The discriminant embedding

Bachelor Thesis

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Contents

0 Introduction

0.1 The Riemann sphere

Riemann projects the complex plane to the sphere S^2, having its north pole N on the complex plane $\mathbb{C}$ [**4,** §B.3, p. 80–81]. Then the point $z \in \mathbb{C}$, its projection $p(z) \in \mathrm{S}^2$ and the south pole S lie on a straight line. To $\mathbb{C}$, a point ∞ is added, which is mapped to the south pole S.

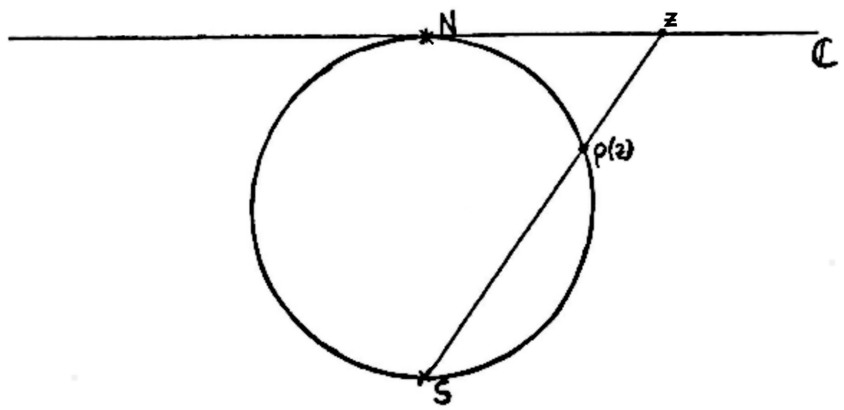


Figure 0.1: Symbolic sketch of the Riemann sphere

As a variant, one can place the sphere such that its equator lies in the complex plane $\mathbb{C}$. In this variant, the point $z \in \mathbb{C}$, its projection $\delta_1(z) \in \mathrm{S}^2$ and the south pole S are still required to lie on a straight line. The point ∞ is still mapped to the south pole S.

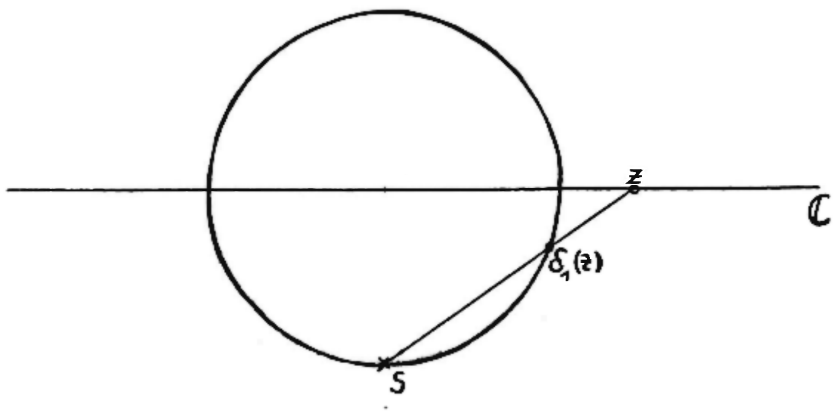


Figure 0.2: Symbolic sketch of the variant of the Riemann sphere

So more formally, identifying $z \in \mathbb{C}$ with $(1 : z) \in \mathrm{P}^1(\mathbb{C})$, this variant gives a map

$$\begin{aligned} \delta_1 : \quad \mathrm{P}^1(\mathbb{C}) &\longrightarrow \mathbb{C} \times \mathbb{R} \\ (z_0 : z_1) &\longmapsto \frac{1}{|z_0|^2 + |z_1|^2} \cdot \Big(2\, \overline{z_0} z_1 \ , \ |z_0|^2 - |z_1|^2 \Big) \end{aligned}$$

restricting to the bijection

$$\delta_1|^{\mathrm{S}^2} : \mathrm{P}^1(\mathbb{C}) \xrightarrow{\sim} \mathrm{S}^2$$

and mapping $\infty := (0:1)$ to the south pole $S = (0, -1)$ of S^2.

0.2 A generalisation: the discriminant embedding

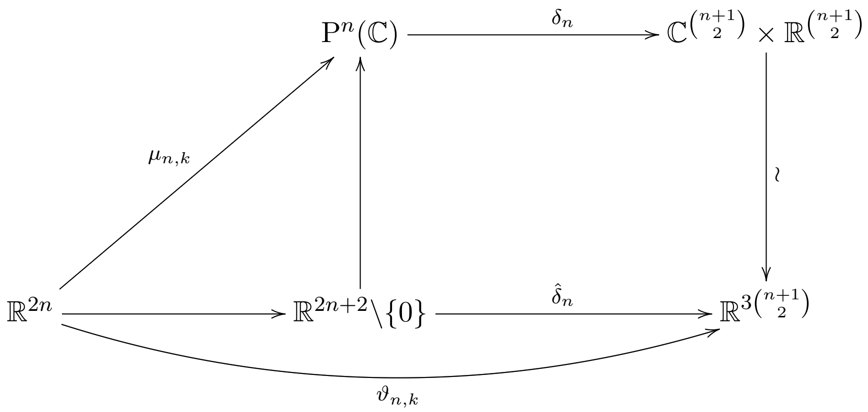
We generalize the variant δ_1 from §0.1 to the map

$$\begin{aligned} \delta_n : \qquad \mathrm{P}^n(\mathbb{C}) &\longrightarrow \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}} \\ (u_0 : \ldots : u_n) &\longmapsto \frac{1}{\sqrt{n} \sum_{j=0}^{n} |u_j|^2} \cdot \bigg(\Big(\sqrt{2}\sqrt{n+1}\, \overline{u_j} u_k \Big)_{0 \leqslant j < k \leqslant n} \ , \ \Big(|u_j|^2 - |u_k|^2 \Big)_{0 \leqslant j < k \leqslant n} \bigg), \end{aligned}$$

called the discriminant embedding.

We show that δ_n is injective; cf. Proposition 12.

We show that δ_n is an immersion; cf. Theorem 43. To do so we use the standard charts $\mu_{n,k}$.



We calculate the Jacobian matrix J on such a standard chart, i.e. the Jacobian matrix of $\vartheta_{n,k}$. We show that this matrix J has full rank. More precisely, we calculate that

$$\det(\mathrm{J}^{\mathrm{T}}\mathrm{J}) \ = \ \left(\frac{2(n+1)}{n} \right)^{2n} \cdot \frac{1}{\Big(1 + \sum_{j \in [0,n] \backslash \{k\}} |u_j|^2 \Big)^{2(n+1)}} > 0 \, .$$

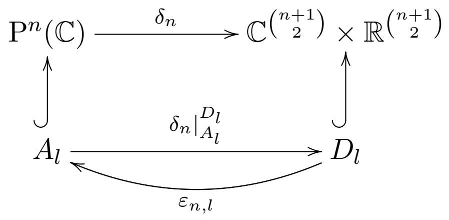
In the process it turned out to be convenient to use the Jacobian matrix of $\hat{\delta}_n$, see diagram.

We show that δ_n maps to $\mathrm{S}^{3\binom{n+1}{2}-1} \subseteq \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}}$; cf. Proposition 11. Since $\mathrm{P}^n(\mathbb{C})$ has real dimension $2n$, which is smaller than $3\binom{n+1}{2} - 1$ for $n \geqslant 2$, the image of δ_n is strictly contained in this sphere in this case.

In addition, we construct partial retractions: For each $l \in [0, n]$, we consider the standard chart $A_l \overset{\text{open}}{\subseteq} \mathrm{P}^n(\mathbb{C})$. We choose $D_l \overset{\text{open}}{\subseteq} \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}}$. We find a continuous map

$$\varepsilon_{n,l} : D_l \longrightarrow A_l$$

such that $\varepsilon_{n,l} \circ \big(\delta_n|_{A_l}^{D_l}\big) = \mathrm{id}_{A_l}$; cf. Proposition 16.



0.3 Similar embeddings by Mannoury and Fuks

G. Mannoury embedded $\mathrm{P}^2(\mathbb{C})$ into $\mathbb{R}^9$ by a map similar to the discriminant embedding, but not directly generalising the Riemann sphere [**3,** eq. (10) on p. 121].

B. A. Fuks constructed a generalisation of the Riemann sphere that resembles closely our discriminant embedding, the difference being that he uses fewer real components [**1,** p. 81–85, Th. 5.3]. So he maps into a smaller space, whereas our map seems to have a more symmetric shape.

0.4 A visual application

We use $\delta_2 : \mathrm{P}^2(\mathbb{C}) \longrightarrow \mathrm{S}^8 \subseteq \mathbb{C}^3 \times \mathbb{R}^3$ to depict the image of the graph of $y = x^3 - x$. More precisely, letting

$$\hat{\Gamma} := \{(x : y : z) \in \mathrm{P}^2(\mathbb{C}) : yz^2 = x^3 - xz^2\} \subseteq \mathrm{P}^2(\mathbb{C})$$

we depict $\delta_2(\hat{\Gamma}) \subseteq \mathrm{S}^8$. Note that even for $|x|$ large, the point $\delta_2\big((x : x^3 - x : 1)\big)$ is still in S^8. Applying δ_2 has the effect of a kind of a fish-eye lens [**2,** §5, §7].

The source [**2**] contains a mistake, which is corrected here; cf. [**2,** §5, footnote 1].

1 Preliminaries

1.1 Conventions

1. Suppose given $a, b \in \mathbb{Z}$. Let $[a, b] := \{z \in \mathbb{Z} : a \leqslant z \leqslant b\}$.
2. We often abbreviate "for all" to "for".
3. Suppose given $p, q \geqslant 0$. For $w = (z_1, \ldots, z_p, x_1, \ldots, x_q) \in \mathbb{C}^p \times \mathbb{R}^q$ we let

$$\|w\| := \sqrt{\sum_{j=1}^{p} |z_j|^2 + \sum_{k=1}^{q} x_k^2}\,.$$

4. Suppose given $n \geqslant 1$.

   We order a tuple $(a_{r,s})_{0 \leqslant r < s \leqslant n}$ using the lexicographical ordering of the indices, that is,

$$(a_{j,k})_{0 \leqslant j < k \leqslant n} = (a_{0,1}\,,\ a_{0,2}\,,\ \ldots\,,\ a_{0,n}\,,\ a_{1,2}\,,\ a_{1,3}\,,\ \ldots\,,\ a_{1,n}\,,\ \ldots\,,\ a_{n-1,n})\,.$$

   Similarly,

$$\begin{aligned}(a_{j,k}, b_{j,k})_{0 \leqslant j < k \leqslant n} = (&a_{0,1}\,,\ b_{0,1}\,,\ a_{0,2}\,,\ b_{0,2}\,,\ \ldots\,,\ a_{0,n}\,,\ b_{0,n}\,,\\ &a_{1,2}\,,\ b_{1,2}\,,\ a_{1,3}\,,\ b_{1,3}\,,\ \ldots\,,\ a_{1,n}\,,\ b_{1,n}\,,\ \ldots\,,\ a_{n-1,n}\,,\ b_{n-1,n})\,.\end{aligned}$$

5. Given tuples $(a_0, a_1, \ldots, a_j)$ and $(b_0, b_1, \ldots, b_k)$ with entries in some set, we write

$$(a_0, a_1, \ldots, a_j) \sqcup (b_0, b_1, \ldots, b_k) := (a_0, a_1, \ldots, a_j, b_0, b_1, \ldots, b_k)$$

   for their concatenation.

   For example, $(1, 2, 5) \sqcup (8, 9) = (1, 2, 5, 8, 9)$.
6. Given $D \overset{\text{open}}{\subseteq} \mathbb{R}^k$ and a differential map

$$f : D \longrightarrow \mathbb{R}^l\,,$$

   we denote by

$$\mathrm{J}(f)(x) \in \mathbb{R}^{l \times k}$$

   the Jacobian matrix of f at $x \in D$.

   We often write $\mathrm{J}(f) := \mathrm{J}(f)(x)$.

1.2 Geometry

Suppose given $p, q \geqslant 0$ and $n \geqslant 1$.

Definition 1. Let

$$\mathrm{S}^{p+q-1} := \{w \in \mathbb{C}^p \times \mathbb{R}^q : \|w\| = 1\}$$

be the $(p+q-1)$-dimensional sphere.

Definition 2. The n-dimensional projective space (over $\mathbb{C}$) is defined to be the set $\mathrm{P}^n(\mathbb{C})$ of the subspaces of $\mathbb{C}$-dimension 1 in $\mathbb{C}^{n+1}$.

Note that as a real manifold, $\mathrm{P}^n(\mathbb{C})$ has dimension $2n$.

Suppose given $(u_0, \ldots, u_n) \in \mathbb{C}^{n+1} \backslash \{(0, \ldots, 0)\}$. The one dimensional subspace generated by $(u_0, \ldots, u_n)$ will be denoted by $(u_0 : \ldots : u_n)$.

Note that for $(u'_0, \ldots, u'_n), (u''_0, \ldots, u''_n) \in \mathbb{C}^{n+1} \backslash \{(0, \ldots, 0)\}$ we have

$$(u'_0 : \ldots : u'_n) = (u''_0 : \ldots : u''_n)$$

if and only if there exists $\lambda \in \mathbb{C} \backslash \{0\}$ with

$$\lambda \cdot (u'_0, \ldots, u'_n) = (u''_0, \ldots, u''_n) \, .$$

Definition 3. Suppose given $(u_0 : \ldots : u_n) \in \mathbb{C}^{n+1}$.

The set $\{j \in [0, n] : u_j \neq 0\}$ is called the *support* of $u := (u_0 : \ldots : u_n)$. So the support is the set of all indices of entries not equal to zero.

Remark 4. Suppose given $(u_0, \ldots, u_n) \in \mathbb{C}^{n+1} \backslash \{0\}$. Write $v_k := \frac{1}{\sqrt{\sum_{j=0}^{n} |u_j|^2}} u_k$ for $k \in [0, n]$. Then

$$(u_0 : \ldots : u_n) = (v_0 : \ldots : v_n)$$

and

$$\sum_{j=0}^{n} |v_j|^2 = 1 \, .$$

1.3 Linear Algebra

Lemma 5. *Suppose given $A \in \mathbb{R}^{m \times n}$. Then*

$$\mathrm{rk}(A) = \mathrm{rk}(A^{\mathrm{T}} A) \, .$$

Proof. We show that

$$\ker(A) = \ker(A^{\mathrm{T}} A) \, .$$

The inclusion $\subseteq$ follows, since $Ax = 0$ implies $A^{\mathrm{T}} Ax = 0$ for $x \in \mathbb{R}^{n \times 1}$.

The inclusion $\supseteq$ follows, since $A^{\mathrm{T}}Ax = 0$ implies $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$, i.e. $(Ax)^{\mathrm{T}} \cdot (Ax) = 0$, which implies $Ax = 0$.

We conclude that

$$\mathrm{rk}(A) = n - \dim(\ker(A)) = n - \dim(\ker(A^{\mathrm{T}}A)) = \mathrm{rk}(A^{\mathrm{T}}A)\ .$$

$\square$

Remark 6. Over $\mathbb{C}$, the assumption of Lemma 5 is false. For instance, for $A = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \in \mathbb{C}^{2\times 1}$ we have

$$\mathrm{rk}(A) = 1\ ,$$

but

$$\mathrm{rk}(A^{\mathrm{T}}A) = 0\ .$$

1.4 Real and complex matrices

Definition 7. We have the injective ring morphism

$$\begin{array}{rccl} \iota: & \mathbb{C} & \longrightarrow & \mathbb{R}^{2\times 2} \\ & a + \mathrm{i}b & \longmapsto & \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\ . \end{array}$$

So given $n \geqslant 0$ we have the injective ring morphism

$$\begin{array}{rcclcl} \iota^{n\times n}: & \mathbb{C}^{n\times n} & \longrightarrow & (\mathbb{R}^{2\times 2})^{n\times n} & = & \mathbb{R}^{2n\times 2n} \\ & (z_{k,l})_{k,l} & \longmapsto & (\iota(z_{k,l}))_{k,l}\ . & & \end{array}$$

Remark 8. *Given $a, b \in \mathbb{R}$, then we have*

$$\begin{aligned} \det(\iota(a + \mathrm{i}b)) &= \det \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ &= a^2 + b^2 \\ &= |a + \mathrm{i}b|^2 \end{aligned}$$

Lemma 9. *For $A \in \mathbb{C}^{n\times n}$ we have*

$$\det(\iota^{n\times n}(A)) = |\det(A)|^2\ .$$

Proof. Choose $S \in \mathbb{C}^{n\times n}$ invertible such that $S^{-1}AS = J$ is in Jordan normal form.

Let

$$J =: \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Then $A = SJS^{-1}$. So

$$\begin{aligned}
|\det(A)|^2 &= \left|\det(SJS^{-1})\right|^2 \\
&= \left|\det(S)\det(J)\det(S)^{-1}\right|^2 \\
&= |\det(J)|^2 \\
&= |\lambda_1 \cdot \ldots \cdot \lambda_n|^2 \\
&= |\lambda_1|^2 \cdot \ldots \cdot |\lambda_n|^2
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\det(\iota^{n\times n}(A)) &= \det\left(\iota^{n\times n}(SJS^{-1})\right) \\
&\overset{\iota^{n\times n}\text{ ring morph.}}{=} \det\left(\iota^{n\times n}(S)\cdot\iota^{n\times n}(J)\cdot\iota^{n\times n}(S^{-1})\right) \\
&= \det\left(\iota^{n\times n}(S)\right)\cdot\det\left(\iota^{n\times n}(J)\right)\cdot\det\left(\iota^{n\times n}(S)\right)^{-1} \\
&= \det\left(\iota^{n\times n}(J)\right) \\
&= \begin{pmatrix} \iota(\lambda_1) & & * \\ & \ddots & \\ 0 & & \iota(\lambda_n) \end{pmatrix} \\
&= \det\left(\iota(\lambda_1)\right)\cdot\ldots\cdot\det\left(\iota(\lambda_n)\right) \\
&\overset{\text{Rem. 8}}{=} |\lambda_1|^2\cdot\ldots\cdot|\lambda_n|^2 \\
&= |\det(A)|^2 \ .
\end{aligned}$$

□

2 The discriminant embedding

2.1 Definition of the discriminant embedding

Suppose given $n \geqslant 1$.

