Some additive galois cohomology rings

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Abstract

Let $p \geqslant 3$ be a prime. We consider the cyclotomic extension $\mathbf{Z}_{(p)}[\zeta_{p^2}] \,|\, \mathbf{Z}_{(p)}$, with galois group $G = (\mathbf{Z}/p^2)^*$. Since this extension is wildly ramified, the $\mathbf{Z}_{(p)}G$-module $\mathbf{Z}_{(p)}[\zeta_{p^2}]$ is not projective. We calculate its cohomology ring $\mathrm{H}^*(G, \mathbf{Z}_{(p)}[\zeta_{p^2}]; \mathbf{Z}_{(p)})$, carrying the cup product induced by the ring structure of $\mathbf{Z}_{(p)}[\zeta_{p^2}]$. Formulated in a somewhat greater generality, our results also apply to certain Lubin-Tate extensions.

Contents

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0 Introduction

0.1 Results

0.1.1 A cohomology ring

Suppose given a purely ramified extension $T|S$ of discrete valuation rings of residue characteristic $p \geqslant 3$, with maximal ideals generated by $s \in S$ and $t \in T$, respectively. Let $K = \operatorname{frac} S$, $L = \operatorname{frac} T$, and assume $L|K$ to be galois with $\operatorname{Gal}(L|K) = C_p = \langle\sigma\rangle$. In particular, $T|S$ is wildly ramified, and so, as Speiser remarked [**29,** §6], T is not projective as a module over SC_p. We are interested in its cohomology ring $\mathrm{H}^*(C_p, T; S) \stackrel{\text{def}}{=} \operatorname{Ext}^*_{SC_p}(S, T)$, carrying the cup product induced by multiplication on T.

We make the assumption that $\mathrm{v}_s(p) \geqslant b - \underline{b}$, where $b \stackrel{\text{def}}{=} -1 + \mathrm{v}_t(t^\sigma - t)$ and $b = p\underline{b} + \overline{b}$ with $\overline{b} \in [0, p-1]$. This assumption is void if K is of characteristic p. As a result, we obtain

$$
\begin{array}{cl}
 & \mathrm{H}^*(C_p, T; S) \\
\stackrel{\text{def}}{=} & \operatorname{Ext}^*_{SC_p}(S, T) \\
= & \chi_0^{(0)} S \\
\oplus & \Big(\chi_0^{(1)}(S/s^{\underline{b}+1}S) \oplus \cdots \oplus \chi_{\overline{b}-1}^{(1)}(S/s^{\underline{b}+1}S)\Big) \oplus \Big(\chi_{\overline{b}+1}^{(1)}(S/s^{\underline{b}}S) \oplus \cdots \oplus \chi_{p-1}^{(1)}(S/s^{\underline{b}}S)\Big) \\
\oplus & \chi_0^{(2)}(S/s^{b-\underline{b}}S) \\
\oplus & \Big(\chi_0^{(3)}(S/s^{\underline{b}+1}S) \oplus \cdots \oplus \chi_{\overline{b}-1}^{(3)}(S/s^{\underline{b}+1}S)\Big) \oplus \Big(\chi_{\overline{b}+1}^{(3)}(S/s^{\underline{b}}S) \oplus \cdots \oplus \chi_{p-1}^{(3)}(S/s^{\underline{b}}S)\Big) \\
\oplus & \chi_0^{(4)}(S/s^{b-\underline{b}}S) \\
\oplus & \cdots\,,
\end{array}
$$

with S-linear generators $\chi_i^{(k)} \in \mathrm{H}^k(C_p, T; S)$; where the multiplication $\chi_0^{(2)} \cup -$ acts as a degree shift by 2, and where multiplication of odd degree elements is given by

$$\chi_j^{(2l+1)} \cup \chi_k^{(2m+1)} \;=\; \partial_{\overline{j+k},\overline{2b}}\, s^{b+\underline{j+k-2b}}\, (\overline{b-j})^{-1} \chi_0^{(2l+2m+2)}$$

for $l, m \geqslant 0$ and $j, k \in [0, p-1] \smallsetminus \{\overline{b}\}$ (4.10), where ∂ denotes the Kronecker delta.

0.1.2 A reduction isomorphism

Let $U|T$ be a further purely ramified extension of discrete valuation rings, and denote $E = \operatorname{frac} U$. Suppose $E|K$ to be finite galois, write $H = \operatorname{Gal}(E|K)$, and suppose that $[E : L]$ is coprime to p. So we are given $K \overset{p}{\subseteq} L \overset{p'}{\subseteq} E$, containing $S \subseteq T \subseteq U$, respectively. We have the reduction isomorphism

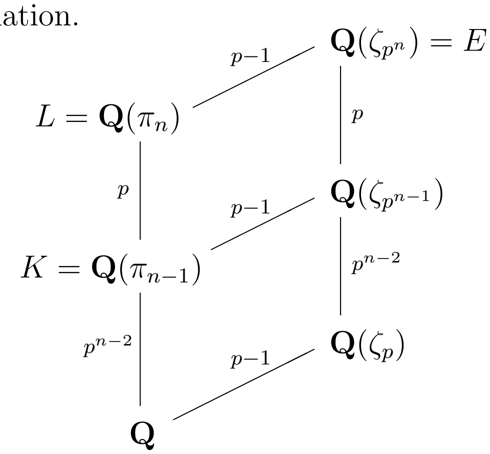
$$\mathrm{H}^*(H, U; S) \;\simeq\; \mathrm{H}^*(C_p, T; S)$$

of graded S-algebras (2.10, 3.1), thus enabling us to disregard the upper p'-part.

0.1.3 A cyclotomic example

The results apply in particular to the following situation.

$$\begin{array}{rcl} S &=& \mathbf{Z}_{(p)}[\pi_{n-1}] \\ T &=& \mathbf{Z}_{(p)}[\pi_n] \\ U &=& \mathbf{Z}_{(p)}[\zeta_{p^n}] \\ H &=& C_{p-1} \times C_p \\ b &=& (p^{n-1}-1)/(p-1) \\ \underline{b} &=& (p^{n-2}-1)/(p-1) \end{array} \qquad \begin{array}{rcl} s &=& \pi_{n-1} \\ t &=& \pi_n \\ u &=& \zeta_{p^n} - 1 \\ && \\ && \\ \bar{b} &=& 1\,, \end{array}$$



where $n \geqslant 2$ and

$$\pi_n \overset{\text{def}}{=} \prod_{j \in [1,\,p-1]} (\zeta_{p^n}^{j^{p^{n-1}}} - 1) \;=\; \mathrm{N}_{E|L}(\zeta_{p^n} - 1)\;,$$

cf. (5.4). In the case $n = 2$, we note that $\pi_1 = p$.

0.2 Method

We reinterpret $\mathrm{H}^*(C_p, T; S) \overset{\text{def}}{=} \mathrm{Ext}^*_{SC_p}(S, T) \simeq \mathrm{Ext}^*_{T \wr C_p}(T, T)$ by adjunction, invoking the *twisted group ring* $T \wr C_p \supseteq SC_p$ which carries the multiplication $(\rho y)(\tau z) = (\rho\tau)(y^\tau z)$, where $\rho, \tau \in C_p$ and $y,\, z \in T$. In this way, we have gained freedom in our choice of a projective resolution — when resolving T over $T \wr C_p$, we are not bound to take a projective resolution of S over SC_p and to tensor it with $T \wr C_p$ over SC_p. The cup product on $\mathrm{Ext}^*_{SC_p}(S, T)$ corresponds to the Yoneda product on $\mathrm{Ext}^*_{T \wr C_p}(T, T)$ (3.1).

Still, we need an interpretation of $T \wr C_p$ that facilitates the Yoneda product calculation. Rationally, there is the Wedderburn isomorphism $L \wr C_p \overset{\omega}{\underset{\sim}{\longrightarrow}} \mathrm{End}_K L = K^{p \times p}$, sending $\rho y \overset{\omega}{\longmapsto} (z \longmapsto z^\rho y)$. Restricting to the locally integral situation and using the resulting

Wedderburn *embedding* $T \wr C_p \overset{\omega}{\hookrightarrow} \operatorname{End}_S T = S^{p\times p}$, we get a workable isomorphic copy $(T \wr C_p)\omega \subseteq S^{p\times p}$ of $T \wr C_p$.

Namely, a matrix in $S^{p\times p}$ is contained in $(T \wr C_p)\omega$ if and only if it satisfies a set of congruences between its entries that can be deduced from the single congruence

$$t^\sigma \;\equiv_{t^{1+b}}\; t \;;$$

see (1.19, 1.17). With this tie description of $(T \wr C_p)\omega$ inside $S^{p\times p}$ at our disposal, it is easy to calculate the cohomology ring.

In particular, we believe that it is easier to use this tie description to resolve projectively than to work with the classical projective resolution of T over $T \wr C_p$. Using the latter, it seems that at some point the operation of σ on T as an element of $\operatorname{End}_S T$, i.e. the matrix $(\sigma)\omega$, enters the picture; for instance, when solving equations occurring in the resolution of cocycles (cf. 4.2). Using our tie description, we circumvent this problem by choosing an S-linear basis of $(T \wr G)\omega$ *without* specifying the coefficients of $(\sigma)\omega$ therein. So it is no longer necessary to determine the matrix $(\sigma)\omega$, which, indeed, we have not been able to control.

0.3 Known results and some historical remarks

0.3.1 Galois module structure, wild case

Let $L|K$ be a finite galois extension with $G = \operatorname{Gal}(L|K)$, and let $S \subseteq K$ be a Dedekind domain with field of fractions K and integral closure T in L. Since T is not isomorphic to SG as a module over SG as soon as a prime ideal of S wildly ramifies in T, Leopoldt split the galois module structure problem in two parts. If $K = \mathbf{Q}$, $S = \mathbf{Z}$ and G is abelian, he determined, firstly, generators for the *associated order*

$$\mathcal{A}_{L|K} \;=\; \{\xi \in KG \;:\; T\xi \subseteq T\}\;;$$

secondly, he showed that $T \simeq \mathcal{A}_{L|K}$ as a module over $\mathcal{A}_{L|K}$ by construction of an isomorphism [**19,** Satz 6].

We recall some of the recent extensions of this result.

(1) If $K = \mathbf{Q}(\zeta_n)$, $K \subseteq L \subseteq \mathbf{Q}(\zeta_{mn})$ and $S = \mathbf{Z}[\zeta_n]$, then Byott and Lettl gave a description of $\mathcal{A}_{L|K}$ and showed that $T \simeq \mathcal{A}_{L|K}$ [**8**].

(2) Aiba showed that the analogue of (1) holds in the Carlitz-Hayes function field case if L equals the analogue of $\mathbf{Q}(\zeta_{mn})$ and if the analogue of m divides n. On the other hand, if this divisibility condition fails, then $T \not\simeq \mathcal{A}_{L|K}$ [**1,** th. 4].

(3) If $L|K$ is an extension of finite extensions of $\mathbf{Q}_p$ with $L|\mathbf{Q}_p$ abelian, and S the valuation ring of K, Lettl gave a description of $\mathcal{A}_{L|K}$ and showed that $T \simeq \mathcal{A}_{L|K}$ [**20**].

(4) As Byott observed, similar phenomena as in (2) occur for Lubin-Tate extensions over $\mathbf{Q}_p$ [**7,** th. 5.1].

(5) If $L|K$ is an extension of finite extensions of $\mathbf{Q}_p$, and if $G = C_{p^n}$, Elder calculated T as a $\mathbf{Z}_pG$-module [**13**].

It is quite possible that such a description of $T \simeq \mathcal{A}_{L|K}$ as a module over SG might be used to calculate cohomology, and also to calculate the cup product on $\mathrm{H}^*(G,T;S)$. Nonetheless, generally speaking, Yoneda products are somewhat easier to calculate than cup products.

In the case of an extension $L|K$ of number fields, with S the ring of algebraic integers in K and $T|S$ at most tamely ramified, Fröhlich conjectured a connection between the class of T in $\mathrm{Cl}(\mathbf{Z}G)$ and the Artin root numbers, which has been proven by Taylor; see e.g. [**31,** th. 3]. The extensions of this result from the tame to the wild case start by replacing T by an object better suited for class group considerations, thus putting the emphasis on the Artin root numbers; see e.g. [**10,** sec. 4.2].

0.3.2 Wedderburn embedding

Suppose given a Dedekind domain R and an R-order Ω which is rationally semisimple, that is, for which $\bar{\Omega} \overset{\mathrm{def}}{=} (\mathrm{frac}\, R) \otimes_R \Omega$ is semisimple.

The Wedderburn embedding method consists of restricting the Wedderburn isomorphism ω from $\bar{\Omega}$ to Ω, and to describe the image $\Omega\omega$ inside a product of rationally simple maximal orders via congruences of matrix entries, called *ties.*

For the first time, this method surfaced in the proof of the Brauer-Nesbitt theorem, in which the assumption of the existence of a certain non-maximal overorder of a quasiblock is led to a contradiction [**6,** eq. (36)]. Here, the *quasi*blocks of Ω are the R-orders $\Omega\varepsilon$ for *rational* primitive central idempotents $\varepsilon \in \bar{\Omega}$.

Around the same time, Higman calculated the ties for $\mathbf{Z}(C_p \rtimes C_{p-1})$ with $C_p \rtimes C_{p-1} = \langle a, b \,:\, a^p, b^{p-1}, a^b = a^r \rangle$, where r is a generator of $\mathbf{F}_p^*$ [**15**]. In particular, he calculated the hereditary quasiblock of size $(p-1) \times (p-1)$, which is isomorphic to the twisted group ring $\mathbf{Z}[\zeta_p] \wr C_{p-1}$. See [**26,** sec. 6].

In the commutative case, descriptions of suborders via congruences have also been given by Leopoldt [**19,** p. 125, p. 134, p. 140].

Milnor gave a pullback description of $\mathbf{Z}C_{p^n}$, the iteration of which yields the ties describing this ring [**3,** p. 601 f.].

The first systematic study to describe Wedderburn embeddings via ties has been undertaken by Plesken [**24**], [**25**], following hints of Zassenhaus. This has been particularly successful when the endomorphism rings of the indecomposable projective modules of the quasiblocks under consideration are isomorphic to the discrete valuation ring R.

We list some subsequent work related to the twisted group ring $T \wr G$, where $T|S$ is a finite galois extension of discrete valuation rings with $G = \mathrm{Gal}(T|S)$.

(1) Auslander and Rim showed that $T|S$ is tamely ramified if and only if $T \wr G$ is hereditary [**2,** prop. 3.5], which in turn corresponds to an upper triangular shape of its single quasiblock.

(2) Benz and Zassenhaus showed that in the wildly ramified case the radical idealisator process, started with $T \wr G$, ends up with a hereditary overorder that is unique with respect to certain conditions [**5,** Satz (a)].

(3) Cliff and Weiss showed that the radical idealisator process of (2) ends up with the unique minimal hereditary overorder of $T \wr G$, they determined its shape, and they calculated the number of steps of this process in terms of the different and the ramification index of $T|S$ [**12**]. As an example, they figured out the ties for $T \wr G$ in the case of a wildly ramified quadratic extension [**11,** p. 98, 97].

(4) Wingen calculated the ties of the associated order for G cyclic [**33,** p. 82].

(5) Weber calculated the ties of $T \wr G$ in the case $T = \mathbf{Z}_{(p)}[\zeta_{p^n}]$, where $n \geqslant 2$, and $G = C_p$ [**32**].

0.4 Acknowledgements

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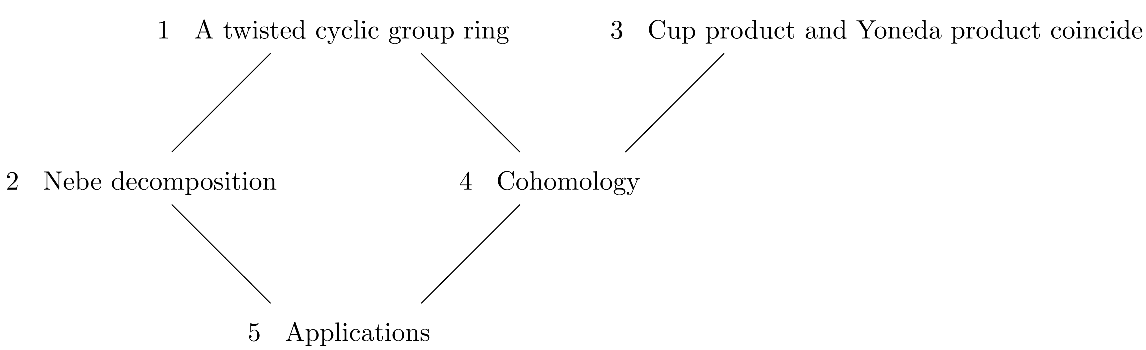
0.5 Leitfaden



0.6 Notations and conventions

(i) Composition of maps is written on the right, $\xrightarrow{a}\xrightarrow{b} = \xrightarrow{ab}$. Exception is made for 'standard' maps, such as traces, characters, derivations ...

(ii) For $a, b \in \mathbf{Z}$, we denote by $[a, b] := \{c \in \mathbf{Z} \mid a \leqslant c \leqslant b\}$ the integral interval.

(iii) Given elements x, y of some set X, we let $\partial_{x,y} = 1$ in case $x = y$ and $\partial_{x,y} = 0$ in case $x \neq y$.

(iv) Given $a, b \in \mathbf{Z}$, the binomial coefficient $\binom{a}{b}$ is defined to be zero unless $0 \leqslant b \leqslant a$.

(v) If R is a discrete valuation ring with maximal ideal generated by r, we write $\mathrm{v}_r(x)$ for the valuation of $x \in R \smallsetminus \{0\}$ at r, i.e. $x/r^{\mathrm{v}_r(x)}$ is a unit in R. Moreover, we let $\mathrm{v}_r(0) = +\infty$.

(vi) If R is a discrete valuation ring and $M \overset{f}{\hookrightarrow} N$ an injective R-linear map between R-modules M and N with cokernel of finite length in the sense of Jordan-Hölder, we refer to this length $\ell_R(N/M)$ as the R-linear colength of M in N.

(vii) Let $n \geqslant 1$, let A be a ring. The ring of $n \times n$-matrices over A is denoted by $A^{n\times n}$.

(viii) Given a ring A, by an A-module we mean a finitely generated right A-module, unless specified otherwise.

(ix) Given a ring A, we denote by $K^-(A)$ the homotopy category of complexes of A-modules bounded to the right.

(x) Given a category $\mathcal{C}$, and objects X, Y in $\mathcal{C}$, we denote the set of morphisms from X to Y by ${}_{\mathcal{C}}(X, Y)$. If $\mathcal{C} = \text{mod-}A$ for a ring A, we abbreviate ${}_A(X, Y) := {}_{\text{mod-}A}(X, Y)$.

1 A twisted cyclic group ring

We give a description of a certain twisted group ring of a cyclic group as a subring of a matrix ring. We proceed in descending generality, ending up with a complete description in the case of a cyclic group of prime order as the galois group of a purely ramified extension of discrete valuation rings. For an attempt in the next larger case C_{p^2}, see appendix A.

For the sake of illustration, a continuing example is included, indicated by (cont.).

1.1 Subrings defined by derivations

Let A be a ring, let $I \subseteq A$ be an ideal. Let $k \geqslant 1$, let $x = (x_j)_{j\in[1,k]}$ be a tuple of elements of A and let the corresponding inner derivations be denoted by

$$\begin{array}{ccc} A & \xrightarrow{D_{x_j}} & A \\ y & \longmapsto & yx_j - x_j y \; . \end{array}$$

We note that if $x_i x_j = x_j x_i$, then $D_{x_i} \circ D_{x_j} = D_{x_j} \circ D_{x_i}$.

Lemma 1.1 *Let $h = (h_j)_{j\in[1,k]}$ be a tuple of positive integers (the* height*). Let $l = (l_j)_{j\in[1,k]}$ be a tuple of positive integers (the* length*). Then the abelian subgroup*

$$A(x, h, l)_I := \Big\{ y \in A \;:\; D_{x_1}^{i_1} \circ \cdots \circ D_{x_k}^{i_k}(y) \in I^{i_1 l_1 + \cdots + i_k l_k} \;\text{ for } (i_j)_{j\in[1,k]} \in \prod_{j\in[1,k]} [0, h_j] \Big\} \subseteq A$$

is a subring of A.

