A genus question for orders

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General

R : principal ideal domain

$p \in R$: prime

$\Lambda \subseteq R \times \cdots \times R =: \Gamma$: orders over R such that

- $\Lambda_{(p)}$ is local
- $p^m \Gamma \subseteq \Lambda \subseteq \Gamma$ for some $m \geqslant 0$

Construct

$$\alpha = (\alpha_i)_i \qquad \Gamma^*_{(p)} = \overbrace{R^*_{(p)} \times \cdots \times R^*_{(p)}}^{\text{normalised idèles}}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\bar{\alpha} \qquad \Gamma^*_{(p)}/\Lambda^*_{(p)}\Gamma^* =: \underbrace{\boxed{\mathrm{Cl}(\Lambda)}}_{\text{idèle class group}}$$

To $\alpha \in \Gamma^*_{(p)}$, we attach the projective indecomposable Λ-module

$$\Lambda(\alpha) \;:=\; \{\, (x_i)_i \in \Gamma \;:\; (x_i \alpha_i^{-1})_i \in \Lambda_{(p)} \,\} \,.$$

- All projective indecomposable Λ-modules are of this form.
- $\Lambda(\alpha) \simeq \Lambda(\beta) \;\Leftrightarrow\; \bar{\alpha} = \bar{\beta}$
- $\mathrm{Hom}_\Lambda(\Lambda(\alpha), \Lambda(\beta)) \simeq \Lambda(\alpha^{-1}\beta)$
- As $\Lambda_{(p)}$-modules, we have $\Lambda(\alpha)_{(p)} \simeq \Lambda_{(p)}$.

A ring Morita-equivalent to Λ is of the form

$$\Xi_{\alpha_1,\ldots,\alpha_t} \;:=\; \mathrm{End}_\Lambda(\Lambda(\alpha_1) \oplus \cdots \oplus \Lambda(\alpha_t))$$

for some $t \geqslant 1$ and some $\alpha_i \in \Gamma^*_{(p)}$ for $i \in [1, t]$.

E.g. $\Xi_{1,\ldots,1} \;\simeq\; \Lambda^{t\times t}$. Note that

$$(\Xi_{\alpha_1,\ldots,\alpha_t})_{(p)} \;\simeq\; (\Xi_{1,\ldots,1})_{(p)}$$

Example

$R := \mathbf{Z}$

$p := 5$

$$\Lambda := \underbrace{\{\, (z, z') \in \mathbf{Z} \times \mathbf{Z} \;:\; z \equiv_5 z' \,\}}_{\text{“}\,\mathbf{Z} -_5- \mathbf{Z}\,\text{”}} \subseteq \mathbf{Z} \times \mathbf{Z} =: \Gamma$$

Calculate

$$\begin{aligned} & \mathrm{Cl}(\Lambda) \\ = \; & \frac{\mathbf{Z}^*_{(5)} \times \mathbf{Z}^*_{(5)}}{\{\, (z, z') \in \mathbf{Z}^*_{(5)} \times \mathbf{Z}^*_{(5)} \;:\; z \equiv_5 z' \,\} \cdot (\{\pm 1\} \times \{\pm 1\})} \\ = \; & \langle\, \overline{(2,1)} \,\rangle \;\simeq\; \mathrm{C}_2 \;\not\simeq\; 1 \,, \end{aligned}$$

whence Krull-Schmidt fails in Λ-proj. Therefore, Λ being a certain endomorphism ring, it also fails in $\mathbf{Z}\mathrm{S}_5$-mod – as is well-known.

