j)^{[1]}} \\ &= (\mathsf{H}_{\mathrm{R}^{k/i}(X), R_{(X^{\mathrm{per}})^{k/i}}}(\kappa_X^{k/i}))^{-1} \, \mathsf{H}_{\mathrm{R}^{k/i}(X), \mathrm{R}^{l/j}(X)}(\operatorname{loc}(X^{k/i, l/j})) \, \mathsf{H}_{\mathrm{R}^{l/j}(X), R_{(X^{\mathrm{per}})^{l/j}}}(\kappa_X^{l/j}) \\ &= \mathsf{H}_{R_{(X^{\mathrm{per}})^{k/i}}, R_{(X^{\mathrm{per}})^{l/j}}}((\kappa_X^{k/i})^{-1} \operatorname{loc}(X^{k/i, l/j}) \, \kappa_X^{l/j}) = \mathsf{H}_{R_{(X^{\mathrm{per}})^{k/i}}, R_{(X^{\mathrm{per}})^{l/j}}}((X^{\mathrm{per}})^{k/i, l/j}) \\ &= ((X^{\mathrm{per}})^{k/i, l/j})^{[1]} \end{aligned}$$

by proposition (5.39)(b) and proposition (5.47), and so X^{per} is periodic.

(b) By (a), the standard n-triangles X^{per} and Y^{per} are periodic n-cosemicomplexes. Moreover, for $k/i \in \#_+^n$, we have

$$\begin{aligned} &(f^{\mathrm{per}})^{(k/i)^{[1]}} = (\kappa_X^{(k/i)^{[1]}})^{-1} \operatorname{loc}(f^{(k/i)^{[1]}}) \, \kappa_Y^{(k/i)^{[1]}} \\ &= (\mathsf{H}_{\mathrm{R}^{k/i}(X), R_{(X^{\mathrm{per}})^{k/i}}}(\kappa_X^{k/i}))^{-1} \, \mathsf{H}_{\mathrm{R}^{k/i}(X), \mathrm{R}^{k/i}(Y)}(\operatorname{loc}(f^{k/i})) \, \mathsf{H}_{\mathrm{R}^{k/i}(Y), R_{(Y^{\mathrm{per}})^{k/i}}}(\kappa_Y^{k/i}) \\ &= \mathsf{H}_{R_{(X^{\mathrm{per}})^{k/i}}, R_{(Y^{\mathrm{per}})^{k/i}}}((\kappa_X^{k/i})^{-1} \operatorname{loc}(f^{k/i}) \, \kappa_Y^{k/i}) = \mathsf{H}_{R_{(X^{\mathrm{per}})^{k/i}}, R_{(Y^{\mathrm{per}})^{k/i}}}((f^{\mathrm{per}})^{k/i}) = ((f^{\mathrm{per}})^{k/i})^{[1]} \end{aligned}$$

by proposition (5.47), and so f^{per} is a periodic morphism of n-cosemicomplexes. □

(5.49) Corollary. Given a morphism of semiquasicyclic types $\alpha \colon \Theta_+^m \to \Theta_+^n$ for $m, n \in \mathbb{N}_0$ and a Heller n-cosemistrip X in $\mathcal{C}$, the isomorphism of m-cosemicomplexes

$$(\mathrm{Strips}_\alpha^{\mathrm{co},+}(\operatorname{Ho}\mathcal{C})\kappa_X)^{-1} \, \kappa_{\mathrm{Strips}_\alpha^{\mathrm{co},+}(\mathcal{C})X} \colon \mathrm{Com}_{\mathrm{per},\alpha}^{\mathrm{co},+}(\operatorname{Ho}\mathcal{C})X^{\mathrm{per}} \to (\mathrm{Strips}_{\mathrm{Heller},\alpha}^{\mathrm{co},+}(\mathcal{C})X)^{\mathrm{per}}$$

is periodic.

Proof. By proposition (5.47) and remark (5.37), we have

$$((\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\kappa_X)^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X})^{(k/i)^{[1]}} = ((\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\kappa_X)^{(k/i)^{[1]}})^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X}^{(k/i)^{[1]}}$$
$$= (\kappa_X^{(k\alpha/i\alpha)^{[1]}})^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X}^{(k/i)^{[1]}}$$
$$= (\mathsf{H}_{\mathrm{R}^{k\alpha/i\alpha}(X),R_{(X^{\mathrm{per}})^{k\alpha/i\alpha}}}(\kappa_X^{k\alpha/i\alpha}))^{-1}\,\mathsf{H}_{\mathrm{R}^{k/i}(\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X),R_{((\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X)^{\mathrm{per}})^{k/i}}}(\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X}^{k/i})$$
$$= \mathsf{H}_{R_{(X^{\mathrm{per}})^{k\alpha/i\alpha}},\mathrm{R}^{k\alpha/i\alpha}(X)}((\kappa_X^{k\alpha/i\alpha})^{-1})\,\mathsf{H}_{\mathrm{R}^{k\alpha/i\alpha}(X),R_{((\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X)^{\mathrm{per}})^{k/i}}}(\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X}^{k/i})$$
$$= \mathsf{H}_{R_{(X^{\mathrm{per}})^{k\alpha/i\alpha}},R_{((\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X)^{\mathrm{per}})^{k/i}}}((\kappa_X^{k\alpha/i\alpha})^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X}^{k/i}) = ((\kappa_X^{k\alpha/i\alpha})^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X}^{k/i})^{[1]}$$
$$= (((\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\kappa_X)^{k/i})^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X}^{k/i})^{[1]} = (((\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\kappa_X)^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})X})^{k/i})^{[1]}$$

for $k/i \in \#_{+}^{m}$. □

The semiquasicyclic category of cosemitriangles

Given a Heller n-cosemistrip X in $\mathcal{C}$ and a morphism of semiquasicyclic types $\alpha\colon \Theta_{+}^{m} \to \Theta_{+}^{n}$ for some $m, n \in \mathbb{N}_0$, we will *not* have $\mathrm{Com}_{\mathrm{per},\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})X^{\mathrm{per}} = (\mathrm{Strips}_{\mathrm{Heller},\alpha}^{\mathrm{co},+}(\mathcal{C})X)^{\mathrm{per}}$ in general. So the sets of standard n-cosemitriangles are not stable under semiquasicyclic operations. But if we consider the standard n-cosemitriangles only up to isomorphism in $\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$, we obtain stability under semiquasicyclic operations, as the following proposition shows.

(5.50) Proposition. We have a full semiquasicyclic zero-pointed subcategory $\mathrm{Tri}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ of $\mathrm{Com}_{\mathrm{per}}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ given by

$$\mathrm{Ob}\,\mathrm{Tri}_{n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C}) = \{X \in \mathrm{Ob}\,\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C}) \mid X \cong_{\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})} \tilde{X}^{\mathrm{per}} \text{ for some } \tilde{X} \in \mathrm{Ob}\,\mathrm{Strips}_{\mathrm{Heller},n}^{\mathrm{co},+}(\mathcal{C})\}$$

for $n \in \mathbb{N}_0$.

Proof. We suppose given a morphism of semiquasicyclic types $\alpha\colon \Theta_{+}^{m} \to \Theta_{+}^{n}$ for $m, n \in \mathbb{N}_0$ and an object X in $\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ such that $X \cong_{\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})} \tilde{X}^{\mathrm{per}}$ for some $\tilde{X} \in \mathrm{Ob}\,\mathrm{Strips}_{\mathrm{Heller},n}^{\mathrm{co},+}(\mathcal{C})$. We choose an isomorphism $\psi\colon X \to \tilde{X}^{\mathrm{per}}$ in $\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$. Then

$$\mathrm{Com}_{\mathrm{per},\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\psi\colon \mathrm{Com}_{\mathrm{per},\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})X \to \mathrm{Com}_{\mathrm{per},\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\tilde{X}^{\mathrm{per}}$$

is an isomorphism in $\mathrm{Com}_{\mathrm{per},m}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$, and

$$(\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\kappa_{\tilde{X}})^{-1}\,\kappa_{\mathrm{Strips}_{\alpha}^{\mathrm{co},+}(\mathcal{C})\tilde{X}}\colon \mathrm{Com}_{\mathrm{per},\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})\tilde{X}^{\mathrm{per}} \to (\mathrm{Strips}_{\mathrm{Heller},\alpha}^{\mathrm{co},+}(\mathcal{C})\tilde{X})^{\mathrm{per}}$$

is an isomorphism in $\mathrm{Com}_{\mathrm{per},m}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ by corollary (5.49). Thus we have

$$\mathrm{Com}_{\mathrm{per},\alpha}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})X \cong_{\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})} (\mathrm{Strips}_{\mathrm{Heller},\alpha}^{\mathrm{co},+}(\mathcal{C})\tilde{X})^{\mathrm{per}}.$$ □

(5.51) Definition (semiquasicyclic category of cosemitriangles)**.** The full semiquasicyclic zero-pointed subcategory $\mathrm{Tri}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ of $\mathrm{Com}_{\mathrm{per}}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ as in proposition (5.50) is called the *semiquasicyclic category of cosemitriangles* in $\mathrm{Ho}\,\mathcal{C}$. For $n \in \mathbb{N}_0$, the category $\mathrm{Tri}_{n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ is called the *category of n-cosemitriangles* in $\mathrm{Ho}\,\mathcal{C}$, an object in $\mathrm{Tri}_{n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ is called an *n-cosemitriangle* in $\mathcal{C}$, and a morphism in $\mathrm{Tri}_{n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$ is called a *morphism of n-cosemitriangles* in $\mathcal{C}$.

Prolongation

In the rest of this section, we are going to prove the main theorem of this chapter, see theorem (5.55), which states that every "potential base" of a cosemitriangle resp. of a morphism of cosemitriangles may be prolonged to a cosemitriangle resp. a morphism of cosemitriangles that actually has this given "potential base" as base, in the following sense.

(5.52) Definition (base)**.** We suppose given $n \in \mathbb{N}_0$.

(a) Given an n-cosemitriangle X in $\operatorname{Ho}\mathcal{C}$, the restriction $X|_{\dot\Delta^n}$ is called the *base* of X.

(b) Given a morphism of n-cosemitriangles $\varphi\colon X \to Y$ in $\operatorname{Ho}\mathcal{C}$, the restriction $\varphi|_{\dot\Delta^n}$ is called the *base* of φ.

The zero-pointedness of $\mathcal{C}$ is not needed in the following proposition, which is a particular case of [9, dual of prop. 2.15]. For the definition and the values of the diagram functor, see definition (3.74) and remark (3.75).

(5.53) Proposition (cf. Cisinski [9, dual of prop. 2.15])**.** We suppose given $n \in \mathbb{N}_0$. The diagram functor

$$\operatorname{dia}\colon \operatorname{Ho}\mathcal{C}^{\dot\Delta^n} \to (\operatorname{Ho}\mathcal{C})^{\dot\Delta^n}$$

is dense and full.

Proof. We proceed by induction on $n \in \mathbb{N}_0$. For $n \in \{0, 1\}$, the diagram functor $\operatorname{dia}_{\Delta^n}\colon \operatorname{Ho}\mathcal{C}^{\Delta^n} \to (\operatorname{Ho}\mathcal{C})^{\Delta^n}$ is an isofunctor, whence in particular dense and full.
So we suppose given $n \in \mathbb{N}$ with $n \geq 2$, and we suppose that $\operatorname{dia}_{\dot\Delta^{n-1}}\colon \operatorname{Ho}\mathcal{C}^{\dot\Delta^{n-1}} \to (\operatorname{Ho}\mathcal{C})^{\dot\Delta^{n-1}}$ is dense and full. To show that $\operatorname{dia}_{\dot\Delta^n}\colon \operatorname{Ho}\mathcal{C}^{\dot\Delta^n} \to (\operatorname{Ho}\mathcal{C})^{\dot\Delta^n}$ is dense, we suppose given an object Y in $(\operatorname{Ho}\mathcal{C})^{\dot\Delta^n}$. By the induction hypothesis, there exist an object X_{res} in $\operatorname{Ho}\mathcal{C}^{\dot\Delta^{n-1}}$ and an isomorphism $\psi_{\mathrm{res}}\colon \operatorname{dia}_{\dot\Delta^{n-1}}(X_{\mathrm{res}}) \to Y|_{\dot\Delta^{n-1}}$. We choose an S-2-arrow $(f, u)\colon X_{\mathrm{res},n-1} \to \tilde{Y}_n \leftarrow Y_n$ in $\mathcal{C}$ with $\psi_{\mathrm{res},n-1} Y_{n-1,n} = \operatorname{loc}(f) \operatorname{loc}(u)^{-1}$, cf. remark (3.129)(a).

$$\begin{tikzcd}
X_{\mathrm{res},1} \arrow[r, "\operatorname{loc}(X_{\mathrm{res},(1,2)})"] \arrow[d, "\psi_{\mathrm{res},1}", "\cong"'] & X_{\mathrm{res},2} \arrow[r, "\operatorname{loc}(X_{\mathrm{res},(2,3)})"] \arrow[d, "\psi_{\mathrm{res},2}", "\cong"'] & \ldots \arrow[r, "\operatorname{loc}(X_{\mathrm{res},(n-2,n-1)})"] & X_{\mathrm{res},n-1} \arrow[r, "\operatorname{loc}(f)"] \arrow[d, "\psi_{\mathrm{res},n-1}", "\cong"'] & \tilde{Y}_n \arrow[d, "\operatorname{loc}(u)^{-1}", "\cong"'] \\
Y_1 \arrow[r, "Y_{1,2}"] & Y_2 \arrow[r, "Y_{2,3}"] & \ldots \arrow[r, "Y_{n-2,n-1}"] & Y_{n-1} \arrow[r, "Y_{n-1,n}"] & Y_n
\end{tikzcd}$$

We let X be the unique $\dot\Delta^n$-commutative diagram in $\mathcal{C}$ with $X|_{\dot\Delta^{n-1}} = X_{\mathrm{res}}$ and $X_{n-1,n} = f$, and we let $\psi\colon \operatorname{dia}_{\dot\Delta^n}(X) \to Y$ be the unique morphism of $\dot\Delta^n$-commutative diagrams in $\operatorname{Ho}\mathcal{C}$ with $\psi|_{\dot\Delta^{n-1}} = \psi_{\mathrm{res}}$ and $\psi_n = \operatorname{loc}(u)^{-1}$. Then ψ is an isomorphism in $(\operatorname{Ho}\mathcal{C})^{\dot\Delta^n}$ from $\operatorname{loc}(X)$ to Y. Thus $\operatorname{dia}_{\dot\Delta^n}\colon \operatorname{Ho}\mathcal{C}^{\dot\Delta^n} \to (\operatorname{Ho}\mathcal{C})^{\dot\Delta^n}$ is dense.
To show that $\operatorname{dia}_{\dot\Delta^n}\colon \operatorname{Ho}\mathcal{C}^{\dot\Delta^n} \to (\operatorname{Ho}\mathcal{C})^{\dot\Delta^n}$ is full, we suppose given objects X, X' in $\operatorname{Ho}\mathcal{C}^{\dot\Delta^n}$ and a morphism $\psi\colon \operatorname{dia}_{\dot\Delta^n}(X) \to \operatorname{dia}_{\dot\Delta^n}(X')$ in $(\operatorname{Ho}\mathcal{C})^{\dot\Delta^n}$. Then by the induction hypothesis, there exists a morphism $\varphi_{\mathrm{res}}\colon X|_{\dot\Delta^{n-1}} \to X|_{\dot\Delta^{n-1}}$ in $\operatorname{Ho}\mathcal{C}^{\dot\Delta^{n-1}}$ with $\psi|_{\dot\Delta^{n-1}} = \operatorname{dia}_{\dot\Delta^{n-1}}(\varphi_{\mathrm{res}})$. As $\mathcal{C}$ is a Brown cofibration category, the diagram category $\mathcal{C}^{\dot\Delta^{n-1}}_{\mathrm{ptw}}$ is also a Brown cofibration category by corollary (3.93). So by theorem (3.128)(a), there exists a Z-2-arrow $(f_{\mathrm{res}}, i_{\mathrm{res}})\colon X|_{\dot\Delta^{n-1}} \to \tilde{X}'_{\mathrm{res}} \leftarrow X'|_{\dot\Delta^{n-1}}$ in $\mathcal{C}^{\dot\Delta^{n-1}}_{\mathrm{ptw}}$ with $\varphi_{\mathrm{res}} = \operatorname{loc}(f_{\mathrm{res}}) \operatorname{loc}(i_{\mathrm{res}})^{-1}$ in $\operatorname{Ho}\mathcal{C}^{\dot\Delta^{n-1}}$. Moreover, there exists a Z-2-arrow $(g, j)\colon X_n \to \bar{X}'_n \leftarrow X'_n$ in $\mathcal{C}$ with $\psi_n = \operatorname{loc}(g) \operatorname{loc}(j)^{-1}$ in $\operatorname{Ho}\mathcal{C}$.

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "f_{\mathrm{res},1}"] & X_2 \arrow[r] \arrow[d, "f_{\mathrm{res},2}"] & \ldots \arrow[r] & X_{n-1} \arrow[r] \arrow[d, "f_{\mathrm{res},n-1}"] & X_n \arrow[d, "g"] \\
\tilde{X}'_{\mathrm{res},1} \arrow[r] & \tilde{X}'_{\mathrm{res},2} \arrow[r] & \ldots \arrow[r] & \tilde{X}'_{\mathrm{res},n-1} & \bar{X}'_n \\
X'_1 \arrow[r] \arrow[u, "i_{\mathrm{res},1}", "\sim"'] & X'_2 \arrow[r] \arrow[u, "i_{\mathrm{res},2}", "\sim"'] & \ldots \arrow[r] & X'_{n-1} \arrow[r] \arrow[u, "i_{\mathrm{res},n-1}", "\sim"'] & X'_n \arrow[u, "j", "\sim"']
\end{tikzcd}$$

We obtain

$$\begin{aligned}
\operatorname{loc}(f_{\mathrm{res},n-1}) \operatorname{loc}(i_{\mathrm{res},n-1})^{-1} \operatorname{loc}(X'_{n-1,n}) &= \operatorname{dia}_{\dot\Delta^{n-1}}(\varphi_{\mathrm{res}})_{n-1} \operatorname{dia}_{\dot\Delta^n}(X')_{n-1,n} = \psi_{n-1} \operatorname{dia}_{\dot\Delta^n}(X')_{n-1,n} \\
&= \operatorname{dia}_{\dot\Delta^n}(X)_{n-1,n}\, \psi_n = \operatorname{loc}(X_{n-1,n}) \operatorname{loc}(g) \operatorname{loc}(j)^{-1},
\end{aligned}$$

so that by theorem (3.128)(c) there exist a Z-2-arrow $(\tilde{g}, \tilde{j})\colon X_n \to A \leftarrow X'_n$ and a normal S-2-arrow $(h, k)\colon \tilde{X}'_{\mathrm{res},n-1} \to A \leftarrow \bar{X}_n$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
X_{n-1} \arrow[r] \arrow[d, "f_{\mathrm{res},n-1}"] & X_n \arrow[r, equal] \arrow[d, dotted, "\tilde{g}"] & X_n \arrow[d, "g"] \\
\tilde{X}'_{\mathrm{res},n-1} \arrow[r, dotted, "h"] & A & \bar{X}'_n \arrow[l, dotted, "k", "\approx"'] \\
X'_{n-1} \arrow[r] \arrow[u, "i_{\mathrm{res},n-1}", "\wr"'] & X'_n \arrow[r, equal] \arrow[u, dotted, "\tilde{j}", "\wr"'] & X'_n \arrow[u, "j", "\wr"']
\end{tikzcd}$$

We let $\tilde{X}'$ be the unique $\dot{\Delta}^n$-commutative diagram in $\mathcal{C}$ with $\tilde{X}'|_{\dot{\Delta}^{n-1}} = \tilde{X}'_{\mathrm{res}}$ and $\tilde{X}'_{n-1,n} = h$, and we let $(f, i)\colon X \to \tilde{X}' \leftarrow X'$ be the unique Z-2-arrow in $\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{ptw}}$ with $f|_{\dot{\Delta}^{n-1}} = f_{\mathrm{res}}$, $f_n = \tilde{g}$, $i|_{\dot{\Delta}^{n-1}} = i_{\mathrm{res}}$, $i_n = \tilde{j}$. Then we have

$$\begin{aligned}
\psi|_{\dot{\Delta}^{n-1}} &= \mathrm{dia}_{\dot{\Delta}^{n-1}}(\varphi_{\mathrm{res}}) = \mathrm{dia}_{\dot{\Delta}^{n-1}}(\mathrm{loc}(f_{\mathrm{res}})\,\mathrm{loc}(i_{\mathrm{res}})^{-1}) = \mathrm{dia}_{\dot{\Delta}^{n-1}}(\mathrm{loc}(f|_{\dot{\Delta}^{n-1}})\,\mathrm{loc}(i|_{\dot{\Delta}^{n-1}})^{-1}) \\
&= \mathrm{dia}_{\dot{\Delta}^n}(\mathrm{loc}(f)\,\mathrm{loc}(i)^{-1})|_{\dot{\Delta}^{n-1}}
\end{aligned}$$

and

$$\psi_n = \mathrm{loc}(g)\,\mathrm{loc}(j)^{-1} = \mathrm{loc}(f_n)\,\mathrm{loc}(i_n)^{-1} = \mathrm{dia}_{\dot{\Delta}^n}(\mathrm{loc}(f)\,\mathrm{loc}(i)^{-1})_n,$$

that is, $\psi = \mathrm{dia}_{\dot{\Delta}^n}(\mathrm{loc}(f)\,\mathrm{loc}(i)^{-1})$. Thus $\mathrm{dia}_{\dot{\Delta}^n}\colon \mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n} \to (\mathrm{Ho}\,\mathcal{C})^{\dot{\Delta}^n}$ is full. □

(5.54) Proposition. We suppose given $n \in \mathbb{N}_0$.

