On Elementary Properties of Crossed Modules

Bachelor's Thesis

Monika Truong
Universität Stuttgart

November 2015

Contents

Introduction

Crossed modules

A crossed module $[\![M, G]\!]$ consists of two groups M and G, an action of G on M and a group morphism $f\colon M \to G$ that satisfies the conditions

$$\text{(CM1) } (m^g)f = (mf)^g \quad \text{and} \quad \text{(CM2) } m^n = m^{nf} \ ,$$

for $m, n \in M$ and $g \in G$.

The category of groups is equivalent to the homotopy category of connected CW-spaces X that have $\pi_k(X) \simeq 1$ for $k \geqslant 2$, i.e. for which only $\pi_1(X)$ is allowed to be nontrivial. Similarly, the category of crossed modules has a homotopy category that is equivalent to the category of CW-spaces X that have $\pi_k(X) \simeq 1$ for $k \geqslant 3$, i.e. for which only $\pi_1(X)$ and $\pi_2(X)$ are allowed to be nontrivial. So just as groups model homotopy types with only $\pi_1(X)$ nontrivial, crossed modules model homotopy types with only $\pi_1(X)$ and $\pi_2(X)$ nontrivial; cf. [**5**], [**1,** Theorem 2.4.8].

Our goal is to transfer some elementary concepts and assertions from group theory to the theory of crossed modules.

Simple crossed modules

A nontrivial crossed module X is called simple if its only normal crossed submodules are 1 and X; cf. Definitions 21, 34. We can sort the simple crossed modules as follows; cf. Theorem 40.

- $[\![G, G]\!]$ with $G \xrightarrow{\text{id}} G$, where G is simple and non-abelian.
- $[\![1, K]\!]$, where K is simple.
- $[\![M, 1]\!]$, where M is cyclic and of prime order.

This proposition is shown using standard short exact sequences for crossed modules; cf. Definition 32. Of course, to classify the crossed modules appearing in these three cases, one would need to know a classification of simple groups, not necessarily finite; we do not treat this problem.

Jordan-Hölder-Schreier-Zassenhaus

The classical procedure for composition series of groups works as follows.

A group G may have no, only one or even more than one composition series. However, the Jordan-Hölder Theorem states that any two composition series of a group G are equivalent. That is, they have the same length and the same composition factors, up to permutation and isomorphism. Note that this theorem does not ensure the existence of a composition series. But if G is finite, then a composition series for G exists.

This assertion is shown by using Schreier's Refinement Theorem; it says that any two subnormal series of a given group G can be refined to equivalent subnormal series by inserting suitable subgroups into the series.

The Zassenhaus Lemma connects the subfactors appearing in Schreier's refinements. This lemma is sometimes called the "Butterfly Lemma" because the diagram that illustrates the relations of the involved subgroups resembles a butterfly.

We show an analogous Jordan-Hölder Theorem for crossed modules; cf. Theorem 53. Its proof runs parallel to the proof of the classical version; cf. [**4,** p. 20–22].

Orbit Lemma

$[\![M, G]\!]$-crossed sets

Analogous to the notion of a G-set for a group G, we define the notion of an $[\![M, G]\!]$-crossed set for a crossed module $[\![M, G]\!]$. To this end, we use the semidirect product $G \ltimes M$. An $[\![M, G]\!]$-crossed set consists of a $(G \ltimes M)$-set U, a G-set V and maps $\sigma, \tau \colon U \to V$, $\iota \colon V \to U$, satisfying certain compatibilities; cf. Definition 59. Such an $[\![M, G]\!]$-crossed set is written $[\![U, V]\!]_{\text{set}}$. For example, if $[\![N, H]\!] \leqslant [\![M, G]\!]$ is a crossed submodule, the $[\![M, G]\!]$-crossed set $[\![N, H]\!] \backslash\!\backslash [\![M, G]\!]$ is given by $U = (H \ltimes N)\backslash(G \ltimes M)$ and $V = H\backslash G$; cf. Lemma 63.

With the notion of an $[\![M, G]\!]$-set for a crossed module $[\![M, G]\!]$, we establish an Orbit Lemma for crossed modules, valid for certain orbits. Suppose given an $[\![M, G]\!]$-crossed set $[\![U, V]\!]_{\text{set}}$. Suppose given $v \in V$. We form the orbit vG in V. We map the element v via ι to U and form the orbit $(v\iota)(G \ltimes M)$ of $v\iota \in U$ under $(G \ltimes M)$. They form the $[\![M, G]\!]$-crossed set $[\![(v\iota)(G \ltimes M), vG]\!]_{\text{set}} =: v[\![M, G]\!]$, called the orbit of v under $[\![M, G]\!]$; cf. Lemma 68.

We obtain an isomorphism of $[\![M,G]\!]$-crossed sets

$$(\zeta,\eta)\colon\ \mathrm{C}_{[\![M,G]\!]}(v)\backslash\!\!\backslash[\![M,G]\!] \xrightarrow{\ \sim\ } v[\![M,G]\!]\ ,$$

where $\mathrm{C}_{[\![M,G]\!]}(v)$ is the centralizer of v in $[\![M,G]\!]$, cf. Lemma 69; cf. Proposition 70.

However, it turns out that the $(G \ltimes M)$-orbits of the elements of the form $v\iota$, where $v \in V$, do not cover the whole $(G \ltimes M)$-set U in general; cf. Example 83. As a consequence, we cannot classify the $[\![M,G]\!]$-sets analogously to the classification of G-sets as disjoint unions of orbits isomorphic to G-sets of the form $U\backslash G$, where $U \leqslant G$.

$[\![M,G]\!]$-crossed categories

An $[\![M,G]\!]$-crossed category is a category $\mathcal{C}$, for which $[\![\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}}$ carries the structure of an $[\![M,G]\!]$-crossed set with $(\boldsymbol{s},\boldsymbol{i},\boldsymbol{t}) = (\sigma,\iota,\tau)$ such that the composition satisfies certain compatibilities; cf. Definition 71. For example, a crossed module $[\![M,G]\!]$ gives rise to a category $\mathcal{C}[\![M,G]\!]$ with $\mathrm{Ob}\big(\mathcal{C}[\![M,G]\!]\big) = G$, $\mathrm{Mor}\big(\mathcal{C}[\![M,G]\!]\big) = G \ltimes M$; cf. [**2**]; cf. also [**6,** (5.25), (5.10)]. This category $\mathcal{C}[\![M,G]\!]$ is in fact an $[\![M,G]\!]$-crossed category in two ways: via multiplication and via conjugation; cf. Remark 73.

Moreover, given a crossed submodule $[\![N,H]\!] \leqslant [\![M,G]\!]$, then a factor construction yields an $[\![M,G]\!]$-crossed category $[\![N,H]\!]\,{}_{\mathcal{C}}\!\backslash\!\!\backslash[\![M,G]\!]$; cf. Lemma 76. For example, we have $[\![1,1]\!]\,{}_{\mathcal{C}}\!\backslash\!\!\backslash[\![M,G]\!] \simeq \mathcal{C}[\![M,G]\!]$.

Just as for $[\![M,G]\!]$-crossed sets, we formulate an Orbit Lemma for $[\![M,G]\!]$-crossed categories, which is an analog to the Orbit Lemma for $[\![M,G]\!]$-crossed sets. Suppose given $v \in V$. The orbit $v[\![M,G]\!]$ carries the structure of an $[\![M,G]\!]$-crossed category; cf. Lemma 81. We have an isomorphism of $[\![M,G]\!]$-crossed categories

$$(\zeta,\eta)\colon\ \mathrm{C}_{[\![M,G]\!]}(v)\,{}_{\mathcal{C}}\!\backslash\!\!\backslash[\![M,G]\!] \xrightarrow{\ \sim\ } v[\![M,G]\!]\ ,$$

where $\mathrm{C}_{[\![M,G]\!]}(v)$ is the centralizer of v in $[\![M,G]\!]$, cf. Lemma 69; cf. Proposition 82.

Conventions

Sets and Mappings

Let X, Y and Z be sets.

- For $a, b \in \mathbb{Z}$, we write $[a, b] := \{z \in \mathbb{Z}\colon a \leqslant z \leqslant b\}$.
- We write $\mathcal{P}(X)$ for the power set of X.
- Let $(X, \leqslant)$ be a partially ordered set. We say that an element $x \in X$ is

  maximal if $\forall y \in X\colon (x \leqslant y \Rightarrow x = y)$,
  terminal if $\forall y \in X\colon (y \leqslant x)$,
  minimal if $\forall y \in X\colon (y \leqslant x \Rightarrow x = y)$,
  initial if $\forall y \in X\colon (x \leqslant y)$.

- Let $f\colon X \to Y$ be a map. We write maps on the right, i.e. f maps $x \in X$ to $xf \in Y$.
- Composition of maps is written on the right, i.e. given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, their composition is written $X \xrightarrow{fg} Z$; cf. also Reminder 4.
- Unary maps are evaluated before binary maps. E.g. for a map $M \xrightarrow{f} G$ from a set M to a group G, we write $g \cdot mf := g \cdot (mf)$, for $m \in M$, $g \in G$.
- Let $f\colon X \to Y$ be a map. Let $X' \subseteq X$ and $Y' \subseteq Y$ such that $X'f \subseteq Y'$. The restriction of f to X' and Y' is written $f\big|_{X'}^{Y'}\colon X' \to Y',\ x' \mapsto x'f$.
  If $Y' = Y$, we also write $f\big|_{X'} := f\big|_{X'}^{Y}$. If $X' = X$, we also write $f\big|^{Y'} := f\big|_{X}^{Y'}$.
- Sometimes, we denote by $X' \hookrightarrow X$ the embedding of a subset X' in a set X.

Groups

Let G be a group.

- The identity element of G is denoted by $1 := 1_G$. The trivial subgroup is denoted by $1 := \{1\}$.

- Suppose given a subset $S \subseteq G$. The subgroup generated by S is denoted by $\langle S \rangle \leqslant G$. If $S = \{s\}$, $s \in G$, we also denote $\langle s \rangle := \langle \{s\} \rangle$.

- We write $g^- := g^{-1}$ for the inverse element of $g \in G$.

- We write $g^h := h^- g\ h$ for $h, g \in G$.

- For $u, v \in G$ we write $[u, v] := u^- v^- uv$ for their commutator. For $U, V \leqslant G$ we write $[U, V] := \big\langle [u, v] \colon u \in U, v \in V \big\rangle \leqslant G$ for their commutator subgroup.

- Given $x \in G$, we write $\mathrm{C}_G(x) = \{g \in G \colon xg = gx\}$ for the centralizer of x in G.

- Sometimes, we denote by $N \overset{\trianglelefteq}{\hookrightarrow} M$, the embedding of N in G and $N \trianglelefteq G$.

- Let $H \leqslant G$ be a subgroup. We denote $H \backslash G := \{Hg \colon g \in G\}$ for the set of right cosets of H in G.

- The symmetric group on a set X is denoted by S_X . If $X = [1, n]$, for some $n \in \mathbb{N}$, then we also denote $\mathrm{S}_n := S_{[1,n]}$.

Reminder 1 (Conjugation map) Suppose given a group G.

(1) For any $h \in G$ we have a group isomorphism

$$\begin{aligned} c_h \colon G &\longrightarrow G \\ g &\longmapsto g^h := h^- g\ h\ , \end{aligned}$$

where $c_{h_1} c_{h_2} = c_{(h_1 h_2)}$ and $(c_h)^- = c_{h^-}$ for $h_1, h_2, h \in G$. We call c_h the *conjugation map* of h on G.

(2) Suppose given a normal subgroup $N \trianglelefteq G$. Then we have a group morphism given by

$$\begin{aligned} c_N \colon G &\longrightarrow \mathrm{Aut}\,(N) \\ h &\longmapsto c_h\big|_N^N\ . \end{aligned}$$

Reminder 2 (Exact sequence) Suppose given $n \in \mathbb{N}$ and groups G_i for $i \in [1, n]$. Suppose given group morphisms $\varphi_i \colon G_i \longrightarrow G_{i+1}$ for $i \in [1, n-1]$.

The sequence of groups and group morphisms

$$G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} G_n$$

is called a exact if $\operatorname{im} \varphi_i = \ker \varphi_{i+1}$ holds for $i \in [1, n-1]$. An exact sequence of the form

$$1 \longrightarrow G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3 \longrightarrow 1$$

is called short exact sequence.

Reminder 3 (Orbit) A set X together with a group G and a group morphism

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{S}_X \\ g & \longmapsto & (x \mapsto x * g) \end{array}$$

is called G-set. The group morphism $G \to \mathrm{S}_X$ is also called (right) action of G on X. For $x \in X$ we call $x * G := \{x * g \mid g \in G\}$ the *orbit of* x under G. Sometimes, we denote for short $xG := x * G$.

Reminder 4 (Category) By a category we understand a small category (with respect to a fixed universe). So a category $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Mor}(\mathcal{C})$, maps

$$\begin{array}{ccc} \mathrm{Mor}(\mathcal{C}) & \xrightarrow{s} & \mathrm{Ob}(\mathcal{C}) \\ \big(X \xrightarrow{f} Y\big) & \longmapsto & X \end{array} \quad \text{(source)}$$

$$\begin{array}{ccc} \mathrm{Mor}(\mathcal{C}) & \xleftarrow{i} & \mathrm{Ob}(\mathcal{C}) \\ \mathrm{id}_X & \longleftarrow\!\shortmid & X \end{array} \quad \text{(identity)}$$

$$\begin{array}{ccc} \mathrm{Mor}(\mathcal{C}) & \xrightarrow{t} & \mathrm{Ob}(\mathcal{C}) \\ \big(X \xrightarrow{f} Y\big) & \longmapsto & Y \end{array} \quad \text{(target)}\ ,$$

and a composition

$$\begin{array}{rcl} \{(f,g) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C}) : \ f\boldsymbol{t} = g\boldsymbol{s}\} & \longrightarrow & \mathrm{Mor}(\mathcal{C}) \\ \big(X \xrightarrow{f} Y,\ Y \xrightarrow{g} Z\big) & \longmapsto & \big(X \xrightarrow{f \blacktriangle g} Z\big) = \big(X \xrightarrow{fg} Z\big) \end{array}$$

which is associative and for which id_X is neutral for $X \in \mathrm{Ob}(\mathcal{C})$.

1 Basics

1.1 Crossed modules

Definition 5 (Crossed module) Suppose given groups M and G. Suppose we are given an action of G on M; namely, we have a group morphism

$$\begin{array}{rcrcl} \alpha & : & G & \longrightarrow & \mathrm{Aut}\,(M) \\ & & g & \longmapsto & g\alpha\ . \end{array}$$

When an element $g \in G$ is applied to an element $m \in M$, we write $m^g := m(g\alpha)$.

Further, let $f\colon M \longrightarrow G$ be a group morphism that satisfies

(CM1) $(m^g)\,f = (mf)^g$
(CM2) $m^n \ = \ m^{nf}$ \quad (*Peiffer identity*),

for $n, m \in M$ and $g \in G$. Such a quadruple (M, G, α, f) is called *crossed module*.

Often, we abbreviate $[\![M, G]\!] := \big(M, G, \alpha, f\big)$. If unambiguous, we denote $1 := [\![1, 1]\!]$ for the *trivial* crossed module.

Remark 6 Note that our notation for conjugation and for the action of G on M coincide. To avoid confusion we note that the axiom (CM1) should be read as $\big(m(g\alpha)\big)f = g^-(mf)g$ and (CM2) should be read as $n^-mn = m\big((nf)\alpha\big)$.

Lemma 7 *Suppose given a crossed module* $\big(M, G, \alpha, f\big)$.

(1) *We have* $[M, \ker f] = 1$. *In particular, the kernel* $\ker f$ *is abelian.*

(2) *We have* $Mf \trianglelefteq G$.

Proof. Ad (1). Suppose given $m \in M$, $k \in \ker f$. We have

$$k^{-}mk = m^{k} \overset{\text{(CM2)}}{=} m^{kf} = m^{1} = m \ .$$

It follows that $m^{-}k^{-}mk = 1$.

Ad (2). Suppose given $mf \in Mf$, $g \in G$. We have

$$g^{-}(mf)g = (mf)^{g} \overset{\text{(CM1)}}{=} (m^{g})f \in Mf \ .$$

It follows that $Mf \trianglelefteqslant G$. $\square$

1.2 Examples of crossed modules

Example 8 Suppose given a group M. Consider the group morphism

$$\begin{aligned} c\colon M &\longrightarrow \mathrm{Aut}\,(M) \\ m &\longmapsto (c_m\colon x \mapsto x^{m}) \ . \end{aligned}$$

Then we have a crossed module given by $[\![M, \mathrm{Aut}\,(M)]\!] = (M, \mathrm{Aut}\,(M), \mathrm{id}_{\mathrm{Aut}\,(M)}, c)$, since the map c satisfies (CM1) and (CM2):

Ad (CM1). Suppose given $g \in \mathrm{Aut}\,(M)$, $m \in M$. Note that we have

$$m^{g} = (m)(g)\mathrm{id}_{\mathrm{Aut}\,(M)} = mg \ .$$

Suppose given $n \in M$. We have

$$\begin{aligned} (n)\,(m^{g})\,c = (n)\,(mg)\,c = (n)c_{mg} &= (mg)^{-}\,n\,\ (mg) = \big(m^{-}g\big)\,n\,(mg) \\ &= (m^{-}g)(ng^{-}g)(mg) = \big(m^{-}(ng^{-})m\big)\,g = (ng^{-})c_m\ g = (n)g^{-}\,(mc)\,g \\ &= (n)\,(mc)^{g} \ . \end{aligned}$$

Ad (CM2). Supose given $m,\ m' \in M$. We have

$$m^{m'} = m'^{-}m\ m' = (m)\,(m'c) = m^{m'c} \ .$$

Example 9 Let G be a group and let $N \trianglelefteqslant G$ be a normal subgroup. Consider the group morphism

$$\begin{aligned} c_N\colon G &\longrightarrow \mathrm{Aut}\,(N) \\ g &\longmapsto (c_g\big|_N^N\colon n \mapsto n^{g}) \ . \end{aligned}$$

Then we have a crossed module $[\![N,G]\!] = (N, G, c_N, \mathrm{id}_G\big|_N)$:

For $g \in G$, the map $c_g\big|_N^N$ is a group isomorphism, hence, c_N is well-defined; cf. Reminder 1.

We have yet to show that the map $\iota := \mathrm{id}_G\big|_N$ satisfies (CM1) and (CM2).

Suppose given $n, m \in M$.

Ad (CM1). We have

$$(n^g)\,\iota = \big(n(gc_N)\big)\iota = g^- ng = \big((n)\iota\big)^g \ .$$

Ad (CM2). We have

$$m^n = m(nc_N) = (m)\big((n\iota)c_N\big)m^{n\iota} \ .$$

Remark 10

(1) Every group G has its trivial subgroup 1 consisting of just the identity element of G. This subgroup is always a normal subgroup. Therefore, according to Example 9, we have the crossed module $[\![1,G]\!] = \big(1, G, c_1, \mathrm{id}_G\big|_1\big)$.

(2) Every group G contains the whole group G as a normal subgroup. Therefore, according to Example 9, we have the crossed module $[\![G,G]\!] = \big(G, G, c, \mathrm{id}_G\big)$.

Example 11 Let M be an abelian group, written multiplicatively. Consider the group morphisms

$$\begin{array}{rlcl} \iota\colon & 1 & \longrightarrow & \mathrm{Aut}\,(M) \\ & 1 & \longmapsto & \mathrm{id}_M \end{array} \qquad \begin{array}{rlcl} \kappa\colon & M & \longrightarrow & 1 \\ & m & \longmapsto & 1\ . \end{array}$$

Then have a crossed module $[\![M,1]\!] = (M, 1, \iota, \kappa)$. We have to show that κ satisfies (CM1) and (CM2). Note that we have $1\iota = \mathrm{id}_M$ and $1 = m\kappa$, for $m \in M$.

Ad (CM1). For $m \in M$ we have

$$(m^1)\kappa = 1 = (1)^1 = (m\kappa)^1 \ .$$

Ad (CM2). For $m, n \in M$ we have

$$m^n = n^- m\ n = m\ n^- n = m = (m)\mathrm{id}_M = m(1\iota) = m\big((n\kappa)\iota\big) = m^{n\kappa} \ .$$

Definition 12

(1) Let G be a group. We define $\mathrm{X}_{\mathrm{contr}}(G) := [\![G, G]\!] = \big(G, G, c, \mathrm{id}_G\big)$; cf. Remark 10.(1).

(2) Let K be a group. We define $\mathrm{X}_1(K) := [\![1, K]\!] = \big(1, K, c_1, \mathrm{id}_K\big|_1\big)$; cf. Remark 10.(2).

(3) Let M be an abelian group. We define $\mathrm{X}_2(M) := [\![M, 1]\!] = \big(M, 1, \iota, \kappa\big)$; cf. Example 11.

1.3 Crossed module morphisms

Definition 13 (Crossed module morphism)
Suppose given crossed modules $\big(M, G, \alpha, f\big)$ and $\big(\tilde{M}, \tilde{G}, \tilde{\alpha}, \tilde{f}\big)$.
Let $\lambda\colon M \longrightarrow \tilde{M}$ and $\mu\colon G \longrightarrow \tilde{G}$ be group morphisms that satisfy the following properties (i) and (ii).

(i) We have $\lambda\tilde{f} = f\mu$, i.e. the following diagram is commutative

$$\begin{tikzcd} M \arrow[r, "\lambda"] \arrow[d, "f"'] & \tilde{M} \arrow[d, "\tilde{f}"] \\ G \arrow[r, "\mu"'] & \tilde{G} \end{tikzcd} .$$

(ii) For $m \in M$ and $g \in G$, we have

$$(m^g)\lambda = (m\lambda)^{g\mu} .$$

We call (λ, μ) a *morphism of crossed modules.*

Lemma 14 (Identity and composition of crossed module morphisms)

(1) *Let* $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ *be a crossed module. Then* $(\mathrm{id}_M, \mathrm{id}_G)$ *is the identity crossed module morphism of* $[\![M, G]\!]$*.*

(2) *Suppose given crossed modules* $[\![M_i, G_i]\!] = \big(M_i, G_i, \alpha_i, f_i\big)$ *for* $i \in [1,3]$.

For $j \in [1,2]$*, suppose given crossed module morphisms*

$$(\lambda_j, \mu_j)\colon [\![M_j, G_j, f_j]\!] \longrightarrow [\![M_{j+1}, G_{j+1}]\!] \ .$$

We have a crossed module morphism given by

$$(\lambda, \mu) := (\lambda_1, \mu_1)(\lambda_2, \mu_2) = (\lambda_1\lambda_2, \mu_1\mu_2)\colon [\![M_1, G_1]\!] \longrightarrow [\![M_3, G_3]\!] \ .$$

Proof. Ad (1). We have

$$\mathrm{id}_M f = f = f \ \mathrm{id}_G \ .$$

Hence the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}_M"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ G \arrow[r, "\mathrm{id}_G"'] & G \ . \end{tikzcd}$$

For $m \in M$ and $g \in G$ we have

$$(m^g)\mathrm{id}_M = (m \ \mathrm{id}_M)^{g \ \mathrm{id}_G} \ .$$

Ad (2). The situation is given as follows.

$$\begin{tikzcd} M_1 \arrow[r, "\lambda_1"] \arrow[d, "f_1"'] & M_2 \arrow[r, "\lambda_2"] \arrow[d, "f_2"'] & M_3 \arrow[d, "f_3"'] \\ G_1 \arrow[r, "\mu_1"'] & G_2 \arrow[r, "\mu_2"'] & G_3 \end{tikzcd}$$

We have commutative squares on the left-hand side and on the right-hand side. Therefore, we have $\lambda_1 f_2 = f_1\mu_1$ and $\lambda_2 f_3 = f_2\mu_2$.

It follows that

$$f_1\mu = f_1\mu_1\mu_2 = \lambda_1 f_2\mu_2 = \lambda_1\lambda_2 f_3 = \lambda f_3 \ .$$

Hence the following diagram commutes

$$\begin{tikzcd}
M_1 \arrow[r, "\lambda"] \arrow[d, "f_1"'] & M_3 \arrow[d, "f_3"] \\
G_1 \arrow[r, "\mu"'] & G_3 \ .
\end{tikzcd}$$

For $m_j \in M_j$ and for $g_j \in G_j$ where $j \in [1,2]$, we have

$$(m_1^{g_1})\lambda = (m_1^{g_1})\lambda_1\lambda_2 = (\underbrace{m_1\lambda_1}_{\in M_2})^{g_1\lambda_1}\lambda_2 = (m_1\lambda_1\lambda_2)^{g_1\mu_1\mu_2} = (m_1\lambda)^{g_1\mu} \ .$$

$\square$

Lemma 15 *Let $[\![M,G]\!] = \big(M,G,\alpha,f\big)$, let $[\![L,E]\!] = \big(L,E,\gamma,d\big)$ be crossed modules. Suppose we have a crossed module morphism $(\lambda,\mu)\colon [\![M,G]\!] \longrightarrow [\![L,E]\!]$ where λ and μ are bijective, i.e. we have*

$$\begin{tikzcd}
M \arrow[r, "\lambda", "\sim"'] \arrow[d, "f"'] & L \arrow[d, "d"] \\
G \arrow[r, "\sim", "\mu"'] & E \ .
\end{tikzcd}$$

Then we have a crossed module morphism from $[\![L,E]\!]$ to $[\![M,G]\!]$ given by (λ^-,μ^-).

Proof. We have

$$\lambda d = f\mu \quad \Leftrightarrow \quad d = \lambda^- f\mu \quad \Leftrightarrow \quad d\mu^- = \lambda^- f \ .$$

Suppose given $p \in L,\ e \in E$. Since λ and μ are bijective there exist $m \in M$ and $g \in G$ such that $m = p\lambda^-$ and $g = e\mu^-$. We have

$$\begin{aligned}
(m^g)\lambda = (m\lambda)^{g\mu} \quad &\Leftrightarrow \quad ((p\lambda^-)^{e\mu^-})\lambda = (p\lambda^-\lambda)^{e\mu^-\mu} \\
&\Leftrightarrow \quad ((p\lambda^-)^{e\mu^-})\lambda = p^e \\
&\Leftrightarrow \quad (p\lambda^-)^{e\mu^-} = (p^e)\lambda^- \ .
\end{aligned}$$

□

Definition 16 (Crossed module isomorphism)
A crossed module morphism $(\lambda, \mu)\colon [\![M, G]\!] \to [\![L, E]\!]$ is called *injective* if λ and μ are injective. It is called *surjective* if λ and μ are surjective. We call (λ, μ) *bijective* if λ and μ are bijective.

We say two crossed modules $[\![M, G]\!]$, $[\![L, E]\!]$ are *isomorphic* if there exists a bijective crossed module morphism between $[\![M, G]\!]$ and $[\![L, E]\!]$.

1.4 Crossed submodules

Definition 17 (Crossed submodule) A crossed module $[\![N, H]\!] = \big(N, H, \beta, k\big)$ is called a *crossed submodule* of a crossed module $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ if the following properties hold.

(i) We have $N \leqslant M$ and $H \leqslant G$.

(ii) We have $k = f\big|_N^H$, i.e. the map k is the restriction of f to N and H.

(iii) We have $n(h\beta) = n(h\alpha)$ for $n \in N,\ h \in H$.

We write $[\![N, H]\!] \leqslant [\![M, G]\!]$ to indicate that $[\![N, H]\!]$ is a crossed submodule of $[\![M, G]\!]$. We write $[\![N, H]\!] < [\![M, G]\!]$ if $[\![N, H]\!] \leqslant [\![M, G]\!]$ and $[\![N, H]\!] \neq [\![M, G]\!]$.

Remark 18 Let $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ be a crossed module. Suppose given $N \leqslant M$, $H \leqslant G$ such that $nf \in H$ and $n^h \in N$, for $n \in N$ and $h \in H$.

Let $\beta\colon H \to \mathrm{Aut}\,(N)$, $h \mapsto \big(h\beta\colon n \mapsto n(n\beta) := n(h\alpha)\big)$. Let $k := f\big|_N^H$.

This defines a crossed module $[\![N, H]\!] = \big(N, H, \beta, k\big)$. We have $[\![N, H]\!] \leqslant [\![M, G]\!]$.

Note that for $n \in N$ and $h \in H$, we have $n^h = n(h\beta) = n(h\alpha) = n^h$, justifying our abuse of notation.

Proof. By assumption, we have $nf \in H$ for $n \in N$. Therefore, $k = f\big|_N^H$ exists.

Suppose given $h \in H$. By assumption, we have $n(h\alpha) = n^h \in N$ for $n \in N$. Therefore $h\alpha\big|_N^N$ exists. Its inverse is given by $(h\alpha\big|_N^N)^- = (h^-\alpha)\big|_N^N$. So $(h\alpha)\big|_N^N$ is bijective. As a restriction of the group morphism $h\alpha$, also $(h\alpha)\big|_N^N$ is a group morphism. So therefore, $h\beta := h\alpha\big|_N^N \in \operatorname{Aut}(N)$ is well-defined. Note that $n(h\beta) = n(h\alpha)$ for $n \in N$.

