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Sheet 9

Problem 21 Let $\mathcal{Z}$ be a grading category. Let $n \in [1,\infty]$. Let $(A,(m_\ell)_\ell)$ be a pre-A_n-algebra. Write $\mathfrak{m} := (({}^\omega m_\ell)_{\ell\in[1,n]\cap\mathbf{Z}})\beta_{\mathrm{Coder},n,A^{[1]}}$.

(1) Suppose $n \in \mathbf{Z}_{\geqslant 1}$. Suppose that $(m_\ell)_\ell$ satisfies the Stasheff equation at each $k \in [1, n-1]$. Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $a \in ((A^{[1]})^{\otimes n})^z$. Consider the following assertions.

(i) We have
$$a\left(\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=n}} (\mathrm{id}^{\otimes r}\otimes {}^\omega m_s \otimes \mathrm{id}^{\otimes t})\cdot {}^\omega m_{r+1+t}\right) \;=\; 0$$

(ii) We have $a\mathfrak{m}^2 = 0$.

Show that (i) and (ii) are equivalent.

(2) Suppose given $p \in [1,n]$. Write $\mathfrak{m}' := (({}^\omega m_\ell)_{\ell\in[1,p]\cap\mathbf{Z}})\beta_{\mathrm{Coder},p,A^{[1]}}$.

Show that $\mathfrak{m}' = \mathfrak{m}|_{\mathrm{T}_{\leqslant p}(A^{[1]})}^{\mathrm{T}_{\leqslant p}(A^{[1]})}$.

Problem 22 Let $\mathcal{Z}$ be a grading category. Let $n \in [1,\infty]$. Let $(\tilde{A},(\tilde{m}_\ell)_\ell)$ and $(A,(m_\ell)_\ell)$ be pre-A_n-algebras. Let $f = (f_\ell)_\ell$ be a pre-A_n-morphism from $\tilde{A}$ to A.
Write
$$\begin{aligned} \tilde{\mathfrak{m}} &:= (({}^\omega \tilde{m}_\ell)_\ell)\beta_{\mathrm{Coder},n,\tilde{A}^{[1]}} \\ \mathfrak{m} &:= (({}^\omega m_\ell)_\ell)\beta_{\mathrm{Coder},n,A^{[1]}} \\ \mathfrak{f} &:= (({}^\omega f_\ell)_\ell)\beta_{\mathrm{Coalg},n,\tilde{A}^{[1]},A^{[1]}} \,. \end{aligned}$$

(1) Suppose $n \in \mathbf{Z}_{\geqslant 1}$. Suppose that $(f_\ell)_\ell$ satisfies the Stasheff equation for morphisms at each $k \in [1, n-1]$.

Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $\tilde{a} \in ((\tilde{A}^{[1]})^{\otimes n})^z$. Consider the following assertions.

(i) We have
$$\tilde{a}\left(\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=n}} (\mathrm{id}^{\otimes r}\otimes {}^\omega \tilde{m}_s \otimes \mathrm{id}^{\otimes t})\cdot {}^\omega f_{r+1+t}\right) \;=\; \tilde{a}\left(\sum_{r\in[1,n]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j = n}} \Big(\bigotimes_{j\in[1,r]} {}^\omega f_{i_j}\Big)\cdot {}^\omega m_r\right)$$

(ii) We have $\tilde{a}(\tilde{\mathfrak{m}}\mathfrak{f} - \mathfrak{f}\mathfrak{m}) = 0$.

Show that (i) and (ii) are equivalent.

(2) Suppose given $p \in [1,n]$. Write $\mathfrak{f}' := (({}^\omega f_\ell)_{\ell\in[1,p]\cap\mathbf{Z}})\beta_{\mathrm{Coalg},p,\tilde{A}^{[1]},A^{[1]}}$.

Show that $\mathfrak{f}' = \mathfrak{f}|_{\mathrm{T}_{\leqslant p}(\tilde{A}^{[1]})}^{\mathrm{T}_{\leqslant p}(A^{[1]})}$.