Proof. Given y and z in $A(x,h,l)_I$, we need to show that the product yz is contained in $A(x,h,l)_I$. So suppose given $(i_j)_{j\in[1,k]} \in \prod_{j\in[1,k]}[0,h_j]$. The term

$$D_{x_1}^{i_1} \circ \cdots \circ D_{x_k}^{i_k}(yz)$$

equals a sum over terms of type

$$\Big(D_{x_1}^{i'_1} \circ \cdots \circ D_{x_k}^{i'_k}(y)\Big) \cdot \Big(D_{x_1}^{i''_1} \circ \cdots \circ D_{x_k}^{i''_k}(z)\Big)$$

where $(i'_j)_j, (i''_j)_j \in \prod_{j\in[1,k]}[0,h_j]$ such that $(i'_j)_j + (i''_j)_j = (i_j)_j$; each such summand is contained in $I^{i_1 l_1 + \cdots + i_k l_k}$. ▫

1.2 A Wedderburn embedding

Let $T|S$ be a finite and purely ramified extension of discrete valuation rings of residue characteristic $p \geqslant 3$. Let $L|K$ be the corresponding extension of fields of fractions, assumed to be galois with galois group G of order g. Let t generate the maximal ideal of T, let $s := (-1)^{g+1}\,\mathrm{N}_{L|K}(t)$ generate the maximal ideal of S. Let

$$\mathfrak{D}_{T|S}^{-1} \;:=\; \{x \in L : \mathrm{Tr}_{L|K}(xT) \subseteq S\} \;\subseteq\; L$$

define the different ideal $\mathfrak{D}_{T|S} \subseteq T$, and let $\Delta_{T|S} := \mathrm{N}_{L|K}(\mathfrak{D}_{T|S}) \subseteq S$ denote the discriminant ideal of $T|S$. Write the minimal polynomial of t over K as

$$\mu_{t,K}(X) \;=\; X^g + \left(\sum_{j\in[1,g-1]} e_j X^j\right) - s \;\in S[X]\,.$$

We recall that $T = S[t]$, that $Tt^g = Ts$, that $S/Ss \xrightarrow{\sim} T/Tt$, that $\mu_{t,K}(X) \equiv_s X^g$ and that $\mathfrak{D}_{T|S} = (\mu'_{t,K}(t)) \subseteq T$ [**28,** III.§6, cor. 2].

By $T \wr G = \{\sum_{\rho\in G} \rho y_\rho \;:\; y_\rho \in T\}$ we denote the *twisted group ring* carrying the multiplication induced by $(\rho y)(\tau z) = (\rho\tau)(y^\tau z)$, where $\rho,\,\tau \in G$ and $y,\,z \in T$. Let Ξ denote the image of the Wedderburn embedding (of S-algebras)

$$\begin{array}{rcl} T\wr G & \xrightarrow{\omega} & \mathrm{End}_S\,T \;=:\; \Gamma \\ y & \longmapsto & (\dot{y} : x \longmapsto xy) \\ \rho & \longmapsto & (\dot{\rho} : x \longmapsto x^\rho) \end{array}$$

We consider the subring

$$\Lambda \;:=\; \Big\{ f \in \Gamma \;:\; (Tt^i)f \subseteq Tt^i \text{ for all } i \geqslant 0 \Big\} \;\subseteq\; \Gamma$$

which contains Ξ, i.e.

$$T\wr G \xrightarrow[\sim]{\omega} \Xi \;\subseteq\; \Lambda \;\subseteq \Gamma\,,$$

by a slight abuse of the notation ω. Since $\dim_L L \otimes_T \Lambda = g$, we have $\ell_T(\Lambda/\dot{t}\Lambda) = \ell_T(\Lambda/\Lambda\dot{t}) = g$, and so we obtain

$$\dot{t}\Lambda \;=\; \Lambda\dot{t} \;=\; \{f \in \Gamma \;:\; (Tt^i)f \subseteq Tt^{i+1} \text{ for all } i \geqslant 0\}\;,$$

which is thus an ideal of Λ. Moreover, note that we may write

$$\Lambda \;=\; \{f \in \Gamma \;:\; (Tt^i)f \subseteq Tt^i \text{ for all } i \in [0, g-1]\}\;.$$

Remark 1.2 Using matrices with respect to the S-linear basis $(t^0, t^1, \ldots, t^{g-1})$ of T to represent elements of $\Gamma = \mathrm{End}_S T$, the subring Λ of the full matrix ring $S^{g\times g}$ is given by the set of matrices with strictly lower triangular entries contained in Ss. The ideal $\dot{t}\Lambda$ is given by the set of matrices with lower triangular entries contained in Ss, including the main diagonal. In this interpretation, we shall refer to matrix positions using the coordinates $[0, g-1] \times [0, g-1]$.

Remark 1.3 (cf. [**2,** prop. 3.5]) *We have* $g \not\equiv_p 0$ *if and only if*

$$T \wr G \xrightarrow[\sim]{\omega} \Xi \;=\; \Lambda\;.$$

In particular, in this case T *is projective as a module over* $T \wr G$ (cf. [**17,** prop. 2.4]).

Proof. By [**28,** IV.§2, cor. 1, cor. 3], $g \not\equiv_p 0$ is equivalent to $\mathrm{v}_t(t^\rho - t) = 1$ for all $\rho \in G \smallsetminus \{1\}$.

Now by [**17,** Cor 2.17], the S-linear colength of Ξ in Γ is given by $\ell_S(\Gamma/\Xi) = g\cdot \mathrm{v}_s(\Delta_{T|S})/2$. On the other hand, $\ell_S(\Gamma/\Lambda) = g(g-1)/2$. Therefore, the inclusion $\Xi \subseteq \Lambda$ is an equality if and only if $\mathrm{v}_s(\Delta_{T|S}) = g-1$. But

$$\mathrm{v}_s(\Delta_{T|S}) \;=\; g^{-1} \sum_{\substack{\rho,\tau \in G,\\ \rho\neq\tau}} \mathrm{v}_t(t^\rho - t^\tau) \;=\; \sum_{\rho \in G\smallsetminus\{1\}} \mathrm{v}_t(t^\rho - t)\;.$$

□

Example 1.4 (cont.) Let $S = \mathbf{Z}_{(3)}$, $s = 3$, $t = \pi_2 := (\zeta_9 - 1)(\zeta_9^{-1} - 1)$, and $T = \mathbf{Z}_{(3)}[\pi_2]$ (cf. §5.2). Let $\sigma : \zeta_9 \longmapsto \zeta_9^4$, restricted from $\mathbf{Q}(\zeta_9)$ to T. We have $G = \{1, \sigma, \sigma^2\} \simeq C_3$. We shall use the matrix interpretation explained in (1.2).

The Wedderburn embedding sends t to $\dot{t} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -9 & 6 \end{bmatrix}$, the last row resulting from $t^3 = 3 - 9t + 6t^2$. Furthermore, it sends σ to $\dot{\sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -5 & 1 \\ 24 & -21 & 4 \end{bmatrix}$, the second row resulting from $t^\sigma = 6 - 5t + t^2$. In particular, $t^\sigma - t = t^2 - 6t + 6$ has valuation 2 at t, so $\Xi \subset \Lambda$ is a proper subring, i.e. the Wedderburn embedding does not induce an isomorphism of $T \wr G$ with Λ.

We have $\Lambda = \begin{bmatrix} S & S & S \\ 3S & S & S \\ 3S & 3S & S \end{bmatrix}$, containing $\dot{\sigma}$, and $\dot{t}\Lambda = \Lambda\dot{t} = \begin{bmatrix} 3S & S & S \\ 3S & 3S & S \\ 3S & 3S & 3S \end{bmatrix}$.

1.3 The intermediate ring Λ^{D}

We assume that

$$b \;:=\; -1 + \min_{\rho\in G} \mathrm{v}_t(t^\rho - t) \;\geqslant\; 1\,.$$

For example, if $G = \langle\sigma\rangle \simeq C_g$, then $b = -1 + \mathrm{v}_t(t^\sigma - t)$ since $\mathrm{v}_t(t^{\sigma^i} - t) \geqslant \mathrm{v}_t(t^\sigma - t)$ for $i \in [0, g-1]$.

Let

$$\Lambda^{\mathrm{D}} \;:=\; \Lambda\big((\dot{t}), (g-1), (1+b)\big)_{\dot{t}\Lambda} \;\subseteq\; \Lambda,$$

where $k = 1$ in the notation of (1.1). Explicitly, we have

$$\Lambda^{\mathrm{D}} \;=\; \left\{ u \in \Lambda \;:\; \sum_{j\in[0,i]} (-1)^j \tbinom{i}{j}\, \dot{t}^j u \dot{t}^{i-j} \equiv_{\dot{t}^{(1+b)i}\Lambda} 0 \text{ for all } i \in [0, g-1] \right\}\,.$$

Lemma 1.5 *We have*

$$\Xi \;\subseteq\; \Lambda^{\mathrm{D}} \quad (\subseteq\; \Lambda \;\subseteq\; \Gamma\,)\,.$$

Proof. Since Λ^{D} is a subring of Λ, it suffices to show that $\dot{t} \in \Lambda^{\mathrm{D}}$, which follows from $D^i_{\dot{t}}(\dot{t}) = 0$ for $i \geqslant 1$, and that $\dot{\rho} \in \Lambda^{\mathrm{D}}$ for $\rho \in G$, i.e. that $D^i_{\dot{t}}(\dot{\rho})$ lies in $\dot{t}^{i(b+1)}\Lambda$ for $\rho \in G$ and $i \in [0, g-1]$. But

$$\begin{array}{rcl} 0 & \equiv_{t^{(1+b)i}} & (t^\rho - t)^i \\ & = & \displaystyle\sum_{j\in[0,i]} (-1)^{i-j} \tbinom{i}{j}\, (t^j)^\rho t^{i-j} \end{array}$$

implies

$$\begin{array}{rcl} 0 & \equiv_{\dot{t}^{(1+b)i}\Lambda} & \displaystyle\sum_{j\in[0,i]} (-1)^{i-j} \tbinom{i}{j}\, (\dot{t}^j)^{\dot{\rho}} \dot{t}^{i-j} \\ & = & \dot{\rho}^{-1} \left(\displaystyle\sum_{j\in[0,i]} (-1)^{i-j} \tbinom{i}{j}\, \dot{t}^j \dot{\rho} \dot{t}^{i-j} \right) \\ & = & \dot{\rho}^{-1}\, D^i_{\dot{t}}(\dot{\rho})\,. \end{array}$$

□

1.4 A description of Λ^{D} via ties

For $i \in \mathbf{Z}$, we write

$$i =: g\underline{i} + \overline{i} \qquad \text{with } \overline{i} \in [0, g-1].$$

We note that $\underline{-i} = -1 - \underline{i-1}$, and that $\overline{-i} = g - 1 - \overline{i-1}$.

For $i \in [0, g-1]$ and $j \geqslant 0$, we consider the element $\varepsilon_{i,j} \in \Lambda \subseteq \Gamma$ defined on the S-linear basis $(t^0, \ldots, t^{g-1})$ of T by

$$\boxed{(t^l)\varepsilon_{i,j} := \partial_{i,l} t^{\overline{l+j}} s^{\underline{l+j}}}$$

for $l \in [0, g-1]$. In the interpretation of (1.2), $\varepsilon_{i,j}$ is the matrix with a single non-zero entry $s^{\underline{i+j}}$ at position $(i, \overline{i+j})$. The tuple $(\varepsilon_{i,j})_{i,j\in[0,g-1]}$ forms an S-linear basis of Λ. For instance,

$$\dot{t} \;=\; \left(\sum_{i\in[0,g-1]} \varepsilon_{i,1}\right) - s^{-1}\left(\sum_{j\in[1,g-2]} e_j\varepsilon_{g-1,j+1}\right) - e_{g-1}\varepsilon_{g-1,0} \;\in\; \Lambda\,.$$

The elements $\dot{\rho}$ seem to be harder to describe in this manner (cf. 1.4). In case $G = C_p$, we shall circumvent this problem by giving a S-linear basis of Ξ *without* giving the transition matrix from the 'initial' S-linear basis $(\dot{t}^i\dot{\sigma}^j)_{i,j\in[0,g-1]}$ of Ξ to it. To do so, we use a description of Ξ by congruences, or *ties,* between the coefficients with respect to the $(\varepsilon_{i,j})_{i,j}$-basis of Λ.

Lemma 1.6 *For $i, i' \in [0, g-1]$ and $j, j' \geqslant 0$, we obtain*

$$\boxed{\varepsilon_{i,j}\varepsilon_{i',j'} = \partial_{i',\overline{i+j}}\varepsilon_{i,j+j'}\;.}$$

Proof. This follows by evaluation on t^l for $l \in [0, g-1]$. ▫

Let

$$\ddot{t} \;:=\; \sum_{i\in[0,g-1]} \varepsilon_{i,1} \;\in\; \Lambda\,.$$

Lemma 1.7 *For $j \geqslant 0$, we obtain*

$$\ddot{t}^j \;=\; \sum_{i\in[0,g-1]} \varepsilon_{i,j}\;.$$

Proof. This follows using induction on j together with (1.6). ▫

Lemma 1.8 *We have $\ddot{t}\Lambda \;=\; \Lambda\ddot{t} \;=\; \dot{t}\Lambda \;=\; \Lambda\dot{t}$. An S-linear basis of $\ddot{t}^k\Lambda$ for $k \geqslant 0$ is given by*

$$(s^{-\underline{j-k}}\varepsilon_{i,j})_{i,j\in[0,g-1]}\;.$$

Proof. To see this, we may use the matrix interpretation of (1.2). ▫

Assumption 1.9 Assume that $\mathrm{v}_s(e_j) \geqslant 1 + b - \underline{b+j}$ for $j \in [1, g-1]$.

For example, if $b = 1$, then the assumption (1.9) reads $e_j \in Ss^2$ for $j \in [1, g-2]$ and $e_{g-1} \in Ss$.

Lemma 1.10 *Assuming* (1.9), *we get* $\ddot{t} - \dot{t} \;\in\; \ddot{t}^{1+b(g-1)}\Lambda$.

Proof. We have $\ddot{t} - \dot{t} = e_{g-1}\varepsilon_{g-1,0} + s^{-1}\sum_{j\in[1,g-2]} e_j\varepsilon_{g-1,j+1}$. Thus, by (1.8), we need to prove the inequalities

$$\left\{\begin{array}{rclcll} \mathrm{v}_s(e_j) & \geqslant & -\,\underline{(j+1)-(1+b(g-1))}+1 & = & 1+b-\underline{j+b} & \text{for } j\in[1,g-2] \\ \mathrm{v}_s(e_{g-1}) & \geqslant & -\,\underline{0-(1+b(g-1))} & = & 1+b-\underline{(g-1)+b}\;, & \end{array}\right.$$

which in turn have been assumed in (1.9). ▫

Using (1.10), we are in position to substitute $\dot{t}$ by $\ddot{t}$ in the construction of the subring Λ^{D}.

Lemma 1.11 *Assume* (1.9). *Given* $\gamma \geqslant 0$, *we obtain*

$$\begin{array}{rcl} \dot{t}^{\gamma}\Lambda^{\mathrm{D}} & \overset{(1)}{=} & \dot{t}^{\gamma}\Lambda\big((\dot{t}),(g-1),(1+b)\big)_{\dot{t}\Lambda} \\ & \overset{(2)}{=} & \left\{ u\in\Lambda \;:\; \sum_{h\in[0,l]}(-1)^h\binom{l}{h}\dot{t}^h u\dot{t}^{l-h} \equiv_{\dot{t}^{(1+b)l+\gamma}\Lambda} 0 \text{ for all } l\in[0,g-1]\right\} \\ & \overset{(3)}{=} & \left\{ u\in\Lambda \;:\; \sum_{h\in[0,l]}(-1)^h\binom{l}{h}\ddot{t}^h u\ddot{t}^{l-h} \equiv_{\ddot{t}^{(1+b)l+\gamma}\Lambda} 0 \text{ for all } l\in[0,g-1]\right\} \\ & \overset{(4)}{=} & \ddot{t}^{\gamma}\Lambda\big((\ddot{t}),(g-1),(1+b)\big)_{\ddot{t}\Lambda}\;. \end{array}$$

Proof. Equation (1) follows by definition of Λ^{D}. Let us prove (3). First, we remark that for $i \geqslant 0$, (1.10, 1.8) give

$$\dot{t}^i \;\equiv_{\ddot{t}^{i+b(g-1)}\Lambda}\; \dot{t}^{i-1}\ddot{t} \;\equiv_{\ddot{t}^{i+b(g-1)}\Lambda}\; \cdots \;\equiv_{\ddot{t}^{i+b(g-1)}\Lambda}\; \ddot{t}^i\;.$$

Now, both sets are subsets of $\dot{t}^{\gamma}\Lambda = \ddot{t}^{\gamma}\Lambda$, as we see by putting $l = 0$. So suppose given $u \in \dot{t}^{\gamma}\Lambda = \ddot{t}^{\gamma}\Lambda$. Using (1.10, 1.8) we obtain

$$\dot{t}^h u\dot{t}^{l-h} \equiv_{\ddot{t}^{l+b(g-1)+\gamma}\Lambda} \ddot{t}^h u\dot{t}^{l-h} \equiv_{\ddot{t}^{l+b(g-1)+\gamma}\Lambda} \ddot{t}^h u\ddot{t}^{l-h}\;.$$

for $0 \leqslant h \leqslant l \leqslant g-1$. Since $l + b(g-1) + \gamma \geqslant (1+b)l + \gamma$, this shows (3). Let us prove (4). Again, both sides are contained in $\ddot{t}^{\gamma}\Lambda$. So if $v \in \Lambda$, then $\ddot{t}^{\gamma}v$ is in the right hand side of (4) if and only if $v \in \Lambda\big((\ddot{t}),(g-1),(1+b)\big)_{\ddot{t}\Lambda}$, for multiplication with $\ddot{t}$ is injective. This in turn is the case if and only if $\ddot{t}^{\gamma}v$ is contained in the left hand side of (4), again by injectivity of the multiplication with $\ddot{t}$. This proves (4). The argument for (2) is analogous. ▫

Lemma 1.12 *For* $x, y \in \mathbf{Z}$ *with* y *coprime to* g, *we obtain*

$$\sum_{i\in[0,g-1]} \underline{x+iy} \;=\; g^{-1}\sum_{i\in[0,g-1]}\big((x+iy)-(\overline{x+iy})\big) \;=\; x+(g-1)(y-1)/2\;.$$

Now we shall give a description of Λ^{D} via *ties;* that is, we give certain congruences between the coefficients with respect to the S-linear ε-basis of Λ that are necessary, and, taken together, also sufficient for an element of Λ to lie in Λ^{D}. Necessity will follow from (1.5), whereas sufficiency needs a comparison of colengths.

Lemma 1.13 *Assume* (1.9). *Given* $\gamma \geqslant 0$*, we get*

$$\begin{array}{rcl} \dot{t}^{\gamma}\Lambda^{\mathrm{D}} & = & \Big\{ \sum_{i,j\in[0,g-1]} a_{i,j}\varepsilon_{i,j} \;:\; a_{i,j}\in S \;,\; \text{and for all } i,j,l\in[0,g-1], \text{ we have} \\ & & \qquad \mathrm{v}_s\Big(\sum_{h\in[0,l]} (-1)^h \binom{l}{h} a_{\overline{i+h},j}\Big) \;\geqslant\; 1+\underline{bl-j-1+\gamma} \Big\} \\ & \subseteq & \Lambda\,. \end{array}$$

Proof. The lth condition of (1.11) ($l\in[0,g-1]$) for an element $u=\sum_{i,j\in[0,g-1]} a_{i,j}\varepsilon_{i,j}\in\Lambda$ ($a_{i,j}\in S$) to lie in $\dot{t}^{\gamma}\Lambda^{\mathrm{D}}$ reads

$$\begin{array}{rl} 0 \;\equiv_{\ddot{t}^{(1+b)l+\gamma}\Lambda} & \sum_{h\in[0,l]}\sum_{i,j\in[0,g-1]}(-1)^h\binom{l}{h}\ddot{t}^{h}a_{i,j}\varepsilon_{i,j}\ddot{t}^{l-h} \\ \overset{(1.7)}{=} & \sum_{h\in[0,l]}\sum_{i,j,i',i''\in[0,g-1]}(-1)^h\binom{l}{h}a_{i,j}\varepsilon_{i',h}\varepsilon_{i,j}\varepsilon_{i'',l-h} \\ \overset{(1.6)}{=} & \sum_{h\in[0,l]}\sum_{i,j,i',i''\in[0,g-1]}(-1)^h\binom{l}{h}a_{i,j}\partial_{i,\overline{i'+h}}\partial_{i'',\overline{i'+h+j}}\varepsilon_{i',l+j} \\ = & \sum_{j,i'\in[0,g-1]}\Big(\sum_{h\in[0,l]}(-1)^h\binom{l}{h}a_{\overline{i'+h},j}\Big)\varepsilon_{i',l+j}\;. \end{array}$$

For $a\in S$ and $k\geqslant 0$, the element $a\varepsilon_{i,j}$ is contained in $\ddot{t}^{k}\Lambda$ if and only if $\mathrm{v}_s(a)\geqslant -\underline{j-k}$ (cf. 1.8). Thus we obtain the set of conditions

$$(*_{i,j,l}) \qquad \mathrm{v}_s\Big(\sum_{h\in[0,l]}(-1)^h\binom{l}{h}a_{\overline{i+h},j}\Big) \;\geqslant\; -(\underline{-bl+j-\gamma}) \;=\; 1+\underline{bl-j-1+\gamma}$$

for $i,j,l\in[0,g-1]$, as claimed. ▫

1.5 The cyclic case of order p

As we have seen, in general we have an inclusion $T\wr G \xrightarrow{\sim} \Xi \subseteq \Lambda^{\mathrm{D}}$, and we dispose of a workable description of Λ^{D}. Now we shall consider a case in which this inclusion will turn out to be an equality.

