Nonisomorphic triangles on a commutative quadrangle

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0 Introduction

An abelian Frobenius category is an abelian category with enough injective and projective objects, and where each injective object is projective and vice versa. For p prime and $k \in \mathbb{N}$, the category of finitely generated $\mathbb{Z}/p^k$-modules, denoted by $\mathbb{Z}/p^k$-mod, is an example of an abelian Frobenius category.

The stable category of an abelian Frobenius category $\mathcal{A}$ is defined as follows. As objects, we take the objects of $\mathcal{A}$. As morphisms, we take the residue classes $[f]$ of the morphisms f in $\mathcal{A}$, modulo those that factorize over injective objects. This stable category is additive, but no longer abelian. As Happel has shown [2, ch. I, sec. 2.6], it is Verdier triangulated [5, ch. I, sec. 1-1].

In a Verdier triangulated category, one can extend any diagram of the form

$$X_1 \longrightarrow X_2$$

to a (distinguished) triangle. We refer to $(X_1 \longrightarrow X_2)$ as the *base* of this triangle. Two triangles on a given base are isomorphic [1, sec. 4.1.4].

This assertion can be extended as follows. Two (generalized) triangles on a base of the form

$$X_1 \longrightarrow X_2 \longrightarrow \ldots \longrightarrow X_n$$

are isomorphic [4, lem. 3.4(5)].

There is an obvious definition of (generalized) triangles on bases of the form

$$\begin{array}{ccc} X_3 & \longrightarrow & X_4 \\ \uparrow & & \uparrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

in the stable category of an abelian Frobenius category. When displayed, such a triangle is a four-dimensional diagram. The question arises whether two such triangles on the same base are necessarily isomorphic.

In this bachelor thesis, I construct two triangles on the base

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \xrightarrow{[0\,1]} & \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2 \\ \uparrow{\scriptstyle [p]} & & \uparrow{\scriptstyle [p\,1]} \\ \mathbb{Z}/p & \xrightarrow{[p]} & \mathbb{Z}/p^2. \end{array}$$

in the stable category $\mathbb{Z}/p^3$-$\underline{\text{mod}}$ of $\mathbb{Z}/p^3$-mod, that are not isomorphic. In particular the functor “restriction to the base” from these triangles to commutative quadrangles is not full.

0.1 Notations

Throughout, let p be a prime number.

- We compose morphisms in the following direction: $\xrightarrow{f}\xrightarrow{g} = \xrightarrow{fg}$. Sometimes we write $f \cdot g = fg$.
- By $\rightarrowtail$ we denote a monomorphism, by $\twoheadrightarrow$ an epimorphism.
- We often refer to a diagram (i.e. quadrangle)

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

  by the tuple of its objects (A, B, C, D).
- By $|A|$ we denote the cardinality of a given set A.
- Given a category $\mathcal{A}$ and objects $X, Y \in \mathrm{Obj}(\mathcal{A})$, we denote the set of morphisms from X to Y by ${}_{\mathcal{A}}(X, Y)$.
- Let $t \in \mathbb{Z}_{\geq 1}$. For $k, l \in \mathbb{Z}_{\geq 1}$, we denote:

$$(a_{i,j})_{i,j} : \bigoplus_{i \in [1,k]} \mathbb{Z}/p^{m_i} \longrightarrow \bigoplus_{j \in [1,l]} \mathbb{Z}/p^{n_j}$$

  in $\mathbb{Z}/p^t$-mod and

$$[a_{i,j}]_{i,j} : \bigoplus_{i \in [1,k]} \mathbb{Z}/p^{m_i} \longrightarrow \bigoplus_{j \in [1,l]} \mathbb{Z}/p^{n_j}$$

  in $\mathbb{Z}/p^t$-$\underline{\mathrm{mod}}$ (see 1.10), where $m_i, n_j \leq t$.

  Remark. If $n_j \geq m_i$ we require that $p^{n_j - m_i} | a_{i,j}$ for welldefinedness of the maps.

  For example,

$$\left(\begin{smallmatrix} 1 & -1 \\ 3 & 1 \end{smallmatrix}\right) : \mathbb{Z}/3 \oplus \mathbb{Z}/3^2 \longrightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/3^2 \text{ in } \mathbb{Z}/3^3\text{-mod}$$

  has the residue class

$$\left[\begin{smallmatrix} 1 & -1 \\ 3 & 1 \end{smallmatrix}\right] : \mathbb{Z}/3 \oplus \mathbb{Z}/3^2 \longrightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/3^2 \text{ in } \mathbb{Z}/3^3\text{-}\underline{\mathrm{mod}}.$$

1 Theoretical preliminaries

1.1 Abelian categories

Definitions 1.1 (additive and abelian categories)**.** Let $\mathcal{C}$ be a category.

1. We call $\mathcal{C}$ an *additive category* if there is a zero object 0 in $\mathrm{Obj}(\mathcal{C})$, if for all objects $X_1, X_2 \in \mathrm{Obj}(\mathcal{C})$ there exists a direct sum $X_1 \oplus X_2$, and if for each object X there is an endomorphism -1_X on X with $1_X + (-1_X) = 0$.

   Remark. For all objects $X_1, X_2 \in \mathrm{Obj}(\mathcal{C})$ the set ${}_{\mathcal{C}}(X_1, X_2)$ is an abelian group, and the composition of morphisms is bilinear.

2. We call $\mathcal{C}$ an *abelian category* if it is additive, if for any morphism in $\mathcal{C}$ there exists a kernel and a cokernel, if any monomorphism in $\mathcal{C}$ is a kernel and if any epimorphism is a cokernel.

Definition 1.2 (additive functor)**.** Let $\mathcal{A}, \mathcal{B}$ be additive categories. A functor $\mathrm{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is called *additive*, if it satisfies the following:

1. F preserves zero objects, i.e. the object $\mathrm{F}0$ is a zero object in $\mathcal{B}$.

2. F preserves binary direct sums, that is, if $X_1 \oplus X_2$ is a direct sum of X_1 and X_2 via $\iota_i : X_i \longrightarrow X_1 \oplus X_2$ and $\pi_i : X_1 \oplus X_2 \longrightarrow X_i, i \in \{1, 2\}$, then $\mathrm{F}(X_1 \oplus X_2)$ is a direct sum of $\mathrm{F}X_1$ and $\mathrm{F}X_2$ via $\mathrm{F}\iota_i : \mathrm{F}X_i \longrightarrow \mathrm{F}(X_1 \oplus X_2)$ and $\mathrm{F}\pi_i : \mathrm{F}(X_1 \oplus X_2) \longrightarrow \mathrm{F}X_i, i \in \{1, 2\}$.

Remark. We apply additive functors summandwise in direct sums and componentwise in matrices:

$$\mathrm{F}\left(\bigoplus_i X_i \xrightarrow{(f_{i,j})_{i,j}} \bigoplus_j Y_j\right) = \left(\bigoplus_i \mathrm{F}X_i \xrightarrow{(\mathrm{F}f_{i,j})_{i,j}} \bigoplus_j \mathrm{F}Y_j\right)$$

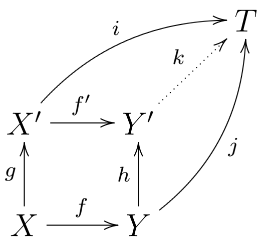
Definition 1.3 (pushout)**.** Let $\mathcal{A}$ be an abelian category. Suppose given the following diagram in $\mathcal{A}$.

$$\begin{array}{ccc} X' & & \\ {\scriptstyle g}\uparrow & & \\ X & \xrightarrow{f} & Y \end{array} \tag{1.1}$$

A commutative diagramm

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h} \\ X & \xrightarrow{f} & Y \end{array} \tag{1.2}$$

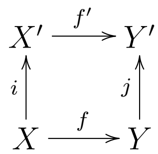
in $\mathcal{A}$ is called a *pushout* of (1.1) if for all $T \in \mathrm{Obj}(\mathcal{A})$ and all morphisms $i : X' \longrightarrow T$, $j : Y \longrightarrow T$ such that $gi = fj$, there exists a unique morphism $k : Y' \longrightarrow T$ such that (1.3) commutes.



(1.3)

Remark. If g in (1.2) is a monomorphism, then so is h.

Lemma 1.4 (a pushout criterion)**.** *Let $t \in \mathbb{Z}_{\geq 1}$. Consider the following diagram in the abelian category $\mathcal{A} := (\mathbb{Z}/p^t)$ -mod.*



The diagram is a pushout if

- *the morphism $(\,i\ f\,) : X \longrightarrow X' \oplus Y$ is a monomorphism,*
- *the morphism $\left(\begin{smallmatrix} f' \\ j \end{smallmatrix}\right) : X' \oplus Y \longrightarrow Y'$ is an epimorphism,*
- *the diagram commutes and*
- $|X|\,|Y'| = |X'|\,|Y|$.

Remark. If i is a monomorphism, then so is $(\,i\ f\,)$.