This defines a map $\beta \colon H \to \operatorname{Aut}(N)$, $h \mapsto h\beta$. It is a group morphism, since given h, $\tilde{h} \in H$ and $n \in N$, we obtain

$$n\big((h\tilde{h})\beta\big) = n\big((h\tilde{h})\alpha\big) = n(h\alpha)(\tilde{h}\alpha) = n(h\alpha)(\tilde{h}\beta) = n(h\beta)(\tilde{h}\beta) \ .$$

Ad (CM1). Suppose given $n \in N$ and $h \in H$. Then

$$(n^h)k = n(h\beta)k = n(h\beta)f = n(h\alpha)f \overset{\text{(CM1)}}{=} (nf)^h = (nk)^h \ .$$

Ad (CM2). Suppose given n, $\tilde{n} \in N$. Then

$$n^{\tilde{n}} \overset{\text{(CM2)}}{=} n^{\tilde{n}f} = n(\tilde{n}f\alpha) = n(\tilde{n}k\alpha) = n(\tilde{n}k\beta) = n^{\tilde{n}k} \ .$$

□

Remark 19 Let $[\![N, H]\!] = \big(N, H, \beta, k\big)$ and $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ be crossed modules. Suppose we have $[\![N, H]\!] \leqslant [\![M, G]\!]$. Then we have a crossed module morphism $(\iota, \kappa) \colon [\![N, H]\!] \to [\![M, G]\!]$, where $\iota := \mathrm{id}_M\big|_N$ and $\kappa := \mathrm{id}_G\big|_H$, called the *inclusion morphism* of $[\![N, H]\!]$ in $[\![M, G]\!]$.

Proof. The diagram

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[d, "k"'] & M \arrow[d, "f"] \\ H \arrow[r, "\kappa"'] & G \end{tikzcd}$$

commutes, since, for $n \in N$, we have

$$(n)(\iota f) = nf = nk = (n)(k\kappa) \ .$$

Further, for $n \in N$, $h \in H$, we have

$$(n^h)\iota = n^h = (n\iota)^{h\kappa} \ .$$

$\square$

Remark 20 Let $[\![M, G]\!] = \big(M, G, \alpha, f\big)$, $[\![N, H]\!] = \big(N, H, \beta, k\big)$, $[\![\tilde{N}, \tilde{H}]\!] = \big(\tilde{N}, \tilde{H}, \tilde{\beta}, \tilde{k}\big)$ be crossed modules. Suppose we have $[\![N, H]\!]$, $[\![\tilde{N}, \tilde{H}]\!] \leqslant [\![M, G]\!]$ and $\tilde{N} \subseteq N$, $\tilde{H} \subseteq H$.

Then $[\![\tilde{N}, \tilde{H}]\!] \leqslant [\![N, H]\!]$.

Proof. From $N \leqslant M$, $\tilde{N} \leqslant M$ and $\tilde{N} \subseteq N$ we infer that $\tilde{N} \leqslant N$. From $H \leqslant G$, $\tilde{H} \leqslant G$ and $\tilde{H} \subseteq H$ we infer that $\tilde{H} \leqslant H$.

For $\tilde{n} \in \tilde{N}$ we have $\tilde{n}\tilde{k} = \tilde{n}f = \tilde{n}k \in H$.

For $\tilde{n} \in \tilde{N}$ and $\tilde{h} \in \tilde{H}$ we have $\tilde{n}(\tilde{h}\tilde{\beta}) = \tilde{n}(\tilde{h}\alpha) = \tilde{n}(\tilde{h}\beta)$. $\square$

Concerning Definition 21, we shall follow [**3,** p. 170].

Definition 21 (Normal crossed submodule) A crossed submodule $[\![N, H]\!]$ of $[\![M, G]\!]$ is called *normal* if the following assertions (i),(ii) and (iii) hold.

(i) We have $N \trianglelefteqslant M$ and $H \trianglelefteqslant G$.

(ii) We have $m^- m^h \in N$ for $m \in M,\ h \in H$.

(iii) We have $n^g \in N$ for $n \in N,\ g \in G$.

We write $[\![N, H]\!] \trianglelefteqslant [\![M, G]\!]$ to indicate that $[\![N, H]\!]$ is a normal crossed submodule of $[\![M, G]\!]$.

Remark 22 From the property (iii) in Definition 21 it follows that $N \trianglelefteqslant M$. Hence this requirement could be dropped from (i) without changing the definition.

Proof. Suppose we have $n^g \in N$ for $n \in N$, $g \in G$. For $m \in M$, we have $mf \in G$, and thus

$$m^- n\ m = n^m \overset{\text{(CM2)}}{=} n^{mf} \in N \ .$$

This shows $N \trianglelefteqslant M$. $\square$

Remark 23 A crossed module $[\![M, G]\!]$ contains the trivial crossed module $[\![1, 1]\!]$ and the whole crossed module $[\![M, G]\!]$ as normal crossed submodules.

Proof. We have $1 \trianglelefteqslant M, G$. In particular, we have $1 \in M, G$. Hence $[\![1, 1]\!] \trianglelefteqslant [\![M, G]\!]$.

We have $M \trianglelefteqslant M$ and $G \trianglelefteqslant G$. We have $m^- m^g \in M$ and $m^g \in M$ for $m \in M$, $g \in G$. Hence $[\![M, G]\!] \trianglelefteqslant [\![M, G]\!]$. $\square$

Remark 24 Suppose we have $[\![N, H]\!] \leqslant [\![M, G]\!] \leqslant [\![L, E]\!]$ with $[\![N, H]\!] \trianglelefteqslant [\![L, E]\!]$. It follows that $[\![N, H]\!] \trianglelefteqslant [\![M, G]\!]$.

Proof. We have $H \leqslant G \leqslant E$ with $H \trianglelefteqslant E$. Thus, we have $H \trianglelefteqslant G$.

For $n \in N$, $h \in H$, $m \in M$ and $g \in G$, we have $m^- m^h \in N$ and $n^g \in N$ because of $[\![N, H]\!] \trianglelefteqslant [\![L, E]\!]$. $\square$

Lemma 25 (Kernel and image of crossed module morphisms) *Let* $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ *and* $[\![L, E]\!] = \big(L, E, \gamma, d\big)$ *be crossed modules. Suppose given a crossed module morphism* $(\lambda, \mu) \colon [\![M, G]\!] \to [\![L, E]\!]$.

(1) *Let* $k := f\big|_{\ker \lambda}^{\ker \mu}$ *be the restriction of* f *to* $\ker \lambda$ *and* $\ker \mu$. *Consider the group morphism*

$$\begin{array}{rccl} \beta \colon & \ker \mu & \longrightarrow & \mathrm{Aut}\,(\ker \lambda) \\ & h & \longmapsto & \big(n \mapsto (n)(h\beta) := (n)(h\alpha)\big) \ . \end{array}$$

We have a normal crossed submodule

$$[\![\ker \lambda, \ker \mu]\!] = \big(\ker \lambda, \ker \mu, \beta, k\big) \trianglelefteqslant [\![M, G]\!].$$

We write $\ker\,(\lambda, \mu) := [\![\ker \lambda, \ker \mu]\!]$.

(2) *Let* $\dot{d} := d\big|_{\mathrm{im}\,\lambda}^{\mathrm{im}\,\mu}$ *be the restriction of* d *to* $\mathrm{im}\,\lambda$ *and* $\mathrm{im}\,\mu$. *Consider the group morphism*

$$\begin{array}{rccl} \dot{\gamma} \colon & \mathrm{im}\,\mu & \longrightarrow & \mathrm{Aut}\,(\mathrm{im}\,\lambda) \\ & \dot{g} & \longmapsto & \big(\dot{m} \mapsto (\dot{m})(\dot{g}\dot{\gamma}) := \dot{m}(\dot{g}\gamma)\big) \ . \end{array}$$

We have a crossed submodule

$$[\![\operatorname{im}\lambda, \operatorname{im}\mu]\!] = \big(\operatorname{im}\lambda, \operatorname{im}\mu, \dot{\gamma}, \dot{d}\big) \leqslant [\![L, E]\!].$$

We write $\operatorname{im}(\lambda,\mu) := [\![\operatorname{im}\lambda, \operatorname{im}\mu]\!]$.

(3) *We have the following diagram.*

$$\begin{tikzcd}
\ker\lambda \arrow[r, hook, "\trianglelefteqslant"] \arrow[d, "k"'] & M \arrow[r, "\lambda|^{\operatorname{im}\lambda}"] \arrow[d, "f"'] & \operatorname{im}\lambda \arrow[r, hook] \arrow[d, "\dot{d}"] & L \arrow[d, "d"] \\
\ker\mu \arrow[r, hook, "\trianglelefteqslant"] & G \arrow[r, "\mu|^{\operatorname{im}\mu}"] & \operatorname{im}\mu \arrow[r, hook] & E
\end{tikzcd}$$

Proof. Ad (1). Since λ and μ are group morphisms, we have $\ker\lambda \trianglelefteqslant M$ and $\ker\mu \trianglelefteqslant G$. Suppose given $n \in \ker\lambda$, $h \in \ker\mu$. We have

$$nf\mu = n\lambda d = 1d = 1\ .$$

Hence $nf \in \ker\mu$. Therefore, the map $k = f|_{\ker\lambda}^{\ker\mu}$ is well-defined. We have

$$n^h\lambda = (n\lambda)^{h\mu} = 1^1 = 1\ .$$

Hence $n^h \in \ker\lambda$. Therefore, the action β is well-defined; cf. Remark 18.

So we have $\ker(\lambda,\mu) \leqslant [\![M, G]\!]$.

Now we show that $\ker(\lambda,\mu)$ is normal in $[\![M, G]\!]$.

For $m \in M,\ h \in \ker\mu$ we have

$$\begin{aligned}(m^- m^h)\lambda &= m^-\lambda m^h\lambda = m^-\lambda(m\lambda)^{h\mu} \\ &= m^-\lambda(m\lambda)^1 = (m^- m)\lambda = 1\lambda = 1\ .\end{aligned}$$

Hence $m^- m^h \in \ker\lambda$.

For $n \in \ker\lambda,\ g \in G$, we have

$$(n^g)\lambda = (n\lambda)^{g\mu} = 1^{g\mu} = 1\ .$$

Hence $n^g \in \ker \lambda$.

This shows $\ker(\lambda, \mu) \trianglelefteqslant [\![M, G]\!]$.

Ad (2). Since λ and μ are group morphisms we have $\operatorname{im} \lambda \leqslant L$ and $\operatorname{im} \mu \leqslant E$.

Suppose given $\dot{m} \in \operatorname{im} \lambda$ and $\dot{g} \in \operatorname{im} \mu$. We can write $\dot{m} = m\lambda$ for some $m \in M$, and $\dot{g} = g\mu$ for some $g \in G$.

We have

$$m\lambda d = mf\mu \ \in \operatorname{im} \mu \ .$$

This shows that the map $\dot{d} = d\big|_{\operatorname{im}\lambda}^{\operatorname{im}\mu}$ is well-defined.

We have

$$\dot{m}^{\dot{g}} = (m\lambda)^{g\mu} = (m^g)\lambda \ \in \operatorname{im} \lambda \ .$$

This shows that the action $\dot{\gamma}$ is well-defined; cf. Remark 18.

So we have $\operatorname{im}(\lambda, \mu) \leqslant [\![L, E]\!]$.

Ad (3). It suffices to show that $\left(\lambda|^{\operatorname{im}\lambda},\ \mu|^{\operatorname{im}\mu}\right) \colon [\![M, G]\!] \to [\![\operatorname{im}\mu, \operatorname{im}\lambda]\!]$ is a crossed module morphism; cf. Remark 19.

Suppose given $m \in M$ and $g \in G$. We have

$$(m)\lambda|^{\operatorname{im}\lambda}\dot{d} = (m)\lambda|^{\operatorname{im}\lambda}d = (m)\lambda d = (m)f\mu = (m)f\mu|^{\operatorname{im}\mu} \ ,$$

and we have

$$(m^g)\lambda|^{\operatorname{im}\lambda} = (m^g)\lambda = (m\lambda)^{g\mu} = \left(m\lambda|^{\operatorname{im}\lambda}\right)^{g\mu|^{\operatorname{im}\mu}} \ .$$

$\square$

1.5 Factor crossed modules

Lemma 26 (Factor crossed module) *Suppose given crossed modules $[\![N, H]\!] = \left(N, H, \beta, k\right)$ and $[\![M, G]\!] = \left(M, G, \alpha, f\right)$ with $[\![N, H]\!] \trianglelefteqslant [\![M, G]\!]$. Let $L := M/N$ be the factor group of M by N, and let $E := G/H$ be the factor group of G by H. Consider the map*

$$d \colon L \to E \ , \ mN \mapsto mfH \ .$$

Consider the map

$$\begin{array}{rccl} \gamma\colon & E & \longrightarrow & \mathrm{Aut}\,(L) \\ & gH & \longmapsto & \big(mN \mapsto (mN)\big((gH)\gamma\big) := \big(m(g\alpha)\big)N\ \big)\ , \end{array}$$

i.e. we have $(mN)^{gH} = (m^g)N$ *for* $m \in M$ *and* $g \in G$.

(1) *We have a crossed module* $[\![L, E]\!] = \big(L, E, \gamma, d\big)$. *We say that* $[\![L, E]\!]$ *is the* factor crossed module *of* $[\![M, G]\!]$ *by* $[\![N, H]\!]$. *We write* $[\![M, G]\!]/[\![N, H]\!] := [\![L, E]\!]$.

(2) *We have a crossed module morphism* $(\bar{\lambda}, \bar{\mu})\colon [\![M, G]\!] \to [\![L, E]\!]$ *with*

$$\bar{\lambda}\colon M \to L = M/N,\ m \mapsto mN \quad \textit{and} \quad \bar{\mu}\colon G \to E = G/H,\ g \mapsto gH\ .$$

So we have the following diagram of crossed module morphisms.

$$\begin{tikzcd} N \arrow[r, hook, "\trianglelefteqslant"] \arrow[d, "k"'] & M \arrow[r, "\bar{\lambda}"] \arrow[d, "f"'] & M/N \arrow[d, "d"] \\ H \arrow[r, hook, "\trianglelefteqslant"] & G \arrow[r, "\bar{\mu}"] & G/H \end{tikzcd}$$

Proof. Ad (1). We have groups $L = M/N$ and $E = G/H$, because we have $N \trianglelefteqslant M$ and $H \trianglelefteqslant G$.

The map γ is independent of representatives:

Suppose given $m \in M$, $n \in N$, $g \in G$, $h \in H$.

We have

$$(mnN)^{gH} = (mn)^g N = m^g \underbrace{n^g}_{\in N} N = m^g N = (mN)^{gH}\ .$$

We have $(m^g)^-(m^g)^h := \tilde{n} \in N$, and hence, $(m^g)^h = m^g\,\tilde{n}$. Therefore, we have

$$(mN)^{ghH} = m^{gh}N = (m^g)^h N = m^g\,\tilde{n}N = m^g N = (mN)^{gH}\ .$$

The map γ is well-defined:

Suppose given $g \in G$ and $m, \tilde{m} \in M$. We have

$$\begin{aligned}(m\tilde{m}N)(gH)\gamma &= \big((m\tilde{m})(g\alpha)\big)N = \big(m(g\alpha)\tilde{m}(g\alpha)\big)N = \big(m(g\alpha)N\big)\big(\tilde{m}(g\alpha)N\big) \\ &= (mN)(gH)\gamma\ (\tilde{m}N)(gH)\gamma\ .\end{aligned}$$

Therefore, $(gH)\gamma$ is a group morphism. We have

$$(mN)(gH)\gamma\ (g^{-}H)\gamma = \big(m(g\alpha)N\big)(g^{-}H)\gamma = m(g\alpha)(g^{-}\alpha)N = m(gg^{-}\alpha)N = mN\ .$$

With a similar calculation we obtain $(mN)(g^{-}H)\gamma\ (gH)\gamma = mN$. Therefore the map $(g^{-}H)\gamma$ is both right inverse and left inverse of $(gH)\gamma$, and hence $(gH)\gamma$ is bijective. So we have indeed $(gH)\gamma \in \operatorname{Aut}(L)$.

The map γ is a group morphism:

Suppose given $g, \tilde{g} \in G$, $m \in M$. We have

$$(mN)(g\tilde{g}H)\gamma = m\big((g\tilde{g})\alpha\big)N = m(g\alpha)(\tilde{g}\alpha\big)N = \big(m(g\alpha)N\big)(\tilde{g}H)\gamma = (mN)(gH)\gamma(\tilde{g}H)\gamma\ .$$

The map d is independent of representatives:

For $m \in M$ and $n \in N$ we have

$$\big((mn)N\big)d = (mn)fH = (mf)\underbrace{(nf)}_{\in H}H = mfH = (mN)d\ .$$

The map d is a group morphism, since f is a group morphism.

Now we prove that d and γ satisfy the axioms (CM1) and (CM2).

For $m \in M$, $g \in G$, we have

$$\begin{aligned}(mN)^{gH}d = (m^g N)d = \big((m^g)f\big)H &\overset{\text{(CM1)}}{=} (mf)^g H \\ &= (g^{-} \cdot (mf) \cdot g)H = g^{-}H \cdot (mf)H \cdot gH = \big((mf)H\big)^{gH} \\ &= \big((mN)d\big)^{gH}\ .\end{aligned}$$

This shows (CM1).

For $m, \tilde{m} \in M$, we have

$$\begin{aligned}(mN)^{\tilde{m}N} &= (\tilde{m}N)^- \cdot (mN) \cdot (\tilde{m}N) = (\tilde{m}^- \cdot m \cdot \tilde{m})N = (m^{\tilde{m}})N \\ &\overset{\text{(CM2)}}{=} (m^{\tilde{m}f})N = (mN)^{\tilde{m}fH} \\ &= (mN)^{(\tilde{m}N)d} \ .\end{aligned}$$

This shows (CM2).

Ad (2). For $m \in M$, we have

$$(m)\bar{\lambda}d = (mN)d = mfH = (m)f\bar{\mu} \ .$$

Hence, the following diagram is commutative.

$$\begin{tikzcd} M \arrow[r, "\bar{\lambda}"] \arrow[d, "f"'] & M/N \arrow[d, "d"] \\ G \arrow[r, "\bar{\mu}"'] & G/H \end{tikzcd}$$

Further, for $m \in M$, $g \in G$, we have

$$m^g\bar{\mu} = m^gN = (mN)^{gH} = (m\bar{\lambda})^{g\bar{\mu}} \ .$$

$\square$

Lemma 27 (Kernel-image lemma)
Let $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ and let $[\![L, E]\!] = \big(L, E, \gamma, d\big)$ be crossed modules. Suppose given a crossed module morphism $(\lambda, \mu)\colon [\![M, G]\!] \to [\![L, E]\!]$. Then we have the following commutative diagram.

$$\begin{tikzcd} \ker(\lambda,\mu) \arrow[r, "(\iota,\kappa)"] & {[\![M,G]\!]} \arrow[r, "(\lambda,\mu)"] \arrow[d, "(\bar{\lambda},\bar{\mu})"'] & {[\![L,E]\!]} \\ & {[\![M,G]\!]}/\ker(\lambda,\mu) \arrow[r, "\sim", "(\tilde{\lambda},\tilde{\mu})"'] & \operatorname{im}(\lambda,\mu) \arrow[u, "(\dot{\iota},\dot{\kappa})"'] \end{tikzcd}$$

Or, more explicitly:

$$
\begin{tikzcd}
\ker\lambda \arrow[r, "\iota"] \arrow[d, "k"'] & M \arrow[rrr, "\lambda"] \arrow[d, "f"'] \arrow[ddr, "\bar{\lambda}"] & & & L \arrow[d, "d"] \\
\ker\mu \arrow[r, "\kappa"'] & G \arrow[rrr, "\mu"'] \arrow[ddr, "\bar{\mu}"'] & & & E \\
& & M/\ker\lambda \arrow[r, "\tilde{\lambda}", "\sim"'] \arrow[d, "\bar{f}"] & \operatorname{im}\lambda \arrow[uur, "\dot{\imath}"] \arrow[d, "\dot{d}"'] & \\
& & G/\ker\mu \arrow[r, "\sim", "\tilde{\mu}"'] & \operatorname{im}\mu \arrow[uur, "\dot{\kappa}"'] &
\end{tikzcd}
$$

Proof. The existence of the crossed modules $\ker(\lambda, \mu)$, $\operatorname{im}(\lambda, \mu)$ and $[\![M, G]\!]/\ker(\lambda, \mu)$ is shown in Lemma 25 and Lemma 26.(1).

By Remark 19, we have the inclusion morphism (ι, κ) and $(\dot{\imath}, \dot{\kappa})$. Lemma 26.(2) yield the crossed module morphism $(\bar{\lambda}, \bar{\mu})$.

By the kernel-image lemma for groups, we have bijective group morphisms

$$
\begin{aligned}
\tilde{\lambda}\colon \quad & M/\ker\lambda \to \operatorname{im}\lambda, \quad m(\ker\lambda) \mapsto m\lambda \\
\tilde{\mu}\colon \quad & G/\ker\mu \to \operatorname{im}\mu, \quad g(\ker\mu) \mapsto g\mu\ .
\end{aligned}
$$

We show that $(\tilde{\lambda}, \tilde{\mu})$ is a (bijective) crossed module morphism.

Let $m(\ker\lambda) \in M/\ker\lambda$ and let $g(\ker\mu) \in G/\ker\mu$. We have

$$\big(m(\ker\lambda)\big)\tilde{\lambda}\dot{d} = m\lambda\dot{d} = m\lambda d = mf\mu = \big(mf(\ker\mu)\big)\tilde{\mu} = \big(m(\ker\lambda)\big)\bar{f}\tilde{\mu}\ .$$

We have

$$\big(m(\ker\lambda)\big)^{g(\ker\mu)}\tilde{\lambda} = \big(m^g(\ker\lambda)\big)\tilde{\lambda} = (m^g)\lambda = (m\lambda)^{g\mu} = \big(m(\ker\lambda)\tilde{\lambda}\big)^{g(\ker\mu)\tilde{\mu}}\ .$$

Therefore, every crossed module and crossed module morphism given in the diagram exist, and hence the diagram commutes. □

Corollary 28 *Let* $[\![M,G]\!] = \big(M,G,\alpha,f\big)$ *and* $[\![L,E]\!] = \big(L,E,\gamma,d\big)$ *be crossed modules. Suppose given a surjective crossed module morphism* $(\lambda,\mu)\colon [\![M,G]\!] \to [\![L,E]\!]$*. Then we have a bijective crossed module morphism given by*

$$(\varphi,\psi)\colon \quad {}^{[\![M,G]\!]}\!\big/_{\ker(\lambda,\mu)} \xrightarrow{\sim} [\![L,E]\!] \ ,$$

with

$$\begin{aligned} \varphi\colon \quad & M/\ker\lambda \to L, \quad m(\ker\lambda) \mapsto m\lambda \\ \psi\colon \quad & G/\ker\mu \to E, \quad g(\ker\mu) \ \mapsto g\mu \ . \end{aligned}$$

Hence, we have the following commutative diagram.

$$\begin{tikzcd}
M \arrow[rr, "\lambda"] \arrow[d, "f"'] \arrow[ddr, "\bar{\lambda}"] & & L \arrow[d, "d"] \\
G \arrow[rr, "\mu"] \arrow[ddr, "\bar{\mu}"'] & & E \\
& M/\ker\lambda \arrow[uur, "\varphi"'] \arrow[d, "\bar{f}"'] & \\
& G/\ker\mu \arrow[uur, "\psi"'] &
\end{tikzcd}$$

Proof. We are in the situation of Lemma 27 with special case of (λ,μ) being surjective. Hence, we have $\operatorname{im}\lambda = L$ and $\operatorname{im}\mu = E$. So the inclusion morphism $(\imath,\dot{\kappa})$ becomes $(\imath,\dot{\kappa}) = (\mathrm{id}_L,\mathrm{id}_E)$. We have

$$\varphi := \tilde{\lambda}\imath = \tilde{\lambda}\ \mathrm{id}_L = \tilde{\lambda} \quad \text{and} \quad \psi := \tilde{\mu}\dot{\kappa} = \tilde{\mu}\ \mathrm{id}_E = \tilde{\mu}\ .$$

Hence, the groups morphisms

$$\varphi\colon M/\ker\lambda \xrightarrow{\sim} L \quad \text{and} \quad \psi\colon G/\ker\mu \xrightarrow{\sim} E$$

are bijective, and therefore, (φ,ψ) is bijective. $\square$

1.6 Examples of crossed submodules

Concerning the notion of a centre of a crossed module, we follow [**3,** p. 171].

Lemma 29 (Centre) *Let* $[\![M,G]\!] = \big(M,G,\alpha,f\big)$ *be a crossed module.*

(1) *Let* $\mathrm{Z}\,(G) = \{z \in G\colon\ gz = zg \quad \text{for } g \in G\}$ *be the centre of* G.
Let $\mathrm{st}_G(M) := \{g \in G\colon\ m^g = m \quad \text{for } m \in M\}$ *be the* stabilizer *of* M *in* G.
We have $\mathrm{Z}\,(G) \cap \mathrm{st}_G(M) \ \trianglelefteqslant\ G$.

(2) *Let* $M^G := \{m \in M\colon\ m^g = m \quad \text{for } g \in G\}$. *We have* $M^G \trianglelefteqslant M$.

(3) *We have* $[\![M^G, \mathrm{Z}\,(G) \cap \mathrm{st}_G(M)]\!] \ \trianglelefteqslant\ [\![M,G]\!]$ *and we call* $[\![M^G, \mathrm{Z}\,(G) \cap \mathrm{st}_G(M)]\!]$ *the* centre *of* $[\![M,G]\!]$. *We write* $\mathrm{Z}\left([\![M,G]\!]\right) := [\![M^G, \mathrm{Z}\,(G) \cap \mathrm{st}_G(M)]\!]$.

Proof. Ad (1). The group $\mathrm{Z}\,(G) \cap \mathrm{st}_G(M)$ is a subgroup of $\mathrm{Z}\,(G)$. Since every subgroup of the centre is normal in G it follows that $\mathrm{Z}\,(G) \cap \mathrm{st}_G(M) \ \trianglelefteqslant\ G$.

Ad (2). We have $1 \in M^G$. Let $\tilde{n}, n \in M^G$. Let $g \in G$. We have

$$\left(\tilde{n}n^-\right)^g = \tilde{n}^g \cdot (n^g)^- = \tilde{n}n^- \ .$$

Hence $\tilde{n}n^- \in M^G$. Therefore $M^G \leqslant M$.

Now let $n \in M^G$. Let $m \in M$. We get

$$n^m \overset{\text{(CM2)}}{=} n^{mf} = n \ .$$

Hence $n^m \in M^G$, and therefore $M^G \trianglelefteqslant M$.

Ad (3). First we show that $[\![M^G, \mathrm{Z}\,(G) \cap \mathrm{st}_G(M)]\!]$ is a crossed submodule of $[\![M,G]\!]$.

Let $n \in M^G$.

Let $g \in G$. We have

$$nf = (n^g)f \overset{\text{(CM1)}}{=} (nf)^g = g^-(nf)\ g \ .$$

Hence $nf \in \mathrm{Z}\,(G)$.

Let $m \in M$. We have

$$m^{nf} \overset{\text{(CM2)}}{=} m^n = n^-m\ n = n^-m\ n^{mf} \overset{\text{(CM2)}}{=} n^-m\ m^-n\ m = n^-n\ m = m \ .$$

Hence $nf \in \mathrm{st}_G(M)$.

Therefore we have $nf \in \mathrm{Z}(G) \cap \mathrm{st}_G(M)$.

Let $h \in \mathrm{Z}(G) \cap \mathrm{st}_G(M)$. Let $n \in M^G$. We have

$$n^h = n \in M^G \ .$$

This shows $[\![M^G, \mathrm{Z}(G) \cap \mathrm{st}_G(M)]\!] \leqslant [\![M, G]\!]$.

Now we want to show that $[\![M^G, \mathrm{Z}(G) \cap \mathrm{st}_G(M)]\!]$ is normal in $[\![M, G]\!]$.

For that, let $m \in M$ and let $h \in \mathrm{Z}(G) \cap \mathrm{st}_G(M)$. We have

$$m^- m^h = m^- m = 1 \in M^G \ .$$

Now let $n \in M^G$ and let $g \in G$. We have

$$n^g = n \in M^G \ .$$

This shows $[\![M^G, \mathrm{Z}(G) \cap \mathrm{st}_G(M)]\!] \trianglelefteqslant [\![M, G]\!]$. $\square$

Example 30 We consider the crossed module defined in [**6,** §1.5.6].

Let $G := \langle a \ : \ a^4 = 1 \rangle$ and let $M := \langle b \ : \ b^4 = 1 \rangle$ be cyclic groups of order 4. Since $(a^2)^4 = 1$, we have a group morphism

$$f \colon M \to G, \ b \mapsto a^2 \ .$$

Further, we can define an action of G on M by

$$\alpha \colon G \longrightarrow \mathrm{Aut}\,(M) \ , \ a \ \longmapsto (b \longmapsto b^a := b^-) \ .$$

This yields a crossed module $[\![M, G]\!] = \big(M, G, \alpha, f\big)$, because we have

$$(b^a)f = (b^-)f = (bf)^- = (a^2)^- = (bf)^a \ ,$$

which shows (CM1), and

$$b^{bf} = b^{(a^2)} = (b^a)^a = (b^-)^a = b = b^- b \ b = b^b \ ,$$

which shows (CM2).

(1) We want to determine the crossed submodules and normal crossed submodules of $[\![M, G]\!]$. We consider all possible candidate pairs (N, H), where $N \leqslant M$ and $H \leqslant G$, and check if they possess the required properties.

Since M and G are abelian, the conditions $N \trianglelefteq M$ and $H \trianglelefteq G$ are always satisfied. Further, the crossed modules $[\![\langle 1\rangle, \langle 1\rangle]\!]$ and $[\![M, G]\!] = [\![\langle b\rangle, \langle a\rangle]\!]$ are trivially normal crossed submodules.