(a) For every $\dot{\Delta}^n$-commutative diagram X_{b} in $\mathcal{C}$ there exists a Heller n-cosemistrip X in $\mathcal{C}$ such that the standard n-cosemitriangle X^{per} has the base $\mathrm{loc}(X_{\mathrm{b}})$.

(b) For all Heller n-cosemistrips X and Y in $\mathcal{C}$ and every morphism $\varphi_{\mathrm{b}}\colon X^{\mathrm{per}}|_{\dot{\Delta}^n} \to Y^{\mathrm{per}}|_{\dot{\Delta}^n}$ in $(\mathrm{Ho}\,\mathcal{C})^{\dot{\Delta}^n}$ there exists an S-2-arrow of Heller n-cosemistrips $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ such that $f^{\mathrm{per}}\,(u^{\mathrm{per}})^{-1}\colon X^{\mathrm{per}} \to Y^{\mathrm{per}}$ has the base φ_{b}.

Proof.

(a) We suppose given a $\dot{\Delta}^n$-commutative diagram X_{b} in $\mathcal{C}$. By the prolongation lemma (5.42)(a) there exists a Heller n-cosemistrip X with $X|_{\dot{\Delta}^n} = X_{\mathrm{b}}$. As $\kappa^i_X = 1_{X^i}$ for $i \in \dot{\Delta}^n$, the standard n-cosemitriangle X^{per} has the base

$$X^{\mathrm{per}}|_{\dot{\Delta}^n} = \mathrm{loc}(X)|_{\dot{\Delta}^n} = \mathrm{loc}(X|_{\dot{\Delta}^n}) = \mathrm{loc}(X_{\mathrm{b}}).$$

(b) We suppose given Heller n-cosemistrips X and Y in $\mathcal{C}$ and a morphism $\varphi_{\mathrm{b}}\colon X^{\mathrm{per}}|_{\dot{\Delta}^n} \to Y^{\mathrm{per}}|_{\dot{\Delta}^n}$ in $(\mathrm{Ho}\,\mathcal{C})^{\dot{\Delta}^n}$. By proposition (5.53), there exists a morphism $\psi_{\mathrm{b}}\colon X^{\mathrm{per}}|_{\dot{\Delta}^n} \to Y^{\mathrm{per}}|_{\dot{\Delta}^n}$ in $\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n}$ with $\varphi_{\mathrm{b}} = \mathrm{dia}(\psi_{\mathrm{b}})$. As $\mathcal{C}^{\dot{\Delta}^n}_{\mathrm{ptw}}$ is a Brown cofibration category by corollary (3.93), there exists an S-2-arrow of $\dot{\Delta}^n$-commutative diagrams $(f_{\mathrm{b}}, u_{\mathrm{b}})\colon X|_{\dot{\Delta}^n} \to \tilde{Y}_{\mathrm{b}} \leftarrow Y|_{\dot{\Delta}^n}$ in $\mathcal{C}$ with $\psi_{\mathrm{b}} = \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n}}(f_{\mathrm{b}})\,\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n}}(u_{\mathrm{b}})^{-1}$ by remark (3.129)(a). Thus we have

$$\begin{aligned}
\varphi_{\mathrm{b}} &= \mathrm{dia}(\psi_{\mathrm{b}}) = \mathrm{dia}(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n}}(f_{\mathrm{b}})\,\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n}}(u_{\mathrm{b}})^{-1}) = \mathrm{dia}(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n}}(f_{\mathrm{b}}))\,\mathrm{dia}(\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n}}(u_{\mathrm{b}}))^{-1} \\
&= \mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}}(f_{\mathrm{b}})\,\mathrm{loc}^{\mathrm{Ho}\,\mathcal{C}}(u_{\mathrm{b}})^{-1}.
\end{aligned}$$

By the prolongation lemma (5.42)(b), there exists an S-2-arrow of Heller n-cosemistrips $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $(f_{\mathrm{b}}, u_{\mathrm{b}}) = (f|_{\dot{\Delta}^n}, u|_{\dot{\Delta}^n})$. As $\kappa^i_X = 1_{X^i}$ for $i \in \dot{\Delta}^n$, the morphism of standard n-cosemitriangles $f^{\mathrm{per}}\,(u^{\mathrm{per}})^{-1}\colon X^{\mathrm{per}} \to Y^{\mathrm{per}}$ has the base

$$\begin{aligned}
(f^{\mathrm{per}}\,(u^{\mathrm{per}})^{-1})|_{\dot{\Delta}^n} &= f^{\mathrm{per}}|_{\dot{\Delta}^n}\,(u^{\mathrm{per}}|_{\dot{\Delta}^n})^{-1} = \mathrm{loc}(f)|_{\dot{\Delta}^n}\,(\mathrm{loc}(u)|_{\dot{\Delta}^n})^{-1} = \mathrm{loc}(f|_{\dot{\Delta}^n})\,\mathrm{loc}(u|_{\dot{\Delta}^n})^{-1} \\
&= \mathrm{loc}(f_{\mathrm{b}})\,\mathrm{loc}(u_{\mathrm{b}})^{-1} = \varphi_{\mathrm{b}}.
\end{aligned}$$

□

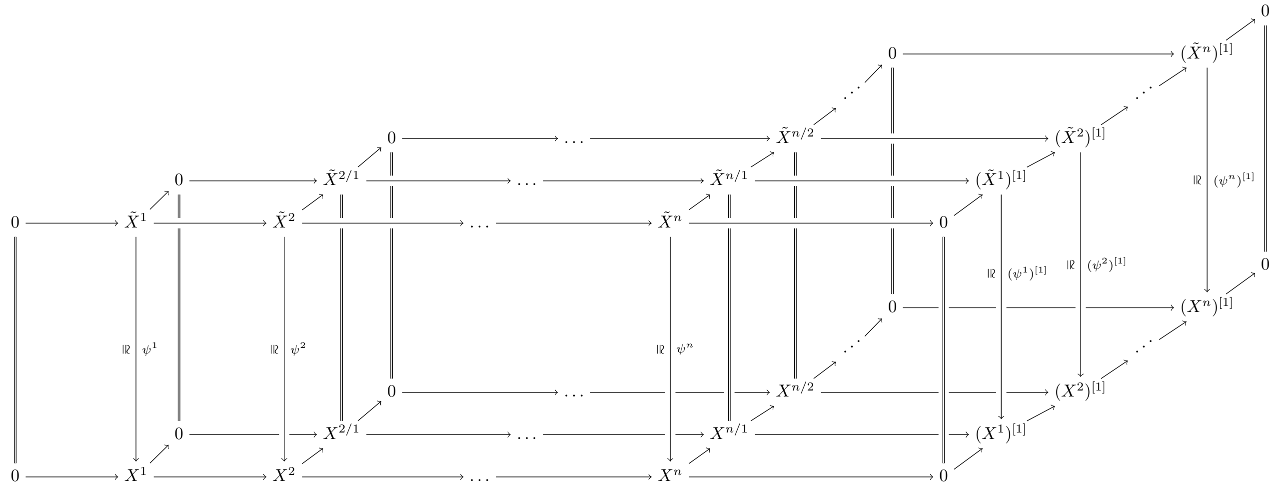


Figure 1: Prolongation to n-cosemitriangles.

(5.55) Theorem (prolongation theorem)**.** We suppose given $n \in \mathbb{N}_0$. The restriction functor

$$(-)|_{\dot{\Delta}^n} \colon \mathrm{Tri}_n^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C}) \to (\mathrm{Ho}\,\mathcal{C})^{\dot{\Delta}^n}$$

is surjective on the objects and full. In other words:

(a) For every $\dot{\Delta}^n$-commutative diagram X_{b} in $\mathrm{Ho}\,\mathcal{C}$ there exists an n-cosemitriangle X in $\mathrm{Ho}\,\mathcal{C}$ with base X_{b}.

(b) For all n-cosemitriangles X, Y and every morphism of $\dot{\Delta}^n$-commutative diagrams $\varphi_{\mathrm{b}} \colon X|_{\Delta^n} \to Y|_{\Delta^n}$ in $\mathrm{Ho}\,\mathcal{C}$ there exists a morphism of n-cosemitriangles $\varphi \colon X \to Y$ in $\mathrm{Ho}\,\mathcal{C}$ with base φ_{b}.

Proof.

(a) We suppose given a $\dot{\Delta}^n$-commutative diagram X_{b} in $\mathrm{Ho}\,\mathcal{C}$. By proposition (5.53), the diagram functor dia: $\mathrm{Ho}\,\mathcal{C}^{\dot{\Delta}^n} \to (\mathrm{Ho}\,\mathcal{C})^{\dot{\Delta}^n}$ is dense, and so there exists a $\dot{\Delta}^n$-commutative diagram $\tilde{X}_{\mathrm{b}}$ in $\mathcal{C}$ with $X_{\mathrm{b}} \cong \mathrm{loc}(\tilde{X}_{\mathrm{b}})$. We choose an isomorphism $\psi_{\mathrm{b}} \colon \mathrm{loc}(\tilde{X}_{\mathrm{b}}) \to X_{\mathrm{b}}$. By proposition (5.54)(a), there exists a Heller n-cosemistrip $\tilde{X}$ in $\mathcal{C}$ such that $\tilde{X}^{\mathrm{per}}$ has the base $\mathrm{loc}(\tilde{X}_{\mathrm{b}})$. We let X be the unique periodic n-cosemicomplex and $\psi \colon \tilde{X}^{\mathrm{per}} \to X$ be the unique morphism of periodic n-cosemicomplexes in $\mathrm{Ho}\,\mathcal{C}$ with

$$\psi^{k/i} = \begin{cases} \psi_{\mathrm{b}}^k & \text{if } i = \text{¡},\ k \in \dot{\Delta}^n, \\ 1_{(\tilde{X}^{\mathrm{per}})^{k/i}} & \text{if } i, k \in \dot{\Delta}^n, \\ (\psi_{\mathrm{b}}^k)^{[1]} & \text{if } i \in \dot{\Delta}^n,\ k = !, \end{cases}$$

for $k/i \in \#_0^n$. Then X is an n-cosemitriangle in $\mathrm{Ho}\,\mathcal{C}$ with base

$$X|_{\dot{\Delta}^n} = \mathrm{Target}\,\psi|_{\dot{\Delta}^n} = \mathrm{Target}\,\psi_{\mathrm{b}} = X_{\mathrm{b}}.$$

(b) We suppose given n-cosemitriangles X, Y and a morphism of $\dot{\Delta}^n$-commutative diagrams $\varphi_{\mathrm{b}} \colon X|_{\dot{\Delta}^n} \to Y|_{\dot{\Delta}^n}$ in $\mathrm{Ho}\,\mathcal{C}$. We choose Heller n-cosemistrips $\tilde{X}$, $\tilde{Y}$ in $\mathcal{C}$ and isomorphisms $\psi \colon \tilde{X}^{\mathrm{per}} \to X$, $\rho \colon \tilde{Y}^{\mathrm{per}} \to Y$ in $\mathrm{Com}_{\mathrm{per},n}^{\mathrm{co},+}(\mathrm{Ho}\,\mathcal{C})$. By proposition (5.54)(b), there exists an S-2-arrow of Heller n-cosemistrips (f, u): $\tilde{X} \to \bar{Y} \leftarrow \tilde{Y}$ in $\mathcal{C}$ such that $f^{\mathrm{per}}\,(u^{\mathrm{per}})^{-1} \colon \tilde{X}^{\mathrm{per}} \to \tilde{Y}^{\mathrm{per}}$ has the base $\psi|_{\dot{\Delta}^n}\,\varphi_{\mathrm{b}}\,(\rho|_{\dot{\Delta}^n})^{-1}$. Moreover, f^{per} and u^{per} are periodic morphisms of n-cosemicomplexes by corollary (5.48)(b), and so $\varphi \colon X \to Y$ defined by $\varphi := \psi^{-1}\,f^{\mathrm{per}}\,(u^{\mathrm{per}})^{-1}\,\rho$ is a periodic morphism of n-cosemicomplexes and therefore a morphism of n-cosemitriangles in $\mathrm{Ho}\,\mathcal{C}$.

$$\begin{tikzcd} \tilde{X}^{\mathrm{per}}|_{\dot{\Delta}^n} \arrow[r, "\psi|_{\dot{\Delta}^n}", "\cong"'] \arrow[d, "(f^{\mathrm{per}}\,(u^{\mathrm{per}})^{-1})|_{\dot{\Delta}^n}"'] & X|_{\dot{\Delta}^n} \arrow[d, "\varphi_{\mathrm{b}}"] \\ \tilde{Y}^{\mathrm{per}}|_{\dot{\Delta}^n} \arrow[r, "\rho|_{\dot{\Delta}^n}", "\cong"'] & Y|_{\dot{\Delta}^n} \end{tikzcd} \qquad \begin{tikzcd} \tilde{X}^{\mathrm{per}} \arrow[r, "\psi", "\cong"'] \arrow[d, "f^{\mathrm{per}}\,(u^{\mathrm{per}})^{-1}"'] & X \arrow[d, "\varphi"] \\ \tilde{Y}^{\mathrm{per}} \arrow[r, "\rho", "\cong"'] & Y \end{tikzcd}$$

Finally, φ has the base

$$\varphi|_{\dot{\Delta}^n} = (\psi\, f^{\mathrm{per}}\, (u^{\mathrm{per}})^{-1}\, \rho^{-1})|_{\dot{\Delta}^n} = \psi|_{\dot{\Delta}^n}\, (f^{\mathrm{per}}\, (u^{\mathrm{per}})^{-1})|_{\dot{\Delta}^n}\, (\rho|_{\dot{\Delta}^n})^{-1} = \varphi_{\mathrm{b}}.$$

□

Appendix A

A construction principle for functors via choices

The construction of a functor $F\colon \mathcal{C} \to \mathcal{D}$ is often done by the following procedure. First, one *chooses* a family $S = (S_X)_{X \in \operatorname{Ob} \mathcal{C}}$ over $\operatorname{Ob} \mathcal{C}$. Second, one constructs $\operatorname{Ob} F\colon \operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D}$, where FX for $X \in \operatorname{Ob} \mathcal{C}$ is supposed to depend on X and S_X, that is, on the pair (X, S_X) – where the choice of S_X is needed to be able to use S_X when only X is given. Third, one constructs induced values on the morphisms, where $Ff\colon FX \to FX'$ for a morphism $f\colon X \to X'$ in $\mathcal{C}$ is supposed to depend on f, S_X and $S_{X'}$, that is, on the triple $(f, S_X, S_{X'})$. Fourth, one checks compatibility with composition and identities.

Different choices $S = (S_X)_{X \in \operatorname{Ob} \mathcal{C}}$ and $S' = (S'_X)_{X \in \operatorname{Ob} \mathcal{C}}$ then often lead to isomorphic functors F and F', where an isotransformation $F \to F'$ is obtained by applying the analogon to the third step from above to the triples $(1_X, S_X, S'_X)$ for $X \in \operatorname{Ob} \mathcal{C}$.

An example is the construction of a pushout functor $F\colon \mathcal{D}^{\llcorner} \to \mathcal{D}$ for some category $\mathcal{D}$, where $\llcorner$ is the full subposet of $\square = \Delta^1 \times \Delta^1$ with underlying set $\{(0,0), (1,0), (0,1)\}$. First, for every object X in $\mathcal{D}^{\llcorner}$, that is, for every span X in $\mathcal{D}$, one chooses a pushout rectangle S_X in $\mathcal{D}$ with $X = S_X|_{\llcorner}$. Second, one stipulates $FX = (S_X)_{1,1}$ for $X \in \operatorname{Ob} \mathcal{C}$. Third, given a morphism $f\colon X \to X'$ in $\mathcal{D}^{\llcorner}$, one stipulates $Ff\colon FX \to FX'$ to be the unique morphism in $\mathcal{D}$ that is induced by the universal property of the pushout rectangle S_X. Fourth, one checks compatibility with composition and identities.

Different choices of pushout rectangles $(S_X)_{X \in \operatorname{Ob} \mathcal{C}}$ and $(S'_X)_{X \in \operatorname{Ob} \mathcal{C}}$ lead to isomorphic functors $F, F'\colon \mathcal{D}^{\llcorner} \to \mathcal{D}$; an isotransformation $\alpha\colon F \to F'$ is given as follows. For $X \in \operatorname{Ob} \mathcal{C}$, one stipulates $\alpha_X\colon FX \to F'X$ to be the unique morphism in $\mathcal{D}$ that is induced by the identity $1_X = 1_{S_X|_{\llcorner}} = 1_{S'_X|_{\llcorner}}$ and the pushout rectangles S_X and S'_X.

The purpose of this chapter is to formalise this procedure. Moreover, we show how the functors constructed via choices arise from functors that do not necessitate choices. To this end, we construct a category that takes all possible choices into account.

The theory is developed in section 1. As an illustration, we reconsider the proof of the characterisation of equivalences of categories as full, faithful and dense functors in section 2.

Further applications of this chapter may be found in appendix B, section 2, where we construct left adjoint functors via choices of couniversal objects, and in chapter V, section 2, where we construct the shift functor on the homotopy category of a Brown cofibration category via choices of Coheller rectangles. The "choiceless variant" of the shift functor also appears in chapter V, section 3 and 4.

1 The structure category

Throughout this section, we suppose given a category $\mathcal{C}$ and a family $\mathfrak{S} = (\mathfrak{S}_X)_{X \in \operatorname{Ob} \mathcal{C}}$ over $\operatorname{Ob} \mathcal{C}$.

The structure category and the forgetful functor

(A.1) Remark.

(a) We have a category $\mathcal{C}_{\mathfrak{S}}$, given as follows. The set of objects of $\mathcal{C}_{\mathfrak{S}}$ is given by

$$\operatorname{Ob} \mathcal{C}_{\mathfrak{S}} = \{(X, S) \mid X \in \operatorname{Ob} \mathcal{C},\, S \in \mathfrak{S}_X\}.$$

For objects (X, S), (Y, T) in $\mathcal{C}_{\mathfrak{S}}$, we have the hom-set

$${}_{\mathcal{C}_{\mathfrak{S}}}((X, S), (Y, T)) = \{(f, S, T) \mid f \in {}_{\mathcal{C}}(X, Y)\}.$$

For morphisms $(f, S, T)\colon (X, S) \to (Y, T)$, $(g, T, U)\colon (Y, T) \to (Z, U)$ in $\mathcal{C}_{\mathfrak{S}}$, the composite is given by

$$(f, S, T)(g, T, U) = (fg, S, U).$$

For an object (X, S) in $\mathcal{C}_{\mathfrak{S}}$, the identity morphism on (X, S) is given by

$$1_{(X,S)} = (1_X, S, S).$$

(b) We have a functor $\mathrm{U}\colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}$, given on the objects by

$$\mathrm{U}(X, S) = X$$

for $(X, S) \in \operatorname{Ob} \mathcal{C}_{\mathfrak{S}}$, and on the morphisms by

$$\mathrm{U}(f, S, T) = f$$

for every morphism $(f, S, T)\colon (X, S) \to (Y, T)$ in $\mathcal{C}_{\mathfrak{S}}$.

(A.2) Definition (structure category)**.** The category $\mathcal{C}_{\mathfrak{S}}$ from remark (A.1)(a) is called the *structure category* of $\mathcal{C}$ with respect to $\mathfrak{S}$. The functor $\mathrm{U}\colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}$ from remark (A.1)(b) is called the *forgetful functor* of $\mathcal{C}_{\mathfrak{S}}$.

For example, we suppose that $\mathcal{C} = \mathbf{Set}_{(\mathfrak{U})}$ for some Grothendieck universe $\mathfrak{U}$, and, for $X \in \operatorname{Ob} \mathcal{C}$, we suppose that $\mathfrak{S}_X$ is the set of all group structures on X. Then $\mathcal{C}_{\mathfrak{S}}$ may be thought of the category whose objects are $\mathfrak{U}$-groups and whose morphisms are *all* maps between $\mathfrak{U}$-groups. In particular, the category $\mathbf{Grp}_{(\mathfrak{U})}$ is a wide subcategory of $\mathcal{C}_{\mathfrak{S}}$.

(A.3) Remark. The forgetful functor $\mathrm{U}\colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}$ is full and faithful. Moreover, U is surjective (on the morphisms and therefore on the objects) if and only if $\mathfrak{S}_X \neq \emptyset$ for every $X \in \operatorname{Ob} \mathcal{C}$. ([1])

(A.4) Notation. We suppose given objects (X, S) and (Y, T) in $\mathcal{C}_{\mathfrak{S}}$. A morphism $(f, S, T)\colon (X, S) \to (Y, T)$ in $\mathcal{C}_{\mathfrak{S}}$ is usually denoted just by $f\colon (X, S) \to (Y, T)$. Moreover, we usually write ${}_{\mathcal{C}_{\mathfrak{S}}}((X, S), (Y, T)) = {}_{\mathcal{C}}(X, Y)$ instead of ${}_{\mathcal{C}_{\mathfrak{S}}}((X, S), (Y, T)) = \{(f, S, T) \mid f \in {}_{\mathcal{C}}(X, Y)\}$.

(A.5) Notation. Given a functor $F\colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{D}$, we usually write $F_S X := F(X, S)$ for $(X, S) \in \operatorname{Ob} \mathcal{C}_{\mathfrak{S}}$ and $F_{S,T} f := F(f, S, T)$ for a morphism $f\colon (X, S) \to (Y, T)$ in $\mathcal{C}_{\mathfrak{S}}$.

Choices of structures

(A.6) Definition (choice of structures)**.** We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$. A *choice of structures* for $\mathcal{U}$ with respect to $\mathfrak{S}$ (or *choice of* $\mathfrak{S}$*-structures* for $\mathcal{U}$) is a family $S = (S_X)_{X \in \operatorname{Ob} \mathcal{U}}$ over $\operatorname{Ob} \mathcal{U}$ such that $S_X \in \mathfrak{S}_X$ for every $X \in \operatorname{Ob} \mathcal{U}$.