Definition 10. We define the *discriminant embedding* to be the following map.

$$\begin{aligned} \delta_n : \qquad \mathrm{P}^n(\mathbb{C}) &\longrightarrow \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}} \\ (u_0 : \ldots : u_n) &\longmapsto \frac{1}{\sqrt{n} \sum\limits_{j=0}^{n} |u_j|^2} \cdot \left(\left(\sqrt{2}\sqrt{n+1}\, \overline{u_j} u_k \right)_{0 \leqslant j < k \leqslant n} , \left(|u_j|^2 - |u_k|^2 \right)_{0 \leqslant j < k \leqslant n} \right) \end{aligned}$$

We have to show that the discriminant embedding is well-defined. So we have to show that, given $(u_0, \ldots, u_n), (v_0, \ldots, v_n) \in \mathbb{C}^{n+1} \backslash \{0\}$ such that $(u_0 : \ldots : u_n) = (v_0 : \ldots : v_n)$, the image using the representative $(u_0, \ldots, u_n)$ equals the image using the representative $(v_0, \ldots, v_n)$.

By definition of $\mathrm{P}^n(\mathbb{C})$, there exists a $\lambda \in \mathbb{C}$ such that

$$(u_0, \ldots, u_n) = (\lambda v_0, \ldots, \lambda v_n).$$

Then we obtain the following.

$$\begin{aligned} & \frac{1}{\sqrt{n} \sum\limits_{j=0}^{n} |u_j|^2} \cdot \left(\left(\sqrt{2}\sqrt{n+1}\, \overline{u_j} u_k \right)_{0 \leqslant j < k \leqslant n} , \left(|u_j|^2 - |u_k|^2 \right)_{0 \leqslant j < k \leqslant n} \right) \\ &= \frac{1}{\sqrt{n} \sum\limits_{j=0}^{n} |\lambda v_j|^2} \cdot \left(\left(\sqrt{2}\sqrt{n+1}\, \overline{\lambda v_j} \lambda v_k \right)_{0 \leqslant j < k \leqslant n} , \left(|\lambda v_j|^2 - |\lambda v_k|^2 \right)_{0 \leqslant j < k \leqslant n} \right) \\ &= \frac{1}{|\lambda|^2 \sqrt{n} \sum\limits_{j=0}^{n} |v_j|^2} \cdot \left(\left(|\lambda|^2 \sqrt{2}\sqrt{n+1}\, \overline{v_j} v_k \right)_{0 \leqslant j < k \leqslant n} , \left(|\lambda|^2 (|v_j|^2 - |v_k|^2) \right)_{0 \leqslant j < k \leqslant n} \right) \\ &= \frac{1}{|\lambda|^2 \sqrt{n} \sum\limits_{j=0}^{n} |v_j|^2} \cdot |\lambda|^2 \left(\left(\sqrt{2}\sqrt{n+1}\, \overline{v_j} v_k \right)_{0 \leqslant j < k \leqslant n} , \left(|v_j|^2 - |v_k|^2 \right)_{0 \leqslant j < k \leqslant n} \right) \\ &= \frac{1}{\sqrt{n} \sum\limits_{j=0}^{n} |v_j|^2} \cdot \left(\left(\sqrt{2}\sqrt{n+1}\, \overline{v_j} v_k \right)_{0 \leqslant j < k \leqslant n} , \left(|v_j|^2 - |v_k|^2 \right)_{0 \leqslant j < k \leqslant n} \right) \end{aligned}$$

Proposition 11. For $x \in \mathrm{P}^n(\mathbb{C})$ we have

$$\delta_n(x) \in \mathrm{S}^{3\binom{n+1}{2}-1} .$$

Proof. Suppose given $x = (u_0 : \ldots : u_n) \in \mathrm{P}^n(\mathbb{C})$. We have to show that $\|\delta_n(x)\| \overset{!}{=} 1$. We square both sides and have to show that

$$1 = \left\| \frac{1}{\sqrt{n}\sum_{j=0}^{n}|u_j|^2} \cdot \left(\left(\sqrt{2}\sqrt{n+1}\,\overline{u_j}u_k\right)_{0\leqslant j<k\leqslant n} , \left(|u_j|^2 - |u_k|^2\right)_{0\leqslant j<k\leqslant n} \right) \right\|^2 ,$$

i.e. that

$$\left(\sqrt{n}\sum_{j=0}^{n}|u_j|^2\right)^2 = \left\| \left(\left(\sqrt{2}\sqrt{n+1}\,\overline{u_j}u_k\right)_{0\leqslant j<k\leqslant n} , \left(|u_j|^2 - |u_k|^2\right)_{0\leqslant j<k\leqslant n} \right) \right\|^2 .$$

We calculate.

$$\begin{aligned}
&\left\| \left(\left(\sqrt{2}\sqrt{n+1}\,\overline{u_j}u_k\right)_{0\leqslant j<k\leqslant n} , \left(|u_j|^2 - |u_k|^2\right)_{0\leqslant j<k\leqslant n} \right) \right\|^2 \\
&= \sum_{0\leqslant j<k\leqslant n} |\sqrt{2}\sqrt{n+1}\,\overline{u_j}u_k|^2 \;+\; \sum_{0\leqslant j<k\leqslant n} \left(|u_j|^2 - |u_k|^2\right)^2 \\
&= \sum_{0\leqslant j<k\leqslant n} 2(n+1)|u_j|^2|u_k|^2 \;+\; \sum_{0\leqslant j<k\leqslant n} \left(|u_j|^4 - 2|u_j|^2|u_k|^2 + |u_k|^4\right) \\
&= \sum_{0\leqslant j<k\leqslant n} 2(n+1)|u_j|^2|u_k|^2 \;+\; \sum_{0\leqslant j<k\leqslant n} |u_j|^4 \;-\; \sum_{0\leqslant j<k\leqslant n} 2|u_j|^2|u_k|^2 \;+\; \sum_{0\leqslant j<k\leqslant n} |u_k|^4 \\
&= \sum_{0\leqslant j<k\leqslant n} 2n|u_j|^2|u_k|^2 \;+\; \sum_{0\leqslant j<k\leqslant n} |u_j|^4 \;+\; \sum_{0\leqslant k<j\leqslant n} |u_j|^4 \\
&= \sum_{0\leqslant j<k\leqslant n} 2n|u_j|^2|u_k|^2 \;+\; \sum_{j,k\in[0,n],j\neq k} |u_j|^4 \\
&= \sum_{0\leqslant j<k\leqslant n} 2n|u_j|^2|u_k|^2 \;+\; n\sum_{0\leqslant j\leqslant n} |u_j|^4 \\
&= n\cdot\left(\left(\sum_{0\leqslant j<k\leqslant n} 2|u_j|^2|u_k|^2\right) \;+\; \left(\sum_{j=0}^{n}|u_j|^4\right) \right) \\
&= \left(\sqrt{n}\sum_{j=0}^{n}|u_j|^2\right)^2
\end{aligned}$$

□

2.2 Injectivity of the discriminant embedding

Proposition 12 (Injectivity of the discriminant embedding)**.**

Recall that we have the discriminant embedding

$$\begin{aligned} \delta_n : \qquad & \mathrm{P}^n(\mathbb{C}) \longrightarrow \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}} \\ (u_0 : \ldots : u_n) & \longmapsto \frac{1}{\sqrt{n} \sum\limits_{j=0}^{n} |u_j|^2} \cdot \left(\left(\sqrt{2}\sqrt{n+1}\, \overline{u_j} u_k \right)_{0 \leqslant j < k \leqslant n} , \left(|u_j|^2 - |u_k|^2 \right)_{0 \leqslant j < k \leqslant n} \right) \end{aligned}$$

from Definition 10.

Then the map δ_n is injective.

Proof. Suppose given $u = (u_0 : \ldots : u_n) \in \mathrm{P}^n(\mathbb{C})$ and $u' = (u'_0 : \ldots : u'_n) \in \mathrm{P}^n(\mathbb{C})$ such that $\delta_n(u) = \delta_n(u')$. Without loss of generality, we may assume that $\sum_{j=1}^n |u_j|^2 = 1$, and $\sum_{j=1}^n |u'_j|^2 = 1$ cf. Remark 4. Multiplication with $\sqrt{n}$ leads to

$$\begin{aligned} & (\sqrt{2}\sqrt{n+1}\, \overline{u_0} u_1, \ldots, \sqrt{2}\sqrt{n+1}\, \overline{u_{n-1}} u_n \, , \, |u_0|^2 - |u_1|^2, \ldots, |u_{n-1}|^2 - |u_n|^2) \\ = \; & (\sqrt{2}\sqrt{n+1}\, \overline{u'_0} u'_1, \ldots, \sqrt{2}\sqrt{n+1}\, \overline{u'_{n-1}} u'_n \, , \, |u'_0|^2 - |u'_1|^2, \ldots, |u'_{n-1}|^2 - |u'_n|^2) \, . \end{aligned}$$

So we get

$$\begin{aligned} \overline{u_0} u_1 &= \overline{u'_0} u'_1 \\ &\vdots \\ \overline{u_{n-1}} u_n &= \overline{u'_{n-1}} u'_n \quad . \end{aligned}$$

By conjugation, we get

$$\begin{aligned} \overline{u_1} u_0 &= \overline{u'_1} u'_0 \\ &\vdots \\ \overline{u_n} u_{n-1} &= \overline{u'_n} u'_{n-1} \quad . \end{aligned}$$

So overall we have

$$(*) \qquad \overline{u_j} u_k = \overline{u'_j} u'_k$$

for $j, k \in [0, n]$.

Moreover we have

$$\begin{aligned} |u_0|^2 - |u_1|^2 &= |u'_0|^2 - |u'_1|^2 \\ &\vdots \\ |u_{n-1}|^2 - |u_n|^2 &= |u'_{n-1}|^2 - |u'_n|^2 \quad . \end{aligned}$$

By multiplication with -1, we get

$$\begin{aligned} |u_1|^2 - |u_0|^2 &= |u'_1|^2 - |u'_0|^2 \\ &\vdots \\ |u_n|^2 - |u_{n-1}|^2 &= |u'_n|^2 - |u'_{n-1}|^2 \quad . \end{aligned}$$

So overall we have

$$(**) \qquad |u_j|^2 - |u_k|^2 = |u'_j|^2 - |u'_k|^2$$

for $j, k \in [0, n]$.

Let $I = \{j \in [0, n] : u_j \neq 0\}$ be the support of u and, analogously, let $I' = \{j \in [0, n] : u'_j \neq 0\}$ be the support of u'; cf. Definition 3. We remark that $I, I' \neq \emptyset$.

We *claim* that $I \overset{!}{=} I'$.

By symmetry, it suffices to show $I \overset{!}{\subseteq} I'$.

Suppose given $j \in I$. We want to show that $j \overset{!}{\in} I'$.

We have $u_j \neq 0$ and we want to show that $u'_j \neq 0$.

Assumption: $u'_j = 0$.

Case 1: $I = \{j\}$.

Then $u_k = 0$ for $k \in [0, n] \backslash \{j\}$. Choose an element $k \in [0, n] \backslash \{j\}$. So

$$0 < |u_j|^2 = |u_j|^2 - |u_k|^2 \overset{(**)}{=} |u'_j|^2 - |u'_k|^2 = -|u'_k|^2 \leqslant 0 \, ,$$

which is impossible.

Case 2: $I \supset \{j\}$.

Choose $k \in I \backslash \{j\}$. So $u_k \neq 0$. Then

$$0 = \overline{u'_j} u'_k \overset{(*)}{=} \overline{u_j} u_k \, ,$$

hence $\overline{u_j} = 0$, which is not possible.

So we have a *contradiction* in both cases. This proves the *claim*, i.e. $I = I'$.

Case 1: $|I| = 1$. Let $I =: \{j\}$.

So

$$u = (0 : \ldots : 0 : u_j : 0 : \ldots : 0) = (0 : \ldots : 0 : u'_j : 0 : \ldots : 0) = u'$$

Case 2: $|I| = 2$. Let $I =: \{j, k\}$, with $j \neq k$.

We have $1 = |u_0|^2 + \ldots + |u_n|^2 = |u_j|^2 + |u_k^2|$ and $1 = |u'_0|^2 + \ldots + |u'_n|^2 = |u'_j|^2 + |u'_k|^2$.

We have $\overline{u_j} u_k \overset{(*)}{=} \overline{u'_j} u'_k$ and $|u_j|^2 - |u_k|^2 \overset{(**)}{=} |u'_j|^2 - |u'_k|^2$.

Thus

$$1 - 2|u_j|^2 = |u_j|^2 + |u_k|^2 - 2|u_k|^2 = |u_j|^2 - |u_k|^2 = |u'_j|^2 - |u'_k|^2 = |u'_j|^2 + |u'_k|^2 - 2|u'_k|^2 = 1 - 2|u'_j|^2 \, .$$

Hence $|u_j|^2 = |u'_j|^2$ and $|u_k|^2 = |u'_k|^2$.
So

$$\frac{u_j}{u'_j} = \frac{u_j\overline{u_j}u'_k}{u'_j\overline{u_j}u'_k} = \frac{u'_j\overline{u'_j}u'_k}{u'_j\overline{u_j}u'_k} = \frac{u'_j\overline{u_j}u_k}{u'_j\overline{u_j}u'_k} = \frac{u_k}{u'_k} .$$

Therefore

$$u = (0 : \ldots : 0 : u_j : 0 : \ldots : 0 : u_k : 0 : \ldots : 0) = (0 : \ldots : 0 : u'_j : 0 : \ldots : 0 : u'_k : 0 : \ldots : 0) = u' .$$

Case 3: Let $|I| \geqslant 3$. Suppose given $j, k \in I$.
We have to show that $\frac{u_k}{u'_k} \overset{!}{=} \frac{u_j}{u'_j}$
Choose $l \in I\backslash\{j, k\}$, which is possible because of $|I| \geqslant 3$. Then

$$\frac{u_j}{u'_j} \overset{(*)}{=} \frac{\overline{u'_l}}{\overline{u_l}} \overset{(*)}{=} \frac{u_k}{u'_k}$$

So

$$(u_0 : \ldots : u_n) = (u'_0 : \ldots : u'_n)$$

□

3 Partial retractions

Suppose given $n \geqslant 1$.

In the following we want to establish a partial retraction for the discriminant embedding δ_n.

Suppose given $l \in [0, n]$.

Definition 13. Consider the open subset

$$A_l := \{(u_0 : \ldots : u_n) \in \mathrm{P}^n(\mathbb{C}) : u_l \neq 0\} \subseteq \mathrm{P}^n(\mathbb{C}) \,.$$

Let

$$N_l := \{((z_{j,k})_{j<k}, (x_{j,k})_{j<k}) \in \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}} : 1 + \sqrt{n} \sum_{k \in [0,n] \setminus \{l\}} x_{l,k} = 0\} \,.$$

Consider the open subset

$$D_l := \left(\mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}}\right) \setminus N_l \subseteq \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}} \,.$$

Definition 14. Consider the continuous map

$$\begin{aligned} \varepsilon_{n,l} : D_l &\longrightarrow A_l \\ ((z_{j,k})_{j<k}, (x_{j,k})_{j<k}) &\longmapsto (z_{l,0} : \ldots : z_{l,l-1} : \\ &\qquad \sqrt{\frac{2}{(n+1)n}} + \sqrt{\frac{2}{(n+1)}} \sum_{k \in [0,n] \setminus \{l\}} x_{l,k} : z_{l,l+1} : \ldots : z_{l,n}) \end{aligned}$$

Comment 15. So N_l consists of all the points for which the formula does not give an element of A_l.

Proposition 16. We have $\varepsilon_{n,l} \circ \delta_n|_{A_l}^{D_l} = \mathrm{id}_{A_l}$.

Given an element $((z_{j,k})_{j<k}, (x_{j,k})_{j<k}) \in \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}}$, we write $x_{k,j} := -x_{j,k}$ and $z_{k,j} := \overline{z_{j,k}}$ for $0 \leqslant j < k \leqslant n$.

Proof. Given $(u_0 : \ldots : u_n) \in A_l$, we obtain the following.

$$(**) \qquad \sqrt{\frac{2}{(n+1)n}} + \sqrt{\frac{2}{(n+1)}} \sum_{k\in[0,n]\setminus\{l\}} \frac{|u_l|^2 - |u_k|^2}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2}$$

$$= \sqrt{\frac{2}{(n+1)n}} \left(1 + \sum_{k\in[0,n]\setminus\{l\}} \frac{|u_l|^2 - |u_k|^2}{\sum\limits_{j=0}^{n}|u_j|^2} \right)$$

$$= \sqrt{\frac{2}{(n+1)n}} \frac{1}{\sum\limits_{j=0}^{n}|u_j|^2} \left(\sum_{j=0}^{n}|u_j|^2 + \sum_{k\in[0,n]\setminus\{l\}} (|u_l|^2 - |u_k|^2) \right)$$

$$= \sqrt{\frac{2}{(n+1)n}} \frac{1}{\sum\limits_{j=0}^{n}|u_j|^2} \left((n+1)|u_l|^2 \right)$$

$$= \frac{\sqrt{2}\sqrt{n+1}\,|u_l|^2}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} = \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_l}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2}$$

We calculate as follows.

$$\varepsilon_{n,l}(\delta_n(u_0 : \ldots : u_n))$$

$$= \varepsilon_{n,l}\left(\frac{1}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} \cdot \left(\left(\sqrt{2}\sqrt{n+1}\,\overline{u_j}u_k\right)_{0\leqslant j<k\leqslant n} , \left(|u_j|^2 - |u_k|^2\right)_{0\leqslant j<k\leqslant n} \right) \right)$$

$$= \left(\frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_0}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} : \ldots : \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_{l-1}}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} : \right.$$

$$\sqrt{\frac{2}{(n+1)n}} + \sqrt{\frac{2}{(n+1)}} \sum_{k\in[0,n]\setminus\{l\}} \frac{|u_l|^2 - |u_k|^2}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} :$$

$$\left. \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_{l+1}}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} : \ldots : \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_n}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} \right)$$

$$
\begin{aligned}
&\overset{(**)}{=} \left(\frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_0}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} : \ldots : \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_{l-1}}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} : \right. \\
&\quad \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_l}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} : \\
&\quad \left. \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_{l+1}}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} : \ldots : \frac{\sqrt{2}\sqrt{n+1}\,\overline{u_l}u_n}{\sqrt{n}\sum\limits_{j=0}^{n}|u_j|^2} \right) \\
&= (u_0 : \ldots : u_{l-1} : u_l : u_{l+1} : \ldots : u_n)
\end{aligned}
$$

. □

4 The Riemann sphere

4.1 Introduction

In this chapter we want to show that the discriminant embedding δ_1 amounts to the standard map from $\mathrm{P}^1(\mathbb{C})$ to the Riemann sphere. In this sense the discriminant embedding δ_n generalises the Riemann sphere construction.

We will consider the variant of the Riemann sphere that has the equator sitting in the complex plane and the south pole, here denoted by U, corresponding to the point $\infty := (0:1)$ at infinity. We write $\varphi(1:z) := (z, 0)$.