We consider the pair $(\langle b^2\rangle, \langle 1\rangle)$. We have

$$b^2 f = (bf)^2 = (a^2)^2 = 1 \in \langle 1\rangle \quad \text{and} \quad (b^2)^1 = b^2 \in \langle b^2\rangle \ .$$

Thus, $[\![\langle b^2\rangle, \langle 1\rangle]\!]$ is a crossed submodule. It is normal as well, because we have

$$b^- b^1 = 1 \in \langle 1\rangle \quad \text{and} \quad (b^2)^a = (b^-)^2 = b^2 \in \langle b^2\rangle \ .$$

Now consider the pair $(\langle b\rangle, \langle 1\rangle)$. We have

$$bf = a^2 \notin \langle 1\rangle \ .$$

Hence, the pair $(\langle b\rangle, \langle 1\rangle)$ does not yield a crossed submodule of $[\![M, G]\!]$.

We proceed through all candidate pairs in the same fashion and obtain the following list.

candidate pair	crossed submodule	normal crossed submodule
$(\langle 1\rangle, \langle 1\rangle)$	$\surd$	$\surd$
$(\langle b^2\rangle, \langle 1\rangle)$	$\surd$	$\surd$
$(\langle b\rangle, \langle 1\rangle)$	$\times$	$\times$
$(\langle 1\rangle, \langle a^2\rangle)$	$\surd$	$\surd$
$(\langle b^2\rangle, \langle a^2\rangle)$	$\surd$	$\surd$
$(\langle b\rangle, \langle a^2\rangle)$	$\surd$	$\surd$
$(\langle 1\rangle, \langle a\rangle)$	$\surd$	$\times$
$(\langle b^2\rangle, \langle a\rangle)$	$\surd$	$\surd$
$(\langle b\rangle, \langle a\rangle)$	$\surd$	$\surd$

(2) Let $X := [\![M, G]\!] = [\![\langle b\rangle, \langle a\rangle]\!]$.

We want to determine the centre $\mathrm{Z}\,(X) = [\![M^G, \mathrm{Z}\,(G) \cap \ \mathrm{st}_G(M)]\!]$; cf. Lemma 29. We have

$$
\begin{aligned}
&M^G = \langle b\rangle^{\langle a\rangle} = \langle b^2\rangle \quad \text{and} \\
&\mathrm{Z}\,(G) \cap \mathrm{st}_G(M) = \mathrm{Z}\,(\langle a\rangle) \cap \mathrm{st}_{\langle a\rangle}(\langle b\rangle) = \langle a\rangle \cap \langle a^2\rangle = \langle a^2\rangle \ .
\end{aligned}
$$

Hence we get $\mathrm{Z}\,(X) = [\![\langle b^2\rangle, \langle a^2\rangle]\!] \trianglelefteqslant X$.

We form the factor crossed module $X/\,\mathrm{Z}\,(X)$ and obtain

$$
X/\,\mathrm{Z}\,(X) = [\![\langle b\rangle/\langle b^2\rangle, \langle a\rangle/\langle a^2\rangle]\!] =: [\![\tilde{M}, \tilde{G}]\!];\ \text{cf. Lemma 26.}
$$

We want to determine the centre $\mathrm{Z}\,(X/\,\mathrm{Z}\,(X)) = [\![\tilde{M}^{\tilde{G}}, \mathrm{Z}\,(\tilde{G}) \cap \mathrm{st}_{\tilde{G}}(\tilde{M})\ ]\!]$.

We have $\tilde{M} = \{1\langle b^2\rangle, b\langle b^2\rangle\} = \langle\ b\langle b^2\rangle\ \rangle$ and $\tilde{G} = \{1\langle a^2\rangle, a\langle a^2\rangle\} = \langle\ a\langle a^2\rangle\ \rangle$.

We have

$$
\begin{aligned}
(\ 1\langle b^2\rangle\ )^{a\langle a^2\rangle} &\overset{26}{=} 1^a\langle b^2\rangle = 1\langle b^2\rangle \\
(\ b\langle b^2\rangle\ )^{a\langle a^2\rangle} &\overset{26}{=} b^a\langle b^2\rangle = b^-\langle b^2\rangle = b^3\langle b^2\rangle = b\langle b^2\rangle \ .
\end{aligned}
$$

Hence we get $\tilde{M}^{\tilde{G}} = \tilde{M}$ and $\mathrm{st}_{\tilde{G}}(\tilde{M}) = \tilde{G}$.

We have

$$
\mathrm{Z}\,(\tilde{G}) \cap \mathrm{st}_{\tilde{G}}(\tilde{M}) = \langle\ a\langle a^2\rangle\ \rangle \cap \langle\ a\langle a^2\rangle\ \rangle = \langle\ a\langle a^2\rangle\ \rangle = \tilde{G}\ .
$$

Therefore, we have $\mathrm{Z}\,(X/\,\mathrm{Z}\,(X)) = [\![\tilde{M}, \tilde{G}]\!] = X/\,\mathrm{Z}\,(X)$.

Example 31 We consider the crossed module $[\![\mathrm{S}_3, \mathrm{S}_3]\!] = \big(\mathrm{S}_3, \mathrm{S}_3, c, \mathrm{id}_{\mathrm{S}_3}\big)$. We want to determine its normal crossed submodules.

We proceed in a similar fashion as in Example 30 and look at all candidate pairs (N, H) with $N, H \trianglelefteqslant \mathrm{S}_3$ and $N \leqslant H$. These are given by $(1, 1)$, $(1, \mathrm{A}_3)$, $(1, \mathrm{S}_3)$, $(\mathrm{A}_3, \mathrm{A}_3)$, $(\mathrm{A}_3, \mathrm{S}_3)$ and $(\mathrm{S}_3, \mathrm{S}_3)$.

Since for all candidate pairs (N, H), the group H acts in N via conjugation, all candidate pairs yield crossed submodules by Remark 18.

We consider $[\![\mathrm{A}_3, \mathrm{S}_3]\!]$. Suppose given $a \in \mathrm{A}_3$, suppose given $s, t \in \mathrm{S}_3$. Since the factor group $\mathrm{S}_3 \,/\, \mathrm{A}_3$ is abelian, the commutator subgroup $[\mathrm{S}_3, \mathrm{S}_3]$ is contained in A_3. We have

$$s^{-}s^{t} = s^{-} \cdot t^{-} \cdot s \cdot t \in [\mathrm{S}_3, \mathrm{S}_3] \subseteq \mathrm{A}_3 \ .$$

Furthermore, we have

$$a^{s} = s^{-} \cdot a \cdot s \in \mathrm{A}_3 \ .$$

Thus, it follows that $[\![\mathrm{A}_3, \mathrm{S}_3]\!] \trianglelefteqslant [\![\mathrm{S}_3, \mathrm{S}_3]\!]$.

We consider $[\![\mathrm{A}_3, \mathrm{A}_3]\!]$. For $a \in \mathrm{A}_3$, $s \in \mathrm{S}_3$ we have

$$s^{-}s^{a} = \underbrace{s^{-}a^{-}s}_{\in \mathrm{A}_3} a \in \mathrm{A}_3 \quad \text{and} \quad a^{s} = s^{-}as \in A_3 \ .$$

Hence, we have $[\![\mathrm{A}_3, \mathrm{A}_3]\!] \trianglelefteqslant [\![\mathrm{S}_3, \mathrm{S}_3]\!]$.

We consider $[\![1, \mathrm{A}_3]\!]$. Let $b := (2,3) \in S_3$, $a := (1,2,3) \in \mathrm{A}_3$. We have

$$b^{-}b^{a} = b^{-}a^{-}b\ a = (2,3)(1,3,2)(2,3)(1,2,3) = (1,3,2) \neq \mathrm{id} \ ,$$

and therefore $[\![1, \mathrm{A}_3]\!] \ntrianglelefteq [\![\mathrm{S}_3, \mathrm{S}_3]\!]$. This calculation also shows that $[\![1, \mathrm{S}_3]\!] \ntrianglelefteq [\![\mathrm{S}_3, \mathrm{S}_3]\!]$.

Note that the crossed submodules $[\![1, 1]\!]$ and $[\![\mathrm{S}_3, \mathrm{S}_3]\!]$ are normal in $[\![\mathrm{S}_3, S_3]\!]$; cf. Remark 23.

We get the following list.

candidate pair	crossed submodule	normal crossed submodule
$(\langle 1\rangle, \langle 1\rangle)$	$\surd$	$\surd$
$(\langle 1\rangle, \mathrm{A}_3)$	$\surd$	$\times$
$(\mathrm{A}_3, \mathrm{A}_3)$	$\surd$	$\surd$
$(\langle 1\rangle, \mathrm{S}_3)$	$\surd$	$\times$
$(\mathrm{A}_3, \mathrm{S}_3)$	$\surd$	$\surd$
$(\mathrm{S}_3, \mathrm{S}_3)$	$\surd$	$\surd$

2 Simple crossed modules

2.1 Sequences

Definition 32 (Short exact sequence for crossed modules) Suppose given crossed modules $[\![M_i, G_i]\!] = \big(M_i, G_i, \alpha_i, f_i\big)$ for $i \in [1, 3]$.

For $i \in [1, 2]$, suppose given crossed module morphisms

$$(\lambda_i, \mu_i)\colon [\![M_i, G_i]\!] \longrightarrow [\![M_{i+1}, G_{i+1}]\!]$$

such that

$$1 \longrightarrow M_1 \xrightarrow{\lambda_1} M_2 \xrightarrow{\lambda_2} M_3 \longrightarrow 1$$

and

$$1 \longrightarrow G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \longrightarrow 1$$

are short exact sequences; cf. Reminder 2. We call

$$1 \longrightarrow [\![M_1, G_1]\!] \xrightarrow{(\lambda_1, \mu_1)} [\![M_2, G_2]\!] \xrightarrow{(\lambda_2, \mu_2)} [\![M_3, G_3]\!] \longrightarrow 1$$

a *short exact sequence (of crossed modules).*

Lemma 33 *Let* $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ *be a crossed module.*

(1) *Let* $[\![N, H]\!] \trianglelefteqslant [\![M, G]\!]$ *and let* $[\![M, G]\!]/[\![N, H]\!]$ *be the factor crossed module; cf. Lemma 26.(1). We have the residue class morphisms*

$$\begin{aligned} p\colon \ & M \to M/N, \quad & m \mapsto mN \\ q\colon \ & G \to G/H, \quad & g \mapsto gH\ . \end{aligned}$$

Further, let $\iota := \mathrm{id}_M\big|_N$ *and let* $\kappa := \mathrm{id}_G\big|_H$ *be the inclusion maps.*
We get a short exact sequence

$$1 \longrightarrow [\![N,H]\!] \xrightarrow{(\iota,\kappa)} [\![M,G]\!] \xrightarrow{(p,q)} [\![M,G]\!]/[\![N,H]\!] \longrightarrow 1\ .$$

(2) *Suppose given crossed modules* $[\![N,H]\!] = \big(N,H,\beta,k\big)$ *and* $[\![L,E]\!] = \big(L,E,\gamma,d\big)$*. Suppose we have a short exact sequence*

$$1 \longrightarrow [\![N,H]\!] \xrightarrow{(\varphi,\psi)} [\![M,G]\!] \xrightarrow{(\lambda,\mu)} [\![L,E]\!] \longrightarrow 1\ .$$

Then, the map $(\varphi|^{\ker\lambda},\psi|^{\ker\mu})\colon [\![N,H]\!] \to \ker(\lambda,\mu)$ *is an isomorphism of crossed modules; cf. Lemma 27.(1).*

Proof. Ad (1). We are given short exact sequences

$$1 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{p} M/N \longrightarrow 1$$

$$1 \longrightarrow H \xrightarrow{\kappa} G \xrightarrow{q} G/H \longrightarrow 1\ .$$

By Remark 19 and Lemma 26.(2), (ι,κ) and (p,q) are crossed module morphisms.

Ad (2). We are given short exact sequences

$$1 \longrightarrow N \xrightarrow{\varphi} M \xrightarrow{\lambda} L \longrightarrow 1$$

$$1 \longrightarrow H \xrightarrow{\psi} G \xrightarrow{\mu} E \longrightarrow 1\ .$$

Therefore, we have bijective group morphisms $\varphi|^{\ker\lambda}\colon N \xrightarrow{\sim} \ker\lambda$ and $\psi|^{\ker\mu}\colon H \xrightarrow{\sim} \ker\mu$. From Lemma 25.(3) we infer that $(\varphi|^{\ker\lambda},\psi|^{\ker\mu})$ is a crossed module morphism. Hence, $(\varphi|^{\ker\lambda},\psi|^{\ker\mu})$ is an isomorphism of crossed modules.

$\square$

2.2 Simplicity

Definition 34 (Simple crossed module) A *simple crossed module* $[\![M, G]\!]$ is a crossed module that is not isomorphic to $[\![1, 1]\!]$ and that has no normal crossed submodules apart from its trivial crossed submodule $[\![1, 1]\!]$ and the crossed module $[\![M, G]\!]$ itself.

Remark 35 Let G be a group. We consider the crossed module $[\![G, G]\!] = \big(G, G, c, \mathrm{id}_G\big)$ from Remark 10.(2). Then, we have a crossed submodule $[\![1, G]\!] \leqslant [\![G, G]\!]$.

We have $[\![1, G]\!] \trianglelefteq [\![G, G]\!]$ if and only if G is abelian.

Proof. We have $[\![1, G]\!] \leqslant [\![G, G]\!]$ since

$$(1)\mathrm{id}_G = 1 \quad \text{and} \quad (1)gc = 1^g = g^{-}1g = 1 \ , \ \text{for } g \in G.$$

"$\Rightarrow$" We suppose that $[\![1, G]\!] \trianglelefteq [\![G, G]\!]$. For $g, h \in G$, we have

$$g^{-}g^{h} = 1 \quad \Leftrightarrow \quad hg = gh \ .$$

Therefore, G is abelian.

"$\Leftarrow$" Suppose that G is abelian. For $g, h \in G$, we get $g^{-}g^{h} = 1$ and $1^g = 1$.

This shows $[\![1, G]\!] \trianglelefteq [\![G, G]\!]$. □

Example 36 Consider the symmetric group S_3. By Remark 35, $[\![1, \mathrm{S}_3]\!] \leqslant [\![\mathrm{S}_3, \mathrm{S}_3]\!]$ is a crossed submodule but not a normal crossed submodule, because S_3 is not abelian. This fact has already been shown in Example 31.

Lemma 37 *Let $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ be a crossed module.*

(1) *We have a crossed module $[\![Mf, G]\!] = \big(Mf, G, c_{Mf}, \mathrm{id}_G\big|_{Mf}\big)$, and a surjective crossed module morphism $(f\big|^{Mf}, \mathrm{id}_G)\colon [\![M, G]\!] \to [\![Mf, G]\!]$. Cf. Example 9.*

(2) *We have a short exact sequence given by*

$$1 \longrightarrow [\![\ker f, 1]\!] \xhookrightarrow{\trianglelefteq} [\![M, G]\!] \xrightarrow{(f|^{Mf}, \mathrm{id}_G)} [\![Mf, G]\!] \longrightarrow 1 \ .$$

Proof. Ad (1). By Lemma 7.(2), we have $Mf \trianglelefteqslant G$. Thus, $[\![Mf, G]\!]$ is a crossed module; cf. Example 9.

Write $\bar{f} := f\big|^{Mf}$ and $\bar{\kappa} := \mathrm{id}_G\big|_{Mf}$. For $m \in M$, $g \in G$, we have

$$(m)\bar{f}\bar{\kappa} = (m)f\bar{\kappa} = (m)f = (m)f\mathrm{id}_G$$

and

$$(m^g)\bar{f} = (m^g)f \overset{\text{(CM1)}}{=} (mf)^g = (m\bar{f})^g \ .$$

This proves that $(f\big|^{Mf}, \mathrm{id}_G)$ is a crossed module morphism. By construction, it is surjective; cf. Definition 16.

Ad (2). The kernel of $(f|^{Mf}, \mathrm{id}_G)\colon [\![M, G]\!] \to [\![Mf, G]\!]$ is given by $[\![\ker f, 1]\!]$.

Since $[\![\ker f, 1]\!] \leqslant [\![M, G]\!]$ is a crossed submodule, we have the inclusion morphism $(\mathrm{id}_M\big|_{\ker f}\ ,\ \mathrm{id}_G\big|_1)\colon [\![\ker f, 1]\!] \to [\![M, G]\!]$; cf. Remark 19.

Altogether, the sequence in question exists and is short exact; cf. Definition 32.

□

Remark 38 Suppose given a group G. Suppose given $M \trianglelefteqslant G$. Consider the crossed module $[\![M, G]\!] = \big(M, G, c|_M, \mathrm{id}_G|_M\big)$; cf. Example 9.

(1) We have a crossed module $[\![1, G/M]\!] = \big(1, G/M, c_1, \mathrm{id}_{G/M}|_1\big)$, and a surjective crossed module morphism $(\kappa, r)\colon [\![M, G]\!] \to [\![1, G/M]\!]$, where

$$\begin{array}{rccl} \kappa\colon & M & \longrightarrow & 1 \\ & m & \longmapsto & 1 = 1_{G/M} \end{array} \qquad \begin{array}{rccl} r\colon & G & \longrightarrow & G/M \\ & g & \longmapsto & gM \ . \end{array}$$

(2) We have a short exact sequence given by

$$1 \longrightarrow [\![M, M]\!] \xhookrightarrow{\ \trianglelefteqslant\ } [\![M, G]\!] \xrightarrow{(\kappa, r)} [\![1, G/M]\!] \longrightarrow 1 \ .$$

Proof. Ad (1). We have a crossed module $[\![1, G/M]\!]$ since it carries the structure of the crossed module given in Example 9.

Let $m \in M$ and let $g \in G$. We have

$$(m)\mathrm{id}_G\big|_M\, r = (m)r = mM = 1M = (1)\mathrm{id}_{G/M}|_1 = (m)\kappa\ \mathrm{id}_{G/M}\big|_1$$

and

$$(m^g)\kappa = 1M = 1^g M = (1M)^{gM} = (m\kappa)^{gr} \ .$$

This proves that (κ, r) is a crossed module morphism. By construction, it is surjective; cf. Definition 16.

Ad (2). The kernel of $(\kappa, r)\colon [\![M, G]\!] \to [\![1, G/M]\!]$ is given by $[\![M, M]\!]$.

Since $[\![M, G]\!] \leqslant [\![1, G/M]\!]$ is a crossed submodule, we have the inclusion morphism $(\mathrm{id}_M, \mathrm{id}_G|_M)\colon [\![M, M]\!] \to [\![M, G]\!]$; cf. Remark 19.

Altogether, the sequence in question exists and is short exact; cf. Definition 32. □

Lemma 39 *Suppose given a group G. Suppose given $M \trianglelefteqslant G$. Consider the crossed module $[\![M, G]\!] = \big(M, G, c_M, \mathrm{id}_G\big|_M\big)$; cf. Example 9.*

A crossed submodule $[\![N, H]\!] \leqslant [\![M, G]\!]$ is normal in $[\![M, G]\!]$ if and only if we have

$$N \trianglelefteqslant M,\ H \trianglelefteqslant G \quad \text{and} \quad N \trianglelefteqslant H,\ N \trianglelefteqslant G,\ [M, H] \leqslant N \ ,$$

i.e. we have the following diagram.

$$\begin{tikzcd}
{[M,H]} \arrow[r, hook] & N \arrow[r, hook, "\trianglelefteqslant"] \arrow[d, hook, "\trianglelefteqslant"'] \arrow[dr, hook, "\trianglelefteqslant"] & M \arrow[d, hook, "\trianglelefteqslant"] \\
 & H \arrow[r, hook, "\trianglelefteqslant"'] & G
\end{tikzcd}$$

Proof. Ad $\Rightarrow$. We assume that $[\![N, H]\!] \trianglelefteqslant [\![M, G]\!]$; cf. Definition 21. Then we have $N \trianglelefteqslant M$ and $H \trianglelefteqslant G$.

For $n \in N$, $g \in G$, we have $n^g = g^- n g \in N$. It follows that $N \trianglelefteqslant G$. Since $[\![N, H]\!]$ carries the morphism $\mathrm{id}_G\big|_M\big|_N^H$, we have $N \leqslant H$. Altogether, we have $N \trianglelefteqslant H$.

For $m \in M$, $h \in H$, we have $[m, h] = m^- h^- m h = m^- m^h \in N$. Hence $[M, H] \leqslant N$.

Ad $\Leftarrow$. By assumption we have $N \trianglelefteqslant M$ and $H \trianglelefteqslant G$. Further, for $n \in N$, $m \in M$ and $h \in H$, $g \in G$, we have $m^- m^h \in [M, H] \leqslant N$, and $n^g \in N$. Hence we have $[\![N, H]\!] \trianglelefteqslant [\![M, G]\!]$; cf. Definition 21. □

Theorem 40 (Simple crossed modules) *Suppose given a simple crossed module C. Then C is simple if and only if* (1) *or* (2) *or* (3) *holds; cf. Definition* 12.

(1) *We have $C \simeq \mathrm{X}_{\mathrm{contr}}(G)$ for some non-abelian and simple group G.*

(2) *We have $C \simeq \mathrm{X}_1(K)$ for some simple group K.*

(3) *We have $C \simeq \mathrm{X}_2(M)$ for some cyclic group M of prime order.*

Proof. Ad $\Leftarrow$. Suppose that (1) holds. We may assume that $C = [\![G, G]\!] = \big(G, G, c, \mathrm{id}_G\big)$, where G a simple and non-abelian group. Furthermore, we may assume that $G \neq 1$, since, by definition, the crossed module $[\![1, 1]\!]$ is not simple; cf. Defintion 34.

Suppose given a normal crossed submodule $[\![N, H]\!] \trianglelefteqslant [\![G, G]\!]$. Then we have $N \trianglelefteqslant G$ and $H \trianglelefteqslant G$. We get $N, H \in \{1, G\}$ because G is simple.

Suppose we have $N = 1$ and $H = G$. By Remark 35, we have $[\![1, G]\!] \ntrianglelefteq [\![G, G]\!]$, since G is non-abelian.

Suppose we have $N = G$ and $H = 1$. By Lemma 39, we have $[\![G, 1]\!] \ntrianglelefteq [\![G, G]\!]$, since we do not have $G \trianglelefteqslant 1$.

Therefore, $[\![1, 1]\!] \trianglelefteqslant [\![G, G]\!]$ and $[\![G, G]\!] \trianglelefteqslant [\![G, G]\!]$ are the only normal crossed submodules we have; cf. Remark 23.

Hence, $[\![G, G]\!]$ is simple.

Suppose that (2) holds. We may assume that $C = [\![1, K]\!] = \big(1, K, c, \mathrm{id}_K\big|_1\big)$, where K is a simple group. Suppose given a normal crossed submodule $[\![N, H]\!] \trianglelefteqslant [\![1, K]\!]$. We obtain $N \trianglelefteqslant 1$ and $H \trianglelefteqslant K$. It follows that $N = 1$, and it follows that $H = 1$ or $H = K$.

Hence, $[\![1, K]\!]$ is simple.

Suppose that (3) holds. We may assume that $C = [\![M, 1]\!] = \big(M, 1, \iota, \kappa\big)$. We need the commutativity of M to define the crossed module $[\![M, 1]\!]$; cf. Example 11. Suppose given a normal crossed submodule $[\![N, H]\!] \trianglelefteqslant [\![M, 1]\!]$. We obtain $N \trianglelefteqslant M$ and $H \trianglelefteqslant 1$. It follows that $N = 1$ or $N = M$, and it follows that $H = 1$.

Hence, $[\![M, 1]\!]$ is simple.

Ad $\Rightarrow$. Suppose given a simple crossed module $C = [\![M, G]\!] = \big(M, G, \alpha, f\big)$. Consider the short exact sequence from Lemma 37.(2).

$$\begin{tikzcd}
1 \arrow[r] & \ker f \arrow[r, hook, "\trianglelefteq"] \arrow[d] & M \arrow[r, "f\big|^{Mf}"] \arrow[d, "f"'] & Mf \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & 1 \arrow[r, hook] & G \arrow[r, "\mathrm{id}_G"] & G \arrow[r] & 1
\end{tikzcd}$$

We have $[\![\ker f, 1]\!] \trianglelefteq [\![M, G]\!]$. Since $[\![M, G]\!]$ is simple, we get $[\![\ker f, 1]\!] = [\![1, 1]\!]$ or $[\![\ker f, 1]\!] = [\![M, G]\!]$.

We consider the case $[\![\ker f, 1]\!] = [\![M, G]\!]$. Lemma 7 states that $\ker f$ is abelian, and therefore M abelian. We show that the group M is simple:

We *assume* that there exists a non-trivial normal subgroup $1 \neq N \vartriangleleft M$. As a subgroup of the abelian group M, the group N is abelian. Therefore, we get a non-trivial normal crossed submodule $1 \neq [\![N, 1]\!] \vartriangleleft [\![M, 1]\!]$, which is a *contradiction* to the simplicity of $[\![M, 1]\!]$.

Hence, M is simple and abelian. Since the simple abelian groups are exactly those groups that are cyclic and of prime order, we obtain $[\![M, G]\!] = [\![M, 1]\!]$ where M is a cyclic group of prime order. So (3) holds.

We consider the case $[\![\ker f, 1]\!] = [\![1, 1]\!]$. We have a trivial kernel $\ker f = 1$. Hence, the map f is injective. Therefore, $\bar{f} := f\big|^{Mf} \colon M \to Mf$ is bijective. Hence $(f|^{Mf}, \mathrm{id}_G)$ is an isomorphism. So it suffices to show that $[\![Mf, G]\!]$ satisfies (1) or (2). Hence, we may assume that $M \trianglelefteq G$, $\alpha = c_M$, $f = \mathrm{id}_G|_M$.

By Remark 38.(2), we are given a short exact sequence

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r, hook, "\mathrm{id}_M"] \arrow[d, hook, "\mathrm{id}_M"'] & M \arrow[r, "\kappa"] \arrow[d, hook, "\trianglelefteq"] & 1 \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & M \arrow[r, hook, "\trianglelefteq"] & G \arrow[r, "r"] & G/M \arrow[r] & 1\,.
\end{tikzcd}$$

Thus, $[\![M, M]\!] \trianglelefteq [\![M, G]\!]$. Since $[\![M, G]\!]$ is simple we get $[\![M, M]\!] = 1$ or $[\![M, M]\!] = [\![M, G]\!]$.

If $[\![M,M]\!] = 1$ then $(\kappa, r)\colon [\![M,G]\!] \to [\![1,G/M]\!]$ is an isomorphism of crossed modules and we obtain $[\![M,G]\!] \simeq [\![1,K]\!]$ with $K := G/M$. We show that K is a simple group:

If we *assume* that there exists a non-trivial normal subgroup $1 \neq N \lhd K$, then we get a non-trivial normal crossed submodule $[\![1,N]\!] \lhd [\![1,K]\!]$. This is a *contradiction* since $[\![M,G]\!] \simeq [\![1,K]\!]$ is assumed to be simple.

Therefore, we obtain $[\![M,G]\!] \simeq [\![1,K]\!]$ where K is a simple group. So (2) holds.

If $[\![M,M]\!] = [\![M,G]\!]$ then we have yet to show that M is a non-abelian and simple group.

If we *assume* that M is abelian then we have a normal crossed submodule $[\![1,M]\!] \trianglelefteq [\![M,M]\!]$; cf. Remark 10, which is a *contradiction* to the simplicity of $[\![M,M]\!]$.

If we *assume* that M is not simple then we have a non-trivial normal subgroup $N \lhd M$. We get a normal crossed submodule $[\![N,N]\!] \lhd [\![M,M]\!]$, which is a contradiction to the simplicity of $[\![M,M]\!]$.

Altogether, we obtain $[\![M,G]\!] = [\![M,M]\!]$ where M is a non-abelian and simple group. So (1) holds. $\square$

3 Jordan-Hölder Theorem

3.1 A preparation

Lemma 41

(1) *Suppose given a group G. Let $N \trianglelefteq G$ be a normal subgroup. For all $g \in G$, $n \in N$, there exists $n^* \in N$ such that $ng = gn^*$.*

(2) *Suppose given a crossed module $[\![M,G]\!]$. Let $[\![N,H]\!] \trianglelefteq [\![M,G]\!]$ be a normal crossed submodule. For all $m \in M$, $h \in H$, there exists $n_0 \in N$ such that $m^h = mn_0$.*

Proof. Ad (1). Suppose given $g \in G, n \in N$. Since $N \trianglelefteq G$, we have

$$n^* := g^{-}n\, g \in N\ , \qquad \text{and so} \quad ng = gg^{-}ng = gn^*\ .$$

Ad (2). Suppose given $m \in M$, $h \in H$. Since $[\![N,H]\!] \trianglelefteqslant [\![M,G]\!]$ we have

$$n_0 := m^- m^h \in N \ , \qquad \text{and so} \quad m^h = m m^- m^h = m n_0 \ .$$

$\square$

3.2 Intersection and product of crossed modules

Lemma 42 *Let $[\![M,G]\!] = \big(M,G,\alpha,f\big)$ be a crossed module. Suppose we are given crossed submodules $[\![N,H]\!] = \big(N,H,\beta,k\big) \leqslant [\![M,G]\!]$ and $[\![\tilde{N},\tilde{H}]\!] = \big(\tilde{N},\tilde{H},\tilde{\beta},\tilde{k}\big) \leqslant [\![M,G]\!]$.*

(1) *Let $l := f\big|_{N\cap\tilde{N}}^{H\cap\tilde{H}}$ be the restriction of f to $N \cap \tilde{N}$ and $H \cap \tilde{H}$. Consider the group morphism*

$$\begin{array}{rccl} \delta\colon & H\cap\tilde{H} & \longrightarrow & \mathrm{Aut}\Big(N\cap\tilde{N}\Big) \\ & h & \longmapsto & \big(n \mapsto (n)(h\delta) := (n)(h\alpha)\big) \ . \end{array}$$

Then we have a crossed submodule given by

$$[\![N\cap\tilde{N}, H\cap\tilde{H}]\!] = \big(N\cap\tilde{N}, H\cap\tilde{H}, \delta, l\big) \leqslant [\![M,G]\!] \ .$$

We write $[\![N,H]\!] \cap [\![\tilde{N},\tilde{H}]\!] := [\![N\cap\tilde{N}, H\cap\tilde{H}]\!]$.