(A.7) Remark. We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$. Every choice of $\mathfrak{S}$-structures $S = (S_X)_{X \in \operatorname{Ob} \mathcal{U}}$ for $\mathcal{U}$ yields a functor

$$\mathrm{I}_S\colon \mathcal{U} \to \mathcal{C}_{\mathfrak{S}},$$

given on the objects by

$$\mathrm{I}_S X = (X, S_X)$$

for $X \in \operatorname{Ob} \mathcal{U}$, and on the morphisms by

$$\mathrm{I}_S f = f\colon (X, S_X) \to (Y, S_Y)$$

for every morphism $f\colon X \to Y$ in $\mathcal{U}$.

[1] So if $\mathfrak{S}_X \neq \emptyset$ for every $X \in \operatorname{Ob} \mathcal{C}$, then the forgetful functor $\mathrm{U}\colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}$ is an equivalence of categories. However, as we will reprove the "full-faithful-dense-criterion" in section 2, we will give a more concrete proof of this result below, see proposition (A.9) and corollary (A.10).

(A.8) Definition (structure choice functor)**.** We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$ and a choice of $\mathfrak{S}$-structures $S = (S_X)_{X \in \operatorname{Ob} \mathcal{U}}$ for $\mathcal{U}$. The functor $\mathrm{I}_S \colon \mathcal{U} \to \mathcal{C}_{\mathfrak{S}}$ from remark (A.7) is called the *structure choice functor* with respect to S.

Given a full subcategory $\mathcal{U}$ of $\mathcal{C}$, the structure category $\mathcal{U}_{\mathfrak{S}|_{\mathcal{U}}}$ is a full subcategory of the structure category $\mathcal{C}_{\mathfrak{S}}$. So in the rest of this section, we are satisfied with the case $\mathcal{U} = \mathcal{C}$.

(A.9) Proposition. We suppose given a choice of $\mathfrak{S}$-structures $S = (S_X)_{X \in \operatorname{Ob} \mathcal{C}}$ for $\mathcal{C}$.

(a) We have

$$\mathrm{U} \circ \mathrm{I}_S = \mathrm{id}_{\mathcal{C}}.$$

(b) We have

$$\mathrm{I}_S \circ \mathrm{U} \cong \mathrm{id}_{\mathcal{C}_{\mathfrak{S}}}.$$

An isotransformation $\mathrm{I}_S \circ \mathrm{U} \to \mathrm{id}_{\mathcal{C}_{\mathfrak{S}}}$ is given by $1_X \colon (X, S_X) \to (X, T)$ for $(X, T) \in \operatorname{Ob} \mathcal{C}_{\mathfrak{S}}$.

In particular, $\mathrm{U} \colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}$ and $\mathrm{I}_S \colon \mathcal{C} \to \mathcal{C}_{\mathfrak{S}}$ are mutually isomorphism inverse equivalences of categories.

Proof.

(a) For every morphism $f \colon X \to Y$ in $\mathcal{C}$, the underlying morphism in $\mathcal{C}$ of $\mathrm{I}_S f \colon (X, S_X) \to (Y, S_Y)$ is given by $f \colon X \to Y$. Thus we have $\mathrm{U} \circ \mathrm{I}_S = \mathrm{id}_{\mathcal{C}}$.

(b) We suppose given a morphism $g \colon (X, T) \to (Y, U)$ in $\mathcal{C}_{\mathfrak{S}}$. Then the following quadrangle in $\mathcal{C}$ commutes.

$$\begin{tikzcd} X \arrow[r, "1_X", "\cong"'] \arrow[d, "g"'] & Y \arrow[d, "g"] \\ Y \arrow[r, "1_Y", "\cong"'] & Y \end{tikzcd}$$

Hence the following quadrangle in $\mathcal{C}_{\mathfrak{S}}$ commutes.

$$\begin{tikzcd} (X, S_X) \arrow[r, "1_X"] \arrow[d, "g"'] & (X, T) \arrow[d, "g"] \\ (Y, S_Y) \arrow[r, "1_Y"] & (Y, U) \end{tikzcd}$$

Thus we have a transformation $\varepsilon \colon \mathrm{I}_S \circ \mathrm{U} \to \mathrm{id}_{\mathcal{C}_{\mathfrak{S}}}$ with components $\varepsilon_{(X,T)} = 1_X \colon (X, S_X) \to (X, T)$ for $(X, T) \in \operatorname{Ob} \mathcal{C}_{\mathfrak{S}}$. Moreover, as $1_X \colon X \to X$ is an isomorphism in $\mathcal{C}$ for every object (X, T) in $\mathcal{C}_{\mathfrak{S}}$, the morphism $\varepsilon_{(X,T)} = 1_X \colon (X, S_X) \to (X, T)$ in $\mathcal{C}_{\mathfrak{S}}$ is an isomorphism in $\mathcal{C}_{\mathfrak{S}}$ with inverse $\varepsilon_{(X,T)}^{-1} = 1_X \colon (X, T) \to (X, S_X)$. Hence ε is an isotransformation. □

(A.10) Corollary. If $\mathfrak{S}_X \neq \emptyset$ for every $X \in \operatorname{Ob} \mathcal{C}$, then the forgetful functor $\mathrm{U} \colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}$ is an equivalence of categories.

(A.11) Corollary. We suppose given choices of $\mathfrak{S}$-structures $S = (S_X)_{X \in \operatorname{Ob} \mathcal{C}}$ and $S' = (S'_X)_{X \in \operatorname{Ob} \mathcal{C}}$ for $\mathcal{C}$. Then we have

$$\mathrm{I}_S \cong \mathrm{I}_{S'}.$$

An isotransformation $\mathrm{I}_S \to \mathrm{I}_{S'}$ is given by $1_X \colon (X, S_X) \to (X, S'_X)$ for $X \in \operatorname{Ob} \mathcal{C}$.

Proof. By proposition (A.9)(b), we have isotransformations

$$\varepsilon \colon \mathrm{I}_S \circ \mathrm{U} \to \mathrm{id}_{\mathcal{C}_{\mathfrak{S}}},$$
$$\varepsilon' \colon \mathrm{I}_{S'} \circ \mathrm{U} \to \mathrm{id}_{\mathcal{C}_{\mathfrak{S}}}$$

given by $\varepsilon_{(X,T)} = 1_X \colon (X, S_X) \to (X, T)$ and $\varepsilon'_{(X,T)} = 1_X \colon (X, S'_X) \to (X, T)$ for $(X, T) \in \operatorname{Ob} \mathcal{C}_{\mathfrak{S}}$. Moreover, by proposition (A.9)(b), we have $\mathrm{U} \circ \mathrm{I}_S = \mathrm{id}_{\mathcal{C}}$. Thus we obtain an isotransformation

$$(\varepsilon\varepsilon'^{-1}) * \mathrm{I}_S \colon \mathrm{I}_S \to \mathrm{I}_{S'}$$

given by

$$((\varepsilon\varepsilon'^{-1}) * \mathrm{I}_S)_X = \varepsilon_{\mathrm{I}_S X}\varepsilon'^{-1}_{\mathrm{I}_S X} = \varepsilon_{(X,S_X)}\varepsilon'^{-1}_{(X,S_X)} = 1_X \colon (X, S_X) \to (X, S'_X).$$ □

The last corollary will often appear in the following form.

(A.12) Corollary. We suppose given a functor $F \colon \mathcal{C}_{\mathfrak{S}} \to \mathcal{D}$ and choices of $\mathfrak{S}$-structures $S = (S_X)_{X \in \operatorname{Ob} \mathcal{C}}$ and $S' = (S'_X)_{X \in \operatorname{Ob} \mathcal{C}}$ for $\mathcal{C}$. Then we have

$$F \circ \mathrm{I}_S \cong F \circ \mathrm{I}_{S'} \colon \mathcal{C} \to \mathcal{D}.$$

An isotransformation $\alpha_{S,S'} \colon F \circ \mathrm{I}_S \to F \circ \mathrm{I}_{S'}$ is given by

$$(\alpha_{S,S'})_X = F_{S_X, S'_X}(1_X) \colon F_{S_X} X \to F_{S'_X} X$$

for $X \in \operatorname{Ob} \mathcal{C}$. The inverse of $\alpha_{S,S'}$ is given by $\alpha_{S,S'}^{-1} = \alpha_{S',S}$.

2 The characterisation of equivalences of categories revisited

In this section, we apply the theory from section 1 in a reconsideration of the proof of the characterisation of equivalences as full, faithful and dense functors.
From definition (A.19) on, we suppose given a functor $F \colon \mathcal{C} \to \mathcal{D}$.

The easy implication

(A.13) Remark. We suppose given functors $F, G \colon \mathcal{C} \to \mathcal{D}$ and an isotransformation $\alpha \colon F \to G$. Moreover, we suppose given $X, X' \in \operatorname{Ob} \mathcal{C}$, and we set

$$F_{X,X'} \colon {}_{\mathcal{C}}(X, X') \to {}_{\mathcal{D}}(FX, FX'),\ f \mapsto Ff,$$
$$G_{X,X'} \colon {}_{\mathcal{C}}(X, X') \to {}_{\mathcal{D}}(GX, GX'),\ f \mapsto Gf,$$
$$\Phi_{X,X'} \colon {}_{\mathcal{D}}(FX, FX') \to {}_{\mathcal{D}}(GX, GX'),\ g \mapsto \alpha_X^{-1} g \alpha_{X'}.$$

(a) We have

$$\Phi_{X,X'} \circ F_{X,X'} = G_{X,X'}.$$

$$\begin{tikzcd} & {}_{\mathcal{D}}(FX, FX') \arrow[dd, "\Phi_{X,X'}"] \\ {}_{\mathcal{C}}(X, X') \arrow[ur, "F_{X,X'}"] \arrow[dr, "G_{X,X'}"'] & \\ & {}_{\mathcal{D}}(GX, GX') \end{tikzcd}$$

(b) The map $\Phi_{X,X'}$ is a bijection with inverse

$$\Phi_{X,X'}^{-1} \colon {}_{\mathcal{D}}(GX, GX') \to {}_{\mathcal{D}}(FX, FX'),\ g \mapsto \alpha_X g \alpha_{X'}^{-1}.$$

Proof.

(a) We have $\alpha_X(Gf) = (Ff)\alpha_{X'}$ for $f \in {}_{\mathcal{C}}(X, X')$ as α is a transformation. But since α is an isotransformation, it follows that

$$\Phi_{X,X'}(F_{X,X'} f) = \alpha_X^{-1}(Ff)\alpha_{X'} = Gf = G_{X,X'} f$$

for $f \in {}_{\mathcal{C}}(X, X')$, that is, we have $\Phi_{X,X'} \circ F_{X,X'} = G_{X,X'}$. □

(A.14) Corollary. We suppose given functors $F, G \colon \mathcal{C} \to \mathcal{D}$ with $F \cong G$.

(a) The functor F is faithful if and only if G is faithful.

(b) The functor F is full if and only if G is full.

Proof. We choose an isotransformation $\alpha \colon F \to G$. Moreover, for $X, X' \in \operatorname{Ob} \mathcal{C}$, we set

$$\begin{aligned}
&F_{X,X'} \colon {}_{\mathcal{C}}(X, X') \to {}_{\mathcal{D}}(FX, FX'),\ f \mapsto Ff,\\
&G_{X,X'} \colon {}_{\mathcal{C}}(X, X') \to {}_{\mathcal{D}}(GX, GX'),\ f \mapsto Gf,\\
&\Phi_{X,X'} \colon {}_{\mathcal{D}}(FX, FX') \to {}_{\mathcal{D}}(GX, GX'),\ g \mapsto \alpha_X^{-1} g \alpha_{X'}.
\end{aligned}$$

Then for $X, X' \in \operatorname{Ob} \mathcal{C}$, the map $\Phi_{X,X'}$ is a bijection by remark (A.13)(b) and we have $\Phi_{X,X'} \circ F_{X,X'} = G_{X,X'}$ by remark (A.13)(a).

(a) We suppose that F is faithful, that is, we suppose that $F_{X,X'}$ is injective for all $X, X' \in \operatorname{Ob} \mathcal{C}$. But then $G_{X,X'} = \Phi_{X,X'} \circ F_{X,X'}$ is also injective for all $X, X' \in \operatorname{Ob} \mathcal{C}$ as $\Phi_{X,X'}$ is a bijection, so G is faithful.

The converse implication follows by symmetry.

(b) We suppose that F is full, that is, we suppose that $F_{X,X'}$ is surjective for all $X, X' \in \operatorname{Ob} \mathcal{C}$. But then $G_{X,X'} = \Phi_{X,X'} \circ F_{X,X'}$ is also surjective for all $X, X' \in \operatorname{Ob} \mathcal{C}$ as $\Phi_{X,X'}$ is a bijection, so G is full.

The converse implication follows by symmetry. □

(A.15) Corollary. We suppose given functors $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{D} \to \mathcal{E}$, $H \colon \mathcal{C} \to \mathcal{E}$ with $G \circ F \cong H$.

(a) (i) If F and G are faithful, then H is faithful.
 (ii) If H is faithful, then F is faithful.

(b) (i) If F and G are full, then H is full.
 (ii) If H is full, then G is full.

(A.16) Remark. We suppose given functors $F, G \colon \mathcal{C} \to \mathcal{D}$ with $F \cong G$. Then F is dense if and only if G is dense.

Proof. We suppose that F is dense, that is, we suppose that for every $Y \in \operatorname{Ob} \mathcal{D}$ there exists an $X \in \operatorname{Ob} \mathcal{C}$ with $Y \cong FX$. As $F \cong G$, we have $FX \cong GX$ for $X \in \operatorname{Ob} \mathcal{C}$, so G is dense.
The converse implication follows by symmetry. □

(A.17) Corollary. We suppose given functors $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{D} \to \mathcal{E}$, $H \colon \mathcal{C} \to \mathcal{E}$ with $G \circ F \cong H$.

(a) If F and G are dense, then H is dense.

(b) If H is dense, then G is dense.

(A.18) Proposition. Every equivalence of categories is faithful, full and dense.

Proof. We suppose given an equivalence of categories $F \colon \mathcal{C} \to \mathcal{D}$. Moreover, we choose a functor $G \colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$. The faithfulness of $\mathrm{id}_{\mathcal{C}}$ implies the faithfulness of F by corollary (A.15)(a)(ii). Moreover, as $\mathrm{id}_{\mathcal{D}}$ is full and dense, it follows that F is full and dense by corollary (A.15)(b)(ii) and corollary (A.17)(b). □

Isomorphic replacements

For the rest of this section, we suppose given a functor $F \colon \mathcal{C} \to \mathcal{D}$.

(A.19) Definition (isomorphic replacement)**.** We suppose given an object Y in $\mathcal{D}$. An *isomorphic replacement* of Y along F is a pair (X, q) such that X is an object in $\mathcal{C}$ and $q \colon FX \to Y$ is an isomorphism in $\mathcal{D}$.

The isomorphic replacement category

(A.20) Definition (isomorphic replacement category)**.** For $Y \in \operatorname{Ob} \mathcal{D}$, we let $\mathfrak{R}_Y$ be the set of isomorphic replacements of Y along F. The structure category

$$\operatorname{Rpl}(F) := \mathcal{D}_{\mathfrak{R}}$$

is called the *isomorphic replacement category* of F.

(A.21) Remark. We have

$$\operatorname{Ob}\operatorname{Rpl}(F) = \{(Y, X, q) \mid Y \in \operatorname{Ob}\mathcal{D},\ X \in \operatorname{Ob}\mathcal{C},\ q\colon FX \to Y \text{ isomorphism in } \mathcal{D}\}.$$

([2]) For objects (Y, X, q), (Y', X', q') in $\operatorname{Rpl}(F)$, we have the hom-set

$${}_{\operatorname{Rpl}(F)}((Y, X, q), (Y', X', q')) = {}_{\mathcal{D}}(Y, Y').$$

For morphisms $g\colon (Y, X, q) \to (Y', X', q')$, $g'\colon (Y', X', q') \to (Y'', X'', q'')$ in $\operatorname{Rpl}(F)$, the composite $gg'\colon (Y, X, q) \to (Y'', X'', q'')$ in $\operatorname{Rpl}(F)$ has the underlying morphism $gg'\colon Y \to Y'$ in $\mathcal{D}$. For an object (Y, X, q) in $\operatorname{Rpl}(F)$, the identity morphism $1_{(Y,X,q)}\colon (Y, X, q) \to (Y, X, q)$ in $\operatorname{Rpl}(F)$ has the underlying morphism $1_Y\colon Y \to Y$ in $\mathcal{D}$.
The forgetful functor $\mathrm{U}\colon \operatorname{Rpl}(F) \to \mathcal{D}$ is given on the objects by

$$\mathrm{U}_{(X,q)}Y = Y$$

for $(Y, X, q) \in \operatorname{Ob}\operatorname{Rpl}(F)$, and on the morphisms by

$$\mathrm{U}_{(X,q),(X',q')}g = g$$

for a morphism $g\colon (Y, X, q) \to (Y', X', q')$ in $\operatorname{Rpl}(F)$.

The canonical lift

(A.22) Remark.

(a) We have a functor $\bar{F}\colon \mathcal{C} \to \operatorname{Rpl}(F)$, given on the objects by

$$\bar{F}X = (FX, X, 1_{FX})$$

for $X \in \operatorname{Ob}\mathcal{C}$, and on the morphisms by

$$\bar{F}f = Ff\colon (FX, X, 1_{FX}) \to (FX', X', 1_{FX'})$$

for every morphism $f\colon X \to X'$ in $\mathcal{C}$.

(b) We have

$$F = \mathrm{U} \circ \bar{F}.$$

(A.23) Definition (canonical lift)**.** The functor $\bar{F}\colon \mathcal{C} \to \operatorname{Rpl}(F)$ from remark (A.22) is called the *canonical lift* of F along the forgetful functor $\mathrm{U}\colon \operatorname{Rpl}(F) \to \mathcal{D}$.

(A.24) Remark. For every object (Y, X, q) in $\operatorname{Rpl}(F)$, we have the isomorphic replacement $(X, \bar{q})$ of (Y, X, q) along the canonical lift $\bar{F}\colon \mathcal{C} \to \operatorname{Rpl}(F)$, where $\bar{q} = q\colon \bar{F}X \to (Y, X, q)$. In particular, $\bar{F}$ is dense.

(A.25) Corollary. The following conditions are equivalent.

(a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is dense.

(b) The forgetful functor $\mathrm{U}\colon \operatorname{Rpl}(F) \to \mathcal{D}$ is dense.

[2] More precisely, the objects in $\operatorname{Rpl}(F)$ are of the form $(Y, (X, q))$ for $Y \in \operatorname{Ob}\mathcal{D}$, $X \in \operatorname{Ob}\mathcal{C}$ and an isomorphism $q\colon FX \to Y$ in $\mathcal{D}$, but we use the simplified notation (Y, X, q) instead.

(c) The forgetful functor $\mathrm{U}\colon \mathrm{Rpl}(F) \to \mathcal{D}$ is surjective on the objects.

(d) The forgetful functor $\mathrm{U}\colon \mathrm{Rpl}(F) \to \mathcal{D}$ is an equivalence of categories.

Proof. The equivalence of condition (a) and condition (b) follows from remark (A.22)(b) and corollary (A.17)(b). Moreover, condition (c) implies condition (d) by corollary (A.10), and condition (d) implies condition (b) by proposition (A.18). So to show that the four conditions are equivalent, it remains to show that condition (b) implies condition (c).
We suppose that condition (b) holds, that is, we suppose that $\mathrm{U}\colon \mathrm{Rpl}(F) \to \mathcal{D}$ is dense. Moreover, we suppose given an object Y in $\mathcal{D}$. As U is dense, there exists an object (Y', X, q) in $\mathrm{Rpl}(F)$ and an isomorphism $g\colon \mathrm{U}(Y', X, q) \to Y$ in $\mathcal{D}$. But then we have the isomorphism $qg\colon FX \to Y$ in $\mathcal{D}$, so (Y, X, qg) is an object in $\mathrm{Rpl}(F)$ with $\mathrm{U}(Y, X, qg) = Y$. Thus $\mathrm{U}\colon \mathrm{Rpl}(F) \to \mathcal{D}$ is surjective on the objects, that is, condition (c) holds.

$$FX \xrightarrow[\cong]{q} Y' \xrightarrow[\cong]{g} Y$$

□

The total isomorphic replacement functor

(A.26) Proposition. We suppose that F is full and faithful. Then we have a functor

$$\bar{G}\colon \mathrm{Rpl}(F) \to \mathcal{C},$$

given on the objects by

$$\bar{G}_{(X,q)}Y = X$$

for $(Y, X, q) \in \mathrm{Ob}\,\mathrm{Rpl}(F)$, and on the morphisms as follows. Given a morphism $g\colon (Y, X, q) \to (Y', X', q')$ in $\mathrm{Rpl}(F)$, then $\bar{G}_{(X,q),(X',q')}g\colon X \to X'$ is the unique morphism in $\mathcal{C}$ with

$$qg = (F\bar{G}_{(X,q),(X',q')}g)q'.$$

Proof. We define a map

$$\bar{G}_0\colon \mathrm{Ob}\,\mathrm{Rpl}(F) \to \mathrm{Ob}\,\mathcal{C},\ (Y, X, q) \mapsto X.$$

We suppose given $(Y, X, q), (Y', X', q') \in \mathrm{Ob}\,\mathrm{Rpl}(F)$. As $F\colon \mathcal{C} \to \mathcal{D}$ is full and faithful, the map

$${}_{\mathcal{C}}(X, X') \to {}_{\mathcal{D}}(FX, FX'),\ f \mapsto Ff$$

is a bijection. Thus we obtain a well-defined map

$$\bar{G}_{(Y,X,q),(Y',X',q')}\colon {}_{\mathrm{Rpl}(F)}((Y, X, q), (Y', X', q')) \to {}_{\mathcal{C}}(X, X'),$$

where $\bar{G}_{(Y,X,q),(Y',X',q')}g \in {}_{\mathcal{C}}(X, X')$ for $g \in {}_{\mathrm{Rpl}(F)}((Y, X, q), (Y', X', q'))$ is the unique element with $qgq'^{-1} = F\bar{G}_{(X,q),(X',q')}g$, that is, with $qg = (F\bar{G}_{(X,q),(X',q')}g)q'$.
Given morphisms $g\colon (Y, X, q) \to (Y', X', q')$ and $g'\colon (Y', X', q') \to (Y'', X'', q'')$ in $\mathrm{Rpl}(F)$, we have

$$\begin{aligned} qgg' &= (F\bar{G}_{(Y,X,q),(Y',X',q')}g)q'g' = (F\bar{G}_{(Y,X,q),(Y',X',q')}g)(F\bar{G}_{(Y',X',q'),(Y'',X'',q'')}g')q'' \\ &= F((\bar{G}_{(Y,X,q),(Y',X',q')}g)(\bar{G}_{(Y',X',q'),(Y'',X'',q'')}g'))\,q'' \end{aligned}$$

and therefore $\bar{G}_{(Y,X,q),(Y'',X'',q'')}(gg') = (\bar{G}_{(Y,X,q),(Y',X',q')}g)(\bar{G}_{(Y',X',q'),(Y'',X'',q'')}g')$. Moreover, for every $(Y, X, q) \in \mathrm{Ob}\,\mathrm{Rpl}(F)$, we have

$$q1_Y = 1_{FX}q = (F1_X)q$$

and therefore $\bar{G}_{(Y,X,q),(Y,X,q)}(1_Y) = 1_X = 1_{\bar{G}_0(Y,X,q)}$. Thus we have a functor $\bar{G}\colon \mathrm{Rpl}(F) \to \mathcal{C}$ given by $\mathrm{Ob}\,\bar{G} = \bar{G}_0$ and by $\bar{G}_{(X,q),(X',q')}g = \bar{G}_{(Y,X,q),(Y',X',q')}g$ for every morphism $g\colon (Y, X, q) \to (Y', X', q')$ in $\mathrm{Rpl}(F)$. □

(A.27) Definition (total isomorphic replacement functor)**.** We suppose that F is full and faithful. The functor $\bar{G}\colon \mathrm{Rpl}(F) \to \mathcal{C}$ from proposition (A.26) is called the *total isomorphic replacement functor* along F.