(2).jpg

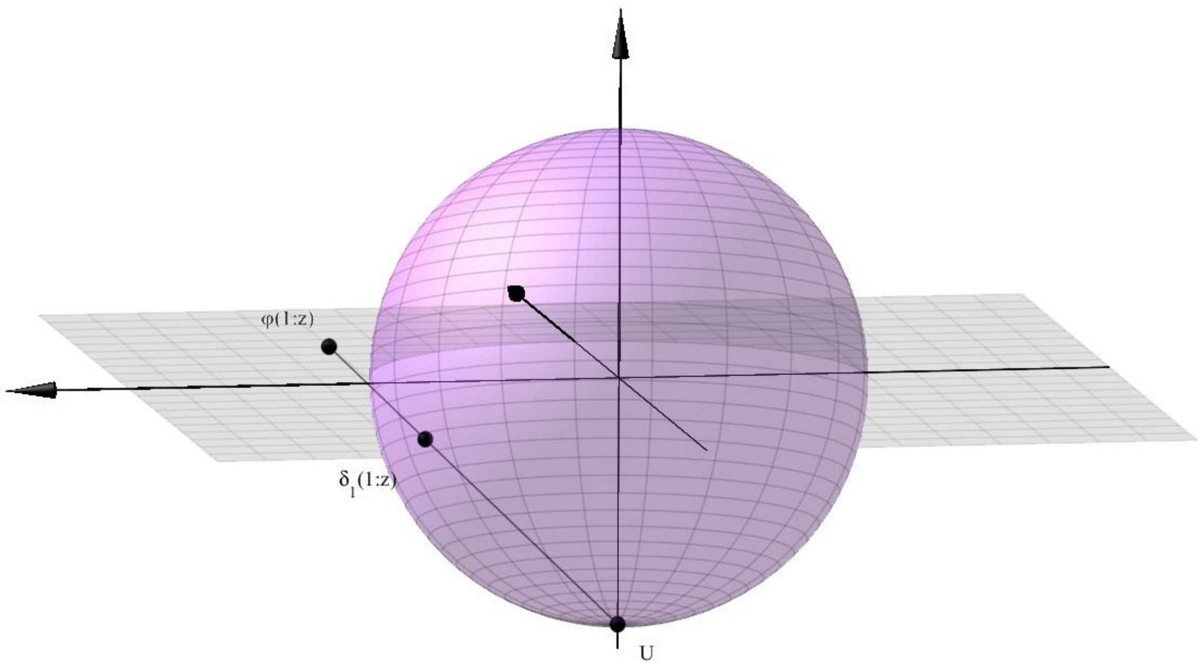


Figure 4.1: Riemann sphere

4.2 The Riemann sphere map

Write $\infty := (0:1)$. Let

$$\begin{aligned}\varphi : \mathrm{P}^1(\mathbb{C})\backslash\{\infty\} &\longrightarrow \mathbb{C}\times\mathbb{R}\\ (1:z) &\longmapsto (z,0)\end{aligned}$$

And let

$$\delta_1(\infty) = (0,-1) =: U\,.$$

We want to show that $\varphi(1:z)$, $\delta_1(1:z)$ and U are on the same real straight line for $z \in \mathbb{C}$.

In coordinates, this means that $(z,0)$, $\frac{1}{1+|z|^2}(2z, 1-|z|^2)$ and $(0,-1)$ are shown to be on the same real straight line.

For $r \in \mathbb{R}^\times$ we have

$$\begin{aligned} r\cdot\Big((z,0)-(0,-1)\Big) &\overset{!}{=} \frac{1}{1+|z|^2}(2z,1-|z|^2)-(0,-1)\\ \Leftrightarrow r\cdot(z,1) &\overset{!}{=} \frac{1}{1+|z|^2}\Big(2z,(1-|z|^2)+(1+|z|^2)\Big)\\ &= \frac{1}{1+|z|^2}(2z,2)\\ \Leftrightarrow r &= \frac{2}{1+|z|^2}\end{aligned}$$

So $\frac{1}{1+|z|^2}(2z,1-|z|^2)$ lies in the straight line passing through $(z,0)$ and $(0,-1)$.

4.3 The partial retraction $\varepsilon_{1,0}$ for δ_1

In §3 we have found a partial retraction $\varepsilon_{n,l}$ of the discriminant embedding. In this section we want to illustrate $\varepsilon_{1,0}$ as an example.

Example 17. We have

$$A_0 = \{(u_0:u_1)\in \mathrm{P}^1(\mathbb{C}) : u_0 \neq 0\}\,.$$

We have

$$D_0 = (\mathbb{C}\times\mathbb{R})\backslash N_0 = \{(z_{0,1},x_{0,1})\in\mathbb{C}\times\mathbb{R} : x_{0,1}\neq -1\}\,.$$

In Definition 14 we have considered the map

$$\begin{aligned}\varepsilon_{1,0} : \qquad D_0 &\longrightarrow A_0\\ (z_{0,1},x_{0,1}) &\longmapsto (1+x_{0,1} : z_{0,1})\,.\end{aligned}$$

For $(u_0 : u_1) \in A_0$, we calculate directly

$$\begin{aligned}
\varepsilon_{1,0}(\delta_1(u_0 : u_1)) &= \varepsilon_{1,0}\left(\frac{1}{|u_0|^2 + |u_1|^2} \cdot (2\overline{u_0}u_1, |u_0|^2 - |u_1|^2)\right) \\
&= \left(1 + \frac{|u_0|^2 - |u_1|^2}{|u_0|^2 + |u_1|^2} : \frac{2\overline{u_0}u_1}{|u_0|^2 + |u_1|^2}\right) \\
&= (|u_0|^2 + |u_1|^2 + |u_0|^2 - |u_1|^2 : 2\overline{u_0}u_1) \\
&= (2|u_0|^2 : 2\overline{u_0}u_1) \\
&= (2\overline{u_0}u_0 : 2\overline{u_0}u_1) \\
&= (u_0 : u_1)\,.
\end{aligned}$$

Remark 18. Note that U, $(z_{0,1}, x_{0,1})$ and $\varphi(\varepsilon_{1,0}(z_{0,1}, x_{0,1})) = (\frac{z_{0,1}}{1+x_{0,1}}, 0)$ lie on a real straight line.

5 The discriminant embedding is an immersion

5.1 Calculation of the matrix entries of the Jacobian matrix $\mathrm{J}(\hat{\delta}_n)$

Suppose given $n \geqslant 1$.

Definition 19. We have the bijective map

$$\begin{aligned} \eta_n : \qquad \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}} &\longrightarrow \mathbb{R}^{3\binom{n+1}{2}} \\ ((z_{r,s})_{0\leqslant r<s\leqslant n}, (u_{r,s})_{0\leqslant r<s\leqslant n}) &\longmapsto \Big((\mathrm{Re}(z_{r,s}), \mathrm{Im}(z_{r,s}))_{0\leqslant r<s\leqslant n} \sqcup (u_{r,s})_{0\leqslant r<s\leqslant n} \Big) . \end{aligned}$$

Definition 20. We have the surjective map

$$\begin{aligned} \rho_n : \qquad \mathbb{R}^{2n+2}\backslash\{0\} &\longrightarrow \mathrm{P}^n(\mathbb{C}) \\ (a_0, b_0, a_1, b_1, \ldots, a_n, b_n) &\longmapsto (a_0 + \mathrm{i}b_0 : a_1 + \mathrm{i}b_1 : \ldots : a_n + \mathrm{i}b_n) . \end{aligned}$$

Definition 21. For $\underline{x} := (a_0, b_0, \ldots, a_n, b_n) \in \mathbb{R}^{2n+2}$, we let

$$\delta_n(\rho_n(\underline{x})) = \delta_n(a_0 + \mathrm{i}b_0 : \ldots : a_n + \mathrm{i}b_n) =: ((f_{r,s}(\underline{x}) + \mathrm{i}g_{r,s}(\underline{x}))_{0\leqslant r<s\leqslant n}, (v_{r,s}(\underline{x}))_{0\leqslant r<s\leqslant n}).$$

This amounts to the following.

$$\begin{aligned} f_{r,s}(\underline{x}) &:= \frac{\sqrt{2(n+1)}(a_r a_s + b_r b_s)}{\sqrt{n}\sum_{j=0}^n (a_j^2 + b_j^2)} \\ g_{r,s}(\underline{x}) &:= \frac{\sqrt{2(n+1)}(a_r b_s - b_r a_s)}{\sqrt{n}\sum_{j=0}^n (a_j^2 + b_j^2)} \\ v_{r,s}(\underline{x}) &:= \frac{(a_r^2 + b_r^2) - (a_s^2 + b_s^2)}{\sqrt{n}\sum_{j=0}^n (a_j^2 + b_j^2)} \end{aligned}$$

If unambiguous we write $f_{r,s}(\underline{x}) =: f_{r,s}$, $g_{r,s}(\underline{x}) =: g_{r,s}$ and $v_{r,s}(\underline{x}) =: v_{r,s}$ for $0 \leqslant r < s \leqslant n$.

In addition we let $f_{s,r} := f_{r,s}$, $g_{s,r} := -g_{r,s}$ and $v_{s,r} := -v_{r,s}$.

That is, we extend the definitions above for $f_{r,s}$, $g_{r,s}$ and $v_{r,s}$ to the case $r, s \in [0, n]$ such that $r \neq s$.

We also abbreviate $\sigma := \sigma(\underline{x}) := \sum_{j=0}^n (a_j^2 + b_j^2)$.

Definition 22. Let

$$\begin{aligned} \hat{\delta}_n := \eta_n \circ \delta_n \circ \rho_n : \qquad \mathbb{R}^{2n+2}\backslash\{0\} &\longrightarrow \mathbb{R}^{3\binom{n+1}{2}} \\ \underline{x} = (a_0, b_0, \ldots, a_n, b_n) &\longmapsto \Big((f_{r,s}, g_{r,s})_{0\leqslant r<s\leqslant n} \sqcup (v_{r,s})_{0\leqslant r<s\leqslant n} \Big) \end{aligned}$$

Definition 23. For $k \in [0, n]$, we let

$$\begin{aligned} \nu_{n,k} : \qquad \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n+2} \backslash \{0\} \\ (a_0, b_0, \ldots, a_{k-1}, b_{k-1}, a_{k+1}, b_{k+1}, \ldots, a_n, b_n) &\longmapsto (a_0, b_0, \ldots, a_{k-1}, b_{k-1}, 1, 0, a_{k+1}, b_{k+1}, \ldots, a_n, b_n) . \end{aligned}$$

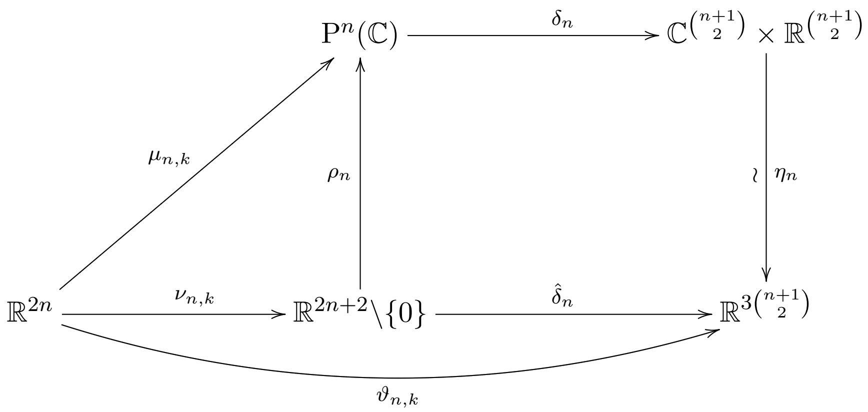
Definition 24. For $k \in [0, n]$, we let

$$\begin{aligned} \mu_{n,k} := \rho_n \circ \nu_{n,k} : \qquad \mathbb{R}^{2n} &\longrightarrow \mathrm{P}^n(\mathbb{C}) \\ (a_0, b_0, \ldots, a_{k-1}, b_{k-1}, a_{k+1}, b_{k+1}, \ldots, a_n, b_n) &\longmapsto (a_0 + \mathrm{i} b_0 : \ldots : a_{k-1} + \mathrm{i} b_{k-1} : 1 \\ &\qquad : a_{k+1} + \mathrm{i} b_{k+1} : \ldots : a_n + \mathrm{i} b_n) . \end{aligned}$$

Definition 25. For $k \in [0, n]$ we let.

$$\begin{aligned} \vartheta_{n,k} := \hat{\delta}_n \circ \nu_{n,k} : \qquad \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{3\binom{n+1}{2}} \\ (a_0, b_0, \ldots, a_{k-1}, b_{k-1}, a_{k+1}, b_{k+1}, \ldots, a_n, b_n) &\longmapsto \Big((f_{r,s}, g_{r,s})_{0 \leqslant r < s \leqslant n} \sqcup (v_{r,s})_{0 \leqslant r < s \leqslant n} \Big) \Big|_{a_k = 1, b_k = 0} . \end{aligned}$$

So we get the commutative diagram:



Remark 26. To show that the discriminant embedding δ_n is an immersion, we have to show that the Jacobian matrix $\mathrm{J}(\vartheta_{n,k})$ of $\vartheta_{n,k}$ has rank $2n$ for $k \in [0, n]$.

To show this we shall prove that $\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}} \cdot \mathrm{J}(\vartheta_{n,k})$ has rank $2n$, i.e. that its determinant is nonzero; cf. 5.

We proceed by a calculation of $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}} \cdot \mathrm{J}(\hat{\delta}_n)$, then using a relation between $\mathrm{J}(\hat{\delta}_n)$ and $\mathrm{J}(\vartheta_{n,k})$.

Remark 27. The Jacobian matrix $\mathrm{J}(\hat{\delta}_n)$ has entries $\frac{\partial}{\partial a_t} f_{r,s}$, $\frac{\partial}{\partial a_t} g_{r,s}$, $\frac{\partial}{\partial a_t} v_{r,s}$, $\frac{\partial}{\partial b_t} f_{r,s}$, $\frac{\partial}{\partial b_t} g_{r,s}$, $\frac{\partial}{\partial b_t} v_{r,s}$ with $0 \leqslant t \leqslant n$ and $0 \leqslant r < s \leqslant n$.

With that we get

$$\frac{\partial}{\partial a_t} f_{r,s} = \begin{cases} \frac{\sqrt{2(n+1)} a_s}{\sqrt{n} \sigma} - \frac{2 a_r}{\sigma} f_{r,s} & \text{if } t = r \\ \frac{\sqrt{2(n+1)} a_r}{\sqrt{n} \sigma} - \frac{2 a_s}{\sigma} f_{r,s} & \text{if } t = s \\ -\frac{2 a_t}{\sigma} f_{r,s} & \text{if } t \in [0, n] \backslash \{r, s\} \end{cases}$$

$$\frac{\partial}{\partial b_t} f_{r,s} = \begin{cases} \frac{\sqrt{2(n+1)} b_s}{\sqrt{n} \sigma} - \frac{2 b_r}{\sigma} f_{r,s} & \text{if } t = r \\ \frac{\sqrt{2(n+1)} b_r}{\sqrt{n} \sigma} - \frac{2 b_s}{\sigma} f_{r,s} & \text{if } t = s \\ -\frac{2 b_t}{\sigma} f_{r,s} & \text{if } t \in [0, n] \backslash \{r, s\} \end{cases}$$

$$\frac{\partial}{\partial a_t} g_{r,s} = \begin{cases} \frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} g_{r,s} & \text{if } t = r \\ -\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} g_{r,s} & \text{if } t = s \\ -\frac{2a_t}{\sigma} g_{r,s} & \text{if } t \in [0,n] \backslash \{r,s\} \end{cases}$$

$$\frac{\partial}{\partial b_t} g_{r,s} = \begin{cases} -\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} g_{r,s} & \text{if } t = r \\ \frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} g_{r,s} & \text{if } t = s \\ -\frac{2b_t}{\sigma} g_{r,s} & \text{if } t \in [0,n] \backslash \{r,s\} \end{cases}$$

$$\frac{\partial}{\partial a_t} v_{r,s} = \begin{cases} \frac{2a_r}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} v_{r,s} & \text{if } t = r \\ -\frac{2a_s}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} v_{r,s} & \text{if } t = s \\ -\frac{2a_t}{\sigma} v_{r,s} & \text{if } t \in [0,n] \backslash \{r,s\} \end{cases}$$

$$\frac{\partial}{\partial b_t} v_{r,s} = \begin{cases} \frac{2b_r}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} v_{r,s} & \text{if } t = r \\ -\frac{2b_s}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} v_{r,s} & \text{if } t = s \\ -\frac{2b_t}{\sigma} v_{r,s} & \text{if } t \in [0,n] \backslash \{r,s\} \end{cases}$$