1.2 Abelian Frobenius categories

Definition 1.5 (bijective object)**.** Let B be an object in an abelian category $\mathcal{A}$. We call B a *bijective* object if the map ${}_{\mathcal{A}}(B, f) : {}_{\mathcal{A}}(B, X) \longrightarrow {}_{\mathcal{A}}(B, Y)$ is surjective for any epimorphism $f : X \longrightarrow Y$ and if the map ${}_{\mathcal{A}}(f, B) : {}_{\mathcal{A}}(Y, B) \longrightarrow {}_{\mathcal{A}}(X, B)$ is surjective for any monomorphism $f : X \longrightarrow Y$.

Remark.

- This condition is equivalent to B being both projective and injective in $\mathcal{A}$.
- The direct sum of bijective objects in $\mathcal{A}$ is bijective.

Definition 1.6 (abelian Frobenius category)**.** Let $\mathcal{A}$ be a abelian category. We call $\mathcal{A}$ an *abelian Frobenius category* if for all $X \in \mathrm{Obj}(\mathcal{A})$ there is an epimorphism $B \twoheadrightarrow X$ and a monomorphism $X \rightarrowtail B'$, where B, B' are bijective objects in $\mathcal{A}$.

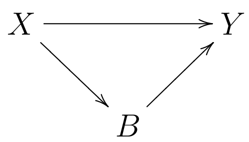
Remark. The category $\mathbb{Z}/p^t$-mod for $t \in \mathbb{Z}_{\geq 1}$ is an abelian Frobenius category.

Definitions 1.7 (stable category, residue class functor)**.** Let $\mathcal{A}$ be an abelian Frobenius category.

1. Let

$$\begin{aligned}{}^{\mathrm{bij}}_{\mathcal{A}}(X,Y) := \{f : X \longrightarrow Y \mid \text{there is a bijective object } B \text{ and morphisms}\\ u : X \longrightarrow B,\ v : B \longrightarrow Y \text{ in } \mathcal{A} \text{ such that } f = uv\}\end{aligned}$$

be the set of all morphisms that factorize over bijective objects in $\mathcal{A}$.



We define the *stable category* $\underline{\mathcal{A}}$ *of* $\mathcal{A}$ as follows. (For welldefinedness see lemma 1.8.1.) We let

$$\begin{aligned}\mathrm{Obj}(\underline{\mathcal{A}}) &:= \mathrm{Obj}(\mathcal{A}),\\ {}_{\underline{\mathcal{A}}}(X,Y) &:=_{\mathcal{A}} (X,Y)/{}^{\mathrm{bij}}_{\mathcal{A}}(X,Y) \text{ for } X,Y \in \mathrm{Obj}(\underline{\mathcal{A}}).\end{aligned}$$

For $f \in {}_{\mathcal{A}}(X,Y)$ we write $[f] := f + {}^{\mathrm{bij}}_{\mathcal{A}}(X,Y)$. Given $f \in {}_{\mathcal{A}}(X,Y)$, $g \in {}_{\mathcal{A}}(Y,Z)$, we define the composite of $[f]$ and $[g]$ in $\underline{\mathcal{A}}$ by $[f][g] := [fg]$. Given $X \in \mathrm{Obj}(\underline{\mathcal{A}})$, we define the identity of X in $\underline{\mathcal{A}}$ by $1_X := [1_X]$.

2. We define the *residue class functor* $\mathrm{R} : \mathcal{A} \longrightarrow \underline{\mathcal{A}}$ by

$$\mathrm{R}X := X, \qquad \mathrm{R}f := [f]$$

for $X \in \mathrm{Obj}(\mathcal{A})$ and $f \in \mathrm{Mor}(\mathcal{A})$. (For welldefinedness see lemma 1.8.2.)

Lemma 1.8. *Let $\mathcal{A}$ be an abelian Frobenius category.*

1. *The stable category $\underline{\mathcal{A}}$ of $\mathcal{A}$ is a welldefined additive category.*

2. *The residue class functor $\mathrm{R} : \mathcal{A} \longrightarrow \underline{\mathcal{A}}$ is a welldefined additive functor.*

Proof.

1. We prove only that the composition in $\underline{\mathcal{A}}$ is independent of the representatives of the composed residue classes. The axioms of a category then follow from the axioms in $\mathcal{A}$.

Consider residue classes $[f] = [f']$, $[g] = [g']$ of morphisms $f, f' : X \longrightarrow Y$, $g, g' : Y \longrightarrow Z$ in $\mathcal{A}$. We have to show that $[fg] = [f'g']$. Since $[f] = [f']$, we have $f - f' \in {}^{\mathrm{bij}}_{\mathcal{A}}(X,Y)$, that is, there exists a bijective object B and morphisms $u : X \longrightarrow B$, $u' : B \longrightarrow Y$ in $\mathcal{A}$ such that $uu' = f - f'$. Analogously, we have $g - g' \in {}^{\mathrm{bij}}_{\mathcal{A}}(Y,Z)$, that is, there exists a bijective object C and morphisms $v : Y \longrightarrow C$, $v' : C \longrightarrow Z$ in $\mathcal{A}$ such that $vv' = g - g'$. We get

$$\begin{aligned}fg - f'g' &= fg - f'g + f'g - f'g'\\ &= (f - f')g + f'(g - g')\\ &= uu'g + f'vv'\\ &= \left(\begin{smallmatrix} u & f'v \end{smallmatrix}\right)\left(\begin{smallmatrix} u'g \\ v' \end{smallmatrix}\right).\end{aligned}$$

Since $B \oplus C$ is a bijective object in $\mathcal{A}$ as a direct sum of such, we get $[fg] = [f'g']$.

$$\begin{array}{ccc} X & \xrightarrow{fg - f'g'} & Z \\ & {\scriptstyle (u\, f'v)} \searrow \quad \nearrow {\scriptstyle \left(\begin{smallmatrix} u'g \\ v' \end{smallmatrix}\right)} & \\ & B \oplus C & \end{array}$$

□

Notation 1.9. The stable category of the abelian Frobenius category $\mathbb{Z}/p^t$-mod for $t \in \mathbb{Z}_{\geq 1}$ will be denoted by

$$\mathbb{Z}/p^t\text{-}\underline{\mathrm{mod}} := \underline{\mathbb{Z}/p^t\text{-mod}}.$$

Lemma 1.10. *For morphisms in* $\mathbb{Z}/p^3$-$\underline{\mathrm{mod}}$ *we have:*

$$\begin{array}{ccc} \mathbb{Z}/p & \xrightarrow{\approx} & {}_{\mathbb{Z}/p^3-\underline{\mathrm{mod}}}(\mathbb{Z}/p, \mathbb{Z}/p) \\ 1 + p\mathbb{Z} & \longmapsto & [1] \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}/p & \xrightarrow{\approx} & {}_{\mathbb{Z}/p^3-\underline{\mathrm{mod}}}(\mathbb{Z}/p, \mathbb{Z}/p^2) \\ 1 + p\mathbb{Z} & \longmapsto & [p] \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}/p & \xrightarrow{\approx} & {}_{\mathbb{Z}/p^3-\underline{\mathrm{mod}}}(\mathbb{Z}/p^2, \mathbb{Z}/p) \\ 1 + p\mathbb{Z} & \longmapsto & [1] \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}/p & \xrightarrow{\approx} & {}_{\mathbb{Z}/p^3-\underline{\mathrm{mod}}}(\mathbb{Z}/p^2, \mathbb{Z}/p^2) \\ 1 + p\mathbb{Z} & \longmapsto & [1] \end{array}$$

For example in the fourth case we have the factorization

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \xrightarrow{(p)} & \mathbb{Z}/p^2 \\ & {\scriptstyle (p)} \searrow \quad \nearrow {\scriptstyle (1)} & \\ & \mathbb{Z}/p^3 & \end{array} .$$

Hence $[p] = [0]$, although $(p) \neq (0)$.

1.3 Co-Heller sequences and shift

Throughout this section, let $\mathcal{A}$ be an abelian Frobenius category.

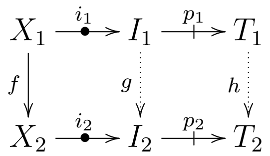
Definition 1.11 (co-Heller sequence)**.** Let $X, I, T \in \mathrm{Obj}(\mathcal{A})$. A *co-Heller sequence of (an object)* X is a short exact sequence

$$X \rightarrowtail I \twoheadrightarrow T$$

where I is bijective in $\mathcal{A}$.

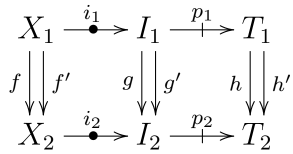
Lemma 1.12 (cf. [3, lemma 5.2]). *Let $\mathcal{A}$ be an additive category. Let $X_1, X_2 \in \mathrm{Obj}(\mathcal{A})$ and consider co-Heller sequences $X_1 \xrightarrow{i_1} I_1 \xrightarrow{p_1} T_1$ for X_1 and $X_2 \xrightarrow{i_2} I_2 \xrightarrow{p_2} T_2$ for X_2.*

1. *For all morphisms $f : X_1 \longrightarrow X_2$ in $\mathcal{A}$ there are morphisms $g : I_1 \longrightarrow I_2$ and $h : I_1 \longrightarrow I_2$ such that the following diagram commutes.*



(1.4)

2. *Consider morphisms f, g, h, f', g', h' in $\mathcal{A}$ such that $fi_2 = i_1 g$, $gp_2 = p_1 h$, $f'i_2 = i_1 g'$, $g'p_2 = p_1 h'$.*

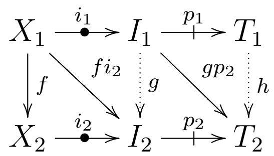


(1.5)

If $[f] = [f']$, then $[h] = [h']$.