Consider the case $g=p$, i.e. assume $G=\langle\sigma\rangle=C_p$ to be of order equal to the residue characteristic p of S. Note that then $b=-1+\mathrm{v}_t(t^{\sigma}-t)$, and recall that we have stipulated $b\geqslant 1$.

Remark 1.14 *We have*

$$\mathrm{v}_t(\mathfrak{D}_{T|S}) \;=\; \mathrm{v}_s(\Delta_{T|S}) \;=\; p^{-1}\,\mathrm{v}_t\left(\prod_{\substack{\rho,\tau\,\in\, G\,,\\ \rho\,\neq\,\tau}} (t^\rho - t^\tau)\right) \;=\; \mathrm{v}_t\left(\prod_{\rho\in G\smallsetminus\{1\}} (t^\rho - t)\right) \;=\; (p{-}1)(1{+}b)$$

(cf. [**28,** V.§3, lem. 3]*).*

Remark 1.15 *In the present case $G = C_p$, assumption* (1.9) *is fulfilled.*

Proof. Since $\mathfrak{D}_{T|S} = (\mu'_{t,K}(t))$, (1.14) implies in particular that

$$t^{(p-1)(1+b)} \mid \mu'_{L|K}(t) \;=\; pt^{p-1} + \sum_{j\in[1,p-1]} je_j t^{j-1}\;,$$

i.e. that $\mathrm{v}_t(e_j) \geqslant (p-1)(1+b) - (j-1)$ for $j \in [1, p-1]$, for the valuations of the summands are pairwise different. But since $e_j \in S$, this implies

$$\mathrm{v}_s(e_j) \;\geqslant\; \underline{(p-1)(1+b) - (j-1) + (p-1)} \;=\; 1 + b - \underline{b + j}$$

for $j \in [1, p-1]$, and thus, assumption (1.9) is fulfilled. ▫

Lemma 1.16 *Suppose given $\alpha_0, \dots, \alpha_{p-1} \geqslant 0$. Consider the S-linear submodule*

$$\begin{array}{rcl} M & := & \Big\{(x_h)_{h\in[0,p-1]} \in S^p \;:\; \text{for all } l\in[0,p-1], \text{ we have } \mathrm{v}_s\Big(\sum_{h\in[0,l]}(-1)^h \tbinom{l}{h}\, x_h\Big) \geqslant \alpha_l\Big\} \\ & \subseteq & S^p\,. \end{array}$$

An S-linear basis of M is given by the tuple $\big((s^{\alpha_l}\binom{i}{l})_{i\in[0,p-1]}\big)_{l\in[0,p-1]}$.

Proof. Let $\tilde{M}$ be the S-linear submodule of S^p spanned by $\big((s^{\alpha_{l'}}\binom{i}{l'})_{i\in[0,p-1]}\big)_{l'\in[0,p-1]}$. Since

$$\sum_{h\in[0,l]} (-1)^h \tbinom{l}{h} \Big(s^{\alpha_{l'}} \tbinom{h}{l'}\Big) \;=\; (-1)^{l'}\partial_{l,l'} s^{\alpha_{l'}}$$

for $l, l' \in [0, p-1]$, we have $\tilde{M} \subseteq M$. The S-linear colength of M in S^p is given by $\sum_{l\in[0,p-1]} \alpha_l$, and so is the colength of $\tilde{M}$. Hence $\tilde{M} = M$. ▫

Proposition 1.17 *Suppose given $\gamma \geqslant 0$ such that* $\mathrm{v}_s(p) \;\geqslant\; b - \underline{b-\gamma}$. *Then*

$$\begin{array}{rcl} \dot{t}^\gamma\Lambda^{\mathrm{D}} & = & \Bigg\{ \displaystyle\sum_{i,j\in[0,p-1]} a_{i,j}\varepsilon_{i,j} \;:\; a_{i,j}\in S\;,\; \text{and for all } j,l\in[0,p-1], \text{ we have} \\ & & \qquad\qquad \mathrm{v}_s\left(\displaystyle\sum_{h\in[0,l]} (-1)^h \tbinom{l}{h}\, a_{h,j}\right) \;\geqslant\; 1 + \underline{bl - j - 1 + \gamma}\;\Bigg\} \\ & \subseteq & \Lambda\,. \end{array}$$

An S-linear basis of $t^\gamma \Lambda^{\mathrm{D}}$ is given by

$$\boxed{(\mu_{l,j}^{(\gamma)})_{l,\,j\,\in\,[0,p-1]} \;:=\; \left(s^{1+\underline{bl-j-1+\gamma}} \sum_{i\in[0,p-1]} \tbinom{i}{l}\,\varepsilon_{i,j}\right)_{l,\,j\,\in\,[0,p-1]}} \;.$$

The according basis of Λ^{D} will also be written $(\mu_{l,j})_{l,j} := (\mu_{l,j}^{(0)})_{l,j}$. The S-linear colength of Λ^{D} in Λ is given by $bp(p-1)/2$.

Proof. Comparing with (1.13), we have to show the redundancy of the conditions $(*_{i,j,l})$ for $i \in [1,p-1]$, $j,l \in [0,p-1]$. First of all, we have $(-1)^h \binom{p-1}{h} \equiv_{s^{1+\underline{b(p-1)-j-1+\gamma}}} 1$ for $h \in [0,p-1]$, since this holds modulo p. Hence the validity of $(*_{i,j,p-1})$ is independent of $i \in [0,p-1]$. By downwards induction on $l \in [0,p-2]$, forming the difference, equivalence of $(*_{i,j,l})$ and $(*_{i+1,j,l})$ ensues from $(*_{i,j,l+1})$, where $i \in [0,p-2]$.

Now (1.16), applied to a fixed $j \in [0,p-1]$, yields the S-linear basis as claimed. The colength of Λ^{D} in Λ is given by

$$\begin{array}{rcl} \sum_{l\in[0,p-1]} \sum_{j\in[0,p-1]}(1+\underline{bl-j-1}) & \overset{(1.12)}{=} & p^2 + \sum_{l\in[0,p-1]}(bl-p) \\ & = & bp(p-1)/2. \end{array}$$

□

Remark 1.18 In (1.17), we can as well fix any $m \in [0,p-1]$ and impose $(*_{m,j,l})$ on elements of Λ for $j,l \in [0,p-1]$ to describe $\dot{t}^\gamma \Lambda^{\mathrm{D}}$ inside. Accordingly, we obtain an S-linear basis $(s^{1+\underline{bl-j-1+\gamma}} \sum_{i\in[0,p-1]} \binom{i}{l}\,\varepsilon_{\overline{i+m},j})_{l,j}$ of $\dot{t}^\gamma \Lambda^{\mathrm{D}}$.

Theorem 1.19 *Recall that we assume $b = -1 + \mathrm{v}_t(t^\sigma - t) \geqslant 1$, where $C_p = \langle\sigma\rangle$. Suppose in addition that $\mathrm{v}_s(p) \geqslant b - \underline{b}$.*

The Wedderburn embedding factors over the S-algebra isomorphism

$$\boxed{T \wr C_p \xrightarrow[\sim]{\omega} \Lambda^{\mathrm{D}} \subseteq \Lambda \subseteq \Gamma} \;.$$

Proof. By (1.5), it suffices to show that the S-linear colengths of $(T \wr G)\omega = \Xi$ and of Λ^{D} in Λ are equal. The colength of Λ in Γ equals $p(p-1)/2$, so that, by (1.17), the colength of Λ^{D} in Γ equals $(1+b)p(p-1)/2$. On the other hand, by **[17,** cor. 2.17], the colength of Ξ in Γ equals $p\,\mathrm{v}_s(\Delta_{T|S})/2 \overset{(1.14)}{=} p(1+b)(p-1)/2$. □

Example 1.20 (cont.) We have $\dot{t} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -9 & 6 \end{bmatrix}$ and $\ddot{t} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$.

We have $g = p = 3$ and $b = 1$. Since $\mu_{t,\mathbf{Q}}(X) = X^3 - 6X^2 + 9X - 3$, assumption (1.9) is satisfied, in accordance with (1.15). Consequently, we have $\dot{t} - \ddot{t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -9 & 6 \end{bmatrix} \in 3\Lambda$ (cf. 1.10).

If $\gamma \leqslant 1$, then $1 = \mathrm{v}_s(p) \geqslant b - \underline{b-\gamma} = 1$, and so (1.17) can be applied to give

$$\begin{array}{lll}
\mathbf{Z}_{(3)}[\pi_2] \wr C_3 & = & T \wr C_3 \\
& \overset{\omega}{\underset{\sim}{\longrightarrow}} & \Lambda^{\mathrm{D}} \\
& = & \left\{ \left[\begin{smallmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ 3a_{1,2} & a_{1,0} & a_{1,1} \\ 3a_{2,1} & 3a_{2,2} & a_{2,0} \end{smallmatrix}\right] : \begin{array}{l} a_{0,0} \in 3^0 S,\ a_{0,0} - a_{1,0} \in 3^1 S,\ a_{0,0} - 2a_{1,0} + a_{2,0} \in 3^1 S; \\ a_{0,1} \in 3^0 S,\ a_{0,1} - a_{1,1} \in 3^0 S,\ a_{0,1} - 2a_{1,1} + a_{2,1} \in 3^1 S; \\ a_{0,2} \in 3^0 S,\ a_{0,2} - a_{1,2} \in 3^0 S,\ a_{0,2} - 2a_{1,2} + a_{2,2} \in 3^0 S; \end{array} \right\} \\
& = & \left\{ \left[\begin{smallmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ 3a_{1,2} & a_{1,0} & a_{1,1} \\ 3a_{2,1} & 3a_{2,2} & a_{2,0} \end{smallmatrix}\right] : a_{0,0} \equiv_3 a_{1,0} \equiv_3 a_{2,0};\ a_{0,1} + a_{1,1} + a_{2,1} \equiv_3 0 \right\} \\
& \subseteq & S^{3\times 3} ,
\end{array}$$

where $S = \mathbf{Z}_{(3)}$. In particular, we have $\dot\sigma = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 6 & -5 & 1 \\ 24 & -21 & 4 \end{smallmatrix}\right] \in \Lambda^{\mathrm{D}}$, as expected. Similarly,

$$\begin{array}{lll}
\dot t \Lambda^{\mathrm{D}} & = & \left\{ \left[\begin{smallmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ 3a_{1,2} & a_{1,0} & a_{1,1} \\ 3a_{2,1} & 3a_{2,2} & a_{2,0} \end{smallmatrix}\right] : \begin{array}{l} a_{0,0} \in 3^1 S,\ a_{0,0} - a_{1,0} \in 3^1 S,\ a_{0,0} - 2a_{1,0} + a_{2,0} \in 3^1 S; \\ a_{0,1} \in 3^0 S,\ a_{0,1} - a_{1,1} \in 3^1 S,\ a_{0,1} - 2a_{1,1} + a_{2,1} \in 3^1 S; \\ a_{0,2} \in 3^0 S,\ a_{0,2} - a_{1,2} \in 3^0 S,\ a_{0,2} - 2a_{1,2} + a_{2,2} \in 3^1 S; \end{array} \right\} \\
& = & \left\{ \left[\begin{smallmatrix} 3a_{0,0} & a_{0,1} & a_{0,2} \\ 3a_{1,2} & 3a_{1,0} & a_{1,1} \\ 3a_{2,1} & 3a_{2,2} & 3a_{2,0} \end{smallmatrix}\right] : a_{0,1} \equiv_3 a_{1,1} \equiv_3 a_{2,1};\quad a_{0,2} + a_{1,2} + a_{2,2} \equiv_3 0 \right\} \\
& \subseteq & S^{3\times 3} .
\end{array}$$

Example 1.21 Suppose that S contains a primitive p-th root of unity ζ_p (so in particular, $\operatorname{char} K = 0$). Let $T := S[\sqrt[p]{s}]$ and $t := \sqrt[p]{s}$, i.e. let $\mu_{t,K}(X) = X^p - s$. Then $L|K$ is galois with galois group C_p, and we have

$$b \;=\; -1 + \mathrm{v}_t(t\zeta_p - t) \;=\; p\,\mathrm{v}_s(1-\zeta_p) = \frac{p}{p-1}\,\mathrm{v}_s(p) ,$$

i.e. $\mathrm{v}_s(p) = b - \underline{b}$, so that (1.19) may be applied.

Corollary 1.22 *Suppose given two discrete valuation rings T and T' over S with $T|S$ and $T'|S$ both purely ramified and galois with galois group C_p. Suppose that*

$$\mathrm{v}_s(\Delta_{T|S}) \;=\; \mathrm{v}_s(\Delta_{T'|S}) \;\leqslant\; p\,\mathrm{v}_s(p) + p - 1 .$$

Then

$$T \wr C_p \;\simeq\; T' \wr C_p$$

as S-algebras.

Proof. Since $\mathrm{v}_s(\Delta_{T|S}) = (p-1)(b+1)$ by (1.14), similarly for T', and since Λ^{D} depends only on S and b (cf. 1.17), we may conclude

$$T \wr C_p \overset{(1.19)}{\simeq} \Lambda^{\mathrm{D}} \overset{(1.19)}{\simeq} T' \wr C_p .$$

□

Example 1.23 (cont.) Let $S = \mathbf{Z}_{(3)}$, $T = \mathbf{Z}_{(3)}[\pi_2]$ and let $T' = S[t']$ with $\mu_{t',K}(X) = X^3 + 3X^2 - 18X + 48$ [**16**]. Then $\Delta_{T|S} = \Delta_{T'|S} = Ss^4$, whence $T \wr C_3 \simeq T' \wr C_3$ by (1.22). The fields of fractions of T and T', however, are not isomorphic.

2 Nebe decomposition

The purpose of this section is, translated to the basic example $\mathbf{Z}_{(p^2)}[\zeta_{p^2}]|\mathbf{Z}_{(p)}$, the reduction of the cohomology calculation from the galois group $C_p \times C_{p-1}$ to its quotient C_p, being the galois group of a subextension.

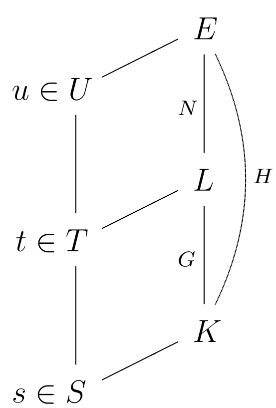
2.1 A block decomposition

Let $T|S$ and $U|T$ be finite purely ramified extensions of discrete valuation rings with maximal ideals generated by $s \in S$, $t \in T$ and $u \in U$, respectively. Let $K = \operatorname{frac} S$, $L = \operatorname{frac} T$ and $E = \operatorname{frac} U$, and suppose that $E|K$ is galois with galois group H, and that $L|K$ is galois with galois group G. In particular, if

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$$

is short exact, then $E|L$ is galois with galois group N. Denote $h := |H|$, $g := |G|$ and $n := |N|$, so $h = gn$.

The situation can be depicted as follows.



Suppose given an S-linear submodule V of U spanned by a T-linear basis of U, i.e. such that $T \otimes_S V \xrightarrow[\sim]{\varphi} U$, $x \otimes y \longmapsto xy$. In general, V is not a subring of U. We write

$$\begin{array}{rcl} \Gamma & := & \operatorname{End}_S U \\ \Gamma' & := & \operatorname{End}_S T \\ \Gamma'' & := & \operatorname{End}_T U \\ \Gamma''_0 & := & \operatorname{End}_S V \,. \end{array}$$

Here we change our notation slightly in that the ring $\operatorname{End}_S T$, which has previously been denoted by Γ, is now denoted by Γ'. Similarly, we denote the Wedderburn embeddings by

$$\begin{array}{rcl} U \wr H & \overset{\omega}{\hookrightarrow} & \Gamma \\ T \wr G & \overset{\omega'}{\hookrightarrow} & \Gamma' \\ U \wr N & \overset{\omega''}{\hookrightarrow} & \Gamma'' \,, \end{array}$$

again switching notation from what has been denoted by ω to ω'. Accordingly, we denote the images by

$$\begin{array}{lcl} \Xi & := & (U \wr H)\omega \\ \Xi' & := & (T \wr G)\omega' \\ \Xi'' & := & (U \wr N)\omega'' \end{array}$$

Usage of the notation Γ', ω', Ξ' pertains only to the present §2, in which we consider the passage from $U \wr H$ to $T \wr G$.

Note that we have an isomorphism of S-algebras

$$\begin{array}{ccccclcl} \Gamma' & \otimes_S & \Gamma''_0 & \xrightarrow{\sim} & \mathrm{End}_S(T \otimes_S V) & & \xrightarrow{\sim} & \Gamma \\ \alpha & \otimes & \beta_0 & \longmapsto & (x \otimes y \longmapsto x\alpha \otimes y\beta_0) & & & \\ & & & & \eta & & \longmapsto & \varphi^{-1}\eta\varphi \;, \end{array}$$

denoted by $\Gamma' \otimes_S \Gamma''_0 \xrightarrow[\sim]{\theta} \Gamma$. Moreover, we have an isomorphism of T-algebras

$$\begin{array}{cccccccccc} T & \otimes_S & \Gamma''_0 & \xrightarrow{\sim} & \mathrm{End}_T(T & \otimes_S & V) & \xrightarrow{\sim} & \Gamma'' \\ 1 & \otimes & \beta_0 & \longmapsto & 1 & \otimes & \beta_0 & \longmapsto & \varphi^{-1}(1 \otimes \beta_0)\varphi \;, \end{array}$$

which shall be denoted by $T \otimes_S \Gamma''_0 \xrightarrow[\sim]{\psi} \Gamma''$. Thus, we have an S-linear isomorphism

$$\Gamma' \otimes_T \Gamma'' \xleftarrow[\sim]{1\otimes\psi} \Gamma' \otimes_T T \otimes_S \Gamma''_0 \xleftarrow{\sim} \Gamma' \otimes_S \Gamma''_0 \xrightarrow[\sim]{\theta} \Gamma \;,$$

denoted by

$$\Gamma' \otimes_T \Gamma'' \xrightarrow[\sim]{\vartheta} \Gamma \;.$$

Lemma 2.1 *If $(x_1, \ldots, x_g)$ is an S-linear basis of T, and $(y_1, \ldots, y_n)$ is a T-linear basis of U lying in V, and given $\alpha \in \Gamma'$ and $\beta \in \Gamma''$, then $(\alpha \otimes \beta)\vartheta$ sends $x_i y_j$ to $(x_i\alpha)(y_j\beta)$, where $i \in [1, g]$, $j \in [1, n]$.*

Proof. By T-bilinearity we may assume that $\beta = (1 \otimes \beta_0)\psi$ with $\beta_0 \in \Gamma''_0$, in which case the assertion follows by construction. ▫

We use ϑ for a transport of the S-algebra-structure from Γ to $\Gamma' \otimes_T \Gamma''$; i.e. given $\gamma_1, \gamma_2 \in \Gamma' \otimes_T \Gamma''$, we let

$$\gamma_1 \cdot \gamma_2 \;:=\; ((\gamma_1\vartheta) \cdot (\gamma_2\vartheta))\vartheta^{-1} \;,$$

so that ϑ becomes an isomorphism of S-algebras.

Now we choose V to be the S-linear span of $(u^0, \ldots, u^{n-1})$ in U. Let

$$\Lambda'' \;:=\; \{\varphi'' \in \Gamma'' \;:\; u^k\varphi'' \in Uu^k \;\text{ for } k \geqslant 0 \;\} \;\subseteq\; \Gamma'' \;,$$

which is a sub-T-algebra.

