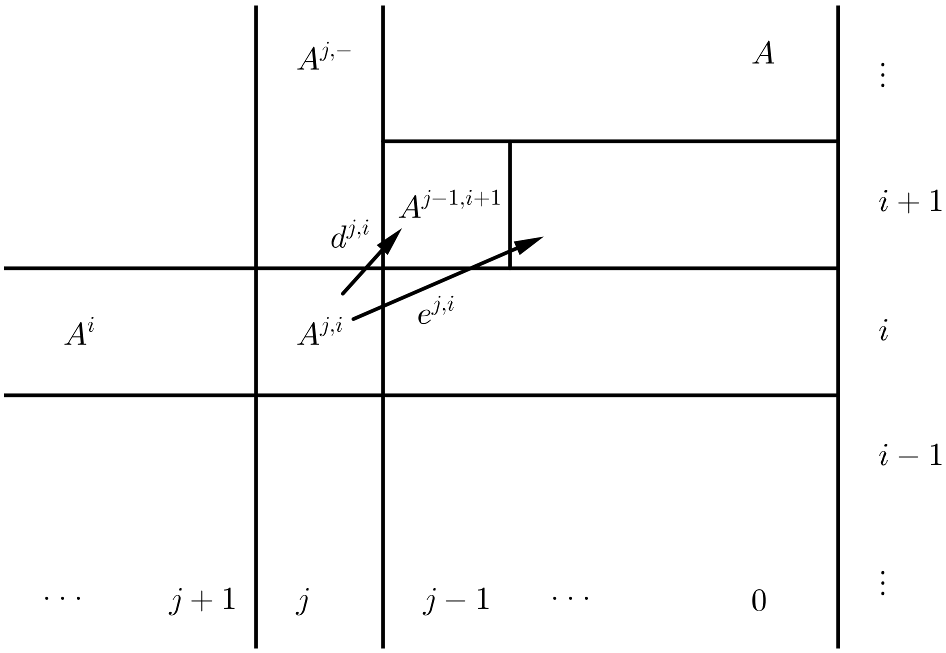
On A_∞-categories

Extended Kadeishvili minimal models and Keller and Lefèvre-Hasegawa's filt construction over arbitrary ground rings



Master thesis (extended version)

Stephan Schmid

February 2015

Universität Stuttgart

Abstract: We generalize Keller and Lefèvre-Hasegawa's filt construction and Kadeishvili's minimality theorem from the case of a ground field $\mathbb{F}$ to the case of an arbitrary commutative ground ring R. Kadeishvili has constructed a minimal model on the homology of a given A_∞-algebra over $\mathbb{F}$. We construct a model on an arbitrary projective resolution of the homology of a given A_∞-algebra over R.

Contents

Preface to extended version: In both my bachelor thesis [20] and my master thesis, I investigated problems in the context of A_∞-theory. For the bachelor thesis, I wrote an introduction to A_∞-theory, which could not be reused directly in the official version of the master thesis to avoid duplication of the bachelor thesis. Instead, only the necessary parts were reused. In particular, the proofs were quoted from [20] and many illustrative parts do not appear at all in the master thesis.

To facilitate reading, the present "extended version" of the master thesis was prepared. It contains all the parts that were cut out in the official version due to formal reasons as explained above. Furthermore, I sketched some further observations that might be useful.

0.1. Introduction

Suppose R is a commutative ring. By graded modules, we denote $\mathbb{Z}$-graded R-modules. By graded maps, we denote R-linear graded maps between graded modules.

Note that at the evaluation of tensor products of graded modules, the Koszul sign rules yields additional signs, cf. Definition 15.

0.1.1. A_∞-algebras

Our first guiding example are differential graded algebras (dg-algebras): A *dg-algebra* over R is a graded module A, a graded map $m_1 : A \to A$ of degree 1 and a graded map $m_2 : A \otimes A \to A$ of degree 0 such that

$$\begin{aligned} m_1 \circ m_1 &= 0 && (m_1 \text{ is a differential}) \\ m_1 \circ m_2 &= m_2(m_1 \otimes 1 + 1 \otimes m_1) && (\text{Leibniz rule}) \\ m_2 \circ (m_2 \otimes 1) &= m_2 \circ (1 \otimes m_2) && (\text{Associativity of the multiplication map } m_2)\ . \end{aligned}$$

Stasheff introduced in [22] a generalization of dg-algebra structures on A to A_∞-algebra structures on A, which consists of a graded "multiplication map" $m_k : A^{\otimes k} \to A$ of degree $2 - k$ for each $k \geq 1$ satisfying the Stasheff identities, cf. Definition 22. The tuple $(A, (m_k)_{k \geq 1})$ is then called an A_∞*-algebra*. The first three of the Stasheff identities are

$$\begin{aligned} 0 &= m_1 \circ m_1 \\ 0 &= m_1 \circ m_2 - m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) \\ 0 &= m_1 \circ m_3 + m_2 \circ (1 \otimes m_2 - m_2 \otimes 1) \\ &\quad + m_3 \circ (m_1 \otimes 1^{\otimes 2} + 1 \otimes m_1 \otimes 1 + 1^{\otimes 2} \otimes m_1). \end{aligned}$$

Note the similarity to the equations for dg-algebras. E.g. the third Stasheff identity is a replacement of associativity. Moreover, we recover the dg-algebras as the A_∞-algebras with $m_k = 0$ for $k \geq 3$, cf. e.g. Example 23. An A_∞-algebra $(A, (m_k)_{k \geq 1})$ is called *minimal* if $m_1 = 0$.

The *bar construction* yields a bijective correspondence between A_∞-structures on A and graded codifferentials of degree 1 on the graded tensor coalgebra $TSA = \oplus_{k \geq 1} (SA)^{\otimes k}$, where SA is the $\mathbb{Z}$-graded module with $(SA)^q = A^{q+1}$ (i.e. we shift the grading of A), cf. section 2.1. For instance, the laborious signs appearing in the Stasheff identities (4)[k] disappear via this bijection. This way, the bar construction can be used to explain the intricate list of Stasheff identities. In the literature, there are slightly different variants of the bar construction in use which yield different variants of the Stasheff identities. We use the variant given e.g. in [14].

Given A_∞-algebras $(A', (m'_k)_{k \geq 1})$ and $(A, (m_k)_{k \geq 1})$, an A_∞*-morphism* or *morphism of* A_∞*-algebras* from A' to A is a tuple $(f_k)_{k \geq 1}$ such that for $k \geq 1$, the map $f_k : A'^{\otimes k} \to A$ is of degree $1 - k$ and such that certain equations hold, cf. Definition 22. The first two of these equations are

$$\begin{aligned} f_1 \circ m'_1 &= m_1 \circ f_1 \\ f_1 \circ m'_2 - f_2 \circ (m'_1 \otimes 1 + 1 \otimes m'_1) &= m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1). \end{aligned}$$

In particular, the first equation implies that $f_1 : (A', m'_1) \to (A, m_1)$ is a morphism of complexes. The A_∞-morphism $(f_k)_{k \geq 1}$ is called a *quasi-isomorphism of* A_∞*-algebras* if $f_1 : (A', m'_1) \to (A, m_1)$ is a quasi-isomorphism. If $(f_k)_{k \geq 1}$ is a quasi-isomorphism, then

A' is called a *model* of A. If additionally $m'_1 = 0$, then A' is called a *minimal model* of A. Similarly to how the bar construction associates A_∞-algebras with differential coalgebras, it associates A_∞-morphisms with morphisms of differential coalgebras, cf. section 2.1. Using composition of morphisms of differential coalgebras, this provides a natural way to define composition of A_∞-morphisms, cf. section 2.3.

0.1.2. A_∞-categories

Our second guiding example are Hom^*-dg-categories, which will illustrate the concept of A_∞-categories.

Suppose given an (ordinary) R-algebra B. Suppose given a set I and a complex $(C^{(i)}, d^{(i)})$ over B for each $i \in I$. We obtain the complex $(C := \oplus_{i\in I} C^{(i)}, d := \oplus_{i \in I} d^{(i)})$ over B. We have the graded module $A := \mathrm{Hom}^*_B(C, C) = \oplus_{i,j\in I} \mathrm{Hom}^*_B(C^{(i)}, C^{(j)})$. Here for $i, j \in I$ and $k \in \mathbb{Z}$, the R-module $\mathrm{Hom}^k_B(C^{(i)}, C^{(j)}) = \prod_{z\in\mathbb{Z}} \mathrm{Hom}_B(C^{(i)}_{z+k}, C^{(j)}_z)$ is the R-module of graded B-linear morphisms of degree k from $C^{(i)}$ to $C^{(j)}$ (which are not necessarily compatible with the differential), which then is the homogeneous component of degree k of the graded R-module $\mathrm{Hom}^*_B(C^{(i)}, C^{(j)}) := \oplus_{q\in\mathbb{Z}} \mathrm{Hom}^q_B(C^{(i)}, C^{(j)})$. On A, we have the differential $d_{\mathrm{Hom}^*_B(C,d)}$ that is given for $f \in A^k = \mathrm{Hom}^k(C, C)$ as

$$d_{\mathrm{Hom}^*_B(C,d)}(f) := d \circ f - (-1)^k f \circ d.$$

On A, there is a dg-algebra structure given by setting $m_1 := d_{\mathrm{Hom}^*_B(C,d)}$ and setting m_2 to be composition, that is for homogeneous $f, g \in A$, we have $m_2(f \otimes g) := f \circ g$. Recall that $m_k := 0$ for $k \geq 3$ for dg-algebras.

Note that m_1 and m_2 respect the decomposition $A = \oplus_{i,j\in I} \mathrm{Hom}^*_B(C^{(i)}, C^{(j)})$: For $i, j, j', k \in I$, we have

$$\begin{array}{lcll} m_1(\mathrm{Hom}^*_B(C^{(i)}, C^{(j)})) & \subseteq & \mathrm{Hom}^*_B(C^{(i)}, C^{(j)}) & \\ m_2(\mathrm{Hom}^*_B(C^{(j)}, C^{(k)}) \otimes \mathrm{Hom}^*_B(C^{(i)}, C^{(j)})) & \subseteq & \mathrm{Hom}^*_B(C^{(i)}, C^{(k)}) & \\ m_2(\mathrm{Hom}^*_B(C^{(j')}, C^{(k)}) \otimes \mathrm{Hom}^*_B(C^{(i)}, C^{(j)})) & = & 0 & \text{if } j \neq j'. \end{array}$$

This behaviour of the dg-algebra $(A, (m_k)_{k\geq 1})$ is the prototype of an A_∞-category. Indeed, by setting $A(i, j) := \mathrm{Hom}^*_B(C^{(j)}, C^{(i)})$ (note the swapped indices), we obtain the tuple $(\mathrm{Obj}\, A := I, A, (m_k)_{k\geq 1})$ which satisfies the following definition of an A_∞-category.

An A_∞-*category* is a tuple $(\mathrm{Obj}\, A, A = \oplus_{i,j\in \mathrm{Obj}\, A} A(i, j), (m_k : A^{\otimes k} \to A)_{k\geq 1})$ such that the following hold.

- $(A, (m_k)_{k\geq 1})$ is an A_∞-algebra.
- For $k \geq 1$ and $j_0, \dots, j_k \in \mathrm{Obj}\, A$, we have

$$m_k(A(j_0, j_1) \otimes \dots \otimes A(j_{k-1}, j_k)) \subseteq A(j_0, j_k).$$

  For $k \geq 1$ and $i_1, \dots, i_k, j_1, \dots, j_k \in \mathrm{Obj}\, A$ such that there exists $x \in [1, k-1]$ with $j_x \neq i_{x+1}$, we have

$$m_k(A(i_1, j_1) \otimes \dots \otimes A(i_k, j_k)) = 0.$$

So an A_∞-category is an A_∞-algebra A together with a decomposition of A into a direct sum such that the m_k respect this decomposition.

Given A_∞-categories $(\operatorname{Obj} A', A', (m'_k)_{k\geq 1})$ and $(\operatorname{Obj} A, A, (m_k)_{k\geq 1})$, a *morphism of* A_∞-*algebras* (or A_∞-*functor*) from A' to A is a tuple $f = (f_{\mathrm{Obj}}, (f_k)_{k\geq 1})$ such that the following hold.

- $(f_k)_{k\geq 1}$ is a morphism of A_∞-algebras from $(A', (m'_k)_{k\geq 1})$ to $(A, (m_k)_{k\geq 1})$.
- $f_{\mathrm{Obj}} : \operatorname{Obj} A' \to \operatorname{Obj} A$ is a map.
- For $k \geq 1$ and $j_0, \ldots, j_k \in \operatorname{Obj} A'$, we have

  $$f_k(A'(j_0, j_1) \otimes \ldots \otimes A'(j_{k-1}, j_k)) \subseteq A(f_{\mathrm{Obj}}(j_0), f_{\mathrm{Obj}}(j_k)).$$

  For $k \geq 1$ and $i_1, \ldots, i_k, j_1, \ldots, j_k \in \operatorname{Obj} A'$ such that there exists $x \in [1, k-1]$ with $j_x \neq i_{x+1}$, we have

  $$f_k(A'(i_1, j_1) \otimes \ldots \otimes A'(i_k, j_k)) = 0.$$

The A_∞-functor f is called a *local quasi-isomorphism* if for $i, j \in \operatorname{Obj} A'$, the complex morphism $f_1|_{A'(i,j)}^{A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j))} : (A'(i,j), m'_1) \to (A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j)), m_1)$ is a quasi-isomorphism.

0.1.3. The filt construction

Suppose given an (ordinary) R-algebra B. Given B-modules S_i for $i \in I$, the filt construction provides a complete description of the full subcategory $\mathtt{filt}(S_i, i \in I)$ of B-$\mathtt{Mod}$ given by the B-modules that have a finite filtration such that each subquotient is isomorphic to some S_i for $i \in I$.

Keller and and Lefèvre-Hasegawa's original version of the filt construction requires that R is a field, cf. [11, Problem 2]. We generalize the filt construction to arbitrary commutative ground rings R such as $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$ for p a prime.

In our notation, the filt construction proceeds as follows. First, we choose a projective resolution $(P^{(i)}, d^{(i)})$ of S_i for each $i \in I$. Then, we define the dg-category $(I, A = \oplus_{i,j\in I} A(j,i) = \oplus_{i,j\in I} \operatorname{Hom}^*(C^{(i)}, C^{(j)}), (m_k)_{k\geq 1})$ as in our second guiding example.

Choose an A_∞-category $(\operatorname{Obj} A' := I, A', (m'_k)_{k\geq 1})$ such that there is a local quasi-isomorphism of A_∞-categories $f = (\mathrm{id}_I, (f_k)_{k\geq 1})$ from A' to A. From A, A' and f, we obtain the A_∞-categories $\operatorname{tw} A$ and $\operatorname{tw} A'$ and the local quasi-isomorphism of A_∞-categories $\operatorname{tw} f$ from $\operatorname{tw} A'$ to $\operatorname{tw} A$. Very loosely speaking, $\operatorname{tw} A$, $\operatorname{tw} A'$ and $\operatorname{tw} f$ are matrix versions of A, A' and f that are twisted by strictly lower triangular matrices obeying the generalized Maurer-Cartan equations, cf. Definition/Lemma 121 and Definition/Lemma 122. The objects of $\operatorname{tw} A$ and $\operatorname{tw} A'$ are prototypes of the modules in $\mathtt{filt}(S_i, i \in I)$, so $\operatorname{tw} A$ and $\operatorname{tw} A'$ typically have a lot more objects than A and A'.

The zeroth homology of an A_∞-category carries the structure of a semicategory, i.e. of a "category without identities", cf. section 1.4.2. In practice, these semicategories are often categories[1]. In our case, we obtain the categories $\mathrm{H}^0 \operatorname{tw} A'$ and $\mathrm{H}^0 \operatorname{tw} A$ and the fully faithful functor $\mathrm{H}^0 \operatorname{tw} f$ from $\mathrm{H}^0 \operatorname{tw} A'$ to $\mathrm{H}^0 \operatorname{tw} A$. There is an equivalence of categories Q from $\mathrm{H}^0 \operatorname{tw} A$ to $\mathtt{filt}(S_i, i \in I)$. The composite functor $Q \circ \mathrm{H}^0 \operatorname{tw} f$ is dense, so we have the

Theorem 1 (cf. Theorem 131)**.** *$Q \circ \mathrm{H}^0 \operatorname{tw} f$ is an equivalence of categories from $\mathrm{H}^0 \operatorname{tw} A'$ to $\mathtt{filt}(S_i, i \in I)$.*

Hence, we may describe the category $\mathtt{filt}(S_i, i \in I)$ by the category $\mathrm{H}^0 \operatorname{tw} A'$.

Note that if A' is minimal, then we have $A' = \oplus_{i,j \in I} \operatorname{Ext}^*_B(S_i, S_j)$ equipped with $m'_1 = 0$, with m'_2 given by the Yoneda product and with some m'_k for $k \geq 3$. Note that A is typically a dg-algebra of enormous size and that A' is comparatively small, and so much better suited for practical purposes, even taking into account the (possibly nonvanishing) higher multiplication maps m'_k for $k \geq 3$. In Keller and Lefèvre-Hasegawa's original variant, A' is chosen in such a way that it is minimal. If R is a field, Kadeishvili's minimality theorem ensures that this is possible. Over arbitrary rings R, finding a suitable A' is more complex: As detailed in section 0.1.4, the minimality theorem does not hold over arbitrary rings R, but it is still possible to find a suitable small model A'.

We generalize Keller and Lefèvre-Hasegawa's filt construction to arbitrary commutative ground rings as follows.

Keller and Lefèvre-Hasegawa use A_∞-modules and factorization of the Yoneda functor (cf. [11, Theorem in section 7.5]). We use a direct approach. The choice to use a direct approach was also influenced by the wish to precisely understand how the assembly of objects of $\mathtt{filt}(S_i, i \in I)$ from the S_i translates to the objects of $\mathrm{H}^0 \operatorname{tw} A$ and of $\mathrm{H}^0 \operatorname{tw} A'$. We prove the fact that $\operatorname{tw} f$ is a local quasi-isomorphism directly from the fact that f is a local quasi-isomorphism. This implies that $\mathrm{H}^0 \operatorname{tw} f$ is fully faithful. Establishing Q as a fully faithful functor from $\mathrm{H}^0 \operatorname{tw} A$ to $\mathtt{filt}(S_i, i \in I)$ combines well-known results on projective resolutions with a straightforward translation to A_∞-terminology. The proof that Q as well as $Q \circ \mathrm{H}^0 \operatorname{tw} f$ are dense is done using explicit constructions with the horseshoe lemma as key ingredient.

0.1.4. Small models of A_∞-algebras and A_∞-categories over arbitrary ground rings. The extended Kadeishvili minimal method.

In the filt construction explained in section 0.1.3, we are given an A_∞-category $(\operatorname{Obj} A, A, (m_k)_{k \geq 1})$ and we want to construct an A_∞-category $(\operatorname{Obj} A', A', (m'_k)_{k \geq 1})$

[1] In our applications, we obtain this almost effortlessly by Lemmas 36 and 39 using the presence of a suitable local quasi-isomorphisms of A_∞-categories. In contrast, there are various concepts of "unital" A_∞-algebras resp. A_∞-categories such as strictly unital A_∞-algebras (cf. [11, section 3.5]) and unital A_∞-categories, (cf. [15, Definition 7.3]). These ensure a priori that the semicategories mentioned above are categories but introduce additional constraints.

together with a local quasi-isomorphism of A_∞-categories $(f_{\mathrm{Obj}}, (f_k)_{k\geq 1})$ from A' to A such that $f_{\mathrm{Obj}} : \mathrm{Obj}\, A' \to \mathrm{Obj}\, A$ is bijective.

We will first discuss the simpler case where we are given an A_∞-algebra $(A, (m_k)_{k\geq 1})$ and we want to construct an A_∞-algebra $(A', (m'_k)_{k\geq 1})$ together with an quasi-isomorphism of A_∞-algebras $(f_k)_{k\geq 1}$ from A' to A. We will return to the more general case of A_∞-categories at the end.

In the context of the filt construction, it is desirable that A' is as small as possible since then $\mathrm{H}^0\,\mathrm{tw}\, A'$ also becomes as small and (hopefully) as simple as possible. If $m'_1 = 0$ then A' is as small as possible since then A' is essentially the homology of A. Recall that in that case, A' is called a minimal model of A. If the ground ring R is a field, then the existence of minimal models is guaranteed by Kadeishvili's minimality theorem, cf. [12] (history), [9], [10]. The original version of the minimality theorem given in [10] uses Kadeishvili's algorithm: After constructing $f_1 : A' \to A$ and $m'_1 = 0$ in the initial step, Kadeishvili's algorithm constructs the f_k and m'_k successively for $k = 2, 3, \ldots$, cf. e.g. Theorem 55.

Over a ground ring R that is not a field, the minimality theorem does not hold in general. If e.g. R is an integral domain but not a field, we may easily obtain an A_∞-algebra that does not have a minimal model, cf. section 4.1.

Kadeishvili's algorithm works if the homology of A is projective over R, cf. e.g. Theorem 55. Hence, one reasonable approach is setting A' to be a direct sum of R-projective resolutions of the $\mathrm{H}^i A$ for $i \in \mathbb{Z}$. One way of concretizing this idea was done by Sagave in [19] by extending the concept of A_∞-algebras to dA_∞-algebras. In addition to the "vertical" grading present in A_∞-algebras, dA_∞-algebras feature a "horizontal" grading. The horizontal rows then contain the projective resolutions that A' is composed of. The multiplication maps of a dA_∞-algebras obey grading conditions and a variant of the Stasheff identities that involve both the horizontal and the vertical grading. Using model categories, Sagave obtains in [19] minimal models in the sense of dA_∞-algebras for dg-algebras over arbitrary commutative rings. However, it is unknown to what extent the projective resolutions occurring in A' can be chosen, cf. [19, Remark 4.14]. In particular, it is not known how large such a minimal model A' in the sense of dA_∞-algebras is.

We introduce eA_∞-algebras. Each eA_∞-algebra has an underlying A_∞-algebra, which facilitates the use of the filt construction. I.e. eA_∞-algebras are A_∞-algebras with additional structure. Using a certain notion of minimality for eA_∞-algebras, we obtain the

Theorem 2 (cf. Theorem 90)**.** *Suppose given an* A_∞*-algebra* $(A, (m_k)_{k\geq 1})$*. Choose projective resolutions* $P^{(z)}$ *of* $\mathrm{H}^z A$ *for* $z \in \mathbb{Z}$*.*

Then there exists a minimal eA_∞*-algebra* $(A', (m'_k)_{k\geq 1})$ *with* $A' := \oplus_{z\in\mathbb{Z}} P^{(z)}$ *and a quasi-isomorphism of* A_∞*-algebras* $(f_k)_{k\geq 1}$ *from* $(A', (m'_k)_{k\geq 1})$ *to* $(A, (m_k)_{k\geq 1})$*.*

The construction of the m'_k and f_k for $k \geq 1$ is done incrementally via an algorithm called the extended Kadeishvili minimal method. This algorithm requires knowledge about how the projective resolutions $P^{(z)}$ decompose into their positions. The need to store this information is the cause for introducing eA$_\infty$-algebras, which are A$_\infty$-algebras that are equipped with an additional structure that is precisely designed to hold that information. Technically, this is done by introducing an additional, "horizontal" $\mathbb{Z}$-grading somewhat similar to Sagave's dA$_\infty$-algebras. The projective resolutions $P^{(z)}$ then run diagonally in contrast to dA$_\infty$-algebras, where they run horizontally. For a detailed comparison of dA$_\infty$-algebras and eA$_\infty$-algebras, see section 4.3.5.

In the same way we have generalized from A$_\infty$-algebras to A$_\infty$-categorise, we may now generalize from eA$_\infty$-algebras to eA$_\infty$-categories.

Suppose given an A$_\infty$-category $(\operatorname{Obj} A, A, (m_k)_{k\geq 1})$. The concept of eA$_\infty$-algebras and the concept of A$_\infty$-categories do not interfere with each other. Thus we can perform the extended Kadeishvili minimal method basically separately on the components of $A = \oplus_{i,j\in \operatorname{Obj} A} A(i,j)$ to obtain the

Theorem 3 (cf. Theorem 98). *Suppose given an* A$_\infty$*-category* $(\operatorname{Obj} A, A, (m_k)_{k\geq 1})$. *Choose a projective resolution* $P^{(z)}_{o_1,o_2}$ *of* $\mathrm{H}^z A(o_1,o_2)$ *for each* $z\in\mathbb{Z}$ *and each* $o_1,o_2\in \operatorname{Obj} A$. *For* $o_1,o_2 \in \operatorname{Obj} A$, *let* $A'(o_1,o_2) = \oplus_{z\in\mathbb{Z}} P^{(z)}$. *Let* $A' := \oplus_{o_1,o_2\in\operatorname{Obj} A} A'(o_1,o_2)$. *Then there exists a minimal* eA$_\infty$*-category* $(\operatorname{Obj} A' := \operatorname{Obj} A, A', (m'_k)_{k\geq 1})$ *and a local quasi-isomorphism of* A$_\infty$*-categories* $(\mathrm{id}, (f_k)_{k\geq 1}) : (\operatorname{Obj} A', A', (m'_k)_{k\geq 1}) \to (\operatorname{Obj} A, A, (m_k)_{k\geq 1})$.

0.1.5. Models for cyclic groups over arbitrary ground rings

Suppose given an (ordinary) R-algebra B, a B-module M and a projective resolution P of M over B. If R is a field, the minimality theorem ensures that there is an A$_\infty$-structure $(m'_k)_{k\geq 1}$ on $\operatorname{Ext}^*_B(M,M) = \mathrm{H}^* \operatorname{Hom}^*_B(P,P)$ such that $(\operatorname{Ext}^*_B(M,M), (m_k)_{k\geq 1})$ becomes a minimal model of the dg-algebra $\operatorname{Hom}^*_B(P,P)$. In the context of this introduction, we call such an A$_\infty$-structure on $\operatorname{Ext}^*_B(M,M)$ a canonical A$_\infty$-structure on $\operatorname{Ext}^*_B(M,M)$. We regard group cohomology algebras as a special cases of Ext*-algebras, so the same terminology applies for group cohomology algebras.

For an arbitrary field $\mathbb{F}$ and $n \in \mathbb{Z}_{\geq 1}$, Madsen computed a canonical A$_\infty$-structures on $\operatorname{Ext}^*_{\mathbb{F}[\alpha]/(\alpha^n)}(\mathbb{F},\mathbb{F})$, where $\mathbb{F}$ is the trivial $\mathbb{F}[\alpha]/(\alpha^n)$-module of the algebra $\mathbb{F}[\alpha]/(\alpha^n)$. For $\mathbb{F} := \mathbb{F}_p$ and $n := p^k$ for a prime p and an integer $k \geq 1$, the algebra $\mathbb{F}_p[\alpha]/(\alpha^{p^k})$ is isomorphic to the group algebra $\mathbb{F}_p\mathrm{C}_{p^k}$ of the cyclic group C_{p^k}. So this yields also a canonical A$_\infty$-structure on the group cohomology $\operatorname{Ext}^*_{\mathbb{F}_p\mathrm{C}_{p^k}}(\mathbb{F}_p,\mathbb{F}_p)$ of the cyclic group C_{p^k} as given by Vejdemo-Johansson in [23, Theorem 4.3.8].

In [23], Vejdemo-Johansson developed an algorithm to compute canonical A$_\infty$-structures on group cohomology algebras partially. This algorithm has become a part of the Magma computing framework, where it can be used to partially compute canonical A$_\infty$-structures on the cohomology algebras of p-groups. In [23], it is used to partially compute canonical A$_\infty$-structure on the cohomology algebras over $\mathbb{F}_2$ of the dihedral groups D_8 and D_{16} as well as the quaternion group Q_8.

In [24] and [23] (note the comments at [23, p. 41]), Vejdemo-Johannson investigated a canonical A_∞-structure $(m_n)_{n\geq 1}$ on the group cohomology $\operatorname{Ext}^*_{\mathbb{F}_p(C_k\times C_l)}(\mathbb{F}_p,\mathbb{F}_p)$, where $k,l\geq 4$ are multiples of the prime p. He showed that in that case, the multiplication maps m_2, m_k, m_l, m_{k+l-2}, $m_{2(k-2)+l}$ and $m_{2(l-2)+k}$ are non-zero, cf. [23, Theorem 3.3.3].

In [13], Klamt applied A_∞-theory to representation theory of Lie-algebras. Given certain direct sums M of parabolic Verma modules, she examined canonical A_∞-structures on $\operatorname{Ext}^*_{\mathcal{O}^{\mathfrak{p}}}(M,M)$. Given such a canonical A_∞-structure $(m_k)_{k\geq 1}$, she proved upper bounds for the maximal k such that m_k is non-zero. In certain cases, she computed complete canonical A_∞-structures.

In [20], a canonical A_∞-structure on the group cohomology $\operatorname{Ext}^*_{\mathbb{F}_p\mathrm{S}_p}(\mathbb{F}_p,\mathbb{F}_p)$ of the symmetric group S_p over $\mathbb{F}_p$ has been computed where p is a prime.

To test the theory of eA_∞-algebras, we examine the case of cyclic groups over an arbitrary ground ring:

Recall that R is a commutative ring. Let $n\geq 1$. Let e be a generator of the cyclic group C_n. We have the following projective resolution of the trivial $R\mathrm{C}_n$-module R.

$$P := (\cdots \to R\mathrm{C}_n \xrightarrow{\sum_{i=0}^{n-1} e^i} R\mathrm{C}_n \xrightarrow{1-e} R\mathrm{C}_n \xrightarrow{\sum_{i=0}^{n-1} e^i} R\mathrm{C}_n \xrightarrow{1-e} R\mathrm{C}_n \to 0 \to \cdots)$$

We want to obtain a model $(A',(m'_k)_{k\geq 1})$ of the dg-algebra $\operatorname{Hom}^*_{R\mathrm{C}_n}(P,P)$ such that $(A',(m'_k)_{k\geq 1})$ is a minimal eA_∞-algebra and such that A' consists of standard projective resolutions of the $\mathrm{H}^i\operatorname{Hom}^*_{R\mathrm{C}_n}(P,P)=\operatorname{Ext}^i_{R\mathrm{C}_n}(R,R)$, $i\in\mathbb{Z}$.

Let A' be the free R-module over the set $\{\overline{\iota^j},\overline{\chi\iota^j}\mid j\in\mathbb{Z}_{\geq 0}\}$.
A' is $\mathbb{Z}$-graded by setting $\overline{\iota^j}$ to be homogeneous of degree $2j$ and setting $\overline{\chi\iota^j}$ to be homogeneous of degree $2j+1$ for $j\in\mathbb{Z}_{\geq 0}$.

Examining models $(A',(m'_k)_{k\geq 1})$ of $\operatorname{Hom}^*_{R\mathrm{C}_n}(P,P)$ on A' that exploit the periodicity of P leads to a certain condition (cf. (67)) on the coefficients of the m'_k, cf. Definition/Remark 104 and Propositions 105 and 110. Some experimentation revealed that this condition is equivalent to the equation

$$(h-e)g=r$$

for formal power series $g\in R\mathrm{C}_n[[X]]$, $h\in R\mathrm{C}_n[[X]]$ and $r\in R[[X]]$ with certain constant and linear terms, cf. Proposition 115. Here, r encodes $(m'_k)_{k\geq 1}$ and g and h encode the quasi-isomorphism of A_∞-algebras from $(A',(m'_k)_{k\geq 1})$ to $\operatorname{Hom}^*_{R\mathrm{C}_n}(P,P)$. Since multiplication with e is a circular shift in C_n, the equation of power series given above is some kind of recurrence relation on g. Indeed, if we restrict us to the case $h\in R[[X]]$, then all solutions can be constructed as follows.

Choose $\breve{g}=\sum_{i\geq 0}\breve{g}_iX^i\in R[[X]]$ and $h=\sum_{i\geq 0}h_iX^i\in R[[X]]$ such that $h_0=\breve{g}_0=1$ and such that h_1 is a unit in R. We then obtain g and r by

$$g := \sum_{i=0}^{n-1} h^{n-1-i}\breve{g}e^i$$

$$r := (h^n - 1)\breve{g}.$$

Given such r, g and h, we may then use r to obtain a model of $\mathrm{Hom}^*_{R\mathrm{C}_n}(P, P)$ on A' as follows. We have $r = \sum_{i \geq 0} r_i X^i$ for some $r_i \in R$, $i \geq 0$.

On A', an A_∞-structure $(m'_k)_{k\geq 1}$ is given by setting

$$m'_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) := \begin{cases} \overline{\chi^{a_1+a_2}\iota^{j_1+j_2}} & \text{if } 0 \in \{a_1, \ldots, a_k\} \text{ and } k = 2 \\ 0 & \text{if } 0 \in \{a_1, \ldots, a_k\} \text{ and } k \neq 2 \\ r_k \overline{\iota^{j_1+\ldots+j_k+1}} & \text{if } a_1 = \ldots = a_k = 1 \end{cases}$$

for $k \geq 1$, $a_1, \ldots, a_k \in \{0, 1\}$ and $j_1, \ldots, j_k \geq 0$. The A_∞-algebra $(A', (m'_k)_{k\geq 1})$ carries the structure of a minimal eA_∞-algebra, cf. Remark 111. We have the following

Proposition 4 (cf. Propositions 110 and 117)**.** *The minimal* eA_∞*-algebra* $(A', (m'_k)_{k\geq 1})$ *is quasi-isomorphic to the dg-algebra* $\mathrm{Hom}^*_{R\mathrm{C}_n}(P, P)$*.*

Note that if $R = \mathbb{F}_p$ for some prime p and $n = p^c$ for $c \in \mathbb{Z}_{\geq 1}$, we recover the model given by Madsen and Vejdemo-Johansson, cf. Remarks 114 and 118.

0.1.6. Connecting Hom^*-dg-algebras with A_∞-morphisms

Suppose given an R-algebra B and two projective resolutions P, Q of the same B-module M. In the situation detailed in section 0.1.7, it was necessary to be able to obtain a sensible A_∞-morphism from the dg-algebra $\mathrm{Hom}^*_B(P, P)$ to the dg-algebra $\mathrm{Hom}^*_B(Q, Q)$. Note that in this situation, the comparison theorem implies that the complexes P and Q are homotopy equivalent. So we are able to use the following

Definition/Lemma 5 (cf. Definition/Lemma 63)**.** Suppose given an R-algebra B. Suppose given complexes (P, d_P), (Q, d_Q) over B. We have the dg-algebras $A' := \mathrm{Hom}^*_B(P, P)$ and $A := \mathrm{Hom}^*_B(Q, Q)$.

Suppose given complex morphisms $g_1 : P \to Q$ and $g_2 : Q \to P$. Suppose given a homotopy $h \in \mathrm{Hom}^{-1}_B(P, P)$ such that $g_2 \circ g_1 = \mathrm{id}_P + d_{\mathrm{Hom}^*_B(P,P)}(h)$.

Then there is a morphism of A_∞-algebras $f_{g_1,g_2,h} = (f_k)_{k\geq 1}$ from $A' = \mathrm{Hom}^*_B(P, P)$ to $A = \mathrm{Hom}^*_B(Q, Q)$ given as follows. For $k \geq 1$ and homogeneous elements $x_i \in (A')^{\mathsf{k}_i}$ for $i \in [1, k]$, we set

$$f_k(x_1 \otimes \ldots \otimes x_k) := (-1)^{\frac{k(k-1)}{2}} (-1)^{\sum_{i\in[1,k]} \mathsf{k}_i(k-i)} g_1 \circ (x_1 \circ h \circ x_2 \circ \ldots \circ h \circ x_k) \circ g_2.$$

Note that f_1 maps an element $x \in \mathrm{Hom}^*_B(P, P)$ to $g_1 \circ x \circ g_2 \in \mathrm{Hom}^*_B(Q, Q)$. So $f_{g_1,g_2,h}$ is in a certain sense induced by g_1 and g_2.

0.1.7. Restriction to a subgroup in terms of minimal models on the group cohomology algebras

Suppose given a field $\mathbb{F}$. Suppose given a finite group G. Suppose given a projective resolution P of the trivial $\mathbb{F}G$-module $\mathbb{F}$. By the minimality theorem, there exists an A_∞-structure on the group cohomology $\operatorname{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}) = \mathrm{H}^* \operatorname{Hom}^*_{\mathbb{F}G}(P,P)$ such that $\operatorname{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$ becomes a minimal model of the dg-algebra $\operatorname{Hom}^*_{\mathbb{F}G}(P,P)$. Recall that in this introduction, we call such an A_∞-structure on $\operatorname{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$ a canonical structure on $\operatorname{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$. Suppose given a subgroup H of G. The restriction from G to H induces an inclusion map $\operatorname{res}_{G,H} : \operatorname{Hom}^*_{\mathbb{F}G}(P,P) \hookrightarrow \operatorname{Hom}^*_{\mathbb{F}H}(P,P)$ and thus a map $\mathrm{H}^* \operatorname{res}_{G,H} : \operatorname{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}) \to \operatorname{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$, cf. e.g. [1, p. 73]. At the presentation of my bachelor thesis [20], Steffen König asked whether it was known if $\operatorname{res}_{G,H}$ somehow provides a connection in the A_∞-sense between canonical A_∞-structures on $\operatorname{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$ and on $\operatorname{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$.

In [20], a canonical A_∞-structures on $\operatorname{Ext}^*_{\mathbb{F}_p \mathrm{S}_p}(\mathbb{F}_p,\mathbb{F}_p)$ has been established, where p is a prime, $\mathbb{F}_p$ is the field with p elements and S_p is the symmetric group with $p!$ elements. Further investigation showed that the canonical A_∞-structure obtained on $\operatorname{Ext}^*_{\mathbb{F}_p \mathrm{S}_p}(\mathbb{F}_p,\mathbb{F}_p)$ (cf. [20, Definition 38, Theorem 39]) bears a striking resemblance to canonical A_∞-structures obtained on $\operatorname{Ext}^*_{\mathbb{F}_p \mathrm{C}_p}(\mathbb{F}_p,\mathbb{F}_p)$, cf. [16, Appendix B Example 2.2] and [23, Theorem 4.3.8]. This resemblance is given as follows. For both cases, there are homogeneous generators a, b such that the group cohomology algebra has the $\mathbb{F}_p$-basis $\{a^j, ba^j \mid j \in \mathbb{Z}_{\geq 0}\} =: B$. Evaluating the multiplication maps m_k for $k \geq 1$ on elements $x_1 \otimes \ldots \otimes x_k$ with $x_i \in B$ for $i \in [1,k]$, the only non-zero images are

$$\begin{aligned}
m_2(a^j \otimes a^{j'}) &= a^{j+j'} \text{ for } j, j' \in \mathbb{Z}_{\geq 0}\\
m_2(ba^j \otimes a^{j'}) &= ba^{j+j'} \text{ for } j, j' \in \mathbb{Z}_{\geq 0}\\
m_2(a^j \otimes ba^{j'}) &= ba^{j+j'} \text{ for } j, j' \in \mathbb{Z}_{\geq 0}
\end{aligned}$$

$$m_p(ba^{j_1} \otimes \ldots \otimes ba^{j_p}) = \begin{cases} a^{1+j_1+\ldots+j_p} \text{ for } j_1,\ldots,j_p \in \mathbb{Z}_{\geq 0} & \text{for the case } \mathrm{C}_p \\ (-1)^p a^{(p-1)+j_1+\ldots+j_p} \text{ for } j_1,\ldots,j_p \in \mathbb{Z}_{\geq 0} & \text{for the case } \mathrm{S}_p. \end{cases}$$

For the cyclic groups, $a =: a_{\mathrm{C}_p}$ has degree 2 and $b =: b_{\mathrm{C}_p}$ has degree 1. For the symmetric groups, $a =: a_{\mathrm{S}_p}$ has degree $2(p-1)$ and $b =: b_{\mathrm{S}_p}$ has degree $2(p-1)-1$. So if we identify a_{S_p} with $a_{\mathrm{C}_p}^{p-1}$ and b_{S_p} with $-b_{\mathrm{C}_p} a_{\mathrm{C}_p}^{p-2}$, the formulas for the m_k are compatible. I.e. we obtain the canonical model on $\operatorname{Ext}^*_{\mathbb{F}_p \mathrm{S}_p}(\mathbb{F}_p,\mathbb{F}_p)$ as a (suitably defined) sub-A_∞-algebra of the canonical model on $\operatorname{Ext}^*_{\mathbb{F}_p \mathrm{C}_p}(\mathbb{F}_p,\mathbb{F}_p)$.

In section 3.2, we develop results which show that this behaviour can partially be generalized to group / subgroup pairs where the index of the subgroup in the group is invertible in the underlying field:

On the one hand using a specialized version of Kadeishvili's algorithm, we obtain the

Proposition 6 (cf. Proposition 66). *Suppose given a field $\mathbb{F}$. Suppose given a finite group G and a subgroup $H \leqslant G$ such that $[G : H]$ is invertible in $\mathbb{F}$. Suppose given a projective resolution P of the trivial $\mathbb{F}G$-module $\mathbb{F}$ over $\mathbb{F}G$.*

Note that the dg-algebra homomorphism $\mathrm{res}_{G,H} : \mathrm{Hom}^*_{\mathbb{F}G}(P,P) \to \mathrm{Hom}^*_{\mathbb{F}H}(P,P)$ *has an* A_∞*-version called* $\mathrm{strict}_\infty(\mathrm{res}_{G,H})$*, cf. Definition 58.*

Suppose given a minimal A_∞*-structure* $({m'}_n^{(G)})_{n\geq 1}$ *on* $\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$ *and a quasi-isomorphism of* A_∞*-algebras* $(f_n^{(G)})_{n\geq 1} : (\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}), ({m'}_n^{(G)})_{n\geq 1}) \to \mathrm{Hom}^*_{\mathbb{F}G}(P,P)$ *such that* $f_1^{(G)}$ *induces the identity in homology.*

Then there is a minimal A_∞*-structure* $(m'_n)_{n\geq 1}$ *on* $\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$ *and a quasi-isomorphism of* A_∞*-algebras* $(f_n)_{n\geq 1} : (\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}), (m'_n)_{n\geq 1}) \to \mathrm{Hom}^*_{\mathbb{F}H}(P,P)$ *such that*

- f_1 *induces the identity in homology,*
- $\mathrm{strict}_\infty(\mathrm{H}^*\,\mathrm{res}_{G,H}) : (\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}), ({m'}_n^{(G)})_{n\geq 1}) \to (\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}), (m'_n)_{n\geq 1})$ *is an* A_∞*-morphism and*
- *the following diagram of* A_∞*-morphisms commutes.*

$$\begin{tikzcd}
(\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}), ({m'}_n^{(G)})_{n\geq 1}) \arrow[r, "(f_n^{(G)})_{n\geq 1}"] \arrow[d, "\mathrm{strict}_\infty(\mathrm{H}^*\,\mathrm{res}_{G,H})"'] & \mathrm{Hom}^*_{\mathbb{F}G}(P,P) \arrow[d, "\mathrm{strict}_\infty(\mathrm{res}_{G,H})"] \\
(\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}), (m'_n)_{n\geq 1}) \arrow[r, "(f_n)_{n\geq 1}"] & \mathrm{Hom}^*_{\mathbb{F}H}(P,P)
\end{tikzcd}$$

On the other hand using a result by Keller and Prouté (cf. [11, Theorem in section 3.7], see also [18, Théorème 4.27] and [21, Corollary 1.14]), we obtain the following

Proposition 7 (cf. Proposition 67)**.** *Suppose given a field* $\mathbb{F}$*. Suppose given finite groups* G, H *with* $H \leqslant G$*. Suppose given a projective resolution* P *of the trivial* $\mathbb{F}G$*-module* $\mathbb{F}$ *and a projective resolution* Q *of the trivial* $\mathbb{F}H$*-module* $\mathbb{F}$*.*

Suppose given minimal A_∞*-algebras*

$$\begin{aligned} M^{(G)} &:= (\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}), ({m'}_k^{(G)})_{k\geq 1}), \\ M^{(H)} &:= (\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}), ({m'}_k^{(H)})_{k\geq 1}) \end{aligned}$$

together with quasi-isomorphisms of A_∞*-algebras*

$$\begin{aligned} f^{(G)} &= (f_k^{(G)})_{k\geq 1} : M^{(G)} \to \mathrm{Hom}^*_{\mathbb{F}G}(P,P) \\ f^{(H)} &= (f_k^{(H)})_{k\geq 1} : M^{(H)} \to \mathrm{Hom}^*_{\mathbb{F}H}(Q,Q). \end{aligned}$$

Suppose given $\mathbb{F}H$*-linear complex morphisms* $g_1 : P \to Q$ *and* $g_2 : Q \to P$ *together with a homotopy* $h \in \mathrm{Hom}^{-1}_{\mathbb{F}H}(P,P)$ *such that* $g_2 \circ g_1 = \mathrm{id}_P + d_{\mathrm{Hom}^*_{\mathbb{F}H}(P,P)}(h)$*.*

From g_1, g_2 *and* h*, we obtain via Definition/Lemma 5 the* A_∞*-morphism* $f_{g_1,g_2,h}$ *from* $\mathrm{Hom}^*_{\mathbb{F}H}(P,P)$ *to* $\mathrm{Hom}^*_{\mathbb{F}H}(Q,Q)$*.*

Then there exists an A_∞*-morphism* $f^{\min}$ *from* $M^{(G)}$ *to* $M^{(H)}$ *such that the following diagram commutes up to homotopy in the sense of [11, section 3.7].*

$$\begin{tikzcd} M^{(G)} \arrow[r, "f^{(G)}"] \arrow[d, "f^{\min}"] & \mathrm{Hom}^*_{\mathbb{F}G}(P,P) \arrow[d, "f_{g_1,g_2,h}\circ \mathrm{strict}_\infty(\mathrm{res}_{G,H})"] \\ M^{(H)} \arrow[r, "f^{(H)}"] & \mathrm{Hom}^*_{\mathbb{F}H}(Q,Q) \end{tikzcd}$$

Comparing the two results, note that in Proposition 6, an explicit construction is used and the morphism between the minimal models is the A_∞-morphism induced by restriction. In Proposition 7, however, the canonical A_∞-structure on $\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$ can be chosen and we have no restriction on the index $[G:H]$, but we know less about the morphism between the minimal models and we obtain commutativity only up to homotopy.

0.2. Acknowledgements

I would like to thank Matthias Künzer for his multitude of helpful comments and his patient reading of the text. His feedback greatly helped improving quality and approachability of this text. I would like to thank my family for their constant support.

0.3. Notations and conventions

Miscellaneous

- For modules M, N, we write $M \leqslant N$ if M is a submodule of N. For groups H, G, we write $H \leqslant G$ if H is a subgroup of G. If G and H are finite groups and $H \leqslant G$, we write $[G:H]$ to denote the index of H in G.
- If we denote a commutative ring R as the *ground ring*, we understand linear maps between R-modules to be R-linear. Furthermore, tensor products are tensor products over R.

  Graded modules are $\mathbb{Z}$-graded modules over R, cf. section 1.1. Graded maps are R-linear graded maps, cf. section 1.1.
- Concerning "∞", we assume the set $\mathbb{Z} \cup \{\infty\}$ to be ordered in such a way that ∞ is greater than any integer, i.e. $\infty > z$ for all $z \in \mathbb{Z}$, and that the integers are ordered as usual.
- For $a \in \mathbb{Z}$, $b \in \mathbb{Z} \cup \{\infty\}$, we denote by $[a,b] := \{z \in \mathbb{Z} \mid a \leq z \leq b\} \subseteq \mathbb{Z}$ the integral interval. In particular, we have $[a,\infty] = \{z \in \mathbb{Z} \mid z \geq a\} \subseteq \mathbb{Z}$ for $a \in \mathbb{Z}$.
- For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}$, let the binomial coefficient $\binom{n}{k}$ be defined by the number of subsets of the set $\{1,\dots,n\}$ that have cardinality k. In particular, if $k < 0$ or $k > n$, we have $\binom{n}{k} = 0$. Then the formula $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ holds for all $k \in \mathbb{Z}$.

- Rings are unital rings.
- For a commutative ring R, an R-module M and $a, b \in M$, $c \in R$, we write

$$b \equiv_c a \qquad :\Longleftrightarrow \qquad a - b \in cM.$$

Often we have $M = R$ as module over itself.
- For a prime q, let $\mathbb{F}_q$ denote the finite field containing q elements.
- Let R be a commutative ring. An R-algebra (A, ρ) is a ring A together with a ring morphism $\rho : R \to A$ such that $\rho(R)$ is a subset of the center of A. By abuse of notation, we often just write A for (A, ρ). A is an R-module via $r \cdot a := \rho(r) \cdot a$ for $r \in R$, $a \in A$.

