Universität Stuttgart

Some local presentations for tensor products of simple modules of the symmetric group

Diploma thesis – Franziska Müller

3 May 2013

Contents

Chapter 1

Introduction

1.1 Aim

Consider the symmetric group S_n for some $n \geq 1$. Let p be a prime dividing $|\mathrm{S}_n| = n!$.

The simple $\mathbb{F}_p\mathrm{S}_n$-modules are parametrized by p-regular partitions; cf. **[5,** Th. 11.5**]**.

Let D and $\tilde{D}$ be simple $\mathbb{F}_p\mathrm{S}_n$-modules. They may be viewed as simple $\mathbb{Z}_{(p)}\mathrm{S}_n$-modules.

Let P be the projective cover of D in mod-$\mathbb{Z}_{(p)}\mathrm{S}_n$. So P is an indecomposable direct summand of the regular module $\mathbb{Z}_{(p)}\mathrm{S}_n$. Let $\mathfrak{r}P \subseteq P$ be the Jacobson radical of P. So we have a short exact sequence

$$\mathfrak{r}P \longrightarrow P \longrightarrow D\,.$$

Let $\tilde{P}$ be the projective cover of $\tilde{D}$ in mod-$\mathbb{Z}_{(p)}\mathrm{S}_n$. We have a short exact sequence

$$\mathfrak{r}\tilde{P} \longrightarrow \tilde{P} \longrightarrow \tilde{D}\,.$$

In mod-$\mathbb{Z}_{(p)}\mathrm{S}_n$, we have the following tensor product.
Given modules M and N, we may form $M \otimes N := M \otimes_{\mathbb{Z}_{(p)}} N$ and equip it with the diagonal action of $\mathbb{Z}_{(p)}\mathrm{S}_n$, that is, $(m \otimes n)\sigma = m\sigma \otimes n\sigma$ for $m \in M$, $n \in N$, $\sigma \in \mathrm{S}_n$.

We consider the canonical right-exact sequence

$$(\mathfrak{r}P \otimes \tilde{P}) \oplus (P \otimes \mathfrak{r}\tilde{P}) \xrightarrow{\left(\begin{smallmatrix} E \\ \tilde{E} \end{smallmatrix}\right)} P \otimes \tilde{P} \longrightarrow D \otimes \tilde{D}\,,$$

where E is the inclusion map from $\mathfrak{r}P \otimes \tilde{P}$ to $P \otimes \tilde{P}$, and where $\tilde{E}$ is the inclusion map from $P \otimes \mathfrak{r}\tilde{P}$ to $P \otimes \tilde{P}$. Since the modules $P \otimes \tilde{P}$, $\mathfrak{r}P \otimes \tilde{P}$ and $P \otimes \mathfrak{r}\tilde{P}$ are projective, the diagram

$$(\mathfrak{r}P \otimes \tilde{P}) \oplus (P \otimes \mathfrak{r}\tilde{P}) \xrightarrow{\left(\begin{smallmatrix} E \\ \tilde{E} \end{smallmatrix}\right)} P \otimes \tilde{P}$$

is a presentation of $D \otimes \tilde{D}$.

Working over $\mathbb{Z}_{(p)}$ instead of $\mathbb{F}_p$, we have

$$\mathfrak{r}P \otimes \tilde{P} \;\cong\; P \otimes \mathfrak{r}\tilde{P} \;\cong\; P \otimes \tilde{P}\,;$$

cf. Lemma 238. So we may find an isomorphic replacement of the diagram

$$\mathfrak{r}P \otimes \tilde{P} \xrightarrow{E} P \otimes \tilde{P} \xleftarrow{\tilde{E}} P \otimes \mathfrak{r}\tilde{P}$$

in mod-$\mathbb{Z}_{(p)}\mathrm{S}_n$ by a diagram of the form

$$Q \xrightarrow{C} Q \xleftarrow{\tilde{C}} Q\,,$$

where Q is a projective $\mathbb{Z}_{(p)}\mathrm{S}_n$-module, and where C and $\tilde{C}$ are injective endomophisms of Q.

We want to find such an isomorphism replacement with C and $\tilde{C}$ of a simple shape.

The aim is twofold.

First, we give a list of examples to explore which shape for C and $\tilde{C}$ can be achieved; cf. Section 1.2 and the Chapters 3, 4 and 5.

Second, we prove that we can achieve C or $\tilde{C}$ to be diagonal; cf. Corollary 159. In general, it is impossible to achieve that both C and $\tilde{C}$ are diagonal; cf. Remark 16.

1.2 Simplifying the canonical presentation

We keep the notation of Section 1.1.

1.2.1 The simplification procedure

Choose a projective $\mathbb{Z}_{(p)}\mathrm{S}_n$-module Q and isomorphisms

$$
\begin{array}{rcl}
P \otimes \tilde{P} & \stackrel{\sim}{\longrightarrow} & Q \\
\mathfrak{t}P \otimes \tilde{P} & \stackrel{\sim}{\longrightarrow} & Q \\
P \otimes \mathfrak{t}\tilde{P} & \stackrel{\sim}{\longrightarrow} & Q \;,
\end{array}
$$

cf. Lemma 238, Chapter 7. In practice, one chooses Q to be a direct sum of standard indecomposable projectives.

We obtain a commutative diagram of the form

$$
\begin{tikzcd}
P \otimes \mathfrak{t}\tilde{P} \arrow[r, "\tilde{B}_0", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "\tilde{C}_0"] \\
P \otimes \tilde{P} \arrow[r, "A_0", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \\
\mathfrak{t}P \otimes \tilde{P} \arrow[r, "B_0", "\sim"'] \arrow[u, hook, "E"] & Q \arrow[u, "C_0"'] \;.
\end{tikzcd}
$$

We may simplify C_0 and $\tilde{C}_0$ using suitable $\mathbb{Z}_{(p)}\mathrm{S}_n$-linear automorphisms of Q, as shown below. Note that A_1 affects both C_0 and $\tilde{C}_0$.

$$
\begin{tikzcd}
P \otimes \mathfrak{t}\tilde{P} \arrow[r, "\tilde{B}_0", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[r, "\tilde{B}_1", "\sim"'] \arrow[d, "\tilde{C}_0"] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "\tilde{C}"] \\
P \otimes \tilde{P} \arrow[r, "A_0", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[r, "A_1", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \\
\mathfrak{t}P \otimes \tilde{P} \arrow[r, "B_0", "\sim"'] \arrow[u, hook, "E"] & Q \arrow[r, "B_1", "\sim"'] \arrow[u, "C_0"] & Q \arrow[u, "C"]
\end{tikzcd}
$$

We obtain the resulting isomorphisms $A := A_0 \cdot A_1$, $B := B_0 \cdot B_1$ and $\tilde{B} := \tilde{B}_0 \cdot \tilde{B}_1$.

If Q is a direct sum of standard indecomposable projectives, then C and $\tilde{C}$ are matrices, having $\mathbb{Z}_{(p)}\mathrm{S}_n$-linear maps between the summands as matrix entries.

We want C and $\tilde{C}$ to be in a simple form – the closer to diagonal form the better. Cf. Construction 33.

It is possible to achieve diagonal form for C or for $\tilde{C}$; cf. Corollary 159.

In general, we cannot achieve both C and $\tilde{C}$ to be diagonal, as we will see on the example of $P_1 \otimes P_1$ of $\mathbb{Z}_{(2)}\mathrm{S}_3$; notation taken from Lemma 14. There, we end up with the endomorphisms

$$C := \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \text{and} \qquad \tilde{C} := \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right)$$

of $Q = P_1 \oplus P_1$, the 2×2-blocks describe $\mathbb{Z}_{(2)}\mathrm{S}_3$-linear maps between the summands. These endomorphisms cannot be simultaneously transformed into diagonal form by a commutative diagram of the form above, as we show in Remark 16.

After the simplification, we may then fold our diagram and add cokernels vertically to obtain the following commutative diagram.

$$\begin{tikzcd}
D \otimes \tilde{D} \arrow[rr, dashed, "\sim"] & & \mathrm{Coker}\begin{pmatrix} C \\ \tilde{C} \end{pmatrix} \\
P \otimes \tilde{P} \arrow[u] \arrow[rr, "A", "\sim"'] \arrow[urr, phantom, "\circlearrowleft"] & & Q \arrow[u] \\
(\mathfrak{r}P \otimes \tilde{P}) \oplus (P \otimes \mathfrak{r}\tilde{P}) \arrow[u, "\begin{pmatrix} E \\ \tilde{E} \end{pmatrix}"] \arrow[rr, "\sim", "\begin{pmatrix} B & 0 \\ 0 & \tilde{B} \end{pmatrix}"'] \arrow[urr, phantom, "\circlearrowleft"] & & Q \oplus Q \arrow[u, "\begin{pmatrix} C \\ \tilde{C} \end{pmatrix}"']
\end{tikzcd}$$

1.2.2 The defect-0 case

We will observe that if P belongs to a defect-0 block, then we can achieve diagonal form for both C and $\tilde{C}$ simultaneously.

More precisely, in this case we have a direct sum decomposition $P \otimes \tilde{P} \xrightarrow{\sim} Q' \oplus Q''$ fitting into a commutative diagram

(Lemma 163)

$$\begin{tikzcd}
P \otimes \mathfrak{r}\tilde{P} \arrow[r, "\sim"] \arrow[d, hook] & Q' \oplus Q'' \arrow[d, "\begin{pmatrix} \mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \end{pmatrix}"] \\
P \otimes \tilde{P} \arrow[r, "\sim"] & Q' \oplus Q'' \\
\mathfrak{r}P \otimes \tilde{P} \arrow[r, "\sim"] \arrow[u, hook] & Q' \oplus Q'' . \arrow[u, "\begin{pmatrix} p\,\mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \end{pmatrix}"']
\end{tikzcd}$$

This is basically a corollary to Theorem 157; cf. also Section 1.5.

1.3 An isomorphic replacement of the comultiplication

Consider the group ring $\mathbb{Z}_{(3)}\mathrm{S}_3$. Consider the Wedderburn isomorphism

$$\omega_{(3)} \;:\; \mathbb{Z}_{(3)}\mathrm{S}_3 \xrightarrow{\sim} \Lambda_{(3)} \;=\; \{(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f) \in \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)} \mid a \equiv_3 b\,,\;\; e \equiv_3 f\,,\;\; d \equiv_3 0\}\;;$$

cf. Definition 21.

Let $\Delta : \mathbb{Z}_{(3)}\mathrm{S}_3 \longrightarrow \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3\,,\ \sigma \longmapsto \sigma\otimes\sigma$ be the comultiplication, where $\sigma \in \mathrm{S}_3$; cf. Definition 37.

We have the commutative diagram of $\mathbb{Z}_{(3)}$-algebras

$$\begin{tikzcd}
\mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[r, "\Delta"] \arrow[d, "\omega_{(3)}", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[d, "\omega_{(3)}\otimes\omega_{(3)}", "\wr"'] \\
\Lambda_{(3)} \arrow[r, "\tilde{\Delta}"] & \Lambda_{(3)} \otimes \Lambda_{(3)}\ .
\end{tikzcd}$$

In Section 3.3, we set out to replace $\tilde{\Delta}$ isomorphically in such a way that the image of an element of $\Lambda_{(3)}$ is in the simplest possible form.

As isomorphic replacement of $\Lambda_{(3)} \otimes \Lambda_{(3)}$, we obtain a $\mathbb{Z}_{(3)}$-subalgebra

$$\Gamma \;\subseteq\; \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)}^{4\times 4} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)}$$

described by ties; cf. Lemma 39. We obtain the commutative diagram of $\mathbb{Z}_{(3)}$-algebras

$$\begin{tikzcd}
\mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[r, "\Delta"] \arrow[d, "\omega_{(3)}", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[d, "\Omega", "\wr"'] \\
\Lambda_{(3)} \arrow[r, "\Delta'"] & \Gamma\ ,
\end{tikzcd}$$

where

$$\begin{array}{ccc}
\Lambda_{(3)} & \xrightarrow{\Delta'} & \Gamma \\
(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f) & \longmapsto & \left(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), a\right)
\end{array}$$

and where the mapping rule for Ω is given in Lemma 38.

Then we will use the ties of Γ to calculate and decompose the tensor product $S^{(2,1)} \otimes S^{(2,1)}$ of Specht modules in Section 3.3.4.

1.4 A Krull-Schmidt argument

Let $\mathcal{B}$ be an additive category in which idempotents split; cf. Definition 140. Suppose given $X_1, \ldots, X_n \in \mathrm{obj}(\mathcal{B})$ with local endomorphism rings; cf. Definition 189. Let

$$e \;\in\; \mathrm{End}_{\mathcal{B}}(X_1 \oplus \ldots \oplus X_n) \setminus \{0\}$$

be an idempotent. We write it as a matrix $e = (e_{ab})_{a,b}$, where $e_{ab} : X_a \longrightarrow X_b$ for $a,\, b \,\in\, [1,n]$.

There exist $i,\, j \,\in\, [1,n]$ such that e_{ij} is an isomorphism; cf. Lemma 139.

Suppose given $Y \in \mathrm{obj}(\mathcal{B})$ such that

$$Y \;\cong\; \bigoplus_{i\in[1,n]} X_i \;.$$

Let Z be a direct summand of Y. Then there exists $I \subseteq [1,n]$

$$Z \;\cong\; \bigoplus_{i\in I} X_i \;,$$

as we will show in Proposition 141.

We will give a construction for this isomorphism. This makes it possible to turn this argument into an algorithm. We will give an implementation in Magma **[3]** in Section 6.2.3.1.

In Section 6.2.4, the algorithm will be applied to the case of a tensor product of a lattice and a direct summand of RG, where R is a localization of the integers at a maximal ideal and where G is a finite group.

In this case, we may also lift a modular isomorphism between the tensor product in question and a standard decomposition into indecomposable projectives, obtained using the MeatAxe of Magma, from the modular to the locally integral case. This lift can be constructed since all modules involved are projective. Then the lifting method is not as memory consuming as the Krull-Schmidt method. Cf. Chapter 7.

1.5 A total decomposability

Let R be a discrete valuation ring with maximal ideal (π).

Let G be a finite group split by R, i.e. suppose that $\Lambda := RG$ is isomorphic to an R-suborder of a finite direct product Γ of matrix rings over R such that the R-linear factor module ${}^{\Gamma}\!/_{\Lambda}$ has finite length over R.

Consider the additive category $\mathcal{A} := (\text{lat-}\Lambda)^{\Delta_1}$, having as objects morphisms of lat-Λ; cf. Definition 153.

We call an object $(M \xrightarrow{f} N)$ in $\mathcal{A}$ **totally decomposable** if it is isomorphic to a finite direct sum of objects of the form

$$\begin{array}{l}(0 \longrightarrow Q)\\ (Q \longrightarrow 0)\\ (Q \xrightarrow{\pi^{\alpha}} Q)\;,\end{array}$$

for $Q \in \mathrm{obj}(\text{lat-}\Lambda)$ an indecomposable direct summand of Λ; cf. Definition 154.

A variant of Heller's Lemma is used to show that the endomorphism rings of these objects are local; cf. Lemmas 231 and 233, used for Lemma 155.

Theorem 157. *Suppose given $(M \xrightarrow{f} N)$ in $\mathrm{obj}(\mathcal{A})$, i.e. M and N are Λ-lattices and f is a Λ-linear map. Let P be a finitely generated projective Λ-module.*

Then the object $(M \xrightarrow{f} N)\otimes P = \big(M\otimes P \xrightarrow{f\otimes P} N\otimes P\big)$ of $\mathcal{A}$ is totally decomposable.

For this theorem, we need to apply Lemma 156, which uses the Krull-Schmidt argument explained in Section 1.4; cf. Proposition 141.

Now suppose f to be injective and $\mathrm{rk}_R M = \mathrm{rk}_R N$. Then we have a decomposition

$$(M \xrightarrow{f} N)\otimes P \;\cong\; \bigoplus_i (Q_i \xrightarrow{\pi^{\alpha_i}} Q_i)\,,$$

where Q_i is an indecomposable direct summand of Λ and $\alpha_i \geq 0$ for all i; cf. Corollary 158.

As an application, suppose given finitely generated projective Λ-modules P and $\tilde{P}$. Recall that $\mathfrak{r}P$ denotes the Jacobson radical of P.

Then we can find projective Λ-modules Q and Q', and a commutative diagram as follows; cf. Corollary 159.

$$\begin{tikzcd}
P \otimes \tilde{P} \arrow[r, "\sim", "\beta"'] & Q \oplus Q' \\
\mathfrak{r}P \otimes \tilde{P} \arrow[u, hook] \arrow[r, "\sim", "\alpha"'] \arrow[ur, phantom, "\circlearrowleft"] & Q \oplus Q' \arrow[u, "\begin{pmatrix} \mathrm{id}_Q & 0 \\ 0 & \pi \mathrm{id}_{Q'} \end{pmatrix}"']
\end{tikzcd}$$

1.6 The Jacobson radical

We give a self-contained introduction to Jacobson radicals of orders over discrete valuation rings, collecting well-known facts in one place; cf. App. C.

Suppose given a discrete valuation ring R. Suppose Λ to be an R-order.

Suppose Λ to be **stable**, i.e. for each primitive idempotent $e \in \Lambda$, the idempotent $\bar{e} \in \bar{\Lambda}$ to be primitive; cf. Definition 207 .

Let $1_\Lambda = \sum_{i \in [1,n]} e_i$ be an orthogonal decomposition into primitive idempotents; cf. Remark 216.

We define an equivalence relation $(\sim)$ on the index set $[1,n]$ by letting

$$i \sim j \;:\Leftrightarrow\; e_i\Lambda \cong e_j\Lambda$$

for $i,\, j \in [1,n]$.

We write $\mathfrak{r}(e_i\Lambda e_j) := e_i\Lambda e_j \cdot \mathfrak{r}(e_j\Lambda e_j) = e_i\,\mathfrak{r}(\Lambda)e_j = \mathfrak{r}(e_i\Lambda e_i) \cdot e_i\Lambda e_j$ for $i,\, j \in [1,n]$ with $i \sim j$; cf. Lemma 185.

Then we get

(Proposition 217) $$\mathfrak{r}\Lambda \;=\; \Big(\bigoplus_{\substack{i,j \in [1,n] \\ i \sim j}} \mathfrak{r}(e_i\Lambda e_j) \Big) \oplus \Big(\bigoplus_{\substack{i,j \in [1,n] \\ i \not\sim j}} e_i\Lambda e_j \Big) \;.$$

Now let $1_{K\Lambda} = \sum_{i \in [1,\ell]} \varepsilon_i$ be an orthogonal decomposition into central idempotents of $K\Lambda$. Then

(Proposition 222) $$\mathfrak{r}\Lambda \;=\; \Lambda \,\cap\, \bigoplus_{i \in [1,\ell]} \mathfrak{r}(\varepsilon_i\Lambda)$$

as R-submodules of $K\Lambda$.

In practice, we use the latter assertion not to calculate the radical of Λ as a whole, but rather to calculate the radical of $e_i\Lambda e_i$, as needed for the former assertion. So we only have to actually compute radicals of the R-orders $\varepsilon_j e_i\Lambda e_i$, which are rather small.

1.7 Conventions

- Maps act on the right. Composition is written in the natural ordering, i.e. $\xrightarrow{a}\xrightarrow{b} = \xrightarrow{ab}$.
- Suppose given sets A and B and a map $f : A \longrightarrow B$.

  Let $X \subseteq A$ and $Y \subseteq B$ be such that $Xf \subseteq Y$.

  Then we write

  $$f|_X^Y : X \longrightarrow Y \text{ for the restriction of } f \text{ to } X \text{ in the domain and to } Y \text{ in the codomain.}$$

  So $xf|_X^Y = xf$ for $x \in X$.
  In addition, if $Y = B$ we also write $f|_X := f|_X^B$, and if $X = A$ we also write $f|^Y := f|_A^Y$.
- Let R be a ring. Unless specified otherwise, by an R-module we understand a right R-module.
- By default, variables without further specification run through the ground ring of the present context. This default is often used in context of matrix entries.
- The identity on a set M we denote as $\mathrm{id}_M = \mathrm{id}$ or as $1_M = 1$.
- Let R be a commutative ring. The identity matrix of $R^{n\times n}$ is denoted $1 = 1_{R^{n\times n}}$ or I_n. Given $x \in R$, we sometimes abbreviate $x = x \cdot I_n$.
- Let $a, b \in \mathbb{Z}$ with $a \leq b$. Let $[a, b] := \{ z \in \mathbb{Z} \mid a \leq z \leq b \}$ be the integral interval ranging from a to b.
- Let R be a commutative ring, G a finite group, M, N be RG-modules. Then $M \otimes_R N$ defines an RG-module via

  $$(m \otimes_R n) \cdot g = mg \otimes_R ng\,.$$
- If R is the commutative ground ring of the present context, we often write $\otimes := \otimes_R$.
- Suppose given a commutative ring R. Suppose given finitely generated free R-modules M and N.

  Suppose we have **fixed** R-linear bases

  $$\begin{array}{rcll} \mathcal{M} & := & (m_1, m_2, \ldots, m_k) & \text{of } M, \\ \mathcal{N} & := & (n_1, n_2, \ldots, n_\ell) & \text{of } N. \end{array}$$

  As basis of $M \oplus N$, we then fix

  $$\mathcal{M} \oplus \mathcal{N} := (m_1, m_2, \ldots, m_k, n_1, n_2, \ldots, n_\ell)\,.$$

  As basis of $M \otimes_R N$, we then fix

  $$\begin{array}{l} \mathcal{M} \otimes_R \mathcal{N} := \\ (\underbrace{m_1 \otimes n_1, m_1 \otimes n_2, \ldots, m_1 \otimes n_\ell}, \underbrace{m_2 \otimes n_1, m_2 \otimes n_2, \ldots, m_2 \otimes n_\ell}, \quad \ldots \quad , \\ \underbrace{m_k \otimes n_1, m_k \otimes n_2, \ldots, m_k \otimes n_\ell})\,. \end{array}$$

  Let $M \xrightarrow{f} N$ be an R-linear map. Once R-linear bases of M and N are fixed, we often use the describing matrix of f instead of f without further mention.

- Let R be a commutative ring. Suppose given $A \in R^{m\times n}$ and $B \in R^{p\times q}$. By
$$A \otimes B \;\in\; R^{mp\times nq}$$
we denote the Kronecker product of A and B, which has as entry at position $((i-1)p+r, (j-1)q+s)$ the product of the entry at position (i,j) of A with the entry at position (r,s) of B, for $i \in [1,m]$, $j \in [1,n]$, $r \in [1,p]$ and $s \in [1,q]$.

  For example,
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 0 & 1\cdot 1 & 2\cdot 0 & 2\cdot 1 & 3\cdot 0 & 3\cdot 1 \\ 1\cdot 0 & 1\cdot 0 & 2\cdot 0 & 2\cdot 0 & 3\cdot 0 & 3\cdot 0 \\ 4\cdot 0 & 4\cdot 1 & 5\cdot 0 & 5\cdot 1 & 6\cdot 0 & 6\cdot 1 \\ 4\cdot 0 & 4\cdot 0 & 5\cdot 0 & 5\cdot 0 & 6\cdot 0 & 6\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Let M be an object of an additive category. Let $k \geq 0$. By $M^{\oplus k}$ we denote the direct sum $\underbrace{M \oplus \ldots \oplus M}_{k \text{ times}}$.

- Let R be a commutative ring. By an **R-order,** we understand an R-algebra that is finitely generated free over R.

  Let Λ be an R-order. By a **Λ-lattice,** we understand a Λ-module that is finitely generated free over R.

- Congruences are also called **ties.** Often, congruences describing a subring of a direct product of matrix rings are referred to as ties.

- Let R be a commutative ring. Let $x \in R$. We often write $(x) := xR$. We also write $0 := (0)$.

- Let R be a discrete valuation ring with maximal ideal (π), and X an R-module. Then we denote $\bar{X} := X/\pi X$.

- The ring A is called **local,** if its set of non-units $A \setminus \mathrm{U}(A)$ is an ideal in A. This is equivalent to having that $0_R \neq 1_R$ and the sum of any two non-units in R is a non-unit; cf. Definition 189, Remark 192.
  For instance, a discrete valuation ring is local.

- Let $n \geq 1$. Let S_n denote the symmetric group on the set $[1,n]$.

  Let $\lambda := (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a partition of n. Then the corresponding Specht module over $R\mathrm{S}_n$ is denoted $S^\lambda = S^\lambda_R$, where R is the ground ring of the present context.

- Let p be a prime. Let P_i denote an indecomposable projective $\mathbb{Z}_{(p)}\mathrm{S}_n$-module. By $\bar{P}_i := P_i/pP_i$ we denote the indecomposable projective $\mathbb{F}_p\mathrm{S}_n$-module belonging to P_i. By D_i we denote the simple $\mathbb{Z}_{(p)}\mathrm{S}_n$-module (also being a $\mathbb{F}_p\mathrm{S}_n$-module) that belongs to P_i , i.e. D_i is the head of P_i , i.e.
$$D_i \;\cong\; P_i/\mathfrak{r}P_i \;\cong\; \bar{P}_i/\mathfrak{r}\bar{P}_i \; .$$
  For p odd, we denote the alternating simple module D_1 and the trivial simple module D_2 .

  For $p = 2$, we denote the trivial simple module D_1 .

- Matrices with blocks as entries are denoted with brackets. For example, $\begin{bmatrix} \text{block 1} & \text{block 2} \\ \text{block 3} & \text{block 4} \end{bmatrix}$.

  Zero blocks are marked by a dot "." . For example, $\begin{bmatrix} \text{block 1} & \text{block 2} \\ . & \text{block 4} \end{bmatrix}$.

- Let R be a ring.

  We denote the set of **left-invertible elements** in R by

  $$\mathrm{U}_{\mathrm{left}}(R) \ := \ \{ \, x \in R \, : \, \text{there exists } y \in R \text{ such that } yx = 1 \, \} \ .$$

  We denote the set of **right-invertible elements** in R by

  $$\mathrm{U}_{\mathrm{right}}(R) \ := \ \{ \, x \in R \, : \, \text{there exists } y \in R \text{ such that } xy = 1 \, \} \ .$$

  We denote the group of **invertible elements** (or **units**) in R by

  $$\mathrm{U}(R) \ := \ \{ \, x \in R \, : \, \text{there exists } y \in R \text{ such that } yx = 1 \text{ and } xy = 1 \, \} \ .$$

  Note that $\mathrm{U}(R) = \mathrm{U}_{\mathrm{left}}(R) \cap \mathrm{U}_{\mathrm{right}}(R)$, as associativity shows.

- Let R be a ring. Idempotents e and f of R are called **equivalent**, written $e \sim f$, if

  $$eR \ \cong \ fR$$

  as R-modules.

Uniform filenames

All proofs using Magma of the Chapters 3, 4 and 5 (except for Section 3.3) can also be found in Magma code in the files named `Diagram_Sn_locp_PioPj`, with P_i and P_j being indecomposable projective $\mathbb{Z}_{(p)}\mathrm{S}_n$-modules, replacing n, p, i, j in the filename by their actual values.

To check the homomorphisms given in these Chapters, the Magma code can be found in the files named `Homs_Sn_locp` for the $\mathbb{Z}_{(p)}\mathrm{S}_n$-linear morphisms, replacing n, p in the filename by their actual values.

Acknowledgements

I thank my parents for their financial and overall support.

Thanks to Juliane Deißler, Simon Klenk and Philipp Ritter for supporting me mentally.

Especially I want to thank my supervisor Dr. Matthias Künzer for his patience, for sharing his knowlegde, his assistance and for his time.

References

[1] Benson, D.J., *Modular Representation Theory,* Springer Lecture Notes 1081, second printing, 2006.

[2] Benson, D.J., *Representation and Cohomology I,* Cambridge Studies in Advanced Mathematics 30, second edition, 2004.

[3] Bosma, W.; Cannon, J.J.; Fieker, C.; Steel, A. (eds.), *Handbook of Magma functions*, Edition 2.19, 2012.

[4] Heller, A., *On group representations over a valuation ring,* Proc. Nat. Am. Soc. 47, p. 1194-1197, 1961.

[5] James, G.D., *The Representation Theory of the Symmetric Groups,* Springer Lecture Notes 682, 1978.

[6] James, G.D.; Kerber, A., *The Representation Theory of the Symmetric Group,* Encyc. Math. Appl. 16, 1981.

[7] Künzer, M.; *Ties for the ingeral group ring of the symmetric group,* Bielefeld, 1999.

[8] Lam, T.Y., *A First Course in Noncommutative Rings,* Springer Graduate Texts in Mathematics 131, 2001.

[9] Roggenkamp, K.W.; *Lattices over Orders II,* Springer Lecture Notes 142, 1970.

[10] Rowen, L.H.; *Ring Theory, Student Edition,* Academic Press, 1993.

[11] Serre, J.-P., *Linear representations of finite groups,* Springer Graduate Texts in Mathematics 42, 1977.

Chapter 2

Prologue

Let R be a discrete valuation ring with maximal ideal (π). Let Λ be an R-order.

Suppose given projective Λ-modules P and $\tilde{P}$. Let D and $\tilde{D}$ be the simple modules belonging to P and to $\tilde{P}$, respectively.

We denote by $\mathfrak{r}P$ the Jacobson radical of P.

Denote by

$$\begin{array}{rcl} \mathfrak{r}P \otimes \tilde{P} & \xrightarrow{E} & P \otimes \tilde{P} \\ P \otimes \mathfrak{r}\tilde{P} & \xrightarrow{\tilde{E}} & P \otimes \tilde{P} \end{array}$$

the respective embeddings.

Denote by

$$\begin{array}{rcl} P & \xrightarrow{\rho} & P/\mathfrak{r}P = D \\ \tilde{P} & \xrightarrow{\tilde{\rho}} & \tilde{P}/\mathfrak{r}\tilde{P} = \tilde{D} \end{array}$$

the respective residue class maps.

Lemma 1 *The sequence*

$$\mathfrak{r}P \otimes \tilde{P} \;\oplus\; P \otimes \mathfrak{r}\tilde{P} \xrightarrow{\begin{bmatrix} E \\ \tilde{E} \end{bmatrix}} P \otimes \tilde{P} \xrightarrow{\rho \otimes \tilde{\rho}} D \otimes \tilde{D}$$

is right-exact ([1]).

Note that

$$\mathfrak{r}P \otimes \tilde{P} \;\cong\; P \otimes \mathfrak{r}\tilde{P} \;\cong\; P \otimes \tilde{P}\,,$$

which is projective; cf. Lemma 238. So we may consider

$$\mathfrak{r}P \otimes \tilde{P} \;\oplus\; P \otimes \mathfrak{r}\tilde{P} \xrightarrow{\begin{bmatrix} E \\ \tilde{E} \end{bmatrix}} P \otimes \tilde{P}$$

to be the **canonical presentation** of $D \otimes \tilde{D}$.

[1] Here brackets are synonymous to parentheses, they just fit into the notational context below.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
P/\mathfrak{r}P \otimes \mathfrak{r}\tilde{P} \arrow[r, "\alpha_1"] & P/\mathfrak{r}P \otimes \tilde{P} \arrow[r, "\alpha_2"] & P/\mathfrak{r}P \otimes \tilde{P}/\mathfrak{r}\tilde{P} \\
P \otimes \mathfrak{r}\tilde{P} \arrow[r, "\beta_1"] \arrow[u, "\lambda_2"] & P \otimes \tilde{P} \arrow[r, "\beta_2"] \arrow[u, "\mu_2"] & P \otimes \tilde{P}/\mathfrak{r}\tilde{P} \arrow[u, "\nu_2"] \\
\mathfrak{r}P \otimes \mathfrak{r}\tilde{P} \arrow[r, "\gamma_1"] \arrow[u, "\lambda_1"] & \mathfrak{r}P \otimes \tilde{P} \arrow[r, "\gamma_2"] \arrow[u, "\mu_1"] & \mathfrak{r}P \otimes \tilde{P}/\mathfrak{r}\tilde{P} \arrow[u, "\nu_1"]
\end{tikzcd}$$

obtained by tensoring inclusion and residue class maps. So $E = \mu_1$, $\tilde{E} = \beta_1$ and $\rho \otimes \tilde{\rho} = \mu_2\alpha_2 = \beta_2\nu_2$.

The tensor functors $(P \otimes_R -)$, $(\mathfrak{r}P \otimes_R -)$, $(- \otimes_R \mathfrak{r}\tilde{P})$ and $(- \otimes_R \tilde{P})$ are exact. So the horizontal sequences (β_1 , β_2) and (γ_1 , γ_2) and the vertical sequences (λ_1 , λ_2) and (μ_1 , μ_2) are short exact.

The tensor functors $P/\mathfrak{r}P \otimes_R -$ and $- \otimes_R \tilde{P}/\mathfrak{r}\tilde{P}$ are right-exact. Thus we know that the sequences (α_1 , α_2) and (ν_1 , ν_2) are right-exact, but not necessarily short exact.

Now we get the commutative diagram

$$\begin{tikzcd}
 & D \otimes \tilde{P} \arrow[r, "\alpha_2"] & D \otimes \tilde{D} \\
P \otimes \mathfrak{r}\tilde{P} \arrow[r, "\beta_1"] \arrow[ur, "\beta_1\mu_2"] & P \otimes \tilde{P} \arrow[r, "\beta_2"] \arrow[u, "\mu_2"] & P \otimes \tilde{D} \arrow[u, "\nu_2"] \\
 & \mathfrak{r}P \otimes \tilde{P} , \arrow[u, "\mu_1"] \arrow[ur, "\mu_1\beta_2"] &
\end{tikzcd}$$

with (μ_1 , μ_2) and (β_1 , β_2) short exact sequences and $(\mu_1\beta_2 , \nu_2)$ a right-exact sequence.

According to Lemma 177, the sequence

$$\mathfrak{r}P \otimes \tilde{P} \ \oplus \ P \otimes \mathfrak{r}\tilde{P} \xrightarrow{\begin{bmatrix} \mu_1 \\ \beta_1 \end{bmatrix}} P \otimes \tilde{P} \xrightarrow{\beta_2\nu_2} D \otimes \tilde{D}$$

is right-exact, as asserted. $\square$

Now we set out to find isomorphisms $P \otimes \tilde{P} \xrightarrow[\sim]{A} Q$, $\mathfrak{r}P \otimes \tilde{P} \xrightarrow[\sim]{B} Q$ and $P \otimes \mathfrak{r}\tilde{P} \xrightarrow[\sim]{\tilde{B}} Q$ to a direct sum Q of standard indecomposable projectives in such a way that the diagram

$$\begin{tikzcd}
P \otimes \mathfrak{r}\tilde{P} \arrow[rr, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] & & Q \arrow[d, "\tilde{C}"] \\
P \otimes \tilde{P} \arrow[rr, "A", "\sim"'] & \circlearrowleft & Q \\
\mathfrak{r}P \otimes \tilde{P} \arrow[rr, "B", "\sim"'] \arrow[u, hook, "E"] & \circlearrowleft & Q \arrow[u, "C"']
\end{tikzcd}$$

commutes and such that C and $\tilde{C}$ are of a simple shape. This amounts to an analysis of the canonical presentation of $D \otimes \tilde{D}$. For more details of how to actually obtain such a diagram, cf. Construction 33.

Then we can complete to the following commutative diagram.

$$\begin{tikzcd}
D \otimes \tilde{D} \arrow[r, dashed, "\simeq"] & \operatorname{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) \\
P \otimes \tilde{P} \arrow[r, "A", "\sim"'] \arrow[u] & Q \arrow[u] \\
\mathfrak{r}P \otimes \tilde{P} \ \oplus \ P \otimes \mathfrak{r}\tilde{P} \arrow[r, "\sim", "\begin{bmatrix} B & 0 \\ 0 & \tilde{B} \end{bmatrix}"'] \arrow[u, "\begin{bmatrix} E \\ \tilde{E} \end{bmatrix}"] & Q \oplus Q \arrow[u, "\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}"']
\end{tikzcd}$$

In this way, we can compare our analysis with the tensor product $D \otimes \tilde{D}$ we started with.

Chapter 3

On localizations of $\mathbb{Z}S_3$

Definition 2 Let $\Lambda := \{(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f) \in \mathbb{Z} \times \mathbb{Z}^{2\times 2} \times \mathbb{Z} \,|\, a \equiv_3 b\,,\, e \equiv_3 f\,,\, d \equiv_3 0\,,\, a \equiv_2 f\}$.

A $\mathbb{Z}$-linear basis of Λ is given by

$$\Big((1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1),\ (0, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, 0),\ (0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0),$$
$$(0, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, 0),\ (0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 4),\ (0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 6) \Big)\ .$$

Lemma 3 *We have the isomorphism of $\mathbb{Z}$-orders*

$$\begin{array}{rlcl} \omega: & \mathbb{Z}S_3 \xrightarrow{\sim} \Lambda, \quad (1,2) & \mapsto & (1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1) \\ & (2,3) & \mapsto & (1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, -1)\ , \end{array}$$

obtained by restriction of a Wedderburn embedding in the codomain to its image; cf. Remark 239.

Note that Λ is in fact a $\mathbb{Z}$-suborder of $\mathbb{Z} \times \mathbb{Z}^{2\times 2} \times \mathbb{Z}$ as the image of the Wedderburn embedding.

The tuple entries belong to the Specht modules $S^{(3)}$, $S^{(2,1)}$ and $S^{(1^3)}$, in the order chosen above.

Proof. We have to calculate the image of the Wedderburn embedding from Remark 239.

With respect to the $\mathbb{Z}$-linear basis $(\mathrm{id}, (1,2), (2,3), (1,2,3), (1,3,2), (1,3))$ of $\mathbb{Z}S_3$ and the standard $\mathbb{Z}$-linear basis of $\mathbb{Z} \times \mathbb{Z}^{2\times 2} \times \mathbb{Z}$, our Wedderburn embedding is described by the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & -2 & -1 & 3 & 2 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -2 & -1 & 3 & 1 & 1 \\ 1 & 1 & 1 & -3 & -2 & 1 \\ 1 & 1 & 0 & -3 & -1 & -1 \end{pmatrix}\ .$$

We see that the image is contained in Λ.

We have $\det B = -54$.

With respect to the $\mathbb{Z}$-linear basis of Λ given above and the standard $\mathbb{Z}$-linear basis of $\mathbb{Z} \times \mathbb{Z}^{2\times 2} \times \mathbb{Z}$, we see that the determinant of the embedding $\Lambda \longrightarrow \mathbb{Z} \times \mathbb{Z}^{2\times 2} \times \mathbb{Z}$ is 54.

So the image of our Wedderburn embedding equals Λ. $\square$

The congruences describing the $\mathbb{Z}$-suborder Λ inside $\mathbb{Z} \times \mathbb{Z}^{2\times 2} \times \mathbb{Z}$ can also be read off the matrix

$$6 \cdot B^{-1} \;=\; \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & -4 & 6 & -2 & 4 & -1 \\ 1 & 2 & 0 & 2 & -2 & -1 \\ 1 & 2 & -6 & 0 & -2 & -1 \\ 1 & 2 & -6 & 2 & -4 & 1 \\ 1 & -4 & 6 & -2 & 2 & 1 \end{pmatrix} \;\in\; \mathbb{Z}^{6\times 6} \; .$$

To wit, each column is a congruence modulo 6. The system of these congruences can be simplified by column operations over $\mathbb{Z}$ to the defining system of congruences for Λ given above. Such congruences are also called **ties.**

3.1 The Localization $\mathbb{Z}_{(2)}\mathrm{S}_3$

The localization $\mathbb{Z}_{(2)}\mathrm{S}_3$ is more of a warm-up. First observations can be made, for instance on defect-0 blocks. To get an insight into the construction techniques used later on, this example is still too small, though.

Write

$$R \;:=\; \mathbb{Z}_{(2)} \; .$$

3.1.1 Idempotents and projectives

Remark 4 The localization of Λ at (2) is given by

$$\begin{aligned} \Lambda_{(2)} \;&=\; \{\big(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\big) \in \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{2\times 2} \times \mathbb{Z}_{(2)} \;\mid\; a \equiv_2 f\} \\ &=\; \{\big(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\big) \in R \times R^{2\times 2} \times R \;\mid\; a \equiv_2 f\} \; . \end{aligned}$$

The isomorphism of R-orders

$$\begin{aligned} \omega_{(2)} : \quad \mathbb{Z}_{(2)}\mathrm{S}_3 = R\mathrm{S}_3 \;\stackrel{\sim}{\longrightarrow}\; \Lambda_{(2)} \, , \quad (1,2) \;&\mapsto\; \big(1, \left(\begin{smallmatrix} -2 & -1 \\ 3 & 2 \end{smallmatrix}\right), -1\big) \\ (2,3) \;&\mapsto\; \big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right), -1\big) \end{aligned}$$

is obtained by localization of the isomorphism ω of Lemma 3.

Letting

$$e_1 \;:=\; \big(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\big) \; , \quad e_2 \;:=\; \big(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\big) \; , \quad e_3 \;:=\; \big(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0\big) \; ,$$

we have an orthogonal decomposition

$$1_{\Lambda_{(2)}} \;=\; e_1 + e_2 + e_3$$

into idempotents of $\Lambda_{(2)}$. They fall into the equivalence classes $\{e_1\}$, $\{e_2\,, e_3\}$.

Remark 5 There are 2 primitive central idempotents of $\Lambda_{(2)}$, namely

$$c_1 \;:=\; \big(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\big) \;=\; e_1 \; , \quad c_2 \;:=\; \big(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0\big) \;=\; e_2 + e_3 \quad .$$

The idempotent c_2 generates a block of defect 0.

Remark 6 Let $E := \{(a, b) \in R \times R \;\mid\; a \equiv_2 b\,\}$.

We have the following isomorphisms of R-orders.

$$\begin{array}{lcl} E & \xrightarrow{\sim} & e_1\Lambda_{(2)}e_1 \\ (a,f) & \longmapsto & \left(a, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), f\right), \\ R & \xrightarrow{\sim} & e_2\Lambda_{(2)}e_2 \\ b & \longmapsto & \left(0, \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right), 0\right), \\ R & \xrightarrow{\sim} & e_3\Lambda_{(2)}e_3 \\ e & \longmapsto & \left(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right), 0\right) \end{array}$$

Lemma 7 *We have an orthogonal decomposition $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3$ into primitive idempotents.*

Proof. We have to show primitivity of e_1, e_2 and e_3.

First we show that e_1 and 0 are the only idempotents of $e_1\Lambda_{(2)}e_1$. By Remark 6 it remains to show that $(0,0)$ and $(1,1)$ are the only idempotents of E. Let $(a,b) \in E \smallsetminus \{(0,0)\}$ be an idempotent. Since $\mathbb{Z}_{(2)}$ is local, Corollary 134 gives $a \in \{0,1\}$ and $b \in \{0,1\}$. Since $a \equiv_2 b$, we conclude that $a = b = 1$. Therefore, using Lemma 136, the idempotent e_1 is primitive.

We show primitivity of e_2. Primitivity of e_3 follows analogously.

We show that e_2 and 0 are the only idempotents of $e_2\Lambda_{(2)}e_2$. By Remark 6 it remains to show that (0) and (1) are the only idempotents of R. Let $0 \neq a \in R$ be an idempotent. Since $\mathbb{Z}_{(2)}$ is local, Corollary 134 gives $a = 1$. Therefore, using Lemma 136, the idempotent e_2 is primitive. □

Corollary 8 *Up to isomorphism, we have the Peirce decomposition*

$$\begin{array}{rcl} \Lambda_{(2)} & = & e_1\Lambda_{(2)} \oplus e_2\Lambda_{(2)} \oplus e_3\Lambda_{(2)} & = & \begin{pmatrix} e_1\Lambda_{(2)}e_1 & e_1\Lambda_{(2)}e_2 & e_1\Lambda_{(2)}e_3 \\ e_2\Lambda_{(2)}e_1 & e_2\Lambda_{(2)}e_2 & e_2\Lambda_{(2)}e_3 \\ e_3\Lambda_{(2)}e_1 & e_3\Lambda_{(2)}e_2 & e_3\Lambda_{(2)}e_3 \end{pmatrix} \\ & & & = & \begin{pmatrix} e_1\Lambda_{(2)}e_1 & 0 & 0 \\ 0 & e_2\Lambda_{(2)}e_2 & e_2\Lambda_{(2)}e_3 \\ 0 & e_3\Lambda_{(2)}e_2 & e_3\Lambda_{(2)}e_3 \end{pmatrix} \\ & \cong & e_1\Lambda_{(2)} \oplus e_2\Lambda_{(2)}^{\oplus 2} & \cong & \begin{pmatrix} (e_1\Lambda_{(2)}e_1)^{1\times 1} & (e_1\Lambda_{(2)}e_2)^{1\times 2} \\ (e_2\Lambda_{(2)}e_1)^{2\times 1} & (e_2\Lambda_{(2)}e_2)^{2\times 2} \end{pmatrix} \\ & & & = & \begin{pmatrix} (e_1\Lambda_{(2)}e_1)^{1\times 1} & 0 \\ 0 & (e_2\Lambda_{(2)}e_2)^{2\times 2} \end{pmatrix}. \end{array}$$

Lemma 9 *We have the following Jacobson radicals of $e_1\Lambda_{(2)}e_1$, $e_2\Lambda_{(2)}e_2$ and $e_3\Lambda_{(2)}e_3$.*

$$\begin{array}{rcl} \mathfrak{r}(e_1\Lambda_{(2)}e_1) & = & \left((2), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (2)\right) \\ \mathfrak{r}(e_2\Lambda_{(2)}e_2) & = & \left(0, \begin{pmatrix} (2) & 0 \\ 0 & 0 \end{pmatrix}, 0\right) \\ \mathfrak{r}(e_3\Lambda_{(2)}e_3) & = & \left(0, \begin{pmatrix} 0 & 0 \\ 0 & (2) \end{pmatrix}, 0\right) \end{array}$$

Proof. By Remark 6, it suffices to show that $\mathfrak{r}(E) \overset{!}{=} 2R \times 2R$, resp. $\mathfrak{r}(R) \overset{!}{=} 2R$. The latter was remarked in Example 179.(i).

Since E is a commutative R-order, we have, by Proposition 212, $\mathfrak{r}(E) = \{x \in E \mid x^n \in 2E \text{ for some } n \geq 0\}$.

Note that $2E = \{(a,b) \in R \times R \mid a \equiv_2 0, b \equiv_2 0, a \equiv_4 b\}$.

Ad $\supseteq$. Given $(a,b) \in 2R \times 2R$, we have $(a,b)^2 \in 2E$, whence $(a,b) \in \mathfrak{r}(E)$.

Ad $\subseteq$. Suppose given $(a,b) \in E$ and $n \geq 0$ such that $(a,b)^n \in 2E$. Then $a^n \equiv_2 0$ and $b^n \equiv_2 0$ in R. Hence $a \equiv_2 0$ and $b \equiv_2 0$ in R. Therefore $(a,b) \in 2R \times 2R$.

We could also have used Example 223. $\square$

Lemma 10 *We have the Jacobson radical* $\mathfrak{r}(\Lambda_{(2)}) = \left((2), \begin{pmatrix} (2) & (2) \\ (2) & (2) \end{pmatrix}, (2)\right)$ *of* $\Lambda_{(2)}$.

Proof. We use Lemma 9 together with Proposition 217, which is possible by Remark 208, to find the Jacobson radical as follows. Cf. also Corollary 8.

$$\begin{aligned}
\mathfrak{r}(\Lambda_{(2)}) &= \mathfrak{r}(e_1\Lambda_{(2)}e_1) \oplus \mathfrak{r}(e_2\Lambda_{(2)}e_2) \oplus \mathfrak{r}(e_2\Lambda_{(2)}e_3) \oplus \mathfrak{r}(e_3\Lambda_{(2)}e_3) \oplus \mathfrak{r}(e_3\Lambda_{(2)}e_2) \\
&= \left((2), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), (2)\right) \oplus \left(0, \left(\begin{smallmatrix} (2) & (2) \\ 0 & 0 \end{smallmatrix}\right), 0\right) \oplus \left(0, \left(\begin{smallmatrix} 0 & 0 \\ (2) & (2) \end{smallmatrix}\right), 0\right) \\
&= \left((2), \left(\begin{smallmatrix} (2) & (2) \\ (2) & (2) \end{smallmatrix}\right), (2)\right)
\end{aligned}$$

For notation cf. Lemma 185. $\square$

Definition 11 Let $P_1 := e_1\Lambda_{(2)}$ and $P_2 := e_2\Lambda_{(2)} \cong e_3\Lambda_{(2)}$ represent the isoclasses of the indecomposable projective modules of $\Lambda_{(2)}$; cf. Remark 208, Lemma 220. So

$$\begin{aligned}
P_1 &= \left\{ \left(a, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), f\right) \in R \times R^{2\times 2} \times R \,\middle|\, a \equiv_2 f \right\}, \\
P_2 &= \left\{ \left(0, \left(\begin{smallmatrix} b & c \\ 0 & 0 \end{smallmatrix}\right), 0\right) \in R \times R^{2\times 2} \times R \right\}.
\end{aligned}$$

We abbreviate

$$\begin{aligned}
[(a),(f)] &:= \left(a, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), f\right) \in P_1 \\
[b,c] &:= \left(0, \left(\begin{smallmatrix} b & c \\ 0 & 0 \end{smallmatrix}\right), 0\right) \in P_2\,,
\end{aligned}$$

and get

$$\begin{aligned}
P_1 &= \{[(a),(f)] \in R \times R \mid a \equiv_2 f\} &\hookrightarrow\;& S^{(3)} \oplus S^{(1,1,1)} \\
P_2 &= \{[b,c] \in R^{1\times 2}\} &\hookrightarrow\;& S^{(2,1)}.
\end{aligned}$$

The radicals of P_1 and P_2 are given by $\mathfrak{r}P_1 = e_1\mathfrak{r}\Lambda_{(2)}$ and $\mathfrak{r}P_2 = e_2\mathfrak{r}\Lambda_{(2)}$. Via Lemma 10, we obtain

$$\begin{aligned}
\mathfrak{r}P_1 &= \{[(2a),(2f)] \in R \times R\} \\
\mathfrak{r}P_2 &= \{[2b,2c] \in R^{1\times 2}\}\,.
\end{aligned}$$

We choose the $\mathbb{Z}_{(2)}$-linear bases

$$\begin{array}{lll}
(\,[(1),(1)], & [(0),(2)]\,) & \text{of } P_1 \\
(\,[1,\ 0], & [0,\ 1]\,) & \text{of } P_2 \\
(\,[(2),(0)], & [(0),(2)]\,) & \text{of } \mathfrak{r}P_1 \\
(\,[2,\ 0], & [0\ ,2]\,) & \text{of } \mathfrak{r}P_2\,.
\end{array}$$

Remark 12 The projective module $\bar{P}_2$ belongs to the defect-0 block and is indecomposable, so we have the Loewy layer

$$D_2 \quad .$$

For $\bar{P}_1$ we have the following Loewy layers.

$$\begin{array}{c} D_1 \\ D_1 \end{array}$$

Remark 13

Recall that $e_1\Lambda_{(2)}e_1 = \left\{(a, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, f) \in R \times R^{2\times 2} \times R \,\middle|\, a \equiv_2 f\right\}$, for which we fix as R-linear basis

$$(e_1, \tilde{h}_1^{11}) := \left((1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1),\ (0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 2)\right).$$

Via the canonical isomorphism from $e_1\Lambda_{(2)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_3}(P_1, P_1)$, it is mapped to the R-linear basis

$$(1, h_1^{11}) = (1_{P_1}, h_1^{11}) := \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{R\mathrm{S}_3}(P_1, P_1)$, using the fixed R-linear basis of P_1 given in Definition 11.

An R-linear basis of $e_2\Lambda_{(2)}e_2 = \left\{(0, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, 0) \in R \times R^{2\times 2} \times R\right\}$ is given by

$$(e_2) := \left((0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0)\right).$$

Via the canonical isomorphism from $e_2\Lambda_{(2)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_3}(P_2, P_2)$, it is mapped to the R-linear basis

$$(1) = (1_{P_2}) := \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{R\mathrm{S}_3}(P_2, P_2)$, using the fixed R-linear basis of P_2, like above.

Because $e_1\Lambda_{(2)}e_2 = e_2\Lambda_{(2)}e_1 = \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)$, we have $\mathrm{Hom}_{R\mathrm{S}_3}(P_1, P_2) = \mathrm{Hom}_{R\mathrm{S}_3}(P_2, P_1) = 0$.

The operating matrices can be found in the file `main_S3_loc2`, the homomorphisms in `Homs_S3_loc2`.

They can e.g. be used to check the $\mathbb{Z}_{(2)}\mathrm{S}_3$-linearity of the maps between P_1 and P_2 derived above.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 are denoted `rhoP1`, `rhoP2`, respectively.

E.g. for the operating matrices on P_1, call

```
rhoP1(S3P!sigma);
```

for an element `sigma` of S_3. Analogously for P_2.

To check that the matrices found above represent $R\mathrm{S}_3$-linear maps between the respective projective modules, follow these steps:

```
load main_S3_loc2;
load Homs_S3_loc2;

[rhoP1(sigma)*Hom_P1P1[i] eq Hom_P1P1[i]*rhoP1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}, i in [1..2]];
[rhoP2(sigma)*Hom_P2P2[i] eq Hom_P2P2[i]*rhoP2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}, i in [1..1]];
```

3.1.2 The tensor product $P_1 \otimes P_1$

Lemma 14 *Let E be the embedding $\mathfrak{t}P_1 \otimes P_1 \longrightarrow P_1 \otimes P_1$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{t}P_1 \longrightarrow P_1 \otimes P_1$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_3$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{t}P_1 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \oplus P_1 \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_1 \arrow[r, "A", "\sim"'] & P_1 \oplus P_1 \\
\mathfrak{t}P_1 \otimes P_1 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_1 \oplus P_1 \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \end{pmatrix},$$

$$E := \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{E} := \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left[\begin{array}{c:c} 2 \cdot 1_{P_1} & . \\ \hdashline . & 1_{P_1} \end{array}\right] =: \begin{bmatrix} 2 & . \\ . & 1 \end{bmatrix},$$

$$\tilde{C} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} = \left[\begin{array}{c:c} 2 \cdot 1_{P_1} & . \\ \hdashline h_1^{1,1} & 1_{P_1} \end{array}\right] =: \begin{bmatrix} 2 & . \\ h_1^{1,1} & 1 \end{bmatrix}.$$

We indicate with brackets that the matrices are to be read blockwise.

Recall that $R = \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
We have to show local invertibility and $R\mathrm{S}_3$-linearity of the maps A, B, $\tilde{B}$, the $R\mathrm{S}_3$-linearity of E, $\tilde{E}$, C and $\tilde{C}$, and the commutativity of the diagram. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_3$-linearity and the commutativity can be found in the file `main_S3_loc2`, the matrices for this diagram in the file `Diagram_S3_loc2_P1oP1` .

The embedding E is defined as the Kronecker product of the embedding $i_1 : \mathfrak{t}P_1 \hookrightarrow P_1$ and id_{P_1}. An embedding in the chosen bases can be found in the files and is denoted `i1`. The embedding $\tilde{E}$ is defined as the Kronecker product of id_{P_1} and the embedding $i_1 : \mathfrak{t}P_1 \hookrightarrow P_1$.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 are denoted `rhoP1`, `rhoP2`, respectively. The representations on $\mathfrak{r}P_1$, $\mathfrak{r}P_2$ are denoted `rhorP1`, `rhorP2`, respectively.

The representations on $P_1 \otimes P_1$, $\mathfrak{r}P_1 \otimes P_1$, $P_1 \otimes \mathfrak{r}P_1$ are denoted `op_p1op1`, `op_rp1op1`, `op_p1orp1`, respectively. The operating matrix of a group element on such a tensor product is defined as the tensor product of the operating matrices on the tensor factors.

For example, for the operating matrices on $P_1 \otimes P_1$ call

```
op_p1op1(S3P!sigma);
```

for an element `sigma` of S_3. The other maps work the same way.

The representation on the direct sum $P_1 \oplus P_1$ is denoted `op_proj_sum_p1p1`. The operating matrix of a group element is defined as the block diagonal matrix containing the operating matrices of the summands. The maps A, B, C, E are denoted `A`, `B1`, `C1`, `E1`, respectively; the maps $\tilde{B}$, $\tilde{C}$, $\tilde{E}$ are denoted `B2`, `C2`, `E2`, respectively.

To verify the Lemma, follow these steps:

```
load "main_S3_loc2";
load "Diagram_S3_loc2_P1oP1";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),2)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),2)!1,i1);

 [op_rp1op1(sigma)*E1 eq E1*op_p1op1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_p1orp1(sigma)*E2 eq E2*op_p1op1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS3-linearity
 [op_p1op1(sigma)*A eq A*op_proj_sum_p1p1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_rp1op1(sigma)*B1 eq B1*op_proj_sum_p1p1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_p1orp1(sigma)*B2 eq B2*op_proj_sum_p1p1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_proj_sum_p1p1(sigma)*C1 eq C1*op_proj_sum_p1p1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_proj_sum_p1p1(sigma)*C2 eq C2*op_proj_sum_p1p1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];

//local invertibility; loc_inv see "main_S3_loc2"
 loc_inv(A,2);
 loc_inv(B1,2);
 loc_inv(B2,2);
```

$\square$

Remark 15 Using Magma, we verify that $\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) \cong D_1 \otimes D_1 \cong D_1$.

Furthermore, $\mathrm{Coker}(C) \cong \mathrm{Coker}(\tilde{C}) \cong D_1 \otimes \bar{P}_1 \cong \bar{P}_1$ with Loewy layers already known.

Remark 16 The matrix $\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}$ of Lemma 14 cannot be transformed into a matrix of the form $\begin{bmatrix} F \\ \tilde{F} \end{bmatrix}$ with F and $\tilde{F}$ both diagonal matrices, i.e. there is no diagram as in Lemma 14 such that

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_1 \arrow[r, "\tilde{B}'", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \oplus P_1 \arrow[d, "\tilde{F}"] \\
P_1 \otimes P_1 \arrow[r, "A'", "\sim"'] & P_1 \oplus P_1 \\
\mathfrak{r}P_1 \otimes P_1 \arrow[r, "B'", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_1 \oplus P_1 \arrow[u, "F"']
\end{tikzcd}$$

is commutative with F, $\tilde{F}$ diagonal matrices.

Proof. The cokernel of $\begin{bmatrix} E \\ \tilde{E} \end{bmatrix}$ is isomorphic to $P_1 \otimes P_1/(\mathfrak{r}P_1 \otimes P_1 + P_1 \otimes \mathfrak{r}P_1) \cong D_1 \otimes D_1 \cong D_1$.

Assume that there exists a commutative diagram as in the Remark. Then the cokernel of $\begin{bmatrix} F \\ \tilde{F} \end{bmatrix}$ is isomorphic to D_1 , too. In particular, $\dim_{\mathbb{F}_2} \mathrm{Coker} \begin{bmatrix} F \\ \tilde{F} \end{bmatrix} = 1$.

The elementary divisors of E are $(1, 1, 2, 2)$, for $P_1 \otimes P_1/\mathfrak{r}P_1 \otimes P_1 \cong D_1 \otimes \bar{P}_1 = \bar{P}_1$, which is of dimension 2.

So F is a diagonal matrix with two diagonal entries having valuation 1 at 2 and two diagonal entries having valuation 0 at 2. Since the upper left 2×2-block of F is an $R\mathrm{S}_3$-linear endomorphism of P_1, we conclude that

$$F \in \{ \mathrm{diag}(u, u, 2v, 2v), \mathrm{diag}(2u, 2u, v, v) \mid u, v \in \mathrm{U}(R) \} .$$

Likewise, we have

$$\tilde{F} \in \{ \mathrm{diag}(\tilde{u}, \tilde{u}, 2\tilde{v}, 2\tilde{v}), \mathrm{diag}(2\tilde{u}, 2\tilde{u}, \tilde{v}, \tilde{v}) \mid \tilde{u}, \tilde{v} \in \mathrm{U}(R) \} .$$

So $\dim_{\mathbb{F}_2} \mathrm{Coker} \begin{bmatrix} F \\ \tilde{F} \end{bmatrix} \in \{0, 2\}$, which is a *contradiction.* □

3.1.3 The tensor product $P_1 \otimes P_2$

Lemma 17 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_2 \longrightarrow P_1 \otimes P_2$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_2 \longrightarrow P_1 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_3$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_2 \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_2 \arrow[r, "A", "\sim"'] & P_2 \oplus P_2 \\
\mathfrak{r}P_1 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_2 \oplus P_2 \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$A := \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -6 & -3 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -6 & -3 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -6 & -3 \end{pmatrix}$$

$$E := \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{E} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$C := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left[\begin{array}{c:c} 2 \cdot 1_{P_2} & \\ \hdashline & 1_{P_2} \end{array}\right] =: \begin{bmatrix} 2 & . \\ . & 1 \end{bmatrix},$$

$$\tilde{C} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \left[\begin{array}{c:c} 2 \cdot 1_{P_2} & . \\ \hdashline . & 2 \cdot 1_{P_2} \end{array}\right] =: \begin{bmatrix} 2 & . \\ . & 2 \end{bmatrix}.$$

The matrices C and $\tilde{C}$ being block diagonal matrices, with blocks of the form id_{P_2} and $2\,\mathrm{id}_{P_2}$, confirms the expected result, shown in Remark 163, for P_2 belongs to a defect-0 block.

Recall that $R = \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the RS_3-linearity and the commutativity can be found in the file `main_S3_loc2`, the matrices for this diagram in the file `Diagram_S3_loc2_P1oP2` .

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 14 .
To verify the Lemma, follow these steps:

```
load "main_S3_loc2";
load "Diagram_S3_loc2_P1oP2";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),2)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),2)!1,i2);

 [op_rp1op2(sigma)*E1 eq E1*op_p1op2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_p1orp2(sigma)*E2 eq E2*op_p1op2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS3-linearity
 [op_p1op2(sigma)*A eq A*op_proj_sum_p1p2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_rp1op2(sigma)*B1 eq B1*op_proj_sum_p1p2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_p1orp2(sigma)*B2 eq B2*op_proj_sum_p1p2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
```

```
 [op_proj_sum_p1p2(sigma)*C1 eq C1*op_proj_sum_p1p2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];
 [op_proj_sum_p1p2(sigma)*C2 eq C2*op_proj_sum_p1p2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}];

//local invertibility; loc_inv see "main_S3_loc2"
 loc_inv(A,2);
 loc_inv(B1,2);
 loc_inv(B2,2);
```

$\square$

Remark 18 Using Magma, we verify that

$$\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) \cong D_1 \otimes D_2 \cong D_1 \otimes \bar{P}_2 \cong \bar{P}_2$$

$$\begin{array}{lclcl} \mathrm{Coker}(C) & \cong & D_1 \otimes \bar{P}_2 & \cong & \bar{P}_2\,, \\ \mathrm{Coker}(\tilde{C}) & \cong & \bar{P}_1 \otimes D_2 & \cong & \bar{P}_1 \otimes \bar{P}_2 \cong \bar{P}_2 \oplus \bar{P}_2\,. \end{array}$$

3.1.4 The tensor product $P_2 \otimes P_2$

Lemma 19 *Let E be the embedding $\mathfrak{t}P_2 \otimes P_2 \longrightarrow P_2 \otimes P_2$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{t}P_2 \longrightarrow P_2 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_3$-linear maps*

$$\begin{tikzcd}
P_2 \otimes \mathfrak{t}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \oplus P_2 \arrow[d, "\tilde{C}"] \\
P_2 \otimes P_2 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \oplus P_2 \\
\mathfrak{t}P_2 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_1 \oplus P_2 \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$A := \begin{pmatrix} 2 & -1 & 1 & 0 \\ -3 & 1 & -3 & -1 \\ -3 & 2 & -3 & -1 \\ 6 & -3 & 6 & 3 \end{pmatrix}, \quad B := \begin{pmatrix} 2 & -1 & 1 & 0 \\ -3 & 1 & -3 & -1 \\ -3 & 2 & -3 & -1 \\ 6 & -3 & 6 & 3 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} 2 & -1 & 1 & 0 \\ -3 & 1 & -3 & -1 \\ -3 & 2 & -3 & -1 \\ 6 & -3 & 6 & 3 \end{pmatrix},$$

$$E := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \tilde{E} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$C := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \left[\begin{array}{c|c} 2 \cdot 1_{P_1} & \\ \hline & 2 \cdot 1_{P_2} \end{array}\right] =: \left[\begin{array}{c|c} 2 & . \\ \hline . & 2 \end{array}\right]$$

$$\tilde{C} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \left[\begin{array}{c|c} 2 \cdot 1_{P_1} & \\ \hline & 2 \cdot 1_{P_2} \end{array}\right] =: \left[\begin{array}{c|c} 2 & . \\ \hline . & 2 \end{array}\right]$$

The matrices C and $\tilde{C}$ being block diagonal matrices, with blocks of the form $2\,\mathrm{id}_{P_1}$ and $2\,\mathrm{id}_{P_2}$, confirms the expected result, shown in Remark 164, for P_2 belongs to a defect-0 block.

Recall that $R = \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_3$-linearity and the commutativity can be found in the file `main_S3_loc2`, the matrices for this diagram in the file `Diagram_S3_loc2_P2oP2` .

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 14 .

To verify the Lemma, follow these steps:

```
load "main_S3_loc2";
load "Diagram_S3_loc2_P2oP2";

 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

E1 := KroneckerProduct(i2,MatrixRing(Rationals(),2)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),2)!1,i2);

 [op_rp2op2(sigma)*E1 eq E1*op_p2op2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p2orp2(sigma)*E2 eq E2*op_p2op2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS3-linearity
 [op_p2op2(sigma)*A eq A*op_proj_sum_p2p2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_rp2op2(sigma)*B1 eq B1*op_proj_sum_p2p2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p2orp2(sigma)*B2 eq B2*op_proj_sum_p2p2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p2p2(sigma)*C1 eq C1*op_proj_sum_p2p2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p2p2(sigma)*C2 eq C2*op_proj_sum_p2p2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

//local invertibility; loc_inv see "main_S3_loc2"
 loc_inv(A,2);
 loc_inv(B1,2);
 loc_inv(B2,2);
```

□

Remark 20 Using Magma, we verify that

$$\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) \cong \mathrm{Coker}(C) \cong \mathrm{Coker}(\tilde{C}) \cong D_2 \otimes D_2 \cong \bar{P}_2 \otimes \bar{P}_2 \cong \bar{P}_1 \oplus \bar{P}_2 \,.$$

3.2 The Localization $\mathbb{Z}_{(3)}\mathrm{S}_3$

In this section, construction techniques are explained in more detail; cf. Construction 33.

Write

$$R \;:=\; \mathbb{Z}_{(3)}\;.$$

3.2.1 Idempotents and projectives

Definition 21 The localization of Λ at (3) is given by

$$\begin{aligned}\Lambda_{(3)} \;&:=\; \{\big(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\big) \in \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)} \mid a \equiv_3 b\,,\;\; e \equiv_3 f\,,\;\; d \equiv_3 0\} \\ &=\; \{\big(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\big) \in R \times R^{2\times 2} \times R \mid a \equiv_3 b\,,\;\; e \equiv_3 f\,,\;\; d \equiv_3 0\}\;.\end{aligned}$$

The isomorphism of R-orders

$$\omega_{(3)}:\;\; \mathbb{Z}_{(3)}\mathrm{S}_3 = R\mathrm{S}_3 \;\xrightarrow{\sim}\; \Lambda_{(3)}\,,\quad \begin{aligned}(1,2) \;&\mapsto\; \big(1, \left(\begin{smallmatrix} -2 & -1 \\ 3 & 2 \end{smallmatrix}\right), -1\big) \\ (2,3) \;&\mapsto\; \big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right), -1\big)\end{aligned}$$

is obtained by localization of the isomorphism ω of Lemma 3.

Letting

$$e_1 \;:=\; \big(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\big), \qquad e_2 \;:=\; \big(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\big)$$

we have an orthogonal decomposition

$$1_{\Lambda_{(3)}} \;=\; e_1 + e_2$$

into idempotents of $\Lambda_{(3)}$. They fall into two equivalence classes $\{e_1\}$, $\{e_2\}$.

Remark 22 Let $E := \{(a,b) \in R \times R \,|\, a \equiv_3 b\}$.

We have the following isomorphisms of R-orders.

$$\begin{aligned} E \;&\longrightarrow\; e_1\Lambda_{(3)}e_1 \\ (e,f) \;&\longmapsto\; \big(\big(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right), f\big) \\[1ex] E \;&\longrightarrow\; e_2\Lambda_{(3)}e_2 \\ (a,b) \;&\longmapsto\; \big(\big(a, \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right), 0\big) \end{aligned}$$

Lemma 23 *We have an orthogonal decomposition* $1_{\Lambda_{(3)}} = e_1 + e_2$ *into primitive idempotents.*

Proof. We have to show primitivity of e_1 and e_2.

We show that e_1 is primitive. Primitivity of e_2 follows analogously.

First we show that e_1 and 0 are the only idempotents of $e_1\Lambda_{(3)}e_1$. By Remark 22 it remains to show that $(0,0)$ and $(1,1)$ are the only idempotents of E. Let $(a,b) \in E \smallsetminus \{(0,0)\}$ be an idempotent. Since $\mathbb{Z}_{(3)}$ is local, Corollary 134 gives $a \in \{0,1\}$ and $b \in \{0,1\}$. Since $a \equiv_3 b$, we conclude that $a = b = 1$. Therefore, using Lemma 136, the idempotents e_1 and e_2 are primitive.

Since this is a finite problem in E, we can use Magma **[3]** instead to list all possible pairs $(a,b) \in R \times R$ with $a,\, b \in \{0,\, 1\}$ and to test the ties defining E in a last step. To do so, we call

```
{<a,b> : a,b in {0,1}| (a mod 3) eq (b mod 3)};
```

and get the following output.

```
> {<a,b> : a,b in {0,1}| (a mod 3) eq (b mod 3)};
{ <1, 1>, <0, 0> }
```

Therefore, using Lemma 136, the idempotent e_1 is primitive.

In this example such a Magma procedure is not necessary, but we can use this trick later on, when the ties are of a complicated shape. $\square$

Corollary 24 *We have the Peirce decomposition*

$$\Lambda_{(3)} \quad = \quad e_1\Lambda_{(3)} \oplus e_2\Lambda_{(3)} \quad = \quad \begin{pmatrix} e_1\Lambda_{(3)}e_1 & e_1\Lambda_{(3)}e_2 \\ e_2\Lambda_{(3)}e_1 & e_2\Lambda_{(3)}e_2 \end{pmatrix} .$$

Lemma 25

We have the Jacobson radicals

$$\begin{aligned} \mathfrak{r}(e_1\Lambda_{(3)}e_1) &= \Big(0, \begin{pmatrix} 0 & 0 \\ 0 & (3) \end{pmatrix}, (3)\Big) , \\ \mathfrak{r}(e_2\Lambda_{(3)}e_2) &= \Big((3), \begin{pmatrix} (3) & 0 \\ 0 & 0 \end{pmatrix}, 0\Big) . \end{aligned}$$

Proof. By Remark 22, it suffices to show that $\mathfrak{r}(E) \overset{!}{=} 3R \times 3R$. Since E is a commutative R-order, we have, by Proposition 212, $\mathfrak{r}(E) = \{x \in E \mid x^n \in 3E \text{ for some } n \geq 0\}$.

Note that $3E = \{(a,b) \in R \times R \mid a \equiv_3 0, b \equiv_3 0, a \equiv_9 b\}$.

Ad $\supseteq$. Given $(a,b) \in 3R \times 3R$, we have $(a,b)^2 \in 3E$, whence $(a,b) \in \mathfrak{r}(E)$.