In particular, we have $[\![N,H]\!] \cap [\![\tilde{N},\tilde{H}]\!] \leqslant [\![N,H]\!]$ and $[\![N,H]\!] \cap [\![\tilde{N},\tilde{H}]\!] \leqslant [\![\tilde{N},\tilde{H}]\!]$.

(2) *If $[\![N,H]\!] \trianglelefteqslant [\![M,G]\!]$ and $[\![\tilde{N},\tilde{H}]\!] \trianglelefteqslant [\![M,G]\!]$ then we have a normal crossed submodule $[\![N,H]\!] \cap [\![\tilde{N},\tilde{H}]\!] \trianglelefteqslant [\![M,G]\!]$.*

In particular, we have $[\![N,H]\!] \cap [\![\tilde{N},\tilde{H}]\!] \trianglelefteqslant [\![N,H]\!]$ and $[\![N,H]\!] \cap [\![\tilde{N},\tilde{H}]\!] \trianglelefteqslant [\![\tilde{N},\tilde{H}]\!]$.

So the situation is given as follows.

$$\begin{tikzcd}
 & \llbracket M,G \rrbracket & \\
\llbracket N,H \rrbracket \arrow[ur, hook] & & \llbracket \tilde{N},\tilde{H} \rrbracket \arrow[ul, hook'] \\
 & \llbracket N,H \rrbracket \cap \llbracket \tilde{N},\tilde{H} \rrbracket \arrow[ul, hook'] \arrow[uu, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Or more explicitly:

$$\begin{tikzcd}
N \arrow[r, hook] \arrow[d, "k"'] & M \arrow[d, "f"] & \tilde{N} \arrow[l, hook'] \arrow[d, "\tilde{k}"] \\
H \arrow[r, hook] & G & \tilde{H} \arrow[l, hook'] \\
 & N \cap \tilde{N} \arrow[uul, hook'] \arrow[uur, hook] \arrow[uu, dotted, bend left] \arrow[d, dashed, "l"] & \\
 & H \cap \tilde{H} \arrow[uul, hook'] \arrow[uur, hook] \arrow[uu, dotted, bend left] &
\end{tikzcd}$$

Proof. Ad (1). We have $N \cap \tilde{N} \leqslant M$ and $H \cap \tilde{H} \leqslant G$. Suppose given $a \in N \cap \tilde{N}$ and $b \in H \cap \tilde{H}$. We have

$$al = af = \underbrace{ak}_{\in H} = \underbrace{a\tilde{k}}_{\in \tilde{H}} \in H \cap \tilde{H} \ ,$$

and we have

$$a^b = a(b\delta) = a(b\alpha) = \underbrace{a(b\beta)}_{\in H} = \underbrace{a(b\tilde{\beta})}_{\in \tilde{H}} \in H \cap \tilde{H} \ .$$

This shows $[\![N \cap \tilde{N}, H \cap \tilde{H}]\!] \leqslant [\![N, H]\!]$, $[\![\tilde{N}, \tilde{H}]\!]$, $[\![M, G]\!]$.

Ad (2). Now we assume that $[\![N, H]\!]$, $[\![\tilde{N}, \tilde{H}]\!] \trianglelefteq [\![M, G]\!]$. We have $N \cap \tilde{N} \trianglelefteq M$ and $H \cap \tilde{H} \trianglelefteq G$.

Since we have $[\![N, H]\!], [\![\tilde{N}, \tilde{H}]\!] \trianglelefteq [\![M, G]\!]$, we get $m^- m^h \in N \cap \tilde{N}$ and $m^g \in N \cap \tilde{N}$ for $m \in M$, $g \in G$. This shows $[\![N \cap \tilde{N}, H \cap \tilde{H}]\!] \trianglelefteq [\![M, G]\!]$. By Remark 24, we also have $[\![N \cap \tilde{N}, H \cap \tilde{H}]\!] \trianglelefteq [\![N, H]\!]$, $[\![\tilde{N}, \tilde{H}]\!]$. $\square$

Lemma 43 *Let $[\![M, G]\!] = \big(M, G, \alpha, f\big)$ be a crossed module.*
Let $[\![\tilde{N}, \tilde{H}]\!] = \big(\tilde{N}, \tilde{H}, \tilde{\beta}, \tilde{k}\big) \trianglelefteq [\![M, G]\!]$ be a normal crossed submodule.

(1) *Suppose given a crossed submodule $[\![N, H]\!] \leqslant [\![M, G]\!]$. Let $l := f\big|_{N\tilde{N}}^{H\tilde{H}}$ be the restriction of f to $N\tilde{N}$ and $H\tilde{H}$. Consider the group morphism*

$$\begin{array}{rrcl} \delta\colon & H\tilde{H} & \longrightarrow & \mathrm{Aut}\Big(N\tilde{N}\Big) \\ & h & \longmapsto & \big(n \mapsto (n)(h\delta) := (n)(h\alpha)\big) \ . \end{array}$$

Then we have a crossed submodule given by

$$[\![N\tilde{N}, H\tilde{H}]\!] = \big(N\tilde{N}, H\tilde{H}, \gamma, l\big) \leqslant [\![M, G]\!] \ .$$

We write $[\![N, H]\!][\![\tilde{N}, \tilde{H}]\!] := [\![N\tilde{N}, H\tilde{H}]\!]$.

In particular, we have $[\![N, H]\!] \leqslant [\![N\tilde{N}, H\tilde{H}]\!]$ and $[\![\tilde{N}, \tilde{H}]\!] \leqslant [\![N\tilde{N}, H\tilde{H}]\!]$.

(2) *If $[\![N, H]\!] \trianglelefteq [\![M, G]\!]$ and $[\![\tilde{N}, \tilde{H}]\!] \trianglelefteq [\![M, G]\!]$ then we have a normal crossed submodule $[\![N\tilde{N}, H\tilde{H}]\!] \trianglelefteq [\![M, G]\!]$.*

In particular, we have $[\![N, H]\!] \trianglelefteq [\![N\tilde{N}, H\tilde{H}]\!]$ and $[\![\tilde{N}, \tilde{H}]\!] \trianglelefteq [\![N\tilde{N}, H\tilde{H}]\!]$.

So the situation is given as follows.

$$
\begin{tikzcd}
 & \llbracket M, G \rrbracket & \\
 & \llbracket N, H \rrbracket \llbracket \tilde{N}, \tilde{H} \rrbracket \arrow[u, hook] & \\
\llbracket N, H \rrbracket \arrow[uur, hook] \arrow[ur, hook] & & \llbracket \tilde{N}, \tilde{H} \rrbracket \arrow[uul, hook] \arrow[ul, hook]
\end{tikzcd}
$$

Or more explicitly:

$$
\begin{tikzcd}
 & M \arrow[d, "f"] & \\
 & G & \\
N \arrow[uur, hook] \arrow[r, hook] \arrow[d, "k"'] & N\tilde{N} \arrow[d, "l"] \arrow[uu, dotted, bend left] & \tilde{N} \arrow[uul, hook] \arrow[l, hook'] \arrow[d, "\tilde{k}"] \\
H \arrow[uur, hook] \arrow[r, hook] & H\tilde{H} \arrow[uu, dotted, bend left] & \tilde{H} \arrow[uul, hook] \arrow[l, hook']
\end{tikzcd}
$$

Proof. Ad (1). We have $N\tilde{N} \leqslant M$ and $H\tilde{H} \leqslant G$. Suppose given $n \in N,\ \tilde{n} \in \tilde{N}$. We have

$$
(n\tilde{n})f = nf \cdot \tilde{n}f = \underbrace{nk}_{\in H} \cdot \underbrace{\tilde{n}\tilde{k}}_{\in \tilde{H}} \in H\tilde{H}\ .
$$

So we may define $l := f\big|_{N\tilde{N}}^{H\tilde{H}}$.

For $n \in N$, $\tilde{n} \in \tilde{N}$ and $h \in H$, $\tilde{h} \in H$, we have

$$\begin{aligned} (n\tilde{n})^{h\tilde{h}} &= n^{h\tilde{h}}\ \tilde{n}^{h\tilde{h}} \\ &\overset{41.(2)}{=} \underbrace{n^h}_{\in N} \underbrace{\tilde{n}_0\ \tilde{n}^{h\tilde{h}}}_{\in \tilde{N}} \in N\tilde{N} \quad (\tilde{n}_0 \in N). \end{aligned}$$

This shows $[\![N\tilde{N}, H\tilde{H}]\!] \leqslant [\![M, G]\!]$. By Remark 20, we also have $[\![N, H]\!] \leqslant [\![N\tilde{N}, H\tilde{H}]\!]$ and $[\![\tilde{N}, \tilde{H}]\!] \leqslant [\![N\tilde{N}, H\tilde{H}]\!]$.

Ad (2). Now we assume that $[\![N, H]\!] \trianglelefteqslant [\![M, G]\!]$. Since we have $N, \tilde{N} \trianglelefteqslant M$ and $H, \tilde{H} \trianglelefteqslant G$ we get $N\tilde{N} \trianglelefteqslant M$ and $H\tilde{H} \trianglelefteqslant G$. For $n \in N,\ \tilde{n} \in \tilde{N},\ g \in G$ we have

$$(n\tilde{n})^g = \underbrace{n^g}_{\in N} \underbrace{\tilde{n}^g}_{\in \tilde{N}} \in N\tilde{N}\ .$$

For $m \in M,\ h \in H,\ \tilde{h} \in \tilde{H}$ we have

$$m^- m^{h\tilde{h}} = m^- (m^h)^{\tilde{h}} = \underbrace{m^- m^h}_{\in N} \underbrace{(m^h)^- (m^h)^{\tilde{h}}}_{\in \tilde{N}} \in N\tilde{N}\ .$$

This shows $[\![N\tilde{N}, H\tilde{H}]\!] \trianglelefteqslant [\![M, G]\!]$. By Remark 24, we also have $[\![N, H]\!] \trianglelefteqslant [\![N\tilde{N}, H\tilde{H}]\!]$ and $[\![\tilde{N}, \tilde{H}]\!] \trianglelefteqslant [\![N\tilde{N}, H\tilde{H}]\!]$.

□

3.3 Zassenhaus

Lemma 44 *Let* $A := [\![A_1, A_2]\!]$, $\tilde{B} := [\![\tilde{B}_1, \tilde{B}_2]\!]$, $B := [\![B_1, B_2]\!]$ *and* $C := [\![C_1, C_2]\!]$ *be crossed modules. Suppose we have* $A \leqslant C$ *and* $\tilde{B} \trianglelefteqslant B \leqslant C$, *i.e. we have the following situation.*

$$\begin{tikzcd}
A = [\![A_1, A_2]\!] \arrow[r, hook] & {[\![C_1, C_2]\!] = C} \\
& {[\![B_1, B_2]\!] = B} \arrow[u, hook] \\
& {[\![\tilde{B}_1, \tilde{B}_2]\!] = \tilde{B}} \arrow[u, hook, "\trianglelefteq"']
\end{tikzcd}$$

(1) *We have* $A \cap \tilde{B} \trianglelefteq A \cap B$.

(2) *If* $A \trianglelefteq C$, *then* $A\tilde{B} \trianglelefteq AB$.

Proof. Ad (1). By Lemma 42.(1) and Remark 20, we have $A \cap \tilde{B} \leqslant A \cap B$.

For $n \in A_1 \cap \tilde{B}_1$ and $g \in A_2 \cap B_2$, we have

$$\begin{aligned}
n^g &\in A_1 \quad \text{since } n \in A_1 \text{ and } g \in A_2, \\
n^g &\in \tilde{B}_1 \quad \text{since } n \in \tilde{B}_1,\ g \in B_2 \text{ and } \tilde{B} \trianglelefteq B\ .
\end{aligned}$$

It follows that $n^g \in A_1 \cap \tilde{B}_1$.

For $m \in A_1 \cap B_1$ and $h \in A_2 \cap \tilde{B}_2$, we have

$$\begin{aligned}
m^- m^h &\in A_1 \quad \text{since } m \in A_1 \text{ and } h \in A_2\ , \\
m^- m^h &\in \tilde{B}_1 \quad \text{since } m \in B_1,\ h \in \tilde{B}_2, \text{ and } \tilde{B} \trianglelefteq B\ .
\end{aligned}$$

It follows that $m^- m^h \in A_1 \cap \tilde{B}_1$.

Therefore, we have $A \cap \tilde{B} \trianglelefteq A \cap B$.

Ad (2). By Lemma 43.(1), we have $A \leqslant AB$ and $\tilde{B} \leqslant AB$. Therefore $A\tilde{B} \leqslant AB$.

Suppose given $n := a_1\tilde{b}_1$ with $a_1 \in A_1$ and $\tilde{b}_1 \in \tilde{B}_1$, and $g := a_2 b_2$ with $a_2 \in A_2$ and $b_2 \in B_2$. We have

$$
\begin{aligned}
n^g &= (a_1\tilde{b}_1)^{a_2b_2} \\
&= a_1^{a_2b_2}\cdot \tilde{b}_1^{\,a_2b_2} \\
&\overset{41.(1)}{=} a_1^{a_2b_2}\cdot (\tilde{b}_1^{\,b_2})^{a_2^*} && (a_2^*\in A_2,\ A_2\trianglelefteq C_2\ni b_2) \\
&\overset{41.(2)}{=} a_1^{a_2b_2}\cdot (\tilde{b}_1^{\,b_2})(a_1)_0 && \big((a_1)_0\in A_1,\ A\trianglelefteq C\big) \\
&\overset{41.(1)}{=} \big(a_1^{a_2b_2}\ (a_1)_0^*\big)\cdot (\tilde{b}_1^{\,b_2})\in A_1\tilde{B}_1 && \big((a_1)_0^*\in A_1\big)\ .
\end{aligned}
$$

Suppose given $m := a_1b_2$ with $a_1 \in\in A_1$ and $b_1 \in B_1$, and $h := a_2\tilde{b}_2$ with $a_2 \in A_2$ and $b_2 \in \tilde{B}_2$. We have

$$
\begin{aligned}
m^-m^h &= (a_1b_1)^-(a_1b_1)^{a_2\tilde{b}_2} \\
&= b_1^-\ a_1^-\ a_1^{a_2\tilde{b}_2}\ b_1^{a_2\tilde{b}_2} \\
&\overset{41.(1)}{=} b_1^-\ a_1^-\ a_1^{a_2\tilde{b}_2}\ (b_1^{\tilde{b}_2})^{a_2^*} && (a_2^*\in A_2,\ A_2\trianglelefteq C_2\ni \tilde{b}_2) \\
&\overset{41.(2)}{=} b_1^-\ a_1^-\ a_1^{a_2\tilde{b}_2}\ (b_1^{\tilde{b}_2})(a_1)_0 && \big((a_1)_0\in A_1,\ A\trianglelefteq C\big) \\
&\overset{41.(1)}{=} b_1^-\ a_1^-\ a_1^{a_2\tilde{b}_2}\ (a_1)_0^*\ (b_1^{\tilde{b}_2}) && \big((a_1)_0^*\in A_1\big) \\
&= b_1^-\ \hat{a}_1\ b_1^{\tilde{b}_2} && \big(\hat{a}_1 := a_1^-a_1^{a_2\tilde{b}_2}\ (a_1)_0^*\in A_1\big) \\
&\overset{41.(1)}{=} (\hat{a}_1^*)\cdot(b_1^-\ b_1^{\tilde{b}_2})\in A_1\tilde{B}_1 && (\hat{a}_1^*\in A_1,\ A_1\trianglelefteq C_1\ni b_1^-)\ .
\end{aligned}
$$

Therefore, we have $A\tilde{B} \trianglelefteq AB$. $\square$

Lemma 45 *(Butterfly Lemma) Let $C := [\![C_1, C_2]\!]$ be a crossed module.*
Let $A := [\![A_1, A_2]\!], B := [\![B_1, B_2]\!] \leqslant C$ be crossed submodules.
Further, let $\tilde{A} := [\![\tilde{A}_1, \tilde{A}_2]\!] \trianglelefteq A$, $\tilde{B} := [\![\tilde{B}_1, \tilde{B}_2]\!] \trianglelefteq B$ be normal crossed submodules.

$$
\begin{tikzcd}
 & C & \\
A \arrow[ur, hook] & & B \arrow[ul, hook'] \\
\tilde{A} \arrow[u, hook, "\trianglelefteq"] & & \tilde{B} \arrow[u, hook, "\trianglelefteq"']
\end{tikzcd}
$$

We have normal crossed submodules

$$\tilde{A}(A \cap \tilde{B}) \trianglelefteqslant \tilde{A}(A \cap B)$$
$$(\tilde{A} \cap B)\tilde{B} \trianglelefteqslant (A \cap B)\tilde{B} \ ,$$

and isomorphic factor crossed modules

$$\tilde{A}(A \cap B) \Big/ \tilde{A}(A \cap \tilde{B}) \ \simeq \ A \cap B \Big/ (\tilde{A} \cap B)(A \cap \tilde{B}) \ \simeq \ (A \cap B)\tilde{B} \Big/ (\tilde{A} \cap B)\tilde{B} \ .$$

If we visualize the involved crossed submodules in a diagram, then the "butterfly" becomes apparent:

$$\begin{tikzcd}
A & & & & & & B \\
& \tilde{A}(A \cap B) \arrow[ul, hook] & & & & (A \cap B)\tilde{B} \arrow[ur, hook] & \\
& & & A \cap B \arrow[ull, hook] \arrow[urr, hook] & & & \\
& \tilde{A}(A \cap \tilde{B}) \arrow[uu, hook] & & & & (\tilde{A} \cap B)\tilde{B} \arrow[uu, hook] & \\
\tilde{A} \arrow[ur, hook] & & & (\tilde{A} \cap B)(A \cap \tilde{B}) \arrow[uu, hook] \arrow[ull, hook] \arrow[urr, hook] & & & \tilde{B} \arrow[ul, hook] \\
& \tilde{A} \cap B \arrow[ul, hook] \arrow[urr, hook] & & & & A \cap \tilde{B} \arrow[ull, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Proof. By assumption, we have $\tilde{B} \trianglelefteqslant B$. With Lemma 44.(1), we get $A \cap \tilde{B} \trianglelefteqslant A \cap B \leqslant A$. With Lemma 44.(2), we get $\tilde{A}(A \cap \tilde{B}) \trianglelefteqslant \tilde{A}(A \cap B) \leqslant A$.

We consider the crossed module morphism

$$(\lambda_1, \lambda_2)\colon A \cap B \longrightarrow {}^{\tilde{A}(A\cap B)}\!\big/_{\tilde{A}(A\cap \tilde{B})}\ ,$$

where, for $i \in [1, 2]$, we have

$$\begin{array}{rcccc} \lambda_i & : & A_i \cap B_i & \longrightarrow & {}^{\tilde{A}_i(A_i\cap B_i)}\!\big/_{\tilde{A}_i(A_i\cap \tilde{B}_i)} \\ & & x_i & \longmapsto & x_i\big(\tilde{A}_i(A_i \cap \tilde{B}_i)\big)\ . \end{array}$$

As a composite of an inclusion and a reduction morphism, (λ_1, λ_2) is in fact a crossed module morphism.

The crossed module morphism (λ_1, λ_2) is surjective:

Suppose given $x \in \tilde{A}_1(A_1 \cap B_1)/\tilde{A}_1(A_1 \cap \tilde{B}_1)$, which can be written as $x = \tilde{a}z\ \tilde{A}_1(A_1 \cap \tilde{B}_1)$, with $\tilde{a} \in \tilde{A}_1$ and $z \in (A_1 \cap B_1)$. We have

$$\begin{array}{rcll} x & = & \tilde{a}z\ \big(\tilde{A}_1(A_1 \cap \tilde{B}_1)\big) & \\ & \overset{41.(1)}{=} & z\ \tilde{a}^*\ \big(\tilde{A}_1(A_1 \cap \tilde{B}_1)\big) & (\tilde{a}^* \in \tilde{A}_1) \\ & = & z\ \big(\tilde{A}_1(A_1 \cap \tilde{B}_1)\big) & \\ & = & z\lambda_1\ . & \end{array}$$

Hence, λ_1 is surjective. In the same way we conclude that λ_2 is surjective. Therefore, (λ_1, λ_2) is surjective.

We have $\ker \lambda_1 = (\tilde{A}_1 \cap B_1)(A_1 \cap \tilde{B}_1)$:

Ad $\subseteq$. Suppose given $k \in \ker \lambda_1 \subseteq (A_1 \cap B_1)$. We have

$$\begin{array}{rcl} k\lambda_1 = 1\ \tilde{A}_1(A_1 \cap \tilde{B}_1) & \Rightarrow & k \in \tilde{A}_1(A_1 \cap \tilde{B}_1) \\ & \Rightarrow & k \in (A_1 \cap B_1) \cap \tilde{A}_1(A_1 \cap \tilde{B}_1)\ . \end{array}$$

We can write $k = \tilde{a}z$ with $\tilde{a} \in \tilde{A}_1$ and $z \in (A_1 \cap \tilde{B}_1)$. We have

$$\tilde{a} = \underbrace{k}_{\in B_1} \cdot \underbrace{z^-}_{\in \tilde{B}_1 \subseteq B_1} \in B_1 \quad \Rightarrow \quad \tilde{a} \in (\tilde{A}_1 \cap B_1) \quad \Rightarrow \quad k = \tilde{a}z \in (\tilde{A}_1 \cap B_1)(A_1 \cap \tilde{B}_1)\ .$$

Ad $\supseteq$. Suppose given $k \in (\tilde{A}_1 \cap B_1)(A_1 \cap \tilde{B}_1) \subseteq \tilde{A}_1(A_1 \cap \tilde{B}_1)$. We have

$$k\lambda_1 = k\ \tilde{A}_1(A_1 \cap \tilde{B}_1) = 1\ \tilde{A}_1(A_1 \cap \tilde{B}_1)\ ,$$

which shows $k \in \ker \lambda_1$.
By the same calculation we get $\ker \lambda_2 = (\tilde{A}_2 \cap B_2)(A_2 \cap \tilde{B}_2)$.
Hence, we have $\ker(\lambda_1, \lambda_2) = (\tilde{A} \cap B) \cap (A \cap \tilde{B})$.
Therefore, the conditions for Corollary 28 are met and we get the isomorphism

$$(\bar{\lambda}_1, \bar{\lambda}_2) \quad : A \cap B \Big/ (\tilde{A} \cap B)(A \cap \tilde{B}) \xrightarrow{\sim} \tilde{A}(A \cap B) \Big/ \tilde{A}(A \cap \tilde{B})\ ,$$

where, for $i \in [1, 2]$, we have

$$\begin{array}{rcccc} \bar{\lambda}_i & : & A_i \cap B_i \Big/ (\tilde{A}_i \cap B_i)(A_i \cap \tilde{B}_i) & \xrightarrow{\sim} & \tilde{A}_i(A_i \cap B_i) \Big/ \tilde{A}_i(A_i \cap \tilde{B}_i) \\ & & x_i(\tilde{A}_i \cap B_i)(A_i \cap \tilde{B}_i) & \longmapsto & x_i\tilde{A}_i(A_i \cap \tilde{B}_i)\ . \end{array}$$

So we obtain

$$A \cap B \Big/ (\tilde{A} \cap B)(A \cap \tilde{B}) \simeq \tilde{A}(A \cap B) \Big/ \tilde{A}(A \cap \tilde{B})\ .$$

For reasons of symmetry, we also have

$$A \cap B \Big/ (\tilde{A} \cap B)(A \cap \tilde{B}) \simeq (A \cap B)\tilde{B} \Big/ (A \cap \tilde{B})\tilde{B}\ .$$

$\square$

3.4 Schreier and Jordan-Hölder

Definition 46 (Composition series) Suppose given a crossed module C. A sequence of crossed submodules

$$\Sigma \,:\; C = C_0 \geqslant C_1 \geqslant \cdots \geqslant C_s = 1$$

is called *subnormal series* if $C_{i+1} \trianglelefteq C_i$ for $i \in [0, s-1]$.

The factor crossed module C_i/C_{i+1} is called the i-th *sub-factor* of this subnormal series, where $i \in [0, s-1]$.

We call s the *length* of Σ.

A subnormal series whose sub-factors are all simple is called a *composition series*. The sub-factors of a composition series are called its *composition factors*.

Definition 47 (Equivalent subnormal series) Suppose we have subnormal series of a crossed module C

$$\Sigma \,:\; C = C_0 \geqslant C_1 \geqslant \cdots \geqslant C_s = 1$$

and

$$\Sigma^* \,:\; C = D_0 \geqslant D_1 \geqslant \cdots \geqslant D_t = 1 \;.$$

We say that Σ is *equivalent* to Σ^* if there exists a bijection $\sigma\colon [0, s-1] \to [0, t-1]$ such that, for $i \in [0, s-1]$,

$$C_i\big/C_{i+1} \simeq D_{i\sigma}\big/D_{i\sigma+1}$$

holds. In particular, if Σ is equivalent to Σ^*, then we have $s = t$.

Definition 48 (Refinement) Let C be a crossed module. Suppose given subnormal series

$$\begin{aligned} \Sigma \,&:\; C = C_0 \geqslant C_1 \geqslant \cdots \geqslant C_s = 1 \;, \\ \Sigma^* \,&:\; C = D_0 \geqslant D_1 \geqslant \cdots \geqslant D_t = 1 \;. \end{aligned}$$

Then Σ^* is called *refinement* of Σ if there exists an injective monotone map $\gamma\colon [0, s] \to [0, t]$ such that $(0)\gamma = 0$, $(s)\gamma = t$ and $C_i = D_{i\gamma}$ for $i \in [0, s]$.

Schreier's theorem generalizes to

Theorem 49 *Two subnormal series of a crossed module C have equivalent refinements.*

Proof. Let

$$\Sigma \ : \ C = A_0 \geqslant A_1 \geqslant \cdots \geqslant A_s = 1$$

and

$$\Sigma^* \ : \ C = B_0 \geqslant B_1 \geqslant \cdots \geqslant B_t = 1$$

be two subnormal series of C. For each $i \in [0, s-1]$ and $j \in [0, t]$ we define

$$A_{i,j} := A_{i+1} \left(A_i \cap B_j \right) \ .$$

Then we have $A_{i,0} = A_i$ and $A_{i,t} = A_{i+1}$ for $i \in [0, s-1]$. A refinement of Σ is given by

$$\begin{array}{llcccccccccc}
\Sigma' \colon & C & = & \overbrace{A_{0,0}}^{A_0} & \geqslant & A_{0,1} & \geqslant & \cdots & \geqslant & A_{0,t-1} & & \\
 & & \geqslant & \overbrace{A_{1,0}}^{=A_{0,t}=A_1} & \geqslant & A_{1,1} & \geqslant & \cdots & \geqslant & A_{1,t-1} & & \\
 & & \geqslant & \overbrace{A_{2,0}}^{=A_{1,t}=A_2} & \geqslant & A_{2,1} & \geqslant & \cdots & \geqslant & A_{2,t-1} & & \\
 & & & \vdots & & & & & & & & \\
 & & \geqslant & \overbrace{A_{s-1,0}}^{=A_{s-2,t}=A_{s-1}} & \geqslant & A_{s-1,1} & \geqslant & \cdots & \geqslant & A_{s-1,t-1} & \geqslant & A_{s-1,t} = 1 \ .
\end{array}$$

One should note that we have indeed normal embeddings at each position of the sequence Σ' because Lemma 45 yields

$$A_{i,j+1} = A_{i+1}(A_i \cap B_{j+1}) \trianglelefteqslant A_{i+1}(A_i \cap B_j) = A_{i,j} \ .$$

Similarly, for each $j \in [0, t-1]$ and $i \in [0, s]$, we define

$$B_{j,i} := (A_i \cap B_j) B_{j+1} \ .$$

This gives us a refinement of Σ^*:

$$
\begin{array}{lllllllllll}
\Sigma^{*\prime}\colon & C & = & \overbrace{B_{0,0}}^{B_0} & \geqslant & B_{0,1} & \geqslant & \cdots & \geqslant & B_{0,s-1} & \\
 & & \geqslant & \overbrace{B_{1,0}}^{=B_{0,s}=B_1} & \geqslant & B_{1,1} & \geqslant & \cdots & \geqslant & B_{1,s-1} & \\
 & & \geqslant & \overbrace{B_{2,0}}^{=B_{1,s}=B_2} & \geqslant & B_{2,1} & \geqslant & \cdots & \geqslant & B_{2,s-1} & \\
 & & & \vdots & & & & & & & \\
 & & \geqslant & \overbrace{B_{t-1,0}}^{=B_{t-2,s}=B_{t-1}} & \geqslant & B_{t-1,1} & \geqslant & \cdots & \geqslant & B_{t-1,s-1} & \geqslant \; B_{t-1,s} = 1 \; .
\end{array}
$$

We have a bijection

$$
\begin{array}{rcrcl}
\varphi & : & [0, st-1] & \to & [0, s-1] \times [0, t-1] \\
 & & k & \mapsto & k\varphi := (i, j) \; ,
\end{array}
$$

where $k = ti + j$ with $j \in [0, t-1]$. So, for $k \in [0, st-1]$ with $k\varphi =: (i, j)$, where $i \in [0, s-1]$, $j \in [0, t-1]$, we have

$$
(k+1)\varphi = \begin{cases} (i+1, 0) & \text{if } t \text{ divides } k+1 \\ (i, j+1) & \text{else .} \end{cases}
$$

Let $A'_k := A_{k\varphi}$ for $k \in [0, st-1]$. Let $A'_{st} := 1$.