To $\alpha := (2, 1)$, $1 = (1, 1) \;\in\; \mathbf{Z}^*_{(5)} \times \mathbf{Z}^*_{(5)} = \Gamma^*_{(p)}$, we attach

$$\begin{aligned} \Lambda(\alpha) &= \{\, (z, z') \in \mathbf{Z} \times \mathbf{Z} \;:\; -2z \equiv_5 z' \,\} \\ \Lambda(1) &= \{\, (z, z') \in \mathbf{Z} \times \mathbf{Z} \;:\; z \equiv_5 z' \,\} \;=\; \Lambda \,. \end{aligned}$$

We obtain

$$\begin{aligned} & \Xi_{\alpha,1} \\ = \; & \mathrm{End}_\Lambda(\Lambda(\alpha) \oplus \Lambda(1)) \\ \simeq \; & \left\{ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \right) \in \mathbf{Z}^{2\times 2} \times \mathbf{Z}^{2\times 2} \;:\; \begin{smallmatrix} a \equiv_5 a', & 2b \equiv_5 b', \\ -2c \equiv_5 c', & d \equiv_5 d' \end{smallmatrix} \right\} \end{aligned}$$

$$\begin{aligned} & \Xi_{1,1} \\ = \; & \mathrm{End}_\Lambda(\Lambda(1) \oplus \Lambda(1)) \\ \simeq \; & \left\{ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \right) \in \mathbf{Z}^{2\times 2} \times \mathbf{Z}^{2\times 2} \;:\; \begin{smallmatrix} a \equiv_5 a', & b \equiv_5 b', \\ c \equiv_5 c', & d \equiv_5 d' \end{smallmatrix} \right\} . \end{aligned}$$

General (cont.)

Write $\mathrm{Cl}(\Lambda)^t := \{ \bar{\alpha}^t \, : \, \bar{\alpha} \in \mathrm{Cl}(\Lambda) \} \leqslant \mathrm{Cl}(\Lambda)$.
Suppose given $t \geqslant 1$ and α_i , $\beta_i \in \Gamma^*_{(p)}$ for $i \in [1, t]$.

Proposition. We have

$$\Xi_{\alpha_1, \dots, \alpha_t} \;\simeq\; \Xi_{\beta_1, \dots, \beta_t}$$

as Λ-algebras if and only if

$$(\bar{\alpha}_1 \cdots \bar{\alpha}_t)(\bar{\beta}_1 \cdots \bar{\beta}_t)^{-1} \;\in\; \mathrm{Cl}(\Lambda)^t \,.$$

Thus we partition the Morita-equivalence class of Λ into isoclasses.

Main ingredient:

Jacobinski's Cancellation Theorem.

In this context it states that given P, Q finitely generated projective Λ-modules, we have the implication

$$P \oplus \Lambda \;\simeq\; Q \oplus \Lambda \;\Longrightarrow\; P \;\simeq\; Q \,.$$

(Without Jacobinski, it is possible to give a complicated proof for the case $t = 2$.)

Example (cont.)

Since $(\bar{\alpha} \cdot \bar{1})(\bar{1} \cdot \bar{1})^{-1} = \bar{\alpha} \notin \underbrace{\mathrm{Cl}(\Lambda)^2}_{\simeq 1} \leqslant \underbrace{\mathrm{Cl}(\Lambda)}_{= \langle \bar{\alpha} \rangle \simeq \mathrm{C}_2}$, we have

$$\Xi_{\alpha,1} \;\not\simeq\; \Xi_{1,1} \,.$$

Since $\mathrm{Cl}(\Lambda)^3 = \mathrm{Cl}(\Lambda)$, we have

$$\Xi_{\alpha,1,1} \;\simeq\; \Xi_{1,1,1} \,.$$

Since $\mathrm{Cl}(\Lambda)^4 \simeq 1$, we have

$$\Xi_{\alpha,1,1,1} \;\not\simeq\; \Xi_{1,1,1,1} \,.$$

Representatives of the isoclasses within the Morita-equivalence class of Λ are given by

$$\Xi_1 \, (= \Lambda), \; \Xi_{1,1} \, , \; \Xi_{\alpha,1} \, , \; \Xi_{1,1,1} \, , \; \Xi_{1,1,1,1} \, , \; \Xi_{\alpha,1,1,1} \, , \; \dots$$