If F is full and faithful, then an isomorphic replacement (X, q) of an object Y becomes a *universal object* over Y along F, that is, it fulfills the universal property dual to that of a *couniversal object* as in definition (B.2). The total isomorphic replacement functor as just introduced may also be defined as the restriction of a *total universal object functor* along F, cf. definition (B.13), to $\mathrm{Rpl}(F)$.

(A.28) Proposition. We suppose that F is full and faithful. Moreover, we let $\bar{F}\colon \mathcal{C} \to \mathrm{Rpl}(F)$ be the canonical lift of F along the forgetful functor $\mathrm{U}\colon \mathrm{Rpl}(F) \to \mathcal{D}$ and we let $\bar{G}\colon \mathrm{Rpl}(F) \to \mathcal{C}$ be the total isomorphic replacement functor along F.

(a) We have
$$\bar{G} \circ \bar{F} = \mathrm{id}_{\mathcal{C}}.$$

(b) We have
$$\bar{F} \circ \bar{G} \cong \mathrm{id}_{\mathrm{Rpl}(F)}.$$

An isotransformation $\bar{F} \circ \bar{G} \to \mathrm{id}_{\mathrm{Rpl}(F)}$ is given by $q\colon (FX, X, 1_{FX}) \to (Y, X, q)$ for $(Y, X, q) \in \mathrm{Ob}\,\mathrm{Rpl}(F)$.

In particular, $\bar{F}\colon \mathcal{C} \to \mathrm{Rpl}(F)$ and $\bar{G}\colon \mathrm{Rpl}(F) \to \mathcal{C}$ are mutually isomorphism inverse equivalences of categories.

Proof.

(a) For every morphism $f\colon X \to X'$ in $\mathcal{C}$, we have $\bar{F}f = Ff\colon (FX, X, 1_{FX}) \to (FX', X', 1_{FX'})$, so $1_{FX}(Ff) = (Ff)1_{FX'}$ and therefore $\bar{G}\bar{F}f = \bar{G}_{(X,1_{FX}),(X',1_{FX'})}Ff = f$. Thus we have $\bar{G} \circ \bar{F} = \mathrm{id}_{\mathcal{C}}$.

(b) We suppose given a morphism $g\colon (Y, X, q) \to (Y', X', q')$ in $\mathrm{Rpl}(F)$. Then the following quadrangle in $\mathcal{D}$ commutes by definition of $\bar{G}\colon \mathrm{Rpl}(F) \to \mathcal{C}$.

$$\begin{tikzcd} FX \arrow[r, "q", "\cong"'] \arrow[d, "F\bar{G}_{(X,q),(X',q')}g"'] & Y \arrow[d, "g"] \\ FX' \arrow[r, "q'", "\cong"'] & Y' \end{tikzcd}$$

Hence the following quadrangle in $\mathrm{Rpl}(F)$ commutes.

$$\begin{tikzcd} (FX, X, 1_{FX}) \arrow[r, "q"] \arrow[d, "F\bar{G}_{(X,q),(X',q')}g"'] & (Y, X, q) \arrow[d, "g"] \\ (FX', X', 1_{FX'}) \arrow[r, "q'"] & (Y', X', q') \end{tikzcd}$$

Thus we have a transformation $\beta\colon \bar{F} \circ \bar{G} \to \mathrm{id}_{\mathrm{Rpl}(F)}$, given by $\beta_{(Y,X,q)} = q\colon (FX, X, 1_{FX}) \to (Y, X, q)$ for $(Y, X, q) \in \mathrm{Ob}\,\mathrm{Rpl}(F)$. Moreover, as $q\colon FX \to Y$ is an isomorphism in $\mathcal{D}$ for every object (Y, X, q) in $\mathrm{Rpl}(F)$, the morphism $\beta_{(Y,X,q)} = q\colon (FX, X, 1_{FX}) \to (Y, X, q)$ in $\mathrm{Rpl}(F)$ is an isomorphism in $\mathrm{Rpl}(F)$ with inverse $\beta^{-1}_{(Y,X,q)} = q^{-1}\colon (Y, X, q) \to (FX, X, 1_{FX})$. Thus β is an isotransformation. □

Isomorphic replacement functors

(A.29) Definition (choice of isomorphic replacements)**.** A *choice of isomorphic replacements* for $\mathcal{D}$ along F is a family $((X_Y, q_Y))_{Y \in \mathrm{Ob}\,\mathcal{D}}$ such that (X_Y, q_Y) is an isomorphic replacement of Y along F for every $Y \in \mathrm{Ob}\,\mathcal{D}$.

(A.30) Remark. A choice of isomorphic replacements for $\mathcal{D}$ along F exists if and only if F is dense.

(A.31) Remark. For $Y \in \mathrm{Ob}\,\mathcal{D}$, we let $\mathfrak{R}_Y$ be the set of isomorphic replacements of Y along F. A choice of isomorphic replacements for $\mathcal{D}$ along F is precisely a choice of structures with respect to $\mathfrak{R} = (\mathfrak{R}_Y)_{Y \in \mathrm{Ob}\,\mathcal{D}}$.

For the definition of the structure choice functor with respect to a choice of structures, see definition (A.8). In the case of a choice of isomorphic replacements, the structure choice functor is given as follows.

(A.32) Remark. We suppose given a choice of S-replacements $R = ((X_Y, q_Y))_{Y \in \operatorname{Ob} \mathcal{D}}$ for $\mathcal{D}$ along F. The structure choice functor $\mathrm{I}_R \colon \mathcal{D} \to \operatorname{Rpl}(F)$ is given on the objects by

$$\mathrm{I}_R Y = (Y, X_Y, q_Y)$$

for $Y \in \operatorname{Ob} \mathcal{D}$, and on the morphisms by

$$\mathrm{I}_R g = g \colon (Y, X_Y, q_Y) \to (Y', X_{Y'}, q_{Y'})$$

for every morphism $g \colon Y \to Y'$ in $\mathcal{D}$.

(A.33) Remark. We suppose given a choice of isomorphic replacements $R = ((X_Y, q_Y))_{Y \in \operatorname{Ob} \mathcal{D}}$ for $\mathcal{D}$ along F.

(a) We have

$$\mathrm{U} \circ \mathrm{I}_R = \mathrm{id}_{\mathcal{D}}.$$

(b) We have

$$\mathrm{I}_R \circ \mathrm{U} \cong \mathrm{id}_{\operatorname{Rpl}(F)}.$$

An isotransformation $\mathrm{I}_R \circ \mathrm{U} \to \mathrm{id}_{\operatorname{Rpl}(F)}$ is given by $1_Y \colon (Y, X_Y, q_Y) \to (Y, X', q')$ for $(Y, X', q') \in \operatorname{Ob} \operatorname{Rpl}(F)$.

In particular, $\mathrm{U} \colon \operatorname{Rpl}(F) \to \mathcal{D}$ and $\mathrm{I}_R \colon \mathcal{D} \to \operatorname{Rpl}(F)$ are mutually isomorphism inverse equivalences of categories.

Proof. This follows from remark (A.31) and proposition (A.9). □

(A.34) Definition (isomorphic replacement functor)**.** We suppose that F is full and faithful, and we let $\bar{G} \colon \operatorname{Rpl}(F) \to \mathcal{C}$ denote the total isomorphic replacement functor along F. Moreover, we suppose given a choice of isomorphic replacements $R = ((X_Y, q_Y))_{Y \in \operatorname{Ob} \mathcal{D}}$ for $\mathcal{D}$ along F. The composite

$$\bar{G} \circ \mathrm{I}_R \colon \mathcal{D} \to \mathcal{C}$$

is called the *isomorphic replacement functor* along F with respect to R.

(A.35) Remark. We suppose that F is full and faithful, and we suppose given a choice of isomorphic replacements $R = ((X_Y, q_Y))_{Y \in \operatorname{Ob} \mathcal{D}}$ for $\mathcal{D}$ along F. Moreover, we let $G \colon \mathcal{D} \to \mathcal{C}$ be the isomorphic replacement functor along F with respect to R. Then G is given on the objects by

$$GY = X_Y$$

for $Y \in \operatorname{Ob} \mathcal{D}$, and on the morphisms as follows. Given a morphism $g \colon Y \to Y'$ in $\mathcal{D}$, then $Gg \colon X_Y \to X_{Y'}$ is the unique morphism in $\mathcal{C}$ with

$$q_Y g = (FGg) q_{Y'}.$$

Proof. We let $\bar{G} \colon \operatorname{Rpl}(F) \to \mathcal{C}$ denote the total isomorphic replacement functor along F, so that $G = \bar{G} \circ \mathrm{I}_R$. For $Y \in \operatorname{Ob} \mathcal{D}$, we have $\mathrm{I}_R Y = (Y, X_Y, q_Y)$ and therefore

$$GY = \bar{G} \mathrm{I}_R Y = \bar{G}_{(X_Y, q_Y)} Y = X_Y.$$

We suppose given a morphism $g \colon Y \to Y'$ in $\mathcal{D}$. Then we have $\mathrm{I}_R g = g \colon (Y, X_Y, q_Y) \to (Y', X_{Y'}, q_{Y'})$, and so $Gg = \bar{G}_{(X_Y, q_Y), (X_{Y'}, q_{Y'})} \mathrm{I}_R g = \bar{G}_{(X_Y, q_Y), (X_{Y'}, q_{Y'})} g \colon X_Y \to X_{Y'}$ is the unique morphism in $\mathcal{C}$ with $q_Y g = (F \bar{G}_{(X_Y, q_Y), (X_{Y'}, q_{Y'})} g) q_{Y'}$, that is, with $q_Y g = (FGg) q_{Y'}$. □

The criterion for equivalences of categories

(A.36) Theorem. If F is full, faithful and dense, then F is an equivalence of categories.
For every choice of isomorphic replacements $R = ((X_Y, q_Y))_{Y \in \operatorname{Ob} \mathcal{D}}$ for $\mathcal{D}$ along F, the isomorphic replacement functor G along F with respect to R is an isomorphism inverse of F. Isotransformations $\alpha \colon G \circ F \rightarrow \mathrm{id}_{\mathcal{C}}$ and $\beta \colon F \circ G \rightarrow \mathrm{id}_{\mathcal{D}}$ are given as follows. For $X' \in \operatorname{Ob} \mathcal{C}$, the component $\alpha_{X'} \colon GFX' \rightarrow X'$ is given by the unique morphism in $\mathcal{C}$ with $q_{FX'} = F\alpha_{X'}$. For $Y \in \operatorname{Ob} \mathcal{D}$, the component $\beta_Y \colon FGY \rightarrow Y$ is given by $\beta_Y = q_Y$.

Proof. We suppose that F is full, faithful and dense.
By remark (A.22)(b), we have $F = \mathrm{U} \circ \bar{F}$, where $\bar{F} \colon \mathcal{C} \rightarrow \mathrm{Rpl}(F)$ denotes the canonical lift of F along the forgetful functor $\mathrm{U} \colon \mathrm{Rpl}(F) \rightarrow \mathcal{D}$.
As F is dense, there exists a choice of isomorphic replacements for $\mathcal{D}$ along F. We suppose given such a choice of isomorphic replacements $R = ((X_Y, q_Y))_{Y \in \operatorname{Ob} \mathcal{D}}$. By remark (A.33), we have $\mathrm{U} \circ \mathrm{I}_R = \mathrm{id}_{\mathcal{D}}$ and an isotransformation $\bar{\alpha} \colon \mathrm{I}_R \circ \mathrm{U} \rightarrow \mathrm{id}_{\mathrm{Rpl}(F)}$ given by $\bar{\alpha}_{(Y, X', q')} = 1_Y \colon (Y, X_Y, q_Y) \rightarrow (Y, X', q')$ for $(Y, X', q') \in \operatorname{Ob} \mathrm{Rpl}(F)$.
As F is full and faithful, the total isomorphic replacement functor $\bar{G} \colon \mathrm{Rpl}(F) \rightarrow \mathcal{C}$ is defined. By proposition (A.28), we have $\bar{G} \circ \bar{F} = \mathrm{id}_{\mathcal{C}}$ and an isotransformation $\bar{\beta} \colon \bar{F} \circ \bar{G} \rightarrow \mathrm{id}_{\mathrm{Rpl}(F)}$ given by $\bar{\beta}_{(Y, X', q')} = q' \colon (FX, X, 1_{FX}) \rightarrow (Y, X', q')$ for $(Y, X', q') \in \operatorname{Ob} \mathrm{Rpl}(F)$.

$$\begin{tikzcd}
& \mathrm{Rpl}(F) \arrow[dl, bend right, "\bar{G}"'] \arrow[dr, "\mathrm{U}"'] & \\
\mathcal{C} \arrow[ur, "\bar{F}"'] \arrow[rr, "F"] & & \mathcal{D} \arrow[ul, bend right, "\mathrm{I}_R"']
\end{tikzcd}$$

We obtain

$$G \circ F = \bar{G} \circ \mathrm{I}_R \circ \mathrm{U} \circ \bar{F} \cong \bar{G} \circ \mathrm{id}_{\mathrm{Rpl}(F)} \circ \bar{F} = \bar{G} \circ \bar{F} = \mathrm{id}_{\mathcal{C}},$$
$$F \circ G = \mathrm{U} \circ \bar{F} \circ \bar{G} \circ \mathrm{I}_R \cong \mathrm{U} \circ \mathrm{id}_{\mathrm{Rpl}(F)} \circ \mathrm{I}_R = \mathrm{U} \circ \mathrm{I}_R = \mathrm{id}_{\mathcal{D}},$$

where isotransformations $\alpha \colon G \circ F \rightarrow \mathrm{id}_{\mathcal{C}}$ and $\beta \colon F \circ G \rightarrow \mathrm{id}_{\mathcal{D}}$ are given by $\alpha := \bar{G} * \bar{\alpha} * \bar{F}$ and $\beta := \mathrm{U} * \bar{\beta} * \mathrm{I}_R$.
Thus $G \colon \mathcal{D} \rightarrow \mathcal{C}$ is an isomorphism inverse of $F \colon \mathcal{C} \rightarrow \mathcal{D}$. In particular, F is an equivalence of categories.
For $X' \in \operatorname{Ob} \mathcal{C}$, we have $\bar{\alpha}_{\bar{F}X'} = \bar{\alpha}_{(FX', X', 1_{FX'})} = 1_{FX'} \colon (FX', X_{FX'}, q_{FX'}) \rightarrow (FX', X', 1_{FX'})$ in $\mathrm{Rpl}(F)$, and so

$$\alpha_{X'} = \bar{G}_{(X_{FX'}, q_{FX'}), (X', 1_{FX'})} \bar{\alpha}_{\bar{F}X'} = \bar{G}_{(X_{FX'}, q_{FX'}), (X', 1_{FX'})} 1_{FX'} \colon GFX' \rightarrow X'$$

is the unique morphism in $\mathcal{C}$ with $q_{FX'} \, 1_{FX'} = (F\bar{G}_{(X_{FX'}, q_{FX'}), (X', 1_{FX'})} 1_{FX'}) \, 1_{FX'}$, that is, with $q_{FX'} = F\alpha_{X'}$.
For $Y \in \operatorname{Ob} \mathcal{D}$, we have $\bar{\beta}_{\mathrm{I}_R Y} = \bar{\beta}_{(Y, X_Y, q_Y)} = q_Y \colon (FX_Y, X_Y, 1_{FX_Y}) \rightarrow (Y, X_Y, q_Y)$ in $\mathrm{Rpl}(F)$, and so

$$\beta_Y = \mathrm{U}\bar{\beta}_{\mathrm{I}_R Y} = q_Y \colon FX_Y \rightarrow Y$$

in $\mathcal{D}$. □

Appendix B

Universal properties

In this appendix, we define couniversal objects and deduce some folklore results. The author does not claim any originality.

1 Couniversal objects

Definition of a couniversal object

(B.1) Remark. We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$ and we choose a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{D}$ is a $\mathfrak{U}$-category. Moreover, we suppose given an object X in $\mathcal{C}$ and an object Y in $\mathcal{D}$.
The map

$$ {}_{\mathrm{Fun}(\mathcal{D},\mathbf{Set})}({}_{\mathcal{D}}(Y,-),{}_{\mathcal{C}}(X,G-)) \to {}_{\mathcal{C}}(X,GY),\ \beta \mapsto 1_Y\beta_Y $$

is a bijection. Its inverse is given by

$$ {}_{\mathcal{C}}(X,GY) \to {}_{\mathrm{Fun}(\mathcal{D},\mathbf{Set})}({}_{\mathcal{D}}(Y,-),{}_{\mathcal{C}}(X,G-)),\ u \mapsto (g \mapsto u(Gg))_{Y' \in \operatorname{Ob}\mathcal{D}}. $$

Proof. This is a particular case of the Yoneda lemma. □

(B.2) Definition (couniversal object)**.** We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$ and an object X in $\mathcal{C}$. A *couniversal object* under X along G (or an *initial object* under X along G) consists of an object U in $\mathcal{D}$ together with a morphism $u\colon X \to GU$ in $\mathcal{C}$ such that for every object Y in $\mathcal{D}$ and every morphism $f\colon X \to GY$ in $\mathcal{C}$ there exists a unique morphism $\hat{f}\colon U \to Y$ in $\mathcal{D}$ with $f = u(G\hat{f})$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & GY \\ GU \arrow[ur, "G\hat{f}"'] & \end{tikzcd}$$

By abuse of notation, we refer to the said couniversal object under X along G as well as to its underlying object just by U. The morphism u is said to be the *universal morphism* of U.
Given a couniversal object under X along G with universal morphism u, we write $\mathrm{uni} = \mathrm{uni}^U := u$.

The defining (universal) property of a couniversal object may be reformulated using isotransformations:

(B.3) Remark. We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$ and an object X in $\mathcal{C}$.

(a) Given a couniversal object U under X along G, then the maps

$$ {}_{\mathcal{D}}(U,Y) \to {}_{\mathcal{C}}(X,GY),\ g \mapsto \mathrm{uni}^U(Gg) $$

for $Y \in \operatorname{Ob}\mathcal{D}$ define an isotransformation

$$ {}_{\mathcal{D}}(U,-) \to {}_{\mathcal{C}}(X,G-). $$

(b) Given an object U in $\mathcal{D}$ and an isotransformation $\Phi\colon {}_{\mathcal{D}}(U, -) \to {}_{\mathcal{C}}(X, G-)$, then U becomes a couniversal object under X along G with

$$\mathrm{uni}^U = 1_U \Phi_U.$$

Proof.

(a) By remark (B.1), the universal morphism $\mathrm{uni}^U : X \to GU$ yields a transformation

$$\Phi\colon {}_{\mathcal{D}}(U, -) \to {}_{\mathcal{C}}(X, G-)$$

given by

$$\Phi_Y : {}_{\mathcal{D}}(U, Y) \to {}_{\mathcal{C}}(X, GY),\ g \mapsto \mathrm{uni}^U(Gg)$$

for $Y \in \operatorname{Ob} \mathcal{D}$. Moreover, Φ_Y is a bijection for every $Y \in \operatorname{Ob} \mathcal{D}$ by the universal property of U, and so $\Phi = (\Phi_Y)_{Y \in \operatorname{Ob} \mathcal{D}}$ is an isotransformation.

