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A_∞-categories, WS 16/17

Sheet 4

Problem 9 Let $q \in \mathbf{Z}_{\geqslant 1}$. Consider the cyclic group $\mathrm{C}_q = \langle\, c \,:\, c^q \,\rangle$.
Abbreviate $\mathrm{K} := \mathrm{K}(R C_q\text{-Mod})$.

(1) Construct a projective resolution P of the trivial $R\mathrm{C}_q$-module R that is periodic of period length 2.

(2) Calculate ${}_{\mathrm{K}}(P, \mathrm{Conc}(R)^{[i]})$ for $i \in \mathbf{Z}$.

(3) Calculate ${}_{\mathrm{K}}(P, P^{[i]})$ for $i \in \mathbf{Z}$.

(4) Calculate the composition map

$$ {}_{\mathrm{K}}(P, P^{[i]}) \otimes {}_{\mathrm{K}}(P^{[i]}, P^{[i+j]}) \;\to\; {}_{\mathrm{K}}(P, P^{[i+j]}) $$

for $i,\ j \,\in\, \mathbf{Z}$.

Problem 10 Let $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ be grading categories.
Let $\mathcal{Z} \xrightarrow{F} \tilde{\mathcal{Z}}$ be a morphism of grading categories; cf. Problem 5.

(1) Construct an isomorphism $F_{\&}(\bigotimes_{i\in[1,n]} M_i) \xrightarrow[\sim]{\sigma_{\underline{M}}} \bigotimes_{i\in[1,n]} F_{\&} M_i$ in $\tilde{\mathcal{Z}}$-grad.

(2) Show that the following quadrangle commutes.

$$
\begin{array}{ccc}
F_{\&}(\bigotimes_{i\in[1,n]} M_i) & \xrightarrow{\sigma_{\underline{M}}} & \bigotimes_{i\in[1,n]} F_{\&} M_i \\
{\scriptstyle F_{\&}(\bigotimes_{i\in[1,n]}(f_i,k_i))}\big\downarrow & & \big\downarrow{\scriptstyle \bigotimes_{i\in[1,n]} F_{\&}(f_i,k_i)} \\
F_{\&}(\bigotimes_{i\in[1,n]} M'_i) & \xrightarrow{\sigma_{\underline{M}'}} & \bigotimes_{i\in[1,n]} F_{\&} M'_i
\end{array}
$$

Problem 11 Let $\mathcal{Z}$ be a grading category.
Let A be a $\mathcal{Z}$-graded module.
Suppose given shift-graded maps $m_1 : A \to A$ of degree 1 and $m_2 : A^{\otimes 2} \to A$ of degree 0. For $n \in \mathbf{Z}_{\geqslant 3}$, we let $m_n := 0$, as shift-graded linear map from $A^{\otimes n}$ to A of degree $2 - n$.
Suppose that $(m_n)_{n\in\mathbf{Z}_{\geqslant 1}}$ satisfies the Stasheff equations for $k \in [1,3]$.
Suppose that for each $X \in \mathrm{Ob}(\mathcal{Z})$, there exists an element $1_X \in A^{\mathrm{id}_X}$ such that for $z,\, w \in \mathrm{Mor}(\mathcal{Z})$ such that $z\mathrm{t}_{\mathcal{Z}} = X = w\mathrm{s}_{\mathcal{Z}}$ and for $a \in A^z$ and $b \in A^w$, we have $(a \otimes 1_X)m_2 = a$ and $(1_X \otimes b)m_2 = b$.
Show that $(A, (m_n)_{n\in\mathbf{Z}_{\geqslant 1}})$ is a differential graded algebra over $\mathcal{Z}$.

Problem 12 Suppose given a grading category $\mathcal{Z}$.
Suppose given A_∞-algebras $\tilde{A}$ and A.
Suppose given a shift-graded linear map $f_1 : \tilde{A} \to A$ of degree 0.
Suppose that $f_1^{\otimes k} \cdot m_k^A = m_k^{\tilde{A}} \cdot f_1$ for $k \in \mathbf{Z}_{\geqslant 1}$.
Let $f_k = 0$ for $k \in \mathbf{Z}_{\geqslant 2}$, as shift-graded linear map from $\tilde{A}^{\otimes k}$ to A of degree $1 - k$.
Show that $(f_k)_{k\in\mathbf{Z}_{\geqslant 1}}$ is a morphism of A_∞-algebras from $\tilde{A}$ to A.