Ad $\subseteq$. Suppose given $(a,b) \in E$ and $n \geq 0$ such that $(a,b)^n \in 3E$. Then $a^n \equiv_3 0$ and $b^n \equiv_3 0$ in R. Hence $a \equiv_3 0$ and $b \equiv_3 0$ in R. Therefore $(a,b) \in 3R \times 3R$.

We could also have used Example 223.

$\square$

Lemma 26 *We have the Jacobson radical* $\mathfrak{r}(\Lambda_{(3)}) = \Big((3), \begin{pmatrix} (3) & R \\ (3) & (3) \end{pmatrix}, (3)\Big)$.

Proof. With Proposition 217, applicable because of Remark 208, and the Peirce decomposition from Corollary 24, the radicals from Lemma 25 yield

$$\begin{aligned} \mathfrak{r}(\Lambda_{(3)}) &= \mathfrak{r}(e_1\Lambda_{(3)}e_1) \oplus \mathfrak{r}(e_2\Lambda_{(3)}e_2) \oplus e_1\Lambda_{(3)}e_2 \oplus e_2\Lambda_{(3)}e_1 \\ &= \Big(0, \begin{pmatrix} 0 & 0 \\ 0 & (3) \end{pmatrix}, (3)\Big) \oplus \Big((3), \begin{pmatrix} (3) & 0 \\ 0 & 0 \end{pmatrix}, 0\Big) \oplus \Big(0, \begin{pmatrix} 0 & 0 \\ (3) & 0 \end{pmatrix}, 0\Big) \oplus \Big(0, \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}, 0\Big) \\ &= \Big((3), \begin{pmatrix} (3) & R \\ (3) & (3) \end{pmatrix}, (3)\Big) . \end{aligned}$$

$\square$

Definition 27 Let $P_1 := e_1\Lambda_{(3)}$ and $P_2 := e_2\Lambda_{(3)}$ represent the isoclasses of the indecomposable projective modules of $\Lambda_{(3)}$; cf. Remark 208, Lemma 220.

So

$$
\begin{aligned}
P_1 &= \left\{ (0, \begin{pmatrix} 0 & 0 \\ d & e \end{pmatrix}, f) \in R \times R^{2\times 2} \times R \,\middle|\, d \equiv_3 0,\ \ e \equiv_3 f \right\} \\
P_2 &= \left\{ (a, \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix}, 0) \in R \times R^{2\times 2} \times R \,\middle|\, a \equiv_3 b \right\}.
\end{aligned}
$$

We abbreviate

$$
\begin{aligned}
[(d\ e), f] &:= (0, \begin{pmatrix} 0 & 0 \\ d & e \end{pmatrix}, f) \ \in\ P_1 \\
[a, (b\ c)] &:= (a, \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix}, 0) \ \in\ P_2 ,
\end{aligned}
$$

and obtain

$$
\begin{aligned}
P_1 &= \{[(d\ e), f] \in R^{1\times 2} \times R \mid d \equiv_3 0,\ \ e \equiv_3 f\} &\hookrightarrow&\ S^{(2,1)} \oplus S^{(1,1,1)} \\
P_2 &= \{[a, (b\ c)] \in R \times R^{1\times 2} \mid a \equiv_3 b\} &\hookrightarrow&\ S^{(3)} \oplus S^{(2,1)}.
\end{aligned}
$$

The radicals of P_1 and P_2 are given by $\mathfrak{r}P_1 = e_1\mathfrak{r}(\Lambda_{(3)})$ and $\mathfrak{r}P_2 = e_2\mathfrak{r}(\Lambda_{(3)})$. Via Lemma 26, we obtain

$$
\begin{aligned}
\mathfrak{r}P_1 &= \{[(d\ e), f] \in R^{1\times 2} \times R \mid d \equiv_3 0, e \equiv_3 f \equiv_3 0\} \\
\mathfrak{r}P_2 &= \{[a, (b\ c)] \in R \times R^{1\times 2} \mid a \equiv_3 b \equiv_3 0\}.
\end{aligned}
$$

We choose the R-linear bases

$$
\begin{array}{llll}
(\,[(3\ 0), 0], & [(0\ 1), 1], & [(0\ 0), 3]\,) & \text{of } P_1 \\
(\,[1, (1\ 0)], & [0, (3\ 0)], & [0, (0\ 1)]\,) & \text{of } P_2 \\
(\,[(3\ 0), 0], & [(0\ 3), 0], & [(0\ 0), 3]\,) & \text{of } \mathfrak{r}P_1 \\
(\,[3, (0\ 0)], & [0, (3\ 0)], & [0, (0\ 1)]\,) & \text{of } \mathfrak{r}P_2 .
\end{array}
$$

Remark 28 The projective modules $\bar{P}_1$ has the Loewy layers

$$
\begin{array}{c} D_1 \\ D_2 \\ D_1 \end{array}
$$

and $\bar{P}_2$ has the Loewy layers

$$
\begin{array}{c} D_2 \\ D_1 \\ D_2 \end{array} .
$$

Remark 29 Recall that $e_1\Lambda_{(3)}e_1 = \left\{ (0, \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, f) \in R \times R^{2\times 2} \times R \,\middle|\, e \equiv_3 f \right\}$, for which we fix as R-linear basis

$$
(e_1, \tilde{h}_1^{11}) := \left((0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1),\ (0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3) \right).
$$

Via the canonical isomorphism from $e_1\Lambda_{(3)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_3}(P_1, P_1)$, it is mapped to the R-linear basis

$$
(1, h_1^{11}) = (1_{P_1}, h_1^{11}) := \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \right)
$$

of $\mathrm{Hom}_{R\mathrm{S}_3}(P_1, P_1)$, using the fixed R-linear basis of P_1 given in Definition 27.

An R-linear basis of $e_2\Lambda_{(3)}e_2 = \left\{(a, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, 0) \in R \times R^{2\times 2} \times R \,\middle|\, a \equiv_3 b\right\}$ is given by

$$(e_2, \tilde{h}_1^{22}) := \left((1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0), (0, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, 0)\right).$$

Via the canonical isomorphism from $e_2\Lambda_{(3)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_3}(P_2, P_2)$, it is mapped to the R-linear basis

$$(1, h_1^{22}) = (1_{P_2}, h_1^{22}) := \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{R\mathrm{S}_3}(P_2, P_2)$, using the fixed R-linear basis of P_2, like above.

For a basis of $\mathrm{Hom}_{R\mathrm{S}_3}(P_2, P_1)$ we consider $e_1\Lambda_{(3)}e_2 = \left\{(0, \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}, 0) \in R \times R^{2\times 2} \times R \,\middle|\, d \equiv_3 0\right\}$, therefore an R-linear basis is given by

$$(\tilde{h}_1^{21}) := \left((0, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, 0)\right).$$

Via the canonical isomorphism from $e_1\Lambda_{(3)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_3}(P_2, P_1)$ it is mapped to

$$(h_1^{21}) := \left(\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix}\right).$$

Now $e_2\Lambda_{(3)}e_1 = \left\{(0, \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, 0) \in R \times R^{2\times 2} \times R\right\}$, therefore an R-linear basis is given by

$$(\tilde{h}_1^{12}) := \left((0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0)\right).$$

Via the canonical isomorphism from $e_2\Lambda_{(3)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_3}(P_1, P_2)$ it is mapped to

$$(h_1^{12}) := \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

The operating matrices can be found in the file `main_S3_loc3`, the homomorphisms in `Homs_S3_loc3`.

They can e.g. be used to check the $\mathbb{Z}_{(3)}\mathrm{S}_3$-linearity of the maps between P_1 and P_2 derived above.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 are denoted `rhoP1`, `rhoP2`, respectively.

E.g. for the operating matrices on P_1, call

```
rhoP1(S3P!sigma);
```

for an element `sigma` of S_3. Analogously for P_2.

To check that the matrices found above represent $R\mathrm{S}_3$-linear maps between the respective projective modules, follow these steps:

```
load main_S3_loc3; // load file containing rhoP1 and rhoP2
load Homs_S3_loc3;

[rhoP1(sigma)*Hom_P1P1[i] eq Hom_P1P1[i]*rhoP1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}, i in [1..2]];
[rhoP2(sigma)*Hom_P2P2[i] eq Hom_P2P2[i]*rhoP2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}, i in [1..2]];

[rhoP1(sigma)*Hom_P1P2[i] eq Hom_P1P2[i]*rhoP2(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}, i in [1..1]];
[rhoP2(sigma)*Hom_P2P1[i] eq Hom_P2P1[i]*rhoP1(sigma):sigma in {S3P!(1,2),S3P!(1,2,3)}, i in [1..1]];
```

3.2.2 The tensor product $P_1 \otimes P_1$

Lemma 30 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_1 \longrightarrow P_1 \otimes P_1$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_1 \longrightarrow P_1 \otimes P_1$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_3$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_1 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_2 \oplus P_1 \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_1 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_2 \oplus P_1 \\
\mathfrak{r}P_1 \otimes P_1 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_2 \oplus P_2 \oplus P_1 \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$A := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 2 & 1 & 2 & -1 \\ 0 & -2 & -4 & 0 & 3 & 6 & 3 & 6 & -2 \\ -1 & 1 & 1 & 0 & 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ -2 & 2 & 2 & 3 & -3 & -3 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 & -3 & -6 & 2 \\ -2 & 0 & -1 & 3 & -1 & 0 & 2 & 3 & -1 \\ -6 & 2 & 0 & 9 & -3 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 0 & 0 & 0 & 3 & 6 & -2 \\ -3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 & 0 & -2 & -3 & 1 \\ 0 & 4 & 6 & 0 & 0 & 0 & -6 & -9 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 & -3 & -6 & 2 \\ -2 & 0 & -1 & 1 & -1 & -1 & 2 & 3 & -1 \\ -6 & 2 & 0 & 3 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -2 & -4 & 0 & 1 & 2 & 3 & 6 & -2 \\ -1 & 1 & 1 & 0 & 0 & 0 & -1 & -2 & 1 \\ 2 & -2 & -2 & 0 & 0 & 0 & 2 & 3 & -1 \\ -2 & 2 & 2 & 1 & -1 & -1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 & -3 & -6 & 2 \\ 0 & -2 & -3 & 0 & 0 & 0 & 6 & 9 & -3 \\ -6 & 2 & 0 & 3 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{E} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C := \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) := \left[\begin{array}{cc|c} 1 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \hline \cdot & h_1^{12} & 1 \end{array}\right], \quad \tilde{C} := \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) := \left[\begin{array}{cc|c} 1 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \hline \cdot & \cdot & 1 \end{array}\right].$$

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
We have to show local invertibility and $R\mathrm{S}_3$-linearity of the maps A, B and $\tilde{B}$, the $R\mathrm{S}_3$-linearity of E, $\tilde{E}$, C and $\tilde{C}$, and the commutativity of the diagram. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_3$-linearity and the commutativity can be found in the file `main_S3_loc3`, the matrices for this diagram in the file `Diagram_S3_loc3_P1oP1` .

The embedding E is defined as the Kronecker product of the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$ and id_{P_1}. An embedding in the chosen bases can be found in the files and is denoted `i1`. The embedding $\tilde{E}$ is defined as the Kronecker product of id_{P_1} and the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 are denoted `rhoP1`, `rhoP2` respectively. The representations on $\mathfrak{r}P_1$, $\mathfrak{r}P_2$ are denoted `rhorP1`, `rhorP2`, respectively.

The representations on $P_1 \otimes P_1$, $\mathfrak{r}P_1 \otimes P_1$, $P_1 \otimes \mathfrak{r}P_1$ are denoted `op_p1op1`, `op_rp1op1`, `op_p1orp1` respectively. The operating matrix of a group element on such a tensor product is defined as the tensor product of the operating matrices on the tensor factors.

For example, for the operating matrices on $P_1 \otimes P_1$ call

```
op_p1op1(S3P!sigma);
```

for an element `sigma` of S_3. The other maps work the same way.

The representation on the direct sum $P_2^{\oplus 2} \oplus P_1$ is denoted `op_proj_sum_p1p1`. The operating matrix of a group element is defined as the block diagonal matrix containing the operating matrices of the summands. The maps A, B, C, E are denoted `A`, `B1`, `C1`, `E1`, respectively; the maps $\tilde{B}$, $\tilde{C}$, $\tilde{E}$ are denoted `B2`, `C2`, `E2`, respectively.

To verify the Lemma, follow these steps:

```
load "main_S3_loc3";
load "Diagram_S3_loc3_P1oP1";

 //RS3-Linearity of i1
 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

//generating the embedding E : (𝔯P1 ⊗ P1) ⟶ (P1 ⊗ P1) as (𝔯P1 ⟶ P1) ⊗ 1_P1
E1 := KroneckerProduct(i1,MatrixRing(Rationals(),3)!1);
//generating the embedding Ẽ : (P1 ⊗ 𝔯P1) ⟶ (P1 ⊗ P1) as 1_P1 ⊗ (𝔯P1 ⟶ P1)
E2 := KroneckerProduct(MatrixRing(Rationals(),3)!1,i1);

//RS3-linearity of E, Ẽ
 [op_rp1op1(sigma)*E1 eq E1*op_p1op1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p1orp1(sigma)*E2 eq E2*op_p1op1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

//commutativity:
 E1*A eq B1*C1;      //upper quadrangle
 E2*A eq B2*C2;      //lower quadrangle

//RS3-linearity of A, B, B̃, C, C̃
 [op_p1op1(sigma)*A eq A*op_proj_sum_p1p1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_rp1op1(sigma)*B1 eq B1*op_proj_sum_p1p1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p1orp1(sigma)*B2 eq B2*op_proj_sum_p1p1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p1p1(sigma)*C1 eq C1*op_proj_sum_p1p1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p1p1(sigma)*C2 eq C2*op_proj_sum_p1p1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

//local invertibility of A, B, B̃; loc_inv see "main_S3_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

$\square$

Remark 31 Using Magma, we verify that

$$\begin{array}{ccccccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_1 \otimes D_1 & \cong & D_2\,, & & \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_1 \otimes \bar{P}_1 & \cong & \bar{P}_2\,, \end{array}$$

with Loewy layers already known.

3.2.3 The tensor product $P_2 \otimes P_2$

Lemma 32 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_2 \longrightarrow P_2 \otimes P_2$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_2 \longrightarrow P_2 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_3$-linear maps*

$$\begin{tikzcd}
P_2 \otimes \mathfrak{r}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_2 \oplus P_1 \arrow[d, "\tilde{C}"] \\
P_2 \otimes P_2 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_2 \oplus P_1 \\
\mathfrak{r}P_2 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_2 \oplus P_2 \oplus P_1 \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$A := \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 4 & -2 & 0 & -2 & 0 & 0 \\ 4 & 0 & 0 & 12 & -5 & 0 & -5 & 0 & 0 \\ -2 & 0 & 0 & -6 & 3 & 1 & 3 & 1 & 0 \\ 4 & 0 & 0 & 12 & -6 & 0 & -4 & 0 & 0 \\ 12 & -2 & 0 & 36 & -15 & 0 & -9 & 0 & 0 \\ -6 & 0 & -2 & -18 & 9 & 3 & 9 & 9 & -2 \\ -2 & 0 & 0 & -6 & 3 & 0 & 3 & 2 & -1 \\ -6 & 0 & -2 & -18 & 9 & 3 & 9 & 9 & -4 \\ 4 & 0 & 2 & 12 & -6 & -3 & -6 & -9 & 3 \end{pmatrix}, \quad B := \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & -6 & 2 \\ 4 & 0 & 0 & 4 & -2 & 0 & -4 & 0 & 0 \\ 12 & -2 & 0 & 12 & -5 & 0 & -9 & 0 & 0 \\ -6 & 0 & -2 & -6 & 3 & 1 & 9 & 9 & -2 \\ -2 & 0 & 0 & -2 & 1 & 0 & 3 & 2 & -1 \\ -6 & 0 & -2 & -6 & 3 & 1 & 9 & 9 & -4 \\ 4 & 0 & 2 & 4 & -2 & -1 & -6 & -9 & 3 \end{pmatrix}, \quad \tilde{B} := \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 & 0 & -5 & 0 & 0 \\ -2 & 0 & 0 & -2 & 0 & 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 12 & -2 & 0 & 12 & -2 & 0 & -9 & 0 & 0 \\ -6 & 0 & -2 & -6 & 0 & -2 & 9 & 9 & -2 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & -3 & 1 \\ -6 & 0 & -2 & -6 & 0 & -2 & 9 & 9 & -4 \\ 4 & 0 & 2 & 4 & 0 & 2 & -6 & -9 & 3 \end{pmatrix},$$

$$E := \begin{pmatrix} 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{E} := \begin{pmatrix} 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C := \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) := \left[\begin{array}{cc|c} 1 & . & . \\ . & 3 & . \\ \hline . & . & 1 \end{array}\right], \quad \tilde{C} := \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) := \left[\begin{array}{cc|c} 1 & . & . \\ . & 3 & . \\ \hline . & h_1^{12} & 1 \end{array}\right].$$

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_3$-linearity and the commutativity can be found in the file `main_S3_loc3`, the matrices for this diagram in the file `Diagram_S3_loc3_P2oP2`.

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 30.
To verify the Lemma, follow these steps:

```
load "main_S3_loc3";
load "Diagram_S3_loc3_P2oP2";

//RS3-Linearity of i2
 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

//generating the embedding E : (rP2 ⊗ P2) ⟶ (P2 ⊗ P2) as (rP2 ⟶ P2) ⊗ 1_P2
E1 := KroneckerProduct(i2,MatrixRing(Rationals(),3)!1);
```

```
//generating the embedding $\tilde{E} : (P_2 \otimes \mathfrak{r}P_2) \hookrightarrow (P_2 \otimes P_2)$ as $1_{P_2} \otimes (\mathfrak{r}P_2 \hookrightarrow P_2)$
E2 := KroneckerProduct(MatrixRing(Rationals(),3)!1,i2);

//$R\mathrm{S}_3$-linearity of $E$, $\tilde{E}$
 [op_rp2op2(sigma)*E1 eq E1*op_p2op2(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p2orp2(sigma)*E2 eq E2*op_p2op2(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];

//commutativity:
 E1*A eq B1*C1;    //upper quadrangle
 E2*A eq B2*C2;    //lower quadrangle

//$R\mathrm{S}_3$-linearity of $A$, $B$, $\tilde{B}$, $C$, $\tilde{C}$
 [op_p2op2(sigma)*A eq A*op_proj_sum_p2p2(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_rp2op2(sigma)*B1 eq B1*op_proj_sum_p2p2(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p2orp2(sigma)*B2 eq B2*op_proj_sum_p2p2(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p2p2(sigma)*C1 eq C1*op_proj_sum_p2p2(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p2p2(sigma)*C2 eq C2*op_proj_sum_p2p2(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];

//local invertibility of $A$, $B$, $\tilde{B}$; loc_inv see "main_S3_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

$\square$

Construction 33 Pars pro toto, we will have a closer look at the method how to construct the commutative diagram for $P_2 \otimes P_2$.

The MeatAxe methods implemented in Magma **[3]** allow to find an isomorphism

$$\bar{P}_2 \otimes \bar{P}_2 \xleftarrow{\sim} \bar{P}_2 \oplus \bar{P}_2 \oplus \bar{P}_1 \,.$$

Since $P_2 \oplus P_2 \oplus P_1$ is projective and the residue class map $P_2 \otimes P_2 \longrightarrow \bar{P}_2 \otimes \bar{P}_2$ is surjective, we may lift as follows.

$$\begin{tikzcd}
P_2 \otimes P_2 \arrow[d] & P_2 \oplus P_2 \oplus P_1 \arrow[l, dashed] \arrow[d] \\
\bar{P}_2 \otimes \bar{P}_2 & \bar{P}_2 \oplus \bar{P}_2 \oplus \bar{P}_1 \arrow[l, "\sim"']
\end{tikzcd}$$

A practical solution to let Magma construct such a lift is given in Chapter 7.

Any such lift is an isomorphism; cf. Lemma 214. So we have an isomorphism $P_2 \otimes P_2 \xrightarrow{\sim} P_2 \oplus P_2 \oplus P_1$ at our disposal.

Now we do the same for $\mathfrak{r}P_2 \otimes P_2$. By Lemma 238, we obtain an isomorphism to the same direct sum, i.e. an isomorphism from $\mathfrak{r}P_2 \otimes P_2$ to $P_2 \oplus P_2 \oplus P_1$.

Going back from $P_2 \oplus P_2 \oplus P_1$ to $\mathfrak{r}P_2 \otimes P_2$, embedding into $P_2 \otimes P_2$ and going forward to $P_2 \oplus P_2 \oplus P_1$ again, we obtain an $R\mathrm{S}_3$-linear embedding of $P_2 \oplus P_2 \oplus P_1$ into $P_2 \oplus P_2 \oplus P_1$, which is isomorphic to the embedding of $\mathfrak{r}P_2 \otimes P_2$ into $P_2 \otimes P_2$.

(i)
$$\begin{tikzcd}
P_2 \otimes P_2 \arrow[r, "\sim"] & P_2 \oplus P_2 \oplus P_1 \\
\mathfrak{r}P_2 \otimes P_2 \arrow[u] \arrow[r, "\sim"] \arrow[ur, phantom, "\circlearrowleft"] & P_2 \oplus P_2 \oplus P_1 \arrow[u, dashed]
\end{tikzcd}$$

This embedding of $P_2 \oplus P_2 \oplus P_1$ into $P_2 \oplus P_2 \oplus P_1$ is now given by a 3×3-matrix consisting of $R\mathrm{S}_3$-linear maps between the indecomposable projectives P_1 and P_2 that occur. This already forces the matrix describing this embedding to be of a not too complicated shape.

Now we do the same for $P_2 \otimes \mathfrak{r}P_2$, yielding another embedding of $P_2 \oplus P_2 \oplus P_1$ into $P_2 \oplus P_2 \oplus P_1$, this time isomorphic to the embedding of $P_2 \otimes \mathfrak{r}P_2$ into $P_2 \otimes P_2$.

Altogether, we have $R\mathrm{S}_3$-linear maps

$$\text{(ii)} \qquad (P_2 \oplus P_2 \oplus P_1) \longrightarrow (P_2 \oplus P_2 \oplus P_1) \longleftarrow (P_2 \oplus P_2 \oplus P_1).$$

This diagram is isomorphic to the diagram

$$\text{(iii)} \qquad (\mathfrak{r}P_2 \otimes P_2) \longrightarrow (P_2 \otimes P_2) \longleftarrow (P_2 \otimes \mathfrak{r}P_2)\ .$$

The essentially arbitrary choice of the horizonal isomorphisms in (i) entails that its vertical right hand side map is usually not in a particularly simple shape. So also (ii) is not in a particularly simple shape.

We aim to simplify further, searching for a diagram isomorphic to (ii), hence also to (iii), but with matrices in a simple form.

We may take the two maps in (iii) together to form the map

$$\text{(iii}'\text{)} \qquad (\mathfrak{r}P_2 \otimes P_2) \oplus (P_2 \otimes \mathfrak{r}P_2) \longrightarrow (P_2 \otimes P_2)\ .$$

The cokernel of this map is isomorphic to $D_2 \otimes D_2$ by Lemma 1.

We may also take the two maps in (ii) together to form the map

$$\text{(ii}'\text{)} \qquad (P_2 \oplus P_2 \oplus P_1) \oplus (P_2 \oplus P_2 \oplus P_1) \longrightarrow (P_2 \oplus P_2 \oplus P_1)\ .$$

The cokernel of this map (ii′) is then of course also isomorphic to $D_2 \otimes D_2$.

Further simplification steps, applied to (ii), can also be applied to (ii′) if we use suitable $R\mathrm{S}_3$-linear automorphisms of $P_2 \oplus P_2 \oplus P_1$ on the right and direct sums of two such automorphisms on the left.

There is no general recipe how to simplify. So this has to be done by hand, often using evident simplification steps, but sometimes also using more intricate sequences of steps.

The result of our simplification is the following matrix, where 1 is the identity on P_1 resp. on P_2, and where h_1^{12} is as given in Remark 29.

$$\begin{bmatrix} C \\ \tilde{C} \end{bmatrix} := \left[\begin{array}{cc|c} 1 & . & . \\ . & 3 & . \\ \hline . & . & 1 \\ \hline\hline 1 & . & . \\ . & 3 & . \\ \hline . & h_1^{12} & 1 \end{array}\right] \ : \ (P_2 \oplus P_2 \oplus P_1) \oplus (P_2 \oplus P_2 \oplus P_1) \longrightarrow (P_2 \oplus P_2 \oplus P_1)\ .$$

The cokernel of this map is then of course also isomorphic to $D_2 \otimes D_2$. But it can also be calculated directly to be D_2. So as a consequence, we get $D_2 \otimes D_2 \cong D_2$.

Of course, in this example, we know that D_2 is the trivial module. So $D_2 \otimes D_2$ is also the trivial module. This is confirmed by our calculation – which thus serves mainly to illustrate the method.

Remark 34 Using Magma, we verify again that

$$\begin{array}{lcccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_2 \otimes D_2 & \cong & D_2\,, \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_2 \otimes \bar{P}_2 \ \cong \ \bar{P}_2\,, \end{array}$$

with Loewy layers already known.

3.2.4 The tensor product $P_2 \otimes P_1$

Lemma 35 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_1 \longrightarrow P_2 \otimes P_1$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_1 \longrightarrow P_2 \otimes P_1$. We have a commutative diagram of $\mathbb{Z}_{(3)}S_3$-linear maps*

$$\begin{tikzcd}
P_2 \otimes \mathfrak{r}P_1 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_1 \oplus P_1 \arrow[d, "\tilde{C}"] \\
P_2 \otimes P_1 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_1 \oplus P_1 \\
\mathfrak{r}P_2 \otimes P_1 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_2 \oplus P_1 \oplus P_1 \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$A := \begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & -1 & -2 & 0 & 0 & 1 & 2 & 4 & -2 \\ 6 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -3 & 0 & -2 & 0 & 0 & 0 & 3 & 5 & -2 \\ 0 & -3 & -6 & 0 & 0 & 0 & 6 & 12 & -4 \\ -3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & -2 & -3 & 1 \\ 0 & 2 & 3 & 0 & 0 & 0 & -4 & -6 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 6 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -3 & 0 & -2 & 0 & 0 & 0 & 3 & 5 & -2 \\ 0 & -3 & -6 & 0 & 0 & 0 & 6 & 12 & -4 \\ -3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & -2 & -3 & 1 \\ 0 & 2 & 3 & 0 & 0 & 0 & -4 & -6 & 2 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & -1 & 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & -1 & -2 & -1 & -2 & 1 & 2 & 4 & -2 \\ 6 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -9 & 3 & 0 & 0 & 0 & 0 & 3 & 3 & -2 \\ 0 & -3 & -6 & -3 & -6 & 2 & 6 & 12 & -4 \\ -3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 6 & -2 & 0 & 0 & 0 & 0 & -2 & -3 & 1 \\ 0 & 2 & 3 & 2 & 3 & -1 & -4 & -6 & 2 \end{pmatrix},$$

$$E := \begin{pmatrix} 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{E} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C := \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) := \left[\begin{array}{c|cc} 1 & . & . \\ \hline . & 3 & . \\ . & . & 1 \end{array}\right], \quad \tilde{C} := \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) := \left[\begin{array}{c|cc} 1 & -h_1^{21} & . \\ \hline . & 3 & . \\ . & . & 1 \end{array}\right].$$

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the RS_3-linearity and the commutativity can be found in the file `main_S3_loc3`, the matrices for this diagram in the file `Diagram_S3_loc3_P2oP1` .

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 30 .
To verify the Lemma, follow these steps:

```
load "main_S3_loc3";
load "Diagram_S3_loc3_P2oP1";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma): sigma in {S3P!(1,2), S3P!(1,2,3)}];

E1 := KroneckerProduct(i2,MatrixRing(Rationals(),3)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),3)!1,i1);
```

```
 [op_rp2op1(sigma)*E1 eq E1*op_p2op1(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p2orp1(sigma)*E2 eq E2*op_p2op1(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS3-linearity
 [op_p2op1(sigma)*A eq A*op_proj_sum_p2p1(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_rp2op1(sigma)*B1 eq B1*op_proj_sum_p2p1(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_p2orp1(sigma)*B2 eq B2*op_proj_sum_p2p1(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p2p1(sigma)*C1 eq C1*op_proj_sum_p2p1(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];
 [op_proj_sum_p2p1(sigma)*C2 eq C2*op_proj_sum_p2p1(sigma):sigma in {S3P!(1,2), S3P!(1,2,3)}];

//local invertibility; loc_inv see "main_S3_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

□

Remark 36 Using Magma, we verify that

$$\begin{array}{rcccl} \mathrm{Coker}(\left[\begin{array}{c} C \\ \tilde{C} \end{array}\right]) & \cong & D_2 \otimes D_1 & \cong & D_1 \\ \mathrm{Coker}(C) & \cong & D_2 \otimes \bar{P}_1 & \cong & \bar{P}_1 \\ \mathrm{Coker}(\tilde{C}) & \cong & \bar{P}_2 \otimes D_1 & \cong & \bar{P}_1 \end{array}$$

with Loewy layers already known.

3.3 An isomorphic replacement of the comultiplication on $\mathbb{Z}_{(3)}\mathrm{S}_3$

3.3.1 Replacing Δ by Δ'

Definition 37 Let $\Xi := \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)}^{4\times 4} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)}$.

A basis of $\Lambda_{(3)}$ is given by

$$\Big((1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0),\ (0, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, 0),\ (0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0),$$
$$(0, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, 0),\ (0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1),\ (0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3) \Big) .$$

Consider the following injective morphism of $\mathbb{Z}_{(3)}$-algebras.

$$\begin{array}{ccc} \Lambda_{(3)} & \xrightarrow{\Delta'_\Xi} & \Xi \\ (a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f) & \longmapsto & \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, a\right). \end{array}$$

Let $\omega_{(3)}$ be the Wedderburn isomorphism given in Definition 21.

Let Δ be the comultiplication

$$\begin{array}{ccc} \mathbb{Z}_{(3)}\mathrm{S}_3 & \xrightarrow{\Delta} & \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \\ \sum_{\sigma\in \mathrm{S}_3} a_\sigma\, \sigma & \longmapsto & \sum_{\sigma\in \mathrm{S}_3} a_\sigma\, \sigma \otimes \sigma\, , \end{array}$$

where $a_\sigma \in \mathbb{Z}_{(3)}$, which is a morphism of $\mathbb{Z}_{(3)}$-algebras.

Definition 38 The morphism of $\mathbb{Z}_{(3)}$-algebras Ω_Ξ is given on generators by

$$
\begin{array}{rcl}
\mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 & \xrightarrow{\Omega_\Xi} & \Xi \\
(1,2)\otimes \mathrm{id} & \longmapsto & \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 \\ 2 & 0 & 0 & 1 \\ -6 & 0 & 0 & -1 \\ 0 & 6 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}, -1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, -1\right) \\
(2,3)\otimes \mathrm{id} & \longmapsto & \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, -1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, -1\right), \\
\mathrm{id}\otimes(1,2) & \longmapsto & \left(1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ -6 & 0 & 0 & 1 \\ 0 & -6 & -2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1, \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}, -1\right) \\
\mathrm{id}\otimes(2,3) & \longmapsto & \left(1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, -1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, -1\right).
\end{array}
$$

Note that

$$
\begin{array}{rcl}
(1,2)\otimes(1,2) & \overset{\Omega_\Xi}{\longmapsto} & \left(1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, 1\right) \\
(2,3)\otimes(2,3) & \overset{\Omega_\Xi}{\longmapsto} & \left(1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, -1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, -1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, 1\right).
\end{array}
$$

Lemma 39 *Let $\Gamma := \mathrm{Im}\,\Omega_\Xi \subseteq \Xi$, which is a $\mathbb{Z}_{(3)}$-subalgebra.*

We have the following description by ties.

$$
\begin{array}{rcl}
\Gamma & = & \Bigg\{\left(a_{11}, \begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} d_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} & 0 & 0 \\ e_{21} & e_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} f_{11} & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} h_{11} & 0 \\ 0 & 0 \end{pmatrix}, i_{11}\right) \in \Xi \mid \\
& & a_{11} - b_{11} \equiv_9 d_{11} + e_{11} - 2e_{12} + e_{21} - 2e_{22} \equiv_3 0, \quad a_{11} \equiv_3 d_{11}, \\
& & h_{11} - i_{11} \equiv_9 2e_{11} - 2e_{12} + e_{21} - e_{22} - f_{11} \equiv_3 0, \quad f_{11} \equiv_3 i_{11}, \\
& & e_{11} \equiv_3 e_{22}, \quad e_{12} \equiv_3 e_{21} \Bigg\} \\
& \oplus & \Bigg\{\left(0, \begin{pmatrix} 0 & 0 \\ 0 & b_{22} \end{pmatrix}, c_{11}, \begin{pmatrix} 0 & 0 \\ 0 & d_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_{33} & e_{34} \\ 0 & 0 & e_{43} & e_{44} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & f_{22} \end{pmatrix}, g_{11}, \begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}, 0\right) \in \Xi \mid \\
& & b_{22} - c_{11} \equiv_9 -e_{33} + 2e_{34} - e_{43} + 2e_{44} - f_{22} \equiv_3 0, \quad c_{11} \equiv_3 f_{22}, \\
& & g_{11} - h_{22} \equiv_9 d_{22} - 2e_{33} + 2e_{34} - e_{43} + e_{44} \equiv_3 0, \quad d_{22} \equiv_3 g_{11}, \\
& & e_{33} \equiv_3 e_{44}, \quad e_{34} \equiv_3 e_{43} \Bigg\} \\
& \oplus & \Bigg\{\left(0, \begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & d_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & e_{13} & e_{14} \\ 0 & 0 & e_{23} & e_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & f_{12} \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & h_{12} \\ 0 & 0 \end{pmatrix}, 0\right) \in \Xi \mid \\
& & b_{12} \equiv_3 e_{13} + e_{14} + e_{23} + e_{24}, \\
& & d_{12} \equiv_3 e_{13} - e_{14} + e_{23} - e_{24}, \\
& & f_{12} \equiv_3 -e_{13} - e_{14} + e_{23} + e_{24}, \\
& & h_{12} \equiv_3 -e_{13} + e_{14} + e_{23} - e_{24} \Bigg\}
\end{array}
$$

$$\begin{aligned}\oplus \quad \Bigg\{ \Bigg(0, \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & 0 \\ d_{21} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_{31} & e_{32} & 0 & 0 \\ e_{41} & e_{42} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ f_{21} & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & 0 \\ h_{21} & 0 \end{pmatrix}, 0 \Bigg) \in \Xi \mid \\ b_{21} \equiv_9 e_{31} - 2e_{32} + e_{41} - 2e_{42} \equiv_3 0, \\ d_{21} \equiv_9 -2e_{31} + 4e_{32} - e_{41} + 2e_{42} \equiv_3 0, \\ f_{21} \equiv_9 -e_{31} + e_{32} - e_{41} + e_{42} \equiv_3 0, \\ h_{21} \equiv_9 2e_{31} - 2e_{32} + e_{41} - e_{42} \equiv_3 0 \Bigg\}\end{aligned}$$

Moreover, Ω_Ξ is injective.

Proof. Let Γ' be the $\mathbb{Z}_{(3)}$-submodule of Ξ occurring as the right hand side of the assertion.

Write elements of Ξ as row vectors with entries in $\mathbb{Z}_{(3)}$, i.e. identify $\Xi = \mathbb{Z}_{(3)}^{1\times 36}$.

Let $T \in \mathbb{Z}_{(3)}^{36\times 28}$ be such that for $\xi \in \Xi$, we have $\xi \in \Gamma'$ if and only if $\xi T \in 9 \cdot \mathbb{Z}_{(3)}^{1\times 28}$. I.e. the columns of T contain the coefficients of the ties defining Γ', written as congruences modulo 9.

The chosen $\mathbb{Z}_{(3)}$-linear basis of $\Lambda_{(3)}$ in Definition 37 yields a $\mathbb{Z}_{(3)}$-linear basis of $\mathbb{Z}_{(3)}\mathrm{S}_3$ via $\omega_{(3)}^{-1}$. Tensoring this basis with itself, we obtain a $\mathbb{Z}_{(3)}$-linear basis of $\mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3$, which we choose.

Let $M \in \mathbb{Z}_{(3)}^{36\times 36}$ be the describing matrix of the $\mathbb{Z}_{(3)}$-linear map Ω_Ξ of Definition 38, with respect to the chosen basis of $\mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3$ and the standard basis of Ξ.

Then

$$M \cdot T \in 9\mathbb{Z}_{(3)}^{36\times 28} \quad ,$$

which shows that $\operatorname{Im}\Omega_\Xi \subseteq \Gamma'$. The index of Γ' equals the index of $\operatorname{Im}\Omega_\Xi$, so that $\operatorname{Im}\Omega_\Xi = \Gamma'$. In particular, Γ' is a $\mathbb{Z}_{(3)}$-subalgebra of Ξ.

To verify the details using Magma **[3]**, we proceed as follows.

```
load "Ties_replacement";

//M · T ∈ 9Z_(3)
//loc_int verifies that a matrix has entries in Z_(3), see file  "Ties_replacement"
 loc_int(1/9*M*T,3);

//index of Im Ω_Ξ is 3^36
 Valuation(Determinant(M),3);

//elementary divisors of T
 [SmithForm(RMatrixSpace(Integers(),36,28)!T)[i,i]:i in [1..NumberOfColumns(T)]];
```

The elementary divisors of T are $1^8 \cdot 3^{20}$. Therefore, the elementary divisors of the embedding of Γ' in Ξ are $9^8 \cdot 3^{20}$. Hence the index of Γ' in Ξ is $3^{2\cdot 8+20} = 3^{36}$.

Finally, $\mathrm{rk}_{\mathbb{Z}_{(3)}}(\mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3) = 6 \cdot 6 = 36 = \mathrm{rk}_{\mathbb{Z}_{(3)}}\,\Xi = \mathrm{rk}_{\mathbb{Z}_{(3)}}\operatorname{Im}\Omega_\Xi$, so that Ω_Ξ is injective. □

Denote the embedding of Γ in Ξ by

$$\Gamma \xrightarrow{i} \Xi \,.$$

Remark 40 By restricting Ω_Ξ in the codomain to Γ, we get an isomorphism

$$\Omega \;:=\; \Omega_\Xi|^\Gamma \;:\; \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \;\xrightarrow{\sim}\; \Gamma \,.$$

$$\begin{tikzcd} \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[r, "\Omega", "\sim"'] \arrow[rr, bend left, "\Omega_\Xi"] & \Gamma \arrow[r, "i"] & \Xi \end{tikzcd}$$

Remark 41 We have $\operatorname{Im}\Delta'_{\Xi} \subseteq \Gamma$, so that we may write

$$\Delta' \;:=\; \Delta'_{\Xi}|^{\Gamma} \;:\; \Lambda_{(3)} \longrightarrow \Gamma\,.$$

$$\begin{tikzcd} \Lambda_{(3)} \arrow[r, "\Delta'"] \arrow[rr, bend left, "\Delta'_{\Xi}"] & \Gamma \arrow[r, "i"] & \Xi \end{tikzcd}$$

Note that the $\mathbb{Z}_{(3)}$-algebra morphism Δ' is only injective.

Proof. We check the ties of Γ given in Lemma 39 for an element of the form

$$\left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, a\right),$$

where $(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f) \in \Lambda_{(3)}$, i.e. where $a \equiv_3 b$, $d \equiv_3 0$ and $e \equiv_3 f$. □

Proposition 42 *We have the commutative diagram of $\mathbb{Z}_{(3)}$-algebras*

$$\begin{tikzcd} \mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[r, "\Delta"] \arrow[d, "\omega_{(3)}", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[d, "\Omega", "\wr"'] \\ \Lambda_{(3)} \arrow[r, "\Delta'"] & \Gamma\,. \end{tikzcd}$$

Proof. We only have to prove commutativity on generators of S_3. We verify that

$$(1,2) \underset{\text{D.37}}{\overset{\Delta}{\longmapsto}} (1,2)\otimes(1,2) \underset{\text{D.38}}{\overset{\Omega}{\longmapsto}} (1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, 1)$$

$$(1,2) \underset{\text{D.21}}{\overset{\omega_{(3)}}{\longmapsto}} (1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1) \underset{\text{D.37}}{\overset{\Delta'}{\longmapsto}} (1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, -1, \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, 1),$$

$$(2,3) \underset{\text{D.37}}{\overset{\Delta}{\longmapsto}} (2,3)\otimes(2,3) \underset{\text{D.38}}{\overset{\Omega}{\longmapsto}} (1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, -1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} -1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, 1)$$

$$(2,3) \underset{\text{D.21}}{\overset{\omega_{(3)}}{\longmapsto}} (1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, -1) \underset{\text{D.37}}{\overset{\Delta'}{\longmapsto}} (1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, -1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} -1, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, 1)\,.$$

Equality of the respective images shows that $\Delta\Omega = \omega_{(3)}\Delta'$; cf. Remarks 40, 41. □

Lemma 43 *We have the orthogonal decomposition*

$$1_\Gamma \;=\; \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$$

into primitive idempotents of Γ, where

$$\begin{aligned} \varepsilon_1 \;&:=\; \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1\right), \\ \varepsilon_2 \;&:=\; \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \\ \varepsilon_3 \;&:=\; \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0\right), \end{aligned}$$

$$\varepsilon_4 \;:=\; \left(0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 1, \begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&-1\\0&0&2&2\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0\right).$$

In the process, this orthogonal decomposition has been used to decompose the ties of Γ.

Proof. We show that ε_2 is primitive. Primitivity of ε_1, ε_3 and ε_4 follows similarly.

A possible orthogonal decomposition of ε_2 necessarily lies in $\varepsilon_2\Gamma\varepsilon_2$. Since, for x, y, z, $w \in \mathbb{Z}_{(3)}$, we have

$$\begin{pmatrix}-1&-1\\2&2\end{pmatrix}\cdot\begin{pmatrix}x&y\\z&w\end{pmatrix}\cdot\begin{pmatrix}-1&-1\\2&2\end{pmatrix} \;\in\; \left\{ \begin{pmatrix}-u&-u\\2u&2u\end{pmatrix} \;\middle|\; u\in\mathbb{Z}_{(3)} \right\},$$

each element of $\varepsilon_2\Gamma\varepsilon_2$ is of the form

$$\left(a_{11}, \begin{pmatrix}b_{11}&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}d_{11}&0\\0&0\end{pmatrix}, \begin{pmatrix}e_{11}&e_{11}&0&0\\-2e_{11}&-2e_{11}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0\right),$$

where a_{11}, b_{11}, d_{11}, $e_{11} \in \mathbb{Z}_{(3)}$.

We obtain the list of ties

$$\begin{aligned}&a_{11}-b_{11}\equiv_9 d_{11}+e_{11}\equiv_3 0,\\ &a_{11}\equiv_3 d_{11}.\end{aligned}$$

For $\Gamma_{22} := \{\,(a,b,d,e)\in\mathbb{Z}_{(3)}^{\times 4} \mid a-b\equiv_9 d-e\equiv_3 0,\ a\equiv_3 d\,\}$, define the $\mathbb{Z}_{(3)}$-linear isomorphism γ by

$$\begin{array}{ccc}\varepsilon_2\Gamma\varepsilon_2 & \xrightarrow[\sim]{\gamma} & \Gamma_{22}\\ \left(a, \begin{pmatrix}b&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}d&0\\0&0\end{pmatrix}, \begin{pmatrix}-e&-e&0&0\\2e&2e&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0\right) & \longmapsto & (a,b,d,e).\end{array}$$

Now γ is a $\mathbb{Z}_{(3)}$-algebra isomorphism.

It remains to show that $1_{\Gamma_{22}} = (1,1,1,1)$ is a primitive idempotent in Γ_{22}.

Let

$$1_{\Gamma_{22}} = (1,1,1,1) = (u,v,w,x) + (1-u, 1-v, 1-w, 1-x)$$

be an orthogonal decomposition into idempotents.

Suppose $(u,v,w,x) \neq (0,0,0,0)$. Then $u^2 = u$, i.e. $u(u-1)=0$ in $\mathbb{Z}_{(3)}$. Therefore $u\in\{0,1\}$. Repeating the argument, we obtain $v,w,x\in\{0,1\}$.

Without loss of generality, let $u=0$. Because $u\equiv_3 v\equiv_3 w\equiv_3 x$ it follows that $u=v=w=x=0$. Therefore $(u,v,w,x) = 0_{\Gamma_{22}}$, and $(1,1,1,1)$ is a primitive idempotent. $\square$

Remark 44 There are infinitly many orthogonal decompositions into primitive idempotents of Γ of the form

$$\begin{aligned}1_\Gamma \;=\;& \left(1, \begin{pmatrix}1&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}a&a&0&0\\1-a&1-a&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0\right)\\ +\;& \left(0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1-a&-a&0&0\\a-1&a&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}1&0\\0&0\end{pmatrix}, 1\right)\\ +\;& \left(0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 1, \begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&b&b\\0&0&1-b&1-b\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0\right)\\ +\;& \left(0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1-b&-b\\0&0&b-1&b\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 1, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 0\right)\end{aligned}$$

for any a, $b \in \mathbb{Z}_{(3)}$ with $a\equiv_3 -1$ and $b\equiv_3 -1$. But there does not seem to exist a simpler looking orthogonal decomposition than $1_\Gamma = \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4$.

Remark 45 The idempotents e_1 and e_2 of $\Lambda_{(3)}$ given in Definition 21 are mapped to the idempotents

$$
\begin{array}{rcll}
e_1 = \left(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right) & \longmapsto & \left(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), 0\right) & := \tilde{e}_1 \\
e_2 = \left(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0\right) & \longmapsto & \left(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), 1\right) & := \tilde{e}_2
\end{array}
$$

of Γ under the map Δ' given in Definition 37.

Orthogonal decompositions of $\tilde{e}_1$ and $\tilde{e}_2$ into the primitive idempotents of Lemma 43 are given by

$$
\begin{array}{rcl}
\tilde{e}_1 & = & \varepsilon_3 + \varepsilon_4 \\
\tilde{e}_2 & = & \varepsilon_1 + \varepsilon_2 \, .
\end{array}
$$

3.3.2 Side remarks on the construction

Definition 46 Let

$$
\begin{array}{rcll}
\Lambda_{(3)} & \otimes & \Lambda_{(3)} & \overset{\mu_\Xi}{\longrightarrow} \quad \Xi \\
\left(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f\right) & \otimes & \left(a', \left(\begin{smallmatrix} b' & c' \\ d' & e' \end{smallmatrix}\right), f'\right) & \longmapsto
\end{array}
$$

$$
\begin{aligned}
&\left(a \otimes a', a \otimes \left(\begin{smallmatrix} b' & c' \\ d' & e' \end{smallmatrix}\right), a \otimes f', \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right) \otimes a', \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} b' & c' \\ d' & e' \end{smallmatrix}\right), \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right) \otimes f', f \otimes a', f \otimes \left(\begin{smallmatrix} b' & c' \\ d' & e' \end{smallmatrix}\right), f \otimes f'\right) \\
&= \left(aa', \left(\begin{smallmatrix} ab' & ac' \\ ad' & ae' \end{smallmatrix}\right), af', \left(\begin{smallmatrix} ba' & ca' \\ da' & ea' \end{smallmatrix}\right), \left(\begin{smallmatrix} bb' & bc' & cb' & cc' \\ bd' & be' & cd' & ce' \\ db' & dc' & eb' & ec' \\ dd' & de' & ed' & ee' \end{smallmatrix}\right), \left(\begin{smallmatrix} bf' & cf' \\ df' & ef' \end{smallmatrix}\right), fa', \left(\begin{smallmatrix} fb' & fc' \\ fd' & fe' \end{smallmatrix}\right), ff'\right) .
\end{aligned}
$$

This is an injective $\mathbb{Z}_{(3)}$-algebra morphism.

Further, let

$$
\tilde{\Omega}_\Xi \; := \; (\omega_{(3)} \otimes \omega_{(3)}) \cdot \mu_\Xi \; : \; \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \; \longrightarrow \; \Xi \, .
$$

It is an injective morphism of $\mathbb{Z}_{(3)}$-algebras mapping the generators as follows.

$$
\begin{array}{lcl}
\mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 & \overset{\tilde{\Omega}_\Xi}{\longrightarrow} & \Xi \\
(1,2) \otimes \mathrm{id} & \longmapsto & \left(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} -2 & -1 \\ 3 & 2 \end{smallmatrix}\right), \begin{pmatrix} -2&0&-1&0 \\ 0&-2&0&-1 \\ 3&0&2&0 \\ 0&3&0&2 \end{pmatrix}, \left(\begin{smallmatrix} -2 & -1 \\ 3 & 2 \end{smallmatrix}\right), -1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), -1\right) \\
(2,3) \otimes \mathrm{id} & \longmapsto & \left(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right), \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}, \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right), -1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), -1\right), \\
\mathrm{id} \otimes (1,2) & \longmapsto & \left(1, \left(\begin{smallmatrix} -2 & -1 \\ 3 & 2 \end{smallmatrix}\right), -1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} -2&-1&0&0 \\ 3&2&0&0 \\ 0&0&-2&-1 \\ 0&0&3&2 \end{pmatrix}, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} -2 & -1 \\ 3 & 2 \end{smallmatrix}\right), -1\right) \\
\mathrm{id} \otimes (2,3) & \longmapsto & \left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right), -1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} 1&1&0&0 \\ 0&-1&0&0 \\ 0&0&1&1 \\ 0&0&0&-1 \end{pmatrix}, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right), -1\right) .
\end{array}
$$

Definition 47 Let

$$
\tilde{\Delta} \; := \; \omega_{(3)}^{-1} \cdot \Delta \cdot (\omega_{(3)} \otimes \omega_{(3)}) \; : \; \Lambda_{(3)} \; \longrightarrow \; \Lambda_{(3)} \otimes \Lambda_{(3)} \, .
$$

Let

$$
\tilde{\Delta}'_\Xi \; := \; \omega_{(3)}^{-1} \cdot \Delta \cdot \tilde{\Omega}_\Xi \; = \; \tilde{\Delta} \cdot \mu_\Xi \; : \; \Lambda_{(3)} \; \longrightarrow \; \Xi \, .
$$

Let $\tilde{\Gamma} := \operatorname{Im} \tilde{\Omega}_\Xi \subseteq \Xi$. Write $\tilde{\Omega} := \tilde{\Omega}_\Xi|^{\tilde{\Gamma}}$ and $\tilde{\Delta}' := \tilde{\Delta}'_\Xi|^{\tilde{\Gamma}}$ and $\mu := \mu_\Xi|^{\tilde{\Gamma}}$.

We obtain the commutative diagram of $\mathbb{Z}_{(3)}$-algebras

$$\begin{tikzcd}
\mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[rr, "\Delta"] \arrow[d, "\omega_{(3)}"', "\wr"] & & \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[d, "\omega_{(3)}\otimes\omega_{(3)}"', "\wr"] \arrow[dd, bend left, "\tilde{\Omega}", "\wr"'] \\
\Lambda_{(3)} \arrow[rr, "\tilde{\Delta}"] \arrow[drr, "\tilde{\Delta}'"] & & \Lambda_{(3)} \otimes \Lambda_{(3)} \arrow[d, "\mu"', "\wr"] \\
& & \tilde{\Gamma}
\end{tikzcd}$$

Remark 48 We obtain

$$\begin{aligned}
\tilde{\Gamma} \;=\; \Bigg\{ \Bigg(& a_{11}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, c_{11}, \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}, \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, g_{11}, \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, i_{11} \Bigg) \in \Xi \;\Bigg| \\
& a_{11} + e_{11} \equiv_9 b_{11} + d_{11}\,, \quad a_{11} \equiv_3 b_{11} \equiv_3 e_{11} \equiv_3 d_{11}\,, \\
& b_{22} + f_{11} \equiv_9 c_{11} + e_{22}\,, \quad b_{22} \equiv_3 c_{11} \equiv_3 f_{11} \equiv_3 e_{22}\,, \\
& d_{22} + h_{11} \equiv_9 e_{33} + g_{11}\,, \quad d_{22} \equiv_3 e_{33} \equiv_3 h_{11} \equiv_3 g_{11}\,, \\
& e_{44} + i_{11} \equiv_9 f_{22} + h_{22}\,, \quad e_{44} \equiv_3 i_{11} \equiv_3 f_{22} \equiv_3 h_{22}\,, \\
& b_{21} \equiv_9 e_{21} \equiv_3 0\,, \\
& d_{21} \equiv_9 e_{31} \equiv_3 0\,, \\
& e_{42} \equiv_9 f_{21} \equiv_3 0\,, \\
& e_{43} \equiv_9 h_{21} \equiv_3 0\,, \\
& b_{12} \equiv_3 e_{12}\,, \\
& d_{12} \equiv_3 e_{13}\,, \\
& e_{24} \equiv_3 f_{12}\,, \\
& e_{34} \equiv_3 h_{12}\,, \\
& e_{23} \equiv_3 0\,, \\
& e_{32} \equiv_3 0\,, \\
& e_{41} \equiv_9 0 \Bigg\}\,.
\end{aligned}$$

Note that the ties of $\tilde{\Gamma}$ are very simple, but that $(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f)\tilde{\Delta}'$ is of a complicated shape; cf. Remark 49 below.

Proof. Let $\tilde{\Gamma}'$ be the $\mathbb{Z}_{(3)}$-submodule of Ξ occurring as the right hand side of the assertion.

Write elements of Ξ as row vectors with entries in $\mathbb{Z}_{(3)}$, i.e. identify $\Xi = \mathbb{Z}_{(3)}^{1\times 36}$.

Let $\tilde{T} \in \mathbb{Z}_{(3)}^{36\times 31}$ be such that for $\xi \in \Xi$, we have $\xi \in \Gamma'$ if and only if $\xi\tilde{T} \in 9 \cdot \mathbb{Z}_{(3)}^{1\times 31}$. I.e. the columns of $\tilde{T}$ contain the coefficients of the ties defining $\tilde{\Gamma}'$, written as congruences modulo 9.

The chosen $\mathbb{Z}_{(3)}$-linear basis of $\Lambda_{(3)}$ in Definition 37 yields a $\mathbb{Z}_{(3)}$-linear basis of $\mathbb{Z}_{(3)}\mathrm{S}_3$ via $\omega_{(3)}^{-1}$. Tensoring this basis with itself, we obtain a $\mathbb{Z}_{(3)}$-linear basis of $\mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3$, which we choose.

Let $\tilde{M} \in \mathbb{Z}_{(3)}^{36\times 36}$ be the describing matrix of the $\mathbb{Z}_{(3)}$-linear map $\tilde{\Omega}_\Xi$ of Definition 46, with respect to the chosen basis of $\mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3$ and the standard basis of Ξ.

Then

$$\tilde{M} \cdot \tilde{T} \in 9\mathbb{Z}_{(3)}^{36\times 31} \quad ,$$

which shows that $\operatorname{Im}\tilde{\Omega}_\Xi \subseteq \tilde{\Gamma}'$. The index of $\tilde{\Gamma}'$ equals the index of $\operatorname{Im}\tilde{\Omega}_\Xi$, so that $\operatorname{Im}\tilde{\Omega}_\Xi = \tilde{\Gamma}'$. In particular, $\tilde{\Gamma}'$ is a $\mathbb{Z}_{(3)}$-subalgebra of Ξ.

To verify the details using Magma, we proceed as follows.

```
load "Ties_replacement_2";

//$\tilde{M}\cdot\tilde{T} \in 9\mathbb{Z}_{(3)}$
//loc_int verifies that a matrix has entries in $\mathbb{Z}_{(3)}$, see file  "Ties_replacement_2"
 loc_int(1/9*M*T,3);

//index of $\mathrm{Im}\,\tilde{\Omega}$ is $3^{36}$
 Valuation(Determinant(M),3);

//elementary divisors of $\tilde{T}$
 [SmithForm(RMatrixSpace(Integers(),36,31)!T)[i,i]:i in [1..NumberOfColumns(T)]];
```

The elementary divisors of $\tilde{T}$ are $1^9 \cdot 3^{18}$. Therefore, the elementary divisors of the embedding of $\tilde{\Gamma}'$ in Ξ are $9^9 \cdot 3^{18}$. Hence the index of $\tilde{\Gamma}'$ in Ξ is $3^{2\cdot 9+18} = 3^{36}$. $\square$

Remark 49 Consider $(a,\, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right),\, f) \in \Lambda_{(3)}$. We have

$$
\begin{array}{l}
(a,\, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right),\, f)\tilde{\Delta}' \;= \\[1ex]
\quad (a,\, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right),\, f,\, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), \\
\quad \begin{pmatrix}
2a-b & a-b+c & a-b+c & \frac{2}{3}a-\frac{2}{3}b+c \\
-3a+3b+d & -\frac{3}{2}a+3b-3c+d-e+\frac{1}{2}f & -\frac{3}{2}a+3b-3c+d-e-\frac{1}{2}f & -a+2b-3c+\frac{2}{3}d-e \\
-3a+3b+d & -\frac{3}{2}a+3b-3c+d-e-\frac{1}{2}f & -\frac{3}{2}a+3b-3c+d-e+\frac{1}{2}f & -a+2b-3c+\frac{2}{3}d-e \\
6a-6b-3d & 3a-6b+6c-3d+3e & 3a-6b+6c-3d+3e & 2a-4b+6c-2d+3e
\end{pmatrix}, \\
\quad \begin{pmatrix} -3b+6c-2d+4e & -2b+3c-\frac{4}{3}d+2e \\ 6b-12c+3d-6e & 4b-6c+2d-3e \end{pmatrix},\, f,\, \begin{pmatrix} -3b+6c-2d+4e & -2b+3c-\frac{4}{3}d+2e \\ 6b-12c+3d-6e & 4b-6c+2d-3e \end{pmatrix},\, a)\,.
\end{array}
$$

Remark 50 We may now conjugate the matrices from the left to receive a simpler form.

Conjugation with

$$
(1,\, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\, 1,\, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\, \frac{1}{6}\begin{pmatrix} 6 & 3 & 3 & 2 \\ 6 & 6 & 6 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & -6 & 6 & 0 \end{pmatrix},\, \begin{pmatrix} 3 & 2 \\ -6 & -3 \end{pmatrix},\, 1,\, \begin{pmatrix} 3 & 2 \\ -6 & -3 \end{pmatrix},\, 1)
$$

from the left defines the $\mathbb{Q}$-algebra isomorphism

$$
\kappa_{\mathbb{Q}\Xi} \;:\; \mathbb{Q}\Xi \;\xrightarrow{\sim}\; \mathbb{Q}\Xi\,.
$$

We notice that we may restrict to the isomorphism

$$
\kappa \;:=\; \kappa_{\mathbb{Q}\Xi}\big|_{\tilde{\Gamma}}^{\Gamma} \;:\; \tilde{\Gamma} \;\xrightarrow{\sim}\; \Gamma\,;
$$

cf. Remark 48, Lemma 39.

For $(a,\, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right),\, f) \in \Lambda_{(3)}$, we obtain

$$
\begin{array}{l}
(a,\, \begin{pmatrix} b & c \\ d & e \end{pmatrix},\, f)\tilde{\Delta}'\kappa = \\
\left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, a\right).
\end{array}
$$

So $\tilde{\Delta}'\kappa = \Delta'$; cf. Remark 41, Definition 37.

In fact, the map Ω introduced in Remark 40, Definition 38 has originally been obtained as $\Omega = \tilde{\Omega} \cdot \kappa$.

So altogether, we have a commutative diagram of $\mathbb{Z}_{(3)}$-algebras

$$\begin{tikzcd}
\mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[r, "\Delta"] \arrow[d, "\omega_{(3)}"', "\wr"] & \mathbb{Z}_{(3)}\mathrm{S}_3 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 \arrow[d, "\omega_{(3)}\otimes\omega_{(3)}"', "\wr"] \arrow[dd, bend left, "\wr\ \tilde{\Omega}"] \arrow[ddd, bend left=60, "\wr\ \Omega"] \\
\Lambda_{(3)} \arrow[r, "\tilde{\Delta}"] \arrow[dr, "\tilde{\Delta}'"] \arrow[ddr, "\Delta'"'] & \Lambda_{(3)} \otimes \Lambda_{(3)} \arrow[d, "\mu"', "\wr"] \\
& \tilde{\Gamma} \arrow[d, "\kappa"', "\wr"] \\
& \Gamma\,,
\end{tikzcd}$$

cf. Definitions 46, 47.

3.3.3 Multiplicities via Δ'

Remark 51 Suppose given a ring A and an idempotent e of A. Let $u \in \mathrm{U}(A)$. Then we have the A-linear isomorphism

$$\begin{array}{rcccccc}
 & & eA & \xrightarrow{\sim} & ueu^{-1}A & & \\
 & & ea & \longmapsto & uea & = & ueu^{-1}ua \\
eu^{-1}b & = & u^{-1}ueu^{-1}b & \longleftarrow\!\shortmid & ueu^{-1}b\,. & &
\end{array}$$

Remark 52 Let e_i be the primitive idempotents of $\Lambda_{(3)}$ given in Definition 21 for $i \in \{1, 2\}$. Suppose given any idempotents e, f of $\Lambda_{(3)}$, not necessarily primitive, and write $e\Lambda_{(3)} \otimes f\Lambda_{(3)} \cong \bigoplus_j P_j^{\oplus\mu_j}$.

We want to calculate the multiplicities μ_j using Δ'.

Let $C := (c_{ij})_{i,j}$ be the Cartan matrix defined in Chapter E.

We have the restriction functor that restricts a $\Lambda_{(3)} \otimes \Lambda_{(3)}$-module along $\tilde{\Delta}$ to obtain a $\Lambda_{(3)}$-module.

Conversely, we have the tensor functor $- \otimes_{\Lambda_{(3)}} (\Lambda_{(3)} \otimes \Lambda_{(3)})$, which applied to a $\Lambda_{(3)}$-module yields a $\Lambda_{(3)} \otimes \Lambda_{(3)}$-module.

This tensor functor is left adjoint to this restriction functor.

In the following calculation (only), we shall distinguish between the $\Lambda_{(3)} \otimes \Lambda_{(3)}$-module $e\Lambda_{(3)} \otimes f\Lambda_{(3)}$ and the $\Lambda_{(3)}$-module $(e\Lambda_{(3)} \otimes f\Lambda_{(3)})|_{\Lambda_{(3)}}$.

We have

$$\begin{array}{rcl}
\sum\limits_j c_{ij}\mu_j & = & \sum\limits_j \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Lambda_{(3)}}(P_i\,, P_j) \cdot \mu_j \\
 & = & \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Lambda_{(3)}}(P_i\,, \bigoplus\limits_j P_j^{\oplus\mu_j}) \\
 & = & \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Lambda_{(3)}}\big(e_i\Lambda_{(3)}\,, (e\Lambda_{(3)} \otimes f\Lambda_{(3)})|_{\Lambda_{(3)}}\big) \\
 & = & \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Lambda_{(3)}\otimes\Lambda_{(3)}}\big(e_i\Lambda_{(3)} \otimes_{\Lambda_{(3)}} (\Lambda_{(3)} \otimes \Lambda_{(3)})\,, e\Lambda_{(3)} \otimes f\Lambda_{(3)}\big) \\
 & = & \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Lambda_{(3)}\otimes\Lambda_{(3)}}\big(e_i\Lambda_{(3)} \otimes_{\Lambda_{(3)}} (\Lambda_{(3)} \otimes \Lambda_{(3)})\,, (e \otimes f)(\Lambda_{(3)} \otimes \Lambda_{(3)})\big) \\
 & = & \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Lambda_{(3)}\otimes\Lambda_{(3)}}\big(e_i\tilde{\Delta}(\Lambda_{(3)} \otimes \Lambda_{(3)})\,, (e \otimes f)(\Lambda_{(3)} \otimes \Lambda_{(3)})\big)
\end{array}$$

$$
\begin{aligned}
&= \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Gamma}\big(e_i\tilde{\Delta}\mu\kappa\Gamma\,,\,(e\otimes f)\mu\kappa\Gamma\big)\\
&= \mathrm{rk}_{\mathbb{Z}_{(3)}} \mathrm{Hom}_{\Gamma}\big(e_i\Delta'\Gamma\,,\,(e\otimes f)\mu\kappa\Gamma\big)\\
&= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}\Gamma}\big(e_i\Delta'\mathbb{Q}\Gamma\,,\,(e\otimes f)\mu\kappa\mathbb{Q}\Gamma\big)\\
&= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}\Xi}\big(e_i\Delta'\mathbb{Q}\Xi\,,\,(e\otimes f)\mu\kappa\mathbb{Q}\Xi\big)\\
&\overset{\text{R.51}}{=} \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}\Xi}\big(e_i\Delta'\mathbb{Q}\Xi\,,\,(e\otimes f)\mu\mathbb{Q}\Xi\big)\\
&= \dim_{\mathbb{Q}} \big((e\otimes f)\mu\cdot\mathbb{Q}\Xi\cdot e_i\Delta'\big)\,.
\end{aligned}
$$

Example 53 We can choose e and f in Remark 52 to be the primitive idempotents given in Definition 21, and calculate the multiplicities of P_1 and P_2 in the tensor products $P_1\otimes P_1$, $P_1\otimes P_2$ and $P_2\otimes P_2$ using the calculation from Remark 52 and the notation from Remark 45.

For $P_1\otimes P_1$, we have to consider

$$
\begin{aligned}
&\dim_{\mathbb{Q}}((e_1\otimes e_1)\mu\cdot\mathbb{Q}\Xi\cdot e_1\Delta')\\
&=\dim_{\mathbb{Q}}((e_1\otimes e_1)\mu\cdot\mathbb{Q}\Xi\cdot\tilde{e}_1)\\
&=\dim_{\mathbb{Q}}\Big(\Big(0,\begin{pmatrix}0&0\\0&0\end{pmatrix},0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},0,\begin{pmatrix}0&0\\0&1\end{pmatrix},1\Big)\mathbb{Q}\Xi\Big(0,\begin{pmatrix}0&0\\0&1\end{pmatrix},1,\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},1,\begin{pmatrix}0&0\\0&1\end{pmatrix},0\Big)\Big)\\
&=\dim_{\mathbb{Q}}\Big(\Big(0,\begin{pmatrix}0&0\\0&0\end{pmatrix},0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&\mathbb{Q}&\mathbb{Q}\end{pmatrix},\begin{pmatrix}0&0\\0&\mathbb{Q}\end{pmatrix},0,\begin{pmatrix}0&0\\0&\mathbb{Q}\end{pmatrix},0\Big)\Big)\\
&=4
\end{aligned}
$$

for $i=1$, and analogously

$$
\begin{aligned}
&\dim_{\mathbb{Q}}((e_1\otimes e_1)\mu\cdot\mathbb{Q}\Xi\cdot e_2\Delta')\\
&=\dim_{\mathbb{Q}}((e_1\otimes e_1)\mu\cdot\mathbb{Q}\Xi\cdot\tilde{e}_2)\\
&=\dim_{\mathbb{Q}}\Big(\Big(0,\begin{pmatrix}0&0\\0&0\end{pmatrix},0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},0,\begin{pmatrix}0&0\\0&1\end{pmatrix},1\Big)\mathbb{Q}\Xi\Big(1,\begin{pmatrix}1&0\\0&0\end{pmatrix},0,\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},0,\begin{pmatrix}1&0\\0&0\end{pmatrix},1\Big)\Big)\\
&=\dim_{\mathbb{Q}}\Big(\Big(0,\begin{pmatrix}0&0\\0&0\end{pmatrix},0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\\mathbb{Q}&\mathbb{Q}&0&0\end{pmatrix},\begin{pmatrix}0&0\\\mathbb{Q}&0\end{pmatrix},0,\begin{pmatrix}0&0\\\mathbb{Q}&0\end{pmatrix},\mathbb{Q}\Big)\Big)\\
&=5
\end{aligned}
$$

for $i=2$. Thus, with $C_{\mathbb{Z}_{(3)}\mathrm{S}_3}=\begin{pmatrix}2&1\\1&2\end{pmatrix}$, we obtain

$$
\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}=\begin{pmatrix}2&1\\1&2\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}.
$$

So $P_1\otimes P_1\cong P_1^{\oplus\mu_1}\oplus P_2^{\oplus\mu_2}=P_1^{\oplus 1}\oplus P_2^{\oplus 2}$, in accordance with Lemma 30.

For $P_2\otimes P_2$, we get

$$\dim_{\mathbb{Q}}((e_2\otimes e_2)\Xi e_1\Delta')=4$$

$$\dim_{\mathbb{Q}}((e_2\otimes e_2)\Xi e_2\Delta')=5$$

and therefore

$$
\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}=\begin{pmatrix}2&1\\1&2\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}.
$$

So $P_2\otimes P_2\cong P_1^{\oplus\mu_1}\oplus P_2^{\oplus\mu_2}=P_1^{\oplus 1}\oplus P_2^{\oplus 2}$, in accordance with Lemma 32.

For $P_1\otimes P_2$ ($\cong P_2\otimes P_1$), we find

$$\dim_{\mathbb{Q}}((e_1\otimes e_2)\Xi e_1\Delta')=5$$

$$\dim_{\mathbb{Q}}((e_1\otimes e_2)\Xi e_2\Delta')=4$$

and thus

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So $P_1 \otimes P_2 \cong P_1^{\oplus \mu_1} \oplus P_2^{\oplus \mu_2} = P_1^{\oplus 2} \oplus P_2^{\oplus 1}$, in accordance with Lemma 35.

3.3.4 Using Δ' and Γ to tensor Specht modules

Remark 54 We can also use the ties of Γ of Lemma 39 to decompose tensor products. We will consider the example of $S^{(2,1)} \otimes S^{(2,1)}$.

Consider the rational idempotents

$$\begin{array}{rcl} \eta^{(3)} & := & (1, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0) \in \mathbb{Q}\Lambda_{(3)}, \\ \eta^{(2,1)} & := & (0, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0) \in \mathbb{Q}\Lambda_{(3)}, \\ \eta^{(1,1,1)} & := & (0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 1) \in \mathbb{Q}\Lambda_{(3)}. \end{array}$$

Note that they are not contained in $\Lambda_{(3)}$.

We obtain the Specht modules

$$\begin{array}{rcll} S^{(3)} & = & \eta^{(3)}\Lambda_{(3)} & \subseteq \mathbb{Q}\Lambda_{(3)} \\ S^{(2,1)} & = & \eta^{(2,1)}\Lambda_{(3)} & \subseteq \mathbb{Q}\Lambda_{(3)} \\ S^{(1,1,1)} & = & \eta^{(1,1,1)}\Lambda_{(3)} & \subseteq \mathbb{Q}\Lambda_{(3)}. \end{array}$$

An element of $S^{(2,1)}$ shall be abbreviated $(u\ v) := (0 \times \left(\begin{smallmatrix} u & v \\ 0 & 0 \end{smallmatrix}\right) \times 0)$, where $u, v \in \mathbb{Z}_{(3)}$. Likewise for elements of $S^{(3)}$ and $S^{(1,1,1)}$.

Tensoring with $\mathbb{Q}$, we obtain a map $\mathbb{Q}(\mu\kappa) : \mathbb{Q}\Lambda_{(3)} \otimes \mathbb{Q}\Lambda_{(3)} \longrightarrow \mathbb{Q}\Gamma = \mathbb{Q}\Xi$.

With the maps given in Definition 47 and Remark 50, we define

$$e' := (\eta^{(2,1)} \otimes \eta^{(2,1)})\,\mathbb{Q}(\mu\kappa) = (0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0) \in \Xi \subseteq \mathbb{Q}\Xi.$$

Note that e' is not contained Γ.

As $\Lambda_{(3)} \otimes \Lambda_{(3)}$-modules, we have

$$\begin{array}{rcl} S^{(2,1)} \otimes S^{(2,1)} & = & \eta^{(2,1)}\Lambda_{(3)} \otimes \eta^{(2,1)}\Lambda_{(3)} \\ & = & (\eta^{(2,1)} \otimes \eta^{(2,1)}) \cdot (\Lambda_{(3)} \otimes \Lambda_{(3)}) \\ & \cong & (\eta^{(2,1)} \otimes \eta^{(2,1)})\,\mathbb{Q}(\mu\kappa) \cdot (\Lambda_{(3)} \otimes \Lambda_{(3)})\,\mathbb{Q}(\mu\kappa) \\ & \cong & (\eta^{(2,1)} \otimes \eta^{(2,1)})\,\mathbb{Q}(\mu\kappa) \cdot (\Lambda_{(3)} \otimes \Lambda_{(3)})\mu\kappa \\ & \cong & e'\Gamma, \end{array}$$

on which an element of $\Lambda_{(3)} \otimes \Lambda_{(3)}$ acts via $\mu\kappa$.

Let $\varepsilon := (0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0) \in \Xi$ be the primitive central idempotent belonging to the fifth entry.

We fix the following $\mathbb{Z}_{(3)}$-linear basis B of $\varepsilon\Gamma$, which we sort according to "regions" of $\mathbb{Z}_{(3)}^{4\times 4}$, and of which we denote only the nontrivial, fifth entry.

$$B := \left(\begin{array}{cc|cc}
\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \\
\begin{pmatrix}0&0&0&0\\3&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&3&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}, \\
\hline
\begin{pmatrix}0&0&0&0\\0&0&0&0\\3&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&3&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \\
\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\3&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&3&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&3&0\end{pmatrix}, & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&3\end{pmatrix}
\end{array}\right)$$

To construct B, we first use the ties describing Γ given in Lemma 39 to obtain a $\mathbb{Z}_{(3)}$-linear basis of Γ, then cut out the relevant block and reduce the resulting generating tuple to a basis. Using the basis B, we then fix a $\mathbb{Z}_{(3)}$-linear basis of $e'\varepsilon\Gamma = e'\Gamma$ as

$$\begin{array}{rcccc}
 & \Bigg(\begin{pmatrix}2&-1&0&0\\2&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&3&0&0\\0&3&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&1&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, & \begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix} \Bigg) \\
=: & (u, & v, & w, & x) \; ,
\end{array}$$

obtained by multiplying the elements of the basis B with e' from the left and then reducing the resulting generating tuple to a basis.

Restricting the $\Lambda_{(3)} \otimes \Lambda_{(3)}$-module $e'\Gamma$ to $\Lambda_{(3)}$ via $\tilde{\Delta}$, an element $(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f) \in \Lambda_{(3)}$ acts on $e'\Gamma$ via $\tilde{\Delta}\mu\kappa = \Delta'$. So on the fifth entry it acts from the right by multiplication with

$$A := \begin{pmatrix} a&0&0&0\\0&b&c&0\\0&d&e&0\\0&0&0&f \end{pmatrix} .$$

Note that $a \equiv_3 b$, $e \equiv_3 f$ and $d \equiv_3 0$.

We calculate the action of $(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f)$, i.e. of A, in the basis fixed above.

$$\begin{array}{rcl}
u \cdot A & = & au + \frac{a-b}{3} v \\
v \cdot A & = & bv + 3cw \\
w \cdot A & = & \frac{d}{3} v + ew \\
x \cdot A & = & fx \; .
\end{array}$$

Notice that all occurring coefficients are in fact contained in $\mathbb{Z}_{(3)}$.

Let

$$\Theta \; := \; \left\{ (m, (n \;\; p), q) \in S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)} \mid m + n \equiv_3 0 \right\} \subseteq S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)} \; ,$$

having the $\mathbb{Z}_{(3)}$-linear basis $\Big((2, (-1 \;\; 0), 0),\; (0, (3 \;\; 0), 0),\; (0, (0 \;\; 1), 0),\; (0, (0 \;\; 0), 1) \Big) =: (u', v', w', x')$.

Now Θ is a $\Lambda_{(3)}$-submodule of $S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)}$. It is isomorphic to $e'\Gamma$ via

$$\varphi \; : \; e'\Gamma \xrightarrow{\sim} \Theta \; : \; u \mapsto u', \; v \mapsto v', \; w \mapsto w', \; x \mapsto x' \; ,$$

since $(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f)$ acts on (u, v, w, x) as on (u', v', w', x').

By multiplication with -1 on $S^{(3)}$ we get the description

$$\begin{aligned}
\Theta &= \Big\{(m,(n\ \ p),q) \in S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)} \mid m+n \equiv_3 0\Big\} \\
&\cong \Big\{(m',(n\ \ p),q) \in S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)} \mid m' \equiv_3 n\Big\} \\
&= \Big\{(m',(n\ \ p),0) \in S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)} \mid m' \equiv_3 n\Big\} \\
&\quad \oplus \Big\{(0,(0\ \ 0),q) \in S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)} \mid m' \equiv_3 n\Big\} \\
&\cong P_2 \oplus S^{(1,1,1)} ,
\end{aligned}$$

where m, m', n, $p \in \mathbb{Z}_{(3)}$; cf. Definition 27.

Altogether, as $\Lambda_{(3)}$-modules we have

$$S^{(2,1)} \otimes S^{(2,1)} \;\cong\; e'\Gamma \;\cong\; \Theta \;\cong\; P_2 \oplus S^{(1,1,1)} .$$

In particular, we have $\bar{S}^{(2,1)} \otimes \bar{S}^{(2,1)} \cong \bar{P}_2 \oplus \bar{S}^{(1,1,1)}$, which we can compare with the result obtained by Magma as follows.