If t divides $k+1$, we have $A'_{k+1} = A_{(k+1)\varphi} = A_{i+1,0} = A_{i,t} = A_{i,j+1}$. If t does not divide $k+1$, we have $A'_{k+1} = A_{i,j+1}$. So we obtain

$$
\Sigma' \; : \; C = A'_0 \geqslant A'_1 \geqslant \cdots \geqslant A'_{st-1} \geqslant A'_{st} = 1 \; .
$$

We have a bijection

$$
\begin{array}{rcrcl}
\varphi^* & : & [0, st-1] & \to & [0, t-1] \times [0, s-1] \\
 & & k & \mapsto & k\varphi^* := (j, i) \; ,
\end{array}
$$

where $k = sj + i$ with $j \in [0, t-1]$. So, for $k \in [0, st-1]$ with $k\varphi^* =: (j, i)$, where $j \in [0, t-1]$, $i \in [0, s-1]$, we have

$$(k+1)\varphi^* = \begin{cases} (j+1, 0) & \text{if } s \text{ divides } k+1 \\ (j, i+1) & \text{else .} \end{cases}$$

Let $B'_k := B_{k\varphi^*}$ for $k \in [0, st-1]$. Let $B'_{st} := 1$. Then

$$\Sigma^{*\prime} \;:\; C = B'_0 \geqslant B'_1 \geqslant \cdots \geqslant B'_{st-1} \geqslant B'_{st} = 1 \;.$$

We conclude that both refinements Σ' and $\Sigma^{*\prime}$ have the same length st.

Now consider the bijective map

$$\begin{array}{rcrcl} \tau & : & [0, s-1] \times [0, t-1] & \to & [0, t-1] \times [0, s-1] \\ & & (i, j) & \mapsto & (j, i) \;. \end{array}$$

Let $\sigma := \varphi\tau(\varphi^*)^- \colon [0, st-1] \to [0, st-1]$. As a composition of bijective maps, σ is bijective. Then, for $k \in [0, st-1]$ with $k\varphi := (i, j)$, we obtain $k\sigma\varphi^* = k\varphi\tau = (i, j)\tau = (j, i)$, and so

$$\begin{aligned} A'_k/A'_{k+1} &= A_{k\varphi}/A_{(k+1)\varphi} = A_{i,j}/A_{i,j+1} = A_{i+1}(A_i \cap B_j)/A_{i+1}(A_i \cap B_{j+1}) \\ &\overset{45}{\simeq} (A_i \cap B_j)B_{j+1}/(A_{i+1} \cap B_j)B_{j+1} = B_{j,i}/B_{j,i+1} \\ &= B_{k\sigma\varphi^*}/B_{(k\sigma+1)\varphi^*} = B'_{k\sigma}/B'_{k\sigma+1} \;. \end{aligned}$$

Hence, Σ' and $\Sigma^{*\prime}$ are equivalent. $\square$

Definition 50 (Reduction) Let C be a crossed module. Let

$$\Sigma \colon C = C_0 \geqslant C_1 \geqslant C_2 \geqslant \cdots \geqslant C_s = 1$$

be a subnormal series of C.

(i) The subnormal series Σ is called *reduced* if $C_i \triangleright C_{i+1}$ holds for all $i \in [0, s-1]$.

(ii) Let $R_\Sigma := \{i \in [0, s-1] \colon C_i > C_{i+1}\} \cup \{s\} \subseteq [0, s]$. Let $u := |R_\Sigma| - 1$. Let $\delta : [0, u] \to R_\Sigma$ be the monotone bijection.

The *reduction* of the subnormal series Σ is given by

$$\Sigma_{\mathrm{red}} \colon C = C_{0\delta} > C_{1\delta} > \cdots > C_{u\delta} > 1 = 1 \;.$$

Lemma 51 *Let Σ and Σ^* be two equivalent subnormal series of a crossed module C. Then their reductions Σ_{red} and Σ^*_{red} are equivalent.*

Proof. Write

$$\begin{aligned}\Sigma \quad &: \quad C = A_0 \geqslant A_1 \geqslant \cdots \geqslant A_s = 1 \\ \Sigma^* \quad &: \quad C = B_0 \geqslant B_1 \geqslant \cdots \geqslant B_s = 1\ .\end{aligned}$$

Let $\delta\colon [0,u] \to R_\Sigma$ be the bijective monotone map. Let $\epsilon\colon [0,v] \to R_{\Sigma^*}$ be the bijective monotone map. Write $A'_i := A_{i\delta}$ for $i \in [0,u]$. Write $B'_j := B_{j\epsilon}$ for $j \in [0,v]$.

The reductions are given by

$$\begin{aligned}\Sigma_{\mathrm{red}} \quad &: \quad C = A'_0 > A'_1 > \cdots > A'_u = 1 \\ \Sigma^*_{\mathrm{red}} \quad &: \quad C = B'_0 > B'_1 > \cdots > B'_v = 1\ .\end{aligned}$$

Since Σ and Σ^* are equivalent we can find a bijection $\sigma\colon [0,s-1] \to [0,s-1]$ such that ${}^{A_i}\!/_{A_{i+1}} \simeq {}^{B_{i\sigma}}\!/_{B_{i\sigma+1}}$ holds for all $i \in [0,s-1]$.

For any index $i \in [0,s-1]$ we have

$$i \in R_\Sigma \quad \Leftrightarrow \quad A_{i+1} < A_i \quad \Leftrightarrow \quad B_{i\sigma+1} < B_{i\sigma} \quad \Leftrightarrow \quad i\sigma \in R_{\Sigma^*}\ ,$$

i.e. we have $(R_\Sigma)\sigma = R_{\Sigma^*}$. With $\rho := \delta\sigma\epsilon^-$, the situation can be depicted as follows.

$$\begin{tikzcd} {[0,u]} \arrow[r, "\rho", "\sim"'] \arrow[d, "\delta"', "\wr"] & {[0,v]} \arrow[d, "\epsilon", "\wr"'] \\ R_\Sigma \arrow[r, "\sim", "\sigma"'] & R_{\Sigma^*} \end{tikzcd}$$

Hence, for any $i \in [0,u-1]$ we have

$${}^{A'_i}\!/_{A'_{i+1}} \;=\; {}^{A_{i\delta}}\!/_{A_{i\delta+1}} \;\simeq\; {}^{B_{i\delta\sigma}}\!/_{B_{i\delta\sigma+1}} \;=\; {}^{B'_{i\delta\sigma\epsilon^-}}\!/_{B'_{i\delta\sigma\epsilon^-+1}} \;=\; {}^{B'_{i\rho}}\!/_{B'_{i\rho+1}}\ .$$

Therefore, Σ_{red} and Σ^*_{red} are equivalent. $\square$

Lemma 52 *Suppose given a composition series Σ of a crossed module C. Let Σ' be a refinement of Σ. Then $\Sigma = \Sigma'_{\mathrm{red}}$.*

Proof. Write

$$\begin{array}{rcccccccccccc} \Sigma & : & C & = & C_0 & > & C_1 & > & C_2 & > & \dots & > & C_s & = & 1\ , \\ \Sigma' & : & C & = & D_0 & \geqslant & D_1 & \geqslant & D_2 & \geqslant & \dots & \geqslant & D_t & = & 1\ . \end{array}$$

By definition of refinement, we have an injective monotone map $\gamma\colon [0,s] \to [0,t]$ such that $C_i = D_{i\gamma}$ for $i \in [0,s]$; cf. Definition 48.

Let C_i/C_{i+1} be a composition factor of Σ, where $i \in [0, s-1]$. We have $C_i/C_{i+1} = D_{i\gamma}/D_{(i+1)\gamma}$. Consider the map

$$\begin{array}{rccl} \tilde{\gamma}\colon & [0,s] & \longrightarrow & [0,t] \\ & i & \longmapsto & i\tilde{\gamma} := \min\{j \in [i\gamma, t-1]\colon D_{i\gamma} \geqslant D_j > D_{j+1}\}\ . \end{array}$$

Note that $D_{i\tilde{\gamma}} = D_{i\gamma} = C_i$ for $i \in [0,s]$. We claim $D_{i\tilde{\gamma}+1} \overset{!}{=} D_{(i+1)\tilde{\gamma}}$ for $i \in [0, s-1]$.

We *assume* the contrary. Namely, there exists an index $k \in [i\tilde{\gamma}+1, (i+1)\tilde{\gamma}-1]$ with $D_{i\tilde{\gamma}} \triangleright D_k \triangleright D_{(i+1)\tilde{\gamma}}$. It follows that

$$C_i/C_{i+1} \;=\; D_{i\tilde{\gamma}}/D_{(i+1)\tilde{\gamma}} \;\triangleright\; D_k/D_{(i+1)\tilde{\gamma}} \;>\; 1\ ,$$

which is a *contradiction* to the simplicity of the composition factor C_i/C_{i+1}. This proves $D_{i\tilde{\gamma}+1} = D_{(i+1)\tilde{\gamma}}$. So, for $i \in [0, s-1]$, we have

$$D_{i\tilde{\gamma}} \;>\; D_{i\tilde{\gamma}+1} \;=\; D_{i\tilde{\gamma}+2} \;=\; \dots \;=\; D_{(i+1)\tilde{\gamma}}\ .$$

Consider the reduction of Σ':

$$\Sigma'_{\mathrm{red}} \;:\; C \;=\; D_{0\delta} \;>\; D_{1\delta} \;>\; D_{2\delta} \;>\; \dots \;>\; D_{u\delta} \;=\; 1\ ,$$

where $\delta\colon [0,u] \to R_{\Sigma'}$ is the monotone bijective map; cf. Definition 50. We have

$$\begin{aligned} \{C_i\colon i \in [0,s]\} &= \{D_{i\tilde{\gamma}}\colon i \in [0,s]\} \\ &= \{D_j\colon j \in [0,t-1],\ D_j > D_{j+1}\} \cup \{1\} \\ &= \{D_j\colon j \in R_{\Sigma'}\}\ . \end{aligned}$$

Hence $\Sigma = \Sigma'_{\mathrm{red}}$. $\square$

Jordan-Hölder's theorem generalizes to

Theorem 53 *Two composition series of a crossed module C are equivalent.*

Proof. Let Σ, Σ^* be two composition series of C.

By Theorem 49, there exist refinements Σ' of Σ and $\Sigma^{*\prime}$ of Σ^* such that Σ' and $\Sigma^{*\prime}$ are equivalent. By Lemma 51, the reduced subnormal series Σ'_{red} and ${\Sigma^{*}}'_{\mathrm{red}}$ are still equivalent. Then, Lemma 52 implies $\Sigma = \Sigma'_{\mathrm{red}}$ and $\Sigma^* = {\Sigma^{*}}'_{\mathrm{red}}$.

Therefore, Σ and Σ^* are equivalent.

$\square$

Definition 54 (Finite crossed module) A crossed module $[\![M, G]\!]$ is said to be *finite* if the groups M and G are finite. The (total) *order* of a finite crossed module $[\![M, G]\!]$ is given by $\big|[\![M, G]\!]\big| := |M| \cdot |G|$.

Lemma 55 *Each finite crossed module has a composition series.*

Proof. Suppose given a finite crossed module $C := [\![M, G]\!]$. We show the assertion by induction on the order $|C|$.

Let $|C| = 1$. Then we have $|M| = |G| = 1$ and a composition series is given by

$$C = 1 \ .$$

Suppose the claim has been proven for all crossed modules $\tilde{C}$ with $|\tilde{C}| < |C|$.

The set $\mathcal{N}$ of proper normal crossed submodules of C is non-empty because of $[\![1, 1]\!] = 1 \leqslant C$. It is finite because $\mathcal{P}(M) \times \mathcal{P}(G)$ is finite. The set $\mathcal{N}$ is partially ordered via inclusion. Therefore, $\mathcal{N}$ contains maximal elements.

Let $\tilde{C}$ be such a maximal element. We have $\big|\tilde{C}\big| < \big|C\big|$. Therefore, by the induction hypothesis, we are given a composition series of $\tilde{C}$:

$$\tilde{C} = C_1 > C_2 > C_3 > \cdots > C_s = 1 \ .$$

Further, the factor C/C_1 is simple because C_1 is maximal in C. Therefore, C has a composition series given by

$$C = C_0 > \underbrace{C_1}_{=\tilde{C}} > C_2 > \cdots > C_s = 1 \ .$$

$\square$

4 Actions of crossed modules

In the following, let $[\![M,G]\!] = \big(M,G,\alpha,f\big)$ be a crossed module.

4.1 Preliminaries

4.1.1 Semidirect Product

We recall the notion of a semidirect product.

Definition 56 (Semidirect product) Let M and G be groups. Suppose given a group morphism

$$\alpha\colon G \to \operatorname{Aut}(M),\ g \mapsto g\alpha\ ,$$

where, for $m \in M$, $g \in G$, we write $m(g\alpha) = m^g$.

The cartesian product $G\times M = \{(g,m)\colon g \in G, m \in M\}$, together with the multiplication

$$\begin{array}{rccl} (\cdot)\colon & (G\times M)\times(G\times M) & \longrightarrow & G\times M \\ & \big((g,m),(\tilde{g},\tilde{m})\big) & \longmapsto & (g,m)\cdot(\tilde{g},\tilde{m}) := (g\tilde{g}, m^{\tilde{g}}\,\tilde{m}) \end{array}$$

is called *semidirect product* of M and G, which we denote by $G \ltimes M$.

Lemma 57 *In the situation of Definition 56, the semidirect product $G \ltimes M$ is a group.*

Proof. For $(g,m),(h,n),(l,k) \in G \ltimes M$ we have

$$\begin{aligned} \big((g,m)\cdot(h,n)\big)\cdot(k,l) &= (gh, m^h\,n)\cdot(k,l) = (ghk, (m^h\,n)^k\,l) = (ghk, m^{hk}\,n^k\,l) \\ &= (g,m)\cdot(hk, n^k\,l) = (g,m)\cdot\big((h,n)\cdot(k,l)\big)\ , \end{aligned}$$

and therefore, $(\cdot)$ is associative. We have the neutral element $1_{G\ltimes M} = (1,1)$. The inverse element of $(g,m) \in G \ltimes M$ is given by $(g,m)^- = (g^-, (m^-)^{g^-})$, because of

$$(g,m)\cdot\big(g^-,(m^-)^{g^-}\big) = \big(gg^-, m^{g^-}(m^-)^{g^-}\big) = \big(gg^-,(mm^-)^{g^-}\big) = (1,1)\ .$$

□

4.1.2 The group morphisms s, i, t

Lemma 58 *We have group morphisms*

$$\begin{array}{rlcc} s\colon & (G \ltimes M) \to G\ , & (g,m) & \mapsto & g\ , \\ i\colon & (G \ltimes M) \leftarrow G\ , & (g,1) & \leftarrow\!\shortmid & g\ , \\ t\colon & (G \ltimes M) \to G\ , & (g,m) & \mapsto & g \cdot mf\ . \end{array}$$

We have $is = \mathrm{id}_G$ and $it = \mathrm{id}_G$.

Proof. Let $(g,m), (\tilde{g},\tilde{m}) \in G \ltimes M$. Let $g, \tilde{g} \in G$.

We have

$$\big((g,m)\cdot(\tilde{g},\tilde{m})\big)s = (g\tilde{g}, m^{\tilde{g}}\,\tilde{m})s = g \cdot \tilde{g} = (g,m)s \cdot (\tilde{g},\tilde{m})s\ .$$

We have

$$(g\tilde{g})i = (g\tilde{g},1) = (g,1)\cdot(\tilde{g},1) = gi \cdot \tilde{g}i\ .$$

We have

$$\begin{aligned} (g,m)\cdot(\tilde{g},\tilde{m})t &= \big((g\tilde{g}, m^{\tilde{g}}\ \tilde{m})\big)t = g\tilde{g}\ (m^{\tilde{g}}\tilde{m})f = g\tilde{g}\ (m^{\tilde{g}})f\ (\tilde{m})f \overset{\text{(CM1)}}{=} g\tilde{g}\ (mf)^{\tilde{g}}\ \tilde{m}f \\ &= g\tilde{g}\ \tilde{g}^{-}(mf)\tilde{g}\ \tilde{m}f = g(mf)\ \tilde{g}(\tilde{m}f) \\ &= (g,m)t\ (\tilde{g},\tilde{m})t\ . \end{aligned}$$

□

4.2 Crossed sets

4.2.1 $[\![M,G]\!]$-crossed sets

Concerning G-sets, cf. Reminder 3.

Definition 59 ($[\![M,G]\!]$-crossed set) Suppose given a crossed module $[\![M,G]\!]$. Let the maps s, i and t be given as in Lemma 58. Suppose given a $(G \ltimes M)$-set U, a G-set V, and maps

$$\sigma\colon U \to V$$
$$\iota\colon U \leftarrow V$$
$$\tau\colon U \to V$$

such that the following axioms (CS1) and (CS2) hold.

(CS1) (i) $\iota\sigma = \mathrm{id}_V$

(ii) $\iota\tau = \mathrm{id}_V$

(CS2) (i) $\big(u \cdot (g,m)\big)\sigma = u\sigma \cdot (g,m)s \quad \forall u \in U,\ (g,m) \in G \ltimes M$

(ii) $\big(u \cdot (g,m)\big)\tau = u\tau \cdot (g,m)t \quad \forall u \in U,\ (g,m) \in G \ltimes M$

(iii) $(v \cdot g)\iota = v\iota \cdot gi \quad \forall v \in V,\ g \in G.$

We call $[\![U,V]\!]_{\mathrm{set}} := \big(U, V, (\sigma, \iota, \tau)\big)$ an $[\![M,G]\!]$-*crossed set*.

Remark 60 Let $U := G \ltimes M$, $V := G$ and $(\sigma, \iota, \tau) := (s, i, t)$.

(1) If we choose the multiplication $(\cdot)$, cf. Definition 56, as the action of $G \ltimes M$ on $G \ltimes M$, respectively of G on G, we obtain an $[\![M,G]\!]$-crossed set.

(2) We have conjugation actions

$$\begin{array}{rrcl} (*)\colon & (G \ltimes M) \times (G \ltimes M) & \longrightarrow & G \ltimes M \\ & \big((g,m),(\tilde{g},\tilde{m})\big) & \longmapsto & (g,m) * (\tilde{g},\tilde{m}) := (g,m)^{(\tilde{g},\tilde{m})} \\ & & & \quad = (\tilde{g}^-, (\tilde{m}^-)^{\tilde{g}^-}) \cdot (g,m) \cdot (\tilde{g},\tilde{m})\ , \end{array}$$

and

$$\begin{array}{rrcl} (*)\colon & G \times G & \longrightarrow & G \\ & (g,\tilde{g}) & \longmapsto & g * \tilde{g} := g^{\tilde{g}} = \tilde{g}^- \cdot g \cdot \tilde{g}\ . \end{array}$$

If we choose $(*)$ as the action of $G \ltimes M$ on $G \ltimes M$, respectively of G on G, we obtain an $[\![M,G]\!]$-crossed set.

Proof. Ad (1). The required properties in (CS1) are given by Lemma 58. Since s, i and t are group morphisms the properties given in (CS2) are satisfied.

Ad (2). Since the maps s, i and t are group morphisms they are compatible with conjugation. Therefore, the identities that are to be verified in (CS2) hold. $\square$

Definition 61 ($\llbracket M, G\rrbracket$-crossed subset) Suppose we are given $\llbracket M, G\rrbracket$-crossed sets $\llbracket U, V\rrbracket_{\text{set}} = \big(U, V, (\sigma, \iota, \tau)\big)$ and $\llbracket X, Y\rrbracket_{\text{set}} = \big(X, Y, (\underline{\sigma}, \underline{\iota}, \underline{\tau})\big)$.

We say that $\llbracket X, Y\rrbracket_{\text{set}}$ is a $\llbracket M, G\rrbracket$*-crossed subset* of $\llbracket U, V\rrbracket_{\text{set}}$, written $\llbracket X, Y\rrbracket_{\text{set}} \leqslant \llbracket U, V\rrbracket_{\text{set}}$, if the following properties hold.

(i) $X \subseteq U$ and $Y \subseteq V$ are subsets.

(ii) We have $\underline{\sigma} = \sigma\big|_X^Y$, $\underline{\tau} = \tau\big|_X^Y$ and $\underline{\iota} = \iota\big|_Y^X$.

(iii) For $(g, m) \in G \ltimes M$, the multiplication map $U \to U,\ u \mapsto u\cdot(g, m)$ given by $\llbracket U, V\rrbracket_{\text{set}}$ restricts to the multiplication map $X \to X,\ x \mapsto x \cdot (g, m)$ given by $\llbracket X, Y\rrbracket_{\text{set}}$.

For $g \in G$, the multiplication map $V \to V,\ v \mapsto v \cdot g$ given by $\llbracket U, V\rrbracket_{\text{set}}$ restricts to the multiplication map $Y \to Y,\ y \mapsto y \cdot g$ given by $\llbracket X, Y\rrbracket_{\text{set}}$.

Remark 62 Let $\llbracket U, V\rrbracket_{\text{set}} = \big(U, V, (\sigma, \iota, \tau)\big)$ be an $\llbracket M, G\rrbracket$-crossed set.

Suppose given subsets $X \subseteq U$, $Y \subseteq V$ such that $x \cdot (G \ltimes M) \subseteq X$ and $y \cdot G \subseteq Y$ holds for all $x \in X$, $y \in Y$. Suppose that we have $X\sigma, X\tau \subseteq Y$ and $X \supseteq Y\iota$.

Thus, X is a $(G \ltimes M)$-subset of U and Y is a G-subset of V.

We can choose $\underline{\sigma} := \sigma\big|_X^Y$, $\underline{\iota} := \iota\big|_Y^X$ and $\underline{\tau} := \tau\big|_X^Y$ to obtain an $\llbracket M, G\rrbracket$-crossed subset $\llbracket X, Y\rrbracket_{\text{set}} = \big(X, Y, (\underline{\sigma}, \underline{\iota}, \underline{\tau})\big)$.

Proof. Since (CS1) holds for $\llbracket U, V\rrbracket_{\text{set}}$, it holds for $\big(X, Y, (\underline{\sigma}, \underline{\iota}, \underline{\tau})\big)$. Since (CS2) holds for $\llbracket U, V\rrbracket_{\text{set}}$, it holds for $\big(X, Y, (\underline{\sigma}, \underline{\iota}, \underline{\tau})\big)$.

$\square$

Lemma 63 ($\llbracket M, G\rrbracket$-crossed right factor set) *Let $\llbracket N, H\rrbracket \leqslant \llbracket M, G\rrbracket$ be a crossed submodule. Let $X := (H \ltimes N)\backslash(G \ltimes M)$, regarded as a $(G \ltimes M) - set$, and let $Y := H\backslash G$, regarded as a G-set. Consider the maps s, i and t from Lemma 58. Let $\bar{s}, \bar{i}, \bar{t}$ be the coset maps induced by s, i, t, i.e. we have*

$$\begin{aligned}
\bar{s}&\colon X \to Y \ , \ (H \ltimes N)(g, m) \mapsto Hg \ , \\
\bar{i}&\colon X \leftarrow Y \ , \ (H \ltimes N)(g, 1) \ \leftarrow\!\shortmid Hg \ , \\
\bar{t}&\colon X \to Y \ , \ (H \ltimes N)(g, m) \mapsto Hg(mf) \ .
\end{aligned}$$

We have an $[\![M,G]\!]$-crossed set given by $[\![X,Y]\!]_{\mathrm{set}} = (X, Y, (\bar{s}, \bar{i}, \bar{t}))$.

*We denote $[\![N,H]\!]\backslash\!\backslash[\![M,G]\!] := [\![X,Y]\!]_{\mathrm{set}}$ and we say that $[\![N,H]\!]\backslash\!\backslash[\![M,G]\!]$ is the $[\![M,G]\!]$-*crossed right factor set of $[\![M,G]\!]$ modulo $[\![N,H]\!]$.

Proof. We show that the maps $\bar{s}$ and $\bar{t}$ are well-defined.

Suppose given (g,m), $(\tilde{g},\tilde{m}) \in (G \ltimes M)$ with

$$(H \ltimes N)(g,m) = (H \ltimes N)(\tilde{g},\tilde{m}) \ .$$

Then there exists $(n,h) \in (H \ltimes N)$ such that

$$(\tilde{g},\tilde{m}) = (h,n) \cdot (g,m) = (hg, n^g\ m) \ .$$

We have

$$\tilde{g} \cdot g^{-} = hg \cdot g^{-} = h \in H \ .$$

It follows that $H\tilde{g} = Hg$. Hence, $\bar{s}$ is well-defined.

We have

$$\begin{aligned}
\tilde{g} \cdot \tilde{m}f \ \cdot \ (g \cdot mf)^{-} &= hg \cdot (n^g\ m)f \cdot (mf)^{-} \cdot g^{-} \\
&= hg \cdot (n^g)f \cdot mf \cdot (mf)^{-} \cdot g^{-} \\
&\overset{\text{(CM1)}}{=} hg \cdot (nf)^g \cdot g^{-} \\
&= hg \cdot g^{-}\ (nf)\ g \cdot g^{-} \\
&= h \cdot \underbrace{nf}_{\in H} \in H \ .
\end{aligned}$$

It follows that $H(\tilde{g} \cdot \tilde{m}f) = H(g \cdot mf)$. Hence, $\bar{t}$ is well-defined.

We show that $\bar{i}$ is well-defined. Suppose given g, $\tilde{g} \in G$ with $Hg = H\tilde{g}$. Then there exists $h \in H$ such that $\tilde{g} = h \cdot g$. We have

$$(\tilde{g},1) \cdot (g,1)^{-} = (hg,1) \cdot (g^{-},1) = (hg \cdot g^{-},1) = (h,1) \in (H \ltimes N) \ .$$

It follows that $(H \ltimes N)(\tilde{g},1) = (H \ltimes N)(g,1)$. Hence, $\bar{i}$ is well-defined.

With the following actions, X is a $(G \ltimes M)$-set and Y is a G-set:

$$\begin{array}{rcll} \big((H \ltimes N)(\tilde{g},\tilde{m})\big) \cdot (g,m) & = & (H \ltimes N)(\tilde{g}g, \tilde{m}^g\ m) & \text{for } g,\tilde{g} \in G,\ m,\tilde{m} \in M, \\ (H\tilde{g}) \cdot g & = & H\tilde{g}g & \text{for } g,\tilde{g} \in G\ . \end{array}$$

We show that all properties required in Definition 59 hold.

Ad (CS1). For $Hg \in Y$ we have

$$(Hg)\bar{i}\bar{s} = Hg\bar{i}\bar{s} = Hg = (Hg)\mathrm{id}_Y\ ,$$

and we have

$$(Hg)\bar{i}\bar{t} = (H \ltimes N)(g,1)\bar{t} = Hg(1f) = Hg = (Hg)\mathrm{id}_Y\ .$$

Ad (CS2). For $(H \ltimes N)(\tilde{g},\tilde{m}) \in X$, $(g,m) \in G \ltimes M$ we have

$$\begin{aligned} \big((H \ltimes N)(\tilde{g},\tilde{m}) \cdot (g,m)\big)\bar{s} &= \big((H \ltimes N)(\tilde{g}g,\tilde{m}^g\ m)\big)\bar{s} \\ &= H\tilde{g}g = H\tilde{g} \cdot g \\ &= \big((H \ltimes N)(\tilde{g},\tilde{m})\big)\bar{s} \cdot (g,m)s\ , \end{aligned}$$

and we have

$$\begin{aligned} \big((H \ltimes N)(\tilde{g},\tilde{m}) \cdot (g,m)\big)\bar{t} &= \big((H \ltimes N)(\tilde{g}g,\tilde{m}^g\, m)\big)\bar{t} \\ &= H\tilde{g}g(\tilde{m}^g\ m)f \\ &= H\tilde{g}g\ (\tilde{m}f)^g\ mf \\ &= H\tilde{g} \cdot \tilde{m}f \cdot g \cdot mf \\ &= (H\tilde{g} \cdot \tilde{m}f) \cdot (g \cdot mf) \\ &= \big((H \ltimes N)(\tilde{g},\tilde{m})\big)\bar{t} \cdot (g,m)t\ . \end{aligned}$$

For $H\tilde{g} \in Y$, $g \in G$, we have

$$(H\tilde{g} \cdot g)\bar{i} = (H\tilde{g}g)\bar{i} = (H \ltimes N)(\tilde{g}g,1) = (H \ltimes N)(\tilde{g},1) \cdot (g,1) = (H\tilde{g})\bar{i} \cdot gi\ .$$

$\square$

4.2.2 $[\![M, G]\!]$-crossed set morphisms

Definition 64 ($[\![M, G]\!]$-crossed set morphism) Let $[\![U, V]\!]_{\text{set}} = \big(U, V, (\sigma, \iota, \tau)\big)$ and let $[\![X, Y]\!]_{\text{set}} = \big(X, Y, (\tilde{\sigma}, \tilde{\iota}, \tilde{\tau})\big)$ be $[\![M, G]\!]$-crossed sets. Suppose given maps $\zeta \colon U \to X$, $\eta \colon V \to Y$ such that in

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
U \arrow[r, "\zeta"] \arrow[d, shift right=3ex, "\sigma"'] \arrow[d, shift left=3ex, "\tau"] & X \arrow[d, shift right=3ex, "\tilde{\sigma}"'] \arrow[d, shift left=3ex, "\tilde{\tau}"] \\
V \arrow[r, "\eta"'] \arrow[u, "\iota"] & Y \arrow[u, "\tilde{\iota}"]
\end{tikzcd}
$$

the pair (ζ, η) is a morphism of diagrams from (σ, ι, τ) to $(\tilde{\sigma}, \tilde{\iota}, \tilde{\tau})$. That is, the following equations hold true

(i) $\sigma\ \eta = \zeta\ \tilde{\sigma}$

(ii) $\iota\ \zeta = \eta\ \tilde{\iota}$

(iii) $\tau\ \eta = \zeta\ \tilde{\tau}$.