Example 28. Suppose that $n = 2$. Then:

$$\begin{aligned} f_{0,1} &= \tfrac{\sqrt{3}(a_0a_1+b_0b_1)}{\sigma}, g_{0,1} = \tfrac{\sqrt{3}(a_0b_1-b_0a_1)}{\sigma}, \\ f_{0,2} &= \tfrac{\sqrt{3}(a_0a_2+b_0b_2)}{\sigma}, g_{0,2} = \tfrac{\sqrt{3}(a_0b_2-b_0a_2)}{\sigma}, \\ f_{1,2} &= \tfrac{\sqrt{3}(a_1a_2+b_1b_2)}{\sigma}, g_{1,2} = \tfrac{\sqrt{3}(a_1b_2-b_1a_2)}{\sigma}, \\ v_{0,1} &= \tfrac{a_0^2+b_0^2-a_1^2-b_1^2}{\sqrt{2}\sigma}, v_{0,2} = \tfrac{a_0^2+b_0^2-a_2^2-b_2^2}{\sqrt{2}\sigma}, v_{1,2} = \tfrac{a_1^2+b_1^2-a_2^2-b_2^2}{\sqrt{2}\sigma} \end{aligned}$$

$$\mathrm{J}(\hat{\delta}_2) = \begin{pmatrix} \frac{\partial}{\partial a_0} f_{0,1} & \frac{\partial}{\partial b_0} f_{0,1} & \frac{\partial}{\partial a_1} f_{0,1} & \frac{\partial}{\partial b_1} f_{0,1} & \frac{\partial}{\partial a_2} f_{0,1} & \frac{\partial}{\partial b_2} f_{0,1} \\ \frac{\partial}{\partial a_0} g_{0,1} & \frac{\partial}{\partial b_0} g_{0,1} & \frac{\partial}{\partial a_1} g_{0,1} & \frac{\partial}{\partial b_1} g_{0,1} & \frac{\partial}{\partial a_2} g_{0,1} & \frac{\partial}{\partial b_2} g_{0,1} \\ \frac{\partial}{\partial a_0} f_{0,2} & \frac{\partial}{\partial b_0} f_{0,2} & \frac{\partial}{\partial a_1} f_{0,2} & \frac{\partial}{\partial b_1} f_{0,2} & \frac{\partial}{\partial a_2} f_{0,2} & \frac{\partial}{\partial b_2} f_{0,2} \\ \frac{\partial}{\partial a_0} g_{0,2} & \frac{\partial}{\partial b_0} g_{0,2} & \frac{\partial}{\partial a_1} g_{0,2} & \frac{\partial}{\partial b_1} g_{0,2} & \frac{\partial}{\partial a_2} g_{0,2} & \frac{\partial}{\partial b_2} g_{0,2} \\ \frac{\partial}{\partial a_0} f_{1,2} & \frac{\partial}{\partial b_0} f_{1,2} & \frac{\partial}{\partial a_1} f_{1,2} & \frac{\partial}{\partial b_1} f_{1,2} & \frac{\partial}{\partial a_2} f_{1,2} & \frac{\partial}{\partial b_2} f_{1,2} \\ \frac{\partial}{\partial a_0} g_{1,2} & \frac{\partial}{\partial b_0} g_{1,2} & \frac{\partial}{\partial a_1} g_{1,2} & \frac{\partial}{\partial b_1} g_{1,2} & \frac{\partial}{\partial a_2} g_{1,2} & \frac{\partial}{\partial b_2} g_{1,2} \\ \frac{\partial}{\partial a_0} v_{0,1} & \frac{\partial}{\partial b_0} v_{0,1} & \frac{\partial}{\partial a_1} v_{0,1} & \frac{\partial}{\partial b_1} v_{0,1} & \frac{\partial}{\partial a_2} v_{0,1} & \frac{\partial}{\partial b_2} v_{0,1} \\ \frac{\partial}{\partial a_0} v_{0,2} & \frac{\partial}{\partial b_0} v_{0,2} & \frac{\partial}{\partial a_1} v_{0,2} & \frac{\partial}{\partial b_1} v_{0,2} & \frac{\partial}{\partial a_2} v_{0,2} & \frac{\partial}{\partial b_2} v_{0,2} \\ \frac{\partial}{\partial a_0} v_{1,2} & \frac{\partial}{\partial b_0} v_{1,2} & \frac{\partial}{\partial a_1} v_{1,2} & \frac{\partial}{\partial b_1} v_{1,2} & \frac{\partial}{\partial a_2} v_{1,2} & \frac{\partial}{\partial b_2} v_{1,2} \end{pmatrix} = \ldots$$

$$\ldots = \frac{1}{\sigma} \begin{pmatrix} \sqrt{3}a_1 - 2a_0 f_{0,1} & \sqrt{3}b_1 - 2b_0 f_{0,1} & \sqrt{3}a_0 - 2a_1 f_{0,1} & \sqrt{3}b_0 - 2b_1 f_{0,1} & -2a_2 f_{0,1} & -2b_2 f_{0,1} \\ \sqrt{3}b_1 - 2a_0 g_{0,1} & -\sqrt{3}a_1 - 2b_0 g_{0,1} & -\sqrt{3}b_0 - 2a_1 g_{0,1} & \sqrt{3}a_0 - 2b_1 g_{0,1} & -2a_2 g_{0,1} & -2b_2 g_{0,1} \\ \sqrt{3}a_2 - 2a_0 f_{0,2} & \sqrt{3}b_2 - 2b_0 f_{0,2} & -2a_1 f_{0,2} & -2b_1 f_{0,2} & \sqrt{3}a_0 - 2a_2 f_{0,2} & \sqrt{3}b_0 - 2b_2 f_{0,2} \\ \sqrt{3}b_2 - 2a_0 g_{0,2} & -\sqrt{3}a_2 - 2b_0 g_{0,2} & -2a_1 g_{0,2} & -2b_1 g_{0,2} & -\sqrt{3}b_0 - 2a_2 g_{0,2} & \sqrt{3}a_0 - 2b_2 g_{0,2} \\ -2a_0 f_{1,2} & -2b_0 f_{1,2} & \sqrt{3}a_2 - 2a_1 f_{1,2} & \sqrt{3}b_2 - 2b_1 f_{1,2} & \sqrt{3}a_1 - 2a_2 f_{1,2} & \sqrt{3}b_1 - 2b_2 f_{1,2} \\ -2a_0 g_{1,2} & -2b_0 g_{1,2} & \sqrt{3}b_2 - 2a_1 g_{1,2} & -\sqrt{3}a_2 - 2b_1 g_{1,2} & -\sqrt{3}b_1 - 2a_2 g_{1,2} & \sqrt{3}a_1 - 2b_2 g_{1,2} \\ \sqrt{2}a_0 - 2a_0 v_{0,1} & \sqrt{2}b_0 - 2b_0 v_{0,1} & -\sqrt{2}a_1 - 2a_1 v_{0,1} & -\sqrt{2}b_1 - 2b_1 v_{0,1} & -2a_2 v_{0,1} & -2b_2 v_{0,1} \\ \sqrt{2}a_0 - 2a_0 v_{0,2} & \sqrt{2}b_0 - 2b_0 v_{0,2} & -2a_1 v_{0,2} & -2b_1 v_{0,2} & -\sqrt{2}a_2 - 2a_2 v_{0,2} & -\sqrt{2}b_2 - 2b_2 v_{0,2} \\ -2a_0 v_{1,2} & -2b_0 v_{1,2} & \sqrt{2}a_1 - 2a_1 v_{1,2} & \sqrt{2}b_1 - 2b_1 v_{1,2} & -\sqrt{2}a_2 - 2a_2 v_{1,2} & -\sqrt{2}b_2 - 2b_2 v_{1,2} \end{pmatrix}$$

5.2 Preparation of the calculation of the matrix entries of $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}}\mathrm{J}(\hat{\delta}_n)$

Remark 29. Given $\underset{k}{\underline{x}_*} := (a_0, b_0, \ldots, a_{k-1}, b_{k-1}, a_{k+1}, b_{k+1}, \ldots, a_n, b_n) \in \mathbb{R}^{2n+2}$, the chain rule gives

$$\mathrm{J}(\vartheta_{n,k})(\underset{k}{\underline{x}_*}) = \mathrm{J}(\hat{\delta}_n)(\nu_{n,k}(\underset{k}{\underline{x}_*})) \cdot \mathrm{J}(\nu_{n,k})$$

In other words, $\mathrm{J}(\vartheta_{n,k}) \in \mathbb{R}^{3\binom{n+1}{2} \times 2n}$ is obtained from $\mathrm{J}(\hat{\delta}_n) \in \mathbb{R}^{3\binom{n+1}{2} \times (2n+2)}$ by deleting the columns belonging to a_k and b_k and then by putting $a_k = 1$ and $b_k = 0$.

So $\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}} \cdot \mathrm{J}(\vartheta_{n,k}) \in \mathbb{R}^{2n \times 2n}$ is obtained from $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}} \cdot \mathrm{J}(\hat{\delta}_n) \in \mathbb{R}^{(2n+2) \times (2n+2)}$ by deleting the columns and rows belonging to a_k and b_k and then by putting $a_k = 1$ and $b_k = 0$.

The result turns out to be of the form $\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}} \cdot \mathrm{J}(\vartheta_{n,k}) = \iota^{n \times n}(K_{n,k})$ for a hermitian matrix $K_{n,k} \in \mathbb{C}^{n \times n}$.

In order to calculate the matrix entries for $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}} \cdot \mathrm{J}(\hat{\delta}_n)$, but first we need two auxiliary calculations.

Remark 30. Let $t \in [0, n]$. Suppose given $c_j \in \mathbb{R}$ for $j \in [0, n]$.

We get

$$\begin{aligned} & \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t = r} c_s && + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t = s} c_r \\ = \; & \textstyle\sum_{t < s \leqslant n} c_s && + \textstyle\sum_{0 \leqslant r < t} c_r \\ \overset{\text{rename } r = s}{=} \; & \textstyle\sum_{t < s \leqslant n} c_s && + \textstyle\sum_{0 \leqslant s < t} c_s \\ = \; & \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } s \neq t} c_s \end{aligned}$$

Remark 31. Let $t, l \in [0, n]$ with $t < l$. Suppose given $d_{j,k} \in \mathbb{R}$ for $j, k \in [0, n]$.

(1) We get

$$\begin{aligned} & \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t = r,\, l \neq s} d_{r,s} && + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t = s} d_{s,r} \\ = \; & \textstyle\sum_{t < s \leqslant n \text{ and } l \neq s} d_{t,s} && + \textstyle\sum_{0 \leqslant r < t} d_{t,r} \\ \overset{\text{rename } r = s}{=} \; & \textstyle\sum_{t < s \leqslant n \text{ and } l \neq s} d_{t,s} && + \textstyle\sum_{0 \leqslant s < t} d_{t,s} \\ = \; & \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } t \neq s,\, l \neq s} d_{t,s} \end{aligned}$$

(2) We get

$$\begin{aligned} & \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } l = r} d_{r,s} && + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t \neq r,\, l = s} d_{s,r} \\ = \; & \textstyle\sum_{l < s \leqslant n} d_{l,s} && + \textstyle\sum_{0 \leqslant r < l \text{ and } t \neq r} d_{l,r} \\ \overset{\text{rename } r = s}{=} \; & \textstyle\sum_{l < r \leqslant n} d_{l,r} && + \textstyle\sum_{0 \leqslant r < l \text{ and } t \neq r} d_{l,r} \\ = \; & \textstyle\sum_{0 \leqslant r \leqslant n \text{ and } t \neq r,\, l \neq r} d_{l,r} \end{aligned}$$

Definition 32. Such given $k \in [0, n]$. The column of $\mathrm{J}(\hat{\delta}_n)$ belonging to a_k is ζ_{a_k}. The column of $\mathrm{J}(\hat{\delta}_n)$ belonging to b_k is ζ_{b_k}.

We have to calculate the following.

(1) $\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{a_t}$ for $t \in [0, n]$

(2) $\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{b_t}$ for $t \in [0, n]$

(3) $\zeta_{b_t}^{\mathrm{T}} \cdot \zeta_{b_t}$ for $t \in [0, n]$

(4) $\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{a_l}$ for $t, l \in [0, n]$ with $t < l$

(5) $\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{b_l}$ for $t, l \in [0, n]$ with $t < l$

(6) $\zeta_{b_t}^{\mathrm{T}} \cdot \zeta_{b_l}$ for $t, l \in [0, n]$ with $t < l$

5.3 Calculation of the matrix entries of $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}}\mathrm{J}(\hat{\delta}_n)$

Calculation 33. Suppose given $t \in [0, n]$. We calculate.

$$
\begin{aligned}
\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{a_t} = \; & \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_t} v_{r,s}) \\
= \; & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_t} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_t} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_t} v_{r,s}) \\
= \; & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} f_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} f_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(-\frac{2a_t}{\sigma} f_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} g_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(-\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} g_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(-\frac{2a_t}{\sigma} g_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{2a_r}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} v_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(-\frac{2a_s}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} v_{r,s})^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(-\frac{2a_t}{\sigma} v_{r,s})^2 \\
= \; & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{2(n+1)a_s^2}{n\sigma^2} - \frac{4a_t a_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s} + \frac{4a_t^2}{\sigma^2} f_{t,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{2(n+1)a_r^2}{n\sigma^2} - \frac{4a_r a_t\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{r,t} + \frac{4a_t^2}{\sigma^2} f_{r,t}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{4a_t^2}{\sigma^2} f_{r,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{2(n+1)b_s^2}{n\sigma^2} - \frac{4\sqrt{2(n+1)}a_t b_s}{\sqrt{n}\sigma^2} g_{t,s} + \frac{4a_t^2}{\sigma^2} g_{t,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{2(n+1)b_r^2}{n\sigma^2} + \frac{4\sqrt{2(n+1)}b_r a_t}{\sqrt{n}\sigma^2} g_{r,t} + \frac{4a_t^2}{\sigma^2} g_{r,t}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{4a_t^2}{\sigma^2} g_{r,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{4a_t^2}{n\sigma^2} - \frac{8a_t^2}{\sqrt{n}\sigma^2} v_{t,s} + \frac{4a_t^2}{\sigma^2} v_{t,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{4a_t^2}{n\sigma^2} + \frac{8a_t^2}{\sqrt{n}\sigma^2} v_{r,t} + \frac{4a_t^2}{\sigma^2} v_{r,t}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{4a_t^2}{\sigma^2} v_{r,s}^2)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{2(n+1)a_s^2}{n\sigma^2} && -\frac{4a_ta_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}f_{t,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{2(n+1)a_r^2}{n\sigma^2} && -\frac{4a_ra_t\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}f_{r,t}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{2(n+1)b_s^2}{n\sigma^2} && -\frac{4\sqrt{2(n+1)}a_tb_s}{\sqrt{n}\sigma^2}g_{t,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{2(n+1)b_r^2}{n\sigma^2} && +\frac{4\sqrt{2(n+1)}b_ra_t}{\sqrt{n}\sigma^2}g_{r,t}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{4a_t^2}{n\sigma^2} && -\frac{8a_t^2}{\sqrt{n}\sigma^2}v_{t,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{4a_t^2}{n\sigma^2} && +\frac{8a_t^2}{\sqrt{n}\sigma^2}v_{r,t}\big) \\
& +\frac{4a_t^2}{\sigma^2}\cdot\textstyle\sum_{0\leqslant r<s\leqslant n}(f_{r,s}^2+g_{r,s}^2+v_{r,s}^2) \\
\overset{\text{Theorem 11}}{=} \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\frac{2(n+1)a_s^2}{n\sigma^2} \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\frac{2(n+1)a_r^2}{n\sigma^2} \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\frac{2(n+1)b_s^2}{n\sigma^2} \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\frac{2(n+1)b_r^2}{n\sigma^2} \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\frac{4a_t^2}{n\sigma^2} \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\frac{4a_t^2}{n\sigma^2} \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\frac{4a_ta_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}f_{t,s}\big) \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\frac{4a_ra_t\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}f_{r,t}\big) \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\frac{4\sqrt{2(n+1)}a_tb_s}{\sqrt{n}\sigma^2}g_{t,s}\big) \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{4\sqrt{2(n+1)}b_ra_t}{\sqrt{n}\sigma^2}g_{r,t}\big) \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\frac{8a_t^2}{\sqrt{n}\sigma^2}v_{t,s}\big) \\
& \quad +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{8a_t^2}{\sqrt{n}\sigma^2}v_{r,t}\big) \\
& +\frac{4a_t^2}{\sigma^2}\cdot 1 \\
\overset{\text{Rem. 30}}{=} \quad & \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\frac{2(n+1)a_s^2}{n\sigma^2} \\
& \quad +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\frac{2(n+1)b_s^2}{n\sigma^2} \\
& \quad +\frac{4a_t^2}{n\sigma^2}\cdot n \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\frac{4a_ta_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}f_{t,s}\big) \\
& \quad +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\frac{4\sqrt{2(n+1)}a_tb_s}{\sqrt{n}\sigma^2}g_{t,s}\big) \\
& \quad +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\frac{8a_t^2}{\sqrt{n}\sigma^2}v_{t,s}\big) \\
& +\frac{4a_t^2}{\sigma^2} \\
= \quad & \frac{2(n+1)}{n\sigma^2}\big((\textstyle\sum_{0\leqslant s\leqslant n}a_s^2)-a_t^2+(\sum_{0\leqslant s\leqslant n}b_s^2)-b_t^2\big) \\
& \quad +\frac{8a_t^2}{\sigma^2} \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\frac{4a_ta_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}f_{t,s}\big) \\
& \quad +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\frac{4\sqrt{2(n+1)}a_tb_s}{\sqrt{n}\sigma^2}g_{t,s}\big) \\
& \quad +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\frac{8a_t^2}{\sqrt{n}\sigma^2}v_{t,s}\big)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{Def. }\sigma}{=} && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& && +\tfrac{8a_t^2}{\sigma^2} \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(-\tfrac{4a_t a_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s}) \\
& && \qquad +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(-\tfrac{4\sqrt{2(n+1)}a_t b_s}{\sqrt{n}\sigma^2} g_{t,s}) \\
& && \qquad +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(-\tfrac{8a_t^2}{\sqrt{n}\sigma^2} v_{t,s}) \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} \\
& && -\tfrac{4a_t}{\sqrt{n}\sigma^2}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(a_s\sqrt{2(n+1)}f_{t,s} + b_s\sqrt{2(n+1)}g_{t,s} + 2a_t v_{t,s}) \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} \\
& && -\tfrac{4a_t}{\sqrt{n}\sigma^2}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(a_s\sqrt{2(n+1)}\tfrac{\sqrt{2(n+1)}(a_t a_s + b_t b_s)}{\sqrt{n}\sigma} \\
& && \qquad\qquad +b_s\sqrt{2(n+1)}\tfrac{\sqrt{2(n+1)}(a_t b_s - b_t a_s)}{\sqrt{n}\sigma} \\
& && \qquad\qquad +2a_t\tfrac{(a_t^2+b_t^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma}) \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} \\
& && -\tfrac{8a_t}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(\;a_s(n+1)(a_t a_s + b_t b_s) \\
& && \qquad\qquad +b_s(n+1)(a_t b_s - b_t a_s) \\
& && \qquad\qquad +a_t((a_t^2+b_t^2)-(a_s^2+b_s^2))) \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} \\
& && -\tfrac{8a_t}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(\;(n+1)(a_t a_s^2 + a_t b_s^2) \\
& && \qquad\qquad +(a_t^3 + a_t b_t^2) - (a_t a_s^2 + a_t b_s^2)) \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} \\
& && -\tfrac{8a_t^2}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(n(a_s^2+b_s^2) + (a_t^2+b_t^2)) \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} \\
& && -\tfrac{8a_t^2}{n\sigma^3}(n(a_t^2+b_t^2) + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} n(a_s^2+b_s^2)) \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} \\
& && -\tfrac{8a_t^2}{n\sigma^3}n\sigma \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) + \tfrac{8a_t^2}{\sigma^2} - \tfrac{8a_t^2}{\sigma^2} \\
&= && \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2)
\end{aligned}
$$

Calculation 34. Suppose given $t \in [0,n]$. We calculate.