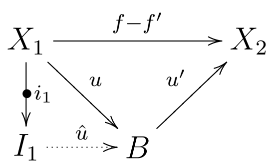
Proof.

1. By the definition of co-Heller sequences I_2 is bijective, so in particular injective. Thus there exists $g : I_1 \longrightarrow I_2$ such that $i_1 g = f i_2$. For the existence of h consider that T_1 is the cokernel of i_1. Since $i_1 g p_2 = f i_2 p_2 = f0 = 0$ it follows that there exists $h : T_1 \longrightarrow T_2$ such that $p_1 h = g p_2$.



(1.6)

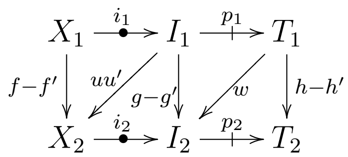
2. We suppose that $[f] = [f']$, that is, $f - f' \in {}^{\mathrm{bij}}_{\mathcal{A}}(X_1, X_2)$. So there exists a bijective object B and morphisms $u : X_1 \longrightarrow B$ and $u' : B \longrightarrow X_2$ in $\mathcal{A}$ such that $f - f' = uu'$. Using the injectivity of B, it follows that there exists $\hat{u} : I_1 \longrightarrow B$ with $u = i_1 \hat{u}$.



From the diagram, we see that

$$i_1 \hat{u} u' i_2 = u u' i_2 = (f - f') i_2 = i_1 (g - g')$$

and hence $i_1((g - g') - \hat{u} u' i_2) = 0$. Since T_1 is a cokernel of i_1, there is a morphism $w : T_1 \longrightarrow I_2$ in $\mathcal{A}$ such that $(g - g') - \hat{u} u' i_2 = p_1 w$.



We get

$$p_1 w p_2 = ((g - g') - \hat{u}u'i_2)p_2 = (g - g')p_2 - \hat{u}u'i_2 p_2 = p_1(h - h').$$

This implies that $wp_2 = h - h'$ as p_1 is an epimorphism. Thus we have $h - h' \in {}^{\mathrm{bij}}_{\mathcal{A}}(T_1, T_2)$, that is, $[h_1] = [h_2]$. □

Definition 1.13.

1. Let $X \in \mathrm{Obj}(\underline{\mathcal{A}})$ and $s = (\, X \rightarrowtail I \twoheadrightarrow T \,)$ be a co-Heller sequence for X. We set $\mathrm{H}_s(X) := T$.

2. Let $\varphi : X_1 \longrightarrow X_2$ be a morphism in $\underline{\mathcal{A}}$ and let $s_i = (\, X_i \rightarrowtail I_i \twoheadrightarrow T_i \,)$ be a co-Heller sequence for X_i, $i \in \{1, 2\}$. We choose a morphism $f : X_1 \longrightarrow X_2$ in $\mathcal{A}$ fulfilling $\varphi = [f]$ and morphisms $g : I_1 \longrightarrow I_2$ and $h : T_1 \longrightarrow T_2$ such that

$$\begin{CD} X_1 @>{i_1}>> I_1 @>{p_1}>> T_1 \\ @V{f}VV @V{g}VV @V{h}VV \\ X_2 @>{i_2}>> I_2 @>{p_2}>> T_2 \end{CD} \qquad (1.7)$$

commutes in $\mathcal{A}$. We set $\mathrm{H}_{s_1,s_2}(\varphi) := [h]$.

Lemma 1.14.

1. *Consider morphisms $\varphi_1 : X_1 \longrightarrow X_2$ and $\varphi_2 : X_2 \longrightarrow X_3$ in $\underline{\mathcal{A}}$ and co-Heller sequences s_i for $X_i, i \in \{1, 2, 3\}$. We then have*

$$\mathrm{H}_{s_1,s_3}(\varphi_1\varphi_2) = \mathrm{H}_{s_1,s_2}(\varphi_1) \cdot \mathrm{H}_{s_2,s_3}(\varphi_2). \qquad (1.8)$$

2. *Let $X \in \mathrm{Obj}(\underline{\mathcal{A}})$ and s be a co-Heller sequence for X. Then*

$$\mathrm{H}_{s,s}(1_X) = 1_{\mathrm{H}_s(X)}. \qquad (1.9)$$

Proof.

1. We write $s_j = (X \xrightarrow{i_j} I_j \xrightarrow{p_j} T_j)$ for $j \in \{1, 2, 3\}$. We choose morphisms $f_1 : X_1 \longrightarrow X_2$, $f_2 : X_2 \longrightarrow X_3$ with $\varphi_1 = [f_1]$, $\varphi_2 = [f_2]$. Moreover, we choose morphisms $g_1 : I_1 \longrightarrow I_2$, $g_2 : I_2 \longrightarrow I_3$, $h_1 : T_1 \longrightarrow T_2$, $h_2 : T_2 \longrightarrow T_3$ such that the following diagram commutes.

$$\begin{CD} X_1 @>{i_1}>> I_1 @>{p_1}>> T_1 \\ @V{f_1}VV @V{g_1}VV @V{h_1}VV \\ X_1 @>{i_2}>> I_1 @>{p_2}>> T_1 \\ @V{f_2}VV @V{g_2}VV @V{h_2}VV \\ X_3 @>{i_3}>> I_3 @>{p_3}>> T_3 \end{CD} \qquad (1.10)$$

We conclude

$$\begin{aligned} \mathrm{H}_{s_1,s_3}(\varphi_1\varphi_2) &= \mathrm{H}_{s_1,s_3}([f_1][f_2]) \\ &= \mathrm{H}_{s_1,s_3}([f_1 f_2]) \\ &= h_1 h_2 \\ &= \mathrm{H}_{s_1,s_2}([f_1]) \cdot \mathrm{H}_{s_2,s_3}([f_2]) \\ &= \mathrm{H}_{s_1,s_2}(\varphi_1) \cdot \mathrm{H}_{s_2,s_3}(\varphi_2). \end{aligned}$$

2. We write $s = (X \xrightarrow{i} I \xrightarrow{p} T)$. As the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & I & \xrightarrow{p} & T \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle 1_I}\downarrow & & {\scriptstyle 1_T}\downarrow \\ X & \xrightarrow{i} & I & \xrightarrow{p} & T \end{array}$$

   commutes, we have

$$\mathrm{H}_{s,s}(1_X) = 1_T = 1_{\mathrm{H}_s(X)}.$$

□

Definition 1.15 (shift functor)**.** For every object X in $\underline{\mathcal{A}}$, choose a co-Heller sequence s_X. (This is possible since $\mathcal{A}$ has enough bijective objects by definition.)

We define the *shift functor* $\mathrm{T} : \underline{\mathcal{A}} \longrightarrow \underline{\mathcal{A}}$ by

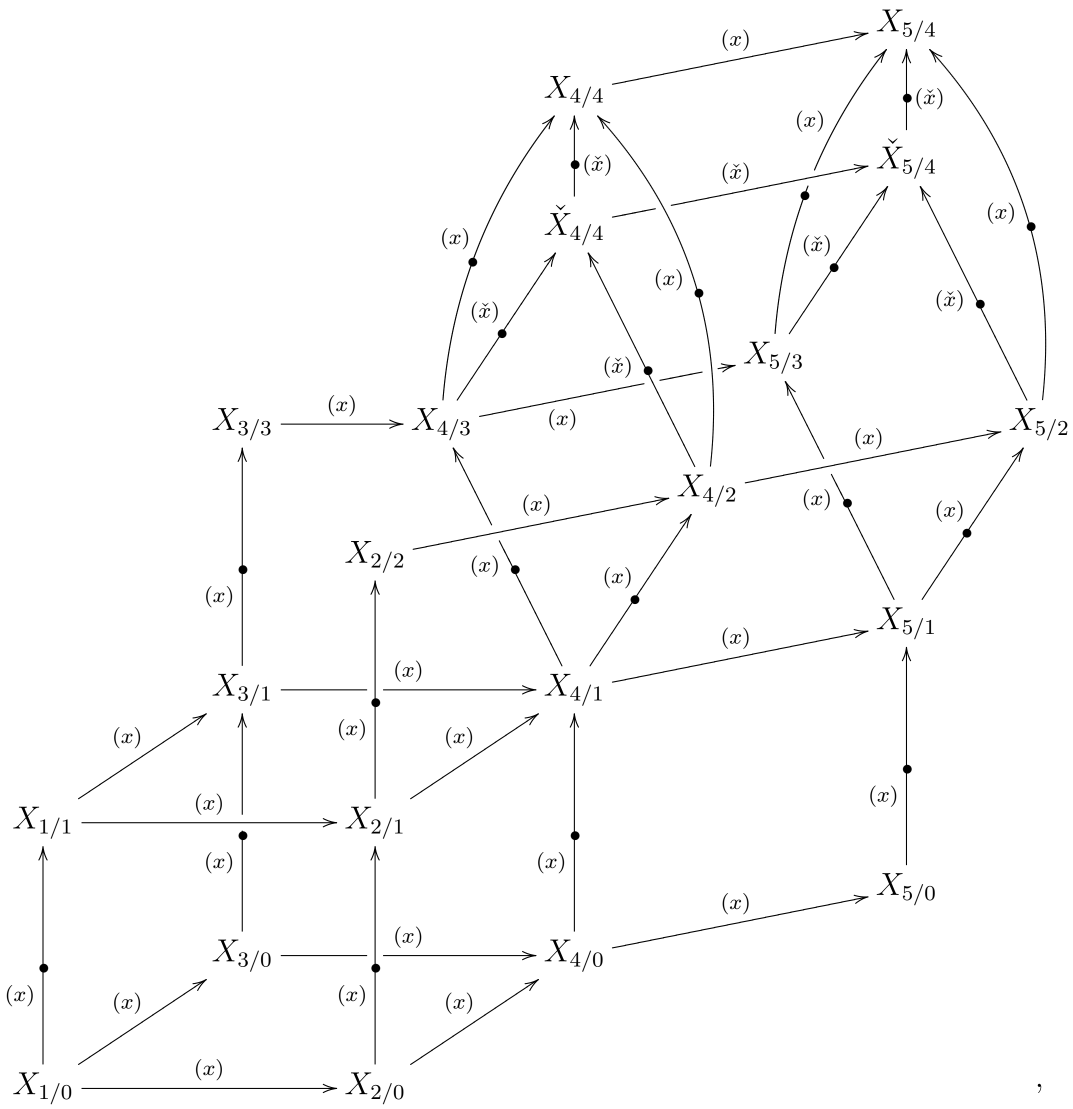
$$\mathrm{T}X := \mathrm{H}_{s_X}(X) \qquad \text{for } X \in \mathrm{Obj}(\underline{\mathcal{A}}) \text{ and}$$
$$\mathrm{T}\varphi := \mathrm{H}_{s_X,s_Y}(\varphi) \qquad \text{for any morphism } \varphi \in {}_{\underline{\mathcal{A}}}(X,Y),\ X, Y \in \mathrm{Obj}(\underline{\mathcal{A}}).$$