Lemma 2.2 *The S-linear submodule $\Xi' \otimes_T \Lambda'' \subseteq \Gamma' \otimes_T \Gamma''$ is a sub-S-algebra.*

Proof. Given $k, l \in [0, n-1]$, we let $e''_{k,l} \in \Gamma''$ be defined by $u^j e''_{k,l} = \partial_{j,k} u^l$ for $j \in [0, n-1]$. Then

$$(*) \qquad \Xi' \otimes_T \Lambda'' \;=\; \left\{ \sum_{k,l \in [0,n-1]} \xi'_{k,l} \otimes e''_{k,l} \;:\; \xi'_{k,l} \in \Xi' \text{ for } k \leqslant l,\, \xi'_{k,l} \in \dot{t}\Xi' \text{ for } k > l \right\} .$$

Since we can write $e''_{k,l} = (1 \otimes e''_{k,l;0})\psi$ with $e''_{k,l;0} \in \Gamma''_0$, i.e. since $e''_{k,l}$ restricts to an S-linear endomorphism of V, multiplication of elements written in the form as in $(*)$ is given by

$$(\xi' \otimes e''_{k,l})(\tilde{\xi}' \otimes e''_{\tilde{k},\tilde{l}}) \;=\; (\xi'\tilde{\xi}') \otimes (\partial_{l,\tilde{k}} e''_{k,\tilde{l}}) ,$$

where $\tilde{\xi}, \tilde{\xi}' \in \Xi'$ and $k, l, \tilde{k}, \tilde{l} \in [0, n-1]$. Now if $k > \tilde{l}$ and $l = \tilde{k}$, then $k > l$ or $\tilde{k} > \tilde{l}$, hence $\xi'\tilde{\xi}' \in \dot{t}\Xi'$. Therefore, the S-linear submodule $\Xi' \otimes_T \Lambda''$ is closed under multiplication. ▫

Lemma 2.3 *We have $\Xi\vartheta^{-1} \subseteq \Xi' \otimes_T \Lambda''$.*

Proof. Since $\Xi' \otimes_T \Lambda''$ is a sub-S-algebra of $\Gamma' \otimes_T \Gamma''$ (2.2), it suffices to show that $u\omega\vartheta^{-1} \in \Xi' \otimes_T \Lambda''$ and that $\rho\omega\vartheta^{-1} \in \Xi' \otimes_T \Lambda''$ for $\rho \in H$.

We have $(1 \otimes u\omega'')\vartheta = u\omega$ by (2.1), hence $u\omega\vartheta^{-1} \in \Xi' \otimes_T \Lambda''$.

Suppose given $\rho \in H$. Let $(x_1, \ldots, x_g)$ be an S-linear basis of T. The element $x_i u^j$ is sent to $x_i^\rho (u^\rho)^j$ by $\rho\omega$, where $i \in [1, g]$, $j \in [0, n-1]$ (2.1). The element of Γ' determined by $x_i \longmapsto x_i^\rho$ is just $\rho\omega'$, which is in Ξ'. The element of Γ'' determined by $u^j \longmapsto (u^\rho)^j$ is contained in Λ''. The tensor product of these elements is thus contained in $\Xi' \otimes_T \Lambda''$, as was to be shown. ▫

Remark 2.4 In general, the element $u^j \longmapsto (u^\rho)^j$, where $j \in [0, n-1]$, is *not* contained in Ξ'' since ρ need not be in N.

By (2.2, 2.3), we have a commutative diagram of S-algebras

$$\begin{array}{ccc} \Gamma' \otimes_T \Gamma'' & \xrightarrow[\sim]{\vartheta} & \Gamma \\ \uparrow & & \uparrow \\ \Xi' \otimes_T \Lambda'' & \xleftarrow{\vartheta^{-1}} & \Xi \, . \end{array}$$

Proposition 2.5 *If $n \not\equiv_p 0$, then*

$$\Xi\vartheta^{-1} \;=\; \Xi' \otimes_T \Lambda'' .$$

In other words, the Wedderburn embedding $U \wr H \overset{\omega}{\hookrightarrow} \Gamma$ factors as

$$U \wr H \xrightarrow[\sim]{\omega\vartheta^{-1}} \Xi' \otimes_T \Lambda'' \hookrightarrow \Gamma' \otimes_T \Gamma'' \xrightarrow[\sim]{\vartheta} \Gamma \,.$$

The tensor product $\Xi' \otimes_T \Lambda''$ is called the *Nebe decomposition* of $U \wr H$.

Proof. First, we remark that if $n \not\equiv_p 0$, then $\Xi'' = \Lambda''$ by (1.3).

We need to show that the S-linear colengths coincide. By **[17,** cor. 2.17], the colength of $\Xi \subseteq \Gamma$ equals

$$h\,\mathrm{v}_s(\Delta_{U|S})/2 \;=\; h\,\mathrm{v}_u(\mathfrak{D}_{U|S})/2 \;=\; h\,\mathrm{v}_u(\mathfrak{D}_{U|T}\mathfrak{D}_{T|S})/2 \;=\; h(n-1)/2 + hn\,\mathrm{v}_s(\Delta_{T|S})/2 \,.$$

On the other hand, by **[17,** cor. 2.17], the colength of the composite embedding $\Xi' \otimes_T \Lambda'' \subseteq \Gamma' \otimes_T \Lambda'' \subseteq \Gamma' \otimes_T \Gamma''$ equals

$$(g\,\mathrm{v}_s(\Delta_{T|S})/2) \cdot n^2 + g \cdot n(n-1)/2 \;=\; hn\,\mathrm{v}_s(\Delta_{T|S})/2 + h(n-1)/2 \,.$$

□

Example 2.6 (cont.) Let $S = \mathbf{Z}_{(3)}$, $s = 3$, $T = \mathbf{Z}_{(3)}[\pi_2]$, $t = \pi_2$, $U = \mathbf{Z}_{(3)}[\zeta_{3^2}]$, $u = \zeta_{3^2} - 1$, so that $H = (\mathbf{Z}/3^2)^* \simeq C_3 \times C_2$, $G = C_3$ and $N = C_2$. We use the S-linear basis (u^0, u^1) of V and the S-linear basis (t^0, t^1, t^2) of T, and thus the S-linear basis $(u^0t^0, u^0t^1, u^0t^2, u^1t^0, u^1t^1, u^1t^2)$ of U. By (2.5) and using $(*)$, the Wedderburn descriptions of Ξ' and $t\Xi'$ obtained in (1.20) can be inserted as blocks into

$$U \wr H \xrightarrow[\sim]{\omega} \Xi \;=$$

$$\left\{ \left[\begin{array}{ccc|ccc} a_{0,0;0,0} & a_{0,0;0,1} & a_{0,0;0,2} & a_{0,1;0,0} & a_{0,1;0,1} & a_{0,1;0,2} \\ 3a_{0,0;1,2} & a_{0,0;1,0} & a_{0,0;1,1} & 3a_{0,1;1,2} & a_{0,1;1,0} & a_{0,1;1,1} \\ 3a_{0,0;2,1} & 3a_{0,0;2,2} & a_{0,0;2,0} & 3a_{0,1;2,1} & 3a_{0,1;2,2} & a_{0,1;2,0} \\ \hline 3a_{1,1;0,0} & a_{1,1;0,1} & a_{1,1;0,2} & a_{1,0;0,0} & a_{1,0;0,1} & a_{1,0;0,2} \\ 3a_{1,1;1,2} & 3a_{1,1;1,0} & a_{1,1;1,1} & 3a_{1,0;1,2} & a_{1,0;1,0} & a_{1,0;1,1} \\ 3a_{1,1;2,1} & 3a_{1,1;2,2} & 3a_{1,1;2,0} & 3a_{1,0;2,1} & 3a_{1,0;2,2} & a_{1,0;2,0} \end{array}\right] : \begin{array}{l} \left\{ \begin{array}{l} a_{0,0;0,0} \equiv_3 a_{0,0;1,0} \equiv_3 a_{0,0;2,0}, \\ a_{0,0;0,1} + a_{0,0;1,1} + a_{0,0;2,1} \equiv_3 0; \end{array}\right. \\ \left\{ \begin{array}{l} a_{0,1;0,0} \equiv_3 a_{0,1;1,0} \equiv_3 a_{0,1;2,0}, \\ a_{0,1;0,1} + a_{0,1;1,1} + a_{0,1;2,1} \equiv_3 0; \end{array}\right. \\ \left\{ \begin{array}{l} a_{1,0;0,0} \equiv_3 a_{1,0;1,0} \equiv_3 a_{1,0;2,0}, \\ a_{1,0;0,1} + a_{1,0;1,1} + a_{1,0;2,1} \equiv_3 0; \end{array}\right. \\ \left\{ \begin{array}{l} a_{1,1;0,1} \equiv_3 a_{1,1;1,1} \equiv_3 a_{1,1;2,1}, \\ a_{1,1;0,2} + a_{1,1;1,2} + a_{1,1;2,2} \equiv_3 0 \end{array}\right. \end{array} \right\}$$

$$\subseteq \; (S^{3\times 3})^{2\times 2} \,.$$

2.2 A reduction isomorphism

We maintain the notation of §2.1, having chosen $V = S\langle u^0, \ldots, u^{n-1}\rangle \subseteq U$.

Definition 2.7 Given a Ξ'-module Y, the S-module $Y \otimes_T U$ decomposes into a direct sum

$$Y \otimes_T U = \bigoplus_{j \in [0, n-1]} Y \otimes u^j ,$$

so that we may endow it with the structure of a $\Xi' \otimes_T \Lambda''$-module by means of

$$(y \otimes u^j)(\xi' \otimes e''_{k,l}) := y\xi' \otimes \partial_{j,k} u^l,$$

where $y \in Y$, $\xi' \in \Xi'$, $j, k, l \in [0, n-1]$, $\xi' \in \dot{t}\Xi$ if $k > l$, and where $e''_{k,l}$ is as in $(*)$ above.

More naturally explained, we use the isomorphism $Y \otimes_T U \simeq Y \otimes_S V$ to transport the structure of an $\Xi' \otimes_S \Gamma''_0$-module from $Y \otimes_S V$ to $Y \otimes_T U$. Restricting the S-algebra isomorphism $\Gamma' \otimes_S \Gamma''_0 \xrightarrow{\sim} \Gamma' \otimes_T \Gamma''$ to $\Xi' \otimes_S \Gamma''_0 \xrightarrow{\sim} \Xi' \otimes_T \Gamma''$, we obtain on $Y \otimes_T U$ the structure of a $\Xi' \otimes_T \Gamma''$-module, which we restrict to $\Xi' \otimes_T \Lambda''$. This also shows how the $\Xi' \otimes_T \Lambda''$-module structure on $Y \otimes_T U$ depends on the choice of V.

We obtain an exact functor

$$\begin{array}{ccc} \text{mod-}\,\Xi' & \xrightarrow{F := (-\otimes_T U)} & \text{mod-}(\Xi' \otimes_T \Lambda'') \\ (Y \xrightarrow{\alpha} Y') & \longmapsto & (Y \otimes_T U \xrightarrow{\alpha \otimes 1} Y' \otimes_T U)\ . \end{array}$$

Lemma 2.8 *The functor F maps projective Ξ'-modules to projective $\Xi' \otimes_T \Lambda''$-modules. Moreover, F is full and faithful.*

Proof. To show that $F\Xi'$ is projective over $\Xi' \otimes_T \Lambda''$, we remark that

$$\begin{array}{rcl} F\Xi' \;=\; \Xi' \otimes_T U & \xrightarrow{\sim} & (1 \otimes e''_{0,0})(\Xi' \otimes_T \Lambda'') \\ \xi' \otimes u^j & \longmapsto & \xi' \otimes e''_{0,j}\ , \end{array}$$

where $j \in [0, n-1]$, is an isomorphism of $\Xi' \otimes_T \Lambda''$-modules.

We shall prove that F is full and faithful. Given $Y_1,\, Y_2 \in \text{mod-}\,\Xi'$, we claim that

$$_{\Xi'}(Y_1, Y_2) \xrightarrow{F} {}_{\Xi' \otimes_T \Lambda''}(FY_1, FY_2)$$

is an isomorphism. Using a two-step free resolution of Y_1 and exactness of F, we may assume that $Y_1 = \Xi'$. Likewise, using projectivity of $F\Xi'$ and a two-step free resolution of Y_2, we may assume $Y_2 = \Xi'$. But in this case, we have an isomorphism

$$\begin{array}{ccccccc} \Xi' & \xrightarrow{\sim} & _{\Xi'}(\Xi', \Xi') & \underset{\sim}{\xrightarrow{F}} & _{\Xi' \otimes_T \Lambda''}(F\Xi', F\Xi') & \xrightarrow{\sim} & (1 \otimes e''_{0,0})(\Xi' \otimes_T \Lambda'')(1 \otimes e''_{0,0}) \\ \xi' & \longmapsto & \xi' \cdot (-) & \longrightarrow & (\xi' \otimes 1) \cdot (-) & \longmapsto & \xi' \otimes e''_{0,0}. \end{array}$$

□

Lemma 2.9 *There is an isomorphism*

$$\text{Ext}^*_{\Xi'}(T, T) \underset{\sim}{\xrightarrow{\text{Ext}^*(F)}} \text{Ext}^*_{\Xi' \otimes_T \Lambda''}(U, U)$$

of graded S-algebras with respect to the Yoneda product, where U is a $\Xi' \otimes_T \Lambda''$-module via $U \simeq T \otimes_T U = FT$.

Proof. Since F preserves projectivity (2.8), we may apply F to a projective resolution of T to obtain a projective resolution of U, and use this resolution to calculate $\text{Ext}^*_{U \wr H}(U, U)$. Application of F to morphisms of complexes modulo homotopy now yields the morphism of graded S-algebras $\text{Ext}^*(F)$. But since F is full and faithful (2.8), this map is an isomorphism. □

Proposition 2.10 *If $n \not\equiv_p 0$, then*

$$\mathrm{Ext}^*_{T\wr G}(T,T) \;\simeq\; \mathrm{Ext}^*_{U\wr H}(U,U)$$

as graded S-algebras.

Proof. By (2.5), we have $\Xi'\otimes_T\Lambda'' \simeq \Xi \simeq U\wr H$. So (2.9) gives the assertion, provided the module structures of $\Xi'\otimes_T\Lambda''$ and of $U\wr H$ on U coincide. Suppose given $\xi'\otimes e''_{k,l} \in \Xi'\otimes_T\Lambda''$, $\xi'\in\Xi'$, $k,\, l \in [0,n-1]$. Let $(x_1,\ldots,x_g)$ be an S-linear basis of T.

In the interpretation $U \simeq T\otimes_T U = FT$, we have $(x_i\otimes u^j)(\xi'\otimes e''_{k,l}) = (x_i\xi')\otimes(\partial_{j,k}u^l) = 1\otimes(x_i\xi')(\partial_{j,k}u^l)$ by definition of F.

On the other hand, by (2.1), the element $\xi'\otimes e''_{k,l}$ is mapped via ϑ to the element in Γ that maps x_iu^j to $(x_i\xi')(\partial_{j,k}u^l)$. So after identifying $T\otimes_T U\simeq U$, the operations coincide. □

Example 2.11 (cont.) We have

$$\mathrm{Ext}^*_{\mathbf{Z}_{(3)}[\zeta_{3^2}]\wr(C_3\times C_2)}(\mathbf{Z}_{(3)}[\zeta_{3^2}],\mathbf{Z}_{(3)}[\zeta_{3^2}]) \;\simeq\; \mathrm{Ext}^*_{\mathbf{Z}_{(3)}[\pi_2]\wr C_3}(\mathbf{Z}_{(3)}[\pi_2],\mathbf{Z}_{(3)}[\pi_2])$$

as graded $\mathbf{Z}_{(3)}$-algebras.

3 Cup product and Yoneda product coincide

In this section, we let T be a commutative ring, we let G be a group acting on T via a group morphism $G\longrightarrow \mathrm{Aut}_{\mathrm{ring}}T$, and define $S := \mathrm{Fix}_G T$ to be the fixed ring of this operation. We assume that T is a free module over S.

We consider the twisted group ring $T\wr G$ with respect to this operation, which is an S-algebra.

We write $\otimes := \otimes_S$. Let

$$P := (\cdots \xrightarrow{d_3} SG^{\otimes 3} \xrightarrow{d_2} SG^{\otimes 2} \xrightarrow{d_1} \underbrace{SG^{\otimes 1}}_{\text{degree } 0} \longrightarrow 0 \longrightarrow \cdots)$$

denote the bar resolution, which is a projective resolution of the trivial module S over SG. The differential is given by

$$\begin{array}{ccc} SG^{\otimes i+1} & \xrightarrow{d_i} & SG^{\otimes i} \\ g_{[0,i]} & \longmapsto & \sum\limits_{l\in[0,i]} (-1)^{i-l} g_{[0,i]\smallsetminus\{l\}} \end{array}$$

where $i\geqslant 1$, and where we write shorthand

$$g_A \;:=\; g_{a_1}\otimes g_{a_2}\otimes\cdots\otimes g_{a_k},$$

for $A\subseteq\mathbf{Z}$, where $k := \#A < \infty$, where $a_i\in A$ and $g_{a_i}\in G$ for $i\in[1,k]$, and where $a_1 < a_2 < \cdots < a_k$. Let $SG\xrightarrow{d_0} S$ denote the augmentation map, sending each element of G to $1 \in S$. We have a quasiisomorphism $P \xrightarrow{d_0} S$, where S is regarded as a complex concentrated in degree 0 and where d_0 also denotes the morphism of complexes given by d_0 in degree 0.

Let

$$\begin{array}{rcl} \text{Mod-}\,SG & \xrightarrow{L} & \text{Mod-}(T \wr G) \\ M & \longmapsto & LM \;:=\; M \otimes_{SG} T \wr G \\ \text{Mod-}\,SG & \xleftarrow{R} & \text{Mod-}(T \wr G) \\ N|_{SG} \;=:\; RN & \longleftarrow\!\shortmid & N\,. \end{array}$$

Then L is left adjoint to R, with adjunction morphisms denoted by

$$\begin{array}{rclc} LRN & \xrightarrow{\eta N} & N & \text{counit} \\ n \otimes x & \longmapsto & nx & \\ RLM & \xleftarrow{\varepsilon M} & M & \text{unit} \\ m \otimes 1 & \longleftarrow\!\shortmid & m & \end{array}$$

We note that $T \simeq LS$, by means of $x \longmapsto 1 \otimes x$, $x^g \longleftarrow\!\shortmid 1 \otimes xg$, $x \in T$, $g \in G$, which we use as identification. Since $T \wr G$ is free as a module over SG, both R and L are exact. To complexes, R and L are applied entrywise. So for instance, RLS is the complex of SG-modules having entry T in degree 0 and entry 0 elsewhere.

We use the description

$$\mathrm{Ext}^i_{T\wr G}(T,T) \;=\; {}_{\mathrm{K}^-(T\wr G)}(LP, LP[i])\;,$$

where $[i]$ denotes the shift of complexes by i steps to the left, i.e. $(X[i])_j := X_{j-i}$, where $i \geqslant 0$, $j \in \mathbf{Z}$. Suppose given $u \in {}_{\mathrm{K}^-(T\wr G)}(LP, LP[i])$, $v \in {}_{\mathrm{K}^-(T\wr G)}(LP, LP[j])$, for some degrees $i, j \geqslant 0$. The *Yoneda product* $u \cdot v$ is given as the composite

$$(LP \xrightarrow{u\cdot v} LP[i+j]) \;:=\; (LP \xrightarrow{u} LP[i] \xrightarrow{v[i]} LP[i+j]) \;\in\; {}_{\mathrm{K}^-(T\wr G)}(LP, LP[i+j])\;.$$

It turns $\mathrm{Ext}^*_{T\wr G}(T,T)$ into a graded S-algebra.

Moreover, we use the description of the cohomology of G with coefficients in T over the ground ring S

$$\mathrm{H}^i(G,T;S) \;:=\; \mathrm{Ext}^i_{SG}(S,T) \;=\; {}_{\mathrm{K}^-(SG)}(P, RLS[i]) \;=\; \mathrm{H}^i(CP)\;,$$

where $i \geqslant 0$, and where $C : \text{Mod-}\,SG \longrightarrow \text{Mod-}\,S : X \longmapsto {}_{SG}(X,T)$ is applied entrywise to complexes. For a morphism f, we denote $f^* := Cf$. Let $\mathrm{Z}^i(CP) := \mathrm{Kern}(CP_i \xrightarrow{d^*_{i+1}} CP_{i+1})$ denote the S-module of i-cocycles.

