A Fourier-Hopf inversion and spectral sequences

1. Fourier-Hopf inversion

Maps on the right : $A \xrightarrow{f} B$, $a \mapsto af = [a]f$.

- R : commutative ring DATA
- H : Hopf algebra free over R, with involutive antipode ; so

$$H \xrightarrow{\varepsilon} R \qquad \text{counit}$$

$$H \xrightarrow{\Delta} H \otimes_R H \qquad \text{comultiplication}$$
$$x \mapsto \sum_i xu_i \otimes xv_i$$

$$H \xrightarrow{S} H \qquad \text{antipode, } S^2 = \mathrm{id}$$

- $K \subseteq H$: normal Hopf subalgebra free over R ; i.e. $K\Delta \subseteq K \otimes_R K$, $KS \subseteq K$, and

$$\left.\begin{array}{l} \sum_i xu_i \cdot k \cdot xv_i S \in K \\ \sum_i xu_i S \cdot k \cdot xv_i \in K \end{array}\right\} \text{ for } k \in K,\ x \in H$$

- M : left H-module

Write $K^+ := \mathrm{Kern}(K \xrightarrow{\varepsilon} R)$.

Then $K^+H = HK^+ \subseteq H$ is a Hopf ideal, so $\bar{H} := H/HK^+$ is a Hopf algebra.

Write $\bar{x} := x + HK^+$ for $x \in H$, etc.

Examples

- G group, $N \trianglelefteq G$, $H = RG$, $K = RN$, $\bar{H} = R(G/N)$ ($\rightsquigarrow$ Lyndon-Hochschild-Serre).
- $\mathfrak{g}$ Lie algebra free$/R$, $\mathfrak{n} \trianglelefteq \mathfrak{g}$ such that $\mathfrak{n}$ and $\mathfrak{g}/\mathfrak{n}$ free$/R$, $H = \mathcal{U}(\mathfrak{g})$, $K = \mathcal{U}(\mathfrak{n})$, $\bar{H} = \mathcal{U}(\mathfrak{g}/\mathfrak{n})$ ($\rightsquigarrow$ Hochschild-Serre).

Two $\bar{H}$-modules

(a) $\mathrm{Hom}_K(H, M)$ is a left $\bar{H}$-module via

$$[h](\bar{x} \cdot f) \;:=\; \sum_i xu_i \cdot [xv_iS \cdot h]f$$

for $f \in \mathrm{Hom}_K(H, M)$, h, $x \in H$.

(b) $\mathrm{Hom}_R(\bar{H}, M)$ is a left $\bar{H}$-module via

$$[\bar{h}](\bar{x} \cdot g) \;:=\; [\bar{h} \cdot \bar{x}]g$$

for $g \in \mathrm{Hom}_R(\bar{H}, M)$, h, $x \in H$.

With help of G. Carnovale :

<u>Lemma</u> (Fourier-Hopf inversion =: FH)

There exist mutually inverse isos of $\bar{H}$-modules :

$$f \longmapsto \left(f\Phi : \bar{h} \mapsto \textstyle\sum_i hu_i \cdot [hv_iS]f\right)$$

$$\mathrm{Hom}_K(H, M) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \sim \; \mathrm{Hom}_R(\bar{H}, M)$$

$$\left(g\Psi : h \mapsto \textstyle\sum_i hv_i \cdot [\overline{hu_iS}]g\right) \longleftarrow\!\shortmid \; g$$

2. Spectral sequence comparisons

Assume : H, K, $\bar{H}$ free over R, and H free over K

Write

$$\begin{array}{ccccl} (H\text{-Mod})^\circ & \times & H\text{-Mod} & \xrightarrow{U} & \bar{H}\text{-Mod} \\ (X & , & X') & \mapsto & U(X,X') := \underbrace{\mathrm{Hom}_K(X,X')}_{\text{cf. 1.(a)}} \end{array}$$

$$\begin{array}{ccccl} (\bar{H}\text{-Mod})^\circ & \times & \bar{H}\text{-Mod} & \xrightarrow{V} & R\text{-Mod} \\ (Y & , & Y') & \mapsto & V(Y,Y') := \mathrm{Hom}_{\bar{H}}(Y,Y') \end{array}$$

Three spectral sequences

(a) The composition

$$H\text{-Mod} \xrightarrow[K\text{-fixed points}]{U(R,-)} \bar{H}\text{-Mod} \xrightarrow[\bar{H}\text{-fixed points}]{V(R,-)} R\text{-Mod}$$

gives rise to the Grothendieck sp. seq.

$$\mathrm{E}_{U(R,-),V(R,-)}(M) \; .$$

(b) The composition

$$(H\text{-Mod})^\circ \xrightarrow{U(-,M)} \bar{H}\text{-Mod} \xrightarrow{V(R,-)} R\text{-Mod}$$

gives rise to the Grothendieck sp. seq.

$$\mathrm{E}_{U(-,M),V(R,-)}(R)$$

But . . .

... existence of this sp. seq. needs : P proj.$/H \overset{!}{\Rightarrow}$ $U(P,M)$ acyclic w.r.t. $V(R,-) = \mathrm{Hom}_{\bar{H}}(R,-)$.

Suffices $P := H$, so $U(P,M) = U(H,M)$. Now

$$U(H,M) \overset{\text{def}}{=} \mathrm{Hom}_K(H,M) \overset{\boxed{\text{FH}}}{\simeq} \mathrm{Hom}_R(\bar{H},M)\ ,$$

and

$$\mathrm{Ext}^i_{\bar{H}}\big(R, \mathrm{Hom}_R(\bar{H},M)\big) \simeq \mathrm{Ext}^i_R(\underbrace{\bar{H}\otimes_{\bar{H}} R}_{=\ R}, M) \simeq 0$$

for $i \geqslant 1$. So existence is ok.

(c) B : proj. res. of R over H.
$\tilde{B}$: proj. res. of R over $\bar{H}$.
Replacing R by B resp. by $\tilde{B}$, we obtain a double complex

$$V(\tilde{B}, U(B,M)) \simeq \mathrm{Hom}_H(\tilde{B}\otimes_R B, M)\ ,$$

whence a Hochschild-Serre type sp. seq.

$$\mathrm{E}\big(\mathrm{Hom}_H(\tilde{B}\otimes_R B, M)\big)\ .$$

Theorem We have

$$\begin{array}{ll}
\mathrm{E}_{U(R,-),V(R,-)}(M) & \text{(a), abstract, using inj.} \\
\wr| & \\
\mathrm{E}_{U(-,M),V(R,-)}(R) & \text{(b), intermediate, } \boxed{\text{FH}}\,! \\
\wr| & \\
\mathrm{E}\big(\operatorname{Hom}_H(\tilde{B} \otimes_R B, M)\big) & \text{(c), concrete, using proj.}
\end{array}$$

Note : both (a) and (c) a priori have

$$\mathrm{E}_2^{p,q} \;=\; \operatorname{Ext}^p_{\bar{H}}\big(\operatorname{Ext}^q_K(R, M)\big)$$

and converge to

$$\operatorname{Ext}^{p+q}_H(R, M) \;.$$

The problem in comparing them is the differentials.

Related work

- Beyl : case of groups
- Haas : naturality of spectral sequences
- Barnes : comparison theorem in different setup