2 □-triangles

2.1 Definition of □-triangles

Throughout this section, let $\mathcal{A}$ be an abelian Frobenius category.

Definition 2.1 (□-triangle model)**.** A □-*triangle model* is a commutative diagram X in $\mathcal{A}$ of the form



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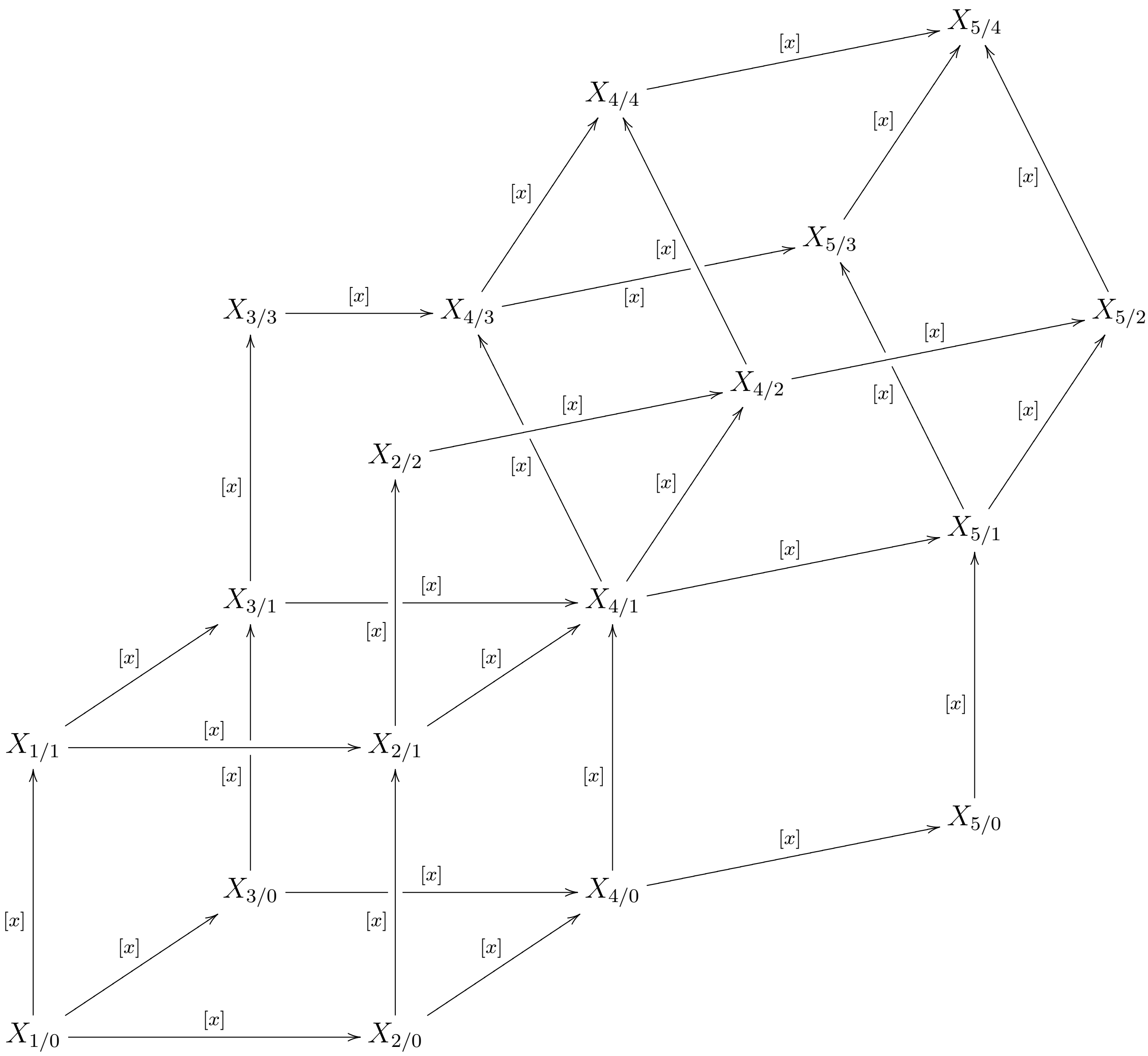
such that $X_{5/0} = 0$ and $X_{i/i}$ is bijective in $\mathcal{A}$ for $i \in \{1, 2, 3, 4\}$ and the following quadruples are pushouts:

$(X_{1/0}, X_{2/0}, X_{1/1}, X_{2/1}), (X_{1/0}, X_{3/0}, X_{1/1}, X_{3/1}), (X_{3/0}, X_{4/0}, X_{3/1}, X_{4/1}), (X_{2/0}, X_{4/0}, X_{2/1}, X_{4/1}),$
$(X_{4/0}, X_{5/0}, X_{4/1}, X_{5/1}), (X_{2/1}, X_{4/1}, X_{2/2}, X_{4/2}), (X_{4/1}, X_{5/1}, X_{4/2}, X_{5/2}), (X_{5/1}, X_{5/2}, X_{5/3}, \check{X}_{5/4}),$
$(X_{3/1}, X_{4/1}, X_{3/3}, X_{4/3}), (X_{4/1}, X_{5/1}, X_{4/3}, X_{5/3}), (X_{4/3}, X_{5/3}, \check{X}_{4/4}, \check{X}_{5/4}), (X_{4/1}, X_{4/2}, X_{4/3}, \check{X}_{4/4}),$
$(X_{4/2}, X_{5/2}, \check{X}_{4/4}, \check{X}_{5/4}), (X_{4/2}, X_{5/2}, X_{4/4}, X_{5/4}), (X_{4/3}, X_{5/3}, X_{4/4}, X_{5/4}), (\check{X}_{4/4}, \check{X}_{5/4}, X_{4/4}, X_{5/4}).$

We call $\check{X}_{4/4}$ and $\check{X}_{5/4}$ *auxiliary objects*. Any morphism that has an auxiliary object as source or target is called *auxiliary morphism* and will be denoted by $\check{x}$, by abuse of notation. Any other morphism will be denoted as x, also by abuse of notation.

Definition 2.2 (□-pretriangle, morphism and base)**.**

1. A □-pretriangle is a commutative diagram X in $\underline{\mathcal{A}}$ of the form



such that

$$\begin{array}{ccc} X_{5/3} & \xrightarrow{[x]} & X_{5/4} \\ {\scriptstyle [x]}\uparrow & & \uparrow{\scriptstyle [x]} \\ X_{5/1} & \xrightarrow{[x]} & X_{5/2} \end{array} \quad = \quad \begin{array}{ccc} \mathrm{T}X_{3/0} & \xrightarrow{\mathrm{T}[x]} & \mathrm{T}X_{4/0} \\ {\scriptstyle \mathrm{T}[x]}\uparrow & & \uparrow{\scriptstyle \mathrm{T}[x]} \\ \mathrm{T}X_{1/0} & \xrightarrow{\mathrm{T}[x]} & \mathrm{T}X_{2/0}. \end{array} \tag{2.1}$$

2. Let X, Y be □-pretriangles in $\underline{\mathcal{A}}$. A *morphism of □-pretriangles* is a diagram morphism $\varphi : X \longrightarrow Y$ in $\underline{\mathcal{A}}$ such that $\varphi_{5/i} = \mathrm{T}\varphi_{i/0}$ for $i \in \{1, 2, 3, 4\}$.