```
 load "main_S3_loc3";

 F := GF(3);
 G := S3P;

 S21  := GModule(G, [Matrix(F,[[-1,1],[-1,0]]),Matrix(F,[[0,1],[1,0]])]);
 S111 := GModule(G, [Matrix(F,[[1]]),Matrix(F,[[-1]])]);
// generating P̄2 :
 PP2  := GModule(G, [MatrixRing(F,3)!rhoP2(S3P!(1,2,3)),MatrixRing(F,3)!rhoP2(S3P!(1,2))]);

 IsIsomorphic(TensorProduct(S21,S21),DirectSum(PP2,S111));
```

We get

```
 > IsIsomorphic(TensorProduct(S21,S21),DirectSum(PP2,S111));
true   .
```

Chapter 4

On localizations of $\mathbb{Z}S_4$

Definition 55

Let

$$\begin{aligned}
\Lambda \quad := \quad & \{(a_{11}, b_{11}, \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}) \mid \\
& a_{11} \equiv_3 e_{22}\,, \quad b_{11} \equiv_3 e_{11}\,, \quad e_{12} \equiv_3 0\,, \\
& c_{31} \equiv_4 c_{32} \equiv_4 d_{31} \equiv_4 d_{32} \equiv_4 0\,, \\
& a_{11} - c_{33} \equiv_8 b_{11} - d_{33} \equiv_4 0\,, \\
& c_{11} \equiv_4 d_{11}\,, \quad c_{12} \equiv_4 d_{12}\,, \quad c_{21} \equiv_4 d_{21}\,, \quad c_{22} \equiv_4 d_{22}\,, \\
& c_{13} \equiv_2 d_{13}\,, \quad c_{23} \equiv_2 d_{23}, \quad c_{33} \equiv_2 d_{33}\,, \\
& c_{31} \equiv_8 d_{31}\,, \quad c_{32} \equiv_8 d_{32}\,, \\
& c_{11} + d_{11} \equiv_8 2e_{11}\,, \quad c_{12} + d_{12} \equiv_8 2e_{12}\,, \quad c_{21} + d_{21} \equiv_8 2e_{21}\,, \\
& c_{22} + d_{22} \equiv_8 2e_{22}\} \\
& \qquad \subseteq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{2\times 2}\,.
\end{aligned}$$

The tuple entries belong to the Specht modules $S^{(1^4)}$, $S^{(4)}$, $S^{(2,1,1)}$, $S^{(3,1)}$ and $S^{(2,2)}$, in the order chosen above.

The $\mathbb{Z}$-order Λ is the image of the Wedderburn embedding of Remark 240.

4.1 The Localization $\mathbb{Z}_{(3)}S_4$

Here, a block of $\mathbb{Z}_{(3)}S_4$ is isomorphic to $\mathbb{Z}_{(3)}S_3$. Also the behaviour of the canonical presentations over $\mathbb{Z}_{(3)}S_4$ parallels that over $\mathbb{Z}_{(3)}S_3$; cf. Remark 72.

Write

$$R := \mathbb{Z}_{(3)}\,.$$

4.1.1 Idempotents and projectives

Definition 56 The localization of Λ at (3) is given by

$$\begin{aligned}
\Lambda_{(3)} \quad := \quad & \{(a_{11}, b_{11}, \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}) \mid \quad a_{11} \equiv_3 e_{22}, \quad b_{11} \equiv_3 e_{11}, \quad e_{12} \equiv_3 0\} \\
\subseteq \quad & \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{3\times 3} \times \mathbb{Z}_{(3)}^{3\times 3} \times \mathbb{Z}_{(3)}^{2\times 2} \\
= \quad & R \times R \times R^{3\times 3} \times R^{3\times 3} \times R^{2\times 2}\,.
\end{aligned}$$

Consider the following idempotents of $\Lambda_{(3)}$.

$$e_1 := \left(1, 0, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0 \\ 0&1 \end{pmatrix}\right), \qquad e_2 := \left(0, 1, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0 \\ 0&0 \end{pmatrix}\right),$$

$$e_3 := \left(0, 0, \begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}\right), \qquad e_4 := \left(0, 0, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}\right)$$

Remark 57 Let $E := \{\, (a,b) \in R \times R \mid a \equiv_3 b \,\}$.

We have isomorphisms of R-orders

$$\begin{array}{rcl}
E & \longrightarrow & e_1\Lambda_{(3)}e_1 \\
(a,b) & \longmapsto & \left(a, 0, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0 \\ 0&b \end{pmatrix}\right), \\
E & \longrightarrow & e_2\Lambda_{(3)}e_2 \\
(a,b) & \longmapsto & \left(0, a, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} b&0 \\ 0&0 \end{pmatrix}\right), \\
R & \longrightarrow & e_3\Lambda_{(3)}e_3 \\
a & \longmapsto & \left(0, 0, \begin{pmatrix} a&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}\right), \\
R & \longrightarrow & e_4\Lambda_{(3)}e_4 \\
a & \longmapsto & \left(0, 0, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} a&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}\right).
\end{array}$$

Lemma 58 *The idempotents e_1, e_2, e_3 and e_4 are primitive.*

They represent the equivalence classes of the primitive idempotents of $\Lambda_{(3)}$.

Proof. First we have to show primitivity of e_1 . Primitivity of e_2 follows analogously.

To this end, we show that e_1 and 0 are the only idempotents of $e_1\Lambda_{(3)}e_1$. By Remark 57 it remains to show that $(0,0)$ and $(1,1)$ are the only idempotents of E. Let $(a,b) \in E$ be an idempotent. Since $\mathbb{Z}_{(3)}$ is local, Corollary 134 gives $a,b \in \{0,1\}$.

Since this is a finite problem in E, we can use Magma **[3]** to list all possible pairs $(a,b) \in R \times R$ with a, $b \in \{0,\ 1\}$ and to test the ties defining E in a last step. To do so, we call

```
i := {<a,b> : a,b in {0,1}| (a mod 3) eq (b mod 3))};
```

and we get the following result.

```
> i;
{ <0, 0>, <1, 1> }
```

Therefore, using Lemma 136, the idempotent e_1 is primitive.

Now we show primitivity of e_3 . Primitivity of e_4 follows analogously.

To this end, we show that e_3 and 0 are the only idempotents of $e_3\Lambda_{(3)}e_3$. By Remark 57 it suffices to show that 0 and 1 are the only idempotents of R. Since $\mathbb{Z}_{(3)}$ is local, this follows from Corollary 134. Therefore, using Lemma 136, the idempotent e_3 is primitive.

Finally, there exists an orthogonal decomposition $1 = e_1 + e_2 + e_3 + e_3' + e_3'' + e_4 + e_4' + e_4''$ into primitive idempotents, which fall into the equivalence classes $\{e_1\}$, $\{e_2\}$, $\{e_3, e_3', e_3''\}$, $\{e_4, e_4', e_4''\}$. Here e_3' and e_3'' are obtained from e_3 by "shifting along the main diagonal". Similarly e_4' and e_4'' . $\square$

Corollary 59 *Up to isomorphism, we have the Peirce decomposition*

$$\Lambda_{(3)} \cong e_1\Lambda_{(3)} \oplus e_2\Lambda_{(3)} \oplus e_3\Lambda_{(3)}^{\oplus 3} \oplus e_4\Lambda_{(3)}^{\oplus 3} \cong \begin{pmatrix} (e_1\Lambda_{(3)}e_1)^{1\times 1} & (e_1\Lambda_{(3)}e_2)^{1\times 1} & 0 & 0 \\ (e_2\Lambda_{(3)}e_1)^{1\times 1} & (e_2\Lambda_{(3)}e_2)^{1\times 1} & 0 & 0 \\ 0 & 0 & (e_3\Lambda_{(3)}e_3)^{3\times 3} & 0 \\ 0 & 0 & 0 & (e_4\Lambda_{(3)}e_4)^{3\times 3} \end{pmatrix}.$$

Lemma 60

We have the radicals

$$\begin{aligned}
\mathfrak{r}(e_1\Lambda_{(3)}e_1) &= \left((3) \times 0 \times \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0\\0&(3) \end{pmatrix}\right) \cap e_1\Lambda_{(3)}e_1\,, \\
\mathfrak{r}(e_2\Lambda_{(3)}e_2) &= \left(0 \times (3) \times \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} (3)&0\\0&0 \end{pmatrix}\right) \cap e_2\Lambda_{(3)}e_2\,, \\
\mathfrak{r}(e_3\Lambda_{(3)}e_3) &= \left(0 \times 0 \times \begin{pmatrix} (3)&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right) \cap e_3\Lambda_{(3)}e_3\,, \\
\mathfrak{r}(e_4\Lambda_{(3)}e_4) &= \left(0 \times 0 \times \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} (3)&0&0\\0&0&0\\0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right) \cap e_4\Lambda_{(3)}e_4\,.
\end{aligned}$$

Proof. This follows by Proposition 222; cf. Remark 208, Example 223. $\square$

Lemma 61 *We have the Jacobson radical*

$$\mathfrak{r}(\Lambda_{(3)}) = \left((3) \times (3) \times \begin{pmatrix} (3)&(3)&(3)\\(3)&(3)&(3)\\(3)&(3)&(3) \end{pmatrix} \times \begin{pmatrix} (3)&(3)&(3)\\(3)&(3)&(3)\\(3)&(3)&(3) \end{pmatrix} \times \begin{pmatrix} (3)&R\\R&(3) \end{pmatrix}\right) \cap \Lambda_{(3)}\,.$$

Proof. This follows by Lemma 60, using Proposition 217 and Remark 208; cf. Corollary 59. $\square$

Definition 62 Let $P_1 := e_1 \cdot \Lambda_{(3)}$, $P_2 := e_2 \cdot \Lambda_{(3)}$, $P_3 := e_3 \cdot \Lambda_{(3)}$ and $P_4 := e_4 \cdot \Lambda_{(3)}$ represent the isoclasses of the indecomposable projective modules of $\Lambda_{(3)}$; cf. Lemma 220, Remark 208.

So we obtain

$$\begin{aligned}
P_1 &= \left\{\left(a_{11}, 0, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\e_{21}&e_{22} \end{pmatrix}\right) \mid a_{11} \equiv_3 e_{22}\right\}, \\
P_2 &= \left\{\left(0, b_{11}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} e_{11}&e_{12}\\0&0 \end{pmatrix}\right) \mid b_{11} \equiv_3 e_{11},\ \ e_{12} \equiv_3 0\right\}, \\
P_3 &= \left\{\left(0, 0, \begin{pmatrix} c_{11}&c_{12}&c_{13}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right)\right\}, \\
P_4 &= \left\{\left(0, 0, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} d_{11}&d_{12}&d_{13}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right)\right\},
\end{aligned}$$

where all entries are to be read as running through R.

We abbreviate

$$\begin{aligned}
\left[a_{11},\ (e_{21}\ \ e_{22})\right] &:= \left(a_{11}, 0, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\e_{21}&e_{22} \end{pmatrix}\right) &\in P_1\,, \\
\left[b_{11},\ (e_{11}\ \ e_{12})\right] &:= \left(0, b_{11}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} e_{11}&e_{12}\\0&0 \end{pmatrix}\right) &\in P_2\,, \\
\left[(c_{11}\ \ c_{12}\ \ c_{13})\right] &:= \left(0, 0, \begin{pmatrix} c_{11}&c_{12}&c_{13}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right) &\in P_3\,, \\
\left[(d_{11}\ \ d_{12}\ \ d_{13})\right] &:= \left(0, 0, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} d_{11}&d_{12}&d_{13}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right) &\in P_4\,.
\end{aligned}$$

So we have $P_1 \longleftrightarrow S^{(1^4)} \oplus S^{(2,2)}$, $P_2 \longleftrightarrow S^{(4)} \oplus S^{(2,2)}$, $P_3 \longleftrightarrow S^{(2,1,1)}$ and $P_4 \longleftrightarrow S^{(3,1)}$.

Using Lemma 61, we fix the following R-linear bases.

$$\begin{array}{llll}
(\,[1,(0\ 1)], & (\,[3,(0\ 0)], & (\,[1,(1\ 0)], & (\,[3,(0\ 0)], \\
\ [0,(1\ 0)], & \ [0,(1\ 0)], & \ [0,(3\ 0)], & \ [0,(3\ 0)], \\
\ [0,(0\ 3)]) \ \text{ of } P_1\,, & \ [0,(0\ 3)]) \ \text{ of } \mathfrak{r}P_1\,, & \ [0,(0\ 3)]) \ \text{ of } P_2\,, & \ [0,(0\ 3)]) \ \text{ of } \mathfrak{r}P_2\,,
\end{array}$$

$$\begin{array}{llll}
(\,[(1\ 0\ 0)], & (\,[(3\ 0\ 0)], & (\,[(1\ 0\ 0)], & (\,[(3\ 0\ 0)], \\
\ [(0\ 1\ 0)], & \ [(0\ 3\ 0)], & \ [(0\ 1\ 0)], & \ [(0\ 3\ 0)], \\
\ [(0\ 0\ 1)]\,) \ \text{ of } P_3\,, & \ [(0\ 0\ 3)]\,) \ \text{ of } \mathfrak{r}P_3\,, & \ [(0\ 0\ 1)]\,) \ \text{ of } P_4\,, & \ [(0\ 0\ 3)]\,) \ \text{ of } \mathfrak{r}P_4
\end{array}$$

Remark 63 Using Magma, we can verify that the indecomposable projective modules $\bar{P}_3$ resp. $\bar{P}_4$ have the Loewy layers D_3 resp. D_4. For $\bar{P}_1$ we have the Loewy layers

$$\begin{array}{l} D_1 \\ D_2 \\ D_1\ , \end{array}$$

and for $\bar{P}_2$ we find

$$\begin{array}{l} D_2 \\ D_1 \\ D_2\ . \end{array}$$

Remark 64 Recall that

$$\begin{aligned}
e_1\Lambda_{(3)}e_1 &= \{\underbrace{\left(a_{11}, 0, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\0&e_{22} \end{pmatrix}\right)}_{=:\,[a_{11},e_{22}]} \ \Big|\ a_{11} \equiv_3 e_{22}\} \\
&= \{\,[a_{11},e_{22}] \ |\ \ a_{11} \equiv_3 e_{22}\}\,,
\end{aligned}$$

for which we fix the R-linear basis

$$(\,e_1\,,\,\tilde{h}_1^{11}\,) := \{\,[1,1],\,[0,3]\,\}\ .$$

Via the canonical isomorphism from $e_1\Lambda_{(3)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_4}(P_1,P_1)$, it is mapped to the fixed R-linear basis

$$(\,1_{P_1}\,,\,h_1^{11}\,) \ := \ \left(\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&3&0\\0&0&3 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{R\mathrm{S}_4}(P_1,P_1)$, using the fixed R-linear basis of P_1 given in Definition 62.

Recall further that

$$\begin{aligned}
e_2\Lambda_{(3)}e_2 &= \{\underbrace{\left(0, b_{11}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} e_{11}&0\\0&0 \end{pmatrix}\right)}_{=:\,[b_{11},e_{11}]} \ \Big|\ b_{11} \equiv_3 e_{11}\,\} \\
&= \{\,[b_{11},e_{11}] \ |\ \ b_{11} \equiv_3 e_{11}\,\}\,,
\end{aligned}$$

for which we fix the R-linear basis

$$(\,e_2,\,\tilde{h}_1^{22}\,) := \{\,[1,1],\,[0,3]\,\}\,,$$

abbreviating analogously.

Via the canonical isomorphism from $e_2\Lambda_{(3)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_4}(P_2,P_2)$, it is mapped to the fixed R-linear basis

$$(\,1_{P_2}\,,\,h_1^{22}\,) \ := \ \left(\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&3&0\\0&0&3 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{RS_4}(P_2, P_2)$, using the fixed R-linear basis of P_2 given in Definition 62.

Moreover,

$$\begin{aligned} e_3\Lambda_{(3)}e_3 &= \{\underbrace{\left(0,0,\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix},\begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right)}_{=:\,[c_{11}]}\} \\ &= \{\,[c_{11}]\,\}, \end{aligned}$$

therefore an R-linear basis is fixed by

$$(\, e_3\,) := \{\,[1]\,\}\,.$$

Via the canonical isomorphism from $e_3\Lambda_{(3)}e_3$ to $\mathrm{Hom}_{RS_4}(P_3, P_3)$, it is mapped to the R-linear basis

$$(\, 1_{P_3}\,) \quad := \quad \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{RS_4}(P_3, P_3)$, using the fixed R-linear basis of P_3 given in Definition 62.

For the idempotent e_4 we get

$$\begin{aligned} e_4\Lambda_{(3)}e_4 &= \{\underbrace{\left(0,0,\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix},\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right)}_{=:\,[d_{11}]}\} \\ &= \{\,[d_{11}]\,\}, \end{aligned}$$

therefore an R-linear basis is fixed by

$$(\, e_4\,) := \{\,[1]\,\}\,.$$

Like before, via the canonical isomorphism from $e_4\Lambda_{(3)}e_4$ to $\mathrm{Hom}_{RS_4}(P_4, P_4)$, it is mapped to the R-linear basis

$$(\, 1_{P_4}\,) \quad := \quad \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{RS_4}(P_4, P_4)$, using the fixed R-linear basis of P_4 given in Definition 62.

Recall that

$$\begin{aligned} e_1\Lambda_{(3)}e_2 &= \{\underbrace{\left(0,0,\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix},\begin{pmatrix} 0&0\\e_{21}&0 \end{pmatrix}\right)}_{=:\,[e_{21}]} \;\Big|\; e_{21} \equiv_3 0\} \\ &= \{\,[e_{21}]\;|\; e_{21}\equiv_3 0\,\}, \end{aligned}$$

for which we fix the R-linear basis

$$(\, \tilde{h}_1^{21}\,) := \{\,[1]\,\}\,.$$

Via the canonical isomorphism from $e_1\Lambda_{(3)}e_2$ to $\mathrm{Hom}_{RS_4}(P_2, P_1)$, it is mapped to the R-linear basis

$$(h_1^{21}\,) \quad := \quad \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

of $\mathrm{Hom}_{RS_4}(P_2, P_1)$, using the fixed R-linear bases.

Finally, the homomorphisms from P_1 to P_2 we get by considering

$$\begin{aligned} e_2\Lambda_{(3)}e_1 &= \{\underbrace{\left(0,0,\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix},\begin{pmatrix} 0&e_{12}\\0&0 \end{pmatrix}\right)}_{=:\,[e_{12}]}\} \\ &= \{\,[e_{12}]\,\} \end{aligned}$$

for which we fix the R-linear basis

$$(\tilde{h}_1^{12}) := \{[3]\}\ .$$

Via the canonical isomorphism from $e_2\Lambda_{(3)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_4}(P_1, P_2)$, it is mapped to the fixed R-linear basis

$$(h_1^{12}) \quad := \quad \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right)$$

of $\mathrm{Hom}_{R\mathrm{S}_4}(P_1, P_2)$, using the fixed R-linear bases.

The operating matrices can be found in the file `main_S4_loc3`, the homomorphisms in `Homs_S4_loc3`.

They can e.g. be used to check the $R\mathrm{S}_4$-linearity of the maps between P_1, P_2, P_3 and P_4 derived above.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2, P_3 and P_4 are denoted `rhoP1`, `rhoP2`, `rhoP3`, `rhoP4`, respectively.

E.g. for the operating matrices on P_1, call

```
rhoP1(S4P!sigma);
```

for an element `sigma` of S_4. Analogously for P_2, P_3 and P_4.

To check that the matrices found above represent $R\mathrm{S}_4$-linear maps between the respective projective modules, follow these steps:

```
load main_S4_loc3;
load Homs_S4_loc3;

[rhoP1(sigma)*Hom_P1P1[i] eq Hom_P1P1[i]*rhoP1(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..2]];
[rhoP2(sigma)*Hom_P2P2[i] eq Hom_P2P2[i]*rhoP2(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..2]];

[rhoP1(sigma)*Hom_P1P2[i] eq Hom_P1P2[i]*rhoP2(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..1]];
[rhoP2(sigma)*Hom_P2P1[i] eq Hom_P2P1[i]*rhoP1(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..1]];

[rhoP3(sigma)*Hom_P3P3[i] eq Hom_P3P3[i]*rhoP3(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..1]];
[rhoP4(sigma)*Hom_P4P4[i] eq Hom_P4P4[i]*rhoP4(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..1]];
```

Remark 65 Let $S^{(1,1,1,1)}$ be the alternating Specht module of $\Lambda_{(3)}$. Then we get the isomorphisms of projective modules of $\Lambda_{(3)}$

$$P_1 \;\cong\; S^{(1,1,1,1)} \;\otimes\; P_2 \;\text{ and }\; P_3 \;\cong\; S^{(1,1,1,1)} \;\otimes\; P_4\ .$$

Therefore, by applying Remark 176, we can reduce our problem to the cases $P_1 \otimes P_1$, $P_1 \otimes P_3$ and $P_3 \otimes P_3$.

4.1.2 The tensor product $P_1 \otimes P_1$

Lemma 66 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_1 \longrightarrow P_1 \otimes P_1$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_1 \longrightarrow P_1 \otimes P_1$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_4$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_1 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \oplus P_2^{\oplus 2} \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_1 \arrow[r, "A", "\sim"'] & P_1 \oplus P_2^{\oplus 2} \\
\mathfrak{r}P_1 \otimes P_1 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_1 \oplus P_2^{\oplus 2} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{c|cc} 1 & . & 2h_1^{12} \\ \hline . & 1 & . \\ . & . & 3 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{c|cc} 1 & . & . \\ \hline . & 1 & . \\ . & . & 3 \end{array}\right].$$

Recall that $R \;=\; \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
We have to show local invertibility and $R\mathrm{S}_4$-linearity of the maps A, B, and $\tilde{B}$, the $R\mathrm{S}_4$-linearity of E, $\tilde{E}$, C and $\tilde{C}$, and the commutativity of the diagram. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_4$-linearity and the commutativity can be found in the file `main_S4_loc3`, the matrices for this diagram in the file `Diagram_S4_loc3_P1oP1` .

The embedding E is defined as the Kronecker product of the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$ and id_{P_1}. An embedding in the chosen bases can be found in the files and is denoted `i1`. The embedding $\tilde{E}$ is defined as the Kronecker product of id_{P_1} and the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2, P_3, P_3 are denoted `rhoP1`, `rhoP2`, `rhoP3`, `rhoP4`, respectively. The representations on $\mathfrak{r}P_1$, $\mathfrak{r}P_2$, $\mathfrak{r}P_3$, $\mathfrak{r}P_4$ are denoted `rhorP1`, `rhorP2`, `rhorP3`, `rhorP4`, respectively.

The representations on $P_1 \otimes P_1$, $\mathfrak{r}P_1 \otimes P_1$, $P_1 \otimes \mathfrak{r}P_1$ are denoted `op_p1op1`, `op_rp1op1`, `op_p1orp1`, respectively. The operating matrix of a group element on such a tensor product is defined as the tensor product of the operating matrices on the tensor factors.

For example, for the operating matrices on $P_1 \otimes P_1$ call

```
op_p1op1(S4P!sigma);
```

for an element `sigma` of S_4. The other maps work the same way.

The representation on the direct sum $P_1^{\oplus 1} \oplus P_2^{\oplus 2}$ is denoted `op_proj_sum_p1p1`. The operating matrix of a group element is defined as the block diagonal matrix containing the operating matrices of the summands. The maps A, B, C, E are denoted `A`, `B1`, `C1`, `E1`, respectively; the maps $\tilde{B}$, $\tilde{C}$, $\tilde{E}$ are denoted `B2`, `C2`, `E2`, respectively.

To verify the Lemma, follow these steps:

```
load "main_S4_loc3";
load "Diagram_S4_loc3_P1oP1";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),3)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),3)!1,i1);

 [op_rp1op1(sigma)*E1 eq E1*op_p1op1(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [op_p1orp1(sigma)*E2 eq E2*op_p1op1(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];

//commutativity:
 E2*A eq B2*C2;
 E1*A eq B1*C1;

//RS4-linearity:
 [op_p1op1(sigma)*A eq A*op_p1p1_sum(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [op_rp1op1(sigma)*B1 eq B1*op_p1p1_sum(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [op_p1orp1(sigma)*B2 eq B2*op_p1p1_sum(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [op_p1p1_sum(sigma)*C1 eq C1*op_p1p1_sum(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [op_p1p1_sum(sigma)*C2 eq C2*op_p1p1_sum(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];

//local invertibility; loc_inv see "main_S4_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

□

Remark 67 Using Magma, we can verify

$$\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) \quad\cong\quad D_1 \otimes D_1 \quad\cong\quad D_2\,,$$

$$\mathrm{Coker}(C) \quad\cong\quad \mathrm{Coker}(\tilde{C}) \quad\cong\quad D_1 \otimes \bar{P}_1 \quad\cong\quad \bar{P}_2\,,$$

with Loewy layers already known.

4.1.3 The tensor product $P_1 \otimes P_3$

Lemma 68 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_3 \longrightarrow P_1 \otimes P_3$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_3 \longrightarrow P_1 \otimes P_3$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_4$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_3 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_3 \oplus P_4^{\oplus 2} \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_3 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_3 \oplus P_4^{\oplus 2} \\
\mathfrak{r}P_1 \otimes P_3 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_3 \oplus P_4^{\oplus 2} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{c|cc} 1 & . & . \\ \hline . & 1 & . \\ . & . & 3 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{c|cc} 3 & . & . \\ \hline . & 3 & . \\ . & . & 3 \end{array}\right].$$

The matrices C and $\tilde{C}$ being block diagonal matrices, with blocks of the form id and $3 \cdot \mathrm{id}$, confirms the expected result, shown in Remark 163, for P_3 belongs to a defect-0 block.

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the RS_4-linearity and the commutativity can be found in the file `main_S4_loc3`, the matrices for this diagram in the file `Diagram_S4_loc3_P1oP3`.

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 66.

To verify the Lemma, follow these steps:

```
load main_S4_loc3;
load "Diagram_S4_loc3_P1oP3";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),3)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),3)!1,i3);

 [op_rp1op3(sigma)*E1 eq E1*op_p1op3(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [op_p1orp3(sigma)*E2 eq E2*op_p1op3(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];

//commutativity:
 E2*A eq B2*C2;
 E1*A eq B1*C1;

//RS4-linearity:
 [op_p1op3(sigma)*A eq A*op_p1p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_rp1op3(sigma)*B1 eq B1*op_p1p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p1orp3(sigma)*B2 eq B2*op_p1p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p1p3_sum(sigma)*C1 eq C1*op_p1p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p1p3_sum(sigma)*C2 eq C2*op_p1p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

//local invertibility; loc_inv see "main_S4_loc3"
  loc_inv(A,3);
  loc_inv(B1,3);
  loc_inv(B2,3);
```

□

Remark 69 Using Magma, we can verify

$$\begin{array}{lclclclcl} \mathrm{Coker}(\left[\begin{array}{c} C \\ \tilde{C} \end{array}\right]) & \cong & D_1 \otimes D_3 & \cong & D_1 \otimes \bar{P}_3 & \cong & D_4 & \cong & \bar{P}_4\,, \\ \mathrm{Coker}(C) & \cong & \bar{P}_1 \otimes D_3 & \cong & D_3 \oplus D_4 \oplus D_4 & \cong & \bar{P}_3 \oplus \bar{P}_4 \oplus \bar{P}_4\,, & & \\ \mathrm{Coker}(\tilde{C}) & \cong & D_1 \otimes \bar{P}_3 & \cong & D_4 & \cong & \bar{P}_4\,, & & \end{array}$$

with Loewy layers already known.

4.1.4 The tensor product $P_3 \otimes P_3$

Lemma 70 *Let E be the embedding $\mathfrak{t}P_3 \otimes P_3 \longrightarrow P_3 \otimes P_3$, and $\tilde{E}$ be the embedding $P_3 \otimes \mathfrak{t}P_3 \longrightarrow P_3 \otimes P_3$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_4$-linear maps*

$$\begin{tikzcd}
P_3 \otimes \mathfrak{t}P_3 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_3 \oplus P_4 \arrow[d, "\tilde{C}"] \\
P_3 \otimes P_3 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \oplus P_3 \oplus P_4 \\
\mathfrak{t}P_3 \otimes P_3 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_2 \oplus P_3 \oplus P_4 \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \begin{bmatrix} 3 & . & . \\ . & 3 & . \\ . & . & 3 \end{bmatrix}, \quad \tilde{C} := \begin{bmatrix} 3 & . & . \\ . & 3 & . \\ . & . & 3 \end{bmatrix}.$$

The matrices C and $\tilde{C}$ being block diagonal matrices, with blocks of the form $3 \cdot \mathrm{id}$, confirms the expected result, shown in Remark 164, for P_3 belongs to a defect-0 block.

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_4$-linearity and the commutativity can be found in the file `main_S4_loc3`, the matrices for this diagram in the file `Diagram_S4_loc3_P3oP3` .

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 66 .
To verify the Lemma, follow these steps:

```
load "main_S4_loc3";
load "Diagram_S4_loc3_P3oP3";

 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];

E1 := KroneckerProduct(i3,MatrixRing(Rationals(),3)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),3)!1,i3);

 [op_rp3op3(sigma)*E1 eq E1*op_p3op3(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];
 [op_p3orp3(sigma)*E2 eq E2*op_p3op3(sigma):sigma in {S4P!(1,2),S4P!(1,2,3,4)}];

//commutativity:
  E2*A eq B2*C2;
  E1*A eq B1*C1;

//RS4-linearity:
 [op_p3op3(sigma)*A eq A*op_p3p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_rp3op3(sigma)*B1 eq B1*op_p3p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p3orp3(sigma)*B2 eq B2*op_p3p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
```

```
 [op_p3p3_sum(sigma)*C1 eq C1*op_p3p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p3p3_sum(sigma)*C2 eq C2*op_p3p3_sum(sigma): sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

//local invertibility; loc_inv see "main_S4_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

$\square$

Remark 71 Using Magma, we can verify

$$\begin{array}{rclclclcl}
\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_3 \otimes D_3 & \cong & \bar{P}_3 \otimes \bar{P}_3 \\
 & & & & & \cong & \bar{P}_2 \oplus D_3 \oplus D_4 & \cong & \bar{P}_2 \oplus \bar{P}_3 \oplus \bar{P}_4 \,,
\end{array}$$

with known Loewy layers.

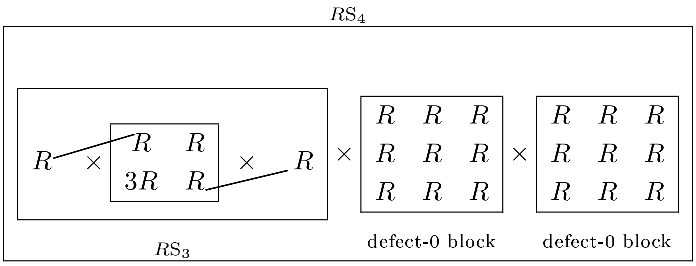
Remark 72 Consider the surjective R-algebra morphism

$$\begin{array}{rcllcl}
\varphi: & RS_4 & \longrightarrow & RS_3, & (1,2) & \longmapsto & (1,2) \\
 & & & & (1,2,3,4) & \longmapsto & (1,3)\ .
\end{array}$$

Since

$$\begin{array}{ccccc}
S_3 & \hookrightarrow & S_4 & \xrightarrow{\varphi} & S_3 \\
(1,2) & \longmapsto & (1,2) & \longmapsto & (1,2) \\
 & & (1,2,3,4) & \longmapsto & (1,3) \\
(1,3) & \longmapsto & (1,3) & \longmapsto & (1,3)\ ,
\end{array}$$

the map φ restricts to the identity on RS_3. Since P_1 and P_2 over RS_4 are obtainable by "restricting" indecomposable projective modules over RS_3 along φ , their behaviour with respect to the tensor product parallels the behaviour of the projective indecomposable RS_3-modules.



As a consequence, the Cartan matrices look very similar:

$$C_{RS_3} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad C_{RS_4} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.2 The Localization $\mathbb{Z}_{(2)}S_4$

In this section, we will meet an indecomposable summand of a tensor product of two simple modules that is neither simple nor projective nor Specht. The corresponding canonical presentation does not simplify as far as it does in the other examples over $\mathbb{Z}_{(2)}S_4$; cf. Section 4.2.3.

4.2.1 Idempotents and projectives

Write

$$R := \mathbb{Z}_{(2)} \ .$$

Definition 73 The localization of Λ at (2) is given by

$$\begin{aligned}
\Lambda_{(2)} \ := \ & \{(a_{11}, b_{11}, \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}) \mid \\
& c_{31} \equiv_4 c_{32} \equiv_4 d_{31} \equiv_4 d_{32} \equiv_4 0 \,, \\
& a_{11} - c_{33} \equiv_8 b_{11} - d_{33} \equiv_4 0 \,, \\
& c_{11} \equiv_4 d_{11} \,, \quad c_{12} \equiv_4 d_{12} \,, \quad c_{21} \equiv_4 d_{21} \,, \quad c_{22} \equiv_4 d_{22} \,, \\
& c_{13} \equiv_2 d_{13} \,, \quad c_{23} \equiv_2 d_{23}, \quad c_{33} \equiv_2 d_{33} \,, \\
& c_{31} \equiv_8 d_{31} \,, \quad c_{32} \equiv_8 d_{32} \,, \\
& c_{11} + d_{11} \equiv_8 2e_{11} \,, \quad c_{12} + d_{12} \equiv_8 2e_{12} \,, \quad c_{21} + d_{21} \equiv_8 2e_{21} \,, \\
& c_{22} + d_{22} \equiv_8 2e_{22}\} \\
& \subseteq \ \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)}^{2\times 2} \\
& = \ R \times R \times R^{3\times 3} \times R^{3\times 3} \times R^{2\times 2} \ .
\end{aligned}$$

Letting

$$e_1 := (1, 1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}),$$

$$e_2 := (0, 0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}),$$

$$e_3 := (0, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \ ,$$

we have an orthogonal deomposition

$$1_{\Lambda_{(2)}} \ = \ e_1 + e_2 + e_3$$

into idempotents of $\Lambda_{(2)}$. They fall into the equivalence classes $\{e_1\}$, $\{e_2, \, e_3\}$.

We choose e_1 and e_2 as representatives of the equivalence classes of the idempotents of $\Lambda_{(2)}$.

Remark 74 *Let*

$$\begin{aligned}
E \ &:= \ \{\, (a, b, c, d) \in R \times R \times R \times R \mid \ a - c \equiv_8 b - d \equiv_4 0, \, c \equiv_2 d \,\} \\
F \ &:= \ \{\, (a, b, c) \in R \times R \times R \mid \ a \equiv_4 b \,, a + b \equiv_8 2c\} \,.
\end{aligned}$$

We have the following isomorphisms of R-orders.

$$\begin{aligned}
E \ &\longrightarrow \ e_1 \Lambda_{(2)} e_1 \\
(a, b, c, d) \ &\longmapsto \ (a, b, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) \\
F \ &\longrightarrow \ e_2 \Lambda_{(2)} e_2 \\
(a, b, c) \ &\longmapsto \ (0, 0, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix})
\end{aligned}$$

Lemma 75 *We have an orthogonal decomposition $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3$ into primitive idempotents.*

Proof. We have to show primitivity of e_1 and e_2.

First we show that e_1 and 0 are the only idempotents of $e_1\Lambda_{(2)}e_1$. By Remark 74 it remains to show that $(0,0,0,0)$ and $(1,1,1,1)$ are the only idempotents of E. Let $(a,b,c,d) \in E$ be an idempotent. Since $\mathbb{Z}_{(2)}$ is local, Corollary 134 gives $a,b,c,d \in \{0,1\}$.

Since this is a finite problem in E, we can use Magma **[3]** to list all possible pairs $(a,b,c,d) \in R\times R\times R\times R$ with a, b, c, $d \in \{0, 1\}$ and to test the ties defining E in a last step. To do so, we call

```
i := {<a,b,c,d> : a,b,c,d in {0,1}| (a-c mod 8) eq (b-d mod 8)
                    and (b-d mod 4) eq 0 and (c mod 2 eq d mod 4)};
```

and we get the following result.

```
> i;
{ <0, 0, 0, 0>, <1, 1, 1, 1> }
```

Now we show that e_2 and 0 are the only idempotents of $e_2\Lambda_{(3)}e_2$. By Remark 74 it remains to show that $(0,0,0)$ and $(1,1,1)$ are the only idempotents of F. Let $(a,b,c) \in F$ be an idempotent. Since $\mathbb{Z}_{(2)}$ is local, Corollary 134 gives $a,b,c \in \{0,1\}$.

Since this is a finite problem in F, we can use Magma **[3]** to list all possible pairs $(a,b,c) \in R \times R \times R$ with a, b, $c \in \{0, 1\}$ and to test the ties defining E in a last step. To do so, we call

```
i := {<a,b,c> : a,b,c in {0,1}|
     (a mod 4) eq (b mod 4) and (a+b mod 8) eq (2*c mod 8)};
```

and we get the following result.

```
> i;
{ <0, 0, 0>, <1, 1, 1> }
```

Therefore, using Lemma 136, the idempotents e_1 and e_2 are primitive. □

Corollary 76 *Up to isomorphism, we have the Peirce decomposition*

$$\Lambda_{(2)} \cong e_1\Lambda_{(2)} \oplus (e_2\Lambda_{(2)})^{\oplus 2} \cong \begin{pmatrix} (e_1\Lambda_{(2)}e_1)^{1\times 1} & (e_1\Lambda_{(2)}e_2)^{1\times 2} \\ (e_2\Lambda_{(2)}e_1)^{2\times 1} & (e_2\Lambda_{(2)}e_2)^{2\times 2} \end{pmatrix}.$$

Lemma 77 *We have the radicals*

$$\begin{aligned} \mathfrak{r}(e_1\Lambda_{(2)}e_1) &= \left((2) \times (2) \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (2) \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (2) \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) \cap e_1\Lambda_{(2)}e_1\,, \\ \mathfrak{r}(e_2\Lambda_{(2)}e_2) &= \left(0 \times 0 \times \begin{pmatrix} (2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} (2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} (2) & 0 \\ 0 & 0 \end{pmatrix}\right) \cap e_2\Lambda_{(2)}e_2\,. \end{aligned}$$

Proof. This follows by Proposition 222; cf. Remark 208. □

Lemma 78 *We have the Jacobson radical*

$$\mathfrak{r}(\Lambda_{(2)}) = \left((2), (2), \begin{pmatrix} (2) & (2) & R \\ (2) & (2) & R \\ R & R & (2) \end{pmatrix}, \begin{pmatrix} (2) & (2) & R \\ (2) & (2) & R \\ R & R & (2) \end{pmatrix}, \begin{pmatrix} (2) & (2) \\ (2) & (2) \end{pmatrix}\right) \cap \Lambda_{(2)}.$$

Proof. This follows by Lemma 77 and Proposition 217, usable by Remark 208; cf. also Corollary 76. $\square$

Definition 79 Let $P_1 := e_1 \cdot \Lambda_{(2)}$, $P_2 := e_2 \cdot \Lambda_{(2)} \cong e_3 \cdot \Lambda_{(2)}$ represent the isoclasses of the indecomposable projective modules of $\Lambda_{(2)}$; cf. Remark 208, Lemma 220. So,

$$\begin{array}{rcl} P_1 & = & \Big\{ \Big(a_{11}, b_{11}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\Big) \mid a_{11} - c_{33} \equiv_8 b_{11} - d_{33} \equiv_4 0\,, \\ & & c_{31} \equiv_8 d_{31}\,,\ c_{32} \equiv_8 d_{32}\,,\ c_{33} \equiv_2 d_{33}\,,\ c_{31} \equiv_4 c_{32} \equiv_4 d_{31} \equiv_4 d_{32} \equiv_4 0 \Big\}\,, \\ P_2 & = & \Big\{ \Big(0, 0, \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} \\ 0 & 0 \end{pmatrix}\Big) \mid \\ & & c_{11} \equiv_4 d_{11}\,,\ c_{12} \equiv_4 d_{12}\,,\ c_{13} \equiv_2 d_{13}\,,\ c_{11} + d_{11} \equiv_8 2e_{11}\,,\ c_{12} + d_{12} \equiv_8 2e_{12} \Big\}\,, \end{array}$$

where all entries are to be read as running through R.

We abbreviate

$$\begin{array}{rcl} \Big[a_{11},\ b_{11},\ (c_{31}\ \ c_{32}\ \ c_{33}),\ (d_{31}\ \ d_{32}\ \ d_{33})\Big] & := & \Big(a_{11}, b_{11}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\Big) \in P_1 \\ \Big[(c_{11}\ \ c_{12}\ \ c_{13}),\ (d_{11}\ \ d_{12}\ \ d_{13}),\ (e_{11}\ \ e_{12})\Big] & := & \Big(0, 0, \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} d_{12} & d_{12} & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} \\ 0 & 0 \end{pmatrix}\Big) \in P_2\,. \end{array}$$

So we have $P_1 \hookrightarrow S^{(1^4)} \oplus S^{(4)} \oplus S^{(2,1,1)} \oplus S^{(3,1)}$ and $P_2 \hookrightarrow S^{(2,1,1)} \oplus S^{(3,1)} \oplus S^{(2,2)}$.

Using Lemma 78, we may fix the R-linear bases

$$\begin{array}{ll} (\,[1,1,(0\ 0\ 1),(0\ 0\ 1)], & (\,[(1\ 0\ 0),(1\ 0\ 0),(1\ 0)] \\ \ [0,2,(0\ 0\ 0),(0\ 0\ 2)], & \ [(0\ 1\ 0),(0\ 1\ 0),(0\ 1)] \\ \ [0,0,(4\ 0\ 0),(4\ 0\ 0)], & \ [(0\ 0\ 1),(0\ 0\ 1),(0\ 0)] \\ \ [0,0,(0\ 4\ 0),(0\ 4\ 0)], & \ [(0\ 0\ 0),(4\ 0\ 0),(2\ 0)] \\ \ [0,0,(0\ 0\ 4),(0\ 0\ 4)], & \ [(0\ 0\ 0),(0\ 4\ 0),(0\ 2)] \\ \ [0,0,(0\ 0\ 0),(8\ 0\ 0)], & \ [(0\ 0\ 0),(0\ 0\ 2),(0\ 0)] \\ \ [0,0,(0\ 8\ 0),(0\ 8\ 0)], & \ [(0\ 0\ 0),(0\ 0\ 0),(4\ 0)] \\ \ [0,0,(0\ 0\ 0),(0\ 0\ 8)]\,)\ \text{ of } P_1\,, & \ [(0\ 0\ 0),(0\ 0\ 0),(0\ 4)]\,)\ \text{ of } P_2\,, \\ & \\ (\,[2,0,(0\ 0\ 2),(0\ 0\ 0)], & (\,[(2\ 0\ 0),(2\ 0\ 0),(2\ 0)] \\ \ [0,2,(0\ 0\ 0),(0\ 0\ 2)], & \ [(0\ 2\ 0),(0\ 2\ 0),(0\ 2)] \\ \ [0,0,(4\ 0\ 0),(4\ 0\ 0)], & \ [(0\ 0\ 1),(0\ 0\ 1),(0\ 0)] \\ \ [0,0,(0\ 4\ 0),(0\ 4\ 0)], & \ [(0\ 0\ 0),(4\ 0\ 0),(2\ 0)] \\ \ [0,0,(0\ 0\ 4),(0\ 0\ 4)], & \ [(0\ 0\ 0),(0\ 4\ 0),(0\ 2)] \\ \ [0,0,(0\ 0\ 0),(8\ 0\ 0)], & \ [(0\ 0\ 0),(0\ 0\ 2),(0\ 0)] \\ \ [0,0,(0\ 8\ 0),(0\ 8\ 0)], & \ [(0\ 0\ 0),(0\ 0\ 0),(4\ 0)] \\ \ [0,0,(0\ 0\ 0),(0\ 0\ 8)]\,)\ \text{ of } \mathfrak{r}P_1\,, & \ [(0\ 0\ 0),(0\ 0\ 0),(0\ 4)]\,)\ \text{ of } \mathfrak{r}P_2\,. \end{array}$$

Remark 80 Using Magma, we can verify that the projective module $\bar{P}_1$ has the Loewy layers

$$\begin{array}{ccc} & D_1 & \\ D_1 & & D_2 \\ D_1 & & D_2 \\ & D_1 & \end{array}$$

and $\bar{P}_2$ has the Loewy layers

$$\begin{array}{ccc} & D_2 & \\ D_1 & & D_2 \\ & D_1 & \\ & D_2 & \end{array}$$

Remark 81 Recall that

$$\begin{aligned} e_1\Lambda_{(2)}e_1 &= \{\underbrace{\left(a, b, \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,c \end{pmatrix}, \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,d \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix}\right)}_{=:\,[a,b,c,d]} \;\Big|\; c \equiv_2 d,\ a - c \equiv_8 b - d \equiv_4 0\} \\ &= \{\, [a,b,c,d] \mid c \equiv_2 d,\ a - c \equiv_8 b - d \equiv_4 0 \,\}\,. \end{aligned}$$

For this set we fix the R-linear basis

$$(\, e_1,\, \tilde{h}_1^{11},\, \tilde{h}_2^{11},\, \tilde{h}_3^{11} \,) := \{\, [1,1,1,1],\ [0,2,0,2],\ [0,0,4,4],\ [0,0,0,8] \,\}\,.$$

Via the canonical isomorphism from $e_1\Lambda_{(2)}e_1$ to $\mathrm{Hom}_{RS_4}(P_1, P_1)$, it is mapped to the fixed R-linear basis

$$(\, 1_{P_1},\, h_1^{11},\, h_2^{11},\, h_3^{11} \,) := \left(\begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&2&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&4&0&0&0&0&0 \\ 0&0&0&4&0&0&0&0 \\ 0&0&0&0&4&0&0&0 \\ 0&0&0&0&0&4&0&0 \\ 0&0&0&0&0&0&4&0 \\ 0&0&0&0&0&0&0&4 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&2 \\ 0&0&0&0&0&4&0&0 \\ 0&0&0&0&0&0&4&0 \\ 0&0&0&0&0&0&0&4 \\ 0&0&0&0&0&8&0&0 \\ 0&0&0&0&0&0&8&0 \\ 0&0&0&0&0&0&0&8 \end{pmatrix} \right)$$

of $\mathrm{Hom}_{RS_4}(P_1, P_1)$, using the fixed R-linear basis of P_1 given in Definition 79.

A R-linear basis of

$$\begin{aligned} e_2\Lambda_{(2)}e_2 &= \{\underbrace{\left(0, 0, \begin{pmatrix} c\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} d\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} e\,0 \\ 0\,0 \end{pmatrix}\right)}_{=:\,[c,\ d,\ e]} \;\Big|\; c \equiv_4 d,\ c + d \equiv_8 2e\} \\ &= \{\, [c,\ d,\ e] \mid c \equiv_4 d,\ c + d \equiv_8 2e \,\} \end{aligned}$$

can be fixed as

$$(\, e_2,\, \tilde{h}_1^{22},\, \tilde{h}_2^{22} \,) := (\, [1,1,1],\ [0,4,2],\ [0,0,4] \,)\,.$$

Via the canonical isomorphism from $e_2\Lambda_{(2)}e_2$ to $\mathrm{Hom}_{RS_4}(P_2, P_2)$, it is mapped to the R-linear basis

$$(\, 1,\, h_1^{22},\, h_2^{22} \,) := \left(\begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&2&0&0 \\ 0&0&0&4&0&0&-1&0 \\ 0&0&0&0&4&0&0&-1 \\ 0&0&0&0&0&4&0&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&2 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&4&0 \\ 0&0&0&0&0&0&0&4 \end{pmatrix} \right)$$

of $\mathrm{Hom}_{RS_4}(P_2, P_2)$, using the fixed R-linear basis of P_2, like above.

Moreover, we have that

$$\begin{aligned} e_2\Lambda_{(2)}e_1 &= \{\underbrace{\left(0, 0, \begin{pmatrix} 0\,0\,c \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0\,d \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix}\right)}_{=:\,[c,d]} \;\Big|\; c \equiv_2 d\} \\ &= \{\, [c,d] \mid c \equiv_2 d \}\,. \end{aligned}$$

Therefore an R-linear basis is given by

$$(\,\tilde{h}_1^{12},\,\tilde{h}_2^{12}\,) \;:=\; \{\,[1,1],\;[0,2]\,\}\;.$$

Via the canonical isomorphism from $e_2\Lambda_{(2)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_4}(P_1,P_2)$, it is mapped to

$$(\,h_1^{12},\,h_2^{12}) \;:=\; \left(\begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 4&0&0&0&0&0&-1&0\\ 0&4&0&0&0&0&0&-1\\ 0&0&4&0&0&0&0&0\\ 0&0&0&2&0&0&-1&0\\ 0&0&0&0&2&0&0&-1\\ 0&0&0&0&0&4&0&0 \end{pmatrix},\; \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&2&0&0&-1&0\\ 0&0&0&0&2&0&0&-1\\ 0&0&0&0&0&4&0&0\\ 0&0&0&4&0&0&-2&0\\ 0&0&0&0&4&0&0&-2\\ 0&0&0&0&0&8&0&0 \end{pmatrix}\right).$$

Finally,

$$\begin{aligned} e_1\Lambda_{(2)}e_2 \quad &= \quad \{\underbrace{\left(0,0,\begin{pmatrix}0&0&0\\0&0&0\\c&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\d&0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix}\right)}_{=:\,[c,d]}\Big|\, c\equiv_4 d\equiv_4 0,\; c\equiv_8 d\} \\ &= \quad \{\,[c,d]\;|\; c\equiv_4 d\equiv_4 0,\; c\equiv_8 d\,\}\,. \end{aligned}$$

Therefore an R-linear basis is given by

$$(\,\tilde{h}_1^{21},\,\tilde{h}_2^{21}\,) \;:=\; \{\,[4,4],\;[0,8]\,\}\;.$$

Via the canonical isomorphism from $e_1\Lambda_{(2)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_4}(P_2,P_1)$, it is mapped to

$$(\,h_1^{21},\,h_2^{21}) \;:=\; \left(\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},\; \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&4&0&0\\ 0&0&0&0&0&0&4&0\\ 0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}\right).$$

The operating matrices necessary to prove the $R\mathrm{S}_4$-linearity can be found in the file `main_S4_loc2`, the homomorphisms in the file `Homs_S4_loc2` .

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 are denoted `rhoP1`, `rhoP2`, respectively.

For the operating matrices on P_1 , call

```
rhoP1(S4P!sigma);
```

for an element `sigma` of S_4 . The operating matrices on P_2 are called analogous.

To check that the matrices found above represent $R\mathrm{S}_4$-linear maps between the respective projective modules, follow these steps:

```
load main_S4_loc2;
load Homs_S4_loc2;

[rhoP1(sigma)*Hom_P1P1[i] eq Hom_P1P1[i]*rhoP1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..4]];
[rhoP2(sigma)*Hom_P2P2[i] eq Hom_P2P2[i]*rhoP2(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..3]];

[rhoP1(sigma)*Hom_P1P2[i] eq Hom_P1P2[i]*rhoP2(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..2]];
[rhoP2(sigma)*Hom_P2P1[i] eq Hom_P2P1[i]*rhoP1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}, i in [1..2]];
```

A file on how to construct those isomorphisms can be found in the digital appendix and is named `generate_Homs_S4_loc2`.

4.2.2 The tensor product $P_1 \otimes P_1$

Lemma 82 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_1 \longrightarrow P_1 \otimes P_1$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_1 \longrightarrow P_1 \otimes P_1$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_4$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_1 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 4} \oplus P_2^{\oplus 4} \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_1 \arrow[r, "A", "\sim"'] & P_1^{\oplus 4} \oplus P_2^{\oplus 4} \\
\mathfrak{r}P_1 \otimes P_1 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_1^{\oplus 4} \oplus P_2^{\oplus 4} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cccc|cccc}
2 & . & . & . & . & . & . & . \\
-h_1^{11} & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
\hline
-h_1^{21} & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & 1
\end{array}\right], \quad \tilde{C} := \left[\begin{array}{cccc|cccc}
2 & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
\hline
. & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & 1
\end{array}\right].$$

Recall that $R = \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
We have to show local invertibility and $R\mathrm{S}_4$-linearity of the maps A, B, and $\tilde{B}$, the $R\mathrm{S}_4$-linearity of E, $\tilde{E}$, C and $\tilde{C}$, and the commutativity of the diagram. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_4$-linearity and the commutativity can be found in the file `main_S4_loc2`, the matrices for this diagram in the file `Diagram_S4_loc2_P1oP1` .

The embedding E is defined as the Kronecker product of the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$ and id_{P_1}. An embedding in the chosen bases can be found in the files and is denoted `i1`. The embedding $\tilde{E}$ is defined as the Kronecker product of id_{P_1} and the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 are denoted `rhoP1`, `rhoP2`, respectively. The representations on $\mathfrak{r}P_1$, $\mathfrak{r}P_2$ are denoted `rhorP1`, `rhorP2`, respectively.

The representations on $P_1 \otimes P_1$, $\mathfrak{r}P_1 \otimes P_1$, $P_1 \otimes \mathfrak{r}P_1$ are denoted `op_p1op1`, `op_rp1op1`, `op_p1orp1` respectively. The operating matrix of a group element on such a tensor product is defined as the tensor product of the operating matrices on the tensor factors.

For example, for the operating matrices on $P_1 \otimes P_1$ call

```
op_p1op1(S4P!sigma);
```

for an element `sigma` of S_4. The other maps work the same way.

The representation on the direct sum $P_1^{\oplus 4} \oplus P_2^{\oplus 4}$ is denoted `op_proj_sum_p1p1`. The operating matrix of a group element is defined as the block diagonal matrix containing the operating matrices of the summands. The maps A, B, C, E are denoted `A`, `B1`, `C1`, `E1`, respectively; the maps $\tilde{B}$, $\tilde{C}$, $\tilde{E}$ are denoted `B2`, `C2`, `E2`, respectively.

To verify the Lemma, follow these steps:

```
load "main_S4_loc2";
load "Diagram_S4_loc2_P1oP1";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),8)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),8)!1,i1);

 [op_rp1op1(sigma)*E1 eq E1*op_p1op1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p1orp1(sigma)*E2 eq E2*op_p1op1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS4-linearity
 [op_p1op1(sigma)*A eq A*op_proj_sum_p1p1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_rp1op1(sigma)*B1 eq B1*op_proj_sum_p1p1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p1orp1(sigma)*B2 eq B2*op_proj_sum_p1p1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_proj_sum_p1p1(sigma)*C1 eq C1*op_proj_sum_p1p1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_proj_sum_p1p1(sigma)*C2 eq C2*op_proj_sum_p1p1(sigma):sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

//local invertibility; loc_inv see "main_S4_loc2"
 loc_inv(A,2);
 loc_inv(B1,2);
 loc_inv(B2,2);
```

□

Remark 83 Using Magma, we can verify

$$\begin{array}{rcccccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_1 \otimes D_1 & \cong & D_1\,, & & \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_1 \otimes \bar{P}_1 & \cong & \bar{P}_1\,. \end{array}$$

with Loewy layers already known.

4.2.3 The tensor product $P_2 \otimes P_2$

Lemma 84 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_2 \longrightarrow P_2 \otimes P_2$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_2 \longrightarrow P_2 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_4$-linear maps*

$$\begin{tikzcd} P_2 \otimes \mathfrak{r}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 3} \oplus P_2^{\oplus 5} \arrow[d, "\tilde{C}"] \\ P_2 \otimes P_2 \arrow[r, "A", "\sim"'] & P_1^{\oplus 3} \oplus P_2^{\oplus 5} \\ \mathfrak{r}P_2 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_1^{\oplus 3} \oplus P_2^{\oplus 5} \arrow[u, "C"'] \end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{ccc|ccccc} 2 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ \hline . & . & . & 2 & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{ccc|ccccc} 2 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & . & 1 & h_1^{12} & . & . & . & . \\ \hline . & . & . & 2 & . & . & . & . \\ h_1^{21} & . & . & . & 1 & . & . & . \\ h_2^{21} & . & . & . & . & 1 & . & . \\ . & . & . & h_1^{22} & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{array}\right].$$

Recall that $R = \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the RS_4-linearity and the commutativity can be found in the file `main_S4_loc2`, the matrices for this diagram in the file `Diagram_S4_loc2_P2oP2`.

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 82.

To verify the Lemma, follow these steps:

```
load "main_S4_loc2";
load "Diagram_S4_loc2_P2oP2";

 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

E1 := KroneckerProduct(i2,MatrixRing(Rationals(),8)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),8)!1,i2);

 [op_rp2op2(sigma)*E1 eq E1*op_p2op2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p2orp2(sigma)*E2 eq E2*op_p2op2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS4-linearity
 [op_p2op2(sigma)*A eq A*op_proj_sum_p2p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_rp2op2(sigma)*B1 eq B1*op_proj_sum_p2p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p2orp2(sigma)*B2 eq B2*op_proj_sum_p2p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_proj_sum_p2p2(sigma)*C1 eq C1*op_proj_sum_p2p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_proj_sum_p2p2(sigma)*C2 eq C2*op_proj_sum_p2p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

//local invertibility; loc_inv see "main_S4_loc2"
 loc_inv(A,2);
 loc_inv(B1,2);
 loc_inv(B2,2);
```

$\square$

Remark 85 Using Magma, we can verify

$$\begin{array}{rcccccl} \mathrm{Coker}(\left[\begin{array}{c} C \\ \tilde{C} \end{array}\right]) & \cong & D_2 \otimes D_2 & \cong & D_2 \oplus X & & \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_2 \otimes \bar{P}_2 & \cong & \bar{P}_1 \oplus \bar{P}_2\,, \end{array}$$

with X a module of rank 2 and with the following Loewy layers.

$$\begin{array}{c} D_1 \\ D_1 \end{array}$$

As operating matrices on X we get

$$\begin{array}{rcl} \mathrm{S}_4 & \longrightarrow & \mathrm{GL}_2(\mathbb{F}_2) \\ (1,2,3,4) & \longmapsto & \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\ (1,2) & \longmapsto & \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) . \end{array}$$

4.2.4 The tensor product $P_1 \otimes P_2$

Lemma 86 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_2 \longrightarrow P_1 \otimes P_2$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_2 \longrightarrow P_1 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_4$-linear maps*

$$\begin{tikzcd} P_1 \otimes \mathfrak{r}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 2} \oplus P_2^{\oplus 6} \arrow[d, "\tilde{C}"] \\ P_1 \otimes P_2 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 2} \oplus P_2^{\oplus 6} \\ \mathfrak{r}P_1 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_1^{\oplus 2} \oplus P_2^{\oplus 6} \arrow[u, "C"'] \end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cc|cccccc} 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ \hline . & . & 2 & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{cc|cccccc} 1 & . & h_1^{12} & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ \hline . & . & 2 & . & . & . & . & . \\ . & . & . & 2 & . & . & . & . \\ . & . & h_1^{22} & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{array}\right].$$

Recall that $R = \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_4$-linearity and the commutativity can be found in the file `main_S4_loc2`, the matrices for this diagram in the file `Diagram_S3_loc2_P1oP2` .

The maps and matrices are denoted analogously to those for $P_1 \otimes P_1$, see proof of Lemma 82 .

To verify the Lemma, follow these steps:

```
load "main_S4_loc2";
load "Diagram_S4_loc2_P1oP2";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),8)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),8)!1,i2);

 [op_rp1op2(sigma)*E1 eq E1*op_p1op2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p1orp2(sigma)*E2 eq E2*op_p1op2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
```

```
//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS4-linearity
 [op_p1op2(sigma)*A eq A*op_proj_sum_p1p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_rp1op2(sigma)*B1 eq B1*op_proj_sum_p1p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_p1orp2(sigma)*B2 eq B2*op_proj_sum_p1p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_proj_sum_p1p2(sigma)*C1 eq C1*op_proj_sum_p1p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];
 [op_proj_sum_p1p2(sigma)*C2 eq C2*op_proj_sum_p1p2(sigma) : sigma in {S4P!(1,2), S4P!(1,2,3,4)}];

//local invertibility; loc_inv see "main_S4_loc2"
 loc_inv(A,2);
 loc_inv(B1,2);
 loc_inv(B2,2);
```

$\square$

Remark 87 Using Magma, we can verify

$$\begin{array}{rcccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_1 \otimes D_2 & \cong & D_2\,, \\ \mathrm{Coker}(C) & \cong & D_1 \otimes \bar{P}_2 & \cong & \bar{P}_2\,, \\ \mathrm{Coker}(\tilde{C}) & \cong & \bar{P}_1 \otimes D_2 & \cong & \bar{P}_2 \oplus \bar{P}_2\,, \end{array}$$

with Loewy layers already known.

Chapter 5

On localizations of $\mathbb{Z}\mathrm{S}_5$

5.1 The Localization $\mathbb{Z}_{(3)}\mathrm{S}_5$

Two blocks of $\mathbb{Z}_{(3)}\mathrm{S}_5$ are Morita-equivalent to $\mathbb{Z}_{(3)}\mathrm{S}_3$, but the behaviour of the tensor product is not parallel; cf. e.g. Remark 105.

Let

$$R := \mathbb{Z}_{(3)} .$$

5.1.1 Idempotents and projectives

Definition 88 Let

$$\begin{array}{rcl}
\Lambda_{(3)} & := & \{\big(a_{11}, b_{11}, \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{45} \\ e_{51} & e_{52} & e_{53} & e_{54} & e_{55} \end{pmatrix}, \\
& & \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} \\ g_{51} & g_{52} & g_{53} & g_{54} & g_{55} & g_{56} \\ g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & g_{66} \end{pmatrix} \mid \\
& & a_{11} \equiv_3 f_{11} ,\ \ b_{11} \equiv_3 e_{11} , \\
& & c_{11} \equiv_3 e_{22} ,\ \ c_{12} \equiv_3 e_{23} ,\ \ c_{13} \equiv_3 e_{24} ,\ \ c_{14} \equiv_3 e_{25} , \\
& & c_{21} \equiv_3 e_{32} ,\ \ c_{22} \equiv_3 e_{33} ,\ \ c_{23} \equiv_3 e_{34} ,\ \ c_{24} \equiv_3 e_{35} , \\
& & c_{31} \equiv_3 e_{42} ,\ \ c_{32} \equiv_3 e_{43} ,\ \ c_{33} \equiv_3 e_{44} ,\ \ c_{34} \equiv_3 e_{45} , \\
& & c_{41} \equiv_3 e_{52} ,\ \ c_{42} \equiv_3 e_{53} ,\ \ c_{43} \equiv_3 e_{54} ,\ \ c_{44} \equiv_3 e_{55} , \\
& & d_{11} \equiv_3 f_{22} ,\ \ d_{12} \equiv_3 f_{23} ,\ \ d_{13} \equiv_3 f_{24} ,\ \ d_{14} \equiv_3 f_{25} , \\
& & d_{21} \equiv_3 f_{32} ,\ \ d_{22} \equiv_3 f_{33} ,\ \ d_{23} \equiv_3 f_{34} ,\ \ d_{24} \equiv_3 f_{35} , \\
& & d_{31} \equiv_3 f_{42} ,\ \ d_{32} \equiv_3 f_{43} ,\ \ d_{33} \equiv_3 f_{44} ,\ \ d_{34} \equiv_3 f_{45} , \\
& & d_{41} \equiv_3 f_{52} ,\ \ d_{42} \equiv_3 f_{53} ,\ \ d_{43} \equiv_3 f_{54} ,\ \ d_{44} \equiv_3 f_{55} , \\
& & e_{21} \equiv_3 e_{31} \equiv_3 e_{41} \equiv_3 e_{51} \equiv_3 0 ,\ \ f_{21} \equiv_3 f_{31} \equiv_3 f_{41} \equiv_3 f_{51} \equiv_3 0\} \\
& \subseteq & \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{4\times 4} \times \mathbb{Z}_{(3)}^{4\times 4} \times \mathbb{Z}_{(3)}^{5\times 5} \times \mathbb{Z}_{(3)}^{5\times 5} \times \mathbb{Z}_{(3)}^{6\times 6} \\
& = & R \times R \times R^{4\times 4} \times R^{4\times 4} \times R^{5\times 5} \times R^{5\times 5} \times R^{6\times 6} .
\end{array}$$

The $\mathbb{Z}_{(3)}$-order $\Lambda_{(3)}$ is the image of $\mathbb{Z}_{(3)}\mathrm{S}_5$ under the Wedderburn isomorphism of Remark 241.

The tuple entries belong to the Specht modules $S^{(1^5)}$, $S^{(5)}$, $S^{(2,1,1,1)}$, $S^{(4,1)}$, $S^{(2,2,1)}$, $S^{(3,2)}$ and $S^{(3,1,1)}$ in the order chosen above.

Consider the following idempotents of $\Lambda_{(3)}$.

$$e_1 := \left(1, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$e_2 := \left(0, 1, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$e_3 := \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$e_4 := \left(0, 0, \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$e_5 := \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)$$

Remark 89 Let $E := \{(a,b) \in R \times R \mid a \equiv_3 b\}$.

We have isomorphisms of R-orders

$$E \longrightarrow e_1\Lambda_{(3)}e_1$$

$$(a, f_{11}) \longmapsto \left(a, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}f_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$E \longrightarrow e_2\Lambda_{(3)}e_2$$

$$(b, e_{11}) \longmapsto \left(0, b, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}e_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$E \longrightarrow e_3\Lambda_{(3)}e_3$$

$$(d_{11}, f_{22}) \longmapsto \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}d_{11}&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&f_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$E \longrightarrow e_4\Lambda_{(3)}e_4$$

$$(c_{11}, e_{22}) \longmapsto \left(0, 0, \begin{pmatrix}c_{11}&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&e_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$$

$$R \longrightarrow e_5\Lambda_{(3)}e_5$$

$$g_{11} \longmapsto \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}g_{11}&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right).$$

Lemma 90 *The idempotents e_1, e_2, e_3, e_4 and e_5 are primitive.*

They represent the equivalence classes of the primitive idempotents of $\Lambda_{(3)}$.

Proof. We have to show primitivity of e_1. Primitivity of e_2, e_3 and e_4 follows analogously.

To this end, we show that e_1 and 0 are the only idempotents of $e_1\Lambda_{(3)}e_1$. By Remark 89 it remains to show that $(0,0)$ and $(1,1)$ are the only idempotents of E. Let $(a,b) \in E$ be an idempotent. Since $\mathbb{Z}_{(3)}$ is local, Corollary 134 gives $a, b \in \{0,1\}$.

Since this is a finite problem in E, we can use Magma **[3]** to list all possible pairs $(a,b) \in R \times R$ with a, $b \in \{0,\ 1\}$ and to test the ties defining E in a last step. To do so, we call

```
i := {<a,b> : a,b in {0,1}| (a mod 3) eq (b mod 3))};
```

and we get the following result.