For R-algebras (A, ρ) and (B, τ), a morphism of R-algebras $g : (A, \rho) \to (B, \tau)$ is a ring morphism $g : A \to B$ such that $g \circ \rho = \tau$.
- Morphisms will be written on the left.
- Modules are left-modules unless otherwise specified. For a ring A, we denote by A-`Mod` the category of left A-modules.
- We denote a tuple by enclosing it in parentheses. I.e. for a set M and $a_i \in M$, $i \in [1, n]$, $n \geq 0$, we have the tuple $(a_1, a_2, \ldots, a_n) = a$. In particular, $()$ is the empty tuple.

For a map $g : M \to N$ from M to a set N, we define

$$g(a) := (g(x)\colon x \in a) := (g(a_1), g(a_2), \ldots, g(a_n)).$$

For a set M', by abuse of notation, we denote by $M' \setminus a$ the set difference between M' and the set of elements of a. Similarly, we write $a \subseteq M'$ if each entry of a is an element of M'.

We will express ordered bases of finite-rank free modules as tuples of pairwise distinct elements.
- For sets, we denote by $\sqcup$ the disjoint union of sets. For tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$, we denote by $\sqcup$ the concatenation:

$$a \sqcup b := (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$$

- $| \cdot |$: For y a real number, $|y|$ denotes its absolute value. For $a = (a_1, \ldots, a_n)$ a tuple, $|a| := n$ is the number of its entries.

For an element x of a graded module, let $|x| := \{k \in \mathbb{Z} \mid x \text{ is homog. of degree } k\}$, cf. section 1.1. Hence, x is homogeneous of degree k iff $|x| \ni k$. For a linear map g between graded modules, let $|g| := \{k \in \mathbb{Z} \mid g \text{ is graded of degree } k\}$, cf. section 1.1.

- Suppose given an $I \times J$-graded module $M = \oplus_{i\in I, j\in J} M^{i,j}$ for sets I, J. We say that we *suppress* the grading along I or say that we *suppress* the grading along i if we consider M as a J-graded module where given $j \in J$, the homogeneous component of M of degree j is $\oplus_{i\in I} M^{i,j}$. In the same way, components of gradings indexed by Cartesian products of more than two sets may be suppressed.

Restricting and extending maps and morphisms

- Suppose given a function $f : A \to B$.
  For a set $A' \subseteq A$, we denote by $f|_{A'} : A' \to B$ the restriction in the domain.
  For a set $B' \subseteq B$ such that $f(A) \subseteq B'$, we denote by $f|^{B'} : A \to B'$ the restriction in the codomain.
  For sets $A' \subseteq A$, $B' \subseteq B$ such that $f(A') \subseteq B'$, let $f|_{A'}^{B'} := (f|_{A'})|^{B'}$.

- Suppose given sets A, B, C. Suppose given a set $B' \subseteq B$. Suppose given maps $f : A \to B'$ and $g : B \to C$. We define the composition $g \circ f$ by $g \circ f := h$, where $h : A \to C$ is given by $h(x) := g(f(x))$ for $x \in A$. I.e. the inclusion map from A' to A is inserted implicitly between g and f.

- Suppose given a commutative ring R. Suppose given R-modules A, B. Suppose given submodules $C, C' \leqslant B$. Suppose given R-linear maps $f : A \to C$, $g : A \to C'$. We define $f + g$ to be the R-linear map $f + g : A \to (C + C')$ given by $(f + g)(x) := f(x) + g(x)$ for $x \in A$. Note that this notion for sums of morphisms is commutative and associative.

- When combining the two previous conventions, there is the following simplification. Suppose given a commutative ring R. Suppose given R-modules A, B, C. Suppose given submodules $B_1', B_2', B'' \leqslant B$. Suppose given R-linear maps $f_1 : A \to B_1'$, $f_2 : A \to B_2'$ and $g : B'' \to C$. The morphism $(f_1 + f_2) : A \to B_1' + B_2'$ is composable with $g : B'' \to C$ iff $B_1' + B_2' \subseteq B''$. This holds iff $B_1' \subseteq B''$ and $B_2' \subseteq B''$. In that case, we have $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2 : A \to C$.

  We conclude that in order to verify that an expression of the form $g \circ (f_1 + f_2)$ with R-linear maps f_1, f_2, g is sensible, we only need to check that f_1 and f_2 have the same domain and that the domains of both f_1 and f_2 are submodules of the domain of g.

Complexes Let R be a commutative ring and B an R-algebra.

- Suppose given a (descending) complex of B-modules

$$(C, d) = (\cdots \to C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \to \cdots).$$

  We often write C instead of (C, d).

The k-th boundaries, cycles and homology groups of C are defined by $\mathrm{B}_k(C) := \operatorname{im} d_{k+1}$, $\mathrm{Z}_k(C) := \ker d_k$ and $\mathrm{H}_k(C) := \mathrm{Z}_k(C)/\mathrm{B}_k(C)$. Write $\mathrm{Z}_*(C) := \oplus_{k\in\mathbb{Z}}\mathrm{Z}_k(C)$, $\mathrm{B}_*(C) := \oplus_{k\in\mathbb{Z}}\mathrm{B}_k(C)$ and $\mathrm{H}_*(C) := \oplus_{k\in\mathbb{Z}}\mathrm{H}_k(C)$.

For a cycle $x \in \mathrm{Z}_k(C)$, we denote by $\overline{x} := x + \mathrm{B}_k(C) \in \mathrm{H}_k(C)$ its equivalence class in homology.

- Suppose given an (ascending) complex of B-modules

$$(C, d) = (\cdots \to C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \to \cdots).$$

We often write C instead of (C, d).

The k-th boundaries, cycles and homology groups of C are defined by $\mathrm{B}^k(C) := \operatorname{im} d^{k-1}$, $\mathrm{Z}^k(C) := \ker d^k$ and $\mathrm{H}^k(C) := \mathrm{Z}^k(C)/\mathrm{B}^k(C)$. Write $\mathrm{Z}^*(C) := \oplus_{k\in\mathbb{Z}}\mathrm{Z}^k(C)$, $\mathrm{B}^*(C) := \oplus_{k\in\mathbb{Z}}\mathrm{B}^k(C)$ and $\mathrm{H}^*(C) := \oplus_{k\in\mathbb{Z}}\mathrm{H}^k(C)$.

For a cycle $x \in \mathrm{Z}^k(C)$, we denote by $\overline{x} := x + \mathrm{B}^k(C) \in \mathrm{H}^k(C)$ its equivalence class in homology.

- Given a descending complex of B-modules

$$\cdots \to C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \to \cdots ,$$

we may obtain an ascending complex of B-modules by setting $C^k := C_{-k}$ and $d^k := d_{-k}$ for $k \in \mathbb{Z}$.

Conversely given an ascending complex of B-modules

$$\cdots \to C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \to \cdots ,$$

we may obtain an descending complex of B-modules by setting $C_k := C^{-k}$ and $d_k := d^{-k}$ for $k \in \mathbb{Z}$.
So we may transform ascending and descending complexes into each other. We will denote both ascending and descending complexes simply as complexes and distinguish them from each other by using upper indices for ascending complexes and lower indices for descending complexes.

- For a complex of B-modules $C = (\cdots \to C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \to)$ and $z \in \mathbb{Z}$, the shifted complex $C[z] =: \tilde{C}$ is defined by $\tilde{C}_k := C_{k+z}$, $\tilde{d}_k := (-1)^z d_{k+z}$.
- Let

$$\begin{aligned} C &= (\cdots \to C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \to \cdots) \\ C' &= (\cdots \to C'_{k+1} \xrightarrow{d'_{k+1}} C'_k \xrightarrow{d'_k} C'_{k-1} \to \cdots) \end{aligned}$$

be complexes of B-modules.

Given $z \in \mathbb{Z}$, let

$$\mathrm{Hom}_B^z(C, C') := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_B(C_{i+z}, C'_i).$$

This is used to define the graded R-module

$$\mathrm{Hom}_B^*(C, C') := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_B^k(C, C').$$

For an additional complex $C'' = (\cdots \to C''_{k+1} \xrightarrow{d''_{k+1}} C''_k \xrightarrow{d''_k} C''_{k-1} \to \cdots)$ and maps $h = (h_i)_{i \in \mathbb{Z}} \in \mathrm{Hom}_B^m(C, C')$, $h' = (h'_i)_{i \in \mathbb{Z}} \in \mathrm{Hom}_B^n(C', C'')$, $m, n \in \mathbb{Z}$, we define the composition by component-wise composition as

$$h' \circ h := (h'_i \circ h_{i+n})_{i \in \mathbb{Z}} \in \mathrm{Hom}_B^{m+n}(C, C'').$$

Furthermore, we define composition on $\mathrm{Hom}_B^*(C', C'') \otimes \mathrm{Hom}_B^*(C, C')$ by linearly extending the definition given on the summands of $\mathrm{Hom}_B^*(C', C'') \otimes \mathrm{Hom}_B^*(C, C') = \oplus_{m,n \in \mathbb{Z}} \mathrm{Hom}^n(C', C'') \otimes \mathrm{Hom}^m(C, C')$.

The graded R-module $\mathrm{Hom}_B^*(C, C') = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_B^k(C, C')$ becomes a complex via the differential $d_{\mathrm{Hom}_B^*(C,C')}$, which is defined on elements $g \in \mathrm{Hom}_B^k(C, C')$, $k \in \mathbb{Z}$ by

$$d_{\mathrm{Hom}_B^*(C,C')}(g) := d' \circ g - (-1)^k g \circ d \in \mathrm{Hom}_B^{k+1}(C, C'),$$

where $d := (d_{i+1})_{i \in \mathbb{Z}} \in \mathrm{Hom}_B^1(C, C)$ and analogously $d' := (d'_{i+1})_{i \in \mathbb{Z}} \in \mathrm{Hom}_B^1(C', C')$.

An element $h \in \mathrm{Hom}_B^0(C, C')$ is called a complex morphism if it satisfies $d_{\mathrm{Hom}_B^*(C,C')}(h) = 0$, i.e. $d' \circ g = g \circ d$.

1. A_∞-algebras

Suppose given a commutative ground ring R.

1.1. Graded modules. The Koszul sign rule. Graded projectivity.

In this subsection, we review basic definitions and results concerning graded modules and the Koszul sign rule.

Definition 8. A *graded R-module* or *graded module* V is an R-module of the form $V = \oplus_{q\in\mathbb{Z}} V^q$. An element $v_q \in V^q$, $q \in \mathbb{Z}$ is said to be of degree q. An element $v \in V$ is called *homogeneous* if there is an integer $q \in \mathbb{Z}$ such that $v \in V^q$. For elements $v \in V$, let $|v| := \{k \in \mathbb{Z} \mid v \text{ is homogeneous of degree } k\}$.

Definition 9. Let $A = \oplus_{q\in\mathbb{Z}} A^q$, $B = \oplus_{q\in\mathbb{Z}} B^q$ be graded R-modules. A *graded map of degree $z \in \mathbb{Z}$* from A to B is a linear map $g : A \to B$ such that $\operatorname{im} g\big|_{A^q} \subseteq B^{q+z}$ for $q \in \mathbb{Z}$. For linear maps $g : A \to B$, let $|g| := \{k \in \mathbb{Z} \mid g \text{ is graded of degree } k\}$.

Definition 10 (Arithmetics of degrees)**.** Given sets $M, M' \subseteq \mathbb{Z}$, we define their sum by $M + M' := \{m + m' \mid m \in M, m' \in M'\}$.

Remark 11. Let A, B be graded modules. Let $f : A \to B$ be a linear map. Let $x \in A$.

If $\mathsf{k}_x \in |x|$ and $\mathsf{k}_f \in |f|$, then $x \in A^{\mathsf{k}_x}$ and f is graded of degree k_f. This implies $f(x) \in B^{\mathsf{k}_x+\mathsf{k}_f}$, so $\mathsf{k}_x + \mathsf{k}_f \in |f(x)|$. We conclude $|x| + |f| \subseteq |f(x)|$.

Definition 12. The category $R\text{-}\mathtt{Mod}^{\mathbb{Z}}$ of graded R-modules is given as follows. Objects are graded R-modules. Morphisms are graded maps of degree 0. The category $R\text{-}\mathtt{Mod}^{\mathbb{Z}}$ is isomorphic to the category of functors from the discrete category $\mathbb{Z}$ to $R\text{-}\mathtt{Mod}$. So $R\text{-}\mathtt{Mod}^{\mathbb{Z}}$ is an abelian category since $R\text{-}\mathtt{Mod}$ is abelian, cf. e.g. [17, II.11].

For $A, B \in \operatorname{Obj}(R\text{-}\mathtt{Mod}^{\mathbb{Z}})$, the direct sum $A \oplus B = \bigoplus_{q\in\mathbb{Z}}(A^q \oplus B^q)$ is then graded by $(A \oplus B)^q = A^q \oplus B^q$. We denote a direct summand in $R\text{-}\mathtt{Mod}^{\mathbb{Z}}$ as a *graded direct summand.*

For $A \in \operatorname{Obj}(R\text{-}\mathtt{Mod}^{\mathbb{Z}})$, a submodule M of A is called a *graded submodule*[2] of A if $M = \oplus_{q\in\mathbb{Z}}(A^q \cap M)$. In that case, the inclusion map $M \hookrightarrow A$ is a graded map of degree 0.

Lemma 13 (cf. e.g. [2, §11.3 Proposition 3(i)])**.** *Suppose given graded modules A, B and a graded map $f : A \to B$. Then* $\operatorname{im}(f) \subseteq B$ *is a graded submodule of B.*

Proof. Choose $\mathsf{k}_f \in |f|$. We have

$$\operatorname{im}(f) = \oplus_{p\in\mathbb{Z}} f(A^p) \overset{f(A^p)\subseteq B^{p+\mathsf{k}_f}}{\subseteq} \oplus_{p\in\mathbb{Z}}(\operatorname{im}(f) \cap B^{p+\mathsf{k}_f}) = \oplus_{q\in\mathbb{Z}}(\operatorname{im}(f) \cap B^q) \subseteq \operatorname{im}(f).$$

Hence we have equality everywhere. In particular, we have $\operatorname{im}(f) = \oplus_{q\in\mathbb{Z}}(\operatorname{im}(f) \cap B^q)$. $\square$

[2] By some authors, a graded submodule resp. a graded direct summand is also called a homogeneous submodule resp. a homogeneous direct summand.

Definition 14. Let $A = \oplus_{q\in\mathbb{Z}} A^q$, $B = \oplus_{q\in\mathbb{Z}} B^q$ be graded R-modules. We have

$$A \otimes B = \bigoplus_{z_1,z_2\in\mathbb{Z}} A^{z_1} \otimes B^{z_2} = \bigoplus_{q\in\mathbb{Z}} \left(\bigoplus_{z_1+z_2=q} A^{z_1} \otimes B^{z_2} \right).$$

As we understand the direct sums to be internal direct sums in $A \otimes B$ and understand $A^{z_1} \otimes B^{z_2}$ to be the linear span of the set $\{a \otimes b \in A \otimes B \mid a \in A^{z_1}, b \in A^{z_2}\}$, we have equations in the above, not just isomorphisms.

We then set $A \otimes B$ to be graded by $A \otimes B = \bigoplus_{q\in\mathbb{Z}} (A \otimes B)^q$, where $(A \otimes B)^q := \bigoplus_{z_1+z_2=q} A^{z_1} \otimes B^{z_2}$.

Note that given $a \in A, b \in B$, the assumption $\mathsf{k}_a \in |a|$, $\mathsf{k}_b \in |b|$ implies $\mathsf{k}_a + \mathsf{k}_b \in |a \otimes b|$. I.e. $|a| + |b| \subseteq |a \otimes b|$.

Definition 15. In the definition of the tensor product of graded maps, we implement the *Koszul sign rule*: Let A_1, A_2, B_1, B_2 be graded R-modules and $g : A_1 \to B_1$, $h : A_2 \to B_2$ graded maps with $\mathsf{k}_g \in |g|$ and $\mathsf{k}_h \in |h|$. Then $g \otimes h$ is given on elements $x \otimes y \in A_1^{\mathsf{k}_x} \otimes A_2^{\mathsf{k}_y}$ by

$$(g \otimes h)(x \otimes y) := (-1)^{\mathsf{k}_h \cdot \mathsf{k}_x} g(x) \otimes h(y). \tag{1}$$

Note that if $\mathsf{k}_g \in |g|$ and $\mathsf{k}_h \in |h|$, then $\mathsf{k}_g + \mathsf{k}_h \in |g \otimes h|$. I.e. $|g| + |h| \subseteq |g \otimes h|$.

Remark 16. It is known that for graded R-modules A, B, C, the map

$$\begin{array}{rccc} \Theta : & (A \otimes B) \otimes C & \longrightarrow & A \otimes (B \otimes C) \\ & (a \otimes b) \otimes c & \longmapsto & a \otimes (b \otimes c) \end{array} \tag{2}$$

is an isomorphism of R-modules. Because of the following, Θ is homogeneous of degree 0.

$$\begin{aligned} ((A \otimes B) \otimes C)^q &= \bigoplus_{y+z_3=q} (A \otimes B)^y \otimes C^{z_3} = \bigoplus_{y+z_3=q} \bigoplus_{z_1+z_2=y} (A^{z_1} \otimes B^{z_2}) \otimes C^{z_3} \\ &= \bigoplus_{z_1+z_2+z_3=q} (A^{z_1} \otimes B^{z_2}) \otimes C^{z_3} \\ (A \otimes (B \otimes C))^q &= \bigoplus_{z_1+y=q} A^{z_1} \otimes (B \otimes C)^y = \bigoplus_{z_1+y=q} \bigoplus_{z_2+z_3=y} A^{z_1} \otimes (B^{z_2} \otimes C^{z_3}) \\ &= \bigoplus_{z_1+z_2+z_3=q} A^{z_1} \otimes (B^{z_2} \otimes C^{z_3}) \end{aligned}$$

Let $A_1, A_2, B_1, B_2, C_1, C_2$ be graded R-modules, $f : A_1 \to A_2$, $g : B_1 \to B_2$, $h : C_1 \to C_2$ graded maps with $\mathsf{k}_f \in |f|$, $\mathsf{k}_g \in |g|$, $\mathsf{k}_h \in |h|$. For homogeneous elements $x \in A_1$, $y \in B_1$, $z \in C_1$ with $\mathsf{k}_x \in |x|$, $\mathsf{k}_y \in |y|$, $\mathsf{k}_z \in |z|$, we have

$$\begin{aligned} ((f \otimes g) \otimes h)((x \otimes y) \otimes z) &= (-1)^{(\mathsf{k}_x + \mathsf{k}_y)\mathsf{k}_h} ((f \otimes g)(x \otimes y)) \otimes h(z) \\ &= (-1)^{(\mathsf{k}_x + \mathsf{k}_y)\mathsf{k}_h + \mathsf{k}_x \mathsf{k}_g} (f(x) \otimes g(y)) \otimes h(z) \end{aligned}$$

$$
\begin{aligned}
(f\otimes(g\otimes h))(x\otimes(y\otimes z)) &= (-1)^{\mathsf{k}_x(\mathsf{k}_g+\mathsf{k}_h)} f(x)\otimes((g\otimes h)(y\otimes z))\\
&= (-1)^{\mathsf{k}_x(\mathsf{k}_g+\mathsf{k}_h)+\mathsf{k}_y\mathsf{k}_h} f(x)\otimes(g(y)\otimes h(z))\\
&= (-1)^{(\mathsf{k}_x+\mathsf{k}_y)\mathsf{k}_h+\mathsf{k}_x\mathsf{k}_g} f(x)\otimes(g(y)\otimes h(z)).
\end{aligned}
$$

Thus we have the following commutative diagram (Θ_1 and Θ_2 are derived from (2))

$$
\begin{tikzcd}
(A_1\otimes B_1)\otimes C_1 \arrow[r, "\Theta_1"] \arrow[d, "(f\otimes g)\otimes h"] & A_1\otimes(B_1\otimes C_1) \arrow[d, "f\otimes(g\otimes h)"]\\
(A_2\otimes B_2)\otimes C_2 \arrow[r, "\Theta_2"] & A_2\otimes(B_2\otimes C_2)
\end{tikzcd}
$$

It is therefore valid to use Θ as an identification and to omit the brackets for the tensorization of graded R-modules and the tensorization of graded maps.

Lemma 17. *Let A_i, B_i, $i\in\{1,2,3\}$ be graded R-modules and $f: A_1\to A_2$, $g: B_1\to B_2$, $h: A_2\to A_3$, $i: B_2\to B_3$ graded maps. Suppose $|f|\ni\mathsf{k}_f$ and $|i|\ni\mathsf{k}_i$. Then*

$$
(h\otimes i)\circ(f\otimes g) = (-1)^{\mathsf{k}_f\cdot\mathsf{k}_i}(h\circ f)\otimes(i\circ g). \tag{3}
$$

Proof. Choose $\mathsf{k}_h\in|h|$ and $\mathsf{k}_g\in|g|$. Let $a\in A_1$ resp. $b\in B_1$ be homogeneous elements of degree k_a resp. k_b. We have

$$
\begin{aligned}
((h\otimes i)\circ(f\otimes g))(a\otimes b) &= (-1)^{\mathsf{k}_a\cdot\mathsf{k}_g}(h\otimes i)(f(a)\otimes g(b))\\
&\overset{\mathsf{k}_f+\mathsf{k}_a\in|f(a)|}{=} (-1)^{\mathsf{k}_a\cdot\mathsf{k}_g+(\mathsf{k}_f+\mathsf{k}_a)\mathsf{k}_i}(h\circ f)(a)\otimes(i\circ g)(b)\\
&= (-1)^{\mathsf{k}_a(\mathsf{k}_g+\mathsf{k}_i)+\mathsf{k}_f\cdot\mathsf{k}_i}(h\circ f)(a)\otimes(i\circ g)(b)\\
&\overset{\mathsf{k}_g+\mathsf{k}_i\in|i\circ g|}{=} (-1)^{\mathsf{k}_f\cdot\mathsf{k}_i}((h\circ f)\otimes(i\circ g))(a\otimes b).
\end{aligned}
$$

□

Repeated application of Lemma 17 yields the following

Corollary 18. *Let $n\geq 1$. Given graded R-modules V_i, W_i, U_i and graded maps $f_i: V_i\to W_i$, $g_i: W_i\to U_i$ with $|f_i|\ni\mathsf{k}_{f_i}$, $|g_i|\ni\mathsf{k}_{g_i}$ for $i\in[1,n]$, we have*

$$
(g_1\otimes\cdots\otimes g_n)\circ(f_1\otimes\cdots\otimes f_n) = (-1)^s(g_1\circ f_1)\otimes\cdots\otimes(g_n\circ f_n),
$$

where $s=\sum_{2\leq i\leq n}\mathsf{k}_{g_i}\cdot\left(\sum_{1\leq j<i}\mathsf{k}_{f_j}\right)=\sum_{1\leq j<i\leq n}\mathsf{k}_{g_i}\cdot\mathsf{k}_{f_j}$.

Definition 19. Let P be a graded module. We denote P to be *graded projective* iff for each surjective graded map $b: D\twoheadrightarrow C$ of degree k_b for some graded module C, D and for each graded map $c: P\to C$ of degree k_c, there exists a graded map $d: P\to D$ with $\mathsf{k}_d\in|d|$ such that $c=b\circ d$ and $\mathsf{k}_c=\mathsf{k}_b+\mathsf{k}_d$.

$$
\begin{tikzcd}
 & P \arrow[dl, dotted, "\exists d"'] \arrow[d, "c"]\\
D \arrow[r, two heads, "b"] & C
\end{tikzcd}
$$

Lemma 20. *Suppose given a graded module P which is projective over R ($\Leftrightarrow$ all P^i are projective over R). Then P is graded projective.*

Proof. Suppose given $C, D, b, c, \mathsf{k}_b, \mathsf{k}_c$ as in Definition 19. Set $\mathsf{k}_d := \mathsf{k}_c - \mathsf{k}_b$. For $x \in \mathbb{Z}$, we construct morphisms $d_x : P^x \to D^{x+\mathsf{k}_d}$ as follows. Since b is surjective and graded, the restricted map $b|_{D^{x+\mathsf{k}_c-\mathsf{k}_b}}^{C^{x+\mathsf{k}_c}} = b|_{D^{x+\mathsf{k}_d}}^{C^{x+\mathsf{k}_c}}$ is surjective. Since P^x is projective, and $c(P^x) \subseteq C^{x+\mathsf{k}_c}$, there is a morphism $d_x : P^x \to D^{x+\mathsf{k}_d}$ such that $b|_{D^{x+\mathsf{k}_d}}^{C^{x+\mathsf{k}_c}} \circ d_x = c|_{P^x}^{C^{x+\mathsf{k}_c}}$. From the maps d_x, we obtain a graded map $d : P \to D$ of degree k_d by setting $d|_{P^x}^{D^x+\mathsf{k}_d} := d_x$ for $x \in \mathbb{Z}$. By the construction of the d_x, we have $b \circ d = c$. $\square$

1.2. A_∞-algebras

Concerning the signs in the definition of A_∞-algebras and A_∞-morphisms, we follow the variant given e.g. in [14].

Definition 21. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(i) Let A be a graded R-module. A *pre-*A_n*-structure on A* is a family of graded maps $(m_k : A^{\otimes k} \to A)_{k\in[1,n]}$ with $|m_k| \ni 2 - k$ for $k \in [1, n]$. The tuple $(A, (m_k)_{k\in[1,n]})$ is called a pre-A_n-algebra.

(ii) Let A', A be graded R-modules. A *pre-*A_n*-morphism from A' to A* is a family of graded maps $(f_k : A'^{\otimes k} \to A)_{k\in[1,n]}$ with $|f_k| \ni 1 - k$ for $k \in [1, n]$.

A pre-A_n-morphism $(f_k)_{k\in[1,n]}$ is called *strict* if $f_k = 0$ for $k \in [2, n]$.

Definition 22. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(i) An A_n*-algebra* is a pre-A_n-algebra $(A, (m_k)_{k\in[1,n]})$ such that for $k \in [1, n]$, the Stasheff identity

$$\sum_{\substack{k=r+s+t,\\ r,t\geq 0, s\geq 1}} (-1)^{rs+t} m_{r+1+t} \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0 \qquad (4)[k]$$

holds.

In abuse of notation, we sometimes abbreviate $A = (A, (m_k)_{k\geq 1})$ for A_∞-algebras.

(ii) Let $(A', (m'_k)_{k\in[1,n]})$ and $(A, (m_k)_{k\in[1,n]})$ be A_n-algebras. An A_n*-morphism* or *morphism of* A_n*-algebras* from $(A', (m'_k)_{k\in[1,n]})$ to $(A, (m_k)_{k\in[1,n]})$ is a pre-A_n-morphism $(f_k)_{k\in[1,n]}$ such that for $k \in [1, n]$, we have

$$\sum_{\substack{k=r+s+t\\ r,t\geq 0, s\geq 1}} (-1)^{rs+t} f_{r+1+t} \circ (1^{\otimes r} \otimes m'_s \otimes 1^{\otimes t}) = \sum_{\substack{1\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} (-1)^v m_r \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r}),$$

$(5)[k]$

where

$$v := \sum_{1 \le t < s \le r} (1 - i_s) i_t. \tag{6}$$

An A_n-morphism is called *strict* if it is a strict pre-A_n-morphism.

Given $\tilde{n} \in \mathbb{Z}_{\ge 0}$, $\tilde{n} < n$, we may forget a part of the structure of an A_n-algebra $(A, (m_k)_{k \in [1,n]})$ to obtain the $\mathrm{A}_{\tilde{n}}$-algebra $(A, (m_k)_{k \in [1,\tilde{n}]})$. This is often done without comment.

Example 23 (dg-algebras)**.** Let $(A, (m_k)_{k \ge 1})$ be an A_∞-algebra. If $m_n = 0$ for $n \ge 3$ then A is called a *differential graded algebra* or *dg-algebra*. In this case the equations (4)[n] for $n \ge 4$ become trivial: We have $(r+1+t)+s = n+1 \Rightarrow (r+1+t)+s \ge 5$ $\Rightarrow m_{r+1+t} = 0$ or $m_s = 0$. So all summands in (4)[n] are zero for $n \ge 4$. Here are the equations for $n \in \{1, 2, 3\}$:

$$\begin{aligned}
(4)[1]: \quad & 0 = m_1 \circ m_1 \\
(4)[2]: \quad & 0 = m_1 \circ m_2 - m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) \\
(4)[3]: \quad & 0 = m_1 \circ m_3 + m_2 \circ (1 \otimes m_2 - m_2 \otimes 1) \\
& \quad + m_3 \circ (m_1 \otimes 1^{\otimes 2} + 1 \otimes m_1 \otimes 1 + 1^{\otimes 2} \otimes m_1) \\
& \overset{m_3 = 0}{=} m_2 \circ (1 \otimes m_2 - m_2 \otimes 1)
\end{aligned}$$

So (4)[1] ensures that m_1 is a differential. Moreover, (4)[3] states that m_2 is an associative binary operation, since for homogeneous $x, y, z \in A$ we have $0 = m_2 \circ (1 \otimes m_2 - m_2 \otimes 1)(x \otimes y \otimes z) = m_2(x \otimes m_2(y \otimes z) - m_2(x \otimes y) \otimes z)$, where because of $|m_2| = 0$ there are no additional signs caused by the Koszul sign rule. Equation (4)[2] is the Leibniz rule which can be motivated by the product rule in the algebra of differential forms on a smooth manifold: We set $m_1 f := \partial f$ and $m_2(f \otimes g) := f \wedge g$ and we have for homogeneous differential forms f, g

$$\partial(f \wedge g) = (\partial f) \wedge g + (-1)^{|f|} f \wedge (\partial g).$$

The signs on the right side also motivate the Koszul sign rule.

Example 24 (A_n-morphisms induce complex morphisms)**.**
Let $n \in \mathbb{Z}_{\ge 1} \cup \{\infty\}$. Let $(A', (m'_k)_{k \in [1,n]})$ and $(A, (m_k)_{k \in [1,n]})$ be A_n-algebras and let $(f_k)_{k \in [1,n]} : (A', (m'_k)_{k \in [1,n]}) \to (A, (m_k)_{k \in [1,n]})$ be an A_n-morphism.

By (4)[1], (A', m'_1) and (A, m_1) are complexes. Equation (5)[1] is

$$f_1 \circ m'_1 = m_1 \circ f_1.$$

Thus $f_1 : (A', m'_1) \to (A, m_1)$ is a complex morphism.

If $n \ge 2$, we have also (5)[2]:

$$f_1 \circ m'_2 - f_2 \circ (m'_1 \otimes 1 + 1 \otimes m'_1) = m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1) \tag{7}$$

Recall the conventions concerning $\mathrm{Hom}_B^k(C, C')$.

Lemma 25. *Let B be an (ordinary) R-algebra and $M = ((M_i)_{i\in\mathbb{Z}}, (d_i)_{i\in\mathbb{Z}})$ a complex of B-modules, that is a sequence $(M_i)_{i\in\mathbb{Z}}$ of B-modules and B-linear maps $d_i : M_i \to M_{i-1}$ such that $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. Let*

$$\begin{aligned}\mathrm{Hom}_B^i(M, M) &:= \prod_{z\in\mathbb{Z}} \mathrm{Hom}_B(M_{z+i}, M_z)\\ &= \{g = (g_z)_{z\in\mathbb{Z}} \mid g_z \in \mathrm{Hom}_B(M_{z+i}, M_z) \text{ for } z \in \mathbb{Z}\}.\end{aligned}$$

We then obtain the graded R-module

$$A = \mathrm{Hom}_B^*(M, M) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_B^i(M, M).$$

We have $d := (d_{z+1})_{z\in\mathbb{Z}} \in \mathrm{Hom}_B^1(M, M)$. We define $m_1 := d_{\mathrm{Hom}^*(M,M)} : A \to A$, that is for homogeneous $g \in A$ of degree k_g, we have

$$m_1(g) = d \circ g - (-1)^{\mathsf{k}_g} g \circ d.$$

We define $m_2 : A^{\otimes 2} \to A$ for homogeneous $g, h \in A$ to be composition, i.e.

$$m_2(g \otimes h) := g \circ h.$$

For $n \geq 3$ we set $m_n : A^{\otimes n} \to A$, $m_n = 0$. Then $(m_n)_{n\geq 1}$ is an A_∞-algebra structure on $A = \mathrm{Hom}_B^*(M, M)$. More precisely, $(A, (m_n)_{n\geq 1})$ is a dg-algebra.

Proof. Since d is homogeneous of degree 1, the map m_1 is graded of degree $1 = 2 - 1$. The graded map m_2 has degree $0 = 2-2$. The other maps m_n are zero and have therefore automatically correct degree. As discussed in Example 23 we only need to check (4)[n] for $n = 1, 2, 3$. Equation (4)[1] holds because for homogeneous $g \in A$ of degree k_g, we have

$$\begin{aligned}m_1(m_1(g)) &= m_1(d \circ g - (-1)^{\mathsf{k}_g} g \circ d)\\ &= d \circ [d \circ g - (-1)^{\mathsf{k}_g} g \circ d] - (-1)^{\mathsf{k}_g+1}[d \circ g - (-1)^{\mathsf{k}_g} g \circ d] \circ d\\ &\overset{d^2=0}{=} - (-1)^{\mathsf{k}_g} d \circ g \circ d - (-1)^{\mathsf{k}_g+1} d \circ g \circ d = 0.\end{aligned}$$

Concerning (4)[2], we have for homogeneous $g, h \in A$ of degrees $\mathsf{k}_g \in |g|$ and $\mathsf{k}_h \in |h|$

$$\begin{aligned}(m_2 \circ (m_1 \otimes 1{+}1 \otimes m_1))(g \otimes h) &= m_2(m_1(g) \otimes h + (-1)^{\mathsf{k}_g} g \otimes m_1(h))\\ &= (d \circ g - (-1)^{\mathsf{k}_g} g \circ d) \circ h + (-1)^{\mathsf{k}_g} g \circ (d \circ h - (-1)^{\mathsf{k}_h} h \circ d)\\ &= d \circ g \circ h - (-1)^{\mathsf{k}_g+\mathsf{k}_h} g \circ h \circ d\\ &= (m_1 \circ m_2)(g \otimes h).\end{aligned}$$

The map m_2 is induced by the composition of morphisms which is associative. As discussed in Example 23, equation (4)[3] holds. $\square$

Definition 26 (Homology of A_∞-algebras, quasi-isomorphisms, minimality, minimal models)**.** Let $n \in \mathbb{Z}_{n\geq 1} \cup \{\infty\}$. Let $(A,(m_k)_{k\in[1,n]})$ be an A_n-algebra. As $m_1^2 = 0$ (cf. (4)[1]) and as m_1 is graded of degree 1, we have the complex

$$\cdots \to A^{i-1} \xrightarrow{m_1|_{A^{i-1}}} A^i \xrightarrow{m_1|_{A^i}} A^{i+1} \to \cdots .$$

We define the homology of the A_n-algebra A to be the homology of this complex (A, m_1). I.e. for $k \in \mathbb{Z}$, we have $\mathrm{H}^k A = \ker(m_1|_{A^k})/\operatorname{im}(m_1|_{A^{k-1}})$. Recall $\mathrm{H}^* A = \bigoplus_{k\in\mathbb{Z}} \mathrm{H}^k A$, which gives the homology of A the structure of a graded R-module.

A morphism of A_n-algebras $(f_k)_{k\in[1,n]} : (A',(m'_k)_{k\in[1,n]}) \to (A,(m_k)_{k\in[1,n]})$ is called a *quasi-isomorphism* if the morphism of complexes $f_1 : (A', m'_1) \to (A, m_1)$ (cf. Example 24) is a quasi-isomorphism.

An A_n-algebra is called *minimal*, if $m_1 = 0$. If A is an A_n-algebra and A' is a minimal A_n-algebra quasi-isomorphic to A, then A' is called a *minimal model of A*.

The existence of minimal models is assured by the following theorem.

Theorem 27 (minimality theorem, cf. [12] (history), [10], [9])**.** *Let $(A,(m_k)_{k\geq 1})$ be an A_∞-algebra such that the homology $\mathrm{H}^* A$ is a projective R-module.*

Then there exists an A_∞-algebra structure $(m'_k)_{k\geq 1}$ on $\mathrm{H}^* A$ and a quasi-isomorphism of A_∞-algebras $(f_k)_{k\geq 1} : (\mathrm{H}^* A,(m'_k)_{k\geq 1}) \to (A,(m_k)_{k\geq 1})$, such that

- *$m'_1 = 0$ and*
- *the complex morphism $f_1 : (\mathrm{H}^* A, m'_1) \to (A, m_1)$ induces the identity in homology. I.e. each element $x \in \mathrm{H}^* A$, which is a homology class of (A, m_1), is mapped by f_1 to a representing cycle.*

We give a proof of Theorem 27 in section 2.2, cf. Theorem 55.

There is a general statement concerning the computation of minimal models of dg-algebras:

Lemma 28 (cf. [25, Theorem 5])**.** *Let R be a commutative ring and $(A,(m_n)_{n\geq 1})$ be a dg-algebra (over R). Suppose given a graded R-module B and graded maps $f_n : B^{\otimes n} \to A$, $m'_n : B^{\otimes n} \to B$ for $n \geq 1$. Suppose given $k \geq 1$ such that*

$$\begin{aligned} f_i &= 0 && \text{for } i \geq k \\ m'_i &= 0 && \text{for } i \geq k+1, \end{aligned}$$

and such that (5)[n] is satisfied for $1 \leq n \leq 2k-2$. Then (5)[n] is satisfied for all $n \geq 1$.

Proof. We need to check (5)[n] for $n \geq 2k-1$:
The left side of (5)[n] is zero: For $f_{r+1+t} \circ (1^{\otimes r} \otimes m'_s \otimes 1^{\otimes t})$ to be non-zero it is necessary that $r+1+t \leq k-1$ and $s \leq k$, so $n+1 = r+s+t+1 \leq 2k-1$, which is not the case. Thus all summands on the left side of (5)[n] are zero.
The right side of (5)[n] is zero: As A is a dg-algebra, we have $m_n = 0$ for $n \geq 3$. So all

non-zero summands on the right side have $r \leq 2$. For a non-zero summand we also have $i_y \leq k-1$ for all $y \in [1, r]$. So for those we have

$$n = \sum_{y=1}^{r} i_y \overset{r\leq 2}{\leq} 2(k-1) = 2k-2.$$

But $n \geq 2k-1$, so all summands on the right side of (5)[n] are zero. □

1.3. A_∞-categories

The following may be found e.g. in [11, sections 7.2 and 7.3].

Definition 29. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. A *pre-*$\mathrm{A}_n$*-category* A is a tuple $(\mathrm{Obj}\, A, A, (m_k)_{k\in[1,n]})$ as follows.

(a) $\mathrm{Obj}\, A =: I$ is the set of objects[3].

(b) $A = \oplus_{(i,j,z)\in I\times I\times\mathbb{Z}} A_0^{i,j,z}$ is an $I^2 \times \mathbb{Z}$-graded R-module.

For $i, j \in I$, let $A(i,j) := \oplus_{z\in\mathbb{Z}} A_0^{i,j,z}$, which is a $\mathbb{Z}$-graded module. The module A becomes a $\mathbb{Z}$-graded module by $A = \oplus_{i,j\in I} A(i,j)$.

(c) For $k \in [1, n]$, the map $m_k : A^{\otimes k} \to A$ satisfies the following.

a) Given $i_y, j_y \in \mathrm{Obj}\, A$ for $y \in [1, k]$ such that there exists $x \in [1, k-1]$ with $j_x \neq i_{x+1}$, we have

$$m_k(A(i_1, j_1) \otimes \ldots \otimes A(i_k, j_k)) = 0.$$

b) Given $i_y \in \mathrm{Obj}\, A$ for $y \in [1, k+1]$, we have

$$m_k(A(i_1, i_2) \otimes A(i_2, i_3) \otimes \ldots \otimes A(i_k, i_{k+1})) \subseteq A(i_1, i_{k+1}).$$

Property (c) ensures that knowledge of $m_k\big|_{A(i_1,i_2)\otimes A(i_2,i_3)\otimes\ldots\otimes A(i_k,i_{k+1})}$ for $i_1, \ldots, i_{k+1} \in \mathrm{Obj}\, A$ is sufficient to obtain m_k.

Definition 30. Let $n, n', n'' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Suppose given a pre-$\mathrm{A}_{n'}$-category $(\mathrm{Obj}\, A', A', (m'_k)_{k\in[1,n']})$ and a pre-A_n-category $(\mathrm{Obj}\, A, A, (m_k)_{k\in[1,n]})$. A *pre-*$\mathrm{A}_{n''}$*-functor* from A' to A is a tuple $(f_{\mathrm{Obj}}, (f_k)_{k\in[1,n'']})$ as follows.

(a) f_{Obj} is a map from $\mathrm{Obj}\, A'$ to $\mathrm{Obj}\, A$.

(b) For $k \in [1, n'']$, the map $f_k : (A')^{\otimes k} \to A$ satisfies the following.

[3] In the literature, it is not always required that $\mathrm{Obj}\, A$ is a set, cf. e.g. [11, 7.2]. Requiring $\mathrm{Obj}\, A$ to be a set allows us to simplify notation. Furthermore, it allows us to perform constructions which require a choice for each pair of objects, cf. e.g. section 4.3.4.

a) Given $i_y, j_y \in \operatorname{Obj} A'$ for $y \in [1,k]$ such that there exists $x \in [1,k-1]$ with $j_x \neq i_{x+1}$, we have

$$f_k(A'(i_1,j_1) \otimes \ldots \otimes A'(i_k,j_k)) = 0.$$

b) Given $i_y \in \operatorname{Obj} A'$ for $y \in [1,k+1]$, we have

$$f_k(A'(i_1,i_2) \otimes A'(i_2,i_3) \otimes \ldots \otimes A'(i_k,i_{k+1})) \subseteq A(f_{\mathrm{Obj}}(i_1), f_{\mathrm{Obj}}(i_{k+1})).$$

Example 31. Suppose given an R-algebra B. Suppose given a set I and complexes $(C^{(i)}, d^{(i)})$ over B for $i \in I$. For $i,j \in I$, we set $A(i,j) := \operatorname{Hom}^*_B(C^{(j)}, C^{(i)})$, which is a graded R-module (Note that i and j are swapped). For $i,j,k \in I$, $f \in A(i,j) = \operatorname{Hom}^*_B(C^{(j)}, C^{(i)})$, $g \in A(j,k) = \operatorname{Hom}^*_B(C^{(k)}, C^{(j)})$, we set

$$\begin{aligned} m_1(f) :=& d_{\operatorname{Hom}^*_B(C^{(j)},C^{(i)})}(f) \\ m_2(f \otimes g) :=& f \circ g. \end{aligned}$$

For $n \geq 3$, set $m_n := 0$. Then $(I, A := \oplus_{i,j \in I} A(i,j), (m_n)_{n \geq 1})$ is a pre-A_∞-category.

Definition 32. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

An A_n-*category* is a pre-A_n-category $(\operatorname{Obj} A, A, (m_k)_{k \in [1,n]})$ such that $(A, (m_k)_{k \in [1,n]})$ is an A_n-algebra.

Suppose given A_n-categories $(\operatorname{Obj} A', A', (m'_k)_{k \in [1,n]})$ and $(\operatorname{Obj} A, A, (m_k)_{k \in [1,n]})$. An A_n-*functor* or *morphism of* A_n-*categories* from A' to A is a pre-A_n-functor $(f_{\mathrm{Obj}}, (f_k)_{k \in [1,n]})$ from A' to A such that $(f_k)_{k \in [1,n]}$ is a morphism of A_n-algebras from $(A', (m'_k)_{k \in [1,n]})$ to $(A, (m_k)_{k \in [1,n]})$.

The A_n-functor $(f_{\mathrm{Obj}}, (f_k)_{k \in [1,n]})$ is called a *quasi-isomorphism* of A_n-categories if the morphism of A_n-algebras $(f_k)_{k \in [1,n]}$ is a quasi-isomorphism.

The A_n-functor $(f_{\mathrm{Obj}}, (f_k)_{k \in [1,n]})$ is called a *local quasi-isomorphism* of A_n-categories if for all $i,j \in \operatorname{Obj} A'$, the complex morphism $f_1|_{A'(i,j)}^{A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j))} : (A'(i,j), m'_1|_{A'(i,j)}^{A'(i,j)}) \to (A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j)), m_1|_{A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j))}^{A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j))})$ is a quasi-isomorphism.

An example is given by the following

Lemma 33. *The pre-*A_∞*-category* $(I, A := \oplus_{i,j \in I} A(i,j), (m_n)_{n \geq 1})$ *given in Example 31 is an* A_∞*-category.*

Proof. We need to show that $(A, (m_n)_{n \geq 1})$ is an A_∞-algebra. By construction, $(A, (m_n)_{n \geq 1})$ is a pre-A_∞-algebra. So we need to verify the Stasheff identities (4)[k] for $k \geq 1$. By Example 23, it suffices to verify (4)[k] for $k \in \{1,2,3\}$. Since $(A, (m_n)_{n \geq 1})$ is a pre-A_∞-category, it suffices to verify (4)[k] on elements of $A(i_1,j_1) \otimes \ldots \otimes A(i_k,j_k)$ where $j_x = i_{x+1}$ for $x \in [1,k-1]$. Eq. (4)[1] holds since m_1 is obtained from a differential. Since m_2 is associative and since $m_3 = 0$, eq. (4)[3] holds, cf. Example 23. It remains to verify (4)[2] on elements of $A(i,i') \otimes A(i',i'') = \mathrm{Hom}^*_B(C^{(i')}, C^{(i)}) \otimes \mathrm{Hom}^*_B(C^{(i'')}, C^{(i')})$. Suppose given $g \in A(i,i')$ and $h \in A(i',i'')$ such that g is homogeneous of degree k_g and h is homogeneous of degree k_h. We have

$$\begin{aligned}
&(m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1)(g \otimes h) \\
&\overset{(1)}{=} m_2(m_1(g) \otimes h + (-1)^{\mathsf{k}_g} g \otimes m_1(h)) \\
&= m_2(d_{\mathrm{Hom}^*_B(C^{(i')},C^{(i)})}(g) \otimes h + (-1)^{\mathsf{k}_g} g \otimes d_{\mathrm{Hom}^*_B(C^{(i'')},C^{(i')})}(h)) \\
&= m_2((d^{(i)} \circ g - (-1)^{\mathsf{k}_g} g \circ d^{(i')}) \otimes h + (-1)^{\mathsf{k}_g} g \otimes (d^{(i')} \circ h - (-1)^{\mathsf{k}_h} h \circ d^{(i'')})) \\
&= (d^{(i)} \circ g - (-1)^{\mathsf{k}_g} g \circ d^{(i')}) \circ h + (-1)^{\mathsf{k}_g} g \circ (d^{(i')} \circ h - (-1)^{\mathsf{k}_h} h \circ d^{(i'')}) \\
&= d^{(i)} \circ g \circ h - (-1)^{\mathsf{k}_g + \mathsf{k}_h} g \circ h \circ d^{(i'')} \\
&= d_{\mathrm{Hom}^*_B(C^{(i'')},C^{(i)})}(g \circ h) = m_1(g \circ h) = (m_1 \circ m_2)(g \otimes h).
\end{aligned}$$

Thus $(m_1 \circ m_2 - m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1))(g \otimes h) = 0$. Hence (4)[2] holds. Thus $(A, (m_n)_{n \geq 1})$ is an A_∞-algebra. $\square$

Lemma 34. *Let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Suppose given A_n-categories $(\mathrm{Obj}\, A', A', (m'_k)_{k \in [1,n']})$, $(\mathrm{Obj}\, A, A, (m_k)_{k \in [1,n]})$ and an A_n-functor $(f_{\mathrm{Obj}}, (f_k)_{k \in [1,n]})$ from A' to A. If f_{Obj} is bijective then the following are equivalent.*

(a) $(f_{\mathrm{Obj}}, (f_k)_{k \in [1,n]})$ is a quasi-isomorphism of A_n-categories.

(b) $(f_{\mathrm{Obj}}, (f_k)_{k \in [1,n]})$ is a local quasi-isomorphism of A_n-categories.