   A morphism of □-pretriangles that is an isomorphism in each component is called an *isomorphism of □-pretriangles.*

3. The *base* of a pretriangle X is the quadrangle $(X_{1/0}, X_{2/0}, X_{3/0}, X_{4/0})$.

We now modify a given □-triangle model to define a standard □-triangle.

Notation 2.3. Suppose given a □-triangle model X in $\mathcal{A}$. Let $i \in \{1, 2, 3, 4\}$. Denote

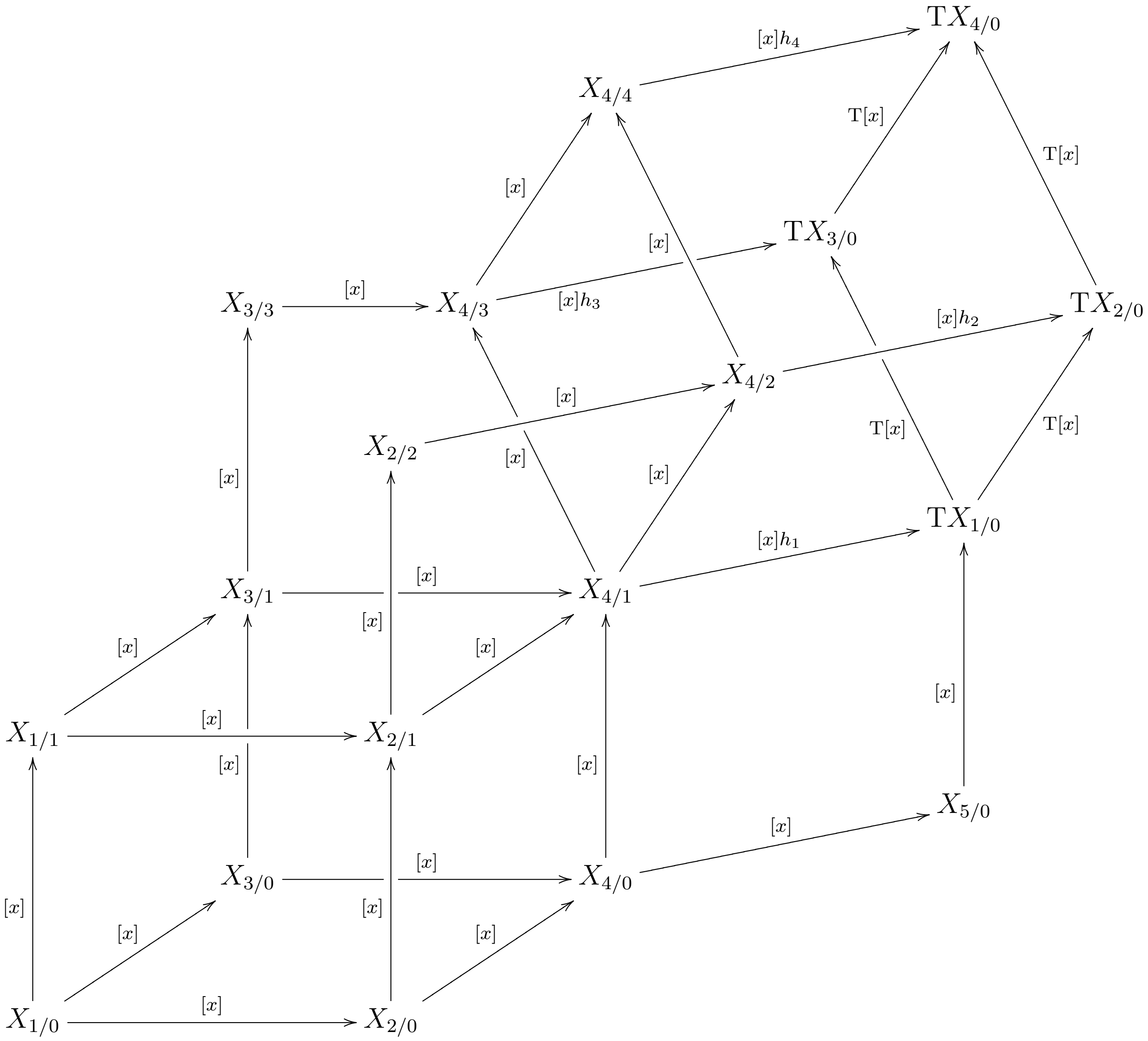
$$s_i^X := (X_{i/0} \longrightarrow X_{i/i} \longrightarrow X_{5/i}).$$

Also denote

$$h_i := \mathrm{H}_{s_i^X, s_{X_{i/0}}}(1_{X_{i/0}})$$

for morphisms from $X_{5/i}$ to $\mathrm{T}X_{i/0}$ in $\underline{\mathcal{A}}$.

Definition 2.4 (standard □-triangle)**.** Consider a □-triangle model X. The *standard □-triangle* $\underline{X}$ obtained from X is defined to be the following diagram in $\underline{\mathcal{A}}$.



Lemma 2.5. *Any standard □-triangle is a □-pretriangle.*

Proof. Suppose given a □-triangle model X. We need to show that the standard □-triangle obtained from X commutes. To this end, we have to show that the quadrangles

$$(X_{4/1}, X_{4/2}, \mathrm{T}X_{1/0}, \mathrm{T}X_{2/0}), (X_{4/1}, X_{4/3}, \mathrm{T}X_{1/0}, \mathrm{T}X_{3/0}),$$
$$(X_{4/2}, X_{4/4}, \mathrm{T}X_{2/0}, \mathrm{T}X_{4/0}), (X_{4/3}, X_{4/4}, \mathrm{T}X_{3/0}, \mathrm{T}X_{4/0})$$

commute. We do this exemplarily for

$$\begin{CD} X_{4/2} @>{[x]h_2}>> \mathrm{T}X_{2/0} \\ @A{[x]}AA @AA{\mathrm{T}[x]}A \\ X_{4/1} @>{[x]h_1}>> \mathrm{T}X_{1/0}. \end{CD}$$

Since $(X_{4/1}, X_{4/2}, X_{5/1}, X_{5/2})$ already commutes as a subdiagram of X in $\mathcal{A}$, its image under the residue class functor $\mathrm{R} : \mathcal{A} \longrightarrow \underline{\mathcal{A}}$ certainly commutes in $\underline{\mathcal{A}}$. Thus it remains to show that the diagram

$$\begin{array}{ccc} X_{5/2} & \xrightarrow{h_2} & \mathrm{T}X_{2/0} \\ {\scriptstyle [x]}\uparrow & & \uparrow{\scriptstyle \mathrm{T}[x]} \\ X_{5/1} & \xrightarrow{h_1} & \mathrm{T}X_{1/0} \end{array}$$

commutes in $\mathcal{A}$. Indeed as

$$(X_{5/1} \xrightarrow{[x]} X_{5/2}) = \mathrm{H}_{s_1, s_2^X}(X_{1/0} \longrightarrow X_{2/0}),$$

we have

$$\begin{aligned} (X_{5/1} \xrightarrow{[x]} X_{5/2})h_2 &= \mathrm{H}_{s_1^X, s_2^X}(X_{1/0} \xrightarrow{[x]} X_{2/0})\mathrm{H}_{s_2^X, s_{X_2}}(1_{X_{2/0}}) \\ &= \mathrm{H}_{s_1^X, s_{X_2}}(X_{1/0} \xrightarrow{[x]} X_{2/0}) \\ &= \mathrm{H}_{s_1^X, s_{X_1}}(1_{X_{1/0}})\mathrm{H}_{s_{X_1}, s_{X_2}}(X_{1/0} \xrightarrow{[x]} X_{2/0}) \\ &= h_1\mathrm{T}(X_{1/0} \xrightarrow{[x]} X_{2/0}) \end{aligned}$$

by lemma 1.14.1. □

Definition 2.6 (□-triangle)**.** Any □-pretriangle isomorphic to a standard □ triangle (in the sense of 2.2) is called a □-*triangle*.

In the following two sections we give two examples Y and Y' of □-triangles in $\mathbb{Z}/p^3$-mod.

2.2 The □-triangle Y

We aim to construct a □-triangle Y having as base the commutative quadrangle

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \xrightarrow{[0\,1]} & \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2 \\ \uparrow{\scriptstyle [p]} & & \uparrow{\scriptstyle [p\,1]} \\ \mathbb{Z}/p & \xrightarrow{[p]} & \mathbb{Z}/p^2. \end{array}$$

First, we construct a □-triangle model X such that

$$\begin{array}{ccccc} & & & & X_{5/0} \\ & & & \nearrow{\scriptstyle x} & \\ X_{3/0} & \xrightarrow{x} & X_{4/0} & & \\ \uparrow{\scriptstyle x} & & \uparrow{\scriptstyle x} & & \\ X_{1/0} & \xrightarrow{x} & X_{2/0} & & \end{array} \quad = \quad \begin{array}{ccccc} & & & & 0 \\ & & & \nearrow & \\ \mathbb{Z}/p^2 & \xrightarrow{(0\,1)} & \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2 & & \\ \uparrow{\scriptstyle (p)} & & \uparrow{\scriptstyle (p\,1)} & & \\ \mathbb{Z}/p & \xrightarrow{(p)} & \mathbb{Z}/p^2. & & \end{array}$$

To this end we construct X levelwise.

