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A_∞-categories, WS 16/17

Sheet 10

Problem 23 Let $n \in [1, \infty]$. Let $\mathcal{Z}$ be a grading category.
Let $(A, (m_\ell)_\ell)$, $(A', (m'_\ell)_\ell)$, $(A'', (m''_\ell)_\ell)$, $(A''', (m'''_\ell)_\ell)$ be A_n-algebras over $\mathcal{Z}$.
Let $f = (f_\ell)_\ell : A \to A'$, $f' = (f'_\ell)_\ell : A' \to A''$, $f'' = (f''_\ell)_\ell : A'' \to A'''$ be A_n-morphisms.
Write ${}^\omega f := ({}^\omega f_\ell)_\ell$.
Define

$$f \cdot f' \;:=\; {}^{\omega^-}\big((({}^\omega f)\beta \cdot ({}^\omega f')\beta)\alpha\big)\,.$$

(1) Show that $f \cdot f'$ is a morphism of A_n-algebras from A to A''.

(2) Write $f \cdot f'$ in terms of $(f_\ell)_\ell$ and $(f'_\ell)_\ell$. What is the entry of $f \cdot f'$ at $\ell = 1$?

(3) Show that $(f \cdot f') \cdot f'' = f \cdot (f' \cdot f'')$.

(4) Suppose given shift-graded linear maps $g : A \to A'$ and $g' : A' \to A''$ of degree 0. Define $\operatorname{strict}_n(g) := (g, 0, 0, \dots)$.
When is $\operatorname{strict}_n(g) : A \to A'$ a morphism of A_n-algebras? Is $\operatorname{strict}_n(\mathrm{id}_A)$ a morphism of A_n-algebras? If $\operatorname{strict}_n(g)$ and $\operatorname{strict}_n(g')$ are morphisms of A_n-algebras, show that $\operatorname{strict}_n(gg') = \operatorname{strict}_n(g) \cdot \operatorname{strict}_n(g')$.

(5) Show that $f \cdot \operatorname{strict}_n(\mathrm{id}_{A'}) = f$ and that $\operatorname{strict}_n(\mathrm{id}_{A'}) \cdot f' = f'$.

(6) Define the category A_n-$\mathcal{Z}$-alg of A_n-algebras over $\mathcal{Z}$ and A_n-morphisms.
Therein, define the subcategory strict-A_n-$\mathcal{Z}$-alg of A_n-algebras over $\mathcal{Z}$ and strict A_n-morphisms.

(7) Show that H is a functor from A_n-$\mathcal{Z}$-alg to $\mathcal{Z}$-grad_0.

Problem 24 Let $n \in [1, \infty]$. Let $\mathcal{Z} \xrightarrow{F} \tilde{\mathcal{Z}}$ be a morphism of grading categories; cf. Problem 5.
Let $(A, (m_k)_k)$ be an A_n-algebra over $\mathcal{Z}$.

(1) Show that $F_\& A = (F_\& A, (\sigma^- \cdot F_\& m_\ell)_\ell)$ is an A_n-algebra over $\tilde{\mathcal{Z}}$, where $\sigma = \sigma_{(A,\dots,A)}$; cf. Problem 10.

(2) Consider the case $n = \infty$, $u \in \mathbf{Z}_{\geqslant 1}$, $\mathcal{Z} = \mathbf{Z} \times [1, u]^{\times 2}$, $\tilde{\mathcal{Z}} = \mathbf{Z}$ and P being the projection, mapping a morphism $(j, (s, t))$ to j.
Given a unital $\mathbf{Z} \times [1, u]^{\times 2}$-algebra A, i.e. an A_∞-category with set of objects $[1, u]$, show that its *total* A_∞*-algebra* $P_\& A$ is unital.