$$
\begin{aligned}
\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{b_t} = \; & \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_t} v_{r,s}) \\
= \; & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_t} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_t} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_t} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_t} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_t} v_{r,s}) \\
= \; & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} f_{r,s}) \cdot (\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} f_{r,s}) \cdot (\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(-\frac{2a_t}{\sigma} f_{r,s}) \cdot (-\frac{2b_t}{\sigma} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} g_{r,s}) \cdot (-\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(-\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} g_{r,s}) \cdot (\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(-\frac{2a_t}{\sigma} g_{r,s}) \cdot (-\frac{2b_t}{\sigma} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{2a_r}{\sqrt{n}\sigma} - \frac{2a_r}{\sigma} v_{r,s}) \cdot (\frac{2b_r}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(-\frac{2a_s}{\sqrt{n}\sigma} - \frac{2a_s}{\sigma} v_{r,s}) \cdot (-\frac{2b_s}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(-\frac{2a_t}{\sigma} v_{r,s}) \cdot (-\frac{2b_t}{\sigma} v_{r,s}) \\
= \; & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{2(n+1)a_s b_s}{n\sigma^2} - \frac{2\sqrt{2(n+1)}(a_t b_s + a_s b_t)}{\sqrt{n}\sigma^2} f_{t,s} + \frac{4a_t b_t}{\sigma^2} f_{t,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{2(n+1)a_r b_r}{n\sigma^2} - \frac{2\sqrt{2(n+1)}(a_t b_r + a_r b_t)}{\sqrt{n}\sigma^2} f_{r,t} + \frac{4a_t b_t}{\sigma^2} f_{r,t}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{4a_t b_t}{\sigma^2} f_{r,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(-\frac{2(n+1)a_s b_s}{n\sigma^2} - \frac{2\sqrt{2(n+1)}(b_s b_t - a_s a_t)}{\sqrt{n}\sigma^2} g_{t,s} + \frac{4a_t b_t}{\sigma^2} g_{t,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(-\frac{2(n+1)a_r b_r}{n\sigma^2} - \frac{2\sqrt{2(n+1)}(a_r a_t - b_r b_t)}{\sqrt{n}\sigma^2} g_{r,t} + \frac{4a_t b_t}{\sigma^2} g_{r,t}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{4a_t b_t}{\sigma^2} g_{r,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\frac{4a_t b_t}{n\sigma^2} - \frac{8a_t b_t}{\sqrt{n}\sigma^2} v_{t,s} + \frac{4a_t b_t}{\sigma^2} v_{t,s}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\frac{4a_t b_t}{n\sigma^2} + \frac{8a_t b_t}{\sqrt{n}\sigma^2} v_{r,t} + \frac{4a_t b_t}{\sigma^2} v_{r,t}^2) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\frac{4a_t b_t}{\sigma^2} v_{r,s}^2)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\tfrac{2(n+1)a_sb_s}{n\sigma^2} - \tfrac{2\sqrt{2(n+1)}(a_tb_s+a_sb_t)}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\tfrac{2(n+1)a_rb_r}{n\sigma^2} - \tfrac{2\sqrt{2(n+1)}(a_tb_r+a_rb_t)}{\sqrt{n}\sigma^2} f_{r,t}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\tfrac{2(n+1)a_sb_s}{n\sigma^2} - \tfrac{2\sqrt{2(n+1)}(b_sb_t-a_sa_t)}{\sqrt{n}\sigma^2} g_{t,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\tfrac{2(n+1)a_rb_r}{n\sigma^2} - \tfrac{2\sqrt{2(n+1)}(a_ra_t-b_rb_t)}{\sqrt{n}\sigma^2} g_{r,t}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\tfrac{4a_tb_t}{n\sigma^2} - \tfrac{8a_tb_t}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\tfrac{4a_tb_t}{n\sigma^2} + \tfrac{8a_tb_t}{\sqrt{n}\sigma^2} v_{r,t}\big) \\
& + \tfrac{4a_tb_t}{\sigma^2} \cdot \textstyle\sum_{0\leqslant r<s\leqslant n}(f_{r,s}^2 + g_{r,s}^2 + v_{r,s}^2) \\
\overset{\text{Theorem 11}}{=} \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \tfrac{2(n+1)a_sb_s}{n\sigma^2} \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \tfrac{2(n+1)a_rb_r}{n\sigma^2} \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\tfrac{2(n+1)a_sb_s}{n\sigma^2}\big) \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\tfrac{2(n+1)a_rb_r}{n\sigma^2}\big) \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \tfrac{4a_tb_t}{n\sigma^2} \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \tfrac{4a_tb_t}{n\sigma^2} \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\tfrac{2\sqrt{2(n+1)}(a_tb_s+a_sb_t)}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\tfrac{2\sqrt{2(n+1)}(a_tb_r+a_rb_t)}{\sqrt{n}\sigma^2} f_{r,t}\big) \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\tfrac{2\sqrt{2(n+1)}(b_sb_t-a_sa_t)}{\sqrt{n}\sigma^2} g_{t,s}\big) \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\tfrac{2\sqrt{2(n+1)}(a_ra_t-b_rb_t)}{\sqrt{n}\sigma^2} g_{r,t}\big) \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\tfrac{8a_tb_t}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
& \quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\tfrac{8a_tb_t}{\sqrt{n}\sigma^2} v_{r,t}\big) \\
& + \tfrac{4a_tb_t}{\sigma^2} \cdot 1 \\
\overset{\text{Rem. 30}}{=} \quad & \tfrac{4a_tb_t}{n\sigma^2} \cdot n \\
& + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\tfrac{2\sqrt{2(n+1)}(a_tb_s+a_sb_t)}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
& \quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\tfrac{2\sqrt{2(n+1)}(b_sb_t-a_sa_t)}{\sqrt{n}\sigma^2} g_{t,s}\big) \\
& \quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\tfrac{8a_tb_t}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
& + \tfrac{4a_tb_t}{\sigma^2} \\
= \quad & \tfrac{8a_tb_t}{\sigma^2} \\
& + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\tfrac{2\sqrt{2(n+1)}(a_tb_s+a_sb_t)}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
& \quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\tfrac{2\sqrt{2(n+1)}(b_sb_t-a_sa_t)}{\sqrt{n}\sigma^2} g_{t,s}\big) \\
& \quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(-\tfrac{8a_tb_t}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
= \quad & \tfrac{8a_tb_t}{\sigma^2} \\
& + \tfrac{-2}{\sqrt{n}\sigma^2} \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(\sqrt{2(n+1)}(a_tb_s+a_sb_t)f_{t,s} \\
& \qquad + \sqrt{2(n+1)}(b_sb_t-a_sa_t)g_{t,s} \\
& \qquad + 4a_tb_tv_{t,s}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{8a_tb_t}{\sigma^2} \\
&\quad + \tfrac{-2}{\sqrt{n}\sigma^2}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}\big(\sqrt{2(n+1)}(a_tb_s+a_sb_t)\tfrac{\sqrt{2(n+1)}(a_ta_s+b_tb_s)}{\sqrt{n}\sigma} \\
&\qquad + \sqrt{2(n+1)}(b_sb_t-a_sa_t)\tfrac{\sqrt{2(n+1)}(a_tb_s-b_ta_s)}{\sqrt{n}\sigma} \\
&\qquad + 4a_tb_t\tfrac{(a_t^2+b_t^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma}\big) \\
&= \tfrac{8a_tb_t}{\sigma^2} \\
&\quad + \tfrac{-2}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}(2(n+1)\cdot(2a_tb_s^2b_t+2a_s^2a_tb_t) \\
&\qquad + 4\cdot(a_t^3b_t+a_tb_t^3-a_s^2a_tb_t-a_tb_s^2b_t)) \\
&= \tfrac{8a_tb_t}{\sigma^2} \\
&\quad + \tfrac{-8a_tb_t}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t}((n+1)\cdot(a_s^2+b_s^2) \\
&\qquad + (a_t^2+b_t^2-a_s^2-b_s^2)) \\
&\overset{\text{Def. }\sigma}{=} \tfrac{8a_tb_t}{\sigma^2} \\
&\quad + \tfrac{-8a_tb_t}{n\sigma^3}((n+1)(\sigma-(a_t^2+b_t^2))+(n+1)(a_t^2+b_t^2)-\sigma) \\
&= \tfrac{8a_tb_t}{\sigma^2} + \tfrac{-8(n+1-1)}{n\sigma^2}a_tb_t \\
&= \tfrac{8a_tb_t}{\sigma^2} - \tfrac{8a_tb_t}{\sigma^2} \\
&= 0
\end{aligned}
$$

The following calculation is analogous to Calculation 33 for $\zeta_{a_t}^{\mathrm{T}}\cdot\zeta_{a_t}$.

Calculation 35. Suppose given $t\in[0,n]$. We calculate.

$$
\begin{aligned}
\zeta_{b_t}^{\mathrm{T}}\cdot\zeta_{b_t} &= \textstyle\sum_{0\leqslant r<s\leqslant n}(\tfrac{\partial}{\partial b_t}f_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n}(\tfrac{\partial}{\partial b_t}g_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n}(\tfrac{\partial}{\partial b_t}v_{r,s})^2 \\
&= \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\tfrac{\partial}{\partial b_t}f_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\tfrac{\partial}{\partial b_t}f_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\tfrac{\partial}{\partial b_t}f_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\tfrac{\partial}{\partial b_t}g_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\tfrac{\partial}{\partial b_t}g_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\tfrac{\partial}{\partial b_t}g_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}(\tfrac{\partial}{\partial b_t}v_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}(\tfrac{\partial}{\partial b_t}v_{r,s})^2 \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}(\tfrac{\partial}{\partial b_t}v_{r,s})^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} f_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} f_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}\big(-\frac{2b_t}{\sigma} f_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(-\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} g_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} g_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}\big(-\frac{2b_t}{\sigma} g_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{2b_r}{\sqrt{n}\sigma} - \frac{2b_r}{\sigma} v_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\frac{2b_s}{\sqrt{n}\sigma} - \frac{2b_s}{\sigma} v_{r,s}\big)^2 \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}\big(-\frac{2b_t}{\sigma} v_{r,s}\big)^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{2(n+1)b_s^2}{n\sigma^2} - \frac{4b_tb_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s} && + \frac{4b_t^2}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{2(n+1)b_r^2}{n\sigma^2} - \frac{4b_rb_t\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{r,t} && + \frac{4b_t^2}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}\big(&& + \frac{4b_t^2}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{2(n+1)a_s^2}{n\sigma^2} + \frac{4\sqrt{2(n+1)}a_sb_t}{\sqrt{n}\sigma^2} g_{t,s} && + \frac{4b_t^2}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{2(n+1)a_r^2}{n\sigma^2} - \frac{4\sqrt{2(n+1)}a_rb_t}{\sqrt{n}\sigma^2} g_{r,t} && + \frac{4b_t^2}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}\big(&& + \frac{4b_t^2}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{4b_t^2}{n\sigma^2} - \frac{8b_t^2}{\sqrt{n}\sigma^2} v_{t,s} && + \frac{4b_t^2}{\sigma^2} v_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{4b_t^2}{n\sigma^2} + \frac{8b_t^2}{\sqrt{n}\sigma^2} v_{r,t} && + \frac{4b_t^2}{\sigma^2} v_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\in[0,n]\setminus\{r,s\}}\big(&& + \frac{4b_t^2}{\sigma^2} v_{r,s}^2\big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{2(n+1)b_s^2}{n\sigma^2} && - \frac{4b_tb_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{2(n+1)b_r^2}{n\sigma^2} && - \frac{4b_rb_t\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{r,t}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{2(n+1)a_s^2}{n\sigma^2} && + \frac{4\sqrt{2(n+1)}a_sb_t}{\sqrt{n}\sigma^2} g_{t,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{2(n+1)a_r^2}{n\sigma^2} && - \frac{4\sqrt{2(n+1)}a_rb_t}{\sqrt{n}\sigma^2} g_{r,t}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r}\big(\frac{4b_t^2}{n\sigma^2} && - \frac{8b_t^2}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{4b_t^2}{n\sigma^2} && + \frac{8b_t^2}{\sqrt{n}\sigma^2} v_{r,t}\big) \\
& + \frac{4b_t^2}{\sigma^2} \cdot \textstyle\sum_{0\leqslant r<s\leqslant n}\big(f_{r,s}^2 + g_{r,s}^2 + v_{r,s}^2\big)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\text{Theorem 11}}{=}\quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \frac{2(n+1)b_s^2}{n\sigma^2} \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \frac{2(n+1)b_r^2}{n\sigma^2} \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \frac{2(n+1)a_s^2}{n\sigma^2} \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \frac{2(n+1)a_r^2}{n\sigma^2} \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \frac{4b_t^2}{n\sigma^2} \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \frac{4b_t^2}{n\sigma^2} \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \big(-\frac{4b_t b_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \big(-\frac{4b_r b_t\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{r,t}\big) \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \frac{4\sqrt{2(n+1)}b_t a_s}{\sqrt{n}\sigma^2} g_{t,s} \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \big(-\frac{4\sqrt{2(n+1)}a_r b_t}{\sqrt{n}\sigma^2} g_{r,t}\big) \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r} \big(-\frac{8b_t^2}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
&\quad + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s} \big(\frac{8b_t^2}{\sqrt{n}\sigma^2} v_{r,t}\big) \\
& + \tfrac{4b_t^2}{\sigma^2} \cdot 1 \\
\overset{\text{Rem. 30}}{=}\quad & \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \frac{2(n+1)b_s^2}{n\sigma^2} \\
&\quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \frac{2(n+1)a_s^2}{n\sigma^2} \\
&\quad + \tfrac{4b_t^2}{n\sigma^2} \cdot n \\
& + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \big(-\frac{4b_t b_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
&\quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \frac{4\sqrt{2(n+1)}b_t a_s}{\sqrt{n}\sigma^2} g_{t,s} \\
&\quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \big(-\frac{8b_t^2}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
& + \tfrac{4b_t^2}{\sigma^2} \\
=\quad & \tfrac{2(n+1)}{n\sigma^2}\big((\textstyle\sum_{0\leqslant s\leqslant n} b_s^2) - b_t^2 + (\sum_{0\leqslant s\leqslant n} a_s^2) - a_t^2\big) \\
&\quad + \tfrac{8b_t^2}{\sigma^2} \\
& + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \big(-\frac{4b_t b_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
&\quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \frac{4\sqrt{2(n+1)}b_t a_s}{\sqrt{n}\sigma^2} g_{t,s} \\
&\quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \big(-\frac{8b_t^2}{\sqrt{n}\sigma^2} v_{t,s}\big) \\
\overset{\text{Def. }\sigma}{=}\quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - b_t^2 - a_t^2) \\
&\quad + \tfrac{8b_t^2}{\sigma^2} \\
& + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \big(-\frac{4b_t b_s\sqrt{2(n+1)}}{\sqrt{n}\sigma^2} f_{t,s}\big) \\
&\quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \frac{4\sqrt{2(n+1)}b_t a_s}{\sqrt{n}\sigma^2} g_{t,s} \\
&\quad + \textstyle\sum_{0\leqslant s\leqslant n \text{ and } s\neq t} \big(-\frac{8b_t^2}{\sqrt{n}\sigma^2} v_{t,s}\big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{4b_t}{\sqrt{n}\sigma^2} \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } s \neq t} (b_s\sqrt{2(n+1)} f_{t,s} - a_s\sqrt{2(n+1)} g_{t,s} + 2b_t v_{t,s}) \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{4b_t}{\sqrt{n}\sigma^2} \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } s \neq t} (\quad b_s\sqrt{2(n+1)}\tfrac{\sqrt{2(n+1)}(b_t b_s + a_t a_s)}{\sqrt{n}\sigma} \\
& \qquad\qquad + a_s\sqrt{2(n+1)}\tfrac{\sqrt{2(n+1)}(b_t a_s - a_t b_s)}{\sqrt{n}\sigma} \\
& \qquad\qquad + 2b_t\tfrac{(b_t^2 + a_t^2) - (b_s^2 + a_s^2)}{\sqrt{n}\sigma}) \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{8b_t}{n\sigma^3} \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } s \neq t} (\quad b_s(n+1)(b_t b_s + a_t a_s) \\
& \qquad\qquad + a_s(n+1)(b_t a_s - a_t b_s) \\
& \qquad\qquad + b_t((b_t^2 + a_t^2) - (b_s^2 + a_s^2))) \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{8b_t}{n\sigma^3} \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } s \neq t} (\quad (n+1)(b_t b_s^2 + b_t a_s^2) \\
& \qquad\qquad + (b_t^3 + b_t a_t^2) - (b_t b_s^2 + b_t a_s^2)) \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{8b_t^2}{n\sigma^3} \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } s \neq t} (n(b_s^2 + a_s^2) + (b_t^2 + a_t^2)) \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{8b_t^2}{n\sigma^3}(n(b_t^2 + a_t^2) + \textstyle\sum_{0 \leqslant s \leqslant n \text{ and } s \neq t} n(b_s^2 + a_s^2)) \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{8b_t^2}{n\sigma^3} n\sigma \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2) \\
& \quad + \tfrac{8b_t^2}{\sigma^2} \\
& - \tfrac{8b_t^2}{\sigma^2} \\
= \quad & \tfrac{2(n+1)}{n\sigma^2}(\sigma - a_t^2 - b_t^2)
\end{aligned}
$$

Calculation 36. Suppose given $t, l \in [0, n]$ with $t < l$. We calculate.