1. Choose a monomorphism from $X_{1/0}$ to a bijective object $X_{1/1}$. Then construct pushouts

$$(X_{i/0}, X_{j/0}, X_{i/1}, X_{j/1})$$

   for $(i, j) \in \{(1, 2), (1, 3), (2, 4), (3, 4), (4, 5)\}$ using lemma 1.4. In fact, we may use the induced morphism $X_{3/1} \longrightarrow X_{4/1}$ for $(X_{3/0}, X_{4/0}, X_{3/1}, X_{4/1})$.

2. Choose a monomorphism from $X_{2/1}$ to a bijective object $X_{2/2}$. Then construct pushouts

$$(X_{i/1}, X_{j/1}, X_{i/2}, X_{j/2})$$

   for $(i, j) \in \{(2, 4), (4, 5)\}$.

3. Choose a monomorphism from $X_{3/1}$ to a bijective object $X_{3/3}$. Then construct pushouts

$$(X_{i/1}, X_{j/1}, X_{i/3}, X_{j/3})$$

   for $(i, j) \in \{(3, 4), (4, 5)\}$.

4. Construct further pushouts

$$(X_{4/1}, X_{4/2}, X_{4/3}, \check{X}_{4/4}), (X_{5/1}, X_{5/2}, X_{5/3}, \check{X}_{5/4}), (X_{4/2}, X_{5/2}, \check{X}_{4/4}, \check{X}_{5/4}).$$

   Then $(X_{4/3}, X_{5/3}, \check{X}_{4/4}, \check{X}_{5/4})$ is also a pushout.

5. Choose a monomorphism from $\check{X}_{4/4}$ to a bijective object $X_{4/4}$. Construct a pushout $(\check{X}_{4/4}, \check{X}_{5/4}, X_{4/4}, X_{5/4})$. Then $(X_{4/2}, X_{5/2}, X_{4/4}, X_{5/4})$ and $(X_{4/3}, X_{5/3}, X_{4/4}, X_{5/4})$ are also pushouts.

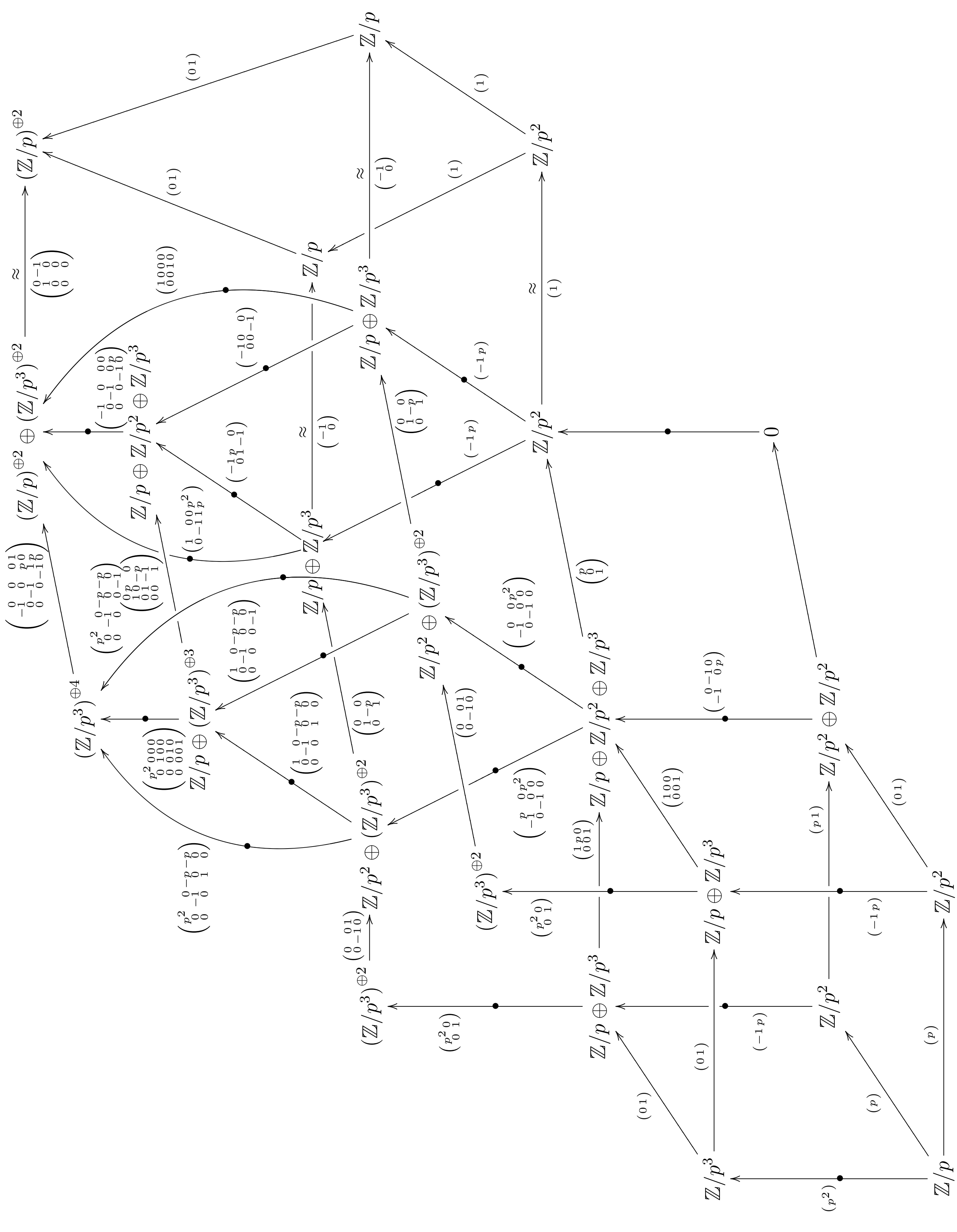
Second, we construct the standard □-triangle $\underline{X}$.

1. Apply the residue class functor $\mathrm{R} : \mathbb{Z}/p^3\text{-mod} \longrightarrow \mathbb{Z}/p^3\text{-}\underline{\text{mod}}$ to the whole diagram.

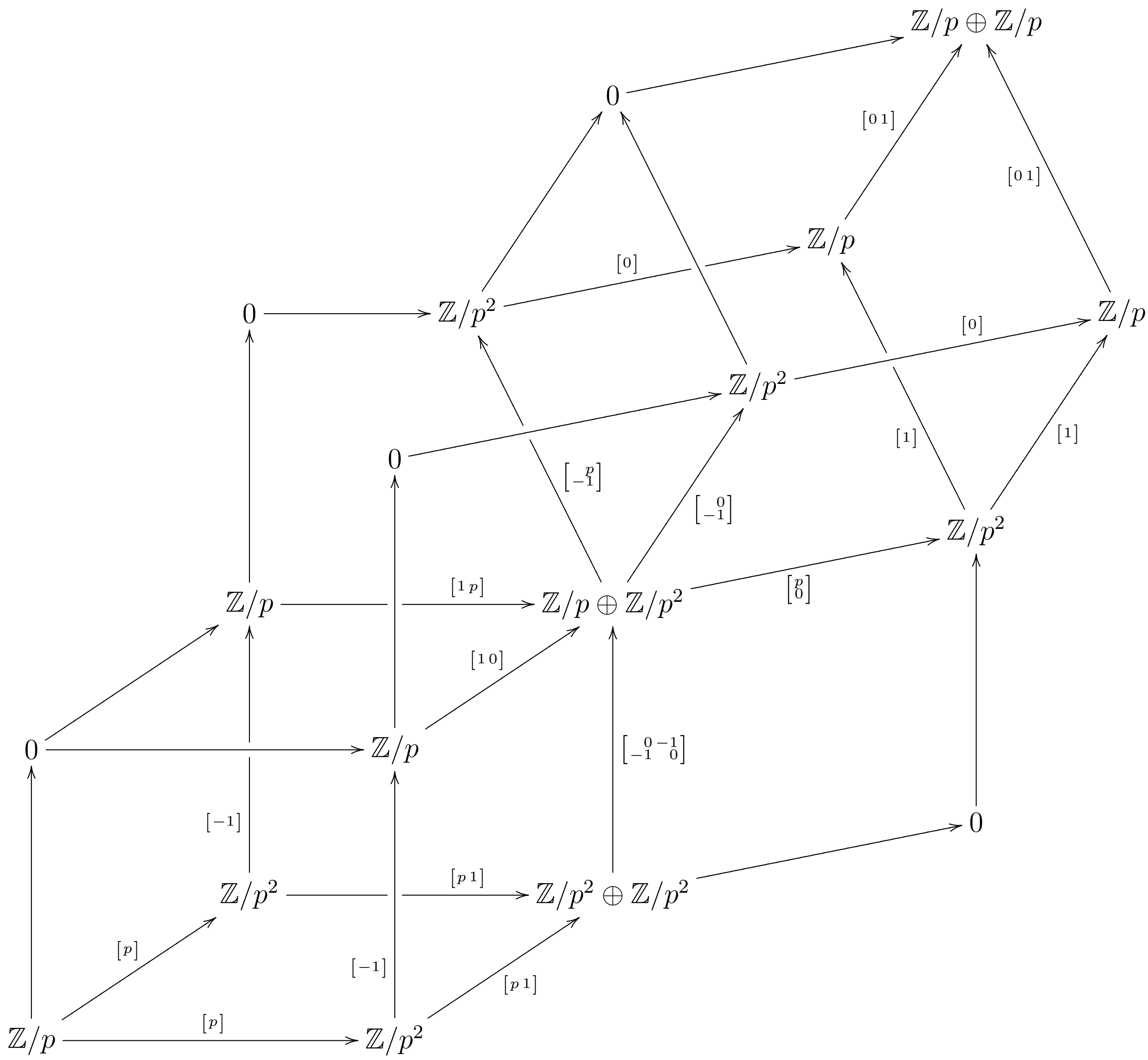
2. For $i \in \{1, 2, 3, 4\}$, replace the object $X_{5/i}$ by $\mathrm{T}X_{i/0}$ and the morphism $X_{4/i} \longrightarrow X_{5/i}$ by its composite with the isomorphism $\mathrm{H}_{s_i^X, s_{X_{i/0}}}(1_{X_{i/0}})$.