Further, suppose that the following properties are satisfied

(iv) For $u \in U$, $(g, m) \in G \ltimes M$, we have $\big(u \cdot (g, m)\big)\zeta = u\zeta \cdot (g, m)$.

(v) For $v \in V$, $g \in G$, we have $(v \cdot g)\eta = (v\eta) \cdot g$.

Then the pair of maps (ζ, η) is called a *morphism of $[\![M, G]\!]$-crossed sets.*

Remark 65 We do not claim that the diagram given in Definition 64 is commutative. For example, for an $u \in U$ it is not always true that $(u)\sigma\iota = u$.

Lemma 66 (Identity and composition of $[\![M,G]\!]$-crossed set morphisms)

(1) *Let* $[\![U,V]\!]_{\mathrm{set}} = \big(U,V,(\sigma,\iota,\tau)\big)$ *be an* $[\![M,G]\!]$*-crossed set. Then* $(\mathrm{id}_U,\mathrm{id}_V)$ *is the identity* $[\![M,G]\!]$*-crossed morphism of* $[\![U,V]\!]_{\mathrm{set}}$.

(2) *Let* $[\![U_i,V_i]\!]_{\mathrm{set}} = \big(U_i,V_i,(\sigma_i,\iota_i,\tau_i)\big)$ *be* $[\![M,G]\!]$*-crossed sets for* $i \in [1,3]$.
For $j \in [1,2]$*, suppose given* $[\![M,G]\!]$*-crossed set morphisms*

$$(\zeta_j,\eta_j)\colon [\![U_j,V_j]\!]_{\mathrm{set}} \to [\![U_{j+1},V_{j+1}]\!]_{\mathrm{set}} \ .$$

We have a crossed set morphism

$$(\zeta,\eta) := (\zeta_1,\eta_1)(\zeta_2,\eta_2) := (\zeta_1\zeta_2,\eta_1\eta_2)\colon [\![U_1,V_1]\!]_{\mathrm{set}} \to [\![U_3,V_3]\!]_{\mathrm{set}} \ .$$

This composition is associative.

Proof. Ad (1). We have

$$\begin{aligned} \sigma\ \mathrm{id}_V &= \sigma = \mathrm{id}_U\ \sigma \\ \iota\ \mathrm{id}_U &= \iota\ = \mathrm{id}_V\ \iota \\ \tau\ \mathrm{id}_V &= \tau = \mathrm{id}_U\ \tau \ . \end{aligned}$$

Hence we get the following morphism of diagrams.

$$\begin{tikzcd} U \arrow[rr, "\mathrm{id}_U"] \arrow[d, shift right=4, "\sigma"'] \arrow[d, shift left=4, "\tau"] & & U \arrow[d, shift right=4, "\sigma"'] \arrow[d, shift left=4, "\tau"] \\ V \arrow[u, "\iota"] \arrow[rr, "\mathrm{id}_V"'] & & V \arrow[u, "\iota"] \end{tikzcd}$$

Let $u \in U$, let $(g,m) \in G \ltimes M$. We have

$$\big(u\cdot(g,m)\big)\mathrm{id}_U = u\cdot(g,m) = \big((u)\mathrm{id}_U\big)\cdot(g,m) \ .$$

Let $v \in V$, let $g \in G$. We have

$$(v\cdot g)\mathrm{id}_V = v\cdot g = \big((v)\mathrm{id}_V\big)\cdot g \ .$$

Ad (2). The situation is given as follows.

$$\begin{tikzcd}
U_1 \arrow[r, "\zeta_1"] \arrow[d, shift right=8pt, "\sigma_1"'] \arrow[d, shift left=8pt, "\tau_1"] & U_2 \arrow[r, "\zeta_2"] \arrow[d, shift right=8pt, "\sigma_2"'] \arrow[d, shift left=8pt, "\tau_2"] & U_3 \arrow[d, shift right=8pt, "\sigma_3"'] \arrow[d, shift left=8pt, "\tau_3"] \\
V_1 \arrow[r, "\eta_1"'] \arrow[u, "\iota_1"] & V_2 \arrow[r, "\eta_2"'] \arrow[u, "\iota_2"] & V_3 \arrow[u, "\iota_3"]
\end{tikzcd}$$

Therefore we have

$$\begin{aligned}
\sigma_1 \cdot \eta = \sigma_1 \cdot \eta_1\eta_2 = \zeta_1\sigma_2 \cdot \eta_2 = \zeta_1\zeta_2 \cdot \sigma_3 = \zeta \cdot \sigma_3 \\
\iota_1 \cdot \zeta = \iota_1 \cdot \zeta_1\zeta_2 = \eta_1\iota_2 \cdot \zeta_2 = \eta_1\eta_2 \cdot \iota_3 = \eta \cdot \iota_3 \\
\tau_1 \cdot \eta = \tau_1 \cdot \eta_1\eta_2 = \zeta_1\tau_2 \cdot \eta_2 = \zeta_1\zeta_2 \cdot \tau_3 = \zeta \cdot \tau_3 \ .
\end{aligned}$$

Hence we get the following diagram of morphisms

$$\begin{tikzcd}
U_1 \arrow[r, "\zeta"] \arrow[d, shift right=8pt, "\sigma_1"'] \arrow[d, shift left=8pt, "\tau_1"] & U_3 \arrow[d, shift right=8pt, "\sigma_3"'] \arrow[d, shift left=8pt, "\tau_3"] \\
V_1 \arrow[r, "\eta"'] \arrow[u, "\iota_1"] & V_3 \arrow[u, "\iota_3"]
\end{tikzcd} \ .$$

Let $u \in U_1$, let $(g, m) \in G \ltimes M$. We have

$$\big(u \cdot (g, m)\big)\zeta = \big(u \cdot (g, m)\big)\zeta_1 \ \zeta_2 = \big(u\zeta_1 \cdot (g, m)\big)\zeta_2 = u\zeta_1\zeta_2 \cdot (g, m) = u\zeta \cdot (g, m) \ .$$

Let $v \in V_1$, let $g \in G$. We have

$$(v \cdot g)\eta = (v \cdot g)\eta_1 \ \eta_2 = (v\eta_1 \cdot g)\eta_2 = v\eta_1\eta_2 \cdot g = v\eta \cdot g \ .$$

$\square$

Lemma 67 ($[\![M,G]\!]$-crossed set isomorphism) *Let* $[\![U,V]\!]_{\mathrm{set}} = \big(U,V,(\sigma,\iota,\tau)\big)$ *and let* $[\![\tilde U,\tilde V]\!]_{\mathrm{set}} = \big(\tilde U,\tilde V,(\tilde\sigma,\tilde\iota,\tilde\tau)\big)$ *be* $[\![M,G]\!]$*-crossed sets. Suppose given a crossed set morphism* $(\zeta,\eta)\colon [\![U,V]\!]_{\mathrm{set}} \to [\![\tilde U,\tilde V]\!]_{\mathrm{set}}$*, where* ζ *and* η *are both bijective.*

Then we have an $[\![M,G]\!]$*-crossed set morphism given by* $(\zeta^-,\eta^-)\colon [\![\tilde U,\tilde V]\!]_{\mathrm{set}} \to [\![U,V]\!]_{\mathrm{set}}$.

We say that (ζ,η) *is an* $[\![M,G]\!]$-crossed set isomorphism*, and we say that* $[\![U,V]\!]_{\mathrm{set}}$ *and* $[\![\tilde U,\tilde V]\!]_{\mathrm{set}}$ *are* isomorphic*.*

Proof. We have

$$\begin{array}{rcccl}
\sigma\eta = \zeta\tilde\sigma & \Leftrightarrow & \zeta^-\sigma\eta = \tilde\sigma & \Leftrightarrow & \zeta^-\sigma = \tilde\sigma\eta^- \\
\iota\zeta = \eta\tilde\iota & \Leftrightarrow & \eta^-\iota\zeta = \tilde\iota & \Leftrightarrow & \eta^-\iota = \tilde\iota\zeta^- \\
\tau\eta = \zeta\tilde\tau & \Leftrightarrow & \zeta^-\tau\eta = \tilde\tau & \Leftrightarrow & \zeta^-\tau = \tilde\tau\eta^- \ .
\end{array}$$

Hence we have the following morphism of diagrams

$$\begin{tikzcd}
\tilde U \arrow[r, "\zeta^-"] \arrow[d, shift right=4, "\tilde\sigma"'] \arrow[d, shift left=4, "\tilde\tau"] & U \arrow[d, shift right=4, "\sigma"'] \arrow[d, shift left=4, "\tau"] \\
\tilde V \arrow[r, "\eta^-"'] \arrow[u, "\tilde\iota"] & V \arrow[u, "\iota"]
\end{tikzcd} .$$

Let $\tilde u \in \tilde U$, let $(g,m) \in G \ltimes M$. Then there exists $u \in U$ such that $\tilde u = u\zeta$ or, equivalently, $u = \tilde u\zeta^-$. We have

$$\big(\tilde u \cdot (g,m)\big)\zeta^- = \big(u\zeta \cdot (g,m)\big)\zeta^- = \big(u \cdot (g,m)\big)\zeta\ \zeta^- = u \cdot (g,m) = \tilde u\zeta^- \cdot (g,m) \ .$$

Let $\tilde v \in \tilde V$, let $g \in G$. Then there exists $v \in V$ such that $\tilde v = v\eta$ or, equivalently, $v = \tilde v\eta^-$. We have

$$(\tilde v \cdot g)\eta^- = (v\eta \cdot g)\eta^- = (v \cdot g)\eta\ \eta^- = v \cdot g = \tilde v\eta^- \cdot g \ .$$

$\square$

4.2.3 Orbit Lemma for $[\![M, G]\!]$-crossed sets

Lemma 68 (Orbit) *Let* $[\![U, V]\!]_{\mathrm{set}} = \big(U, V, (\sigma, \iota, \tau)\big)$ *be an* $[\![M, G]\!]$*-crossed set.*

Suppose given $v \in V$*. Let* vG *be the orbit of* v *under* G*, let* $(v\iota)(G \ltimes M)$ *be the orbit of* $v\iota$ *under* $G \ltimes M$*; cf. Reminder 3.*

Then $[\![(v\iota)(G \ltimes M), vG]\!]_{\mathrm{set}}$ *is an* $[\![M, G]\!]$*-crossed subset of* $[\![U, V]\!]_{\mathrm{set}}$*.*

We write $v \cdot [\![M, G]\!] := [\![(v\iota)(G \ltimes M), vG]\!]_{\mathrm{set}}$ *and we say that* $v \cdot [\![M, G]\!]$ *is the* orbit of v *under* $[\![M, G]\!]$*. Sometimes, we abbreviate* $v[\![M, G]\!] := v \cdot [\![M, G]\!]$*.*

Proof. We have $vG \subseteq V$ and $(v\iota)(G \ltimes M) \subseteq U$.

Let $(g, m) \in G \ltimes M$. We have

$$\big(v\iota \cdot (g, m)\big)\tau = v\iota\tau \cdot (g, m)s = v \cdot g(mf) \in vG \ .$$

This shows $\big((v\iota)(G \ltimes M)\big)\tau \subseteq vG$.

With a similiar calculation, we get $\big((v\iota)(G \ltimes M)\big)\sigma \subseteq vG$.

Let $g \in G$. We have

$$(vg)\iota = v\iota \cdot gi = v\iota \cdot (g, 1) \in (v\iota)(G \ltimes M) \ .$$

This shows $(vG)\iota \subseteq (v\iota)(G \ltimes M)$.

Let $(v\iota) \cdot (\tilde{g}, \tilde{m}) \in (v\iota)(G \ltimes M)$, let $(g, m) \in G \ltimes M$. We have

$$\big((v\iota) \cdot (\tilde{g}, \tilde{m})\big) \cdot (g, m) = (v\iota) \cdot (\tilde{g}, \tilde{m}) \cdot (g, m) = (v\iota) \cdot \underbrace{(\tilde{g}g, \tilde{m}^g\ m)}_{\in G \ltimes M} \in (v\iota)(G \ltimes M) \ .$$

Let $v \cdot \tilde{g} \in vG$, let $g \in G$. We have

$$(v \cdot \tilde{g}) \cdot g = v \cdot \underbrace{\tilde{g}g}_{\in G} \in vG \ .$$

Hence the subsets $(v\iota)(G \ltimes M)$ and vG are closed under the actions of $G \ltimes M$ and G respectively.

This shows $v[\![M, G]\!] = [\![(v\iota)(G \ltimes M), vG]\!]_{\mathrm{set}} \leqslant [\![U, V]\!]_{\mathrm{set}}$; cf. Remark 62. $\square$

Lemma 69 (Centralizer) *Let* $[\![U,V]\!]_{\mathrm{set}} = \big(U, V, (\sigma, \iota, \tau)\big)$ *be an* $[\![M,G]\!]$*-crossed set. Let* $v \in V$. *Let*

$$\mathrm{C}_{G \ltimes M}(v\iota) = \{(g,m) \in G \ltimes M \colon (v\iota)\cdot(g,m) = v\iota\}$$

be the centralizer of $v\iota$ *in* $G \ltimes M$. *Let*

$$\mathrm{C}_G(v) = \{g \in G \colon v \cdot g = v\}$$

be the centralizer of v *in* G.

(1) *We are given restricted maps*

$$\begin{aligned}
\underset{\sim}{s} := s\big|_{\mathrm{C}_{G\ltimes M}(v\iota)}^{\mathrm{C}_G(v)} &\quad:\quad \mathrm{C}_{G\ltimes M}(v\iota) \to \mathrm{C}_G(v) \quad,\quad (g,m) \mapsto g \ , \\
\underset{\sim}{i} := i\big|_{\mathrm{C}_G(v)}^{\mathrm{C}_{G\ltimes M}(v\iota)} &\quad:\quad \mathrm{C}_{G\ltimes M}(v\iota) \leftarrow \mathrm{C}_G(v) \quad,\quad (g,1) \leftarrow\!\shortmid g \ , \\
\underset{\sim}{t} := t\big|_{\mathrm{C}_{G\ltimes M}(v\iota)}^{\mathrm{C}_G(v)} &\quad:\quad \mathrm{C}_{G\ltimes M}(v\iota) \to \mathrm{C}_G(v) \quad,\quad (g,m) \mapsto g \cdot mf \ .
\end{aligned}$$

(2) *Let* $N_{\mathrm{C}}(v) := \{m \in M \colon (1,m) \in \mathrm{C}_{G\ltimes M}(v\iota)\} = \{m \in M \colon (v\iota)(1,m) = v\iota\}$. *Let* $H_{\mathrm{C}}(v) := \mathrm{C}_G(v)$.
We have a crossed submodule $\mathrm{C}_{[\![M,G]\!]}(v) := [\![\mathrm{C}_C(v), \mathrm{C}_C(v)]\!] \leqslant [\![M,G]\!]$. *We call* $\mathrm{C}_{[\![M,G]\!]}(v)$ *the* centralizer *of* v *in* $[\![M,G]\!]$.

(3) *We have* $\mathrm{C}_{G\ltimes M}(v\iota) = H_{\mathrm{C}}(v) \ltimes N_{\mathrm{C}}(v)$.

Proof. Ad (1). For $(g,m) \in \mathrm{C}_{G\ltimes M}(v\iota)$, we have

$$(v\iota\sigma)\cdot(g,m)s = \big((v\iota)\cdot(g,m)\big)\sigma = (v\iota)\sigma = v \ .$$

This shows $(g,m)s \in \mathrm{C}_{G\ltimes M}(v\iota)$. With a similar calculation we get $(g,m)t \in \mathrm{C}_{G\ltimes M}(v\iota)$. Hence, the restricted maps $\underset{\sim}{s}$ and $\underset{\sim}{t}$ exist. For $g \in \mathrm{C}_G(v)$, we have

$$(v\iota)\cdot(gi) = (v\cdot g)\iota = v\iota = v \ .$$

This shows $gi \in \mathrm{C}_G(v)$. Hence, the restricted map $\underset{\sim}{i}$ exist.

Ad (2). We have $N_{\mathrm{C}}(v) \leqslant M$ and $H_{\mathrm{C}}(v) \leqslant G$. For $n \in N_{\mathrm{C}}(v)$, we have

$$nf = 1 \cdot nf = (1,n)t = (1,n)\underline{t} \overset{(1)}{\in} \mathrm{C}_G(v) = H_{\mathrm{C}}(v) \ .$$

For $n \in N_{\mathrm{C}}(v)$ and $h \in H_{\mathrm{C}}(v)$, we have

$$\begin{aligned}(v\iota) \cdot (1, n^h) &= (v\iota) \cdot (h^- \cdot h, n^h \cdot 1) = (v\iota) \cdot (h^-, n) \cdot (h, 1) \\ &= (v\iota) \cdot (h^-, 1) \cdot (1, n) \cdot (h, 1) = (v\iota) \cdot (h^- i) \cdot (1, n) \cdot (hi) \\ &= (vh^-)\iota \cdot (1, n) \cdot (hi) = (v\iota) \cdot (1, n) \cdot (hi) \\ &= (v\iota) \cdot (hi) = (vh)\iota \\ &= v\iota \ .\end{aligned}$$

Hence $(1, n^h) \in \mathrm{C}_{G \ltimes M}(v\iota)$, and therefore $n^h \in N_{\mathrm{C}}(v)$.

This shows $\mathrm{C}_{[\![M,G]\!]}(v) = [\![N_{\mathrm{C}}(v), H_{\mathrm{C}}(v)]\!] \leqslant [\![M, G]\!]$; cf. Remark 18.

Ad (3). Ad $\subseteq$. Suppose given $(g, m) \in \mathrm{C}_{G \ltimes M}(v\iota)$. We have to show $g \overset{!}{\in} \mathrm{C}_G(v)$ and $(1, m) \overset{!}{\in} \mathrm{C}_{G \ltimes M}(v\iota)$. We have

$$v \cdot g = (v\iota)\sigma \cdot (g, m)s = \big(v\iota \cdot (g, m)\big)\sigma = (v\iota)\sigma = v \ .$$

Hence $g \in \mathrm{C}_G(v) = H_{\mathrm{C}}(v)$. Further, we obtain

$$v\iota = (v \cdot g)\iota = (v\iota) \cdot gi = (v\iota) \cdot (g, 1) \quad \Leftrightarrow \quad (v\iota) \cdot (g, 1)^- = v\iota \ ,$$

which implies $(g, 1)^- \in \mathrm{C}_{G \ltimes M}(v\iota)$. We have

$$(v\iota) \cdot (1, m) = (v\iota) \cdot (g^-, 1) \cdot (g, m) = (v\iota) \cdot (g, m) = v\iota \ .$$

Therefore we have $(1, m) \in \mathrm{C}_{G \ltimes M}(v\iota)$, and hence $m \in N_{\mathrm{C}}(v)$.

Ad $\supseteq$. Suppose given $n \in N_{\mathrm{C}}(v)$ and $h \in H_{\mathrm{C}}(v)$. We have

$$(v\iota) \cdot (h, n) = (v\iota) \cdot (h, 1) \cdot (1, n) = (v\iota) \cdot (hi) \cdot (1, n) = (vh)\iota \cdot (1, n) = v\iota \cdot (1, n) = v\iota \ .$$

Hence, we have $(h, n) \in \mathrm{C}_{G \ltimes M}(v\iota)$. □

Proposition 70 (Orbit Lemma for $[\![M,G]\!]$-crossed sets)
Suppose given an $[\![M,G]\!]$-crossed set $[\![U,V]\!]_{\mathrm{set}} = \big(U,V,(\sigma,\iota,\tau)\big)$. Suppose given $v \in V$.

Recall that the orbit $v[\![M,G]\!] = [\![(v\iota)(G \ltimes M), vG]\!]_{\mathrm{set}}$ is an $[\![M,G]\!]$-crossed set; cf. Lemma 68.

Consider the centralizer $\mathrm{C}_{[\![M,G]\!]}(v) = [\![N_{\mathrm{C}}(v), H_{\mathrm{C}}(v)]\!]$, where we have

$$N_{\mathrm{C}}(v) = \{m \in M \colon (v\iota)\cdot(1,m) = v\iota\} \text{ and } H_{\mathrm{C}}(v) = \mathrm{C}_G(v);\ cf.\ \text{Lemma 69.(2).}$$

Recall that $\mathrm{C}_{G\ltimes M}(v\boldsymbol{i}) = H_{\mathrm{C}}(v) \ltimes N_{\mathrm{C}}(v)$ and that $\mathrm{C}_G(v) = H_{\mathrm{C}}(v)$; cf. Lemma 69.(2,3).

Recall that we may form the $[\![M,G]\!]$-crossed set

$$\mathrm{C}_{[\![M,G]\!]}(v\iota)\backslash\!\!\backslash[\![M,G]\!] = [\![\mathrm{C}_{G\ltimes M}(v\iota)\backslash(G \ltimes M)\,,\,\mathrm{C}_G(v)\backslash G\,]\!]_{\mathrm{set}}\,;$$

cf. Lemma 63 and Lemma 69.(3). Then we have an isomorphism of $[\![M,G]\!]$-crossed sets given by

$$(\zeta,\eta)\colon\quad \mathrm{C}_{[\![M,G]\!]}(v\iota)\backslash\!\!\backslash[\![M,G]\!] \quad\longrightarrow\quad v[\![M,G]\!]\ ,$$

where

$$\begin{array}{rcrcl} \zeta & : & \mathrm{C}_{G\ltimes M}(v\iota)\backslash(G\ltimes M) & \longrightarrow & (v\iota)(G\ltimes M) \\ & & \big(\mathrm{C}_{G\ltimes M}(v\iota)\big)(g,m) & \longmapsto & (v\iota)(g,m) \end{array}$$

and

$$\begin{array}{rcrcl} \eta & : & \mathrm{C}_G(v)\backslash G & \longrightarrow & vG \\ & & \big(\mathrm{C}_G(v)\big)g & \longmapsto & vg\ . \end{array}$$

Proof. By the Orbit Lemma for groups, ζ and η are bijective. We have yet to show that (ζ,η) is a morphism of $[\![M,G]\!]$-crossed sets.

Recall that $\mathrm{C}_{G\ltimes M}(v\iota) = H_{\mathrm{C}}(v) \ltimes N_{\mathrm{C}}(v)$; cf. Lemma 69.(3).

In the following, we write $N := N_{\mathrm{C}}(v)$ and $H := H_{\mathrm{C}}(v) = \mathrm{C}_G(v)$.

For $u := \big(H \ltimes N\big)(g,m) \in \big(H \ltimes N\big)(G \ltimes M)$ we have

$$\begin{aligned}(u)\bar{s}\ \eta &= \big((H \ltimes N)(g,m)\big)\bar{s}\eta = (Hg)\eta = v \cdot g = v\iota\sigma \cdot (g,m)s \\ &= \big((v\iota)\cdot(g,m)\big)\sigma = \big((v\iota)\cdot(g,m)\big)\underline{\sigma} = \big((H \ltimes N)(g,m)\big)\zeta\underline{\sigma} \\ &= (u)\zeta\ \underline{\sigma}\ ,\end{aligned}$$

and we have

$$\begin{aligned}(u)\bar{t}\ \eta &= \big((H \ltimes N)(g,m)\big)\bar{s}\eta = \big(Hg(mf)\big)\eta = v \cdot g(mf) = v\iota\tau \cdot (g,m)t \\ &= \big((v\iota)\cdot(g,m)\big)\tau = \big((v\iota)\cdot(g,m)\big)\underline{\tau} = \big((H \ltimes N)(g,m)\big)\zeta\underline{\tau} \\ &= (u)\zeta\ \underline{\tau}\ .\end{aligned}$$

This shows $\bar{s}\ \eta = \zeta\ \underline{\sigma}$ and $\bar{t}\ \eta = \zeta\ \underline{\tau}$.

For $Hg \in H\backslash G$ we have

$$\begin{aligned}(Hg)\bar{i}\ \zeta &= \big((H \ltimes N)(g,1)\big)\zeta = (v\iota)\cdot(g,1) = v\iota \cdot gi = (vg)\iota = (vg)\underline{\iota} \\ &= (Hg)\eta\ \underline{\iota}\ .\end{aligned}$$

This shows $\bar{i}\ \zeta = \eta\ \underline{\iota}$.

Let $u := \big(H \ltimes N\big)(\tilde{g},\tilde{m}) \in (H \ltimes N)\backslash(G \ltimes M)$, let $(g,m) \in G \ltimes M$. We have

$$\begin{aligned}\big(u \cdot (g,m)\big)\zeta &= \Big(\Big(\big(H \ltimes N\big)(\tilde{g},\tilde{m})\Big) \cdot (g,m)\Big)\zeta \\ &= \Big(\big(H \ltimes N\big)\big((\tilde{g},\tilde{m})\cdot(g,m)\big)\Big)\zeta = (v\iota)\Big((\tilde{g},\tilde{m})\cdot(g,m)\Big) \\ &= \Big((v\iota)(\tilde{g},\tilde{m})\Big)\cdot(g,m) = \Big((H \ltimes N)(\tilde{g},\tilde{m})\Big)\zeta \cdot (g,m) \\ &= u\zeta \cdot (g,m)\ .\end{aligned}$$

Let $Hg \in H\backslash G$, let $\tilde{g} \in G$. We have

$$\big((H\tilde{g})\cdot g\big)\eta = (H\tilde{g}g)\eta = v\tilde{g}g = (v\tilde{g})\cdot g = (H\tilde{g})\eta \cdot g\ .$$

This shows that (ζ,η) is an $[\![M,G]\!]$-crossed set morphism. $\square$

4.3 Crossed categories

4.3.1 $[\![M,G]\!]$-crossed categories

Concerning categories, cf. Reminder 4.

Definition 71 ($[\![M,G]\!]$-crossed category) Let $[\![M,G]\!]$ be a crossed module.

Let $\mathcal{C} = \big(\operatorname{Mor}(\mathcal{C})\,, \operatorname{Ob}(\mathcal{C})\,, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t}), (\,\blacktriangle\,)\big)$ be a category together with the structure of an $[\![M,G]\!]$-crossed set on

$$[\![\operatorname{Mor}(\mathcal{C})\,, \operatorname{Ob}(\mathcal{C})]\!]_{\text{set}} = \big(\operatorname{Mor}(\mathcal{C})\,, \operatorname{Ob}(\mathcal{C})\,, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t})\big)\ .$$

We call $\mathcal{C}$ a $[\![M,G]\!]$-*crossed category* if (CC1) and (CC2) hold.

(CC1) For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ and $g \in G$, we have

$$(a \blacktriangle b) \cdot (g,1) = \big(a \cdot (g,1)\big) \blacktriangle \big(b \cdot (g,1)\big)\ .$$

(CC2) For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ and $m \in M$, we have

$$(a \blacktriangle b) \cdot (1,m) = a \blacktriangle \big(b \cdot (1,m)\big)\ .$$

Remark 72 So altogether, as data for an $[\![M,G]\!]$-crossed category $\mathcal{C}$, we need

- sets $\operatorname{Mor}(\mathcal{C})$, $\operatorname{Ob}(\mathcal{C})$,
- maps $\boldsymbol{s}$, $\boldsymbol{t}$: $\operatorname{Mor}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{C})$, $\boldsymbol{i}$: $\operatorname{Ob}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{C})$,
- a map $(\,\blacktriangle\,)\colon \{(a,b) \in \operatorname{Mor}(\mathcal{C}) \times \operatorname{Mor}(\mathcal{C}) : a\boldsymbol{t} = b\boldsymbol{s}\} \to \operatorname{Mor}(\mathcal{C})$,
- a map $G \to \mathrm{S}_{\operatorname{Ob}(\mathcal{C})}$,
- a map $G \ltimes M \to \mathrm{S}_{\operatorname{Mor}(\mathcal{C})}$.

Remark 73 Let $[\![M, G]\!]$ be a crossed module.

(0) We shall define a category $\mathcal{C}[\![M, G]\!]$. Let

$$\begin{aligned}
\mathrm{Ob}\big(\mathcal{C}[\![M,G]\!]\big) &:= G \\
\mathrm{Mor}\big(\mathcal{C}[\![M,G]\!]\big) &:= G \ltimes M \ .
\end{aligned}$$

Let source, identity and target map be given by

$$\begin{array}{ccc} G \ltimes M & \xrightarrow{\ s\ } & G \\ (g,m) & \longmapsto & g \end{array} \qquad \text{(source)}$$

$$\begin{array}{ccc} G \ltimes M & \xleftarrow{\ i\ } & G \\ (g,1) & \longleftarrow\!\!\shortmid & g \end{array} \qquad \text{(identity)}$$

$$\begin{array}{ccc} G \ltimes M & \xrightarrow{\ t\ } & G \\ (g,m) & \longmapsto & g \cdot mf \end{array} \qquad \text{(target)} \ .$$

Note that $is = \mathrm{id}_G$ and $it = \mathrm{id}_G$; cf. Lemma 58.

Given $(g,m), (\tilde{g}, \tilde{m}) \in \mathrm{Mor}\big(\mathcal{C}[\![M,G]\!]\big) = G \ltimes M$, we have $(g,m)t = (\tilde{g},\tilde{m})s$ if and only if $g \cdot mf = \tilde{g}$.