Proof. To shorten notation, we write $f_{\mathrm{Obj}} =: g$. The complexes (A', m'_1) and (A, m_1) decompose as

$$(A', m_1) = \bigoplus_{i,j \in \mathrm{Obj}\, A'} (A'(i,j), m'_1|^{A'(i,j)}_{A'(i,j)}) \qquad \text{and}$$
$$(A, m_1) = \bigoplus_{i,j \in \mathrm{Obj}\, A} (A(i,j), m_1|^{A(i,j)}_{A(i,j)}).$$

For $i, j \in \mathrm{Obj}\, A'$, we have $f_1(A'(i,j)) \subseteq A(g(i), g(j))$. So the complex morphism $f_1 : (A', m'_1) \to (A, m_1)$ decomposes as

$$\begin{array}{c}
\bigoplus_{i,j \in \mathrm{Obj}\, A'} (A'(i,j), m'_1|^{A'(i,j)}_{A'(i,j)}) \\
{\scriptstyle f_1 = \bigoplus_{i,j \in \mathrm{Obj}\, A'} f_1|^{A(g(i),g(j))}_{A'(i,j)}} \Big\downarrow \\
\bigoplus_{i,j \in \mathrm{Obj}\, A'} (A(g(i),g(j)), m_1|^{A(g(i),g(j))}_{A(g(i),g(j))}) = \bigoplus_{i,j \in \mathrm{Obj}\, A} (A(i,j), m_1|^{A(i,j)}_{A(i,j)}).
\end{array}$$

Hence, $f_1 : (A', m'_1) \to (A, m_1)$ is a quasi-isomorphism iff all components
$f_1|^{A(g(i),g(j))}_{A'(i,j)} : (A'(i,j), m'_1|^{A'(i,j)}_{A'(i,j)}) \to (A(g(i),g(j)), m_1|^{A(g(i),g(j))}_{A(g(i),g(j))})$ are quasi-isomorphisms. $\square$

1.4. Homology of A_∞-categories

It is well-known that the homology of dg-algebras and A_∞-algebras resp. of dg-categories and A_∞-categories carries the structure an algebra resp. a category if we somehow ensure the existence of a unit resp. of identity morphisms, cf. e.g. [11, sections 3.5, 7.2 and 7.7] and [21, Definition (1a)]. In the following, we present a variant which initially does not require the homology of an A_∞-category to have identity morphisms. I.e. it is not necessarily a category. Instead there is a simple way to use suitable context for proving the existence of identity morphisms, cf. Lemmas 36 and 39.

1.4.1. Semicategories

We will see that, similarly to a category, the homology of an A_∞-category features an associative composition map for morphisms but it may lack identity morphisms. Such a structure is called a semicategory:

Definition/Remark 35 (cf. e.g. [6])**.** A *semicategory* $\mathtt{C}$ consists of

- a set of objects $\operatorname{Obj}\mathtt{C}$
- for each pair $A, B \in \operatorname{Obj}\mathtt{C}$, a set of morphisms $\mathtt{C}(A, B)$
- for each triple $A, B, C \in \operatorname{Obj}\mathtt{C}$, a function

$$\begin{array}{rcl}(\cdot) : \mathtt{C}(A,B) \times \mathtt{C}(B,C) & \longrightarrow & \mathtt{C}(A,C) \\ (f,g) & \longmapsto & f \cdot g\end{array}$$

  called composition such that composition is associative. Note that in the context of semicategories, we write composition *on the right*. We use a dot ("$\cdot$") to distinguish it from composition on the left, which is denoted by a circle ("$\circ$").

  Composition on the right is used to simplify notation when constructing semicategories as homologies of A_∞-categories.

For $A \in \operatorname{Obj}\mathtt{C}$, we call a morphism $\mathrm{id}_A \in \mathtt{C}(A, A)$ an identity morphism on A if for all $B \in \operatorname{Obj}\mathtt{C}$ and for all $f \in \mathtt{C}(A, B)$, $g \in \mathtt{C}(B, A)$, we have $\mathrm{id}_A \cdot f = f$ and $g \cdot \mathrm{id}_A = g$.

If there are identity morphisms $\mathrm{id}_{A,(1)}, \mathrm{id}_{A,(2)}$ on $A \in \mathtt{C}$, we have $\mathrm{id}_{A,(1)} = \mathrm{id}_{A,(1)} \cdot \mathrm{id}_{A,(2)} = \mathrm{id}_{A,(2)}$. So identity morphisms are unique, which justifies the notation id_A for an identity map on A. We say that $\mathtt{C}$ *has identities* iff for each object $A \in \operatorname{Obj}\mathtt{C}$, there is an identity morphism on A. In that case, $\mathtt{C}$ is a category.

If $\mathtt{C}$ and $\mathtt{D}$ are categories, then a semifunctor $F : \mathtt{C} \to \mathtt{D}$ consists of

- a function $F : \operatorname{Obj}\mathtt{C} \to \operatorname{Obj}\mathtt{D}$
- for each pair $A, B \in \operatorname{Obj}\mathtt{C}$, a function

$$F_{AB} : \mathtt{C}(A,B) \to \mathtt{D}(F(A), F(B))$$

such that for $A, B, C \in \operatorname{Obj} \mathtt{C}$, $f \in \mathtt{C}(A, B)$ and $g \in \mathtt{D}(B, C)$, we have $F_{AB}(f) \cdot F_{BC}(g) = F_{AC}(f \cdot g)$.

We call the semifunctor F *surjective* iff $F : \operatorname{Obj} \mathtt{C} \to \operatorname{Obj} \mathtt{D}$ is surjective.
We call F *faithful* iff for all $A, B \in \operatorname{Obj} \mathtt{C}$, the map F_{AB} is injective.
We call F *full* iff for all $A, B \in \operatorname{Obj} \mathtt{C}$, the map F_{AB} is surjective.

Lemma 36. *Suppose given a fully faithful semifunctor $F : \mathtt{C} \to \mathtt{D}$, where $\mathtt{D}$ has identities. Then $\mathtt{C}$ has identities and $F : \mathtt{C} \to \mathtt{D}$ is a fully faithful functor between categories.*

Proof. Since F is fully faithful, the maps F_{AB} for $A, B \in \operatorname{Obj} \mathtt{C}$ are all bijective. For $A \in \operatorname{Obj} \mathtt{C}$, we set $\mathrm{id}_A := F_{AA}^{-1}(\mathrm{id}_{F(A)})$. For $B \in \operatorname{Obj} \mathtt{C}$, $f \in \mathtt{C}(A, B)$ and $g \in \mathtt{C}(B, A)$, we have

$$\begin{aligned} F_{AB}(\mathrm{id}_A \cdot f) &= F_{AA}(\mathrm{id}_A) \cdot F_{AB}(f) = \mathrm{id}_{F(A)} \cdot F_{AB}(f) = F_{AB}(f) \\ F_{BA}(g \cdot \mathrm{id}_A) &= F_{BA}(g) \cdot F_{AA}(\mathrm{id}_A) = F_{BA}(g) \cdot \mathrm{id}_{F(A)} = F_{BA}(g). \end{aligned}$$

Since F_{AB} and F_{BA} are injective, we have $\mathrm{id}_A \cdot f = f$ and $g \cdot \mathrm{id}_A = g$. Hence, id_A is an identity on A for $A \in \operatorname{Obj} \mathtt{C}$. I.e. $\mathtt{C}$ is a category. Since F is a semifunctor, it preserves composition. By the definition of the id_A, the semifunctor F preserves identities, so it is actually a functor. □

1.4.2. A_∞-categories

Definition/Remark 37. Let $n \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$. Suppose given an A_n-category $(\operatorname{Obj} A, A(i,j)_{i,j \in \operatorname{Obj} A}, (m_k)_{k \in [1,n]})$. We define its zeroth homology $\mathrm{H}^0 A$, which is a semicategory.

The objects of $\mathrm{H}^0 A$ are the same as those of A. I.e. $\operatorname{Obj}(\mathrm{H}^0 A) := \operatorname{Obj} A$.

For $i, j \in \operatorname{Obj} A$, $(A(i,j), m_1)$ is a complex. We set $(\mathrm{H}^0 A)(i,j) := \mathrm{H}^0(A(i,j), m_1)$.

For $i_1, i_2, i_3 \in \operatorname{Obj} A$, we define composition by

$$\begin{array}{cccc} (\cdot) : & (\mathrm{H}^0 A)(i_1, i_2) \times (\mathrm{H}^0 A)(i_2, i_3) & \longrightarrow & (\mathrm{H}^0 A)(i_1, i_3) \\ & (g + \mathrm{B}^0(A(i_1, i_2), m_1)\,,\ f + \mathrm{B}^0(A(i_2, i_3), m_1)) & \longmapsto & m_2(g \otimes f) + \mathrm{B}^0(A(i_1, i_3), m_1). \end{array}$$

Recall here that we have $\mathrm{H}^0 C = \mathrm{Z}^0 C / \mathrm{B}^0 C$.

We need to check that $(\cdot)$ defined above is well-defined and associative. By (4)[2], we have

$$\begin{aligned} & m_1(m_2(\mathrm{Z}^0(A(i_1,i_2),m_1) \otimes \mathrm{Z}^0(A(i_2,i_3),m_1))) \\ =& m_2((m_1 \otimes 1 + 1 \otimes m_1)(\mathrm{Z}^0(A(i_1,i_2),m_1) \otimes \mathrm{Z}^0(A(i_2,i_3),m_1))) = 0, \end{aligned}$$

so pairs of cycles are mapped by m_2 to cycles. Once more by (4)[2], we have

$$m_2(\mathrm{B}^0(A(i_1,i_2),m_1) \otimes \mathrm{Z}^0(A(i_2,i_3),m_1))$$

$$
\begin{aligned}
&= m_2((m_1 \otimes 1)(A(i_1,i_2)^{-1} \otimes \mathrm{Z}^0(A(i_2,i_3),m_1))) \\
&= (m_1 \circ m_2 - m_2 \circ (1 \otimes m_1))(A(i_1,i_2)^{-1} \otimes \mathrm{Z}^0(A(i_2,i_3),m_1)) \\
&= m_1(m_2(A(i_1,i_2)^{-1} \otimes \mathrm{Z}^0(A(i_2,i_3),m_1))) \subseteq \mathrm{B}^0(A(i_1,i_3),m_1).
\end{aligned}
$$

Similarly, we have

$$
m_2(\mathrm{Z}^0(A(i_1,i_2),m_1) \otimes \mathrm{B}^0(A(i_2,i_3),m_1)) \subseteq \mathrm{B}^0(A(i_1,i_3),m_1),
$$

so $(\cdot)$ is well-defined.

(4)[3] is

$$
m_2(m_2 \otimes 1 - 1 \otimes m_2) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes 1^{\otimes 2} + 1 \otimes m_1 \otimes 1 + 1^{\otimes 2} \otimes m_1)
$$

Hence, for $i_1, i_2, i_3, i_4 \in \operatorname{Obj} A$ and for $f \in \mathrm{Z}^0(A(i_1,i_2),m_1)$, $g \in \mathrm{Z}^0(A(i_2,i_3),m_1)$ and $h \in \mathrm{Z}^0(A(i_3,i_4),m_1)$, we have

$$
m_2(m_2 \otimes 1 - 1 \otimes m_2)(f \otimes g \otimes h) = m_1(m_3(f \otimes g \otimes h)) \subseteq \mathrm{B}^0(A(i_1,i_4),m_1).
$$

Hence $(\bar{f} \cdot \bar{g}) \cdot \bar{h} - \bar{f} \cdot (\bar{g} \cdot \bar{h}) = 0$. Thus the composition $(\cdot)$ is associative.

1.4.3. A_∞-functors

Definition/Remark 38. Let $n' \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$. Let $n \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Suppose given $\mathrm{A}_{n'}$-categories $(\operatorname{Obj} A', A', (m'_k)_{k \in [1,n']})$, $(\operatorname{Obj} A, A, (m_k)_{k \in [1,n']})$. Suppose given an A_n-functor $f = (f_{\mathrm{Obj}}, f_{k \in [1,n]})$ from A' to A. Then f induces a semifunctor $\mathrm{H}^0 f$ from $\mathrm{H}^0 A'$ to $\mathrm{H}^0 A$ as follows.

We set

$$
(\mathrm{H}^0 f)_{\mathrm{Obj}} := f_{\mathrm{Obj}} : \operatorname{Obj} A' \to \operatorname{Obj} A.
$$

For $i, j \in \operatorname{Obj} A'$, we set

$$
(\mathrm{H}^0 f)_{ij} := \mathrm{H}^0 \left(f_1|_{A'(i,j)}^{A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j))} : (A'(i,j), m'_1) \to (A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j)), m_1) \right).
$$

By (5)[2] (cf. (7)), we have

$$
f_1 \circ m'_2 - m_2 \circ (f_1 \otimes f_1) = m_1 \circ f_2 + f_2 \circ (m'_1 \otimes 1 + 1 \otimes m'_1).
$$

Hence for $i, j, k \in \operatorname{Obj} A'$ and $\bar{g}_1 \in (\mathrm{H}^0 A')(i,j)$, $\bar{g}_2 \in (\mathrm{H}^0 A')(j,k)$ with representatives $g_1 \in \mathrm{Z}^0(A'(i,j),m'_1)$, $g_2 \in \mathrm{Z}^0(A'(j,k),m'_1)$, we have

$$
\begin{aligned}
&(\mathrm{H}^0 f)_{ik}(\bar{g}_1 \cdot \bar{g}_2) - (\mathrm{H}^0 f)_{ij}(\bar{g}_1) \cdot (\mathrm{H}^0 f)_{jk}(\bar{g}_2) \\
&\quad = \overline{(f_1 \circ m'_2)(g_1 \otimes g_2) - (m_2 \circ (f_1 \otimes f_1))(g_1 \otimes g_2)} \\
&\quad = \overline{(m_1 \circ f_2 + f_2 \circ (m'_1 \otimes 1 + 1 \otimes m'_1))(g_1 \otimes g_2)}
\end{aligned}
$$

$$=\overline{\underbrace{(m_1 \circ f_2)(g_1 \otimes g_2)}_{\in \mathrm{B}^0(A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(k)), m_1)} + \underbrace{(f_2 \circ (m'_1 \otimes 1 + 1 \otimes m'_1))(g_1 \otimes g_2)}_{=0}} = 0\,.$$

Hence

$$(\mathrm{H}^0 f)_{ij}(\bar{g}_1) \cdot (\mathrm{H}^0 f)_{jk}(\bar{g}_2) = (\mathrm{H}^0 f)_{ik}(\bar{g}_1 \cdot \bar{g}_2),$$

so $\mathrm{H}^0 f$ is a semifunctor.

Lemma 39. *Let $n' \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$. Let $n \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Suppose given $\mathrm{A}_{n'}$-categories $(\mathrm{Obj}\, A', A', (m'_k)_{k \in [1,n']})$, $(\mathrm{Obj}\, A, A, (m_k)_{k \in [1,n']})$ and an A_n-functor $f = (f_{\mathrm{Obj}}, f_{k \in [1,n]})$ from A' to A. Suppose that f is a local quasi-isomorphism. Then $\mathrm{H}^0 f$ is fully faithful.*

Proof. For $i, j \in \mathrm{Obj}\, A'$, the complex morphism $f_1|_{A'(i,j)}^{A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j))} : (A'(i,j), m'_1) \to (A(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j)), m_1)$ is a quasi-isomorphism since f is a local quasi-isomorphism. Hence for $i, j \in \mathrm{Obj}\, A'$, the map $(\mathrm{H}^0 f)_{ij} : (\mathrm{H}^0 A')(i,j) \to (\mathrm{H}^0 A)(f_{\mathrm{Obj}}(i), f_{\mathrm{Obj}}(j))$ is an isomorphism. Thus $(\mathrm{H}^0 f)$ is fully faithful. $\square$

2. On the bar construction

Suppose given a commutative ground ring R.

2.1. The bar construction

The following may be found e.g. in [14, 1.2.2].

Definition 40.

(i) A *R-coalgebra* (B, Δ) is an R-module B equipped with a linear and coassociative *comultiplication* $\Delta : B \to B \otimes B$. *Coassociativity* means that $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$. We will denote R-coalgebras simply as "coalgebras".

(ii) A *coderivation of a coalgebra* (B, Δ) is a linear map $b : B \to B$ such that $\Delta \circ b = (b \otimes 1 + 1 \otimes b) \circ \Delta$.

(iii) A *codifferential of a coalgebra* (B, Δ) is a coderivation $b : B \to B$ satisfying $b^2 = 0$.

(iv) A *coalgebra morphism* $F : (B', \Delta') \to (B, \Delta)$ between coalgebras (B', Δ'), (B, Δ) is a linear map $F : B' \to B$ such that $\Delta \circ F = (F \otimes F) \circ \Delta'$.

(v) A *differential coalgebra* (B, Δ, b) is a coalgebra (B, Δ) with a codifferential b on (B, Δ).

(vi) A *morphism of differential coalgebras* $F : (B', \Delta', b') \to (B, \Delta, b)$ is a coalgebra morphism $F : (B', \Delta') \to (B, \Delta)$ that commutes with the differentials, i.e. $b \circ F = F \circ b'$.

Lemma 41.

(a) A morphism of coalgebras is an isomorphism if and only if it is bijective.

(b) A morphism of differential coalgebras is an isomorphism if and only if it is bijective.

Proof. Each isomorphism of (differential) coalgebras is bijective as it is also an isomorphism in the category of sets.

Now let $F : (B', \Delta') \to (B, \Delta)$ be a bijective morphism of coalgebras. Then we have an R-linear inverse F'. We have

$$\Delta' \circ F' = (F' \otimes F') \circ (F \otimes F) \circ \Delta' \circ F' = (F' \otimes F') \circ \Delta \circ F \circ F' = (F' \otimes F') \circ \Delta$$

so F' is a morphism of coalgebras and F an isomorphism of coalgebras.

For a bijective morphism of differential coalgebras $F : (B', \Delta', b') \to (B, \Delta, b)$, we need to check that its inverse coalgebra morphism F' commutes with the differentials. In fact,

$$F' \circ b = F' \circ b \circ F \circ F' = F' \circ F \circ b \circ F' = b \circ F'.$$

So F is an isomorphism of differential coalgebras. $\square$

Definition/Remark 42. Let V be a graded R-module. We shall define the structure of a (graded) coalgebra on the graded module $TV := \bigoplus_{k\geq 1} V^{\otimes k}$ which then will be called the *tensor coalgebra of* V. The grading on TV is given by the grading of tensor products and sums of graded R-modules, i.e. for homogeneous elements $v_1, \ldots, v_k$ of degrees $\mathsf{k}_1, \ldots, \mathsf{k}_k$, the element $v_1 \otimes \cdots \otimes v_k$ is graded of degree $\mathsf{k}_1 + \ldots + \mathsf{k}_k$. The coalgebra structure is given by the comultiplication $\Delta : TV \to TV \otimes TV$ defined for elements $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ by

$$
\begin{aligned}
\Delta(v_1 \otimes \cdots \otimes v_k) &:= \sum_{1 \leq i \leq k-1} (v_1 \otimes \cdots \otimes v_i) \bigotimes (v_{i+1} \otimes \cdots \otimes v_k) \\
&= \sum_{\substack{i_1+i_2=k \\ i_1,i_2 \geq 1}} (v_1 \otimes \cdots \otimes v_{i_1}) \bigotimes (v_{i_1+1} \otimes \cdots \otimes v_{i_1+i_2}).
\end{aligned}
$$

Δ is coassociative, as for $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ we have

$$
\begin{aligned}
&((\Delta \otimes 1) \circ \Delta)(v_1 \otimes \cdots \otimes v_k) \\
&\qquad = \sum_{\substack{i_1+i_2+i_3=k \\ i_1,i_2,i_3 \geq 1}} (v_1 \otimes \cdots \otimes v_{i_1}) \bigotimes (v_{i_1+1} \otimes \cdots \otimes v_{i_1+i_2}) \bigotimes (v_{i_1+i_2+1} \otimes \cdots \otimes v_k) \\
&\qquad = ((1 \otimes \Delta) \circ \Delta)(v_1 \otimes \cdots \otimes v_k).
\end{aligned}
$$

So (TV, Δ) is indeed a coalgebra. The map Δ is graded of degree 0.

We have the canonical inclusions and projections for $k \geq 1$:

$$
\begin{aligned}
\iota_k : V^{\otimes k} &\longrightarrow TV \\
\pi_k : TV &\longrightarrow V^{\otimes k}
\end{aligned}
$$

If we have several graded R-modules V, V', we will usually distinguish the comultiplications, inclusions and projections on TV resp. TV' by Δ resp. Δ', ι_k resp. ι'_k and π_k resp. π'_k etc.

We will prove $\Delta x = 0 \Leftrightarrow x \in V$ for $x \in TV$, i.e.

$$
\ker \Delta = V. \tag{8}
$$

We have readily $V \subseteq \ker \Delta$. To prove equality, we first compose Δ with the projection $\pi_1 \otimes \mathrm{id} : TV \otimes TV \to V \otimes TV$ which maps $TV \otimes TV = \bigoplus_{k\geq 1}(V^{\otimes k} \otimes TV)$ onto its first component. Secondly we compose with the multiplication $\mu : V \otimes TV \to TV, v_1 \bigotimes (v_2 \otimes \cdots \otimes v_k) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_k$. Application to $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$, $k \geq 2$, gives

$$
\begin{array}{cccccc}
TV & \xrightarrow{\Delta} & TV \otimes TV & & & \\
v_1 \otimes \cdots \otimes v_k & \mapsto & \displaystyle\sum_{\substack{i_1+i_2=k \\ i_1,i_2 \geq 1}} (v_1 \otimes \cdots \otimes v_{i_1}) \bigotimes (v_{i_1+1} \otimes \cdots \otimes v_{i_1+i_2}) & & & \\
& \xrightarrow{\pi_1 \otimes \mathrm{id}} & V \otimes TV & \xrightarrow{\mu} & TV \\
& \longmapsto & v_1 \bigotimes (v_2 \otimes \cdots \otimes v_k) & \mapsto & v_1 \otimes v_2 \otimes \cdots \otimes v_k\,.
\end{array}
$$

So Δ is injective on $\bigoplus_{k\geq 2} V^{\otimes k}$ and zero on V, which proves (8).

For $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we set

$$TV_{\leq n} := \bigoplus_{k\in[1,n]} V^{\otimes k} \subseteq TV.$$

In particular $TV_{\leq\infty} = TV$.

Note that for $k \in \mathbb{Z}_{\geq 1}$, we have

$$\operatorname{im}\left(\Delta\big|_{V^{\otimes k}}\right) \subseteq TV_{\leq k-1} \otimes TV_{\leq k-1} \subseteq TV_{\leq k} \otimes TV_{\leq k}\,, \tag{9}$$

so $\left(TV_{\leq n}, \Delta\big|_{TV_{\leq n}}\right)$ is a subcoalgebra of (TV, Δ).

Lemma 43 (Lifting to coderivations)**.** *Let V be a graded R-module. Let $n\in\mathbb{Z}_{\geq 1}\cup\{\infty\}$. Then the map from the set of graded coderivations of $TV_{\leq n}$ of degree 1 to the set of families of graded maps $(b_k : V^{\otimes k} \to V)_{k\in[1,n]}$ with $|b_k| \ni 1$ for $k \in [1, n]$ that is given by*

$$b \longmapsto (\pi_1 \circ b\big|_{V^{\otimes k}})_{k\in[1,n]} = (\pi_1 \circ b \circ \iota_k)_{k\in[1,n]}$$

is bijective. Its inverse is given by $(b_k)_{k\in[1,n]} \mapsto b$, where b is defined by

$$b\big|_{V^{\otimes k}} := \sum_{\substack{r+s+t=k\\ r,t\geq 0,\ s\geq 1}} 1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}\ . \tag{10}$$

Proof. To show that $b \mapsto (b_k)_{k\in[1,n]}$ is surjective, let $(b_k : V^{\otimes k} \to V)_{k\in[1,n]}$ be a family of graded maps with $|b_k| \ni 1$ for $k \in [1, n]$ and construct b as given in (10). The properties $|b| \ni 1$, $\operatorname{im} b \subseteq TV_{\leq n}$ and $\pi_1 \circ b\big|_{V^{\otimes k}} = b_k$ follow immediately. We show that b is a coderivation:

$$\begin{aligned}
\Delta \circ b|_{V^{\otimes k}} &= \Delta \circ \sum_{\substack{r+s+t=k\\ r,t\geq 0, s\geq 1}} 1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}\\
&= \sum_{\substack{r_1+r_2+s+t=k\\ r_2,t\geq 0\\ r_1,s\geq 1}} 1^{\otimes r_1} \otimes (1^{\otimes r_2} \otimes b_s \otimes 1^{\otimes t}) + \sum_{\substack{r+s+t_1+t_2=k\\ r,t_1\geq 0\\ t_2,s\geq 1}} (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t_1}) \otimes 1^{\otimes t_2}\\
&= \sum_{\substack{r_1+t_2=k\\ r_1,t_2\geq 1}} \left(\sum_{\substack{r_2+s+t=t_2\\ r_2,t\geq 0,s\geq 1}} 1^{\otimes r_1} \otimes (1^{\otimes r_2} \otimes b_s \otimes 1^{\otimes t}) + \sum_{\substack{r+s+t_1=r_1\\ r,t_1\geq 0,s\geq 1}} (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t_1}) \otimes 1^{\otimes t_2} \right)\\
&= (1 \otimes b + b \otimes 1) \circ \Delta
\end{aligned}$$

So $b \mapsto (b_k)_{k\in[1,n]}$ is surjective and we find a preimage as indicated by (10). For injectivity, we use the fact that set of graded coderivations of degree 1 is closed under addition, i.e.

it is an R-module. So we only need to check that the kernel of $b \mapsto (b_k)_{k\in[1,n]}$ is zero: Let $b : TV_{\le n} \to TV_{\le n}$ be a graded coderivation of degree 1 such that $\pi_1 \circ b\big|_{V^{\otimes k}} = 0$ for all $k \in [1,n]$. We prove by induction on $k \ge 0$ that $b\big|_{TV_{\le k}} = 0$ thus $b = 0$: For $k = 0$ there is nothing to prove. So suppose for the induction step that $b\big|_{TV_{\le k}} = 0$ and $k+1 \in [1,n]$. Then $\Delta \circ b \circ \iota_{k+1} = (1 \otimes b + b \otimes 1) \circ \Delta \circ \iota_{k+1} \overset{(9),\text{ind.hyp.}}{=} 0$. So by (8), we have $b \circ \iota_{k+1} = \iota_1 \circ (\pi_1 \circ b \circ \iota_{k+1}) = 0$ and we have proven $b\big|_{TV_{\le k+1}} = 0$.
Thus the map $b \mapsto (b_k)_{k\in[1,n]}$ is bijective and its inverse images are given by (10). $\square$

Lemma 44 (Lifting to coalgebra morphisms)**.**
Let V, V' be graded R-modules. Let $n \in \mathbb{Z}_{\ge 1} \cup \{\infty\}$.
The map from the set of graded coalgebra morphisms $F : TV'_{\le n} \to TV_{\le n}$ of degree 0 to the set of families of graded maps $(F_k : V'^{\otimes k} \to V)_{k\in[1,n]}$ with $|F_k| \ni 0$ for $k \in [1,n]$ given by

$$F \mapsto (\pi_1 \circ F\big|_{V'^{\otimes k}})_{k\in[1,n]} = (\pi_1 \circ F \circ \iota'_k)_{k\in[1,n]}$$

is bijective. Its inverse is given by $(F_k)_{k\in[1,n]} \mapsto F$, where F is defined by

$$F\big|_{V'^{\otimes k}} := \sum_{\substack{i_1+\ldots+i_s=k \\ \text{all } i_j\ge 1}} F_{i_1} \otimes \cdots \otimes F_{i_s} \ . \tag{11}$$

Proof. To show that $F \mapsto (F_k)_{k\in[1,n]}$ is surjective, let $(F_k : V'^{\otimes k} \to V)_{k\in[1,n]}$ be a family of graded maps with $|F_k| \ni 0$ for all $k \in [1,n]$ and construct F be as in (11). The properties $\pi_1 \circ F|_{V'^{\otimes k}} = F_k$, $\operatorname{im} F \subseteq TV_{\le n}$ and $|F| \ni 0$ follow immediately. We show that F is a coalgebra morphism:

$$\begin{aligned}
\Delta \circ F|_{V'^{\otimes k}} &= \sum_{\substack{i_1+\ldots+i_{s+s'}=k \\ s,s'\ge 1,\ \text{all } i_j\ge 1}} (F_{i_1} \otimes \cdots \otimes F_{i_s}) \bigotimes (F_{i_{s+1}} \otimes \cdots \otimes F_{i_{s+s'}}) \\
&= \sum_{\substack{y_1+y_2=k \\ y_1,y_2\ge 1}} \sum_{\substack{i_1+\ldots+i_s=y_1 \\ i_{s+1}+\ldots+i_{s+s'}=y_2 \\ \text{all } i_j\ge 1}} (F_{i_1} \otimes \cdots \otimes F_{i_s}) \bigotimes (F_{i_{s+1}} \otimes \cdots \otimes F_{i_{s+s'}}) \\
&= (F \otimes F) \circ \Delta'
\end{aligned}$$

So $F \mapsto (F_k)_{k\in[1,n]}$ is surjective and we obtain a preimage as indicated by (11). To prove that $F \mapsto (F_k)_{k\in[1,n]}$ is injective, let $(F_k)_{k\in[1,n]}$ be as before and let $F, F' : TV'_{\le n} \to TV_{\le n}$ be coalgebra morphisms of degree 1 satisfying $\pi_1 \circ F\big|_{V'^{\otimes k}} = \pi_1 \circ F'\big|_{V'^{\otimes k}} = F_k$ for all $k \in [1,n]$. We prove by induction on $k \ge 0$ that $F\big|_{TV'_{\le k}} = F'\big|_{TV'_{\le k}}$, so $F = F'$. For $k = 0$, there is nothing to prove. So suppose $F\big|_{TV'_{\le k}} = F'\big|_{TV'_{\le k}}$ and $k+1 \in [1,n]$ for the induction step. We have

$$\begin{aligned}
\Delta \circ (F - F') \circ \iota'_{k+1} &= (F \otimes F - F' \otimes F') \circ \Delta' \circ \iota'_{k+1} \\
&= (F \otimes (F - F') - (F' - F) \otimes F') \circ \Delta' \circ \iota'_{k+1} = 0
\end{aligned}$$

as $\Delta'(V'^{\otimes k+1}) \subseteq TV'_{\leq k} \otimes TV'_{\leq k}$. As $\ker \Delta = V$, we have

$$(F - F') \circ \iota'_{k+1} = \iota_1 \circ \pi_1 \circ (F - F') \circ \iota'_{k+1} = \iota_1 \circ (F_{k+1} - F_{k+1}) = 0.$$

Thus we have $F\big|_{TV'_{\leq k+1}} = F'\big|_{TV'_{\leq k+1}}$ and the induction is complete. We have $F = F'$ so $F \mapsto (F_k)_{k\in[1,n]}$ is bijective and its inverse images are given by (11). □

Corollary 45. *Let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Let V, V' be graded modules. Let $F : TV'_{\leq n} \to TV_{\leq n}$ be a morphism of coalgebras of degree 0. Then $F(TV'_{\leq k}) \subseteq TV_{\leq k}$ for $k \in [0, n]$.*

Proof. This follows from (11) in Lemma 44. □

Lemma 46. *Let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Let $k \in [0, n]$ such that $k + 1 \in [1, n]$.*

(i) Let V be a graded R-module and $b\colon TV_{\leq n} \to TV_{\leq n}$ be a graded coderivation of degree 1. Then $b^2\big|_{TV_{\leq k}} = 0$ implies $\operatorname{im}(b^2 \circ \iota_{k+1}) \subseteq V$.

(ii) Let V, V' be graded R-modules and $b\colon TV_{\leq n} \to TV_{\leq n}$, $b'\colon TV'_{\leq n} \to TV'_{\leq n}$ be graded coderivations. Let $F : TV'_{\leq n} \to TV_{\leq n}$ be a graded coalgebra morphism of degree 0. Then $(b \circ F - F \circ b')\big|_{TV'_{\leq k}} = 0$ implies $\operatorname{im}\big((b \circ F - F \circ b') \circ \iota'_{k+1}\big) \subseteq V$.

Proof. At the steps marked by "$*$" in the following, we use (9), and $b^2\big|_{TV_{\leq k}} = 0$ respectively $(F \circ b' - b \circ F)\big|_{TV'_{\leq k}} = 0$.

$$\begin{aligned}
\Delta \circ b^2 \circ \iota_{k+1} &= (1 \otimes b + b \otimes 1) \circ (1 \otimes b + b \otimes 1) \circ \Delta \circ \iota_{k+1} \\
&\overset{(3),|b|\ni 1}{=} [1 \otimes b^2 - b \otimes b + b \otimes b + b^2 \otimes 1] \circ \Delta \circ \iota_{k+1} \\
&= [1 \otimes b^2 + b^2 \otimes 1] \circ \Delta \circ \iota_{k+1} \overset{*}{=} 0
\end{aligned}$$

$$\begin{aligned}
\Delta \circ (F \circ b' - b \circ F) \circ \iota'_{k+1} &= [(F \otimes F) \circ \Delta' \circ b' - (1 \otimes b + b \otimes 1) \circ \Delta \circ F] \circ \iota'_{k+1} \\
&= [(F \otimes F) \circ (1 \otimes b' + b' \otimes 1) - (1 \otimes b + b \otimes 1) \circ (F \otimes F)] \circ \Delta' \circ \iota'_{k+1} \\
&\overset{(3),|F|\ni 0}{=} [F \otimes (F \circ b' - b \circ F) + (F \circ b' - b \circ F) \otimes F] \circ \Delta' \circ \iota'_{k+1} \overset{*}{=} 0
\end{aligned}$$

The lemma now follows from $\ker \Delta = V$, cf. (8). □

Definition/Remark 47. For a graded R-module A, we define the R-module SA with shifted grading by $SA = A$ and $(SA)^q := A^{q+1}$. We have the shift map $\omega : SA \to A$, $\omega(x) = x$ which is a graded map of degree 1. If we have multiple graded modules, say A and A', we usually distinguish the shift maps accordingly as ω and ω'.

We write $SA^{\otimes k} := (SA)^{\otimes k}$ for $k \geq 1$.

Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. A *corresponding pre-$\mathrm{A}_n$-triple on A* is defined as a triple $((m_k)_{k\in[1,n]}, (b_k)_{k\in[1,n]}, b)$ consisting of

(i) a pre-A_n-structure $(m_k)_{k\in[1,n]}$ on A,

(ii) a family of graded maps $(b_k : SA^{\otimes k} \to SA)_{k\in[1,n]}$ with $|b_k| \ni 1$ for $k \in [1,n]$ and

(iii) a graded coderivation $b : TSA_{\leq n} \to TSA_{\leq n}$ of degree 1

such that [4] $b_k = \omega^{-1} \circ m_k \circ \omega^{\otimes k}$ for $k \in [1,n]$ and $\pi_1 \circ b\big|_{SA^{\otimes k}} = b_k$ for $k \in [1,n]$.
Given a pre-A_n-structure $(m_k)_{k\in[1,n]}$ on A, a family of graded maps $(b_k : SA^{\otimes k} \to SA)_{k\in[1,n]}$ with $|b_k| \ni 1$ for $k \in [1,n]$ or a graded coderivation $b : TSA_{\leq n} \to TSA_{\leq n}$ of degree 1, i.e. a datum of type (i), (ii) or (iii), it can be uniquely extended to a corresponding pre-A_n-triple on A: The condition $b_k = \omega^{-1} \circ m_k \circ \omega^{\otimes k}$ for $k \in [1,n]$ induces a bijection between data of type (i) and of type (ii). Similarly, Lemma 43 gives a bijection between data of types (ii) and (iii).

Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Let A, A' be graded R-modules. A *corresponding pre-*A_n*-morphism triple from A' to A* is defined as a triple $((f_k)_{k\in[1,n]}, (F_k)_{k\in[1,n]}, F)$ consisting of

(i) a pre-A_n-morphism $(f_k)_{k\in[1,n]}$ from A' to A,

(ii) a family of graded maps $(F_k : SA'^{\otimes k} \to SA)_{k\in[1,n]}$ with $|F_k| \ni 0$ for $k \in [1,n]$ and

(iii) a graded coalgebra morphism $F : TSA'_{\leq n} \to TSA_{\leq n}$ of degree 0

such that $F_k = \omega^{-1} \circ f_k \circ \omega'^{\otimes k}$ for $k \in [1,n]$ and $\pi_1 \circ F\big|_{SA'^{\otimes k}} = F_k$ for $k \in [1,n]$. Analogous to corresponding pre-A_n-triples, given a datum of type (i), (ii) or (iii), it can be uniquely extended to a corresponding pre-A_n-morphism triple via Lemma 44 and the bijection induced by the condition $F_k = \omega^{-1} \circ f_k \circ \omega'^{\otimes k}$.

We write an asterisk ("$*$") in place of an entry of a corresponding triple to denote that the value of that entry is uninteresting.

Theorem 48 (Stasheff [22]). *Let A be a graded R-module. Let $\tilde{n} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Let $((m_k)_{k\in[1,\tilde{n}]}, (b_k)_{k\in[1,\tilde{n}]}, b)$ be a corresponding pre-$\mathrm{A}_{\tilde{n}}$-triple on A.*
Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $n \leq \tilde{n}$. The following are equivalent:

(a) Equation (4)[k] holds for $k \in [1,n]$, i.e. $(m_k)_{k\in[1,n]}$ is an A_n-structure on A.

(b) For $k \in [1,n]$, we have

$$\sum_{\substack{k=r+s+t,\\ r,t\geq 0,\ s\geq 1}} b_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) = 0. \tag{12)[k}$$

(c) $b^2\big|_{TSA_{\leq n}} = 0$, i.e. $b\big|_{TSA_{\leq n}}$ is a coalgebra differential on $TSA_{\leq n}$.

Proof. We prove (a) $\Leftrightarrow$ (b): Recall $|\omega| \ni 1$. Recall $|b_i| \ni 1$ and $|m_i| \ni 2 - i$ for $i \in [1,k]$.

[4] Note that we have $m_k = (-1)^{\frac{k(k-1)}{2}} b_k$. I.e. we get an additional sign in situations where the m_k are inferred from the b_k such as in FIXME 63 and 121. There are other versions of the bar construction in use (with suitable versions of the Stasheff identities) where inferring the m_k from the b_k is easier. E.g. in [11], we have the variant $m_k = \omega \circ b_k \circ (\omega^{-1})^{\otimes k}$.

We have

$$\begin{aligned}
&\sum_{\substack{k=r+s+t,\\ r,t\geq 0,\ s\geq 1}} b_{r+1+t}\circ(1^{\otimes r}\otimes b_s\otimes 1^{\otimes t})\\
=&\sum_{\substack{k=r+s+t,\\ r,t\geq 0,\ s\geq 1}} \omega^{-1}\circ m_{r+1+t}\circ(\omega^{\otimes r}\otimes\omega\otimes\omega^{\otimes t})\circ(1^{\otimes r}\otimes b_s\otimes 1^{\otimes t})\\
\overset{\text{C.18}}{=}&\,\omega^{-1}\circ\sum_{\substack{k=r+s+t,\\ r,t\geq 0,\ s\geq 1}}(-1)^{t\cdot 1}m_{r+1+t}\circ(\omega^{\otimes r}\otimes(\omega\circ b_s)\otimes\omega^{\otimes t})\\
=&\,\omega^{-1}\circ\sum_{\substack{k=r+s+t,\\ r,t\geq 0,\ s\geq 1}}(-1)^{t}m_{r+1+t}\circ(\omega^{\otimes r}\otimes(m_s\circ\omega^{\otimes s})\otimes\omega^{\otimes t})\\
\overset{\text{C.18}}{=}&\,\omega^{-1}\circ\sum_{\substack{k=r+s+t,\\ r,t\geq 0,\ s\geq 1}}(-1)^{t}(-1)^{r(2-s)}m_{r+1+t}\circ(1^{\otimes r}\otimes m_s\otimes 1^{\otimes t})\circ(\omega^{\otimes r}\otimes\omega^{\otimes s}\otimes\omega^{\otimes t})\\
=&\,\omega^{-1}\circ\sum_{\substack{k=r+s+t,\\ r,t\geq 0,\ s\geq 1}}(-1)^{rs+t}m_{r+1+t}\circ(1^{\otimes r}\otimes m_s\otimes 1^{\otimes t})\circ\omega^{\otimes k}. \qquad (13)
\end{aligned}$$

So (4)[k] $\Leftrightarrow$ (12)[k], whence (a) $\Leftrightarrow$ (b).

We prove (b) $\Leftrightarrow$ (c): We first prove for finite n that ((12)[k] for $k\in[1,n]$) $\Leftrightarrow b^2|_{TSA_{\leq n}}=0$. We proceed by induction on $n\geq 0$.
For $n=0$ we have $[1,n]=\emptyset$ and $TSA_{\leq n}=\{0\}$, so there is nothing to prove. So now assume for induction that $b^2|_{TSA_{\leq n}}=0\Leftrightarrow$ (12)[k] for $k\in[1,n]$. We have to show that $b^2|_{TSA_{\leq n+1}}=0\Leftrightarrow$ (12)[k] for $k\in[1,n+1]$. It is sufficient to prove under the assumption $b^2|_{TSA_{\leq n}}=0$ the equivalence $b^2|_{SA^{\otimes n+1}}=0\Leftrightarrow$ (12)[$n+1$]. So we assume $b^2|_{TSA_{\leq n}}=0$. By Lemma 46(i), we have

$$b^2\circ\iota_{n+1}=\iota_1\circ\pi_1\circ b^2\circ\iota_{n+1}\overset{(10)}{=}\iota_1\circ\sum_{\substack{n+1=r+s+t,\\ r,t\geq 0,\ s\geq 1}} b_{r+1+t}\circ(1^{\otimes r}\otimes b_s\otimes 1^{\otimes t}).$$

So $b^2|_{SA^{\otimes n+1}}=0\Leftrightarrow$ (12)[$n+1$] and the induction step is complete.

The case $n=\infty$ follows by

$$\begin{aligned}
&\forall k\in\mathbb{Z}_{\geq 1}:(12)[k] &&\Leftrightarrow\ \forall k\in\mathbb{Z}_{\geq 0}\,\forall k'\in[1,k]:(12)[k']\\
\Leftrightarrow\ &\forall k\in\mathbb{Z}_{\geq 0}:b^2|_{TSA_{\leq k}}=0 &&\Leftrightarrow\ b^2=0.
\end{aligned}$$

□

We need a pointwise version of the Theorem 48:

Theorem 49. *Let A be a graded R-module. Let $\tilde{n}\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$.
Let $((m_k)_{k\in[1,\tilde{n}]},(b_k)_{k\in[1,\tilde{n}]},b)$ be a corresponding pre-$\mathrm{A}_{\tilde{n}}$-triple on A.
Let $n\in[1,\tilde{n}]$. Let $x\in SA^{\otimes n}$. The following are equivalent:*

(a) Equation (4)[n] *holds on* $\omega^{\otimes n}(x)$*, that is*

$$\sum_{\substack{n=r+s+t,\\ r,t\geq 0, s\geq 1}} (-1)^{rs+t}(m_{r+1+t} \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) \circ \omega^{\otimes n})(x) = 0.$$

(b) Eq. (12)[n] *holds on* x*, that is*

$$\sum_{\substack{n=r+s+t,\\ r,t\geq 0,\ s\geq 1}} (b_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}))(x) = 0.$$

If additionally (4)[k] *holds for all* $k \in [1, n-1]$ *(*$\Leftrightarrow$ (12)[k] *holds for all* $k \in [1, n-1]$ $\Leftrightarrow$ $b^2|_{TSA_{\leq n-1}} = 0$*, cf. Theorem 48), then (a) and (b) are equivalent to*

(c) $b^2(x) = 0$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from (13).

So suppose $b^2|_{TSA_{\leq n-1}} = 0$. By Lemma 46(i), we have

$$b^2(x) = \iota_1(\pi_1(b^2(x))) \overset{(10)}{=} \iota_1\left(\sum_{\substack{n=r+s+t,\\ r,t\geq 0,\ s\geq 1}} (b_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}))(x)\right).$$

This proves the equivalence (b) $\Leftrightarrow$ (c). $\square$

Lemma 50. *Let* A, A' *be graded* R*-modules. Let* $\tilde{n} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$*.*
Let $((m_k)_{k\in[1,\tilde{n}]}, (b_k)_{k\in[1,\tilde{n}]}, b)$ *resp.* $((m'_k)_{k\in[1,\tilde{n}]}, (b'_k)_{k\in[1,\tilde{n}]}, b')$ *be corresponding pre-*$\mathrm{A}_{\tilde{n}}$*-triples on* A *resp.* A'*. Let* $((f_k)_{k\in[1,\tilde{n}]}, (F_k)_{k\in[1,\tilde{n}]}, F)$ *be a corresponding pre-*$\mathrm{A}_{\tilde{n}}$*-morphism triple from* A' *to* A*.*
Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ *be such that* $n \leq \tilde{n}$*. The following are equivalent:*

(a) Assertion (5)[k] *holds for* $k \in [1, n]$*.*

(b) For $k \in [1, n]$*, we have*

$$\sum_{\substack{k=r+s+t\\ r,t\geq 0,\ s\geq 1}} F_{r+1+t} \circ (1^{\otimes r} \otimes b'_s \otimes 1^{\otimes t}) = \sum_{\substack{1\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} b_r \circ (F_{i_1} \otimes F_{i_2} \otimes \cdots \otimes F_{i_r}). \quad (14)[k]$$

(c) $F \circ b'\big|_{TSA'_{\leq n}} = b \circ F\big|_{TSA'_{\leq n}}$

Note that we only require conditions on the grading of $(m_n)_{n\geq 1}$ and $(m'_n)_{n\geq 1}$. However if A and A' are actually A_n-algebras, then condition (a) holds iff $(f_k)_{k\in[1,n]}$ is an A_n-morphism.

Proof. We prove (a) $\Leftrightarrow$ (b): Analogously to the proof of (a) $\Leftrightarrow$ (b) of Theorem 48 we obtain for the left side of (14)[k]

$$\sum_{\substack{k=r+s+t\\ r,t\geq 0,\ s\geq 1}} F_{r+1+t}\circ(1^{\otimes r}\otimes b'_s\otimes 1^{\otimes t}) = \omega^{-1}\circ \sum_{\substack{k=r+s+t\\ r,t\geq 0,\ s\geq 1}} (-1)^{rs+t} f_{r+1+t}\circ(1^{\otimes r}\otimes m'_s\otimes 1^{\otimes t})\circ\omega'^{\otimes k}. \tag{15}$$

It remains to examine the right side:

$$\begin{aligned}
\sum_{\substack{1\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} b_r\circ(F_{i_1}\otimes\cdots\otimes F_{i_r}) &= \sum_{\substack{1\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} \omega^{-1}\circ m_r\circ\omega^{\otimes r}\circ(F_{i_1}\otimes\cdots\otimes F_{i_r})\\
&\overset{\text{C.18}}{=} \omega^{-1}\circ \sum_{\substack{1\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} (-1)^0 m_r\circ((\omega\circ F_{i_1})\otimes\cdots\otimes(\omega\circ F_{i_r}))\\
&= \omega^{-1}\circ \sum_{\substack{1\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} m_r\circ((f_{i_1}\circ\omega'^{\otimes i_1})\otimes\cdots\otimes(f_{i_r}\circ\omega'^{\otimes i_r}))\\
&= \omega^{-1}\circ \sum_{\substack{1\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} (-1)^v m_r\circ(f_{i_1}\otimes\cdots\otimes f_{i_r})\circ\omega'^{\otimes k}
\end{aligned} \tag{16}$$

In the last step, Corollary 18 gives the exponent

$$v = \sum_{2\leq s\leq r}\Big(\underbrace{(1-i_s)}_{\in|f_{i_s}|}\sum_{1\leq t<s}\underbrace{i_t}_{\in|\omega'^{\otimes i_t}}\Big) = \sum_{1\leq t<s\leq r}(1-i_s)i_t. \tag{17}$$

So we have (5)[k] $\Leftrightarrow$ (14)[k], whence (a) $\Leftrightarrow$ (b).

We prove (b) $\Leftrightarrow$ (c).
We first prove (b) $\Leftrightarrow$ (c) for finite n. We proceed by induction on $n\in[0,\tilde{n}]$: For $n=0$ we have $[1,n]=\emptyset$ and $TSA'_{\leq n}=\{0\}$, so there is nothing to prove. Now suppose given n. As induction hypothesis, suppose the equivalence $F\circ b'\big|_{TSA'_{\leq n}} = b\circ F\big|_{TSA'_{\leq n}} \Leftrightarrow$ ((14)[k] for $k\in[1,n]$) holds. For the induction step we need to prove that $F\circ b'\big|_{TSA'_{\leq n+1}} = b\circ F\big|_{TSA'_{\leq n+1}} \Leftrightarrow$ ((14)[k] for $k\in[1,n+1]$). Suppose that $F\circ b'\big|_{TSA'_{\leq n}} = b\circ F\big|_{TSA'_{\leq n}}$. It suffices to show the equivalence $F\circ b'\big|_{SA'^{\otimes n+1}} = b\circ F\big|_{SA'^{\otimes n+1}} \Leftrightarrow$ (14)[$n+1$].
By Lemma 46(ii), we have $(F\circ b'-b\circ F)\circ\iota'_{n+1} = \iota_1\circ[\pi_1\circ(F\circ b'-b\circ F)\circ\iota'_{n+1}]$. Now $\pi_1\circ(F\circ b'-b\circ F)\circ\iota'_{n+1}$ is exactly the difference of the sides of (14)[$n+1$], cf. (10),(11). So $F\circ b'\big|_{SA'^{\otimes n+1}} = b\circ F\big|_{SA'^{\otimes n+1}} \Leftrightarrow$ (14)[$n+1$] and the induction step is complete.