$$
\begin{aligned}
\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{a_l} = \;& \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_l} v_{r,s}) \\
= \;& \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}(\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial a_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}(\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial a_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}(\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial a_l} v_{r,s})
\end{aligned}
$$

$$
\begin{aligned}
= \ & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l=s}\big(\frac{\sqrt{2(n+1)}a_l}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} f_{r,s}\big) \cdot \big(\frac{\sqrt{2(n+1)}a_t}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l\neq s}\big(\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} f_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\, l\neq s}\big(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} f_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=r}\big(-\frac{2a_t}{\sigma} f_{r,s}\big) \cdot \big(\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=s}\big(-\frac{2a_t}{\sigma} f_{r,s}\big) \cdot \big(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(-\frac{2a_t}{\sigma} f_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l=s}\big(\frac{\sqrt{2(n+1)}b_l}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} g_{r,s}\big) \cdot \big(-\frac{\sqrt{2(n+1)}b_t}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l\neq s}\big(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} g_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\, l\neq s}\big(-\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} g_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=r}\big(-\frac{2a_t}{\sigma} g_{r,s}\big) \cdot \big(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=s}\big(-\frac{2a_t}{\sigma} g_{r,s}\big) \cdot \big(-\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(-\frac{2a_t}{\sigma} g_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l=s}\big(\frac{2a_t}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} v_{r,s}\big) \cdot \big(-\frac{2a_l}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l\neq s}\big(\frac{2a_t}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} v_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\, l\neq s}\big(-\frac{2a_t}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma} v_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=r}\big(-\frac{2a_t}{\sigma} v_{r,s}\big) \cdot \big(\frac{2a_l}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=s}\big(-\frac{2a_t}{\sigma} v_{r,s}\big) \cdot \big(-\frac{2a_l}{\sqrt{n}\sigma} - \frac{2a_l}{\sigma} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(-\frac{2a_t}{\sigma} v_{r,s}\big) \cdot \big(-\frac{2a_l}{\sigma} v_{r,s}\big)
\end{aligned}
$$

$$
\begin{aligned}
= \ & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l=s}\big(\frac{2(n+1)a_t a_l}{n\sigma^2} - \frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t^2 + 2a_l^2) f_{r,s} && + \frac{4a_t a_l}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l\neq s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_s a_l) f_{r,s} && + \frac{4a_t a_l}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\, l\neq s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_r a_l) f_{r,s} && + \frac{4a_t a_l}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=r}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_s a_t) f_{r,s} && + \frac{4a_t a_l}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_r a_t) f_{r,s} && + \frac{4a_t a_l}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(&& + \frac{4a_t a_l}{\sigma^2} f_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l=s}\big(-\frac{2(n+1)b_t b_l}{n\sigma^2} + \frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t b_t - 2a_l b_l) g_{r,s} && + \frac{4a_t a_l}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l\neq s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_l b_s) g_{r,s} && + \frac{4a_t a_l}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\, l\neq s}\big(\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_l b_r) g_{r,s} && + \frac{4a_t a_l}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=r}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t b_s) g_{r,s} && + \frac{4a_t a_l}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=s}\big(\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t b_r) g_{r,s} && + \frac{4a_t a_l}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(&& + \frac{4a_t a_l}{\sigma^2} g_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l=s}\big(-\frac{4a_t a_l}{n\sigma^2} && + \frac{4a_t a_l}{\sigma^2} v_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\, l\neq s}\big(-\frac{4a_t a_l}{\sqrt{n}\sigma^2} v_{r,s} && + \frac{4a_t a_l}{\sigma^2} v_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\, l\neq s}\big(\frac{4a_t a_l}{\sqrt{n}\sigma^2} v_{r,s} && + \frac{4a_t a_l}{\sigma^2} v_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=r}\big(-\frac{4a_t a_l}{\sqrt{n}\sigma^2} v_{r,s} && + \frac{4a_t a_l}{\sigma^2} v_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\, l=s}\big(\frac{4a_t a_l}{\sqrt{n}\sigma^2} v_{r,s} && + \frac{4a_t a_l}{\sigma^2} v_{r,s}^2\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(&& + \frac{4a_t a_l}{\sigma^2} v_{r,s}^2\big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{2(n+1)a_ta_l}{n\sigma^2} && -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t^2+2a_l^2)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(&& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sa_l)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(&& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_ra_l)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(&& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sa_t)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(&& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_ra_t)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2(n+1)b_tb_l}{n\sigma^2} && +\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(&& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_lb_s)g_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(&& \frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_lb_r)g_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(&& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_s)g_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(&& \frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_r)g_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{4a_ta_l}{n\sigma^2}\big) && \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(&& -\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(&& \frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(&& -\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(&& \frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s}\big) \\
& +\frac{4a_ta_l}{\sigma^2}\cdot\textstyle\sum_{0\leqslant r<s\leqslant n}(f_{r,s}^2+g_{r,s}^2+v_{r,s}^2) &&
\end{aligned}
$$

$$
\begin{aligned}
\overset{\text{Theorem 11}}{=} \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t^2+2a_l^2)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sa_l)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_ra_l)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sa_t)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_ra_t)f_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{r,s} \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_lb_s)g_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_lb_r)g_{r,s} \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_s)g_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_r)g_{r,s} \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s} \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(-\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s}\big) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,s} \\
& +\frac{4a_ta_l}{\sigma^2}\cdot 1 \\
& +\frac{2(n+1)a_ta_l}{n\sigma^2}-\frac{2(n+1)b_tb_l}{n\sigma^2}-\frac{4a_ta_l}{n\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{Rem. 31}}{=} && \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t^2+2a_l^2)f_{r,s}\right) \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sa_l)f_{t,s}\right) \\
& && +\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_ra_t)f_{r,l}\right) \\
& && +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{r,s} \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_lb_s)g_{t,s}\right) \\
& && +\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_r)g_{r,l} \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{t,s}\right) \\
& && +\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{r,l} \\
& && +\frac{(6n-2)a_ta_l}{n\sigma^2}-\frac{2(n+1)b_tb_l}{n\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{rename}}{=} && \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t^2+2a_l^2)f_{r,s}\right) \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sa_l)f_{t,s}\right) \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sa_t)f_{l,s}\right) \\
& && +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{r,s} \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_lb_s)g_{t,s}\right) \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_s)g_{l,s}\right) \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{t,s}\right) \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4a_ta_l}{\sqrt{n}\sigma^2}v_{l,s}\right) \\
& && +\frac{(6n-2)a_ta_l}{n\sigma^2}-\frac{2(n+1)b_tb_l}{n\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
&= && -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t^2+2a_l^2)f_{t,l} \\
& && +\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{t,l} \\
& && +2\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_sa_lf_{t,s}+a_sa_tf_{l,s})\right) \\
& && +2\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_lb_sg_{t,s}+a_tb_sg_{l,s})\right) \\
& && +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4a_ta_l}{\sqrt{n}\sigma^2}(v_{t,s}+v_{l,s})\right) \\
& && +\frac{(6n-2)a_ta_l}{n\sigma^2}-\frac{2(n+1)b_tb_l}{n\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
&= && -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t^2+2a_l^2)\frac{\sqrt{2(n+1)}(a_ta_l+b_tb_l)}{\sqrt{n}\sigma} \\
& && +\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)\frac{\sqrt{2(n+1)}(a_tb_l-b_ta_l)}{\sqrt{n}\sigma} \\
& && -\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(a_sa_l\frac{\sqrt{2(n+1)}(a_ta_s+b_tb_s)}{\sqrt{n}\sigma}+a_sa_t\frac{\sqrt{2(n+1)}(a_la_s+b_lb_s)}{\sqrt{n}\sigma}\right) \\
& && -\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(a_lb_s\frac{\sqrt{2(n+1)}(a_tb_s-b_ta_s)}{\sqrt{n}\sigma}+a_tb_s\frac{\sqrt{2(n+1)}(a_lb_s-b_la_s)}{\sqrt{n}\sigma}\right) \\
& && -\frac{4a_ta_l}{\sqrt{n}\sigma^2}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(\frac{(a_t^2+b_t^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma}+\frac{(a_l^2+b_l^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma}\right) \\
& && +\frac{(6n-2)a_ta_l}{n\sigma^2}-\frac{2(n+1)b_tb_l}{n\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\tfrac{2(n+1)}{n\sigma^3}(2a_t^2 + 2a_l^2)(a_t a_l + b_t b_l) \\
& +\tfrac{2(n+1)}{n\sigma^3}(2a_t b_t - 2a_l b_l)(a_t b_l - b_t a_l) \\
& -\tfrac{4(n+1)}{n\sigma^3} \cdot \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(a_s a_l(a_t a_s + b_t b_s) + a_s a_t(a_l a_s + b_l b_s)) \\
& -\tfrac{4(n+1)}{n\sigma^3} \cdot \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(a_l b_s(a_t b_s - b_t a_s) + a_t b_s(a_l b_s - b_l a_s)) \\
& -\tfrac{4a_t a_l}{n\sigma^3} \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}((a_t^2 + b_t^2) - (a_s^2 + b_s^2) + (a_l^2 + b_l^2) - (a_s^2 + b_s^2)) \\
& +\tfrac{(6n-2)a_t a_l}{n\sigma^2} - \tfrac{2(n+1)b_t b_l}{n\sigma^2} \\
= \quad & -\tfrac{2(n+1)}{n\sigma^3}(2a_t^2 + 2a_l^2)(a_t a_l + b_t b_l) \\
& +\tfrac{2(n+1)}{n\sigma^3}(2a_t b_t - 2a_l b_l)(a_t b_l - b_t a_l) \\
& -\tfrac{4(n+1)}{n\sigma^3} \cdot \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(2a_s^2 a_t a_l) \\
& -\tfrac{4(n+1)}{n\sigma^3} \cdot \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(2a_t a_l b_s^2) \\
& -\tfrac{4a_t a_l}{n\sigma^3} \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(a_t^2 + b_t^2 + a_l^2 + b_l^2 - 2(a_s^2 + b_s^2)) \\
& +\tfrac{(6n-2)a_t a_l}{n\sigma^2} - \tfrac{2(n+1)b_t b_l}{n\sigma^2} \\
= \quad & -\tfrac{4(n+1)}{n\sigma^3}(a_t^3 a_l + a_t a_l^3 + a_t a_l b_t^2 + a_t a_l b_l^2) \\
& -\tfrac{8(n+1)}{n\sigma^3} \cdot \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(a_s^2 a_l a_t + a_l a_t b_s^2) \\
& -\tfrac{4a_t a_l}{n\sigma^3} \textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(a_t^2 + b_t^2 + a_l^2 + b_l^2 - 2(a_s^2 + b_s^2)) \\
& +\tfrac{(6n-2)a_t a_l}{n\sigma^2} - \tfrac{2(n+1)b_t b_l}{n\sigma^2} \\
\overset{\text{Def.}\sigma}{=} \quad & -\tfrac{4(n+1)}{n\sigma^3} a_t a_l(a_t^2 + a_l^2 + b_t^2 + b_l^2) \\
& -\tfrac{8(n+1)a_t a_l}{n\sigma^3}(\sigma - (a_t^2 + b_t^2 + a_l^2 + b_l^2)) \\
& -\tfrac{4a_t a_l}{n\sigma^3}((n+1)(a_t^2 + b_t^2 + a_l^2 + b_l^2) - 2\sigma) \\
& +\tfrac{(6n-2)a_t a_l}{n\sigma^2} - \tfrac{2(n+1)b_t b_l}{n\sigma^2} \\
= \quad & -\tfrac{8(n+1)a_t a_l}{n\sigma^2} \\
& +\tfrac{8a_t a_l}{n\sigma^2} \\
& +\tfrac{(6n-2)a_t a_l}{n\sigma^2} - \tfrac{2(n+1)b_t b_l}{n\sigma^2} \\
= \quad & -\tfrac{2(n+1)}{n\sigma^2}(a_t a_l + b_t b_l)
\end{aligned}
$$

Calculation 37. Suppose given $t, l \in [0, n]$ with $t < l$. We calculate.

$$
\begin{aligned}
\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{b_l} = \;& \textstyle\sum_{0 \leqslant r < s \leqslant n} (\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n} (\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n} (\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
= \;& \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=r,\ l=s} (\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=r,\ l\neq s} (\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=s,\ l\neq s} (\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t\neq r,\ l=r} (\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t\neq r,\ l=s} (\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset} (\frac{\partial}{\partial a_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=r,\ l=s} (\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=r,\ l\neq s} (\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=s,\ l\neq s} (\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t\neq r,\ l=r} (\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t\neq r,\ l=s} (\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset} (\frac{\partial}{\partial a_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=r,\ l=s} (\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=r,\ l\neq s} (\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t=s,\ l\neq s} (\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t\neq r,\ l=r} (\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } t\neq r,\ l=s} (\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0 \leqslant r < s \leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset} (\frac{\partial}{\partial a_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s})
\end{aligned}
$$

$$
\begin{aligned}
= \;& \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{\sqrt{2(n+1)}a_l}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}f_{r,s}\big)\cdot\big(\frac{\sqrt{2(n+1)}b_t}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}f_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}f_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(-\frac{2a_t}{\sigma}f_{r,s}\big)\cdot\big(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{2a_t}{\sigma}f_{r,s}\big)\cdot\big(\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(-\frac{2a_t}{\sigma}f_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{\sqrt{2(n+1)}b_l}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}g_{r,s}\big)\cdot\big(\frac{\sqrt{2(n+1)}a_t}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}g_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(-\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}g_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(-\frac{2a_t}{\sigma}g_{r,s}\big)\cdot\big(-\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{2a_t}{\sigma}g_{r,s}\big)\cdot\big(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(-\frac{2a_t}{\sigma}g_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{2a_t}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}v_{r,s}\big)\cdot\big(-\frac{2b_l}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\frac{2a_t}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}v_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(-\frac{2a_t}{\sqrt{n}\sigma} - \frac{2a_t}{\sigma}v_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(-\frac{2a_t}{\sigma}v_{r,s}\big)\cdot\big(\frac{2b_l}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{2a_t}{\sigma}v_{r,s}\big)\cdot\big(-\frac{2b_l}{\sqrt{n}\sigma} - \frac{2b_l}{\sigma}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(-\frac{2a_t}{\sigma}v_{r,s}\big)\cdot\big(-\frac{2b_l}{\sigma}v_{r,s}\big)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{2(n+1)a_lb_t}{n\sigma^2} - \frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_tb_t + a_lb_l)f_{r,s} && +\frac{4a_tb_l}{\sigma^2}f_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}a_sb_l}{\sqrt{n}\sigma^2}f_{r,s} && +\frac{4a_tb_l}{\sigma^2}f_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}a_rb_l}{\sqrt{n}\sigma^2}f_{r,s} && +\frac{4a_tb_l}{\sigma^2}f_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(-\frac{2\sqrt{2(n+1)}a_tb_s}{\sqrt{n}\sigma^2}f_{r,s} && +\frac{4a_tb_l}{\sigma^2}f_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}a_tb_r}{\sqrt{n}\sigma^2}f_{r,s} && +\frac{4a_tb_l}{\sigma^2}f_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(&& \frac{4a_tb_l}{\sigma^2}f_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{2(n+1)a_tb_l}{n\sigma^2} - \frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t^2 + b_l^2)g_{r,s} && +\frac{4a_tb_l}{\sigma^2}g_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}b_lb_s}{\sqrt{n}\sigma^2}g_{r,s} && +\frac{4a_tb_l}{\sigma^2}g_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\frac{2\sqrt{2(n+1)}b_lb_r}{\sqrt{n}\sigma^2}g_{r,s} && +\frac{4a_tb_l}{\sigma^2}g_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(\frac{2\sqrt{2(n+1)}a_sa_t}{\sqrt{n}\sigma^2}g_{r,s} && +\frac{4a_tb_l}{\sigma^2}g_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}a_ra_t}{\sqrt{n}\sigma^2}g_{r,s} && +\frac{4a_tb_l}{\sigma^2}g_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(&& \frac{4a_tb_l}{\sigma^2}g_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{4a_tb_l}{n\sigma^2} && +\frac{4a_tb_l}{\sigma^2}v_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{r,s} && +\frac{4a_tb_l}{\sigma^2}v_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{r,s} && +\frac{4a_tb_l}{\sigma^2}v_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(-\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{r,s} && +\frac{4a_tb_l}{\sigma^2}v_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{r,s} && +\frac{4a_tb_l}{\sigma^2}v_{r,s}^2\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\big(&& \frac{4a_tb_l}{\sigma^2}v_{r,s}^2\big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{2(n+1)a_l b_t}{n\sigma^2} \quad - \frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t b_t + a_l b_l) f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\quad - \frac{2\sqrt{2(n+1)}a_s b_l}{\sqrt{n}\sigma^2} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\quad - \frac{2\sqrt{2(n+1)}a_r b_l}{\sqrt{n}\sigma^2} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(\quad - \frac{2\sqrt{2(n+1)}a_t b_s}{\sqrt{n}\sigma^2} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\quad - \frac{2\sqrt{2(n+1)}a_t b_r}{\sqrt{n}\sigma^2} f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{2(n+1)a_t b_l}{n\sigma^2} \quad - \frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t^2 + b_l^2) g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\quad - \frac{2\sqrt{2(n+1)}b_l b_s}{\sqrt{n}\sigma^2} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\quad \frac{2\sqrt{2(n+1)}b_l b_r}{\sqrt{n}\sigma^2} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(\quad \frac{2\sqrt{2(n+1)}a_s a_t}{\sqrt{n}\sigma^2} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\quad - \frac{2\sqrt{2(n+1)}a_r a_t}{\sqrt{n}\sigma^2} g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{4a_t b_l}{n\sigma^2}\ \big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\quad -\frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\quad \frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(\quad -\frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\quad \frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s}\big) \\
& + \tfrac{4a_t b_l}{\sigma^2} \cdot \textstyle\sum_{0\leqslant r<s\leqslant n}\big(f_{r,s}^2 + g_{r,s}^2 + v_{r,s}^2\big) \\
\overset{\text{Theorem 11}}{=} \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t b_t + a_l b_l) f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_s b_l) f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_r b_l) f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t b_s) f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t b_r) f_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t^2 + b_l^2) g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(b_l b_s) g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\ \frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(b_l b_r) g_{r,s} \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\ \frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_s a_t) g_{r,s} \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_r a_t) g_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\ \frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s} \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(-\frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s}\big) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\ \frac{4a_t b_l}{\sqrt{n}\sigma^2} v_{r,s} \\
& + \tfrac{4a_t b_l}{\sigma^2} \cdot 1 \\
& + \tfrac{2(n+1)a_l b_t}{n\sigma^2} + \tfrac{2(n+1)a_t b_l}{n\sigma^2} - \tfrac{4a_t b_l}{n\sigma^2}
\end{aligned}
$$

$$\begin{aligned}
\overset{\text{Rem. 31}}{=}\ & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_tb_t+a_lb_l)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_sb_l)f_{t,s}\big)\\
&+\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_tb_r)f_{r,l}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t^2+b_l^2)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(b_lb_s)g_{t,s}\big)\\
&+\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_ra_t)g_{r,l}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{t,s}\big)\\
&+\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{r,l}\\
&+\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2}
\end{aligned}$$

$$\begin{aligned}
\overset{\text{rename}}{=}\ & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_tb_t+a_lb_l)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_sb_l)f_{t,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_tb_s)f_{l,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t^2+b_l^2)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(b_lb_s)g_{t,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_sa_t)g_{l,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{t,s}\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{4a_tb_l}{\sqrt{n}\sigma^2}v_{l,s}\big)\\
&+\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2}
\end{aligned}$$