```
> i;
{ <0, 0>, <1, 1> }
```

Therefore, using Lemma 136, the idempotent e_1 is primitive.

Now we show primitivity of e_5.

To this end, we show that e_5 and 0 are the only idempotents of $e_5\Lambda_{(3)}e_5$. By Remark 89 it suffices to show that 0 and 1 are the only idempotents of R. Since $\mathbb{Z}_{(3)}$ is local, this follows from Corollary 134. Therefore, using Lemma 136, the idempotent e_5 is primitive.

Finally, there exists an orthogonal decomposition

$$1 = e_1 + e_2 + e_3 + e_3' + e_3'' + e_3''' + e_4 + e_4' + e_4'' + e_4''' + e_5 + e_5' + e_5'' + e_5''' + e_5''''$$

into primitive idempotents, which fall into the equivalence classes

$$\{e_1\},\ \{e_2\},\ \{e_3, e_3', e_3'', e_3'''\},\ \{e_4, e_4', e_4'', e_4'''\},\ \{e_5, e_5', e_5'', e_5''', e_5''''\}\,.$$

Here e_3', e_3'' and e_3''' are obtained from e_3 by "shifting along the main diagonal". Similarly e_4', e_4'' and e_4''', and also e_5', e_5'', e_5''' and e_5''''. □

Corollary 91 *Up to isomorphism, we have the Peirce decomposition*

$$\begin{array}{rcl} \Lambda_{(3)} & \cong & e_1\Lambda_{(3)} \oplus e_3\Lambda_{(3)}{}^{\oplus 4} \oplus e_2\Lambda_{(3)} \oplus e_4\Lambda_{(3)}{}^{\oplus 4} \oplus e_5\Lambda_{(3)}{}^{\oplus 6} \\ & \cong & \begin{pmatrix} (e_1\Lambda_{(3)}e_1)^{1\times 1} & (e_1\Lambda_{(3)}e_3)^{1\times 4} & 0 & 0 & 0 \\ (e_3\Lambda_{(3)}e_1)^{4\times 1} & (e_3\Lambda_{(3)}e_3)^{4\times 4} & 0 & 0 & 0 \\ 0 & 0 & (e_2\Lambda_{(3)}e_2)^{1\times 1} & (e_2\Lambda_{(3)}e_4)^{1\times 4} & 0 \\ 0 & 0 & (e_4\Lambda_{(3)}e_2)^{4\times 1} & (e_4\Lambda_{(3)}e_4)^{4\times 4} & 0 \\ 0 & 0 & 0 & 0 & (e_5\Lambda_{(3)}e_5)^{6\times 6} \end{pmatrix}. \end{array}$$

Lemma 92

Recall that $R = \mathbb{Z}_{(3)}$. We have the following radicals.

$$\mathfrak{r}(e_1\Lambda_{(3)}e_1) =$$

$$e_1\Lambda_{(3)}e_1 \cap \left((3)\times 0\times \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} (3)&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\right)$$

$$\mathfrak{r}(e_2\Lambda_{(3)}e_2)=$$

$$e_2\Lambda_{(3)}e_2\cap\Big(0\times(3)\times\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}(3)&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Big)$$

$$\mathfrak{r}(e_3\Lambda_{(3)}e_3)=$$

$$e_3\Lambda_{(3)}e_3\cap\Big(0\times0\times\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}(3)&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0\\0&(3)&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Big)$$

$$\mathfrak{r}(e_4\Lambda_{(3)}e_4)=$$

$$e_4\Lambda_{(3)}e_4\cap\Big(0\times0\times\begin{pmatrix}(3)&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0\\0&(3)&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Big)$$

$$\mathfrak{r}(e_5\Lambda_{(3)}e_5)=$$

$$e_5\Lambda_{(3)}e_5\cap\Big(0\times0\times\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\times\begin{pmatrix}(3)&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Big)$$

Altogether, we obtain the Jacobson radical

$$\mathfrak{r}(\Lambda_{(3)})=$$

$$\Big((3)\times(3)\times\begin{pmatrix}(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)\end{pmatrix}\times\begin{pmatrix}(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)\end{pmatrix}\times\begin{pmatrix}(3)&R&R&R&R\\R&(3)&(3)&(3)&(3)\\R&(3)&(3)&(3)&(3)\\R&(3)&(3)&(3)&(3)\\R&(3)&(3)&(3)&(3)\end{pmatrix}\times$$

$$\begin{pmatrix}(3)&R&R&R&R\\R&(3)&(3)&(3)&(3)\\R&(3)&(3)&(3)&(3)\\R&(3)&(3)&(3)&(3)\\R&(3)&(3)&(3)&(3)\end{pmatrix}\times\begin{pmatrix}(3)&(3)&(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)&(3)&(3)\\(3)&(3)&(3)&(3)&(3)&(3)\end{pmatrix}\Big)\cap\Lambda_{(3)}\,.$$

Proof. The first follows by Proposition 222; cf. Remark 208. The latter then follows by using Proposition 217 and Remark 208; cf. Corollary 91. □

Definition 93 Let $P_1:=e_1\cdot\Lambda_{(3)}$, $P_2:=e_2\cdot\Lambda_{(3)}$, $P_3:=e_3\cdot\Lambda_{(3)}$, $P_4:=e_4\cdot\Lambda_{(3)}$ and $P_5:=e_5\cdot\Lambda_{(3)}$ represent the isoclasses of indecomposable projective modules of $\Lambda_{(3)}$; cf. Remark 208, Lemma 220.

$$P_1=\Bigg\{\Big(a_{11},0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}f_{11}&f_{12}&f_{13}&f_{14}&f_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Big)\Bigg|$$

$$a_{11}\equiv_3 f_{11}\Bigg\},$$

$$P_2=\Bigg\{\Big(0,b_{11},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}e_{11}&e_{12}&e_{13}&e_{14}&e_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Big)\Bigg|$$

$$b_{11}\equiv_3 e_{11}\Bigg\},$$

$$P_3 = \left\{ \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}d_{11}&d_{12}&d_{13}&d_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\f_{21}&f_{22}&f_{23}&f_{24}&f_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \right| $$
$$\left. f_{21} \equiv_3 0\,, \;\; d_{11} \equiv_3 f_{22}\,, \;\; d_{12} \equiv_3 f_{23}\,, \;\; d_{13} \equiv_3 f_{24}\,, \;\; d_{14} \equiv_3 f_{25} \right\},$$

$$P_4 = \left\{ \left(0, 0, \begin{pmatrix}c_{11}&c_{12}&c_{13}&c_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\e_{21}&e_{22}&e_{23}&e_{24}&e_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \right| $$
$$\left. e_{21} \equiv_3 0\,, \;\; c_{11} \equiv_3 e_{22}\,, \;\; c_{12} \equiv_3 e_{23}\,, \;\; c_{13} \equiv_3 e_{24}\,, \;\; c_{14} \equiv_3 e_{25} \right\},$$

$$P_5 = \left\{ \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}g_{11}&g_{12}&g_{13}&g_{14}&g_{15}&g_{16}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \right\},$$

where all entries are to be read as running through R.

We abbreviate

$$\Big[a_{11},\ (f_{11}\ \ f_{12}\ \ f_{13}\ \ f_{14}\ \ f_{15})\Big]$$
$$:= \quad \left(a_{11}, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}f_{11}&f_{12}&f_{13}&f_{14}&f_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \in P_1$$

$$\Big[b_{11},\ (e_{11}\ \ e_{12}\ \ e_{13}\ \ e_{14}\ \ e_{15})\Big]$$
$$:= \quad \left(0, b_{11}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}e_{11}&e_{12}&e_{13}&e_{14}&e_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \in P_2$$

$$\Big[(d_{11}\ \ d_{12}\ \ d_{13}\ \ d_{14}),\ (f_{21}\ \ f_{22}\ \ f_{23}\ \ f_{24}\ \ f_{25})\Big]$$
$$:= \quad \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}d_{11}&d_{12}&d_{13}&d_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\f_{21}&f_{22}&f_{23}&f_{24}&f_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \in P_3$$

$$\Big[(c_{11}\ \ c_{12}\ \ c_{13}\ \ c_{14}), (e_{21}\ \ e_{22}\ \ e_{23}\ \ e_{24}\ \ e_{25})\Big]$$
$$:= \quad \left(0, 0, \begin{pmatrix}c_{11}&c_{12}&c_{13}&c_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\e_{21}&e_{22}&e_{23}&e_{24}&e_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \in P_4$$

$$\Big[(g_{11}\ \ g_{12}\ \ g_{13}\ \ g_{14}\ \ g_{15}\ \ g_{16})\Big]$$
$$:= \quad \left(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}g_{11}&g_{12}&g_{13}&g_{14}&g_{15}&g_{16}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right) \in P_5\,.$$

So we have $P_1 \hookrightarrow S^{(1^5)} \oplus S^{(3,2)}$, $P_2 \hookrightarrow S^{(5)} \oplus S^{(2,2,1)}$, $P_3 \hookrightarrow S^{(4,1)} \oplus S^{(3,2)}$, $P_4 \hookrightarrow S^{(2,1,1,1)} \oplus S^{(2,2,1)}$ and $P_5 \hookrightarrow S^{(3,1,1)}$.

Using Lemma 92, we may fix the following R-linear bases.

$$\begin{array}{ll}
(\,[1,(1\ 0\ 0\ 0\ 0)], & (\,[3,(0\ 0\ 0\ 0\ 0)],\\
\ [0,(3\ 0\ 0\ 0\ 0)], & \ [0,(3\ 0\ 0\ 0\ 0)],\\
\ [0,(0\ 1\ 0\ 0\ 0)], & \ [0,(0\ 1\ 0\ 0\ 0)],\\
\ [0,(0\ 0\ 1\ 0\ 0)], & \ [0,(0\ 0\ 1\ 0\ 0)],\\
\ [0,(0\ 0\ 0\ 1\ 0)], & \ [0,(0\ 0\ 0\ 1\ 0)],\\
\ [0,(0\ 0\ 0\ 0\ 1)]\,) \ \text{ of } P_1 \text{ and of } P_2, & \ [0,(0\ 0\ 0\ 0\ 1)]\,) \ \text{ of } \mathfrak{r}P_1 \text{ and of } \mathfrak{r}P_2\,,
\end{array}$$

$$\begin{array}{ll}
(\,[(1\ 0\ 0\ 0),(0\ 1\ 0\ 0\ 0)] & (\,[(3\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0)]\\
\ [(0\ 1\ 0\ 0),(0\ 0\ 1\ 0\ 0)] & \ [(0\ 3\ 0\ 0),(0\ 0\ 0\ 0\ 0)]\\
\ [(0\ 0\ 1\ 0),(0\ 0\ 0\ 1\ 0)] & \ [(0\ 0\ 3\ 0),(0\ 0\ 0\ 0\ 0)]\\
\ [(0\ 0\ 0\ 1),(0\ 0\ 0\ 0\ 1)] & \ [(0\ 0\ 0\ 3),(0\ 0\ 0\ 0\ 0)]\\
\ [(0\ 0\ 0\ 0),(3\ 0\ 0\ 0\ 0)] & \ [(0\ 0\ 0\ 0),(3\ 0\ 0\ 0\ 0)]\\
\ [(0\ 0\ 0\ 0),(0\ 3\ 0\ 0\ 0)] & \ [(0\ 0\ 0\ 0),(0\ 3\ 0\ 0\ 0)]\\
\ [(0\ 0\ 0\ 0),(0\ 0\ 3\ 0\ 0)] & \ [(0\ 0\ 0\ 0),(0\ 0\ 3\ 0\ 0)]\\
\ [(0\ 0\ 0\ 0),(0\ 0\ 0\ 3\ 0)] & \ [(0\ 0\ 0\ 0),(0\ 0\ 0\ 3\ 0)]\\
\ [(0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 3)]\,) \ \text{ of } P_3 \text{ and of } P_4, & \ [(0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 3)]\,) \ \text{ of } \mathfrak{r}P_3 \text{ and of } \mathfrak{r}P_4,
\end{array}$$

$$\begin{array}{ll}
(\,[(1\ 0\ 0\ 0\ 0\ 0)], & (\,[(3\ 0\ 0\ 0\ 0\ 0)],\\
\ [(0\ 1\ 0\ 0\ 0\ 0)], & \ [(0\ 3\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 1\ 0\ 0\ 0)], & \ [(0\ 0\ 3\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 1\ 0\ 0)], & \ [(0\ 0\ 0\ 3\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 1\ 0)], & \ [(0\ 0\ 0\ 0\ 3\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0\ 1)]\,) \ \text{ of } P_5, & \ [(0\ 0\ 0\ 0\ 0\ 3)]\,) \ \text{ of } P_5,
\end{array}$$

Remark 94 Using Magma we verify that the indecomposable projective module $\bar{P}_5$ has the Loewy layer D_5, for $\bar{P}_1$ we obtain the Loewy layers $\begin{array}{c} D_1 \\ D_3 \\ D_1 \end{array}$, for $\bar{P}_2$ we find $\begin{array}{c} D_2 \\ D_4 \\ D_2 \end{array}$, the Loewy layers of $\bar{P}_3$ are calculated to be $\begin{array}{c} D_3 \\ D_1 \\ D_3 \end{array}$, and for $\bar{P}_4$ we get $\begin{array}{c} D_4 \\ D_2 \\ D_4 \end{array}$.

Remark 95 Recall that

$$\begin{aligned}
e_1\Lambda_{(3)}e_1 &= \Bigg\{ \underbrace{\left(a_{11}, 0, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} f_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\right)}_{=:\,[\,a_{11},\,f_{11}\,]} \Bigg|\, a_{11} \equiv_3 f_{11} \Bigg\}\\
&= \{[\,a_{11},\,f_{11}\,] \mid a_{11} \equiv_3 f_{11}\}
\end{aligned}$$

for which we fix as R-linear basis

$$(e_1, \tilde{h}_1^{11}) := \{\,[1,1],\,[0,3]\,\}\,.$$

Via the canonical isomorphism from $e_1\Lambda_{(3)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_1, P_1)$, it is mapped to the R-linear basis

$$(1_{P_1}, h_1^{11}) := \left(\begin{pmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0\\0&3&0&0&0&0\\0&0&3&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&3 \end{pmatrix} \right)$$

of $\mathrm{Hom}_{R\mathrm{S}_5}(P_1, P_1)$, using the fixed R-linear basis of P_1 given in Definition 93.

An R-linear basis of

$$\begin{aligned} e_2\Lambda_{(3)}e_2 &= \Big\{ \underbrace{\Big(0, b_{11}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} e_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\Big)}_{=:\,[\,b_{11},\, e_{11}\,]} \Big|\, b_{11} \equiv_3 e_{11} \Big\} \\ &= \{[\,b_{11},\, e_{11}\,] \mid b \equiv_3 e_{11}\,\} \end{aligned}$$

is given by

$$(e_2,\, \tilde{h}_1^{22}) := (\,[1,1],\, [0,3]\,)\,.$$

Via the canonical isomorphism from $e_2\Lambda_{(3)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_2, P_2)$, it is mapped to the R-linear basis

$$(1,\, h_1^{22}) := \left(\begin{pmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0\\0&3&0&0&0&0\\0&0&3&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&3 \end{pmatrix} \right)$$

of $\mathrm{Hom}_{R\mathrm{S}_5}(P_2, P_2)$, using the fixed R-linear basis of P_2, like above.

As an R-linear basis of

$$\begin{aligned} e_3\Lambda_{(3)}e_3 &= \Big\{ \underbrace{\Big(0, 0, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} d_{11}&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&f_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\Big)}_{=:\,[\,d_{11},\, f_{22}\,]} \Big|\, d_{11} \equiv_3 f_{22} \Big\} \\ &= \{\,[\,d_{11},\, f_{22}\,] \mid d_{11} \equiv_3 f_{22}\} \end{aligned}$$

we fix

$$(e_3,\, \tilde{h}_1^{33}) := (\,[1,1],\, [0,3]\,)\,.$$

Via the canonical isomorphism from $e_3\Lambda_{(3)}e_3$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_3, P_3)$, it is mapped to the R-linear basis

$$(1,\, h_1^{33}) := \left(\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\\0&0&0&0&3&0&0&0&0\\0&0&0&0&0&3&0&0&0\\0&0&0&0&0&0&3&0&0\\0&0&0&0&0&0&0&3&0\\0&0&0&0&0&0&0&0&3 \end{pmatrix} \right)$$

of $\mathrm{Hom}_{R\mathrm{S}_5}(P_3, P_3)$, using the fixed R-linear basis of P_3.

An R-linear basis of

$$\begin{aligned} e_4\Lambda_{(3)}e_4 &= \Big\{ \underbrace{\Big(0, 0, \begin{pmatrix} c_{11}&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&e_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\Big)}_{=:\,[\,c_{11},\, e_{22}\,]} \Big|\, c_{11} \equiv_3 e_{22} \Big\} \\ &= \{[\,c_{11},\, e_{22}\,] \mid c_{11} \equiv_3 e_{22}\} \end{aligned}$$

is given by

$$(e_4,\, \tilde{h}_1^{44}) := (\,[1,1],\, [0,3]\,)\,.$$

Via the canonical isomorphism from $e_4\Lambda_{(3)}e_4$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_4, P_4)$, it is mapped to the R-linear basis

$$(1,\, h_1^{44}) := \left(\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\\0&0&0&0&3&0&0&0&0\\0&0&0&0&0&3&0&0&0\\0&0&0&0&0&0&3&0&0\\0&0&0&0&0&0&0&3&0\\0&0&0&0&0&0&0&0&3 \end{pmatrix} \right)$$

of $\mathrm{Hom}_{R\mathrm{S}_5}(P_4, P_4)$, using the fixed R-linear basis of P_4.

An R-linear basis of

$$\begin{aligned} e_5\Lambda_{(3)}e_5 &= \Bigg\{ \underbrace{\left(0,0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}g_{11}&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)}_{=:\,[\,g_{11}\,]} \Bigg\} \\ &= \{[\,g_{11}\,]\} \end{aligned}$$

is given by

$$(e_5) := (\,[1]\,)\,.$$

Via the canonical isomorphism from $e_5\Lambda_{(3)}e_5$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_5,P_5)$, it is mapped to the R-linear basis

$$(1) := \left(\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\right)$$

of $\mathrm{Hom}_{R\mathrm{S}_5}(P_5,P_5)$, using the fixed R-linear basis of P_5.

Now

$$\begin{aligned} e_2\Lambda_{(3)}e_4 &= \Bigg\{ \underbrace{\left(0,0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&e_{12}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)}_{=:\,[\,e_{12}\,]} \Bigg\} \\ &= \{[\,e_{12}\,]\} \end{aligned}$$

therefore an R-linear basis is given by

$$(\tilde{h}_1^{42}) := \{\,[1]\,\}\,.$$

Via the canonical isomorphism from $e_2\Lambda_{(3)}e_4$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_4,P_2)$ it is mapped to

$$(h_1^{42}) := \left(\begin{pmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&3&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&3\end{pmatrix}\right).$$

Further

$$\begin{aligned} e_1\Lambda_{(3)}e_3 &= \Bigg\{ \underbrace{\left(0,0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&f_{12}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)}_{=:\,[\,f_{12}\,]} \Bigg\} \\ &= \{\,[\,f_{12}\,]\,\}, \end{aligned}$$

therefore an R-linear basis is given by

$$(\tilde{h}_1^{31}) := \{\,[1]\,\}\,.$$

Via the canonical isomorphism from $e_1\Lambda_{(3)}e_3$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_3,P_1)$ it is mapped to

$$(h_1^{31}) := \left(\begin{pmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&3&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&3\end{pmatrix}\right).$$

Moreover,

$$\begin{aligned} e_3\Lambda_{(3)}e_1 &= \Bigg\{ \underbrace{\left(0,0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\f_{21}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)}_{=:\,[\,f_{21}\,]} \;\Bigg|\; f_{21} \equiv_3 0 \Bigg\} \\ &= \{\,[\,f_{21}\,]\,\}, \end{aligned}$$

therefore an R-linear basis is given by

$$(\tilde{h}_1^{13}) \;:=\; \{\; [3] \;\}\,.$$

Via the canonical isomorphism from $e_3\Lambda_{(3)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_1, P_3)$ it is mapped to

$$(h_1^{13}) \;:=\; \left(\begin{pmatrix} 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&3&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}\right).$$

We have

$$\begin{aligned} e_4\Lambda_{(3)}e_2 \;&=\; \Bigg\{\underbrace{\left(0,0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\e_{21}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)}_{=:\,[\,e_{21}\,]} \;\Big|\; e_{21} \equiv_3 0 \Bigg\} \\ &=\; \{\,[\,e_{21}\,]\,\}\,, \end{aligned}$$

therefore an R-linear basis is given by

$$(\tilde{h}_1^{24}) \;:=\; \{\; [3] \;\}\,.$$

Via the canonical isomorphism from $e_4\Lambda_{(3)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_2, P_4)$ it is mapped to

$$(h_1^{24}) \;:=\; \left(\begin{pmatrix} 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&3&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}\right).$$

The operating matrices can be found in the file `main_S5_loc3`, the homomorphisms `Homs_S5_loc3`.

They can e.g. be used to check the $R\mathrm{S}_5$-linearity of the maps between P_1, P_2, P_3, P_4 and P_5 derived above.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 are denoted `rhoP1`, `rhoP2`,`rhoP3`, `rhoP4` and `rhoP5`, respectively.

E.g. for the operating matrices on P_1, call

```
rhoP1(S5P!sigma);
```

for an element `sigma` of S_5. Analogously for P_2, P_3, P_4 and P_5.

To check that the matrices found above represent $R\mathrm{S}_5$-linear maps between the respective projective modules, follow these steps:

```
load main_S5_loc3;
load Homs_S5_loc3;

[rhoP1(sigma)*Hom_P1P1[i] eq Hom_P1P1[i]*rhoP1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..2]];
[rhoP2(sigma)*Hom_P2P2[i] eq Hom_P2P2[i]*rhoP2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..2]];
[rhoP2(sigma)*Hom_P2P4[i] eq Hom_P2P4[i]*rhoP4(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..1]];

[rhoP1(sigma)*Hom_P1P3[i] eq Hom_P1P3[i]*rhoP3(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..1]];
[rhoP3(sigma)*Hom_P3P1[i] eq Hom_P3P1[i]*rhoP1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..1]];
[rhoP3(sigma)*Hom_P3P3[i] eq Hom_P3P3[i]*rhoP3(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..2]];

[rhoP4(sigma)*Hom_P4P2[i] eq Hom_P4P2[i]*rhoP2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..1]];
[rhoP4(sigma)*Hom_P4P4[i] eq Hom_P4P4[i]*rhoP4(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..2]];
[rhoP5(sigma)*Hom_P5P5[i] eq Hom_P5P5[i]*rhoP5(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}, i in [1..1]];
```

Remark 96 There are 3 primitive central idempotents of $\Lambda_{(3)}$,

$c_1 :=$

$$\left(1, 0, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\right),$$

$c_2 :=$

$$\left(0, 1, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\right),$$

$c_3 :=$

$$\left(0, 0, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{pmatrix}\right)$$

The idempotent c_3 generates a block of defect 0.

Remark 97 Let $S^{(1^5)}$ be the alternating Specht module of $\Lambda_{(3)}$. Then we get the isomorphisms of projective modules of $\Lambda_{(3)}$

$$P_1 \;\cong\; S^{(1^5)} \;\otimes\; P_2\,, \quad P_3 \;\cong\; S^{(1^5)} \;\otimes\; P_4 \quad \text{and} \quad P_5 \;\cong\; S^{(1^5)} \;\otimes\; P_5.$$

Therefore, by applying Remark 176, we can reduce our problem of the projective modules of $R\mathrm{S}_5$ to the tensor products $P_2 \otimes P_2$, $P_2 \otimes P_3$, $P_2 \otimes P_5$, $P_3 \otimes P_3$, $P_3 \otimes P_5$ and $P_5 \otimes P_5$.

5.1.2 The tensor product $P_2 \otimes P_2$

Lemma 98 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_2 \longrightarrow P_2 \otimes P_2$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_2 \longrightarrow P_2 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_5$-linear map*

$$\begin{tikzcd}
P_2 \otimes \mathfrak{r}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2^{\oplus 2} \oplus P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 1} \arrow[d, "\tilde{C}"] \\
P_2 \otimes P_2 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2^{\oplus 2} \oplus P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 1} \\
\mathfrak{r}P_2 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_2^{\oplus 2} \oplus P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 1} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cc|c|c|c} 3 & . & . & . & . \\ . & 1 & . & . & . \\ \hline . & . & 1 & . & . \\ \hline . & . & . & 1 & . \\ \hline . & . & . & . & 1 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{cc|c|c|c} 3 & . & . & . & . \\ . & 1 & . & . & . \\ \hline . & . & 1 & . & . \\ \hline -h_1^{42} & . & . & 1 & . \\ \hline . & . & . & . & 1 \end{array}\right].$$

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.

We have to show local invertibility and $R\mathrm{S}_5$-linearity of the maps A, B and $\tilde{B}$, the $R\mathrm{S}_5$-linearity of E, $\tilde{E}$, C and $\tilde{C}$, and the commutativity of the diagram. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc3`, the matrices for this diagram in the file `Diagram_S5_loc3_P2oP2` .

The embedding E is defined as the Kronecker product of the embedding $i_2 : \mathfrak{r}P_2 \hookrightarrow P_2$ and id_{P_2}. An embedding in the chosen bases can be found in the files and is denoted `i1`. The embedding $\tilde{E}$ is defined as the Kronecker product of id_{P_2} and the embedding $i_2 : \mathfrak{r}P_2 \hookrightarrow P_2$.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2, P_3, P_4, P_5 are denoted `rhoP1`, `rhoP2`, `rhoP3`, `rhoP4`, `rhoP5`, respectively.

The representations on $\mathfrak{r}P_1$, $\mathfrak{r}P_2$, $\mathfrak{r}P_3$, $\mathfrak{r}P_4$, $\mathfrak{r}P_5$ are denoted `rhorP1`, `rhorP2`, `rhorP3`, `rhorP4`, `rhorP5`, respectively.

The representations on $P_2 \otimes P_2$, $\mathfrak{r}P_2 \otimes P_2$, $P_2 \otimes \mathfrak{r}P_2$ are denoted `op_p2op2`, `op_rp2op2`, `op_p2orp2`, respectively. The operating matrix of a group element on such a tensor product is defined as the tensor product of the operating matrices on the tensor factors.

For example, for the operating matrices on $P_2 \otimes P_2$ call

```
op_p2op2(S5P!sigma);
```

for an element `sigma` of S_5. The other maps work the same way.

The representation on the direct sum $P_2^{\oplus 2} \oplus P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 1}$ is denoted `op_proj_sum_p2p2`. The operating matrix of a group element is defined as the block diagonal matrix containing the operating matrices of the summands. The maps A, B, C, E are denoted `A`, `B1`, `C1`, `E1`, respectively; the maps $\tilde{B}$, $\tilde{C}$, $\tilde{E}$ are denoted `B2`, `C2`, `E2`, respectively.

To verify the Lemma, follow these steps:

```
load "main_S5_loc3";
load "main_S5_loc3_P2oP2";
load "Diagram_S5_loc3_P2oP2";

 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i2,MatrixRing(Rationals(),6)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),6)!1,i2);

 [op_rp2op2(sigma)*E1 eq E1*op_p2op2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp2(sigma)*E2 eq E2*op_p2op2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity
 [op_p2op2(sigma)*A eq A*op_proj_sum_p2p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp2op2(sigma)*B1 eq B1*op_proj_sum_p2p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp2(sigma)*B2 eq B2*op_proj_sum_p2p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p2p2(sigma)*C1 eq C1*op_proj_sum_p2p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p2p2(sigma)*C2 eq C2*op_proj_sum_p2p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility; loc_inv see "main_S5_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

□

Remark 99 Using Magma, we can verify

$$\begin{array}{rcccccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_2 \otimes D_2 & \cong & D_2\,, & & \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_2 \otimes \bar{P}_2 & \cong & \bar{P}_2\,, \end{array}$$

with Loewy layers already known.

5.1.3 The tensor product $P_2 \otimes P_3$

Lemma 100 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_3 \longrightarrow P_2 \otimes P_3$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_3 \longrightarrow P_2 \otimes P_3$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd} P_2 \otimes \mathfrak{r}P_3 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 1} \oplus P_3^{\oplus 2} \oplus P_4^{\oplus 2} \oplus P_5^{\oplus 2} \arrow[d, "\tilde{C}"] \\ P_2 \otimes P_3 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 1} \oplus P_3^{\oplus 2} \oplus P_4^{\oplus 2} \oplus P_5^{\oplus 2} \\ \mathfrak{r}P_2 \otimes P_3 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_1^{\oplus 1} \oplus P_3^{\oplus 2} \oplus P_4^{\oplus 2} \oplus P_5^{\oplus 2} \arrow[u, "C"'] \end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{c|cc|cc|cc} 1 & . & . & . & . & . & . \\ \hline . & 3 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ \hline . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ \hline . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{c|cc|cc|cc} 1 & -h_1^{13} & . & . & . & . & . \\ \hline . & 3 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ \hline . & . & . & 3 & . & . & . \\ . & . & . & . & 1 & . & . \\ \hline . & . & . & . & . & 3 & . \\ . & . & . & . & . & . & 1 \end{array}\right].$$

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc3` and `main_S5_loc3_P2oP3` the matrices for this diagram in the file `Diagram_S5_loc3_P2oP3` .

The maps and matrices are denoted analogously to those for $P_2 \otimes P_2$, see proof of Lemma 98 .

To verify the Lemma, follow these steps:

```
load "main_S5_loc3";
load "main_S5_loc3_P2oP3";
load "Diagram_S5_loc3_P2oP3";

 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
```

```
E1 := KroneckerProduct(i2,MatrixRing(Rationals(),9)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),6)!1,i3);

 [op_rp2op3(sigma)*E1 eq E1*op_p2op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp3(sigma)*E2 eq E2*op_p2op3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity
 [op_p2op3(sigma)*A eq A*op_proj_sum_p2p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp2op3(sigma)*B1 eq B1*op_proj_sum_p2p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp3(sigma)*B2 eq B2*op_proj_sum_p2p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p2p3(sigma)*C1 eq C1*op_proj_sum_p2p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p2p3(sigma)*C2 eq C2*op_proj_sum_p2p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility; loc_inv see "main_S5_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

□

Remark 101 Using Magma, we can verify

$$\begin{array}{rcccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_2 \otimes D_3 & \cong & D_3\,, \\ \mathrm{Coker}(C) & \cong & D_2 \otimes \bar{P}_3 & \cong & \bar{P}_3\,, \\ \mathrm{Coker}(\tilde{C}) & \cong & \bar{P}_2 \otimes D_3 & \cong & \bar{P}_3 \oplus \bar{P}_4 \oplus \bar{P}_5\,, \end{array}$$

with Loewy layers already known.

5.1.4 The tensor product $P_2 \otimes P_5$

Lemma 102 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_5 \longrightarrow P_2 \otimes P_5$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_5 \longrightarrow P_2 \otimes P_5$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd} P_2 \otimes \mathfrak{r}P_5 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 3} \arrow[d, "\tilde{C}"] \\ P_2 \otimes P_5 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 3} \\ \mathfrak{r}P_2 \otimes P_5 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 3} \arrow[u, "C"'] \end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{c|c|ccc} 1 & . & . & . & . \\ \hline . & 1 & . & . & . \\ \hline . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 3 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{c|c|ccc} 3 & . & . & . & . \\ \hline . & 3 & . & . & . \\ \hline . & . & 3 & . & . \\ . & . & . & 3 & . \\ . & . & . & . & 3 \end{array}\right].$$

The matrices C and $\tilde{C}$ being block diagonal matrices, with blocks of the form id and $3 \cdot \mathrm{id}$, confirms the expected result, shown in Remark 163, for P_5 belongs to a defect-0 block.

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the RS_5-linearity and the commutativity can be found in the file `main_S5_loc3` and `main_S5_loc3_P2oP5` the matrices for this diagram in the file `Diagram_S5_loc3_P2oP5` .

The maps and matrices are denoted analogously to those for $P_2 \otimes P_2$, see proof of Lemma 98 .

To verify the Lemma, follow these steps:

```
load "main_S5_loc3";
load "main_S5_loc3_P2oP5";
load "Diagram_S5_loc3_P2oP5";

 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [rhorP5(sigma)*i5 eq i5*rhoP5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i2,MatrixRing(Rationals(),6)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),6)!1,i5);

 [op_rp2op5(sigma)*E1 eq E1*op_p2op5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp5(sigma)*E2 eq E2*op_p2op5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity
 [op_p2op5(sigma)*A eq A*op_proj_sum_p2p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp2op5(sigma)*B1 eq B1*op_proj_sum_p2p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp5(sigma)*B2 eq B2*op_proj_sum_p2p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p2p5(sigma)*C1 eq C1*op_proj_sum_p2p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p2p5(sigma)*C2 eq C2*op_proj_sum_p2p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

 //local invertibility; loc_inv see "main_S5_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

□

Remark 103 Using Magma, we can verify

$$\begin{array}{rcccccccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_2 \otimes D_5 & \cong & D_2 \otimes \bar{P}_5 & \cong & D_5 & \cong & \bar{P}_5\,, \\ \mathrm{Coker}(C) & \cong & D_2 \otimes \bar{P}_5 & \cong & D_5 & \cong & \bar{P}_5\,, & & \\ \mathrm{Coker}(\tilde{C}) & \cong & \bar{P}_2 \otimes D_5 & \cong & \bar{P}_2 \otimes \bar{P}_5 & \cong & \bar{P}_3 \oplus \bar{P}_4 \oplus \bar{P}_5^{\oplus 3} & & \end{array}$$

with Loewy layers already known.

5.1.5 The tensor product $P_3 \otimes P_3$

Lemma 104 *Let E be the embedding $\mathfrak{t}P_3 \otimes P_3 \longrightarrow P_3 \otimes P_3$, and $\tilde{E}$ be the embedding $P_3 \otimes \mathfrak{t}P_3 \longrightarrow P_3 \otimes P_3$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd}
P_3 \otimes \mathfrak{t}P_3 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2^{\oplus 2} \oplus P_3^{\oplus 4} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 4} \arrow[d, "\tilde{C}"] \\
P_3 \otimes P_3 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2^{\oplus 2} \oplus P_3^{\oplus 4} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 4} \\
\mathfrak{t}P_3 \otimes P_3 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_2^{\oplus 2} \oplus P_3^{\oplus 4} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 4} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cc|cccc|c|cccc}
3 & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . \\
\hline
. & . & 3 & . & . & . & . & . & . & . & . \\
. & . & . & 3 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
\hline
. & . & . & . & . & . & 1 & . & . & . & . \\
\hline
. & . & . & . & . & . & . & 3 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 3 & . \\
. & . & . & . & . & . & . & . & . & . & 1
\end{array}\right], \quad \tilde{C} := \left[\begin{array}{cc|cccc|c|cccc}
3 & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . \\
\hline
. & . & 3 & . & . & . & . & . & . & . & . \\
. & . & . & 3 & . & . & . & . & . & . & . \\
. & . & 1 & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
\hline
2h_1^{42} & . & . & . & . & . & 1 & . & . & . & . \\
\hline
. & . & . & . & . & . & . & 3 & . & . & . \\
. & . & . & . & . & . & . & -1 & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 3 & . \\
. & . & . & . & . & . & . & . & . & . & 1
\end{array}\right].$$

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc3` and `main_S5_loc3_P3oP3` the matrices for this diagram in the file `Diagram_S5_loc3_P3oP3`.

The maps and matrices are denoted analogously to those for $P_2 \otimes P_2$, see proof of Lemma 98.

To verify the Lemma, follow these steps:

```
load "main_S5_loc3";
load "main_S5_loc3_P3oP3";
load "Diagram_S5_loc3_P3oP3";

 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i3,MatrixRing(Rationals(),9)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),9)!1,i3);

 [op_rp3op3(sigma)*E1 eq E1*op_p3op3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p3orp3(sigma)*E2 eq E2*op_p3op3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
```

```
//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity
 [op_p3op3(sigma)*A eq A*op_proj_sum_p3p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp3op3(sigma)*B1 eq B1*op_proj_sum_p3p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p3orp3(sigma)*B2 eq B2*op_proj_sum_p3p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p3p3(sigma)*C1 eq C1*op_proj_sum_p3p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p3p3(sigma)*C2 eq C2*op_proj_sum_p3p3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility; loc_inv see "main_S5_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

$\square$

Remark 105 Using Magma, we can verify

$$\begin{array}{ccccccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_3 \otimes D_3 & \cong & D_2 \oplus \bar{P}_3 \oplus \bar{P}_5 \,, & & \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_3 \otimes \bar{P}_3 & \cong & \bar{P}_2 \oplus \bar{P}_3^{\oplus 2} \oplus \bar{P}_5^{\oplus 2} \,, \end{array}$$

with Loewy layers already known.

5.1.6 The tensor product $P_3 \otimes P_5$

Lemma 106 *Let E be the embedding $\mathfrak{r}P_3 \otimes P_5 \longrightarrow P_3 \otimes P_5$, and $\tilde{E}$ be the embedding $P_3 \otimes \mathfrak{r}P_5 \longrightarrow P_3 \otimes P_5$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd}
P_3 \otimes \mathfrak{r}P_5 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_3^{\oplus 2} \oplus P_4^{\oplus 2} \oplus P_5^{\oplus 3} \arrow[d, "\tilde{C}"] \\
P_3 \otimes P_5 \arrow[r, "A", "\sim"'] & P_3^{\oplus 2} \oplus P_4^{\oplus 2} \oplus P_5^{\oplus 3} \\
\mathfrak{r}P_3 \otimes P_5 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_3^{\oplus 2} \oplus P_4^{\oplus 2} \oplus P_5^{\oplus 3} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cc|cc|ccc} 3 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ \hline . & . & 3 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ \hline . & . & . & . & 3 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{cc|cc|ccc} 3 & . & . & . & . & . & . \\ . & 3 & . & . & . & . & . \\ \hline . & . & 3 & . & . & . & . \\ . & . & . & 3 & . & . & . \\ \hline . & . & . & . & 3 & . & . \\ . & . & . & . & . & 3 & . \\ . & . & . & . & . & . & 3 \end{array}\right].$$

The matrices C and $\tilde{C}$ being block diagonal matrices, with blocks of the form id and $3 \cdot \mathrm{id}$, confirms the expected result, shown in Remark 163, for P_5 belongs to a defect-0 block.

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc3` and `main_S5_loc3_P3oP5` the matrices for this diagram in the file `Diagram_S5_loc3_P3oP5` .

The maps and matrices are denoted analogously to those for $P_2 \otimes P_2$, see proof of Lemma 98 .
To verify the Lemma, follow these steps:

```
load "main_S5_loc3";
load "main_S5_loc3_P3oP5";
load "Diagram_S5_loc3_P3oP5";

 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [rhorP5(sigma)*i5 eq i5*rhoP5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i3,MatrixRing(Rationals(),6)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),9)!1,i5);

 [op_rp3op5(sigma)*E1 eq E1*op_p3op5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p3orp5(sigma)*E2 eq E2*op_p3op5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity
 [op_p3op5(sigma)*A eq A*op_proj_sum_p3p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp3op5(sigma)*B1 eq B1*op_proj_sum_p3p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p3orp5(sigma)*B2 eq B2*op_proj_sum_p3p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p3p5(sigma)*C1 eq C1*op_proj_sum_p3p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p3p5(sigma)*C2 eq C2*op_proj_sum_p3p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

 //local invertibility; loc_inv see "main_S5_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

□

Remark 107 Using Magma, we can verify

$$\begin{aligned}
\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) &\cong D_3 \otimes D_5 \cong D_3 \otimes \bar{P}_5 \cong \bar{P}_3 \oplus \bar{P}_4 \oplus D_5 \cong \bar{P}_3 \oplus \bar{P}_4 \oplus \bar{P}_5\,, \\
\mathrm{Coker}(C) &\cong D_3 \otimes \bar{P}_5 \cong \bar{P}_3 \oplus \bar{P}_4 \oplus D_5 \cong \bar{P}_3 \oplus \bar{P}_4 \oplus \bar{P}_5\,, \\
\mathrm{Coker}(\tilde{C}) &\cong \bar{P}_3 \otimes D_5 \cong \bar{P}_5 \otimes \bar{P}_3 \cong \bar{P}_3^{\oplus 2} \oplus \bar{P}_4^{\oplus 2} \oplus \bar{D}_5^{\oplus 3} \cong \bar{P}_3^{\oplus 2} \oplus \bar{P}_4^{\oplus 2} \oplus \bar{P}_5^{\oplus 3}\,,
\end{aligned}$$

with Loewy layers already known.

5.1.7 The tensor product $P_5 \otimes P_5$

Lemma 108 *Let E be the embedding $\mathfrak{t}P_5 \otimes P_5 \longrightarrow P_5 \otimes P_5$, and $\tilde{E}$ be the embedding $P_5 \otimes \mathfrak{t}P_5 \longrightarrow P_5 \otimes P_5$. We have a commutative diagram of $\mathbb{Z}_{(3)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd}
P_5 \otimes \mathfrak{t}P_5 \arrow[rr, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[drr, phantom, "\circlearrowleft"] & & P_2^{\oplus 1} \oplus P_1^{\oplus 1} \oplus P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 1} \arrow[d, "\tilde{C}"] \\
P_5 \otimes P_5 \arrow[rr, "A", "\sim"'] & & P_2^{\oplus 1} \oplus P_1^{\oplus 1} \oplus P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 1} \\
\mathfrak{t}P_5 \otimes P_5 \arrow[rr, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[urr, phantom, "\circlearrowleft"] & & P_2^{\oplus 1} \oplus P_1^{\oplus 1} \oplus P_3^{\oplus 1} \oplus P_4^{\oplus 1} \oplus P_5^{\oplus 1} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{c|c|c|c|c} 3 & . & . & . & . \\ \hline . & 3 & . & . & . \\ \hline . & . & 3 & . & . \\ \hline . & . & . & 3 & . \\ \hline . & . & . & . & 3 \end{array}\right] \quad \tilde{C} := \left[\begin{array}{c|c|c|c|c} 3 & . & . & . & . \\ \hline . & 3 & . & . & . \\ \hline . & . & 3 & . & . \\ \hline . & . & . & 3 & . \\ \hline . & . & . & . & 3 \end{array}\right]$$

The matrices C and $\tilde{C}$ being block diagonal matrices, with blocks of the form $3 \cdot \mathrm{id}$, confirms the expected result, shown in Remark 164, for P_5 belongs to a defect-0 block.

Recall that $R = \mathbb{Z}_{(3)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc3` and `main_S5_loc3_P5oP5` the matrices for this diagram in the file `Diagram_S5_loc3_P5oP5` .

The maps and matrices are denoted analogously to those for $P_2 \otimes P_2$, see proof of Lemma 98 .

To verify the Lemma, follow these steps:

```
load "main_S5_loc3";
load "main_S5_loc3_P5oP5";
load "Diagram_S5_loc3_P5oP5";

 [rhorP5(sigma)*i5 eq i5*rhoP5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i5,MatrixRing(Rationals(),6)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),6)!1,i5);

 [op_rp5op5(sigma)*E1 eq E1*op_p5op5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p5orp5(sigma)*E2 eq E2*op_p5op5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//commutativity
 E1*A eq B1*C1;
 E2*A eq B2*C2;
```

```
//RS5-linearity:
 [op_p5op5(sigma)*A eq A*op_proj_sum_p5p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp5op5(sigma)*B1 eq B1*op_proj_sum_p5p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p5orp5(sigma)*B2 eq B2*op_proj_sum_p5p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p5p5(sigma)*C1 eq C1*op_proj_sum_p5p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p5p5(sigma)*C2 eq C2*op_proj_sum_p5p5(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility; loc_inv see "main_S5_loc3"
 loc_inv(A,3);
 loc_inv(B1,3);
 loc_inv(B2,3);
```

□

Remark 109 Using Magma, we can verify

$$\begin{aligned} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) \;&\cong\; \mathrm{Coker}(C) \;\cong\; \mathrm{Coker}(\tilde{C}) \\ &\cong\; D_5 \otimes D_5 \;\cong\; \bar{P}_5 \otimes \bar{P}_5 \;\cong\; \bar{P}_1 \oplus \bar{P}_2 \oplus \bar{P}_3 \oplus \bar{P}_4 \oplus D_5 \\ &\cong\; \bar{P}_1 \oplus \bar{P}_2 \oplus \bar{P}_3 \oplus \bar{P}_4 \oplus \bar{P}_5 \,, \end{aligned}$$

with Loewy layers already known.

5.2 The Localization $\mathbb{Z}_{(2)}\mathrm{S}_5$

In this section, we will meet twice the phenomenon that a tensor product of two simple modules may have an indecomposable summand that is neither simple nor projective nor Specht. We consider the corresponding canonical presentations in Sections 5.2.5 and 5.2.7.

Let

$$R \;:=\; \mathbb{Z}_{(2)} \;.$$

5.2.1 Idempotents and projectives

Definition 110 Let

$$\begin{aligned} \Lambda_{(2)} \;:=\; \{\big(&a_{11}, b_{11}, \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}, \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{45} \\ e_{51} & e_{52} & e_{53} & e_{54} & e_{55} \end{pmatrix}, \\ &\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} \\ g_{51} & g_{52} & g_{53} & g_{54} & g_{55} & g_{56} \\ g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & g_{66} \end{pmatrix} \mid \end{aligned}$$

$$\begin{aligned} & a_{11} \equiv_2 b_{11}\,,\;\; e_{11} \equiv_2 g_{66}\,,\;\; a_{11} - f_{11} \equiv_4 2g_{61}\,,\;\; a_{11} + b_{11} + e_{11} + f_{11} \equiv_8 2g_{11} + 2g_{66} \equiv_4 0\,, \\ & e_{12} + f_{12} \equiv_8 2g_{12}\,,\;\; e_{13} + f_{13} \equiv_8 2g_{13}\,,\;\; e_{14} + f_{14} \equiv_8 2g_{14}\,,\;\; e_{15} + f_{15} \equiv_8 2g_{15}\,, \\ & e_{22} + f_{22} \equiv_8 2g_{22}\,,\;\; e_{23} + f_{23} \equiv_8 2g_{23}\,,\;\; e_{24} + f_{24} \equiv_8 2g_{24}\,,\;\; e_{25} + f_{25} \equiv_8 2g_{25}\,, \\ & e_{32} + f_{32} \equiv_8 2g_{32}\,,\;\; e_{33} + f_{33} \equiv_8 2g_{33}\,,\;\; e_{34} + f_{34} \equiv_8 2g_{34}\,,\;\; e_{35} + f_{35} \equiv_8 2g_{35}\,, \\ & e_{42} + f_{42} \equiv_8 2g_{42}\,,\;\; e_{43} + f_{43} \equiv_8 2g_{43}\,,\;\; e_{44} + f_{44} \equiv_8 2g_{44}\,,\;\; e_{45} + f_{45} \equiv_8 2g_{45}\,, \\ & e_{52} + f_{52} \equiv_8 2g_{52}\,,\;\; e_{53} + f_{53} \equiv_8 2g_{53}\,,\;\; e_{54} + f_{54} \equiv_8 2g_{54}\,,\;\; e_{55} + f_{55} \equiv_8 2g_{55}\,, \\ & f_{21} \equiv_2 g_{21}\,,\;\; f_{22} \equiv_2 g_{22}\,,\;\; f_{23} \equiv_2 g_{23}\,,\;\; f_{24} \equiv_2 g_{24}\,,\;\; f_{25} \equiv_2 g_{25}\,, \\ & f_{31} \equiv_2 g_{31}\,,\;\; f_{32} \equiv_2 g_{32}\,,\;\; f_{33} \equiv_2 g_{33}\,,\;\; f_{34} \equiv_2 g_{34}\,,\;\; f_{35} \equiv_2 g_{35}\,, \\ & f_{41} \equiv_2 g_{41}\,,\;\; f_{42} \equiv_2 g_{42}\,,\;\; f_{43} \equiv_2 g_{43}\,,\;\; f_{44} \equiv_2 g_{44}\,,\;\; f_{45} \equiv_2 g_{45}\,, \\ & f_{51} \equiv_2 g_{51}\,,\;\; f_{52} \equiv_2 g_{52}\,,\;\; f_{53} \equiv_2 g_{53}\,,\;\; f_{54} \equiv_2 g_{54}\,,\;\; f_{55} \equiv_2 g_{55}\,, \end{aligned}$$

$$
\begin{array}{rl}
& c_{11} \equiv_2 d_{11}\,,\ \ c_{12} \equiv_2 d_{12}\,,\ \ c_{13} \equiv_2 d_{13}\,,\ \ c_{14} \equiv_2 d_{14}\,,\\
& c_{21} \equiv_2 d_{21}\,,\ \ c_{22} \equiv_2 d_{22}\,,\ \ c_{23} \equiv_2 d_{23}\,,\ \ c_{24} \equiv_2 d_{24}\,,\\
& c_{31} \equiv_2 d_{31}\,,\ \ c_{32} \equiv_2 d_{32}\,,\ \ c_{33} \equiv_2 d_{33}\,,\ \ c_{34} \equiv_2 d_{34}\,,\\
& c_{41} \equiv_2 d_{41}\,,\ \ c_{42} \equiv_2 d_{42}\,,\ \ c_{43} \equiv_2 d_{43}\,,\ \ c_{44} \equiv_2 d_{44}\,,\\
& e_{11} - f_{11} \equiv_4 g_{16}\,,\ \ e_{21} - f_{21} \equiv_4 g_{26}\,,\ \ e_{31} - f_{31} \equiv_4 g_{36}\,,\ \ e_{41} - f_{41} \equiv_4 g_{46}\,,\ \ e_{51} - f_{51} \equiv_4 g_{56}\,,\\
& f_{12} \equiv_4 g_{62}\,,\ \ f_{13} \equiv_4 g_{63}\,,\ \ f_{14} \equiv_4 g_{64}\,,\ \ f_{15} \equiv_4 g_{65}\,\}\\
\subseteq & \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{6\times 6}\\
= & R \times R \times R^{4\times 4} \times R^{4\times 4} \times R^{5\times 5} \times R^{5\times 5} \times R^{6\times 6}\,.
\end{array}
$$

The $\mathbb{Z}_{(2)}$-order $\Lambda_{(2)}$ is the image of $\mathbb{Z}_{(2)}\mathrm{S}_5$ under the Wedderburn isomorphism of Remark 242.

The entries of tuples belong to the Specht modules $S^{(1^5)}$, $S^{(5)}$, $S^{(2,1,1,1)}$, $S^{(4,1)}$, $S^{(2,2,1)}$, $S^{(3,2)}$ and $S^{(3,1,1)}$ in the order chosen above.

Consider the following idempotents of $\Lambda_{(2)}$.

$$
e_1 := \left(1, 1, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1 \end{pmatrix}\right),
$$

$$
e_2 := \left(0, 0, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\right),
$$

$$
e_3 := \left(0, 0, \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\right)
$$

Remark 111 Let

$$
\begin{array}{rcl}
E & := & \{(a, b, c, d, \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)) \mid \\
& & \quad a \equiv_2 b\,,\ c \equiv_2 h\,,\ a - d \equiv_4 2g\,,\ a + b + c + d \equiv_8 2e + 2h \equiv_4 0,\ c - d \equiv_4 f\ \}\\
& \subseteq & R \times R \times R \times R \times R^{2\times 2}\,,\\
F & := & \{(a, b, c)\ \mid a + b \equiv_8 2c,\ a \equiv_2 c\} \subseteq R \times R \times R,\\
G & := & \{(a, b)\ \mid a \equiv_2 b\} \subseteq R \times R.
\end{array}
$$

We have isomorphisms of R-orders

$$
\begin{array}{rcl}
E & \longrightarrow & e_1 \Lambda_{(2)} e_1\\
(a, b, e_{11}, f_{11}, \left(\begin{smallmatrix} g_{11} & g_{16} \\ g_{61} & g_{66} \end{smallmatrix}\right)) & \longmapsto &
\end{array}
$$

$$
\left(a, b, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} e_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} f_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} g_{11}&0&0&0&0&g_{16}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\g_{61}&0&0&0&0&g_{66} \end{pmatrix}\right),
$$

$$
\begin{array}{rcl}
F & \longrightarrow & e_2 \Lambda_{(2)} e_2\\
(e_{22}, f_{22}, g_{22}) & \longmapsto &
\end{array}
$$

$$
\left(0, 0, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&e_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&f_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&g_{22}&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}\right),
$$

$$\begin{array}{rcl} G & \longrightarrow & e_3\Lambda_{(2)}e_3 \\ (c_{11},d_{11}) & \longmapsto & \end{array}$$

$$\left(0,0,\begin{pmatrix} c_{11} & 0&0&0\\ 0 & 0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix},\begin{pmatrix} d_{11} & 0&0&0\\ 0 & 0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right).$$

Lemma 112 *The idempotents* e_1 , e_2 *and* e_3 *are primitive.*

They represent the equivalence classes of the primitive idempotents of $\Lambda_{(2)}$.

Proof. We have to show primitivity of e_1, e_2 and e_3 .

First we show that e_2 and 0 are the only idempotents of $e_2\Lambda_{(2)}e_2$. By Remark 111 it remains to show that $(0,0,0)$ and $(1,1,1)$ are the only idempotents of F. Let $(a,b,c) \in F$ be an idempotent. Since R is local, Corollary 134 gives $a,b,c \in \{0,1\}$. Since this is a finite problem, we can use Magma to list all possible tuples $(a,b,c) \in R\times R\times R$ with $a,b,c \in \{0,1\}$ and to test the ties of F in a last step. To do so, we call

```
i := {<a,b,c> : a,b,c in {0,1}| (a+b mod 8) eq (2*c mod 8) and (a mod 2) eq (c mod 2) };
```

and we get

```
> i;
{ <1, 1, 1> , <0, 0, 0> }  .
```

Next, we show that e_3 and 0 are the only idempotents of $e_3\Lambda_{(2)}e_3$. By Remark 111 it remains to show that $(0,0)$ and $(1,1)$ are the only idempotents of G. Let $(a,b) \in G$ be an idempotent. Since R is local, Corollary 134 gives $a,b \in \{0,1\}$. Since this is a finite problem, we can use Magma to list all possible tuples $(a,b) \in R\times R$ with $a,b \in \{0,1\}$ and to test the ties of G in a last step. To do so, we call

```
i := {<a,b> : a,b in {0,1}| (a mod 2) eq (b mod 2) };
```

and we get

```
> i;
{ <1, 1> , <0, 0> }  .
```

Therefore, using Lemma 136, the idempotents e_2 and e_3 are primitive.

For E, we cannot reduce the ties to a finite problem that way. We construct a basis of $e_1\Lambda_{(2)}e_1$ and $\mathfrak{r}(e_1\Lambda_{(2)}e_1)$, e.g. by projection of the bases of P_1 and $\mathfrak{r}P_1$ given in Definition 116 below, and obtain

$$\begin{array}{ll}
[(1,\ 1,\ 1,\ 1,\ \begin{pmatrix}1&0\\0&1\end{pmatrix}), & [(2,\ 0,\ 0,\ 2,\ \begin{pmatrix}0&2\\1&2\end{pmatrix}),\\
(0,\ 2,\ 0,\ 2,\ \begin{pmatrix}0&2\\0&2\end{pmatrix}), & (0,\ 2,\ 0,\ 2,\ \begin{pmatrix}0&2\\0&2\end{pmatrix}),\\
(0,\ 0,\ 2,\ 2,\ \begin{pmatrix}0&0\\1&2\end{pmatrix}), & (0,\ 0,\ 2,\ 2,\ \begin{pmatrix}0&0\\1&2\end{pmatrix}),\\
(0,\ 0,\ 0,\ 4,\ \begin{pmatrix}0&0\\0&2\end{pmatrix}), & (0,\ 0,\ 0,\ 4,\ \begin{pmatrix}0&0\\0&2\end{pmatrix}),\\
(0,\ 0,\ 0,\ 0,\ \begin{pmatrix}2&0\\0&2\end{pmatrix}), & (0,\ 0,\ 0,\ 0,\ \begin{pmatrix}2&0\\0&2\end{pmatrix}),\\
(0,\ 0,\ 0,\ 0,\ \begin{pmatrix}0&4\\0&0\end{pmatrix}), & (0,\ 0,\ 0,\ 0,\ \begin{pmatrix}0&4\\0&0\end{pmatrix}),\\
(0,\ 0,\ 0,\ 0,\ \begin{pmatrix}0&0\\2&0\end{pmatrix}), & (0,\ 0,\ 0,\ 0,\ \begin{pmatrix}0&0\\2&0\end{pmatrix}),\\
(0,\ 0,\ 0,\ 0,\ \begin{pmatrix}0&0\\0&4\end{pmatrix})] \quad \text{as a basis of } e_1\Lambda_{(2)}e_1, & (0,\ 0,\ 0,\ 0,\ \begin{pmatrix}0&0\\0&4\end{pmatrix})] \quad \text{as a basis of } \mathfrak{r}(e_1\Lambda_{(2)}e_1).
\end{array}$$

We observe that the index $\mathfrak{r}(e_1\Lambda_{(2)}e_1) \subseteq e_1\Lambda_{(2)}e_1$ is 2, and therefore $E/_{\mathfrak{r}E} \cong \mathbb{F}_2$. Thus, the only orthogonal decomposition into idempotents of 1 in $E/_{\mathfrak{r}E}$ is the trivial decomposition. Using Lemma 131 below, the only orthogonal decomposition into idempotents of 1 in E is the trivial decomposition.

Finally, there exists an orthogonal decomposition

$$1 = e_1 + e_2 + e_2' + e_2'' + e_2''' + e_3 + e_3' + e_3'' + e_3'''$$

into primitive idempotents, which fall into the equivalence classes $\{e_1\}$, $\{e_2, e_2', e_2'', e_2'''\}$ and $\{e_3, e_3', e_3'', e_3'''\}$. Here e_2', e_2'' and e_2''' are obtained from e_2 by "shifting along the main diagonal". Similarly e_3', e_3'' and e_3'''. $\square$

Corollary 113 *Up to isomorphism, we have the Peirce decomposition*

$$\begin{aligned}
\Lambda_{(2)} &\cong e_1\Lambda_{(2)} \oplus e_2\Lambda_{(2)}^{\oplus 4} \oplus e_3\Lambda_{(2)}^{\oplus 4} \\
&\cong \begin{pmatrix} (e_1\Lambda_{(2)}e_1)^{1\times 1} & (e_1\Lambda_{(2)}e_2)^{1\times 4} & 0 \\ (e_2\Lambda_{(2)}e_1)^{4\times 1} & (e_2\Lambda_{(2)}e_2)^{4\times 4} & 0 \\ 0 & 0 & (e_3\Lambda_{(2)}e_3)^{4\times 4} \end{pmatrix} .
\end{aligned}$$

Remark 114 Let $\varepsilon := (0, 0, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) \in \mathbb{Q}E$.

We consider the quasiblock εE and drop the first two zeroes in notation; i.e. $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) := (0, 0, \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right))$ is in εE if and only if there exist a and b such that $(a, b, \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right))$ is in E.

Then εE has $(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$ as an R-linear basis. So

$$\varepsilon E \;=\; \{ \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \mid e \equiv_2 h \,,\; f \equiv_2 0 \} \;\subseteq\; R^{2\times 2} \,.$$

We *claim* that

$$\mathfrak{r}(\varepsilon E) \;\stackrel{!}{=}\; \{ \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \mid e \equiv_2 h \equiv_2 0 \,,\; f \equiv_2 0 \} \;=\; \left(\begin{smallmatrix} (2) & (2) \\ R & (2) \end{smallmatrix}\right) \;=:\; I \,.$$

Both-sided multiplication with basis elements of εE shows that I is an ideal in εE.

We have $I^2 \subseteq 2R^{2\times 2}$, hence $I^4 \subseteq 4R^{2\times 2} \subseteq 2\varepsilon E$. By Lemma 213.(ii), we conclude that $I \subseteq \mathfrak{r}(\varepsilon E) \subseteq \varepsilon E$. Since the index of I in εE is 2 and $\mathfrak{r}(\varepsilon E) \subsetneq \varepsilon E$, we have that $I = \mathfrak{r}(\varepsilon E)$.

This proves the *claim.*

Lemma 115

Recall that $R = \mathbb{Z}_{(2)}$. We have the radicals

$$\mathfrak{r}(e_1\Lambda_{(2)}e_1) =$$

$$e_1\Lambda_{(2)}e_1 \cap \left((2)\times(2)\times \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \times \begin{pmatrix} (2)&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} (2)&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} (2)&0&0&0&0&(2) \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ R&0&0&0&0&(2) \end{pmatrix} \right) ,$$

$$\mathfrak{r}(e_2\Lambda_{(2)}e_2) =$$

$$e_2\Lambda_{(2)}e_2 \cap \left(0\times 0\times \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0 \\ 0&(2)&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0 \\ 0&(2)&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0&0 \\ 0&(2)&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \right) ,$$

$$\mathfrak{r}(e_3\Lambda_{(2)}e_3) =$$

$$e_3\Lambda_{(2)}e_3 \cap \left(0\times 0\times \begin{pmatrix} (2)&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \times \begin{pmatrix} (2)&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \right) ,$$

where all entries are to be read as running through R.

Altogether, we obtain the Jacobson radical

$$
\begin{aligned}
&\mathfrak{r}(\Lambda_{(2)}) = \\
&\Bigg((2)\times(2)\times\begin{pmatrix}(2)&(2)&(2)&(2)\\(2)&(2)&(2)&(2)\\(2)&(2)&(2)&(2)\\(2)&(2)&(2)&(2)\end{pmatrix}\times\begin{pmatrix}(2)&(2)&(2)&(2)\\(2)&(2)&(2)&(2)\\(2)&(2)&(2)&(2)\\(2)&(2)&(2)&(2)\end{pmatrix}\times\begin{pmatrix}(2)&R&R&R&R\\R&(2)&(2)&(2)&(2)\\R&(2)&(2)&(2)&(2)\\R&(2)&(2)&(2)&(2)\\R&(2)&(2)&(2)&(2)\end{pmatrix}\times \\
&\qquad\begin{pmatrix}(2)&R&R&R&R\\R&(2)&(2)&(2)&(2)\\R&(2)&(2)&(2)&(2)\\R&(2)&(2)&(2)&(2)\\R&(2)&(2)&(2)&(2)\end{pmatrix}\times\begin{pmatrix}(2)&R&R&R&R&(2)\\R&(2)&(2)&(2)&(2)&R\\R&(2)&(2)&(2)&(2)&R\\R&(2)&(2)&(2)&(2)&R\\R&(2)&(2)&(2)&(2)&R\\R&R&R&R&R&(2)\end{pmatrix}\Bigg)\cap\Lambda_{(2)}\,.
\end{aligned}
$$

Proof. The first follows by Proposition 222; cf. Remarks 114 and 208. The latter then follows by using Proposition 217 and Remark 208; cf. Corollary 113. $\square$

Definition 116 Let $P_1 := e_1{\cdot}\Lambda_{(2)}$, $P_2 := e_2{\cdot}\Lambda_{(2)}$, $P_3 := e_3{\cdot}\Lambda_{(2)}$ represent the isoclasses of indecomposable projective modules of $\mathbb{Z}_{(2)}\mathrm{S}_5$; cf. Remark 208, Lemma 220.

So we have

$$
P_1 = \{\Bigg(a_{11}, b_{11}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}e_{11}&e_{12}&e_{13}&e_{14}&e_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}f_{11}&f_{12}&f_{13}&f_{14}&f_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}g_{11}&g_{12}&g_{13}&g_{14}&g_{15}&g_{16}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\g_{61}&g_{62}&g_{63}&g_{64}&g_{65}&g_{66}\end{pmatrix}\Bigg)\Bigg|
$$

$$
\begin{aligned}
&a_{11}\equiv_2 b_{11}\,,\quad e_{11}\equiv_2 g_{66}\,,\quad a_{11}-f_{11}\equiv_4 2g_{61}\,,\quad a_{11}+b_{11}+e_{11}+f_{11}\equiv_8 2g_{11}+2g_{66}\equiv_4 0\,,\quad e_{12}+f_{12}\equiv_8 2g_{12}\,,\\
&e_{13}+f_{13}\equiv_8 2g_{13}\,,\quad e_{14}+f_{14}\equiv_8 2g_{14}\,,\quad e_{15}+f_{15}\equiv_8 2g_{15}\,,\quad f_{12}\equiv_4 g_{62}\,,\quad f_{13}\equiv_4 g_{63}\,,\quad f_{14}\equiv_4 g_{64}\,,\\
&f_{15}\equiv_4 g_{65}\,\}\,,
\end{aligned}
$$

$$
P_2 = \{\Bigg(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\e_{21}&e_{22}&e_{23}&e_{24}&e_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\f_{21}&f_{22}&f_{23}&f_{24}&f_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\g_{21}&g_{22}&g_{23}&g_{24}&g_{25}&g_{26}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Bigg)\Bigg|
$$

$$
\begin{aligned}
&e_{22}+f_{22}\equiv_8 2g_{22},\quad e_{23}+f_{23}\equiv_8 2g_{23},\quad e_{24}+f_{24}\equiv_8 2g_{24},\quad e_{25}+f_{25}\equiv_8 2g_{25}\,,\\
&f_{21}\equiv_2 g_{21},\quad f_{22}\equiv_2 g_{22},\quad f_{23}\equiv_2 g_{23},\quad f_{24}\equiv_2 g_{24},\quad f_{25}\equiv_2 g_{25},\quad e_{21}-f_{21}\equiv_4 g_{26}\,\},
\end{aligned}
$$

$$
P_3 = \{\Bigg(0, 0, \begin{pmatrix}c_{11}&c_{12}&c_{13}&c_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}d_{11}&d_{12}&d_{13}&d_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Bigg)\Bigg|
$$

$$
c_{11}\equiv_2 d_{11},\quad c_{12}\equiv_2 d_{12},\quad c_{13}\equiv_2 d_{13},\quad c_{14}\equiv_2 d_{14}\,\}\,.
$$

We abbreviate

$$
\begin{aligned}
&\Big[a_{11}, b_{11}, (e_{11}\ e_{12}\ e_{13}\ e_{14}\ e_{15}), (f_{11}\ f_{12}\ f_{13}\ f_{14}\ f_{15}), (g_{11}\ g_{12}\ g_{13}\ g_{14}\ g_{15}\ g_{16}), (g_{61}\ g_{62}\ g_{63}\ g_{64}\ g_{65}\ g_{66})\Big]\\
&:= \Bigg(a_{11}, b_{11}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}e_{11}&e_{12}&e_{13}&e_{14}&e_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}f_{11}&f_{12}&f_{13}&f_{14}&f_{15}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}g_{11}&g_{12}&g_{13}&g_{14}&g_{15}&g_{16}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\g_{61}&g_{62}&g_{63}&g_{64}&g_{65}&g_{66}\end{pmatrix}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&\Big[(e_{21}\ e_{22}\ e_{23}\ e_{24}\ e_{25}), (f_{21}\ f_{22}\ f_{23}\ f_{24}\ f_{25}), (g_{21}\ g_{22}\ g_{23}\ g_{24}\ g_{25}\ g_{26})\Big]\\
&:= \Bigg(0, 0, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\e_{21}&e_{22}&e_{23}&e_{24}&e_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\f_{21}&f_{22}&f_{23}&f_{24}&f_{25}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\g_{21}&g_{22}&g_{23}&g_{24}&g_{25}&g_{26}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&\Big[(c_{11}\ c_{12}\ c_{13}\ c_{14}), (d_{11}\ d_{12}\ d_{13}\ d_{14})\Big]\\
&:= \Bigg(0, 0, \begin{pmatrix}c_{11}&c_{12}&c_{13}&c_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}d_{11}&d_{12}&d_{13}&d_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\Bigg).
\end{aligned}
$$

So we have $P_1 \hookrightarrow S^{(1^5)}\oplus S^{(5)}\oplus S^{(2,2,1)}\oplus S^{(3,2)}\oplus S^{(3,1,1)}\oplus S^{(3,1,1)}$, $P_2 \hookrightarrow S^{(2,2,1)}\oplus S^{(3,2)}\oplus S^{(3,1,1)}$ and $P_3 \hookrightarrow S^{(2,1,1,1)}\oplus S^{(4,1)}$.