The case $n=\infty$ follows by

$$\begin{aligned}
&\forall k\in\mathbb{Z}_{\geq 1}: (14)[k] &&\Leftrightarrow\ \forall k\in\mathbb{Z}_{\geq 0}\,\forall k'\in[1,k]: (14)[k']\\
\Leftrightarrow\ &\forall k\in\mathbb{Z}_{\geq 0}: F\circ b'\big|_{TSA'_{\leq k}} = b\circ F\big|_{TSA'_{\leq k}} &&\Leftrightarrow\ F\circ b' = b\circ F.
\end{aligned}$$

$\square$

Similarly to Theorem 48 and Theorem 49, we have a pointwise version of Lemma 50 given as follows.

Lemma 51. *Let A, A' be graded R-modules. Let $\tilde{n} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.*
Let $((m_k)_{k\in[1,\tilde{n}]}, (b_k)_{k\in[1,\tilde{n}]}, b)$ resp. $((m'_k)_{k\in[1,\tilde{n}]}, (b'_k)_{k\in[1,\tilde{n}]}, b')$ be corresponding pre-$\mathrm{A}_{\tilde{n}}$-triples on A resp. A'. Let $((f_k)_{k\in[1,\tilde{n}]}, (F_k)_{k\in[1,\tilde{n}]}, F)$ be a corresponding pre-$\mathrm{A}_{\tilde{n}}$-morphism triple from A' to A.
Let $n \in [1, \tilde{n}]$. Let $x \in SA'^{\otimes n}$. The following are equivalent:

(a) Assertion (5)[n] *holds on $\omega'^{\otimes n}(x)$, that is*

$$\sum_{\substack{n=r+s+t\\ r,t\geq 0, s\geq 1}} (-1)^{rs+t}(f_{r+1+t} \circ (1^{\otimes r} \otimes m'_s \otimes 1^{\otimes t}) \circ \omega'^{\otimes n})(x)$$
$$= \sum_{\substack{1\leq r\leq n\\ i_1+\ldots+i_r=n\\ \text{all } i_s\geq 1}} (-1)^v (m_r \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r}) \circ \omega'^{\otimes n})(x),$$

where v is given by (6).

(b) Assertion (14)[n] *holds on x, that is*

$$\sum_{\substack{n=r+s+t\\ r,t\geq 0,\ s\geq 1}} (F_{r+1+t} \circ (1^{\otimes r} \otimes b'_s \otimes 1^{\otimes t}))(x) = \sum_{\substack{1\leq r\leq n\\ i_1+\ldots+i_r=n\\ \text{all } i_s\geq 1}} (b_r \circ (F_{i_1} \otimes F_{i_2} \otimes \cdots \otimes F_{i_r}))(x). \quad (18)$$

If additionally (5)[k] *holds for all $k \in [1, n-1]$ ($\Leftrightarrow$* (14)[k] *holds for all $k \in [1, n-1] \Leftrightarrow (F \circ b')|_{TSA'_{\leq n-1}} = (b \circ F)|_{TSA'_{\leq n-1}}$, cf. Lemma 50), then (a) and (b) are equivalent to*

(c) $(F \circ b')(x) = (b \circ F)(x)$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from (15), (16) and (17).

So suppose $(F\circ b')|_{TSA'_{\leq n-1}} = (b\circ F)|_{TSA'_{\leq n-1}}$. By Lemma 46(ii), we have $(F\circ b' - b\circ F)(x) = (\iota_1 \circ [\pi_1 \circ (F \circ b' - b \circ F)])(x)$. Now $(\pi_1 \circ (F \circ b' - b \circ F))(x)$ is exactly the difference of the sides of (18), cf. (10),(11). This proves the equivalence (b) $\Leftrightarrow$ (c). ☐

2.2. Applications. Kadeishvili's algorithm and the minimality theorem.

In this subsection we will discuss the construction of minimal models of A_∞-algebras. Firstly, Lemma 52 states that certain pre-A_n-structures and pre-A_n-morphisms that arise in the construction of minimal models are actually A_n-structures and A_n-morphisms. Secondly, we give a proof of Theorem 27. We will review KADEISHVILI's original proof of [10] as it gives a an algorithm for constructing minimal models which can be used for the direct calculation of examples.

Lemma 52. *Let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Let $(A', (m'_k)_{k\in[1,n]})$ be a pre-A_n-algebra. Let $(A, (m_k)_{k\in[1,n]})$ be an A_n-algebra. Let $(f_k)_{k\in[1,n]}$ be a pre-A_n-morphism from A' to A such that (5)$[k]$ holds for $k \in [1, n]$. Suppose that f_1 is injective.*
Then $(A', (m'_k)_{k\in[1,n]})$ is an A_n-algebra and $(f_k)_{k\in[1,n]}$ is a morphism of A_n-algebras from $(A', (m'_k)_{k\in[1,n]})$ to $(A, (m_k)_{k\in[1,n]})$.

Proof. We have the corresponding pre-A_n-triple $((m'_k)_{k\in[1,n]}, (b'_k)_{k\in[1,n]}, b')$, the corresponding pre-$\mathrm{A}_n$-triple $((m_k)_{k\in[1,n]}, (b_k)_{k\in[1,n]}, b)$ and the corresponding pre-A_n-morphism triple $((f_k)_{k\in[1,n]}, (F_k)_{k\in[1,n]}, F)$. It suffices to prove by induction on $k \in [0, n]$ that $(b')^2\big|_{TSA'_{\leq k}} = 0$, cf. Theorem 48.
For $k = 0$, we have $TSA'_{\leq k} = 0$ so there is nothing to prove. For the induction step, suppose that $b'^{\,2}\big|_{TSA'_{\leq k}} = 0$ for some $k \geq 0$ with $k + 1 \in [0, n]$. By the induction hypothesis and Lemma 46(i), we have

$$\mathrm{im}(b'^{\,2} \circ \iota'_{k+1}) \subseteq SA'. \tag{19}$$

Thus $0 \overset{\text{T.48}}{=} b^2 \circ F \circ \iota'_{k+1} \overset{\text{L.50}}{=} F \circ b'^{\,2} \circ \iota'_{k+1} \overset{(19)}{=} F \circ \iota'_1 \circ \pi'_1 \circ b'^{\,2} \circ \iota'_{k+1} \overset{(11)}{=} \iota_1 \circ F_1 \circ \pi'_1 \circ b'^{\,2} \circ \iota'_{k+1}$. As the injectivity of f_1 implies the injectivity of F_1, we have $0 = \iota'_1 \circ \pi'_1 \circ b'^{\,2} \circ \iota'_{k+1} \overset{(19)}{=} b'^{\,2} \circ \iota'_{k+1}$. Together with the induction hypothesis, we obtain $b'^{\,2}\big|_{TSA'_{\leq k+1}} = 0$, which completes the induction step. □

The following two lemmas give the incremental step in Kadeishvili's algorithm. By a quasi-monomorphism of complexes we will denote a complex morphism that induces monomorphisms on homology.

Lemma 53. *Let $n \in \mathbb{Z}_{\geq 1}$. Let A, A' be graded R-modules.*
Let $((m'_k)_{k\in[1,n+1]}, (b'_k)_{k\in[1,n+1]}, b')$ be a corresponding pre-A_{n+1}-triple on A'.
Let $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, b)$ be a corresponding pre-A_∞-triple on A.
Let $((f_k)_{k\in[1,n+1]}, (F_k)_{k\in[1,n+1]}, F)$ be a corresponding pre-A_{n+1}-morphism triple from A' to A.
Suppose that the following hold.

(i) We have $b'^{\,2}\big|_{TSA'_{\leq n}} = 0$, $b^2 = 0$ and $F \circ b'\big|_{TSA'_{\leq n}} = b \circ F\big|_{TSA'_{\leq n}}$.

(ii) We have $b'_1 = 0$ and F_1 is a quasi-monomorphism from the complex (SA', b'_1) to the complex (SA, b_1).

We set $h : SA'^{\otimes n+1} \to SA$,

$$h := \sum_{\substack{n+1=r+s+t \\ r,t\geq 0, s\in[2,n]}} F_{r+1+t} \circ (1^{\otimes r} \otimes b'_s \otimes 1^{\otimes t}) \quad - \sum_{\substack{r\in[2,n+1] \\ i_1+\ldots+i_r=n+1 \\ \text{all } i_s\geq 1}} b_r \circ (F_{i_1} \otimes F_{i_2} \otimes \cdots \otimes F_{i_r}).$$

Then

(a) $b'^{\,2} = 0$, i.e. $(A', (m'_k)_{k\in[1,n+1]})$ is an A_{n+1}-algebra[5], cf. Theorem 48.

(b) $b_1 \circ h = 0$.

(c) $F \circ b' = b \circ F \Leftrightarrow F_1 \circ b'_{n+1} - b_1 \circ F_{n+1} + h = 0$.

Proof. By Lemma 50, we have $F \circ b' = b \circ F \Leftrightarrow (14)[n+1]$. The difference of the sides of $(14)[n+1]$ is given by

$$\begin{aligned} G := &\sum_{\substack{n+1=r+s+t\\ r,t\geq 0, s\geq 1}} F_{r+1+t} \circ (1^{\otimes r} \otimes b'_s \otimes 1^{\otimes t}) \quad - \sum_{\substack{1\leq r\leq n+1\\ i_1+\ldots+i_r=n+1\\ \text{all } i_s\geq 1}} b_r \circ (F_{i_1} \otimes F_{i_2} \otimes \cdots \otimes F_{i_r}) \\ \overset{b'_1=0}{=} &F_1 \circ b'_{n+1} - b_1 \circ F_{n+1} + h \end{aligned}$$

Thus we have $F \circ b' = b \circ F \Leftrightarrow (14)[n+1] \Leftrightarrow F_1 \circ b'_{n+1} - b_1 \circ F_{n+1} + h = 0$, which proves (c). Note that by (10) and (11), we have $\pi_1 \circ (F \circ b' - b \circ F) \circ \iota'_{n+1} = G = F_1 \circ b'_{n+1} - b_1 \circ F_{n+1} + h$. Thus we have

$$\begin{aligned} b_1 \circ h &= b_1 \circ \pi_1 \circ (F \circ b' - b \circ F) \circ \iota'_{n+1} - b_1 \circ F_1 \circ b'_{n+1} + (b_1)^2 \circ F_{n+1} \\ &\overset{(i),(14)[1]}{=} b_1 \circ \pi_1 \circ (F \circ b' - b \circ F) \circ \iota'_{n+1} - F_1 \circ b'_1 \circ b'_{n+1} \\ &\overset{b'_1=0}{=} b_1 \circ \pi_1 \circ (F \circ b' - b \circ F) \circ \iota'_{n+1} \\ &\overset{\text{D./R.47}}{=} \pi_1 \circ b \circ \iota_1 \circ \pi_1 \circ (F \circ b' - b \circ F) \circ \iota'_{n+1} \\ &\overset{\text{L.46}(ii)}{=} \pi_1 \circ b \circ (F \circ b' - b \circ F) \circ \iota'_{n+1} \\ &\overset{(i)}{=} \pi_1 \circ b \circ F \circ b' \circ \iota'_{n+1} \end{aligned}$$

As $b'_1 = 0$, we obtain $\mathrm{im}(b' \circ \iota'_{n+1}) \subseteq TSA'_{\leq n}$, cf. (10). By $b \circ F\big|_{TSA'_{\leq n}} = F \circ b'\big|_{TSA'_{\leq n}}$, we conclude

$$\begin{aligned} b_1 \circ h &= \pi_1 \circ F \circ b'^{\,2} \circ \iota'_{n+1} \\ &\overset{\text{L.46}(i)}{=} \pi_1 \circ F \circ \iota'_1 \circ \pi'_1 \circ b'^{\,2} \circ \iota'_{n+1} \overset{\text{D./R.47}}{=} F_1 \circ \pi'_1 \circ b'^{\,2} \circ \iota'_{n+1}. \end{aligned} \tag{20}$$

For $x \in SA'^{\otimes n+1}$, the element $y := (\pi'_1 \circ b'^{\,2} \circ \iota'_{n+1})(x)$ is a cycle as $b'_1 = 0$. Now $F_1(y) = (F_1 \circ \pi'_1 \circ b'^{\,2} \circ \iota'_{n+1})(x) \overset{(20)}{=} (b_1 \circ h)(x)$ is a boundary. As F_1 is a quasi-monomorphism, y is a boundary. As $b'_1 = 0$, this implies $y = 0$. Hence $\pi'_1 \circ b'^{\,2} \circ \iota'_{n+1} = 0$. Applying Lemma 46(i) via (i), we obtain

$$(b'^{\,2} \circ \iota'_{n+1})(x) = 0. \tag{21}$$

Together with (i), we obtain $b'^{\,2} = 0$, whence $(m'_k)_{k\in[1,n+1]}$ is an A_{n+1}-structure on A' as claimed in (a). Thus, $b_1 \circ h \overset{(20)}{=} F_1 \circ \pi_1 \circ b'^{\,2} \circ \iota'_{n+1} \overset{(21)}{=} 0$ as claimed in (b). $\square$

[5]Note that $(4)[n+1]$ does not depend on m'_{n+1} or f_{n+1}, as $m'_1 = \omega' \circ b'_1 \circ (\omega')^{-1} = 0$.

Lemma 54. *Let $n \in \mathbb{Z}_{\geq 1}$. Let $(A, (m_k)_{k\geq 1})$ be an A_∞-algebra. Let $(A', (m'_k)_{k\in[1,n]})$ be an A_n-algebra. Let $(f_k)_{k\in[1,n]}$ be an A_n-morphism from $(A', (m'_k)_{k\in[1,n]})$ to $(A, (m_k)_{k\in[1,n]})$. Suppose the following hold.*

(i) We have $m'_1 = 0$ and f_1 is a quasi-isomorphism from the complex (A', m'_1) to the complex (A, m_1).

(ii) A' is a projective R-module.

Then there exist f_{n+1} and m'_{n+1} such that $(A', (m'_k)_{k\in[1,n+1]})$ is an A_{n+1}-algebra and $(f_k)_{k\in[1,n+1]}$ is an A_{n+1}-morphism from $(A', (m'_k)_{k\in[1,n+1]})$ to $(A, (m_k)_{k\in[1,n+1]})$.

Note that $(A')^k \cong H^k(A, m_1)$ for $k \in \mathbb{Z}$.

Proof. We have the corresponding triples $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, b)$, $((m'_k)_{k\in[1,n]}, (b'_k)_{k\in[1,n]}, b')$ and $((f_k)_{k\in[1,n]}, (F_k)_{k\in[1,n]}, F)$. For the complexes (SA, b_1) and (SA', b'_1), we will use the usual notation for boundaries, cycles and homology. As $f_1 : (A', m'_1) \to (A, m_1)$ is a quasi-isomorphism, the complex morphism $F_1 : (SA', b'_1) \to (SA, b_1)$ is a quasi-isomorphism. We have $b'_1 = 0$ since $m'_1 = 0$. Note that the term h of Lemma 53 does not depend on b'_{n+1} or F_{n+1}, so h can be unambiguously defined even when m'_{n+1} and F_{n+1} are not yet defined and we have $b_1 \circ h = 0$. Furthermore, h is graded of degree 1 since for $k \in [1, n]$, we have $1 \in |b_k|, 1 \in |b'_k|$ and $0 \in |F_k|$. Motivated by Lemma 53(c), we seek (properly graded) morphisms $b'_{n+1} : SA'^{\otimes n+1} \to SA'$ and $F_{n+1} : SA'^{\otimes n+1} \to SA$ such that the following holds.

$$h \overset{!}{=} b_1 \circ F_{n+1} - F_1 \circ b'_{n+1}$$

The module A' is projective, so SA' and thus also $SA^{\otimes n+1}$ is projective. So Lemma 20 implies that $SA'^{\otimes n+1}$ is graded projective. Since $b_1 \circ h = 0$, we have $h(SA'^{\otimes n+1}) \subseteq \mathrm{Z}^*SA$. Since $b'_1 = 0$, we have $\mathrm{Z}^*SA' = SA'$. We have the following diagram.

$$\begin{tikzcd}
SA'^{\otimes n+1} \arrow[r, "h|^{\mathrm{Z}^*SA}"] \arrow[dr, dotted, "\exists b'_{n+1}"'] & \mathrm{Z}^*SA \arrow[r, "p"] & \mathrm{H}^*SA \\
& SA' = \mathrm{Z}^*SA' \arrow[u, "-F_1|^{\mathrm{Z}^*SA}_{\mathrm{Z}^*SA'}"] \arrow[ur, two heads, "-p\circ(F_1|^{\mathrm{Z}^*SA}_{\mathrm{Z}^*SA'})"'] &
\end{tikzcd}$$

Here, $p : \mathrm{Z}^*SA \to \mathrm{H}^*SA$ is the residue class map. Since $F_1 : (SA', b'_1) \to (SA, b_1)$ is a quasi-isomorphism, the map $-p \circ (F_1|^{\mathrm{Z}^*SA}_{\mathrm{Z}^*SA'})$ is surjective. The map $p \circ (h|^{\mathrm{Z}^*SA})$ is graded of degree 1. The map $p \circ (F_1|^{\mathrm{Z}^*SA}_{\mathrm{Z}^*SA'})$ is graded of degree 0. So since $SA'^{\otimes n+1}$ is graded projective, there is a graded map $b'_{n+1} : SA'^{\otimes n+1} \to SA'$ of degree 1 such that $p\circ(h|^{\mathrm{Z}^*SA}) = -p\circ(F_1|^{\mathrm{Z}^*SA}_{\mathrm{Z}^*SA'})\circ b'_{n+1}|^{\mathrm{Z}^*SA'}$. Hence, $\mathrm{im}((h+F_1\circ b'_{n+1})|^{\mathrm{Z}^*SA}) \subseteq \ker p = \mathrm{B}^*SA$. Since $(h + F_1 \circ b'_{n+1})|^{\mathrm{B}^*SA}$ is graded of degree 1, since $b_1|^{\mathrm{B}^*SA}$ is a graded epimorphism of degree 1 and since $SA'^{\otimes n+1}$ is graded projective, there exists a graded map $F_{n+1} : SA'^{\otimes n+1} \to SA$ of degree 0 such that $(h + F_1 \circ b'_{n+1})^{\mathrm{B}^*SA} = b_1|^{\mathrm{B}^*SA} \circ F_{n+1}$. Hence, we have

$$h = b_1 \circ F_{n+1} - F_1 \circ b'_{n+1} \tag{22}$$

Using b'_{n+1} and F_{n+1}, we extend the corresponding triples $((m'_k)_{k\in[1,n]}, (b'_k)_{k\in[1,n]}, b')$ and $((f_k)_{k\in[1,n]}, (F_k)_{k\in[1,n]}, F)$ to corresponding triples $((m'_k)_{k\in[1,n+1]}, (b'_k)_{k\in[1,n+1]}, \hat{b}')$ and $((f_k)_{k\in[1,n+1]}, (F_k)_{k\in[1,n+1]}, \hat{F})$. By Lemma 53(a), $(A', (m'_k)_{k\in[1,n+1]})$ is an A_{n+1}-algebra. By (22) and Lemma 53(c), we obtain $\hat{F} \circ \hat{b}' = b \circ \hat{F}$. Hence Lemma 50 yields that $(f_k)_{k\in[1,n+1]} : (A', (m'_k)_{k\in[1,n+1]}) \to (A, (m_k)_{k\in[1,n+1]})$ is a morphism of A_{n+1}-algebras. □

Concerning Lemma 54, we may now also construct m'_{m+1} and f_{m+1} directly: We construct (properly graded) maps m'_{m+1} and f_{m+1} such that (5)$[m+1]$ holds. Such m'_{m+1} and f_{m+1} exist by Lemma 54. Then Lemma 52 ensures that all other requirements are met.

Theorem 55 (Kadeishvili's algorithm for the minimality theorem)**.** *Let $(A, (m_k)_{k\geq 1})$ be an A_∞-algebra. Let H^*A be its homology. Suppose H^*A is a projective R-module. Then we construct a minimal model as follows:*

The residue class map $p : \mathrm{Z}^*A \to \mathrm{H}^*A$ is graded of degree 0 and surjective. Since H^*A is by Lemma 20 graded projective, there is a graded map $g : \mathrm{H}^*A \to \mathrm{Z}^*A$ of degree 0 such that $p \circ g = \mathrm{id}_{\mathrm{H}^*A}$. Let $f_1 : \mathrm{H}^*A \to A$ be the composite of $g : \mathrm{H}^*A \to \mathrm{Z}^*A$ with the inclusion map of the inclusion $\mathrm{Z}^*A \subseteq A$. The relation $p \circ g = \mathrm{id}_{\mathrm{H}^*A}$ implies that g (and thus also f_1) maps each homology class in H^*A to a representing cycle in Z^*A.

We set $m'_1 : \mathrm{H}^*A \to \mathrm{H}^*A$, $m'_1 := 0$. We have $f_1 \circ m'_1 \overset{m'_1=0}{=} 0 \overset{\mathrm{im}\, f_1 \subseteq \mathrm{Z}^*A = \ker m_1}{=} m_1 \circ f_1$, so $f_1 : (\mathrm{H}^*A, m'_1) \to (A, m_1)$ is a complex morphism. I.e. it is a morphism of A_1-algebras. Since $f_1 : (\mathrm{H}^*A, m'_1) \to (A, m_1)$ maps each element of H^*A to a representing cycle, it induces the identity in homology. In particular, f_1 is a quasi-isomorphism of A_1-algebras.

We then use Lemma 54 and the construction principle given in Lemma 134 to successively construct an A_∞-structure $(m'_k)_{k\geq 1}$ on H^*A and a quasi-isomorphism $(f_k)_{k\geq 1}$ of A_∞-algebras from $(\mathrm{H}^*A, (m'_k)_{k\geq 1})$ to $(A, (m_k)_{k\geq 1})$.

2.3. More on A_∞-morphisms. The category of A_∞-algebras.

The following may be found e.g. in [11, section 3.4].

Definition/Remark 56. Suppose $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Suppose given graded modules A, A', A''. Suppose given a pre-A_n-morphism $f = (f_k)_{k\in[1,n]}$ from A to A'. Suppose given a pre-A_n-morphism $f' = (f'_k)_{k\in[1,n]}$ from A' to A''. We define the composite $f' \circ f := (g_k)_{k\in[1,n]}$ to be the pre-A_n-morphism from A to A'' given by

$$g_k := \sum_{\substack{1\leq r\leq k \\ i_1+\ldots+i_r=k \\ \text{all } i_s\geq 1}} (-1)^v f'_r \circ (f_{i_1} \otimes \ldots \otimes f_{i_r}) \tag{23}$$

where

$$v := \sum_{1\leq t<s\leq r} (1 - i_s) i_t.$$

(This is the same sign as in (5)[k]). Since f_i and f'_i are graded of degree $1-i$, the term $f'_r \circ (f_{i_1} \otimes \ldots \otimes f_{i_r})$ in (23) is graded of degree $(1-r) + \sum_{j\in[1,r]}(1-i_j) = 1 - \sum_{j\in[1,r]} i_j = 1-k$. Thus g_k is graded of degree $1-k$. Hence, $(g_k)_{k\in[1,n]}$ is a pre-A_n-morphism from A to A''.

This definition is motivated as follows. We have seen that the bar construction relates morphisms of (pre-)A_n-algebras bijectively to certain morphisms of graded coalgebras of degree 0, cf. Definition/Remark 47 and Lemma 50. Composition of coalgebra morphisms is given by composition of the underlying maps. Hence, composition of A_n-morphisms is defined in such a way that it coincides with the composition induced by the bar construction and composition of coalgebra morphisms:

Lemma 57. *Suppose $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Suppose given graded modules A, A', A''. Suppose given a pre-A_n-morphism $f = (f_k)_{k\in[1,n]}$ from A to A' and a pre-A_n-morphism $f' = (f'_k)_{k\in[1,n]}$ from A' to A''. We have the corresponding A_n-morphism triples $((f_k)_{k\in[1,n]}, (F_k)_{k\in[1,n]}, F)$ and $((f'_k)_{k\in[1,n]}, (F'_k)_{k\in[1,n]}, F')$, cf. Definition/Remark 47 .*

Then for $f' \circ f = (g_k)_{k\in[1,n]}$ as given in Definition/Remark 56, we have the corresponding pre-A_n-morphism triple $((g_k)_{k\in[1,n]}, (G_k)_{k\in[1,n]}, G)$, where the G_k are given by

$$G_k := \sum_{\substack{1\leq r\leq k \\ i_1+\ldots+i_r=k \\ \textit{all } i_s\geq 1}} F'_r \circ (F_{i_1} \otimes \ldots \otimes F_{i_r})$$

and where $G := F' \circ F$.

Proof. Concerning the grading of the maps, we have $|F_i| \ni 0$ and $|F'_i| \ni 0$ for $i \in [1,n]$. Hence, we have $|G_k| \ni 0$ for $k \in [1,n]$. Since $|F| \ni 0$ and $|F'| \ni 0$, we have $|F' \circ F| \ni 0$. Since F and F' are coalgebra morphisms, $G = F' \circ F$ is a coalgebra morphism. It remains to show that $G_k = (\omega'')^{-1} \circ g_k \circ \omega^{\otimes k}$ and $\pi''_1 \circ G \circ \iota_k = G_k$ for $k \in [1,n]$.

For $k \in [1,n]$, we have

$$\begin{aligned}
G_k &= \sum_{\substack{1\leq r\leq k \\ i_1+\ldots+i_r=k \\ \text{all } i_s\geq 1}} F'_r \circ (F_{i_1} \otimes \ldots \otimes F_{i_r}) \\
&= \sum_{\substack{1\leq r\leq k \\ i_1+\ldots+i_r=k \\ \text{all } i_s\geq 1}} (\omega'')^{-1} \circ f'_r \circ (\omega')^{\otimes r} \circ (F_{i_1} \otimes \ldots \otimes F_{i_r}) \\
&\overset{*}{=} (\omega'')^{-1} \circ \sum_{\substack{1\leq r\leq k \\ i_1+\ldots+i_r=k \\ \text{all } i_s\geq 1}} f'_r \circ ((\omega' \circ F_{i_1}) \otimes \ldots \otimes (\omega' \circ F_{i_r})) \\
&= (\omega'')^{-1} \circ \sum_{\substack{1\leq r\leq k \\ i_1+\ldots+i_r=k \\ \text{all } i_s\geq 1}} f'_r \circ ((f_{i_1} \circ \omega^{\otimes i_1}) \otimes \ldots \otimes (f_{i_r} \circ \omega^{\otimes i_r}))
\end{aligned}$$

$$\overset{**}{=} (\omega'')^{-1} \circ \sum_{\substack{1 \le r \le k \\ i_1 + \ldots + i_r = k \\ \text{all } i_s \ge 1}} (-1)^v f'_r \circ (f_{i_1} \otimes \ldots \otimes f_{i_r}) \circ \omega^{\otimes k}.$$

Here at $*$, we use (3) and the fact that $|F_i| \ni 0$. At $**$, we use (3) and the fact that $|f_i| \ni 1 - i$ and $|\omega^{\otimes j}| \ni j$.

Hence, we have $G_k = (\omega'')^{-1} \circ g_k \circ \omega^{\otimes k}$ for $k \in [1, n]$.

Let $k \in [1, n]$. By Corollary 45, we have $F(TSA_{\le k}) \subseteq TSA'_{\le k}$. Hence, we have

$$\begin{aligned} \pi''_1 \circ G \circ \iota_k &= \pi''_1 \circ F' \circ F \circ \iota_k = \sum_{1 \le r \le k} \pi''_1 \circ F' \circ \iota'_r \circ \pi'_r \circ F \circ \iota_k \\ &= \sum_{1 \le r \le k} F'_r \circ \pi'_r \circ F \circ \iota_k \overset{(11)}{=} \sum_{\substack{1 \le r \le k \\ i_1 + \ldots + i_r = k \\ \text{all } i_s \ge 1}} F'_r \circ (F_{i_1} \otimes \ldots \otimes F_{i_r}) = G_k \,. \end{aligned}$$

Thus $((g_k)_{k \in [1,n]}, (G_k)_{k \in [1,n]}, G)$ is a corresponding pre-A_n-morphism triple. □

Definition 58. Suppose given graded modules A, A'. Suppose given a graded map $g : A \to A'$ of degree 0. Suppose given $n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. We define the strict pre-A_n-morphism

$$\mathrm{strict}_n(g) = (f_k)_{k \in [1,n]}$$

from A to A' by

$$f_k := \begin{cases} g & \text{if } k = 1 \\ 0 & \text{else.} \end{cases}$$

Example 59. Suppose given $n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. Suppose given graded modules A, A', A''. Suppose given a graded map $g : A \to A'$ of degree 0. Suppose given a pre-A_n-morphism $f = (f_k)_{k \in [1,n]}$ from A' to A'' and a pre-A_n-morphism $f' = (f'_k)_{k \in [1,n]}$ from A'' to A. Then we have

$$f \circ \mathrm{strict}_n(g) = (f_k \circ g^{\otimes k})_{k \in [1,n]}$$

and

$$\mathrm{strict}_n(g) \circ f' = (g \circ f'_k)_{k \in [1,n]} \,.$$

In particular if we replace f by a strict pre-A_n-morphism $\mathrm{strict}_n(g')$ for some graded map $g' : A' \to A''$ of degree 0, we have

$$\mathrm{strict}_n(g') \circ \mathrm{strict}_n(g) = \mathrm{strict}_n(g' \circ g).$$

Lemma 60. *Suppose given $n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$.*

(i) Composition of pre-A_n-morphisms is associative.

(ii) *For a graded module A, the triple* $(\mathrm{strict}_n(\mathrm{id}_A), (F_k)_{k\in[1,n]}, \mathrm{id}_{TSA_{\leq n}})$ *is a pre-*A_n*-morphism triple, where*

$$F_k := \begin{cases} \mathrm{id}_{SA} & \textit{if } k = 1 \\ 0 & \textit{else.} \end{cases}$$

(iii) *For a graded module A, the pre-*A_n*-morphism* $\mathrm{strict}_n(\mathrm{id}_A)$ *is the identity pre-*A_n*-morphism on A.*

Proof. (i): This follows by Lemma 57 from the associativity of the composition of coalgebra morphisms.

(ii): We need to show that $(\mathrm{strict}_n(\mathrm{id}_A), (F_k)_{k\in[1,n]}, \mathrm{id}_{TSA_{\leq n}})$ is a pre-A_n-morphism triple. The pre-A_n-morphism $\mathrm{strict}_n(\mathrm{id}_A)$ is well-defined since $|\,\mathrm{id}_A\,| \ni 0$. We have $|F_k| \ni 0$ for all $k \in [1, n]$. We have $|\,\mathrm{id}_{TSA_{\leq n}}\,| \ni 0$.

For $k \in [2, n]$, we have $\omega^{-1} \circ \mathrm{strict}_n(\mathrm{id}_A)_k \circ \omega^{\otimes k} = \omega^{-1} \circ 0 \circ \omega^{\otimes k} = 0 = F_k$.
Furthermore, we have $\omega^{-1} \circ \mathrm{strict}_n(\mathrm{id}_A)_1 \circ \omega^{\otimes 1} = \omega^{-1} \circ \mathrm{id}_A \circ \omega = \mathrm{id}_{SA} = F_1$.
Hence, we have $\omega^{-1} \circ \mathrm{strict}_n(\mathrm{id}_A)_k \circ \omega^{\otimes k} = F_k$ for $k \in [1, n]$.

Recall that the π_i, ι_i are the projections and inclusions of the direct sum $TSA_{\leq n} = \oplus_{k\in[1,n]}(SA)^{\otimes k}$. Hence for $k \in [1, n]$, we have

$$\begin{aligned} \pi_1 \circ \mathrm{id}_{TSA_{\leq n}} \circ \iota_k &= \begin{cases} \mathrm{id}_{SA} & \text{if } k = 1 \\ 0 & \text{else.} \end{cases} \\ &= F_k. \end{aligned}$$

(iii): This follows from (ii) and Lemma 57. □

Definition/Lemma 61. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We define the composite of A_n-morphisms to be the composite of the underlying pre-A_n-morphisms, cf. Definition/Remark 56.

We have

(i) The composite of two A_n-morphisms is an A_n-morphism.

(ii) Composition of A_n-morphisms is associative.

(iii) Given an A_n-algebra $(A, (m_k)_{k\in[1,n]})$, the pre-A_n-morphism $\mathrm{strict}_n(\mathrm{id}_A)$ is the identity A_n-morphism on A.

Proof. (i): Suppose given A_n-algebras $(A, (m_k)_{k\in[1,n]})$, $(A', (m'_k)_{k\in[1,n]})$ and $(A'', (m''_k)_{k\in[1,n]})$, with corresponding triples $((m_k)_{k\in[1,n]}, *, b)$, $((m'_k)_{k\in[1,n]}, *, b')$ and $((m''_k)_{k\in[1,n]}, *, b'')$. Suppose given A_n-morphisms $f = (f_k)_{k\in[1,n]} : A \to A'$ and $f' = (f'_k)_{k\in[1,n]} : A' \to A''$, with corresponding triples $((f_k)_{k\in[1,n]}, *, F)$ and $((f'_k)_{k\in[1,n]}, *, F')$.

By Lemma 57, the pre-A_n-morphism $f' \circ f$ has the corresponding triple $(f' \circ f, *, F' \circ F)$.

By Lemma 50, we have $b' \circ F = F \circ b$ and $b'' \circ F' = F' \circ b'$.
This implies $b'' \circ (F' \circ F) = F' \circ b' \circ F = (F' \circ F) \circ b$. Hence by Lemma 50, $f' \circ f$ is an A_n-morphism from $(A, (m_k)_{k \in [1,n]})$ to $(A'', (m''_k)_{k \in [1,n]})$.

(ii): This results directly from Lemma 60(i).

(iii): We have the corresponding triples $((m_k)_{k \in [1,n]}, *, b)$ and (by Lemma 60(ii)) $(\mathrm{strict}_n(\mathrm{id}_A), *, \mathrm{id}_{TSA_{\le n}})$. We have $b \circ \mathrm{id}_{TSA_{\le n}} = b = \mathrm{id}_{TSA_{\le n}} \circ b$. Hence, Lemma 50 implies that $\mathrm{strict}_n(\mathrm{id}_A)$ is an A_n-morphism from $(A, (m_k)_{k \in [1,n]})$ to $(A, (m_k)_{k \in [1,n]})$. By Lemma 60(iii), $\mathrm{strict}_n(\mathrm{id}_A)$ is the identity morphism on A with respect to pre-A_n-morphisms. Each A_n-morphism is also a pre-A_n-morphism, cf. Definition 22. So the A_n-morphism $\mathrm{strict}_n(\mathrm{id}_A)$ is also the identity on $(A, (m_k)_{k \in [1,n]})$ with respect to A_n-morphisms. □

Example 62. Suppose given dg-algebras $(A', (m'_k)_{k \ge 1})$ and $(A, (m_k)_{k \ge 1})$, cf. Example 23. Suppose given a graded map $f : A' \to A$ of degree 0 such that

$$\begin{aligned} f \circ m'_1 &= m_1 \circ f && \text{and} \\ f \circ m'_2 &= m_2 \circ (f \otimes f)\,. && \end{aligned} \tag{24}$$

I.e. f is a morphism of dg-algebras.

Then $\mathrm{strict}_\infty(f)$ is an A_∞-morphism from $(A', (m'_k)_{k \ge 1})$ to $(A, (m_k)_{k \ge 1})$.

Proof. Let $(f_k)_{k \ge 1} = \mathrm{strict}_\infty(f)$. Recall $f_1 = f$ and $f_k = 0$ for $k \ge 2$. Recall that for $k \ge 3$, we have $m_k = 0$ and $m'_k = 0$. Thus for $k \ge 3$, eq. (5)[k] holds since all summands in (5)[k] are zero. Since $f_2 = 0$ and $f_1 = f$, the equations (5)[1] and (5)[2] are the equations in (24), cf. also Example 24. □

3. Restriction to a subgroup in terms of minimal models on the group cohomology algebras

3.1. An A_∞-morphism between Hom^*-dg-algebras

Definition/Lemma 63. Suppose given a commutative ground ring R. Suppose given an R-algebra B. Suppose given complexes (P, d_P), (Q, d_Q) over B. Lemma 25 yields the dg-algebras $(A' := \mathrm{Hom}^*_B(P,P), (m'_k)_{k\geq 1})$ and $(A := \mathrm{Hom}^*_B(Q,Q), (m_k)_{k\geq 1})$.

Suppose given complex morphisms $g_1 : P \to Q$ and $g_2 : Q \to P$. Suppose given a homotopy $h \in \mathrm{Hom}^{-1}_B(P,P)$ such that $g_2 \circ g_1 = \mathrm{id}_P + d_{\mathrm{Hom}^*_B(P,P)}(h)$.

Then there is an A_∞-morphism $f_{g_1,g_2,h} = (f_k)_{k\geq 1}$ from A' to A given as follows. For $k \geq 1$ and homogeneous elements $x_i \in (A')^{\mathsf{k}_i}$ for $i \in [1,k]$, we set

$$f_k(x_1 \otimes \ldots \otimes x_k) := (-1)^{\frac{k(k-1)}{2}} (-1)^{\sum_{i\in[1,k]} \mathsf{k}_i(k-i)} g_1 \circ (x_1 \circ h \circ x_2 \circ \ldots \circ h \circ x_k) \circ g_2.$$

Note that e.g. if $P = Q$, $g_1 = g_2 = \mathrm{id}_P$ and $h = 0$, then $f_{g_1,g_2,h}$ is the identity-A_∞-morphism on $\mathrm{Hom}^*_B(P,P)$, cf. Definition/Lemma 61(iii).

Proof. We need to prove that $f_{g_1,g_2,h}$ is actually an A_∞-morphism. It is readily checked that it is a pre-A_∞-morphism. We have the corresponding triples $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, *)$, $((m'_k)_{k\geq 1}, (b'_k)_{k\geq 1}, *)$ and $((f_k)_{k\geq 1}, (F_k)_{k\geq 1}, *)$. By Lemma 50, it suffices to show (14)[k] for $k \geq 1$.

Suppose given homogeneous elements $X_1 \in SA^{\mathsf{k}_{X_1}}, X_2 \in SA^{\mathsf{k}_{X_2}}$. We have

$$\begin{aligned}
b_1(X_1) &= (\omega^{-1} \circ m_1 \circ \omega)(X_1) = \omega^{-1}(m_1(\omega(X_1))) \\
&= \omega^{-1}(d_Q \circ \omega(X_1) - (-1)^{\mathsf{k}_{X_1}+1}\omega(X_1) \circ d_Q) \\
&= \omega^{-1}(d_Q \circ \omega(X_1) + (-1)^{\mathsf{k}_{X_1}}\omega(X_1) \circ d_Q) \\
b_2(X_1 \otimes X_2) &= (\omega^{-1} \circ m_2 \circ \omega^{\otimes 2})(X_1 \otimes X_2) \overset{(1)}{=} (-1)^{\mathsf{k}_{X_1}}\omega^{-1}(m_2(\omega(X_1) \otimes \omega(X_2))) \\
&= (-1)^{\mathsf{k}_{X_1}}\omega^{-1}(\omega(X_1) \circ \omega(X_2)).
\end{aligned}$$

Analogous identities hold for b'_1 and b'_2.

Suppose given $k \geq 1$. Suppose given homogeneous elements $X_i \in (SA')^{\mathsf{k}_i}$ for $i \in [1,k]$. For convenience, let $x_i := \omega'(X_i) \in (A')^{\mathsf{k}_i+1}$ for $i \in [1,k]$. We use the symbol "$\diamond$" as an abbreviation of "$\circ h \circ$", i.e. as a symbol for composition with h inserted in between. We have

$$\begin{aligned}
&(\omega \circ F_k)(X_1 \otimes \ldots \otimes X_k) \\
&= (\omega \circ \omega^{-1} \circ f_k \circ (\omega')^{\otimes k})(X_1 \otimes \ldots \otimes X_k) \overset{(1)}{=} (-1)^{\sum_{i\in[1,k]}(k-i)\mathsf{k}_i} f_k(x_1 \otimes \ldots \otimes x_k) \\
&= \underbrace{(-1)^{\sum_{i\in[1,k]} \mathsf{k}_i(k-i)}(-1)^{\sum_{i\in[1,k]}(\mathsf{k}_i+1)(k-i)}}_{=(-1)^{\sum_{i\in[1,k]}(k-i)} = (-1)^{\frac{k(k-1)}{2}}} (-1)^{\frac{k(k-1)}{2}} g_1 \circ (x_1 \diamond x_2 \diamond \ldots \diamond x_k) \circ g_2
\end{aligned}$$

$$= g_1 \circ (x_1 \diamond \ldots \diamond x_k) \circ g_2.$$

Thus we have

$$\sum_{\substack{k=r+s+t\\ r,t\geq 0, s\geq 1}} (\omega \circ F_{r+1+t} \circ (1^{\otimes r} \otimes b'_s \otimes 1^{\otimes t}))(X_1 \otimes \ldots \otimes X_k)$$
$$= \left(\sum_{r=0}^{k-2} \omega \circ F_{k-1} \circ (1^{\otimes r} \otimes b'_2 \otimes 1^{\otimes k-2-r}) + \sum_{r=0}^{k-1} \omega \circ F_k \circ (1^{\otimes r} \otimes b'_1 \otimes 1^{\otimes k-1-r}) \right)$$
$$(X_1 \otimes \ldots \otimes X_k)$$
$$\overset{(1)}{=} \sum_{r=0}^{k-2} (-1)^{\sum_{i=1}^{r} \mathsf{k}_i} (\omega \circ F_{k-1})(X_1 \otimes \ldots \otimes X_r \otimes b'_2(X_{r+1} \otimes X_{r+2}) \otimes X_{r+3} \otimes \ldots \otimes X_k)$$
$$+ \sum_{r=0}^{k-1} (-1)^{\sum_{i=1}^{r} \mathsf{k}_i} (\omega \circ F_k)(X_1 \otimes \ldots \otimes X_r \otimes b'_1(X_{r+1}) \otimes X_{r+2} \otimes \ldots \otimes X_k)$$
$$= \sum_{r=0}^{k-2} (-1)^{\sum_{i=1}^{r} \mathsf{k}_i} (\omega \circ F_{k-1})(X_1 \otimes \ldots \otimes X_r \otimes$$
$$(-1)^{\mathsf{k}_{r+1}} (\omega')^{-1}(x_{r+1} \circ x_{r+2}) \otimes X_{r+3} \otimes \ldots \otimes X_k)$$
$$+ \sum_{r=0}^{k-1} (-1)^{\sum_{i=1}^{r} \mathsf{k}_i} (\omega \circ F_k)(X_1 \otimes \ldots \otimes X_r \otimes$$
$$(\omega')^{-1}(d_P \circ x_{r+1} + (-1)^{\mathsf{k}_{r+1}} x_{r+1} \circ d_P) \otimes X_{r+2} \otimes \ldots \otimes X_k)$$
$$= \sum_{r=0}^{k-2} (-1)^{\sum_{i=1}^{r+1} \mathsf{k}_i} g_1 \circ (x_1 \diamond \ldots \diamond x_r \diamond (x_{r+1} \circ x_{r+2}) \diamond x_{r+3} \diamond \ldots \diamond x_k) \circ g_2$$
$$+ \sum_{r=0}^{k-1} (-1)^{\sum_{i=1}^{r} \mathsf{k}_i} g_1 \circ (x_1 \diamond \ldots \diamond x_r \diamond$$
$$(d_P \circ x_{r+1} + (-1)^{\mathsf{k}_{r+1}} x_{r+1} \circ d_P) \diamond x_{r+2} \diamond \ldots \diamond x_k) \circ g_2$$
$$= \sum_{r=0}^{k-2} (-1)^{\sum_{i=1}^{r+1} \mathsf{k}_i} g_1 \circ ((x_1 \diamond \ldots \diamond x_{r+1}) \circ \mathrm{id}_P \circ (x_{r+2} \diamond \ldots \diamond x_k)) \circ g_2$$
$$+ \sum_{r=0}^{k-1} (-1)^{\sum_{i=1}^{r} \mathsf{k}_i} g_1 \circ (x_1 \diamond \ldots \diamond x_r \diamond d_P \circ x_{r+1} \diamond \ldots \diamond x_k) \circ g_2$$
$$+ \sum_{r=0}^{k-1} (-1)^{\sum_{i=1}^{r+1} \mathsf{k}_i} g_1 \circ (x_1 \diamond \ldots \diamond x_{r+1} \circ d_P \diamond x_{r+2} \diamond \ldots \diamond x_k) \circ g_2$$
$$\overset{r'=r+1}{=} \sum_{r'=1}^{k-1} (-1)^{\sum_{i=1}^{r'} \mathsf{k}_i} g_1 \circ ((x_1 \diamond \ldots \diamond x_{r'}) \circ \mathrm{id}_P \circ (x_{r'+1} \diamond \ldots \diamond x_k)) \circ g_2$$

$$
\begin{aligned}
&+\sum_{r=0}^{k-1}(-1)^{\sum_{i=1}^{r}\mathsf{k}_i} g_1\circ(x_1\diamond\ldots\diamond x_r \underbrace{\diamond\, d_P}_{\circ h\circ d_P \text{ if } r\neq 0}\circ x_{r+1}\diamond\ldots\diamond x_k)\circ g_2\\
&+\sum_{r'=1}^{k}(-1)^{\sum_{i=1}^{r'}\mathsf{k}_i} g_1\circ(x_1\diamond\ldots\diamond x_{r'}\circ \underbrace{d_P\diamond}_{d_P\circ h\circ \text{ if } r'\neq k} x_{r'+1}\diamond\ldots\diamond x_k)\circ g_2\\
\overset{*}{=}&\sum_{r=1}^{k-1}(-1)^{\sum_{i=1}^{r}\mathsf{k}_i} g_1\circ(x_1\diamond\ldots\diamond x_r\circ g_2\circ g_1\circ x_{r+1}\diamond\ldots\diamond x_k)\circ g_2\\
&+g_1\circ d_P\circ x_1\diamond\ldots\diamond x_k\circ g_2+(-1)^{\sum_{i=1}^{k}\mathsf{k}_i} g_1\circ x_1\diamond\ldots\diamond x_k\circ d_P\circ g_2\\
\overset{**}{=}&\sum_{r=1}^{k-1}(-1)^{\sum_{i=1}^{r}\mathsf{k}_i}(\omega\circ F_r)(X_1\otimes\ldots\otimes X_r)\circ(\omega\circ F_{k-r})(X_{r+1}\otimes\ldots\otimes X_k)\\
&+d_Q\circ g_1\circ x_1\diamond\ldots\diamond x_k\circ g_2+(-1)^{\sum_{i=1}^{k}\mathsf{k}_i} g_1\circ x_1\diamond\ldots\diamond x_k\circ g_2\circ d_Q\\
=&\sum_{r=1}^{k-1}(-1)^{\sum_{i=1}^{r}\mathsf{k}_i}(\omega\circ F_r)(X_1\otimes\ldots\otimes X_r)\circ(\omega\circ F_{k-r})(X_{r+1}\otimes\ldots\otimes X_k)\\
&+d_Q\circ(\omega\circ F_k)(X_1\otimes\ldots\otimes X_k)+(-1)^{\sum_{i=1}^{k}\mathsf{k}_i}(\omega\circ F_k)(X_1\otimes\ldots\otimes X_k)\circ d_Q\\
=&\sum_{r=1}^{k-1}(\omega\circ b_2)(F_r(X_1\otimes\ldots\otimes X_r)\otimes F_{k-r}(X_{r+1}\otimes\ldots\otimes X_k))\\
&+(\omega\circ b_1)(F_k(X_1\otimes\ldots\otimes X_k))\\
\overset{(1)}{=}&\left(\omega\circ b_1\circ F_k+\sum_{r=1}^{k-1}\omega\circ b_2\circ(F_r\otimes F_{k-r})\right)(X_1\otimes\ldots\otimes X_k)\\
=&\sum_{\substack{1\le r\le k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\ge 1}}(\omega\circ b_r\circ(F_{i_1}\otimes\ldots\otimes F_{i_r}))(X_1\otimes\ldots\otimes X_k).
\end{aligned}
$$

At the step marked by "$*$", we use $g_2\circ g_1=\mathrm{id}_P+d_{\mathrm{Hom}_B^*(P,P)}(h)=\mathrm{id}_P+d_P\circ h+h\circ d_P$. At "$**$", we use the fact that $g_1: P\to Q, g_2: Q\to P$ are complex morphisms, hence $g_1\circ d_P=d_Q\circ g_1$ and $d_P\circ g_2=g_2\circ d_Q$.