$$\begin{aligned}
=\ & -\tfrac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_tb_t+a_lb_l)f_{t,l}\\
&-\tfrac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t^2+b_l^2)g_{t,l}\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_sb_lf_{t,s}+a_tb_sf_{l,s})\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(b_lb_sg_{t,s}-a_sa_tg_{l,s})\big)\\
&+\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\frac{4a_tb_l}{\sqrt{n}\sigma^2}(v_{t,s}+v_{l,s})\big)\\
&+\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2}
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\tfrac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_tb_t+a_lb_l)\tfrac{\sqrt{2(n+1)}(a_ta_l+b_tb_l)}{\sqrt{n}\sigma} \\
& -\tfrac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_t^2+b_l^2)\tfrac{\sqrt{2(n+1)}(a_tb_l-b_ta_l)}{\sqrt{n}\sigma} \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\tfrac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_sb_l\tfrac{\sqrt{2(n+1)}(a_ta_s+b_tb_s)}{\sqrt{n}\sigma}+a_tb_s\tfrac{\sqrt{2(n+1)}(a_la_s+b_lb_s)}{\sqrt{n}\sigma})\big) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\tfrac{2\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(b_lb_s\tfrac{\sqrt{2(n+1)}(a_tb_s-b_ta_s)}{\sqrt{n}\sigma}-a_sa_t\tfrac{\sqrt{2(n+1)}(a_lb_s-b_la_s)}{\sqrt{n}\sigma})\big) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\big(-\tfrac{4a_tb_l}{\sqrt{n}\sigma^2}(\tfrac{(a_t^2+b_t^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma}+\tfrac{(a_l^2+b_l^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma})\big) \\
& +\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2} \\[1ex]
= \quad & -\tfrac{4(n+1)}{n\sigma^3}(a_tb_t+a_lb_l)(a_ta_l+b_tb_l) \\
& -\tfrac{4(n+1)}{n\sigma^3}(a_t^2+b_l^2)(a_tb_l-b_ta_l) \\
& -\tfrac{4(n+1)}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(a_sb_l(a_ta_s+b_tb_s)+a_tb_s(a_la_s+b_lb_s)) \\
& -\tfrac{4(n+1)}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(b_lb_s(a_tb_s-b_ta_s)-a_sa_t(a_lb_s-b_la_s)) \\
& -\tfrac{4a_tb_l}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}((a_t^2+b_t^2)-(a_s^2+b_s^2)+(a_l^2+b_l^2)-(a_s^2+b_s^2)) \\
& +\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2} \\[1ex]
= \quad & -\tfrac{4(n+1)}{n\sigma^3}(a_tb_t+a_lb_l)(a_ta_l+b_tb_l) \\
& -\tfrac{4(n+1)}{n\sigma^3}(a_t^2+b_l^2)(a_tb_l-b_ta_l) \\
& -\tfrac{4(n+1)}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(2a_s^2a_tb_l+2a_tb_s^2b_l) \\
& -\tfrac{4a_tb_l}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}((a_t^2+b_t^2)+(a_l^2+b_l^2)-2(a_s^2+b_s^2)) \\
& +\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2} \\[1ex]
= \quad & -\tfrac{4(n+1)}{n\sigma^3}(a_l^2a_tb_l+a_tb_t^2b_l+a_t^3b_l+a_tb_l^3) \\
& -\tfrac{8(n+1)}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}(a_s^2a_tb_l+a_tb_s^2b_l) \\
& -\tfrac{4a_tb_l}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}((a_t^2+b_t^2)+(a_l^2+b_l^2)-2(a_s^2+b_s^2)) \\
& +\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2} \\[1ex]
\overset{\text{Def.}\sigma}{=} \quad & -\tfrac{4(n+1)}{n\sigma^3}a_tb_l(a_l^2+b_t^2+a_t^2+b_l^2) \\
& -\tfrac{8(n+1)a_tb_l}{n\sigma^3}(\sigma-(a_t^2+b_t^2+a_l^2+b_l^2)) \\
& -\tfrac{4a_tb_l}{n\sigma^3}((n+1)(a_t^2+b_t^2+a_l^2+b_l^2)-2\sigma) \\
& +\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2} \\[1ex]
= \quad & -\tfrac{8(n+1)a_tb_l}{n\sigma^2} \\
& +\tfrac{8a_tb_l}{n\sigma^2} \\
& +\tfrac{(6n-2)a_tb_l}{n\sigma^2}+\tfrac{2(n+1)a_lb_t}{n\sigma^2} \\[1ex]
= \quad & \tfrac{2(n+1)}{n\sigma^2}(-a_tb_l+a_lb_t)
\end{aligned}
$$

The following calculation is analogous to Calculation 36 for $\zeta_{a_t}^{\mathrm{T}} \cdot \zeta_{a_l}$.

Calculation 38. Suppose given $t, l \in [0, n]$ with $t < l$. We calculate.

$$
\begin{aligned}
\zeta_{b_t}^{\mathrm{T}} \cdot \zeta_{b_l} = \;& \textstyle\sum_{0\leqslant r<s\leqslant n} (\frac{\partial}{\partial b_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n} (\frac{\partial}{\partial b_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n} (\frac{\partial}{\partial b_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
= \;& \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s} (\frac{\partial}{\partial b_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s} (\frac{\partial}{\partial b_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s} (\frac{\partial}{\partial b_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r} (\frac{\partial}{\partial b_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s} (\frac{\partial}{\partial b_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset} (\frac{\partial}{\partial b_t} f_{r,s}) \cdot (\frac{\partial}{\partial b_l} f_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s} (\frac{\partial}{\partial b_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s} (\frac{\partial}{\partial b_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s} (\frac{\partial}{\partial b_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r} (\frac{\partial}{\partial b_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s} (\frac{\partial}{\partial b_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset} (\frac{\partial}{\partial b_t} g_{r,s}) \cdot (\frac{\partial}{\partial b_l} g_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s} (\frac{\partial}{\partial b_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s} (\frac{\partial}{\partial b_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s} (\frac{\partial}{\partial b_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r} (\frac{\partial}{\partial b_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s} (\frac{\partial}{\partial b_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s}) \\
& + \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset} (\frac{\partial}{\partial b_t} v_{r,s}) \cdot (\frac{\partial}{\partial b_l} v_{r,s})
\end{aligned}
$$

$$
\begin{aligned}
= &\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\Big(\frac{\sqrt{2(n+1)}b_l}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}f_{r,s}\Big)\cdot\Big(\frac{\sqrt{2(n+1)}b_t}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}f_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\Big(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}f_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}f_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\Big(\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}f_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}f_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\Big(-\frac{2b_t}{\sigma}f_{r,s}\Big)\cdot\Big(\frac{\sqrt{2(n+1)}b_s}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}f_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\Big(-\frac{2b_t}{\sigma}f_{r,s}\Big)\cdot\Big(\frac{\sqrt{2(n+1)}b_r}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}f_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\Big(-\frac{2b_t}{\sigma}f_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}f_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\Big(-\frac{\sqrt{2(n+1)}a_l}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}g_{r,s}\Big)\cdot\Big(\frac{\sqrt{2(n+1)}a_t}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}g_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\Big(-\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}g_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}g_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\Big(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}g_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}g_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\Big(-\frac{2b_t}{\sigma}g_{r,s}\Big)\cdot\Big(-\frac{\sqrt{2(n+1)}a_s}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}g_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\Big(-\frac{2b_t}{\sigma}g_{r,s}\Big)\cdot\Big(\frac{\sqrt{2(n+1)}a_r}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}g_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\Big(-\frac{2b_t}{\sigma}g_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}g_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\Big(\frac{2b_t}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}v_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}v_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\Big(\frac{2b_t}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}v_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}v_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\Big(-\frac{2b_t}{\sqrt{n}\sigma}-\frac{2b_t}{\sigma}v_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}v_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\Big(-\frac{2b_t}{\sigma}v_{r,s}\Big)\cdot\Big(\frac{2b_l}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}v_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\Big(-\frac{2b_t}{\sigma}v_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sqrt{n}\sigma}-\frac{2b_l}{\sigma}v_{r,s}\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\Big(-\frac{2b_t}{\sigma}v_{r,s}\Big)\cdot\Big(-\frac{2b_l}{\sigma}v_{r,s}\Big)
\end{aligned}
$$

$$
\begin{aligned}
= &\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\Big(\frac{2(n+1)b_tb_l}{n\sigma^2}-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_t^2+2b_l^2)f_{r,s} &&+\frac{4b_tb_l}{\sigma^2}f_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\Big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_sb_l)f_{r,s} &&+\frac{4b_tb_l}{\sigma^2}f_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\Big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_rb_l)f_{r,s} &&+\frac{4b_tb_l}{\sigma^2}f_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\Big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_sb_t)f_{r,s} &&+\frac{4b_tb_l}{\sigma^2}f_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\Big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_rb_t)f_{r,s} &&+\frac{4b_tb_l}{\sigma^2}f_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\Big(&&\frac{4b_tb_l}{\sigma^2}f_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\Big(-\frac{2(n+1)a_ta_l}{n\sigma^2}-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{r,s} &&+\frac{4b_tb_l}{\sigma^2}g_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\Big(\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sb_l)g_{r,s} &&+\frac{4b_tb_l}{\sigma^2}g_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\Big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_rb_l)g_{r,s} &&+\frac{4b_tb_l}{\sigma^2}g_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\Big(\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sb_t)g_{r,s} &&+\frac{4b_tb_l}{\sigma^2}g_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\Big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_rb_t)g_{r,s} &&+\frac{4b_tb_l}{\sigma^2}g_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\Big(&&\frac{4b_tb_l}{\sigma^2}g_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\Big(-\frac{4b_tb_l}{n\sigma^2} &&+\frac{4b_tb_l}{\sigma^2}v_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\Big(-\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s} &&+\frac{4b_tb_l}{\sigma^2}v_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\Big(\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s} &&+\frac{4b_tb_l}{\sigma^2}v_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\Big(-\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s} &&+\frac{4b_tb_l}{\sigma^2}v_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\Big(\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s} &&+\frac{4b_tb_l}{\sigma^2}v_{r,s}^2\Big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } \{t,l\}\cap\{r,s\}=\emptyset}\Big(&&\frac{4b_tb_l}{\sigma^2}v_{r,s}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(\frac{2(n+1)b_tb_l}{n\sigma^2} \quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_t^2+2b_l^2)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_sb_l)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_rb_l)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(\quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_sb_t)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_rb_t)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{2(n+1)a_ta_l}{n\sigma^2} \quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\quad \frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sb_l)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_rb_l)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(\quad \frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sb_t)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\quad -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_rb_t)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{4b_tb_l}{n\sigma^2}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\quad -\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s,\ l\neq s}\big(\quad \frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=r}\big(\quad -\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\quad \frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\frac{4b_tb_l}{\sigma^2}\cdot\textstyle\sum_{0\leqslant r<s\leqslant n}(f_{r,s}^2+g_{r,s}^2+v_{r,s}^2)\\
\overset{\text{Theorem 11}}{=} \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_t^2+2b_l^2)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_sb_l)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_rb_l)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_sb_t)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_rb_t)f_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_tb_t-2a_lb_l)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sb_l)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_rb_l)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_sb_t)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_rb_t)g_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l\neq s}\big(-\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=s}\big(\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } l=r}\big(-\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t\neq r,\ l=s}\big(\frac{4b_tb_l}{\sqrt{n}\sigma^2}v_{r,s}\big)\\
&+\frac{4b_tb_l}{\sigma^2}\cdot 1\\
&+\frac{2(n+1)b_tb_l}{n\sigma^2}-\frac{2(n+1)a_ta_l}{n\sigma^2}-\frac{4b_tb_l}{n\sigma^2}
\end{aligned}
$$

$$\begin{aligned}
\overset{\text{Rem. 31}}{=} \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_t^2+2b_l^2)f_{r,s}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_s b_l)f_{t,s}\right) \\
& +\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_r b_t)f_{r,l}\right) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t b_t-2a_l b_l)g_{r,s}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_s b_l)g_{t,s} \\
& +\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_r b_t)g_{r,l}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4b_t b_l}{\sqrt{n}\sigma^2}v_{t,s}\right) \\
& +\textstyle\sum_{0\leqslant r\leqslant n \text{ and } t\neq r,\ l\neq r}\frac{4b_t b_l}{\sqrt{n}\sigma^2}v_{r,l} \\
& +\frac{(6n-2)b_t b_l}{n\sigma^2}-\frac{2(n+1)a_t a_l}{n\sigma^2}
\end{aligned}$$

$$\begin{aligned}
\overset{\text{rename}}{=} \quad & \textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_t^2+2b_l^2)f_{r,s}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_s b_l)f_{t,s}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_s b_t)f_{l,s}\right) \\
& +\textstyle\sum_{0\leqslant r<s\leqslant n \text{ and } t=r,\ l=s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t b_t-2a_l b_l)g_{r,s}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_s b_l)g_{t,s} \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_s b_t)g_{l,s} \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4b_t b_l}{\sqrt{n}\sigma^2}v_{t,s}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4b_t b_l}{\sqrt{n}\sigma^2}v_{l,s}\right) \\
& +\frac{(6n-2)b_t b_l}{n\sigma^2}-\frac{2(n+1)a_t a_l}{n\sigma^2}
\end{aligned}$$

$$\begin{aligned}
= \quad & -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_t^2+2b_l^2)f_{t,l} \\
& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t b_t-2a_l b_l)g_{t,l} \\
& +2\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(b_s b_l f_{t,s}+b_s b_t f_{l,s})\right) \\
& +2\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(a_s b_l g_{t,s}+a_s b_t g_{l,s}) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4b_t b_l}{\sqrt{n}\sigma^2}(v_{t,s}+v_{l,s})\right) \\
& +\frac{(6n-2)b_t b_l}{n\sigma^2}-\frac{2(n+1)a_t a_l}{n\sigma^2}
\end{aligned}$$

$$\begin{aligned}
= \quad & -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2b_t^2+2b_l^2)\frac{\sqrt{2(n+1)}(a_t a_l+b_t b_l)}{\sqrt{n}\sigma} \\
& -\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}(2a_t b_t-2a_l b_l)\frac{\sqrt{2(n+1)}(a_t b_l-b_t a_l)}{\sqrt{n}\sigma} \\
& +2\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}\left(b_s b_l\frac{\sqrt{2(n+1)}(a_t a_s+b_t b_s)}{\sqrt{n}\sigma}+b_s b_t\frac{\sqrt{2(n+1)}(a_l a_s+b_l b_s)}{\sqrt{n}\sigma}\right)\right) \\
& +2\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\frac{\sqrt{2(n+1)}}{\sqrt{n}\sigma^2}\left(a_s b_l\frac{\sqrt{2(n+1)}(a_t b_s-b_t a_s)}{\sqrt{n}\sigma}+a_s b_t\frac{\sqrt{2(n+1)}(a_l b_s-b_l a_s)}{\sqrt{n}\sigma}\right) \\
& +\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\ l\neq s}\left(-\frac{4b_t b_l}{\sqrt{n}\sigma^2}\left(\frac{(a_t^2+b_t^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma}+\frac{(a_l^2+b_l^2)-(a_s^2+b_s^2)}{\sqrt{n}\sigma}\right)\right) \\
& +\frac{(6n-2)b_t b_l}{n\sigma^2}-\frac{2(n+1)a_t a_l}{n\sigma^2}
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\tfrac{4(n+1)}{n\sigma^3}(b_t^2+b_l^2)(a_ta_l+b_tb_l) \\
& -\tfrac{4(n+1)}{n\sigma^3}(a_tb_t-a_lb_l)(a_tb_l-b_ta_l) \\
& -\tfrac{4(n+1)}{n\sigma^3}\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\, l\neq s}(b_sb_l(a_ta_s+b_tb_s)+b_sb_t(a_la_s+b_lb_s)) \\
& +\tfrac{4(n+1)}{n\sigma^3}\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\, l\neq s}(a_sb_l(a_tb_s-b_ta_s)+a_sb_t(a_lb_s-b_la_s)) \\
& -\tfrac{4b_tb_l}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\, l\neq s}((a_t^2+b_t^2)-(a_s^2+b_s^2)+(a_l^2+b_l^2)-(a_s^2+b_s^2)) \\
& +\tfrac{(6n-2)b_tb_l}{n\sigma^2}-\tfrac{2(n+1)a_ta_l}{n\sigma^2} \\
= \quad & -\tfrac{4(n+1)}{n\sigma^3}(b_t^3b_l+b_tb_l^3+a_t^2b_tb_l+a_l^2b_tb_l) \\
& -\tfrac{8(n+1)}{n\sigma^3}\cdot\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\, l\neq s}(a_s^2b_lb_t+b_lb_tb_s^2) \\
& -\tfrac{4b_tb_l}{n\sigma^3}\textstyle\sum_{0\leqslant s\leqslant n \text{ and } t\neq s,\, l\neq s}(a_t^2+b_t^2+a_l^2+b_l^2-2(a_s^2+b_s^2)) \\
& +\tfrac{(6n-2)b_tb_l}{n\sigma^2}-\tfrac{2(n+1)a_ta_l}{n\sigma^2} \\
\overset{\text{Def. }\sigma}{=} \quad & -\tfrac{4(n+1)b_tb_l}{n\sigma^3}(a_t^2+a_l^2+b_t^2+b_l^2) \\
& -\tfrac{8(n+1)b_tb_l}{n\sigma^3}(\sigma-(a_t^2+b_t^2+a_l^2+b_l^2)) \\
& -\tfrac{4b_tb_l}{n\sigma^3}((n+1)(a_t^2+b_t^2+a_l^2+b_l^2)-2\sigma) \\
& +\tfrac{(6n-2)b_tb_l}{n\sigma^2}-\tfrac{2(n+1)a_ta_l}{n\sigma^2} \\
= \quad & -\tfrac{8(n+1)b_tb_l}{n\sigma^2} \\
& +\tfrac{8b_tb_l}{n\sigma^2} \\
& +\tfrac{(6n-2)b_tb_l}{n\sigma^2}-\tfrac{2(n+1)a_ta_l}{n\sigma^2} \\
= \quad & -\tfrac{2(n+1)}{n\sigma^2}(a_ta_l+b_tb_l)
\end{aligned}
$$

Example 39. Now that we calculated our matrix entries for $\mathrm{J}(\hat{\delta}_2)^{\mathrm{T}}\mathrm{J}(\hat{\delta}_2)$ we want to continue our Example 28.

$$
\mathrm{J}(\hat{\delta}_2)^{\mathrm{T}}\mathrm{J}(\hat{\delta}_2)
$$

$$
= \frac{1}{\sigma^2}\begin{pmatrix}
3(\sigma-a_0^2-b_0^2) & 0 & -3(a_0a_1+b_0b_1) & -3(a_0b_1-a_1b_0) & -3(a_0a_2+b_0b_2) & -3(a_0b_2-a_2b_0) \\
0 & 3(\sigma-a_0^2-b_0^2) & 3(a_0b_1-a_1b_0) & -3(a_0a_1+b_0b_1) & 3(a_0b_2-a_2b_0) & -3(a_0a_2+b_0b_2) \\
-3(a_0a_1+b_0b_1) & 3(a_0b_1-a_1b_0) & 3(\sigma-a_1^2-b_1^2) & 0 & -3(a_1a_2+b_1b_2) & -3(a_1b_2-a_2b_1) \\
-3(a_0b_1-a_1b_0) & -3(a_0a_1+b_0b_1) & 0 & 3(\sigma-a_1^2-b_1^2) & 3(a_1b_2-a_2b_1) & -3(a_1a_2+b_1b_2) \\
-3(a_0a_2+b_0b_2) & 3(a_0b_2-a_2b_0) & -3(a_1a_2+b_1b_2) & 3(a_1b_2-a_2b_1) & 3(\sigma-a_2^2-b_2^2) & 0 \\
-3(a_0b_2-a_2b_0) & -3(a_0a_2+b_0b_2) & -3(a_1b_2-a_2b_1) & -3(a_1a_2+b_1b_2) & 0 & 3(\sigma-a_2^2-b_2^2)
\end{pmatrix}
$$

Here, $\sigma=\sum_{j=0}^{2}(a_j^2+b_j^2)$.