We choose the $\mathbb{Z}_{(2)}$-linear bases

of P_1

$$
\begin{array}{l}
([1,1,(1\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 1)],\\
\ [0,2,(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(2\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(0\ 2\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0),(0\ 1\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 2\ 0\ 0),(0\ 0\ 2\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0),(0\ 0\ 1\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 2\ 0\ 0),(0\ 0\ 0\ 1\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 0\ 1\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(4\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0),(0\ 0\ 0\ 2\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4),(0\ 0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 4),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 2\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 2\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 4)]\ ,
\end{array}
$$

of $\mathfrak{r}P_1$,

$$
\begin{array}{l}
([2,0,(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2),(1\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,2,(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(2\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(0\ 2\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0),(0\ 1\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 2\ 0\ 0),(0\ 0\ 2\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0),(0\ 0\ 1\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 2\ 0\ 0),(0\ 0\ 0\ 1\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 0\ 1\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(4\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0),(0\ 0\ 0\ 2\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4),(0\ 0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4\ 0),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 4),(0\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 2\ 0\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 2\ 0)],\\
\ [0,0,(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 4)]\ ,
\end{array}
$$

of P_2,

$$
\begin{array}{l}
([(1\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0\ 0)],\\
\ [(0\ 1\ 0\ 0\ 0),(0\ 1\ 0\ 0\ 0),(0\ 1\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 1\ 0\ 0),(0\ 0\ 1\ 0\ 0),(0\ 0\ 1\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 1\ 0),(0\ 0\ 0\ 1\ 0),(0\ 0\ 0\ 1\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 1),(0\ 0\ 0\ 0\ 1),(0\ 0\ 0\ 0\ 1\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0),(0\ 0\ 0\ 2\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4),(0\ 0\ 0\ 0\ 2\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 4)])\ ,
\end{array}
$$

of $\mathfrak{r}P_2$,

$$
\begin{array}{l}
([(1\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0),(1\ 0\ 0\ 0\ 0\ 0)],\\
\ [(0\ 2\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 2\ 0\ 0),(0\ 0\ 2\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 2\ 0),(0\ 0\ 0\ 2\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 2),(0\ 0\ 0\ 0\ 2\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 2)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0),(0\ 2\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0),(0\ 0\ 2\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0),(0\ 0\ 0\ 2\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4),(0\ 0\ 0\ 0\ 2\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(2\ 0\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 4\ 0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 4\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 4\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 4\ 0)],\\
\ [(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0),(0\ 0\ 0\ 0\ 0\ 4)])
\end{array}
$$

$$\begin{array}{ll}
(\,[(1\ 0\ 0\ 0),(1\ 0\ 0\ 0)], & (\,[(2\ 0\ 0\ 0),(0\ 0\ 0\ 0)],\\
\ [(0\ 1\ 0\ 0),(0\ 1\ 0\ 0)], & \ [(0\ 2\ 0\ 0),(0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 1\ 0),(0\ 0\ 1\ 0)], & \ [(0\ 0\ 2\ 0),(0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 1),(0\ 0\ 0\ 1)], & \ [(0\ 0\ 0\ 2),(0\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0),(2\ 0\ 0\ 0)], & \ [(0\ 0\ 0\ 0),(2\ 0\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0),(0\ 2\ 0\ 0)], & \ [(0\ 0\ 0\ 0),(0\ 2\ 0\ 0)],\\
\ [(0\ 0\ 0\ 0),(0\ 0\ 2\ 0)], & \ [(0\ 0\ 0\ 0),(0\ 0\ 2\ 0)],\\
\ [(0\ 0\ 0\ 0),(0\ 0\ 0\ 2)]\,) \quad \text{of } P_3, & \ [(0\ 0\ 0\ 0),(0\ 0\ 0\ 2)]\,) \quad \text{of } \mathfrak{r}P_3.
\end{array}$$

Remark 117 Using Magma, we can verify the following Loewy layers:

$$\bar{P}_1 : \begin{array}{ccc} & D_1 & \\ D_1 & & D_2 \\ D_1 & & D_2 \\ D_1 & & D_1 \\ D_1 & & D_2 \\ D_1 & & D_2 \\ & D_1 & \end{array}, \qquad \bar{P}_2 : \begin{array}{c} D_2 \\ D_1 \\ D_1 \\ D_2 \\ D_1 \\ D_1 \\ D_2 \end{array}, \qquad \bar{P}_3 : \begin{array}{c} D_3 \\ D_3 \end{array}$$

Remark 118 Recall that

$$\begin{aligned}
e_1\Lambda_{(2)}e_1 &= \Bigg\{ \overbrace{\left(a_{11}, b_{11}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} e_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} f_{11}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} g_{11}&0&0&0&0&g_{16}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\g_{61}&0&0&0&0&g_{66} \end{pmatrix} \right)}^{=:\,[\,a_{11}, b_{11}, e_{11}, f_{11}, g_{11}, g_{16}, g_{61}, g_{66}\,]} \Bigg| \\
&\qquad a_{11}+b_{11}+e_{11}+f_{11} \equiv_8 2g_{11}+2g_{66} \equiv_4 0, \quad e_{11}-f_{11} \equiv_4 g_{16} \Bigg\} \\
&= \{[\,a_{11}, b_{11}, e_{11}, f_{11}, g_{11}, g_{16}, g_{61}, g_{66}\,] \mid a_{11}+b_{11}+e_{11}+f_{11} \equiv_8 2g_{11}+2g_{66} \equiv_4 0, \quad e_{11}-f_{11} \equiv_4 g_{16}\,\}
\end{aligned}$$

for which we fix the R-linear basis

$$(e_1, \tilde{h}_1^{11}, \tilde{h}_2^{11}, \tilde{h}_3^{11}, \tilde{h}_4^{11}, \tilde{h}_5^{11}, \tilde{h}_6^{11}, \tilde{h}_7^{11}\,) := \begin{array}{l} \{\,[1,1,1,1,1,0,0,1], \\ \ [0,2,0,2,0,2,0,2], \\ \ [0,0,2,2,0,0,1,2], \\ \ [0,0,0,4,0,0,0,2], \\ \ [0,0,0,0,2,0,0,2], \\ \ [0,0,0,0,0,4,0,0], \\ \ [0,0,0,0,0,0,2,0], \\ \ [0,0,0,0,0,0,0,4]\,\} \end{array} .$$

Our fixed R-linear basis $(1, h_1^{11}, h_2^{11}, h_3^{11}, h_4^{11}, h_5^{11}, h_6^{11}, h_7^{11}\,)$ of $\mathrm{Hom}_{R\mathrm{S}_5}(P_1, P_1)$ we get via the canonical isomorphism from $e_1\Lambda_{(2)}e_1$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_1, P_1)$.

For $e_2\Lambda_{(2)}e_2$ we get

$$\begin{aligned}
e_2\Lambda_{(2)}e_2 &= \Bigg\{ \overbrace{\left(0, 0, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&e_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&f_{22}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&g_{22}&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right)}^{=:\,[\,e_{22}, f_{22}, g_{22}\,]} \Bigg| \\
&\qquad e_{22}+f_{22} \equiv_8 2g_{22}, \quad f_{22} \equiv_2 g_{22} \Bigg\} \\
&= \{\,[\,e_{22}, f_{22}, g_{22}\,] \mid e_{22}+f_{22} \equiv_8 2g_{22}, \quad f_{22} \equiv_2 g_{22}\,\},
\end{aligned}$$

for which we fix the R-linear basis

$$(e_2, \tilde{h}_1^{22}, \tilde{h}_2^{22}, \tilde{h}_3^{22}\,) := \{[1,1,1],\ [0,4,2],\ [0,0,4]\}\,.$$

Our fixed R-linear basis $(1, h_1^{22}, h_2^{22}, h_3^{22}\,)$ of $\mathrm{Hom}_{R\mathrm{S}_5}(P_2, P_2)$ we get via the canonical isomorphism from $e_2\Lambda_{(2)}e_2$ to $\mathrm{Hom}_{R\mathrm{S}_5}(P_2, P_2)$.

As an R-linear basis of

$$
\begin{aligned}
e_3\Lambda_{(2)}e_3 &= \Bigg\{ \overbrace{\left(0,0,\begin{pmatrix} c_{11} & 0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix},\begin{pmatrix} d_{11} & 0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)}^{=:\,[\,c_{11},d_{11}\,]} \Bigg|\, c_{11} \equiv_2 d_{11} \Bigg\} \\
&= \{\,[\,c_{11},d_{11}\,] \mid c_{11} \equiv_2 d_{11}\,\}
\end{aligned}
$$

we fix

$$(e_3, \tilde{h}_1^{33}\,) := \{[1,1],\ [0,2]\,\}\,.$$

Our fixed R-linear basis $(1, h_1^{33}\,)$ of $\mathrm{Hom}_{RS_5}(P_3,P_3)$ we get via the canonical isomorphism from $e_3\Lambda_{(2)}e_3$ to $\mathrm{Hom}_{RS_5}(P_3,P_3)$.

Now, as R-linear basis of

$$
\begin{aligned}
e_2\Lambda_{(2)}e_1 &= \Bigg\{ \overbrace{\left(0,0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\ f_{21}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\ e_{21}&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0&0\\ g_{21}&0&0&0&0&g_{26}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)}^{=:\,[\,e_{21},f_{21},g_{21},g_{26}\,]} \Bigg| \\
&\qquad e_{21}+f_{21} \equiv_4 2g_{26},\quad f_{21} \equiv_2 g_{21} \Bigg\} \\
&= \{\,[\,e_{21},f_{21},g_{21},g_{26}\,] \mid e_{21}+f_{21} \equiv_4 2g_{26},\quad f_{21} \equiv_2 g_{21}\,\}
\end{aligned}
$$

we fix

$$(\tilde{h}_1^{12},\, \tilde{h}_2^{12},\, \tilde{h}_3^{12},\, \tilde{h}_4^{12}\,) := \{[1,1,1,0],\ [0,2,0,2],\ [0,0,2,0],\ [0,0,0,4]\,\}\,.$$

Our fixed R-linear basis $(h_1^{12}, h_2^{12}, h_3^{12}, h_4^{12}\,)$ of $\mathrm{Hom}_{RS_5}(P_1,P_2)$ we get via the canonical isomorphism from $e_2\Lambda_{(2)}e_1$ to $\mathrm{Hom}_{RS_5}(P_1,P_2)$.

Now, for

$$
\begin{aligned}
e_1\Lambda_{(2)}e_2 &= \Bigg\{ \overbrace{\left(0,0,\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&f_{12}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&e_{12}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&g_{12}&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&g_{62}&0&0&0&0\end{pmatrix}\right)}^{=:\,[\,e_{12},f_{12},g_{12},g_{62}\,]} \Bigg| \\
&\qquad e_{12}+f_{12} \equiv_8 2g_{12},\quad f_{12} \equiv_2 g_{26} \Bigg\} \\
&= \{\,[\,e_{12},f_{12},g_{12},g_{62}\,] \mid e_{12}+f_{12} \equiv_8 2g_{12},\quad f_{12} \equiv_2 g_{26}\,\}
\end{aligned}
$$

we fix the R-linear basis

$$(\tilde{h}_1^{21},\, \tilde{h}_2^{21},\, \tilde{h}_3^{21},\, \tilde{h}_4^{21}\,) := \{[2,2,2,1],\ [0,4,2,0],\ [0,0,4,0],\ [0,0,0,2]\,\}\,.$$

Our fixed R-linear basis $(h_1^{21}, h_2^{21}, h_3^{21}, h_4^{21}\,)$ of $\mathrm{Hom}_{RS_5}(P_2,P_1)$ we get via the canonical isomorphism from $e_1\Lambda_{(2)}e_2$ to $\mathrm{Hom}_{RS_5}(P_2,P_1)$.

Describing matrices of the homomorphisms occurring in these bases can be found in the digital appendix. To view them, load `Homs_S5_loc2`.

For example, the elements of the basis $(h_1^{21},\, h_2^{21},\, h_3^{21},\, h_4^{21})$ of $\mathrm{Hom}_{RS_5}(P_2,P_1)$ are denoted `Hom[2,1][1]`, `Hom[2,1][2]`, `Hom[2,1][3]`, `Hom[2,1][4]`, the first two entries labelling the projective modules, the last one the number of the homomorphism in the ordered basis. The elements of the basis $(1, h_1^{33}\,)$ of $\mathrm{Hom}_{RS_5}(P_3,P_3)$ are denoted `Hom[3,3][1]`, `Hom[3,3][2]`, where the former is the identity.

For $\mathrm{Hom}_{RS_5}(P_1,P_1)$, $\mathrm{Hom}_{RS_5}(P_2,P_2)$ and $\mathrm{Hom}_{RS_5}(P_1,P_2)$, the notation works analogously.

The operating matrices can be found in the file `main_S5_loc2`.

They can e.g. be used to check the RS_5-linearity of the maps between P_1, P_2 and P_3 derived above.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2 and P_3 are denoted `rhoP1`, `rhoP2`,`rhoP3`, respectively.

E.g. for the operating matrices on P_1, call

```
rhoP1(S5P!sigma);
```

for an element `sigma` of S_5. Analogously for P_2, P_3, P_4 and P_5.

To check that the matrices found above represent $R\mathrm{S}_5$-linear maps between the respective projective modules, follow these steps:

```
load main_S5_loc2;
load Homs_S5_loc2;

[rhoP1(sigma)*Homs[1,1][i] eq Homs[1,1][i]*rhoP1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)},i in [1..8]];
[rhoP2(sigma)*Homs[2,2][i] eq Homs[2,2][i]*rhoP2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)},i in [1..3]];

[rhoP1(sigma)*Homs[1,2][i] eq Homs[1,2][i]*rhoP2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)},i in [1..4]];
[rhoP2(sigma)*Homs[2,1][i] eq Homs[2,1][i]*rhoP1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)},i in [1..4]];

[rhoP3(sigma)*Homs[3,3][i] eq Homs[3,3][i]*rhoP3(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)},i in [1..2]];
```

A file on how to construct those isomorphisms can be found in the digital appendix and is named `generate_Homs_S5_loc2`.

5.2.2 The tensor product $P_1 \otimes P_1$

Lemma 119 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_1 \longrightarrow P_1 \otimes P_1$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_1 \longrightarrow P_1 \otimes P_1$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_1 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 8} \oplus P_2^{\oplus 16} \oplus P_3^{\oplus 16} \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_1 \arrow[r, "A", "\sim"'] & P_1^{\oplus 8} \oplus P_2^{\oplus 16} \oplus P_3^{\oplus 16} \\
\mathfrak{r}P_1 \otimes P_1 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_1^{\oplus 8} \oplus P_2^{\oplus 16} \oplus P_3^{\oplus 16} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cccccccc|cccccccccccccccc|cccccccccccccccc}
2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{array}\right],$$

$$
\tilde{C} := \left[\begin{array}{cccccccc|cccccccccccccccc|cccccccccccccccc}
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
h_7^{11} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
h_1^{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
h_1^{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right].
$$

Recall that $R := \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.

Let $Q := P_1^{\oplus 8} \oplus P_2^{\oplus 16} \oplus P_3^{\oplus 16}$. We *claim* that we have a commutative diagram as follows.

$$
\begin{tikzcd}
P_1 \otimes \mathfrak{t}P_1 \arrow[d, hook, "\tilde{E}"] & Q \arrow[l, "\vartheta_1"', "\sim"] \arrow[r, "\sim", "S_1"'] \arrow[d, "\tilde{C}_1"'] \arrow[dl, phantom, "\circlearrowleft"] & Q \arrow[d, "\tilde{C}_2"'] \arrow[dl, phantom, "\circlearrowleft"] & Q \arrow[l, "T_1"', "\sim"] \arrow[d, "\tilde{C}"'] \arrow[dl, phantom, "\circlearrowleft"] \\
P_1 \otimes P_1 & Q \arrow[l, "\sim"', "\vartheta_2"] \arrow[r, "\sim", "\mathrm{id}"'] \arrow[dl, phantom, "\circlearrowleft"] & Q \arrow[r, "\sim", "T_2"'] \arrow[dl, phantom, "\circlearrowleft"] & Q \arrow[dl, phantom, "\circlearrowleft"] \\
\mathfrak{t}P_1 \otimes P_1 \arrow[u, hook, "E"] & Q \arrow[l, "\vartheta_3"', "\sim"] \arrow[r, "\sim", "S_3"'] \arrow[u, "C_1"'] & Q \arrow[u, "C_2"'] & Q \arrow[l, "T_3"', "\sim"] \arrow[u, "C"']
\end{tikzcd}
$$

Then the isomorphisms A, B and $\tilde{B}$ are given as the composites $A := \vartheta_2^{-1}T_2$, $B := \vartheta_1^{-1}S_1T_1^{-1}$ and $\tilde{B} := \vartheta_3^{-1}S_3T_3^{-1}$. We refrain from calculating the occurring inverses explicitly, for their entries are rather big.

So we have to show commutativity of all six quadrangles, $R\mathrm{S}_5$-linearity for all maps involved, and local invertibility for all horizontal maps. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the files `main_S5_loc2` and main_S5_loc2_P1oP1. The matrices for this diagram are contained in Diagram_S5_loc2_P1oP1_part1, Diagram_S5_loc2_P1oP1_part2 and Diagram_S5_loc2_P1oP1_part3.

We denote the maps ϑ_1, ϑ_2, ϑ_3 as theta_1, theta_2, theta_3, respectively; the maps E, C_1, C_2, C we denote as E1, C1_1, C1_2, C1; the maps $\tilde{E}$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}$ as E2, C2_1, C2_2, C2. The maps S_1, S_3, T_1, T_2, T_3 we denote as S1, S3, T1, T2, T3, respectively.

The embedding E is defined as the Kronecker product of the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$ and id_{P_1}. An embedding in the chosen bases can be found in the files and is denoted i1. The embedding $\tilde{E}$ is defined as the Kronecker product of id_{P_1} and the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2, P_3 are denoted rhoP1, rhoP2, rhoP3, respectively. The representations on $\mathfrak{r}P_1$, $\mathfrak{r}P_2$, $\mathfrak{r}P_3$ are denoted rhorP1, rhorP2, rhorP3, respectively.

The representations on $P_1 \otimes P_1$, $\mathfrak{r}P_1 \otimes P_1$, $P_1 \otimes \mathfrak{r}P_1$ are denoted op_p1op1, op_rp1op1, op_p1orp1, respectively. The operating matrix of a group element on such a tensor product is defined as the tensor product of the operating matrices on the tensor factors.

For example, for the operating matrices on $P_1 \otimes P_1$ call

```
op_p1op1(S5P!sigma);
```

for an element sigma of S_5. The other maps work the same way.

The representation on the direct sum $Q = P_1^{\oplus 8} \oplus P_2^{\oplus 16} \oplus P_3^{\oplus 16}$ is denoted op_proj_sum_p1p1. The operating matrix of a group element is defined as the block diagonal matrix containing the operating matrices of the summands.

To verify the *claim,* follow these steps:

```
load "main_S5_loc2";
load "main_S5_loc2_P1oP1";
load "Diagram_S5_loc2_P1oP1";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),24)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),24)!1,i1);

 [op_rp1op1(sigma)*E1 eq E1*op_p1op1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_p1orp1(sigma)*E2 eq E2*op_p1op1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];

//first column of the diagram:

//commutativity:
 theta_3*E1 eq C1_1*theta_2;
 theta_1*E2 eq C2_1*theta_2;

//RS5-linearity
 [op_proj_sum_p1p1(sigma)*theta_3 eq theta_3*op_rp1op1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p1(sigma)*theta_1 eq theta_1*op_p1orp1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p1(sigma)*theta_2 eq theta_2*op_p1op1(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p1(sigma)*C1_1 eq C1_1*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p1(sigma)*C2_1 eq C2_1*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility; loc_inv see "main_S5_loc2"
 loc_inv(theta_1,2);
 loc_inv(theta_2,2);
 loc_inv(theta_3,2);
```

```
 //second column of the diagram:

 //commutativity:
  C1_1 eq S3*C1_2;
  C2_1 eq S1*C2_2;

 //RS5-linearity
  [op_proj_sum_p1p1(sigma)*C1_2 eq C1_2*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
  [op_proj_sum_p1p1(sigma)*C2_2 eq C2_2*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
  [op_proj_sum_p1p1(sigma)*S1 eq S1*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
  [op_proj_sum_p1p1(sigma)*S3 eq S3*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

 //local invertibility
  loc_inv(S1,2);
  loc_inv(S3,2);

//third column of the diagram:

//commutativity:
  T3*C1_2*T2 eq C1;
  T1*C2_2*T2 eq C2;

 //RS5-linearity
  [op_proj_sum_p1p1(sigma)*C1 eq C1*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
  [op_proj_sum_p1p1(sigma)*C2 eq C2*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
  [op_proj_sum_p1p1(sigma)*T1 eq T1*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
  [op_proj_sum_p1p1(sigma)*T2 eq T2*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
  [op_proj_sum_p1p1(sigma)*T3 eq T3*op_proj_sum_p1p1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

 //local invertibility
  loc_inv(T1,2);
  loc_inv(T2,2);
  loc_inv(T3,2);
```

$\square$

Remark 120 Using Magma, we can verify that

$$\begin{array}{lcccccc} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_1 \otimes D_1 & \cong & D_1\,, & & \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_1 \otimes \bar{P}_1 & \cong & \bar{P}_1\,, \end{array}$$

with Loewy layers already known.

5.2.3 The tensor product $P_1 \otimes P_2$

Lemma 121 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_2 \longrightarrow P_1 \otimes P_2$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_2 \longrightarrow P_1 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_5$-linear maps*

$$
\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 4} \oplus P_2^{\oplus 12} \oplus P_3^{\oplus 12} \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_2 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 4} \oplus P_2^{\oplus 12} \oplus P_3^{\oplus 12} \\
\mathfrak{r}P_1 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_1^{\oplus 4} \oplus P_2^{\oplus 12} \oplus P_3^{\oplus 12} \arrow[u, "C"']
\end{tikzcd}
$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$
C := \left[\begin{array}{cccc|cccccccccccc|cccccccccccc}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{array}\right],
$$

$$
\tilde{C} := \left[\begin{array}{cccc|cccccccccccc|cccccccccccc}
1 & . & . & . & h_1^{12} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{array}\right] .
$$

Recall that $R := \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.

Let $Q := P_1^{\oplus 4} \oplus P_2^{\oplus 12} \oplus P_3^{\oplus 12}$. We *claim* that we have a commutative diagram as follows.

$$
\begin{tikzcd}
P_1 \otimes \mathfrak{t}P_2 \arrow[d, hook, "\tilde{E}"] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[l, "\vartheta_1", "\sim"'] \arrow[r, "\sim", "S_1"'] \arrow[d, "\tilde{C}_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "\tilde{C}_2"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[l, "T_1", "\sim"'] \arrow[d, "\tilde{C}"'] \\
P_1 \otimes P_2 \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[l, "\sim", "\vartheta_2"'] \arrow[r, "\sim", "\mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[r, "\sim", "T_2"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \\
\mathfrak{t}P_1 \otimes P_2 \arrow[u, hook, "E"] & Q \arrow[l, "\vartheta_3", "\sim"'] \arrow[r, "\sim", "S_3"'] \arrow[u, "C_1"'] & Q \arrow[u, "C_2"'] & Q \arrow[l, "T_3", "\sim"'] \arrow[u, "C"']
\end{tikzcd}
$$

Then the isomorphisms A, B and $\tilde{B}$ are given as the composites $A := \vartheta_2^{-1}T_2$, $B := \vartheta_1^{-1}S_1T_1^{-1}$ and $\tilde{B} := \vartheta_3^{-1}S_3T_3^{-1}$. We refrain from calculating the occurring inverses explicitly, for their entries are rather big.

So we have to show commutativity of all six quadrangles, $R\mathrm{S}_5$-linearity for all maps involved, and local invertibility for all horizontal maps. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the files `main_S5_loc2` and `main_S5_loc2_P1oP2`. The matrices for this diagram are contained in `Diagram_S5_loc2_P1oP2_part1`, `Diagram_S5_loc2_P1oP2_part2` and `Diagram_S5_loc2_P1oP2_part3`.

For notations, see proof of Lemma 119.

To verify the *claim,* follow these steps:

```
load "main_S5_loc2";
load "main_S5_loc2_P1oP2";
```

```
load "Diagram_S5_loc2_P1oP2";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),16)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),24)!1,i2);

 [op_rp1op2(sigma)*E1 eq E1*op_p1op2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_p1orp2(sigma)*E2 eq E2*op_p1op2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];

//first column of the diagram:

//commutativity:
  theta_3*E1 eq C1_1*theta_2;
  theta_1*E2 eq C2_1*theta_2;

//RS5-linearity
 [op_proj_sum_p1p2(sigma)*theta_3 eq theta_3*op_rp1op2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*theta_1 eq theta_1*op_p1orp2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*theta_2 eq theta_2*op_p1op2(sigma):sigma in {S5P!(1,2),S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*C1_1 eq C1_1*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*C2_1 eq C2_1*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility; loc_inv see "main_S5_loc2"
 loc_inv(theta_1,2);
 loc_inv(theta_2,2);
 loc_inv(theta_3,2);

//second column of the diagram:

//commutativity:
 C1_1 eq S3*C1_2;
 C2_1 eq S1*C2_2;

//RS5-linearity
 [op_proj_sum_p1p2(sigma)*C1_2 eq C1_2*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*C2_2 eq C2_2*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*S1 eq S1*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*S3 eq S3*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility
 loc_inv(S1,2);
 loc_inv(S3,2);

//third column of the diagram:

//commutativity:
 T3*C1_2*T2 eq C1;
 T1*C2_2*T2 eq C2;

//RS5-linearity
 [op_proj_sum_p1p2(sigma)*C1 eq C1*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*C2 eq C2*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*T1 eq T1*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*T2 eq T2*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p1p2(sigma)*T3 eq T3*op_proj_sum_p1p2(sigma):sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility
 loc_inv(T1,2);
 loc_inv(T2,2);
 loc_inv(T3,2);
```

□

Remark 122 Using Magma, we can verify that

$$\begin{array}{lcccc}
\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_1 \otimes D_2 & \cong & D_2\,, \\
\mathrm{Coker}(C) & \cong & D_1 \otimes \bar{P}_2 & \cong & \bar{P}_2\,, \\
\mathrm{Coker}(\tilde{C}) & \cong & \bar{P}_1 \otimes D_2 & \cong & \bar{P}_2^{\oplus 4} \oplus \bar{P}_3^{\oplus 4}\,,
\end{array}$$

with Loewy layers already known.

5.2.4 The tensor product $P_1 \otimes P_3$

Lemma 123 *Let E be the embedding $\mathfrak{r}P_1 \otimes P_3 \longrightarrow P_1 \otimes P_3$, and $\tilde{E}$ be the embedding $P_1 \otimes \mathfrak{r}P_3 \longrightarrow P_1 \otimes P_3$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd}
P_1 \otimes \mathfrak{r}P_3 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2^{\oplus 8} \oplus P_3^{\oplus 8} \arrow[d, "\tilde{C}"] \\
P_1 \otimes P_3 \arrow[r, "A", "\sim"'] & P_2^{\oplus 8} \oplus P_3^{\oplus 8} \\
\mathfrak{r}P_1 \otimes P_3 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_2^{\oplus 8} \oplus P_3^{\oplus 8} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cccccccc|cccccccc}
1&.&.&.&.&.&.&.&.&.&.&.&.&.&.&. \\
.&1&.&.&.&.&.&.&.&.&.&.&.&.&.&. \\
.&.&1&.&.&.&.&.&.&.&.&.&.&.&.&. \\
.&.&.&1&.&.&.&.&.&.&.&.&.&.&.&. \\
.&.&.&.&1&.&.&.&.&.&.&.&.&.&.&. \\
.&.&.&.&.&1&.&.&.&.&.&.&.&.&.&. \\
.&.&.&.&.&.&1&.&.&.&.&.&.&.&.&. \\
.&.&.&.&.&.&.&1&.&.&.&.&.&.&.&. \\
\hline
.&.&.&.&.&.&.&.&2&.&.&.&.&.&.&. \\
.&.&.&.&.&.&.&.&.&1&.&.&.&.&.&. \\
.&.&.&.&.&.&.&.&.&.&1&.&.&.&.&. \\
.&.&.&.&.&.&.&.&.&.&.&1&.&.&.&. \\
.&.&.&.&.&.&.&.&.&.&.&.&1&.&.&. \\
.&.&.&.&.&.&.&.&.&.&.&.&.&1&.&. \\
.&.&.&.&.&.&.&.&.&.&.&.&.&.&1&. \\
.&.&.&.&.&.&.&.&.&.&.&.&.&.&.&1
\end{array}\right], \quad
\tilde{C} := \left[\begin{array}{cccccccc|cccccccc}
1&.&.&.&.&.&.&.&.&.&.&.&.&.&.&. \\
.&2&.&.&.&.&.&.&.&.&.&.&.&.&.&. \\
.&.&2&.&.&.&.&.&.&.&.&.&.&.&.&. \\
.&.&.&1&.&.&.&.&.&.&.&.&.&.&.&. \\
.&.&.&.&2&.&.&.&.&.&.&.&.&.&.&. \\
.&.&.&.&.&2&.&.&.&.&.&.&.&.&.&. \\
.&.&.&.&.&.&1&.&.&.&.&.&.&.&.&. \\
.&.&.&.&.&.&.&1&.&.&.&.&.&.&.&. \\
\hline
.&.&.&.&.&.&.&.&2&.&.&.&.&.&.&. \\
.&.&.&.&.&.&.&.&.&2&.&.&.&.&.&. \\
.&.&.&.&.&.&.&.&.&.&2&.&.&.&.&. \\
.&.&.&.&.&.&.&.&.&.&.&1&.&.&.&. \\
.&.&.&.&.&.&.&.&.&.&.&.&2&.&.&. \\
.&.&.&.&.&.&.&.&h_2^{33}&.&.&.&.&1&.&. \\
.&.&.&.&.&.&.&.&.&.&.&.&.&.&1&. \\
.&.&.&.&.&.&.&.&.&.&.&.&.&.&.&1
\end{array}\right].$$

Recall that $R := \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.

We have to show local invertibility and $R\mathrm{S}_5$-linearity of the maps A, B, and $\tilde{B}$, the $R\mathrm{S}_5$-linearity of E, $\tilde{E}$, C and $\tilde{C}$, and the commutativity of the diagram. The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc2`, the matrices for this diagram in the file `Diagram_S5_loc2_P1oP3` .

The embedding E is defined as the Kronecker product of the embedding $i_1 : \mathfrak{r}P_1 \hookrightarrow P_1$ and id_{P_3}. An embedding in the chosen bases can be found in the files and is denoted `i1`. The embedding $\tilde{E}$ is defined as the Kronecker product of id_{P_1} and the embedding $i_3 : \mathfrak{r}P_3 \hookrightarrow P_3$, deoted `i3`.

The representations, i.e. the maps sending group elements to operating matrices, on P_1, P_2, P_3 are denoted `rhoP1`, `rhoP2`, `rhoP3`, respectively. The representations on $\mathfrak{r}P_1$, $\mathfrak{r}P_2$, $\mathfrak{r}P_3$ are denoted `rhorP1`, `rhorP2`, `rhorP3`, respectively.

The representations on $P_1 \otimes P_3$, $\mathfrak{r}P_1 \otimes P_3$, $P_1 \otimes \mathfrak{r}P_3$ are denoted `op_p1op3`, `op_rp1op3`, `op_p1orp3`, respectively. The operating matrix of a group element on such a tensor product is defined as the tensor product of the operating matrices on the tensor factors.

For example, for the operating matrices on $P_1 \otimes P_3$ call

```
op_p1op3(S5P!sigma);
```

for an element `sigma` of S_5. The other maps work the same way.

The representation on the direct sum $P_2^{\oplus 8} \oplus P_3^{\oplus 8}$ is denoted `op_proj_sum_p1p3`. The operating matrix of a group element is defined as the block diagonal matrix containing the operating matrices of the summands. The maps A, B, C, E are denoted `A`, `B1`, `C1`, `E1`, respectively; the maps $\tilde{B}$, $\tilde{C}$, $\tilde{E}$ are denoted `B2`, `C2`, `E2`, respectively.

To verify the Lemma, follow these steps:

```
load "main_S5_loc2";
load "main_S5_loc2_P1oP3";
load "Diagram_S5_loc2_P1oP3";

 [rhorP1(sigma)*i1 eq i1*rhoP1(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i1,MatrixRing(Rationals(),8)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),24)!1,i3);

 [op_rp1op3(sigma)*E1 eq E1*op_p1op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p1orp3(sigma)*E2 eq E2*op_p1op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity
 [op_proj_sum_p1p3(sigma)*C1 eq C1*op_proj_sum_p1p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_proj_sum_p1p3(sigma)*C2 eq C2*op_proj_sum_p1p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp1op3(sigma)*B1 eq B1*op_proj_sum_p1p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p1orp3(sigma)*B2 eq B2*op_proj_sum_p1p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p1op3(sigma)*A eq A*op_proj_sum_p1p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];

//local invertibility; loc_inv see "main_S5_loc2"
 loc_inv(B1,2);
 loc_inv(B2,2);
 loc_inv(A,2);
```

□

Remark 124 Using Magma, we can verify that

$$\begin{array}{lcccl} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_1 \otimes D_3 & \cong & D_3\,, \\[1em] \mathrm{Coker}(C) & \cong & D_1 \otimes \bar{P}_3 & \cong & \bar{P}_3\,, \\[1em] \mathrm{Coker}(\tilde{C}) & \cong & \bar{P}_1 \otimes D_3 & \cong & \bar{P}_2^{\oplus 4} \oplus \bar{P}_3^{\oplus 4}\,, \end{array}$$

with Loewy layers already known.

5.2.5 The tensor product $P_2 \otimes P_2$

Lemma 125 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_2 \longrightarrow P_2 \otimes P_2$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_2 \longrightarrow P_2 \otimes P_2$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd} P_2 \otimes \mathfrak{r}P_2 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 3} \oplus P_2^{\oplus 7} \oplus P_3^{\oplus 9} \arrow[d, "\tilde{C}"] \\ P_2 \otimes P_2 \arrow[r, "A", "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^{\oplus 3} \oplus P_2^{\oplus 7} \oplus P_3^{\oplus 9} \\ \mathfrak{r}P_2 \otimes P_2 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] & P_1^{\oplus 3} \oplus P_2^{\oplus 7} \oplus P_3^{\oplus 9} \arrow[u, "C"'] \end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{ccc|ccccccc|ccccccccc}
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right], \quad
\tilde{C} := \left[\begin{array}{ccc|ccccccc|ccccccccc}
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-h_1^{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
h_3^{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right].$$

Recall that $R := \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc2` and `main_S5_loc2_P2oP2` the matrices for this diagram in the file `Diagram_S5_loc2_P2oP2`.

The maps and matrices are denoted analogously to those for $P_1 \otimes P_3$, see proof of Lemma 123.

To verify the Lemma, follow these steps:

```
load "main_S5_loc2";
load "main_S5_loc2_P2oP2";
load "Diagram_S5_loc2_P2oP2";

 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i2,MatrixRing(Rationals(),16)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),16)!1,i2);

 [op_rp2op2(sigma)*E1 eq E1*op_p2op2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp2(sigma)*E2 eq E2*op_p2op2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity
 [op_proj_sum_p2p2(sigma)*C1 eq C1*op_proj_sum_p2p2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_proj_sum_p2p2(sigma)*C2 eq C2*op_proj_sum_p2p2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_rp2op2(sigma)*B1 eq B1*op_proj_sum_p2p2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2orp2(sigma)*B2 eq B2*op_proj_sum_p2p2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [op_p2op2(sigma)*A eq A*op_proj_sum_p2p2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

//local invertibility; loc_inv see "main_S5_loc2"
 loc_inv(B1,2);
 loc_inv(B2,2);
 loc_inv(A,2);
```

□

Remark 126 Using Magma, we can verify that

$$\begin{aligned}
\mathrm{Coker}\left(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}\right) &\cong D_2 \otimes D_2 \cong \bar{P}_3 \oplus X\,, \\
\mathrm{Coker}(C) &\cong \mathrm{Coker}(\tilde{C}) \cong D_2 \otimes \bar{P}_2 \cong \bar{P}_1^{\oplus 1} \oplus \bar{P}_2^{\oplus 1} \oplus \bar{P}_3^{\oplus 3}\,.
\end{aligned}$$

with X a module of dimension 8, with Loewy layers $\begin{matrix} D_1 \\ D_1 \\ D_2 \\ D_1 \\ D_1 \end{matrix}$.

As operating matrices in lower block-triangular form, we get for X

$$
\begin{aligned}
\mathrm{S}_5 \;\longrightarrow\; \mathrm{GL}_8(\mathbb{F}_2)\,, \quad (1,2,3,4,5) \;&\longmapsto\; \left(\begin{array}{c|c|cccc|c|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right) \\
(1,2) \;&\longmapsto\; \left(\begin{array}{c|c|cccc|c|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1
\end{array}\right) .
\end{aligned}
$$

5.2.6 The tensor product $P_2 \otimes P_3$

Lemma 127 *Let E be the embedding $\mathfrak{r}P_2 \otimes P_3 \longrightarrow P_2 \otimes P_3$, and $\tilde{E}$ be the embedding $P_2 \otimes \mathfrak{r}P_3 \longrightarrow P_2 \otimes P_3$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_5$-linear maps*

$$
\begin{tikzcd}
P_2 \otimes \mathfrak{r}P_3 \arrow[r, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2^{\oplus 6} \oplus P_3^{\oplus 4} \arrow[d, "\tilde{C}"] \\
P_2 \otimes P_3 \arrow[r, "A", "\sim"'] & P_2^{\oplus 6} \oplus P_3^{\oplus 4} \\
\mathfrak{r}P_2 \otimes P_3 \arrow[r, "B", "\sim"'] \arrow[u, hook, "E"] \arrow[ur, phantom, "\circlearrowleft"] & P_2^{\oplus 6} \oplus P_3^{\oplus 4} \arrow[u, "C"']
\end{tikzcd}
$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$
C := \left[\begin{array}{cccccc|cccc}
2 & . & . & . & . & . & . & . & . & . \\
. & 2 & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . \\ \hline
. & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & 1
\end{array}\right], \quad
\tilde{C} := \left[\begin{array}{cccccc|cccc}
1 & . & . & . & . & . & . & . & . & . \\
. & 2 & . & . & . & . & . & . & . & . \\
. & . & 2 & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & 2 & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . \\ \hline
. & . & . & . & . & . & 2 & . & . & . \\
. & . & . & . & . & . & . & 2 & . & . \\
. & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & 1
\end{array}\right] .
$$

Recall that $R := \mathbb{Z}_{(2)}$.

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the

commutativity can be found in the file main_S5_loc2 and main_S5_loc2_P2oP3 the matrices for this diagram in the file Diagram_S5_loc2_P2oP3 .

The maps and matrices are denoted analogously to those for $P_1 \otimes P_3$, see proof of Lemma 123 .

To verify the Lemma, follow these steps:

```
load "main_S5_loc2";
load "main_S5_loc2_P2oP3";
load "Diagram_S5_loc2_P2oP3";

 [rhorP2(sigma)*i2 eq i2*rhoP2(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i2,MatrixRing(Rationals(),8)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),16)!1,i3);

 [op_rp2op3(sigma)*E1 eq E1*op_p2op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p2orp3(sigma)*E2 eq E2*op_p2op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

//RS5-linearity:
 [op_proj_sum_p2p3(sigma)*C1 eq C1*op_proj_sum_p2p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_proj_sum_p2p3(sigma)*C2 eq C2*op_proj_sum_p2p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp2op3(sigma)*B1 eq B1*op_proj_sum_p2p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p2orp3(sigma)*B2 eq B2*op_proj_sum_p2p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p2op3(sigma)*A eq A*op_proj_sum_p2p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];

//local invertibility; loc_inv see "main_S5_loc2"
 loc_inv(B1,2);
 loc_inv(B2,2);
 loc_inv(A,2);
```

□

Remark 128 Using Magma, we can verify that

$$\begin{aligned} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) &\cong D_2 \otimes D_3 \cong \bar{P}_2\,, \\ \mathrm{Coker}(C) &\cong D_2 \otimes \bar{P}_3 \cong \bar{P}_2^{\oplus 2}\,, \\ \mathrm{Coker}(\tilde{C}) &\cong \bar{P}_2 \otimes D_3 \cong \bar{P}_2^{\oplus 3} \oplus \bar{P}_3^{\oplus 2}\,, \end{aligned}$$

with Loewy layers already known. Note that $\mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix})$ is projective over $\bar{\Lambda}_{(2)}$, although neither P_2 nor P_3 belong to a defect-0 block.

5.2.7 The tensor product $P_3 \otimes P_3$

Lemma 129 *Let E be the embedding $\mathfrak{r}P_3 \otimes P_3 \longrightarrow P_3 \otimes P_3$, and $\tilde{E}$ be the embedding $P_3 \otimes \mathfrak{r}P_3 \longrightarrow P_3 \otimes P_3$. We have a commutative diagram of $\mathbb{Z}_{(2)}\mathrm{S}_5$-linear maps*

$$\begin{tikzcd}
P_3 \otimes \mathfrak{r}P_3 \arrow[rr, "\tilde{B}", "\sim"'] \arrow[d, hook, "\tilde{E}"'] \arrow[drr, phantom, "\circlearrowleft"] & & P_1^{\oplus 2} \oplus P_3^{\oplus 2} \arrow[d, "\tilde{C}"] \\
P_3 \otimes P_3 \arrow[rr, "A", "\sim"'] \arrow[drr, phantom, "\circlearrowleft"] & & P_1^{\oplus 2} \oplus P_3^{\oplus 2} \\
\mathfrak{r}P_3 \otimes P_3 \arrow[rr, "B", "\sim"'] \arrow[u, hook, "E"] & & P_1^{\oplus 2} \oplus P_3^{\oplus 2} \arrow[u, "C"']
\end{tikzcd}$$

with A, B, $\tilde{B}$ isomorphisms, and the describing matrices

$$C := \left[\begin{array}{cc|cc} 2 & . & . & . \\ . & 1 & . & . \\ \hline . & . & 2 & . \\ . & . & . & 1 \end{array}\right], \quad \tilde{C} := \left[\begin{array}{cc|cc} 2 & . & . & . \\ h_1^{11} - h_3^{11} & 1 & . & . \\ \hline . & . & 2 & . \\ . & . & h_1^{33} & 1 \end{array}\right].$$

Proof. The assertion results from a Magma calculation **[3]**. We explain the necessary steps to verify the result via Magma.
The functions and operating matrices necessary to prove the local invertibility, the $R\mathrm{S}_5$-linearity and the commutativity can be found in the file `main_S5_loc2` and `main_S5_loc2_P3oP3` the matrices for this diagram in the file `Diagram_S5_loc2_P3oP3` .

The maps and matrices are denoted analogously to those for $P_1 \otimes P_3$, see proof of Lemma 123 .

To verify the Lemma, follow these steps:

```
load "main_S5_loc2";
load "main_S5_loc2_P3oP3";
load "Diagram_S5_loc2_P3oP3";

 [rhorP3(sigma)*i3 eq i3*rhoP3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];

E1 := KroneckerProduct(i3,MatrixRing(Rationals(),8)!1);
E2 := KroneckerProduct(MatrixRing(Rationals(),8)!1,i3);

 [op_rp3op3(sigma)*E1 eq E1*op_p3op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p3orp3(sigma)*E2 eq E2*op_p3op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];

//commutativity:
 E1*A eq B1*C1;
 E2*A eq B2*C2;

 //ℤ(2)S5-linearity
 [op_proj_sum_p3p3(sigma)*C1 eq C1*op_proj_sum_p3p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_proj_sum_p3p3(sigma)*C2 eq C2*op_proj_sum_p3p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_rp3op3(sigma)*B1 eq B1*op_proj_sum_p3p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p3orp3(sigma)*B2 eq B2*op_proj_sum_p3p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
 [op_p3op3(sigma)*A eq A*op_proj_sum_p3p3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)} ];
```

```
//local invertibility; loc_inv see "main_S5_loc2"
  loc_inv(B1,2);
  loc_inv(B2,2);
  loc_inv(A,2);
```

□

Remark 130 Using Magma, we can verify that

$$\begin{array}{lclclcl} \mathrm{Coker}(\begin{bmatrix} C \\ \tilde{C} \end{bmatrix}) & \cong & D_3 \otimes D_3 & \cong & D_3 \oplus X\,, & & \\ \mathrm{Coker}(C) & \cong & \mathrm{Coker}(\tilde{C}) & \cong & D_3 \otimes \bar{P}_3 & \cong & \bar{P}_1 \oplus \bar{P}_3\,, \end{array}$$

with X a module of dimension 12, with Loewy layers $\begin{array}{c} D_1 \\ D_2 \\ D_1 \\ D_1 \\ D_2 \\ D_1 \end{array}$.

As operating matrices in lower block-triangular form, we get for X

$$\mathrm{S}_5 \longrightarrow \mathrm{GL}_{12}(\mathbb{F}_2)\,, \quad (1,2,3,4,5) \longmapsto \left(\begin{array}{c|cccc|c|c|cccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$(1,2) \longmapsto \left(\begin{array}{c|cccc|c|c|cccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Chapter 6

The Krull-Schmidt Algorithm

6.1 Lemmas on idempotents

Let E be a ring.

Lemma 131 *If* $E/_{\mathfrak{r}E}$ *does not contain any nontrivial idempotents, then neither does* E.

Proof. Assume there exists an orthogonal decomposition $1 = e' + e''$ into idempotents in E such that $e' \neq 0$ and $e'' \neq 0$.

We have $e' \notin \mathfrak{r}E$, because otherwise $1 - e'$ is invertible by Lemma 182, whence $(1 - e')e' = 0$ implies $e' = 0$, which is not the case.

Likewise $e'' \notin \mathfrak{r}E$.

Denote by $\varphi : E \longrightarrow E/\mathfrak{r}E$, $x \longmapsto x + \mathfrak{r}E$ the residue class map. Then $1\varphi = e'\varphi + e''\varphi$ is an orthogonal decomposition into idempotents, and $e'\varphi$, $e''\varphi \neq 0$, which is a *contradiction.* □

Lemma 132 *Let* $e \in E$ *be an idempotent. If* e *is a unit in* E*, then* $e = 1_E$ *.*

Proof. Since e is an idempotent, we have $(1-e)e = 0$. Because e is a unit, we obtain that $1-e = 0$, and thus $e = 1$. □

Lemma 133 *Let* $e \in E$ *be an idempotent. If* $(1_E - e)$ *is a unit in* E*, then* $e = 0_E$ *.*

Proof. Since e is an idempotent, we have $(1-e)e = 0$. Because $1-e$ is an isomorphism, we obtain that $e = 0$. □

Corollary 134 *Let* E *be local. Suppose given an idempotent* $e \in E$*. Then* $e = 1_E$ *or* $e = 0_E$ *.*

Proof. Assume e to be invertible. Then $e = 1$ with Lemma 132.

Now assume e to be not invertible. Then $1-e$ is invertible, since E is local. Then $e = 0$ with Lemma 133. □

Lemma 135 *Suppose* E *to be an integral domain. Suppose given an idempotent* $e \in E$*. Then* $e = 1_E$ *or* $e = 0_E$ *.*

Proof. Since e is an idempotent, we have $e(1-e) = 0$. Since E is an integral domain, we have that $e = 0$ or $e - 1 = 0$. □

Lemma 136 *Suppose given an idempotent $e \in E$. Then e is primitive if and only if 0_E and e are the only idempotents in eEe.*

Note that eEe is not required to be local.

Proof. *Ad "$\Rightarrow$". Assume* that there exists an idempotent $f \in eEe$ with $f \neq 0$ and $f \neq e$. Since

$$\begin{array}{rclclcl} (e-f)^2 & = & e^2 - ef - fe + f^2 & = & e - f - f + f & = & e - f \\ (e-f)f & = & ef - f^2 & = & f - f & = & 0 \\ f(e-f) & = & fe - f^2 & = & f - f & = & 0 \end{array}$$

the decomposition $e = f + (e-f)$ is an orthogonal decomposition into idempotents. This is a *contradiction* to the primitivity of e.

Ad "$\Leftarrow$". Let $e = e' + e''$ be an orthogonal decomposition into idempotents.

Then $e' = (e' + e'')e'(e' + e'') = ee'e \in eEe$. Therefore $e' \in \{0, e\}$, and so e is primitive. □

6.2 A Krull-Schmidt-type decomposition method

6.2.1 Detecting an isomorphism

Let $\mathcal{B}$ be an additive category.

Lemma 137 *Let X, Y, $Z \in \mathrm{obj}(\mathcal{B})$ with*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and g a nonisomorphism. Further, suppose $X \not\cong 0$ and $\mathrm{End}_{\mathcal{B}}(Y)$ to be local. Then the composite fg is a nonisomorphism.

Proof. Assume that $X \xrightarrow{fg} Z$ is an isomorphism. Then $e := g \cdot (fg)^{-1} \cdot f$ is an idempotent endomorphism on Y, since $e^2 = g\,\underbrace{(fg)^{-1}f \cdot g}_{=1}(fg)^{-1}f = g(fg)^{-1}f = e$.

Assume further that e is an isomorphism. Then with Lemma 132, we have $e = \mathrm{id}_Y$. Thus, we obtain $e = g \cdot ((fg)^{-1}f) = \mathrm{id}_Y$ and $((fg)^{-1}f) \cdot g = (fg)^{-1} \cdot (fg) = \mathrm{id}_Z$ by definition, so that g is an isomorphism. This is a *contradiction*. Therefore, e is a nonisomorphism.

Now assume that $\mathrm{id} - e$ is an isomorphism. Then with Lemma 133, $e = 0_Y$. Thus,

$$0 \;=\; feg(fg)^{-1} \;=\; fg(fg)^{-1}fg(fg)^{-1} \;=\; \mathrm{id}_X \ .$$

This is a *contradiction*, for $X \not\cong 0$. Therefore, $\mathrm{id} - e$ is a nonisomorphism.

Alltogether, we now have $\mathrm{id}_Y = e + (\mathrm{id}_Y - e)$, with $\mathrm{id}_Y \in \mathrm{U}(\mathrm{End}(Y))$ and e and $(\mathrm{id}_Y - e)$ non-units. This is a *contradiction to our initial assumption* with Remark 192, for $\mathrm{End}(Y)$ is local. Therefore, fg is a nonisomorphism. □

Lemma 138 *Let X, $Y \in \mathrm{obj}(\mathcal{B})$ with*

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

and f, g nonisomorphisms and $\mathrm{End}_{\mathcal{B}}(X)$ local. Then $f + g$ is a nonisomorphism.

Proof. If $X \not\cong Y$, then there is nothing to show, since no isomorphism exists.

So let $X \cong Y$. Let $X \xrightarrow[\sim]{u} Y$ be an isomorphism. For f, g are nonisomorphisms, so are fu^{-1} and gu^{-1}, which are in $\mathrm{End}(X)$. Thus, $fu^{-1} + gu^{-1} = (f+g) \cdot u^{-1}$ is a nonisomorphism with Remark 192, since $\mathrm{End}(X)$ is local. Hence, $f + g$ is a nonisomorphism. □

The following method is a variant of the proof of Krull-Schmidt found in **[2,** Th. 1.4.3**]**.

Lemma 139 *Let $X_1, \ldots, X_n \in \mathrm{obj}(\mathcal{B})$ with local endomorphism rings; cf. Definition 189. Let*

$$e \;\in\; \mathrm{End}_{\mathcal{B}}(X_1 \oplus \ldots \oplus X_n) \setminus \{0\}$$

be an idempotent. We write it as a matrix $e = (e_{ab})_{a,b}$, where $e_{ab} : X_a \longrightarrow X_b$ for a, $b \in [1, n]$.

Then there exist i, $j \in [1, n]$ such that e_{ij} is an isomorphism.

Proof. First, we remark that $X_i \not\cong 0$ for $i \in [1, n]$.

Now, *assume* that e_{ab} is not an isomorphism for all a, $b \in [1, n]$.

Let $f := \mathrm{id}_{X_1 \oplus \ldots \oplus X_n} - e = 1 - e$. Writing $f = (f_{ab})_{a,b} \in \mathrm{End}_{\mathcal{B}}(X_1 \oplus \ldots \oplus X_n)$, we obtain

$$f_{ab} = \begin{cases} \mathrm{id}_{X_a} - e_{aa} & \text{for } a = b \\ -e_{ab} & \text{for } a \neq b\ . \end{cases}$$

If f is an isomorphism then $e \neq 0$ implies $fe \neq 0$, but $fe = (1-e)e = 0$. This is a *contradiction.*

So it remains to show that f is an isomorphism.

Now f_{aa} is an isomorphism for every a since $\mathrm{End}_{\mathcal{B}}(X_a)$ is local; cf. Remark 192. In contrast, f_{ab} is not an isomorphism if $a \neq b$. So we have a matrix f with isomorphisms on the main diagonal, and all other entries consisting of nonisomorphisms.

By row and column operations, corresponding to composition with automorphism from left and right, this matrix can be transformed into the identity endomorphism, as we shall describe in what follows. This then shows that f was already an isomorphism.

We begin with the last row, and, by multiplying the row with the isomorphism f_{nn}^{-1}, produce the entry $\tilde{f}_{nn} = 1$. Off the diagonal, we have the composite of an isomorphism and a nonisomorphism, so we still obtain nonisomorphisms there; cf. Lemma 137. We now use $\tilde{f}_{nn} = 1$ to clear the last column by row operations. Off the diagonal, we obtain sums of the form $kl + m$ with k, l and m nonisomorphisms. Using Lemma 137, kl is a nonisomorphism, and therefore according to Lemma 138, $kl + m$ is a nonisomorphism. On the diagonal, we obtain sums of the form $kl + m$ with k, l nonisomorphisms and m an isomorphism. Using Lemma 137, kl is a nonisomorphism, and thus $kl + m$ is an isomorphism again.

Then, we use $\tilde{f}_{nn} = 1$ to clear the last row by column operations, without any effect on the other entries of f, since we operate with a column that has zero entries everywhere but on the diagonal.

By repeating this steps for every row, our matrix simplifies to $\mathrm{id}_{X_1 \oplus \ldots \oplus X_n}$. □

Definition 140 We say that **idempotents split** in $\mathcal{B}$ if for $V \in \mathrm{obj}(\mathcal{B})$ every idempotent $\varepsilon \in \mathrm{End}_{\mathcal{B}}(V)$ **splits,** meaning that there are $W, W' \in \mathrm{obj}(\mathcal{B})$ and an isomorphism $f : V \xrightarrow{\sim} W \oplus W'$ such that

$$\begin{tikzcd}
V \arrow[r, "f =: (g\ g')", "\sim"'] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\circlearrowleft"] & W \oplus W' \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] \\
V \arrow[r, shift left, "f", "\sim"'] & W \oplus W' \arrow[l, shift left, "f^{-1} =: \left(\begin{smallmatrix} h \\ h' \end{smallmatrix}\right)"]
\end{tikzcd}$$

commutes.

6.2.2 The method in general

Proposition 141 *Let $\mathcal{B}$ be an additive category in which idempotents split; cf. Definition 140.*

Suppose given $Y \in \mathrm{obj}(\mathcal{B})$.

Let $n \geq 1$. Suppose given $X_i \in \mathrm{obj}(\mathcal{B})$ such that $\mathrm{End}_{\mathcal{B}}(X_i)$ is local for $i \in [1,n]$.

Suppose that

$$Y \;\cong\; \bigoplus_{i\in[1,n]} X_i \,.$$

Let Z be a direct summand of Y. Then there is $I \subseteq [1,n]$ with

$$Z \;\cong\; \bigoplus_{i\in I} X_i \,.$$

Proof. Choose

$$\begin{array}{rccc} q: & Y & \xrightarrow{(q_1,\,\dots,\,q_n)} & \bigoplus_{i\in[1,n]} X_i \\ & y & \longmapsto & (yq_1,\dots,yq_n)\;, \end{array}$$

$$\begin{array}{rccc} j: & \bigoplus_{i\in[1,n]} X_i & \xrightarrow{\begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix}} & Y \\ & (x_1,\dots,x_n) & \longmapsto & x_1 j_1 + \ldots + x_n j_n\;, \end{array}$$

with

$$jq = \mathrm{id}_{\bigoplus_{i\in[1,n]} X_i} \quad \text{and} \quad qj = \mathrm{id}_Y \,.$$

We consider the case $Z \ncong 0$. Choose

$$\begin{tikzcd} Y \arrow[dr, "\pi"] & & Y \\ & Z \arrow[ur, "\iota"] & \end{tikzcd}$$

such that $\iota\pi = 1_Z$. Writing $e := \pi\iota \in \mathrm{End}_{\mathcal{B}}\, Y$, we have $e^2 = \pi\iota\pi\iota = \pi\iota = e$, so e is an idempotent. We have $e \neq 0$, since otherwise $\mathrm{id}_Z = \iota\pi\iota\pi = \iota e\pi = 0$, whence Z would be isomorphic to 0.

Let $C := \{\, i \in [1,n] \,:\, j_c\pi = 0 \,\} \subseteq [1,n]$.

For a, $b \in [1,n]$, we let j'_a be the embedding of X_a to $\bigoplus_{i\in[1,n]} X_i$ and q'_b be the projection of $\bigoplus_{i\in[1,n]} X_i$ to X_b.

According to Lemma 139 there are a, $b \in [1,n]$ such that we obtain the following commutative diagram with an isomorphism $\tilde{e}_{ab} = j_a\, e\, q_b$.

$$\begin{tikzcd} Y \arrow[r, "e"] & Y \arrow[d, "q", "\wr"'] \arrow[dd, bend left=60, "q_b"] \\ \bigoplus_{i\in[1,n]} X_i \arrow[u, "\wr", "j"'] \arrow[r, "\tilde{e}"] & \bigoplus_{i\in[1,n]} X_i \arrow[d, "q'_b"] \\ X_a \arrow[u, "j'_a"] \arrow[uu, bend left=60, "j_a"] \arrow[r, "\tilde{e}_{ab}", "\sim"'] & X_b \end{tikzcd}$$

Hence, we get the diagram

$$\begin{tikzcd} X_a \arrow[r, "j_a"'] \arrow[rrr, bend left, "\tilde{e}_{ab}", "\sim"'] & Y \arrow[r, "e"'] \arrow[dr, "\pi"'] & Y \arrow[r, "q_b"'] & X_b\,. \\ & & Z \arrow[u, "\iota"'] & \end{tikzcd}$$

Let $\varepsilon_1 := \iota\, q_b\, \tilde{e}_{ab}^{-1}\, j_a\, \pi$. Since

$$\varepsilon_1^2 \;=\; \iota\, q_b\, \tilde{e}_{ab}^{-1}\, j_a\, \pi\, \iota\, q_b\, \tilde{e}_{ab}^{-1}\, j_a\, \pi \;=\; \iota\, q_b\, \tilde{e}_{ab}^{-1}\, \tilde{e}_{ab}\, \tilde{e}_{ab}^{-1}\, j_a\, \pi \;=\; \iota\, q_b\, \tilde{e}_{ab}^{-1}\, j_a\, \pi \;=\; \varepsilon_1\ ,$$

ε_1 is an idempotent endomorphism of Z. It splits, and so we have the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, shift right, "(g'\; g'')"', "\sim"] \arrow[d, "\varepsilon_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Z' \oplus Z'' \arrow[l, shift right, "\left(\begin{smallmatrix} h' \\ h'' \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] \\
Z \arrow[r, shift left, "(g'\; g'')", "\sim"'] & Z' \oplus Z'' \arrow[l, shift left, "\left(\begin{smallmatrix} h' \\ h'' \end{smallmatrix}\right)"]\ .
\end{tikzcd}$$

In particular, $\begin{pmatrix} h' \\ h'' \end{pmatrix}(g'\, g'') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(g'\, g'')\begin{pmatrix} h' \\ h'' \end{pmatrix} = 1$.

We *claim* that $j_a \pi g' : X_a \overset{!}{\xrightarrow{\sim}} Z'$ and that $j_a \pi g'' \overset{!}{=} 0$.

Considering the diagram

$$\begin{tikzcd}
 & Z \arrow[dr, "\iota\, q_b\, \tilde{e}_{ab}^{-1}"] & \\
Z' \arrow[ur, "h'"] & & X_a\ , \arrow[dl, "j_a\, \pi"] \\
 & Z \arrow[ul, "g'"] &
\end{tikzcd}$$

we have

$$\begin{aligned}
(h' \iota\, q_b\, \tilde{e}_{ab}^{-1})(j_a \pi g') &= h' \varepsilon_1 g' \\
&= h'\,(g'\, g'')\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h' \\ h'' \end{pmatrix} g' \\
&= (1\, 0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= 1
\end{aligned}$$

and on the other hand, using that $1 = (g'\, g'')\begin{pmatrix} h' \\ h'' \end{pmatrix} = g'h' + g''h''$, i.e. that $g'h' = 1 - g''h''$,

$$\begin{aligned}
(j_a \pi g')(h' \iota\, q_b\, \tilde{e}_{ab}^{-1}) &= j_a \pi (1 - g''h'')\iota\, q_b\, \tilde{e}_{ab}^{-1} \\
&= \underbrace{\overbrace{j_a \pi \iota\, q_b}^{\tilde{e}_{ab}}\, \tilde{e}_{ab}^{-1}}_{=1} - \underbrace{j_a \pi g'' h'' \iota\, q_b\, \tilde{e}_{ab}^{-1}}_{\overset{!}{=}0}\ .
\end{aligned}$$

So it suffices to show that $j_a \pi g'' \overset{!}{=} 0$.

We find $\varepsilon_1\,(g'\, g'') = (g'\, g'')\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, i.e. $(\varepsilon_1 g'\; \varepsilon_1 g'') = (g'\; 0)$. In particular, $\iota\, q_b\, \tilde{e}_{ab}^{-1}\, j_a \pi g'' = \varepsilon_1 g'' = 0$. So

$$j_a \pi g'' \;=\; \underbrace{j_a \pi \iota\, q_b}_{=\tilde{e}_{ab}}\, \tilde{e}_{ab}^{-1}\, j_a \pi g'' \;=\; 0\,.$$

This proves the *claim*.

We consider the general case again, allowing for $Z \cong 0$ or $Z \not\cong 0$.

We now proceed by *descending induction* over $|C|$ to show that there is an isomorphism $Z \overset{!}{\cong} \bigoplus_{i \in [1,n] \setminus C} X_i$.

Basis. Let $|C| = n$. Then $j_c \pi = 0$ for all $c \in [1,n]$ and therefore $\underbrace{(q_1 j_1 + \ldots + q_n j_n)}_{=1}\pi = 0$. Thus we have $0 = \iota\pi = 1_Z$, whence $Z \cong 0$.

Inductive step. Suppose $|C| \leq n-1$. Then there exists $c \in [1,n] \setminus C$, so that $j_c\pi \neq 0$. Hence $\pi \neq 0$. So $Z \not\cong 0$, and our claim above is applicable.

Let

$$\begin{tikzcd}
Y \arrow[dr, "\pi"] \arrow[ddrr, bend right, "\pi''"'] & & & & Y \\
& Z \arrow[dr, "g''"] & & Z \arrow[ur, "\iota"] & \\
& & Z'' \arrow[ur, "h''"] \arrow[uurr, bend right, "\iota''"'] & &
\end{tikzcd}$$

be a commutative diagram.

We have $\iota''\pi'' = h''\iota\pi g'' = h''g'' = 1_{Z''}$.

For any $c \in C$ we get $j_c\pi = 0$ and thus $j_c\pi'' = j_c\pi g'' = 0$.

For $c = a$ our claim yields $j_a\pi'' = j_a\pi g'' = 0$; moreover, $X_a \not\cong 0$ and $j_a\pi g' : X_a \xrightarrow{\sim} Z'$ imply that $j_a\pi \neq 0$, i.e. $a \notin C$.

So $j_c\pi'' = 0$ for $c \in C \,\dot\cup\, \{a\}$.

By induction, there exists $Z'' \underset{\sim}{\xrightarrow{\varphi''}} \bigoplus\limits_{i\in[1,n]\setminus(C\dot\cup\{a\})} X_i$. Altogether, using our claim, we obtain

$$\begin{tikzcd}[column sep=large]
Z \arrow[r, "(g'\ g'')", "\sim"'] \arrow[rrrr, bend right=15, "\sim", "=:\varphi"'] & Z' \oplus Z'' \arrow[r, "\left(\begin{smallmatrix} j_a\pi g' & 0 \\ 0 & 1 \end{smallmatrix}\right)", "\sim"'] & X_a \oplus Z'' \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varphi'' \end{smallmatrix}\right)", "\sim"'] & X_a \oplus \bigoplus\limits_{i\in[1,n]\setminus(C\dot\cup\{a\})} X_i \arrow[r, "\sim"'] & \bigoplus\limits_{i\in[1,n]\setminus C} X_i\,.
\end{tikzcd}$$

$\square$

6.2.3 The Krull-Schmidt Algorithm

6.2.3.1 The algorithm

The procedure described here will be referred to as the **Krull-Schmidt Algorithm**. For the assertions needed in the procedure, we refer to the proof of Proposition 141.

Let Λ be a ring.

Suppose given $k \geq 0$. Suppose given nonzero Λ-modules W_α for $\alpha \in [1,k]$. Let $X := \bigoplus\limits_{\alpha\in[1,k]} W_\alpha$. Let U be a direct summand of X.

We want to find $I \subseteq [1,k]$ such that $U \cong \bigoplus\limits_{\alpha\in I} W_\alpha$.

If W_α has a local endomorphism ring for $\alpha \in [1,k]$, such a subset I exists by Proposition 141. Otherwise, we may attempt to use the same algorithm, which then is not guaranteed to succeed – and so the procedure is not an algorithm any longer. It will return a warning in case it breaks down.

Let i be the embedding $U \hookrightarrow X$. Let p be the projection of X onto U. Then $i \cdot p$ is the identity on U. Let j_α be the embedding $W_\alpha \hookrightarrow X$ and q_α be the projection of X onto W_α for $\alpha \in [1,k]$.

$$\begin{tikzcd}
U \arrow[r, shift left, "i"] & X \arrow[l, shift left, "p"] \arrow[r, shift left, "q_\alpha"] & W_\alpha \arrow[l, shift left, "j_\alpha"]
\end{tikzcd}$$

Step 1.

Write $g_0 := p \cdot i$, which is an idempotent endomorphism of X.

If $U = 0$, then we are done. If $U \neq 0$, then $g_0 \neq 0$.

We search for $\alpha_1 \in [1, k]$ and $\beta_1 \in [1, k]$ such that $f_1 := j_{\alpha_1} \cdot p \cdot i \cdot q_{\beta_1}$ is an isomorphism.

If W_α has a local endomorphism ring for $\alpha \in [1, k]$, then such elements α_1 and β_1 exist by Lemma 139.

If such elements α_1 and β_1 do not exist, we break the algorithm and return a warning.

$$\begin{tikzcd}
W_{\alpha_1} \arrow[r, "j_{\alpha_1}"'] \arrow[rrrr, bend left, "f_1", "\sim"'] & X \arrow[rr, "g_0"] \arrow[dr, "p"] & & X \arrow[r, "q_{\beta_1}"'] & W_{\beta_1} \\
& & U \arrow[ur, "i"] & &
\end{tikzcd}$$

Then $\varepsilon_1 := i \cdot q_{\beta_1} \cdot f_1^{-1} \cdot j_{\alpha_1} \cdot p$ is an idempotent endomorphism on U. It splits the module U into two direct summands

$$U \;=\; U\varepsilon_1 \,\oplus\, U(1 - \varepsilon_1)\,,$$

where $U\varepsilon_1$ is isomorphic to W_{α_1}.

It remains to find $I_1 \subseteq [1, k] \setminus \{\alpha_1\}$ such that $U(1 - \varepsilon_1) \cong \bigoplus_{\alpha \in I_1} W_\alpha$. Then we may choose $I = \{\alpha_1\} \,\dot\cup\, I_1$.

Step 2.

Write $g_1 := p \cdot (1 - \varepsilon_1) \cdot i$, which is an idempotent endomorphism of X.

If $U(1 - \varepsilon_1) = 0$, then we are done. If $U(1 - \varepsilon_1) \neq 0$, then $g_1 \neq 0$.

We search for $\alpha_2 \in [1, k] \setminus \{\alpha_1\}$ and $\beta_2 \in [1, k]$ such that $f_2 := j_{\alpha_2} \cdot p \cdot (1 - \varepsilon_1) \cdot i \cdot q_{\beta_2}$ is an isomorphism.

If W_α has a local endomorphism ring for $\alpha \in [1, k]$, then such elements α_2 and β_2 exist by Lemma 139, remarking that $j_{\alpha_1} \cdot p \cdot (1 - \varepsilon_1) = 0$.

If such elements α_2 and β_2 do not exist, we break the algorithm and return a warning.

Let i_1 be the embedding $U(1-\varepsilon_1) \hookrightarrow U$. Let p_1 be the projection of U onto $U(1-\varepsilon_1)$. Then $i_1 \cdot p_1 = \mathrm{id}$.

$$\begin{tikzcd}
W_{\alpha_2} \arrow[r, "j_{\alpha_2}"'] \arrow[rrrrrr, bend left, "f_2", "\sim"'] & X \arrow[rrrr, "g_1"] \arrow[dr, "p"] & & & & X \arrow[r, "q_{\beta_2}"'] & W_{\beta_2} \\
& & U \arrow[rr, "1-\varepsilon_1"] \arrow[dr, "p_1"] & & U \arrow[ur, "i"] & & \\
& & & U(1-\varepsilon_1) \arrow[ur, "i_1"] & & &
\end{tikzcd} .$$

Then $\varepsilon_2 := p_1 \cdot i_1 \cdot i \cdot q_{\beta_2} \cdot f_2^{-1} \cdot j_{\alpha_2} \cdot p \cdot p_1 \cdot i_1$ is an idempotent endomorphism on U, orthogonal to $\varepsilon_1 = 1 - p_1 \cdot i_1$.

It splits the module U into three direct summands

$$U \;=\; U\varepsilon_1 \,\oplus\, U\varepsilon_2 \,\oplus\, U(1 - \varepsilon_1 - \varepsilon_2)\,,$$

where $U\varepsilon_s$ is isomorphic to W_{α_s} for $s \in \{1, 2\}$.

It remains to find $I_2 \subseteq [1, k] \setminus \{\alpha_1, \alpha_2\}$ such that $U(1 - \varepsilon_1 - \varepsilon_2) \cong \bigoplus_{\alpha \in I_2} W_\alpha$ Then we may choose $I = \{\alpha_1\} \,\dot\cup\, \{\alpha_2\} \,\dot\cup\, I_2$.

Step 3.

Write $g_2 := p \cdot (1 - \varepsilon_1 - \varepsilon_2) \cdot i$, which is an idempotent endomorphism of X.

If $U(1 - \varepsilon_1 - \varepsilon_2) = 0$, then we are done. If $U(1 - \varepsilon_1 - \varepsilon_2) \neq 0$, then $g_2 \neq 0$.

We search for $\alpha_3 \in [1, k] \setminus \{\alpha_1, \alpha_2\}$ and $\beta_3 \in [1, k]$ such that $f_3 := j_{\alpha_3} \cdot p \cdot (1 - \varepsilon_1 - \varepsilon_2) \cdot i \cdot q_{\beta_3}$ is an isomorphism.

If W_α has a local endomorphism ring for $\alpha \in [1, k]$, then such elements α_3 and β_3 exist by Lemma 139, remarking that $j_{\alpha_s} \cdot p \cdot (1 - \varepsilon_1 - \varepsilon_2) = 0$ for $s \in \{1, 2\}$.

If such elements α_3 and β_3 do not exist, we break the algorithm and return a warning.

Let i_2 be the embedding $U(1 - \varepsilon_1 - \varepsilon_2) \hookrightarrow U$. Let p_2 be the projection of U onto $U(1 - \varepsilon_1 - \varepsilon_2)$. Then $i_2 \cdot p_2 = \mathrm{id}$.

$$\begin{tikzcd}
W_{\alpha_3} \arrow[r, "j_{\alpha_3}"'] \arrow[rrrrrr, bend left, "f_3", "\sim"'] & X \arrow[rrrr, "g_2"] \arrow[dr, "p"] & & & & X \arrow[r, "q_{\beta_3}"'] & W_{\beta_3} \, . \\
& & U \arrow[rr, "1-\varepsilon_1-\varepsilon_2"] \arrow[dr, "p_2"] & & U \arrow[ur, "i"] & & \\
& & & U(1-\varepsilon_1-\varepsilon_2) \arrow[ur, "i_2"] & & &
\end{tikzcd}$$

Then $\varepsilon_3 := p_2 \cdot i_2 \cdot i \cdot q_{\beta_3} \cdot f_3^{-1} \cdot j_{\alpha_3} \cdot p \cdot p_2 \cdot i_2$ is an idempotent endomorphism on U, orthogonal to $\varepsilon_1 + \varepsilon_2 = 1 - p_2 \cdot i_2$.

It splits the module U into four direct summands

$$U \;=\; U\varepsilon_1 \,\oplus\, U\varepsilon_2 \,\oplus\, U\varepsilon_3 \,\oplus\, U(1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)\,,$$

where $U\varepsilon_s$ is isomorphic to W_{α_s} for $s \in \{1, 2, 3\}$.

It remains to find $I_3 \subseteq [1, k] \setminus \{\alpha_1, \alpha_2, \alpha_3\}$ such that $U(1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3) \cong \bigoplus_{\alpha \in I_3} W_\alpha$. Then we may choose $I = \{\alpha_1\} \,\dot\cup\, \{\alpha_2\} \,\dot\cup\, \{\alpha_3\} \,\dot\cup\, I_3$.

Step ≥ 4. Etc.

This procedure ends after $\leq k$ steps.

If W_α has a local endomorphism ring for $\alpha \in [1, k]$, then the procedure ends once $U(1 - \varepsilon_1 - \ldots - \varepsilon_\ell) = 0$ for some $\ell \in [0, k]$, i.e. once $1 = \varepsilon_1 + \ldots + \varepsilon_\ell$ is an orthogonal decomposition into idempotents. This yields

$$U \;=\; \bigoplus_{s \in [1, \ell]} U\varepsilon_s \;\cong\; \bigoplus_{s \in [1, \ell]} W_{\alpha_s}\,,$$

for $U\varepsilon_s \cong W_{\alpha_s}$ for $s \in [1, \ell]$. So we may choose $I = \{\alpha_1\,,\,\ldots\,,\,\alpha_\ell\} \subseteq [1, k]$.