Hence, (14)[k] holds for $k\ge 1$, so $f_{g_1,g_2,h}$ is by Lemma 50 a morphism of A_∞-algebras. $\square$

We have the following well-known application of the comparison theorem.

Lemma 64. *Suppose given a commutative ring R. Suppose given an R-algebra B. Suppose given a B-module M. Suppose given projective resolutions P,Q of M.*

There exist B-linear quasi-isomorphisms of complexes $g_1: P\to Q$ and $g_2: Q\to P$ together with homotopies $h_P\in\mathrm{Hom}_B^{-1}(P,P)$, $h_Q\in\mathrm{Hom}_B^{-1}(Q,Q)$ such that

$$
\begin{aligned}
g_2\circ g_1&=\mathrm{id}_P+d_{\mathrm{Hom}_B^*(P,P)}(h_P)\\
g_1\circ g_2&=\mathrm{id}_Q+d_{\mathrm{Hom}_B^*(Q,Q)}(h_Q).
\end{aligned}
$$

Proof. Since P and Q are projective resolutions of M, we have $\mathrm{H}_i P = \mathrm{H}_i Q = 0$ for $i \in \mathbb{Z} \setminus \{0\}$ and $\mathrm{H}_0 P = \mathrm{H}_0 Q = M$.

By the comparison theorem (cf. e.g. [26, Comparison Theorem 2.2.6]), we obtain B-linear morphisms $g_1 : P \to Q$, $g_2 : Q \to P$ such that $\mathrm{H}_0 g_1 = \mathrm{id}_M$ and $\mathrm{H}_0 g_2 = \mathrm{id}_M$. In particular, g_1 and g_2 induce the identity on $\mathrm{H}_* P = \mathrm{H}_* Q = M$. Thus g_1 and g_2 are quasi-isomorphisms.

Since $\mathrm{H}_0(g_2 \circ g_1) = \mathrm{id}_M = \mathrm{H}_0 \,\mathrm{id}_P$, the comparison theorem provides a homotopy $h_P \in \mathrm{Hom}^{-1}(P,P)$ such that $g_2 \circ g_1 = \mathrm{id}_P + d_{\mathrm{Hom}^*_{\mathbb{F}H}(P,P)}(h)$. Analogously, we obtain the homotopy h_Q. □

3.2. Restriction to subgroups

We use term 'canonical A_∞-structure on a group cohomology algebra' as in section 0.1.7. There is well-known theory of how restriction to subgroups interacts with group cohomology, cf. e.g. [1, p. 73]. As outlined in section 0.1.7, there is evidence that suggests that restriction to subgroups behaves similarly in the context of canonical A_∞-structures on group cohomology. This is intriguing since obtaining canonical A_∞-structures on group cohomology algebras directly is an arduous process, so it would be extremely helpful if we were somehow able to infer canonical A_∞-structures on group cohomology from canonical A_∞-structures on the group cohomology algebras of subgroups of the group we are examining. Below, we explain some of the observations made in section 0.1.7, which can be considered a first attempt in this direction.

Remark 65 (Restriction)**.** Suppose given a field $\mathbb{F}$. Suppose given a finite group G. Suppose $H \leqslant G$ is a subgroup of G. Suppose P is a projective resolution of the trivial $\mathbb{F}G$-module $\mathbb{F}$ over $\mathbb{F}G$, with augmentation $\varepsilon : P_0 \to \mathbb{F}$.

Since projective $\mathbb{F}G$-modules are also projective over $\mathbb{F}H$ and since $\mathbb{F}G$-linear maps are also $\mathbb{F}H$-linear, the complex P is also a projective resolution of $\mathbb{F}$ over $\mathbb{F}H$.

Since $\mathbb{F}G$-linear maps are also $\mathbb{F}H$-linear, we obtain the canonical inclusion $\mathrm{res}_{G,H} : \mathrm{Hom}^*_{\mathbb{F}G}(P,P) \to \mathrm{Hom}^*_{\mathbb{F}H}(P,P)$. The map $\mathrm{res}_{G,H}$ is a morphism of dg-algebras. Thus it induces a morphism on the Ext-algebras $\mathrm{H}^* \mathrm{res}_{G,H} : \mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}) \to \mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$.

If $[G : H]$ is invertible in $\mathbb{F}$, then $\mathrm{H}^* \mathrm{res}_{G,H}$ is injective, cf. e.g. [1, Corollary 3.6.18].

Proposition 66. *Suppose given a field $\mathbb{F}$. Let $\mathbb{F}$ be the ground ring. Suppose given a finite group G and a subgroup $H \leqslant G$ such that $[G : H]$ is invertible in $\mathbb{F}$. Suppose given a projective resolution P of the trivial $\mathbb{F}G$-module $\mathbb{F}$ over $\mathbb{F}G$.*

Let $(m_n^{(G)})_{n\geq 1}$ resp. $(m_n)_{n\geq 1}$ be the dg-algebra structure given by Lemma 25 on $\mathrm{Hom}^*_{\mathbb{F}G}(P,P)$ resp. $\mathrm{Hom}^*_{\mathbb{F}H}(P,P)$.

Suppose given a minimal A_∞-structure $(m_n'^{(G)})_{n\geq 1}$ on $\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$ and a quasi-isomorphism of A_∞-algebras $(f_n^{(G)})_{n\geq 1} : (\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}), (m_n'^{(G)})_{n\geq 1}) \to (\mathrm{Hom}^*_{\mathbb{F}G}(P,P), (m_n^{(G)})_{n\geq 1})$ such that $f_1^{(G)}$ induces the identity in homology.

Then there is a minimal A_∞*-structure* $(m'_n)_{n\geq 1}$ *on* $\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$ *and a quasi-isomorphism of* A_∞*-algebras* $(f_n)_{n\geq 1} : (\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}),(m'_n)_{n\geq 1}) \to (\mathrm{Hom}^*_{\mathbb{F}H}(P,P),(m_n)_{n\geq 1})$ *such that*

- f_1 *induces the identity in homology,*
- $\mathrm{strict}_\infty(\mathrm{H}^*\mathrm{res}_{G,H}) : (\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}),({m'_n}^{(G)})_{n\geq 1}) \to (\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}),(m'_n)_{n\geq 1})$ *is an* A_∞*-morphism and*
- *the following diagram commutes.*

$$\begin{tikzcd}
(\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}),({m'_n}^{(G)})_{n\geq 1}) \arrow[r, "(f_n^{(G)})_{n\geq 1}"] \arrow[d, "\mathrm{strict}_\infty(\mathrm{H}^*\mathrm{res}_{G,H})"'] & (\mathrm{Hom}^*_{\mathbb{F}G}(P,P),(m_n^{(G)})_{n\geq 1}) \arrow[d, "\mathrm{strict}_\infty(\mathrm{res}_{G,H})"] \\
(\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}),(m'_n)_{n\geq 1}) \arrow[r, "(f_n)_{n\geq 1}"] & (\mathrm{Hom}^*_{\mathbb{F}H}(P,P),(m_n)_{n\geq 1})
\end{tikzcd} \tag{25}$$

Note that $\mathrm{strict}_\infty(\mathrm{res}_{G,H})$ *is an* A_∞*-morphism by Example 62.*

Proof. Set $B := \mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$, $A := \mathrm{Hom}^*_{\mathbb{F}H}(P,P)$. Since the inclusion $\mathrm{res}_{G,H}$ is graded and injective, we may identify $\mathrm{Hom}^*_{\mathbb{F}G}(P,P) =: A^{(G)}$ with the image of $\mathrm{res}_{G,H}$, which is a graded subalgebra of A. Since $\mathrm{H}^*\mathrm{res}_{G,H}$ is graded and by [1, Corollary 3.6.18] injective, we may identify $\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}) =: B^{(G)}$ with the image of $\mathrm{H}^*\mathrm{res}_{G,H}$, which is a graded direct submodule of B. I.e. $B = B^{(G)} \oplus B'$ for some graded submodule B' of B.

To construct $(m_n)_{n\geq 1}$ and $(f_n)_{n\geq 1}$, we modify Kadeishvili's algorithm suitably:

By Definition 58 and Definition 22(ii), the pre-A_∞-morphism $\mathrm{strict}_\infty(\mathrm{H}^*\mathrm{res}_{G,H})$ is an A_∞-morphism iff $m'_n|_{(B^{(G)})^{\otimes n}} = {m'_n}^{(G)}$ for $n \geq 1$. Similarly, Example 59 implies that the diagram (25) commutes iff $f_n|_{(B^{(G)})^{\otimes n}} = f_n^{(G)}$ for $n \geq 1$.

We set $m'_1 : B \to B$, $m'_1 = 0$, which is a graded map of degree 1. We construct $f_1 : B \to A$ as follows. There is a graded map $g : B' \to A$ of degree 0 such that each element of $B' \subseteq B = \mathrm{H}^*A$ is mapped to a representing cycle. Note that $\mathrm{H}^* f_1^{(G)} = \mathrm{id}_{\mathrm{H}^*A^{(G)}} = \mathrm{id}_{B^{(G)}}$, so $f_1^{(G)}$ maps elements of $B^{(G)}$ to representing cycles. We set $f_1|_{B'} := g$ and $f_1|_{B^{(G)}} := f_1^{(G)}$. Thus f_1 maps each element of B to a representing cycle. Hence, $f_1 : (B, m'_1) \to (A, m_1)$ is a complex morphism and $\mathrm{H}^* f_1 = \mathrm{id}_{\mathrm{H}^*A}$. In particular, $f_1 : (B, m'_1) \to (A, m_1)$ is a quasi-isomorphism of complexes.

We construct the f_n and m'_n for $n \geq 2$ successively using the construction principle given in Lemma 134.

So suppose $n \in [1,\infty]$. Suppose there are f_k and m'_k for $k \in [1,n]$ such that $(f_k)_{k\in[1,n]} : (B,(m'_k)_{k\in[1,n]}) \to (A,(m_k)_{k\geq 1})$ is a quasi-isomorphism of A_n-algebras and such that $m_k|_{(B^{(G)})^{\otimes k}} = m_k^{(G)}$ and $f_k|_{(B^{(G)})^{\otimes k}} = f_k^{(G)}$ for $k \in [1,n]$.

We have the corresponding triples $((m_k)_{k\geq 1},(b_k)_{k\geq 1},b)$, $((m'_k)_{k\in[1,n]},(b'_k)_{k\in[1,n]},b')$, $((f_k)_{k\in[1,n]},(F_k)_{k\in[1,n]},F)$, $((m_k^{(G)})_{k\geq 1},(b_k^{(G)})_{k\geq 1},b^{(G)})$, $(({m'_k}^{(G)})_{k\geq 1},({b'_k}^{(G)})_{k\geq 1},{b'}^{(G)})$ and $((f_k^{(G)})_{k\geq 1},(F_k^{(G)})_{k\geq 1},F^{(G)})$. Note that $b_k|_{(SB^{(G)})^{\otimes k}} = b_k^{(G)}$ and $F_k|_{(SB^{(G)})^{\otimes k}} = F_k^{(G)}$ for $k \in [1,n]$. Recall Theorem 48 and Lemma 50.

Consider Lemma 54. There, we let B resp. A take the role of A' resp. A. The proof of Lemma 54 provides graded maps ${}^0b'_{n+1} : (SB)^{\otimes n+1} \to SB$ resp. ${}^0F_{n+1} : (SB)^{\otimes n+1} \to SA$ of degree 1 resp. 0 such that

$$h = b_1 \circ {}^0F_{n+1} - F_1 \circ {}^0b'_{n+1},$$

where the term h is given in Lemma 53.

Note that application of Lemma 53(c) to the quasi-isomorphism of A_{n+1}-algebras $(f_k^{(G)})_{k\in[1,n+1]} : (\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}), ({m'_k}^{(G)})_{k\in[1,n+1]}) \to (\mathrm{Hom}^*_{\mathbb{F}G}(P,P), (m_k^{(G)})_{k\geq 1})$ yields

$$h|^{SA^{(G)}}_{(SB^{(G)})^{\otimes n+1}} = b_1^{(G)} \circ F_{n+1}^{(G)} - F_1^{(G)} \circ {b'_{n+1}}^{(G)} = b_1|^{SA^{(G)}}_{SA^{(G)}} \circ F_{n+1}^{(G)} - F_1|^{SA^{(G)}}_{SB^{(G)}} \circ {b'_{n+1}}^{(G)}.$$

Since $B^{(G)}$ is a graded direct summand of B, the module $(SB^{(G)})^{\otimes n+1}$ is a graded direct summand of $(SB)^{\otimes n+1}$. Hence, we have $(SB)^{\otimes n+1} = (SB^{(G)})^{\otimes n+1} \oplus B_{n+1}$ for some graded submodule B_{n+1} of $(SB)^{\otimes n+1}$.

We define the graded maps $b'_{n+1} : (SB)^{\otimes n+1} \to SB$ resp. $F_{n+1} : (SB)^{\otimes n+1} \to SA$ of degree 1 resp. 0 by setting

$$\begin{aligned} b'_{n+1}|_{(SB^{(G)})^{\otimes n+1}} &:= {b'_{n+1}}^{(G)} & \qquad b'_{n+1}|_{B_{n+1}} &:= {}^0b'_{n+1}|_{B_{n+1}} \\ F_{n+1}|_{(SB^{(G)})^{\otimes n+1}} &:= {F_{n+1}}^{(G)} & \qquad F_{n+1}|_{B_{n+1}} &:= {}^0F_{n+1}|_{B_{n+1}}. \end{aligned}$$

Now $h = b_1 \circ F_{n+1} - F_1 \circ b'_{n+1}$ holds since it holds on the summands of the direct sum of $(SB)^{\otimes n+1} = (SB^{(G)})^{\otimes n+1} \oplus B_{n+1}$.

Using b'_{n+1} and F_{n+1}, we extend the corresponding triples $((m'_k)_{k\in[1,n]}, (b'_k)_{k\in[1,n]}, b')$ and $((f_k)_{k\in[1,n]}, (F_k)_{k\in[1,n]}, F)$ to corresponding triples $((m'_k)_{k\in[1,n+1]}, (b'_k)_{k\in[1,n+1]}, \hat{b}')$ and $((f_k)_{k\in[1,n+1]}, (F_k)_{k\in[1,n+1]}, \hat{F})$.

By Lemma 53(a), the tuple $(B, (m'_k)_{k\in[1,n+1]})$ is an A_{n+1}-algebra. Lemma 53(c) and Lemma 50 imply (5)$[n+1]$. Hence, $(f_k)_{k\in[1,n+1]} : (B, (m'_k)_{k\in[1,n+1]}) \to (A, (m_k)_{k\in[1,n+1]})$ is a quasi-isomorphism of A_{n+1}-algebras. Since $b'_{n+1}|_{(SB^{(G)})^{\otimes n+1}} = {b'_{n+1}}^{(G)}$ and $F_{n+1}|_{(SB^{(G)})^{\otimes n+1}} = {F_{n+1}}^{(G)}$, we have $m'_{n+1}|_{(B^{(G)})^{\otimes n+1}} = {m'_{n+1}}^{(G)}$ and $f_{n+1}|_{(B^{(G)})^{\otimes n+1}} = {f_{n+1}}^{(G)}$. This completes the successive step. $\square$

Using a result by Keller and Prouté (cf. [11, Theorem in section 3.7], see also [18, Théorème 4.27] and [21, Corollary 1.14]), we obtain the following proposition. In contrast to Proposition 66, we may choose the canonical A_∞-structure on $\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F})$ and we have no restriction on the index $[G : H]$, but we cannot control the morphism between the minimal models as well and we have commutativity only up to homotopy in the sense of [11, section 3.7].

Proposition 67. *Suppose given a field $\mathbb{F}$. Let $\mathbb{F}$ be the ground ring. Suppose given finite groups G, H with $H \leqslant G$. Suppose given a projective resolution P of the trivial $\mathbb{F}G$-module $\mathbb{F}$ and a projective resolution Q of the trivial $\mathbb{F}H$-module $\mathbb{F}$.*

Let $(m_n^{(G)})_{n\geq 1}$ *resp.* $(m_n^{(H)})_{n\geq 1}$ *be the dg-algebra structure on* $\mathrm{Hom}^*_{\mathbb{F}G}(P,P)$ *resp.* $\mathrm{Hom}^*_{\mathbb{F}H}(Q,Q)$*, cf. Lemma 25.*

Suppose given minimal A_∞*-algebras*

$$\begin{aligned} M^{(G)} &:= (\mathrm{Ext}^*_{\mathbb{F}G}(\mathbb{F},\mathbb{F}),({m'_k}^{(G)})_{k\geq 1}),\\ M^{(H)} &:= (\mathrm{Ext}^*_{\mathbb{F}H}(\mathbb{F},\mathbb{F}),({m'_k}^{(H)})_{k\geq 1}) \end{aligned}$$

together with quasi-isomorphisms of A_∞*-algebras*

$$\begin{aligned} f^{(G)} &= (f_k^{(G)})_{k\geq 1} : M^{(G)} \to (\mathrm{Hom}^*_{\mathbb{F}G}(P,P),(m_k^{(G)})_{k\geq 1})\\ f^{(H)} &= (f_k^{(H)})_{k\geq 1} : M^{(H)} \to (\mathrm{Hom}^*_{\mathbb{F}H}(Q,Q),(m_k^{(H)})_{k\geq 1}). \end{aligned}$$

Suppose given $\mathbb{F}H$*-linear complex morphisms* $g_1 : P \to Q$ *and* $g_2 : Q \to P$ *together with a homotopy* $h \in \mathrm{Hom}^{-1}_{\mathbb{F}H}(P,P)$ *such that* $g_2 \circ g_1 = \mathrm{id}_P + d_{\mathrm{Hom}^*_{\mathbb{F}H}(P,P)}(h)$ *(cf. e.g. Lemma 64).*

From g_1, g_2 *and* h*, we obtain via Definition/Lemma 63 the* A_∞*-morphism* $f_{g_1,g_2,h}$ *from* $\mathrm{Hom}^*_{\mathbb{F}H}(P,P)$ *to* $\mathrm{Hom}^*_{\mathbb{F}H}(Q,Q)$*.*

Then there exists an A_∞*-morphism* $f^{\min}$ *from* $M^{(G)}$ *to* $M^{(H)}$ *such that the following diagram commutes up to homotopy in the sense of [11, section 3.7], cf. also [18, Définition 4.1].*

$$\begin{tikzcd} M^{(G)} \arrow[r, "f^{(G)}"] \arrow[d, "f^{\min}"] & (\mathrm{Hom}^*_{\mathbb{F}G}(P,P),(m_k^{(G)})_{k\geq 1}) \arrow[d, "f_{g_1,g_2,h}\circ \mathrm{strict}_\infty(\mathrm{res}_{G,H})"] \\ M^{(H)} \arrow[r, "f^{(H)}"] & (\mathrm{Hom}^*_{\mathbb{F}H}(Q,Q),(m_k^{(H)})_{k\geq 1}) \end{tikzcd} \tag{26}$$

Proof. Using a result of Keller and Prouté (cf. [11, Theorem in section 3.7], see also [18, Théorème 4.27] and [21, Corollary 1.14]), we obtain a quasi-isomorphism of A_∞-algebras

$$\tilde{f}^{(H)} : (\mathrm{Hom}^*_{\mathbb{F}H}(Q,Q),(m_k^{(H)})_{k\geq 1}) \to M^{(H)}$$

which inverts $f^{(H)}$ up to homotopy in the sense of [11, section 3.7]. So by setting $f^{\min} := \tilde{f}^{(H)} \circ \mathrm{strict}_\infty(\mathrm{res}_{G,H}) \circ f^{(G)}$, we obtain the diagram (26), which is commutative up to homotopy in the sense of [11, section 3.7]. □

Question 68. Is it possible to use or improve these results in such a way that we can obtain canonical structures on the group cohomology algebra of a group G from canonical structures on the group cohomology algebras of subgroups of G ?

4. Extended Kadeishvili minimal method

Suppose given a commutative ground ring R.

4.1. A counterexample to the existence of Kadeishvili-styled minimal models over arbitrary rings

In this subsection, suppose that R be an integral domain such that there is an element $n \in R$, $n \neq 0$ which is not a multiplicative unit (this facilitates the construction of counterexamples).

Let the R-module $A := R^{2\times 2}$ be graded by setting $A^j := \{0\}$ for $j \in \mathbb{Z} \setminus \{-1, 0, 1\}$, $A^{-1} := \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \middle| c \in R \right\}$, $A^0 := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a, d \in R \right\}$ and $A^1 := \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in R \right\}$.

On A, there is a dg-algebra structure given by

$$m_1 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) := n \cdot \begin{pmatrix} c & d-a \\ 0 & c \end{pmatrix}$$

$$m_2 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}.$$

Remark 71 will show that A is in fact the dg-algebra $\mathrm{Hom}^*(C, C)$, where C is the complex $C := (\ldots \to 0 \to R \xrightarrow{x \mapsto nx} R \to 0 \to \ldots)$. But for the sake of self-containedness of this example, we manually check that A is a dg-algebra. The maps m_1 resp. m_2 are graded of degree $|m_1| = 1$ resp. $|m_2| = 0$. We have $m_1^2 = 0$. Since m_2 is matrix multiplication, it is associative. By Example 23, it remains to check the Leibniz rule (4)[2]:

$$\begin{aligned}
&(m_1 \circ m_2) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = m_1 \left(\begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix} \right) \\
&= n \cdot \begin{pmatrix} ca'+dc' & cb'+dd'-aa'-bc' \\ 0 & ca'+dc' \end{pmatrix} \\
&= n \cdot m_2 \left(\begin{pmatrix} c & d-a \\ 0 & c \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} + \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \otimes \begin{pmatrix} c' & d'-a' \\ 0 & c' \end{pmatrix} \right) \\
&\overset{(1)}{=} m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right)
\end{aligned}$$

Hence, A is a dg-algebra.

As n is not a zero divisor, the set of cycles in A is $\mathrm{Z}^* A = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \middle| x, y \in R \right\}$. The set of boundaries is $\mathrm{B}^* A = \left\{ n \cdot \begin{pmatrix} z & w \\ 0 & z \end{pmatrix} \middle| z, w \in R \right\}$. Thus we have $\mathrm{H}^1 A \simeq \mathrm{H}^0 A \simeq (R/nR)$ and $\mathrm{H}^{-1} A = \{0\}$, hence $\mathrm{H}^* A \simeq (R/nR) \oplus (R/nR)$.

For the following proposition, we recall from Examples 23 and 24 that the notion of an A_1-algebra is the same as the notion of a complex. Furthermore minimality of an A_1-algebra simply means that its differential m_1 is zero.

Proposition 69. *(a) There is no minimal* A_1*-structure on* H^*A *such that there is a quasi-isomorphism of* A_1*-algebras from* H^*A *to* A.

(b) There is no minimal A_2*-structure on* H^*A *such that there is a quasi-isomorphism of* A_2*-algebras from* A *to* H^*A.

This immediately yields

Corollary 70. *(a) There is no minimal* A_∞*-structure on* H^*A *such that there is a quasi-isomorphism of* A_∞*-algebras from* H^*A *to* A.

(b) There is no minimal A_∞*-structure on* H^*A *such that there is a quasi-isomorphism of* A_∞*-algebras from* A *to* H^*A.

Proof of Proposition 69. By the choice of n, the module R/nR is a non-zero torsion module that is annihilated by $n \neq 0$. The same holds for H^*A.

As A is torsion-free, there is no non-zero R-linear map $f_1 : \mathrm{H}^*A \to A$. In particular, there is no quasi-isomorphism of complexes $f_1 : (\mathrm{H}^*A, 0) \to (A, m_1)$, which proves (a).

We prove (b) by contradiction. Assume that there is a minimal A_2-algebra structure $(m_i)_{i\in[1,2]}$ on H^*A and a quasi-isomorphism of A_2-algebras $(f_k)_{k\in[1,2]} : (A, (m_k)_{k\in[1,2]}) \to (\mathrm{H}^*A, (m'_k)_{k\in[1,2]})$. Hence equation (5)[2] must hold, that is

$$f_1 \circ m_2 - f_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) = m'_1 \circ f_2 + m'_2 \circ (f_1 \otimes f_1).$$

We have $m_1 = n\cdot g$, where $g : A \to A$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} c & d-a \\ 0 & c \end{pmatrix}$. Thus $f_2\circ(m_1\otimes 1+1\otimes m_1) = n \cdot f_2 \circ (g \otimes 1 + 1 \otimes g)$. As n annihilates H^*A, we have $f_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) = 0$ for any R-linear map $f_2 : A^{\otimes 2} \to \mathrm{H}^*A$. Furthermore $m'_1 = 0$, so equation (5)[2] reduces to

$$f_1 \circ m_2 = m'_2 \circ (f_1 \otimes f_1). \tag{27}$$

As $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: \zeta$ resp. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =: \eta$ generate Z^0A resp. Z^1A and as $f_1 : (A, m_1) \to (\mathrm{H}^*A, 0)$ is a quasi-isomorphism, $f_1(\zeta)$ generates H^0A and $f_1(\eta)$ generates H^1A. So there is an $r \in R$ with $\mathrm{H}^0A \ni -f_1\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = rf_1\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$, whence $f_1\left(\begin{pmatrix} 1+r & 0 \\ 0 & r \end{pmatrix}\right) = 0$.

For $\omega := \begin{pmatrix} 1+r & 0 \\ 0 & r \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1+r & 0 \\ 0 & r \end{pmatrix} \in A^{\otimes 2}$, we thus have

$m'_2 \circ (f_1 \otimes f_1)(\omega)$

$$\overset{(1)}{=} m_2'\Bigg(\underbrace{f_1\left(\begin{pmatrix}1+r & 0\\ 0 & r\end{pmatrix}\right)}_{=0}\otimes f_1\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right) - f_1\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\otimes \underbrace{f_1\left(\begin{pmatrix}1+r & 0\\ 0 & r\end{pmatrix}\right)}_{=0}\Bigg) = 0.$$

But $(f_1 \circ m_2)(\omega) = f_1\left(\begin{pmatrix}0 & 1+r\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & r\\ 0 & 0\end{pmatrix}\right) = f_1(\eta) \neq 0$ since $f_1(\eta)$ generates $\mathrm{H}^1 A \simeq R/nR \neq \{0\}$. Thus we have a contradiction to (27). □

Remark 71. For the complex

$$\begin{aligned} C &:= (\ldots \to 0 \to C_1 \longrightarrow C_0 \to 0 \to \ldots) \\ &:= (\ldots \to 0 \to R \xrightarrow{x \mapsto nx} R \to 0 \to \ldots), \end{aligned}$$

we will examine $\mathrm{Hom}^*(C,C) =: \tilde{A}$ as a dg-algebra, cf. Lemma 25. We will show that $\tilde{A} \simeq A$.

Let β_0 resp. β_1 be generators of C_0 resp. C_1. We have the graded module $\tilde{C} := \oplus_{i\in\mathbb{Z}} C_i$ with ordered basis $B := (\beta_0, \beta_1)$. For $i \in \mathbb{Z}$, we may identify the elements of $\tilde{A}^i = \mathrm{Hom}^i(C,C)$ with the graded maps $\tilde{C} \to \tilde{C}$ of degree $-i$, cf. Lemma 25. This way, $\tilde{A} = \oplus_{i\in\mathbb{Z}} \tilde{A}^i$ is identified with the endomorphism ring $\mathrm{Hom}_R(\tilde{C},\tilde{C})$. Hence, we may use the ordered basis B of $\tilde{C}$ to identify the R-algebra $\tilde{A}$ with the R-matrix algebra $R^{2\times 2}$. I.e. $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in R^{2\times 2}$ is the morphism that maps β_0 to $a\beta_0 + c\beta_1$ and β_1 to $b\beta_0 + d\beta_1$.

Elements of $\tilde{A}^{-1}$ are morphisms that send β_0 to some multiple of β_1 and send β_1 to zero. Hence, $\tilde{A}^{-1} = \left\{\begin{pmatrix}0 & 0\\ c & 0\end{pmatrix} \middle| c \in R\right\}$. Similarly, we have $\tilde{A}^0 = \left\{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} \middle| a, d \in R\right\}$ and $\tilde{A}^1 = \left\{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix} \middle| b \in R\right\}$.

The map m_2 is defined by composition, so in the matrix picture, m_2 is given by matrix multiplication.

The differential $d \in \mathrm{Hom}^1(C,C)$ of C maps β_0 to 0 and β_1 to $n\beta_0$, so $d = \begin{pmatrix}0 & n\\ 0 & 0\end{pmatrix}$. For homogeneous elements $g \in \tilde{A}$ of degree k_g, we have $m_1(g) = d \circ g - (-1)^{\mathsf{k}_g} g \circ d$, cf. Lemma 25. Hence, for $a, b, c, d \in R$, we have

$$\begin{aligned} m_1\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right) &= m_1\Bigg(\underbrace{\begin{pmatrix}0 & 0\\ c & 0\end{pmatrix}}_{\in A^{-1}} + \underbrace{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}}_{\in A^0} + \underbrace{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}}_{\in A^1}\Bigg) = d \circ \begin{pmatrix}a & b\\ c & d\end{pmatrix} - \begin{pmatrix}a & -b\\ -c & d\end{pmatrix} \circ d \\ &= \begin{pmatrix}0 & n\\ 0 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} - \begin{pmatrix}a & -b\\ -c & d\end{pmatrix}\begin{pmatrix}0 & n\\ 0 & 0\end{pmatrix} = \begin{pmatrix}nc & nd\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & na\\ 0 & -nc\end{pmatrix} \\ &= n\begin{pmatrix}c & d-a\\ 0 & c\end{pmatrix}. \end{aligned}$$

We conclude that the dg-algebra $\tilde{A} := \mathrm{Hom}^*(C,C)$ is isomorphic to the dg-algebra A given above.

4.2. Posets

In this subsection, we review well-known basic facts concerning partially ordered sets (posets), cf. e.g. [5]. In Lemma 75, we obtain information about the partially ordered set $(\mathbb{Z}^n_{\geq n}, \leq_n)$, which we will use in the sequel.

Definition 72 (Posets)**.** We use the abbreviation *poset* for *partially ordered set.* Let $(X, \leq)$ be a poset. Abusing notation, we often write X instead of $(X, \leq)$.

The poset $(X, \leq)$ is called *artinian* if every descending chain $x_0 \geq x_1 \geq x_2 \geq x_3 \geq \ldots$ in X becomes stationary, i.e. there is a $N \in \mathbb{Z}_{\geq 0}$ such that $x_k = x_N$ for all $k \geq N$.

A set $D \subseteq X$ is called *discrete* if for any $x, y \in D$ such that $x \neq y$, we have $x \nleq y$ and $x \ngeq y$. In that situation, the pair (x, y) is called *incomparable.*

The poset $(X, \leq)$ is called *narrow* if every discrete set $D \subseteq X$ is finite.

Given $x \in X$, we write $X_{\leq x} := (X, \leq)_{\leq x} := \{y \in X, y \leq x\}$ and $X_{\geq x} := (X, \leq)_{\geq x} := \{y \in X, y \geq x\}$.

A set $M \subseteq X$ is called a *lower set* if $X_{\leq x} \subseteq M$ for all $x \in M$. I.e. M is a lower set if for all $x \in M$, all elements of X smaller than x are also contained in M.

A morphism of posets $f : (A, \leq_A) \to (B, \leq_B)$ is a map $f : A \to B$ such that $a_1, a_2 \in A$, $a_1 \leq_A a_2$ implies $f(a_1) \leq_B f(a_2)$. Hence, an isomorphism of posets from $(A, \leq_A)$ to $(B, \leq_B)$ is a bijective map $f : A \to B$ such that for all $a_1, a_2 \in A$, we have $a_1 \leq_A a_2 \Leftrightarrow f(a_1) \leq_B f(a_2)$.

For $M \subseteq X$, the restricted relation $\leq|_{M\times M}$ is a partial order on M. We denote by $(M, \leq)$ the subposet $(M, \leq|_{M\times M})$.

Lemma 73. *Let $(X, \leq)$ be a poset. The following are equivalent.*

(i) The poset $(X, \leq)$ is artinian.

(ii) Every nonempty subset of X has a minimal element.

Proof. (i) $\Rightarrow$ (ii): Suppose $(X, \leq)$ is artinian. Suppose given $\emptyset \neq M \subseteq X$. We need to prove that M has a minimal element. Assume to the contrary that there exist no minimal elements in M. I.e. for each $x \in M$ there is $y \in M$ with $x > y$. Since $M \neq \emptyset$, we can construct a strictly descending chain in M. But this is impossible since $(X, \leq)$ is artinian. Hence the assumption is false and M has a minimal element.

(ii) $\Rightarrow$ (i): Suppose every nonempty subset of X has a minimal element. Suppose given a chain $x_0 \geq x_1 \geq x_2 \geq \ldots$ in X. By hypothesis, the set $M = \cup_{i=0}^{\infty}\{x_i\} \neq \emptyset$ has a minimal element, say x_N for some $N \in \mathbb{Z}_{\geq 0}$. Since the chain is descending, we have $x_j \leq x_N$ for all $j \in \mathbb{Z}_{\geq N}$. Since x_N is minimal, we have $x_N \leq x_j$, hence $x_N = x_j$ for all $j \in \mathbb{Z}_{\geq N}$. So any descending chain in X becomes stationary, i.e. $(X, \leq)$ is artinian. $\square$

Definition 74. Let $n \in \mathbb{Z}_{\geq 1}$. Let the partial order $\leq_n$ on $\mathbb{Z}^n$ be defined by

$$(\alpha_1, \ldots, \alpha_n) \leq_n (\beta_1, \ldots, \beta_n) \quad :\Leftrightarrow \quad \alpha_i \leq \beta_i \text{ for all } i \in [1, n]$$

for $(\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$. Defining $\mathbb{Z}^n_{\geq 0} := (\mathbb{Z}_{\geq 0})^n \subseteq \mathbb{Z}^n$, we obtain the subposet $(\mathbb{Z}^n_{\geq 0}, \leq_n)$ of $(\mathbb{Z}^n, \leq_n)$.

Lemma 75.

(a) For $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n_{\geq 0}$, the set $(\mathbb{Z}^n_{\geq 0})_{\leq x} = \{(y_1, \ldots, y_n) \mid y_i \in [0, x_i]$ for $i \in [1, n]\}$ is finite.

(b) The poset $(\mathbb{Z}^n_{\geq 0}, \leq_n)$ is artinian.

(c) The poset $(\mathbb{Z}^n_{\geq 0}, \leq_n)$ is narrow.

Proof. Assertion (a) follows by construction. Assertion (b) results from (a). We show (c): We have to show that any discrete subset of $(\mathbb{Z}^n_{\geq 0}, \leq_n)$ is finite. We proceed by induction on $n \geq 1$. Since the poset $(\mathbb{Z}^1_{\geq 0}, \leq_1) \simeq (\mathbb{Z}_{\geq 0}, \leq)$ is, in fact, totally ordered, the claim holds for $n = 1$. Suppose given $n \in \mathbb{Z}_{\geq 2}$ such that the claim holds for $n - 1$. Suppose D is a discrete subset of $(\mathbb{Z}^n_{\geq 0}, \leq_n)$. Since $\emptyset$ is finite, we may suppose $D \neq \emptyset$. Choose $x = (x_1, \ldots, x_n) \in D$. For $y = (y_1, \ldots, y_n) \in D$ with $x \neq y$, there exists $i \in [1, n]$ such that $x_i > y_i$ since $x \not\leq_n y$. Setting $M_{i,l} := \{(z_1, \ldots, z_n) \in \mathbb{Z}^n_{\geq 0} \mid z_i = l\}$ for $i \in [1, n]$ and $l \in \mathbb{Z}_{\geq 0}$, we obtain

$$D = \cup_{i \in [1,n]} \cup_{l \in [0, x_i]} (D \cap M_{i,l}). \tag{28}$$

The sub-poset $(M_{i,l}, \leq_n)$ of $(\mathbb{Z}^n_{\geq 0}, \leq_n)$ is as a poset isomorphic to $(\mathbb{Z}^{n-1}_{\geq 0}, \leq_{n-1})$. Hence as $(D \cap M_{i,l})$ is a discrete subset of $M_{i,l}$, it is finite by the induction hypothesis. So by (28), the set D is a finite union of finite sets and thus finite. So the claim holds for n, which completes the proof. $\square$

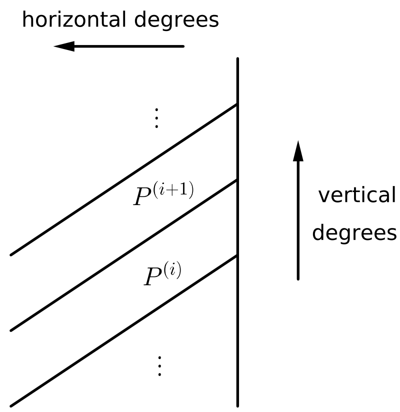
4.3. Extended Kadeishvili Method

The purpose of this section is to obtain minimal or "almost" minimal models of a given A_∞-algebra $(\check{A}, (\check{m}_k)_{k \geq 1})$ over arbitrary rings R. Corollary 70 shows that it is generally impossible to obtain a minimal A_∞-structure $(m_k)_{k \geq 1}$ on $\mathrm{H}^*\check{A}$ such that there is a quasi-isomorphism $(f_k)_{k \geq 1} : (\mathrm{H}^*\check{A}, (m_k)_{k \geq 1}) \to (\check{A}, (\check{m}_k)_{k \geq 1})$ or a quasi-isomorphism $(\tilde{f}_k)_{k \geq 1} : (\check{A}, (\check{m}_k)_{k \geq 1}) \to (\mathrm{H}^*\check{A}, (m_k)_{k \geq 1})$.

Note that in the examples discussed in Corollary 70, the module $\mathrm{H}^*\check{A}$ is non-projective over R since it has torsion over R. In contrast, Theorem 55 shows that Kadeishvili's algorithm works if $\mathrm{H}^*\check{A}$ is projective. So it is reasonable to replace $\mathrm{H}^*\check{A}$ by a the direct sum $A := \oplus_{i \in \mathbb{Z}} P^{(i)}$, where for $i \in \mathbb{Z}$, the complex $P^{(i)}$ is a projective resolution of $\mathrm{H}^i\check{A}$, since A is projective as an R-module and its homology is isomorphic to $\mathrm{H}^*\check{A}$.

We ask whether A can be made a model of $\check{A}$. Each $P^{(i)}$ carries a (generally non-zero) differential, so if $(m_k)_{k \geq 1}$ is an A_∞-structure on A such that A becomes a model of $\check{A}$ and such that m_1 contains the differentials of the $P^{(i)}$ as components, then $(A, (m_k)_{k \geq 1})$ cannot be a minimal A_∞-algebra and in particular not a minimal model of $\check{A}$.

Thus we will need to modify the notion of minimality and to that end, we introduce the notion of eA$_\infty$-algebras, which are A$_\infty$-algebras with additional structure. The module A sketched above is composed of the $P^{(i)}$ and each $P^{(i)}$ is composed of its positions. A major feature of eA$_\infty$-categories is that they can capture this twofold decomposition of A. To that end, eA$_\infty$-categories feature a "horizontal" and a "vertical" grading. We may then assign the positions of the $P^{(i)}$ in such a way to the horizontally and vertically homogeneous components that the $P^{(i)}$ run diagonally as illustrated below (cf. also Proposition 81).



The "vertical" grading of an eA$_\infty$-algebra is the grading that is known from A$_\infty$-algebras and which interacts with the Koszul sign rule. The "horizontal" grading is used for bookkeeping. Minimality of eA$_\infty$-algebras requires in particular that homogeneous elements of a certain horizontal degree are mapped by m_1 to a sum of homogeneous elements of strictly lower horizontal degree. By Remark 77, the notion of minimality of eA$_\infty$-algebras generalizes in a certain way the notion of minimality on A$_\infty$-algebras.

Definition 76. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.
Suppose given a $\mathbb{Z} \times \mathbb{Z}$-graded R-module $A = \oplus_{i,j\in\mathbb{Z}} A^{j,i}$ and suppose given R-linear maps $m_k : A^{\otimes k} \to A$ for $k \in [1, n]$. Whenever A is treated as a $\mathbb{Z}$-graded module (in particular concerning the Koszul sign rule), we refer to the grading of $A = \oplus_{i,j\in\mathbb{Z}} A^{j,i}$ obtained by suppressing the grading along j. For convenience, let $A^i := \oplus_{j\in\mathbb{Z}} A^{j,i}$ for $i \in \mathbb{Z}$ and $A^{j,-} := \oplus_{i\in\mathbb{Z}} A^{j,i}$ for $j \in \mathbb{Z}$.

We call the tuple $(A, (m_k)_{k\in[1,n]})$ an eA$_n$-*algebra* if (EA1), (EA2) and (EA3) hold, which are given as follows.

(EA1) $(A, (m_k)_{k\in[1,n]})$ is an A$_n$-algebra (Recall that the $\mathbb{Z}$-grading of $A = \oplus_{i,j\in\mathbb{Z}} A^{j,i}$ is obtained by suppressing the grading along j).

(EA2) $A^{j,i}$ is the zero module for all $j < 0$. I.e. $A = \oplus_{j\in\mathbb{Z}_{\geq 0}, i\in\mathbb{Z}} A^{j,i}$.

(EA3) For $k \in [1, n]$ and $j_1, \ldots, j_k \in \mathbb{Z}_{\geq 0}$, we have[6]

$$m_k(A^{j_1,-} \otimes \ldots \otimes A^{j_k,-}) \subseteq \oplus_{j' \leq (j_1+\ldots+j_k)+(2k-2)} A^{j',-}.$$

[6] In an earlier version of this text, the axioms (EA3) and (EA3′) were not present. Instead, an

We call the tuple $(A, (m_k)_{k\in[1,n]})$ a *minimal* eA_n*-algebra* if (EA1), (EA2) and (EA3′) hold, where (EA3′) is given as follows.

(EA3′) For $k \in [1, n]$ and $j_1, \ldots, j_k \in \mathbb{Z}_{\geq 0}$, we have

$$m_k(A^{j_1,-} \otimes \ldots \otimes A^{j_k,-}) \subseteq \oplus_{j' \leq (j_1+\ldots+j_k)+(2k-3)} A^{j',-}.$$

Since eA_n-algebras are A_n-algebras with additional structure, the notations and conventions given for A_n-algebras will also be used for eA_n-algebras.

Given eA_n-algebras $(A, (m_k)_{k\in[1,n]})$ and $(A', (m'_k)_{k\in[1,n]})$, a *morphism of* eA_n*-algebras* or eA_n*-morphism* from A to A' is a morphism of A_n-algebras $(f_k)_{k\in[1,n]} : (A, (m_k)_{k\in[1,n]}) \to (A'(m'_k)_{k\in[1,n]})$. Composition of eA_n-morphisms is composition of A_n-morphisms.

Remark 77 (Relation of A_∞- and eA_∞-algebras. Some functors.)**.** Suppose given $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Suppose given an A_n-algebra $(A, (m_k)_{k\in[1,n]})$. For $i, j \in \mathbb{Z}$, we define

$$A^{j,i} := \begin{cases} A^i & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

Thus $A^{j,-} = 0$ for $j \in \mathbb{Z} \setminus \{0\}$.

For $k \in [1, n]$ and $j_1, \ldots, j_k \in \mathbb{Z}_{\geq 0}$, we have

$$m_k(A^{j_1,-} \otimes \ldots \otimes A^{j_k,-}) \subseteq A = A^{0,-} \overset{2k-2\geq 0}{\subseteq} \oplus_{j' \leq (j_1+\ldots+j_k)+(2k-2)} A^{j',-}.$$

I.e. by defining $A^{j,i}$ as above, $(A, (m_k)_{k\in[1,n]})$ becomes an eA_n-algebra.

Suppose $n \geq 1$. For $k \in [2, n]$, we have

$$m_k(A^{j_1,-} \otimes \ldots \otimes A^{j_k,-}) \subseteq A = A^{0,-} \overset{2k-3\geq 0}{\subseteq} \oplus_{j' \leq (j_1+\ldots+j_k)+(2k-3)} A^{j',-}.$$

Since additionally $A^{j,-} = 0$ for $j \in \mathbb{Z} \setminus \{0\}$, we have the following equivalence.
$(A, (m_k)_{k\in[1,n]})$ is a minimal eA_n-algebra
$\Leftrightarrow m_1(A^{0,-}) \subseteq \oplus_{j' \leq 0+2\cdot 1-3} A^{j',-} = \oplus_{j' \in \mathbb{Z}_{<0}} A^{j',0} = 0$
$\Leftrightarrow m_1 = 0$
$\Leftrightarrow (A, (m_k)_{k\in[1,n]})$ is a minimal A_n-algebra.

Denote the category of A_n-algebras by $\mathtt{Alg}_n$. Denote the category of eA_n-algebras by $\mathtt{eAlg}_n$. Axiom (EA1) yields the forgetful functor $\mathtt{forget} : \mathtt{eAlg}_n \to \mathtt{Alg}_n$. Since the

eA_∞-algebra was called minimal if $m_1(A^{j,-}) \subseteq \oplus_{j' \leq j-1} A^{j,-}$ for all $j \in \mathbb{Z}$. The axioms (EA3) and (EA3′) in their present version are inspired by Sagave's dA_∞-algebras, for which a satisfy similar condition holds. Ultimately, (EA3) and (EA3′) were strengthened to their present form since it was easily possible and since this yields strengthened results for the extended Kadeishvili minimal method.

eA_n-morphisms are the A_n-morphisms, the functor $\mathtt{forget} : \mathtt{eAlg}_n \to \mathtt{Alg}_n$ is fully faithful.

By the construction given above, we may obtain eA_n-algebras from A_n-algebras. This yields a functor $\mathtt{F} : \mathtt{Alg}_n \to \mathtt{eAlg}_n$. Since the eA_n-morphisms are the A_n-morphisms, the functor $\mathtt{F} : \mathtt{Alg}_n \to \mathtt{eAlg}_n$ is fully faithful.

The functor $\mathtt{forget} : \mathtt{eAlg}_n \to \mathtt{Alg}_n$ is a left inverse to the functor $\mathtt{F} : \mathtt{Alg}_n \to \mathtt{eAlg}_n$. I.e. $\mathtt{forget} \circ \mathtt{F} = \mathtt{id}_{\mathtt{Alg}_n}$. Hence, the fully faithful functor $\mathtt{forget}$ is dense and thus an equivalence from $\mathtt{eAlg}_n$ to $\mathtt{Alg}_n$

4.3.1. Structure of the successive construction

Remark 78 (setup of the incremental step)**.** The incremental step will be performed in the following situation:

- $n \in \mathbb{Z}_{\geq 2}$.
- $(\check{A}, (\check{m}_k)_{k\geq 1})$ is an A_∞-algebra.
- $(A, (m_k)_{k\in[1,n-1]})$ is a minimal eA_{n-1}-algebra.
- $(f_k)_{k\in[1,n-1]}$ is a quasi-isomorphism of A_{n-1}-algebras from A to $\check{A}$.
- Assertions (P1) - (P3) hold:
  - (P1) A is projective over R.
  - (P2) For all $j \in \mathbb{Z}$, we have $m_1(\oplus_{j'\leq j} A^{j',-}) = (\mathrm{B}^* A) \cap (\oplus_{j'\leq j-1} A^{j',-})$.
  - (P3) $p \circ (f_1|_{A^{0,-}}^{\mathrm{Z}^*\check{A}})$ is surjective, where $p : \mathrm{Z}^*\check{A} \to \mathrm{H}^*\check{A}$ is the residue class map.

4.3.2. The initial step

For the initial step, we will show in Proposition 81 that the setup defined in Remark 78 is attainable for $n = 2$.

