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A_∞-categories, WS 16/17

Sheet 1

Problem 1 Consider the commutative ring $\mathbf{Z}$. Consider the $\mathbf{Z}$-algebra $\mathbf{Z}$.
Determine the isoclasses of the $\mathbf{Z}$-modules M that have a chain of submodules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 = 0$$

such that

$$\begin{aligned} M_0/M_1 &\simeq \mathbf{Z}/(2) \\ M_1/M_2 &\simeq \mathbf{Z}/(4) \\ M_2/M_3 &\simeq \mathbf{Z}/(2) \,. \end{aligned}$$

Problem 2
Let Cat denote the (1-)category of categories, (1-)morphisms being functors.
Let Set denote the category of sets, morphisms being maps.

(1) Given a set X, how many isoclasses does the pair category $X^{\times 2}$ have?

(2) Construct a full and faithful functor $P : \mathrm{Set} \to \mathrm{Cat}$ sending X to $X^{\times 2}$.

(3) Show that the functor $\mathrm{Ob} : \mathrm{Cat} \to \mathrm{Set}$ has P as a right adjoint, i.e. $\mathrm{Ob} \dashv P$.

(4) Determine unit and counit of the adjunction in (3).

Problem 3 Let Poset denote the category of posets and monotone maps.

(1) Suppose given a poset X. Show that we have a subcategory CX of the pair category $X^{\times 2}$ with $\mathrm{Ob}(CX) = X$ and $\mathrm{Mor}(CX) = \{ (x, y) \in X^{\times 2} \,:\, x \leqslant y \}$.

(2) Construct a functor $C : \mathrm{Poset} \longrightarrow \mathrm{Cat}$.

(3) Given $n \in \mathbf{Z}_{\geqslant 0}$, we write $\Delta_n := C[0, n]$.
We have the monotone map $\omega : [0, 1] \to [0, n]$, $0 \mapsto 0$, $1 \mapsto n$.
Suppose given a category $\mathcal{Z}$ and $z \in \mathrm{Mor}(\mathcal{Z})$. Let $F_z : \Delta_1 \to \mathcal{Z}$, $(0, 1) \mapsto z$.
Let $n \geqslant 1$. Show that $\mathrm{fact}_n(z)$ is in bijection to

$$\{ \Delta_n \xrightarrow{G} \mathcal{Z} \,:\, G \text{ is a functor such that } G \circ (C\omega) = F_z \} \,.$$