(b) By remark (B.1), Φ is given by

$$\Phi_Y : {}_{\mathcal{D}}(U, Y) \to {}_{\mathcal{C}}(X, GY), g \mapsto (1_U \Phi_U)(Gg)$$

for $Y \in \operatorname{Ob} \mathcal{D}$. So since Φ is an isotransformation, given an object Y in $\mathcal{D}$ and a morphism $f\colon X \to GY$ in $\mathcal{C}$, there exists a unique morphism $\hat{f}\colon U \to Y$ in $\mathcal{D}$ with $\hat{f}\Phi_Y = f$, that is, with $(1_U \Phi_U)(G\hat{f}) = f$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_U \Phi_U"'] & GY \\
GU \arrow[ur, "G\hat{f}"'] &
\end{tikzcd}$$

Thus we have shown that U becomes a couniversal object under X along G with $\mathrm{uni}^U = 1_U \Phi_U$. □

Simple properties of couniversal objects

(B.4) Remark. We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$, an object X in $\mathcal{C}$ and couniversal objects U, U' under X along G. We let $\hat{u}\colon U' \to U$ denote the unique morphism in $\mathcal{D}$ with $\mathrm{uni}^U = \mathrm{uni}^{U'}(G\hat{u})$, and we let $\hat{u}'\colon U \to U'$ denote the unique morphism in $\mathcal{D}$ with $\mathrm{uni}^{U'} = \mathrm{uni}^U(G\hat{u}')$. Then $\hat{u}$ and $\hat{u}'$ are mutually inverse isomorphisms.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{uni}^{U'}"] \arrow[d, "\mathrm{uni}^U"'] & GU' \arrow[dl, bend right, "G\hat{u}"'] \arrow[dl, phantom, "\cong"] \\
GU \arrow[ur, bend right, "G\hat{u}'"'] &
\end{tikzcd}$$

Proof. We have

$$\mathrm{uni}^U = \mathrm{uni}^{U'}(G\hat{u}) = \mathrm{uni}^U(G\hat{u}')(G\hat{u}) = \mathrm{uni}^U G(\hat{u}'\hat{u}),$$
$$\mathrm{uni}^{U'} = \mathrm{uni}^U(G\hat{u}') = \mathrm{uni}^{U'}(G\hat{u})(G\hat{u}') = \mathrm{uni}^{U'} G(\hat{u}\hat{u}').$$

So since we also have

$$\mathrm{uni}^U = \mathrm{uni}^U 1_{GU} = \mathrm{uni}^U(G1_U),$$
$$\mathrm{uni}^{U'} = \mathrm{uni}^{U'} 1_{G'U} = \mathrm{uni}^{U'}(G1_{U'}),$$

we get $\hat{u}'\hat{u} = 1_U$ by the universal property of U and $\hat{u}\hat{u}' = 1_{U'}$ by the universal property of U'. □

(B.5) Remark. We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$, a morphism $f\colon X_1 \to X_2$ in $\mathcal{C}$, a couniversal object U_1 under X_1 along G and a couniversal object U_2 under X_2 along G. There exists a unique morphism $\hat{f}\colon U_1 \to U_2$ in $\mathcal{D}$ with $f \mathrm{uni}^{U_2} = \mathrm{uni}^{U_1}(G\hat{f})$.

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "\mathrm{uni}^{U_1}"'] & X_2 \arrow[d, "\mathrm{uni}^{U_2}"] \\ GU_1 \arrow[r, "G\hat{f}"] & GU_2 \end{tikzcd}$$

Proof. This follows from the universal property of U_1.

$$\begin{tikzcd} X_1 \arrow[r, "f\mathrm{uni}^{U_2}"] \arrow[d, "\mathrm{uni}^{U_1}"'] & GU_2 \\ GU_1 \arrow[ur, "G\hat{f}"'] & \end{tikzcd}$$

□

(B.6) Proposition. We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$, an object X in $\mathcal{C}$ and a couniversal object U under X along G. Moreover, we suppose given a retraction $p\colon X \to X'$ in $\mathcal{C}$ with corresponding coretraction $i\colon X' \to X$, a retraction $q\colon U \to U'$ in $\mathcal{D}$ with corresponding coretraction $j\colon U' \to U$, and a morphism $u'\colon X' \to GU'$ such that $\mathrm{uni}^U(Gq) = pu'$ and $i\,\mathrm{uni}^U = u'(Gj)$.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\mathrm{uni}^U"'] & X' \arrow[l, bend left, "i"] \arrow[d, "u'"] \\ GU \arrow[r, "Gq"] & GU' \arrow[l, bend left, "Gj"] \end{tikzcd}$$

Then U' becomes a couniversal object under X' along G with universal morphism

$$\mathrm{uni}^{U'} = u'.$$

Given an object Y in $\mathcal{D}$ and a morphism $f'\colon X' \to GY$ in $\mathcal{C}$, the unique morphism $\hat{f}'\colon U' \to Y$ in $\mathcal{D}$ with $f' = \mathrm{uni}^{U'}(G\hat{f}')$ is given by

$$\hat{f}' = j\hat{f},$$

where $\hat{f}\colon U \to Y$ denotes the unique morphism in $\mathcal{D}$ with $pf' = \mathrm{uni}^U(G\hat{f})$.

Proof. To show that U' becomes a couniversal object under X' with universal morphism u', we suppose given an object Y in $\mathcal{D}$ and a morphism $f'\colon X' \to GY$ in $\mathcal{C}$. Since U is a couniversal object under X along G, there exists a unique morphism $\hat{f}\colon U \to Y$ with $pf' = \mathrm{uni}^U(G\hat{f})$.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\mathrm{uni}^U"'] & X' \arrow[r, "f'"] & GY \\ GU \arrow[urr, "G\hat{f}"'] & & \end{tikzcd}$$

Hence we get

$$u'\,G(j\hat{f}) = u'(Gj)(G\hat{f}) = i\,\mathrm{uni}^U(G\hat{f}) = ipf' = f'$$

Conversely, we suppose given an arbitrary morphism $g'\colon U' \to Y$ in $\mathcal{D}$ with $f' = u'(Gg')$. We obtain

$$\mathrm{uni}^U G(qg) = \mathrm{uni}^U(Gq)(Gg) = pu'(Gg) = pf'$$

and therefore $qg = \hat{f}$. But then we necessarily have $g = jqg = j\hat{f}$.
Altogether, U' becomes a couniversal object under X' along G with $\mathrm{uni}^{U'} = u'$. □

(B.7) Corollary. We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$, an object X in $\mathcal{C}$ and a couniversal object U under X along G. Moreover, we suppose given an isomorphism $a\colon X \to X'$ in $\mathcal{C}$ and an isomorphism $b\colon U \to U'$ in $\mathcal{D}$. Then U' becomes a couniversal object under X' along G with universal morphism $\mathrm{uni}^{U'} = a^{-1}\,\mathrm{uni}^{U}\,(Gb)$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[r, swap, "\cong"] \arrow[d, swap, "\mathrm{uni}^U"] & X' \arrow[d, "\mathrm{uni}^{U'}"] \\ GU \arrow[r, "Gb"] \arrow[r, swap, "\cong"] & GU' \end{tikzcd}$$

Given an object Y in $\mathcal{D}$ and a morphism $f'\colon X' \to GY$ in $\mathcal{C}$, the unique morphism $\hat{f}'\colon U' \to Y$ in $\mathcal{D}$ such that $f' = \mathrm{uni}^{U'}\,(G\hat{f}')$ is given by

$$\hat{f}' = b^{-1}\hat{f},$$

where $\hat{f}\colon U \to Y$ is the unique morphism in $\mathcal{D}$ with $af' = \mathrm{uni}^{U}\,(G\hat{f})$.

(B.8) Proposition. We suppose given a full functor $G\colon \mathcal{D} \to \mathcal{C}$, an object X in $\mathcal{C}$ and a couniversal object U under X along G. The following assertions are equivalent.

(a) The universal morphism $\mathrm{uni}\colon X \to GU$ is an isomorphism in $\mathcal{C}$.

(b) We have $X \cong GU$ in $\mathcal{C}$.

(c) There exists an object Y in $\mathcal{D}$ with $X \cong GY$ in $\mathcal{C}$.

Proof. If $\mathrm{uni}\colon X \to GU$ is an isomorphism in $\mathcal{C}$, then in particular $X \cong GU$ in $\mathcal{C}$. Moreover, if $X \cong GU$, then in particular there exists an object Y in $\mathcal{D}$ with $X \cong GY$ in $\mathcal{C}$. So we suppose that there exists an object Y in $\mathcal{D}$ and an isomorphism $f\colon X \to GY$ in $\mathcal{C}$. By the universal property of U, there exists a unique morphism $\hat{f}\colon U \to Y$ in $\mathcal{D}$ with $f = \mathrm{uni}\,(G\hat{f})$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[r, swap, "\cong"] \arrow[d, swap, "\mathrm{uni}"] & GY \\ GU \arrow[ur, swap, "G\hat{f}"] & \end{tikzcd}$$

As f is invertible, we get $1_X = \mathrm{uni}\,(G\hat{f})f^{-1}$. Moreover, we have $\mathrm{uni}\,(G\hat{f})f^{-1}\,\mathrm{uni} = \mathrm{uni}$.

$$\begin{tikzcd} X \arrow[r, "\mathrm{uni}"] \arrow[d, swap, "\mathrm{uni}"] & GU \\ GU \arrow[ur, swap, "(G\hat{f})f^{-1}\,\mathrm{uni}"] & \end{tikzcd}$$

Since G is full, there exists a morphism $e\colon U \to U$ with $(G\hat{f})f^{-1}\,\mathrm{uni} = Ge$, and the universal property of U implies that $e = 1_U$. Thus we have $(G\hat{f})f^{-1}\,\mathrm{uni} = G1_U = 1_{GU}$.
Altogether, uni is an isomorphism in $\mathcal{C}$ with $\mathrm{uni}^{-1} = (G\hat{f})f^{-1}$. □

Composition of functors

(B.9) Proposition. We suppose given a functor $G\colon \mathcal{D} \to \mathcal{C}$, an object X in $\mathcal{C}$ and a couniversal object U_G in $\mathcal{D}$ under X along G. Moreover, we suppose given a functor $K\colon \mathcal{E} \to \mathcal{D}$.

(a) Given a couniversal object U_K in $\mathcal{E}$ under U_G along K, then the underlying object of U_K becomes a couniversal object $U_{G \circ K}$ in $\mathcal{E}$ under X along $G \circ K$ with universal morphism

$$\mathrm{uni}^{U_{G\circ K}} = \mathrm{uni}^{U_G}(G\mathrm{uni}^{U_K}).$$

(b) Given a couniversal object $U_{G \circ K}$ in $\mathcal{E}$ under X along $G \circ K$, then the underlying object of $U_{G \circ K}$ becomes a couniversal object U_K in $\mathcal{E}$ under U_G along K, where the universal morphism $\mathrm{uni}^{U_K} \colon U_G \to KU_K$ is the unique morphism in $\mathcal{D}$ with

$$\mathrm{uni}^{U_{G \circ K}} = \mathrm{uni}^{U_G}(G\mathrm{uni}^{U_K}).$$

$$\begin{tikzcd} X \arrow[r, "\mathrm{uni}^{U_{G \circ K}}"] \arrow[d, "\mathrm{uni}^{U_G}"'] & GKU_K = GKU_{G \circ K} \\ GU_G \arrow[ur, "G\mathrm{uni}^{U_K}"'] & \end{tikzcd}$$

Proof. We freely use remark (B.3) in this proof. As U_G is a universal object under X along G, we have an isotransformation $\Phi \colon {}_{\mathcal{D}}(U_G, -) \to {}_{\mathcal{C}}(X, G-)$ given by

$$\Phi_Y \colon {}_{\mathcal{D}}(U_G, Y) \to {}_{\mathcal{C}}(X, GY),\ g \mapsto \mathrm{uni}^{U_G}(Gg)$$

for $Y \in \operatorname{Ob} \mathcal{D}$.

(a) We suppose given a couniversal object U_K under U_G along K, so that we have an isotransformation $\Psi \colon {}_{\mathcal{E}}(U_K, -) \to {}_{\mathcal{D}}(U_G, K-)$ given by

$$\Psi_Z \colon {}_{\mathcal{E}}(U_K, Z) \to {}_{\mathcal{D}}(U_G, KZ),\ h \mapsto \mathrm{uni}^{U_K}(Kh)$$

for $Z \in \operatorname{Ob} \mathcal{E}$. We let $\Theta \colon {}_{\mathcal{E}}(U_K, -) \to {}_{\mathcal{C}}(X, GK-)$ be the isotransformation defined by $\Theta := \Psi\,(\Phi * K)$. Then the underlying object of U_K becomes a couniversal object $U_{G \circ K}$ under X along $G \circ K$ with universal morphism

$$\mathrm{uni}^{U_{G \circ K}} = 1_{U_{G \circ K}} \Theta_{U_{G \circ K}} = 1_{U_{G \circ K}} \Psi_{U_{G \circ K}} \Phi_{KU_{G \circ K}} = \mathrm{uni}^{U_G}\, G(1_{U_K} \Psi_{U_K}) = \mathrm{uni}^{U_G}\, G(\mathrm{uni}^{U_K}).$$

(b) We suppose given a couniversal object $U_{G \circ K}$ under X along $G \circ K$, so that we have an isotransformation $\Theta \colon {}_{\mathcal{E}}(U_{G \circ K}, -) \to {}_{\mathcal{C}}(X, GK-)$ given by

$$\Theta_Z \colon {}_{\mathcal{E}}(U_{G \circ K}, Z) \to {}_{\mathcal{C}}(X, GKZ),\ h \mapsto \mathrm{uni}^{U_{G \circ K}}(GKh)$$

for $Z \in \operatorname{Ob} \mathcal{E}$. We let $\Psi \colon {}_{\mathcal{E}}(U_{G \circ K}, -) \to {}_{\mathcal{D}}(U_G, K-)$ be the isotransformation defined by $\Psi := \Theta\,(\Phi * K)^{-1}$. Then the underlying object of $U_{G \circ K}$ becomes a couniversal object U_K under U_G along K with universal morphism

$$\mathrm{uni}^{U_K} = 1_{U_K} \Psi_{U_K} = 1_{U_K} \Theta_{U_K} \Phi^{-1}_{KU_K} = 1_{U_{G \circ K}} \Theta_{U_{G \circ K}} \Phi^{-1}_{KU_{G \circ K}} = \mathrm{uni}^{U_{G \circ K}} \Phi^{-1}_{KU_{G \circ K}}.$$

But this means that uni^{U_K} is the unique morphism in $\mathcal{D}$ with

$$\mathrm{uni}^{U_{G \circ K}} = \mathrm{uni}^{U_K} \Phi_{KU_{G \circ K}} = \mathrm{uni}^{U_G}(G\mathrm{uni}^{U_K}).$$ □

2 From couniversal objects to left adjoint functors

Throughout this section, we suppose given a functor $G \colon \mathcal{D} \to \mathcal{C}$.

The couniversal object category

(B.10) Definition (couniversal object category)**.** For $X \in \operatorname{Ob} \mathcal{C}$, we let $\mathfrak{U}_X$ be the set of couniversal objects under X along G. The structure category $\mathrm{Uni}^{\mathrm{co}}(G) := \mathcal{C}_{\mathfrak{U}}$ is called the *couniversal object category* of G.

(B.11) Remark. We have

$$\operatorname{Ob} \mathrm{Uni}^{\mathrm{co}}(G) = \{(X, U) \mid X \in \operatorname{Ob} \mathcal{C},\ U \text{ couniversal object under } X \text{ along } G\}.$$

For objects (X, U), (X', U') in $\mathrm{Uni}^{\mathrm{co}}(G)$, we have the hom-set

$$_{\mathrm{Uni}^{\mathrm{co}}(G)}((X, U), (X', U')) = {}_{\mathcal{C}}(X, X').$$

For morphisms $f\colon (X, U) \to (X', U')$, $f'\colon (X', U') \to (X'', U'')$ in $\mathrm{Uni}^{\mathrm{co}}(G)$, the composite $ff'\colon (X, U) \to (X'', U'')$ in $\mathrm{Uni}^{\mathrm{co}}(G)$ has the underlying morphism $ff'\colon X \to X'$ in $\mathcal{C}$. For an object (X, U) in $\mathrm{Uni}^{\mathrm{co}}(G)$, the identity morphism $1_{(X,U)}\colon (X, U) \to (X, U)$ in $\mathrm{Uni}^{\mathrm{co}}(G)$ has the underlying morphism $1_X\colon X \to X$ in $\mathcal{C}$.
The forgetful functor $\mathrm{U}\colon \mathrm{Uni}^{\mathrm{co}}(G) \to \mathcal{C}$ is given on the objects by

$$\mathrm{U}_U X = X$$

for $(X, U) \in \mathrm{Ob}\,\mathrm{Uni}^{\mathrm{co}}(G)$, and on the morphisms by

$$\mathrm{U}_{U,U'} f = f$$

for a morphism $f\colon (X, U) \to (X', U')$ in $\mathrm{Uni}^{\mathrm{co}}(G)$.

The total couniversal object functor

(B.12) Proposition. We have a functor $\bar{F}\colon \mathrm{Uni}^{\mathrm{co}}(G) \to \mathcal{D}$, given on the objects by

$$\bar{F}_U X = U$$

for $(X, U) \in \mathrm{Ob}\,\mathrm{Uni}^{\mathrm{co}}(G)$, and on the morphisms as follows. Given a morphism $f\colon (X, U) \to (X', U')$ in $\mathrm{Uni}^{\mathrm{co}}(G)$, then $\bar{F}_{U,U'} f\colon U \to U'$ is the unique morphism in $\mathcal{D}$ with

$$f \mathrm{uni}^{U'} = \mathrm{uni}^U (G\bar{F}_{U,U'} f).$$

Proof. We define a map

$$\bar{F}_0\colon \mathrm{Ob}\,\mathrm{Uni}^{\mathrm{co}}(G) \to \mathrm{Ob}\,\mathcal{D},\ (X, U) \mapsto U.$$

We suppose given $(X, U), (X', U') \in \mathrm{Ob}\,\mathrm{Uni}^{\mathrm{co}}(G)$. By remark (B.5), we obtain a well-defined map

$$\bar{F}_{(X,U),(X',U')}\colon {}_{\mathrm{Uni}^{\mathrm{co}}(G)}((X, U), (X', U')) \to {}_{\mathcal{D}}(U, U'),$$

where $\bar{F}_{U,U'} f\colon \in {}_{\mathcal{D}}(U, U')$ for $f \in {}_{\mathrm{Uni}^{\mathrm{co}}(G)}((X, U), (X', U'))$ is the unique element with $f \mathrm{uni}^{U'} = \mathrm{uni}^U (G\bar{F}_{U,U'} f)$.
Given morphisms $f\colon (X, U) \to (X', U')$ and $f'\colon (X', U') \to (X'', U'')$ in $\mathrm{Uni}^{\mathrm{co}}(G)$, we have

$$\begin{aligned} ff' \mathrm{uni}^{U''} &= f \mathrm{uni}^{U'} (G\bar{F}_{(X',U'),(X'',U'')} f') = \mathrm{uni}^U (G\bar{F}_{(X,U),(X',U')} f)(G\bar{F}_{(X',U'),(X'',U'')} f') \\ &= \mathrm{uni}^U\, G((\bar{F}_{(X,U),(X',U')} f)(\bar{F}_{(X',U'),(X'',U'')} f')) \end{aligned}$$

and therefore $\bar{F}_{(X,U),(X'',U'')}(ff') = (\bar{F}_{(X,U),(X',U')} f)(\bar{F}_{(X',U'),(X'',U'')} f')$. Moreover, for $(X, U) \in \mathrm{Ob}\,\mathrm{Uni}^{\mathrm{co}}(G)$, we have

$$1_X \mathrm{uni}_U = \mathrm{uni}_U 1_{GU} = \mathrm{uni}_U (G1_U)$$

and therefore $\bar{F}_{(X,U),(X,U)}(1_X) = 1_U = 1_{\bar{F}_0(X,U)}$. Thus we have a functor $\bar{F}\colon \mathrm{Uni}^{\mathrm{co}}(G) \to \mathcal{D}$ given by $\mathrm{Ob}\,\bar{F} = \bar{F}_0$ and by $\bar{F}_{U,U'} f = \bar{F}_{(X,U),(X',U')} f$ for every morphism $f\colon (X, U) \to (X', U')$ in $\mathrm{Uni}^{\mathrm{co}}(G)$. □

(B.13) Definition (total couniversal object functor)**.** The functor $\bar{F}\colon \mathrm{Uni}^{\mathrm{co}}(G) \to \mathcal{D}$ from proposition (B.12) is called the *total couniversal object functor* along G.

Choices of couniversal objects

(B.14) Definition (choice of couniversal objects)**.** We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$. A *choice of couniversal objects* for $\mathcal{U}$ along G is a family $(U_X)_{X \in \mathrm{Ob}\,\mathcal{U}}$ over $\mathrm{Ob}\,\mathcal{U}$ such that U_X is a couniversal object under X along G for every $X \in \mathrm{Ob}\,\mathcal{C}$.

(B.15) Remark. We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$. For $X \in \operatorname{Ob} \mathcal{C}$, we let $\mathfrak{U}_X$ be the set of couniversal objects under X along G. A choice of couniversal objects for $\mathcal{U}$ along G is precisely a choice of structures for $\mathcal{U}$ with respect to $\mathfrak{U} = (\mathfrak{U}_X)_{X \in \operatorname{Ob} \mathcal{U}}$.

For the definition of the structure choice functor with respect to a choice of structures, see definition (A.8). In the case of a choice of couniversal objects, the structure choice functor is given as follows.

(B.16) Remark. We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$ and a choice of couniversal objects $U = (U_X)_{X \in \operatorname{Ob} \mathcal{C}}$ for $\mathcal{U}$ along G. The structure choice functor $\mathrm{I}_U \colon \mathcal{U} \to \operatorname{Uni}^{\mathrm{co}}(G)$ is given on the objects by

$$\mathrm{I}_U X = (X, U_X)$$

for $X \in \operatorname{Ob} \mathcal{C}$, and on the morphisms by

$$\mathrm{I}_U f = f \colon (X, U_X) \to (X', U_{X'})$$

for every morphism $f \colon X \to X'$ in $\mathcal{C}$.

(B.17) Remark. We suppose given a choice of couniversal objects $U = (U_X)_{X \in \operatorname{Ob} \mathcal{C}}$ for $\mathcal{C}$ along G.