6.2.3.2 The Magma code

We will perform the Krull-Schmidt Algorithm via Magma **[3]** in a particular case.

Let R be a localization of $\mathbb{Z}$ at a maximal ideal. Let Λ be an R-order.

We always view localizations of the integers Magma-internally as subrings of $\mathbb{Q}$, without defining that subring explicitly.

Suppose given $k \geq 0$. Suppose given nonzero Λ-lattices W_α for $\alpha \in [1,k]$. Let $X := \bigoplus_{\alpha \in [1,k]} W_\alpha$. Let U be a direct summand of X.

We want to find $I \subseteq [1,k]$ such that $U \cong \bigoplus_{\alpha \in I} W_\alpha$.

Let i be the embedding $U \hookrightarrow X$. Let p be the projection of X onto U. Let j_α be the embedding $W_\alpha \hookrightarrow X$ and q_α be the projection of X onto W_α for $\alpha \in [1,k]$.

$$\begin{tikzcd} U \arrow[r, shift left, "i"] & X \arrow[l, shift left, "p"] \arrow[r, shift left, "q_\alpha"] & W_\alpha \arrow[l, shift left, "j_\alpha"] \end{tikzcd}$$

The `Krull_Schmidt_Algorithm` is a procedure that yields the subset $I = \{\alpha_1, \ldots, \alpha_\ell\}$, together with the idempotent endomorphisms ε_s of U such that $U\varepsilon_s \cong W_{\alpha_s}$ and the maps $j_{\alpha_s} \cdot p : W_{\alpha_s} \longrightarrow U$ for $s \in [1,\ell]$.

As input data we need the following. A tuple `emb` containing describing matrices of the embeddings $W_\alpha \xrightarrow{j_\alpha} X$. A tuple `proj` containing describing matrices of the projections $X \xrightarrow{q_\alpha} W_\alpha$. The variable `rank_tuple` is a tuple of the ranks of the direct summands W_α of X. Then `rank_summand` is the rank of U. The projection $X \xrightarrow{p} U$ is denoted `p` and is given by a matrix, the embedding $U \xrightarrow{i} X$ is denoted `i` and is given by a matrix. Finally, $R = \mathbb{Z}_{(\texttt{prime})}$.

Calling

```
Krull_Schmidt_Algorithm(emb,proj,rank_tuple,rank_summand,p,i,prime);
```

the output is given as a sequence of tuples of the form

`<idempotent,number,embedding>` ,

where `idempotent` is the idempotent endomorphisms ε_s of U, where `number` is the number α_s such that $U\varepsilon_s \cong W_{\alpha_s}$, in such a way that $1 = \sum_{s \in [1,\ell]} \varepsilon_s$ is an orthogonal decomposition into idempotents and where `embedding` is the monomorphism from W_{α_s} to U, given as a matrix in the choses bases.

If there exists an α such that $\operatorname{End} W_\alpha$ is not local, then the algorithm might not yield a decompostion into summands isomorphic to certain W_α. In this case, the procedure is broken after Step t and it returns the idempotents $(\varepsilon_1, \ldots, \varepsilon_t, 1 - \sum_{s \in [1,t]} \varepsilon_s)$, the numbers $(\alpha_1, \ldots, \alpha_t, 0)$ and the maps $(j_{\alpha_1} \cdot p, \ldots, j_{\alpha_t} \cdot p, 0)$, together with the

```
WARNING: idempotent decomposition incomplete .
```

The algorithm can also be found in the file `Krull_Schmidt_Algorithm`.

```
Q := Rationals();

Krull_Schmidt_Algorithm := function(emb,proj,rank_tuple,rank_summand,p,i,prime)
 // zero just to fix the size of a sequence entry, removed at the end :
 idem_vec := [*<MatrixRing(Q,rank_summand)!0,0,0>*];
 r := 0;                                         // counting variable for while-loop
 // while loop done once the sum over the constructed idempotents equals 1
 while &+[x[1] : x in idem_vec] ne MatrixRing(Q,rank_summand)!1 do
  // success means: composed with maps in emb and proj an isomorphism has been successfully found,
  // which then yields another idempotent
  success := false;
  r +:= 1;
  next := &+[x[1] : x in idem_vec];                        // sum of all idempotents constructed so far
  // remaining endomorphism, turned into an idempotent of X :
  g_next := p * (MatrixRing(Q,rank_summand)!1 - next) * i;
  // (critical point for usage of larger base rings than localizations of integers)
```

```
  D_idem, S_idem, T_idem := SmithFormRat(1 - next);
  // embedding of the image of 1 - next
  Emb_Image_1_minus_idem :=
      ((RMatrixSpace(Q,Rank(D_idem),rank_summand)!(RowSubmatrixRange(D_idem, 1, Rank(D_idem))) *
      (RMatrixSpace(Q,rank_summand,rank_summand)!T_idem^-1)));
  // projection to the image of 1 - next
  Proj_Image_1_minus_idem :=
      RMatrixSpace(Q,rank_summand,Rank(D_idem))!ColumnSubmatrixRange(S_idem^-1, 1, Rank(D_idem));
  // attempt to find an isomorphism of the form emb[j] * g_next * proj[l], for some j and some l
  // must be successful if End W_α is local for every α
  for j in [1..#emb] do
    for l in [1..#proj] do
       if rank_tuple[j] eq rank_tuple[l] then // no isomorphism if ranks not equal
        F_next := emb[j] * g_next * proj[l];  // only quadratic F_next remain under consideration
        // test whether F_next, considered as an endomorphism over the ground ring, is invertible :
        if loc_inv(F_next,prime) then
         emb_next := emb[j];                 // fix data at j and l
         proj_next := proj[l];
         a := j;                             // remember number of chosen summand
         success := true;
         break j;                            // attempt finished, since successful
        end if;
       end if;
      end for;
     end for;
  if success then
   // generating the actual idempotent on U, using the found isomorphism :
   idem_next :=
         Proj_Image_1_minus_idem * Emb_Image_1_minus_idem * i * proj_next * (F_next^-1) *
         emb_next * p * Proj_Image_1_minus_idem * Emb_Image_1_minus_idem;
   idem_vec cat:= [*<idem_next, a, emb[a] * p>*];
  else  // if End W_α is not local for every α, the algorithm might not find an isomorphism - and if not :
   print "WARNING: idempotent decomposition incomplete";
   idem_vec cat:= [*<MatrixRing(Q,rank_summand)!1 - next,0,0>*];
  end if;
 end while;
 idem_vec := idem_vec[2..r+1];
 return idem_vec;
end function;
```

6.2.4 Decomposition of a tensor product of a lattice with a projective lattice in the local case

6.2.4.1 The algorithm

We will now apply the Krull-Schmidt Algorithm to our problem and decompose the tensor product of a lattice and a direct summand of RG, where G is a finite group and R is a localization of the integers at a maximal ideal.

Note that R is a non-complete discrete valuation ring.

Let Λ be the image of RG under the Wedderburn isomorphism ω.

Let M be an RG-lattice. Let P be a direct summand of RG.

Suppose an orthogonal decomposition $1_{RG} = \sum\limits_{i\in[1,\kappa]} e'_i$ into idempotents, not necessarily primitive, to be known. This can be obtained e.g. using Λ and ω^{-1}.

We want to use the Krull-Schmidt-Algorithm of Section 6.2.3.1 to decompose the tensor product $M \otimes P$, using the fact that a decompsition of $M \otimes RG$ is easy to find.

With the isomorphism φ of Lemma 167, we define an isomorphism $\tilde{\varphi}$ as follows.

$$\begin{tikzcd} M \otimes RG \arrow[r, "\sim", "\varphi"'] \arrow[rrr, bend left, "=:\,\tilde{\varphi}", "\sim"'] & M \vec{\otimes} RG \arrow[r, "\sim"] & R^{\oplus \operatorname{rk} M} \vec{\otimes} RG \arrow[r, "\sim"] & RG^{\oplus \operatorname{rk} M} \end{tikzcd}$$

Further, P being a summand of RG, we have an embedding $i : M \otimes P \longrightarrow M \otimes RG$ and an projection $p : M \otimes RG \longrightarrow M \otimes P$, so that $ip = 1_{M \otimes P}$.

Use the orthogonal decomposition $1_{RG} = \sum\limits_{i \in [1,\kappa]} e'_i$ into idempotents of RG to obtain a direct sum decomposition $RG^{\oplus \operatorname{rk} M} = \bigoplus\limits_{\alpha \in [1,k]} W_\alpha$, so that $k = \kappa \cdot \operatorname{rk} M$. We denote by $q_\alpha : RG^{\oplus \operatorname{rk} M} \longrightarrow W_\alpha$ the projection and by $j_\alpha : W_\alpha \longrightarrow RG^{\oplus \operatorname{rk} M}$ the embedding for every α.

$$\begin{tikzcd} M \otimes P \arrow[r, shift left, "i"] & M \otimes RG \arrow[l, shift left, "p"] \arrow[r, "\tilde{\varphi}", "\sim"'] & RG^{\oplus \operatorname{rk} M} \arrow[r, shift left, "q_\alpha"] & W_\alpha \arrow[l, shift left, "j_\alpha"] \end{tikzcd} .$$

With the isomorphism $\tilde{\varphi}$ just defined, we obtain the embedding $\iota := i \cdot \tilde{\varphi} : M \otimes P \longrightarrow RG^{\oplus \operatorname{rk} M}$ and the projection $\pi := \tilde{\varphi}^{-1} \cdot p : RG^{\oplus \operatorname{rk} M} \longrightarrow M \otimes P$. We define $g_0 := \pi \cdot \iota$.

Since $\iota \cdot \pi = i \cdot \tilde{\varphi} \cdot \tilde{\varphi}^{-1} \cdot p = 1$ and since g_0 is an idempotent endomorphism, we may, if $\operatorname{End} W_\alpha$ is local for $\alpha \in [1,k]$, apply Lemma 139, which yields α_1, β_1 such that $f_1 := j_{\alpha_1} \cdot g_0 \cdot q_{\beta_1}$ is an isomorphism.

If there exists α such that $\operatorname{End} W_\alpha$ is not local, we still attempt to find such an isomorphism, which might turn out to be unsuccessful, however, yielding a break and a warning.

$$\begin{tikzcd} W_{\alpha_1} \arrow[r, "j_{\alpha_1}"'] \arrow[rrr, bend left, "f_1", "\sim"'] & RG^{\oplus \operatorname{rk} M} \arrow[r, "g_0"] \arrow[dr, "\pi"] & RG^{\oplus \operatorname{rk} M} \arrow[r, "q_{\beta_1}"'] & W_{\beta_1} \\ & & M \otimes P \arrow[u, "\iota"'] & \end{tikzcd}$$

We obtain an idempotent endomorphism on $M \otimes P$ as $\varepsilon_1 := \iota \cdot q_{\beta_1} \cdot f_1^{-1} \cdot j_{\alpha_1} \cdot \pi$.

By iteration, as described in Section 6.2.3.1, we find an orthogonal decomposition into idempotents $1_{M \otimes P} = \sum\limits_i \varepsilon_i$, yielding a decomposition into indecomposable projectives of $M \otimes P$, provided an isomorphism f_s can be found in every Step s. Again, if $\operatorname{End} W_\alpha$ is local for $\alpha \in [1,k]$, then this is possible.

Remark 142 Suppose that the Wedderburn isomorphism maps RG into a finite direct product Γ of matrix rings over R, so that $\Lambda \subseteq \Gamma$ is an R-suborder such that ${}^{\Gamma}\!/_{\Lambda}$ is of finite length as an R-module, i.e. such that $\operatorname{rk}_R \Lambda = \operatorname{rk}_R \Gamma = |G|$.

Let W_α be indecomposable for $\alpha \in [1,k]$. Then by Lemma 233, each endomorphism ring $\operatorname{End}(W_\alpha)$ is local, so that the requirements of Lemma 139 (and of Proposition 141) are met.

Remark 143 Suppose the requirements of Remark 142 to hold for RG.

Let P be a finitely generated indecomposable projective Λ-module, let $1_\Lambda = \sum\limits_{i \in [1,k]} e_i$ be an orthogonal decomposition into primitive idempotents.

Then we can find $s \in [1,k]$ so that $P \cong e_s \Lambda$.

Proof. First proof, using Proposition 141.

Since P is projective, we can find $\ell \geq 1$ so that P is a summand of $\Lambda^{\oplus \ell} \cong \bigoplus\limits_i (e_i \Lambda)^{\oplus \ell}$. With Proposition 141, we can find direct summands such that $P \cong e_{\gamma_1} \Lambda \oplus \ldots \oplus e_{\gamma_t} \Lambda$ with $\gamma_j \in \{1, \ldots, k\}$. Since P is indecomposable, it is isomorphic to only one summand.

Second proof, using only Lemma 139.

Since P is projective, we can find $\ell \geq 1$ so that P is a summand of $\Lambda^{\oplus \ell} \cong \bigoplus_i (e_i\Lambda)^{\oplus \ell}$. Let $W_i := e_i\Lambda$ for every $i \in [1,k]$. Write $\Lambda^{\oplus \ell} = (\bigoplus_{i\in[1,k]} W_i)^{\oplus \ell} = \bigoplus_{i\in[1,k]} W_i^{\oplus \ell}$.

Then we have the projection and the embedding

$$\bigoplus_{i\in[1,k]} W_i^{\oplus \ell} \xrightarrow{\pi} P \xrightarrow{\iota} \bigoplus_{i\in[1,k]} W_i^{\oplus \ell}.$$

Write $\pi =: \begin{pmatrix} u_1 \\ \vdots \\ u_s \end{pmatrix}$ and $\iota =: (\, v_1 \cdots v_s \,)$.

We have the idempotent matrix $e := \pi\iota = (u_iv_j)_{ij} : \bigoplus_{i\in[1,k]} W_i^{\oplus \ell} \longrightarrow \bigoplus_{i\in[1,k]} W_i^{\oplus \ell}$.

With Lemma 139, we can find i, j such that u_iv_j is an isomorphism.

$$\begin{tikzcd} e_i\Lambda \arrow[r, "u_i"] \arrow[rr, bend right, "\sim", "=:f"'] & P \arrow[r, "v_j"] & e_j\Lambda \arrow[r, "f^{-1}", "\sim"'] & e_i\Lambda \end{tikzcd}$$

Thus, $e_i\Lambda$ is a summand of P. Since P is indecomposable, P is isomorphic to $e_i\Lambda$ with isomorphism u_i.

Third proof, using Lemma 220 from the appendix.

By Remark 208, RG is stable. So the assertion follows from Lemma 220. $\square$

Remark 144 Krull-Schmidt is also true in a more general form. For this, see **[9,** Cor. VI.3.3**]**.

6.2.4.2 The Magma code

We continue our Magma **[3]** implementation.

The function `Application` generates the information for the algorithm `Krull_Schmidt_Algorithm`. It needs the Wedderburn isomorphism `rho`, being a function that maps each group element to a tuple of matrices, having image Λ; a tuple `ez` containing an orthogonal decomposition $1_\Lambda = \sum_{i\in[1,\kappa]} e_i$ into idempotents of Λ, written as vectors in the standard basis of the direct product of the matrix rings; an idempotent `f` of Λ; a function `rhoM` mapping each group element to a matrix acting on an RG-lattice M, the group `G` given as a permutation group, and the `prime` at which we localize the integers.

Write $e_i\omega := e_i'$ for $i \in [1,\kappa]$.

```
Q := Rationals();

Application := function(rho,ez,f,rhoM,G,prime)
Gsorted := [sigma : sigma in G];            // fixing an order on the elements of G
// numbering the elements of G :
Gsorted_bij := map< G -> [1..#Gsorted] | [<Gsorted[i],i> : i in [1..#Gsorted]]>;
Ggen := [G.i : i in [1..NumberOfGenerators(G)]];
// the Wedderburn isomorphism as a matrix :
omega := MatrixRing(Q,#Gsorted)!&cat[ElementToSequence(rho(sigma)[i]):
                              i in [1..#rho(sigma)], sigma in Gsorted];
rkM := NumberOfRows(rhoM(G!1));
// generating M as a GModule :
M :=  GModule(G, [MatrixRing(Q,rkM)!rhoM(x) : x in Ggen]);
// generating RG as a GModule :
RG := GModule(G, [PermutationMatrix(Q,[Gsorted_bij(x * y) : x in Gsorted]) : y in Ggen]);
op_left := [PermutationMatrix(Q,[Gsorted_bij(y * x) : x in Gsorted]) : y in Gsorted];
```

```
 op_left_f := &+[(f * omega^-1)[1,i] * op_left[i] : i in [1..#Gsorted]];
 P,ii,pp := Image(AHom(RG,RG)!op_left_f); //embedding $P \xrightarrow{ii} RG$, projection $RG \xrightarrow{pp} P$
 idM := AHom(M,M)!(MatrixRing(Q,rkM)!1);
 ppp := TensorProduct(idM,MapToMatrix(pp,Q));  //projection $M \otimes RG \xrightarrow{ppp} M \otimes P$
 iii := TensorProduct(idM,MapToMatrix(ii,Q));  //embedding $M \otimes P \xrightarrow{iii} M \otimes RG$
 MoRG := TensorProduct(M,RG);
 phi_prel := DirectSum([rhoM(sigma^-1): sigma in Gsorted]);
 tup1 := [<i,j> : i in [1..rkM] , j in [1..#Gsorted]];
 tup2 := [<i,j> : j in [1..#Gsorted] , i in [1..rkM]];
 tup1_bij := map< tup1 -> [1..#tup1] | [<tup1[i],i> : i in [1..#tup1]]>;
 tup2_bij := map< tup2 -> [1..#tup2] | [<tup2[i],i> : i in [1..#tup1]]>;
 PM_aux_1 := PermutationMatrix(Q,[tup1_bij(x) : x in tup2]);
 PM_aux_2 := PermutationMatrix(Q,[tup2_bij(x) : x in tup1]); // PM_aux_2 = PM_aux_1^-1
 //$M \otimes RG \xrightarrow[\sim]{phi = \varphi} M \vec{\otimes} RG$; and Magma-internally $\varphi = \tilde{\varphi}$ :
 phi := PM_aux_1 * phi_prel * PM_aux_2;
 phiinv := PM_aux_1 * phi_prel^-1 * PM_aux_2;
 vect := [];
 for i in [1..rkM] do
   vec := RMatrixSpace(Q,1,rkM)!0;
   vec[1,i] := 1;
   vect cat:= [vec];
 end for;
 op_left_ez := [&+[(x * omega^-1)[1,i] * op_left[i] : i in [1..#Gsorted]] : x in ez];
 // tuple containing $(e'_1 RG)^{\oplus \operatorname{rk} M} \oplus \ldots \oplus (e'_\kappa RG)^{\oplus \operatorname{rk} M}$ :
 PP   := [Image(AHom(RG,RG)!op) : j in [1..rkM], op in op_left_ez];
 emb:= [* TensorProduct(vec,MapToMatrix(x,Q))* phiinv where
          _, x := Image(AHom(RG,RG)!op) : vec in vect, op in op_left_ez*];
 proj := [* phi*TensorProduct(Transpose(vec),MapToMatrix(x,Q)) where
          _,_, x := Image(AHom(RG,RG)!op) : vec in vect, op in op_left_ez*];
 rank_tuple := [Rank(PP[i]): i in [1..#PP]];
 rank_summand := rkM * Rank(P);
 //calling the main algorithm; cf. file Krull_Schmidt_Algorithm :
 vec := Krull_Schmidt_Algorithm(emb,proj,rank_tuple,rank_summand,ppp,iii,prime);
 return vec;
end function;
```

6.2.5 Examples for the Krull-Schmidt Algorithm

We keep the notation from Sections 6.2.4.1 and 6.2.4.2. We have

$$M \otimes RG \cong M \vec{\otimes} RG \cong RG^{\oplus \operatorname{rk} M} \cong (e'_1 RG \oplus \ldots \oplus e'_\kappa RG)^{\oplus \operatorname{rk} M} \cong (e'_1 RG)^{\oplus \operatorname{rk} M} \oplus \ldots \oplus (e'_\kappa RG)^{\oplus \operatorname{rk} M}.$$

We fix the latter as the order of the summands W_α of $RG^{\oplus \operatorname{rk} M}$, i.e., provided $\operatorname{rk} M \geq 2$, we choose $W_1 = e'_1 RG,\ W_2 = e'_1 RG,\ \ldots,\ W_{k-1} = e'_\kappa RG,\ W_k = e'_\kappa RG$.

Example 145 We will test the algorithm with a decomposition we already know well.

Let $\mathbb{Z}_{(3)}\mathrm{S}_3 \cong \Lambda_{(3)} \cong P_1 \oplus P_2$ with an orthogonal decomposition $1_{\Lambda_{(3)}} = e_1 + e_2$ into primitive idempotents, like given in Definitions 21 and 27.

We can now use the Krull-Schmidt Algorithm to decompose the tensor product

$$M \otimes P \;=\; P_1 \otimes P_1\,,$$

well-known from Lemma 30. We write

$$\begin{array}{ccccccccc} P_1 \otimes \Lambda_{(3)} & \cong & P_1 \otimes \mathbb{Z}_{(3)}\mathrm{S}_3 & \cong & P_1 \vec{\otimes} \mathbb{Z}_{(3)}\mathrm{S}_3 & \cong & \mathbb{Z}_{(3)}\mathrm{S}_3^{\oplus \operatorname{rk} P_1} & & \\ & \cong & \Lambda_{(3)}^{\oplus \operatorname{rk} P_1} & \cong & (e_1\Lambda_{(3)} \oplus e_2\Lambda_{(3)})^{\oplus \operatorname{rk} P_1} & \cong & (P_1 \oplus P_2)^{\oplus 3} & \cong & P_1^{\oplus 3} \oplus P_2^{\oplus 3}\,. \end{array}$$

We prepare the input and start the algorithm.

```
load "Krull_Schmidt_Algorithm";
load "main_S3_loc3";            //loading file containing the map rho

G  := S3P; // S_3 as permutation group
// [e_1, e_2] as vectors in the standard basis of a direct product of matrix rings over Q :
ez := [RMatrixSpace(Q,1,6)![0,0,0,0,1,1],RMatrixSpace(Q,1,6)![1,1,0,0,0,0]];
f  := ez[1];                                                      // choose P := e_1 Λ_(3) ≅ P_1
rhoM := rhoP1;                                                    // M := P_1
prime := 3;

idempotents := Application(rho,ez,f,rhoM,G,prime);                // starting the algorithm
```

We obtain the following.

```
[*
    <
        [   1    0    0    0    0    0    0    0    0]
        [ 3/2 -1/2    3    0 -3/2  9/2    0  1/2 -3/2]
        [ 3/2 -3/2    1    0 -9/2  3/2    0  3/2 -1/2]
        [-1/2  1/2    1    0  3/2  3/2    0 -1/2 -1/2]
        [-1/2  1/2   -1    0  3/2 -3/2    0 -1/2  1/2]
        [  -1    1    0    0    3    0    0   -1    0]
        [   0    0    0    0    0    0    0    0    0]
        [   0    0    0    0    0    0    0    0    0]
        [   0    0    0    0    0    0    0    0    0],

        1,

        [ 1  1  2  0  3  3  0 -1 -1]
        [-1  0 -2  0  0 -3  0  0  1]
        [ 0 -1 -2  0 -3 -3  0  1  1]
    >,

    <
        [   0    0    0    0    0    0    0    0    0]
        [-3/2  3/2   -3    0  3/2 -9/2    0 -1/2  3/2]
        [-3/2  3/2    0    0  9/2 -3/2    0 -3/2  1/2]
        [   0    0    0    0    0    0    0    0    0]
        [   0    0    0    0    0    0    0    0    0]
        [ 3/2 -3/2    0    0 -9/2  3/2    0  3/2 -1/2]
        [-3/2  3/2    3   -3  9/2  9/2    1 -3/2 -3/2]
        [-3/2  3/2   -3    0  3/2 -9/2    0 -1/2  3/2]
        [ 3/2 -3/2    0    0 -9/2  3/2    0  3/2 -1/2],

        4,

        [   0    0   -3    0   -3   -3    0    1    1]
        [   0    0    3 -3/2  3/2  9/2  1/2 -1/2 -3/2]
        [-3/2  3/2  3/2 -3/2  9/2  3/2  1/2 -3/2 -1/2]
    >,

    <
        [   0    0    0    0    0    0    0    0    0]
        [   0    0    0    0    0    0    0    0    0]
        [   0    0    0    0    0    0    0    0    0]
        [ 1/2 -1/2   -1    1 -3/2 -3/2    0  1/2  1/2]
        [ 1/2 -1/2    1    0 -1/2  3/2    0  1/2 -1/2]
        [-1/2  1/2    0    0  3/2 -1/2    0 -1/2  1/2]
        [ 3/2 -3/2   -3    3 -9/2 -9/2    0  3/2  3/2]
        [ 3/2 -3/2    3    0 -3/2  9/2    0  3/2 -3/2]
        [-3/2  3/2    0    0  9/2 -3/2    0 -3/2  3/2],

        5,
```

```
        [ 0  0  2  0  2  2  1 -1 -1]
        [ 0  0 -2  1 -1 -3 -1  1  1]
        [ 1 -1 -1  1 -3 -1 -1  1  1]
    >
*]
```

This tells us that $P_1 \otimes P_1$ is isomorphic to the direct sum of the first, fourth and fifth summand of the direct sum $P_1^{\oplus 3} \oplus P_2^{\oplus 3} = P_1 \oplus P_1 \oplus P_1 \oplus P_2 \oplus P_2 \oplus P_2$, i.e. to $P_1 \oplus P_2 \oplus P_2$. The corresponding idempotent endomorphisms of $P_1 \otimes P_1$ and the embeddings of these summands into $P_1 \otimes P_1$ are returned as well.

Now we want to test this result independently.

First we verify that the found tuple contains an orthogonal decomposition into idempotents.

For this and further tests, we need

```
Gsorted := [sigma : sigma in G];
Gsorted_bij := map< G -> [1..#Gsorted] | [<Gsorted[i],i> : i in [1..#Gsorted]]>;
Ggen := [G.i : i in [1..NumberOfGenerators(G)]];
RG := GModule(G, [PermutationMatrix(Q,[Gsorted_bij(x * y) : x in Gsorted]) : y in Ggen]);
omega := MatrixRing(Q,#Gsorted)!&cat[ElementToSequence(rho(sigma)[i]): i in [1..#rho(sigma)],
                                                                          sigma in Gsorted];
rkM := NumberOfRows(rhoM(G!1));
M :=  GModule(G, [MatrixRing(Q,rkM)!rhoM(x) : x in Ggen]);
op_left := [PermutationMatrix(Q,[Gsorted_bij(y * x) : x in Gsorted]) : y in Gsorted];
op_left_f := &+[(f * omega^-1)[1,i] * op_left[i] : i in [1..#Gsorted]];
op_left_ez := [&+[(x * omega^-1)[1,i] * op_left[i] : i in [1..#Gsorted]] : x in ez];
P := Image(AHom(RG,RG)!op_left_f);
MoP := TensorProduct(M,P);
Homs := AHom(MoP,MoP);

Iop := [* Image_local(AHom(RG,RG)!op,RG) : op in op_left_ez*]; // Iop[1] ≅ P_1, Iop[2] ≅ P_2

P1 := GModule(G, [rhoP1(S3P!(1,2,3)),rhoP1(S3P!(1,2))]);   // P_1 as GModule
P2 := GModule(G, [rhoP2(S3P!(1,2,3)),rhoP2(S3P!(1,2))]);   // P_2 as GModule

// testing the idempotents (entry 1):
// RG-linearity:
[idempotents[i,1] * ActionGenerator(MoP,j) eq ActionGenerator(MoP,j)*idempotents[i,1]:
                          i in [1..#idempotents], j in [1..NumberOfGenerators(S3P)]];
// idempotent:
[idempotents[i,1]^2 eq idempotents[i,1]: i in [1..#idempotents]];
//orthogonal:
[idempotents[i,1]*idempotents[j,1] eq 0: i,j in [1..#idempotents]| i ne j];
// decomposition of 1;
&+[id[1] : id in idempotents] eq MatrixRing(Rationals(),NumberOfRows(idempotents[1,1]))!1;

// for which i do we have e_i(M ⊗ P) ≅ P_1 :
[IsIsomorphic(Image_local(Homs!idempotents[i,1],MoP),P1): i in [1..#idempotents]];
// for which i do we have e_i(M ⊗ P) ≅ P_2 :
[IsIsomorphic(Image_local(Homs!idempotents[i,1],MoP),P2): i in [1..#idempotents]];
//⊕ e_i(M ⊗ P) ≅ M ⊗ P
IsIsomorphic(DirectSum([Image_local(Homs!idempotents[i,1],MoP): i in [1..#idempotents]]),MoP);

// testing the embeddings (entry 3):
// yielding altogether an isomorphism (loc_inv cf. main_S3_loc3) :
loc_inv(VerticalJoin([idempotents[i,3]: i in [1..#idempotents]]),prime);
// RG-linearity:
o := [1,2,2]; // M ⊗ P ≅ P_1 ⊕ P_2 ⊕ P_2
[[ActionGenerator(Iop[o[i]],j)*idempotents[i,3] eq idempotents[i,3]*ActionGenerator(MoP,j):
                                                i in [1..#idempotents]]: j in [1..2]];
```

We identify the summands up to isomorphism:

```
>  [IsIsomorphic(Image_local(Homs!idempotents[i,1],MoP),P1): i in [1..#idempotents]];
```

```
[ true, false, false ]
>  [IsIsomorphic(Image_local(Homs!idempotents[i,1],MoP),P2): i in [1..#idempotents]];
[ false, true, true ]
> IsIsomorphic(DirectSum([Image_local(Homs!idempotents[i,1],MoP): i in [1..#idempotents]]),MoP);
true
```

So $P_1 \otimes P_1 \cong P_1 \oplus P_2^{\oplus 2}$, in accordance with Lemma 30.

Example 146 Let $\mathbb{Z}_{(3)}\mathrm{S}_3 \cong \Lambda_{(3)} \cong P_1 \oplus P_2$ with an orthogonal decomposition $1_{\Lambda_{(3)}} = e_1 + e_2$ into primitive idempotents, like given in Definitions 21 and 27..

We can now use the Krull-Schmidt Algorithm to decompose the tensor product

$$M \otimes P \;=\; S^{(2,1)} \otimes P_1 \;,$$

where $S^{(2,1)}$ is the Specht module to the partition $(2,1)$ over $\mathbb{Z}_{(3)}$.

We write $S^{(2,1)} \otimes \Lambda_{(3)} \cong \Lambda_{(3)}^{\oplus\,\mathrm{rk}\,S^{(2,1)}} \cong P_1^{\oplus 2} \oplus P_2^{\oplus 2}$.

We prepare the input and start the algorithm.

```
load "Krull_Schmidt_Algorithm";
load "main_S3_loc3";              // loading file containing the map rho
load "Krull_Schmidt_Example_146";   // loading file containing the map rhoS21

G := S3P;                                  // S_3 as permutation group
// [e_1,e_2] as vectors in the standard basis of a direct product of matrix rings over Q :
ez :=  [RMatrixSpace(Q,1,6)![0,0,0,0,1,1],RMatrixSpace(Q,1,6)![1,1,0,0,0,0]];
f := ez[1];                                // choose P  :=  e_1Λ_(3) ≅  P_1
prime := 3;

idempotents := Application(rho,ez,f,rhoS21,G,prime);
```

We obtain the following.

```
[*
    <
        [   1    0   -1    0   -1    1]
        [ 1/2  1/2   -1    0 -1/2  1/2]
        [ 1/2 -1/2    0    0 -1/2  1/2]
        [ 1/2 -1/2    0    0 -1/2  1/2]
        [ 1/2 -1/2    0   -1  1/2  1/2]
        [   1   -1    0   -1    0    1],

        1,

        [ 1  1 -2  0 -1  1]
        [ 0 -1  1 -1  1  0]
        [-1  0  1  0  1 -1]
    >,

    <
        [   0    0    1    0    1   -1]
        [-1/2  1/2    1    0  1/2 -1/2]
        [-1/2  1/2    1    0  1/2 -1/2]
        [-1/2  1/2    0    1  1/2 -1/2]
        [-1/2  1/2    0    1  1/2 -1/2]
        [  -1    1    0    1    0    0],

        3,

        [   0    0    1    0    1   -1]
        [   0    0   -1  1/2 -1/2  1/2]
```

```
        [ 1/2 -1/2 -1/2 -1/2 -1/2  1/2]
    >
*]
```

This means that $S^{(2,1)} \otimes P_1$ is isomorphic to the direct sum of the first and third summand of the direct sum $P_1^{\oplus 2} \oplus P_2^{\oplus 2}$, i.e. to $P_1 \oplus P_2$. The corresponding idempotent endomorphisms of $S^{(2,1)} \otimes P_1$ and the embeddings of these summands into $S^{(2,1)} \otimes P_1$ are returned as well.

In accordance, we verify that over $\mathbb{Q}$ we have

$$\begin{array}{rclcl} S^{(2,1)} \otimes P_1 & \cong & S^{(2,1)} \otimes (S^{(2,1)} \oplus S^{(1,1,1)}) & \cong & (S^{(2,1)} \otimes S^{(2,1)}) \oplus (S^{(2,1)} \otimes S^{(1,1,1)}) \\ & \cong & S^{(3)} \oplus S^{(2,1)} \oplus S^{(1,1,1)} \oplus S^{(2,1)} & \cong & P_1 \oplus P_2 \, . \end{array}$$

To verify the result independently, see file `Krull_Schmidt_Example_146`.

Example 147 Let $\mathbb{Z}_{(2)}\mathrm{S}_4 \cong \Lambda_{(2)} \cong P_1 \oplus P_2^{\oplus 2}$.

We have an orthogonal decomposition $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3$ into primitive idempotents, like given in Definition 73. Let $M = S^{(211)} \oplus S^{(22)}$. We can now decompose the tensor product

$$M \otimes P \;=\; (S^{(2,1,1)} \oplus S^{(2,2)}) \otimes P_1 \, .$$

We write $(S^{(2,1,1)} \oplus S^{(2,2)}) \otimes \mathbb{Z}_{(2)}\mathrm{S}_4 \cong P_1^{\oplus 5} \oplus P_2^{\oplus 10}$.

```
load "Krull_Schmidt_Algorithm";
load main_S4_loc2;                // file containing S4P
load "Krull_Schmidt_Example_147";   // file containing the map rhoS211_S22

G := S4P;                                     //symmetric group S_4 as permutation group
// [e_1,e_2,e_3] as vectors in the standard basis of a direct product of matrix rings over Q :
ez := [RMatrixSpace(Q,1,24)![1,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0],
       RMatrixSpace(Q,1,24)![0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0],
       RMatrixSpace(Q,1,24)![0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1]];
f := ez[1];                                   // choose P := e_1 Λ_(3) ≅ P_1
prime := 2;

idempotents := Application(rho,ez,f,rhoS211_S22,G,prime);  // starting the algorithm
```

We get the following result.

```
> [idempotents[i,2]: i in [1..#idempotents]];
[ 1, 6, 7, 9, 10 ]
```

This means that $(S^{(2,1,1)} \oplus S^{(2,2)}) \otimes P_1$ is isomorphic to the direct sum of the summands 1, 6, 7, 9 and 10 of the direct sum $P_1^{\oplus 5} \oplus P_2^{\oplus 10}$, i.e. to $P_1 \oplus P_2^{\oplus 4}$. The corresponding idempotent endomorphisms of $(S^{(2,1,1)} \oplus S^{(2,2)}) \otimes P_1$ and the embeddings of these summands into $(S^{(2,1,1)} \oplus S^{(2,2)}) \otimes P_1$ are returned as well, but not printed here.

We verify over $\mathbb{Q}$ that

$$\begin{array}{rl} & (S^{(2,1,1)} \oplus S^{(2,2)}) \otimes P_1 \\ \cong & (S^{(2,1,1)} \oplus S^{(2,2)}) \otimes (S^{(1,1,1,1)} \oplus S^{(4)} \oplus S^{(2,1,1)} \oplus S^{(3,1)}) \\ \cong & (S^{(2,1,1)} \otimes S^{(1,1,1,1)}) \oplus (S^{(2,1,1)} \otimes S^{(4)}) \oplus (S^{(2,1,1)} \otimes S^{(2,1,1)}) \oplus (S^{(2,1,1)} \otimes S^{(3,1)}) \\ & \oplus (S^{(2,2)} \otimes S^{(1,1,1,1)}) \oplus (S^{(2,2)} \otimes S^{(4)}) \oplus (S^{(2,2)} \otimes S^{(2,1,1)}) \oplus (S^{(2,2)} \otimes S^{(3,1)}) \\ \cong & (S^{(3,1)}) \oplus (S^{(2,1,1)}) \oplus (S^{(4)} \oplus S^{(2,1,1)} \oplus S^{(3,1)} \oplus S^{(2,2)}) \oplus (S^{(1,1,1,1)} \oplus S^{(2,1,1)} \\ & \oplus S^{(3,1)} \oplus S^{(2,2)}) \oplus (S^{(2,2)}) \oplus (S^{(2,2)}) \oplus (S^{(2,1,1)} \oplus S^{(3,1)}) \oplus (S^{(2,1,1)} \oplus S^{(3,1)}) \\ \cong & P_1 \oplus P_2^{\oplus 4} \quad . \end{array}$$

Example 148 Let $\mathbb{Z}_{(2)}\mathrm{S}_4 \cong \Lambda_{(2)} \cong P_1 \oplus P_2^{\oplus 2}$.

We have an orthogonal decomposition $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3$ into primitive idempotents, like given in Definition 73.

Let $M := P_2$. We choose the orthogonal decomposition into idempotents $1_{\Lambda_{(2)}} = \varepsilon + e_3$, with $\varepsilon := e_1 + e_2$, which is no decompostion into primitive idempotents, and so the endomorphism ring of $\varepsilon\Lambda_{(2)}$ is not local. The algorithm nevertheless decomposes the tensor product

$$M \otimes P \;=\; P_2 \otimes P_2$$

into summands of the form W_α, i.e. from $\{P_1 \oplus P_2\,,\, P_2\}$.

```
load "Krull_Schmidt_Algorithm";
load main_S4_loc2;              //file containing the map rhoP2

G := S4P;                                      //symmetric group S_4 as permutation group

rhoM := rhoP2;                                 // M := P_2
ez := [RMatrixSpace(Q,1,24)![1,1,1,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1,0,0,0],    // e_1 + e_2
       RMatrixSpace(Q,1,24)![0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]];   // e_3
f := ez[2];                                    // choose P := e_2Λ_(3) ≅ P_2
prime := 2;

idempotents := Application(rho,ez,f,rhoM,G,prime);
```

We obtain the following.

```
> [idempotents[i,2]: i in [1..#idempotents]];
[ 1, 4, 7, 9, 11 ]
```

This means we obtain a decomposition into the summands 1, 4, 7, 11 and 14 of the direct sum $(P_1 \oplus P_2)^{\oplus 8} \oplus P_2^{\oplus 8}$, i.e. $P_2 \otimes P_2 \cong (P_1 \oplus P_2)^{\oplus 3} \oplus P_2^{\oplus 2}$ in accordance to Lemma 84.

Example 149 Let $\mathbb{Z}_{(2)}\mathrm{S}_4 \cong \Lambda_{(2)} \cong P_1 \oplus P_2^{\oplus 2}$.

We have an orthogonal decomposition $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3$ into primitive idempotents, like given in Definition 73.

Let $M := P_2$. We choose the trivial orthogonal decomposition into idempotents $1_{\Lambda_{(2)}} = 1_{\Lambda_{(2)}}$, which is no decompostion into primitive idempotents, and so the endomorphism ring of $\Lambda_{(2)}$ is not local. The algorithm attempts to decompose the tensor product

$$M \otimes P \;=\; P_2 \otimes P_2$$

into modules of the form W_α, i.e. from $\{P_1 \oplus P_2^{\oplus 2}\}$, finds one such summand, fails to continue, breaks, returns a warning and an incomplete decomposition.
Note that $P_2 \otimes P_2 \cong P_1^{\oplus 3} \oplus P_2^{\oplus 5}$ cannot be decomposed using only copies of $P_1 \oplus P_2^{\oplus 2}$ as summands.

```
load "Krull_Schmidt_Algorithm";
load main_S4_loc2;

G := S4P;                                      //symmetric group S_4 as permutation group

rhoM := rhoP2;                                 // M := P_2
ez := [RMatrixSpace(Q,1,24)![1,1,1,0,0,0,1,0,0,0,1,1,0,0,0,1,0,0,0,1,1,0,0,1]]; //e_1 + e_2 + e_3
f  :=  RMatrixSpace(Q,1,24)![0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]; //choose P := e_2Λ_(3) ≅ P_2
prime := 2;
idempotents := Application(rho,ez,f,rhoM,G,prime);
```

We obtain the following.

```
>  idempotents := Application(rho,ez,f,rhoM,G,prime);
WARNING: idempotent decomposition incomplete

> [idempotents[i,2]: i in [1..#idempotents]];
[ 1, 0 ]
```

This shows that the algorithm was able to split the tensor product into two summands. The first one is isomorphic to $W_1 \cong P_1 \oplus P_2^{\oplus 2}$. Then the algorithm is not able to find another isomorphism, and therefore returns as second idempotent the unity matrix minus the first idempotent. The embedding of this remaining summand is not returned, instead we see a dummy zero (and a dummy zero in entry 3, which is not printed here).

Chapter 7

Lifting isomorphisms

Let R be a discrete valuation ring with maximal ideal (π). Let Λ be a stable R-order; cf. Definition 207, Remark 208.

Let P be a finitely generated projective Λ-modules. Let M be a Λ-lattice. Let $\rho_M : M \longrightarrow \bar{M}$ denote the residue class map.

Let $1_\Lambda = \sum\limits_{i\in[1,k]} e_i$ be an orthogonal decomposition into primitive idempotents. By Lemma 220, there exists an isomorphism

$$P \otimes M \stackrel{\sim}{\longrightarrow} \bigoplus_{i\in[1,k]} e_i\Lambda^{\oplus a_i} \;=:\; Q$$

for some $a_i \geq 0$; cf. Lemma 168. We aim to construct such an isomorphism.

Concerning the multiplicities a_i in the case of $\mathbb{Z}_{(p)}\mathrm{S}_n$, cf. Section E.1. But it is also possible to do a finite search, for Magma **[3]** gives an isomorphism $\bar{P} \otimes \bar{M} \stackrel{\sim}{\longrightarrow} \bigoplus\limits_{i\in[1,k]} \bar{e}_i\bar{\Lambda}^{\oplus a_i} = \bar{Q}$ via `IsIsomorphic` if one plugs in the correct a_i.

Remark 150 We have the situation

$$\begin{tikzcd}
P \otimes M \arrow[d, "\rho_P \otimes \rho_M"'] & Q \arrow[d, "\rho_Q"] \\
\bar{P} \otimes \bar{M} & \bar{Q} \arrow[l, "\bar{\vartheta}"', "\sim"]
\end{tikzcd}$$

Since Q is projective over Λ, we can lift the isomorphism $\bar{\vartheta}$ to an isomorphism $\vartheta : P \otimes M \longrightarrow Q$ with Lemma 214. We want to do this constructively.

Since $Q = \bigoplus_{i\in[1,k]} e_i\Lambda^{\oplus a_i}$, it suffices to find, for a given $\ell \in [1,k]$ and a given Λ-linear map $e_\ell\Lambda \stackrel{g}{\longrightarrow} \bar{Q}$, a completion

$$\begin{tikzcd}
P \otimes M \arrow[d, "\rho_P \otimes \rho_M"'] \arrow[dr, phantom, "\circlearrowleft"] & e_\ell\Lambda \arrow[l, dashed, "h"'] \arrow[d, "g"] \\
\bar{P} \otimes \bar{M} & \bar{Q} \arrow[l, "\bar{\vartheta}", "\sim"']
\end{tikzcd}$$

to a commutative quadrangle of Λ-linear maps.

Choose an element $\xi \in P \otimes M$ such that

$$\xi(\rho_P \otimes \rho_M) \;=\; e_\ell g\bar{\vartheta}\,.$$

Define

$$\begin{array}{rcl} e_\ell\Lambda & \xrightarrow{h} & P\otimes M \\ e_\ell\cdot y & \longmapsto & \xi\cdot e_\ell\cdot y \end{array}$$

Then the quadrangle commutes because

$$e_\ell h(\rho_P\otimes\rho_M) \;=\; (\xi\cdot e_\ell)(\rho_P\otimes\rho_M) \;=\; (\xi)(\rho_P\otimes\rho_M)\cdot e_\ell \;=\; e_\ell g\bar{\vartheta}\cdot e_\ell \;=\; (e_\ell\cdot e_\ell)g\bar{\vartheta} \;=\; e_\ell g\bar{\vartheta}\,.$$

Construction 151 On the example of the tensor product of projective modules $P_1\otimes P_3$ of $\mathbb{Z}_{(2)}\mathrm{S}_5$, the main step of the lifting process will be descibed here in detail.

Let $\Lambda=\mathbb{Z}_{(2)}\mathrm{S}_5$. We will calculate the isomorphism $P_1\otimes P_3 \xleftarrow[\sim]{\vartheta} P_2^{\oplus 8}\oplus P_3^{\oplus 8}$ by lifting the isomorphism $\bar{P}_1\otimes\bar{P}_3 \xleftarrow[\sim]{\bar{\vartheta}} \bar{P}_2^{\oplus 8}\oplus\bar{P}_3^{\oplus 8}$.

To generate $\bar{P}_1$, $\bar{P}_2$, and $\bar{P}_3$ as `GModule`, we use the representations of P_1, P_2 and P_3, given in the file `main_S5_loc2`, denoted `rhoP1`, `rhoP2` and `rhoP3`.

The file `Bases_for_lift` contains bases of P_1, P_2 and P_3 written as vectors in the standard basis of the matrix rings, where the first row represents the idempotent generating the respective projective module.

With Magma **[3]**, the lifting process can be realized as follows.

```
load "main_S5_loc2";
load Bases_for_lift;

// generating the symmetric group $S_5$ :
G := SymmetricGroup(5);
// fixing an order on the elements of $S_5$ :
S5Ptup := [sigma : sigma in S5P];
// generating $\mathbb{F}_2$ :
F := GF(2);

// the Wedderburn isomorphism, as used in Section 5.2 and as explicitly
// given in Remark 242, written as a matrix :
omega := Matrix(120,120,&cat[ElementToSequence(rho(sigma)[i]) : i in [1..7], sigma in S5Ptup]);
omega_inv := MatrixRing(Rationals(),120)!omega^-1;

// generating $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ as GModules over $\mathbb{F}_2$
PP1 := GModule(G, [MatrixRing(F,24)!rhoP1(S5P!(1,2,3,4,5)), MatrixRing(F,24)!rhoP1(S5P!(1,2))]);
PP2 := GModule(G, [MatrixRing(F,16)!rhoP2(S5P!(1,2,3,4,5)), MatrixRing(F,16)!rhoP2(S5P!(1,2))]);
PP3 := GModule(G, [MatrixRing(F,8)!rhoP3(S5P!(1,2,3,4,5)), MatrixRing(F,8)!rhoP3(S5P!(1,2))]);

// generating the tensor product $\bar{P}_1\otimes\bar{P}_3$
PP1oPP3 := TensorProduct(PP1,PP3);

// operating matrices on the tensor product $P_1\otimes P_3$ :
op_p1op3 :=  map<S5P -> RMatrixSpace(Rationals(),192,192) |
[<sigma, RMatrixSpace(Rationals(),192,192)!KroneckerProduct(rhoP1(sigma),rhoP3(sigma))>:sigma in S5P]>;

PP := [*PP1,PP2,PP3*];

// matrices containing the coefficients of basis elements of $P_1$, $P_2$, $P_3$, viewed
// as submodules of $\Lambda=\mathbb{Z}_{(2)}\mathrm{S}_5$, at group elements;
// found by taking the preimages of the bases of the projectives as constructed in Definition 116,
// under the Wedderburn isomorphism used there :
BasisP1GR := RMatrixSpace(Rationals(),24,120)!BasisP1*omega_inv;
BasisP2GR := RMatrixSpace(Rationals(),16,120)!BasisP2*omega_inv;
BasisP3GR := RMatrixSpace(Rationals(),8,120)!BasisP3*omega_inv;

BasisPPGR := [*BasisP1GR,BasisP2GR,BasisP3GR*];
```

```
// using Magma's MeatAxe to find a decomposition of $\bar{P}_1 \otimes \bar{P}_3$ (cf. also Section E.1):
DSD := DirectSumDecomposition(PP1oPP3);

// counting multiplicities of $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ in DSD :
cn := [ #[1 : Q in DSD | IsIsomorphic(Q,P)]: P in PP];

// sequence of the form [ $\underbrace{1,\dots,1}_{a_1 \text{ times } \bar{P}_1}$ , $\underbrace{2,\dots,2}_{a_2 \text{ times } \bar{P}_2}$ ,..., $\underbrace{k,\dots,k}_{a_k \text{ times } \bar{P}_k}$ ] :
//
//
proj_num := [j : i in [1..cn[j]], j in [1..#cn]];

// generating $\bar{Q} = \bar{P}_2^{\oplus 8} \oplus \bar{P}_3^{\oplus 8}$ :
QQ := DirectSum([PP[proj_num[i]] : i in [1..#proj_num]]);

// when taken as a basis for QQ a disjoint union of bases of its summands, the following
// sequence row_choice contains the position numbers of the respective first basis elements,
// that is, of the generating idempotents of the summands :
row_choice := [&+([0] cat [Rank(PP[proj_num[i]]) : i in [1..#proj_num]])[1..j] + 1 : j in [1..#proj_num]];

// calculating $\bar{\vartheta}$ :
dummy, theta_bar := IsIsomorphic(QQ,PP1oPP3);

// $\mathbb{Z}_{(2)}$-linear lift, realized by coercing the matrix entries of theta_bar into $\mathbb{Z}$ :
iso_local := MatrixRing(Integers(),Dimension(PP1oPP3))!theta_bar;

// For the lift $\xi \in P_1 \otimes P_3$ of the image of $\bar{e}_\ell$ under $\bar{\vartheta}$,
// as used above in Remark 150, we may use $\xi$=iso_local[row_choice[l]].
// We will map an element of the basis of $e_\ell\Lambda$, which is of the form $e_\ell \cdot y$, to $\xi \cdot e_\ell \cdot y$.
// We know that $\xi$ is multiplied with an element $\sigma \in \mathrm{S}_5$ via op_p1op3.
// We need to write $e_\ell y = \sum_{\sigma \in \mathrm{S}_5} u_\sigma \sigma$, where $u_\sigma \in \mathbb{Z}_{(2)}$, to obtain $\xi \cdot e_\ell \cdot y = \sum_{\sigma \in \mathrm{S}_5} \xi \cdot \sigma u_\sigma$.
// We have numbered the elements of $\mathrm{S}_5$. Say, $\sigma$ has number $i$.
// We have numbered the basis elements of $e_\ell\Lambda$. Say, $e_\ell y$ has number $j$.
// Then the coefficient $u_\sigma$ of $e_\ell y$ is given by BasisPPGR[proj_num[l]][j][i].
// Moreover, the product of $\xi$ with $\sigma$ is iso_local[row_choice[l]]*op_p1op3(S5Ptup[i]).
// So :
theta := MatrixRing(Rationals(),192)!
 &cat[
      [
      &+[iso_local[row_choice[l]]*op_p1op3(S5Ptup[i])*BasisPPGR[proj_num[l]][j][i] : i in [1..Order(S5P)]
        ] : j in [1..NumberOfRows(BasisPPGR[proj_num[l]])]
      ] : l in [1..#proj_num]
     ];
```

To check the map theta, load the file main_S5_loc2_P1oP3 containing the representation on the tensor product $P_1 \otimes P_3$, denoted op_p1op3, and the representation on the direct sum $P_2^{\oplus 8} \oplus P_3^{\oplus 8}$, denoted op_proj_sum_p1p3.

```
load "main_S5_loc2_P1oP3";

// local invertibility:
loc_inv(theta,2);

// $\mathbb{Z}_{(2)}\mathrm{S}_5$-linearity :
[op_proj_sum_p1p3(sigma)*theta eq theta*op_p1op3(sigma) : sigma in {S5P!(1,2), S5P!(1,2,3,4,5)}];
```

Chapter 8

Diagonalizing partially

8.1 A total decomposability

Let R be a discrete valuation ring with maximal ideal (π). Recall that $\otimes_R = \otimes$.

Let G be a finite group split by R, i.e. suppose that $\Lambda := RG$ is isomorphic to an R-suborder of a finite direct product Γ of matrix rings over R such that the R-linear factor module ${}^{\Gamma}\!/_{\Lambda}$ has finite length over R.

Example 152 We may choose $R = \mathbb{Z}_{(p)}$ for some prime p and $G = \mathrm{S}_n$ for some $n \geq 1$; cf. **[5,** Th. 4.12**]**.

Definition 153

(i) Consider the category lat-Λ of Λ-lattices and Λ-linear maps.

Consider the category $\Delta_1 := (\bullet \longrightarrow \bullet)$, having two objects and a single nonidentical morphism.

Consider the category $\mathcal{A} := (\text{lat-}\Lambda)^{\Delta_1}$ of diagrams of shape Δ_1 with values in lat-Λ.

Then $\mathcal{A}$ is an additive category. It is a full additive subcategory of the abelian category $(\text{mod-}\Lambda)^{\Delta_1}$ closed under summands. Therefore, idempotents split in $\mathcal{A}$.

The category $\mathcal{A}$ has as objects diagrams of the form $M \xrightarrow{f} N$, with M, N Λ-lattices and f a Λ-linear map. As morphisms in $\mathcal{A}$ we have those pairs of Λ-linear maps (m, n) that make the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "m"] & N \arrow[d, "n"] \\ M' \arrow[r, "g"] & N' \end{tikzcd}$$

commutative.

Suppose given two composable morphisms

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "m"] & N \arrow[d, "n"] \\ M' \arrow[r, "g"] \arrow[d, "m'"] & N' \arrow[d, "n'"] \\ M'' \arrow[r, "h"] & N'' \end{tikzcd}$$

in $\mathcal{A}$. The composite of (m, n) and (m', n') is given entrywise by $(m, n)(m', n') = (mm', nn')$.

Also addition of morphisms is given entrywise.

The direct sum of two objects is given as

$$(M \xrightarrow{f} N) \oplus (M' \xrightarrow{g} N') \;:=\; \big(M \oplus N \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)} M' \oplus N'\big)\;.$$

(ii) Recall that given U, V in obj(lat-Λ), we get $U \otimes V \in$ obj(lat-Λ) via $(u \otimes v)g = (ug \otimes vg)$ for $u \in U$, $v \in V$ and $g \in G$.

Given $(M \xrightarrow{f} N)$ in obj($\mathcal{A}$) and $X \in$ obj(lat-Λ), we define

$$(M \xrightarrow{f} N) \otimes X \;:=\; (M \otimes X \xrightarrow{f \otimes X} N \otimes X) \;\in\; \mathrm{obj}(\mathcal{A})\;.$$

Definition 154 Recall from Definition 153.(i) that $\mathcal{A} = (\text{lat-}\Lambda)^{\Delta_1}$.

We call an object $(M \xrightarrow{f} N) \in \mathrm{Ob}(\mathcal{A})$ **elementary** if it is isomorphic in $\mathcal{A}$ to an object of the form

$$\begin{array}{l} (0 \longrightarrow Q) \\ (Q \longrightarrow 0) \\ (Q \xrightarrow{\pi^\alpha} Q)\;, \end{array}$$

where $Q \in$ obj(lat-Λ) is isomorphic to an indecomposable direct summand of Λ, where $\alpha \geq 0$ and where $Q \xrightarrow{\pi^\alpha} Q$, $q \longmapsto \pi^\alpha q$.

We call an object $(M \xrightarrow{f} N) \in \mathrm{Ob}(\mathcal{A})$ **totally decomposable** if it is isomorphic in $\mathcal{A}$ to a finite direct sum of elementary objects.

Lemma 155 *The endomorphism ring of each elementary object of $\mathcal{A}$ is local.*

Proof. Suppose given a finitely generated indecomposable projective Λ-module Q. We know by Lemma 233 that $\mathrm{End}\, Q$ is local.

Suppose given $\alpha \geq 0$.

We remark that we have ring isomorphisms

$$\begin{array}{rcl} \mathrm{End}_{\mathcal{A}}(0 \longrightarrow Q) & \xrightarrow{\sim} & \mathrm{End}_\Lambda Q \\ \mathrm{End}_{\mathcal{A}}(Q \longrightarrow 0) & \xrightarrow{\sim} & \mathrm{End}_\Lambda Q \\ \mathrm{End}_{\mathcal{A}}(Q \xrightarrow{\pi^\alpha} Q) & \xrightarrow{\sim} & \mathrm{End}_\Lambda Q\,, \end{array}$$

given by projection to the second resp. to the first resp. to the first (or second) component. Since the right hand side is local, so is the left hand side in each case. □

Lemma 156 *Suppose given $(M \xrightarrow{f} N) \in \mathrm{Ob}(\mathcal{A})$.*

Suppose given a finitely generated projective Λ-module P.

Then, if $(M \xrightarrow{f} N) \otimes \Lambda$ is totally decomposable, so is $(M \xrightarrow{f} N) \otimes P$.

Proof. There exist $k \geq 0$ and $\tilde{P} \in$ obj(lat-Λ) such that $P \oplus \tilde{P} \cong \Lambda^{\oplus k}$.

Suppose that $(M \xrightarrow{f} N) \otimes \Lambda$ is totally decomposable.

Then

$$(M \xrightarrow{f} N) \otimes \Lambda^{\oplus k} \;\cong\; (M \xrightarrow{f} N) \otimes (P \oplus \tilde{P}) \;\cong\; (M \xrightarrow{f} N) \otimes P \,\oplus\, (M \xrightarrow{f} N) \otimes \tilde{P}$$

is totally decomposable as a direct sum of totally decomposable summands.

We apply Proposition 141, where, in the notation used there, $\mathcal{B}$ is $\mathcal{A}$, where Y is $(M \xrightarrow{f} N) \otimes \Lambda^{\oplus k}$, where Z is $(M \xrightarrow{f} N) \otimes P$ and where the X_i are elementary objects in $\mathcal{A}$, which, by Lemma 155, have local endomorphism rings. We obtain that $Z = (M \xrightarrow{f} N) \otimes P$ is isomorphic to a direct sum of certain of the X_i and thus totally decomposable. □

Theorem 157 *Recall that R is a discrete valuation ring with maximal ideal (π), that G is a finite group split by R and that $\Lambda = RG$.*

Suppose given $(M \xrightarrow{f} N)$ in $\mathrm{obj}(\mathcal{A})$, i.e. M and N are Λ-lattices and f is a Λ-linear map.

Let P be a finitely generated projective Λ-module.

Then the object $(M \xrightarrow{f} N) \otimes P = \big(M \otimes P \xrightarrow{f \otimes P} N \otimes P\big)$ of $\mathcal{A}$ is totally decomposable; cf. Definition 154.

Proof. By Lemma 156, we may assume that $P = \Lambda$.

Let $\vec{\otimes}$ be the tensor product and φ the isomorphism given in Lemma 167. Then we have the commutative diagram

$$\begin{tikzcd} M \otimes \Lambda \arrow[r, "f\otimes\Lambda"] \arrow[d, "\varphi", "\wr"'] & N \otimes \Lambda \arrow[d, "\varphi", "\wr"'] \\ M\vec{\otimes}\Lambda \arrow[r, "f\,\vec{\otimes}\,\Lambda"] & N\vec{\otimes}\Lambda\,. \end{tikzcd}$$

Hence we are reduced to show that $\big(M\vec{\otimes}\Lambda \xrightarrow{f\,\vec{\otimes}\,\Lambda} N\vec{\otimes}\Lambda\big) = (M \xrightarrow{f} N)\vec{\otimes}\Lambda$ is totally decomposable.

We can use the R-linear elementary divisor theorem to decompose

$$(M \xrightarrow{f} N) \;\cong\; (R \longrightarrow 0)^{\oplus k} \oplus (0 \longrightarrow R)^{\oplus \ell} \oplus \bigoplus_{\alpha \geq 0} (R \xrightarrow{\pi^\alpha} R)^{\oplus m_\alpha}\,,$$

where $k,\, \ell \geq 0$, where $m_\alpha \geq 0$ for all α and $m_\alpha = 0$ for almost all α.

So $(M \longrightarrow N)\vec{\otimes}\Lambda$ splits into a direct sum

$$(M \xrightarrow{f} N)\vec{\otimes}\Lambda \;\cong\; (\Lambda \longrightarrow 0)^{\oplus k} \oplus (0 \longrightarrow \Lambda)^{\oplus l} \oplus \bigoplus_{\alpha \geq 0} (\Lambda \xrightarrow{\pi^\alpha} \Lambda)^{\oplus m_\alpha}\,,$$

and is therefore totally decomposable. □

Corollary 158 *Suppose given $(M \xrightarrow{f} N)$ in $\mathrm{obj}(\mathcal{A})$ such that $\mathrm{rk}_R M = \mathrm{rk}_R N$ and such that f is injective.*

Let P be a finitely generated projective Λ-module.

Then we have a decomposition

$$(M \xrightarrow{f} N) \otimes P \;\cong\; \bigoplus_i (Q_i \xrightarrow{\pi^{\alpha_i}} Q_i)\,,$$

where Q_i is an indecomposable direct summand of Λ and $\alpha_i \geq 0$ for all i.

Proof. By Theorem 157, the object $(M \xrightarrow{f} N) \otimes P$ of $\mathcal{A}$ is totally decomposable. It remains to show that its direct sum decomposition into elementary objects contains neither elementary objects of the form $(0 \longrightarrow Q)$ nor of the form $(Q \longrightarrow 0)$.

Since f is injective, so is $f \otimes P$, and thus this decomposition does not contain objects of the form $(Q \longrightarrow 0)$.

Since $\mathrm{rk}_R M = \mathrm{rk}_R N$, this decomposition does not contain objects of the form $(0 \longrightarrow Q)$. □

Corollary 159 *Let P and $\tilde{P}$ be finitely generated projective Λ-modules. Recall that $\mathfrak{r}P$ denotes the Jacobson radical of P. Let δ denote the embedding of $\mathfrak{r}P$ into P. Then we can find projective Λ-modules*

Q and Q', not necessarily indecomposable, and Λ-linear isomorphisms α, β such that

$$\begin{tikzcd} P \otimes \tilde{P} \arrow[r, "\sim", "\beta"'] & Q \oplus Q' \\ \mathfrak{r}P \otimes \tilde{P} \arrow[u, hook, "\delta\otimes\tilde{P}"] \arrow[r, "\sim", "\alpha"'] \arrow[ur, phantom, "\circlearrowleft"] & Q \oplus Q' \arrow[u, "{\begin{bmatrix} \mathrm{id}_Q & 0 \\ 0 & \pi\mathrm{id}_{Q'} \end{bmatrix}}"'] \end{tikzcd}$$

commutes.

Proof. Note that $\mathrm{rk}_R\, \mathfrak{r}P = \mathrm{rk}_R\, P$ since $P/\mathfrak{r}P$ is a torsion module.

So by Corollary 158, we have a decomposition

$$(\mathfrak{r}P \hookrightarrow P) \otimes \tilde{P} \;\cong\; \bigoplus_i (Q_i \xrightarrow{\pi^{\alpha_i}} Q_i)\,,$$

where Q_i is an indecomposable direct summand of Λ and $\alpha_i \geq 0$ for all i.

Let $Q := \bigoplus_{\alpha_i=0} Q_i$ and $Q' := \bigoplus_{\alpha_i=1} Q_i$.

We have to show that $\alpha_i \overset{!}{\leq} 1$ for all i. It suffices to show that the R-linear elementary divisors of $\bigoplus_i (Q_i \xrightarrow{\pi^{\alpha_i}} Q_i)$ are contained in $\{\pi^0,\, \pi^1\}$. That is to say, we have to show that the elementary divisors of $\delta \otimes \tilde{P}$ are contained in $\{\pi^0,\, \pi^1\}$.

It suffices to show that the elementary divisors of $\delta : \mathfrak{r}P \hookrightarrow P$ are contained in $\{\pi^0,\, \pi^1\}$. In fact, $P/\mathfrak{r}P$ is a (semisimple) $\bar{\Lambda}$-module, thus annihilated by π^1. $\square$

Remark 160 Note that Corollary 159 does not assert that both matrices C and $\tilde{C}$ of the examples stated in the Chapters 3, 4 and 5 can be transformed into diagonal form **simultaneously**.

To this end, in the proof of Theorem 157, we would need a generalization of the R-linear elementary divisor theorem to simultaneously diagonalize f and f' in a diagram $M \xrightarrow{f} N \xleftarrow{f'} M'$ of R-lattices, using a triple of R-linear automorphisms on M, N resp. M', which we do not have at our disposal.

An example where the matrices C and $\tilde{C}$ cannot be transformed simultaneously is given in Remark 16.

Example 161 Suppose that $R = \mathbb{Z}_{(2)}$ and that $\Lambda = \mathbb{Z}_{(2)}\mathrm{S}_5 \cong P_1^{\oplus 1} \oplus P_2^{\oplus 4} \oplus P_3^{\oplus 4}$; cf. Definition 116 and Corollary 113. Suppose that $M = \mathfrak{r}P_1$ and $N = P_1$.

We then have a total decomposition of $(M \xrightarrow{f} N) \otimes \Lambda$ as follows.

$$(*) \qquad \begin{array}{rcl} (\mathfrak{r}P_1 \hookrightarrow P_1) \otimes \Lambda & \cong & (\mathfrak{r}P_1 \hookrightarrow P_1) \otimes \Lambda \\ & \cong & (\mathfrak{r}P_1 \hookrightarrow P_1) \vec{\otimes} \Lambda \\ & \cong & \Big((R \xrightarrow{2} R)^{\oplus 1} \oplus (R \xrightarrow{1} R)^{\oplus 23} \Big) \vec{\otimes} \Lambda \\ & \cong & (\Lambda \xrightarrow{2} \Lambda)^{\oplus 1} \oplus (\Lambda \xrightarrow{1} \Lambda)^{\oplus 23} \\ & \cong & (P_1 \xrightarrow{2} P_1)^{\oplus 1\cdot 1} \oplus (P_2 \xrightarrow{2} P_2)^{\oplus 4\cdot 1} \oplus (P_3 \xrightarrow{2} P_3)^{\oplus 4\cdot 1} \\ & & \oplus (P_1 \xrightarrow{1} P_1)^{\oplus 1\cdot 23} \oplus (P_2 \xrightarrow{1} P_2)^{\oplus 4\cdot 23} \oplus (P_3 \xrightarrow{1} P_3)^{\oplus 4\cdot 23} \end{array}$$

The matrix E of Lemma 119 has been brought into diagonal form C; the possibility to diagonalize E is guaranteed by Corollary 159. We have obtained

$$(1) \qquad (\mathfrak{r}P_1 \hookrightarrow P_1) \otimes P_1 \;\;\cong\;\; (P_1 \xrightarrow{2} P_1)^{\oplus 1} \oplus (P_1 \xrightarrow{1} P_1)^{\oplus 7} \oplus (P_2 \xrightarrow{1} P_2)^{\oplus 16} \oplus (P_3 \xrightarrow{1} P_3)^{\oplus 16}.$$

This is what can be expected, once the decomposition of $P_1 \otimes P_1$ is known, for $\mathrm{Coker}\,((\mathfrak{r}P_1 \longrightarrow P_1) \otimes P_1) \cong D_1 \otimes \bar{P}_1 \cong {P_1}\big/{2P_1} = \bar{P}_1$.

Moreover, in Lemma 121 we have obtained

(2) $\quad (\mathfrak{r}P_1 \hookrightarrow P_1)\otimes P_2 \;\cong\; (P_1 \xrightarrow{1} P_1)^{\oplus 4}\oplus(P_2 \xrightarrow{2} P_2)^{\oplus 1}\oplus(P_2 \xrightarrow{1} P_2)^{\oplus 11}\oplus(P_3 \xrightarrow{1} P_3)^{\oplus 12}$,

and in Lemma 123, we have obtained

(3) $\quad (\mathfrak{r}P_1 \hookrightarrow P_1)\otimes P_3 \;\cong\; (P_2 \xrightarrow{1} P_2)^{\oplus 8}\oplus(P_3 \xrightarrow{2} P_3)^{\oplus 1}\oplus(P_3 \xrightarrow{1} P_3)^{\oplus 7}$.

Note that in fact $(*) = (1)^{\oplus 1}\oplus(2)^{\oplus 4}\oplus(3)^{\oplus 4}$.

In particular, $(\mathfrak{r}P_1 \hookrightarrow P_1)\otimes P_1 \not\cong (P_1 \xrightarrow{2} P_1)^{\oplus 1}\oplus(P_1 \xrightarrow{1} P_1)^{\oplus 23}$, so that the argument using $\vec{\otimes}$ that worked for the tensor factor Λ does not work for the tensor factor P_1 .

Example 162 Suppose that $R = \mathbb{Z}_{(2)}$ and that $\Lambda = \mathbb{Z}_{(2)}\mathrm{S}_5 \cong P_1^{\oplus 1}\oplus P_2^{\oplus 4}\oplus P_3^{\oplus 4}$; cf. Definition 116 and Corollary 113. Suppose that $M = \mathfrak{r}P_3$ and $N = P_3$.

We then have a total decomposition of $(M \xrightarrow{f} N)\otimes\Lambda$ by

$$(*)\qquad \begin{array}{rcl}(\mathfrak{r}P_3 \hookrightarrow P_3)\otimes\Lambda & \cong & (\mathfrak{r}P_3 \hookrightarrow P_3)\,\vec{\otimes}\,\Lambda \\ & \cong & \Big((R \xrightarrow{2} R)^{\oplus 4}\oplus(R \xrightarrow{1} R)^{\oplus 4}\Big)\,\vec{\otimes}\,\Lambda \\ & \cong & (\Lambda \xrightarrow{2} \Lambda)^{\oplus 4}\oplus(\Lambda \xrightarrow{1} \Lambda)^{\oplus 4} \\ & \cong & (P_1 \xrightarrow{2} P_1)^{\oplus 1\cdot 4}\oplus(P_2 \xrightarrow{2} P_2)^{\oplus 4\cdot 4}\oplus(P_3 \xrightarrow{2} P_3)^{\oplus 4\cdot 4} \\ & & \oplus(P_1 \xrightarrow{1} P_1)^{\oplus 1\cdot 4}\oplus(P_2 \xrightarrow{1} P_2)^{\oplus 4\cdot 4}\oplus(P_3 \xrightarrow{1} P_3)^{\oplus 4\cdot 4}\,.\end{array}$$

Transforming the Matrix $\tilde{C}$ of Lemma 123 into diagonal form, which is possible by Corollary 159, by using column transformations from the right, we can see that

(1) $\quad (\mathfrak{r}P_3 \longrightarrow P_3)\otimes P_1 \;\cong\; (P_2 \xrightarrow{2} P_2)^{\oplus 4}\oplus(P_2 \xrightarrow{1} P_2)^{\oplus 4}\oplus(P_3 \xrightarrow{2} P_3)^{\oplus 4}\oplus(P_3 \xrightarrow{1} P_3)^{\oplus 4}$.