3. Omit the auxiliary morphisms and objects (cf. definition 2.1), composing where necessary.

The following diagram in $\mathbb{Z}/p^3$-mod displays all construction steps so far. It contains the □-triangle model X, which commutes in $\mathbb{Z}/p^3$-mod. The whole diagram commutes only after application of the residue class functor $\mathrm{R} : \mathbb{Z}/p^3\text{-mod} \longrightarrow \mathbb{Z}/p^3\text{-}\underline{\text{mod}}$.



Bijective objects in $\mathcal{A}$ are mapped to zero objects in $\underline{\mathcal{A}}$ under the residue class functor $\mathrm{R} : \mathcal{A} \longrightarrow \underline{\mathcal{A}}$. We omit the summands of the form $(\mathbb{Z}/p^3)^{\oplus k}$ from the $\underline{X}$, writing 0 for the empty sum. The resulting diagram Y, shown below, is isomorphic to $\underline{X}$ and therefore a □-triangle.



2.3 The □-triangle Y'

We construct a second □-triangle Y' analogous to Y on the base

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \xrightarrow{[0\,1]} & \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2 \\ \uparrow{\scriptstyle [p]} & & \uparrow{\scriptstyle [0\,1]} \\ \mathbb{Z}/p & \xrightarrow{[p]} & \mathbb{Z}/p^2. \end{array}$$

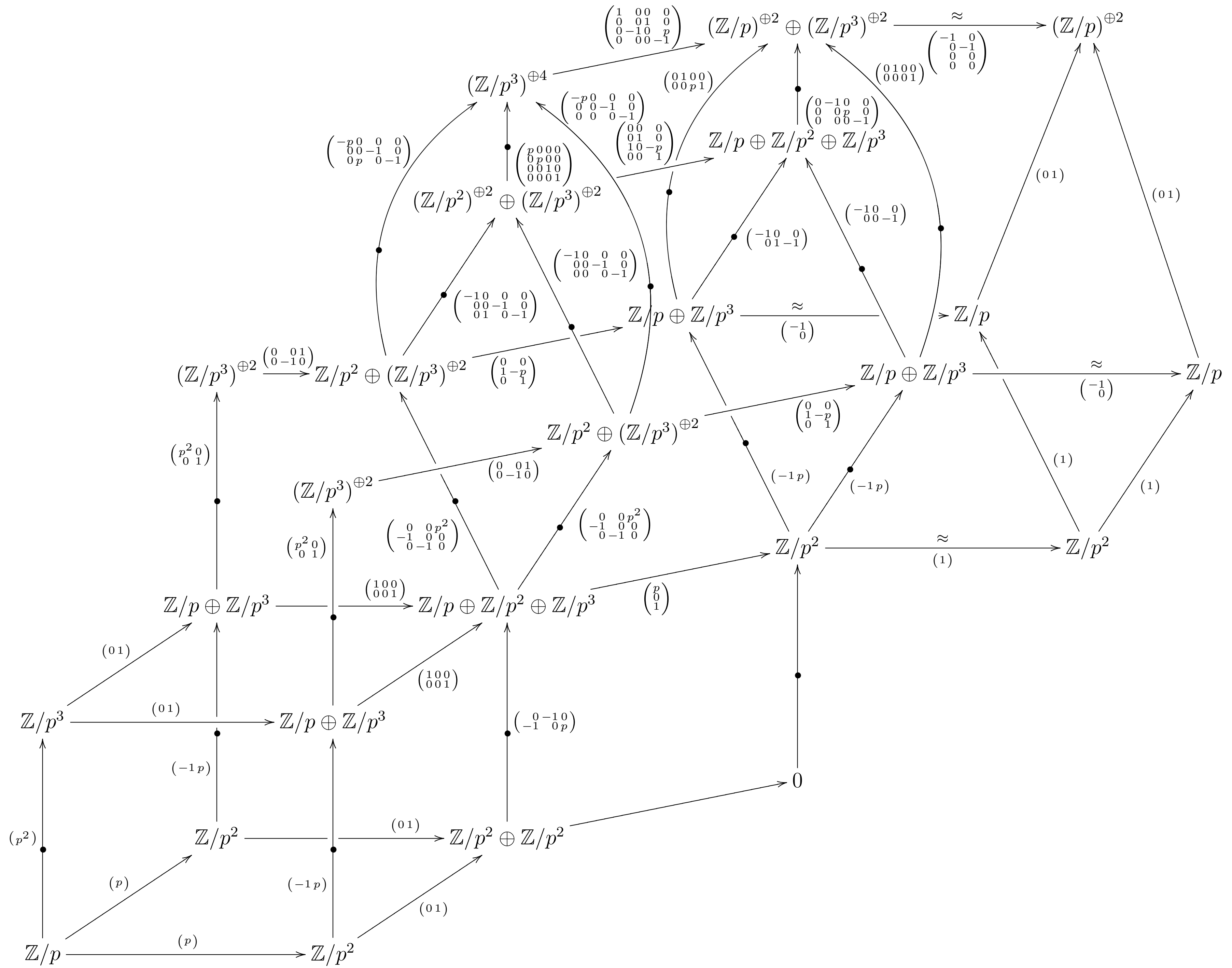
This time, we construct a $\square$-triangle model X' such that

$$
\begin{array}{ccccc}
 & & X'_{5/0} \\
 & & \nearrow{\scriptstyle x'} \\
X'_{3/0} & \xrightarrow{x'} & X'_{4/0} & & \\
\uparrow{\scriptstyle x'} & & \uparrow{\scriptstyle x'} \\
X'_{1/0} & \xrightarrow{x'} & X'_{2/0}
\end{array}
\quad = \quad
\begin{array}{ccccc}
 & & 0 \\
 & & \nearrow \\
\mathbb{Z}/p^2 & \xrightarrow{(p\,1)} & \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2 \\
\uparrow{\scriptstyle (p)} & & \uparrow{\scriptstyle (p\,1)} \\
\mathbb{Z}/p & \xrightarrow{(p)} & \mathbb{Z}/p^2.
\end{array}
$$

Note that the morphism $x' : X'_{3/0} \longrightarrow X'_{4/0}$ differs from $x : X_{3/0} \longrightarrow X_{4/0}$, but their images $[x']$ and $[x]$ under the residue class functor are equal.

First, we construct X' levelwise, analogously to X. Second, we pass to $\underline{X}'$, analogously to $\underline{X}$.

The following diagram in $\mathbb{Z}/p^3$-mod displays the construction steps. It contains the $\square$-triangle model X'. The whole diagram commutes after application of the residue class functor $\mathrm{R} : \mathbb{Z}/p^3\text{-mod} \longrightarrow \mathbb{Z}/p^3\text{-}\underline{\text{mod}}$. It commutes in $\mathbb{Z}/p^3$-mod only incidentally.