Definition 79. Suppose given a minimal eA_1-algebra $(A, (m_k)_{k\in[1,1]})$. We have the filtration $A^{\leq j,-} := \oplus_{j'\in\mathbb{Z}_{\leq j}} A^{j',-}$ for $j \in \mathbb{Z}$. Note that for $j \in \mathbb{Z}_{<0}$, we have $A^{\leq j,-} = 0$ by (EA2). By (EA3′), we have $m_1(A^{j,-}) \subseteq \oplus_{j'\leq j-1} A^{j',-}$ for $j \in \mathbb{Z}$. Hence we have $m_1(A^{\leq j,-}) \subseteq A^{\leq j-1,-}$ for $j \in \mathbb{Z}$. We obtain the complex

$$\Big(\dots \xrightarrow{\bar{m}_1^{(j+2)}} \underbrace{A^{\leq j+1,-}/A^{\leq j,-}}_{j+1} \xrightarrow{\bar{m}_1^{(j+1)}} \underbrace{A^{\leq j,-}/A^{\leq j-1,-}}_{j} \xrightarrow{\bar{m}_1^{(j)}} \dots$$
$$\dots \xrightarrow{\bar{m}_1^{(1)}} \underbrace{A^{\leq 0,-}/A^{\leq -1,-}}_{0} \xrightarrow{m_1^{(0)}} \underbrace{A^{\leq -1,-}/A^{\leq -2,-}}_{-1} \to \dots\Big)$$

$$= (\dots \xrightarrow{\bar m_1^{(j+2)}} \underbrace{A^{\le j+1,-}/A^{\le j,-}}_{j+1} \xrightarrow{\bar m_1^{(j+1)}} \underbrace{A^{\le j,-}/A^{\le j-1,-}}_{j} \xrightarrow{\bar m_1^{(j)}} \dots \xrightarrow{\bar m_1^{(1)}} \underbrace{A^{0,-}}_{0} \to \underbrace{0}_{-1} \to \dots), \tag{29}$$

where $\bar m_1^{(j)}$ is the map given by $\bar m_1^{(j)} : A^{\le j,-}/A^{\le j-1,-} \to A^{\le j-1,-}/A^{\le j-2,-}, x + A^{\le j-1,-} \mapsto m_1(x) + A^{\le j-2,-}$ and where the positions are written underneath the entries. We call $(A, (m_k)_{k\in[1,1]})$ *filtered-exact* if the complex (29) is exact at all positions $j \in \mathbb{Z}_{\ge 1}$.

Lemma 80. *Suppose given a filtered-exact minimal* eA_1*-algebra* $(A, (m_k)_{k\in[1,1]})$*. Then*

(i) $\mathrm{Z}^*A = ((\mathrm{Z}^*A) \cap A^{0,-}) + \mathrm{B}^*A = A^{0,-} + \mathrm{B}^*A$

(ii) $m_1(\oplus_{j'\le j} A^{j',-}) = (\mathrm{B}^*A) \cap (\oplus_{j'\le j-1} A^{j',-})$ *for* $j \in \mathbb{Z}$.

Proof. Ad (i). We use the notation given in Definition 79. By (EA3′) and (EA2), we have $m_1(A^{0,-}) \subseteq \oplus_{j'\le 0+2-3} A^{j',-} = 0$. Hence $((\mathrm{Z}^*A) \cap A^{0,-}) + \mathrm{B}^*A = A^{0,-} + \mathrm{B}^*A$.
Since $A = \bigcup_{j=0}^{\infty} A^{\le j,-}$, we have $\mathrm{Z}^*A = \bigcup_{j=0}^{\infty} (A^{\le j,-} \cap \mathrm{Z}^*A)$. So since $A^{\le 0,-} = A^{0,-}$, it suffices to prove $(A^{\le j,-} \cap \mathrm{Z}^*A) + \mathrm{B}^*A \subseteq (A^{\le j-1,-} \cap \mathrm{Z}^*A) + \mathrm{B}^*A$ for $j \ge 1$. Since B^*A is a summand on both sides, it suffices to prove $A^{\le j,-} \cap \mathrm{Z}^*A \subseteq (A^{\le j-1,-} \cap \mathrm{Z}^*A) + \mathrm{B}^*A$ for $j \ge 1$.

So suppose given $x \in A^{\le j,-} \cap \mathrm{Z}^*A$ for some $j \ge 1$. Since $x \in \mathrm{Z}^*A$, we have $\bar m_1^{(j)}(x + A^{\le j-1,-}) = 0$. Since (29) is exact at position $j \ge 1$, there exists $y \in A^{\le j+1,-}$ such that $\bar m_1^{(j+1)}(y + A^{\le j,-}) = x + A^{\le j-1,-}$. I.e. $m_1(y) - x \in A^{\le j-1,-}$. Since $m_1(y) - x \in \mathrm{Z}^*A$, we have $m_1(y) - x \in A^{\le j-1,-} \cap \mathrm{Z}^*A$. Hence, $x \in m_1(y) + (A^{\le j-1,-} \cap \mathrm{Z}^*A) \subseteq \mathrm{B}^*A + (A^{\le j-1,-} \cap \mathrm{Z}^*A)$.

Ad (ii). By (EA3′), we have $m_1(\oplus_{j'\le j} A^{j',-}) \subseteq \oplus_{j'\le j-1} A^{j',-}$ for $j \in \mathbb{Z}$. Hence

$$\underbrace{m_1(\oplus_{j'\le j} A^{j',-})}_{=m_1(A^{\le j,-})} \subseteq \underbrace{\mathrm{B}^*A}_{=m_1(A)} \cap \underbrace{(\oplus_{j'\le j-1} A^{j',-})}_{=A^{\le j-1,-}} \quad \text{for } j \in \mathbb{Z}. \tag{30}$$

We need to show equality in (30) for $j \in \mathbb{Z}$. For $j \le 0$, this follows from the fact that $A^{j',-} = 0$ for $j' \in \mathbb{Z}_{<0}$, so the right hand side in (30) is the zero module.

So we may suppose $j \in \mathbb{Z}_{>0}$. We show that

$$m_1(A^{\le k,-}) \cap A^{\le j-1,-} \subseteq m_1(A^{\le k-1,-}) \cap A^{\le j-1,-} \quad \text{for } k \in \mathbb{Z}_{>j}\,. \tag{31}$$

Suppose given $w \in m_1(A^{\le k,-}) \cap A^{\le j-1,-}$. There is $x \in A^{\le k,-}$ such that $m_1(x) = w$. We have $m_1(x) = w \in A^{\le j-1,-} \subseteq A^{\le k-2,-}$. Hence $\bar m_1^{(k)}(x + A^{\le k-1,-}) = 0$. Since A is filtered-exact, the complex (29) is exact at position $k > 0$ and so there is $y \in A^{\le k+1,-}$ such that $\bar m_1^{(k+1)}(y + A^{\le k,-}) = x + A^{\le k-1,-}$. I.e. $m_1(y) = x - z$ for some $z \in A^{\le k-1,-}$. Hence, $w = m_1(x) = m_1(m_1(y)) + m_1(z) = m_1(z) \in m_1(A^{\le k-1,-})$ which concludes the proof of (31).

Now suppose given $i \in \mathbb{Z}_{\ge j}$. Successive application of (31) yields $m_1(A^{\le i,-}) \cap A^{\le j-1,-} \subseteq m_1(A^{\le j,-}) \cap A^{\le j-1,-}$. Since $A = \cup_{i\ge j} A^{\le i,-}$, we have $\mathrm{B}^*A \cap A^{\le j-1,-} = m_1(A) \cap A^{\le j-1,-} = m_1(\cup_{i\ge j} A^{\le i,-}) \cap A^{\le j-1,-} = \bigcup_{i\ge j} (m_1(A^{\le i,-}) \cap A^{\le j-1,-}) \subseteq m_1(A^{\le j,-}) \cap A^{\le j-1,-}$. This proves equality in (30). □

Proposition 81. *Suppose given an* A_1*-algebra* $(\check{A}, (\check{m}_k)_{k\in[1,1]})$*. Suppose given a projective resolution* $P^{(i)} = (\ldots \to P^{2,i-2} \xrightarrow{d^{2,i-2}} P^{1,i-1} \xrightarrow{d^{1,i-1}} P^{0,i} \xrightarrow{d^{0,i}} 0 \to \ldots)$ *of* $\mathrm{H}^i\check{A}$ *with augmentation* $\varepsilon_i : P^{0,i} \to \mathrm{H}^i\check{A}$ *for each* $i \in \mathbb{Z}$*. Note that* $P^{j,i} = 0$ *for* $i \in \mathbb{Z}$*,* $j \in \mathbb{Z}_{<0}$*.*

Let $A = \oplus_{j,i\in\mathbb{Z}} A^{j,i}$ *be given by* $A^{j,i} := P^{j,i}$*.*

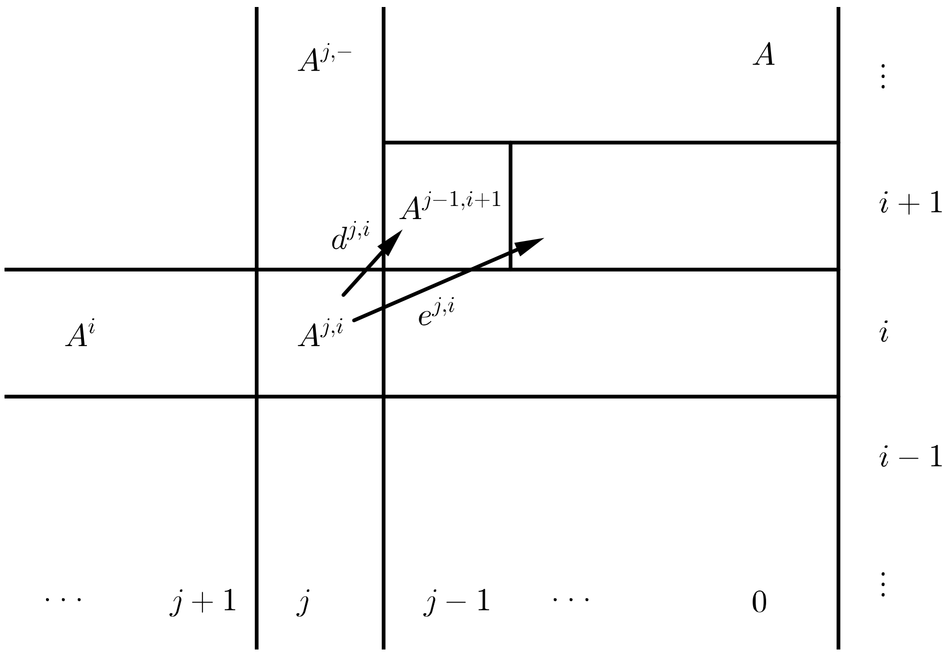
Then for $i, j \in \mathbb{Z}$*, there are morphisms* $e^{j,i} : A^{j,i} \to \oplus_{j'\in[0,j-2]} A^{j',i+1}$ *and there is a morphism* $f_1 : A \to \check{A}$ *such that the following hold.*

- *The pair* $(A, (m_k)_{k\in[1,1]})$*, where* $m_1 : A \to A$ *is given by*

$$m_1|_{A^{j,i}} := d^{j,i} + e^{j,i} \quad \text{for } j, i \in \mathbb{Z},$$

  is a minimal eA_1*-algebra.*
- $(f_k)_{k\in[1,1]} : (A, (m_k)_{k\in[1,1]}) \to (\check{A}, (\check{m}_k)_{k\in[1,1]})$ *is a quasi-isomorphism of* A_1*-algebras.*
- *(P1), (P2) and (P3) hold.*

We write $(m_k)_{k\in[1,1]}$ for (m_1) to emphasize that Proposition 81 is the initial part of a successive construction. The incremental step is given by Proposition 88.



Proof. As in Definition 76, the module A is graded as $A = \oplus_{i\in\mathbb{Z}} A^i$, where $A^i := \oplus_{j\in\mathbb{Z}} A^{i,j}$. So A is the graded direct sum of the graded modules $A^{j,-}$, where $A^{j,-} := \oplus_{i\in\mathbb{Z}} A^{j,i}$ and $(A^{j,-})^i = A^{j,i}$. For $j \in \mathbb{Z}$, we define $A^{\leq j,-}$ as given in Definition 79. For $j \in \mathbb{Z}$, we set

$$d^{(j)} := \oplus_{i\in\mathbb{Z}} d^{j,i} : A^{j,-} = \oplus_{i\in\mathbb{Z}} A^{j,i} \to \oplus_{i\in\mathbb{Z}} A^{j-1,i+1} = A^{j-1,-}.$$

For $j \in \mathbb{Z}$, the map $d^{(j)}$ is graded of degree 1. We define

$$\varepsilon = \oplus_{i\in\mathbb{Z}} \varepsilon_i : A^{0,-} = \oplus_{i\in\mathbb{Z}} P^{0,i} \to \oplus_{i\in\mathbb{Z}} \mathrm{H}^i\check{A} = \mathrm{H}^*\check{A},$$

which is a graded map of degree 0. By construction, the complex

$$\ldots \to A^{2,-} \xrightarrow{d^{(2)}} A^{1,-} \xrightarrow{d^{(1)}} A^{0,-} \xrightarrow{\varepsilon} \mathrm{H}^*\check{A} \to 0 \tag{32}$$

is an exact sequence. In particular, ε is surjective.

The residue class map $p : \mathrm{Z}^*\check{A} \to \mathrm{H}^*\check{A}$ is surjective and graded of degree 0. By Lemma 20, the module $A^{0,-}$ is graded projective, so there exists a graded map $f^{(0)} : A^{0,-} \to \check{A}$ of degree 0 such that $\operatorname{im} f^{(0)} \subseteq \mathrm{Z}^*\check{A}$ and $p \circ (f^{(0)}|^{\mathrm{Z}^*\check{A}}) = \varepsilon$.

$$\begin{tikzcd} A^{0,-} \arrow[r, two heads, "\varepsilon"] \arrow[dr, dotted, "f^{(0)}|^{\mathrm{Z}^*\check{A}}"'] & \mathrm{H}^*\check{A} \\ & \mathrm{Z}^*\check{A} \arrow[u, two heads, "p"'] \end{tikzcd} \tag{33}$$

For $j \in \mathbb{Z}_{<0}$, we set $f^{(j)} : A^{j,-} = 0 \to \check{A}$ to be the zero morphism. For $j \in \mathbb{Z}_{\leq 0}$, we set $e^{(j)} : A^{j,-} \to \oplus_{j' \in [0,j-2]} A^{j',-} = 0$ to be the zero morphism.

Using the construction principle given in Lemma 134, we will successively construct morphisms $e^{(j)} : A^{j,-} \to A^{\leq j-2,-}$ and $f^{(j)} : A^{j,-} \to \check{A}$ for $j \in \mathbb{Z}_{\geq 1}$ satisfying conditions (i)-(vi) given below. We call this the *outer iteration.* For given $j \in \mathbb{Z}$ and given $e^{(j')}$ and $f^{(j')}$ for $j' \in [1, j]$, we define

$$\begin{aligned} &m_1^{(\leq j)} : A^{\leq j,-} \longrightarrow A^{\leq j-1,-}\,, \\ &m_1^{(\leq j)}|_{A^{j',-}} := e^{(j')} + d^{(j')} \quad \text{for } j' \in \mathbb{Z}_{\leq j} \end{aligned} \tag{34}$$

and

$$\begin{aligned} &f^{(\leq j)} : A^{\leq j,-} \longrightarrow \check{A}\,, \\ &f^{(\leq j)}|_{A^{j',-}} := f^{(j')} \quad \text{for } j' \in \mathbb{Z}_{\leq j}. \end{aligned} \tag{35}$$

They are to satisfy the following conditions

(i) $e^{(j)}$ is graded of degree 1.

(ii) $f^{(j)}$ is graded of degree 0.

(iii) $m_1^{(\leq j-1)} \circ (e^{(j)} + d^{(j)}) = 0$.

(iv) $\check{m}_1 \circ f^{(j)} = f^{(\leq j-1)} \circ (e^{(j)} + d^{(j)})$.

(v) $m_1^{(\leq j)} \circ m_1^{(\leq j)} = 0$.

(vi) $\check{m}_1 \circ f^{(\leq j)} = f^{(\leq j)} \circ m_1^{(\leq j)}$.

Note that (i)-(vi) hold for $j \leq 0$.

So suppose given $k \in \mathbb{Z}_{\geq 1}$ and suppose given $f^{(1)}, \ldots, f^{(k-1)}$ and $e^{(1)}, \ldots, e^{(k-1)}$ such that (i)-(vi) hold for $j \in [1, k-1]$. We need to show that there are maps $f^{(k)} : A^{k,-} \to \check{A}$ and $e^{(k)} : A^{k,-} \to A^{\leq k-2,-}$ such that (i)-(vi) hold for $j = k$. To that end, we will prove the following

Claim: $m_1^{(\le k-1)}(d^{(k)}(A^{k,-})) \overset{!}{\subseteq} \operatorname{im} m_1^{(\le k-2)}$.

So suppose given $x \in A^{k,-}$. We have $y := m_1^{(\le k-1)}(d^{(k)}(x)) = (d^{(k-1)} + e^{(k-1)})(d^{(k)}(x)) = e^{(k-1)}(d^{(k)}(x)) \in A^{\le k-3,-}$.

For $k \le 2$, we have $A^{\le k-3,-} = 0$. So in that case, we have $y = 0 \in \operatorname{im} m_1^{(\le k-2)}$. So suppose $k \ge 3$.

Since (v) holds for $j = k-1$, we have $m_1^{(\le k-3)}(y) = (m_1^{(\le k-1)})^2(d^{(k)}(x)) = 0$. Hence $y = y_{k-3} + \tilde{y}_{k-3}$, where $y_{k-3} := y \in \ker m_1^{(\le k-3)}$ and $\tilde{y}_{k-3} := 0 \in \operatorname{im} m_1^{(\le k-2)}$.

We show by induction on $i \in [0, k-3]$ that there exist $y_{k-3-i} \in \ker m_1^{(\le k-3-i)}$ and $\tilde{y}_{k-3-i} \in \operatorname{im} m_1^{(\le k-2)}$ such that $y = y_{k-3-i} + \tilde{y}_{k-3-i}$.

We have already proven the initial step $i = 0$.
So suppose given $i \in [0, k-4]$, $y_{k-3-i} \in \ker m_1^{(\le k-3-i)}$ and $\tilde{y}_{k-3-i} \in \operatorname{im} m_1^{(\le k-2)}$ such that $y = y_{k-3-i} + \tilde{y}_{k-3-i}$. We have $y_{k-3-i} = \hat{y}_{k-3-i} + \check{y}_{k-3-i}$ for some $\hat{y}_{k-3-i} \in A^{k-3-i,-}$ and $\check{y}_{k-3-i} \in A^{\le k-3-(i+1),-}$. We have

$$\begin{aligned} 0 &= m_1^{(\le k-3-i)}(y_{k-3-i}) = m_1^{(\le k-3-i)}(\hat{y}_{k-3-i}) + m_1^{(\le k-3-i)}(\check{y}_{k-3-i}) \\ &= \underbrace{d^{(k-3-i)}(\hat{y}_{k-3-i})}_{\in A^{k-3-i-1}} + \underbrace{e^{(k-3-i)}(\hat{y}_{k-3-i}) + m_1^{(\le k-3-(i+1))}(\check{y}_{k-3-i})}_{\in A^{\le k-3-i-2,-}} . \end{aligned}$$

So in particular $d^{(k-3-i)}(\hat{y}_{k-3-i}) = 0$. We have $k-3-i \ge k-3-(k-4) = 1$. So since (32) is exact, we have $\ker d^{(k-3-i)} = \operatorname{im} d^{(k-2-i)}$. Hence there is $z \in A^{k-2-i,-}$ such that $d^{(k-2-i)}(z) = \hat{y}_{k-3-i}$. We have $e^{(k-2-i)}(z) \in A^{\le k-3-(i+1),-}$. We set $y_{k-3-(i+1)} := -e^{(k-2-i)}(z) + \check{y}_{k-3-i} \in A^{\le k-3-(i+1),-}$ and $\tilde{y}_{k-3-(i+1)} := m_1^{(\le k-2)}(z) + \tilde{y}_{k-3-i} \in \operatorname{im} m_1^{(\le k-2)}$. We have

$$\begin{aligned} y &= y_{k-3-i} + \tilde{y}_{k-3-i} = \hat{y}_{k-3-i} + \check{y}_{k-3-i} + \tilde{y}_{k-3-i} \\ &= (d^{(k-2-i)} + e^{(k-2-i)})(z) - e^{(k-2-i)}(z) + \check{y}_{k-3-i} + \tilde{y}_{k-3-i} \\ &= m_1^{(\le k-2)}(z) - e^{(k-2-i)}(z) + \check{y}_{k-3-i} + \tilde{y}_{k-3-i} \\ &= y_{k-3-(i+1)} + \tilde{y}_{k-3-(i+1)}. \end{aligned}$$

We have

$$m_1^{(\le k-3-(i+1))}(y_{k-3-(i+1)}) = m_1^{(\le k-3)}(\underbrace{y}_{\in \ker m_1^{(\le k-3)}} - \underbrace{\tilde{y}_{k-3-(i+1)}}_{\in \operatorname{im} m_1^{(\le k-2)}}) \overset{(v)}{=} 0.$$

Hence $y_{k-3-(i+1)} \in \ker m_1^{(\le k-3-(i+1))}$. This completes the induction step of the induction over i.

In particular, for $i = k-3$ we obtain $y_0 \in \ker m_1^{(\le 0)} \subseteq A^{0,-}$ and $\tilde{y}_0 \in \operatorname{im} m_1^{(\le k-2)}$ such that $y = y_0 + \tilde{y}_0$. I.e. $y = y_0 + m_1^{(\le k-2)}(w)$ for some $w \in A^{\le k-2,-}$.

Recall that $p : \mathrm{Z}^*\check{A} \to \mathrm{H}^*\check{A}$ is the residue class map. We have

$$\begin{aligned} f^{(\leq k-1)}(y) &= (f^{(\leq k-1)} \circ m_1^{(\leq k-1)} \circ d^{(k)})(x) \\ &\overset{(vi)}{=} (\check{m}_1 \circ f^{(\leq k-1)} \circ d^{(k)})(x). \end{aligned}$$

Thus we have

$$\begin{aligned} 0 = p(f^{(\leq k-1)}(y)) &= p(f^{(\leq k-1)}(y_0)) + p(f^{(\leq k-1)}(m_1^{(\leq k-2)}(w))) \\ &\overset{(vi)}{=} \varepsilon(y_0) + p(\check{m}_1(f^{(\leq k-1)}(w))) = \varepsilon(y_0). \end{aligned}$$

By exactness of (32), we have $\ker\varepsilon = \operatorname{im} d^{(1)}$. Combined with $\operatorname{im} e^{(1)} \subseteq A^{\leq 1-2,-} = 0$, we obtain $y_0 \in \operatorname{im} d^{(1)} = \operatorname{im} m_1^{(\leq 1)} \subseteq \operatorname{im} m_1^{(\leq k-2)}$. Thus $y = y_0 + \tilde{y}_0 \in \operatorname{im} m_1^{(\leq k-2)}$.

This proves the *claim*. I.e. we have proven $m_1^{(\leq k-1)}(d^{(k)}(A^{k,-})) \subseteq \operatorname{im} m_1^{(\leq k-2)}$.

Since $m_1^{(\leq k-2)}|^{\operatorname{im} m_1^{(\leq k-2)}}$ is graded of degree 1 and surjective, since the map $m_1^{(\leq k-1)} \circ d^{(k)}$ is graded of degree 2 and since $A^{k,-}$ is by Lemma 20 graded projective, there is a graded map $\tilde{e} : A^{k,-} \to A^{\leq k-2,-}$ of degree 1 such that $m_1^{(\leq k-2)} \circ \tilde{e} = m_1^{(\leq k-1)} \circ d^{(k)}$.

$$\begin{tikzcd} A^{k,-} \arrow[r, "(m_1^{(\leq k-1)} \circ d^{(k)})|^{\operatorname{im} m_1^{(\leq k-2)}}"] \arrow[dr, dotted, "\exists \tilde{e}"'] & \operatorname{im} m_1^{(\leq k-2)} \\ & A^{\leq k-2,-} \arrow[u, two heads, "m_1^{(\leq k-2)}|^{\operatorname{im} m_1^{(\leq k-2)}}"'] \end{tikzcd}$$

Thus

$$m_1^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}) = 0. \tag{36}$$

Hence, $(\check{m}_1 \circ f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}))(A^{k,-}) \overset{(vi)}{=} (f^{(\leq k-1)} \circ m_1^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}))(A^{k,-}) = 0$. So $(f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}))(A^{k,-}) \subseteq \mathrm{Z}^*\check{A}$.

Recall that $\operatorname{im} f^{(0)} \subseteq \mathrm{Z}^*\check{A}$. We will construct a graded map $\hat{e} : A^k \to A^{0,-}$ of degree 1 such that

$$p \circ (f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}))|^{\mathrm{Z}^*\check{A}} = p \circ (f^{(0)}|^{\mathrm{Z}^*\check{A}}) \circ \hat{e}. \tag{37}$$

Case $k = 1$: We set $\hat{e} := 0$. We have $\operatorname{im}\tilde{e} \subseteq \oplus_{j' \leq 1-2} A^{j',0} = 0$. Thus $p \circ (f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}))|^{\mathrm{Z}^*\check{A}} = p \circ (f^{(0)} \circ d^{(1)})|^{\mathrm{Z}^*\check{A}} = p \circ (f^{(0)}|^{\mathrm{Z}^*\check{A}}) \circ d^{(1)} = \varepsilon \circ d^{(1)} = 0 = p \circ (f^{(0)}|^{\mathrm{Z}^*\check{A}}) \circ \hat{e}$.

Case $k \geq 2$: The map $p \circ (f^{(0)}|^{\mathrm{Z}^*\check{A}}) = \varepsilon : A^{0,-} \to \mathrm{H}^*\check{A}$ is a graded epimorphism of degree 0. So since $A^{k,-}$ is graded projective and since $p \circ (f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}))|^{\mathrm{Z}^*\check{A}}$ is graded of degree 1, there is a graded map $\hat{e} : A^{k,-} \to A^{0,-}$ of degree 1 such that (37) holds.

$$\begin{tikzcd} A^{0,-} \arrow[r, two heads, "\varepsilon"] & \mathrm{H}^*A \\ & A^{k,-} \arrow[u, "p \circ (f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}))|^{\mathrm{Z}^*\check{A}}"'] \arrow[ul, dotted, "\exists \hat{e}"] \end{tikzcd}$$

Eq. (37) yields

$$\operatorname{im}(f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e} - \hat{e})) = \operatorname{im}(f^{(\leq k-1)} \circ (d^{(k)} - \tilde{e}) - f^{(0)} \circ \hat{e}) \subseteq \ker p = \mathrm{B}^* \check{A}. \quad (38)$$

We set $e^{(k)} := (-\tilde{e} - \hat{e})|^{A^{\leq k-2}} : A^{k,-} \to A^{\leq k-2,-}$ (recall $\hat{e} = 0$ in case $k = 1$), which is a graded map of degree 1. Thus property (i) holds for $j = k$. We have

$$m_1^{(\leq k-1)} \circ (d^{(k)} + e^{(k)}) = m_1^{(\leq k-1)} \circ (d^{(k)} - \tilde{e} - \hat{e}) \overset{(36)}{=} -m_1^{(\leq k-1)} \circ \hat{e} \overset{*}{=} 0,$$

where in the step marked by $*$, we use $\operatorname{im} \hat{e} \subseteq A^{0,-}$ and $m_1^{(\leq k-1)}|_{A^{0,-}} = d^{(0)} + e^{(0)} = 0 + 0$. Thus (iii) holds for $j = k$. By (38), we have $\operatorname{im}(f^{(\leq k-1)} \circ (d^{(k)} + e^{(k)})) \subseteq \mathrm{B}^* \check{A}$. The maps $f^{(\leq k-1)} \circ (d^{(k)} + e^{(k)})$ and $\check{m}_1|^{\mathrm{B}^* \check{A}}$ are both graded of degree 1. So since $\check{m}_1|^{\mathrm{B}^* \check{A}}$ is surjective and since $A^{k,0}$ is graded projective, there is a graded map $f^{(k)} : A^{k,-} \to \check{A}$ of degree 0 such that

$$\check{m}_1 \circ f^{(k)} = f^{(\leq k-1)} \circ (d^{(k)} + e^{(k)}).$$

$$\begin{tikzcd}
\check{A} \arrow[r, two heads, "\check{m}_1|^{\mathrm{B}^* \check{A}}"] & \mathrm{B}^* \check{A} \\
 & A^{k,-} \arrow[u, "(f^{(\leq k-1)} \circ (d^{(k)} + e^{(k)}))|^{\mathrm{B}^* \check{A}}"'] \arrow[ul, dotted, "\exists f^{(k)}"]
\end{tikzcd}$$

I.e. (ii) and (iv) hold for $j = k$. Having constructed $e^{(k)}$ and $f^{(k)}$, we define $m_1^{(\leq k)}$ and $f^{(\leq k)}$ by (34) and (35). Since (v) and (vi) hold for $j = k - 1$, we have

$$\begin{aligned}
(m_1^{(\leq k)} \circ m_1^{(\leq k)})|_{A^{\leq k-1,-}} &= m_1^{(\leq k-1)} \circ m_1^{(\leq k-1)} = 0 \text{ and} \\
\check{m}_1 \circ f^{(\leq k)}|_{A^{\leq k-1,-}} = \check{m}_1 \circ f^{(\leq k-1)} &= f^{(\leq k-1)} \circ m_1^{(\leq k-1)} = (f^{(\leq k)} \circ m_1^{(\leq k)})|_{A^{\leq k-1,-}}.
\end{aligned}$$

Since (iii) and (iv) hold for $j = k$, we have

$$\begin{aligned}
(m_1^{(\leq k)} \circ m_1^{(\leq k)})|_{A^{k,-}} &= m_1^{(\leq k-1)} \circ (d^{(k)} + e^{(k)}) = 0 \text{ and} \\
\check{m}_1 \circ f^{(\leq k)}|_{A^{k,-}} = \check{m}_1 \circ f^{(k)} &= f^{(\leq k-1)} \circ (e^{(k)} + d^{(k)}) = (f^{(\leq j)} \circ m_1^{(\leq j)})_{A^{k,-}}.
\end{aligned}$$

Hence, (v) and (vi) hold for $j = k$. This concludes the *outer iteration*.

We define the graded map $m_1 : A \to A$ of degree 1 by setting

$$m_1|_{A^{j,-}} := d^{(j)} + e^{(j)} \qquad \text{for } j \in \mathbb{Z}.$$

We define the graded map $f_1 : A \to \check{A}$ of degree 0 by setting

$$f_1|_{A^{j,-}} := f^{(j)} \qquad \text{for } j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, we have $m_1|_{A^{\leq j,-}}^{A^{\leq j-1,-}} = m_1^{(\leq j)}$ and $f_1|_{A^{\leq j,-}} = f^{(\leq j)}$. So since $A = \bigcup_{j \geq 0} A^{\leq j,-}$ and since (v) and (vi) hold for $j \in \mathbb{Z}_{\geq 0}$, we have

$$m_1 \circ m_1 = 0 \qquad\qquad \text{and}$$

$$\check{m}_1 \circ f_1 = f_1 \circ m_1 .$$

I.e. $(A, (m_1))$ is an A_1-algebra and (f_1) is a morphism of A_1-algebras from A to $\check{A}$. We prove that $(A, (m_1))$ is a minimal eA_1-algebra: We have just proven (EA1). Assertion (EA2) holds by construction. For $j \in \mathbb{Z}$, we have $m_1(A^{j,-}) \subseteq (d^{(j)} + e^{(j)})(A^{j,-}) \subseteq A^{j-1,-} + A^{\leq j-2,-} = A^{\leq j-1,-}$. Thus (EA3′) holds.

For $j \in \mathbb{Z}$, define $\bar{m}_1^{(j)}$ as given in Definition 79. For $j \in \mathbb{Z}$, the composite $t^{(j)} : A^{j,-} \hookrightarrow A^{\leq j,-} \twoheadrightarrow A^{\leq j,-}/A^{\leq j-1,-}$ is an isomorphism of graded modules. For $j \in \mathbb{Z}$ and $x \in A^{j,-}$, we have

$$\begin{aligned}\bar{m}_1^{(j)}(t^{(j)}(x)) &= m_1(x) + A^{\leq j-2,-} = d^{(j)}(x) + \underbrace{e^{(j)}(x)}_{\in A^{\leq j-2,-}} + A^{\leq j-2,-} \\ &= d^{(j)}(x) + A^{\leq j-2,-} = t^{(j-1)}(d^{(j)}(x)).\end{aligned}$$

Hence, the complex (29) is via the complex morphism $(t^{(j)})_{j\in\mathbb{Z}}$ isomorphic to the complex

$$\to \underbrace{A^{j+1,-}}_{j+1} \xrightarrow{d^{(j+1)}} \underbrace{A^{j,-}}_{j} \xrightarrow{d^{(j)}} \dots \xrightarrow{d^{(1)}} \underbrace{A^{0,-}}_{0} \to \underbrace{0}_{-1} \to \dots\ , \tag{39}$$

where the positions are written underneath the entries. The complex (39) (and thus also the complex (29)) is exact at all positions $j \geq 1$ since (32) is an exact sequence. Hence, $(A, (m_1))$ is a filtered-exact minimal eA_1-algebra.

Assertion (P1) holds since all $A^{j,i} = P^{j,i}$ are projective. Lemma 80(ii) implies (P2).

We have $A^{0,-} \subseteq \mathrm{Z}^*A$. We have $p \circ (f_1|_{A^{0,-}}^{\mathrm{Z}^*\check{A}}) = p \circ (f^{(0)}|^{\mathrm{Z}^*\check{A}}) \overset{(33)}{=} \varepsilon$. So since $\varepsilon : A^{0,-} \to \mathrm{H}^*\check{A}$ is surjective, the map $p \circ (f_1|_{A^{0,-}}^{\mathrm{Z}^*\check{A}})$ is surjective, which proves (P3). In particular, the map $\mathrm{H}^*f_1 : \mathrm{H}^*A \to \mathrm{H}^*\check{A}$ is surjective. For injectivity of H^*f_1, suppose given $x \in \mathrm{H}^*A$ such that $(\mathrm{H}^*f_1)(x) = 0$. By Lemma 80(i), we may represent x by an element $y \in A^{0,-} \subseteq \mathrm{Z}^*A$. We have $0 = (\mathrm{H}^*f_1)(x) = (p \circ (f_1|_{A^{0,-}}^{\mathrm{Z}^*\check{A}}))(y) = (p \circ (f^{(0)}|^{\mathrm{Z}^*\check{A}}))(y) \overset{(33)}{=} \varepsilon(y)$. Since (32) is exact, we have $\ker \varepsilon = \operatorname{im} d^{(1)}$. So there is $z \in A^{1,-}$ such that $d^{(1)}(z) = y$. Since $\operatorname{im} e^{(1)} \subseteq A^{\leq 1-2,-} = 0$, we have $y = d^{(1)}(z) = (d^{(1)} + e^{(1)})(z) = m_1(z)$. I.e. $y \in \mathrm{B}^*A$, whence $x = \bar{y} = 0$. Hence $\mathrm{H}^*f_1 : \mathrm{H}^*A \to \mathrm{H}^*\check{A}$ is injective, thus bijective. Thus we have proven that $(f_1) : (A, (m_1)) \to (\check{A}, (m_1'))$ is a quasi-isomorphism of A_1-algebras.

Finally, we have for $i, j \in \mathbb{Z}$

$$m_1|_{A^{j,i}}^{\oplus_{j' \in [0,j-1]} A^{j',i+1}} = d^{j,i} + e^{j,i},$$

where $e^{j,i} : A^{j,i} \to \oplus_{j' \in [0,j-2]} A^{j',i+1}$ is given by $e^{j,i} := e^{(j)}|_{A^{j,i}}^{\oplus_{j' \in [0,j-2]} A^{j',i+1}}$. □

Lemma 82 (cf. e.g. [3, VII §3]). *Suppose R is a principal ideal domain. Then every module over R has projective dimension ≤ 1.*

Proof. Suppose given an R-module M. Choose a cover $\varepsilon : P \twoheadrightarrow M$ such that P is free. The sequence

$$\ker\varepsilon \overset{\subseteq}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} M \tag{40}$$

is short exact. By [3, VII §3 Corollaire 2], $\ker\varepsilon \subseteq P$ is free. So (40) is a free (so in particular a projective) resolution of M of length ≤ 1. $\square$

Corollary 83 given below shows that if in Proposition 81, the projective resolutions $P^{(i)}$ all have length ≤ 1, we obtain $e^{j,i} = 0$ for all $i, j \in \mathbb{Z}$. This may be used in case R is a principal ideal domain, since then all R-modules have projective dimension ≤ 1, cf. Lemma 82.

Corollary 83. *Suppose given an* A_1*-algebra* $(\check{A}, (\check{m}_k)_{k\in[1,1]})$. *Suppose given a projective resolution* $P^{(i)} = (\ldots \to P^{2,i-2} \xrightarrow{d^{2,i-2}} P^{1,i-1} \xrightarrow{d^{1,i-1}} P^{0,i} \xrightarrow{d^{0,i}} 0 \to \ldots)$ *of* $\mathrm{H}^i\check{A}$ *with augmentation* $\varepsilon_i : P^{0,i} \to \mathrm{H}^i\check{A}$ *with length* ≤ 1 *for each* $i \in \mathbb{Z}$.

Let $A = \oplus_{j,i\in\mathbb{Z}} A^{j,i}$ *be given by* $A^{j,i} := P^{j,i}$.

Then there is a morphism $f_1 : A \to \check{A}$ *such that the following hold.*

- *The pair* $(A, (m_k)_{k\in[1,1]})$, *where* $m_1 : A \to A$ *is given by*

$$m_1|_{A^{j,i}} := d^{j,i} \quad \text{for } j, i \in \mathbb{Z},$$

  is a minimal eA_1*-algebra.*
- $(f_k)_{k\in[1,1]} : (A, (m_k)_{k\in[1,1]}) \to (\check{A}, (\check{m}_k)_{k\in[1,1]})$ *is a quasi-isomorphism of* A_1*-algebras.*
- *(P1), (P2) and (P3) hold.*

Proof. We apply Proposition 81 to obtain $e^{j,i} : A^{j,i} \to \oplus_{j'\in[0,j-2]} A^{j',i+1}$ for $i, j \in \mathbb{Z}$ and $f_1 : A \to \check{A}$. Given the assertions of Proposition 81, it suffices to prove $e^{j,i} = 0$ for $j, i \in \mathbb{Z}$. For $i \in \mathbb{Z}$, $j \in \mathbb{Z} \setminus \{0, 1\}$, the domain of $e^{j,i}$ is $A^{j,i} = 0$, hence $e^{j,i} = 0$. For $i \in \mathbb{Z}$, $j \in \{0, 1\}$, the codomain of $e^{j,i}$ is $\oplus_{j'\in[0,j-2]} A^{j',i+1} = 0$, hence $e^{j,i} = 0$. Thus $e^{j,i} = 0$ for $j, i \in \mathbb{Z}$. $\square$

4.3.3. The incremental step

Suppose we have the situation given in Remark 78.

We will show that there exist $m_n : A^{\otimes n} \to A$ and $f_n : A^{\otimes n} \to \check{A}$ such that $(A, (m_k)_{k\in[1,n]})$ is a minimal eA_n-algebra and $(f_k)_{k\in[1,n]}$ is a quasi-isomorphism of A_n-algebras from A to $\check{A}$ (Proposition 88).

By the bar construction, we have corresponding triples $((m_k)_{k\in[1,n-1]}, (b_k)_{k\in[1,n-1]}, *)$, $((f_k)_{k\in[1,n-1]}, (F_k)_{k\in[1,n-1]}, *)$ and $((\check{m}_k)_{k\in[1,n]}, (\check{b}_k)_{k\in[1,n]}, \check{b})$, cf. Definition/Remark 47. We will solve the equivalent problem of constructing suitable maps $b_n : (SA)^{\otimes n} \to SA$

and $F_n : (SA)^{\otimes n} \to S\check{A}$, cf. Theorem 48 and Lemma 50. For the complexes (SA, b_1) and $(S\check{A}, \check{b}_1)$, we will use the usual notation for boundaries, cycles and homology. As $f_1 : (A, m_1) \to (\check{A}, \check{m}_1)$ is a quasi-isomorphism, the complex morphism $F_1 : (SA, b_1) \to (S\check{A}, \check{b}_1)$ is a quasi-isomorphism. For $k \in [0, n]$, let

$$\begin{aligned} \pi_{\leq k} : TSA_{\leq n} = \oplus_{i\in[1,n]} SA^{\otimes i} &\longrightarrow \oplus_{i\in[1,k]} SA^{\otimes i} = TSA_{\leq k} \quad \text{and} \\ \iota_{\leq k} : TSA_{\leq k} = \oplus_{i\in[1,k]} SA^{\otimes i} &\longrightarrow \oplus_{i\in[1,n]} SA^{\otimes i} = TSA_{\leq n} \end{aligned}$$

be the projection and inclusion maps. Let

$$b_1^{(n)} := \sum_{r\in[0,n-1]} 1^{\otimes r} \otimes b_1 \otimes 1^{\otimes n-r-1} : (SA)^{\otimes n} \to (SA)^{\otimes n}. \tag{41}$$

For $j \in \mathbb{Z}$, we define $SA^{j,-} := \oplus_{i\in\mathbb{Z}} SA^{j,i}$, which is a graded direct summand of SA. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we define

$$SA^\alpha := SA^{\alpha_1,-} \otimes \ldots \otimes SA^{\alpha_n,-},$$

which is projective and a graded direct summand of $(SA)^{\otimes n}$. By (EA2), the module SA^α is the zero module unless $\alpha \in \mathbb{Z}^n_{\geq 0}$. Furthermore, $(SA)^{\otimes n} = \oplus_{\alpha\in\mathbb{Z}^n} SA^\alpha = \oplus_{\alpha\in\mathbb{Z}^n_{\geq 0}} SA^\alpha$. For $I \subseteq \mathbb{Z}^n$, we define

$$SA^I := \oplus_{\alpha\in I} SA^\alpha.$$

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, we define

$$R_\alpha := \{\beta \in \mathbb{Z}^n_{\geq 0} \mid \beta <_n \alpha\}.$$

We aim to construct F_n and b_n by constructing them on each summand of the direct sum $(SA)^{\otimes n} = \oplus_{\alpha\in\mathbb{Z}^n_{\geq 0}} SA^\alpha$ such that (12)[n] and (14)[n] hold. More explicitly, these equations are as follows.

$$(12)[n]: \qquad b_n \circ b_1^{(n)} + b_1 \circ b_n + \sum_{\substack{n=r+s+t,\\ r,t\geq 0,\ s\in[2,n-1]}} b_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) = 0$$

$$\begin{aligned} (14)[n]: \qquad & F_n \circ b_1^{(n)} + F_1 \circ b_n + \sum_{\substack{n=r+s+t\\ r,t\geq 0,\ s\in[2,n-1]}} F_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) \\ &= \check{b}_1 \circ F_n + \sum_{\substack{2\leq r\leq n\\ i_1+\ldots+i_r=n\\ \text{all } i_s\geq 1}} \check{b}_r \circ (F_{i_1} \otimes F_{i_2} \otimes \cdots \otimes F_{i_r}) \end{aligned}$$

Suppose given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Condition (EA3$'$) asserts for $k = 1$ that

$$b_1(SA^{j,-}) \subseteq \oplus_{j'\leq j-1} SA^{j',-} \tag{42}$$

for all $j \in \mathbb{Z}$, hence

$$
\begin{aligned}
b_1^{(n)}(SA^\alpha) &\subseteq \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b_1 \otimes 1^{n-i-1})(SA^{\alpha_1,-} \otimes \ldots \otimes SA^{\alpha_n,-}) \\
&\subseteq \sum_{i=0}^{n-1} SA^{(\alpha_1,\ldots,\alpha_i)} \otimes b_1(SA^{\alpha_{i+1},-}) \otimes SA^{(\alpha_{i+2},\ldots,\alpha_n)} \\
&\subseteq \sum_{i=0}^{n-1} \sum_{j' \leq \alpha_{i+1}-1} SA^{(\alpha_1,\ldots,\alpha_i,j',\alpha_{i+2},\ldots,\alpha_n)} \subseteq \bigoplus_{\beta \in R_\alpha} SA^\beta = SA^{R_\alpha}. \qquad (43)
\end{aligned}
$$

Hence, if (12)[n] and (14)[n] are evaluated at an element of SA^α, the b_n and F_n that occur are evaluated only at elements of $\oplus_{\beta \in (\mathbb{Z}^n_{\geq 0})_{\leq \alpha}} SA^\beta$. I.e. minimality of A, in particular the part of (EA3′) where $k = 1$, allows us to decouple the problem.

Definition 84. We call a triple (L, b_n, F_n) *admissible*, if (i)-(vi) hold.

(i) $L \subseteq \mathbb{Z}^n_{\geq 0}$ is a lower set.

(ii) $b_n : SA^L \to SA$ is a graded map of degree 1.

(iii) $F_n : SA^L \to S\check{A}$ is a graded map of degree 0.

(iv) Eq. (12)[n] holds on SA^L. That is

$$
\left[b_n \circ b_1^{(n)} + b_1 \circ b_n + \sum_{\substack{n=r+s+t, \\ r,t \geq 0,\ s \in [2,n-1]}} b_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) \right](x) = 0
$$

for $x \in SA^L$.

(v) Eq. (14)[n] holds on SA^L. That is

$$
\begin{aligned}
&\left[F_n \circ b_1^{(n)} + F_1 \circ b_n + \sum_{\substack{n=r+s+t \\ r,t \geq 0,\ s \in [2,n-1]}} F_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) \right](x) \\
&\qquad = \left[\check{b}_1 \circ F_n + \sum_{\substack{2 \leq r \leq n \\ i_1+\ldots+i_r=n \\ \text{all } i_s \geq 1}} \check{b}_r \circ (F_{i_1} \otimes F_{i_2} \otimes \cdots \otimes F_{i_r}) \right](x)
\end{aligned}
$$

for $x \in SA^L$.

(vi) For $(j_1, \ldots, j_n) \in L$, we have $b_n(SA^{(j_1,\ldots,j_n)}) \subseteq \oplus_{j' \leq (j_1+\ldots+j_n)+(2n-3)} SA^{j',-}$.

Let M be the set of admissible triples. The set M is partially ordered by the relation

$$
(\hat{L}, \hat{b}_n, \hat{F}_n) \leq (\tilde{L}, \tilde{b}_n, \tilde{F}_n) \quad :\Leftrightarrow \quad \hat{L} \subseteq \tilde{L} \text{ and } \tilde{b}_n|_{SA^{\hat{L}}} = \hat{b}_n \text{ and } \tilde{F}_n|_{SA^{\hat{L}}} = \hat{F}_n.
$$

The set M is nonempty since $(\emptyset, b_n : \{0\} \to SA, F_n : \{0\} \to S\check{A}) \in M$.

Lemma 85. *Suppose given $(\hat{L}, \hat{b}_n, \hat{F}_n) \in M$. Suppose given $\alpha \in \mathbb{Z}^n_{\geq 0}$ such that $R_\alpha \subseteq \hat{L}$. Then there exists $(\hat{L} \cup \{\alpha\}, \tilde{b}_n, \tilde{F}_n) \in M$ such that $(\hat{L} \cup \{\alpha\}, \tilde{b}_n, \tilde{F}_n) \geq (\hat{L}, \hat{b}_n, \hat{F}_n)$.*

Proof. We may assume $\alpha = (\alpha_1, \ldots, \alpha_n) \notin \hat{L}$. We write $\hat{L} \cup \{\alpha\} =: \tilde{L}$, which is a lower set. Suppose given a graded map $g : SA^{\otimes n} \to SA$ of degree 1 such that $g|_{SA^{\hat{L}}} = \hat{b}_n$ and such that

$$g(SA^\alpha) \subseteq \oplus_{j' \leq (\alpha_1 + \ldots + \alpha_n) + (2n-3)} SA^{j', -}. \tag{44}$$

Suppose given a graded map $h : SA^{\otimes n} \to S\check{A}$ of degree 0 such that $h|_{SA^{\hat{L}}} = \hat{F}_n$. We remark that such maps g, h exist since $SA^{\hat{L}} = \oplus_{\beta \in \hat{L}} SA^\beta$ is a graded direct summand of $SA^{\otimes n}$.

We define $b[g] : TSA_{\leq n} \to TSA_{\leq n}$ as the unique graded coderivation of degree 1 such that

$$\begin{aligned} \pi_1 \circ b[g] \circ \iota_k &= b_k \text{ for } k \in [1, n-1] \text{ and} \\ \pi_1 \circ b[g] \circ \iota_n &= g, \end{aligned} \tag{45}$$

cf. Lemma 43. So $(*, (b_k)_{k \in [1,n-1]} \sqcup (g), b[g])$ is a corresponding pre-A$_n$-triple.

We define $F[h] : TSA_{\leq n} \to TS\check{A}_{\leq n}$ as the unique graded coalgebra morphism of degree 0 such that

$$\begin{aligned} \check{\pi}_1 \circ F[h] \circ \iota_k &= F_k \text{ for } k \in [1, n-1] \text{ and} \\ \check{\pi}_1 \circ F[h] \circ \iota_n &= h, \end{aligned} \tag{46}$$

cf. Lemma 44. So $(*, (F_k)_{k \in [1,n-1]} \sqcup (h), F[h])$ is a corresponding pre-A$_n$-morphism triple.