5.4 Calculation of $K_{n,k}$

Example 40. We continue Example 39. To calculate $\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}}\mathrm{J}(\vartheta_{n,k})$, we eliminate in $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}}\mathrm{J}(\hat{\delta}_n)$ all the rows and columns belonging to a_0 and b_0 and set in the remaining matrix $a_0=1$ and $b_0=0$.

$$\mathrm{J}(\vartheta_{2,0})^{\mathrm{T}}\mathrm{J}(\vartheta_{2,0}) = \frac{1}{\sigma_0^2}\begin{pmatrix} 3(a_2^2+b_2^2+1) & 0 & -3(a_1a_2+b_1b_2) & -3(a_1b_2-a_2b_1) \\ 0 & 3(a_2^2+b_2^2+1) & 3(a_1b_2-a_2b_1) & -3(a_1a_2+b_1b_2) \\ -3(a_1a_2+b_1b_2) & 3(a_1b_2-a_2b_1) & 3(a_1^2+b_1^2+1) & 0 \\ -3(a_1b_2-a_2b_1) & -3(a_1a_2+b_1b_2) & 0 & 3(a_1^2+b_1^2+1) \end{pmatrix}$$

Here, $\sigma_0 = 1 + \sum_{j=1}^{2}(a_j^2 + b_j^2)$.

Remark 41. Suppose given $k \in [0,n]$. We want to convert our real matrix $\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}}\mathrm{J}(\vartheta_{n,k}) \in \mathbb{R}^{2n\times 2n}$ into a complex matrix $K_{n,k} \in \mathbb{C}^{n\times n}$.

To that end, we use the injective ring morphism ι; cf. Definition 7. We proceed as follows.

Write

$$\sigma_k = 1 + \sum_{j\in[0,n]\backslash\{k\}} (a_j^2 + b_j^2)\,.$$

Suppose given $j, l \in [0,n]\backslash\{k\}$. We get the 2×2-block of the real matrix $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}}\mathrm{J}(\hat{\delta}_n)$.

$$\begin{pmatrix} \zeta_{a_j}^{\mathrm{T}}\zeta_{a_l} & \zeta_{a_j}^{\mathrm{T}}\zeta_{b_l} \\ \zeta_{b_j}^{\mathrm{T}}\zeta_{a_l} & \zeta_{b_j}^{\mathrm{T}}\zeta_{b_l} \end{pmatrix}$$

If $j = l$, we obtain, using Calculations 33, 34 and 35,

$$\begin{pmatrix} \zeta_{a_j}^{\mathrm{T}}\zeta_{a_j} & \zeta_{a_j}^{\mathrm{T}}\zeta_{b_j} \\ \zeta_{b_j}^{\mathrm{T}}\zeta_{a_j} & \zeta_{b_j}^{\mathrm{T}}\zeta_{b_j} \end{pmatrix} = \begin{pmatrix} \frac{2(n+1)}{n\sigma_k^2}(\sigma_k - a_j^2 - b_j^2) & 0 \\ 0 & \frac{2(n+1)}{n\sigma_k^2}(\sigma_k - a_j^2 - b_j^2) \end{pmatrix} = \iota\left(\frac{2(n+1)}{n\sigma_k^2}(\sigma_k - a_j^2 - b_j^2) + \mathrm{i}\cdot 0\right)\,.$$

If $j \neq l$, we obtain, using Calculations 36, 37 and 38,

$$\begin{aligned} \begin{pmatrix} \zeta_{a_j}^{\mathrm{T}}\zeta_{a_l} & \zeta_{a_j}^{\mathrm{T}}\zeta_{b_l} \\ \zeta_{b_j}^{\mathrm{T}}\zeta_{a_l} & \zeta_{b_j}^{\mathrm{T}}\zeta_{b_l} \end{pmatrix} &= \begin{pmatrix} -\frac{2(n+1)}{n\sigma_k^2}(a_ja_l + b_jb_l) & \frac{2(n+1)}{n\sigma_k^2}(-a_jb_l + a_lb_j) \\ \frac{2(n+1)}{n\sigma_k^2}(-a_lb_j + a_jb_l) & -\frac{2(n+1)}{n\sigma_k^2}(a_ja_l + b_jb_l) \end{pmatrix} \\ &= \iota\left(-\frac{2(n+1)}{n\sigma_k^2}(a_ja_l + b_jb_l) + \mathrm{i}\cdot\frac{2(n+1)}{n\sigma_k^2}(-a_jb_l + a_lb_j)\right)\,. \end{aligned}$$

Recall that $\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}}\cdot\mathrm{J}(\vartheta_{n,k}) \in \mathbb{R}^{2n\times 2n}$ is obtained from $\mathrm{J}(\hat{\delta}_n)^{\mathrm{T}}\cdot\mathrm{J}(\hat{\delta}_n) \in \mathbb{R}^{(2n+2)\times(2n+2)}$ by deleting the columns and rows belonging to a_k and b_k and then by putting $a_k = 1$ and $b_k = 0$; cf. Remark 29.

This amounts to

$$\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}}\mathrm{J}(\vartheta_{n,k}) = \iota^{n\times n}(K_{n,k})$$

with

$$K_{n,k} := \frac{2(n+1)}{n\sigma_k^2}\left(\sigma_k \cdot \mathrm{E}_n - \begin{pmatrix} z_0 \\ \vdots \\ z_{k-1} \\ z_{k+1} \\ \vdots \\ z_n \end{pmatrix}\begin{pmatrix}\overline{z_0} & \dots & \overline{z_{k-1}} & \overline{z_{k+1}} & \dots & \overline{z_n}\end{pmatrix}\right)$$

with $z_j := a_j + \mathrm{i}b_j$ for $j \in [0,n]\backslash\{k\}$ and $\sigma_k = 1 + \sum_{j\in[0,n]\backslash\{k\}}(a_j^2 + b_j^2)$, note that

$$z_j\overline{z_l} = (a_j + \mathrm{i}b_j)(a_l - \mathrm{i}b_l) = (a_j a_l + b_j b_l) + \mathrm{i}(-a_j b_l + a_l b_j)\,.$$

Example 42. Now we continue the Example 40 and calculate the complex matrix $K_{2,0} \in \mathbb{C}^{2\times 2}$.

$$K_{2,0} = \frac{3}{\sigma_0^2}\left(\sigma_0 \cdot \mathrm{E}_2 - \begin{pmatrix} a_1 + \mathrm{i}b_1 \\ a_2 + \mathrm{i}b_2 \end{pmatrix}\begin{pmatrix}\overline{a_1 + \mathrm{i}b_1} & \overline{a_2 + \mathrm{i}b_2}\end{pmatrix}\right)$$

Here, $\sigma_0 = 1 + \sum_{j=1}^{2}(a_j^2 + b_j^2)$.

5.5 Calculation of the determinant of $K_{n,k}$, leading to δ_n being an immersion

Theorem 43. *The discriminant embedding δ_n is an immersion.*
More precisely,

$$\det(\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}}\mathrm{J}(\vartheta_{n,k})) = \left(\frac{2(n+1)}{n}\right)^{2n} \cdot \frac{1}{\sigma_k^{2(n+1)}} > 0$$

for $k \in [0,n]$, where $\sigma_k = 1 + \sum_{j\in[0,n]\backslash\{k\}}(a_j^2 + b_j^2)$.

Proof. We have

$$\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}}\mathrm{J}(\vartheta_{n,k}) = K_{n,k} = \frac{2(n+1)}{n\sigma_k^2}\left(\sigma_k \cdot \mathrm{E}_n - \begin{pmatrix} z_0 \\ \vdots \\ z_{k-1} \\ z_{k+1} \\ \vdots \\ z_n \end{pmatrix}\begin{pmatrix}\overline{z_0} & \dots & \overline{z_{k-1}} & \overline{z_{k+1}} & \dots & \overline{z_n}\end{pmatrix}\right).$$

Write

$$u := \begin{pmatrix} z_0 \\ \vdots \\ z_{k-1} \\ z_{k+1} \\ \vdots \\ z_n \end{pmatrix} .$$

The matrix $u\overline{u}^{\mathrm{T}}$ has rank 1.

Since it is hermitian, it has eigenvalue 0 to the algebraic multiplicity $n-1$. It has the eigenvector u to the eigenvalue $\overline{u}^{\mathrm{T}} \cdot u = \sigma_k - 1$ since $(u \cdot \overline{u}^{\mathrm{T}}) \cdot u = u \cdot (\overline{u}^{\mathrm{T}} \cdot u)$.

Therefore, $K_{n,k}$ has eigenvalue

$$\frac{2(n+1)}{n\sigma_k^2} \cdot \sigma_k = \frac{2(n+1)}{n\sigma_k}$$

with algebraic multiplicity $n-1$ and

$$\frac{2(n+1)}{n\sigma_k^2} \cdot (\sigma_k - (\sigma_k - 1)) = \frac{2(n+1)}{n\sigma_k^2}$$

with algebraic multiplicity 1.

So

$$\det(K_{n,k}) = \left(\frac{2(n+1)}{n}\right)^n \cdot \frac{1}{\sigma_k^{n+1}} .$$

So

$$\det(\mathrm{J}(\vartheta_{n,k})^{\mathrm{T}}\mathrm{J}(\vartheta_{n,k})) \overset{\text{Lemma 9}}{=} |\det(K_{n,k})|^2 = \left(\frac{2(n+1)}{n}\right)^{2n} \cdot \frac{1}{\sigma_k^{2(n+1)}} > 0 .$$

□

6 A visual application

We consider the graph

$$\Gamma = \{(x, y) \in \mathbb{C}^2 : y = x^3 - x\} \subseteq \mathbb{C}^2 .$$

In Figure 6.1 we display a projection of a part of Γ to $\mathbb{R}^3$ (and then to $\mathbb{R}^2$).

The real x-axis is shown in green, the real y-axis is shown in blue.

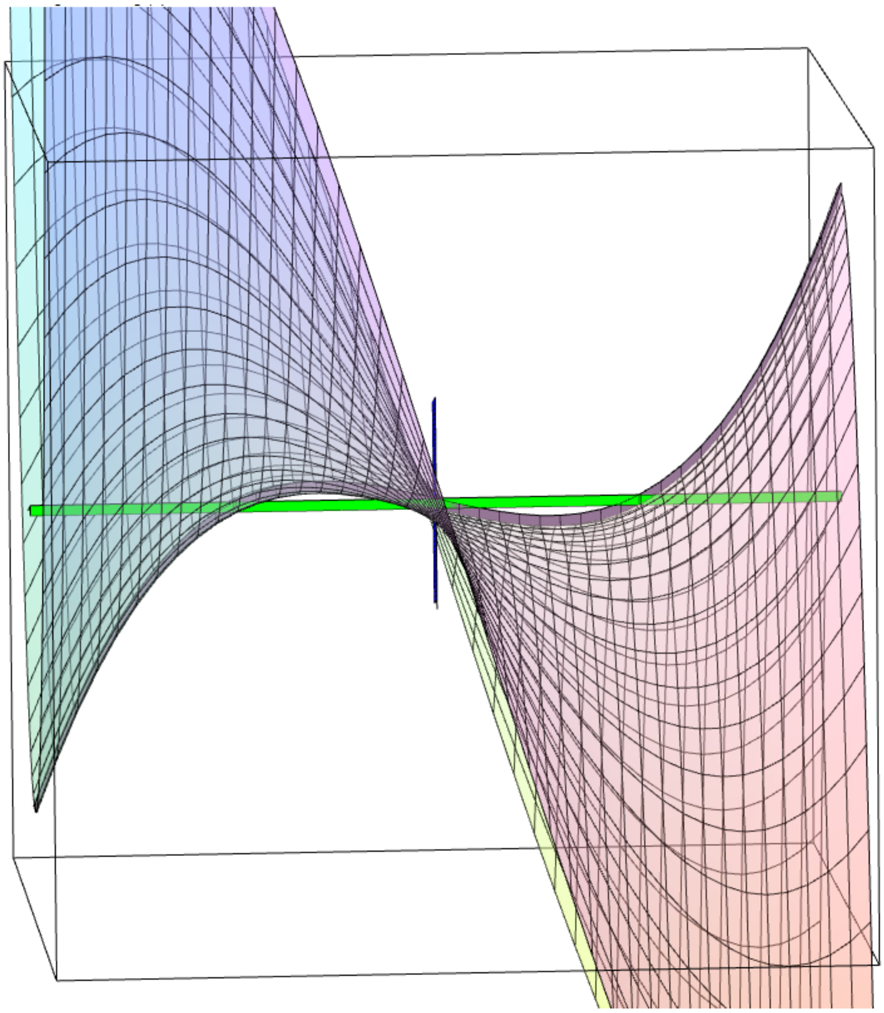


Figure 6.1: Γ, without the discriminant embedding

Now we consider the graph

$$\hat{\Gamma} = \{(x : y : z) \in \mathrm{P}^2(\mathbb{C}) : yz^2 = x^3 - xz^2\} \subseteq \mathrm{P}^2(\mathbb{C}) \, .$$

The discriminant embedding $\delta_2 : \mathrm{P}^2(\mathbb{C}) \longrightarrow \mathbb{C}^3 \times \mathbb{R}^3$ maps it to

$$\delta_2(\hat{\Gamma}) \subseteq \mathrm{S}^8 \subseteq \mathbb{C}^3 \times \mathbb{R}^3 \, .$$

In Figure 6.2, we display a projection of a part of $\delta_2(\hat{\Gamma})$ to $\mathbb{R}^3$ (and then to $\mathbb{R}^2$).

The real x-axis is shown in green, the real y-axis is shown in blue. The real part of the graph is emphasized with a thick line.

We obtain a kind of a "fish-eye effect".

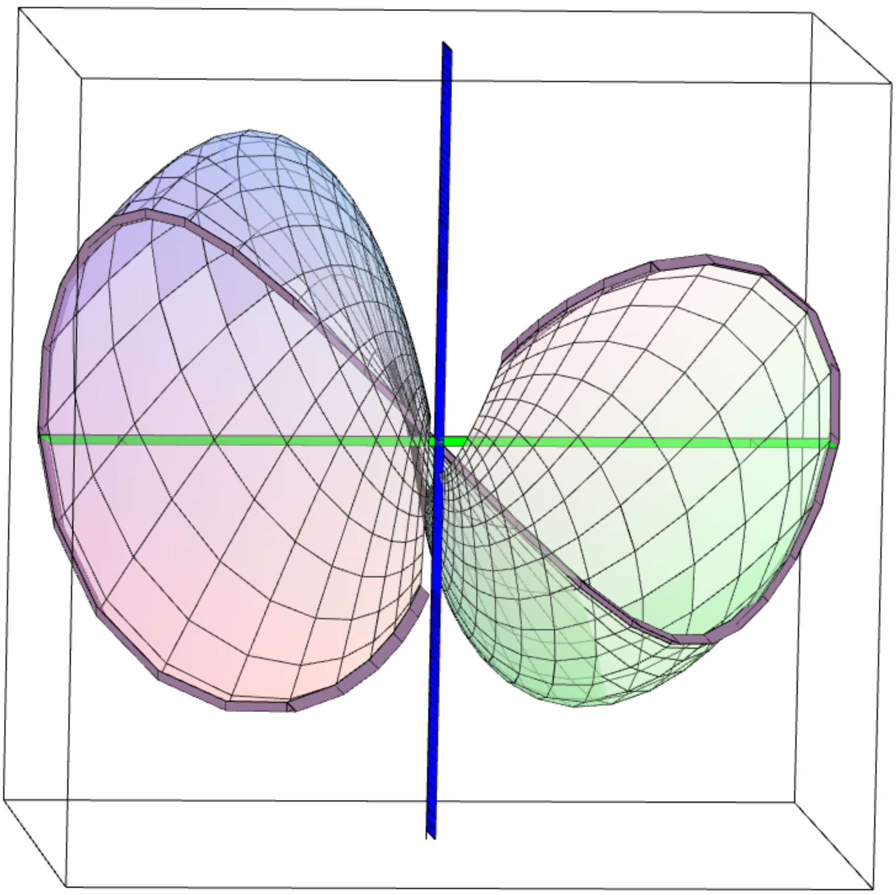


Figure 6.2: The image $\delta_2(\hat{\Gamma})$ of $\hat{\Gamma}$ under the discriminant embedding δ_2

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German summary

Die Riemannsche Zahlenkugel ist eine Abbildung

$$\delta_1 : \mathrm{P}^1(\mathbb{C}) \longrightarrow \mathrm{S}^2 \subset \mathbb{C} \times \mathbb{R},$$

die den unendlichen Punkt auf den Südpol der Sphäre S^2 abbildet. Wir verallgemeinern dies zu einer Abbildung

$$\delta_n : \ \mathrm{P}^n(\mathbb{C}) \longrightarrow \mathrm{S}^{3\binom{n+1}{2}-1} \subset \mathbb{C}^{\binom{n+1}{2}} \times \mathbb{R}^{\binom{n+1}{2}},$$

genannt Diskriminanteneinbettung.
Hierbei ist $\mathrm{P}^n(\mathbb{C})$ der n-dimensionale komplex-projektive Raum und $\mathrm{S}^{3\binom{n+1}{2}-1}$ die $(3\binom{n+1}{2} - 1)$-dimensionale Sphäre.

Es wird gezeigt, dass δ_n eine injektive Immersion ist. Die Injektivität ergibt sich aus einer direkten Rechnung. Um zu erhalten, dass δ_n eine Immersion ist, wird die Jacobimatrix J der Einschränkung von δ_n auf die k-te Standardkarte herangezogen. Hierzu wird die Determinante an der Stelle $(u_0 : \ldots : u_{k-1} : 1 : u_{k+1} : \ldots : u_n) \in \mathrm{P}^n(\mathbb{C})$ von $\mathrm{J}^{\mathrm{T}}\mathrm{J}$ berechnet und als positive reelle Zahl

$$\left(\frac{2(n+1)}{n}\right)^{2n} \cdot \frac{1}{\left(1 + \sum_{j\in[0,n]\setminus\{k\}} |u_j|^2\right)^{2(n+1)}} > 0$$

erkannt.

Versicherung

Hiermit versichere ich,

1. dass ich meine Arbeit selbstständig verfasst habe,
2. dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,
3. dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und
4. dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, den 05.07.2021

Svea Döring