This is what can be expected, once the decomposition of $P_1\otimes P_3$ is known, for $\operatorname{Coker}((\mathfrak{r}P_3 \longrightarrow P_3)\otimes P_1) \cong D_3\otimes\bar{P}_1 \cong \bar{P}_2^{\oplus 4}\oplus\bar{P}_3^{\oplus 4}$, as is calculated via Magma **[3]**.

Moreover, in Lemma 127 we have obtained

(2) $\quad (\mathfrak{r}P_3 \longrightarrow P_3)\otimes P_2 \;\cong\; (P_2 \xrightarrow{2} P_2)^{\oplus 3}\oplus(P_2 \xrightarrow{1} P_2)^{\oplus 3}\oplus(P_3 \xrightarrow{2} P_3)^{\oplus 2}\oplus(P_3 \xrightarrow{1} P_3)^{\oplus 2}$

and in Lemma 129, we have obtained

(3) $\quad (\mathfrak{r}P_3 \longrightarrow P_3)\otimes P_3 \;\cong\; (P_1 \xrightarrow{2} P_1)^{\oplus 1}\oplus(P_1 \xrightarrow{1} P_1)^{\oplus 1}\oplus(P_3 \xrightarrow{2} P_3)^{\oplus 1}\oplus(P_3 \xrightarrow{1} P_3)^{\oplus 1}$.

Note that in fact $(*) = (1)^{\oplus 1}\oplus(2)^{\oplus 4}\oplus(3)^{\oplus 4}$.

8.2 On defect-0 blocks

Let p be a prime. Let $n \geq 1$.

Lemma 163 *Let D be a simple $\mathbb{Z}_{(p)}\mathrm{S}_n$-module that belongs to a defect-0 block. Let P be the corresponding indecomposable projective $\mathbb{Z}_{(p)}\mathrm{S}_n$-module; cf. Proposition 225. So $D = \bar{P}$.*

Let $\tilde{D}$ be a simple $\mathbb{Z}_{(p)}\mathrm{S}_n$-module. Let $\tilde{P}$ be the corresponding indecomposable projective module $\tilde{P}$.

Then we have a direct sum decomposition $P\otimes\tilde{P} \xrightarrow{\sim} Q'\oplus Q''$ fitting into a commutative diagram

$$\begin{tikzcd} P\otimes\mathfrak{r}\tilde{P} \arrow[r, "\sim"] \arrow[d, hook] & Q'\oplus Q'' \arrow[d, "\begin{pmatrix}\mathrm{id}_{Q'} & 0\\ 0 & p\,\mathrm{id}_{Q''}\end{pmatrix}"] \\ P\otimes\tilde{P} \arrow[r, "\sim"] & Q'\oplus Q'' \\ \mathfrak{r}P\otimes\tilde{P} \arrow[r, "\sim"] \arrow[u, hook] & Q'\oplus Q''\,. \arrow[u, "\begin{pmatrix}p\,\mathrm{id}_{Q'} & 0\\ 0 & p\,\mathrm{id}_{Q''}\end{pmatrix}"'] \end{tikzcd}$$

In particular, the diagram

$$(Q' \oplus Q'') \oplus (Q' \oplus Q'') \xrightarrow{\begin{pmatrix} p\,\mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \\ \mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \end{pmatrix}} (Q' \oplus Q'')$$

is isomorphic to the canonical presentation of $D \otimes \tilde{D}$. *So* $D \otimes \tilde{D} \cong \bar{Q}''$.

Proof. Since P belongs to a defect-0 block, we have that

$$\mathfrak{r}P \otimes \tilde{P} \;=\; pP \otimes \tilde{P} \;=\; p(P \otimes \tilde{P})\ .$$

We have a commutative triangle

$$\begin{tikzcd} P \otimes \tilde{P} & \\ p(P \otimes \tilde{P}) \arrow[u, hook] \arrow[r, "\sim", "\psi"'] & P \otimes \tilde{P}\ , \arrow[ul, "p\,\mathrm{id}"'] \end{tikzcd}$$

where $(p(x \otimes \tilde{x}))\psi = x \otimes \tilde{x}$ for $x \in P$ and $\tilde{x} \in \tilde{P}$.

By Corollary 159 from Section 8.1, we can find projective modules Q' and Q'', together with isomorphisms $\mu = (\mu' \;\ \mu'') : Q \xrightarrow{\sim} Q' \oplus Q''$ and $\xi = (\xi' \;\ \xi'') : Q \xrightarrow{\sim} Q' \oplus Q''$ such that

$$\begin{tikzcd} P \otimes \mathfrak{r}\tilde{P} \arrow[r, "(\mu'\ \mu'')", "\sim"'] \arrow[d, hook] & Q' \oplus Q'' \arrow[d, "\begin{pmatrix} \mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \end{pmatrix}"] \\ P \otimes \tilde{P} \arrow[r, "(\xi'\ \xi'')", "\sim"'] & Q' \oplus Q'' \end{tikzcd}$$

Altogether, we have the following commutative diagram.

$$\begin{tikzcd} P \otimes \mathfrak{r}\tilde{P} \arrow[rr, "(\mu'\ \mu'')", "\sim"'] \arrow[d, hook] & & Q' \oplus Q'' \arrow[d, "\begin{pmatrix} \mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \end{pmatrix}"] \\ P \otimes \tilde{P} \arrow[rr, "(\xi'\ \xi'')", "\sim"'] & & Q' \oplus Q'' \\ p(P \otimes \tilde{P}) \arrow[u, hook] \arrow[r, "\sim", "\psi"'] & P \otimes \tilde{P} \arrow[ul, "p\,\mathrm{id}"'] \arrow[r, "(\xi'\ \xi'')", "\sim"'] & Q' \oplus Q'' \arrow[u, "\begin{pmatrix} p\,\mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \end{pmatrix}"'] \end{tikzcd}$$

Finally, note that we have a right exact sequence

$$(Q' \oplus Q'') \oplus (Q' \oplus Q'') \xrightarrow{\begin{pmatrix} p\,\mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \\ \mathrm{id}_{Q'} & 0 \\ 0 & p\,\mathrm{id}_{Q''} \end{pmatrix}} (Q' \oplus Q'') \xrightarrow{\begin{pmatrix} 0 \\ \rho_{Q''} \end{pmatrix}} \bar{Q}''\ .$$

□

Lemma 164 *Let* D *and* $\tilde{D}$ *be simple* $\mathbb{Z}_{(p)}\mathrm{S}_n$*-modules that belong to defect-0 blocks. Let* P *and* $\tilde{P}$ *be the respective corresponding indecomposable projective* $\mathbb{Z}_{(p)}\mathrm{S}_n$*-modules. So* $D = \bar{P}$ *and* $\tilde{D} = \bar{\tilde{P}}$.

Then we have $P \otimes \tilde{P} \stackrel{\sim}{\longrightarrow} Q$ fitting into a commutative diagram

$$\begin{tikzcd}
P \otimes \mathfrak{r}\tilde{P} \arrow[r, "\sim"] \arrow[d, hook] & Q \arrow[d, "p\,\mathrm{id}_Q"] \\
P \otimes \tilde{P} \arrow[r, "\sim"] & Q \\
\mathfrak{r}P \otimes \tilde{P} \arrow[r, "\sim"] \arrow[u, hook] & Q . \arrow[u, "p\,\mathrm{id}_Q"']
\end{tikzcd}$$

In particular, the diagram

$$Q \oplus Q \xrightarrow{\left(\begin{smallmatrix} p\,\mathrm{id} \\ p\,\mathrm{id} \end{smallmatrix}\right)} Q$$

is isomorphic to the canonical presentation of $D \otimes \tilde{D}$.

Proof. Since P and $\tilde{P}$ belong to defect-0 blocks, we have

$$\mathfrak{r}P \otimes \tilde{P} \;=\; pP \otimes \tilde{P} \;=\; p(P \otimes \tilde{P})$$

and

$$P \otimes \mathfrak{r}\tilde{P} \;=\; P \otimes p\tilde{P} \;=\; p(P \otimes \tilde{P}) \; .$$

We have a commutative triangle

$$\begin{tikzcd}
P \otimes \tilde{P} & \\
p(P \otimes \tilde{P}) \arrow[u, hook] \arrow[r, "\sim", "\psi"'] & P \otimes \tilde{P} \, , \arrow[ul, "p\,\mathrm{id}"']
\end{tikzcd}$$

where $(p(x \otimes \tilde{x}))\psi = x \otimes \tilde{x}$ for $x \in P$ and $\tilde{x} \in \tilde{P}$.

Letting $Q := P \otimes \tilde{P}$, we have a commutative diagram

$$\begin{tikzcd}
p(P \otimes \tilde{P}) \arrow[r, "\psi", "\sim"'] \arrow[d, hook] & Q \arrow[d, "p\,\mathrm{id}_Q"] \\
P \otimes \tilde{P} \arrow[r, "\mathrm{id}", "\sim"'] & Q \\
p(P \otimes \tilde{P}) \arrow[r, "\sim", "\psi"'] \arrow[u, hook] & Q . \arrow[u, "p\,\mathrm{id}_Q"']
\end{tikzcd}$$

$\square$

Appendix A

Some facts on completion

For the following facts, let R be a discrete valuation ring with maximal ideal (π). Let $K := \mathrm{frac}(R)$ be the field of fractions of R,

$$\hat{R} := \varprojlim_n R\big/_{\pi^n} := \{(a_i + \pi^i R)_{i \geq 1} : a_{i+1} \equiv_{\pi^i} a_i\}$$

the completion of R. Let Λ be an R-order, X a Λ-lattice, i.e. a free Λ-module finitely generated over R. The following well-known facts are stated without proof.

Lemma 165

(i) *R is a subset of its completion $R \subseteq \hat{R}$; more precisely, $R \xhookrightarrow{i_R} \hat{R} : r \mapsto (r)_{i \geq 1}$ is injective.*

(ii) *$\bar{i}_R : R\big/_{\pi R} \longrightarrow \hat{R}\big/_{\pi \hat{R}} : r + \pi R \longmapsto i_R(r) + \pi \hat{R}$ is an isomorphism.*

(iii) *$\hat{R}$ is a complete discrete valuation ring with maximal ideal π, and $i_{\hat{R}}$ is an isomorphism.*

(iv) *The completion $\hat{\Lambda} := \hat{R} \otimes_R \Lambda$ is an $\hat{R}$-order, and we have $\Lambda \hookrightarrow \hat{\Lambda} : \lambda \mapsto 1 \otimes \lambda$.*

(v) *The completion $\hat{X} := \hat{R} \otimes_R X$ is a $\hat{\Lambda}$-lattice.*

(vi) *Recall that $\bar{X} = X\big/_{\pi X}$. We have $\bar{X} \cong \bar{\hat{X}}$ with (i) and (ii).*

(vii) *If P is projective over Λ, then $\hat{P}$ is projective over $\hat{\Lambda}$.*

(viii) *We have $K \cap \hat{R} = R$.*

(ix) *Given $x \in \hat{K}$ and $N \geq 0$, there exists $y \in K$ with $x - y \in \pi^N \hat{R}$.*

Lemma 166 *Let Δ be an $\hat{R}$-order. Then Δ contains a nontrivial idempotent if and only if $\bar{\Delta}$ contains a nontrivial idempotent.*

Note that one needs completeness of $\hat{R}$ in order to show "$\Leftarrow$", and that this is the only passage where we really need the completion.

Appendix B

General tools

B.1 Two tensor products

Let R be a commutative ring. Let G be a finite group. Tensor products over R are denoted by $\otimes := \otimes_R$.

Suppose given RG-modules M and N.

Let the RG-module $M \otimes N$ be defined as the R-module $M \otimes N$, equipped with the diagonal action of G, i.e.

$$(m \otimes n)g \;=\; (mg \otimes ng) \;,$$

where $m \in M$, $n \in N$ and $g \in G$. Note that $M \otimes N \xrightarrow{\sim} N \otimes M$ via $m \otimes n \longmapsto n \otimes m$, where $m \in M$ and $n \in N$.

In particular, on the RG-module $M \otimes RG$ we have

$$(m \otimes x)g \;=\; (mg \otimes xg) \;,$$

where $m \in M$, $x \in RG$ and $g \in G$.

Let the RG-module $M \vec{\otimes} RG$ be defined as the R-module $M \otimes RG$, equipped with the action of G on the right factor, i.e.

$$(m \otimes x)g \;=\; (m \otimes xg) \;,$$

where $m \in M$, $x \in RG$ and $g \in G$.

Lemma 167 *The RG-modules $M \otimes RG$ and $M \vec{\otimes} RG$ are isomorphic via*

$$\begin{array}{rlcl} \varphi : & M \otimes RG & \xrightarrow{\sim} & M \vec{\otimes} RG \\ & m \otimes g & \longmapsto & m \cdot g^{-1} \otimes g \\ & m \cdot g \otimes g & \longleftarrow & m \otimes g \;. \end{array}$$

Lemma 168 *Let M be an RG-lattice. Let P be a finitely generated projective RG-module.*

Then $P \otimes M$ is a projective RG-lattice.

Proof. There is an RG-module Q such that $P \oplus Q \cong RG^{\oplus k}$ for some $k \geq 0$. Therefore, without loss of generality, we can assume that $P \cong RG^{\oplus k}$. We may moreover assume that $k = 1$. So it remains to show that $RG \otimes M$ is projective.

We have $RG \otimes M \cong M \otimes RG \cong M \vec{\otimes} RG$; cf. Lemma 167. As R-modules, we have $M \cong R^{\oplus l}$ for some $l \geq 0$. Since the action of G on M does not play a role in $M \vec{\otimes} RG$, we have $M \vec{\otimes} RG \cong R^{\oplus l} \vec{\otimes} RG \cong (R \vec{\otimes} RG)^{\oplus l} \cong RG^{\oplus l}$ as RG-modules. Now RG is projective, and we are done. $\square$

B.2 Reducing matrix entries and resulting transformation

Let R be a discrete valuation ring with maximal ideal (π). Let $K := \mathrm{frac}(R)$ be its field of fractions.

Let Λ be an R-order.

Remark 169 Suppose given $m \geq 1$ and $A \in R^{m\times m}$. Then $I_m + \pi \cdot A$ is invertible in $R^{m\times m}$.

Proof. With the Leibniz formula, we see that $\det(I_m + \pi \cdot A) \equiv_\pi \det I_m = 1$. So $\det(I_m + \pi \cdot A)$ is a unit in R. Hence $I_m + \pi \cdot A$ is invertible in $R^{m\times m}$. $\square$

Definition 170 Suppose given Λ-lattices M and N. Suppose given $s \geq 0$.

(i) Let α, $\beta \in \mathrm{Hom}_\Lambda(M, N)$. Then we write

$$\alpha \equiv_{\pi^s} \beta$$

if there exists $\gamma \in \mathrm{Hom}_R(M, N)$ with $\alpha = \beta + \pi^s\gamma$.

Note that in this case, we have $\gamma \in \mathrm{Hom}_\Lambda(M, N)$, for $(x\lambda)(\pi^s\gamma) = (x)(\pi^s\gamma)\lambda$ implies, N being R-free, that $(x\lambda)\gamma = (x)\gamma\lambda$ for $x \in M$ and $\lambda \in \Lambda$.

(ii) Fix R-linear bases of M and N. Write $m := \mathrm{rk}_R M$ and $n := \mathrm{rk}_R N$. Let A, $B \in R^{m\times n}$ be two describing matrices of Λ-linear maps $M \longrightarrow N$. Then we write

$$A \equiv_{\pi^s} B$$

if there exists $C \in R^{m\times n}$ with $A = B + \pi^s C$. Note that in this case, C describes a Λ-linear map $M \longrightarrow N$, too.

Lemma 171 *Suppose given Λ-lattices M and N with $n := \mathrm{rk}_R M = \mathrm{rk}_R N$.*

Fix R-linear bases of M and N.

Suppose A, $B \in R^{n\times n} \subseteq K^{n\times n}$ to be the describing matrices of Λ-linear maps $M \longrightarrow N$.

Suppose A to be injective. Let ℓ be the maximal valuation of an elementary divisor of A.

Suppose that

$$A \equiv_{\pi^{\ell+1}} B.$$

We can invert A as a matrix of $K^{n\times n}$. Then $B \cdot A^{-1} : M \longrightarrow M$ and $A^{-1} \cdot B : N \longrightarrow N$ are Λ-linear automorphisms. Moreover, B is injective.

Proof. Without loss of generality, we have $A = \begin{pmatrix} \pi^{\vartheta_1} & & & \\ & \pi^{\vartheta_2} & & \\ & & \ddots & \\ & & & \pi^{\vartheta_n} \end{pmatrix}$, where $\vartheta_i \in [0, \ell]$ for $i \in [1, n]$.

In $K^{n\times n}$, we may invert A. We *claim* that $\pi^\ell A^{-1} \overset{!}{\in} \mathrm{Hom}_\Lambda(N, M)$.

It is an R-linear map from N to M, since $\pi^\ell A^{-1} \in R^{n\times n}$. So it remains to show that for $y \in N$ and $\lambda \in N$ we have that

$$(y\lambda)\pi^\ell A^{-1} \overset{!}{=} y\pi^\ell A^{-1}\lambda\,.$$

Since A is injective, it suffices to show that

$$(y\lambda)\pi^\ell A^{-1}A \overset{!}{=} y\pi^\ell A^{-1}\lambda A\,.$$

But $(y\lambda)\pi^\ell A^{-1}A = y\lambda\pi^\ell = y\pi^\ell A^{-1}A\lambda = y\pi^\ell A^{-1}\lambda A$. This proves the *claim.*

We have $B = A + \pi^{\ell+1} \cdot C_1$ for some $C_1 \in \mathrm{Hom}_\Lambda(M, N)$; cf. Definition 170.(ii). So $B = A(I_n + \pi \cdot C_2)$ for $C_2 := A^{-1}\pi^\ell C_1 \in \mathrm{Hom}_\Lambda(N, N)$; cf. the claim above.

Note that $I_n + \pi \cdot C_2$ is an automorphism of N by Remark 169. In particular, injectivity of A implies injectivity of $B = A(I_n + \pi \cdot C_2)$.

We get

$$A^{-1} \cdot B = A^{-1}(A(I_n + \pi \cdot C_2)) = (I_n + \pi \cdot C_2)\ ,$$

which is an automorphism.

For $B \cdot A^{-1}$, the proof works analogously. $\square$

Lemma 172 *Suppose given Λ-lattices M and N with $\operatorname{rk}_R M = \operatorname{rk}_R N$. Suppose $\alpha,\ \beta\ :\ M \longrightarrow N$ to be Λ-linear maps.*

Suppose α to be injective. Let $\ell \geq 0$ be minimal such that $\pi^{\ell} \operatorname{Coker}(\alpha) = 0$.

Suppose that

$$\alpha \equiv_{\pi^{\ell+1}} \beta.$$

Then $\mu := ((K\beta) \cdot (K\alpha)^{-1})\big|_M^M : M \longrightarrow M$ and $\nu := ((K\alpha)^{-1} \cdot (K\beta))\big|_N^N : N \longrightarrow N$ exist and are Λ-linear automorphisms.

Proof. This follows by Lemma 171. $\square$

Construction 173 Let P and $\tilde{P}$ be projective Λ-modules. Consider a commutative quadrangle of the form

$$\begin{tikzcd}
\mathfrak{r}P \otimes \tilde{P} \arrow[r, "\beta", "\sim"'] \arrow[d, hook, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "\gamma"] \\
P \otimes \tilde{P} \arrow[r, "\alpha", "\sim"'] & Q\ ,
\end{tikzcd}$$

where Q is a projective module; like observed in the examples of the Chapters 3, 4 and 5.

Then $\gamma \in \operatorname{End}(Q)$ is an Λ-linear injective map, but in general not an isomorphism.

Let $\ell \geq 0$ be minimal such that $\pi^{\ell} \operatorname{Coker}(\gamma) = 0$. Note that $\ell \leq 1$; cf. proof of Corollary 159.

Let $\tilde{\gamma} \in \operatorname{End}(Q)$ with $\gamma \equiv_{\pi^{\ell+1}} \tilde{\gamma}$. With Lemma 172 we have the isomorphism

$$\kappa\ :=\ ((K\tilde{\gamma})(K\gamma)^{-1})\big|_Q^Q\ :\ Q \xrightarrow{\sim} Q$$

with $\kappa\gamma = \tilde{\gamma}$.

Thus, we obtain a commutative diagram

$$(*) \qquad \begin{tikzcd}
\mathfrak{r}P \otimes \tilde{P} \arrow[r, "\beta", "\sim"'] \arrow[d, hook, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "\gamma"] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[l, "\kappa"', "\sim"] \arrow[d, "\tilde{\gamma}"] \\
P \otimes \tilde{P} \arrow[r, "\alpha", "\sim"'] & Q & Q\ , \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

and thus the commutative quadrangle

$$(**) \qquad \begin{tikzcd}
\mathfrak{r}P \otimes \tilde{P} \arrow[r, "\beta\kappa^{-1}", "\sim"'] \arrow[d, hook, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "\tilde{\gamma}"] \\
P \otimes \tilde{P} \arrow[r, "\alpha", "\sim"'] & Q\ .
\end{tikzcd}$$

In practice, we may use this construction to pick a Λ-linear map $\tilde{\gamma}$ that is of a simpler shape than γ. Then, we may also state the existence of the commutative quadrangle $(**)$ without inverting κ explicitly, but rather by constructing a diagram of the form $(*)$.

B.3 The radical and the sign

Let R be a discrete valuation ring. Let $n \geq 1$.

Notation 174 Let A be the alternating module of $R\mathrm{S}_n$, belonging to the partition (1^n). We denote the functor

$$A \otimes_R - \;=:\; (-)^- \;,$$

applicable to $R\mathrm{S}_n$-modules and $R\mathrm{S}_n$-linear maps.

Remark 175 *Let P be a projective module over $R\mathrm{S}_n$. Then we have*

$$(\mathfrak{r}P)^- \;=\; \mathfrak{r}(P^-) \;.$$

Proof. We have

$$(P/\mathfrak{r}P)^- \overset{\text{Def.}}{=} A \otimes (P/\mathfrak{r}P) \;\cong\; A \otimes P/A \otimes \mathfrak{r}P \overset{\text{Def.}}{=} P^-/(\mathfrak{r}P)^-$$

The module $(P/\mathfrak{r}P)^-$ is simple, so $(\mathfrak{r}P)^-$ must be a maximal submodule of P^-. For P^- is indecomposable projective, there is only one maximal submodule, namely the radical $\mathfrak{r}(P^-)$. □

Remark 176 Let P, $\tilde{P}$ be projective $R\mathrm{S}_n$-modules.

Denote by $\mathfrak{r}P \xrightarrow{u} P$ and by $\mathfrak{r}\tilde{P} \xrightarrow{\tilde{u}} \tilde{P}$ the respective embeddings.

Decompose $P \otimes \tilde{P} \cong \bigoplus_i Q_i$ into indecomposable projective $R\mathrm{S}_n$-modules Q_i.

Suppose given a commutative diagram

$$\begin{tikzcd}
P \otimes \tilde{P} \arrow[r, "\sim"] & \bigoplus_i Q_i \\
\mathfrak{r}P \otimes \tilde{P} \,\oplus\, P \otimes \mathfrak{r}\tilde{P} \arrow[u, "\left(\begin{smallmatrix} u\otimes 1 \\ 1\otimes\tilde{u} \end{smallmatrix}\right)"] \arrow[r, "\sim"] \arrow[ur, phantom, "\circlearrowleft"] & \bigoplus_i Q_i \oplus \bigoplus_i Q_i \;. \arrow[u, "\left(\begin{smallmatrix} C \\ \tilde{C} \end{smallmatrix}\right)"']
\end{tikzcd}$$

Then there exists a commutative diagram as follows.

$$\begin{tikzcd}
P^- \otimes \tilde{P} \arrow[r, "\sim"] & (P \otimes \tilde{P})^- \arrow[r, "\sim"] & \bigoplus_i Q_i^- \\
\mathfrak{r}P^- \otimes \tilde{P} \,\oplus\, P^- \otimes \mathfrak{r}\tilde{P} \arrow[u, "\left(\begin{smallmatrix} u^-\otimes 1 \\ 1\otimes\tilde{u} \end{smallmatrix}\right)"] \arrow[r, "\sim"] \arrow[ur, phantom, "\circlearrowleft"] & (\mathfrak{r}P \otimes \tilde{P})^- \oplus (P \otimes \mathfrak{r}\tilde{P})^- \arrow[u, "\left(\begin{smallmatrix} (u\otimes 1)^- \\ (1\otimes\tilde{u})^- \end{smallmatrix}\right)"] \arrow[r, "\sim"] \arrow[ur, phantom, "\circlearrowleft"] & \bigoplus_i Q_i^- \oplus \bigoplus_i Q_i^- \arrow[u, "\left(\begin{smallmatrix} C \\ \tilde{C} \end{smallmatrix}\right)"']
\end{tikzcd}$$

Writing $C = (\gamma_{i,j})_{ij}$, where $\gamma_{i,j} : Q_i \longrightarrow Q_j$, and $\tilde{C} = (\tilde{\gamma}_{i,j})_{ij}$, where $\tilde{\gamma}_{i,j} : Q_i \longrightarrow Q_j$, we have $C^- = (\gamma_{i,j}^-)_{ij}$ and $\tilde{C}^- = (\tilde{\gamma}_{i,j}^-)_{ij}$.

B.4 A lemma on exact sequences

Lemma 177 *Suppose given a commutative diagram*

$$\begin{tikzcd}
& B'' \arrow[r, "p''"] & C'' \\
A \arrow[ur, "i\beta''"] \arrow[r, "i"] & B \arrow[u, "\beta''"] \arrow[r, "p"] & C \arrow[u, "\gamma''"] \\
& B' \arrow[u, "\beta'"] \arrow[ur, "\beta' p"] \; . &
\end{tikzcd}$$

of abelian groups with (B', B, B''), (A, B, C) *short exact sequences, and* (B', C, C'') *a right-exact sequence. Then* $A \oplus B' \xrightarrow{\begin{pmatrix} i \\ \beta' \end{pmatrix}} B \xrightarrow{p\gamma''=\beta''p''} C''$ *is a right-exact sequence.*

Proof. The composite $p\gamma''$ is surjective as a composite of surjective maps.
We have to show that $\operatorname{im}\begin{pmatrix} i \\ \beta' \end{pmatrix} = \ker(p\gamma'')$.

We have $\operatorname{im}\begin{pmatrix} i \\ \beta' \end{pmatrix} \subseteq \ker(p\gamma'')$, because

$$\begin{aligned}
&ip\gamma'' = 0\,, \quad \text{for} \;\; ip = 0\,, \\
&\beta' p\gamma'' = 0\,, \quad \text{for} \;\; \beta' p\gamma'' = \beta'\beta'' p'' = 0p'' = 0\,.
\end{aligned}$$

Further we have $\operatorname{im}\begin{pmatrix} i \\ \beta' \end{pmatrix} \supseteq \ker(p\gamma'')$, because given $b \in B$ with $(bp)\gamma'' = 0$, right-exactness of the sequence (B', C, C'') gives a $b' \in B'$ with $b'\beta' p = bp$, and therefore $(b'\beta' - b)p = 0$, so that there exists $a \in A$ with $ai = b'\beta' - b$, whence $(-a, b')\begin{pmatrix} i \\ \beta' \end{pmatrix} = -ai + b'\beta' = b$. □

Appendix C

The Jacobson radical

We want to give a self-contained introduction to Jacobson radicals of orders over discrete valuation rings, collecting well-known facts in one place.

C.1 Jacobson radical of rings

Let A be a ring. Recall that

$$\mathrm{U}_{\mathrm{left}}(A)\ ,\ \ \mathrm{U}_{\mathrm{right}}(A)\ ,\ \ \mathrm{U}(A)$$

denote the set of left-invertible, right-invertible and invertible elements in A, respectively.

Definition 178 The intersection of the annihilator ideals of the simple right modules of A is called the **Jacobson radical** and denoted $\mathfrak{r}A$.

Note that $\mathfrak{r}A$ is an ideal of A.

Example 179

(i) Let R be a discrete valuation ring with maximal ideal generated by π. Then πR is the unique maximal (left) ideal of R, whence $\mathfrak{r}R = \pi R$.

(ii) We have $\mathfrak{r}\mathbb{Z} = \bigcap_{p\,>\,0\text{ prime}} p\mathbb{Z} = 0$.

(iii) We have $\mathfrak{r}\left(\begin{smallmatrix}\mathbb{Q}&\mathbb{Q}\\0&\mathbb{Q}\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&\mathbb{Q}\\0&0\end{smallmatrix}\right) \not\subseteq \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) = \mathfrak{r}\left(\begin{smallmatrix}\mathbb{Q}&\mathbb{Q}\\\mathbb{Q}&\mathbb{Q}\end{smallmatrix}\right)$.

Definition 180 For a finitely generated projective A-module P, we denote its **radical** by $\mathfrak{r}P := P\cdot\mathfrak{r}A$.

In particular, given an idempotent e of A, we have $\mathfrak{r}(eA) = eA\cdot\mathfrak{r}A = e\,\mathfrak{r}A$.

Lemma 181 (**[8,** 4.1**]**) *For $a \in A$, the following statements are equivalent.*

(i) a is in the intersection of all maximal right ideals of A.

(ii) $1 - ax$ is right-invertible for all $x \in A$.

(iii) $a \in \mathfrak{r}A$.

Proof. *Ad* (i) $\Rightarrow$ (ii). Let a be in the intersection of all maximal right ideals of A.

Assume that there exists an element x for which $1 - ax \notin \mathrm{U}_{\mathrm{right}}(A)$. Then $(1 - ax)A \subsetneq A$ is contained in a maximal right ideal I of A. Since $(1 - ax) \in I$ and $a \in I$, we get that $1 = (1 - ax)\cdot 1 + a\cdot x \in I$. That is a *contradiction*.

Ad (ii) $\Rightarrow$ (iii). *Assume* that $a \notin \mathfrak{r}A$, i.e. that there exists a simple right module M and $m \in M$ such that $ma \neq 0$. By simplicity of M, we must have $ma \cdot A = M$. In particular, we have that $m = ma \cdot x$ for some $x \in A$. This yields that $m(1 - ax) = 0$. However, $(1 - ax) \in \mathrm{U}_{\mathrm{right}}(A)$ by (ii). So there exists $y \in A$ with $(1 - ax)y = 1$. Hence $0 = m(1 - ax)ya = ma$, which is a *contradiction*.

Ad (iii) $\Rightarrow$ (i). For each maximal right ideal I, the right module A/I is simple. Therefore by (iii), $(A/I) \cdot a = 0$, i.e. $Aa \subseteq I$, so in particular, $a \in I$. So a is in the intersection of all maximal right ideals of A. $\square$

Lemma 182 (**[8,** 4.3**]**) *For $a \in A$, the following statements are equivalent.*

(i) a is in the intersection of all maximal right ideals of A.

(i′) a is in the intersection of all maximal left ideals of A.

(ii) $1 - yax$ is invertible for all $y \in A$ and $x \in A$.

(iii) $a \in \mathfrak{r}A$.

Proof. *Ad* (i) $\Rightarrow$ (ii). First, we recall that $a \in \mathfrak{r}A$, which is an ideal of A; cf. Lemma 181. Suppose given $y \in A$ and $x \in A$. We have to show that $1 - yax \overset{!}{\in} \mathrm{U}_{\mathrm{left}}(A) \cap \mathrm{U}_{\mathrm{right}}(A)$. Since $ya \in \mathfrak{r}A$, we have that $1 - yax \in \mathrm{U}_{\mathrm{right}}(A)$; cf. Lemma 181. So there exists $u \in A$ such that $(1 - yax)u = 1$. So $u = 1 + yaxu$. Since $yaxu \in \mathfrak{r}A$, we have that $u = 1 - (yaxu) \cdot (-1) \in \mathrm{U}_{\mathrm{right}}(A)$; cf. Lemma 181. So there exists $v \in A$ with $uv = 1$. Hence $1 - yax = (1 - yax)uv = v$. Hence $1 = uv = u(1 - yax)$.

Ad (ii) $\Rightarrow$ (i). Letting $y = 1$, we see that $1 - ax$ is invertible, hence right-invertible, for all $x \in A$. So by Lemma 181, the element a is contained in the intersection of all maximal right ideals.

Ad (i′) $\Leftrightarrow$ (ii). Since assertion (ii) is left-right-symmetric, the equivalence (i′) $\Leftrightarrow$ (ii) is the symmetric assertion to the equivalence (i) $\Leftrightarrow$ (ii), and the latter has already been shown.

Ad (i) $\Leftrightarrow$ (iii). See Lemma 181. $\square$

Lemma 183 (**[10,** 2.5.14**]**) *Let $e \in A$ be an idempotent. Then*

$$\mathfrak{r}(eAe) \;=\; e\,\mathfrak{r}(A)e \;=\; eAe \cap \mathfrak{r}A\,.$$

Moreover, we have an isomorphism of rings

$$\begin{array}{rcl} eAe\big/\mathfrak{r}(eAe) & \xrightarrow{\sim} & e'\left(A\big/\mathfrak{r}A\right)e' \\ exe + \mathfrak{r}(eAe) & \longmapsto & e'(x + \mathfrak{r}A)e'\,, \end{array}$$

where we abbreviate $e' := e + \mathfrak{r}A$.

Proof.

We treat the first equality.

Ad $\supseteq$. Suppose given $a \in e\mathfrak{r}(A)e$. By Lemma 181, we have to show that $e - aexe \in \mathrm{U}_{\mathrm{right}}(eAe)$ for all $x \in A$. Note that $ae = a = ea$. Since $a \in \mathfrak{r}(A)$ we get by Lemma 181, that $1 - axe \in \mathrm{U}_{\mathrm{right}}(A)$. So there exists $y \in A$ such that $(1 - axe)y = 1$. We get

$$(e - axe)eye \;=\; eye - axeye \;=\; e(1 - axe)ye \;=\; e^2 \;=\; e\,.$$

Ad $\subseteq$. Conversely, suppose given $x \in \mathfrak{r}(eAe) \subseteq eAe$. It suffices to show that $x \overset{!}{\in} \mathfrak{r}A$, for $x = exe$. Suppose given a simple right A-module M. We have to show that $Mx \overset{!}{=} 0$. *Assume* that $Mx \neq 0$. Then $0 \neq Mx = Mex$, so $0 \neq Me$. Note that Me is a right eAe-submodule of M.

Given $m \in M$ such that $me \neq 0$, we get $me \cdot eAe = (meA)e = Me$, since M is a simple right A-module. Hence Me is generated as a right eAe-module by every nonzero element, and so it is a simple right eAe-module.

But $x \in \mathfrak{r}(eAe)$, so it annihilates every simple right eAe-module. Therefore $Me \cdot x = 0$, which is a *contradiction.*

We treat the second equality.

Ad $\subseteq$. We have $e\,\mathfrak{r}(A)e \subseteq \mathfrak{r}(A)$ and $e\,\mathfrak{r}(A)e \subseteq eAe$.

Ad $\supseteq$. Suppose given $x \in \mathfrak{r}(A) \cap eAe$. Then we have $x = exe \in e\,\mathfrak{r}(A)e$.

We consider the claimed isomorphism.

The kernel of the surjective ring morphism $eAe \longrightarrow e'\,(A/\mathfrak{r}A)\,e'$, $exe \longmapsto e'(x+\mathfrak{r}A)e'$ is given by $eAe\cap\mathfrak{r}A$, which we know to be equal to $e\,\mathfrak{r}(A)e$. $\square$

Remark 184 Note that in general for an idempotent e we have

$$(\mathfrak{r}(eAe))^2 \neq e(\mathfrak{r}A)^2 e\,.$$

For example, let $A := \begin{pmatrix} R & R \\ (3) & R \end{pmatrix}$ with $R := \mathbb{Z}_{(3)}$.

Then we have

$$\begin{array}{rcl} \mathfrak{r}A & = & \begin{pmatrix} (3) & R \\ (3) & (3) \end{pmatrix}, \\ (\mathfrak{r}A)^2 & = & \begin{pmatrix} (3) & (3) \\ (9) & (3) \end{pmatrix}. \end{array}$$

Let $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We get

$$\begin{array}{rcl} e(\mathfrak{r}A)^2 e & = & \begin{pmatrix} (3) & 0 \\ 0 & 0 \end{pmatrix}, \\ (\mathfrak{r}(eAe))^2 & = & \begin{pmatrix} (9) & 0 \\ 0 & 0 \end{pmatrix}. \end{array}$$

Lemma 185 (and Definition) *Suppose given idempotents $e,\ f \in A$ such that $eA \cong fA$ as A-modules. Then*

$$eAf \cdot \mathfrak{r}(fAf) \;=\; e\,\mathfrak{r}Af \;=\; \mathfrak{r}(eAe)\cdot eAf\,.$$

As abelian group, we define

$$\mathfrak{r}(eAf) \;:=\; eAf \cdot \mathfrak{r}(fAf) \;=\; e\,\mathfrak{r}(A)f \;=\; \mathfrak{r}(eAe)\cdot eAf\,.$$

Proof. Since $eA \cong fA$, we may choose $u = euf \in eAf$ and $v = fve \in fAe$ such that $uv = e$ and $vu = f$. We want to show that $e\,\mathfrak{r}(A)f \overset{!}{=} \mathfrak{r}(eAe)\cdot eAf$, the other equation being symmetric.

Ad $\supseteq$. Suppose given $a = eae \in \mathfrak{r}(eAe)$ and $b = ebf \in eAf$. We have to show that $ab \overset{!}{\in} e\,\mathfrak{r}(A)f$. Since $ab \in eAf$, it suffices to show that $ab \overset{!}{\in} \mathfrak{r}A$. So it suffices to show that $a \overset{!}{\in} \mathfrak{r}A$. But $a \in \mathfrak{r}(eAe) = e\,\mathfrak{r}(A)e \subseteq \mathfrak{r}A$ by Lemma 183.

Ad $\subseteq$. Suppose given $x = exf \in e\,\mathfrak{r}(A)f$. It suffices to show that $xv \overset{!}{\in} \mathfrak{r}(eAe)$, for then $x = xvu \in \mathfrak{r}(eAe)\cdot eAf$. However, $xv \in e\,\mathfrak{r}(A)f \cdot fAe \subseteq e\,\mathfrak{r}(A)e = \mathfrak{r}(eAe)$ by Lemma 183. $\square$

Lemma 186 *We have*

$$\mathfrak{r}({}^{A}/_{\mathfrak{r}A}) = 0\,.$$

Proof. We abbreviate $A' := A/\mathfrak{r}A$. Let M be a simple A-module. We have $M \cdot \mathfrak{r}A = 0$; cf. Definition 178. So M may be viewed as an A'-module via $m \cdot (x + \mathfrak{r}A) := m \cdot x$ for $x \in A$. As an A'-module, M is again simple.

Now let $x \in A$ be such that $x + \mathfrak{r}A \in \mathfrak{r}A'$. Then $M \cdot x = M \cdot (x + \mathfrak{r}A) = 0$ for every simple A-module M. Hence $x \in \mathfrak{r}A$, i.e. $x + \mathfrak{r}A = 0$. This shows that $\mathfrak{r}A' = 0$. □

Lemma 187 (Nakayama) *Let X be a finitely generated right module over A.*

If $X \neq 0$, then $X \cdot \mathfrak{r}A$ is a proper submodule of X.

Proof. We show that $X \cdot \mathfrak{r}A = X$ implies $X \overset{!}{=} 0$.

Write $X = \langle x_1, \ldots, x_k \rangle$ with a minimal number k of generators.

Assume that $k \geq 1$. Since $X \cdot \mathfrak{r}A = X$, we have $x_k \in X \cdot \mathfrak{r}A$. So

$$x_k \;=\; x_1 a_1 + \ldots + x_k a_k$$

for some $a_1, \ldots, a_k \in \mathfrak{r}A$. Hence

$$x_k(1 - a_k) \;=\; x_1 a_1 + \ldots + x_{k-1} a_{k-1} \,.$$

By Lemma 182, $(1 - a_k) \in \mathrm{U}(A)$. So there exists $w \in A$ with $(1 - a_k)w = 1$. We conclude that

$$x_k \;=\; x_1 a_1 w + \ldots + x_{k-1} a_{k-1} w \,.$$

Hence each element $y \in X$ may be written as

$$y \;=\; x_1 b_1 + \ldots + x_k b_k \;=\; x_1 b_1 + \ldots + x_{k-1} b_{k-1} + (x_1 a_1 w + \ldots + x_{k-1} a_{k-1} w) b_k \;\in\; \langle x_1, \ldots, x_{k-1} \rangle \,,$$

where $b_i \in A$ for $i \in [1, k]$. Hence $X = \langle x_1, \ldots, x_{k-1} \rangle$, *contradicting* the minimality of k.

So $k = 0$, whence $X = 0$. □

Lemma 188 *Let P and Q be finitely generated projective A-modules.*
Suppose that $P/\mathfrak{r}P \cong Q/\mathfrak{r}Q$ as A-modules.
Then $P \cong Q$ as A-modules.

Proof. We may choose a diagram as follows, with vertical residue class maps

$$\begin{tikzcd} P \arrow[r, dashed, "\exists\,\tilde{\varphi}"] \arrow[d, "\rho_P"] & Q \arrow[d, "\rho_Q"] \\ P/\mathfrak{r}P \arrow[r, "\sim", "\varphi"'] & Q/\mathfrak{r}Q \,, \end{tikzcd}$$

where the Λ-linear map $\tilde{\varphi}$ exists, because P is projective and ρ_Q is surjective.

It remains to show that $\tilde{\varphi}$ is an isomorphism.

Ad surjectivity. We have that $(P\tilde{\varphi})\rho_Q = P\rho_P\,\varphi = Q/\mathfrak{r}Q$, and therefore $P\tilde{\varphi} + \mathfrak{r}Q = Q$.

We obtain

$$(Q/P\tilde{\varphi}) \cdot \mathfrak{r}A \;=\; (Q \cdot \mathfrak{r}A + P\tilde{\varphi})/P\tilde{\varphi} \;=\; (\mathfrak{r}Q + P\tilde{\varphi})/P\tilde{\varphi} \;=\; Q/P\tilde{\varphi} \,.$$

So Lemma 187 yields $Q/P\tilde{\varphi} = 0$, i.e. $Q = P\tilde{\varphi}$.

Ad injectivity. Let $K := \operatorname{Ker} \tilde{\varphi}$. We aim to show that $K \overset{!}{=} 0$.

We have a short exact sequence $K \xrightarrow{i} P \xrightarrow{\tilde{\varphi}} Q$, where i is the inclusion map. Since Q is projective, this sequence splits, i.e. there exists a Λ-linear map $t : Q \longrightarrow P$ such that $t\tilde{\varphi} = \mathrm{id}_Q$. Let $Q' := Qt$.

We *claim* that $P \overset{!}{=} K \oplus Q'$. Given $x \in P$, we may write $x = (x - x\tilde{\varphi}) + x\tilde{\varphi} \in K + Q'$. Moreover, given $y \in K \cap Q'$, we may write $y = qt$ for some $q \in Q$ to obtain $0 = y\tilde{\varphi} = qt\tilde{\varphi} = q$, so that we get $y = qt = 0$. This proves the *claim*. In particular, we have the projection map from P to K, which shows that K is finitely generated.

We have $K\rho_P\,\varphi = K\tilde{\varphi}\rho_Q = 0$, and thus $K\rho_P = 0$, i.e. $K \subseteq \mathfrak{r}P = P \cdot \mathfrak{r}A = K \cdot \mathfrak{r}A \oplus Q' \cdot \mathfrak{r}A$. So $K \subseteq K \cdot \mathfrak{r}A \subseteq K$, whence $K = 0$ by Lemma 187. □

C.2 Local rings

Let A be a ring.

Definition 189 The ring A is called **local,** if its set of non-units $A \setminus \mathrm{U}(A)$ is an ideal in A.

Remark 190 (cf. **[8,** 4.8**]**) Suppose given $a \in A$. The following hold.

(i) We have $a \in \mathrm{U}_{\mathrm{left}}(A)$ if and only if $a + \mathfrak{r}A \in \mathrm{U}_{\mathrm{left}}({}^{A}/_{\mathfrak{r}A})$.

(ii) We have $a \in \mathrm{U}_{\mathrm{right}}(A)$ if and only if $a + \mathfrak{r}A \in \mathrm{U}_{\mathrm{right}}({}^{A}/_{\mathfrak{r}A})$.

(iii) We have $a \in \mathrm{U}(A)$ if and only if $a + \mathfrak{r}A \in \mathrm{U}({}^{A}/_{\mathfrak{r}A})$.

Proof. *Ad* (i). We have to show the reverse implication. Suppose that $a + \mathfrak{r}A \in \mathrm{U}_{\mathrm{left}}(A/\mathfrak{r}A)$. Then there exists $b \in A$ such that $(b+\mathfrak{r}A)(a+\mathfrak{r}A) = 1+\mathfrak{r}A$, i.e. such that $ba = 1+z$ for some $z \in \mathfrak{r}A$. By Lemma 182, there exists $w \in A$ such that $w(1 + z) = 1$. So $(wb)a = 1$.

Ad (iii). Follows from (i) and its symmetric assertion (ii). □

Remark 191 Suppose given a ring B. If the zero ideal is a maximal left ideal in B, then B is a skewfield.

Proof. Since B has a maximal ideal, we have $B \neq 0$.

Suppose given $x \in B$ with $x \neq 0$. Then Bx is a nonzero left ideal in B. If Bx was a proper left ideal of B, then it would be contained in a maximal left ideal, which is not the case. Hence $Bx = B$. So there exists $y \in B$ such that $yx = 1$. Note that $y \neq 0$. So, likewise, there exists $z \in B$ such that $zy = 1$. Hence $z = zyx = x$. Therefore $yx = 1$ and $xy = zy = 1$. □

Remark 192 (cf. **[8,** 19.1**]**) The following assertions are equivalent.

(i) A is a local ring.

(ii) We have $0_A \neq 1_A$ and the sum of any two non-units in A is a non-unit.

(iii) ${}^{A}/_{\mathfrak{r}A}$ is a skewfield.

(iv) $\mathfrak{r}A$ is the unique maximal left ideal in A.

(iv′) $\mathfrak{r}A$ is the unique maximal right ideal in A.

(v) We have $A = \mathrm{U}(A) \,\dot{\cup}\, \mathfrak{r}A$.

Proof. Write $I := A \setminus \mathrm{U}(A)$.

Ad (i) $\Rightarrow$ (ii). By assumption, I is an ideal in A. So it is closed under addition in A. Moreover, $0 \in I$. Since $1 \in \mathrm{U}(A)$, we have $1 \notin I$, so $0 \neq 1$.

Ad (ii) $\Rightarrow$ (iii). Since $0 \neq 1$, the left ideal $\{0\}$ is properly contained in A, whence there exists a maximal left ideal in A, which does not contain 1. Hence $1 \notin \mathfrak{r}A$, i.e. $0 + \mathfrak{r}A \neq 1 + \mathfrak{r}A$; cf. Lemma 182.

Suppose given $a \in A \setminus \mathfrak{r}A$. Condition (ii) being left-right symmetric, it suffices to show that there exists $b \in A$ such that $(b + \mathfrak{r}A)(a + \mathfrak{r}A) = 1 + \mathfrak{r}A$. Since $a \notin \mathfrak{r}A$, we may choose a maximal left ideal $L \subseteq A$ not containing a. As $L + Aa$ is a left ideal of A properly containing L, we have $L + Aa = A$. So there exists $\ell \in L$ and $c \in A$ such that $\ell + ca = 1$. By (ii), ca is a unit in A. So there exists $w \in A$ such that $(wc)a = 1$. In particular, $(wc + \mathfrak{r}A)(a + \mathfrak{r}A) = 1 + \mathfrak{r}A$.

Ad (iii) $\Rightarrow$ (iv). Since $A/\mathfrak{r}A \neq 0$, there exists a maximal left ideal; cf. Lemma 182. Suppose given a maximal left ideal $L \subseteq A$. Note that $\mathfrak{r}A \subseteq L$; cf. Lemma 182. Therefore $L/\mathfrak{r}A$ is a maximal left ideal in $A/\mathfrak{r}A$. The latter being a skewfield, we conclude that $L/\mathfrak{r}A = 0$, i.e. $L = \mathfrak{r}A$.

Ad (iv) $\Rightarrow$ (iii). Since $\mathfrak{r}A$ is a maximal left ideal in A, we have $A/\mathfrak{r}A \neq 0$.

Since $\mathfrak{r}A$ is the unique maximal left ideal in A, the ring $A/\mathfrak{r}A$ has the unique maximal left ideal $\mathfrak{r}A/\mathfrak{r}A = 0$. By Remark 191, the ring $A/\mathfrak{r}A$ is a skewfield.

Ad (iv) $\Rightarrow$ (v). We have to show that $I \overset{!}{=} \mathfrak{r}A$. We may use (iii).

Ad $I \overset{!}{\supseteq} \mathfrak{r}A$. Since $\mathfrak{r}A$ is a maximal left ideal of A, it does not contain a unit of A.

Ad $I \overset{!}{\subseteq} \mathfrak{r}A$. Suppose given $a \in A \setminus \mathfrak{r}A$. We have to show that a is not contained in I, i.e. that $a \overset{!}{\in} \mathrm{U}(A)$.

Since $a + \mathfrak{r}A \neq 0$ and since $A/\mathfrak{r}A$ is a skewfield by (iii), we have $a + \mathfrak{r}A \in \mathrm{U}(A/\mathfrak{r}A)$. By Remark 190.(iii), we have $a \in \mathrm{U}(A)$.

Ad (v) $\Rightarrow$ (i). We conclude that $I = \mathfrak{r}A$, which is an ideal in A; cf. Definition 178. □

Remark 193 A local ring has, up to isomorphism, only one simple module, viz. ${}^{A}\!/_{\mathfrak{r}A}$.

Proof. Note that $\mathfrak{r}A$ is the unique maximal right ideal in A; cf. Remark 192.

So the A-module $A/\mathfrak{r}A$ is simple.

Conversely, suppose given a simple A-module M. We may choose $m \in M \setminus \{0\}$. Consider the surjective A-linear map $\varphi : A \longrightarrow M$, $a \longmapsto ma$. Then $A/\operatorname{Ker}\varphi$ is isomorphic to M as an A-module. So $A/\operatorname{Ker}\varphi$ is simple, whence $\operatorname{Ker}\varphi$ is a maximal right ideal in A. Therefore $\operatorname{Ker}\varphi = \mathfrak{r}A$. Altogether, $A/\mathfrak{r}A$ is isomorphic to M as an A-module. □

C.3 Jacobson radical of K-algebras

Let K be a field. Let A be a finite-dimensional K-algebra.

Recall that an ideal $I \subseteq A$ is nilpotent if there exists $\ell \geq 0$ such that each product of length ℓ with factors in I is zero.

Lemma 194 *Let $S \subseteq A$ be a subset such that $s \cdot a \in S$ for all $s \in S$ and $a \in A$.*

Suppose that S consists of nilpotent elements.

Then $S \subseteq \mathfrak{r}A$.

Proof. Suppose given $s \in S$. It suffices to show that $1 - sa$ is right-invertible for all $a \in A$; cf. Lemma 181.

Write $t := sa$. By assumption, $t \in S$. So there exists $\ell \geq 0$ such that $t^\ell = 0$. Then

$$(1-t) \cdot \sum_{i \in [0,\ell-1]} t^i \;=\; 1 - t^\ell \;=\; 1\,.$$

□

Lemma 195 ([**8**, 4.11])

(i) There exists $k \geq 1$ such that $(\mathfrak{r}A)^k = 0$.

(ii) Given an ideal $N \subseteq A$ such that there exists $k \geq 1$ such that $N^k = 0$, then $N \subseteq \mathfrak{r}A$.

Proof.

Ad (i). Given $k \geq 0$, the A-module $(\mathfrak{r}A)^k$ is finitely generated, since it is even finite-dimensional over K. Hence $(\mathfrak{r}A)^k \cdot \mathfrak{r}A \subsetneq (\mathfrak{r}A)^k$ by Lemma 187. Since A is finite-dimensional over K, we conclude that there exists $\ell \geq 0$ such that $(\mathfrak{r}A)^\ell = 0$.

Ad (ii). This follows by Lemma 194, since N consists of nilpotent elements and since $N \cdot A \subseteq N$. □

Lemma 196 *Suppose A to be commutative. Then*

$$\mathfrak{r}A \;=\; \{\, a \in A \mid a \text{ is nilpotent} \,\}.$$

Proof. Write $N := \{\, a \in A \mid a \text{ is nilpotent} \,\}$.

We *claim* that N is an additive subgroup of A. Suppose given $x,\, y \in N$. We have to show that $x - y \overset{!}{\in} N$. Pick $k \geq 1$ with $x^k = 0$ and $\ell \geq 1$ with $y^\ell = 0$. Then $(x-y)^{k+\ell} = 0$ since A is commutative. This proves the *claim.*

Since A is commutative, we have $A \cdot N \subseteq N$. Together with our claim, this shows that N is an ideal in A.

Ad $\mathfrak{r}A \subseteq N$. By Lemma 195.(i), each element of $\mathfrak{r}A$ is nilpotent.

Ad $\mathfrak{r}A \supseteq N$. By Lemma 195.(ii), it suffices to show that the ideal N is nilpotent. Let $(x_1\,, \,\dots\,, \,x_m)$ be a K-linear basis of N. Pick $s \geq 1$ such that $x_i^s = 0$ for all $i \in [1,m]$. Then

$$x_{j_1} \cdot x_{j_2} \cdot \ldots \cdot x_{j_{sm}} \;=\; 0$$

for each choice of indices $j_t \in [1,m]$ for $t \in [1,sm]$, because we may reorder the factors to get

$$x_{j_1} \cdot x_{j_2} \cdot \ldots \cdot x_{j_{sm}} \;=\; x_1^{\alpha_1} \cdot \ldots \cdot x_m^{\alpha_m}$$

with some $\alpha_i \geq 0$ for $i \in [1,m]$ and observe that there exists an $i \in [1,m]$ with $\alpha_i \geq s$.

So multiplying K-linear combinations and expanding, this shows that any product of length sm with factors in N is zero. Hence N is nilpotent. □

Lemma 197 (Fitting, cf. e.g. [**1**, 1.3.2])

Suppose given an A-module M that is finite dimensional over K. Suppose given $f \in \operatorname{End} M$.

Then there exists $n \geq 1$ such that

$$M \;=\; \operatorname{Im} f^n \oplus \operatorname{Ker} f^n\;.$$

If M is indecomposable, then f is an automorphism or nilpotent.

Proof. There exists an $n \geq 1$ such that $\dim_K \operatorname{Im} f^n = \dim_K \operatorname{Im} f^{2n}$. Then the surjective map

$$\varphi \;:=\; f^n\Big|_{\operatorname{Im} f^n}^{\operatorname{Im} f^{2n}} \;:\; \operatorname{Im} f^n \;\longrightarrow\; \operatorname{Im} f^{2n}$$

is also injective. Moreover, $\operatorname{Im} f^n = \operatorname{Im} f^{2n}$.

Given $u \in \operatorname{Im} f^n \cap \operatorname{Ker} f^n$, we get $u\varphi = uf^n = 0$ and thus $u = 0$. This shows that $\operatorname{Im} f^n \cap \operatorname{Ker} f^n = 0$.

Suppose given $m \in M$. Then $mf^n \in \operatorname{Im} f^n = \operatorname{Im} f^{2n}$, so that we may write $mf^n = xf^{2n}$ for some $x \in M$. Hence $m = xf^n + (m - xf^n)$ with $xf^n \in \operatorname{Im} f^n$ and $(m - xf^n)f^n = 0$, i.e. $m - xf^n \in \operatorname{Ker} f^n$. This shows that $M = \operatorname{Im} f^n + \operatorname{Ker} f^n$.

If M is indecomposable, we conclude that there exists $n \geq 1$ such that $M = \operatorname{Im} f^n$ or $M = \operatorname{Ker} f^n$. In the former case, f is an automorphism; in the latter case, f is nilpotent. □

Lemma 198 *Suppose given an indecomposable A-module M that is finite dimensional over K.*

Then $\operatorname{End}_A M$ is a local ring.

Proof. Write $E := \operatorname{End}_A M$. We have to show that $E \overset{!}{=} \mathrm{U}(E) \,\dot\cup\, \mathfrak{r}E$. Since $E \neq 0$, we have $\mathfrak{r}E \subsetneq E$ and thus $\mathrm{U}(E) \cap \mathfrak{r}E = \emptyset$. So it suffices to show that $E \setminus \mathrm{U}(E) \overset{!}{\subseteq} \mathfrak{r}E$.

The subset $E \setminus \mathrm{U}(E)$ consists of nilpotent elements by Lemma 197. It is closed under right-multiplication with elements of E, as it consists of non-invertible K-linear endomorphisms. So $E \setminus \mathrm{U}(E) \subseteq \mathfrak{r}E$ by Lemma 194. □

Lemma 199 *Suppose given a simple A-module S.*

(i) There exists an indecomposable projective A-module P and a surjective A-linear map $P \xrightarrow{p} S$.

(ii) Given indecomposable projective A-modules P and Q and surjective A-linear maps $P \xrightarrow{p} S \xleftarrow{q} Q$, then there exists an A-linear isomorphism $P \xrightarrow[\sim]{f} Q$ such that $fq = p$.

$$\begin{tikzcd} P \arrow[rr, "f", "\sim"'] \arrow[dr, "p"'] & & Q \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

Proof. Ad (i). Choose $s \in S \setminus \{0\}$. We have a surjective A-linear map $\varphi : A \longrightarrow S$, $a \longmapsto sa$. Decompose $A = \bigoplus_{i \in [1,k]} P_i$ with each P_i indecomposable projective. There exists $j \in [1, k]$ such that $\varphi|_{P_j} \neq 0$. Since S is simple, $\varphi|_{P_j} : P_j \longrightarrow S$ is surjective. So we may choose $P := P_j$ and $p := \varphi|_{P_j}$.

Ad (ii). Using projectivity of P, we find an A-linear map $P \xrightarrow{f} Q$ such that $fq = p$. Using projectivity of Q, we find an A-linear map $Q \xrightarrow{g} P$ such that $gp = q$. Since $(fg)^k p = p \neq 0$ for $k \geq 0$, the endomorphism fg of P is not nilpotent. Likewise, the endomorphism gf of Q is not nilpotent. So using Lemma 197, we get $fg \in U(\operatorname{End}(P))$ and $gf \in U(\operatorname{End}(Q))$. Therefore, f and g are isomorphisms. □

Lemma 200 *Suppose given primitive idempotents $e, f \in A$. If eA is not isomorphic to fA as A-modules, then $eAf \subseteq \mathfrak{r}A$.*

Proof. Given a simple A-module S, we have to show that $SeAf = 0$; cf. Definition 178.

We *assume* that $SeAf \neq 0$. Then $Se \neq 0$. Moreover, $0 \subsetneq SeAf \subseteq Sf$, so $Sf \neq 0$.

Then we can find an element $s \in S$ such that $se \neq 0$. Thus, there exists an A-linear map $eA \xrightarrow{\varphi} S$, $ea \mapsto sea$, which is nonzero since $se \neq 0$, and thus surjective since S is simple.

Analogously, we get a surjective A-linear map $fA \longrightarrow S$.

According to Lemma 199, this implies $eA \cong fA$, which is a *contradiction*. □

Proposition 201 *Recall that A is a finite-dimensional K-algebra. Let $1_A = \sum_{i \in [1,n]} e_i$ be an orthogonal decomposition into primitive idempotents.*

We define an equivalence relation $(\sim)$ *on the index set* $[1, n]$ *by letting*

$$i \sim j \;:\Leftrightarrow\; e_i A \cong e_j A$$

for $i,\, j \in [1, n]$.

Recall from Lemma 185 that we write $\mathfrak{r}(e_i A e_j) := e_i A e_j \cdot \mathfrak{r}(e_j A e_j) = e_i\,\mathfrak{r}(A) e_j = \mathfrak{r}(e_i A e_i) \cdot e_i A e_j$ *for* $i,\, j \in [1, n]$ *with* $i \sim j$.

Then we get

$$\mathfrak{r}A \;=\; \Big(\bigoplus_{\substack{i,j \in [1,n] \\ i \sim j}} \mathfrak{r}(e_i A e_j) \Big) \oplus \Big(\bigoplus_{\substack{i,j \in [1,n] \\ i \not\sim j}} e_i A e_j \Big)\;.$$

Proof. Write RS for the right-hand side of the equation in question.

Ad $\mathfrak{r}A \supseteq \mathrm{RS}$. Suppose given $i,\, j \in [1, n]$. We have to show that the corresponding summand in RS is contained in $\mathfrak{r}A$.

If $i \not\sim j$, then $e_i A e_j \subseteq \mathfrak{r}A$ by Lemma 200.

If $i \sim j$, then $\mathfrak{r}(e_i A e_j) = e_i\,\mathfrak{r}(A) e_j \subseteq \mathfrak{r}A$.

Ad $\mathfrak{r}A \subseteq \mathrm{RS}$. Suppose given $a \in \mathfrak{r}A$. Suppose given $i,\, j \in [1, n]$. We have to show that $e_i a e_j$ is contained in the corresponding summand in RS, for then $a = \sum\limits_{i,\, j \,\in\, [1,n]} e_i a e_j$ is contained in RS.

If $i \not\sim j$, there is nothing to show.

If $i \sim j$, then $e_i a e_j \in e_i\,\mathfrak{r}(A) e_j = \mathfrak{r}(e_i A e_j)$. □

Example 202

Suppose given an orthogonal decomposition into primitive idempotents $1_A = e_1 + e_2 + e_3 + e_4$ with $e_1 \not\sim e_2$, $e_1 \not\sim e_3$, $e_1 \not\sim e_4$, $e_2 \not\sim e_3$, $e_2 \not\sim e_4$ and $e_3 \sim e_4$, i.e. with $(\sim)$-classes $\{e_1\}$, $\{e_2\}$ and $\{e_3\,,\, e_4\}$.

We have the Peirce decomposition

$$A = \begin{pmatrix} e_1Ae_1 & e_1Ae_2 & e_1Ae_3 & e_1Ae_4 \\ e_2Ae_1 & e_2Ae_2 & e_2Ae_3 & e_2Ae_4 \\ e_3Ae_1 & e_3Ae_2 & e_3Ae_3 & e_3Ae_4 \\ e_4Ae_1 & e_4Ae_2 & e_4Ae_3 & e_4Ae_4 \end{pmatrix}\;.$$

With Proposition 201, we get the following matrix.

$$\begin{aligned} \mathfrak{r}A \;&=\; \begin{pmatrix} \mathfrak{r}(e_1Ae_1) & e_1Ae_2 & e_1Ae_3 & e_1Ae_4 \\ e_2Ae_1 & \mathfrak{r}(e_2Ae_2) & e_2Ae_3 & e_2Ae_4 \\ e_3Ae_1 & e_3Ae_2 & \mathfrak{r}(e_3Ae_3) & \mathfrak{r}(e_3Ae_4) \\ e_4Ae_1 & e_4Ae_2 & \mathfrak{r}(e_4Ae_3) & \mathfrak{r}(e_4Ae_4) \end{pmatrix} \\ &=\; \begin{pmatrix} \mathfrak{r}(e_1Ae_1) & e_1Ae_2 & e_1Ae_3 & e_1Ae_4 \\ e_2Ae_1 & \mathfrak{r}(e_2Ae_2) & e_2Ae_3 & e_2Ae_4 \\ e_3Ae_1 & e_3Ae_2 & \mathfrak{r}(e_3Ae_3) & e_3Ae_4\mathfrak{r}(e_4Ae_4) \\ e_4Ae_1 & e_4Ae_2 & e_4Ae_3\mathfrak{r}(e_3Ae_3) & \mathfrak{r}(e_4Ae_4) \end{pmatrix} \end{aligned}$$

So informally speaking, to get the radical of A we have to take radicals on its Peirce block main diagonal.

Remark 203 *Suppose given a primitive idempotent* $e \in A$*. Then the ring* eAe *is local.*

Proof. We have $eAe \cong \mathrm{End}_A(eA)$. Since e is primitive, eA is indecomposable and finite-dimensional, and so $\mathrm{End}_A(eA)$ is local by Lemma 198. □

Lemma 204 *Let e and f be primitive idempotents of A. Each A-linear map*

$$eA \xrightarrow{\psi} fA\big/ f\mathfrak{r}A$$

is surjective or the zero map.

Proof. Using projectivity of eA, we obtain a commutative triangle of A-linear maps as follows.

$$\begin{tikzcd} eA \arrow[r, dashed, "\hat\psi"] \arrow[dr, "\psi"'] & fA \arrow[d] \\ & fA/f\mathfrak{r}A \end{tikzcd}$$

Then $\hat\psi$ is the multiplication from the left with some $w \in fAe$.

If $\hat\psi$ is an isomorphism, then ψ is surjective. So from now on we can assume that $\hat\psi$ is not an isomorphism.

We want to show that $\hat\psi \overset{!}{=} 0$. We have to show that $fweA = weA = (eA)\psi \overset{!}{\subseteq} f\mathfrak{r}A$. It suffices to show that $w \overset{!}{\in} \mathfrak{r}A$.

Case $eA \not\cong fA$. We have $w \in fAe \subseteq \mathfrak{r}A$ by Lemma 200.

Case $eA \cong fA$. We have $u \in eAf$ and $v \in fAe$ with $uv = e$ and $vu = f$. Consider the following commutative triangle of A-linear maps.

$$\begin{tikzcd} eA \arrow[rr, "w(-)"] \arrow[dr, "v(-)"'] & & fA \\ & fA \arrow[ur, "wu(-)"'] & \end{tikzcd}$$

Since $w(-)$ is not an isomorphism, neither is $wu(-)$, i.e. wu is not a unit in fAf. Note that fAf is a local ring by Remark 203, so that $fAf = \mathrm{U}(fAf) \,\dot\cup\, \mathfrak{r}(fAf)$ by Remark 192. So $wu \in fAf \setminus \mathrm{U}(fAf) = \mathfrak{r}(fAf) = f\mathfrak{r}(A)f \subseteq \mathfrak{r}A$; cf. Lemma 183. □

Proposition 205 *Let $e \in A$ be a primitive idempotent. Then the A-module $eA\big/ e\mathfrak{r}A$ is simple.*

Proof. Write $X := eA/e\mathfrak{r}A$. *Assume* that X is not simple. Let $0 \subsetneq X' \subsetneq X$ be a proper submodule. Choose $x' \in X' \setminus \{0\}$. Then the A-linear map $\varphi : A \longrightarrow X$, $a \longmapsto x'a$ is neither zero nor surjective.

Let $1_A = \sum_{i\in[1,n]} e_i$ be an orthogonal decomposition into primitive idempotents. Then $\varphi|_{e_iA}$ is zero or surjective for $i \in [1,n]$ by Lemma 204. Since φ is not surjective, $\varphi|_{e_iA}$ is not surjective for $i \in [1,n]$. Hence $\varphi|_{e_iA} = 0$ for $i \in [1,n]$. Since $A = \bigoplus_{i\in[1,n]} e_iA$, this implies that $\varphi = 0$. We have reached a *contradiction.* □

Lemma 206 *Suppose given an A-module M that is a finite direct sum of certain simple submodules. Suppose given an A-module N and a surjective A-linear map $f : M \longrightarrow N$. Then N is a finite direct sum of certain simple submodules.*

Proof. Note that each simple A-module is an epimorphic image of the regular module A, hence finite-dimensional. In particular, M is finite-dimensional.

We choose a decomposition $M = \bigoplus_{i\in[1,k]} S_i$ with S_i simple for $i \in [1,k]$. Then

$$N \;=\; Mf \;=\; \Big(\sum_{i\in[1,k]} S_i\Big)f \;=\; \sum_{i\in[1,k]} (S_if)\;.$$

Let $I := \{ i \in [1,k] \mid S_i f \neq 0\}$. For $i \in I$, the surjective A-linear map $f\big|_{S_i}^{S_i f} : S_i \longrightarrow S_i f$ is nonzero, hence, S_i being simple, injective. So $S_i f$ is simple for $i \in I$. We have $N = \sum_{i \in I}(S_i f)$.

Let $N' \subseteq N$ be a submodule of maximal dimension with respect to being a direct sum of certain simple submodules. *Assume* that $N' \subsetneq N$. We may choose $j \in I$ such that $S_j f \not\subseteq N'$, for otherwise we would have $\sum_{i \in I}(S_i f) \subseteq N' \subsetneq N$. Then $S_j f \cap N' \subsetneq S_j f$. Since $S_j f$ is simple, this implies $S_j f \cap N' = 0$. So we have found the submodule $N' \oplus S_j f$, which is again a direct sum of certain simple submodules and which is of bigger dimension than N'. This is a *contradiction.* □

C.4 Jacobson radical of R-orders

Let R be a discrete valuation ring with maximal ideal (π). Recall that for reduction modulo π, we use the bar-notation.

Let Λ be an R-order. Denote the residue class ring morphism by

$$\begin{array}{ccc} \Lambda & \xrightarrow{\rho} & \bar{\Lambda} \\ x & \longmapsto & \bar{x}\,. \end{array}$$

Definition 207 The R-order Λ is called **stable,** if for each primitive idempotent $e \in \Lambda$, the idempotent $\bar{e} \in \bar{\Lambda}$ is primitive.

Cf. Lemma 220 below.

Remark 208 If Λ is an R-suborder in a finite direct product Γ of matrix rings over R such that Γ/Λ is of finite length as an R-module, then Λ is stable, as we will see in Lemma 232 below.

Lemma 209 *We have* $\pi\Lambda \subseteq \mathfrak{r}\Lambda$.

Proof. It suffices to show that $\pi\Lambda$ annihilates each simple Λ-module X. It suffices to show that π annihilates each simple Λ-module X.

Note that $X\pi \subseteq X$ is a Λ-submodule. So to show that $X\pi \overset{!}{=} 0$, it suffices to show that $X\pi \overset{!}{\subsetneq} X$.

We consider X as an R-module now. Note that $\mathfrak{r}R = \pi R$; cf. Example 179.(i).

Note that X is a factor module of Λ, via $\Lambda \longrightarrow X$, $\lambda \longmapsto x\lambda$ for any chosen element $x \in X \setminus \{0\}$, and therefore finitely generated as a module over R.

Since $X \neq 0$ and since X is finitely generated over R, we obtain $X\pi \subsetneq X$ by Lemma 187. □

Lemma 210 *We have*

$$\mathfrak{r}\Lambda \;=\; \rho^{-1}(\mathfrak{r}\bar{\Lambda})\,,$$

i.e.

$$\mathfrak{r}\Lambda/\pi\Lambda \;=\; \mathfrak{r}\bar{\Lambda}\,.$$

We obtain a ring isomorphism

$$\begin{array}{ccc} \Lambda/\mathfrak{r}\Lambda & \xrightarrow{\sim} & \bar{\Lambda}/\mathfrak{r}\bar{\Lambda} \\ x + \mathfrak{r}\Lambda & \longmapsto & \bar{x} + \mathfrak{r}\bar{\Lambda}\,. \end{array}$$

Proof. For $I \subseteq \Lambda$, we apply the bar elementwise and write $\bar{I} := I\rho = \{ \bar{x} \mid x \in I \} = \{ x + \pi\Lambda \mid x \in I \}$.

Recall that

$$\begin{array}{ccc} \{ \pi\Lambda \subseteq I \subseteq \Lambda \mid I \text{ is a right ideal in } \Lambda\} & \longrightarrow & \{ J \subseteq \bar{\Lambda} \mid J \text{ is a right ideal in } \bar{\Lambda}\} \\ I & \longmapsto & \bar{I} \\ \rho^{-1}(J) & \longleftarrow & J \end{array}$$

are mutually inverse bijections preserving inclusions.