For (g,m), $(g \cdot mf, \tilde{m}) \in \mathrm{Mor}\big(\mathcal{C}[\![M,G]\!]\big)$, composition is defined by

$$(g,m) \blacktriangle (g \cdot mf, \tilde{m}) := (g, m\tilde{m}) \ .$$

Suppose we are given

$$g \xrightarrow{(g,m)} g \cdot mf \xrightarrow{(g \cdot mf, \tilde{m})} g \cdot (m\tilde{m})f \xrightarrow{(g \cdot (m\tilde{m})f, \hat{m})} g \cdot (m\tilde{m}\hat{m})f \ .$$

The composition is associative since

$$\begin{aligned}
\Big((g,m) \blacktriangle (g \cdot mf, \tilde{m})\Big) \blacktriangle \big(g \cdot (m\tilde{m})f, \hat{m}\big) &= (g, m\tilde{m}) \blacktriangle \big(g \cdot (m\tilde{m})f, \hat{m}\big) \\
&= (g, m\tilde{m}\hat{m}) \\
&= (g,m) \blacktriangle (g \cdot mf, \tilde{m}\hat{m}) \\
&= (g,m) \blacktriangle \Big((g \cdot mf, \tilde{m}) \blacktriangle \big(g \cdot (m\tilde{m})f, \hat{m}\big)\Big) \ .
\end{aligned}$$

For $(g,m) \in \operatorname{Mor}\big(\mathcal{C}[\![M,G]\!]\big)$, we have

$$
\begin{array}{rclclcl}
(g,m) \blacktriangle (g,m)ti & = & (g,m) \blacktriangle (g(mf),1) & = & (g, m\cdot 1) & = & (g,m) \\
(g,m)si \blacktriangle (g,m) & = & (g,1) \blacktriangle (g,m) & = & (g, 1\cdot m) & = & (g,m)\ .
\end{array}
$$

This shows that $\mathcal{C}[\![M,G]\!]$ is a category.

(1) We consider the structure of an $[\![M,G]\!]$-crossed set on $\big(G \ltimes M, G, (s,i,t)\big)$ given in Remark 60.(1), i.e. by right multiplication of $G \ltimes M$ on $G \ltimes M$ and by right multiplication of G on G.

We *claim* that the category $\mathcal{C}[\![M,G]\!]$, equipped with this structure of an $[\![M,G]\!]$-crossed set, is an $[\![M,G]\!]$-crossed category.

Suppose given

$$
\tilde{g} \xrightarrow{(\tilde{g},\tilde{m})} \tilde{g}(\tilde{m}f) \xrightarrow{(\tilde{g}(\tilde{m}f),\hat{m})} \tilde{g}(\tilde{m}\hat{m})f\ .
$$

Ad (CC1). For $(g,1) \in \operatorname{Mor}\big(\mathcal{C}[\![M,G]\!]\big)$, we have

$$
\begin{array}{rcl}
\big((\tilde{g},\tilde{m}) \blacktriangle (\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(g,1) & = & (\tilde{g},\tilde{m}\hat{m})\cdot(g,1) \\
& = & \big(\tilde{g}g, (\tilde{m}\hat{m})^g\big) \\
& = & \big(\tilde{g}g, \tilde{m}^g\ \hat{m}^g\big) \\
& = & (\tilde{g}g,\tilde{m}^g) \blacktriangle \big(\tilde{g}g\cdot(\tilde{m}^g)f, \hat{m}^g\big) \\
& = & \big((\tilde{g},\tilde{m})\cdot(g,1)\big) \blacktriangle \big(\tilde{g}g\cdot(\tilde{m}^g)f, \hat{m}^g\big) \\
& \stackrel{\text{(CM1)}}{=} & \big((\tilde{g},\tilde{m})\cdot(g,1)\big) \blacktriangle \big(\tilde{g}g\cdot(\tilde{m}f)^g, \hat{m}^g\big) \\
& = & \big((\tilde{g},\tilde{m})\cdot(g,1)\big) \blacktriangle \big(\tilde{g}gg^-\ (\tilde{m}f)\ g, \hat{m}^g\big) \\
& = & \big((\tilde{g},\tilde{m})\cdot(g,1)\big) \blacktriangle \big(\tilde{g}\ (\tilde{m}f)\ g, \hat{m}^g\big) \\
& = & \big((\tilde{g},\tilde{m})\cdot(g,1)\big) \blacktriangle \big((\tilde{g}\cdot\tilde{m}f,\hat{m})\cdot(g,1)\big)\ .
\end{array}
$$

Ad (CC2). For $(1,m) \in \operatorname{Mor}\big(\mathcal{C}[\![M,G]\!]\big)$, we have

$$\begin{aligned}
\big((\tilde{g},\tilde{m}) \blacktriangle (\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(1,m) &= (\tilde{g},\tilde{m}\hat{m})\cdot(1,m) \\
&= (\tilde{g},\tilde{m}\hat{m}m) \\
&= (\tilde{g},\tilde{m}) \blacktriangle \big(\tilde{g}\cdot\tilde{m}f,\hat{m}m\big) \\
&= (\tilde{g},\tilde{m}) \blacktriangle \big((\tilde{g}\cdot\tilde{m}f,\hat{m})\cdot(1,m)\big)\ .
\end{aligned}$$

This shows the *claim*.

(2) We consider the structure of an $[\![M,G]\!]$-crossed set on $(G \ltimes M, G)$ given in Remark 60.(2), i.e. by conjugation of $G \ltimes M$ on $G \ltimes M$ and by conjugation of G on G.

We *claim* that the category $\mathcal{C}[\![M,G]\!]$, equipped with the structure of an $[\![M,G]\!]$-crossed set, is an $[\![M,G]\!]$-crossed category.

Recall that for $(\tilde{g},\tilde{m}) \in \operatorname{Mor}\big(\mathcal{C}[\![M,G]\!]\big)$ and $g \in G$, we have

$$(\tilde{g},\tilde{m}) * (g,1) = (g^-,1)\cdot(\tilde{g},\tilde{m})\cdot(g,1) = (g^-\ \tilde{g}\ g,\tilde{m}^g) = (\tilde{g}^g,\tilde{m}^g)\ .$$

Suppose given

$$\tilde{g} \xrightarrow{(\tilde{g},\tilde{m})} \tilde{g}(\tilde{m}f) \xrightarrow{(\tilde{g}(\tilde{m}f),\hat{m})} \tilde{g}(\tilde{m}\hat{m})f\ .$$

Ad (CC1). For $(g,1) \in \operatorname{Mor}\big(\mathcal{C}[\![M,G]\!]\big)$ we have

$$\begin{aligned}
\big((\tilde{g},\tilde{m}) \blacktriangle (\tilde{g}\cdot\tilde{m}f,\hat{m})\big) * (g,1) &= (\tilde{g},\tilde{m}\hat{m}) * (g,1) \\
&= \big(\tilde{g}^g,(\tilde{m}\hat{m})^g\big) \\
&= \big(\tilde{g}^g,\tilde{m}^g\ \hat{m}^g\big) \\
&= (\tilde{g}^g,\tilde{m}^g) \blacktriangle \big(\tilde{g}^g\cdot(\tilde{m}^g)f,\hat{m}^g\big) \\
&\overset{\text{(CM1)}}{=} (\tilde{g}^g,\tilde{m}^g) \blacktriangle \big(\tilde{g}^g\cdot(\tilde{m}f)^g,\hat{m}^g\big) \\
&= \big((\tilde{g},\tilde{m}) * (g,1)\big) \blacktriangle \big((\tilde{g}\cdot\tilde{m}f,\hat{m}) * (g,1)\big)\ .
\end{aligned}$$

Ad (CC2). For $(1,m) \in \mathrm{Mor}\big(\mathcal{C}[\![M,G]\!]\big)$, we have

$$\begin{aligned}
\big((\tilde g,\tilde m) \blacktriangle (\tilde g \cdot \tilde m f, \hat m)\big) * (1,m) &= (\tilde g, \tilde m \hat m) * (1,m) \\
&= (1, m^-) \cdot (\tilde g, \tilde m \hat m) \cdot (1,m) \\
&= \big(\tilde g, (m^{\tilde g})^- \ \tilde m \hat m \ m\big) \\
&= \big(\tilde g, (m^-)^{\tilde g} \ \tilde m \hat m \ m\big) \\
&= \big(\tilde g, \tilde m \ \tilde m^- (m^-)^{\tilde g} \ \tilde m \hat m \ m\big) \\
&= \big(\tilde g, \tilde m \big((m^-)^{\tilde g}\big)^{\tilde m} \ \hat m \ m\big) \\
&\overset{\text{(CM2)}}{=} \big(\tilde g, \tilde m \big((m^-)^{\tilde g}\big)^{\tilde m f} \ \hat m \ m\big) \\
&= \big(\tilde g, \tilde m (m^-)^{\tilde g(\tilde m f)} \ \hat m \ m\big) \\
&= (\tilde g, \tilde m) \blacktriangle \big(\tilde g \cdot \tilde m f, (m^-)^{\tilde g(\tilde m f)} \ \hat m \ m\big) \\
&= (\tilde g, \tilde m) \blacktriangle \big((1, m^-) \cdot (\tilde g \cdot \tilde m f, \hat m) \cdot (1,m)\big) \\
&= (\tilde g, \tilde m) \blacktriangle \big((\tilde g \cdot \tilde m f, \hat m) * (1,m)\big) \ .
\end{aligned}$$

This shows the *claim*.

Definition 74 ($[\![M,G]\!]$-crossed subcategory) Let $\mathcal{C}$, $\mathcal{D}$ be $[\![M,G]\!]$-crossed categories; cf. Definition 71. So we have categories $\mathcal{C}$, $\mathcal{D}$, and $[\![M,G]\!]$-crossed sets

$$\begin{aligned}
[\![\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})]\!]_{\text{set}} &= \big(\,\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})\,,(\boldsymbol{s},\boldsymbol{i},\boldsymbol{t})\big) \\
[\![\mathrm{Mor}(\mathcal{D})\,,\mathrm{Ob}(\mathcal{D})]\!]_{\text{set}} &= \big(\,\mathrm{Mor}(\mathcal{D})\,,\mathrm{Ob}(\mathcal{D})\,,(\underline{\boldsymbol{s}},\underline{\boldsymbol{i}},\underline{\boldsymbol{t}})\big) \ .
\end{aligned}$$

We say that $\mathcal{D}$ is an $[\![M,G]\!]$-*crossed subcategory* of $\mathcal{C}$ if the properties (i) and (ii) hold.

(i) We have $[\![\mathrm{Mor}(\mathcal{D})\,,\mathrm{Ob}(\mathcal{D})]\!]_{\text{set}} \leqslant [\![\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})]\!]_{\text{set}}$, i.e. $[\![\mathrm{Mor}(\mathcal{D})\,,\mathrm{Ob}(\mathcal{D})]\!]_{\text{set}}$ is an $[\![M,G]\!]$-crossed subset of $[\![\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})]\!]_{\text{set}}$; cf. Definition 61.

(ii) The category $\mathcal{D}$ is a subcategory of $\mathcal{C}$.

We write $\mathcal{D} \leqslant \mathcal{C}$ to denote that $\mathcal{D}$ is an $[\![M,G]\!]$-crossed subcategory of $\mathcal{C}$.

Remark 75 Let $\mathcal{C}$ be an $[\![M,G]\!]$-crossed category, so that we have an $[\![M,G]\!]$-crossed set $[\![\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}} = \big(\,\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})\,,(\boldsymbol{s},\boldsymbol{i},\boldsymbol{t})\big)$.

Suppose we are given an $[\![M,G]\!]$-crossed subset $[\![X,Y]\!]_{\mathrm{set}} \leqslant [\![\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}}$ such that $a \blacktriangle b \in X$ holds, for $a,b \in X$ with $a\boldsymbol{t} = b\boldsymbol{s}$.

Then we have an $[\![M,G]\!]$-crossed subcategory $\mathcal{D} \leqslant \mathcal{C}$ with

$$\mathrm{Mor}(\mathcal{D}) = X, \quad \mathrm{Ob}(\mathcal{D}) = Y, \quad (\underline{\boldsymbol{s}},\underline{\boldsymbol{i}},\underline{\boldsymbol{t}}) = \big(\boldsymbol{s}\big|_X^Y,\ \boldsymbol{i}\big|_Y^X,\ \boldsymbol{t}\big|_X^Y\big),$$

and where the composition $(\,\blacktriangle\,)$ in $\mathcal{C}$ restricts to the composition in $\mathcal{D}$. In fact, (CC1) and (CC2) for $\mathcal{D}$ are inherited from $\mathcal{C}$.

Lemma 76 *Let $[\![N,H]\!] \leqslant [\![M,G]\!]$ be a crossed submodule. Then $H \ltimes N \leqslant G \ltimes M$. Consider the $[\![M,G]\!]$-crossed right factor set $[\![N,H]\!]\backslash\!\backslash[\![M,G]\!] = \big(X,Y,(\bar{s},\bar{\imath},\bar{t})\big)$, with $X = (H \ltimes N)\backslash(G \ltimes M)$, $Y = H\backslash G$, given in Lemma 63.*

We have an $[\![M,G]\!]$-crossed category $[\![N,H]\!]\,{}_{\mathcal{C}}\!\backslash\!\backslash[\![M,G]\!]$ with

$$\begin{aligned} \mathrm{Ob}\left([\![N,H]\!]\,{}_{\mathcal{C}}\!\backslash\!\backslash[\![M,G]\!]\right) &= H\backslash G \\ \mathrm{Mor}\left([\![N,H]\!]\,{}_{\mathcal{C}}\!\backslash\!\backslash[\![M,G]\!]\right) &= (H \ltimes N)\backslash(G \ltimes M)\ , \end{aligned}$$

maps

$$\begin{array}{ccc} (H \ltimes N)\backslash(G \ltimes M) & \xrightarrow{\ \bar{s}\ } & H\backslash G \\ (H \ltimes N)(g,m) & \longmapsto & Hg \end{array} \qquad \textit{(source)}$$

$$\begin{array}{ccc} (H \ltimes N)\backslash(G \ltimes M) & \xleftarrow{\ \bar{\imath}\ } & H\backslash G \\ (H \ltimes N)(g,1) & \longleftarrow\!\shortmid & Hg \end{array} \qquad \textit{(identity)}$$

$$\begin{array}{ccc} (H \ltimes N)\backslash(G \ltimes M) & \xrightarrow{\ \bar{t}\ } & H\backslash G \\ (H \ltimes N)(g,m) & \longmapsto & H(g \cdot mf) \end{array} \qquad \textit{(target)}\ ,$$

and a composition given by

$$\begin{aligned}(H \ltimes N)(g,m) \blacktriangle (H \ltimes N)(\tilde{g},\tilde{m}) &= (H \ltimes N)(g,m) \blacktriangle (H \ltimes N)(g \cdot mf,\tilde{m}) \\ &:= (H \ltimes N)(g,m\tilde{m}) \ ,\end{aligned}$$

for $(H \ltimes N)(g,m)$, $(H \ltimes N)(\tilde{g},\tilde{m}) \in \mathrm{Mor}(\mathcal{D})$ *such that*

$$\big((H \ltimes N)(g,m)\big)\boldsymbol{t} = \big((H \ltimes N)(\tilde{g},\tilde{m})\big)\boldsymbol{s}.$$

Proof. We abbreviate $\chi := [\![N,H]\!]\backslash\!\backslash[\![M,G]\!]$.

Suppose given $(H \ltimes N)(g,m)$, $(H \ltimes N)(\tilde{g},\tilde{m}) \in \mathrm{Mor}\big(\mathcal{C}\chi\big)$ with

$$\big((H \ltimes N)(g,m)\big)\bar{t} = \big((H \ltimes N)(\tilde{g},\tilde{m})\big)\bar{s}.$$

Then we obtain $H(g \cdot mf) = H\tilde{g}$ by definition of the maps $\bar{t}$ and $\bar{s}$.

Suppose we are given $m \in M$, $g \in G$ with $H(g \cdot mf) = H\tilde{g}$. Then we have

$$(H \ltimes N)(\tilde{g},\tilde{m}) = (H \ltimes N)(g \cdot mf,\tilde{m}), \text{ since}$$

$$\begin{aligned}(g \cdot mf,\tilde{m})(\tilde{g},\tilde{m})^{-} &= (g \cdot mf,\tilde{m})(\tilde{g}^{-}(\tilde{m}^{-})^{\tilde{g}^{-}} = (g \cdot mf \cdot \tilde{g}^{-},\tilde{m}^{\tilde{g}^{\cdot}}(\tilde{m}^{-})^{\tilde{g}^{-}}) \\ &= (g \cdot mf \cdot \tilde{g}^{-},1) \in H \ltimes N \ .\end{aligned}$$

So, for $g,\tilde{g} \in G$, $m,\tilde{m} \in M$, we have

$$\big((H \ltimes N)(g,m)\big)\bar{t} = \big((H \ltimes N)(\tilde{g},\tilde{m})\big)\bar{s} \quad \text{if and only if} \quad H(g \cdot mf) = H\tilde{g}.$$

The composition is well-defined:

Given $(g,m),(\tilde{g},\tilde{m})$ and $(g',m'),(\tilde{g}',\tilde{m}') \in G \ltimes M$ such that the following compatibilities hold.

$$\begin{aligned}H(g \cdot mf) = H\tilde{g} \quad &, \quad (H \ltimes N)(g,m) = (H \ltimes N)(g',m') \\ H(g' \cdot m'f) = H\tilde{g}' \quad &, \quad (H \ltimes N)(\tilde{g},\tilde{m}) = (H \ltimes N)(\tilde{g}',\tilde{m}')\end{aligned}$$

Then there exist $\hat{h}$, $\hat{h}' \in H$ such that

$$\begin{aligned}\tilde{g} &= \hat{h} \cdot g \cdot mf \\ \tilde{g}' &= \hat{h}' \cdot g' \cdot m'f \ .\end{aligned}$$

There exist (h, n), $(\tilde{h}, \tilde{n}) \in (H \ltimes N)$ such that

$$\begin{array}{rcccl} (g', m') & = & (h, n) \cdot (g, m) & = & (h \cdot g,\ n^g \cdot m) \\ (\tilde{g}', \tilde{m}') & = & (\tilde{h}, \tilde{n}) \cdot (\tilde{g}, \tilde{m}) & = & (\tilde{h} \cdot \tilde{g},\ \tilde{n}^{\tilde{g}} \cdot \tilde{m})\ . \end{array}$$

Hence

$$\begin{array}{rcl} g' & = & h \cdot g \\ \tilde{g}' & = & \tilde{h} \cdot \tilde{g} \\ m' & = & n^g \cdot m \\ \tilde{m}' & = & \tilde{n}^{\tilde{g}} \cdot \tilde{m}\ . \end{array}$$

We have to show that

$$(H \ltimes N)(g, m\tilde{m}) \overset{!}{=} (H \ltimes N)(g', m'\tilde{m}')\ .$$

We have

$$\begin{array}{rcl} (g', m'\tilde{m}') \cdot (g, m\tilde{m})^- & = & (h\ g,\ n^g\ m\ \tilde{n}^{\tilde{g}}\ \tilde{m}) \cdot \big(g^-, (\tilde{m}^-)^{g^-}\ (m^-)^{g^-}\big) \\ & = & \Big(h\ g\ g^-,\ n^{gg^-}\ m^{g^-}\ \tilde{n}^{\tilde{g}g^-}\ \tilde{m}^{g^-}\ (\tilde{m}^-)^{g^-}(m^-)^{g^-}\Big) \\ & = & \big(h,\ n\ m^{g^-}\ \tilde{n}^{\tilde{g}g^-}\ (m^-)^{g^-}\big) \\ & = & \big(h,\ n\ m^{g^-}\ \tilde{n}^{(\hat{h}\cdot g\cdot mf)g^-}\ (m^-)^{g^-}\big) \\ & = & \big(h,\ n\ m^{g^-}\ (\tilde{n}^{\hat{h}})^{(mf)^{g^-}}\ (m^-)^{g^-}\big) \\ & \overset{\text{(CM1)}}{=} & \big(h,\ n\ m^{g^-}\ (\tilde{n}^{\hat{h}})^{(m^{g^-})f}\ (m^-)^{g^-}\big) \\ & \overset{\text{(CM2)}}{=} & \big(h,\ n\ m^{g^-}\ (\tilde{n}^{\hat{h}})^{(m^{g^-})}\ (m^-)^{g^-}\big) \\ & = & \big(h,\ n\ m^{g^-}\ (m^{g^-})^-\ \tilde{n}^{\hat{h}}\ m^{g^-}\ (m^-)^{g^-}\big) \\ & = & \big(h,\ n\ \tilde{n}^{\hat{h}}\big) \in (H \ltimes N)\ . \end{array}$$

The composition is associative and has identity elements:

Suppose given

$$Hg \xrightarrow{(H\ltimes N)(g,m)} H(g\cdot mf) \xrightarrow{(H\ltimes N)(g\cdot mf,\tilde{m})} H\big(g\cdot (m\tilde{m})f\big) \xrightarrow{(H\ltimes N)(g\cdot (m\tilde{m})f,\hat{m})} H\big(g\cdot (m\tilde{m}\hat{m})f\big)\ .$$

We have

$$\begin{aligned}
& \big((H\ltimes N)(g,m)\blacktriangle (H\ltimes N)(g\cdot mf,\tilde{m})\big)\blacktriangle (H\ltimes N)(g\cdot (m\tilde{m})f,\hat{m}) \\
= \quad & (H\ltimes N)\Big(\big((g,m)\blacktriangle (g\cdot mf,\tilde{m})\big)\blacktriangle (g\cdot (m\tilde{m})f,\hat{m})\Big) \\
\overset{73.(0)}{=} \quad & (H\ltimes N)\Big((g,m)\blacktriangle \big((g\cdot mf,\tilde{m})\blacktriangle (g\cdot (m\tilde{m})f,\hat{m})\big)\Big) \\
= \quad & (H\ltimes N)(g,m)\blacktriangle \big((H\ltimes N)(g\cdot mf,\tilde{m})\blacktriangle (H\ltimes N)(g\cdot (m\tilde{m})f,\hat{m})\big)\ .
\end{aligned}$$

For $(H\ltimes N)(g,m)\in \operatorname{Mor}\big(\mathcal{C}\chi\big)$, we have

$$\begin{aligned}
\big((H\ltimes N)(g,m)\big)\blacktriangle \big((H\ltimes N)(g,m)\big)\bar{t}\,\bar{i} \quad &= \quad \big((H\ltimes N)(g,m)\big)\blacktriangle \big((H\ltimes N)(g\cdot mf,1)\big) \\
&= \quad (H\ltimes N)(g,m\cdot 1) \\
&= \quad (H\ltimes N)(g,m)\ , \\
\big((H\ltimes N)(g,m)\big)\bar{s}\,\bar{i}\ \blacktriangle \big((H\ltimes N)(g,m)\big) \quad &= \quad \big((H\ltimes N)(g,1)\big)\blacktriangle \big((H\ltimes N)(g,m)\big) \\
&= \quad (H\ltimes N)(g,1\cdot m) \\
&= \quad (H\ltimes N)(g,m)\ .
\end{aligned}$$

This shows that $\mathcal{C}\chi$ is a category.

Suppose given

$$H\tilde{g} \xrightarrow{(H\ltimes N)(\tilde{g},\tilde{m})} H(\tilde{g}\cdot \tilde{m}f) \xrightarrow{(H\ltimes N)(\tilde{g}\cdot\tilde{m}f,\hat{m})} H\big(\tilde{g}\cdot (\tilde{m}\hat{m})f\big)\ .$$

Ad (CC1). For $(g,1) \in (G \ltimes M)$, we have

$$\begin{aligned}
& \big((H \ltimes N)(\tilde{g},\tilde{m}) \;\blacktriangle\; (H \ltimes N)(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(g,1) \\
= \;& \Big((H \ltimes N)\big((\tilde{g},\tilde{m})\blacktriangle(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\Big)\cdot(g,1) \\
= \;& (H \ltimes N)\Big(\big((\tilde{g},\tilde{m})\blacktriangle(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(g,1)\Big) \\
\overset{73.(1)}{=} \;& (H \ltimes N)\Big(\big((\tilde{g},\tilde{m})\cdot(g,1)\big)\blacktriangle\big((\tilde{g}\cdot\tilde{m}f,\hat{m})\cdot(g,1)\big)\Big) \\
= \;& (H \ltimes N)\big((\tilde{g},\tilde{m})\cdot(g,1)\big) \;\blacktriangle\; (H \ltimes N)\big((\tilde{g}\cdot\tilde{m}f,\hat{m})\cdot(g,1)\big) \\
= \;& \big((H \ltimes N)(\tilde{g},\tilde{m})\big)\cdot(g,1) \;\blacktriangle\; \big((H \ltimes N)(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(g,1)\ .
\end{aligned}$$

Ad (CC2). For $(1,m) \in (G \ltimes M)$, we have

$$\begin{aligned}
& \big((H \ltimes N)(\tilde{g},\tilde{m}) \;\blacktriangle\; (H \ltimes N)(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(1,m) \\
= \;& \Big((H \ltimes N)\big((\tilde{g},\tilde{m})\blacktriangle(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\Big)\cdot(1,m) \\
= \;& (H \ltimes N)\Big(\big((\tilde{g},\tilde{m})\blacktriangle(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(1,m)\Big) \\
\overset{73.(2)}{=} \;& (H \ltimes N)\Big((\tilde{g},\tilde{m})\blacktriangle\big((\tilde{g}\cdot\tilde{m}f,\hat{m})\cdot(1,m)\big)\Big) \\
= \;& (H \ltimes N)(\tilde{g},\tilde{m}) \;\blacktriangle\; (H \ltimes N)\big((\tilde{g}\cdot\tilde{m}f,\hat{m})\cdot(1,m)\big) \\
= \;& (H \ltimes N)(\tilde{g},\tilde{m}) \;\blacktriangle\; \big((H \ltimes N)(\tilde{g}\cdot\tilde{m}f,\hat{m})\big)\cdot(1,m)\ .
\end{aligned}$$

This shows that $\mathcal{C}\chi$ is an $[\![M,G]\!]$-crossed category.

□

4.3.2 $[\![M,G]\!]$-crossed category morphisms

Definition 77 ($[\![M,G]\!]$-crossed category morphism) Let $\mathcal{C}$ and $\mathcal{D}$ be $[\![M,G]\!]$-crossed categories. Let

$$(\zeta,\eta)\colon \big(\operatorname{Mor}(\mathcal{C}),\operatorname{Ob}(\mathcal{C}),(s,i,t)\big) \to \big(\operatorname{Mor}(\mathcal{D}),\operatorname{Ob}(\mathcal{D}),(\underset{\sim}{s},\underset{\sim}{i},\underset{\sim}{t})\big)$$

be an $[\![M,G]\!]$-crossed set morphism; cf. Definition 64.

For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$, suppose that

$$(a \blacktriangle b)\zeta = a\zeta \blacktriangle b\zeta$$

holds true.

Then (ζ, η) is called $[\![M, G]\!]$-*crossed category morphism.*

Remark 78 An $[\![M, G]\!]$-crossed category morphism $(\zeta, \eta)\colon \mathcal{C} \to \mathcal{D}$ yields a functor from $\mathcal{C}$ to $\mathcal{D}$.

Lemma 79 (Identity and composition of $[\![M, G]\!]$-crossed category morphisms)

(1) *Let $\mathcal{C}$ be an $[\![M, G]\!]$-crossed category. The mapping* $(\mathrm{id}_{\mathrm{Mor}(\mathcal{C})}, \mathrm{id}_{\mathrm{Ob}(\mathcal{C})})$ *is an $[\![M, G]\!]$-crossed category morphism, called the* identity *of* $\big(\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}), (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t})\big)$.

(2) *Let $\mathcal{C}_i$ be $[\![M, G]\!]$-crossed categories for $i \in [1, 3]$. We have $[\![M, G]\!]$-crossed sets*

$$[\![\mathrm{Mor}\,(\mathcal{C}_i), \mathrm{Ob}\,(\mathcal{C}_i)]\!]_{\mathrm{set}} = \big(\mathrm{Mor}\,(\mathcal{C}_i), \mathrm{Ob}\,(\mathcal{C}_i), (\boldsymbol{s}_i, \boldsymbol{i}_i, \boldsymbol{t}_i)\big)$$

for $i \in [1, 3]$; cf. Definition 71.
For $j \in [1, 2]$, suppose given $[\![M, G]\!]$-crossed category morphisms

$$(\zeta_j, \eta_j)\colon [\![\mathrm{Mor}\,(\mathcal{C}_j), \mathrm{Ob}\,(\mathcal{C}_j)]\!]_{\mathrm{set}} \to [\![\mathrm{Mor}\,(\mathcal{C}_{j+1}), \mathrm{Mor}\,(\mathcal{C}_{j+1})]\!]_{\mathrm{set}} \ .$$

We have an $[\![M, G]\!]$-crossed category morphism

$$(\zeta, \eta) := (\zeta_1\zeta_2, \eta_1\eta_2)\colon [\![\mathrm{Mor}\,(\mathcal{C}_1), \mathrm{Ob}\,(\mathcal{C}_1)]\!]_{\mathrm{set}} \to [\![\mathrm{Mor}\,(\mathcal{C}_3), \mathrm{Ob}\,(\mathcal{C}_3)]\!]_{\mathrm{set}} \ .$$

Proof. Both $(\mathrm{id}_{\mathrm{Mor}(\mathcal{C})}, \mathrm{id}_{\mathrm{Ob}(\mathcal{C})})$ and (ζ, η) are $[\![M, G]\!]$-crossed set morphisms; cf. Lemma 66. We have yet to show the property stated in Definition 77.