Motivation

1. Data

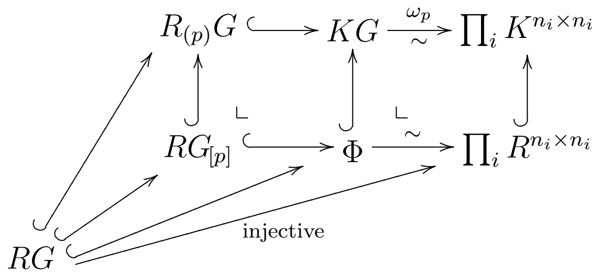
R : principal ideal domain with $\operatorname{char} R = 0$

G : finite group such that R is large enough for G

$\pi(G)$: set of prime divisors of $|G|$

$p \in \pi(G)$: prime divisor of $|G|$

2. Using simple RG-lattices for a Wedderburn isomorphism ω_p, we obtain



A hook in a quadrangle indicates a pullback.

Here $RG_{[p]} := \Phi \cap R_{(p)}G$ is called the *naive localisation* of RG at p. It depends on the choice of the maximal order Φ, i.e. on the choice of RG-lattices used for the Wedderburn isomorphism ω_p.

3. We aim to find a "good" Wedderburn embedding $\omega_q : RG_{[q]} \to \prod_i R^{n_i\times n_i}$ for each $q \in \pi(G)$ separately, using the same RG-lattices for all $q \in \pi(G)$, but different R-linear bases (with base change matrices of determinant 1 between them). "Good" means that its image allows a simple description via congruences.

Since $\mathrm{SL}_{n_i}(R) \to \prod_{q\in\pi(G)} \mathrm{SL}_{n_i}(R/q^{\mathrm{v}_q(|G|)}R)$ is surjective, the various ω_q can be patched together to yield

$$RG \;\simeq\; \bigcap_{q\in\pi(G)} \omega_q(RG_{[q]}) \;\subseteq\; \prod_i R^{n_i\times n_i}$$

4. Two $RG_{[p]}$-lattices are *in the same genus* if they are isomorphic after localisation at (p).

Let $1_{RG_{[p]}} = e_1 + \cdots + e_n$ be an orthogonal decomposition into primitive idempotents in $RG_{[p]}$, ordered such that indecomposable projective modules $RG_{[p]}e_i$ in the same genus occur in intervals.

Say, $RG_{[p]}e_1, \ldots, RG_{[p]}e_t$ are in the same genus, for some $t \in [1, n]$. Assume that the decomposition numbers that occur for $R_{(p)}Ge_1$ are all in $\{0, 1\}$.

Let

$$\Lambda \;:=\; e_1 RG_{[p]} e_1$$

be the endomorphism ring of the indecomposable projective $RG_{[p]}$-module $RG_{[p]}e_1$.

Now the proposition above allows to decide whether

$$\Xi \;:=\; (e_1 + \cdots + e_t)RG_{[p]}(e_1 + \cdots + e_t) \;\overset{?}{\simeq}\; \Lambda^{t\times t},$$

provided the left hand side is sufficiently explicitly known.

An affirmative answer is "good" for our Wedderburn embedding ω_p.

5. Using Specht lattices and their duals for the Wedderburn isomorphisms, the answers were all affirmative for $(\mathbf{Z}\mathrm{S}_n)_{[q]}$ for $n \in [1,6]$, $q \in \pi(\mathrm{S}_n)$.

To illustrate, there exists an endomorphism ring of an indecomposable projective $(\mathbf{Z}\mathrm{S}_6)_{[2]}$-module isomorphic to

$$\Lambda \;:=\; \{\,(a,b,c,d,e,f) \in \mathbf{Z}^{\times 6} \;:\; a \equiv_2 e,\; a+d-2f \equiv_{16} b+c-2e \equiv_8 0,\; e-f \equiv_4 c-d \equiv_2 0\,\}\;,$$

having $\mathrm{Cl}(\Lambda) \simeq \mathrm{C}_4$.