(a) We have

$$\mathrm{U} \circ \mathrm{I}_U = \mathrm{id}_{\mathcal{C}}.$$

(b) We have

$$\mathrm{I}_U \circ \mathrm{U} \cong \mathrm{id}_{\operatorname{Uni}^{\mathrm{co}}(G)}.$$

An isotransformation $\mathrm{I}_U \circ \mathrm{U} \to \mathrm{id}_{\operatorname{Uni}^{\mathrm{co}}(G)}$ is given by $1_X \colon (X, U_X) \to (X, U')$ for $(X, U') \in \operatorname{Ob} \operatorname{Uni}^{\mathrm{co}}(G)$.

In particular, $\mathrm{U} \colon \operatorname{Uni}^{\mathrm{co}}(G) \to \mathcal{C}$ and $\mathrm{I}_U \colon \mathcal{C} \to \operatorname{Uni}^{\mathrm{co}}(G)$ are mutually isomorphism inverse equivalences of categories.

Proof. This follows from remark (A.31) and proposition (A.9). □

Couniversal object functors

(B.18) Definition (couniversal object functor)**.** We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$ and a choice of couniversal objects $U = (U_X)_{X \in \operatorname{Ob} \mathcal{U}}$ for $\mathcal{U}$ along G. Moreover, we let $\bar{F} \colon \operatorname{Uni}^{\mathrm{co}}(G) \to \mathcal{D}$ denote the total couniversal object functor along G. The composite

$$\bar{F} \circ \mathrm{I}_U \colon \mathcal{C} \to \mathcal{D}$$

is called the *couniversal object functor* along G with respect to U.

(B.19) Remark. We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$ and a choice of couniversal objects $U = (U_X)_{X \in \operatorname{Ob} \mathcal{U}}$ for $\mathcal{U}$ along G. Moreover, we let $F \colon \mathcal{C} \to \mathcal{D}$ be the couniversal object functor along G with respect to U. Then F is given on the objects by

$$FX = U_X$$

for $X \in \operatorname{Ob} \mathcal{U}$, and on the morphisms as follows. Given a morphism $f \colon X \to X'$ in $\mathcal{C}$, then $Ff \colon U_X \to U_{X'}$ is the unique morphism in $\mathcal{D}$ with

$$f \operatorname{uni}^{U_{X'}} = \operatorname{uni}^{U_X}(GFf).$$

Proof. We let $\bar{F} \colon \operatorname{Uni}^{\mathrm{co}}(G) \to \mathcal{D}$ denote the total couniversal object functor along G, so that $F = \bar{F} \circ \mathrm{I}_U$. For $X \in \operatorname{Ob} \mathcal{U}$, we have $\mathrm{I}_U X = (X, U_X)$ and therefore

$$FX = \bar{F} \mathrm{I}_U X = \bar{F}_{U_X} X = U_X.$$

We suppose given a morphism $f \colon X \to X'$ in $\mathcal{C}$. Then we have $\mathrm{I}_U f = f \colon (X, U_X) \to (X', U_{X'})$, and so $Ff = \bar{F} \mathrm{I}_U f = \bar{F}_{U_X, U_{X'}} f \colon U_X \to U_{X'}$ is the unique morphism in $\mathcal{D}$ with $f \operatorname{uni}^{U_{X'}} = \operatorname{uni}^{U_X}(G \bar{F}_{U_X, U_{X'}} f)$, that is, with $f \operatorname{uni}^{U_{X'}} = \operatorname{uni}^{U_X}(GFf)$. □

(B.20) Remark. We suppose given a full subcategory $\mathcal{U}$ of $\mathcal{C}$ and choices of couniversal objects $U = (U_X)_{X \in \operatorname{Ob} \mathcal{U}}$ and $U' = (U'_X)_{X \in \operatorname{Ob} \mathcal{U}}$ for $\mathcal{U}$ along G. Moreover, we let $F \colon \mathcal{U} \to \mathcal{D}$ resp. $F' \colon \mathcal{U} \to \mathcal{D}$ be the couniversal object functor along G with respect to U resp. U'. Then we have

$$F \cong F'.$$

An isotransformation $F \to F'$ is given as follows. For $X \in \operatorname{Ob} \mathcal{U}$, the component $\alpha_X \colon FX \to F'X$ is given by the unique morphism in $\mathcal{D}$ with $\operatorname{uni}^{U'_X} = \operatorname{uni}^{U_X}(G\alpha_X)$.

Proof. We let $\bar{F} \colon \operatorname{Uni}^{\mathrm{co}}(G) \to \mathcal{D}$ denote the total couniversal object functor along G, so that $F = \bar{F} \circ \mathrm{I}_U$ and $F' = \bar{F} \circ \mathrm{I}_{U'}$. By corollary (A.12), we have

$$F = \bar{F} \circ \mathrm{I}_U \cong \bar{F} \circ \mathrm{I}_{U'} = F',$$

and an isotransformation $\alpha \colon F \to F'$ is given by $\alpha_X = \bar{F}_{U_X, U'_X}(1_X) \colon FX \to F'X$ for $X \in \operatorname{Ob} \mathcal{U}$, that is, by the unique morphism in $\mathcal{D}$ with $\operatorname{uni}^{U'_X} = \operatorname{uni}^{U_X}(G\alpha_X)$. $\square$

Adjointness

(B.21) Theorem. If there exists a couniversal object under every object in $\mathcal{C}$ along G, then G has a left adjoint.
For every choice of couniversal objects $U = (U_X)_{X \in \operatorname{Ob} \mathcal{C}}$ for $\mathcal{C}$ along G, the couniversal object functor $F \colon \mathcal{C} \to \mathcal{D}$ along G with respect to U is left adjoint to G. An adjunction $\Phi \colon F \dashv G$ is given by

$$\Phi_{X,Y} \colon {}_{\mathcal{D}}(FX, Y) \to {}_{\mathcal{C}}(X, GY),\ g \mapsto \operatorname{uni}^{FX}(Gg)$$

for $X \in \operatorname{Ob} \mathcal{C}$, $Y \in \operatorname{Ob} \mathcal{D}$.

Proof. We suppose given a choice of couniversal objects $U = (U_X)_{X \in \operatorname{Ob} \mathcal{C}}$ for $\mathcal{C}$ along G and we let $F \colon \mathcal{C} \to \mathcal{D}$ be the couniversal object functor along G with respect to U. By remark (B.3)(a), the maps

$$\Phi_{X,Y} \colon {}_{\mathcal{D}}(FX, Y) \to {}_{\mathcal{C}}(X, GY),\ g \mapsto \operatorname{uni}^{FX}(Gg).$$

for $X \in \operatorname{Ob} \mathcal{C}$, $Y \in \operatorname{Ob} \mathcal{D}$ define isotransformations

$$\Phi_{X,=} \colon {}_{\mathcal{D}}(FX, =) \to {}_{\mathcal{C}}(X, G=)$$

for every $X \in \operatorname{Ob} \mathcal{C}$. So to show that $\Phi = (\Phi_{X,Y})_{X \in \operatorname{Ob} \mathcal{C}, Y \in \operatorname{Ob} \mathcal{D}}$ is an adjunction, it remains to prove naturality in X. Indeed, given a morphism $f \colon X' \to X$ in $\mathcal{C}$, we have

$$\begin{aligned} g \Phi_{X,Y}\, {}_{\mathcal{C}}(f, GY) &= (\operatorname{uni}^{FX}(Gg))\, {}_{\mathcal{C}}(f, GY) = f \operatorname{uni}^{FX}(Gg) = \operatorname{uni}^{FX'}(GUf)(Gg) = \operatorname{uni}^{FX'} G((Ff)g) \\ &= ((Ff)g)\Phi_{X',Y} = g\, {}_{\mathcal{D}}(Ff, Y)\Phi_{X',Y} \end{aligned}$$

for $g \in {}_{\mathcal{D}}(FX, Y)$, $Y \in \operatorname{Ob} \mathcal{D}$, that is, $\Phi_{X,Y}\, {}_{\mathcal{C}}(f, GY) = {}_{\mathcal{D}}(Ff, Y)\Phi_{X',Y}$ for $Y \in \operatorname{Ob} \mathcal{D}$.

$$\begin{tikzcd} {}_{\mathcal{D}}(FX, Y) \arrow[r, "\Phi_{X,Y}", "\cong"'] \arrow[d, "{}_{\mathcal{D}}(Ff, Y)"'] & {}_{\mathcal{C}}(X, GY) \arrow[d, "{}_{\mathcal{C}}(f, GY)"] \\ {}_{\mathcal{D}}(FX', Y) \arrow[r, "\Phi_{X',Y}", "\cong"'] & {}_{\mathcal{C}}(X', GY) \end{tikzcd}$$

$\square$

Appendix C

Another proof of the Z-2-arrow calculus

To prove the Z-2-arrow calculus for Brown cofibration categories (3.128), we roughly proceeded in three steps: First, we introduced the notion of a Z-fractionable category and constructed a localisation, the S-Ore localisation, in this framework ab ovo. Second, we proved the Z-2-arrow calculus as a consequence of this particular construction. Third, we showed that every Brown cofibration category gives rise to the structure of a Z-fractionable category, so in particular the said Z-2-arrow calculus holds, see theorem (3.128). The first two steps were treated in chapter II, (mainly) section 4 to 6, the third one in chapter III, section 9. As a consequence, we gave an alternative proof for Brown's homotopy S-2-arrow calculus, see theorem (3.132).

In this appendix, we give an alternative proof of theorem (3.128)(b), using Brown's homotopy S-2-arrow calculus in the sense of theorem (C.16), which is a consequence of [7, dual of prop. 2], cf. [7, dual of th. 1]. To this end, we introduce a variant of the cylinder notion, see definition (C.4), and develop some further results. The main step is the imitation of the *mapping cylinder construction* from classical homotopy theory and its application to S-2-arrows in a suitable way, see proposition (C.12) and remark (C.11).

Finite coproducts of cylinders

We show that the notion of a cylinder, see definition (3.108), and of a cylinder homotopy, see definition (3.130), is compatible with finite coproducts.

(C.1) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and $n \in \mathbb{N}_0$. Moreover, for $k \in [1, n]$, we suppose given an S-2-arrow $(f_k, u_k)\colon X_k \to \tilde{Y}_k \leftarrow Y_k$ in $\mathcal{C}_{\mathrm{cof}}$ and a cylinder Z_k of (f_k, u_k). Then $\coprod_{k\in[1,n]} Z_k$ becomes a cylinder of $(\coprod_{k\in[1,n]} f_k, \coprod_{k\in[1,n]} u_k)\colon \coprod_{k\in[1,n]} X_k \to \coprod_{k\in[1,n]} \tilde{Y}_k \leftarrow \coprod_{k\in[1,n]} Y_k$ having $\mathrm{ins}_0^{\coprod_{k\in[1,n]} Z_k} = \coprod_{k\in[1,n]} \mathrm{ins}_0^{Z_k}$, $\mathrm{ins}_1^{\coprod_{k\in[1,n]} Z_k} = \coprod_{k\in[1,n]} \mathrm{ins}_1^{Z_k}$, $\mathrm{s}^{\coprod_{k\in[1,n]} Z_k} = \coprod_{k\in[1,n]} \mathrm{s}^{Z_k}$.

Proof. We have

$$\begin{aligned}
(\coprod_{k\in[1,n]} \mathrm{ins}_0^{Z_k})(\coprod_{k\in[1,n]} \mathrm{s}^{Z_k}) &= \coprod_{k\in[1,n]} (\mathrm{ins}_0^{Z_k}\mathrm{s}^{Z_k}) = \coprod_{k\in[1,n]} f_k, \\
(\coprod_{k\in[1,n]} \mathrm{ins}_1^{Z_k})(\coprod_{k\in[1,n]} \mathrm{s}^{Z_k}) &= \coprod_{k\in[1,n]} (\mathrm{ins}_1^{Z_k}\mathrm{s}^{Z_k}) = \coprod_{k\in[1,n]} u_k.
\end{aligned}$$

Moreover,

$$\begin{pmatrix} \coprod_{k\in[1,n]} \mathrm{emb}_0^{X_k \amalg Y_k} \\ \coprod_{k\in[1,n]} \mathrm{emb}_1^{X_k \amalg Y_k} \end{pmatrix} : (\coprod_{k\in[1,n]} X_k) \amalg (\coprod_{k\in[1,n]} Y_k) \to \coprod_{k\in[1,n]} (X_k \amalg Y_k)$$

is an isomorphism in $\mathcal{C}_{\mathrm{cof}}$, whence a cofibration. So as

$$\coprod_{k\in[1,n]} \begin{pmatrix} \mathrm{ins}_0^{Z_k} \\ \mathrm{ins}_1^{Z_k} \end{pmatrix} : \coprod_{k\in[1,n]} (X_k \amalg Y_k) \to \coprod_{k\in[1,n]} Z_k$$

is a cofibration by proposition (3.26)(b), it follows that

$$\begin{aligned}
\begin{pmatrix} \coprod_{k\in[1,n]} \mathrm{emb}_0^{X_k \amalg Y_k} \\ \coprod_{k\in[1,n]} \mathrm{emb}_1^{X_k \amalg Y_k} \end{pmatrix} (\coprod_{k\in[1,n]} \begin{pmatrix} \mathrm{ins}_0^{Z_k} \\ \mathrm{ins}_1^{Z_k} \end{pmatrix}) &= \begin{pmatrix} (\coprod_{k\in[1,n]} \mathrm{emb}_0^{X_k \amalg Y_k})(\coprod_{k\in[1,n]} \begin{pmatrix} \mathrm{ins}_0^{Z_k} \\ \mathrm{ins}_1^{Z_k} \end{pmatrix}) \\ (\coprod_{k\in[1,n]} \mathrm{emb}_1^{X_k \amalg Y_k})(\coprod_{k\in[1,n]} \begin{pmatrix} \mathrm{ins}_0^{Z_k} \\ \mathrm{ins}_1^{Z_k} \end{pmatrix}) \end{pmatrix} \\
&= \begin{pmatrix} \coprod_{k\in[1,n]} (\mathrm{emb}_0^{X_k \amalg Y_k} \begin{pmatrix} \mathrm{ins}_0^{Z_k} \\ \mathrm{ins}_1^{Z_k} \end{pmatrix}) \\ \coprod_{k\in[1,n]} (\mathrm{emb}_1^{X_k \amalg Y_k} \begin{pmatrix} \mathrm{ins}_0^{Z_k} \\ \mathrm{ins}_1^{Z_k} \end{pmatrix}) \end{pmatrix} = \begin{pmatrix} \coprod_{k\in[1,n]} \mathrm{ins}_0^{Z_k} \\ \coprod_{k\in[1,n]} \mathrm{ins}_1^{Z_k} \end{pmatrix}
\end{aligned}$$

is a cofibration by closedness under composition.

$$\begin{tikzcd}
\coprod_{k\in[1,n]} (X_k \amalg Y_k) \arrow[r, "\coprod_{k\in[1,n]} \begin{pmatrix} \mathrm{ins}_0^{Z_k} \\ \mathrm{ins}_1^{Z_k} \end{pmatrix}", "\bullet"{marking}] & \coprod_{k\in[1,n]} Z_k \arrow[d, equal] \\
(\coprod_{k\in[1,n]} X_k) \amalg (\coprod_{k\in[1,n]} Y_k) \arrow[u, "\begin{pmatrix} \coprod_{k\in[1,n]} \mathrm{emb}_0^{X_k \amalg Y_k} \\ \coprod_{k\in[1,n]} \mathrm{emb}_1^{X_k \amalg Y_k} \end{pmatrix}", "\cong"', "\bullet"{marking}] \arrow[r, "\begin{pmatrix} \coprod_{k\in[1,n]} \mathrm{ins}_0^{Z_k} \\ \coprod_{k\in[1,n]} \mathrm{ins}_1^{Z_k} \end{pmatrix}", "\bullet"{marking}] & \coprod_{k\in[1,n]} Z_k
\end{tikzcd}$$

Altogether, $\coprod_{k\in[1,n]} Z_k$ becomes a cylinder of $(\coprod_{k\in[1,n]} f_k, \coprod_{k\in[1,n]} u_k)$ having $\mathrm{ins}_0^{\coprod_{k\in[1,n]} Z_k} = \coprod_{k\in[1,n]} \mathrm{ins}_0^{Z_k}$, $\mathrm{ins}_1^{\coprod_{k\in[1,n]} Z_k} = \coprod_{k\in[1,n]} \mathrm{ins}_1^{Z_k}$, $\mathrm{s}^{\coprod_{k\in[1,n]} Z_k} = \coprod_{k\in[1,n]} \mathrm{s}^{Z_k}$. □

(C.2) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and $n \in \mathbb{N}_0$. Moreover, for $k \in [1,n]$, we suppose given a cofibrant object X_k in $\mathcal{C}$ and a cylinder $\dot{X}_k$ of X_k. Given morphisms $f_0, f_1 \colon \coprod_{k\in[1,n]} X_k \to Y$, $f \colon \coprod_{k\in[1,n]} \dot{X}_k \to Y$ in $\mathcal{C}$, we have

$$f \colon f_0 \overset{\mathrm{c}}{\sim}_{\coprod_{k\in[1,n]} \dot{X}_k} f_1$$

if and only if

$$\mathrm{emb}_k f \colon \mathrm{emb}_k f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}_k} \mathrm{emb}_k f_1$$

for every $k \in [1,n]$.

Proof. We have $f \colon f_0 \overset{\mathrm{c}}{\sim}_{\coprod_{k\in[1,n]} \dot{X}_k} f_1$ if and only if $\mathrm{ins}_0^{\coprod_{k\in[1,n]} \dot{X}_k} f = f_0$ and $\mathrm{ins}_0^{\coprod_{k\in[1,n]} \dot{X}_k} f = f_1$, that is, if and only if $\mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} \mathrm{ins}_0^{\coprod_{k\in[1,n]} \dot{X}_k} f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} f_0$ and $\mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} \mathrm{ins}_1^{\coprod_{k\in[1,n]} \dot{X}_k} f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} f_1$ for every $k \in [1,n]$. So as

$$\mathrm{ins}_l^{\dot{X}_k} \mathrm{emb}_k^{\coprod_{k\in[1,n]} \dot{X}_k} f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} (\coprod_{k\in[1,n]} \mathrm{ins}_l^{\dot{X}_k}) f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} \mathrm{ins}_l^{\coprod_{k\in[1,n]} \dot{X}_k} f,$$

for $k \in [1,n]$, $l \in \{0,1\}$, the condition $\mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} \mathrm{ins}_l^{\coprod_{k\in[1,n]} \dot{X}_k} f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} f_l$ is equivalent to $\mathrm{ins}_l^{\dot{X}_k} \mathrm{emb}_k^{\coprod_{k\in[1,n]} \dot{X}_k} f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} f_l$. Altogether, we have $f \colon f_0 \overset{\mathrm{c}}{\sim}_{\coprod_{k\in[1,n]} \dot{X}_k} f_1$ if and only if $\mathrm{ins}_0^{\dot{X}_k} \mathrm{emb}_k^{\coprod_{k\in[1,n]} \dot{X}_k} f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} f_0$ and $\mathrm{ins}_1^{\dot{X}_k} \mathrm{emb}_k^{\coprod_{k\in[1,n]} \dot{X}_k} f = \mathrm{emb}_k^{\coprod_{k\in[1,n]} X_k} f_1$ for every $k \in [1,n]$, that is, to $\mathrm{emb}_k f \colon \mathrm{emb}_k f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}_k} \mathrm{emb}_k f_1$ for every $k \in [1,n]$. □

(C.3) Corollary. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$, cofibrant objects X and Y in $\mathcal{C}$, a cylinder $\dot{X}$ of X and a cylinder $\dot{Y}$ of Y. Given S-2-arrows $(f_0, u_0), (f_1, u_1) \colon X \to \tilde{Y} \leftarrow Y$, $(f, u) \colon \dot{X} \to \tilde{Y} \leftarrow \dot{Y}$ in $\mathcal{C}$, we have $(f, u) \colon (f_0, u_0) \overset{\mathrm{c}}{\sim}_{\dot{X}, \dot{Y}} (f_1, u_1)$ if and only if $\binom{f}{u} \colon \binom{f_0}{u_0} \overset{\mathrm{c}}{\sim}_{\dot{X} \amalg \dot{Y}} \binom{f_1}{u_1}$.

Proof. We have $(f, u) \colon (f_0, u_0) \overset{\mathrm{c}}{\sim}_{\dot{X}, \dot{Y}} (f_1, u_1)$ if and only if $f \colon f_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f_1$ and $u \colon u_0 \overset{\mathrm{c}}{\sim}_{\dot{Y}} u_1$. By remark (C.2), this is equivalent to $\binom{f}{u} \colon \binom{f_0}{u_0} \overset{\mathrm{c}}{\sim}_{\dot{X} \amalg \dot{Y}} \binom{f_1}{u_1}$. □

Cylinders with mid insertion

A cylinder, see definition (3.108), is a structure consisting of an object together with three morphisms, the start insertion, the end insertion and the cylinder equivalence. Next, we will introduce a variant of this notion where one has a fourth morphism at hand, the so-called mid insertion.