(ℤ/p)^⊕2 ⊕ (ℤ/p^3)^⊕2
(ℤ/p)^⊕2
(ℤ/p^3)^⊕4
ℤ/p ⊕ ℤ/p^2 ⊕ ℤ/p^3
(ℤ/p^2)^⊕2 ⊕ (ℤ/p^3)^⊕2
ℤ/p ⊕ ℤ/p^3
ℤ/p
(ℤ/p^3)^⊕2
ℤ/p^2 ⊕ (ℤ/p^3)^⊕2
ℤ/p ⊕ ℤ/p^3
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ℤ/p^2 ⊕ (ℤ/p^3)^⊕2
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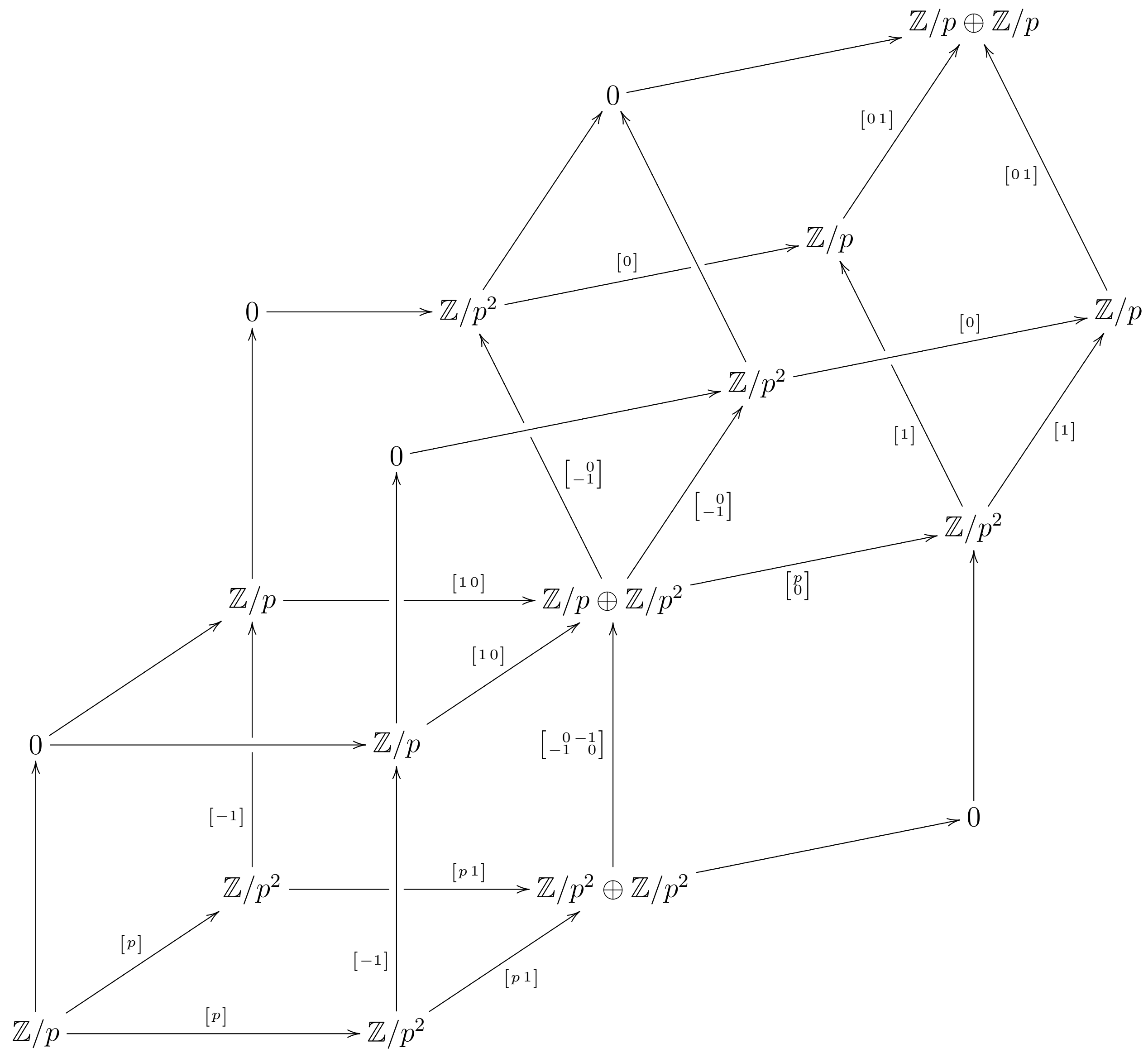
ℤ/p ⊕ ℤ/p^3
ℤ/p ⊕ ℤ/p^2 ⊕ ℤ/p^3
ℤ/p^3
ℤ/p ⊕ ℤ/p^3
0
ℤ/p^2
ℤ/p^2 ⊕ ℤ/p^2
ℤ/p
ℤ/p^2
≈

Analogously to Y, we obtain the desired □-triangle Y', shown below, by an isomorphic replacement of $\underline{X}'$.



2.4 The □-triangles Y and Y' are not isomorphic

Theorem 2.7. *There exists an abelian Frobenius category $\mathcal{A}$ and two □-triangles in $\underline{\mathcal{A}}$ that both have the same base, but that are not isomorphic to each other.*

Proof. Let $\mathcal{A} = \mathbb{Z}/p^3$-mod. Concerning morphisms in $\underline{\mathcal{A}}$, see lemma 1.10.

Consider Y from section 2.2 and Y' from section 2.3. We observe that Y and Y' have the same basis

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \xrightarrow{[p\,1]} & \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2\ . \\ \uparrow{\scriptstyle [p]} & & \uparrow{\scriptstyle [p\,1]} \\ \mathbb{Z}/p & \xrightarrow{[p]} & \mathbb{Z}/p^2 \end{array}$$

We claim that they are not isomorphic in $\underline{\mathcal{A}}$. To prove this, it suffices to show that the subdiagrams $(Y_{2/1}, Y_{4/1}, Y_{3/1})$ and $(Y'_{2/1}, Y'_{4/1}, Y'_{3/1})$ are not isomorphic.

Assume that they are isomorphic. That means there are $a, b, c, d, e, f \in \mathbb{Z}$ such that

$$\begin{array}{ccccc} \mathbb{Z}/p & \xrightarrow{[1\,0]} & \mathbb{Z}/p \oplus \mathbb{Z}/p^2 & \xleftarrow{[1\,0]} & \mathbb{Z}/p \\ \downarrow{\scriptstyle [a]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} b & pc \\ d & e \end{smallmatrix}\right]} & & \downarrow{\scriptstyle [f]} \\ \mathbb{Z}/p & \xrightarrow{[1\,p]} & \mathbb{Z}/p \oplus \mathbb{Z}/p^2 & \xleftarrow{[1\,0]} & \mathbb{Z}/p \end{array} \tag{2.2}$$

commutes and the vertical morphisms are isomorphisms.

Since the left quadrangle in diagram 2.2 commutes we have:

$$\left[\,b\ pc\,\right] = \left[\,1\ 0\,\right]\begin{bmatrix} b & pc \\ d & e \end{bmatrix} = \left[\,a\,\right]\left[\,1\ p\,\right] = \left[\,a\ pa\,\right]. \tag{2.3}$$

It follows that $b \overset{\circledast}{\equiv}_p a$ and $pc \equiv_{p^2} pa$, and therefore $c \overset{\circledast\circledast}{\equiv}_p a$.

Since the right quadrangle in diagram 2.2 also commutes we have

$$\left[\,b\ pc\,\right] = \left[\,1\ 0\,\right]\begin{bmatrix} b & pc \\ d & e \end{bmatrix} = \left[\,f\,\right]\left[\,1\ 0\,\right] = \left[\,f\ 0\,\right]. \tag{2.4}$$

We get $b \equiv_p f$ and $pc \equiv_{p^2} 0$, and therefore $c \equiv_p 0$.

Together with $\circledast$ and $\circledast\circledast$ we have:

$$0 \equiv_p c \overset{\circledast\circledast}{\equiv}_p a \overset{\circledast}{\equiv}_p b \equiv_p f. \tag{2.5}$$

Hence $[a] = [0]$ is not an isomorphism and $[f] = [0]$ is not an isomorphism, which is a contradiction. □

Corollary 2.8. *There exist a □-triangle X with base $\dot{X}$, a □-triangle Y with base $\dot{Y}$ and a diagram morphism $\dot{f} : \dot{X} \longrightarrow \dot{Y}$ in $\underline{\mathcal{A}}$ such that there does not exist a morphism of triangles $f : X \longrightarrow Y$ that restricts to $\dot{f}$.*

Proof. By theorem 2.7, there exist a □-triangle X with base $\dot{X}$, a □-triangle Y with base $\dot{Y}$ such that $\dot{X} = \dot{Y}$ and such that X and Y are not isomorphic as □-triangles.

Let $\dot{f} := 1_{\dot{X}} = 1_{\dot{Y}} : \dot{X} \longrightarrow \dot{Y}$. Now assume that there exists a morphism of □-triangles $f : X \longrightarrow Y$ that restricts to $\dot{f}$. Then

$$\begin{array}{ccccccc} X_{i/0} & \longrightarrow & X_{k/0} & \longrightarrow & X_{k/i} & \longrightarrow & X_{5/i} \\ \downarrow{\scriptstyle f_{i/0}} & & \downarrow{\scriptstyle f_{k/0}} & & \downarrow{\scriptstyle f_{k/i}} & & \downarrow{\scriptstyle f_{5/i}} \\ Y_{i/0} & \longrightarrow & Y_{k/0} & \longrightarrow & Y_{k/i} & \longrightarrow & Y_{5/i} \end{array}$$

is a morphism of ordinary (Verdier) triangles [2, section 2.5] in $\underline{\mathcal{A}}$ for all $i, k \in \{1, \ldots, 4\}$ with $i \leq k$. Since $f_{i/0} = 1_{X_{i/0}} = 1_{Y_{i/0}}$ for all $i \in \{1, \ldots, 4\}$, it follows from [1, sec. 4.1.4] that $f_{k/i}$ is an isomorphism for all $i, k \in \{1, \ldots, 4\}$ with $i \leq k$. But then f is an isomorphism of □-triangles in contradiction to X and Y being not isomorphic. □

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Erklärung

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Aachen, 27.7.2012