The definition of $b[g]$ in (10) and the equation (41) yield

$$b[g] = \iota_n \circ b_1^{(n)} \circ \pi_n + \iota_{\leq n-1} \circ \pi_{\leq n-1} \circ b[g]. \tag{47}$$

In particular, we have

$$\pi_n \circ b[g] \circ \iota_n = b_1^{(n)}. \tag{48}$$

We also remark that since $R_\alpha \subseteq \hat{L}$, eq. (43) implies

$$b_1^{(n)}(SA^\alpha) \subseteq SA^{R_\alpha} \subseteq SA^{\hat{L}}. \tag{49}$$

Since $(\hat{L}, \hat{b}_n, \hat{F}_n)$ is admissible, property (iv) together with Theorem 49 yields

$$b[g]^2|_{SA^{\hat{L}}} = 0, \tag{50}$$

and property (v) together with Lemma 51 yields

$$\check{b} \circ F[h]|_{SA^{\hat{L}}} = F[h] \circ b[g]|_{SA^{\hat{L}}}. \tag{51}$$

Furthermore, Theorem 48 yields

$$b[g]^2|_{TSA_{\leq n-1}} = 0 \quad \text{and} \tag{52}$$
$$\check{b}^2 = 0. \tag{53}$$

Lemma 50 yields

$$\check{b} \circ F[h]|_{TSA_{\leq n-1}} = F[h] \circ b[g]|_{TSA_{\leq n-1}}\,. \tag{54}$$

We show that $b[g]^2(SA^\alpha) \subseteq \mathrm{B}^*SA$. We have

$$\begin{aligned} b[g]^3|_{SA^\alpha} &\overset{(47)}{=} \big(b[g]^2 \circ \iota_n \circ b_1^{(n)} \circ \pi_n + b[g]^2 \circ \iota_{\leq n-1} \circ \pi_{\leq n-1} \circ b[g]\big)\big|_{SA^\alpha} \\ &\overset{(52)}{=} b[g]^2 \circ \iota_n \circ b_1^{(n)} \circ \pi_n|_{SA^\alpha} \overset{(49),(50)}{=} 0. \end{aligned}$$

By (52) and Lemma 46(i), we have $b[g]^2(SA^\alpha) \subseteq b[g]^2(SA^{\otimes n}) \subseteq SA$.
Hence, $b_1 \circ \pi_1 \circ b[g]^2(SA^\alpha) = \pi_1 \circ b[g] \circ \iota_1 \circ \pi_1 \circ b[g]^2(SA^\alpha) = \pi_1 \circ b[g]^3(SA^\alpha) = 0$. So we conclude that $b[g]^2(SA^\alpha) \subseteq \mathrm{Z}^*SA$. We have

$$\begin{aligned} \check{\iota}_1 \circ \check{b}_1 \circ \check{\pi}_1 \circ (F[h] \circ b[g] - \check{b} \circ F[h])|_{SA^\alpha} &\underset{(10)}{\overset{(54),\mathrm{L}.46(ii)}{=}} \check{b} \circ (F[h] \circ b[g] - \check{b} \circ F[h])|_{SA^\alpha} \\ &\overset{(53),(47)}{=} \check{b} \circ F[h] \circ (\iota_n \circ b_1^{(n)} \circ \pi_n + \iota_{\leq n-1} \circ \pi_{\leq n-1} \circ b[g])|_{SA^\alpha} \\ &= \Big(\check{b} \circ F[h] \circ \iota_n \circ b_1^{(n)} \circ \pi_n + \check{b} \circ F[h] \circ \iota_{\leq n-1} \circ \pi_{\leq n-1} \circ b[g]\Big)\,|_{SA^\alpha} \\ &\overset{(54)}{=} \Big(\check{b} \circ F[h] \circ \iota_n \circ b_1^{(n)} \circ \pi_n + F[h] \circ b[g] \circ \iota_{\leq n-1} \circ \pi_{\leq n-1} \circ b[g]\Big)\,|_{SA^\alpha} \\ &\overset{(49),(51)}{=} \Big(F[h] \circ b[g] \circ \iota_n \circ b_1^{(n)} \circ \pi_n + F[h] \circ b[g] \circ \iota_{\leq n-1} \circ \pi_{\leq n-1} \circ b[g]\Big)\,|_{SA^\alpha} \\ &\overset{(47)}{=} F[h] \circ b[g]^2|_{SA^\alpha} \overset{(52),\mathrm{L}.46(i)}{=} F[h] \circ \iota_1 \circ \pi_1 \circ b[g]^2|_{SA^\alpha} \\ &\overset{(11)}{=} \check{\iota}_1 \circ F_1 \circ \pi_1 \circ b[g]^2|_{SA^\alpha}. \end{aligned} \tag{55}$$

So $b[g]^2(SA^\alpha) \subseteq \mathrm{Z}^*SA$ and F_1 maps $b[g]^2(SA^\alpha)$ to boundaries. Thus H^*F_1 maps the homology classes of the elements of $b[g]^2(SA^\alpha)$ to zero. Since F_1 is a quasi-isomorphism, the homology classes of the elements of $b[g]^2(SA^\alpha)$ vanish. Hence, $b[g]^2(SA^\alpha) \subseteq \mathrm{B}^*SA$.

We have

$$\begin{aligned} \pi_1 \circ b[g]^2 \circ \iota_n &= \sum_{k \in [1,n]} \pi_1 \circ b[g] \circ \iota_k \circ \pi_k \circ b[g] \circ \iota_n \\ &\overset{(45)}{=} b_1 \circ \pi_1 \circ b[g] \circ \iota_n + g \circ \pi_n \circ b[g] \circ \iota_n + \sum_{k \in [2,n-1]} b_k \circ \pi_k \circ b[g] \circ \iota_n \\ &\overset{(45),(48),(10)}{=} b_1 \circ g + g \circ b_1^{(n)} + \sum_{k \in [2,n-1]} \sum_{\substack{r+s+t=n,\\ r+1+t=k,\\ r,t \geq 0, s \geq 1}} b_k \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}). \end{aligned} \tag{56}$$

Up to now, g and h were arbitrary. Our strategy is the following. We fix some g and h, which do not necessarily satisfy (12)[n] and (14)[n] on SA^α. Then, we add some correction terms to g and h to cancel out the defect in (12)[n] and (14)[n] on SA^α.

So choose a $g =: g_1$ and a $h =: h_1$ as above. Recall $b[g_1]^2(SA^\alpha) \subseteq \mathrm{B}^*SA$.

Set $\tau := (2n-3) + \sum_{i=1}^n \alpha_i$. By (56), we have

$$
\begin{aligned}
\pi_1 \circ b[g_1]^2(SA^\alpha) &\subseteq b_1(g_1(SA^\alpha)) + g_1(b_1^{(n)}(SA^\alpha)) \\
&\quad + \sum_{k\in[2,n-1]} \sum_{\substack{r+s+t=n,\\ r+1+t=k,\\ r,t\geq 0, s\geq 1}} b_k((1^{\otimes r}\otimes b_s \otimes 1^{\otimes t})(SA^\alpha)) \\
&\overset{(44),(49),(\mathrm{EA3}')}{\subseteq} b_1(\oplus_{j\leq\tau} SA^{j,-}) + g_1\left(SA^{R_\alpha}\right) \\
&\quad + \sum_{k\in[2,n-1]} \sum_{\substack{r+s+t=n,\\ r+1+t=k,\\ r,t\geq 0, s\geq 1}} \sum_{j\leq(\alpha_{r+1}+\ldots+\alpha_{r+s})+(2s-3)} b_k\left(SA^{(\alpha_1,\ldots,\alpha_r,j,\alpha_{r+s+1},\ldots,\alpha_n)}\right) \\
&\overset{g_1|_{SA^{\hat{L}}}=\hat{b}_n}{=} b_1(\oplus_{j\leq\tau} SA^{j,-}) + \hat{b}_n\left(\sum_{i\in[1,n]}\sum_{j\leq\alpha_i-1} SA^{(\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\ldots,\alpha_n)}\right) \\
&\quad + \sum_{k\in[2,n-1]} \sum_{\substack{r+s+t=n,\\ r+1+t=k,\\ r,t\geq 0, s\geq 1}} \sum_{j\leq(\alpha_{r+1}+\ldots+\alpha_{r+s})+(2s-3)} b_k\left(SA^{(\alpha_1,\ldots,\alpha_r,j,\alpha_{r+s+1},\ldots,\alpha_n)}\right) \\
&\overset{(\mathrm{EA3}'),(vi),(\mathrm{EA3}')}{\subseteq} \oplus_{\tilde{j}\leq\tau-1} SA^{\tilde{j},-} + \oplus_{\tilde{j}\leq(\alpha_1+\ldots+\alpha_n)+(2n-4)} SA^{\tilde{j},-} \\
&\quad + \sum_{k\in[2,n-1]} \sum_{\substack{r+s+t=n,\\ r+1+t=k,\\ r,t\geq 0, s\geq 1}} \oplus_{\tilde{j}\leq(\alpha_1+\ldots+\alpha_n)+(2s-3)+(2k-3)} SA^{\tilde{j},-} \\
&\overset{k+s=n+1}{=} \oplus_{\tilde{j}\leq\tau-1} SA^{\tilde{j},-} + \oplus_{\tilde{j}\leq(\alpha_1+\ldots+\alpha_n)+(2(n+1)-3-3)} SA^{\tilde{j},-} \\
&= \oplus_{\tilde{j}\leq\tau-1} SA^{\tilde{j},-}.
\end{aligned}
$$

Hence, $b[g_1]^2(SA^\alpha) \subseteq \mathrm{B}^*SA \cap (\oplus_{j\leq\tau-1} SA^{j,-})$.

By (P2), the map $b_1|_{\oplus_{j\leq\tau}SA^{j,-}}^{\mathrm{B}^*SA\cap(\oplus_{j\leq\tau-1}SA^{j,-})}$ is an epimorphism. Since $\oplus_{j\leq\tau}SA^{j,-}$ is a graded direct summand of SA and since $b_1|_{\oplus_{j\leq\tau}SA^{j,-}}$ is a graded map, $\mathrm{B}^*SA \cap (\oplus_{j\leq\tau-1}SA^{j,-}) = b_1(\oplus_{j\leq\tau}SA^{j,-})$ is by Lemma 13 a graded submodule of SA. Hence, the map $b_1|_{\oplus_{j\leq\tau}SA^{j,-}}^{\mathrm{B}^*SA\cap(\oplus_{j\leq\tau-1}SA^{j,-})}$ is a graded epimorphism. Combined with the fact that SA^α is by Lemma 20 graded projective, we obtain a graded map $w : SA^\alpha \to SA$ of degree 1 with $\operatorname{im} w \subseteq \oplus_{j\leq\tau}SA^{j,-}$ such that $b_1 \circ w = -\pi_1 \circ b[g_1]^2|_{SA^\alpha}$. We define the graded map $g_2 : SA^{\otimes n} \to SA$ of degree 1 by $g_2|_{SA^\alpha} := g_1|_{SA^\alpha} + w$ and $g_2|_{SA^\beta} := g_1|_{SA^\beta}$ for $\beta \in \mathbb{Z}^n_{\geq 0} \setminus \{\alpha\}$. Note that g_2 satisfies the stipulations for g given above. We have

$$
\pi_1 \circ (b[g_2]^2 - b[g_1]^2)|_{SA^\alpha} \overset{(56)}{=} b_1 \circ (g_2 - g_1)|_{SA^\alpha} + (g_2 - g_1) \circ b_1^{(n)}|_{SA^\alpha}
$$

$$\stackrel{*}{=} b_1 \circ (g_2 - g_1)|_{SA^\alpha} = b_1 \circ w = -\pi_1 \circ b[g_1]^2|_{SA^\alpha},$$

where for $*$, we use (49) and $g_2|_{SA^{\hat{L}}} = g_1|_{SA^{\hat{L}}} = \hat{b}_n$. Hence,

$$\pi_1 \circ b[g_2]^2|_{SA^\alpha} = 0. \tag{57}$$

We have

$$\check{b}_1 \circ \check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1])(SA^\alpha) \stackrel{(55)}{=} F_1 \circ \pi_1 \circ b[g_2]^2(SA^\alpha) \stackrel{(57)}{=} 0.$$

I.e. $\check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1])(SA^\alpha) \subseteq \mathrm{Z}^* S\check{A}$. We have the following diagram.

$$\begin{tikzcd}
SA^\alpha \arrow[r, "(\check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1]))|_{SA^\alpha}^{\mathrm{Z}^* S\check{A}}"] \arrow[dr, dotted, "\exists w'"'] & \mathrm{Z}^* S\check{A} \arrow[r, "p"] & \mathrm{H}^* S\check{A} \\
& SA^{0,-} \arrow[u, "F_1|_{SA^{0,-}}^{\mathrm{Z}^* S\check{A}}"] \arrow[ur, "p \circ \left(F_1|_{SA^{0,-}}^{\mathrm{Z}^* S\check{A}}\right)"'] &
\end{tikzcd} \tag{58}$$

Here, p is the residue class map. By (P3), the map $p \circ (F_1|_{SA^{0,-}}^{\mathrm{Z}^* S\check{A}})$ is surjective. Note that (42) implies $SA^{0,-} \subseteq \mathrm{Z}^* SA$, hence $\mathrm{Z}^* SA \cap SA^{0,-} = SA^{0,-}$. We have $|p \circ F_1|_{SA^{0,-}}^{\mathrm{Z}^* S\check{A}}| \ni 0$ and $|p \circ \check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1])| \ni 1$. Since SA^α is by Lemma 20 graded projective, there is a graded map $w' : SA^\alpha \to SA$ of degree 1 with $w'(SA^\alpha) \subseteq SA^{0,-} = \mathrm{Z}^* SA \cap SA^{0,-}$ such that

$$p \circ (\check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1]))|_{SA^\alpha}^{\mathrm{Z}^* S\check{A}} = p \circ (F_1|_{SA^{0,-}}^{\mathrm{Z}^* S\check{A}}) \circ w'|^{SA^{0,-}}.$$

Hence, $[F_1 \circ w' - \check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1])](SA^\alpha) \subseteq \ker p = \mathrm{B}^* S\check{A}$. Since $\check{b}_1|^{\mathrm{B}^* S\check{A}}$ is surjective with $|\check{b}_1| \ni 1$, the graded projectivity of SA^α provides a graded map $v : SA^\alpha \to S\check{A}$ of degree 0 such that $\check{b}_1 \circ v = F_1 \circ w' - \check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1])|_{SA^\alpha}$. I.e.

$$\check{\pi}_1 \circ F[h_1] \circ b[g_2]|_{SA^\alpha} - F_1 \circ w' = \check{\pi}_1 \circ \check{b} \circ F[h_1]|_{SA^\alpha} - \check{b}_1 \circ v. \tag{59}$$

We define the graded maps $g_3 : SA^{\otimes n} \to SA$ resp. $h_2 : SA^{\otimes n} \to S\check{A}$ of degree 1 resp. 0 by

$$\begin{array}{lll} g_3|_{SA^\alpha} := g_2|_{SA^\alpha} - w', & h_2|_{SA^\alpha} := h_1|_{SA^\alpha} - v & \text{and} \\ g_3|_{SA^\beta} := g_2|_{SA^\beta}, & h_2|_{SA^\beta} := h_1|_{SA^\beta} & \text{for } \beta \in \mathbb{Z}_{\geq 0}^n \setminus \{\alpha\}. \end{array}$$

Note that g_3 resp. h_2 satisfy the stipulations for g resp. h given above. In particular, assertion (44) holds since $w'(SA^\alpha) \subseteq SA^{0,-} \stackrel{n \geq 2,\ \text{all } \alpha_i \geq 0}{\subseteq} \oplus_{j \leq (\alpha_1 + \ldots + \alpha_n) + (2n-3)} SA^{j,-}$. We have

$$\begin{aligned}
\check{\pi}_1 \circ F[h_2] \circ b[g_3]|_{SA^\alpha} &= \sum_{k \in [1,n]} \check{\pi}_1 \circ F[h_2] \circ \iota_k \circ \pi_k \circ b[g_3]|_{SA^\alpha} \\
&\stackrel{(46)}{=} h_2 \circ \pi_n \circ b[g_3]|_{SA^\alpha} + F_1 \circ \pi_1 \circ b[g_3]|_{SA^\alpha} + \sum_{k \in [2,n-1]} F_k \circ \pi_k \circ b[g_3]|_{SA^\alpha}
\end{aligned}$$

$$\overset{(48),(45),(10)}{=} h_2 \circ b_1^{(n)}|_{SA^\alpha} + F_1 \circ g_3|_{SA^\alpha} + \sum_{k\in[2,n-1]} \sum_{\substack{r+s+t=n\\ r+1+t=k,\\ r,t\geq 0,\, s\geq 1}} F_k \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t})|_{SA^\alpha}$$

$$= h_2 \circ b_1^{(n)}|_{SA^\alpha} + F_1 \circ (g_2|_{SA^\alpha} - w') + \sum_{k\in[2,n-1]} \sum_{\substack{r+s+t=n\\ r+1+t=k,\\ r,t\geq 0,\, s\geq 1}} F_k \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t})|_{SA^\alpha}$$

$$\overset{*}{=} h_1 \circ b_1^{(n)}|_{SA^\alpha} + F_1 \circ (g_2|_{SA^\alpha} - w') + \sum_{k\in[2,n-1]} \sum_{\substack{r+s+t=n\\ r+1+t=k,\\ r,t\geq 0,\, s\geq 1}} F_k \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t})|_{SA^\alpha}$$

$$\overset{**}{=} \check{\pi}_1 \circ F[h_1] \circ b[g_2]|_{SA^\alpha} - F_1 \circ w' \overset{(59)}{=} \check{\pi}_1 \circ \check{b} \circ F[h_1]|_{SA^\alpha} - \check{b}_1 \circ v. \tag{60}$$

For $*$, we have used (49) and $h_1|_{SA^{\hat{L}}} = h_2|_{SA^{\hat{L}}}$. The step marked by $**$ is just the reversal of the first three steps with h_2 replaced by h_1 and g_3 replaced by g_2. We have

$$\check{\pi}_1 \circ \check{b} \circ F[h_2]|_{SA^\alpha} = \check{\pi}_1 \circ \check{b} \circ \check{\iota}_1 \circ \check{\pi}_1 \circ F[h_2]|_{SA^\alpha} + \sum_{k\in[2,n]} \check{\pi}_1 \circ \check{b} \circ \check{\iota}_k \circ \check{\pi}_k \circ F[h_2]|_{SA^\alpha}$$

$$\overset{(10),(46)}{=} \check{b}_1 \circ h_2|_{SA^\alpha} + \sum_{k\in[2,n]} \check{b}_k \circ \check{\pi}_k \circ F[h_2]|_{SA^\alpha}$$

$$\overset{(11)}{=} \check{b}_1 \circ h_2|_{SA^\alpha} + \sum_{k\in[2,n]} \sum_{\substack{i_1+\ldots+i_k=n,\\ \text{all } i_j\geq 1}} \check{b}_k \circ (F_{i_1} \otimes \ldots \otimes F_{i_k})|_{SA^\alpha}$$

$$= \check{b}_1 \circ (h_1|_{SA^\alpha} - v) + \sum_{k\in[2,n]} \sum_{\substack{i_1+\ldots+i_k=n,\\ \text{all } i_j\geq 1}} \check{b}_k \circ (F_{i_1} \otimes \ldots \otimes F_{i_k})|_{SA^\alpha}$$

$$= -\check{b}_1 \circ v + \check{b}_1 \circ h_1|_{SA^\alpha} + \sum_{k\in[2,n]} \sum_{\substack{i_1+\ldots+i_k=n,\\ \text{all } i_j\geq 1}} \check{b}_k \circ (F_{i_1} \otimes \ldots \otimes F_{i_k})|_{SA^\alpha}$$

$$\overset{*}{=} -\check{b}_1 \circ v + \check{\pi}_1 \circ \check{b} \circ F[h_1]|_{SA^\alpha} \overset{(60)}{=} \check{\pi}_1 \circ F[h_2] \circ b[g_3]|_{SA^\alpha}.$$

The step marked by $*$ is just the reversal of the first three steps with h_2 replaced by h_1. So we have

$$\check{\pi}_1 \circ \check{b} \circ F[h_2]|_{SA^\alpha} = \check{\pi}_1 \circ F[h_2] \circ b[g_3]|_{SA^\alpha}.$$

Using Lemma 46(ii) and (54), this yields $F[h_2] \circ b[g_3]|_{SA^\alpha} = \check{b} \circ F[h_2]|_{SA^\alpha}$. Since g_3 and h_2 satisfy the stipulations given for g and h, eq. (51) implies $F[h_2] \circ b[g_3]|_{SA^{\hat{L}}} = \check{b} \circ F[h_2]|_{SA^{\hat{L}}}$. Hence, we have $F[h_2] \circ b[g_3]|_{SA^{\check{L}}} = \check{b} \circ F[h_2]|_{SA^{\check{L}}}$.

We have

$$\pi_1 \circ b[g_3]^2|_{SA^\alpha} \overset{(57)}{=} \pi_1 \circ (b[g_3]^2 - b[g_2]^2)|_{SA^\alpha}$$

$$\overset{(56)}{=} b_1 \circ (g_3 - g_2)|_{SA^\alpha} + (g_3 - g_2) \circ b_1^{(n)}|_{SA^\alpha}$$

$$\overset{*}{=} b_1 \circ (g_3 - g_2)|_{SA^\alpha} = -b_1 \circ w' \overset{\operatorname{im} w' \subseteq \mathrm{Z}^* SA}{=} 0,$$

where for $*$, we use (49) and $g_3|_{SA^{\hat{L}}} = \hat{b}_n = g_2|_{SA^{\hat{L}}}$. Lemma 46(i) and (52) yield $b[g_3]^2|_{SA^\alpha} = 0$. Since g_3 satisfies the assumptions given for g, eq. (50) yields $b[g_3]^2|_{SA^{\hat{L}}} = 0$. Thus $b[g_3]^2|_{SA^{\tilde{L}}} = 0$.

By Theorem 49 and Lemma 51, we conclude that $(\tilde{L}, g_3|_{SA^{\tilde{L}}}, h_2|_{SA^{\tilde{L}}})$ is admissible. $\square$

Recall the poset M defined in Definition 84.

Lemma 86. *Every ascending chain in M has an upper bound.*

Proof. Suppose given a chain $C = \{({}_iL, {}_ib_n, {}_iF_n) : i \in I\} \subseteq M$ for a set I. We set $\tilde{L} := \cup_{i\in I}\, {}_iL$, which is a lower set. Since $\{{}_iL : i \in I\}$ is totally ordered by inclusion, the set of submodules $\{\oplus_{\beta\in {}_iL} SA^\beta : i \in I\}$ of $(SA)^{\otimes n}$ is totally ordered by inclusion. For each $a \in \oplus_{\beta\in\tilde{L}} SA^\beta$, there exist $\beta_1, \ldots, \beta_m \in \tilde{L}$ such that $a \in \oplus_{j=1}^m SA^{\beta_j}$. We conclude that

$$\bigoplus_{\beta\in\tilde{L}} SA^\beta = \bigcup_{i\in I} \left(\bigoplus_{\beta\in {}_iL} SA^\beta \right). \tag{61}$$

I.e. $\{\oplus_{\beta\in {}_iL} SA^\beta : i \in I\}$ is a set of submodules totally ordered by inclusion whose union is $\oplus_{\beta\in\tilde{L}} SA^\beta$. Hence since C is totally ordered, there are $\tilde{b}_n : \oplus_{\beta\in\tilde{L}} SA^\beta \to SA$ and $\tilde{F}_n : \oplus_{\beta\in\tilde{L}} SA^\beta \to S\check{A}$ such that $\tilde{b}_n|_{\oplus_{\beta\in {}_iL} SA^\beta} = {}_ib_n$ and $\tilde{F}_n|_{\oplus_{\beta\in {}_iL} SA^\beta} = {}_iF_n$ for all $i \in I$. The $({}_iL, {}_ib_n, {}_iF_n)$ are admissible for $i \in I$, so assertions (ii)-(vi) hold for $({}_iL, {}_ib_n, {}_iF_n)$ for $i \in I$. So because of (61), assertions (ii)-(vi) hold for $(\tilde{L}, \tilde{b}_n, \tilde{F}_n)$. Thus $(\tilde{L}, \tilde{b}_n, \tilde{F}_n) \in M$ is an upper bound of C. $\square$

Lemma 87. *For every $x = (L, b_n, F_n) \in M$, there exists $(\tilde{L}, \tilde{b}_n, \tilde{F}_n) \in M_{\geq x}$ with $\tilde{L} = \mathbb{Z}^n_{\geq 0}$.*

This is a stronger statement than Proposition 88. Lemma 87 may be useful for computation, since it shows that any "partial solution" may be extended to a complete solution. Note also that Lemma 87 may easily deduced from Zorn's lemma: Lemma 86 and Zorn's lemma show that there exists a maximal element $(\tilde{L}, \tilde{b}_n, \tilde{F}_n) \in M_{\geq x}$. Then Lemma 85 and Lemma 75(b) show that $\tilde{L} = \mathbb{Z}^n_{\geq 0}$. However, the uncomplicated structure of the poset $(\mathbb{Z}^n_{\geq 0}, \leq_n)$ allows us to give a more explicit proof as follows.

Proof. By Lemma 86, it suffices to construct an ascending chain $\{({}_kL, {}_kb_n, {}_kF_n) : k \in \mathbb{Z}_{\geq 0}\} \subseteq M_{\geq x}$ such that $\cup_{k\in\mathbb{Z}_{\geq 0}}\, {}_kL = \mathbb{Z}^n_{\geq 0}$.

Firstly, we construct a certain sequence $({}_kL)_{k\geq 0}$ with ${}_0L = L$ and $\cup_{k\in\mathbb{Z}_{\geq 0}}\, {}_kL = \mathbb{Z}^n_{\geq 0}$ by induction on $k \geq 0$: For $k = 0$, set ${}_0L := L$. For the induction step, suppose given ${}_kL$ for some $k \geq 0$. Let D_k be the set of minimal elements in $\mathbb{Z}^n_{\geq 0} \setminus {}_kL$. We set ${}_{k+1}L := D_k \cup {}_kL$. This completes the induction. We postpone the proof of $\cup_{k\in\mathbb{Z}_{\geq 0}}\, {}_kL = \mathbb{Z}^n_{\geq 0}$.

Using the construction principle given in Lemma 134, we construct ${}_kb_n$ and ${}_kF_n$ successively on $k \geq 0$ such that $\{({}_kL, {}_kb_n, {}_kF_n) : k \in \mathbb{Z}_{\geq 0}\} \subseteq M_{\geq x}$ is an ascending chain.

For $k = 0$, set ${}_0b_n := b_n$ and ${}_0F_n := F_n$. I.e. $({}_0L, {}_0b_n, {}_0F_n) = (L, b_n, F_n) =: x_0$. For the incremental step, suppose given $({}_{k'}L, {}_{k'}b_n, {}_{k'}F_n) =: x_{k'}$ for $k' \in [1, k]$ such that $x_0 \leq \ldots \leq x_k$. We have $D_k = {}_{k+1}L \setminus {}_kL$, which is the set of minimal elements in $\mathbb{Z}^n_{\geq 0} \setminus {}_kL$. In particular, we have $R_\alpha \subseteq {}_kL$ for $\alpha \in D_k$. Since two distinct minimal elements are incomparable, D_k is discrete. Lemma 75(c) states that $(\mathbb{Z}^n_{\geq 0}, \leq_n)$ is narrow, so D_k is finite. So by successively applying Lemma 85 for each element of D_k, we obtain $({}_{k+1}L, {}_{k+1}b_n, {}_{k+1}F_n) \geq ({}_kL, {}_kb_n, {}_kF_n)$. This completes the incremental step.

It remains to show that $\cup_{k \in \mathbb{Z}_{\geq 0}} {}_kL = \mathbb{Z}^n_{\geq 0}$. Suppose given $z \in \mathbb{Z}^n_{\geq 0}$. By Lemma 75(a), the set $Q_k := \{y \in \mathbb{Z}^n_{\geq 0} \setminus {}_kL \mid y \leq_n z\} \subseteq (\mathbb{Z}^n_{\geq 0})_{\leq z}$ is finite for $k \in \mathbb{Z}_{\geq 0}$. Let $\tilde{D}_k$ be the set of minimal elements in Q_k. We show that $\tilde{D}_k = D_k \cap Q_k$. It suffices to show that $\tilde{D}_k \subseteq D_k \cap Q_k$. So suppose given $y \in \tilde{D}_k$. Suppose given $y' \in \mathbb{Z}^n_{\geq 0} \setminus {}_kL$ with $y' \leq_n y$. So $y' \leq_n y \leq_n z$, hence $y' \in Q_k$. So $y' = y$, since y is minimal in Q_k. Thus, y is minimal in $\mathbb{Z}^n_{\geq 0} \setminus {}_kL$. So $y \in D_k \cap Q_k$. We conclude that $\tilde{D}_k = D_k \cap Q_k$. Since ${}_{k+1}L = {}_kL \cup D_k$, we have $Q_{k+1} = Q_k \setminus D_k = Q_k \setminus (Q_k \cap D_k) = Q_k \setminus \tilde{D}_k$.
If $Q_k \neq \emptyset$, we have $\tilde{D}_k \neq \emptyset$ by Lemma 73. Hence, $|Q_k| > |Q_{k+1}|$ if $|Q_k| \neq 0$. Since $|Q_0|$ is finite, there is a k such that $Q_k = \emptyset$. In particular, we have $z \in {}_kL$. We conclude that $\cup_{k \in \mathbb{Z}_{\geq 0}} {}_kL = \mathbb{Z}^n_{\geq 0}$.

□

Application of Lemma 87 to $(\emptyset, b_n : \{0\} \to SA, F_n : \{0\} \to SA) \in M$ yields

Proposition 88. *Let $n \in \mathbb{Z}_{\geq 2}$. Let $(\check{A}, (\check{m}_k)_{k \geq 1})$ be an A_∞-algebra. Let $(A, (m_k)_{k \in [1,n-1]})$ be a minimal eA_{n-1}-algebra. Suppose there is a quasi-isomorphism of A_{n-1}-algebras $(f_k)_{k \in [1,n-1]}$ from A to $\check{A}$. Suppose that (P1), (P2) and (P3) hold.*

Then there exist $m_n : A^{\otimes n} \to A$ and $f_n : A^{\otimes n} \to \check{A}$ such that $(A, (m_k)_{k \in [1,n]})$ is a minimal eA_n-algebra and $(f_k)_{k \in [1,n]}$ is a quasi-isomorphism of A_n-algebras from A to $\check{A}$.

Successive application of Proposition 88, using the construction principle given in Lemma 134, yields

Proposition 89. *Let $n \in \mathbb{Z}_{\geq 1}$. Let $(\check{A}, (\check{m}_k)_{k \geq 1})$ be an A_∞-algebra. Let $(A, (m_k)_{k \in [1,n]})$ be a minimal eA_n-algebra. Suppose there is a quasi-isomorphism of A_n-algebras $(f_k)_{k \in [1,n]}$ from A to $\check{A}$. Suppose that (P1), (P2) and (P3) hold.*

Then there exist $(m_k)_{k \in [n+1,\infty]}$ and $(f_k)_{k \in [n+1,\infty]}$ such that $(A, (m_k)_{k \in [1,\infty]})$ is a minimal eA_∞-algebra and $(f_k)_{k \in [1,\infty]}$ is a quasi-isomorphism of A_∞-algebras from A to $\check{A}$.

Combination of Propositions 81 and 89 yields

Theorem 90. *Suppose given an A_∞-algebra $(\check{A}, (\check{m}_k)_{k \geq 1})$. Suppose given a projective resolution $(P^{(i)}, d^{(i)}) = (\ldots \to P^{2,i-2} \xrightarrow{d^{2,i-2}} P^{1,i-1} \xrightarrow{d^{1,i-1}} P^{0,i} \xrightarrow{d^{0,i}} 0 \to \ldots)$ of $\mathrm{H}^i\check{A}$ with augmentation $\varepsilon_i : P^{0,i} \to \mathrm{H}^i\check{A}$ for each $i \in \mathbb{Z}$. Let $A := \oplus_{i,j \in \mathbb{Z}} P^{j,i}$.*

Then there exists a minimal eA_∞*-structure* $(m_k)_{k\geq 1}$ *on* A *(cf. Proposition 81 for the relationship of* m_1 *and the* $d^{i,j}$*) such that there is a quasi-isomorphism of* A_∞*-algebras* $(f_k)_{k\geq 1} : (A, (m_k)_{k\geq 1}) \to (\check{A}, (\check{m}_k)_{k\geq 1})$.

This implies

Theorem 91. *Let* $(\check{A}, (\check{m}_k)_{k\geq 1})$ *be an* A_∞*-algebra.*

Then there exists a minimal eA_∞*-algebra* $(A, (m_k)_{k\geq 1})$ *and a quasi-isomorphism of* A_∞*-algebras* $(f_k)_{k\geq 1} : (A, (m_k)_{k\geq 1}) \to (\check{A}, (\check{m}_k)_{k\geq 1})$.

Question 92. What is the relationship between different minimal eA_∞-models ? Is there something like the uniqueness property given e.g. in [11, Theorem in section 3.3] ? Confer also [18, Théorème 4.27] and [11, Theorem in section 3.7].

Proposition 93. *Suppose given an* A_∞*-algebra* $(\check{A}, (\check{m}_k)_{k\geq 1})$*. Suppose given a complex* (A, m_1) *and a quasi-isomorphism of* A_1*-algebras* $f_1 : (A, m_1) \to (\check{A}, \check{m}_1)$*. Suppose that* A *and* B^*A *are projective.*

Then there exist $(m_k : A^{\otimes k} \to A)_{k\geq 2}$ *and* $(f_k : A^{\otimes k} \to \check{A})_{k\geq 2}$ *such that* $(A, (m_k)_{k\geq 1})$ *is an* A_∞*-algebra and* $(f_k)_{k\geq 1} : (A, (m_k)_{k\geq 1}) \to (\check{A}, (\check{m}_k)_{k\geq 1})$ *is a quasi-isomorphism of* A_∞*-algebras.*

Proof. The map $m_1|^{\mathrm{B}^*A} : A \to \mathrm{B}^*A$ is graded of degree 1 and surjective. The map $\mathrm{id}_{\mathrm{B}^*A} : \mathrm{B}^*A \to \mathrm{B}^*A$ is graded of degree 0. By Lemma 20, B^*A is graded projective, so there is a graded map $t : \mathrm{B}^*A \to A$ of degree -1 such that $(m_1|^{\mathrm{B}^*A}) \circ t = \mathrm{id}_{\mathrm{B}^*A}$.

$$\begin{tikzcd} A \arrow[r, two heads, "m_1|^{\mathrm{B}^*A}"] & \mathrm{B}^*A \\ & \mathrm{B}^*A \arrow[u, "\mathrm{id}_{\mathrm{B}^*A}"'] \arrow[ul, dotted, "\exists t"] \end{tikzcd}$$

Let $C := \operatorname{im} t$, which is a graded submodule of A. The map t is a section of $m_1|^{\mathrm{B}^*A}$ in the short exact sequence $\mathrm{Z}^*A \xrightarrow{\subseteq} A \xrightarrow{m_1|^{\mathrm{B}^*A}} \mathrm{B}^*A$. Thus by the splitting lemma, we have $A = \mathrm{Z}^*A \oplus (\operatorname{im} t) = \mathrm{Z}^*A \oplus C$. Since Z^*A and C are both graded submodules of A, the direct sum $A = \mathrm{Z}^*A \oplus C$ is a graded direct sum. For $i \in \mathbb{Z}$, we set $A^{0,i} := \mathrm{Z}^iA$ and $A^{1,i} := C^i$. For $i \in \mathbb{Z}$ and $j \in \mathbb{Z} \setminus \{0, 1\}$, we set $A^{j,i} := 0$. By construction, we have $A = \oplus_{i,j\in\mathbb{Z}} A^{j,i} = \oplus_{i\in\mathbb{Z}}(A^{0,i} \oplus A^{1,i})$. Since $\oplus_{i\in\mathbb{Z}} A^{0,i} = \mathrm{Z}^*A$, we have $m_1(\oplus_{i\in\mathbb{Z}} A^{0,i}) = 0$. Since $\oplus_{i\in\mathbb{Z}} A^{1,i} = C$ and since $m_1^2 = 0$, we have $m_1(\oplus_{i\in\mathbb{Z}} A^{1,i}) \subseteq \ker m_1 = \mathrm{Z}^*A = \oplus_{i\in\mathbb{Z}} A^{0,i}$. Hence, $(A, (m_k)_{k\in[1,1]})$ is a minimal eA_1-algebra. We have the quasi-isomorphism of A_1-algebras $(f_k)_{k\in[1,1]} : (A, (m_k)_{k\in[1,1]}) \to (\check{A}, (\check{m}_k)_{k\in[1,1]})$. We show (P1), (P2) and (P3). Assertion (P1) holds since A is projective. Consider (P2). For $j < 0$, we have

$$m_1(\oplus_{j'\leq j} A^{j',-}) = m_1(\{0\}) = \{0\} = (\mathrm{B}^*A) \cap \{0\} = (\mathrm{B}^*A) \cap (\oplus_{j'\leq j-1} A^{j',-}).$$

For $j = 0$, we have

$$m_1(\oplus_{j'\leq j} A^{j',-}) = m_1(\mathrm{Z}^*A) = \{0\} = (\mathrm{B}^*A) \cap \{0\} = (\mathrm{B}^*A) \cap (\oplus_{j'\leq j-1} A^{j',-}).$$

For $j = 1$, we have

$$\begin{aligned} m_1(\oplus_{j' \le j} A^{j',-}) &= m_1(A) = \mathrm{B}^*A = (\mathrm{B}^*A) \cap (\mathrm{Z}^*A) = (\mathrm{B}^*A) \cap A^{0,-} \\ &= (\mathrm{B}^*A) \cap (\oplus_{j' \le j-1} A^{j',-}) \, . \end{aligned}$$

For $j \ge 2$, we have

$$m_1(\oplus_{j' \le j} A^{j',-}) = m_1(A) = \mathrm{B}^*A = (\mathrm{B}^*A) \cap A = (\mathrm{B}^*A) \cap (\oplus_{j' \le j-1} A^{j',-}).$$

This proves (P2). Since $f_1 : (A, m_1) \to (\check{A}, m_1)$ is a quasi-isomorphism, the composite $p \circ (f_1|_{\mathrm{Z}^*A}^{\mathrm{Z}^*\check{A}})$ is surjective, where $p : \mathrm{Z}^*\check{A} \to \mathrm{H}^*\check{A}$ is the residue class map. So since $\mathrm{Z}^*A = A^{0,-}$, we have (P3). Application of Proposition 89 completes the proof.

□

As a special case of Proposition 93, we obtain the well-known

Proposition 94 (cf. e.g. [14, Théorème 1.4.1.1])**.** *Suppose that R is a field. Suppose given an A_∞-algebra $(\check{A}, (\check{m}_k)_{k \ge 1})$. Suppose given a complex (A, m_1) and a quasi-isomorphism of A_1-algebras $f_1 : (A, m_1) \to (\check{A}, \check{m}_1)$.*

Then there exist $(m_k : A^{\otimes k} \to A)_{k \ge 2}$ and $(f_k : A^{\otimes k} \to \check{A})_{k \ge 2}$ such that $(A, (m_k)_{k \ge 1})$ is an A_∞-algebra and $(f_k)_{k \ge 1} : (A, (m_k)_{k \ge 1}) \to (\check{A}, (\check{m}_k)_{k \ge 1})$ is a quasi-isomorphism of A_∞-algebras.

Proof. Since R is a field, all R-modules are projective over R. In particular, A and B^*A are projective. Apply Proposition 93. □

4.3.4. eA_n-categories

Definition 95. Let $n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. An eA_n*-category* is a triple $(\mathrm{Obj}\, A, A, (m_k)_{k \in [1,n]})$, where $A = \oplus_{(k,l,j,i) \in \mathrm{Obj}\, A \times \mathrm{Obj}\, A \times \mathbb{Z} \times \mathbb{Z}} A(k,l)^{j,i}$ is a $\mathrm{Obj}\, A \times \mathrm{Obj}\, A \times \mathbb{Z} \times \mathbb{Z}$-graded R-module such that the following hold.

(a) $(A, (m_k)_{k \in [1,n]})$ is an eA_n-algebra if the grading of A along $\mathrm{Obj}\, A \times \mathrm{Obj}\, A$ is suppressed.

(b) $(\mathrm{Obj}\, A,\ A,\ (m_k)_{k \in [1,n]})$ is an A_n-category if we suppress the grading along j of $A = \oplus_{(k,l,j,i) \in \mathrm{Obj}\, A \times \mathrm{Obj}\, A \times \mathbb{Z} \times \mathbb{Z}} A(k,l)^{j,i}$.

An eA_n-category is called *minimal* if the underlying eA_n-algebra (cf. (a)) is a minimal eA_n-algebra, cf. Definition 76.

We define eA_n*-functors* (also called *morphisms of* eA_n*-categories*) between two eA_n-categories to be the A_n-functors between the underlying A_n-categories (cf. (b)). An eA_n-functor is a *(local) quasi-isomorphism* if the underlying A_n-functor is a (local) quasi-isomorphism.

In sections 4.3.1 to 4.3.3, we developed methods for constructing minimal eA_n-algebras but we did not cover eA_n-categories at all. This approach was chosen in order to be able to introduce the new concept of eA_n-algebras without the additional notational complexity of eA_n-categories. All results of sections 4.3.2 and 4.3.3 can be adapted to eA_n-categories by performing the constructions component-wise on suitable directs sums:

For the incremental step, we modify Remark 78 to

Remark 96 (setup of the incremental step for eA_n-categories)**.** The incremental step will be performed in the following situation:

- $n \in \mathbb{Z}_{\geq 2}$.
- $(\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ is an A_∞-category.
- $(\operatorname{Obj} A, A, (m_k)_{k\in[1,n-1]})$ is a minimal eA_{n-1}-category.
- $(f_{\mathrm{Obj}}, (f_k)_{k\in[1,n-1]})$ is a quasi-isomorphism of A_{n-1}-categories from A to $\check{A}$.
- Assertions (P1) - (P3) hold. Recall that they are given as follows.
  - (P1) A is projective over R.
  - (P2) For all $j \in \mathbb{Z}$, we have $m_1(\oplus_{j'\leq j} A^{j',-}) = (\mathrm{B}^* A) \cap (\oplus_{j'\leq j-1} A^{j',-})$.
  - (P3) $p \circ (f_1|^{\mathrm{Z}^*\check{A}}_{A^{0,-}})$ is surjective, where $p : \mathrm{Z}^*\check{A} \to \mathrm{H}^*\check{A}$ is the residue class map.
- $\operatorname{Obj} A = \operatorname{Obj} \check{A}$ and f_{Obj} is the identity map.

The initial step is performed as follows. Set $\operatorname{Obj} A := \operatorname{Obj} \check{A}$, $f_{\mathrm{Obj}} = \mathrm{id}$. Then for each $i, j \in \operatorname{Obj} A$, we obtain the eA_1-algebra $(A(i,j), m_1|^{A(i,j)}_{A(i,j)})$ and the morphism of A_1-algebras $f_1|^{\check{A}(i,j)}_{A(i,j)} : (A(i,j), m_1|^{A(i,j)}_{A(i,j)}) \to (\check{A}(i,j), \check{m}_1|^{\check{A}(i,j)}_{\check{A}(i,j)})$ by applying Proposition 81 (resp. Corollary 83 if there are projective resolutions of the $\mathrm{H}^k\check{A}(i,j)$, $k \in \mathbb{Z}$ with length ≤ 1) to the A_1-algebra $(\check{A}(i,j), \check{m}_1|^{\check{A}(i,j)}_{\check{A}(i,j)})$. So we have attained the setup given in Remark 96 for $n = 2$.

The incremental step is performed as follows. Given A_0-categories $(\operatorname{Obj} A, A, ())$ and $(\operatorname{Obj} A, A', ())$, we denote an R-linear map $f : D \subseteq A^{\otimes k} \to A'$ for some $k \geq 1$ and some R-module $D \subseteq A^{\otimes k}$ to be *category-compatible* if it satisfies the following (cf. e.g. Definition 29).

(1) Given $i_y, j_y \in \operatorname{Obj} A$ for $y \in [1,k]$ such that there exists $x \in [1, k-1]$ with $j_x \neq i_{x+1}$, we have

$$f((A(i_1, j_1) \otimes \ldots \otimes A(i_k, j_k)) \cap D) = 0.$$

(2) Given $i_y \in \operatorname{Obj} A$ for $y \in [1, k+1]$, we have

$$f((A(i_1, i_2) \otimes A(i_2, i_3) \otimes \ldots \otimes A(i_k, i_{k+1})) \cap D) \subseteq A'(i_1, i_{k+1}).$$

Note that composites and sums of category-compatible maps are also category-compatible. E.g. the left hand side of (4) is category-compatible if all m_k are category-compatible.

In Definition 84, we need to add the requirement that b_n and F_n are category-compatible to the definition of admissibility. We now examine Lemma 85. In the proof we additionally require that g and h and thus also g_1, g_2, g_3, h_1, h_2 are category-compatible. For existence of category-compatible g and h, note that the category-compatible maps $\hat{b}_n$ and $\hat{F}_n$ are defined on

$$SA^{\hat{L}} = \bigoplus_{\substack{(\alpha_1,\ldots,\alpha_n)\in\hat{L}, k_1,\ldots,k_n\in\mathbb{Z},\\ i_1,\ldots,i_n,j_1,\ldots,j_n \operatorname{Obj} A}} SA(i_1,j_1)^{\alpha_1,k_1} \otimes \ldots \otimes SA(i_n,j_n)^{\alpha_n,k_n}.$$

Thus we obtain suitable g resp. h on

$$SA^{\otimes n} = \bigoplus_{\substack{\alpha_1,\ldots,\alpha_n\in\mathbb{Z}, k_1,\ldots,k_n\in\mathbb{Z},\\ i_1,\ldots,i_n,j_1,\ldots,j_n \operatorname{Obj} A}} SA(i_1,j_1)^{\alpha_1,k_1} \otimes \ldots \otimes SA(i_n,j_n)^{\alpha_n,k_n}$$

by extending $\hat{b}_n$ resp. $\hat{F}_n$ by zero to the remaining summands.

Consider the construction of g_2. Since g_1 and h_1 are category-compatible and since b_k is category-compatible for $k \in [1, n-1]$, the map $\pi_1 \circ b[g_1]^2 \circ \iota_n$ is by (56) category-compatible. Since $b_1|_{\oplus_{j\leq\tau} SA^{j,-}}^{\mathrm{B}^* SA \cap (\oplus_{j\leq\tau-1} SA^{j,-})}$ is by (P2) a graded epimorphism and since b_1 is category-compatible, the map $b_1|_{\oplus_{j\leq\tau} SA(o_1,o_2)^{j,-}}^{\mathrm{B}^* SA \cap (\oplus_{j\leq\tau-1} SA(o_1,o_2)^{j,-})}$ is a graded epimorphism for all $o_1, o_2 \in \operatorname{Obj} A$. We have the direct sum $SA^\alpha = \oplus_{i_1,\ldots i_k, j_1,\ldots,j_k \in \operatorname{Obj} A} SA^\alpha \cap (SA(i_1,j_1) \otimes \ldots \otimes SA(i_k,j_k))$, so we can construct a graded category-compatible map $w : SA^\alpha \to SA$ of degree 1 with $\operatorname{im} w \subseteq \oplus_{j\leq\tau} SA^{j,-}$ such that $b_1 \circ w = -\pi_1 \circ b[g_1]^2|_{SA^\alpha}$ by constructing it on each summand of that direct sum: On summands of type (1) of the definition of category-compatibility, we set w to be zero. On summands of type (2), we use the graded projectivity of that summand to obtain w.

Since g_2 is obtained from the category-compatible maps g_1 and w, it is category-compatible.

Consider now the construction of g_3 and h_2. The map $p \circ \check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1])$ used in the construction of w' is category-compatible, cf. (10), (11). Since F_1 is category-compatible and since $p \circ (F_1|_{SA^{0,-}}^{\mathrm{Z}^* SA})$ is surjective, the map $(p|_{\mathrm{Z}^* SA(o_1,o_2)}^{\mathrm{H}^* SA(o_1,o_2)}) \circ (F_1|_{SA(o_1,o_2)^{0,-}}^{\mathrm{Z}^* SA(o_1,o_2)})$ is surjective for all $o_1, o_2 \in \operatorname{Obj} A$. So analogous to the way we refined the construction of w to ensure category-compatibility, we obtain a category-compatible w'. The map $\big(F_1 \circ w' - \check{\pi}_1 \circ (F[h_1] \circ b[g_2] - \check{b} \circ F[h_1])|_{SA^\alpha}\big)|^{\mathrm{B}^* S\check{A}}$ used in the construction of v is category-compatible. For the construction of w, we have already shown that $b_1|_{\oplus_{j\leq\tau} SA(o_1,o_2)^{j,-}}^{\mathrm{B}^* SA \cap (\oplus_{j\leq\tau-1} SA(o_1,o_2)^{j,-})}$ is a graded epimorphism for all $o_1, o_2 \in \operatorname{Obj} A$. So analogous to the refinement of the construction of w to ensure category-compatibility of w, we obtain a category-compatible v. Since g_3 and h_2 are obtained from the category-compatible maps g_2, h_1, w' and v, they are category-compatible. This concludes our examination of Lemma 85.