Ad (1). For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ we have

$$(a \blacktriangle b)\mathrm{id}_{\mathrm{Mor}(\mathcal{C})} = (a \blacktriangle b) = \big((a)\mathrm{id}_{\mathrm{Mor}(\mathcal{C})} \blacktriangle (b)\mathrm{id}_{\mathrm{Mor}(\mathcal{C})}\big) \ .$$

Ad (2). For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}_1$ we have

$$(a \blacktriangle b)\zeta = (a \blacktriangle b)\zeta_1\zeta_2 = (a\zeta_1 \blacktriangle b\zeta_1)\zeta_2 = (a\zeta_1\zeta_2 \blacktriangle b\zeta_1\zeta_2) = (a\zeta \blacktriangle b\zeta) \ .$$

$\square$

Lemma 80 ($[\![M,G]\!]$-crossed category isomorphism) *Let $\mathcal{C}$ and let $\mathcal{D}$ be $[\![M,G]\!]$-crossed categories. We write $[\![\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}} = \big(\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}), (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t})\big)$ and we write $[\![\mathrm{Mor}(\mathcal{D}), \mathrm{Ob}(\mathcal{D})]\!]_{\mathrm{set}} = \big(\mathrm{Mor}(\mathcal{D}), \mathrm{Ob}(\mathcal{D}), (\underline{\boldsymbol{s}}, \underline{\boldsymbol{i}}, \underline{\boldsymbol{t}})\big)$.*

Given a crossed category morphism $(\zeta, \eta)\colon [\![\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}} \to [\![\mathrm{Mor}(\mathcal{D}), \mathrm{Ob}(\mathcal{D})]\!]_{\mathrm{set}}$, where ζ and η are both bijective.

Then we have a crossed category morphism given by

$$(\zeta^-, \eta^-)\colon [\![\mathrm{Mor}(\mathcal{D}), \mathrm{Ob}(\mathcal{D})]\!]_{\mathrm{set}} \to [\![\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}} \ .$$

We say that (ζ, η) is an $[\![M,G]\!]$-crossed category isomorphism*, and we say that $[\![\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}}$ and $[\![\mathrm{Mor}(\mathcal{D}), \mathrm{Ob}(\mathcal{D})]\!]_{\mathrm{set}}$ are* isomorphic*.*

Proof. By Lemma 67, $(\zeta^-, \eta^-)\colon [\![\mathrm{Mor}(\mathcal{D}), \mathrm{Ob}(\mathcal{D})]\!]_{\mathrm{set}} \to [\![\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}}$ is an $[\![M,G]\!]$-crossed set morphism.

Given $\tilde{X} \xrightarrow{e} \tilde{Y} \xrightarrow{b} \tilde{Z}$ in $\mathcal{D}$. We have to show $(e \blacktriangle f)\zeta^- \stackrel{!}{=} e\zeta^- \blacktriangle f\zeta^-$.

Since ζ is bijective there exist $a, b \in \mathrm{Mor}(\mathcal{C})$ such that $e = a\zeta$ and $f = b\zeta$. Equivalently, we have $a = e\zeta^-$ and $b = f\zeta^-$. Then

$$a\boldsymbol{t} = e\zeta^- \boldsymbol{t} = e\underline{\boldsymbol{t}}\eta^- = f\underline{\boldsymbol{s}}\eta^- = f\zeta^- \boldsymbol{s} = b\boldsymbol{s} \ .$$

We have

$$(e \blacktriangle f)\zeta^- = (a\zeta \blacktriangle b\zeta)\zeta^- = (a \blacktriangle b)\zeta\ \zeta^- = (a \blacktriangle b) = e\zeta^- \blacktriangle f\zeta^- \ .$$

□

4.3.3 Orbit Lemma for $[\![M,G]\!]$-crossed categories

Lemma 81 (Orbit) *Let $\mathcal{C}$ be an $[\![M,G]\!]$-crossed category; cf. Definition 71. In particular, we have an $[\![M,G]\!]$-crossed set*

$$[\![\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}} = \big(\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}), (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t})\big) \ .$$

Suppose given $v \in \mathrm{Ob}(\mathcal{C})$.

Let $v[\![M,G]\!] = [\![(v\boldsymbol{i})(G \ltimes M), vG]\!]_{\mathrm{set}}$ be the orbit of v under $[\![M,G]\!]$; cf. Lemma 68.

We have an $[\![M,G]\!]$-crossed subcategory $\mathcal{D} \leqslant \mathcal{C}$ with

$$\mathrm{Mor}(\mathcal{D}) = (v\boldsymbol{i})(G \ltimes M), \quad \mathrm{Ob}(\mathcal{D}) = vG \ ,$$

and a composition given by

$$(v\boldsymbol{i})(g,m) \ \blacktriangle \ (v\boldsymbol{i})(\tilde{g},\tilde{m}) = (v\boldsymbol{i})(g,m\tilde{m}) \ ,$$

for $(v\boldsymbol{i})(g,m)$, $(v\boldsymbol{i})(\tilde{g},\tilde{m}) \in \mathrm{Mor}(\mathcal{D})$ such that $\big((v\boldsymbol{i})(g,m)\big)\boldsymbol{t} = \big((v\boldsymbol{i})(\tilde{g},\tilde{m})\big)\boldsymbol{s}$.

By abuse of notation, we also denote by $v \cdot [\![M,G]\!] = v[\![M,G]\!]$ the $[\![M,G]\!]$-crossed subcategory $\mathcal{D}$, called the orbit *of v under $[\![M,G]\!]$.*

Proof. The orbit $v[\![M,G]\!]$ is an $[\![M,G]\!]$-crossed subset of $[\![\mathrm{Mor}(\mathcal{C})\,,\mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}}$; cf. Lemma 68. We shall verify that $\mathcal{D}$ is an $[\![M,G]\!]$-crossed subcategory using Remark 75.

Suppose given $a := v\boldsymbol{i} \cdot (g,m), b := v\boldsymbol{i} \cdot (\tilde{g},\tilde{m}) \in (v\boldsymbol{i})(G \ltimes M)$ with $a\boldsymbol{t} = b\boldsymbol{s}$. We have

$$\begin{aligned} a\boldsymbol{t} &= \big(v\boldsymbol{i} \cdot (g,m)\big)\boldsymbol{t} = v\boldsymbol{i}\boldsymbol{t} \cdot (g,m)t = v \cdot (g \cdot mf) \ , \\ b\boldsymbol{s} &= \big(v\boldsymbol{i} \cdot (\tilde{g},\tilde{m})\big)\boldsymbol{s} = v\boldsymbol{i}\boldsymbol{s} \cdot (\tilde{g},\tilde{m})s = v \cdot \tilde{g} \ . \end{aligned}$$

It follows that $v \cdot (g \cdot mf) = v \cdot \tilde{g}$. We have

$$\begin{aligned} b &= v\boldsymbol{i} \cdot (\tilde{g},\tilde{m}) = v\boldsymbol{i} \cdot (\tilde{g},1) \cdot (1,\tilde{m}) = v\boldsymbol{i} \cdot \tilde{g}i \cdot (1,\tilde{m}) = (v \cdot \tilde{g})\boldsymbol{i} \cdot (1,\tilde{m}) \\ &= \big(v \cdot (g \cdot mf)\big)\boldsymbol{i} \cdot (1,\tilde{m}) = \big(v\boldsymbol{i}\boldsymbol{t} \cdot (g,m)t\ \big)\boldsymbol{i} \cdot (1,\tilde{m}) \\ &= \big(v\boldsymbol{i} \cdot (g,m)\big)\boldsymbol{t}\boldsymbol{i} \cdot (1,\tilde{m}) \ . \end{aligned}$$

Hence, we get

$$\begin{aligned} a \blacktriangle b &= \big(v\boldsymbol{i} \cdot (g,m)\big) \ \blacktriangle \ \big(v\boldsymbol{i} \cdot (\tilde{g},\tilde{m})\big) \\ &= \big(v\boldsymbol{i} \cdot (g,m)\big) \ \blacktriangle \ \Big(\underbrace{\big(v\boldsymbol{i} \cdot (g,m)\big)\boldsymbol{t}}_{=:Y \in \mathrm{Ob}(\mathcal{D})}\,\boldsymbol{i} \cdot (1,\tilde{m})\Big) \\ &= \big(v\boldsymbol{i} \cdot (g,m)\big) \ \blacktriangle \ \big(Y\boldsymbol{i} \cdot (1,\tilde{m})\big) \\ &= \big(v\boldsymbol{i} \cdot (g,m)\big) \ \blacktriangle \ \big(\mathrm{id}_Y \cdot (1,\tilde{m})\big) \\ &\overset{\text{(CC2)}}{=} \Big(v\boldsymbol{i} \cdot (g,m) \ \blacktriangle \ \mathrm{id}_Y\Big) \cdot (1,\tilde{m}) \\ &= v\boldsymbol{i} \cdot (g,m) \cdot (1,\tilde{m}) \\ &= v\boldsymbol{i} \cdot (g,m\tilde{m}) \ \in \ (v\boldsymbol{i})(G \ltimes M) \ . \end{aligned}$$

$\square$

Proposition 82 (Orbit Lemma for $\llbracket M, G\rrbracket$-crossed categories)
Suppose given an $\llbracket M, G\rrbracket$-crossed category $\mathcal{C}$. Suppose given $v \in \mathrm{Ob}(\mathcal{C})$.

Recall that the orbit $v\llbracket M, G\rrbracket = \llbracket (v\iota)(G \ltimes M), vG\rrbracket$ is an $\llbracket M, G\rrbracket$-crossed subcategory of $\mathcal{C}$; cf. Lemma 81.

Consider the centralizer $\mathrm{C}_{\llbracket M,G\rrbracket}(v) = \llbracket N_{\mathrm{C}}(v), H_{\mathrm{C}}(v)\rrbracket$, where we have

$$N_{\mathrm{C}}(v) = \{m \in M \colon (v\boldsymbol{i}) \cdot (1, m) = (v\boldsymbol{i})\} \text{ and } H_{\mathrm{C}}(v) = \{g \in G \colon v \cdot g = v\}, \text{ Lemma 69.}$$

Recall that $\mathrm{C}_{G\ltimes M}(v\boldsymbol{i}) = H_{\mathrm{C}}(v) \ltimes N_{\mathrm{C}}(v)$ and that $\mathrm{C}_G(v) = H_{\mathrm{C}}(v)$; cf. Lemma 69.(2,3).

Recall that we may form the $\llbracket M, G\rrbracket$-crossed category $\mathrm{C}_{\llbracket M,G\rrbracket}(v)\,{}_{\mathcal{C}}\!\backslash\!\!\backslash \llbracket M, G\rrbracket$; cf. Lemma 76.

Then we have an isomorphism of $\llbracket M, G\rrbracket$-crossed categories given by

$$(\zeta, \eta)\colon \quad \mathrm{C}_{\llbracket M,G\rrbracket}(v)\,{}_{\mathcal{C}}\!\backslash\!\!\backslash \llbracket M, G\rrbracket \quad \longrightarrow \quad v\llbracket M, G\rrbracket\ ,$$

where

$$\begin{array}{rcrcl} \zeta & : & \big(\mathrm{C}_{G\ltimes M}(v\iota)\big)\backslash(G \ltimes M) & \longrightarrow & (v\boldsymbol{i})(G \ltimes M) \\ & & \big(\mathrm{C}_{G\ltimes M}(v\iota)\big)(g, m) & \longmapsto & (v\boldsymbol{i}) \cdot (g, m) \end{array}$$

and

$$\begin{array}{rcrcl} \eta & : & \mathrm{C}_G(v)\backslash G & \longrightarrow & vG \\ & & \big(\mathrm{C}_G(v)\big)g & \longmapsto & v \cdot g\ . \end{array}$$

Proof. In the following, we write $N := N_{\mathrm{C}}(v)$ and $H := H_{\mathrm{C}}(v) = \mathrm{C}_G(v)$.

Let $a := (H \ltimes N)(g, m)$, $b := (H \ltimes N)(\tilde{g}, \tilde{m}) \in (H \ltimes N)\backslash(G \ltimes M)$ with $a\boldsymbol{t} = b\boldsymbol{s}$, i.e. $H(g \cdot mf) = H\tilde{g}$. We have to show $(a \blacktriangle b)\zeta \overset{!}{=} a\zeta \blacktriangle b\zeta$.

Since (ζ, η) is a morphism of $\llbracket M, G\rrbracket$-crossed sets by Proposition 70, $a\zeta$ and $b\zeta$ are composable, for we have

$$(a\zeta)\boldsymbol{t} = (a\boldsymbol{t})\eta = (b\boldsymbol{s})\eta = (b\zeta)\boldsymbol{s}\ .$$

We have

$$
\begin{aligned}
(a \blacktriangle b)\zeta &= \big((H \ltimes N)(g,m) \;\blacktriangle\; (H \ltimes N)(\tilde{g},\tilde{m})\big)\zeta \\
&\overset{76}{=} \Big((H \ltimes N)(g,m\tilde{m})\Big)\zeta \\
&= (v\boldsymbol{i})(g,m\tilde{m}) \\
&\overset{81}{=} (v\boldsymbol{i})(g,m) \;\blacktriangle\; (v\boldsymbol{i})(\tilde{g},\tilde{m}) \\
&= \big((H \ltimes N)(g,m)\big)\zeta \;\blacktriangle\; \big((H \ltimes N)(\tilde{g},\tilde{m})\big)\zeta \\
&= a\zeta \;\blacktriangle\; b\zeta\ .
\end{aligned}
$$

$\square$

4.4 Example

Example 83 We consider the crossed module $\big(\langle b\rangle, \langle a\rangle, \alpha, f\big)$ from Example 30. We have

$$
\begin{aligned}
&\langle b\rangle := \langle b\colon b^4 = 1\rangle \\
&\langle a\rangle := \langle a\colon a^4 = 1\rangle \\
&\alpha\colon \langle a\rangle \to \mathrm{Aut}\,(\langle b\rangle)\ ,\ a \mapsto (b \mapsto b^a := b^-) \\
&f\colon \langle b\rangle \to \langle a\rangle\ ,\ b \mapsto a^2\ .
\end{aligned}
$$

We have group morphisms

$$
\begin{aligned}
&s\colon (\langle a\rangle \ltimes \langle b\rangle) \to \langle a\rangle\ ,\ (a^j, b^k) \mapsto a^j\ , \\
&i\colon (\langle a\rangle \ltimes \langle b\rangle) \leftarrow \langle a\rangle\ ,\ (a^j, 1) \ \hookleftarrow a^j\ , \\
&t\colon (\langle a\rangle \ltimes \langle b\rangle) \to \langle a\rangle\ ,\ (a^j, b^k) \mapsto a^j \cdot (b^k)f = a^{j+2k},\ \text{where } j,k \in [0,3].
\end{aligned}
$$

Then we have a $[\![\langle a\rangle, \langle b\rangle]\!]$-crossed set given by $[\![\langle a\rangle \ltimes \langle b\rangle, \langle a\rangle]\!]_{\text{set}} = \big(\langle a\rangle \ltimes \langle b\rangle, \langle a\rangle, (s,i,t)\big)$ via conjugation of $\langle a\rangle \ltimes \langle b\rangle$ on $\langle a\rangle \ltimes \langle b\rangle$ and via conjugation of $\langle a\rangle$ on $\langle a\rangle$ (which is trivial); cf. Remark 60.(2). This $[\![\langle a\rangle, \langle b\rangle]\!]$-crossed set then becomes a $[\![\langle a\rangle, \langle b\rangle]\!]$-crossed category by Remark 73.(2).

(1) We want to determine the orbits $v * [\![\langle b\rangle, \langle a\rangle]\!] = [\![(vi) * (\langle a\rangle \ltimes \langle b\rangle), v * \langle a\rangle]\!]_{\text{set}}$ for all $v \in \langle a\rangle$.

Let $j \in [0,3]$. Let $v := a^j \in \langle a\rangle$. Note that we have $vi = (v,1) = (a^j,1)$. Note that we have $v * \langle a\rangle = \{v\}$ since $\langle a\rangle$ is abelian.

We have

$$\begin{aligned}(a^j, b^k) * (a, 1) &= (a^j, b^k)^{(a,1)} = (a^-, 1) \cdot (a^j, b^k) \cdot (a, 1) = (a^{j-1}, b^k) \cdot (a, 1) \\ &= (a^j, b^{-k}) \ ,\end{aligned}$$

and

$$\begin{aligned}(a^j, b^k) * (1, b) &= (a^j, b^k)^{(1,b)} = (1, b) \cdot (a^j, b^k) \cdot (1, b) = (a^j, b^{a^j} \cdot b^k) \cdot (1, b) \\ &= (a^j, b^{a^j} \cdot b^{k+1}) = (a^j, b^{(-1)^{j+1}} \cdot b^{k+1}) \\ &= (a^j, b^{(-1)^{j+1}+k+1}) \ .\end{aligned}$$

For $j = 1$, we get

$$\begin{aligned}(a, 1) * (a, 1) &= (a, 1) \\ (a, 1) * (1, b) &= (a, b^{(-1)^{1+1}+0+1}) = (a, b^2) \ , \\ (a, b^2) * (a, 1) &= (a, b^{-2}) = (a, b^2) \\ (a, b^2) * (1, b) &= (a, b^{(-1)^{1+1}+2+1}) = (a, b^4) = (a, 1) \ .\end{aligned}$$

We obtain:

$$\begin{tikzcd}
 & (a,1) & \\
(a,1) \arrow[ur, "*(a,1)"] \arrow[dr, "*(1,b)"'] & & (a,b^2) \\
 & (a,b^2) \arrow[ur, "*(a,1)"] \arrow[dr, "*(1,b)"'] & \\
 & & (a,1)
\end{tikzcd} \ .$$

We proceed similarly for $v \in \{1, a^2, a^3\}$ and obtain:

$$
\begin{tikzcd}
 & (1,1) \\
(1,1) \arrow[ur, "*(a,1)"] \arrow[dr, "*(1,b)"'] & \\
 & (1,1)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & (a^2,1) \\
(a^2,1) \arrow[ur, "*(a,1)"] \arrow[dr, "*(1,b)"'] & \\
 & (a^2,1)
\end{tikzcd}
.
$$

$$
\begin{tikzcd}
 & (a^3,1) & \\
(a^3,1) \arrow[ur, "*(a,1)"] \arrow[dr, "*(1,b)"'] & & (a^3,b^2) \\
 & (a^3,b^2) \arrow[ur, "*(a,1)"] \arrow[dr, "*(1,b)"'] & \\
 & & (a^3,1)
\end{tikzcd}
.
$$

We obtain orbits

$$
\begin{aligned}
1 \ * [\![\langle b\rangle, \langle a\rangle]\!] &= [\![\{(1,1)\},\{1\}]\!]_{\text{set}} \\
a \ * [\![\langle b\rangle, \langle a\rangle]\!] &= [\![\{(a,1),(a,b^2)\},\{a\}]\!]_{\text{set}} \\
a^2 * [\![\langle b\rangle, \langle a\rangle]\!] &= [\![\{(a^2,1)\},\{a^2\}]\!]_{\text{set}} \\
a^3 * [\![\langle b\rangle, \langle a\rangle]\!] &= [\![\{(a^3,1),(a^3,b^2)\},\{a^3\}]\!]_{\text{set}} \ .
\end{aligned}
$$

(2) Recall that the centralizer for a $v \in \langle a \rangle$ in $[\![\langle b \rangle, \langle a \rangle]\!]$ is given by

$$\mathrm{C}_{[\![\langle b \rangle, \langle a \rangle]\!]}(v) = [\![N_{\mathrm{C}}(v), H_{\mathrm{C}}(v)]\!] \ ,$$

with $N_{\mathrm{C}}(v) := \{m \in \langle b \rangle \colon (vi) * (1, m) = (vi)\}$ and $H_{\mathrm{C}}(v) := \{g \in \langle a \rangle \colon v * g = v\}$; cf. Lemma 69.

We consider $v = a$. Then $ai = (a, 1)$. We have

$$\begin{array}{rcl} (a,1) * (1,1) & = & (a,1) \\ (a,1) * (1,b) & = & (a,b^2) \\ (a,1) * (1,b^2) & = & (a,1) \\ (a,1) * (1,b^3) & = & (a,b^2) \ . \end{array}$$

We get $N_{\mathrm{C}}(a) = \langle b^2 \rangle$.

Since $\langle a \rangle$ is abelian, we have $H_{\mathrm{C}}(a) = \langle a \rangle$.

We proceed similarly for $v \in \{1, a^2, a^3\}$ and obtain centralizers

$$\begin{array}{rcl} \mathrm{C}_{[\![\langle b \rangle, \langle a \rangle]\!]}(1) & = & [\![\langle b \rangle, \langle a \rangle]\!] \\ \mathrm{C}_{[\![\langle b \rangle, \langle a \rangle]\!]}(a) & = & [\![\langle b^2 \rangle, \langle a \rangle]\!] \\ \mathrm{C}_{[\![\langle b \rangle, \langle a \rangle]\!]}(a^2) & = & [\![\langle b \rangle, \langle a \rangle]\!] \\ \mathrm{C}_{[\![\langle b \rangle, \langle a \rangle]\!]}(a^3) & = & [\![\langle b^2 \rangle, \langle a \rangle]\!] \ . \end{array}$$

(3) We want to apply the Orbit Lemma for crossed modules to our example; cf. Proposition 70. We form the $[\![\langle b \rangle, \langle a \rangle]\!]$-crossed set

$$\mathrm{C}_{[\![\langle b \rangle, \langle a \rangle]\!]}(v) \backslash\!\!\backslash [\![\langle b \rangle, \langle a \rangle]\!] = [\![\big(\mathrm{C}_C(v) \ltimes N_{\mathrm{C}}(v)\big) \backslash (\langle a \rangle \ltimes \langle b \rangle), H_{\mathrm{C}}(v) \backslash \langle a \rangle]\!]_{\mathrm{set}} \,.$$

For $v = a$ we obtain

$$\begin{aligned} \mathrm{C}_{[\![\langle b \rangle, \langle a \rangle]\!]}(a) \backslash\!\!\backslash [\![\langle b \rangle, \langle a \rangle]\!] &= [\![(\langle a \rangle \ltimes \langle b^2 \rangle) \backslash (\langle a \rangle \ltimes \langle b \rangle), \langle a \rangle \backslash \langle a \rangle]\!]_{\mathrm{set}} \\ &= [\![\{\ \big(\langle a \rangle \ltimes \langle b^2 \rangle\big)(1,1), \big(\langle a \rangle \ltimes \langle b^2 \rangle\big)(1,b)\ \}, \{\ \langle a \rangle 1\ \}]\!]_{\mathrm{set}} \,. \end{aligned}$$

The maps ζ and η yield

$$\begin{array}{rcccl} \zeta & : & \{ \ \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,1), \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b) \ \} & \xrightarrow{\sim} & (a,1) * (\langle a\rangle \ltimes \langle b\rangle) \\ & & \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,1) & \longmapsto & (a,1) * (1,1) = (a,1) \\ & & \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b) & \longmapsto & (a,1) * (1,b) = (a,b^2) \ , \end{array}$$

$$\begin{array}{rcccl} \eta & : & \{ \ \langle a\rangle 1 \ \} & \xrightarrow{\sim} & a * \langle a\rangle \\ & & \langle a\rangle 1 & \longmapsto & a * 1 \ = \ a \ . \end{array}$$

So we obtain an isomorphism (ζ,η) from $\mathrm{C}_{[\![\langle b\rangle,\langle a\rangle]\!]}(a)\backslash\!\!\backslash[\![\langle b\rangle,\langle a\rangle]\!]$ to $a * [\![\langle b\rangle,\langle a\rangle]\!]$, the latter as calculated in (1).

(4) By Lemma 76, $\mathrm{C}_{[\![\langle b\rangle,\langle a\rangle]\!]}(a) \,{}_{\mathcal{C}}\!\backslash\!\!\backslash[\![\langle b\rangle,\langle a\rangle]\!]$ is an $[\![M,G]\!]$-crossed category.

Let $\big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b) \ \in \ \mathrm{Mor}\Big(\mathrm{C}_{[\![\langle b\rangle,\langle a\rangle]\!]}(a) \,{}_{\mathcal{C}}\!\backslash\!\!\backslash[\![\langle b\rangle,\langle a\rangle]\!]\Big) \ = \ (\langle a\rangle \ltimes \langle b^2\rangle)\backslash(\langle a\rangle \ltimes \langle b\rangle)$ $= \{ \ \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,1), \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b) \ \}$.
We have

$$\begin{aligned} \Big(\big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b)\Big)\bar{s} &= \langle a\rangle(1,b)s = \langle a\rangle 1 \\ \Big(\big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b)\Big)\bar{t} &= \langle a\rangle(1,b)t = \langle a\rangle a^2 = \langle a\rangle 1 \ , \end{aligned}$$

and thus, $\big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b)$ is composable with itself. We have

$$\big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b) \blacktriangle \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b) \overset{76}{=} \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b^2) = \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,1) \ .$$

Write $x := \big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,b)$. By Proposition 82, we must have $(x \blacktriangle x)\zeta = x\zeta \blacktriangle x\zeta$.
In fact, we have

$$(x \blacktriangle x)\zeta = \Big(\big(\langle a\rangle \ltimes \langle b^2\rangle\big)(1,1)\Big)\zeta \overset{(3)}{=} (a,1) = (a, b^2 \cdot b^2) \overset{73.(0)}{=} (a,b^2) \blacktriangle (a,b^2) \overset{(3)}{=} x\zeta \blacktriangle x\zeta \ .$$

5 References

[1] R. Brown, P. J. Higgins, R. Sivera, *Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubic Homotopy Grupoids*, European Mathematical Society, 2011.

[2] R. Brown, C. B. Spencer, *G-groupoids, crossed modules and the fundamental groupoid of a topological group*, Indagationes Mathematicae, Vol. 79, Issue 4, pp. 296–302, 1976.

[3] A. R. Grandjeán, M. Ladra, *$H_2(T, G, \delta)$ and Central Extension for Crossed Modules*, Proceedings of the Edinburgh Mathematical Society, Vol. 42, No. 1, pp. 169–177, 1999.

[4] S. Lang, *Algebra*, 3rd edition, Addison-Wesley Publishing Company, 1993.

[5] S. MacLane, J. H. C. Whitehead, *On the 3-Type of a Complex*, Proceedings of the National Academy of Sciences of the United States of America, Vol. 36, No. 1, pp. 41–48, 1950.

[6] S. Thomas, *(Co)homology of crossed modules*, diploma thesis, Aachen, 2007.

Zusammenfassung

Seien M und G Gruppen. Sei $f\colon M \to G$ ein Gruppenmorphismus. Sei $\alpha\colon G \to \mathrm{Aut}\,(M)$ eine Operation von G auf M so, dass für $m, n \in M$ und $g \in G$ die Bedingungen

$$\text{(CM1) } (m^g)f = (mf)^g \quad \text{und} \quad \text{(CM2) } m^n = m^{nf}$$

erfüllt sind. Dann nennen wir das Quadrupel $[\![M, G]\!] := \big(M, G, \alpha, f\big)$ einen verschränkten Modul.

Ein nichttrivialer verschränkter Modul X heißt einfach, falls er nur 1 und X als normale verschränkte Untermoduln enthält. Ein einfacher verschränkter Modul ist stets von einer der drei folgenden Formen.

- $[\![G, G]\!]$, mit $G \xrightarrow{\mathrm{id}} G$, und G einfach und nichtabelsch.
- $[\![1, K]\!]$, mit K einfach.
- $[\![M, 1]\!]$, mit M zyklisch von Primordnung.

Ähnlich wie für Gruppen lassen sich Kompositionsreihen für verschränkte Moduln definieren. Das Jordan-Hölder-Theorem sagt aus, dass die Kompositionsreihe einer Gruppe bis auf Reihenfolge und Isomorphie der Kompositionsfaktoren eindeutig bestimmt ist. Dieses Theorem lässt sich auf verschränkte Moduln verallgemeinern, und wir erhalten eine analoge Aussage: Die Kompositionsreihe eines verschränkten Moduls ist eindeutig bis auf Reihenfolge und Isomorphie der Kompositionsfaktoren.

Wir definieren analog zum Begriff einer G-Menge einer Gruppe G den Begriff einer $[\![M, G]\!]$-verschränkten Menge eines verschränkten Moduls $[\![M, G]\!]$. Eine solche besteht aus einer $(G \ltimes M)$-Menge U, einer G-Menge V und gewissen Abbildungen. Für ein $v \in V$ definieren wir die Bahn $v[\![M, G]\!]$ von v unter $[\![M, G]\!]$. Wir bilden den Zentralisator $\mathrm{C}_{[\![M,G]\!]}(v)$ von v in $[\![M, G]\!]$ sowie dessen verschränkte Faktormenge $\mathrm{C}_{[\![M,G]\!]}(v)\backslash\!\backslash[\![M, G]\!]$ und erhalten einen Isomorphismus von $[\![M, G]\!]$-verschränkten Mengen.

$$(\zeta, \eta)\colon\ \mathrm{C}_{[\![M,G]\!]}(v)\backslash\!\backslash[\![M, G]\!] \xrightarrow{\ \sim\ } v[\![M, G]\!]$$

Anders als im Fall des klassischen Bahnenlemmas aus der Gruppentheorie wird auf diese Weise im Allgemeinen die gesamte verschränkte $[\![M, G]\!]$-Menge nicht durch disjunkte Bahnen überdeckt.

Auf einer Kategorie $\mathcal{C}$ können wir die Struktur einer solchen $[\![M,G]\!]$-verschränkten Menge definieren, indem wir $\mathrm{Mor}(\mathcal{C})$ als $(G \ltimes M)$-Menge, und $\mathrm{Ob}(\mathcal{C})$ als G-Menge auffassen; wir erhalten eine $[\![M,G]\!]$-verschränkte Kategorie. Analog zu den $[\![M,G]\!]$-verschränkten Mengen haben wir für ein $v \in \mathrm{Ob}(\mathcal{C})$ einen Isomorphismus von $[\![M,G]\!]$-verschränkten Kategorien.

$$(\zeta,\eta)\colon\ \mathrm{C}_{[\![M,G]\!]}(v)\,{}_{\mathcal{C}}\!\!\setminus\!\!\setminus[\![M,G]\!] \xrightarrow{\sim} v[\![M,G]\!]$$

Erklärung

Hiermit versichere ich, dass ich meine Arbeit selbstständig verfasst und keine andere als die angegebenen Quellen benutzt habe. Alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen habe ich als solche gekennzeichnet. Weiterhin versichere ich, dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, November 2015

Monika Truong