Write $M := \{ I \subseteq \Lambda \mid I \text{ is a maximal right ideal in } \Lambda\}$. By Lemmas 181 and 209, we have

$$M \;=\; \{ \pi\Lambda \subseteq I \subseteq \Lambda \mid I \text{ is a maximal right ideal in } \Lambda\}\;.$$

Given $I \in M$, the image $\bar{I}$ is a maximal right ideal in $\bar{\Lambda}$.

Conversely, for each maximal right ideal $J \subseteq \bar{\Lambda}$ we have $J = \bar{I}$ for $I = \rho^{-1}(J) \in M$. So by Lemma 181 we have

$$\mathfrak{r}\bar{\Lambda} \;=\; \bigcap_{I\in M} \bar{I}\;.$$

By Lemma 181, we conclude

$$\mathfrak{r}\Lambda \;=\; \bigcap_{I\in M} I \;=\; \bigcap_{I\in M} \rho^{-1}(\bar{I}) \;=\; \rho^{-1}(\bigcap_{I\in M} \bar{I}) \;=\; \rho^{-1}(\mathfrak{r}\bar{\Lambda})\;.$$

Applying ρ to both sides, this implies

$$\mathfrak{r}\Lambda/\pi\Lambda \;=\; \mathfrak{r}\bar{\Lambda}\;.$$

Finally, we get

$$\begin{array}{rclcl} \Lambda/\mathfrak{r}\Lambda & \xrightarrow{\sim} & (\Lambda/\pi\Lambda)/(\mathfrak{r}\Lambda/\pi\Lambda) & \xrightarrow{\sim} & \bar{\Lambda}/\mathfrak{r}\bar{\Lambda} \\ x+\mathfrak{r}\Lambda & \longmapsto & \bar{x} + (\mathfrak{r}\Lambda/\pi\Lambda) & \longmapsto & \bar{x}+\mathfrak{r}\bar{\Lambda}\;. \end{array}$$

□

Corollary 211 *Λ is a local ring if and only if $\bar{\Lambda}$ is a local ring.*

Proof. By Lemma 210, we have $\Lambda/\mathfrak{r}\Lambda \cong \bar{\Lambda}/\mathfrak{r}\bar{\Lambda}$. So the result follows by Remark 192. □

Proposition 212 *Suppose Λ to be commutative. Then*

$$\mathfrak{r}\Lambda \;=\; \{\, x \in \Lambda \mid \bar{x} \text{ is nilpotent}\,\}\;.$$

Proof. By Lemmas 210 and 196, we obtain

$$\mathfrak{r}\Lambda \;=\; \rho^{-1}(\mathfrak{r}\bar{\Lambda}) \;=\; \rho^{-1}(\{\, \bar{x} \mid x \in \Lambda \text{ such that } \bar{x} \text{ is nilpotent}\,\}) \;=\; \{\, x\in\Lambda \mid \bar{x} \text{ is nilpotent}\,\}\;.$$

□

Lemma 213

(i) There exists $k \geq 1$ such that $(\mathfrak{r}\Lambda)^k \subseteq \pi\Lambda$.

(ii) Given an ideal $N \subseteq \Lambda$ such that there exists $k \geq 1$ such that $N^k \subseteq \pi\Lambda$, then $N \subseteq \mathfrak{r}\Lambda$.

Proof.

Ad (i). We have to show that there exists $k \geq 1$ such that $((\mathfrak{r}\Lambda)^k)\rho \overset{!}{=} 0$. But $(\mathfrak{r}\Lambda)\rho \overset{\text{L.210}}{=} \mathfrak{r}\bar{\Lambda}$; and we may choose $k \geq 1$ such that $(\mathfrak{r}\bar{\Lambda})^k = 0$ by Lemma 195.(i). So

$$((\mathfrak{r}\Lambda)^k)\rho \;=\; ((\mathfrak{r}\Lambda)\rho)^k \;=\; (\mathfrak{r}\bar{\Lambda})^k \;=\; 0\;.$$

Ad (ii). Choose k such that $N^k \subseteq \pi\Lambda$. Then $(N\rho)^k = (N^k)\rho = 0$. By Lemma 195.(ii), we obtain $N\rho \subseteq \mathfrak{r}\bar{\Lambda}$, so that $N \subseteq \rho^{-1}(\mathfrak{r}\bar{\Lambda}) \overset{\text{L.210}}{=} \mathfrak{r}\Lambda$. □

Lemma 214 *Suppose given Λ-lattices P and Q.*

Suppose that P is projective. Suppose that $\bar{P} \cong \bar{Q}$ as $\bar{\Lambda}$-modules.

Then $P \cong Q$ as Λ-lattices.

Given an isomorphism $\bar{P} \xrightarrow{\sim} \bar{Q}$, any morphism $P \longrightarrow Q$ lifting it is an isomorphism.

Cf. also Lemma 188.

Proof. We may choose a commutative diagram as follows, with vertical residue class maps

$$\begin{tikzcd} P \arrow[r, dashed, "\exists\,\tilde\varphi"] \arrow[d, "\rho_P"] & Q \arrow[d, "\rho_Q"] \\ \bar P \arrow[r, "\sim", "\varphi"'] & \bar Q \end{tikzcd} ,$$

where the Λ-linear map $\tilde\varphi$ exists, because P is projective and ρ_Q is surjective. It is an isomorphism, because

$$\det\tilde\varphi + \pi R = \det\varphi \in \mathrm{U}(\bar R)\,,$$

for φ is an isomorphism. Hence we know that $\det\tilde\varphi \in \mathrm{U}(R)$, so that $\tilde\varphi$ is an isomorphism. □

Remark 215 Let e and f be idempotents of Λ.

Then $e\Lambda \cong f\Lambda$ as Λ-modules if and only if $\bar e\bar\Lambda \cong \bar f\bar\Lambda$ as $\bar\Lambda$-modules.

Proof. This is a particular case of Lemma 214. □

Remark 216 (**[10,** p. 211**]**) There exists an orthogonal decomposition

$$1_\Lambda \;=\; \sum_{i\in[1,n]} e_i$$

into primitive idempotents e_i of Λ.

Proof. We have to show that Λ may be written as a finite direct sum of indecomposable submodules.

Note that Λ is right-noetherian, for it is even noetherian as an R-module. In fact, it is finitely generated as a module over the noetherian ring R. So each nonempty subset of submodules in Λ contains a submodule not included in any other submodule of this set.

Write $M_0 := \Lambda$.

If $M_0 \neq 0$ then choose a maximal submodule M_1 in M_0 that is a direct summand therein. Choose a submodule N_1 of M_0 such that $M_1 \oplus N_1 = M_0$. Then N_1 is indecomposable, for a decomposition would allow to enlarge M_1 by one of the summands of N_1, which is impossible by maximality of M_1.

If $M_1 \neq 0$ then choose a maximal submodule M_2 in M_1 that is a direct summand therein. Choose a submodule N_2 of M_1 such that $M_2 \oplus N_2 = M_1$. Then N_2 is indecomposable, for a decomposition would allow to enlarge M_2 by one of the summands of N_2, which is impossible by maximality of M_2.

Etc.

We obtain a strictly ascending chain of submodules

$$N_1 \;\subsetneq\; N_1\oplus N_2 \;\subsetneq\; N_1\oplus N_2\oplus N_3 \;\subsetneq\; \ldots$$

Since Λ is right-noetherian, this chain cannot be of infinite length. So there exists $n \geq 0$ such that $M_n = 0$. Hence we have found a finite direct sum decomposition $\Lambda = \bigoplus_{i\in[1,n]} N_i$ into indecomposable submodules. □

Proposition 217 *Recall that Λ is an R-order. Suppose Λ to be stable; cf. Definition 207.*

Let $1_\Lambda = \sum_{i\in[1,n]} e_i$ be an orthogonal decomposition into primitive idempotents; cf. Remark 216.

We define an equivalence relation $(\sim)$ on the index set $[1,n]$ by letting

$$i \sim j \;:\Leftrightarrow\; e_i\Lambda \cong e_j\Lambda$$

for $i,\, j \in [1,n]$.

Recall from Lemma 185 that we write $\mathfrak{r}(e_i\Lambda e_j) := e_i\Lambda e_j \cdot \mathfrak{r}(e_j\Lambda e_j) = e_i\,\mathfrak{r}(\Lambda)e_j = \mathfrak{r}(e_i\Lambda e_i)\cdot e_i\Lambda e_j$ for $i,\, j \in [1,n]$ with $i \sim j$.

Then we get

$$\mathfrak{r}\Lambda \;=\; \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\sim j}} \mathfrak{r}(e_i\Lambda e_j)\Big) \oplus \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\nsim j}} e_i\Lambda e_j\Big)\;.$$

Proof. Since Λ is stable, $1_{\bar\Lambda} = \sum_{i\in[1,n]} \bar e_i$ is an orthogonal decomposition into primitive idempotents.

For $i,\, j \in [1,n]$, we have $e_i\Lambda \cong e_j\Lambda$ if and only if $\bar e_i\bar\Lambda \cong \bar e_j\bar\Lambda$ by Remark 215. So the equivalence relation $(\sim)$ for the decomposition $1_\Lambda = \sum_{i\in[1,n]} e_i$ coincides with the equivalence relation $(\sim)$ for the decomposition $1_{\bar\Lambda} = \sum_{i\in[1,n]} \bar e_i$ in the sense of Proposition 201.

By Proposition 201, we have

$$\mathfrak{r}\bar\Lambda \;=\; \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\sim j}} \mathfrak{r}(\bar e_i\bar\Lambda\bar e_j)\Big) \oplus \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\nsim j}} \bar e_i\bar\Lambda\bar e_j\Big)\;.$$

Suppose given $i,\, j \in [1,n]$ such that $i \sim j$. Suppose given $x \in \Lambda$. We *claim* that $\bar e_i\bar x\bar e_j \in \mathfrak{r}(\bar e_i\bar\Lambda\bar e_j)$ if and only if $e_ixe_j \in \mathfrak{r}(e_i\Lambda e_j)$. We have

$$\begin{array}{rcl} \mathfrak{r}(\bar e_i\bar\Lambda\bar e_j) & = & \bar e_i\mathfrak{r}\bar\Lambda\bar e_j \\ & \overset{\text{L.210}}{=} & \bar e_i(\mathfrak{r}\Lambda)\rho\,\bar e_j \\ & = & (e_i\mathfrak{r}\Lambda e_j)\rho \\ & = & (\mathfrak{r}(e_i\Lambda e_j))\rho\;. \end{array}$$

Now if $e_ixe_j \in \mathfrak{r}(e_i\Lambda e_j)$, then $\bar e_i\bar x\bar e_j = (e_ixe_j)\rho \in (\mathfrak{r}(e_i\Lambda e_j))\rho = \mathfrak{r}(\bar e_i\bar\Lambda\bar e_j)$.

Conversely, if $\bar e_i\bar x\bar e_j = (e_ixe_j)\rho$ is in $\mathfrak{r}(\bar e_i\bar\Lambda\bar e_j) = (\mathfrak{r}(e_i\Lambda e_j))\rho$, then $e_ixe_j \in \mathfrak{r}(e_i\Lambda e_j) + \pi\Lambda$, whence $e_ixe_j \in e_i(\mathfrak{r}(e_i\Lambda e_j) + \pi\Lambda)e_j = \mathfrak{r}(e_i\Lambda e_j) + e_i\pi\Lambda e_j = \mathfrak{r}(e_i\Lambda e_j)$ by Lemma 209.

This proves the *claim*.

By Lemma 210, we obtain

$$\begin{array}{rcl} \mathfrak{r}\Lambda & = & \rho^{-1}(\mathfrak{r}\bar\Lambda) \\ & = & \Big\{\, x\in\Lambda \mid \bar x = \sum_{i,j\in[1,n]} \bar e_i\bar x\bar e_j \text{ is in } \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\sim j}} \mathfrak{r}(\bar e_i\bar\Lambda\bar e_j)\Big) \oplus \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\nsim j}} \bar e_i\bar\Lambda\bar e_j\Big) \Big\} \\ & = & \{\, x\in\Lambda \mid \bar e_i\bar x\bar e_j \in \mathfrak{r}(\bar e_i\bar\Lambda\bar e_j) \text{ for } i,\, j \in [1,n] \text{ with } i\sim j \,\} \\ & \overset{\text{Claim}}{=} & \{\, x\in\Lambda \mid e_ixe_j \in \mathfrak{r}(e_i\Lambda e_j) \text{ for } i,\, j \in [1,n] \text{ with } i\sim j \,\} \\ & = & \Big\{\, x\in\Lambda \mid x = \sum_{i,j\in[1,n]} e_ixe_j \text{ is in } \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\sim j}} \mathfrak{r}(e_i\Lambda e_j)\Big) \oplus \Big(\bigoplus_{\substack{i,j\in[1,n]\\ i\nsim j}} e_i\Lambda e_j\Big) \Big\}\;. \end{array}$$

$\square$

Let ε be a central idempotent of $K\Lambda$. Then $\varepsilon\Lambda$ is a subring of $K\Lambda$, which is an R-order.

We have a surjective morphism of R-orders $\Lambda \longrightarrow \varepsilon\Lambda$, $x \longmapsto \varepsilon x$.

Lemma 218 *Suppose Λ to be stable.*

Let e be a primitive idempotent of Λ.

Let ε be a central idempotent of $K\Lambda$.

(i) The Λ-module ${}^{e\Lambda}\!/_{e\,\mathfrak{r}\Lambda}$ is simple.

(ii) We have a surjective Λ-linear map

$$\begin{array}{rcl} {}^{e\Lambda}\!/_{e\,\mathfrak{r}\Lambda} & \longrightarrow & {}^{\varepsilon e\Lambda}\!/_{\varepsilon e\,\mathfrak{r}\Lambda} \\ ex + e\,\mathfrak{r}\Lambda & \longmapsto & \varepsilon ex + \varepsilon e\,\mathfrak{r}\Lambda\ . \end{array}$$

It is an isomorphism if $\varepsilon e \neq 0$.

(iii) The $\varepsilon\Lambda$-module $\varepsilon e\Lambda/\varepsilon e\,\mathfrak{r}\Lambda$ is simple if $\varepsilon e \neq 0$, it is zero if $\varepsilon e = 0$.

Proof.

Ad (i). Since Λ is stable, $\bar{e}$ is a primitive idempotent of $\bar{\Lambda}$.

We remark first that we have the following isomorphism of Λ-modules.

$$\begin{array}{rcl} e\Lambda/e\pi\Lambda & \stackrel{\sim}{\longrightarrow} & \bar{e}\bar{\Lambda} \\ ex + e\pi\Lambda & \longmapsto & \bar{e}\bar{x} \\ ex + e\pi\Lambda & \longleftarrow\!\shortmid & \bar{e}\bar{x} \end{array}$$

Well-definedness in both directions follows from $e(x - \tilde{x}) \in e\pi\Lambda$ being equivalent to $e(x - \tilde{x}) \in \pi\Lambda$ for $x,\, \tilde{x} \in \Lambda$.

So we have the following isomorphisms of Λ-modules.

$$\begin{array}{rcl} e\Lambda/e\,\mathfrak{r}\Lambda \;\cong\; (e\Lambda/e\pi\Lambda)/(e\,\mathfrak{r}\Lambda/e\pi\Lambda) & \stackrel{\sim}{\longrightarrow} & \bar{e}\bar{\Lambda}/\bar{e}(\mathfrak{r}\Lambda/\pi\Lambda) \stackrel{\text{L.210}}{\cong} \bar{e}\bar{\Lambda}/\bar{e}\,\mathfrak{r}\bar{\Lambda} \\ (ex + e\pi\Lambda) + (e\,\mathfrak{r}\Lambda/e\pi\Lambda) & \longmapsto & \bar{e}\bar{x} + \bar{e}(\mathfrak{r}\Lambda/\pi\Lambda) \end{array}$$

By Proposition 205 the latter is simple as a module over $\bar{\Lambda}$, hence over Λ. Hence so is the former.

Ad (ii, iii). The Λ-linear map

$$\begin{array}{rcl} e\Lambda/e\,\mathfrak{r}\Lambda & \longrightarrow & \varepsilon e\Lambda/\varepsilon e\,\mathfrak{r}\Lambda \\ ex + e\,\mathfrak{r}\Lambda & \longmapsto & \varepsilon ex + \varepsilon e\,\mathfrak{r}\Lambda \end{array}$$

is well-defined since $e(x - \tilde{x}) \in e\,\mathfrak{r}\Lambda$ implies $\varepsilon e(x - \tilde{x}) \in \varepsilon e\,\mathfrak{r}\Lambda$ for $x,\, \tilde{x} \in \Lambda$.

By construction, this map is surjective. So with (i), the Λ-module $\varepsilon e\Lambda/\varepsilon e\,\mathfrak{r}\Lambda$ is simple or zero. By Lemma 187, it is nonzero if $\varepsilon e \neq 0$. It is zero if $\varepsilon e = 0$.

As a surjective Λ-linear map between simple modules, our map is an isomorphism if $\varepsilon e \neq 0$.

We have a surjective ring morphism $\Lambda \longrightarrow \varepsilon\Lambda$, $x \longmapsto \varepsilon x$. So also over $\varepsilon\Lambda$, the module $\varepsilon e\Lambda/\varepsilon e\,\mathfrak{r}\Lambda$ is simple or zero, for we know this fact for its restriction along $\Lambda \longrightarrow \varepsilon\Lambda$. □

Lemma 219 *Suppose given a simple Λ-module S.*

Then S is isomorphic to ${}^{e\Lambda}\!/_{e\,\mathfrak{r}\Lambda}$ for some primitive idempotent e of Λ.

Proof. Suppose given a simple Λ-module S.

We choose an epimorphism $\Lambda \stackrel{\varphi}{\longrightarrow} S$, $x \longmapsto xs$ for some $s \in S \setminus \{0\}$.

We have an orthogonal decomposition $1_\Lambda = \sum\limits_{i\in[1,n]} e_i$ into primitive idempotents, and therefore a decomposition $\Lambda = \bigoplus\limits_{i\in[1,n]} e_i\Lambda$ of Λ into indecomposable projective modules; cf. Remark 216.

We consider the restrictions $\varphi|_{e_i\Lambda} : e_i\Lambda \longrightarrow S$ of φ on the summands $e_i\Lambda$. We can find $i \in [1, n]$ such that the map $\varphi|_{e_i\Lambda}$ is not the zero map. Therefore it is an epimorphism, since S is simple.

Write $K := \operatorname{Ker}\varphi|_{e_i\Lambda}$. Consider the short exact sequence

$$K \longrightarrow e_i\Lambda \longrightarrow S\ .$$

By definition, $S\mathfrak{r}\Lambda = 0$. Thus, we get with $S \cong e_i\Lambda/K$ that $(e_i\Lambda/K)\mathfrak{r}\Lambda = 0$, i.e. $e_i\Lambda\mathfrak{r}\Lambda = e_i\mathfrak{r}\Lambda \subseteq K$. So altogether we have $e_i\mathfrak{r}\Lambda \subseteq K \subsetneq e_i\Lambda$, since $S \neq 0$.

With Lemma 218.(i), $e_i\Lambda/e_i\mathfrak{r}\Lambda$ is simple, i.e. $e_i\mathfrak{r}\Lambda \subseteq e_i\Lambda$ is maximal submodule. Thus, $e_i\mathfrak{r}\Lambda = K$ and so $S \cong e_i\Lambda/e_i\mathfrak{r}\Lambda$. □

Lemma 220 *Suppose Λ to be stable. Suppose given a finitely generated indecomposable projective Λ-module P. Suppose given an orthogonal decomposition $1_\Lambda = \sum\limits_{i\in[1,n]} e_i$ into primitive idempotents; cf. Remark 216.*

Then there exists $j \in [1, n]$ such that $P \cong e_j\Lambda$.

Proof. There exists $k \geq 0$ and a surjective Λ-linear map $\varphi : \Lambda^{\oplus k} \longrightarrow P$. We obtain an induced surjective Λ-linear map $\check{\varphi} : (\Lambda/\mathfrak{r}\Lambda)^{\oplus k} \longrightarrow P/\mathfrak{r}P$, $(x_i + \mathfrak{r}\Lambda)_{i\in[1,k]} \longmapsto (x_i)_{i\in[1,k]}\varphi + \mathfrak{r}P$.

Note that $\Lambda/\mathfrak{r}\Lambda$ is a $\bar{\Lambda}$-module by Lemma 209. By Lemma 218.(i), the module $e_i\Lambda/e_i\mathfrak{r}\Lambda$ is simple over Λ for $i \in [1, n]$, hence over $\bar{\Lambda}$.

The $\bar{\Lambda}$-module $(\Lambda/\mathfrak{r}\Lambda)^{\oplus k} \cong (\bigoplus\limits_{i\in[1,n]} e_i\Lambda/e_i\mathfrak{r}\Lambda)^{\oplus k}$ is a direct sum of certain simple submodules. So by Lemma 206, so is its epimorphic image $P/\mathfrak{r}P$. Hence by Lemma 219, we have that $P/\mathfrak{r}P$ is isomorphic to $\bigoplus\limits_{i\in[1,n]} (e_i\Lambda/e_i\mathfrak{r}\Lambda)^{\oplus\alpha_i}$ for some $\alpha_i \geq 0$. Then by Lemma 188 P is isomorphic to a sum of projective modules $\bigoplus\limits_{i\in[1,n]} (e_i\Lambda)^{\oplus\alpha_i}$. But P is indecomposable. Hence there exists $j \in [1, n]$ such that $\alpha_j = 1$ and $\alpha_i = 0$ for $i \in [1, n] \setminus \{j\}$, i.e. P is isomorphic to $e_j\Lambda$. □

Lemma 221 *Suppose Λ to be stable. Let ε be a central idempotent of $K\Lambda$.*

We have

$$\varepsilon(\mathfrak{r}\Lambda) = \mathfrak{r}(\varepsilon\Lambda)\,.$$

Proof.

Ad $\subseteq$. Suppose given a maximal right ideal $M \subseteq \varepsilon\Lambda$. By Lemma 181, we have to show that $\varepsilon(\mathfrak{r}\Lambda) \overset{!}{\subseteq} M$. The factor module $\varepsilon\Lambda/M$ is simple over $\varepsilon\Lambda$, hence over Λ. Thus it is annihilated by $\mathfrak{r}\Lambda$. So $\varepsilon(\mathfrak{r}\Lambda) = \varepsilon\Lambda \cdot \mathfrak{r}\Lambda \subseteq M$.

Ad $\supseteq$. Let e be a primitive idempotent of Λ. We *claim* that $e\,\mathfrak{r}(\varepsilon\Lambda) \overset{!}{\subseteq} \varepsilon(\mathfrak{r}\Lambda)$. With Lemma 218.(iii), the module $\varepsilon e\Lambda/\varepsilon e\,\mathfrak{r}\Lambda$ is simple or zero over $\varepsilon\Lambda$. In any case, it is annihilated by $\mathfrak{r}(\varepsilon\Lambda)$. Consequently,

$$e\,\mathfrak{r}(\varepsilon\Lambda) \;\subseteq\; \varepsilon e\Lambda \cdot \mathfrak{r}(\varepsilon\Lambda) \;\subseteq\; \varepsilon e\,\mathfrak{r}\Lambda \;\subseteq\; \varepsilon(\mathfrak{r}\Lambda)\ ,$$

which proves the *claim*.

Let $1_\Lambda = \sum\limits_{i=1}^{n} e_i$ be an orthogonal decomposition into idempotents. Suppose given $\xi \in \mathfrak{r}(\varepsilon\Lambda)$. We have to show that $\xi \overset{!}{\in} \varepsilon(\mathfrak{r}\Lambda)$. We get

$$\xi \;=\; 1_\Lambda \cdot \xi \;=\; \sum_{i\in[1,n]} e_i \cdot \xi \;\overset{\text{Claim}}{=}\; \varepsilon(\mathfrak{r}\Lambda)\ .$$

□

Proposition 222 *Suppose Λ to be stable.*

Let $1_{K\Lambda} = \sum_{i\in[1,\ell]} \varepsilon_i$ be an orthogonal decomposition into central idempotents of $K\Lambda$. Then

$$\mathfrak{r}\Lambda \;=\; \Lambda \,\cap\, \bigoplus_{i\in[1,\ell]} \mathfrak{r}(\varepsilon_i\Lambda)$$

as R-submodules of $K\Lambda$.

Proof.

Ad $\subseteq$. We have $\mathfrak{r}\Lambda \subseteq \bigoplus_{i\in[1,\ell]} \varepsilon_i\mathfrak{r}(\Lambda) \overset{\text{L.221}}{=} \bigoplus_{i\in[1,\ell]} \mathfrak{r}(\varepsilon_i\Lambda)$.

Ad $\supseteq$. By Lemma 213.(ii), we have to show that there exists $k \geq 1$ such that

$$\Big(\Lambda \,\cap\, \bigoplus_{i\in[1,\ell]} \mathfrak{r}(\varepsilon_i\Lambda)\Big)^k \overset{!}{\subseteq} \pi\Lambda\ .$$

Consider the subrings $\Lambda \subseteq \bigoplus_{i\in[1,\ell]} \varepsilon_i\Lambda \subseteq K\Lambda$. Consider the ideal $\bigoplus_{i\in[1,\ell]} \mathfrak{r}(\varepsilon_i\Lambda)$ in $\bigoplus_{i\in[1,\ell]} \varepsilon_i\Lambda$.

It suffices to show that there exists $k \geq 1$ such that

$$\Big(\bigoplus_{i\in[1,\ell]} \mathfrak{r}(\varepsilon_i\Lambda)\Big)^k \overset{!}{\subseteq} \pi\Lambda\ .$$

By Lemma 213.(i), we may choose $s \geq 1$ such that $\mathfrak{r}(\varepsilon_i\Lambda)^s \subseteq \pi\varepsilon_i\Lambda$ for $i \in [1,\ell]$.

Choose $t \geq 1$ such that $\pi^t\varepsilon_i \in \pi\Lambda$ for $i \in [1,\ell]$.

Let $k := st$. Note that

$$\prod_{j\in[1,k]} \Big(\sum_{i\in[1,\ell]} \varepsilon_i x_{ij}\Big) \;=\; \sum_{i\in[1,\ell]} \prod_{j\in[1,k]} \varepsilon_i x_{ij}$$

for $x_{ij} \in K\Lambda$. Hence

$$\Big(\bigoplus_{i\in[1,\ell]} \mathfrak{r}(\varepsilon_i\Lambda)\Big)^k \;=\; \bigoplus_{i\in[1,\ell]} \big(\mathfrak{r}(\varepsilon_i\Lambda)\big)^k \;=\; \bigoplus_{i\in[1,\ell]} \Big(\big(\mathfrak{r}(\varepsilon_i\Lambda)\big)^s\Big)^t \;\subseteq\; \bigoplus_{i\in[1,\ell]} \big(\pi\varepsilon_i\Lambda\big)^t \;=\; \bigoplus_{i\in[1,\ell]} \pi^t\varepsilon_i\Lambda \;\subseteq\; \pi\Lambda\ .$$

□

Example 223 Suppose given $s \geq 1$. Let

$$\Lambda \;:=\; \{\,(a,b) \in R\times R \mid a \equiv_{\pi^s} b\,\} \subseteq R\times R\ .$$

We have an orthogonal decomposition $1_{K\Lambda} = (1,0) + (0,1)$ into central idempotents $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$ of $K\Lambda = K\times K$. Then $\varepsilon_1\Lambda = R\times 0$, which is a subring of $K\Lambda$ and which has $\mathfrak{r}\varepsilon_1\Lambda = (\pi)\times 0$. Likewise, $\mathfrak{r}\varepsilon_2\Lambda = 0\times(\pi)$. So Proposition 222 gives

$$\mathfrak{r}\Lambda \;=\; \Lambda\cap(\mathfrak{r}\varepsilon_1\Lambda \oplus \mathfrak{r}\varepsilon_2\Lambda) \;=\; \Lambda\cap\big(((\pi)\times 0)\oplus(0\times(\pi))\big) \;=\; \{\,(a,b)\in R\times R \mid a \equiv_{\pi^s} b \equiv_\pi 0\,\}\ .$$

Proposition 224 *Suppose Λ to be stable. Let ε be a central idempotent of $K\Lambda$. Suppose given primitive idempotents e and f of Λ.*

We may form the $\varepsilon\Lambda$-submodules $\varepsilon e\Lambda$ and $\varepsilon f\Lambda$ of $K\Lambda$.

Suppose that $\varepsilon e\Lambda$ and $\varepsilon f\Lambda$ are nonzero $\varepsilon\Lambda$-modules isomorphic to each other.

Then $e\Lambda$ and $f\Lambda$ are isomorphic Λ-modules.

Proof. We write $P := e\Lambda$ and $Q := f\Lambda$. With Lemma 218.(i), $P/\mathfrak{r}P$ and $Q/\mathfrak{r}Q$ are simple.

Note that $\varepsilon e\Lambda = \varepsilon e\cdot\varepsilon\Lambda$ is a projective $\varepsilon\Lambda$-module, generated by the idempotent εe of $\varepsilon\Lambda$. So $\varepsilon e\Lambda \cong \varepsilon f\Lambda$ entails

$$\varepsilon e\Lambda/\mathfrak{r}(\varepsilon e\Lambda) \;\cong\; \varepsilon f\Lambda/\mathfrak{r}(\varepsilon f\Lambda)\ .$$

We obtain the following isomorphism of Λ-modules.

$$\begin{array}{rcl} P/\mathfrak{r}P & = & e\Lambda/e\,\mathfrak{r}\Lambda \\ & \underset{\text{L.218.(ii)}}{\xrightarrow{\sim}} & \varepsilon e\Lambda/\varepsilon e\,\mathfrak{r}\Lambda \\ & = & \varepsilon e\Lambda/e\,\varepsilon(\mathfrak{r}\Lambda) \\ & \overset{\text{L. 221}}{=} & \varepsilon e\Lambda/e\,\mathfrak{r}(\varepsilon\Lambda) \\ & = & \varepsilon e\cdot\varepsilon\Lambda/\varepsilon e\cdot\mathfrak{r}(\varepsilon\Lambda) \\ & = & \varepsilon e\cdot\varepsilon\Lambda/\mathfrak{r}(\varepsilon e\cdot\varepsilon\Lambda) \\ & = & \varepsilon e\Lambda/\mathfrak{r}(\varepsilon e\Lambda) \end{array}$$

Altogether, we get $P/\mathfrak{r}P \cong Q/\mathfrak{r}Q$. By Lemma 188, this implies $P \cong Q$. □

Proposition 225 *Suppose Λ to be stable.*

Write $[X]$ for the isoclass of a Λ-module X.

Write IsoP *for the set of isoclasses of the finitely generated indecomposable projective Λ-modules.*

Write IsoS *for the set of isoclasses of the simple Λ-modules.*

We have the bijection

$$\begin{array}{rcl} \mathrm{IsoP} & \xrightarrow{\mathrm{red}} & \mathrm{IsoS} \\ [P] & \longmapsto & \left[{}^{P}\!/\!{}_{\mathfrak{r}P}\right] . \end{array}$$

Proof. The map red is well-defined, since $P \cong e\Lambda$ for some primitive idempotent e of Λ by Lemma 220, and therefore $P/\mathfrak{r}P \cong e\Lambda/e\,\mathfrak{r}\Lambda$ is simple with Lemma 218.(i).

With Lemma 188, red is injective.

Surjectivity of red follows from Lemma 219. □

Appendix D

Heller's Lemma

Let R be a discrete valuation ring with maximal ideal (π). Let $K := \operatorname{frac} R$ be its field of fractions.

Tensor products over R are denoted by $\otimes := \otimes_R$.

Given an R-module X, we denote by $\bar{X} := X/\pi X$ its reduction modulo π, which is a module over $\bar{R}$.

D.1 Preparations

Let Λ be an R-order.

Reduction modulo π yields the $\bar{R}$-algebra $\bar{\Lambda}$. Given a Λ-module Y, this yields a $\bar{\Lambda}$-module $\bar{Y}$.

Lemma 226 *Let P be a projective Λ-lattice. Then $\bar{P}$ is finitely generated projective over $\bar{\Lambda}$.*

Proof. There exists a Λ-module Q such that $P \oplus Q \cong \Lambda^{\oplus k}$ for some $k \geq 0$. Hence $\bar{P} \oplus \bar{Q} \cong \bar{\Lambda}^{\oplus k}$. So $\bar{P}$ is projective over $\bar{\Lambda}$. $\square$

Lemma 227 *Let P be a projective Λ-lattice. Let X be a Λ-lattice. Consider the R-linear map*

$$\begin{array}{ccc} \overline{\operatorname{Hom}_\Lambda(P, X)} & \xrightarrow{\rho} & \operatorname{Hom}_{\bar{\Lambda}}(\bar{P}, \bar{X}) \\ f + \pi \operatorname{Hom}_\Lambda(P, X) & \longmapsto & \bar{f} \end{array} ,$$

where $f : P \longrightarrow X$ is a Λ-linear map and $\bar{f} : \bar{P} \longrightarrow \bar{X}$ its induced $\bar{\Lambda}$-linear map.

The map ρ is an R-linear isomorphism.

Proof. For both sides are additive in P, without loss of generality, we can assume that $P = \Lambda^{\oplus k}$ for some $k \geq 0$, and further we can assume that $k = 1$. So $P = \Lambda$.
Then we have the commutative triangle

$$\begin{tikzcd}
(\bar\lambda \mapsto \bar x \bar\lambda) & & \operatorname{Hom}_{\bar\Lambda}(\bar\Lambda, \bar X) & & \overline{\operatorname{Hom}_\Lambda(\Lambda, X)} \arrow[ll, "\rho"'] & & (\lambda \mapsto x\lambda) + \pi \operatorname{Hom}_\Lambda(\Lambda, X) \arrow[llllll, bend right, mapsto] \\
& & \bar x \arrow[ull, mapsto] & \bar X \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \bar x := x + \pi X \arrow[urr, mapsto] & &
\end{tikzcd} ,$$

where the right hand side map is induced by $X \xrightarrow{\sim} \operatorname{Hom}_\Lambda(\Lambda, X)$, $x \longmapsto (\lambda \mapsto x\lambda)$.

Therefore, ρ has to be an isomorphism. $\square$

Lemma 228 *Let P be a projective Λ-lattice. Then the R-linear isomorphism*

$$\begin{array}{ccc} \overline{\mathrm{End}_\Lambda(P)} & \xrightarrow[\sim]{\rho} & \mathrm{End}_{\bar\Lambda}(\bar P) \\ f + \pi\,\mathrm{End}_\Lambda(P) & \longmapsto & \bar f \end{array}$$

from Lemma 227 is an isomorphism of $\bar R$-algebras.

D.2 Heller's Lemma on R-orders

Suppose given $\mu \geq 1$ and $n_i \geq 1$ for $i \in [1,\mu]$. Let $\Gamma := \prod_{i\in[1,\mu]} R^{n_i\times n_i} = \prod_i R^{n_i\times n_i}$.

Suppose $\Lambda \subseteq \Gamma$ to be an R-suborder such that ${}^{\Gamma}/_{\Lambda}$ is of finite length as an R-module, i.e. such that $K\Lambda = K\Gamma$, i.e. such that $\mathrm{rk}_R\,\Lambda = \mathrm{rk}_R\,\Gamma$.

Reduction modulo π yields the $\bar R$-algebra $\bar\Lambda$. Given a Λ-module Y, this yields a $\bar\Lambda$-module $\bar Y$.

Given a finitely generated R-module X, we denote by $\hat X := \hat R \otimes_R X$ its completion, which is a module over $\hat R$. We obtain the $\hat R$-order $\hat\Lambda$. Given a Λ-lattice Y, this yields a $\hat\Lambda$-lattice $\hat Y$. Cf. Lemma 165.(v).

Notation 229 We denote $K\Lambda := K \otimes_R \Lambda$, and further $\hat K\Lambda := \hat K \otimes_R \Lambda = \hat K \otimes_{\hat R} \hat R \otimes_R \Lambda = \hat K \otimes_{\hat R} \hat\Lambda$.

Note that

$$\hat K \otimes_K K\Lambda = \hat K \otimes_K K \otimes_R \Lambda = \hat K \otimes_R \Lambda = \hat K\Lambda.$$

So given a finitely generated $K\Lambda$-module X, we obtain a finitely generated $\underbrace{\hat K \otimes_K K\Lambda}_{\hat K\Lambda}$-module $\hat K \otimes_K X$.

Lemma 230 *Let $K := \mathrm{frac}\,R$ and $\hat K := \mathrm{frac}\,\hat R$, so that $K \subseteq \hat K$.*

Suppose given a finitely generated $\hat K\Lambda$-module M.

Then there is a finitely generated $K\Lambda$-module N and an isomorphism of $\hat K\Lambda$-modules $\hat K \otimes_K N \xrightarrow{\sim} M$.

Proof. We have the short exact sequence

$$\Lambda \hookrightarrow \prod_i R^{n_i\times n_i} \longrightarrow C\,.$$

The functor $(\hat K \otimes_R -) = (\hat K \otimes_K -)\circ(K \otimes_R -)$ is exact, so that we get the short exact sequence

$$\underbrace{\hat K \otimes_R \Lambda}_{=\hat K\Lambda} \hookrightarrow \underbrace{\hat K \otimes_R \prod_i R^{n_i\times n_i}}_{\cong \prod_i \hat K^{n_i\times n_i}} \longrightarrow \underbrace{\hat K \otimes_R C}_{\substack{\cong\, 0\,, \\ \text{for } C \text{ is f.g. torsion}}}\,.$$

Hence $\hat K\Lambda = \prod_i \hat K^{n_i\times n_i}$. Thus, the module M is isomorphic to a finite sum of rows of $\prod_i K^{n_i\times n_i}$; so that, without loss of generality, M is the right ideal generated by a primitive main diagonal idempotent $\varepsilon \in \prod_i K^{n_i\times n_i} \subseteq \prod_i \hat K^{n_i\times n_i}$,

$$\begin{aligned} M &= (\prod_i \hat K^{n_i\times n_i})\varepsilon \\ &= (\hat K \otimes_K \prod_i K^{n_i\times n_i})(1\otimes\varepsilon) \\ &= \hat K \otimes_K \underbrace{\big((\prod_i K^{n_i\times n_i})\varepsilon\big)}_{=:\,N}\,. \end{aligned}$$

$\square$

Lemma 231 (Heller, cf. **[4,** Prop. 2.5**]**)

Let U be a $\hat{\Lambda}$-lattice.

Then there is a Λ-lattice X and an isomorphism $\hat{X} \xrightarrow{\sim} U$ of $\hat{\Lambda}$-lattices.

Proof. The tensor product $\hat{K} \otimes_{\hat{R}} U$ is a finitely generated module over $\hat{K} \otimes_{\hat{R}} \hat{\Lambda} = \hat{K}\Lambda$.

Then, using Lemma 230 there is a $K\Lambda$-module V and an isomorphism

$$\hat{K} \otimes_{\hat{R}} U \underset{\sim}{\xrightarrow{\psi}} \hat{K} \otimes_K V$$

of $\hat{K}\Lambda$-modules.

We choose an $\hat{R}$-linear basis $(u_1, \ldots, u_n)$ of U, with

$$n \;:=\; \mathrm{rk}_{\hat{R}}\, U \;=\; \dim_{\hat{K}} \hat{K} \otimes_{\hat{R}} U \;=\; \dim_{\hat{K}} \hat{K} \otimes_K V \;=\; \dim_K V\ .$$

We choose a K-linear basis $(v_1, \ldots, v_n)$ of V.

We further define the injective maps α and β

$$\begin{tikzcd}
1 \otimes u & \hat{K} \otimes_{\hat{R}} U \arrow[r, "\psi", "\sim"'] & \hat{K} \otimes_K V & 1 \otimes v \\
u \arrow[u, mapsto] & U \arrow[u, "\alpha"] & V \arrow[u, "\beta"'] & v\ . \arrow[u, mapsto]
\end{tikzcd}$$

Let $\tilde{v}_i := (1 \otimes v_i)\psi^{-1} = v_i\beta\psi^{-1}$ for $i \in [1, n]$. Then $(\tilde{v}_1, \ldots, \tilde{v}_n)$ is a $\hat{K}$-linear basis of $\hat{K} \otimes_{\hat{R}} U$.

Let $\tilde{u}_i := (1 \otimes u_i) = u_i\alpha$ for $i \in [1, n]$. Then $(\tilde{u}_1, \ldots, \tilde{u}_n)$ is a $\hat{K}$-linear basis of $\hat{K} \otimes_{\hat{R}} U$.

Let

$$X \;:=\; U\alpha \cap V\beta\psi^{-1}\ .$$

Let $A = (a_{ij})_{i,j} \in \mathrm{GL}_n(\hat{K})$ be such that $\tilde{v}_i = \sum_{j=1}^{n} a_{ij}\tilde{u}_j$.

Further, let $N > \max_{i,j}(\text{-}\,\mathrm{val}_\pi(a_{ij}))$, i.e. $\pi^N a_{i,j} \in \pi\hat{R}$ for all $i,\ j\ \in\ [1, n]$.

Denote $A_0 := \pi^{N-1}A \in \hat{R}^{n\times n}$.

Using Lemma 165.*(ix)*, there is a $B \in K^{n\times n}$ with $B \equiv_{\pi^N} A^{-1}$; i.e., writing $B =: (b_{ij})_{i,j}$ and $A^{-1} =: (\tilde{a}_{ij})_{i,j}$, with $b_{ij} - \tilde{a}_{ij} \in \pi^N\hat{R}$ for all $i, j \in [1, n]$.

Then $BA_0 \equiv_{\pi^N} A^{-1}A_0$, and therefore $BA = B(\pi^{1-N}A_0) \equiv_\pi A^{-1}(\pi^{1-N}A_0) = A^{-1}A = I_n$.

Thus, $BA = I_n + \pi C$ for some $C \in \hat{R}^{n\times n}$, so $BA \in \mathrm{GL}_n(\hat{R})$, by considering the determinant.

In particular, $\det(B) \neq 0$, so that $B \in \mathrm{GL}_n(K)$.

Let $\tilde{x}_i \;:=\; \sum_{j,k} b_{ij}a_{jk}\tilde{u}_k \;=\; \sum_j b_{ij}\tilde{v}_j$ for $i \in [1, n]$. Note that $\tilde{x}_i = \sum_k \big(\sum_j b_{ij}a_{jk}\big)\tilde{u}_k$ with the matrix $\big(\sum_j b_{ij}a_{jk}\big)_{i,k} = BA \in \mathrm{GL}_n(\hat{R})$.

We now show that

$${}_R\langle \tilde{x}_1, \ldots, \tilde{x}_n\rangle \;\overset{!}{=}\; X\ .$$

Since $(\tilde{v}_1, \ldots, \tilde{v}_n)$ is a K-linear basis of $V\beta\psi^{-1}$ and since $B \in \mathrm{GL}_n(K)$, the tuple $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is a K-linear basis of $V\beta\psi^{-1}$.

Since $(\tilde{u}_1, \ldots, \tilde{u}_n)$ is an $\hat{R}$-linear basis of $U\alpha$ and since $BA \in \mathrm{GL}_n(\hat{R})$, the tuple $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is an $\hat{R}$-linear basis of $U\alpha$.

" $\subseteq$ ":

Since $R \subseteq \hat{R} \cap K$, we have ${}_R\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle \subseteq U\alpha \cap V\beta\psi^{-1} = X$.

" $\supseteq$ ":

Suppose given $\xi \in X = U\alpha \cap V\beta\psi^{-1}$. We have to show that $\xi \overset{!}{\in} {}_R\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle$.

Because $\xi \in V\beta\psi^{-1}$, we can write $\xi = \sum\limits_i k_i \tilde{x}_i$, for some $k_i \in K$.

Because $\xi \in U\alpha$, we can write $\xi = \sum\limits_i \hat{r}_i \tilde{x}_i$, for some $\hat{r}_i \in \hat{R}$.

The tuple $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is linearly independent over $\hat{R}$ and over K. Thus, $\hat{r}_i = k_i \in \hat{R} \cap K = R$ for $i \in [1, n]$; cf. Lemma 165.*(viii)*. We conclude that $\xi = \sum\limits_i k_i \tilde{x}_i \in {}_R\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle$.

Hence equality is shown, and $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is an R-linear basis of X.

Since $X = U\alpha \cap V\beta\psi^{-1}$, it is closed under multiplication with $\hat{\Lambda} \cap K\Lambda$ in $\hat{K} \otimes_{\hat{R}} U$. Since $\Lambda \subseteq \hat{\Lambda} \cap K\Lambda$, the abelian subgroup X is a Λ-linear submodule of $\hat{K} \otimes_{\hat{R}} U$.

Having a finite R-linear basis, X is thus shown to be a Λ-lattice.

By definition, $\hat{X} = \hat{R} \otimes_R X$ as $\hat{\Lambda}$-modules. Via α, the $\hat{\Lambda}$-lattices U and $U\alpha$ are isomorphic.

So we have to show that $\hat{R} \otimes_R X$ and $U\alpha$ are isomorphic.

The Λ-linear inclusion map

$$\begin{array}{ccc} X & \hookrightarrow & U\alpha \\ \tilde{x}_i & \longmapsto & \tilde{x}_i \end{array}$$

induces the $\hat{\Lambda}$-linear map

$$\begin{array}{ccc} \hat{R} \otimes_R X & \longrightarrow & U\alpha \\ 1 \otimes \tilde{x}_i & \longmapsto & \tilde{x}_i \end{array}$$

sending an $\hat{R}$-linear basis to an $\hat{R}$-linear basis. Hence it is a $\hat{\Lambda}$-linear bijection $\hat{R} \otimes_R X \xrightarrow{\sim} U\alpha$. $\square$

Lemma 232 *Let P be an indecomposable finitely generated projective Λ-module, i.e. an indecomposable projective Λ-lattice.*

Then $\hat{P}$ is indecomposable projective over $\hat{\Lambda}$.

Moreover, $\bar{P}$ is indecomposable projective over $\bar{\Lambda}$.

In the language of Definition 207, the latter property means that Λ is stable; cf. Lemma 220.

Proof. Assume known that $\hat{P}$ is indecomposable over $\hat{\Lambda}$. Then there are no nontrivial idempotents in $\mathrm{End}_{\hat{\Lambda}}(\hat{P})$. Using Lemma 166 we know that then there are no nontrivial idempotents in $\overline{\mathrm{End}_{\hat{\Lambda}}(\hat{P})} \cong \mathrm{End}_{\bar{\hat{\Lambda}}}(\bar{\hat{P}}) \cong \mathrm{End}_{\bar{\Lambda}}(\bar{P})$; cf. Lemma 228, Lemma 165.(vi). Therefore, once we have shown that $\hat{P}$ is indecomposable over $\hat{\Lambda}$, we are done.

Let P be indecomposable over Λ. We *assume* that $\hat{P}$ is decomposable over $\hat{\Lambda}$, so that there exists a decomposition $\hat{P} \cong U \oplus V$ into $\hat{\Lambda}$-lattices U and V such that $U \neq 0 \neq V$. With Heller's Lemma 231 we know that we can find Λ-lattices X and Y in such a way that $U \cong \hat{X}$ and $V \cong \hat{Y}$ as $\hat{\Lambda}$-lattices. In particular, $X \neq 0 \neq Y$. So we have a decomposition $\hat{P} \cong \hat{X} \oplus \hat{Y}$. We then have

$$\bar{P} \;\cong\; \bar{\hat{P}} \;\cong\; \overline{\hat{X} \oplus \hat{Y}} \;\cong\; \bar{\hat{X}} \oplus \bar{\hat{Y}} \;\cong\; \bar{X} \oplus \bar{Y} \;\cong\; \overline{X \oplus Y}\,;$$

cf. Lemma 165.(v). Therefore Lemma 214 yields $P \cong X \oplus Y$, which is a *contradiction* to P being indecomposable. Therefore, $\hat{P}$ must be indecomposable over $\hat{\Lambda}$. $\square$

Lemma 233 *Let P be an indecomposable finitely generated projective Λ-module. Then $\mathrm{End}_\Lambda(P)$ is local.*

Proof. To prove this lemma, we will proceed as follows.

P indecomposable over $\Lambda \stackrel{(1)}{\Longrightarrow} \bar{P}$ indecomposable over $\bar{\Lambda} \stackrel{(2)}{\Longrightarrow} \mathrm{End}_{\bar{\Lambda}}(\bar{P})$ local $\stackrel{(3)}{\Longrightarrow} \mathrm{End}_\Lambda(P)$ local

$\stackrel{(1)}{\Longrightarrow}$: This is Lemma 232.

$\stackrel{(2)}{\Longrightarrow}$: Note that $\bar{P}$ is indecomposable and finite-dimensional over $\bar{R}$. Then, (2) follows by Lemma 198.

$\stackrel{(3)}{\Longrightarrow}$: Note that by Lemma 228 and Lemma 209, $\mathrm{End}_{\bar{\Lambda}}(\bar{P}) = \overline{\mathrm{End}_\Lambda(P)}$ and $\pi \,\mathrm{End}_\Lambda(P) \subseteq \mathfrak{r}(\mathrm{End}_\Lambda(P))$. Let ρ be the residue class map.
Then $\mathfrak{r}(\mathrm{End}_\Lambda(P)) = \rho^{-1}(\mathfrak{r}(\overline{\mathrm{End}_\Lambda(P)}))$.

□

Appendix E

The Cartan matrix

E.1 Multiplicities of indecomposable summands in a tensor product of indecomposable projectives

Let p be a prime. Let $n \geq 1$. Write $\Lambda = \mathbb{Z}_{(p)}\mathrm{S}_n$.

Let e, f be idempotents of Λ.

Suppose given an orthogonal decomposition $1_\Lambda = \sum\limits_{i=1}^{k} e_i$ into primitive idempotents.

We aim to determine the multiplicity of $e_i\Lambda$ as a summand of $e\Lambda \otimes f\Lambda$. There are (at least) two ways to do this via Magma **[3]**. On the one hand, we may directly apply `DirectSumDecomposition` to $\bar{e}\bar{\Lambda} \otimes \bar{f}\bar{\Lambda}$. On the other hand, we may use `AHom` to calculate $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\bar{\Lambda}}(\bar{e}_j\bar{\Lambda}, \bar{e}\bar{\Lambda} \otimes \bar{f}\bar{\Lambda})$; then the result may be derived using the Cartan matrix. The latter way is a bit faster. For example, in the case $n = 5$, $p = 2$, $\mathrm{rk}_{\mathbb{Z}_{(p)}} e\Lambda = \mathrm{rk}_{\mathbb{Z}_{(p)}} f\Lambda = 24$, the former way takes 39 sec, the latter way takes 1.5 sec.

Definition 234 Let the **Cartan matrix** be

$$\begin{array}{rcl} C_{\mathbb{Z}_{(p)}\mathrm{S}_n} & := & \big(\mathrm{rk}_{\mathbb{Z}_{(p)}} \mathrm{Hom}_\Lambda(e_i\Lambda, e_j\Lambda)\big)_{i,j} \\ & = & \big(\mathrm{rk}_{\mathbb{Z}_{(p)}} e_j\Lambda e_i\big)_{i,j} \\ & = & \big(\dim_{\mathbb{F}_p} \bar{e}_j\bar{\Lambda}\bar{e}_i\big)_{i,j} \\ & = & \big(\dim_{\mathbb{F}_p} \mathrm{Hom}_{\bar{\Lambda}}(\bar{e}_i\bar{\Lambda}, \bar{e}_j\bar{\Lambda})\big)_{i,j} \ \in \ \mathbb{Z}^{k\times k} \ . \end{array}$$

It is well-known that the Cartan matrix of $\mathbb{Z}_p\mathrm{S}_n$ is symmetric and regular **[11,** §16.1, Th. 35, Cor. 3, p. 132**]**. We shall verify that the Cartan matrices of $\mathbb{Z}_p\mathrm{S}_n$ and of $\mathbb{Z}_{(p)}\mathrm{S}_n$ coincide to get the same assertion for the latter.

Lemma 235 *The Cartan matrix of* $\mathbb{Z}_{(p)}\mathrm{S}_n$ *is symmetric and regular.*

Proof. Let $1_{\mathbb{Z}_{(p)}\mathrm{S}_n} = e_1 + \ldots + e_k$ be an orthogonal decomposition into primitive idempotents. Note that e_i remains primitive in $\mathbb{Z}_p\mathrm{S}_n$ by Lemma 232; cf. beginning of Section D.2. Hence $1_{\mathbb{Z}_p\mathrm{S}_n} = e_1 + \ldots + e_k$ remains an orthogonal decomposition into primitive idempotents.

Given i, $j \in [1, n]$, we have

$$\begin{array}{rcl} \mathrm{rk}_{\mathbb{Z}_{(p)}} e_i\mathbb{Z}_{(p)}\mathrm{S}_n e_j & = & \mathrm{rk}_{\mathbb{Z}_p} \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} (e_i\mathbb{Z}_{(p)}\mathrm{S}_n e_j) \\ & = & \mathrm{rk}_{\mathbb{Z}_p} e_i(\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}\mathrm{S}_n)e_j \\ & = & \mathrm{rk}_{\mathbb{Z}_p} e_i\mathbb{Z}_p\mathrm{S}_n e_j \ . \end{array}$$

So the Cartan matrices of $\mathbb{Z}_{(p)}\mathrm{S}_n$ and $\mathbb{Z}_p\mathrm{S}_n$ coincide. The latter is symmetric and regular by **[11,** §16.1, Th. 35, Cor. 3, p. 132**]**; cf. **[5,** Th. 4.12**]**. □

The tensor product $e\Lambda$ and $f\Lambda$ is, as tensor product of two projective modules, isomorphic to a direct sum of indecomposable projective modules. Using Lemma 220 and Remark 208, we can find

$$e\Lambda \otimes f\Lambda \;\cong\; \bigoplus_{i\in[1,k]} e_i\Lambda^{\oplus a_i}\;.$$

Certainly,

$$\begin{array}{rcl}
(\dim_{\mathbb{F}_p}\mathrm{Hom}_{\bar\Lambda}(\bar e_j\bar\Lambda,\bar e\bar\Lambda\otimes\bar f\bar\Lambda))_j & = & (\dim_{\mathbb{F}_p}(\bar e\bar\Lambda\otimes\bar f\bar\Lambda)\bar e_j)_j \\
& = & (\dim_{\mathbb{F}_p}\overline{(e\Lambda\otimes f\Lambda)e_j})_j \\
& = & (\mathrm{rk}_{\mathbb{Z}_{(p)}}(e\Lambda\otimes f\Lambda)e_j)_j \\
& = & (\mathrm{rk}_{\mathbb{Z}_{(p)}}\mathrm{Hom}_\Lambda(e_j\Lambda, e\Lambda\otimes f\Lambda))_j \\
& = & (\mathrm{rk}_{\mathbb{Z}_{(p)}}\mathrm{Hom}_\Lambda(e_j\Lambda, \bigoplus_{i\in[1,k]} e_i\Lambda^{\oplus a_i}))_j \\
& = & \big(\sum_{i=1}^{k}\mathrm{rk}_{\mathbb{Z}_{(p)}}\mathrm{Hom}_\Lambda(e_j\Lambda,e_i\Lambda)\cdot a_i\big)_j \quad = \quad C_{\mathbb{Z}_{(p)}\mathrm{S}_n}\cdot\begin{pmatrix} a_1\\ \vdots\\ a_k\end{pmatrix}.
\end{array}$$

So knowing the dimension vector $(\dim_{\mathbb{F}_p}\mathrm{Hom}_{\bar\Lambda}(\bar e_j\bar\Lambda,\bar e\bar\Lambda\otimes\bar f\bar\Lambda))_j$ and the Cartan matrix, we are able to find the multiplicities a_i of the modules $e_i\Lambda$ in the direct sum decomposition by solving a linear equation system; cf. Lemma 235.

Example 236 Applying this to $\Lambda=\mathbb{Z}_{(3)}\mathrm{S}_3$, we have the orthogonal decomposition into primitive idempotents, given by e_1 and e_2 from Definition 21, and the projective modules $P_1 := e_1\Lambda$ and $P_2 := e_2\Lambda$. Note that the ring denoted by $\Lambda_{(3)}$ in Definition 21 is, up to isomorphism, denoted Λ here.

The Cartan matrix can be found by counting the elements of the bases given in Remark 29. Write $\mathrm{rk}=\mathrm{rk}_{\mathbb{Z}_{(3)}}$. We get

$$\begin{array}{rcl}
\mathrm{rk}\,\mathrm{Hom}_\Lambda(P_1,P_1) & = & 2\\
\mathrm{rk}\,\mathrm{Hom}_\Lambda(P_2,P_2) & = & 2\\
\mathrm{rk}\,\mathrm{Hom}_\Lambda(P_1,P_2) & = & 1\\
\mathrm{rk}\,\mathrm{Hom}_\Lambda(P_2,P_1) & = & 1\,,
\end{array}$$

and therefore the Cartan matrix is

$$C_{\mathbb{Z}_{(3)}\mathrm{S}_3}=\begin{pmatrix}2&1\\1&2\end{pmatrix}.$$

Consider the example of $P_1\otimes P_1$. Knowing the vector of ranks $(\dim_{\mathbb{F}_3}\mathrm{Hom}_{\bar\Lambda}(\bar e_j\bar\Lambda,\bar P_1\otimes\bar P_1))_j=\binom{4}{5}$ and the Cartan matrix, we are able to find the multiplicities a_i of P_1 and P_2 in the direct sum decomposition as follows.

$$\begin{pmatrix}a_1\\a_2\end{pmatrix} = C^{-1}_{\mathbb{Z}_{(3)}\mathrm{S}_3}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

Example 237 We show how our Magma **[3]** routine works in the example of $\mathbb{Z}_{(3)}\mathrm{S}_3$ and $P_1\otimes P_2$, keeping the notation of Example 236.

First, we need to define $\bar P_1$ = `PP1` and $\bar P_2$ = `PP2` via Magma. We use the representations `rhoP1` and `rhoP2` which can be found in the file `main_S3_loc3` as input for `GModule`.

The next step is to find the Cartan matrix. For that, we use `AHom` to generate the spaces of homomorphisms $\mathrm{Hom}_{\mathbb{F}_3\mathrm{S}_3}(\bar P_i,\bar P_j)$.

The last thing we need is $\dim_{\mathbb{F}_3} \mathrm{Hom}_{\mathbb{F}_3 \mathrm{S}_3}(\bar{e}_j \bar{\Lambda}, \bar{P}_1 \otimes \bar{P}_2)$ for $j \in \{1, 2\}$, which we again get via `AHom`. The resulting dimension vector is called `d`.

All we have to do now to find the vector a containing the multiplicities of P_1 and P_2 in the direct sum decomposition is to solve the linear equation system $C \cdot a = d$.

```
load main_S3_loc3;

//the finite field $\mathbb{F}_3$ :
F := GF(3);
//the symmetric group $\mathrm{S}_3$ :
G := SymmetricGroup(3);

//$\bar{P}_1$ in a Magma-compatible form, using the generators $(1,2,3)$ and $(1,2)$ :
PP1 := GModule(G, [MatrixRing(F,3)!rhoP1(S3P!(1,2,3)), MatrixRing(F,3)!rhoP1(S3P!(1,2))]);
//$\bar{P}_2$ in a Magma-compatible form, using the generators $(1,2,3)$ and $(1,2)$ :
PP2 := GModule(G, [MatrixRing(F,3)!rhoP2(S3P!(1,2,3)), MatrixRing(F,3)!rhoP2(S3P!(1,2))]);

proj := [PP1,PP2];

//generating the Cartan matrix :
C := MatrixRing(Integers(),#proj)![Dimension(AHom(proj[i],proj[j])):j,i in [1..#proj]];

//generating the tensor product of PP1 and PP2 :
PP1oPP2 := TensorProduct(PP1,PP2);

//$\dim \mathrm{Hom}(\bar{P}_1, \bar{P}_1 \otimes \bar{P}_2)$ :
dim1 := Dimension(AHom(PP1,PP1oPP2));
//$\dim \mathrm{Hom}(\bar{P}_2, \bar{P}_1 \otimes \bar{P}_2)$ :
dim2 := Dimension(AHom(PP2,PP1oPP2));

d := RMatrixSpace(Integers(),1,#proj)![dim1,dim2];

//we find a vector $a$ with $C \cdot a = d$ :
a := Solution(C,d);
```

Having followed the steps above, a contains the multiplicities of P_1 and P_2 in the direct sum decomposition of $P_1 \otimes P_2$. And indeed, we get

```
> a;
[2 1]
```

meaning that $P_1 \otimes P_2 \cong P_1^{\oplus 2} \oplus P_2^{\oplus 1}$, in accordance with Lemma 35.

E.2 Rational identification of projectives

Let p be a prime. Let $n \geq 1$.

We pull down an identification assertion from $\mathbb{Z}_p \mathrm{S}_n$, where it is known by **[11,** Ch. 16, Th. 35, Cor. 2, p. 132**]**, to $\mathbb{Z}_{(p)} \mathrm{S}_n$.

Lemma 238 *Suppose given a finitely generated indecomposable projective $\mathbb{Z}_{(p)} \mathrm{S}_n$-module P. Suppose given $\mathbb{Z}_{(p)} \mathrm{S}_n$-lattices M and N such that $\mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} M \cong \mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} N$.*

Then $P \otimes_{\mathbb{Z}_{(p)}} M \cong P \otimes_{\mathbb{Z}_{(p)}} N$ as $\mathbb{Z}_{(p)} \mathrm{S}_n$-modules.

In other words, to identify $P \otimes_{\mathbb{Z}_{(p)}} M$, it suffices to know the rational multiplicities of the Specht modules in M.

Proof. According to Lemma 168, $P \otimes_{\mathbb{Z}_{(p)}} M$ and $P \otimes_{\mathbb{Z}_{(p)}} N$ are both projective $\mathbb{Z}_{(p)}\mathrm{S}_n$-modules.

We have

$$\mathbb{Q}\otimes_{\mathbb{Z}_{(p)}}(P\otimes_{\mathbb{Z}_{(p)}}M) \;\cong\; (\mathbb{Q}\otimes_{\mathbb{Z}_{(p)}}P)\otimes_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}_{(p)}}M) \;\cong\; (\mathbb{Q}\otimes_{\mathbb{Z}_{(p)}}P)\otimes_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}_{(p)}}N) \;\cong\; \mathbb{Q}\otimes_{\mathbb{Z}_{(p)}}(P\otimes_{\mathbb{Z}_{(p)}}M)\ .$$

Write $X := P \otimes_{\mathbb{Z}_{(p)}} M$ and $Y := P \otimes_{\mathbb{Z}_{(p)}} N$. Note that $\hat{X}$ and $\hat{Y}$ are projective $\mathbb{Z}_p\mathrm{S}_n$-modules; cf. Lemma 165.(v).

Since

$$\begin{array}{rcl}
\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{X} & = & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} X \\
& \cong & \mathbb{Q}_p \otimes_{\mathbb{Z}_{(p)}} X \\
& \cong & \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} X \\
& \cong & \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} Y \\
& \cong & \mathbb{Q}_p \otimes_{\mathbb{Z}_{(p)}} Y \\
& \cong & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} Y \\
& = & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{Y}
\end{array}$$

as $\mathbb{Q}_p\mathrm{S}_n$-modules, we may conclude by **[11,** Ch. 16, Th. 35, Cor. 2, p. 132**]** that $\hat{X} \cong \hat{Y}$ as $\mathbb{Z}_p\mathrm{S}_n$-modules.

Therefore

$$\begin{array}{rcl}
\bar{X} & \cong & \mathbb{F}_p \otimes_{\mathbb{Z}_{(p)}} X \\
& \cong & \mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} X \\
& = & \mathbb{F}_p \otimes_{\mathbb{Z}_p} \hat{X} \\
& \cong & \mathbb{F}_p \otimes_{\mathbb{Z}_p} \hat{Y} \\
& = & \mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} Y \\
& \cong & \mathbb{F}_p \otimes_{\mathbb{Z}_{(p)}} Y \\
& \cong & \bar{Y}
\end{array}$$

as $\mathbb{F}_p\mathrm{S}_n$-modules. By Lemma 214, we conclude that $X \cong Y$ as $\mathbb{Z}_{(p)}\mathrm{S}_n$-modules. $\square$

Appendix F

The Wedderburn isomorphisms

The Wedderburn embeddings are taken from **[7]**.

Remark 239 The Wedderburn embedding used in Chapter 3 for $\mathbb{Z}_{(2)}\mathrm{S}_3$ and $\mathbb{Z}_{(3)}\mathrm{S}_3$.

$$\begin{array}{ccccccc}
\mathbb{Z}\mathrm{S}_3 & \longrightarrow & \mathbb{Z} & \times & \mathbb{Z}^{2\times 2} & \times & \mathbb{Z} \\
(1,2) & \longmapsto & 1 & \times & \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix} & \times & -1 \\
(1,2,3) & \longmapsto & 1 & \times & \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix} & \times & 1
\end{array}$$

For this embedding, load the Magma-files `main_S3_loc2` or `main_S3_loc3`.

To use the representations, call for example

```
load main_S3_loc2;

rho(S3P!(1,2)); // tuple of 3 matrices
```

Remark 240 The Wedderburn embedding used in Chapter 4 for $\mathbb{Z}_{(2)}\mathrm{S}_4$ and $\mathbb{Z}_{(3)}\mathrm{S}_4$.

$$\begin{array}{ccccccccccc}
\mathbb{Z}\mathrm{S}_4 & \longrightarrow & \mathbb{Z} & \times & \mathbb{Z} & \times & \mathbb{Z}^{3\times 3} & \times & \mathbb{Z}^{3\times 3} & \times & \mathbb{Z}^{2\times 2} \\
(1,2) & \longmapsto & -1 & \times & 1 & \times & \begin{pmatrix} -11 & -24 & 2 \\ 5 & 11 & -1 \\ 0 & 0 & -1 \end{pmatrix} & \times & \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \times & \begin{pmatrix} -5 & 24 \\ -1 & 5 \end{pmatrix} \\
(1,2,3,4) & \longmapsto & 1 & \times & 1 & \times & \begin{pmatrix} 16 & 41 & -9 \\ -7 & -18 & 4 \\ -8 & -20 & 3 \end{pmatrix} & \times & \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & -2 \\ 0 & 4 & -3 \end{pmatrix} & \times & \begin{pmatrix} 4 & -15 \\ 1 & -4 \end{pmatrix}
\end{array}$$

For this embedding, load `main_S4_loc2` or `main_S4_loc3`. Then `rho(S4P!(1,2,4))` will return the image tuple, consisting of 5 representing matrices.

Remark 241 The Wedderburn embedding used in Section 5.1 for $\mathbb{Z}_{(3)}\mathrm{S}_5$.

$$\begin{array}{rcccccll}
\mathbb{Z}_{(3)}\mathrm{S}_5 \longrightarrow & \mathbb{Z}_{(3)} & \times & \mathbb{Z}_{(3)} & \times & \mathbb{Z}_{(3)}^{4\times 4} & \times\, \mathbb{Z}_{(3)}^{4\times 4} & \\
& & & & & \mathbb{Z}_{(3)}^{5\times 5} & \times\, \mathbb{Z}_{(3)}^{5\times 5} & \\
& & & & & & \times\, \mathbb{Z}_{(3)}^{6\times 6} & \\
(12) \longmapsto & -1 & \times & 1 & \times & \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \times \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \\
& & & & \times & \begin{pmatrix} -2 & -1 & -1 & -1 & 1 \\ -3 & -4 & -3 & -3 & 8 \\ -3 & -3 & -4 & -3 & 7 \\ 9 & 9 & 9 & 8 & -16 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \times \begin{pmatrix} 2 & 1 & 1 & 1 & -1 \\ 3 & 4 & 3 & 3 & -8 \\ 3 & 3 & 4 & 3 & -7 \\ -9 & -9 & -9 & -8 & 16 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} & \\
& & & & & & \times \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \\
(12345) \longmapsto & 1 & \times & 1 & \times & \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} & \times \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} & \\
& & & & \times & \begin{pmatrix} 1 & 3 & 3 & 2 & -6 \\ -12 & -15 & -15 & -12 & 28 \\ -9 & -16 & -15 & -12 & 29 \\ 12 & 15 & 14 & 12 & -26 \\ -6 & -9 & -9 & -7 & 17 \end{pmatrix} & \times \begin{pmatrix} 1 & 3 & 3 & 2 & -6 \\ -12 & -15 & -15 & -12 & 28 \\ -9 & -16 & -15 & -12 & 29 \\ 12 & 15 & 14 & 12 & -26 \\ -6 & -9 & -9 & -7 & 17 \end{pmatrix} & \\
& & & & & & \times \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. &
\end{array}$$

For this embedding, load `main_S5_loc3`. Then `rho(S5P!(1,2)(3,5,4))` will return the image tuple, consisting of 7 representing matrices.

Remark 242 The Wedderburn embedding used in Section 5.2 for $\mathbb{Z}_{(2)}\mathrm{S}_5$.

$$\begin{array}{rcccccll}
\mathbb{Z}_{(2)}\mathrm{S}_5 \longrightarrow & \mathbb{Z}_{(2)} & \times & \mathbb{Z}_{(2)} & \times & \mathbb{Z}_{(2)}^{4\times 4} & \times\, \mathbb{Z}_{(2)}^{4\times 4} & \\
& & & & & \mathbb{Z}_{(2)}^{5\times 5} & \times\, \mathbb{Z}_{(2)}^{5\times 5} & \\
& & & & & & \times\, \mathbb{Z}_{(2)}^{6\times 6} & \\
(12) \longmapsto & -1 & \times & 1 & \times & \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \times \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \\
& & & & \times & \begin{pmatrix} 3 & 4 & 2 & 4 & -4 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -2 & 1 \\ 1 & 1 & 0 & 1 & -2 \end{pmatrix} & \times \begin{pmatrix} -3 & -64 & 42 & -12 & -28 \\ 0 & 11 & -5 & 0 & 0 \\ 0 & 24 & -11 & 0 & 0 \\ 3 & 67 & -41 & 10 & 21 \\ -1 & -11 & 8 & -3 & -6 \end{pmatrix} & \\
& & & & & & \times \begin{pmatrix} -5 & -1850 & -294 & -860 & -600 & -110 \\ 2 & 1025 & 161 & 476 & 328 & 64 \\ -4 & -1680 & -265 & -780 & -540 & -100 \\ -5 & -2627 & -413 & -1220 & -841 & -164 \\ 3 & 1419 & 224 & 659 & 456 & 86 \\ 0 & 134 & 21 & 62 & 42 & 9 \end{pmatrix} & \\
(12345) \longmapsto & 1 & \times & 1 & \times & \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} & \times \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} & \\
& & & & \times & \begin{pmatrix} 3 & 4 & 6 & 6 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -2 & 1 \\ 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 2 & 2 & -1 \end{pmatrix} & \times \begin{pmatrix} 3 & 60 & -38 & 10 & 22 \\ 2 & 40 & -28 & 9 & 20 \\ 5 & 99 & -69 & 22 & 49 \\ 4 & 104 & -73 & 23 & 55 \\ 1 & 9 & -6 & 2 & 3 \end{pmatrix} & \\
& & & & & & \times \begin{pmatrix} -7 & -3540 & -560 & -1644 & -1138 & -212 \\ 8 & 4408 & 698 & 2049 & 1422 & 270 \\ -13 & -6987 & -1103 & -3246 & -2243 & -426 \\ -18 & -9984 & -1581 & -4641 & -3221 & -612 \\ 7 & 3861 & 610 & 1794 & 1241 & 236 \\ 3 & 1668 & 263 & 775 & 535 & 103 \end{pmatrix}. &
\end{array}$$

For this embedding, load `main_S5_loc2`. Then `rho(S5P!(1,5))` will return the image tuple, consisting of 7 representing matrices.