(C.4) Definition (cylinder with mid insertion)**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$. A *cylinder with mid insertion* (or *cylinder object with mid insertion*) of (f, u) consists of a cylinder Z of (f, u) together with a weak equivalence $i_{0.5}\colon \tilde{Y} \to Z$ in $\mathcal{C}$ such that $i_{0.5}\,\mathrm{s} = 1_{\tilde{Y}}$, and such that there exists a coproduct C of X, $\tilde{Y}$, Y such that $\begin{pmatrix} \mathrm{ins}_0 \\ i_{0.5} \\ \mathrm{ins}_1 \end{pmatrix} : C \to Z$ is a cofibration. By abuse of notation, we denote the cylinder with mid insertion as well as its underlying cylinder by Z. The morphism $i_{0.5}$ is called *mid insertion* (or *insertion at* 0.5) of Z.
Given a cylinder with mid insertion Z of (f, u) having the mid insertion $i_{0.5}$, we write $\mathrm{ins}_{0.5} := i_{0.5}$.

$$\begin{tikzcd} X \arrow[r, "\mathrm{ins}_0"] \arrow[d, equal] & Z \arrow[d, shift right, "\mathrm{s}"'] & Y \arrow[l, "\mathrm{ins}_1"', "\approx"] \arrow[d, equal] \\ X \arrow[r, "f"] & \tilde{Y} \arrow[u, shift right, bend right, "\mathrm{ins}_{0.5}"', "\wr"] & Y \arrow[l, "u"', "\approx"] \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dr, "\mathrm{s}"] & \\ C \arrow[ur, rightarrowtail, "\mathrm{ins}"] \arrow[rr, "\left(\begin{smallmatrix} f \\ 1 \\ u \end{smallmatrix}\right)"'] & & \tilde{Y} \end{tikzcd}$$

(C.5) Notation. In the context of cylinders with mid insertion, we use a different notation for the embeddings into a tertiary coproduct. Given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$, a cylinder with mid insertion Z of (f, u) and a coproduct C of X, $\tilde{Y}$, Y, we write $\mathrm{emb}_0 = \mathrm{emb}_0^C\colon X \to C$, $\mathrm{emb}_{0.5} = \mathrm{emb}_{0.5}^C\colon \tilde{Y} \to C$, $\mathrm{emb}_1 = \mathrm{emb}_1^C\colon Y \to C$ (instead of emb_1, emb_2, emb_3).

(C.6) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$, an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}_{\mathrm{cof}}$ and a cylinder with mid insertion Z of (f, u). For every coproduct C of X, $\tilde{Y}$, Y, the induced morphism $\begin{pmatrix} \mathrm{ins}_0 \\ \mathrm{ins}_{0.5} \\ \mathrm{ins}_1 \end{pmatrix}^C : C \to Z$ is a cofibration in $\mathcal{C}$.

Proof. This is proven analogously to remark (3.110). $\square$

(C.7) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$, an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}_{\mathrm{cof}}$ and a cylinder with mid insertion Z of (f, u). Then the start insertion ins_0 is a cofibration and the mid insertion $\mathrm{ins}_{0.5}$ and the end insertion ins_1 are acyclic cofibrations in $\mathcal{C}$.

Proof. This follows from corollary (3.27). $\square$

The following remark states that cylinders with mid insertions are closely related to (ordinary) cylinders.

(C.8) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ and an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with X, $\tilde{Y}$, Y cofibrant. Given a cylinder Z of $\left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right) : X \amalg Y \to \tilde{Y}$, the underlying object of Z becomes a cylinder with mid insertion $Z_{(f,u)}$ of (f, u) having $\mathrm{ins}_0^{Z_{(f,u)}} = \mathrm{emb}_0 \,\mathrm{ins}_0^Z$, $\mathrm{ins}_{0.5}^{Z_{(f,u)}} = \mathrm{ins}_1^Z$, $\mathrm{ins}_1^{Z_{(f,u)}} = \mathrm{emb}_1 \,\mathrm{ins}_0^Z$, $\mathrm{s}^{Z_{(f,u)}} = \mathrm{s}^Z$.

Proof. We have

$$\begin{aligned} &\mathrm{emb}_0^{X \amalg Y} \,\mathrm{ins}_0^Z \,\mathrm{s}^Z = \mathrm{emb}_0^{X \amalg Y} \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right) = f, \\ &\mathrm{ins}_1^Z \,\mathrm{s}^Z = 1_{\tilde{Y}}, \\ &\mathrm{emb}_1^{X \amalg Y} \,\mathrm{ins}_0^Z \,\mathrm{s}^Z = \mathrm{emb}_1^{X \amalg Y} \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right) = u. \end{aligned}$$

Moreover,

$$\begin{pmatrix} \mathrm{emb}_0^{X \amalg Y} \,\mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \\ \mathrm{emb}_1^{(X \amalg Y) \amalg \tilde{Y}} \\ \mathrm{emb}_1^{X \amalg Y} \,\mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \end{pmatrix} : X \amalg \tilde{Y} \amalg Y \to (X \amalg Y) \amalg \tilde{Y}$$

is an isomorphism in $\mathcal{C}_{\mathrm{cof}}$, whence a cofibration. So as

$$\begin{pmatrix} \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \end{pmatrix} : (X \amalg Y) \amalg \tilde{Y} \to Z$$

is a cofibration, it follows that

$$\begin{pmatrix} \mathrm{emb}_0^{X \amalg Y} \, \mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \\ \mathrm{emb}_1^{(X \amalg Y) \amalg \tilde{Y}} \\ \mathrm{emb}_1^{X \amalg Y} \, \mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \end{pmatrix} \begin{pmatrix} \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \end{pmatrix} = \begin{pmatrix} \mathrm{emb}_0^{X \amalg Y} \, \mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \begin{pmatrix} \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \end{pmatrix} \\ \mathrm{emb}_1^{(X \amalg Y) \amalg \tilde{Y}} \begin{pmatrix} \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \end{pmatrix} \\ \mathrm{emb}_1^{X \amalg Y} \, \mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \begin{pmatrix} \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \mathrm{emb}_0^{X \amalg Y} \, \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \\ \mathrm{emb}_1^{X \amalg Y} \, \mathrm{ins}_0^Z \end{pmatrix}$$

is a cofibration by closedness under composition.

$$\begin{tikzcd}
(X \amalg Y) \amalg \tilde{Y} \arrow[r, "\begin{pmatrix} \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \end{pmatrix}", "\bullet" marking] & Z \arrow[d, equal] \\
X \amalg \tilde{Y} \amalg Y \arrow[u, "\begin{pmatrix} \mathrm{emb}_0^{X \amalg Y} \, \mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \\ \mathrm{emb}_1^{(X \amalg Y) \amalg \tilde{Y}} \\ \mathrm{emb}_1^{X \amalg Y} \, \mathrm{emb}_0^{(X \amalg Y) \amalg \tilde{Y}} \end{pmatrix}", "\bullet" marking, "\cong"'] \arrow[r, "\begin{pmatrix} \mathrm{emb}_0^{X \amalg Y} \, \mathrm{ins}_0^Z \\ \mathrm{ins}_1^Z \\ \mathrm{emb}_1^{X \amalg Y} \, \mathrm{ins}_0^Z \end{pmatrix}", "\bullet" marking] & Z
\end{tikzcd}$$

Altogether, the underlying object of Z becomes a cylinder with mid insertion $Z_{(f,u)}$ of (f, u) having $\mathrm{ins}_0^{Z_{(f,u)}} = \mathrm{emb}_0 \, \mathrm{ins}_0^Z$, $\mathrm{ins}_{0.5}^{Z_{(f,u)}} = \mathrm{ins}_1^Z$, $\mathrm{ins}_1^{Z_{(f,u)}} = \mathrm{emb}_1 \, \mathrm{ins}_0^Z$, $\mathrm{s}^{Z_{(f,u)}} = \mathrm{s}^Z$. □

Corresponding cylinders

In the following, we construct a cylinder with mid insertion of an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ from given cylinders of X and Y.

(C.9) Definition (corresponding cylinder (with mid insertion))**.** We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$.

(a) We suppose given a morphism $f\colon X \to Y$ in $\mathcal{C}$ and a cylinder $\dot{X}$ of X. A *cylinder of f corresponding to $\dot{X}$* consists of a cylinder Z of f and a cylinder homotopy $H\colon \mathrm{ins}_0^Z \overset{\mathrm{c}}{\sim}_{\dot{X}} f \, \mathrm{ins}_1^Z$ such that the following holds.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ins}_0", "\bullet" marking] \arrow[d, "\bullet" marking, "\mathrm{ins}_0", "\wr"'] & Z \arrow[d, equal] \\
\dot{X} \arrow[r, "H"] & Z \\
X \arrow[u, "\bullet" marking, "\mathrm{ins}_1", "\wr"'] \arrow[r, "f"] & Y \arrow[u, "\bullet" marking, "\mathrm{ins}_1", "\wr"']
\end{tikzcd}$$

- For every morphism $g_0\colon X \to \bar{Y}$ and every weak equivalence $g_1\colon Y \to \bar{Y}$ in $\mathcal{C}$ and for every cylinder homotopy $K\colon g_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f g_1$ there exists a unique morphism $\hat{g}\colon Z \to \bar{Y}$ with $g_1 = \mathrm{ins}_1^Z \hat{g}$ and $K = H\hat{g}$.
- The cylinder equivalence $\mathrm{s}^Z\colon Z \to \tilde{Y}$ is the unique morphism in $\mathcal{C}$ with $1_{\tilde{Y}} = \mathrm{ins}_1^Z \mathrm{s}^Z$ and $\mathrm{s}^{\dot{X}} f = H \mathrm{s}^Z$.

By abuse of notation, we refer to the said cylinder of f corresponding to $\dot{X}$ as well as to its underlying cylinder by Z. The cylinder homotopy H is called the *universal cylinder homotopy* of Z.

Given a cylinder Z of f corresponding to $\dot{X}$ with universal cylinder homotopy H, we write $\mathrm{H} = \mathrm{H}^Z := H$.

(b) We suppose given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$, a cylinder $\dot{X}$ of X and a cylinder $\dot{Y}$ of Y. A *cylinder with mid insertion of (f, u) corresponding to $(\dot{X}, \dot{Y})$* consists of a cylinder with mid insertion Z

of (f, u) and a cylinder homotopy $(H_0, H_1)\colon (\mathrm{ins}_0^Z, \mathrm{ins}_1^Z) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (f\,\mathrm{ins}_{0.5}^Z, u\,\mathrm{ins}_{0.5}^Z)$ such that the following holds.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ins}_0"] \arrow[d, "\mathrm{ins}_0", "\wr"'] & Z \arrow[d, equal] & Y \arrow[l, "\mathrm{ins}_1"', "\approx"] \arrow[d, "\mathrm{ins}_0", "\wr"'] \\
\dot{X} \arrow[r, "H_0"] & Z & \dot{Y} \arrow[l, "H_1"', "\approx"] \\
X \arrow[r, "f"] \arrow[u, "\mathrm{ins}_1"', "\wr"] & \tilde{Y} \arrow[u, "\mathrm{ins}_{0.5}"', "\wr"] & Y \arrow[l, "u"', "\approx"] \arrow[u, "\mathrm{ins}_1"', "\wr"]
\end{tikzcd}$$

- For all S-2-arrows $(g_0, g_1)\colon X \to \bar{Y} \leftarrow Y$ and every weak equivalence $g_{0.5}\colon \tilde{Y} \to \bar{Y}$ in $\mathcal{C}$ and for every cylinder homotopy $(K_0, K_1)\colon (g_0, g_1) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (fg_{0.5}, ug_{0.5})$ there exists a unique morphism $\hat{g}\colon Z \to \bar{Y}$ with $g_{0.5} = \mathrm{ins}_{0.5}^Z \hat{g}$ and $(K_0, K_1) = (H_0\hat{g}, H_1\hat{g})$.
- The cylinder equivalence $\mathrm{s}^Z\colon Z \to \tilde{Y}$ is the unique morphism in $\mathcal{C}$ with $1_{\tilde{Y}} = \mathrm{ins}_{0.5}^Z \mathrm{s}^Z$ and $(\mathrm{s}^{\dot{X}} f, \mathrm{s}^{\dot{Y}} u) = (H_0 \mathrm{s}^Z, H_1 \mathrm{s}^Z)$.

By abuse of notation, we refer to the said cylinder with mid insertion of (f, u) corresponding to $(\dot{X}, \dot{Y})$ as well as to its underlying cylinder by Z. The cylinder homotopy (H_0, H_1) is called the *universal cylinder homotopy* of Z.

Given a cylinder Z of (f, u) corresponding to $(\dot{X}, \dot{Y})$ with universal cylinder homotopy (H_0, H_1), we write $\mathrm{H}_0 = \mathrm{H}_0^Z := H_0$ and $\mathrm{H}_1 = \mathrm{H}_1^Z := H_1$.

(C.10) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$.

(a) We suppose given a morphism $f\colon X \to Y$ in $\mathcal{C}$, a cylinder $\dot{X}$ of X and a cylinder Z of f corresponding to $\dot{X}$. Moreover, we suppose given a morphism $g_0\colon X \to \bar{Y}$ and a weak equivalence $g_1\colon Y \to \bar{Y}$ in $\mathcal{C}$ and a cylinder homotopy $K\colon g_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} fg_1$, and we let $\hat{g}\colon Z \to \bar{Y}$ be the unique morphism in $\mathcal{C}$ with $g_1 = \mathrm{ins}_1^Z \hat{g}$ and $K = \mathrm{H}\hat{g}$. Then we have

$$g_0 = \mathrm{ins}_0^Z \hat{g}.$$

(b) We suppose given an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$, a cylinder $\dot{X}$ of X, a cylinder $\dot{Y}$ of Y and a cylinder Z of (f, u) corresponding to $(\dot{X}, \dot{Y})$. Moreover, we suppose given an S-2-arrow $(g_0, g_1)\colon X \to \bar{Y} \leftarrow Y$ and a weak equivalence $g_{0.5}\colon \tilde{Y} \to \bar{Y}$ in $\mathcal{C}$ and a cylinder homotopy $(K_0, K_1)\colon (g_0, g_1) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (fg_{0.5}, ug_{0.5})$, and we let $\hat{g}\colon Z \to \bar{Y}$ be the unique morphism in $\mathcal{C}$ with $g_{0.5} = \mathrm{ins}_{0.5}^Z \hat{g}$ and $(K_0, K_1) = (\mathrm{H}_0\hat{g}, \mathrm{H}_1\hat{g})$. Then we have

$$(g_0, g_1) = (\mathrm{ins}_0^Z \hat{g}, \mathrm{ins}_1^Z \hat{g}).$$

Proof.

(a) We have

$$g_0 = \mathrm{ins}_0^{\dot{X}} K = \mathrm{ins}_0^{\dot{X}} \mathrm{H}\,\hat{g} = \mathrm{ins}_0^Z \hat{g}.$$

(b) We have

$$(g_0, g_1) = (\mathrm{ins}_0^{\dot{X}} K_0, \mathrm{ins}_0^{\dot{Y}} K_1) = (\mathrm{ins}_0^{\dot{X}} \mathrm{H}_0\hat{g}, \mathrm{ins}_0^{\dot{Y}} \mathrm{H}_1\hat{g}) = (\mathrm{ins}_0^Z \hat{g}, \mathrm{ins}_1^Z \hat{g}).$$ □

(C.11) Remark. We suppose given an S-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$, an S-2-arrow $(f, u)\colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with X, $\tilde{Y}$, Y cofibrant, a cylinder $\dot{X}$ of X, a cylinder $\dot{Y}$ of Y, and a cylinder Z of $\left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right)\colon X \amalg Y \to \tilde{Y}$ corresponding to $\dot{X} \amalg \dot{Y}$. Then the underlying object of Z becomes a cylinder with mid insertion $Z_{(f,u)}$ of (f, u) corresponding to $(\dot{X}, \dot{Y})$ having $\mathrm{ins}_0^{Z_{(f,u)}} = \mathrm{emb}_0^{X \amalg Y} \mathrm{ins}_0^Z$, $\mathrm{ins}_{0.5}^{Z_{(f,u)}} = \mathrm{ins}_1^Z$, $\mathrm{ins}_1^{Z_{(f,u)}} = \mathrm{emb}_1^{X \amalg Y} \mathrm{ins}_0^Z$, $\mathrm{s}^{Z_{(f,u)}} = \mathrm{s}^Z$, $(\mathrm{H}_0^{Z_{(f,u)}}, \mathrm{H}_1^{Z_{(f,u)}}) = (\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z, \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z)$.

Proof. By remark (C.8), the underlying object of Z becomes a cylinder with mid insertion $Z_{(f,u)}$ of (f,u) having $\mathrm{ins}_0^{Z_{(f,u)}} = \mathrm{emb}_0^{X \amalg Y} \mathrm{ins}_0^Z$, $\mathrm{ins}_{0.5}^{Z_{(f,u)}} = \mathrm{ins}_1^Z$, $\mathrm{ins}_1^{Z_{(f,u)}} = \mathrm{emb}_1^{X \amalg Y} \mathrm{ins}_0^Z$, $\mathrm{s}^{Z_{(f,u)}} = \mathrm{s}^Z$. Moreover, we have $\mathrm{H}^Z \colon \mathrm{ins}_0^Z \overset{\mathrm{c}}{\sim}_{\dot{X} \amalg \dot{Y}} \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right) \mathrm{ins}_1^Z$, that is, we have $\begin{pmatrix} \mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \\ \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \end{pmatrix} \colon \begin{pmatrix} \mathrm{emb}_0^{X \amalg Y} \mathrm{ins}_0^Z \\ \mathrm{emb}_1^{X \amalg Y} \mathrm{ins}_0^Z \end{pmatrix} \overset{\mathrm{c}}{\sim}_{\dot{X} \amalg \dot{Y}} \begin{pmatrix} f \, \mathrm{ins}_1^Z \\ u \, \mathrm{ins}_1^Z \end{pmatrix}$. As $\mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z$ is a weak equivalence by S-semisaturatedness, corollary (C.3) yields

$$(\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z, \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z) \colon (\mathrm{emb}_0^{X \amalg Y} \mathrm{ins}_0^Z, \mathrm{emb}_1^{X \amalg Y} \mathrm{ins}_0^Z) \overset{\mathrm{c}}{\sim}_{\dot{X}, \dot{Y}} (f \, \mathrm{ins}_1^Z, u \, \mathrm{ins}_1^Z).$$

To show that $Z_{(f,u)}$ becomes a cylinder with mid insertion of (f,u) corresponding to $(\dot{X}, \dot{Y})$ with universal cylinder homotopy $(\mathrm{H}_0^{Z_{(f,u)}}, \mathrm{H}_1^{Z_{(f,u)}}) = (\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z, \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z)$, we suppose given an S-2-arrow $(g_0, g_1) \colon X \to \bar{Y} \leftarrow Y$, a weak equivalence $g_{0.5} \colon \tilde{Y} \to \bar{Y}$ and a cylinder homotopy $(K_0, K_1) \colon (g_0, g_1) \overset{\mathrm{c}}{\sim}_{\dot{X}, \dot{Y}} (f g_{0.5}, u g_{0.5})$. By corollary (C.3), we have $\left(\begin{smallmatrix} K_0 \\ K_1 \end{smallmatrix}\right) \colon \left(\begin{smallmatrix} g_0 \\ g_1 \end{smallmatrix}\right) \overset{\mathrm{c}}{\sim}_{\dot{X} \amalg \dot{Y}} \left(\begin{smallmatrix} f g_{0.5} \\ u g_{0.5} \end{smallmatrix}\right) = \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right) g_{0.5}$. By the universal property of H^Z, there exists a unique morphism $\hat{g} \colon Z \to \bar{Y}$ with $g_{0.5} = \mathrm{ins}_1^Z \hat{g}$ and $\left(\begin{smallmatrix} K_0 \\ K_1 \end{smallmatrix}\right) = \mathrm{H}^Z \hat{g}$, that is, with $g_{0.5} = \mathrm{ins}_{0.5}^{Z_{(f,u)}} \hat{g}$ and $(K_0, K_1) = (\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \hat{g}, \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \hat{g})$.
Finally, as $\mathrm{H}^Z \mathrm{s}^Z = \mathrm{s}^{\dot{X} \amalg \dot{Y}} \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right)$, we have

$$\begin{aligned}
(\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \mathrm{s}^{Z_{(f,u)}}, \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \mathrm{s}^{Z_{(f,u)}}) &= (\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \mathrm{s}^Z, \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z \mathrm{s}^Z) \\
&= (\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{s}^{\dot{X} \amalg \dot{Y}} \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right), \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{s}^{\dot{X} \amalg \dot{Y}} \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right)) \\
&= (\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} (\mathrm{s}^{\dot{X}} \amalg \mathrm{s}^{\dot{Y}}) \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right), \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} (\mathrm{s}^{\dot{X}} \amalg \mathrm{s}^{\dot{Y}}) \left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right)) \\
&= (\mathrm{s}^{\dot{X}} f, \mathrm{s}^{\dot{Y}} u).
\end{aligned}$$

Altogether, the cylinder with mid insertion $Z_{(f,u)}$ of (f,u) corresponds to $(\dot{X}, \dot{Y})$ having $(\mathrm{H}_0^{Z_{(f,u)}}, \mathrm{H}_1^{Z_{(f,u)}}) = (\mathrm{emb}_0^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z, \mathrm{emb}_1^{\dot{X} \amalg \dot{Y}} \mathrm{H}^Z)$. □

(C.12) Proposition. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the incision axiom, a morphism $f \colon X \to Y$ in $\mathcal{C}_{\mathrm{cof}}$ and a cylinder $\dot{X}$ of X. Then $\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y$ becomes a cylinder of f corresponding to $\dot{X}$ having

$$\begin{aligned}
&\mathrm{ins}_0^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y} = \mathrm{ins}_0^{\dot{X}} \mathrm{emb}_0^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y}, \\
&\mathrm{ins}_1^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y} = \mathrm{emb}_1^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y}, \\
&\mathrm{s}^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y} = \begin{pmatrix} \mathrm{s}^{\dot{X}} f \\ 1_Y \end{pmatrix}^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y}, \\
&\mathrm{H}^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y} = \mathrm{emb}_0^{\dot{X} \,{}_{\mathrm{ins}_1}\!\amalg_f^X Y}.
\end{aligned}$$

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ins}_0", "\approx"'] & \dot{X} \arrow[r, "\mathrm{H}^{\dot{X} \amalg^X Y}"] & \dot{X} \amalg^X Y \\
& X \arrow[u, "\mathrm{ins}_1", "\wr"'] \arrow[r, "f"] & Y \arrow[u, "\mathrm{ins}_1", "\wr"']
\end{tikzcd}$$

Proof. As $\mathrm{ins}_1^{\dot{X}} \mathrm{s}^{\dot{X}} f = f = f \, 1_Y$, the induced morphism $\begin{pmatrix} \mathrm{s}^{\dot{X}} f \\ 1_Y \end{pmatrix}^{\dot{X} \amalg^X Y} \colon \dot{X} \amalg^X Y \to Y$ is well-defined. Moreover, we have

$$\begin{aligned}
&\mathrm{ins}_0^{\dot{X}} \mathrm{emb}_0^{\dot{X} \amalg^X Y} \begin{pmatrix} \mathrm{s}^{\dot{X}} f \\ 1_Y \end{pmatrix}^{\dot{X} \amalg^X Y} = \mathrm{ins}_0^{\dot{X}} \mathrm{s}^{\dot{X}} f = f, \\
&\mathrm{emb}_1^{\dot{X} \amalg^X Y} \begin{pmatrix} \mathrm{s}^{\dot{X}} f \\ 1_Y \end{pmatrix}^{\dot{X} \amalg^X Y} = 1_Y.
\end{aligned}$$

As $\dot{X}$ is a cylinder, $\begin{pmatrix} \mathrm{ins}_0^{\dot{X}} \\ \mathrm{ins}_1^{\dot{X}} \end{pmatrix} \colon X \amalg X \to \dot{X}$ is a cofibration, whence $\begin{pmatrix} \mathrm{ins}_0^{\dot{X}} \mathrm{emb}_0^{\dot{X} \amalg^X Y} \\ \mathrm{emb}_1^{\dot{X} \amalg^X Y} \end{pmatrix} \colon X \amalg Y \to \dot{X} \amalg^X Y$ is a cofibration by proposition (3.28). So $\dot{X} \amalg^X Y$ becomes a cylinder of f having $\mathrm{ins}_0^{\dot{X} \amalg^X Y} = \mathrm{ins}_0^{\dot{X}} \mathrm{emb}_0^{\dot{X} \amalg^X Y}$, $\mathrm{ins}_1^{\dot{X} \amalg^X Y} = \mathrm{emb}_1^{\dot{X} \amalg^X Y}$, $\mathrm{s}^{\dot{X} \amalg^X Y} = \begin{pmatrix} \mathrm{s}^{\dot{X}} f \\ 1_Y \end{pmatrix}^{\dot{X} \amalg^X Y}$.