We will also make use of the alternative interpretation

$$\begin{array}{rcccl} \mathrm{H}^i(G,T;S) & = & {}_{\mathrm{K}^-(SG)}(P, RLS[i]) & \xleftarrow[\sim]{\lambda} & {}_{\mathrm{K}^-(SG)}(P, RLP[i]) \\ & & u(RLd_0[i]) & \longleftarrow\!\shortmid & u\,. \end{array}$$

Given $a \in CP_i = {}_{SG}(SG^{\otimes i+1}, T)$, $b \in CP_j = {}_{SG}(SG^{\otimes j+1}, T)$, for some degrees $i, j \geqslant 0$, their *cup product* $a \cup b \in CP_{i+j} = {}_{SG}(SG^{\otimes i+j+1}, T)$ is defined by

$$(g_{[0,i+j]})(a \cup b) \ := \ (g_{[0,i]})a \cdot (g_{[i,i+j]})b \ ,$$

where $g_l \in G$ for $l \in [0, i+j]$. Because of the Leibniz rule

$$(a \cup b)d^*_{i+j+1} \ = \ (-1)^j (ad^*_{i+1} \cup b) + (a \cup bd^*_{j+1}) \ ,$$

the cup product restricts to $\mathrm{Z}^i(CP) \times \mathrm{Z}^j(CP) \xrightarrow{\cup} \mathrm{Z}^{i+j}(CP)$ and induces a map

$$\mathrm{H}^i(G, T; S) \times \mathrm{H}^j(G, T; S) \ \xrightarrow{\cup} \ \mathrm{H}^{i+j}(G, T; S) \ .$$

Given cocycles $a \in \mathrm{Z}^i(CP)$, $b \in \mathrm{Z}^j(CP)$, where $i, j \geqslant 0$, we let $c_{a,b} \in CP_{i+j-1}$ be defined by

$$h_{[0,i+j-1]}c_{a,b} \ := \sum_{m \in [0,j-1]} (-1)^{m(i+j-1)} (h_{[m,m+i]})a \cdot (h_{[m+i,i+j-1]} \otimes h_{[0,m]})b \ ,$$

where $h_l \in G$ for $l \in [0, i+j-1]$. We obtain

$$(g_{[0,i+j]})(c_{a,b}d^*_{i+j}) \ = \ g_{[0,i]}a \cdot g_{[i,i+j]}b - (-1)^{ij} g_{[0,j]}b \cdot g_{[j,i+j]}a \ ,$$

whence $a \cup b = (-1)^{ij}\, b \cup a$ as elements of $\mathrm{H}^{i+j}(G, T; S)$. Thus the cup product turns $\mathrm{H}^*(G, T; S)$ into a graded commutative S-algebra.

The following proposition we owe to B. Keller.

Proposition 3.1 *The isomorphism $\lambda^{-1}\kappa$ of graded S-modules, given by*

$$\begin{array}{rcccl} \mathrm{H}^*(G, T; S) & \xleftarrow[\sim]{\lambda} & {}_{\mathrm{K}^-(SG)}(P, RLP[*]) & \xrightarrow[\sim]{\kappa} & {}_{\mathrm{K}^-(T\wr G)}(LP, LP[*]) \ = \ \mathrm{Ext}^*_{T\wr G}(T, T) \\ & & u & \longmapsto & (Lu)(\eta LP[*]) \\ & & (\varepsilon P)(Ru') & \longleftarrow\!\shortmid & u' \ , \end{array}$$

is in fact an isomorphism of graded S-algebras, with respect to the cup product on $\mathrm{H}^*(G, T; S)$ and with respect to the Yoneda product on $\mathrm{Ext}^*_{T\wr G}(T, T)$.

In particular, $\mathrm{Ext}^*_{T\wr G}(T, T)$ is a graded commutative S-algebra.

Note that $\mathrm{H}^0(G, T; S) \simeq \mathrm{Hom}_{T\wr G}(T, T) \simeq S$ as S-algebras.

Proof. Given $u \in {}_{\mathrm{K}^-(SG)}(P, RLP[i])$, $v \in {}_{\mathrm{K}^-(SG)}(P, RLP[j])$, where $i, j \geqslant 0$, we calculate

$$\begin{array}{rcl} ((u\kappa) \cdot (v\kappa))\kappa^{-1} & = & \Big((Lu)(\eta LP[i])(Lv[i])(\eta LP[i+j])\Big)\kappa^{-1} \\ & = & (\varepsilon P)(RLu)(R\eta LP[i])(RLv[i])(R\eta LP[i+j]) \\ & = & u(RLv[i])(R\eta LP[i+j]) \ . \end{array}$$

Now suppose given $a \in \mathrm{Z}^i(CP)$, $b \in \mathrm{Z}^j(CP)$. Letting $P_{i+l} \xrightarrow{\tilde{a}_{i+l}} RLP_l$ be defined by

$$(g_{[0,i+l]})\tilde{a}_{i+l} \ := \ (-1)^{il} g_{[0,l]} \otimes (g_{[l,i+l]}a) \ ,$$

where $l \geqslant 0$ and $g_k \in G$ for $k \in [0, i+l]$, we obtain a morphism of complexes $P \xrightarrow{\tilde{a}} RLP[i]$ that is mapped by λ to $P \xrightarrow{a} RLS[i]$. Hence

$$\begin{array}{rl} & (g_{[0,i+j]})\Big(((a\lambda^{-1}\kappa) \cdot (b\lambda^{-1}\kappa))\kappa^{-1}\lambda\Big) \\ = & (g_{[0,i+j]})\Big(((\tilde{a}\kappa) \cdot (\tilde{b}\kappa))\kappa^{-1}\lambda\Big) \\ = & (g_{[0,i+j]})\tilde{a}_{i+j}(RL\tilde{b}_j)(R\eta L(SG))(RLd_0[i+j]) \\ = & (-1)^{ij}(g_{[0,j]} \otimes (g_{[j,i+j]}a))(RL\tilde{b}_j)(R\eta L(SG))(RLd_0[i+j]) \\ = & (-1)^{ij}(g_0 \otimes (g_{[0,j]}b) \otimes (g_{[j,i+j]}a))(R\eta L(SG))(RLd_0[i+j]) \\ = & (-1)^{ij}(g_0 \otimes (g_{[0,j]}b) \cdot (g_{[j,i+j]}a))(RLd_0[i+j]) \\ = & (-1)^{ij}(g_{[0,j]}b) \cdot (g_{[j,i+j]}a) \\ = & (-1)^{ij}(g_{[0,i+j]})(b \cup a) \; , \end{array}$$

i.e.

$$(a\lambda^{-1}\kappa) \cdot (b\lambda^{-1}\kappa) \;=\; (a \cup b)\lambda^{-1}\kappa \; .$$

□

Remark 3.2 The graded commutativity of $\mathrm{Ext}^*_{T \wr G}(T,T)$ can also be obtained by **[30,** 2.1].

4 Cohomology

4.0 A classical approach

Example 4.1 We shall calculate the cohomology directly in an example, still disregarding the cup product, however.

Let $\pi_2 = \prod_{j \in [1,p-1]}(\zeta_{p^2}^{j^p} - 1)$, cf. §5.2 below. Let

$$\begin{array}{rclcrcl} S & = & \mathbf{Z}_{(p)} \; , & & K & = & \mathbf{Q} \; , \\ T & = & \mathbf{Z}_{(p)}[\pi_2] \; , & & L & = & \mathbf{Q}(\pi_2) \; . \end{array}$$

Then $G = C_p = \langle \sigma \rangle$, where σ is the restriction to $\mathbf{Q}(\pi_2)$ to the automorphism $\zeta_{p^2} \longmapsto \zeta_{p^2}^{1+p}$ of $\mathbf{Q}(\zeta_{p^2})$. Let $e := \frac{1}{p}\sum_{j \in [0,p-1]} \sigma^j \in \mathbf{Q}G$. We have a split short exact sequence of SG-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & T & \longrightarrow & S & \longrightarrow & 0 \\ & & & & x & \longmapsto & xe & & \\ & & & & y & \longleftarrow\!\shortmid & y \; , & & \end{array}$$

which is welldefined since $\mathrm{Tr}_{L|K}(x)$ is divisible by p for all $x \in T$ because $T|S$ is wildly ramified.

Now M is an $SG(1-e)$-lattice of rank $\mathrm{rk}_S\, M = p-1$. Since $SG(1-e)$ is isomorphic to $S[\zeta_p]$ via $\sigma(1-e) \overset{\sim}{\longmapsto} \zeta_p$ as an S-algebra, we have $M \simeq SG(1-e)$. Using the 2-periodic projective resolution

$$\cdots \xrightarrow{b} SG \xrightarrow{a} SG \xrightarrow{b} SG \longrightarrow 0$$

of S, where $a : 1 \longmapsto pe$ and $b : 1 \longmapsto \sigma - 1$, we obtain

$$\begin{array}{rcl} \mathrm{H}^j(G,T;S) & \simeq & \mathrm{H}^j(G, SGe \oplus SG(1-e); S) \\ & \simeq & \mathrm{H}^j(G, SGe; S) \oplus \mathrm{H}^j(G, SG(1-e); S) \\ & \simeq & \left\{ \begin{array}{ll} 0 & j \text{ odd} \\ \mathbf{F}_p & j \geqslant 2 \text{ even} \\ S & j = 0 \end{array} \right\} \oplus \left\{ \begin{array}{ll} \mathbf{F}_p & j \text{ odd} \\ 0 & j \text{ even} \end{array} \right\} \\ & \simeq & \left\{ \begin{array}{ll} \mathbf{F}_p & j \geqslant 1 \\ S & j = 0 \end{array} \right\} \end{array}$$

as an S-module. This will be confirmed by (4.6, 5.4) below, since $\mathrm{H}^j(G,T;S) \simeq \mathrm{Ext}^j_{T\wr G}(T,T)$ by adjunction.

Remark 4.2 We attempt to explain why in the case of a cyclic galois group, and in presence of a classical periodic resolution, the description in (1.19) is actually useful to calculate products in cohomology.

For the purpose of this remark, let $G = \langle \sigma \,|\, \sigma^g = 1 \rangle$ be a cyclic group of order $g \geqslant 1$ acting on the discrete valuation ring T, let $S = \mathrm{Fix}_G T$ be the fixed ring in T under G, and let $L = \mathrm{frac}\, T$, $K = \mathrm{frac}\, S$. Let

$$\begin{array}{ccccccl} T\wr G & \xrightarrow{\alpha_0} & T\wr G & : & 1 & \longmapsto & \left(\sum_{i\in\mathbf{Z}/g} \sigma^i\right) \\ T\wr G & \xrightarrow{\beta_0} & T\wr G & : & 1 & \longmapsto & (\sigma - 1) \\ T\wr G & \xrightarrow{\varepsilon_0} & T & : & 1 & \longmapsto & 1 \end{array}$$

be $T\wr G$-linear maps. We obtain a periodic projective resolution of period 2

$$P_0 := \left(\cdots \longrightarrow T\wr G \xrightarrow{\alpha_0} T\wr G \xrightarrow{\beta_0} T\wr G \xrightarrow{\alpha_0} T\wr G \xrightarrow{\beta_0} \underbrace{T\wr G}_{\text{degree } 0} \longrightarrow 0 \longrightarrow \cdots \right),$$

mapping quasiisomorphically to T via ε_0 in degree 0 since the image of α_0 is isomorphic to T via $T \xrightarrow{\sim} \mathrm{Im}\,\alpha_0$, $x \longmapsto \sum_{i\in\mathbf{Z}/g} \sigma^i x$. So we have

$$\mathrm{RHom}_{T\wr G}(T,T) = \left(\cdots \longleftarrow T \xleftarrow{\mathrm{Tr}_{L|K}} T \xleftarrow{x^\sigma - x \,\leftarrow\!\dashv\, x} T \xleftarrow{\mathrm{Tr}_{L|K}} T \xleftarrow{x^\sigma - x \,\leftarrow\!\dashv\, x} \underbrace{T}_{\text{degree } 0} \longleftarrow 0 \longleftarrow \cdots \right),$$

whence for instance $\mathrm{Ext}^1_{T\wr G}(T,T) \simeq \{x \in T : \mathrm{Tr}_{L|K}(x) = 0\}/\{y^\sigma - y : y \in T\}$, or $\mathrm{Ext}^2_{T\wr G}(T,T) \simeq S/\mathrm{Tr}_{L/K}(T)$. Suppose given an element of, say, $\mathrm{Ext}^1_{T\wr G}(T,T)$, represented by an element $e \in T$ with trace 0. To apply Yoneda multiplication, we need to represent it as an element of ${}_{\mathrm{K}^-(T\wr G)}(P_0, P_0[1])$. Let us consider the necessary construction of a morphism of complexes. We attempt to construct a periodic resolution of e with period 2. If in odd degrees, the morphism of complexes is given by $1 \longmapsto \sum_{i\in\mathbf{Z}/g} \sigma^i x_i$, and in even degrees $\geqslant 2$ by $1 \longmapsto \sum_{i\in\mathbf{Z}/g} \sigma^i y_i$, where x_i, $y_i \in T$, then these coefficients are subject to the following conditions.

$$\begin{array}{rcl} \sum_i x_i & = & e \\ (\sum_i \sigma^i x_i)(\sum_j \sigma^j) & = & (\sigma - 1)(\sum_i \sigma^i y_i) \\ (\sum_j \sigma^j)(\sum_i \sigma^i x_i) & = & (\sum_i \sigma^i y_i)(\sigma - 1) \,, \end{array}$$

i.e.

$$\begin{array}{rcll} \sum_i x_i & = & e & \\ \sum_{i+j\equiv_g k} x_i^{\sigma^j} & = & y_{k-1} - y_k & \text{for all } k \in \mathbf{Z}/g \\ e & = & y_{k-1}^{\sigma} - y_k & \text{for all } k \in \mathbf{Z}/g\ . \end{array}$$

As far as we can see, solving this system requires knowledge of the operation of σ on T in terms of an S-linear basis of T. Cf. e.g. (1.4) or appendix A.2.

4.1 A projective resolution

We maintain the notation and assumptions of §1.5. In particular, $G = C_p = \langle\sigma\rangle$ is cyclic of order equal to the residue characteristic p of S, and $b = -1 + \mathrm{v}_t(t^\sigma - t) \geqslant 1$. To dispose of the equality $\Xi = \Lambda^{\mathrm{D}}$, we assume that $\mathrm{v}_s(p) \geqslant b - \underline{b}$ (1.19).

Let

$$\begin{array}{ccccccll} \Xi & \xrightarrow{\alpha} & \Xi & : & 1 & \longmapsto & \tilde{\mu}_{p-1,p-\bar{b}} & := s^{b-\underline{b}-1}\,\varepsilon_{\bar{b},p-\bar{b}} \\ \Xi & \xrightarrow{\beta} & \Xi & : & 1 & \longmapsto & \mu_{1,\bar{b}} & = s^{\underline{b}} \sum_{i\in[0,p-1]} i\varepsilon_{i,\bar{b}} \\ \Xi & \xrightarrow{\chi_k} & T & : & 1 & \longmapsto & t^k & \text{for } k \in [0,p-1] \end{array}$$

be Ξ-linear maps (cf. 1.17). Here $\tilde{\mu}_{p-1,p-\bar{b}}$ is a 'shifted version' of $\mu_{p-1,p-\bar{b}}$ (cf. 1.18).

Proposition 4.3 *The complex of Ξ-linear maps*

$$\cdots \xrightarrow{\beta} \Xi \xrightarrow{\alpha} \Xi \xrightarrow{\beta} \Xi \xrightarrow{\alpha} \Xi \xrightarrow{\beta} \Xi \xrightarrow{\alpha} \cdots\ ,$$

periodic of period 2, is acyclic. The image of α is isomorphic to T as a module over Ξ; more precisely, we have a factorization

$$(\Xi \xrightarrow{\alpha} \Xi) \;=\; (\Xi \xrightarrow{\chi_0} T \longrightarrow \Xi),$$

with χ_0 surjective and $T \longrightarrow \Xi$ injective.

Denote by

$$P \;:=\; (\cdots \longrightarrow \Xi \xrightarrow{\alpha} \Xi \xrightarrow{\beta} \underbrace{\Xi}_{\text{degree } 0} \longrightarrow 0 \longrightarrow \cdots) \;\in\; \mathrm{K}^-(\Xi)$$

the resulting projective resolution of T.

Proof. We claim exactness of $\Xi \xrightarrow{\alpha} \Xi \xrightarrow{\beta} \Xi$. To prove that $\alpha\beta = 0$, we calculate

$$\begin{array}{rcl} \mu_{1,\bar{b}}\,\tilde{\mu}_{p-1,p-\bar{b}} & = & s^{b-1} \sum_{i\in[0,p-1]} i\varepsilon_{i,\bar{b}}\,\varepsilon_{\bar{b},p-\bar{b}} \\ & \overset{(1.6)}{=} & s^{b-1} \sum_{i\in[0,p-1]} i\partial_{\bar{b},\overline{i+b}}\,\varepsilon_{i,p} \\ & = & 0\ . \end{array}$$

Now, let $B \subseteq \Lambda$ be the kernel of the Λ-linear map $\Lambda \longrightarrow \Lambda$, $1 \longmapsto \mu_{1,\bar{b}}$. To prove that $\operatorname{Im}\alpha = \operatorname{Kern}\beta$, we shall show that the S-linear colengths of both submodules in B coincide. For $j, k \in [0, p-1]$, we have

$$\begin{array}{rcl} \mu_{1,\bar{b}}\,\varepsilon_{j,k} & = & s^{\underline{b}} \sum_{i\in[0,p-1]} i\varepsilon_{i,\bar{b}}\,\varepsilon_{j,k} \\ & \overset{(1.6)}{=} & s^{\underline{b}} \sum_{i\in[0,p-1]} i\partial_{j,\overline{i+b}}\,\varepsilon_{i,k+\bar{b}} \\ & = & s^{\underline{b}}\,\overline{j-b}\,\varepsilon_{\overline{j-b},k+\bar{b}}\ , \end{array}$$

and thus B has the S-linear basis $(\varepsilon_{\bar{b},k})_{k\in[0,p-1]}$. By (1.17), an element $\sum_{k\in[0,p-1]} x_k\varepsilon_{\bar{b},k}$, where $x_k \in S$, is in Ξ if and only if

$$\mathrm{v}_s(x_k) \;\geqslant\; 1 + \underline{b(p-1)-k-1}$$

for $k \in [0, p-1]$. By (1.12), we obtain the colength of $\operatorname{Kern}\beta$ in B to be equal to $\sum_{k\in[0,p-1]}(1 + \underline{b(p-1)-k-1}) = b(p-1)$. On the other hand we obtain

$$\begin{array}{rcl} \mu_{l,j}\alpha & = & \tilde{\mu}_{p-1,p-\bar{b}}\,\mu_{l,j} \\ & = & s^{b-\underline{b}+\underline{bl-j-1}} \sum_{i\in[0,p-1]} \binom{i}{l}\,\varepsilon_{\bar{b},p-\bar{b}}\,\varepsilon_{i,j} \\ & \overset{(1.6)}{=} & s^{b-\underline{b}+\underline{bl-j-1}} \sum_{i\in[0,p-1]} \binom{i}{l}\,\partial_{i,0}\,\varepsilon_{\bar{b},j+p-\bar{b}} \\ & = & \partial_{l,0}s^{b-\underline{b}-1}\varepsilon_{\bar{b},j+p-\bar{b}} \\ & = & \begin{cases} s^{b-\underline{b}-1}\varepsilon_{\bar{b},j+p-\bar{b}} & \text{for } l = 0 \text{ and } j \in [0, \bar{b}-1] \\ s^{b-\underline{b}}\,\varepsilon_{\bar{b},j-\bar{b}} & \text{for } l = 0 \text{ and } j \in [\bar{b}, p-1] \\ 0 & \text{for } l \in [1, p-1]\ , \end{cases} \end{array}$$

where $l, j \in [0, p-1]$, whence the colength of $\operatorname{Im}\alpha$ in B equals $(b-\underline{b}-1)p+(p-\bar{b}) = b(p-1)$, too. Thus $\operatorname{Im}\alpha = \operatorname{Kern}\beta$.

