Does "Quillen A with an extra direction" hold?

1. Some notation

A *bicategory* X is a bisimplicial set such that the simplicial sets $X_{i,*}$ and $X_{*,j}$ are (nerves of) categories for all i, $j \geqslant 0$. A *bifunctor* between bicategories is simply a bisimplicial map.

We write maps on the right.

The face operators in a bisimplicial set are denoted by $\mathrm{d}_i^{(1)}$ in the first, and by $\mathrm{d}_j^{(2)}$ in the second direction. Analogously in a trisimplicial set.

Write $\mathrm{d}^{(2)}_{\lfloor m+n+1,m+1\rfloor} := \mathrm{d}^{(2)}_{m+n+1} \cdots \mathrm{d}^{(2)}_{m+1}$. Etc.

If X is a bisimplicial set, denote by $X\,\mathrm{const}_2$ the trisimplicial set that has $(X\,\mathrm{const}_2)_{i,j,k} = X_{i,k}$. Etc.

A bisimplicial map is called a weak homotopy equivalence if its diagonalisation is a weak homotopy equivalence. Likewise for a trisimplicial map.

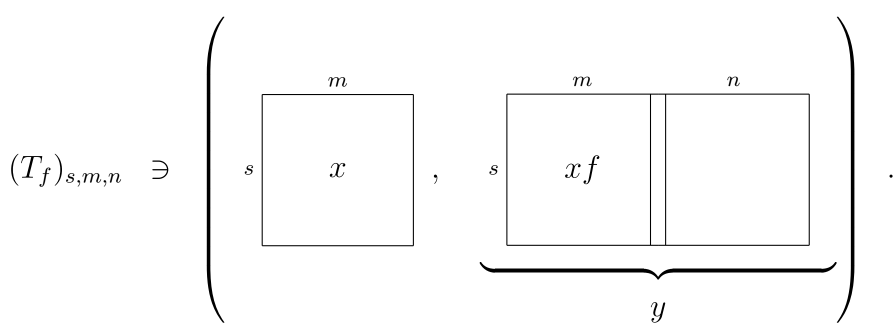
2. The question on "Quillen A with an extra direction"

Let X and Y be bicategories, and let $X \xrightarrow{f} Y$ be a bifunctor.

Form the trisimplicial set T_f that has

$$(T_f)_{s,m,n} \;=\; \{(x,y) \in X_{s,m} \times Y_{s,m+n+1} \;:\; xf = y\mathrm{d}^{(2)}_{\lfloor m+n+1,m+1\rfloor}\}$$

for s, m, $n \geqslant 0$. To sketch a schematic picture,



The trisimplicial operation in the first (aka s-) direction on (x,y) is the operation on x and on y in the first direction.

The trisimplicial operation in the second (aka m-) direction on (x,y) is the operation on x in the second direction and the operation on y in the "front part" in the second direction. So e.g. if $(x,y) \in (T_f)_{s,m,n}$, then $(x,y)\mathrm{d}_i^{(2)} = (x\mathrm{d}_i^{(2)}, y\mathrm{d}_i^{(2)})$.

The trisimplicial operation in the third (aka n-) direction on (x,y) is the identical operation on x and the operation on y in the "back part" in the second direction. So e.g. if $(x,y) \in (T_f)_{s,m,n}$, then $(x,y)\mathrm{d}_i^{(3)} = (x, y\mathrm{d}^{(2)}_{m+1+i})$.

We have trisimplicial "projection" maps

$$\begin{array}{rcl} T_f & \xrightarrow{p_{1,f}} & X\,\mathrm{const}_3 \\ (x,y) & \longmapsto & x \\ & & \\ T_f & \xrightarrow{p_{2,f}} & Y\,\mathrm{const}_2 \\ (x,y) & \longmapsto & y\mathrm{d}^{(2)}_{\lfloor m,0\rfloor} \end{array}$$

For a schematic picture of $p_{2,f}$, cf. §3 (upper row of picture).

Remark. Imitating Quillen's proof of Theorem A, it is not difficult to show that f is a weak homotopy equivalence if and only if $p_{2,f}$ is a weak homotopy equivalence.

Consider the bisimplicial map $p_{2,f}|_{n=0}$ that is given at (s,m) by $(T_f)_{s,m,0} \xrightarrow{(p_{2,f})_{s,m,0}} Y_{s,0}$. For a schematic picture of $p_{2,f}|_{n=0}$, cf. §3 (lower row of picture).

Question. If $p_{2,f}|_{n=0}$ is a weak homotopy equivalence, is then $p_{2,f}$ (and thus f) a weak homotopy equivalence?

Remark. If X and Y are constant in the first (aka s-) direction (that is, if this simplicial direction is "not there"), then the answer is affirmative by Quillen A. In fact, $(T_f)_{0,*,0}$ is the disjoint union of the over-categories $f_{0,*}/y$, indexed by $y \in Y_{0,0}$, and $p_{2,f}|_{n=0}$ maps an element in that disjoint union just to its indexing element.