As $\mathrm{ins}_0^{\dot{X}} \mathrm{emb}_0^{\dot{X} \amalg^X Y} = \mathrm{ins}_0^{\dot{X} \amalg^X Y}$ and $\mathrm{ins}_1^{\dot{X}} \mathrm{emb}_0^{\dot{X} \amalg^X Y} = f \, \mathrm{emb}_1^{\dot{X} \amalg^X Y} = f \, \mathrm{ins}_1^{\dot{X} \amalg^X Y}$, we have the cylinder homotopy $\mathrm{emb}_0^{\dot{X} \amalg^X Y} \colon \mathrm{ins}_0^{\dot{X} \amalg^X Y} \overset{\mathrm{c}}{\sim}_{\dot{X}} f \, \mathrm{ins}_1^{\dot{X} \amalg^X Y}$. To show that $\dot{X} \amalg^X Y$ becomes a cylinder of f corresponding to $\dot{X}$ with universal cylinder homotopy $\mathrm{H}^{\dot{X} \amalg^X Y} = \mathrm{emb}_0^{\dot{X} \amalg^X Y}$, we suppose given a morphism $g_0 \colon X \to \bar{Y}$, a weak equivalence $g_1 \colon Y \to \bar{Y}$ and a cylinder homotopy $K \colon g_0 \overset{\mathrm{c}}{\sim}_{\dot{X}} f g_1$. Then we in particular have $\mathrm{ins}_1^{\dot{X}} K = f g_1$, and so $\left(\begin{smallmatrix} K \\ g_1 \end{smallmatrix}\right)^{\dot{X} \amalg^X Y} \colon \dot{X} \amalg^X Y \to \bar{Y}$ is well-defined, which is the unique morphism with $\mathrm{ins}_1^{\dot{X} \amalg^X Y} \left(\begin{smallmatrix} K \\ g_1 \end{smallmatrix}\right)^{\dot{X} \amalg^X Y} = \mathrm{emb}_1^{\dot{X} \amalg^X Y} \left(\begin{smallmatrix} K \\ g_1 \end{smallmatrix}\right)^{\dot{X} \amalg^X Y} = g_1$ and $\mathrm{emb}_0^{\dot{X} \amalg^X Y} \left(\begin{smallmatrix} K \\ g_1 \end{smallmatrix}\right)^{\dot{X} \amalg^X Y} = K$. Finally, we have $\mathrm{s}^{\dot{X} \amalg^X Y} = \left(\begin{smallmatrix} \mathrm{s}^{\dot{X}} f \\ 1_{\bar{Y}} \end{smallmatrix}\right)^{\dot{X} \amalg^X Y}$. Thus the cylinder Z of f corresponds to $\dot{X}$ having $\mathrm{H}^Z = \mathrm{emb}_0^{\dot{X} \amalg^X Y}$. □

(C.13) Corollary. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the incision axiom.

(a) For every morphism $f \colon X \to Y$ in $\mathcal{C}_{\mathrm{cof}}$ and every cylinder $\dot{X}$ of X, there exists a cylinder of f corresponding to $\dot{X}$.

(b) If $\mathcal{C}$ is S-semisaturated, then for every S-2-arrow $(f, u) \colon X \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}_{\mathrm{cof}}$, every cylinder $\dot{X}$ of X and every cylinder $\dot{Y}$ of Y, there exists a cylinder with mid insertion of (f, u) corresponding to $(\dot{X}, \dot{Y})$.

Proof.

(a) This follows from proposition (C.12).

(b) This follows from (a) and remark (C.11). □

(C.14) Corollary. We suppose given an S-semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the incision axiom. The following conditions are equivalent.

(a) There exists a cylinder of every object in $\mathcal{C}_{\mathrm{cof}}$.

(b) There exists a cylinder of every morphism in $\mathcal{C}_{\mathrm{cof}}$.

(c) There exists a cylinder of every S-2-arrow in $\mathcal{C}_{\mathrm{cof}}$.

(d) There exists a cylinder with mid insertion of every S-2-arrow in $\mathcal{C}_{\mathrm{cof}}$.

Proof. Condition (a) is a particular case of condition (b), condition (b) is a particular case of condition (c), and condition (c) is a particular case of condition (d). So it suffices to show that condition (a) implies condition (d). But if there exists a cylinder of every object in $\mathcal{C}_{\mathrm{cof}}$, then there also exists a cylinder with mid insertion of every S-2-arrow in $\mathcal{C}_{\mathrm{cof}}$ by corollary (C.13)(b). □

(C.15) Corollary. We suppose given a semisaturated category with cofibrations and weak equivalences $\mathcal{C}$ that fulfills the incision axiom and the cofibrancy axiom. Then $\mathcal{C}$ is a Brown cofibration category if and only if there exists a cylinder of every object in $\mathcal{C}$.

Proof. By definition (3.52)(a), the category with cofibrations and weak equivalences $\mathcal{C}$ is a Brown cofibration category if and only if it fulfills the factorisation axiom for cofibrations. If $\mathcal{C}$ fulfills the factorisation axiom for cofibrations, then there exists a cylinder of every object in $\mathcal{C}$ by the Brown factorisation lemma (3.113)(a). Conversely, if there exists a cylinder of every object in $\mathcal{C}$, then there exists a cylinder of every morphism in $\mathcal{C}$, and so $\mathcal{C}$ fulfills the factorisation axiom for cofibrations: Given a morphism f in $\mathcal{C}$ and a cylinder Z of f, we have $f = \mathrm{ins}_0^Z \mathrm{s}^Z$, where ins_0^Z is a cofibration by remark (3.111) and s^Z is a weak equivalence. □

From Brown's homotopy S-2-arrow calculus to the Z-2-arrow calculus

Finally, we will give an alternative proof for the main part of the Z-2-arrow calculus, namely theorem (3.128)(b). To this end, we make use of Brown's homotopy S-2-arrow calculus, which has been proven in the main text as a *consequence* of the Z-2-arrow calculus, see theorem (3.132).

More precisely, we use the following form of Brown's homotopy S-2-arrow calculus, which is slightly weaker than that of theorem (3.132)(b).

(C.16) Theorem (Brown's homotopy S-2-arrow calculus [7, dual of th. 1 and proof], cf. theorem (3.132)(b))**.** We suppose given a Brown cofibration category $\mathcal{C}$.

(a) We have

$$\operatorname{Mor} \operatorname{Ho} \mathcal{C} = \{\operatorname{loc}(f) \operatorname{loc}(u)^{-1} \mid (f, u) \text{ is an S-2-arrow in } \mathcal{C}\}.$$

(b) Given S-2-arrows $(f_0, u_0)\colon X \to \tilde{Y}_0 \leftarrow Y$, $(f_1, u_1)\colon X \to \tilde{Y}_1 \leftarrow Y$ in $\mathcal{C}$, we have

$$\operatorname{loc}(f_0) \operatorname{loc}(u_0)^{-1} = \operatorname{loc}(f_1) \operatorname{loc}(u_1)^{-1}$$

in $\operatorname{Ho} \mathcal{C}$ if and only if there exist weak equivalences $c_0\colon \tilde{Y}_0 \to \tilde{Y}$, $c_1\colon \tilde{Y}_1 \to \tilde{Y}$ with $(f_0 c_0, u_0 c_0) \overset{\mathrm{c}}{\sim} (f_1 c_1, u_1 c_1)$.

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}"] & \tilde{Y}_0 \arrow[d, dotted, "\wr", "c_0"'] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}"] & Y \arrow[l, "u_0", "\approx"'] \arrow[d, equal] \\
X \arrow[r, dotted, "f_1 c_1"] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, dotted, "u_1 c_1", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "f_1"] & \tilde{Y}_1 \arrow[u, dotted, "\wr", "c_1"'] & Y \arrow[l, "u_1", "\approx"']
\end{tikzcd}$$

Proof. This follows from [7, dual of prop. 2] and theorem (2.35). ([1]) □

(C.17) Remark. We suppose given a category with cofibrations and weak equivalences $\mathcal{C}$, S-2-arrows $(f, u)\colon X \to \tilde{Y} \leftarrow Y$, $(g, v)\colon X \to \bar{Y} \leftarrow Y$ and a weak equivalence $c\colon \tilde{Y} \to \bar{Y}$. Moreover, we suppose given a cylinder $\dot{X}$ of X and a cylinder $\dot{Y}$ of Y such that $(g, v) \overset{\mathrm{c}}{\sim}_{\dot{X}, \dot{Y}} (fc, uc)$, and we suppose given a cylinder with mid insertion Z of (f, u) corresponding to $(\dot{X}, \dot{Y})$. Then there exists a morphism $\hat{c}\colon Z \to \bar{Y}$ such that $(g, v) = (\operatorname{ins}_0^Z \hat{c}, \operatorname{ins}_1^Z \hat{c})$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}"] & \tilde{Y} \arrow[d, "\wr", "c"'] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}"] & Y \arrow[l, "u", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "g"] & \bar{Y} & Y \arrow[l, "v", "\approx"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, "u", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "\operatorname{ins}_0"] \arrow[d, equal] & Z \arrow[u, "\wr", "\mathrm{s}"'] \arrow[d, dotted, "\hat{c}"] & Y \arrow[l, "\operatorname{ins}_1", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "g"] & \bar{Y} & Y \arrow[l, "v", "\approx"']
\end{tikzcd}$$

Proof. We let $(K_0, K_1)\colon (g, v) \overset{\mathrm{c}}{\sim}_{\dot{X}, \dot{Y}} (fc, uc)$. As Z corresponds to $(\dot{X}, \dot{Y})$, there exists a unique morphism $\hat{c}\colon Z \to \bar{Y}$ with $c = \operatorname{ins}_{0.5}^Z \hat{c}$ and $(K_0, K_1) = (\mathrm{H}_0 \hat{c}, \mathrm{H}_1 \hat{c})$. But then we in particular have $(g, v) = (\operatorname{ins}_0^Z \hat{c}, \operatorname{ins}_1^Z \hat{c})$ by remark (C.10)(b). □

Alternative proof of theorem (3.128)(b). We suppose that $\operatorname{loc}(f_1) \operatorname{loc}(u_1)^{-1} = \operatorname{loc}(f_2) \operatorname{loc}(u_2)^{-1}$ in $\operatorname{Ho} \mathcal{C}$. By Brown's homotopy S-2-arrow calculus (C.16)(b) there exist an S-2-arrow $(c_1, c_2)\colon \tilde{Y}_1 \to \tilde{Y} \leftarrow \tilde{Y}_2$ in $\mathcal{C}$ with weak equivalence c_1 and such that $(f_1 c_1, u_1 c_1) \overset{\mathrm{c}}{\sim} (f_2 c_2, u_2 c_2)$.

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}"] & \tilde{Y}_1 \arrow[d, dotted, "\wr", "c_1"'] \arrow[dr, phantom, "\overset{\mathrm{c}}{\sim}"] & Y \arrow[l, "u_1", "\approx"'] \arrow[d, equal] \\
X \arrow[r, dotted, "f_2 c_2"] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, dotted, "u_2 c_2", "\approx"'] \arrow[d, equal] \\
X \arrow[r, "f_2"] & \tilde{Y}_2 \arrow[u, dotted, "\wr", "c_2"'] & Y \arrow[l, "u_2", "\approx"']
\end{tikzcd}$$

[1] In particular, this proof *avoids* the Z-2-arrow calculus.

So there exists a cylinder $\dot{X}$ of X and a cylinder $\dot{Y}$ of Y such that $(f_1 c_1, u_1 c_1) \overset{\mathrm{c}}{\sim}_{\dot{X},\dot{Y}} (f_2 c_2, u_2 c_2)$. By corollary (C.13)(b), there exists a cylinder Z_1 of (f_1, u_1) corresponding to $(\dot{X}, \dot{Y})$, and so remark (C.17) implies that there exists a morphism $c\colon Z_1 \to \tilde{Y}$ such that $(f_2 c_2, u_2 c_2) = (\mathrm{ins}_0^{Z_1} c, \mathrm{ins}_1^{Z_1} c)$.

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & \tilde{Y}_1 & Y \arrow[l, "u_1"', "\approx"] \arrow[d, equal] \\
X \arrow[r, dotted, "\mathrm{ins}_0"] \arrow[d, equal] & Z_1 \arrow[u, dotted, "\mathrm{s}"', "\wr"] \arrow[d, dotted, "c", "\wr"'] & Y \arrow[l, dotted, "\mathrm{ins}_1"', "\approx"] \arrow[d, equal] \\
X \arrow[r, "f_2 c_2"] \arrow[d, equal] & \tilde{Y} & Y \arrow[l, "u_2 c_2"', "\approx"] \arrow[d, equal] \\
X \arrow[r, "f_2"] & \tilde{Y}_2 \arrow[u, "c_2"', "\wr"] & Y \arrow[l, "u_2"', "\approx"]
\end{tikzcd}$$

In particular, we have $(f_1, u_1) \equiv_{\mathrm{S}} (f_2, u_2)$, see definition (2.14)(a), and so theorem (2.60)(c) and the Brown factorisation lemma (3.113) yield the asserted commutative diagram in $\mathcal{C}$.
The converse implication follows from remark (2.17). □

Bibliography

[1] Artin, Michael; Grothendieck, Alexander; Verdier, Jean-Louis. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA4). With the collaboration of N. Bourbaki, P. Deligne and B. Saint-Donat.

[2] Asano, Keizo. *Über die Quotientenbildung von Schiefringen.* Journal of the Mathematical Society of Japan **1**(2) (1949), pp. 73–78.

[3] Barwick, Clark; Kan, Daniel M. *Relative categories: another model for the homotopy theory of homotopy theories.* Preprint, 2010 (vers. 2, January 4, 2011). `arXiv:1011.1691v2 [math.AT]`.

[4] Bass, Hyman (editor). *Algebraic K-theory. I: Higher K-theories.* Lecture Notes in Mathematics, vol. 341. Springer-Verlag, Berlin, 1973. Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institut, Seattle, 1972.

[5] Beĭlinson, A. A.; Bernstein, J.; Deligne, P. *Faisceaux pervers.* 1982. Published in [35, pp. 5–171]

[6] Beligiannis, Apostolos; Marmaridis, Nikolaos. *Left triangulated categories arising from contravariantly finite subcategories.* Communications in Algebra **22**(12) (1994), pp. 5021–5036.

[7] Brown, Kenneth S. *Abstract homotopy theory and generalized sheaf cohomology.* Transactions of the American Mathematical Society **186** (1974), pp. 419–458.

[8] Bühler, Theo. *Exact categories.* Expositiones Mathematicae **28**(1) (2010), pp. 1–69.

[9] Cisinski, Denis-Charles. *Catégories dérivables.* Bulletin de la Société Mathématique de France **138**(3) (2010), pp. 317–393.

[10] Deligne, Pierre. *Cohomologie étale.* Lecture Notes in Mathematics, vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie (SGA4$\frac{1}{2}$). With the collaboration of J.-F. Boutot, A. Grothendieck, L. Illusie and J.-L. Verdier.

[11] Dwyer, William G.; Hirschhorn, Philip S.; Kan, Daniel M.; Smith, Jeffrey H. *Homotopy limit functors on model categories and homotopical categories.* Mathematical Surveys and Monographs, vol. 113. American Mathematical Society, Providence (RI), 2004.

[12] Gabriel, Peter; Zisman, Michel. *Calculus of Fractions and Homotopy Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag, New York, 1967.

[13] Gelfand, Sergei I.; Manin, Yuri I. *Methods of homological algebra.* Springer Monographs in Mathematics, second edition. Springer-Verlag, Berlin, 2003.

[14] Gunnarsson, Thomas E.W. *Abstract Homotopy Theory and Related Topics.* Doctoral thesis, Chalmers University of Technology Göteborg, 1978.

[15] Hartshorne, Robin. *Residues and duality.* Lecture Notes in Mathematics, vol. 20. Springer-Verlag, Berlin, 1966. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64.

[16] Heller, Alex. *The loop-space functor in homological algebra.* Transactions of the American Mathematical Society **96** (1960), pp. 382–394.

[17] HELLER, ALEX. *Stable homotopy categories.* Bulletin of the American Mathematical Society **74** (1968), pp. 28–63.

[18] HOUZEL, C. (editor). *Séminaire Banach.* Lecture Notes in Mathematics, vol. 277. Springer-Verlag, Berlin, 1972. Given at École Normale Supérieure 1962/63.

[19] HOVEY, MARK. *Model Categories.* Mathematical Surveys and Monographs, vol. 63. American Mathematical Society, Providence (RI), 1999.

[20] KELLER, BERNHARD. *Chain complexes and stable categories.* Manuscripta Mathematica **67**(4) (1990), pp. 379–417.

[21] KELLER, BERNHARD; VOSSIECK, DIETER. *Sous les catégories dérivées.* Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique **305**(6) (1987), pp. 225–228.

[22] KÜNZER, MATTHIAS. *On derived categories.* Diplomarbeit, Universität Stuttgart, 1996.

[23] KÜNZER, MATTHIAS. *Heller triangulated categories.* Homotopy, Homology and Applications **9**(2) (2007), pp. 233–320.

[24] KÜNZER, MATTHIAS. *Nonisomorphic Verdier octahedra on the same base.* Journal of Homotopy and Related Structures **4**(1) (2009), pp. 7–38.

[25] MALTSINIOTIS, GEORGES. *Catégories triangulées supérieures.* Preprint, 2005.

[26] MALTSINIOTIS, GEORGES. *La théorie du calcul des trois flèches: catégories d'objets fibrants.* Handwritten manuscript, 2012.

[27] ORE, ØYSTEIN. *Linear equations in non-commutative fields.* Annals of Mathematics **32**(3) (1931), pp. 463–477.

[28] QUILLEN, DANIEL G. *Homotopical Algebra.* Lecture Notes in Mathematics, vol. 43. Springer-Verlag, Berlin-New York, 1967.

[29] QUILLEN, DANIEL G. *Higher algebraic K-theory: I.* Published in [4, pp. 85–147], 1973.

[30] RĂDULESCU-BANU, ANDREI. *Cofibrations in Homotopy Theory.* Preprint, 2006 (vers. 4, February 8, 2009). `arXiv:math/0610009v4 [math.AT]`.

[31] RANICKI, A.; LEVITT, N.; QUINN, F.. *Algebraic and geometric topology.* Lecture Notes in Mathematics, vol. 1126. Springer-Verlag, Berlin, 1985. Proceedings of the conference on surgery theory held at Rutgers University, New Brunswick (NJ), July 6–13, 1983.

[32] SCHIFFMANN, GÉRARD. *Théorie élémentaire des catégories.* Published as chapter 0 in [18, pp. 1–33], 1962.

[33] SCHWEDE, STEFAN. *Topological triangulated categories.* Preprint, 2012. `arXiv:1201.0899v1 [math.AT]`.

[34] SERRE, JEAN-PIERRE. *Groupes d'homotopie et classes de groupes abéliens.* Annals of Mathematics **58**(2) (1953), pp. 258–294.

[35] TEISSIER, B.; VERDIER, J.-L. (organisers). *Analyse et topologie sur les espaces singuliers. I.* Astérisque, vol. 100. Société Mathématique de France, Paris, 1982. Proceedings of the colloquium held at Luminy, July 6–11, 1981.

[36] THOMAS, SEBASTIAN. *On the 3-arrow calculus for homotopy categories.* Homology, Homotopy and Applications **13**(1) (2011), pp. 89–119.

[37] VERDIER, JEAN-LOUIS. *Catégoriés derivées.* 1963. Published as appendix in [10, pp. 266–315], 1977.

[38] WALDHAUSEN, FRIEDHELM. *Algebraic K-theory of spaces.* 1985. Published in [31, pp. 318–419].

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