Moreover, since

$$\begin{array}{rcl} \mu_{l,j}\chi_0 & = & s^{1+\underline{bl-j-1}} \sum_{i\in[0,p-1]} \binom{i}{l}\,t^0\varepsilon_{i,j} \\ & = & \partial_{l,0}t^j\ , \end{array}$$

where $l, j \in [0, p-1]$, the Ξ-linear isomorphism

$$\begin{array}{rcl} T & \overset{\sim}{\longrightarrow} & \Xi\alpha \\ t^j & \longmapsto & s^{b-\underline{b}-1}\varepsilon_{\underline{b},j+p-\underline{b}} \end{array}$$

yields the commutativity

$$(\ \Xi \xrightarrow{\ \alpha\ } \Xi\alpha\ ) \;=\; (\ \Xi \xrightarrow{\ \chi_0\ } T \xrightarrow{\ \sim\ } \Xi\alpha\ )\ .$$

We claim exactness of $\Xi \xrightarrow{\beta} \Xi \xrightarrow{\alpha} \Xi$. To prove that $\beta\alpha = 0$, we calculate

$$\begin{array}{rcl} \tilde{\mu}_{p-1,p-\bar{b}}\,\mu_{1,\bar{b}} & = & s^{b-1} \sum_{i\in[0,p-1]} i\varepsilon_{\bar{b},p-\bar{b}}\,\varepsilon_{i,\bar{b}} \\ & \overset{(1.6)}{=} & s^{b-1} \sum_{i\in[0,p-1]} i\partial_{i,0}\,\varepsilon_{\bar{b},p} \\ & = & 0\ . \end{array}$$

Now, let $A \subseteq \Lambda$ be the kernel of the Λ-linear map $\Lambda \longrightarrow \Lambda$, $1 \longmapsto \tilde{\mu}_{p-1,p-\bar{b}}$. To prove that $\operatorname{Im}\beta = \operatorname{Kern}\alpha$, we shall show that the S-linear colengths of both submodules in A coincide. For $j, k \in [0, p-1]$, we have

$$\varepsilon_{j,k}\alpha \;=\; s^{b-\underline{b}-1}\varepsilon_{\bar{b},p-\bar{b}}\,\varepsilon_{j,k} \;\overset{(1.6)}{=}\; s^{b-\underline{b}-1}\partial_{j,0}\,\varepsilon_{\bar{b},p-\bar{b}+k}\;,$$

and therefore A has the S-linear basis $(\varepsilon_{j,k})_{j\in[1,p-1],\,k\in[0,p-1]}$. By (1.17), $\operatorname{Kern}\alpha = \Xi \cap A$ has the S-linear basis $(\mu_{l,j})_{l\in[1,p-1],\,j\in[0,p-1]}$, whence the colength of $\operatorname{Kern}\alpha$ in A equals $bp(p-1)/2$ (cf. pf. of 1.17). On the other hand, using the S-linear basis

$$\left(s^{1+\underline{bl-j-1}}\sum_{k\in[0,p-1]}\binom{k}{l}\,\varepsilon_{\overline{k+b+1},j}\right)_{l,j\in[0,p-1]}$$

of Ξ (1.18), we obtain

$$\begin{array}{rl} & \left(s^{1+\underline{bl-j-1}}\sum_{k\in[0,p-1]}\binom{k}{l}\,\varepsilon_{\overline{k+b+1},j}\right)\beta \\ = & s^{\underline{b}+1+\underline{bl-j-1}}\sum_{i\in[0,p-1]}\sum_{k\in[0,p-1]} i\binom{k}{l}\,\varepsilon_{i,\bar{b}}\,\varepsilon_{\overline{k+b+1},j} \\ \overset{(1.6)}{=} & s^{\underline{b}+1+\underline{bl-j-1}}\sum_{i\in[0,p-1]}\sum_{k\in[0,p-1]} i\binom{k}{l}\,\partial_{\overline{k+b+1},\overline{i+b}}\,\varepsilon_{i,\bar{b}+j} \\ = & s^{\underline{b}+1+\underline{bl-j-1}}\sum_{i\in[l+1,p-1]} i\binom{i-1}{l}\,\varepsilon_{i,\bar{b}+j} \\ = & s^{\underline{b}+1+\underline{bl-j-1}+\underline{\bar{b}+j}}\sum_{i\in[l+1,p-1]} i\binom{i-1}{l}\,\varepsilon_{i,\overline{b+j}}\;. \end{array}$$

This yields the colength of $\operatorname{Im}\beta$ in A to be equal to

$$\begin{array}{rl} & \sum_{j\in[0,p-1]}\sum_{l\in[0,p-2]}(\underline{\bar{b}+j}+\underline{b}+1+\underline{bl-j-1}) \\ \overset{(1.12)}{=} & \bar{b}(p-1)+(\underline{b}+1)p(p-1)+\sum_{l\in[0,p-2]}(bl-p) \\ = & bp(p-1)/2\;, \end{array}$$

too. Thus $\operatorname{Im}\beta = \operatorname{Kern}\alpha$. ▫

Example 4.4 (cont.) For $\Xi \simeq \mathbf{Z}_{(3)}[\pi_2] \wr C_3$, we have $b = 1$ and $\underline{b} = 0$, thus obtaining the projective resolution

$$P \;=\; \left(\cdots \longrightarrow \Xi \xrightarrow{\left[\begin{smallmatrix} 0&0&0\\ 0&0&1\\ 3\cdot 2&0&0 \end{smallmatrix}\right]\cdot(-)} \Xi \xrightarrow{\left[\begin{smallmatrix} 0&0&0\\ 3\cdot 1&0&0\\ 0&0&0 \end{smallmatrix}\right]\cdot(-)} \Xi \xrightarrow{\left[\begin{smallmatrix} 0&0&0\\ 0&0&1\\ 3\cdot 2&0&0 \end{smallmatrix}\right]\cdot(-)} \underbrace{\Xi}_{\text{degree }0} \longrightarrow 0 \longrightarrow \cdots\right)$$

of T, with quasiisomorphism given by $\Xi \xrightarrow{\chi_0} T$ in degree 0, sending $\xi = \left[\begin{smallmatrix} a_{0,0} & a_{0,1} & a_{0,2}\\ 3a_{1,2} & a_{1,0} & a_{1,1}\\ 3a_{2,1} & 3a_{2,2} & a_{2,0} \end{smallmatrix}\right] \in \Xi$ to $t^0\xi = a_{0,0}t^0 + a_{0,1}t^1 + a_{0,2}t^2$. Representing elements of T as row vectors with entries in S with respect to the basis (t^0, t^1, t^2), the map χ_0 is given by $\left[\begin{smallmatrix} 1&0&0 \end{smallmatrix}\right]\cdot(-)$.

4.2 The Yoneda ring

Remark 4.5 *An S-linear basis of ${}_{\Xi}(\Xi, T)$ is given by $(\chi_i)_{i\in[0,p-1]}$. The restriction map*

$${}_{\Lambda}(\Lambda, T) \;\longrightarrow\; {}_{\Xi}(\Xi, T)$$

is an isomorphism.

Proposition 4.6 *We have*

$$\begin{array}{rcll}
\mathrm{Ext}^0_{T\wr G}(T,T) & = & S\langle\chi_0\rangle & \\
 & \simeq & S\ , & \\
\mathrm{Ext}^{2i}_{T\wr G}(T,T) & = & S\langle\chi_0\rangle/S\langle s^{b-\underline{b}}\cdot\chi_0\rangle & \\
 & \simeq & S/s^{b-\underline{b}} & \textit{for } i\geqslant 1\ ,\ \textit{and} \\
\mathrm{Ext}^{2i+1}_{T\wr G}(T,T) & = & \dfrac{S\langle\quad \chi_0,\dots,\quad \chi_{\bar b-1},\quad \chi_{\bar b+1},\dots,\quad \chi_{p-1}\rangle}{S\langle s^{\underline{b}+1}\chi_0,\dots,s^{\underline{b}+1}\chi_{\bar b-1},s^{\underline{b}}\chi_{\bar b+1},\dots,s^{\underline{b}}\chi_{p-1}\rangle} & \\
 & \simeq & \left(\bigoplus\limits_{k\in[0,\bar b-1]} S/s^{\underline{b}+1}\right)\oplus\left(\bigoplus\limits_{k\in[\bar b+1,p-1]} S/s^{\underline{b}}\right) & \textit{for } i\geqslant 0\ .
\end{array}$$

The element represented by χ_0 in $\mathrm{Ext}^{2i}_{T\wr G}(T,T)$ shall be written $\chi_0^{(2i)}$, the element represented by χ_j in $\mathrm{Ext}^{2i+1}_{T\wr G}(T,T)$ shall be written $\chi_j^{(2i+1)}$, where $i\geqslant 0$, $j\in[0,p-1]\smallsetminus\{1\}$.

Proof. We calculate ${}_{\Xi}(\Xi,T)\xrightarrow{\alpha^*}{}_{\Xi}(\Xi,T)$, $\chi\longmapsto\alpha\chi$. Given $k\in[0,p-1]$, we get

$$\begin{array}{rcl}
1_\Xi(\chi_k\alpha^*) & = & (1_\Xi\alpha)\chi_k \\
 & = & (s^{b-\underline{b}-1}\varepsilon_{\bar b,p-\bar b})\chi_k \\
 & = & s^{b-\underline{b}-1}(1_\Xi\,\chi_k)\varepsilon_{\bar b,p-\bar b} \\
 & = & \partial_{\bar b,k}s^{b-\underline{b}} \\
 & = & \partial_{\bar b,k}s^{b-\underline{b}}(1_\Xi\,\chi_0)\ ,
\end{array}$$

i.e.

$$\chi_k\alpha^* \;=\; \partial_{\bar b,k}s^{b-\underline{b}}\cdot\chi_0\ .$$

We calculate ${}_{\Xi}(\Xi,T)\xrightarrow{\beta^*}{}_{\Xi}(\Xi,T)$, $\chi\longmapsto\beta\chi$. Given $k\in[0,p-1]$, we get

$$\begin{array}{rcl}
1_\Xi(\chi_k\beta^*) & = & (1_\Xi\beta)\chi_k \\
 & = & (s^{\underline{b}}\sum_{i\in[0,p-1]} i\varepsilon_{i,\bar b})\chi_k \\
 & = & s^{\underline{b}}\sum_{i\in[0,p-1]} i\partial_{i,k}t^{\overline{k+b}}s^{\underline{k+\bar b}} \\
 & = & ks^{\underline{k+\bar b}+\underline{b}}(1_\Xi\,\chi_{\overline{k+b}})\ ,
\end{array}$$

i.e.

$$\chi_k\beta^* \;=\; ks^{\underline{k+\bar b}+\underline{b}}\cdot\chi_{\overline{k+b}}\ .$$

The shape of the Ext-groups now follows by (4.3). ▫

Lemma 4.7 *We have*

$$s^{b+\underline{j-2b}}\left((\overline{b-j})^{-1}\varepsilon_{\overline{2b-j},\overline{j-2b}}+(\overline{j-b})^{-1}\varepsilon_{\bar b,\overline{j-2b}}\right)\;\in\;\Xi$$

for $j\in[0,p-1]\smallsetminus\{\bar b\}$.

Proof. First of all, we have $b+\underline{j-2b} \geqslant b+\underline{-2b} = b-1-\underline{2b-1}$ and $p(b-1-\underline{2b-1}) = (p-2)(b-1)+\overline{2b-1}-1 \geqslant 0$, both if $b=1$ or if $b>1$. So the element in question is in Λ.

By (1.17), we have to prove that for $l \in [0,p-1]$,

$$\mathrm{v}_s\Big((-1)^{\overline{2b-j}}\tbinom{l}{\overline{2b-j}}(\overline{b-j})^{-1}+(-1)^{\bar{b}}\tbinom{l}{\bar{b}}(\overline{j-b})^{-1}\Big) \;\geqslant\; 1+\underline{bl-\overline{j-2b}-1}-(b+\underline{j-2b})\,.$$

If $l \in [0,p-2]$, then

$$1+\underline{bl-\overline{j-2b}-1} \;\leqslant\; 1+\underline{b(p-2)-\overline{j-2b}-1} \;=\; b+\underline{j-2b}\,.$$

If $l = p-1$ then

$$1+\underline{b(p-1)-\overline{j-2b}-1}-(b+\underline{j-2b}) \;=\; -\underline{j-b}\,.$$

Now $(-1)^h\binom{p-1}{h} \equiv_p 1$ for $h \in [0,p-1]$ and $(\overline{b-j})^{-1}+(\overline{j-b})^{-1} \equiv_p 0$ together with $\mathrm{v}_s(p) \geqslant b-\underline{b} \geqslant -\underline{j-b}$, both if $b=1$ or if $b>1$, yield the result. ▫

Let

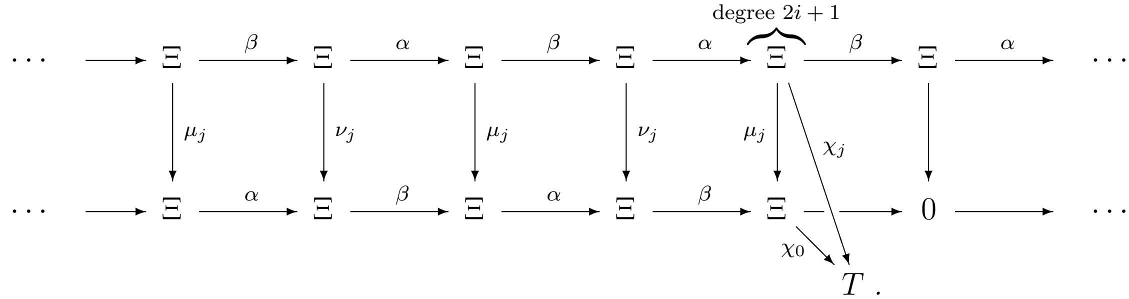
$$\begin{array}{lclcl} \Xi & \xrightarrow{\mu_j} & \Xi & : & 1 \longmapsto \mu_{0,j} \;=\; \sum_{i\in[0,p-1]}\varepsilon_{i,j} \\ \Xi & \xrightarrow{\nu_j} & \Xi & : & 1 \longmapsto s^{b+\underline{j-2b}}\big((\overline{b-j})^{-1}\varepsilon_{\overline{2b-j},\overline{j-2b}}+(\overline{j-b})^{-1}\varepsilon_{\bar{b},\overline{j-2b}}\big) \end{array}$$

be Ξ-linear maps for $j \in [0,p-1] \smallsetminus \{\bar{b}\}$. By (4.7), the map ν_j is welldefined.

Lemma 4.8 *For $j \in [0,p-1] \smallsetminus \{\bar{b}\}$, we obtain*

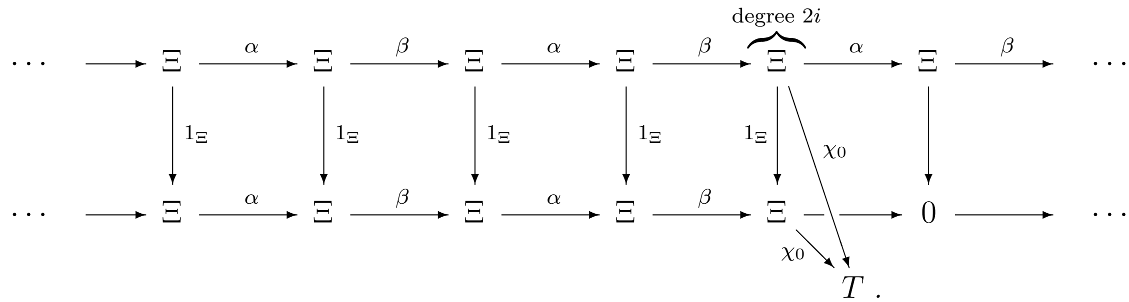
$$\begin{array}{rcl} \nu_j\beta & = & \alpha\mu_j \\ \mu_j\alpha & = & \beta\nu_j \\ \mu_j\chi_0 & = & \chi_j\,. \end{array}$$

That is, for $i \geqslant 0$, we obtain a representative



in ${}_{\mathrm{K}^-(\Xi)}(P,P[2i+1])$ *of* $\chi_j^{(2i+1)} \in \mathrm{Ext}^{2i+1}_{T\wr G}(T,T)$.

Moreover, for $i \geqslant 0$, we obtain a representative



in $_{\mathrm{K}^-(\Xi)}(P, P[2i])$ *of* $\chi_0^{(2i)} \in \mathrm{Ext}^{2i}_{T\wr G}(T,T)$.

Proof. We claim that $\nu_j\beta = \alpha\mu_j$. On the one hand, we obtain

$$\begin{array}{rcl}
1_\Xi(\nu_j\beta) & = & s^{\underline{b}+b+\underline{j-2b}} \sum_{i\in[0,p-1]} \big(i(\overline{b-j})^{-1}\varepsilon_{i,\bar{b}}\,\varepsilon_{\overline{2b-j},\overline{j-2b}} + i(\overline{j-b})^{-1}\varepsilon_{i,\bar{b}}\,\varepsilon_{\bar{b},\overline{j-2b}}\big) \\
& \overset{(1.6)}{=} & s^{\underline{b}+b+\underline{j-2b}} \sum_{i\in[0,p-1]} i(\overline{b-j})^{-1}\partial_{\overline{2b-j},\overline{i+b}}\,\varepsilon_{i,\bar{b}+\overline{j-2b}} \\
& = & s^{\underline{b}+b+\underline{j-2b}}\,\varepsilon_{\overline{b-j},\bar{b}+\overline{j-2b}} \\
& = & s^{b-\underline{b}-1}\varepsilon_{\overline{b-j},j+p-\bar{b}}\ .
\end{array}$$

On the other hand, we obtain

$$\begin{array}{rcl}
1_\Xi(\alpha\mu_j) & = & s^{b-\underline{b}-1}\sum_{i\in[0,p-1]}\varepsilon_{i,j}\,\varepsilon_{\bar{b},p-\bar{b}} \\
& \overset{(1.6)}{=} & s^{b-\underline{b}-1}\sum_{i\in[0,p-1]}\partial_{\bar{b},\overline{i+j}}\,\varepsilon_{i,j+p-\bar{b}} \\
& = & s^{b-\underline{b}-1}\varepsilon_{\overline{b-j},j+p-\bar{b}}\ .
\end{array}$$

We claim that $\mu_j\alpha = \beta\nu_j$. On the one hand, we obtain

$$\begin{array}{rcl}
1_\Xi(\beta\nu_j) & = & s^{\underline{b}+b+\underline{j-2b}} \sum_{i\in[0,p-1]} \big(i(\overline{b-j})^{-1}\varepsilon_{\overline{2b-j},\overline{j-2b}}\,\varepsilon_{i,\bar{b}} + i(\overline{j-b})^{-1}\varepsilon_{\bar{b},\overline{j-2b}}\,\varepsilon_{i,\bar{b}}\big) \\
& \overset{(1.6)}{=} & s^{\underline{b}+b+\underline{j-2b}} \sum_{i\in[0,p-1]} i(\overline{j-b})^{-1}\partial_{i,\overline{j-b}}\,\varepsilon_{\bar{b},\bar{b}+\overline{j-2b}} \\
& = & s^{\underline{b}+b+\underline{j-2b}}\,\varepsilon_{\bar{b},\bar{b}+\overline{j-2b}} \\
& = & s^{b-\underline{b}-1}\varepsilon_{\bar{b},j-\bar{b}+p}\ .
\end{array}$$

On the other hand, we obtain

$$\begin{array}{rcl}
1_\Xi(\mu_j\alpha) & = & s^{b-\underline{b}-1}\sum_{i\in[0,p-1]}\varepsilon_{\bar{b},p-\bar{b}}\varepsilon_{i,j} \\
& \overset{(1.6)}{=} & s^{b-\underline{b}-1}\sum_{i\in[0,p-1]}\partial_{i,0}\varepsilon_{\bar{b},j-\bar{b}+p} \\
& = & s^{b-\underline{b}-1}\varepsilon_{\bar{b},j-\bar{b}+p}\ .
\end{array}$$

We claim that $\mu_j\chi_0 = \chi_j$. In fact,

$$\begin{array}{rcl}
1_\Xi(\mu_j\chi_0) & = & \sum_{i\in[0,p-1]} t^0\varepsilon_{i,j} \\
& = & \sum_{i\in[0,p-1]}\partial_{i,0}t^j \\
& = & t^j \\
& = & 1_\Xi\,\chi_j\ .
\end{array}$$

□

Example 4.9 (cont.) If $\Xi \simeq \mathbf{Z}_{(3)}[\pi_2] \wr C_3$, then $b = \bar{b} = 1$; $\nu_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 3 & 0 & 0 \end{bmatrix} \cdot (-)$, $\mu_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot (-)$; $\nu_2 = \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot (-)$, $\mu_2 = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \cdot (-)$.