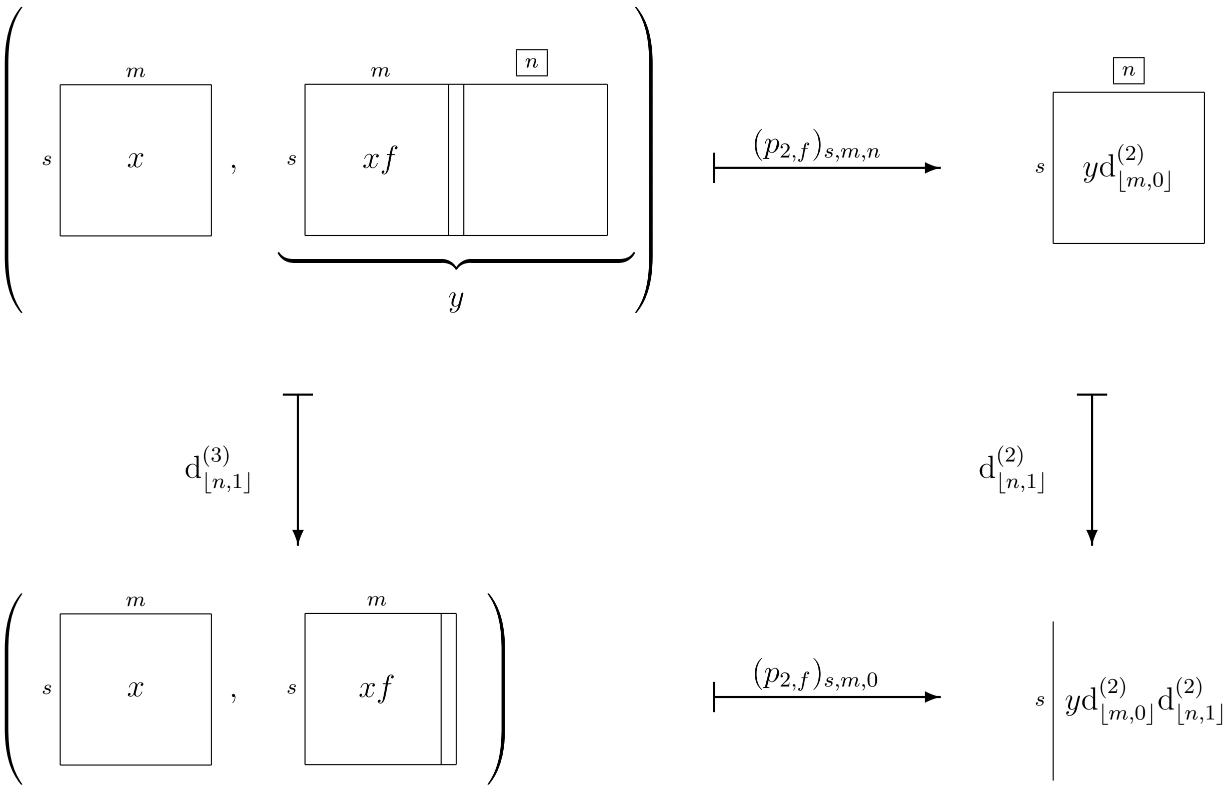
Remark. If the simplicial subset $p_{2,f}^{-1}(\tilde{y})$ of $(T_f)_{s,*,0}$ is weakly contractible for all $s \geqslant 0$ and all $\tilde{y} \in Y_{s,0}$, then $p_{2,f}$ and $p_{2,f}|_{n=0}$ are both weak homotopy equivalences. In fact, in this case it follows that $p_{2,f}^{-1}(\tilde{y})$ is weakly contractible for all s, $n \geqslant 0$ and all $\tilde{y} \in Y_{s,n}$, for $p_{2,f}^{-1}(\tilde{y}) \simeq p_{2,f}^{-1}(\tilde{y}\mathrm{d}^{(2)}_{\lfloor n,1\rfloor})$ (isomorphism of simplicial sets).

3. A question for a homotopy pullback

Fix $n \geqslant 0$. Consider the following commutative quadrangle of bisimplicial sets.

$$(*) \qquad \begin{array}{ccc} (T_f)_{*,\tilde{*},n} & \xrightarrow{(p_{2,f})_{*,\tilde{*},n}} & Y_{*,n} \\ \Big\downarrow{\scriptstyle \mathrm{d}^{(3)}_{\lfloor n,1\rfloor}} & & \Big\downarrow{\scriptstyle \mathrm{d}^{(2)}_{\lfloor n,1\rfloor}} \\ (T_f)_{*,\tilde{*},0} & \xrightarrow{(p_{2,f})_{*,\tilde{*},0}} & Y_{*,0} \end{array}$$

To sketch a schematic picture,



Question. Is $(*)$ a homotopy pullback?

Remark. If the answer to this question is affirmative, so is the answer to the question in §2.

Remark. If X and Y are constant in the first direction, then, as far as I can see, this is true.

> *Speculation.* Is there a categorical model for the homotopy pullback (of categories, to begin with; then of bicategories)? With objects like in the comma category, only with an eventually bothsided constant zigzag instead of simply a morphism? Such a model could then be used to compare – one would need to get rid of the zigzag again somehow.

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