Theorem 4.10 *We have isomorphisms of graded S-algebras*

$$\frac{S\left[h_0^{(1)},\ldots,h_{\bar b-1}^{(1)};h_{\bar b+1}^{(1)},\ldots,h_{p-1}^{(1)};h_0^{(2)}\right]}{\left(\begin{array}{l} s^{\underline{b}+1}h_0^{(1)},\ldots,s^{\underline{b}+1}h_{\bar b-1}^{(1)};s^{\underline{b}}h_{\bar b+1}^{(1)},\ldots,s^{\underline{b}}h_{p-1}^{(1)};s^{b-\underline{b}}h_0^{(2)}; \\ h_j^{(1)}\cdot h_k^{(1)} - \partial_{\overline{j+k},\overline{2b}}\, s^{b+\underline{j+k-2b}}\,(\overline{b-j})^{-1}h_0^{(2)} \\ \qquad\qquad \text{for } j,k\in[0,p-1]\smallsetminus\{\bar b\} \end{array}\right)} \xrightarrow{\sim} \operatorname{Ext}^*_{T\wr G}(T,T) \xrightarrow{\sim} \mathrm{H}^*(G,T;S)$$

$$\begin{array}{rcll} h_j^{(1)} & \longmapsto & \chi_j^{(1)}\ , & j\in[0,p-1]\smallsetminus\{\bar b\} \\ h_0^{(2)} & \longmapsto & \chi_0^{(2)}\ , & \end{array}$$

as quotient of the graded commutative *polynomial ring* $S[h_0^{(1)},\ldots,h_{\bar b-1}^{(1)};h_{\bar b+1}^{(1)},\ldots,h_{p-1}^{(1)};h_0^{(2)}]$ *with grading determined by* $\deg h_j^{(1)}=1$ *for* $j\in[0,p-1]\smallsetminus\{\bar b\}$ *and* $\deg h_0^{(2)}=2$.

Proof. The isomorphism $\operatorname{Ext}^*_{T\wr G}(T,T)\xrightarrow{\sim}\mathrm{H}^*(G,T;S)$ of graded S-algebras with respect to the Yoneda product resp. to the cup product is a consequence of (3.1).

We shall exhibit the ring structure on the graded S-module $\operatorname{Ext}^*_{T\wr G}(T,T)$ (cf. 4.6). By (4.8), we obtain

$$\begin{array}{ccccccc} \chi_0^{(2i)} & \cdot & \chi_0^{(2j)} & = & \chi_0^{(2i+2j)} & \in & \operatorname{Ext}^{2i+2j}_{T\wr G}(T,T) \\ \chi_0^{(2i)} & \cdot & \chi_k^{(2j+1)} & = & \chi_k^{(2i+2j+1)} & \in & \operatorname{Ext}^{2i+2j+1}_{T\wr G}(T,T) \end{array}$$

for $i,\ j\geqslant 0$ and $k\in[0,p-1]\smallsetminus\{\bar b\}$. It remains to calculate $\chi_j^{(1)}\cdot\chi_k^{(1)}\in\operatorname{Ext}^2_{T\wr G}(T,T)$ for $j,\ k\in[0,p-1]\smallsetminus\{\bar b\}$. Using (4.8) to represent χ_j in ${}_{\mathrm{K}^-(\Xi)}(P,P[1])$, this product is represented by the 2-cocycle $\nu_j\chi_k\in{}_{\Xi}(\Xi,T)$. We obtain

$$\begin{array}{rcl} 1_\Xi(\nu_j\chi_k) & = & t^k s^{b+\underline{j-2b}}\left((\overline{b-j})^{-1}\varepsilon_{\overline{2b-j},\overline{j-2b}}+(\overline{j-b})^{-1}\varepsilon_{\bar b,\overline{j-2b}}\right) \\ & = & t^k s^{b+\underline{j-2b}}\,(\overline{b-j})^{-1}\varepsilon_{\overline{2b-j},\overline{j-2b}} \\ & = & s^{b+\underline{j-2b}}\,(\overline{b-j})^{-1}\partial_{k,\overline{2b-j}}\,t^{\overline{k+j-2b}}s^{\underline{k+\overline{j-2b}}} \\ & = & s^{b+\underline{j+k-2b}}\,(\overline{b-j})^{-1}\partial_{\overline{j+k},\overline{2b}}\,t^0\ , \end{array}$$

i.e. $\nu_j\chi_k=\partial_{\overline{j+k},\overline{2b}}\,s^{b+\underline{j+k-2b}}\,(\overline{b-j})^{-1}\chi_0$. Hence

$$\chi_j^{(1)}\cdot\chi_k^{(1)} \;=\; \partial_{\overline{j+k},\overline{2b}}\,s^{b+\underline{j+k-2b}}\,(\overline{b-j})^{-1}\chi_0^{(2)}\ .$$

□

Remark 4.11 Since $\overline{j+k}=\overline{2b}$ implies $(\overline{b-j})^{-1}\equiv_p-(\overline{b-k})^{-1}$, and since $\mathrm{v}_s(p)\geqslant b-\underline{b}$, we obtain the graded commutativity of $\operatorname{Ext}^*_{T\wr G}(T,T)$ without reverting to the graded commutativity of the cup product.

For instance, if $b=p+1$, then $\chi_0^{(1)}\cdot\chi_2^{(1)}=s^{p-1}\chi_0^{(2)}\neq 0$. In particular, $\operatorname{Ext}^*_{T\wr G}(T,T)$ is not commutative in this case.

Corollary 4.12 *If $b = 1$, then we have isomorphisms of graded S-algebras*

$$S[h^{(1)}, h^{(2)}]/(sh^{(1)},\ sh^{(2)},\ (h^{(1)})^2) \xrightarrow{\sim} \mathrm{Ext}^*_{T\wr G}(T,T) \xrightarrow{\sim} \mathrm{H}^*(G,T;S)\ ,$$

as quotient of the commutative *polynomial ring $S[h^{(1)}, h^{(2)}]$ with grading determined by* $\deg h^{(1)} = 1$, $\deg h^{(2)} = 2$.

Proof. In fact, we have $\underline{b} = 0$. There are no nonzero products of homogeneous elements of odd degree, so we may use the commutative polynomial ring. ▫

Example 4.13 (cont.) We have $\mathrm{H}^*(C_3, \mathbf{Z}_{(3)}[\pi_2]; \mathbf{Z}_{(3)}) \simeq \mathbf{Z}_{(3)}[h^{(1)}, h^{(2)}]/(3h^{(1)}, 3h^{(2)}, (h^{(1)})^2)$. Note that $\mathrm{H}^0(C_3, \mathbf{Z}_{(3)}[\pi_2]; \mathbf{Z}_{(3)}) \simeq \mathbf{Z}_{(3)}$, and that $\mathrm{H}^i(C_3, \mathbf{Z}_{(3)}[\pi_2]; \mathbf{Z}_{(3)}) \simeq \mathbf{F}_3$ for $i \geqslant 1$.

5 Applications

We give some applications, refraining, however, from a repetition of (4.10) in different instances.

5.1 Lubin-Tate extensions

The results apply to certain of the extensions of local fields described by Lubin and Tate **[21]**. An introduction to their theory is also given in **[28,** p. 146 ff.].

Let $p \geqslant 3$ be a prime. Let B be a local field with discrete valuation ring R, whose maximal ideal is generated by π. Assume that $R/\pi \simeq \mathbf{F}_p$. We choose the Lubin-Tate series $f(X) = X^p + \pi X \in R[[X]]$, and obtain the unique commutative formal group

$$\begin{array}{rcl} F(X,Y) & = & X + Y \\ & - & (\pi - \pi^p)^{-1}((X+Y)^p - (X^p + Y^p)) \\ & - & p(\pi - \pi^p)^{-1}(\pi - \pi^{2p-1})^{-1} \left(\begin{array}{rl} & \pi^{p-1}(X+Y)^{p-1}(X^p + Y^p) \\ - & (X+Y)^{p-1}((X+Y)^p - (X^p + Y^p)) \\ - & \pi^{p-1}(X^{2p-1} + Y^{2p-1}) \end{array} \right) \\ & + & \mathcal{O}(\text{degree } 3p-2) \ \in \ R[[X,Y]] \end{array}$$

such that $F(f(X), f(Y)) = f(F(X,Y))$. There is an injective ring morphism

$$\begin{array}{rcl} R & \longrightarrow & \mathrm{End}\, F \\ a & \longmapsto & [a](X)\ , \end{array}$$

where $[a](X) \in R[[X]]$ is uniquely determined by $[a](X) \equiv_{X^2} aX$ and the endomorphism property $F([a](X), [a](Y)) = [a](F(X,Y))$. So for instance, $[\pi](X) = f(X) = X^p + \pi X$. We write $P_n(X) := [\pi^n](X) \in R[X]$ for $n \geqslant 0$, so that $P_0(X) = X$, $P_1(X) = f(X) = X^p + \pi X$ and $P_n(X) = P_{n-1}(X)^p + \pi P_{n-1}(X)$. Moreover, $P_n(0) = 0$, $\deg P_n(X) = p^n$, $P_n(X) \equiv_\pi X^{p^n}$ and $P_n'(X) \equiv_p \pi^n$.

Let $\bar{B}$ be an algebraic closure of B, and let $\bar{\mathfrak{m}} = \{x \in \bar{B} : \mathrm{N}_{B(x)|B}(x) \in R\pi\} \subseteq \bar{B}$ be the maximal ideal of its valuation ring, which becomes an abelian group $(\bar{\mathfrak{m}}, *)$ via $x * y := F(x,y)$. Moreover, $\bar{\mathfrak{m}}$ becomes an R-module via

$$\begin{array}{rcl} R & \longrightarrow & \mathrm{End}(\bar{\mathfrak{m}}, *) \\ a & \longmapsto & (\ [a]\ :\ x \longmapsto [a]\cdot x := [a](x)\ )\ . \end{array}$$

For $n \geqslant 1$, we let

$$\boldsymbol{\mu}_n \ := \ \mathrm{ann}_{[\pi^n]}\bar{\mathfrak{m}} \ = \ \{x \in \bar{K}\ :\ P_n(x) = 0\}\ .$$

By separability of $P_n(X)$, we have $\#\boldsymbol{\mu}_n = p^n$ for each $n \geqslant 1$, whence $\boldsymbol{\mu}_n \simeq R/\pi^n$ as R-modules. Let ϑ_n be an R-linear generator of $\boldsymbol{\mu}_n$, chosen in such a way that $[\pi](\vartheta_n) = \vartheta_{n-1}$. We have $\mu_{\vartheta_n,B}(X) = P_n(X)/P_{n-1}(X) = P_{n-1}(X)^{p-1} + \pi$, whence $B(\boldsymbol{\mu}_n) = B(\vartheta_n)$ is galois over B with

$$\begin{array}{rcl} (R/\pi^n)^* & \overset{\sim}{\longrightarrow} & \mathrm{Gal}(B(\boldsymbol{\mu}_n)|B) \\ u & \longmapsto & (\ \langle u\rangle\ :\ \vartheta_n \longmapsto [u](\vartheta_n)\ )\ . \end{array}$$

Now $R[\vartheta_n]|R$ is purely ramified, and as different we obtain

$$\begin{array}{rcl} \mathfrak{D}_{R[\vartheta_n]|R} & = & \big(\mu'_{\vartheta_n,B}(\vartheta_n)\big) \\ & = & (P'_n(\vartheta_n)/P_{n-1}(\vartheta_n)) \\ & = & \big(P'_{n-1}(\vartheta_n)(p\vartheta_1^{p-1} + \pi)/\vartheta_1\big) \\ & = & \big(P'_{n-2}(\vartheta_n)(p\vartheta_2^{p-1} + \pi)(p\vartheta_1^{p-1} + \pi)/\vartheta_1\big) \\ & = & \cdots \\ & = & \Big(\vartheta_1^{-1} \prod_{i\in[1,n]} (p\vartheta_i^{p-1} + \pi)\Big) \\ & = & \big(\vartheta_1^{-1}\pi^n\big)\ , \end{array}$$

whence $\mathfrak{D}_{R[\vartheta_n]|R[\vartheta_{n-1}]} = (\pi) = (\vartheta_n^{p^{n-1}(p-1)})$.

Example 5.1 Let $n \geqslant 2$. We may apply (1.19, 4.10) to

$$\begin{array}{rclcrcl} S & = & R[\vartheta_{n-1}] & \quad & s & = & \vartheta_{n-1} \\ T & = & R[\vartheta_n] & & t & = & \vartheta_n \\ b & = & p^{n-1} - 1 & & & & \\ \underline{b} & = & p^{n-2} - 1 & & \bar{b} & = & p - 1\ . \end{array}$$

The value of b results from the different by the formula $\mathrm{v}_t(\mathfrak{D}_{T|S}) = (p-1)(b+1)$ (1.14). Moreover, $\mathrm{v}_s(p) \geqslant \mathrm{v}_s(\pi) = p^{n-2}(p-1) = b - \underline{b}$.

We have $\mathbf{F}_p^* \hookrightarrow (R/\pi^n)^*$ by sending $j \longmapsto j^{p^{n-1}}$. For $n \geqslant 1$, we let

$$\pi_n \ := \ \prod_{j\in\mathbf{F}_p^*} \vartheta_n^{\langle j^{p^{n-1}}\rangle} \ = \ \prod_{j\in[1,p-1]} [j^{p^{n-1}}](\vartheta_n)\ .$$

Then $R[\pi_n]$ is purely ramified over R, of degree p^{n-1} and with maximal ideal generated by π_n. In particular, $\pi_1 = \pi$. Moreover, $\pi_n = \mathrm{N}_{B(\vartheta_n)|B(\pi_n)}(\vartheta_n)$.

As different, we obtain

$$
\begin{array}{rcl}
\mathfrak{D}_{R[\pi_n]|R[\pi_{n-1}]} & = & \mathfrak{D}^{-1}_{R[\vartheta_n]|R[\pi_n]}\mathfrak{D}_{R[\vartheta_n]|R[\vartheta_{n-1}]}\mathfrak{D}_{R[\vartheta_{n-1}]|R[\pi_{n-1}]} \\
& = & (\vartheta_n^{p-2})^{-1}(\vartheta_n^{p^{n-1}(p-1)})(\vartheta_n^{p(p-2)}) \\
& = & (\pi_n^{p^{n-1}+p-2})
\end{array}
$$

[**28,** III.§3, prop. 13].

Example 5.2 Let $n \geqslant 2$, $p \geqslant 3$. We may apply (1.19, 4.10, 2.10) to

$$
\begin{array}{rclcrcl}
S & = & R[\pi_{n-1}] & & s & = & \pi_{n-1} \\
T & = & R[\pi_n] & & t & = & \pi_n \\
U & = & R[\vartheta_n] & & u & = & \vartheta_n \\
b & = & (p^{n-1}-1)/(p-1) & & & & \\
\underline{b} & = & (p^{n-2}-1)/(p-1) & & \bar{b} & = & 1\ ,
\end{array}
$$

The value of b results from the different by the formula $\mathrm{v}_t(\mathfrak{D}_{T|S}) = (p-1)(b+1)$ (1.14). Moreover, $\mathrm{v}_s(p) \geqslant \mathrm{v}_s(\pi) = p^{n-2} = b - \underline{b}$.

In particular, (3.1, 2.10) yield isomorphisms of graded S-algebras

$$
\mathrm{H}^*(C_p \times C_{p-1}, U; S) \;\simeq\; \mathrm{Ext}^*_{U\wr(C_p\times C_{p-1})}(U,U) \;\simeq\; \mathrm{Ext}^*_{T\wr C_p}(T,T) \;\simeq\; \mathrm{H}^*(C_p, T; S)\ .
$$

5.2 Cyclotomic number field extensions

Passing to completions without changing cohomology, we may consider cyclotomic number field extensions as particular Lubin-Tate extensions. For sake of illustration, we recall the cyclotomic framework; in it, there is no need for completion, since the formal group law is given by the polynomial $F(X,Y) = X + Y + XY$. Strictly speaking, since in §5.2 we choose a different Lubin-Tate series as in §5.1, viz. $(X+1)^p - 1$ instead of $X^p + pX$, we are not directly specializing to this cyclotomic case. So keeping the notation of §5.1 is a slight abuse.

Let $p \geqslant 3$ be a prime. For $n \geqslant 1$, we let ζ_{p^n} be a primitive p^nth root of unity over $\mathbf{Q}$. We make choices in such a manner that $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for $n \geqslant 2$ and denote $\vartheta_n := \zeta_{p^n} - 1$. Let

$$
\pi_n \;=\; \prod_{j\in[1,\,p-1]} (\zeta_{p^n}^{j^{p^{n-1}}} - 1)\ .
$$

Then $\mathbf{Q}(\vartheta_n) = \mathbf{Q}(\zeta_{p^n})$, $\mathbf{Q}(\pi_n) = \mathrm{Fix}_{C_{p-1}}\,\mathbf{Q}(\vartheta_n)$ and $\pi_n = \mathrm{N}_{\mathbf{Q}(\vartheta_n)|\mathbf{Q}(\pi_n)}(\vartheta_n)$.

We have $\mathrm{N}_{\mathbf{Q}(\vartheta_n)|\mathbf{Q}(\vartheta_{n-1})}(\vartheta_n) = \vartheta_{n-1}$ and $\mathrm{N}_{\mathbf{Q}(\pi_n)|\mathbf{Q}(\pi_{n-1})}(\pi_n) = \pi_{n-1}$. Note that $\pi_1 = p$.

The integral closure of $\mathbf{Z}_{(p)}$ in $\mathbf{Q}(\vartheta_n)$ is given by the discrete valuation ring $\mathbf{Z}_{(p)}[\vartheta_n]$, with maximal ideal generated by ϑ_n, purely ramified over $\mathbf{Z}_{(p)}$; the integral closure of $\mathbf{Z}_{(p)}$ in $\mathbf{Q}(\pi_n)$ is given by the discrete valuation ring $\mathbf{Z}_{(p)}[\pi_n]$, with maximal ideal generated by π_n, purely ramified over $\mathbf{Z}_{(p)}$.

Example 5.3 Let $n \geqslant 2$. We may apply (1.19, 4.10) to

$$\begin{array}{rclcrcl} S &=& \mathbf{Z}_{(p)}[\vartheta_{n-1}] & & s &=& \vartheta_{n-1} \\ T &=& \mathbf{Z}_{(p)}[\vartheta_n] & & t &=& \vartheta_n \\ b &=& p^{n-1}-1 & & & & \\ \underline{b} &=& p^{n-2}-1 & & \overline{b} &=& p-1\ . \end{array}$$

We remark that $\mathrm{v}_s(p) = p^{n-2}(p-1) = b - \underline{b}$.

Example 5.4 Let $n \geqslant 2$, $p \geqslant 3$. We may apply (1.19, 4.10, 2.10) to

$$\begin{array}{rclcrcl} S &=& \mathbf{Z}_{(p)}[\pi_{n-1}] & & s &=& \pi_{n-1} \\ T &=& \mathbf{Z}_{(p)}[\pi_n] & & t &=& \pi_n \\ U &=& \mathbf{Z}_{(p)}[\vartheta_n] & & u &=& \vartheta_n \\ b &=& (p^{n-1}-1)/(p-1) & & & & \\ \underline{b} &=& (p^{n-2}-1)/(p-1) & & \overline{b} &=& 1\ , \end{array}$$

where b is e.g. calculated using [**28,** VI.§1, prop. 3]. We remark that $\mathrm{v}_s(p) = p^{n-2} = b - \underline{b}$. In particular, (3.1, 2.10) yield

$$\mathrm{H}^*(C_p \times C_{p-1}, U; S) \;\simeq\; \mathrm{H}^*(C_p, T; S)\ .$$

Hence, for instance,

$$\mathrm{H}^*((\mathbf{Z}/p^2)^*, \mathbf{Z}_{(p)}[\zeta_{p^2}]; \mathbf{Z}_{(p)}) \;\simeq\; \mathbf{Z}_{(p)}[h^{(1)}, h^{(2)}]/(ph^{(1)},\, ph^{(2)},\, (h^{(1)})^2)$$

(cf. 4.12).

Remark 5.5 In the same manner, (1.19, 4.10, 2.10) may be applied to certain cyclotomic function field extensions as defined by Carlitz and Hayes (cf. [**9**], [**14**]; see also [**18,** sec. 6.1]). Up to completion, these also form a particular case of Lubin-Tate-extensions; again, there is no need for completion, the formal group law being given by $F(X,Y) = X + Y$.

A The case C_{p^2}: a conjecture and an experiment

So far, we have essentially only treated the case of an extension with galois group C_p. The galois group C_{p^2} seems to yield a somewhat more involved twisted group ring, which we would like to illustrate in the case of $\mathbf{Z}_{(p)}[\pi_3] \wr C_{p^2}$. The calculations were carried out using Magma [**22**].

A.1 The conjectural situation

Suppose given a prime $p \geqslant 3$. We maintain the notation of §5.2 concerning π_n. Let

$$\begin{array}{rclcrcl} S &=& \mathbf{Z}_{(p)}\ , & & s &=& \pi_1 \;=\; p\ , \\ T &=& \mathbf{Z}_{(p)}[\pi_2]\ , & & t &=& \pi_2\ , \\ U &=& \mathbf{Z}_{(p)}[\pi_3]\ , & & u &=& \pi_3\ , \end{array}$$

and let $G = C_{p^2}$ be generated by the restriction of $\zeta_{p^3} \overset{\sigma}{\longmapsto} \zeta_{p^3}^{p+1}$ from $\mathbf{Q}(\zeta_{p^3})$ to $\mathbf{Q}(\pi_3)$. (The role of U in this appendix differs from the role of U in §2, where it has been a 'cohomologically inessential' extension of T.)

For some peculiar reason, t will not play a role at all. Instead, we consider a *Sen element* (cf. [**27,** Lem. 1])

$$v \ := \prod_{i\in[0,p-1]} u^{\sigma^i} \ .$$

The S-linear colength of the Wedderburn embedding

$$U \wr C_{p^2} \overset{\omega}{\hookrightarrow} \Gamma \ := \ \mathrm{End}_S\, U$$

is $p^2(p^2 + (p^2 - p - 2)/2)$ [**17,** (2.17)]. We fix the S-linear basis

$$(u^i v^j)_{i\in[0,p-1],\, j\in[0,p-1]} \ = \ (u^0v^0, u^0v^1, \ldots, u^0v^{p-1}, u^1v^0, \ldots, u^1v^{p-1},\ \ldots\ , u^{p-1}v^0, \ldots, u^{p-1}v^{p-1})$$

of U with respect to which we represent elements of Γ as matrices, i.e. by means of which we identify $\Gamma = S^{p^2\times p^2}$.

Remark A.1 *We have*

$$\begin{array}{rcl} \mathrm{v}_u(u^\sigma - u) & = & 1+1 \\ \mathrm{v}_u(v^\sigma - v) & = & 1+2p\ . \end{array}$$

Proof. The second congruence is equivalent to $\mathrm{v}_u(u^{\sigma^p} - u) = 1 + (p+1)$, so that both assertions follow from [**28,** VI.§1, prop. 3]. ▫

As usual, let Ξ denote the image of the Wedderburn embedding

$$\begin{array}{rcl} U \wr C_{p^2} & \overset{\omega}{\longrightarrow} & \Gamma \\ u & \longmapsto & (\dot u : x \longmapsto xu) \\ \sigma & \longmapsto & (\dot\sigma : x \longmapsto x^\sigma)\ , \end{array}$$

and let

$$\Lambda \ := \ \Big\{ f \in \Gamma \ :\ (Uu^i)f \subseteq Uu^i \text{ for all } i\in[0,p-1] \Big\} \ \subseteq\ \Gamma\ .$$

By (A.1) we obtain the intermediate ring

$$\begin{array}{rcl} \Xi \ \subseteq \ \Lambda^{\mathrm{D}} & := & \Lambda((\dot u, \dot v), (p-1,p-1), (2, 2p+1))_{\dot u\Lambda} \\ & = & \{ f\in\Lambda \ :\ D_{\dot u}^i \circ D_{\dot v}^j(f) \equiv_{\dot u^{2i+(2p+1)j}\Lambda} 0 \text{ for all } i,j\in[0,p-1]\} \\ \subseteq \ \Lambda\ , & & \end{array}$$

cf. (1.1). Presumably, this is the smallest intermediate ring between Ξ and Λ that can be defined by derivations.

Conjecture A.2

(i) *Given $\tau\in C_{p^2}$, we conjecture that*

$$2(v - v^\tau) + (u^\tau)^p(u - u^\tau)u^{p-1} + ((u^\tau)^{2p-1} - (u^\tau)^{2p})(u - u^\tau) \ \equiv_{u^{2p+3}}\ 0\ .$$

(ii) *Moreover, we conjecture that*

$$\Lambda^{\mathrm{D,\,E}} \;:=\; \{f \in \Lambda^{\mathrm{D}} \;:\; D_{\dot u}^{i} \circ E_{\dot u,\dot v}^{j}(f) \equiv_{\dot u^{2i+(2p+3)j}\Lambda} 0 \text{ for all } i, j \in [0, p-1]\}$$

contains Ξ, *where*

$$E_{\dot u,\dot v}(f) \;:=\; 2D_{\dot v}(f) + \dot u^{p} D_{\dot u}(f)\dot u^{p-1} + (\dot u^{2p-1} - \dot u^{2p})D_{\dot u}(f)\;.$$

Remark A.3

(i) Conjecture (A.2.i) holds for $p \in \{3, 5, 7\}$.

(ii) If $p = 3$, we obtain the colengths

$$\Xi \;=\; \Lambda^{\mathrm{D,\,E}} \overset{9\cdot(3-1)}{\subseteq} \Lambda^{\mathrm{D}} \overset{9\cdot(3\cdot(3+1)/2-1)}{\subseteq} \Lambda \overset{9\cdot(9-1)/2}{\subseteq} \Gamma\;.$$

(iii) If $p = 5$, we obtain the colengths

$$\Xi \overset{25\cdot(5-1)(5-3)/2}{\subseteq} \Lambda^{\mathrm{D,\,E}} \overset{25\cdot(5-1)}{\subseteq} \Lambda^{\mathrm{D}} \overset{25\cdot(5\cdot(5+1)/2-1)}{\subseteq} \Lambda \overset{25\cdot(25-1)/2}{\subseteq} \Gamma\;.$$

(iv) According to our wishful thinking, (A.2.i) should be part of a series of congruences in U that completely describes Ξ by adding the according congruences in Λ to the provisional definition of $\Lambda^{\mathrm{D,\,E}}$ given in (A.2.ii).

(v) Since $E_{\dot u,\dot v}^{j}$ is not a derivation, we do not know whether (A.2.i) implies (A.2.ii).

A.2 Simplifying $\dot u$, $\dot v$

A.2.1 The case $p = 3$

Suppose $p = 3$, i.e. $S = \mathbf{Z}_{(3)}$, $u = \pi_3 = (\zeta_{27} - 1)(\zeta_{27}^{-1} - 1)$, $v = u\, u^{\sigma} u^{\sigma^2} = (\zeta_{27} - 1)(\zeta_{27}^{-1} - 1)(\zeta_{27}^{4} - 1)\cdot$ $\cdot(\zeta_{27}^{-4} - 1)(\zeta_{27}^{16} - 1)(\zeta_{27}^{-16} - 1)$ and $U = \mathbf{Z}_{(3)}[\pi_3]$.

With respect to the basis $(u^0v^0, u^0v^1, u^0v^2, u^1v^0, u^1v^1, u^1v^2, u^2v^0, u^2v^1, u^2v^2)$, the multiplication by u on U is given by

$$\dot u \;=\; \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -\frac{2433}{7217} & \frac{3119}{1031} & -\frac{2151}{7217} & -\frac{6249}{7217} & -\frac{12225}{1031} & \frac{1168}{1031} & \frac{19680}{7217} & \frac{25540}{7217} & -\frac{306}{1031} \\ -\frac{4050}{7217} & \frac{3777}{1031} & \frac{593}{7217} & \frac{18804}{7217} & -\frac{20163}{1031} & \frac{987}{1031} & -\frac{6885}{7217} & \frac{64005}{7217} & -\frac{278}{1031} \\ -\frac{111}{7217} & -\frac{1026}{1031} & \frac{50145}{7217} & \frac{18189}{7217} & -\frac{14604}{1031} & -\frac{12306}{1031} & -\frac{6684}{7217} & \frac{29259}{7217} & \frac{6603}{1031} \end{array}\right]$$

$$\equiv_{\dot u^6\Lambda} \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & -1 \\ 3 & 0 & 0 & 0 & -3 & 0 & -3 & 0 & 0 \end{array}\right] =: \dddot u \;\equiv_{\dot u^3\Lambda} \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] =: \ddot u\;,$$

the multiplication by the Sen-element v by

$$\dot v \;=\; \left[\begin{array}{ccc|ccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2256}{1031} & -\frac{19548}{1031} & \frac{31788}{1031} & \frac{2484}{1031} & -\frac{3456}{1031} & -\frac{31626}{1031} & -\frac{495}{1031} & -\frac{7083}{1031} & \frac{10224}{1031} \\ \hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{27891}{7217} & -\frac{37620}{1031} & \frac{494226}{7217} & \frac{69981}{7217} & -\frac{19980}{1031} & -\frac{97587}{1031} & -\frac{11043}{7217} & -\frac{186021}{7217} & \frac{35910}{1031} \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{104247}{7217} & -\frac{137862}{1031} & \frac{1734570}{7217} & \frac{186300}{7217} & -\frac{7866}{1031} & -\frac{385245}{1031} & -\frac{15474}{7217} & -\frac{809595}{7217} & \frac{140031}{1031}
\end{array}\right]$$

$$\equiv_{\dot u^{9+3\cdot(3+1)/2}\Lambda} \left[\begin{array}{ccc|ccc|ccc}
0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
3&0&0&0&0&0&9&0&0\\ \hline
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&9&0&3&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&9&0&3&0&0
\end{array}\right] \;=:\; \dddot v \;\equiv_{\dot u^{9+3}\Lambda} \left[\begin{array}{ccc|ccc|ccc}
0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
3&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&3&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&3&0&0
\end{array}\right] \;=:\; \ddot v\,,$$

and the operation of σ by

$$\dot\sigma \;=\; \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{24876}{7217} & \frac{25976}{1031} & -\frac{31737}{7217} & \frac{55068}{7217} & -\frac{25392}{1031} & \frac{5540}{1031} & -\frac{16308}{7217} & \frac{36132}{7217} & -\frac{1176}{1031} \\
-\frac{62766}{1031} & \frac{413856}{1031} & -\frac{149176}{1031} & \frac{117030}{1031} & -\frac{647484}{1031} & \frac{228306}{1031} & -\frac{34326}{1031} & \frac{147615}{1031} & -\frac{51159}{1031} \\ \hline
0 & 0 & 0 & 4 & 0 & 0 & -1 & 0 & 0 \\
-\frac{35412}{7217} & \frac{29460}{1031} & -\frac{75624}{7217} & \frac{164160}{7217} & -\frac{57244}{1031} & \frac{17712}{1031} & -\frac{51540}{7217} & \frac{97432}{7217} & -\frac{4033}{1031} \\
-\frac{845298}{7217} & \frac{815757}{1031} & -\frac{3392322}{7217} & \frac{2508327}{7217} & -\frac{2001978}{1031} & \frac{810182}{1031} & -\frac{796137}{7217} & \frac{3442200}{7217} & -\frac{184868}{1031} \\ \hline
-\frac{210}{1031} & -\frac{2776}{1031} & -\frac{206}{1031} & \frac{13095}{1031} & \frac{563}{1031} & \frac{679}{1031} & -\frac{2419}{1031} & -\frac{347}{1031} & -\frac{162}{1031} \\
-\frac{9945}{1031} & \frac{52791}{1031} & -\frac{35236}{1031} & \frac{72018}{1031} & -\frac{171069}{1031} & \frac{62570}{1031} & -\frac{21546}{1031} & \frac{43586}{1031} & -\frac{14447}{1031} \\
-\frac{2471781}{7217} & \frac{2428305}{1031} & -\frac{12112977}{7217} & \frac{8656818}{7217} & -\frac{7025124}{1031} & \frac{2951589}{1031} & -\frac{2779557}{7217} & \frac{12296484}{7217} & -\frac{676060}{1031}
\end{array}\right] .$$

We observe that we may replace $\dot u$, $\dot v$ by $\ddot u$, $\ddot v$ resp. by $\dddot u$, $\dddot v$ to obtain

$$\Lambda^{\mathrm{D}} \;=\; \Lambda((\ddot u,\ddot v),(2,2),(2,7))_{\dot u\Lambda} \;=\; \{f\in\Lambda \;:\; D^i_{\ddot u}\circ D^j_{\ddot v}(f)\equiv_{\dot u^{2i+7j}\Lambda} 0 \text{ for all } i,j\in[0,2]\}$$

and

$$\Xi \;=\; \Lambda^{\mathrm{D,\,E}} \;=\; \{f\in\Lambda^{\mathrm{D}} \;:\; D^i_{\dddot u}\circ E^j_{\dddot u\,,\,\dddot v}(f)\equiv_{\dot u^{2i+9j}\Lambda} 0 \text{ for all } i,j\in[0,2]\}\,.$$

A.2.2 The case $p=5$

Suppose $p=5$, i.e. $S=\mathbf{Z}_{(5)}$, $u=\pi_3=(\zeta_{125}-1)(\zeta_{125}^{-1}-1)(\zeta_{125}^{57}-1)(\zeta_{125}^{-57}-1)$, $\zeta_{125}^{\sigma}=\zeta_{125}^{6}$, $v=u\,u^{\sigma}u^{\sigma^2}u^{\sigma^3}u^{\sigma^4}$ and $U=\mathbf{Z}_{(5)}[\pi_3]$. With respect to the basis

$$\begin{array}{ll}
(& u^0v^0, u^0v^1, u^0v^2, u^0v^3, u^0v^4, \\
 & u^1v^0, u^1v^1, u^1v^2, u^1v^3, u^1v^4, \\
 & u^2v^0, u^2v^1, u^2v^2, u^2v^3, u^2v^4, \\
 & u^3v^0, u^3v^1, u^3v^2, u^3v^3, u^3v^4, \\
 & u^4v^0, u^4v^1, u^4v^2, u^4v^3, u^4v^4, \quad)\,,
\end{array}$$

the matrix describing the multiplication by u on U reduces to

$$
\dot u \equiv_{\dot u^{31}\Lambda} \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
\hline
-5&-9&0&3&3&10&0&0&3&1&-10&10&0&1&2&0&-10&-1&1&1&-5&-6&1&0&2\\
15&-5&-9&0&3&5&10&0&0&3&10&-10&10&0&1&5&0&10&-1&1&10&-5&-6&1&0\\
15&15&-5&-9&0&15&5&10&0&0&5&10&-10&10&0&5&5&0&10&-1&0&10&-5&-6&1\\
0&15&15&-5&-9&0&15&5&10&0&0&5&10&-10&10&-5&5&5&0&10&5&0&10&-5&-6\\
-45&0&15&15&-5&0&0&15&5&10&50&0&5&10&-10&50&-5&5&5&0&-5&5&0&10&-5
\end{array}\right] =: \dddot{u}
$$

$$
\equiv_{\dot u^{5}\Lambda} \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
\hline
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
5&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right] =: \ddot{u}\,,
$$

and the matrix describing the multiplication by the Sen-element v reduces to

$$
\dot v \equiv_{\dot u^{25+5\cdot(5+1)/2}\Lambda} \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
5&75&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-25&0&0&0&25&25&0&0&0\\
\hline
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&25&25&0&0&5&75&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-50&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&-50&0&0&0&25&25&0&0&5&75&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&-50&0&0&0&25&25&0&0&5&75&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&-50&0&0&0&25&25&0&0&5&75&0&0&0
\end{array}\right] =: \dddot{v}
$$

$$\equiv_{\dot{u}^{25+5}\Lambda} \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
5&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&5&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&5&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&0&0&0&0
\end{array}\right] =: \;\ddot{v}\,.$$

We observe that we may replace $\dot{u}$, $\dot{v}$ by $\ddot{u}$, $\ddot{v}$ resp. by $\dddot{u}$, $\dddot{v}$ to obtain

$$\Lambda^{\mathrm{D}} \;=\; \Lambda((\ddot{u},\ddot{v}),(4,4),(2,11))_{\dot{u}\Lambda} \;=\; \{f \in \Lambda \;:\; D^{i}_{\ddot{u}} \circ D^{j}_{\ddot{v}}(f) \equiv_{\dot{u}^{2i+11j}\Lambda} 0 \text{ for all } i,j \in [0,4]\}$$

and

$$\Lambda^{\mathrm{D,\,E}} \;=\; \{f \in \Lambda^{\mathrm{D}} \;:\; D^{i}_{\dddot{u}} \circ E^{j}_{\dddot{u},\,\dddot{v}}(f) \equiv_{\dot{u}^{2i+13j}\Lambda} 0 \text{ for all } i,j \in [0,4]\}\;.$$

A.3 A spectral sequence

Alternatively, there is a Lyndon-Hochschild-Serre-Grothendieck spectral sequence that might perhaps help in calculating cohomology in the case C_{p^2} using the result in the case C_p instead of using the Wedderburn embedding of $U \wr C_{p^2}$ (cf. preceding sections). Due to varying ground rings, we have to apply a (hardly visible) modification to the usual Lyndon-Hochschild-Serre spectral sequence.

Let $S \subseteq T \subseteq U$ be an iterated finite extension of discrete valuation rings, $U|S$ galois with $H = \mathrm{Gal}(U|S)$, $T|S$ galois with $G = \mathrm{Gal}(T|S)$. Let N be the kernel of the restriction map $H \longrightarrow G$, so $N = \mathrm{Gal}(U|T)$ and $G \simeq H/N$. In this section, modules are not necessarily finitely generated.

The Grothendieck spectral sequence of the composition

$$\text{Mod-}U \wr H \xrightarrow{\;{}_{U\wr N}(U,-)\;} \text{Mod-}T \wr G \xrightarrow{\;{}_{T\wr G}(T,-)\;} \text{Mod-}S$$

is given by

$$(*) \qquad E_2^{m,n} \;:=\; \mathrm{Ext}^m_{T\wr G}(T, \mathrm{Ext}^n_{U\wr N}(U,X))\;,$$

where $X \in \text{mod-}U \wr H$, $m,n \geqslant 0$. For $X \in \text{Mod-}U \wr H$, the $T \wr G$-module structure on the image ${}_{U\wr N}(U,X)$ is induced by the ${}_{U\wr N}(U,X) \simeq {}_{TN}(T,X)$ and the left $T \wr G$-module structure on T.

To prove that it converges to $\mathrm{Ext}^{m+n}_{U\wr H}(U,X)$, it suffices to show that an injective $U \wr H$-module I is mapped to an injective module. In fact, for $Y \in \text{Mod-}T \wr G$ we calculate

$$\begin{array}{rcl}
{}_{T\wr G}(Y,\, {}_{U\wr N}(U,I)) & \simeq & {}_{T\wr G}(Y,\, {}_{TN}(T,I)) \\
 & \simeq & {}_{TN}(Y \otimes_{T\wr G} T, I) \\
 & \simeq & {}_{TN}(Y \otimes_{T\wr G} T,\, {}_{U\wr H}(U \wr H, I)) \\
 & \simeq & {}_{U\wr H}(Y \otimes_{T\wr G} T \otimes_{TN} U \wr H, I)\;,
\end{array}$$

so that the assertion follows by injectivity of I and by projectivity as a left $T \wr G$-module of

$$\begin{array}{rcl}
T \otimes_{TN} U \wr H & {}_{T\wr G}\!\simeq & T \otimes_{TN} (T \wr H)^{(\#N)} \\
 & {}_{T\wr G}\!\simeq & (T \wr G)^{(\#N)}\;.
\end{array}$$

Using adjunction, we may rewrite $(*)$ in the familiar shape

$$(**) \qquad E_2^{m,n} \;=\; \mathrm{H}^m(G, \mathrm{H}^n(N, X; T); S) \;\Longrightarrow\; \mathrm{H}^{m+n}(H, X; S)\ ,$$

applicable to $X \in \text{Mod-}U \wr H$ — so e.g. to $X = U$. Concerning the cup product, cf. [**4,** sec. 3.9].

Now, if $H = C_{p^2}$, $N = C_p$, $G = C_p$ and $X = U$, and our remaining conditions are satisfied (pure ramification, $\mathrm{v}(p)$ big enough), then (4.6) already calculates $E_2^{m,0}$ for $m \geqslant 0$. The first step to take when pursuing this spectral sequence approach, using $(*)$ rather than $(**)$, would be to calculate $\mathrm{Ext}^n_{U \wr N}(U, U)$ *as a* $T \wr G$*-module* for $n \geqslant 1$. We do not know whether this approach is actually viable.

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