In Lemma 86, we remark that in its proof, admissibility of the $({}_iL, {}_ib_n, {}_iF_n)$ implies category-compatibility of the ${}_iL$ and ${}_ib_n$ for $i \in I$. This implies the category-compatibility of $\tilde{b}_n$ and $\tilde{F}_n$ and thus the admissibility of $(\tilde{L}, \tilde{b}_n, \tilde{F}_n)$.

All other parts of section 4.3.3 can be adapted directly for eA_n-categories. In particular, the adapted versions of Proposition 89 to Proposition 94 are given below. Note that by Lemma 34, the quasi-isomorphisms of A_∞-categories obtained below are also local quasi-isomorphisms.

Proposition 97. *Let $n \in \mathbb{Z}_{\geq 1}$. Let $(\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ be an A_∞-category. Let $(\operatorname{Obj} A, A, (m_k)_{k\in[1,n]})$ be a minimal eA_n-category. Suppose there is a quasi-isomorphism of A_n-categories $(f_{\mathrm{Obj}}, (f_k)_{k\in[1,n]})$ from A to $\check{A}$ such that f_{Obj} is bijective. Suppose (P1), (P2) and (P3) hold.*

Then there exist $(m_k)_{k\in[n+1,\infty]}$ and $(f_k)_{k\in[n+1,\infty]}$ such that $(\operatorname{Obj} A, (m_k)_{k\in[1,\infty]})$ is a minimal eA_∞-category and $(f_{\mathrm{Obj}}, (f_k)_{k\in[1,\infty]})$ is a quasi-isomorphism of A_∞-categories from A to $\check{A}$.

Theorem 98. *Suppose given an A_∞-category $(\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$. Suppose given a projective resolution $(P^{(i)}_{o_1,o_2}, d^{(i)}_{o_1,o_2}) = (\ldots \to P^{2,i-2}_{o_1,o_2} \xrightarrow{d^{2,i-2}_{o_1,o_2}} P^{1,i-1}_{o_1,o_2} \xrightarrow{d^{1,i-1}_{o_1,o_2}} P^{0,i}_{o_1,o_2} \xrightarrow{d^{0,i}_{o_1,o_2}} 0 \to \ldots)$ of $\mathrm{H}^i \check{A}(o_1, o_2)$ with augmentation $\varepsilon_{i,o_1,o_1} : P^{0,i}_{o_1,o_2} \to \mathrm{H}^i \check{A}(o_1, o_2)$ for $i \in \mathbb{Z}$ and $o_1, o_2 \in \operatorname{Obj} \check{A}$. We have the eA_0-category $(\operatorname{Obj} \check{A}, A, ())$ given by $A := \oplus_{o_1,o_2\in \operatorname{Obj} \check{A}, j,i\in\mathbb{Z}} P^{j,i}_{o_1,o_2}$.*

Then there exist $(m_k)_{k\geq 1}$, $(f_k)_{k\geq 1}$ (cf. Proposition 81 for the relationship of the $m_1|^{A(o_1,o_2)}_{A(o_1,o_2)}$ and the $d^{i,j}_{o_1,o_2}$) such that $(\operatorname{Obj} \check{A}, A, (m_k)_{k\geq 1})$ is a minimal eA_∞-category and $(\mathrm{id}, (f_k)_{k\geq 1}) : (\operatorname{Obj} \check{A}, A, (m_k)_{k\geq 1}) \to (\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ is a quasi-isomorphism of A_∞-categories.

Theorem 99. *Let $(\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ be an A_∞-category.*

Then there exists a minimal eA_∞-category $(\operatorname{Obj} \check{A}, A, (m_k)_{k\geq 1})$ and a quasi-isomorphism of A_∞-categories $(\mathrm{id}, (f_k)_{k\geq 1}) : (\operatorname{Obj} \check{A}, A, (m_k)_{k\geq 1}) \to (\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$.

Proposition 100. *Suppose given an A_∞-category $(\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$. Suppose given an A_1-category $(\operatorname{Obj} A, A, (m_1))$. Suppose given a quasi-isomorphism of A_1-categories $(f_{\mathrm{Obj}}, (f_1)) : (\operatorname{Obj} A, A, (m_1)) \to (\operatorname{Obj} \check{A}, \check{A}, (\check{m}_1))$ such that f_{Obj} is bijective. Suppose that A and $\mathrm{B}^* A$ are projective.*

Then there exist $(m_k : A^{\otimes k} \to A)_{k\geq 2}$ and $(f_k : A^{\otimes k} \to \check{A})_{k\geq 2}$ such that $(\operatorname{Obj} A, A, (m_k)_{k\geq 1})$ is an A_∞-category and $(f_{\mathrm{Obj}}, (f_k)_{k\geq 1}) : (\operatorname{Obj} A, A, (m_k)_{k\geq 1}) \to (\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ is a quasi-isomorphism of A_∞-categories.

Proposition 101. *Suppose R is a field. Suppose given an A_∞-category $(\operatorname{Obj}\check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$. Suppose given an A_1-category $(\operatorname{Obj} A, A, (m_1))$ and a quasi-isomorphism of A_1-categories $(f_{\mathrm{Obj}}, (f_1)) : (\operatorname{Obj} A, A, (m_1)) \to (\operatorname{Obj} \check{A}, \check{A}, (\check{m}_1))$ such that f_{Obj} is bijective.*

Then there exist $(m_k : A^{\otimes k} \to A)_{k\geq 2}$ and $(f_k : A^{\otimes k} \to \check{A})_{k\geq 2}$ such that $(\operatorname{Obj} A, A, (m_k)_{k\geq 1})$ is an A_∞-category and $(f_{\mathrm{Obj}}, (f_k)_{k\geq 1}) : (\operatorname{Obj} A, A, (m_k)_{k\geq 1}) \to (\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ is a quasi-isomorphism of A_∞-categories.

4.3.5. Comparison with Sagave's dA_∞-algebras

Remark 102 (Comparison with Sagave's dA_∞-algebras)**.** The eA_∞-algebras are a variant of Sagave's dA_∞-algebras, cf. Definition 76 and [19, Definition 2.1]. Before we examine their relationship more closely, we will motivate the introduction of eA_∞-algebras.

We would like to apply Keller and Lefèvre-Hasegawa's filt construction (cf. section 6, [11, section 7.7]) to the representation theory of group rings RG, where R is not necessarily a field. But over arbitrary commutative ground rings, minimal models of A_∞-algebras may not exist, cf. Corollary 70. Sagave's dA_∞-algebras provide minimal models in the dA_∞-sense for arbitrary dg-algebras over arbitrary ground rings R, but it is unknown if the filt construction can be adapted for dA_∞-algebras. Thus, a generalization of A_∞-algebras was sought that has the following properties.

- In a certain sense, minimal models exist.
- Generalized A_∞-algebras have A_∞-algebras as underlying structure. Hence, the filt construction can be applied directly.

These generalized A_∞-algebras are named eA_∞-algebras reminiscent of the name of dA_∞-algebras.

So let us examine dA_∞-algebras and eA_∞-algebras more closely.

Suppose given an eA_∞-algebra $(A, (m_k)_{k\geq 1})$. For the remainder of this remark, we will denote the $\mathbb{Z}$-grading on A given by $A = \oplus_{j\in\mathbb{Z}} A^{j,-}$ the *horizontal grading* of A. The horizontal grading on A induces a grading on $A^{\otimes k}$, $k \geq 1$, which we also call horizontal. We will distinguish the usual gradings on A and $A^{\otimes k}$ from the horizontal gradings by calling the usual gradings *vertical*.

For dA_∞-algebras, the multiplication maps consist of horizontally and vertically graded maps $m_{ij} : A^{\otimes j} \to A$. A similar approach can be pursued for eA_∞-algebras:

Suppose given $k \geq 1$. For $j \in \mathbb{Z}$, denote by B^j the horizontally homogeneous component of degree j of $A^{\otimes k}$. For $j \in \mathbb{Z}$, denote by $p_j : A \to A^{j,-}$ the projection to the horizontally homogeneous component $A^{j,-}$ of degree j of A.
Suppose given $j' \in \mathbb{Z}$. We define the R-linear map $m_{j',k} : A^{\otimes k} = \oplus_{j\in\mathbb{Z}} B^j \to A$ by

$$m_{j',k}|_{B^j} := p_{j-j'} \circ m_k|_{B^j}.$$

The map $m_{j',k}$ is horizontally graded of degree $-j'$ and vertically graded of degree $2-k$. Note that given $x \in A^{\otimes k}$, we have $m_{j,k}(x) = 0$ for almost all $j \in \mathbb{Z}$. So abusing notation, we have

$$m_k = \sum_{j\in\mathbb{Z}} m_{j,k},$$

where for $x \in A^k$, we define $(\sum_{j\in\mathbb{Z}} m_{j,k})(x)$ to be $\sum_{j\in\mathbb{Z}, m_{j,k}(x)\neq 0} m_{j,k}(x)$.

In this formulation, eq. (4)[k] means that for $j \in \mathbb{Z}$, we have

$$0 = \sum_{\substack{k=r+s+t \\ r,t\geq 0, s\geq 1 \\ j'+j''=j}} (-1)^{rs+t} m_{j',r+1+t} \circ (1^{\otimes r} \otimes m_{j'',s} \otimes 1^{\otimes t}). \tag{62}$$

We note the following differences between dA$_\infty$-algebras and eA$_\infty$-algebras.

(i) The signs in (62) differ from the signs that appear in the defining equation of dA$_\infty$-algebras, cf. [19, eq. (2.2)].

(ii) The map $m_{j,k}$ of the eA$_\infty$-algebra given above has horizontal degree $-j$ and vertical degree $2-k$, whereas for a dA$_\infty$-algebra, the map m_{jk} has horizontal degree $-j$ and vertical degree $-(2-j-k)$.

(iii) For dA$_\infty$-algebras, the horizontal grading interacts with the Koszul sign rule. That is not the case for eA$_\infty$-algebras.

(iv) For dA$_\infty$-algebras, all maps m_{jk} with $j < 0$ vanish. For eA$_\infty$-algebras, due to (EA3) all maps $m_{j,k}$ with $j < -(2k-2)$ vanish.

One question that arises naturally is whether dA$_\infty$-algebras and eA$_\infty$-algebras are in some way compatible or whether somehow these differences might be mitigated:

Difference (ii) is due to different layouts of the gradings. It can be avoided by a suitable reparametrisation of the degrees. For (iv), the bounds for eA$_\infty$-algebras and in particular for minimal eA$_\infty$-algebras are chosen in such a way that they fit the bounds that are achieved by the extended Kadeishvili minimal method. Hence, difference (iv) is caused on the side of eA$_\infty$-algebras by mere convenience and not by some intentional decision.

Difference (i) together with (iii) effects that the bar construction for dA$_\infty$-algebras yields a twisted chain complex, cf. [19, Lemma 4.1], while the bar construction for eA$_\infty$-algebras (which is the bar construction of the underlying A$_\infty$-algebras) yields a chain complex. It is unknown to me if there exists a way to make twisted chain complexes into chain complexes and vice versa.

Finally, we briefly discuss some of the results available for dA$_\infty$-algebras and for eA$_\infty$-algebras. Sagave constructs in [19] minimal models in the dA$_\infty$-sense for arbitrary dg-algebras over arbitrary base rings, cf. [19, Theorem 1.1]. For eA$_\infty$-algebras, we obtain minimal models in the sense of eA$_\infty$-algebras for arbitrary A$_\infty$-algebras over arbitrary base rings, cf. Theorem 91, but the bounds for the degrees of the non-zero parts of the multiplications maps are weaker than those of dA$_\infty$-algebras, cf. difference (iv). Sagave uses model categories to obtain minimal models in the dA$_\infty$-sense, cf. [19, Theorem 3.4]. It is unknown to what extent the underlying modules of minimal models in the dA$_\infty$-sense can be chosen, cf. [19, Remark 4.14]. Hence, it is unknown how large such minimal models become in practice. For eA$_\infty$-algebras, the approach for constructing minimal models in the sense of eA$_\infty$-algebras is the following. Given an A$_\infty$-algebra $(A, (m_k)_{k\geq 1})$, we fix arbitrary projective resolutions $P^{(i)}$ of $\mathrm{H}^i A$ for $i \in \mathbb{Z}$ and form their direct sum $\oplus_{i\in\mathbb{Z}} P^{(i)} =: P$. Similar to Kadeishvili's algorithm, the desired minimal eA$_\infty$-structure is then constructed on P in an incremental way, cf. Theorem 90 and Propositions 81 and 89. In fact, this method is based on Kadeishvili's algorithm.

5. Models for cyclic groups over arbitrary ground rings

Suppose given a commutative ground ring R. Suppose $n \in \mathbb{Z}_{\geq 1}$.

We denote the cyclic group of order n by C_n. In C_n, we fix a generator e of C_n.

Definition 103. Let $a := 1 - e \in R\mathrm{C}_n$ and $b := \sum_{i=0}^{n-1} e^i \in R\mathrm{C}_n$. We define the $R\mathrm{C}_n$-linear maps

$$\begin{aligned} \alpha : R\mathrm{C}_n &\longrightarrow R\mathrm{C}_n, \quad x \mapsto a \cdot x \\ \beta : R\mathrm{C}_n &\longrightarrow R\mathrm{C}_n, \quad x \mapsto b \cdot x \\ \varepsilon : R\mathrm{C}_n &\longrightarrow R, \qquad \sum_{i=0}^{n-1} x_i e^i \mapsto \sum_{i=0}^{n-1} x_i\,, \end{aligned}$$

where the codomain R of ε is the trivial $R\mathrm{C}_n$-module.

We have $\ker \beta = \ker \varepsilon = \operatorname{im} \alpha = \{\sum_{i=0}^{n-1} x_i e^i \in R\mathrm{C}_n \mid \sum_{i=0}^{n-1} x_i = 0\}$.
We have $\ker \alpha = \operatorname{im} \beta = \{\sum_{i=0}^{n-1} x_i e^i \in R\mathrm{C}_n \mid x_0 = x_1 = \ldots = x_{n-1}\}$.
Hence, the sequence

$$P := (\cdots \to \underbrace{R\mathrm{C}_n}_{4} \xrightarrow{\beta} \underbrace{R\mathrm{C}_n}_{3} \xrightarrow{\alpha} \underbrace{R\mathrm{C}_n}_{2} \xrightarrow{\beta} \underbrace{R\mathrm{C}_n}_{1} \xrightarrow{\alpha} \underbrace{R\mathrm{C}_n}_{0} \to \underbrace{0}_{-1} \to \cdots), \tag{63}$$

where the positions are written underneath, is a projective resolution of the trivial $R\mathrm{C}_n$-module R, with augmentation ε.

Let $(A, (m_k)_{k \geq 1})$ be the dg-algebra on $A := \operatorname{Hom}^*_{R\mathrm{C}_n}(P, P)$ as given in Lemma 25.

Let A' be the free R-module on the set

$$\mathfrak{B} := \{\overline{\iota^j}, \overline{\chi\iota^j} \mid j \in \mathbb{Z}_{\geq 0}\}.$$

By stipulating that the basis element $\overline{\iota^j}$ is to be homogeneous of degree $|\overline{\iota^j}| := 2j$ and that the basis element $\overline{\chi\iota^j}$ is to be homogeneous of degree $|\overline{\chi\iota^j}| := 2j + 1$ for $j \in \mathbb{Z}_{\geq 0}$, the free module A' becomes a $\mathbb{Z}$-graded R-module. For convenience, let $\overline{\chi^0\iota^j} := \overline{\iota^j}$ and $\overline{\chi^1\iota^j} := \overline{\chi\iota^j}$ for $j \in \mathbb{Z}_{\geq 0}$.

Note that for $k \in \mathbb{Z}_{\geq 1}$, the set

$$\mathfrak{B}^{\otimes k} := \{\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \mid \text{ all } a_i \in \{0,1\}, \text{ all } j_i \in \mathbb{Z}_{\geq 0}\} \subseteq (A')^{\otimes k} \tag{64}$$

is an R-basis of $(A')^{\otimes k}$ consisting of homogeneous elements.

In A, we fix the following 2-periodic elements. For convenience, we highlight position 0 in the complexes by underlining it.

$$\iota := \left(\begin{tikzcd} \cdots \arrow[r] & R\mathrm{C}_n \arrow[r, "\beta"] \arrow[d, "1"] & R\mathrm{C}_n \arrow[r, "\alpha"] \arrow[d, "1"] & R\mathrm{C}_n \arrow[r, "\beta"] \arrow[d, "1"] & R\mathrm{C}_n \arrow[r, "\alpha"] \arrow[d, "1"] & R\mathrm{C}_n \arrow[r, "\beta"] \arrow[d, "1"] & R\mathrm{C}_n \arrow[r, "\alpha"] \arrow[d] & \underline{R\mathrm{C}_n} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & R\mathrm{C}_n \arrow[r, "\beta"] & R\mathrm{C}_n \arrow[r, "\alpha"] & R\mathrm{C}_n \arrow[r, "\beta"] & R\mathrm{C}_n \arrow[r, "\alpha"] & \underline{R\mathrm{C}_n} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd} \right) \in A^2$$

$$\iota_0 := \left(\begin{tikzcd} \cdots \arrow[r] & RC_n \arrow[r, "\beta"] \arrow[d, "0"] & RC_n \arrow[r, "\alpha"] \arrow[d, "1"] & RC_n \arrow[r, "\beta"] \arrow[d, "0"] & RC_n \arrow[r, "\alpha"] \arrow[d, "1"] & RC_n \arrow[r, "\beta"] \arrow[d, "0"] & RC_n \arrow[r, "\alpha"] \arrow[d] & \underline{RC_n} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & \underline{RC_n} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}\right) \in A^2$$

$$\iota_1 := \left(\begin{tikzcd} \cdots \arrow[r] & RC_n \arrow[r, "\beta"] \arrow[d, "1"] & RC_n \arrow[r, "\alpha"] \arrow[d, "0"] & RC_n \arrow[r, "\beta"] \arrow[d, "1"] & RC_n \arrow[r, "\alpha"] \arrow[d, "0"] & RC_n \arrow[r, "\beta"] \arrow[d, "1"] & RC_n \arrow[r, "\alpha"] \arrow[d] & \underline{RC_n} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & \underline{RC_n} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}\right) \in A^2$$

$$\chi_0 := \left(\begin{tikzcd} \cdots \arrow[r] & RC_n \arrow[r, "\beta"] \arrow[d, "1"] & RC_n \arrow[r, "\alpha"] \arrow[d, "0"] & RC_n \arrow[r, "\beta"] \arrow[d, "1"] & RC_n \arrow[r, "\alpha"] \arrow[d, "0"] & RC_n \arrow[r, "\beta"] \arrow[d, "1"] & RC_n \arrow[r, "\alpha"] \arrow[d, "0"] & \underline{RC_n} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & RC_n \arrow[r, "\alpha"] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & \underline{RC_n} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}\right) \in A^1$$

$$\chi_1 := \left(\begin{tikzcd} \cdots \arrow[r] & RC_n \arrow[r, "\beta"] \arrow[d, "0"] & RC_n \arrow[r, "\alpha"] \arrow[d, "1"] & RC_n \arrow[r, "\beta"] \arrow[d, "0"] & RC_n \arrow[r, "\alpha"] \arrow[d, "1"] & RC_n \arrow[r, "\beta"] \arrow[d, "0"] & RC_n \arrow[r, "\alpha"] \arrow[d, "1"] & \underline{RC_n} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & RC_n \arrow[r, "\alpha"] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & RC_n \arrow[r, "\beta"] & RC_n \arrow[r, "\alpha"] & \underline{RC_n} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}\right) \in A^1$$

Note that multiplication with ι is "shift by a period" in the 2-periodic projective resolution P. Hence, the 2-periodic elements ι_0, ι_1, χ_0 and χ_1 commute with ι and its powers ι^j for $j \geq 0$.

The ring RC_n is commutative, so given RC_n-modules M and N, the R-module structure on $\mathrm{Hom}_{RC_n}(M, N)$ canonically extends to an RC_n-module structure on $\mathrm{Hom}_{RC_n}(M, N)$, that is, given $f \in \mathrm{Hom}_{RC_n}(M, N)$ and $x \in RC_n$, we have $xf := (y \mapsto x \cdot f(y)) \in \mathrm{Hom}_{RC_n}(M, N)$. In this way, the $\mathbb{Z}$-graded R-module $A = \mathrm{Hom}^*_{RC_n}(P, P)$ becomes a $\mathbb{Z}$-graded RC_n-module.

Note the following relations. Given $j, j' \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} m_1(\chi_0\iota^j) &= a\iota^{j+1} & \qquad m_2(\chi_0\iota^j \otimes \chi_0\iota^{j'}) &= \chi_0\iota^j \circ \chi_0\iota^{j'} = 0 \\ m_1(\chi_1\iota^j) &= b\iota^{j+1} & \qquad m_2(\chi_1\iota^j \otimes \chi_1\iota^{j'}) &= \chi_1\iota^j \circ \chi_1\iota^{j'} = 0 \\ m_1(\iota^j) &= 0 & \qquad m_2(\chi_0\iota^j \otimes \chi_1\iota^{j'}) &= \chi_0\iota^j \circ \chi_1\iota^{j'} = \iota_0\iota^{j+j'} \\ (\iota_0 + \iota_1)\iota^j &= \iota^{j+1} & \qquad m_2(\chi_1\iota^j \otimes \chi_0\iota^{j'}) &= \chi_1\iota^j \circ \chi_0\iota^{j'} = \iota_1\iota^{j+j'}\,. \end{aligned} \tag{65}$$

Definition/Remark 104. Suppose given sequences $(r_k)_{k\geq 1}, (g_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ such that $r_k \in R$ and $g_k, h_k \in RC_n$ for $k \geq 1$.

We define the pre-A_∞-structure $(m'_k)_{k\geq 1}$ on A' and the pre-A_∞ morphism $(f_k)_{k\geq 1}$ from A' to A as follows. For $k \in \mathbb{Z}_{\geq 1}$, we define m'_k and f_k by defining them on the elements $\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}$ of the R-bases $\mathfrak{B}^{\otimes k}$ of $(A')^{\otimes k}$, cf.(64).

Case 1: Elements $\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \in \mathfrak{B}^{\otimes k}$ such that $0 \in \{a_1, \ldots, a_k\}$.

We set

$$m'_2(\overline{\chi^{a_1}\iota^{j_1}} \otimes \overline{\chi^{a_2}\iota^{j_2}}) := \overline{\chi^{a_1+a_2}\iota^{j_1+j_2}} \qquad \text{(Note that } a_1 + a_2 \in \{0,1\}.\text{)}$$
$$m'_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) := 0 \text{ for } k \in \mathbb{Z}_{\geq 1} \setminus \{2\}.$$

We set

$$\begin{aligned} f_1(\overline{\chi^0\iota^{j_1}}) = f_1(\overline{\iota^{j_1}}) &:= \iota^{j_1} \\ f_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) &:= 0 \text{ for } k \in \mathbb{Z}_{\geq 2}\,. \end{aligned} \tag{66}$$

Case 2: Elements $\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \in \mathfrak{B}^{\otimes k}$ such that $a_1 = \ldots = a_k = 1$.

For $k \in \mathbb{Z}_{\geq 1}$, we set

$$\begin{aligned} m'_k(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) &:= r_k \overline{\iota^{(j_1+\ldots+j_k)+1}} \\ f_k(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) &:= (g_k\chi_0 + h_k\chi_1)\iota^{j_1+\ldots+j_k}\,. \end{aligned}$$

Proof. We need to show that $(A', (m'_k)_{k\geq 1})$ is a pre-A_∞-algebra and that $(f_k)_{k\geq 1}$ is a pre-A_∞-morphism from A' to A. For this, we need to verify that the m'_k are graded of degree $2-k$ and the f_k are graded of degree $1-k$. This is done by comparing the degrees of the elements of the bases $\mathfrak{B}^{\otimes k}$ with the degrees of their images under m'_k respectively f_k. □

We say that an element x of an R-module is *R-torsion-free* if $rx \neq 0$ for $r \in R \setminus \{0\}$. Note that in this case, $\langle x\rangle_R$ is free over R with basis $\{x\}$.

Proposition 105. *Suppose that g_1 or h_1 is R-torsion-free. Suppose that*

$$g_k a + h_k b + \sum_{i\in[1,k-1]} g_i h_{k-i} = r_k \textit{ for } k \in \mathbb{Z}_{\geq 1}\,. \tag{67}$$

Then the tuple $(A', (m'_k)_{k\geq 1})$ given in Definition/Remark 104 is an A_∞-algebra and the tuple $(f_k)_{k\geq 1}$ is a morphism of A_∞-algebras from $(A', (m'_k)_{k\geq 1})$ to $(A, (m_k)_{k\geq 1})$.

The proof of Proposition 105 is similar to the proof of [20, Theorem 39].

For the proof of Proposition 105, we will need the Lemmas 106, 108 and 109.

Lemma 106. *The equations* (5)[1] *and* (5)[2] *hold.*

Proof. We check (5)[1] resp. (5)[2] by checking them on $\mathfrak{B}$ resp. $\mathfrak{B}^{\otimes 2}$: Concerning the Koszul sign rule, note that $|m'_1| \ni 1$ and $|f_1| \ni 0$. For $j \in \mathbb{Z}_{\geq 0}$, we have

$$(f_1 \circ m'_1)(\overline{\iota^j}) = f_1(0) \overset{(65)}{=} m_1(\iota^j) = (m_1 \circ f_1)(\overline{\iota^j})$$

$$(f_1\circ m_1')(\overline{\chi\iota^j}) = f_1(r_1\overline{\iota^{j+1}}) = r_1\iota^{j+1} \overset{(67)}{=} (g_1a+h_1b)\iota^{j+1}$$
$$\overset{(65)}{=} m_1((g_1\chi_0+h_1\chi_1)\iota^j) = (m_1\circ f_1)(\overline{\chi\iota^j}).$$

For $j,j'\in\mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}
&(f_1\circ m_2' - f_2\circ(m_1'\otimes 1 + 1\otimes m_1'))(\overline{\iota^j}\otimes\overline{\iota^{j'}}) = f_1(\overline{\iota^{j+j'}}) - f_2(0+0)\\
&\quad = \iota^{j+j'} = m_1(0) + m_2(\iota^j\otimes\iota^{j'}) = (m_1\circ f_2 + m_2\circ(f_1\otimes f_1))(\overline{\iota^j}\otimes\overline{\iota^{j'}})\\
&(f_1\circ m_2' - f_2\circ(m_1'\otimes 1 + 1\otimes m_1'))(\overline{\iota^j}\otimes\overline{\chi\iota^{j'}}) = f_1(\overline{\chi\iota^{j+j'}}) - f_2(0+\overline{\iota^j}\otimes r_1\overline{\iota^{j'+1}})\\
&\quad = (g_1\chi_0+h_1\chi_1)\iota^{j+j'} + 0 = m_1(0) + m_2(\iota^j\otimes(g_1\chi_0+h_1\chi_1)\iota^{j'})\\
&\quad = (m_1\circ f_2 + m_2\circ(f_1\otimes f_1))(\overline{\iota^j}\otimes\overline{\chi\iota^{j'}})\\
&(f_1\circ m_2' - f_2\circ(m_1'\otimes 1 + 1\otimes m_1'))(\overline{\chi\iota^j}\otimes\overline{\iota^{j'}}) = f_1(\overline{\chi\iota^{j+j'}}) - f_2(r_1\overline{\iota^{j+1}}\otimes\overline{\iota^{j'}} - 0)\\
&\quad = (g_1\chi_0+h_1\chi_1)\iota^{j+j'} + 0 = m_1(0) + m_2((g_1\chi_0+h_1\chi_1)\iota^j\otimes\iota^{j'})\\
&\quad = (m_1\circ f_2 + m_2\circ(f_1\otimes f_1))(\overline{\chi\iota^j}\otimes\overline{\iota^{j'}})\\
&(f_1\circ m_2' - f_2\circ(m_1'\otimes 1 + 1\otimes m_1'))(\overline{\chi\iota^j}\otimes\overline{\chi\iota^{j'}})\\
&\quad = f_1(r_2\overline{\iota^{j+j'+1}}) - f_2(r_1\overline{\iota^{j+1}}\otimes\overline{\chi\iota^{j'}} - \overline{\chi\iota^j}\otimes r_1\overline{\iota^{j'+1}}) = r_2\iota^{j+j'+1} + 0\\
&\quad \overset{(67)}{=} (g_2a+h_2b+g_1h_1)\iota^{j+j'+1} \overset{(65)}{=} m_1((g_2\chi_0+h_2\chi_1)\iota^{j+j'}) + g_1h_1(\iota_0+\iota_1)\iota^{j+j'}\\
&\quad \overset{(65)}{=} m_1((g_2\chi_0+h_2\chi_1)\iota^{j+j'}) + m_2((g_1\chi_0+h_1\chi_1)\iota^j\otimes(g_1\chi_0+h_1\chi_1)\iota^{j'})\\
&\quad = (m_1\circ f_2 + m_2\circ(f_1\otimes f_1))(\overline{\chi\iota^j}\otimes\overline{\chi\iota^{j'}}).
\end{aligned}$$

□

Let $k\geq 3$.

Equation (5)[k] can be reformulated as

$$\underbrace{f_1\circ m_k' + \sum_{\substack{k=r+s+t\\ r,t\geq 0, s\geq 1\\ s\leq k-1}} (-1)^{rs+t} f_{r+1+t}\circ(1^{\otimes r}\otimes m_s'\otimes 1^{\otimes t})}_{=:\Phi_k}$$
$$= m_1\circ f_k + \underbrace{\sum_{\substack{2\leq r\leq k\\ i_1+\ldots+i_r=k\\ \text{all } i_s\geq 1}} (-1)^v m_r\circ(f_{i_1}\otimes f_{i_2}\otimes\ldots\otimes f_{i_r})}_{=:\Xi_k},$$

where $v=\sum_{1\leq t<s\leq r}(1-i_s)i_t$.

A term of the form $f_{r+1+t}\circ(1^{\otimes r}\otimes m_s'\otimes 1^{\otimes t})$, $s\in\mathbb{Z}_{\geq 1}\setminus\{2\}$, $r+t\geq 1$, is zero because of (66) and the definition of m_s'. Thus

$$\Phi_k = \sum_{\substack{k=r+2+t\\ r,t\geq 0}} (-1)^{2r+t} f_{k-1}\circ(1^{\otimes r}\otimes m_2'\otimes 1^{\otimes t})$$

$$= \sum_{r \in [0,k-2]} (-1)^{k-r} f_{k-1} \circ (1^{\otimes r} \otimes m_2' \otimes 1^{\otimes k-r-2}). \tag{68}$$

Since $m_r = 0$ for $r \geq 3$, we have

$$\Xi_k = \sum_{\substack{i_1+i_2=k \\ i_1,i_2 \geq 1}} (-1)^{(1-i_2)i_1} m_2 \circ (f_{i_1} \otimes f_{i_2}) = \sum_{i \in [1,k-1]} (-1)^{ki} m_2 \circ (f_i \otimes f_{k-i}). \tag{69}$$

We have proven:

Lemma 107. *For $k \geq 3$, condition (5)[k] is equivalent to $f_1 \circ m_k' + \Phi_k = m_1 \circ f_k + \Xi_k$ where Φ_k and Ξ_k are as in (68) and (69).*

Lemma 108. *Condition (5)[k] holds for $k \geq 3$ and arguments $\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \in \mathfrak{B}^{\otimes k} = \{\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \in (A')^{\otimes k} \mid a_i \in \{0,1\}$ and $j_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [1,k]\}$ such that $0 \in \{a_1, \ldots, a_k\}$.*

Proof. Because of Definition/Remark 104, we have $m_k'(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$ and $f_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$. So by Lemma 107, it suffices to show that

$$\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = \Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}})$$

if there exists $z \in [1,k]$ such that $a_z = 0$.

Case 1 $|\{z \in [1,k] \mid a_z = 0\}| \geq 2$.

To show $\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$, we show $f_{k-1}(1^{\otimes r} \otimes m_2' \otimes 1^{\otimes k-r-2})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$ for $r \in [0, k-2]$: In case both factors of the argument of m_2' are of the form $\overline{\chi^0\iota^j}$, the result of m_2' is a multiple of some $\overline{\iota^{j'}}$ (see Definition/Remark 104). Since $2 \leq k-1$, eq. (66) implies that the result of f_{k-1} is zero. Otherwise, at least one of the factors of the argument of f_{k-1} must be of the form $\overline{\iota^j}$ and the result of f_{k-1} is zero as well. So $\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$.

To show $\Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$, we show $m_2(f_i \otimes f_{k-i})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$ for $i \in [1, k-1]$:

- Suppose that $i \in [2, k-2]$: The statements $a_1 = \ldots = a_i = 1$ and $a_{i+1} = \ldots = a_k = 1$ cannot be both true, so f_i or f_{k-i} evaluates to 0. Hence, we have $m_2(f_i \otimes f_{k-i})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$.
- Suppose that $i = 1$. Since $|\{z \in [1,k] \mid a_z = 0\}| \geq 2$, the statement $a_2 = \ldots = a_k = 1$ cannot be true. Since $k - 1 \geq 2$, the map f_{k-1} evaluates to 0 and we have $m_2(f_1 \otimes f_{k-1})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$.
- The case $i = k-1$ is analogous to the case $i = 1$.

So we have $\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0 = \Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}})$.

Case 2 $|\{z \in [1,k] \mid a_z = 0\}| = 1$.

Case 2a $\{z \in [1,k] \mid a_z = 0\} = \{x\}$, where $x \in [2, k-1]$.

We have $\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$: We have $m_2'(\overline{\chi\iota^j} \otimes \overline{\chi\iota^{j'}}) = r_2\overline{\iota^{j+j'+1}}$ for $j, j' \in \mathbb{Z}_{\geq 0}$, so (66) implies $f_{k-1}(1^{\otimes r} \otimes m_2' \otimes 1^{\otimes k-r-2})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$ unless $r \in \{x-2, x-1\}$. So

$$\begin{aligned}
&\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) \\
&\quad = (-1)^{k-x+2} f_{k-1}(1^{\otimes x-2} \otimes m_2' \otimes 1^{\otimes k-x} - 1^{\otimes x-1} \otimes m_2' \otimes 1^{k-x-1}) \\
&\qquad (\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) \\
&\quad = (-1)^{k-x} f_{k-1}(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_{x-2}}} \otimes m_2'(\overline{\chi\iota^{j_{x-1}}} \otimes \overline{\iota^{j_x}}) \otimes \overline{\chi\iota^{j_{x+1}}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}} \\
&\qquad - \overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_{x-1}}} \otimes m_2'(\overline{\iota^{j_x}} \otimes \overline{\chi\iota^{j_{x+1}}}) \otimes \overline{\chi\iota^{j_{x+2}}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) \\
&\quad = (-1)^{k-x} f_{k-1}(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_{x-2}}} \otimes \overline{\chi\iota^{j_{x-1}+j_x}} \otimes \overline{\chi\iota^{j_{x+1}}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}} \\
&\qquad - \overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_{x-1}}} \otimes \overline{\chi\iota^{j_x+j_{x+1}}} \otimes \overline{\chi\iota^{j_{x+2}}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) \\
&\quad = (-1)^{k-x}((g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_1+\ldots+j_n} - (g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_1+\ldots+j_n}) = 0.
\end{aligned}$$

To show $\Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$, we show $m_2(f_i \otimes f_{k-i})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$ for $i \in [1, k-1]$: The element $\overline{\chi^{a_x}\iota^{j_x}}$ is a tensor factor of the argument of f_i or of f_{k-i}. Since $x \notin \{1, k\}$, (66) implies that f_i or f_{k-i} evaluates to 0. Thus $m_2(f_i \otimes f_{k-i})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$.

So $\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0 = \Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}})$.

Case 2b $\{z \in [1,k] \mid a_z = 0\} = \{1\}$.

We have $f_{k-1}(1^{\otimes r} \otimes m_2' \otimes 1^{\otimes k-r-2})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_n}\iota^{j_k}}) = 0$ unless $r = 0$. So

$$\begin{aligned}
\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) &= (-1)^k f_{k-1}(1^{\otimes 0} \otimes m_2' \otimes 1^{\otimes k-2})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) \\
&= (-1)^k f_{k-1}(m_2'(\overline{\iota^{j_1}} \otimes \overline{\chi\iota^{j_2}}) \otimes \overline{\chi\iota^{j_3}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) \\
&= (-1)^k f_{k-1}(\overline{\chi\iota^{j_1+j_2}} \otimes \overline{\chi\iota^{j_3}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) \\
&= (-1)^k (g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_1+\ldots+j_k}.
\end{aligned}$$

We have $(f_i \otimes f_{k-i})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$ if $i \geq 2$. So

$$\begin{aligned}
\Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) &= (-1)^{1\cdot k} m_2(f_1 \otimes f_{k-1})(\overline{\iota^{j_1}} \otimes \overline{\chi\iota^{j_2}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) \\
&\overset{(1)}{=} (-1)^k m_2\left(f_1(\overline{\iota^{j_1}}) \otimes f_{k-1}(\overline{\chi\iota^{j_2}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}})\right) \\
&= (-1)^k m_2\left(\iota^{j_1} \otimes (g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_2+\ldots+j_k}\right) \\
&= (-1)^k (g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_1+\ldots+j_k}.
\end{aligned}$$

So $\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = \Xi_n(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}})$.

Case 2c $\{z \in [1,k] \mid a_z = 0\} = \{k\}$.

We have $f_{k-1}(1^{\otimes r} \otimes m_2' \otimes 1^{\otimes k-r-2})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_n}\iota^{j_k}}) = 0$ unless $r = k-2$. So

$$\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = (-1)^2 f_{k-1}(1^{\otimes k-2} \otimes m_2' \otimes 1^{\otimes 0})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}})$$

$$
\begin{aligned}
&\overset{(1)}{=} f_{k-1}(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_{k-2}}} \otimes m_2'(\overline{\chi\iota^{j_{k-1}}} \otimes \overline{\iota^{j_k}}))\\
&= f_{k-1}(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_{k-2}}} \otimes \overline{\chi\iota^{j_{k-1}+j_k}})\\
&= (g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_1+\ldots+j_k}.
\end{aligned}
$$

We have $(f_i \otimes f_{k-i})(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = 0$ if $i \leq k-2$. So

$$
\begin{aligned}
\Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) &= (-1)^{k(k-1)} m_2(f_{k-1} \otimes f_1)(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}})\\
&\overset{(1)}{=} m_2\left(f_{k-1}(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_{k-1}}}) \otimes f_1(\overline{\iota^{j_k}})\right)\\
&= m_2((g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_1+\ldots+j_{k-1}} \otimes i^{j_k})\\
&= (g_{k-1}\chi_0 + h_{k-1}\chi_1)\iota^{j_1+\ldots+j_k}.
\end{aligned}
$$

So $\Phi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) = \Xi_k(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}})$.

□

Now we examine the case where $a_1 = \ldots = a_n = 1$:

Lemma 109. *Condition* (5)[k] *holds for* $k \geq 3$ *and arguments* $\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}} \in \mathfrak{B}^{\otimes k} = \{\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \in (A')^{\otimes k} \mid a_i \in \{0,1\}$ *and* $j_i \in \mathbb{Z}_{\geq 0}$ *for all* $i \in [1,k]\}$.

Proof. We have $m_2'(\overline{\chi\iota^j} \otimes \overline{\chi\iota^{j'}}) = r_2\overline{\iota^{j+j'+1}}$ for $j, j' \in \mathbb{Z}_{\geq 0}$, so (66) implies $f_{k-1}(1^{\otimes r} \otimes m_2' \otimes 1^{\otimes k-r-2})(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) = 0$ for $r \in [0, k-2]$. Hence, we have $\Phi_k(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) = 0$.

We have

$$
\begin{aligned}
&(m_1 \circ f_k + \Xi_k)(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}})\\
&\quad= m_1((g_k\chi_0 + h_k\chi_1)\iota^{j_1+\ldots+j_k}) + \sum_{i\in[1,k-1]} (-1)^{ki} m_2((f_i \otimes f_{k-i})(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}))\\
&\quad\overset{(65),(1)}{=} (ag_k + bh_k)\iota^{(j_1+\ldots+j_k)+1}\\
&\qquad+ \sum_{i\in[1,k-1]} (-1)^{ki+(1-k+i)i} m_2(f_i(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_i}}) \otimes f_{k-i}(\overline{\chi\iota^{j_{i+1}}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}))\\
&\quad= (ag_k + bh_k)\iota^{(j_1+\ldots+j_k)+1}\\
&\qquad+ \sum_{i\in[1,k-1]} m_2((g_i\chi_0 + h_i\chi_1)\iota^{j_1+\ldots+j_i} \otimes (g_{k-i}\chi_0 + h_{k-i}\chi_1)\iota^{j_{i+1}+\ldots+j_k})\\
&\quad\overset{(65)}{=} (ag_k + bh_k)\iota^{(j_1+\ldots+j_k)+1} + \sum_{i\in[1,k-1]} (g_ih_{k-i}\iota_0 + h_ig_{k-i}\iota_1)\iota^{j_1+\ldots+j_k}\\
&\quad= (ag_k + bh_k)\iota^{(j_1+\ldots+j_k)+1} + \sum_{i\in[1,k-1]} (g_ih_{k-i}\iota_0 + h_{k-i}g_i\iota_1)\iota^{j_1+\ldots+j_k}\\
&\quad\overset{(65)}{=} \left(ag_k + bh_k + \sum_{i\in[1,k-1]} g_ih_{k-i}\right)\iota^{(j_1+\ldots+j_k)+1}
\end{aligned}
$$

$$\overset{(67)}{=} r_k \iota^{(j_1+\ldots+j_k)+1} = f_1(r_k \overline{\iota^{(j_1+\ldots+j_k)+1}}) + 0$$
$$= (f_1 \circ m'_k + \Phi_k)(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}})\,.$$

Application of Lemma 107 completes the proof. □

Proof of Proposition 105. Lemmas 106, 108 and 109 ensure that (5)[k] holds for $k \in \mathbb{Z}_{\geq 1}$. We show that f_1 is injective. Suppose given $j \in \mathbb{Z}_{\geq 0}$. The element ι^j is R-torsion free. Since h_1 or g_1 is R-torsion-free, the element $(g_1\chi_0 + h_1\chi_1)\iota^j$ is R-torsion-free.

So since the set $X := \{\iota^j \mid j \in \mathbb{Z}_{\geq 0}\} \cup \{(g_1\chi_0 + h_1\chi_1)\iota^j \mid j \in \mathbb{Z}_{\geq 0}\} \subseteq A$ consists of R-torsion-free elements of different summands of the direct sum $A = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(P,P)$, it is linearly independent. The set $\mathfrak{B}$, which is a basis of A', is mapped bijectively to X by f_1, so f_1 is injective.

Lemma 52 proves that $(A', (m'_k)_{k\geq 1})$ is an A_∞-algebra and $(f_k)_{k\geq 1}$ is an A_∞-morphism from $(A', (m'_k)_{k\geq 1})$ to $(A, (m_k)_{k\geq 1})$. □

Proposition 110. *Suppose that $\varepsilon(h_1)$ is a unit in R and that* (67) *holds. Then the tuple $(A', (m'_k)_{k\geq 1})$ given in Definition/Remark 104 is an A_∞-algebra and the tuple $(f_k)_{k\geq 1}$ is a quasi-isomorphism of A_∞-algebras from $(A', (m'_k)_{k\geq 1})$ to $(A, (m_k)_{k\geq 1})$.*

Proof. Since $\varepsilon(h_1)$ is a unit in R and since ε is R-linear, the element h_1 is R-torsion free. Proposition 105 shows that $(A', (m'_k)_{k\geq 1})$ is an A_∞-algebra and $(f_k)_{k\geq 1}$ is an A_∞-morphism from $(A', (m'_k)_{k\geq 1})$ to $(A, (m_k)_{k\geq 1})$. It remains to show that $(f_k)_{k\geq 1}$ is actually a quasi-isomorphism of A_∞-algebras. I.e. we need to show that the complex morphism $f_1 : (A', m'_1) \to (A, m_1)$ is a quasi-isomorphism.

From the augmentation ε, we obtain the quasi-isomorphism of complexes

$$\tilde{\varepsilon} := \left(\begin{array}{c}\begin{tikzcd} \cdots \arrow[r] & RC_n \arrow[r, "\beta"] \arrow[d] & RC_n \arrow[r, "\alpha"] \arrow[d] & RC_n \arrow[r, "\beta"] \arrow[d] & RC_n \arrow[r, "\alpha"] \arrow[d] & RC_n \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & R \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}\end{array}\right) \in \mathrm{Hom}^0_{RC_n}(P, \tilde{R}),$$

where $\tilde{R}$ is the complex that has the module R at position 0 and zero in all other positions. By [4, §5 Proposition 4], the map

$$\begin{aligned} \hat{\varepsilon} : A = \mathrm{Hom}^*_{RC_n}(P,P) &\longrightarrow \mathrm{Hom}^*_{RC_n}(P,\tilde{R}) \\ g \in A^i &\longmapsto \tilde{\varepsilon} \circ g \quad \text{for } i \in \mathbb{Z} \end{aligned}$$

is a quasi-isomorphism of complexes.

For $j \in \mathbb{Z}_{\geq 0}$, we have

$$\hat{\varepsilon}(f_1(\overline{\iota^j})) = \hat{\varepsilon}(\iota^j) = \left(\begin{tikzcd} \cdots \arrow[r] & \overbrace{R\mathrm{C}_n}^{2j+1} \arrow[r, "\alpha"] \arrow[d] & \overbrace{R\mathrm{C}_n}^{2j} \arrow[r, "\beta"] \arrow[d, "\varepsilon"] & \cdots \arrow[r, "\beta"] & \overbrace{R\mathrm{C}_n}^{1} \arrow[r, "\alpha"] \arrow[d] & \overbrace{R\mathrm{C}_n}^{0} \arrow[r] \arrow[d] & \overbrace{0}^{-1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \underbrace{0}_{1} \arrow[r] & \underbrace{R}_{0} \arrow[r] & \cdots \arrow[r] & \underbrace{0}_{1-2j} \arrow[r] & \underbrace{0}_{-2j} \arrow[r] & \underbrace{0}_{-1-2j} \arrow[r] & \cdots \end{tikzcd}\right)$$

$$\in \mathrm{Hom}^{2j}_{R\mathrm{C}_n}(P, \tilde{R})$$

$$\begin{aligned} \hat{\varepsilon}(f_1(\overline{\chi\iota^j})) &= \hat{\varepsilon}((g_1\chi_0 + h_1\chi_1)\iota^j) \\ &= \left(\begin{tikzcd} \cdots \arrow[r] & \overbrace{R\mathrm{C}_n}^{2j+2} \arrow[r, "\beta"] \arrow[d] & \overbrace{R\mathrm{C}_n}^{2j+1} \arrow[r, "\alpha"] \arrow[d, "\varepsilon(h_1)\varepsilon"] & \cdots \arrow[r, "\beta"] & \overbrace{R\mathrm{C}_n}^{1} \arrow[r, "\alpha"] \arrow[d] & \overbrace{R\mathrm{C}_n}^{0} \arrow[r] \arrow[d] & \overbrace{0}^{-1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \underbrace{0}_{1} \arrow[r] & \underbrace{R}_{0} \arrow[r] & \cdots \arrow[r] & \underbrace{0}_{2j} \arrow[r] & \underbrace{0}_{-2j-1} \arrow[r] & \underbrace{0}_{-2j-2} \arrow[r] & \cdots \end{tikzcd}\right) \\ &\in \mathrm{Hom}^{2j+1}_{R\mathrm{C}_n}(P, \tilde{R}), \end{aligned}$$

where the positions are written above resp. below the complexes. The R-module $\mathrm{Hom}_{R\mathrm{C}_n}(R\mathrm{C}_n, R)$ is free over R of rank 1 with basis $\{\varepsilon\}$. Since $\varepsilon(h_1)$ is a unit in R, we conclude that the R-basis $\mathfrak{B}$ of A' is mapped by $\hat{\varepsilon} \circ f_1$ to an R-basis of $\mathrm{Hom}^*_{R\mathrm{C}_n}(P, \tilde{R})$. Hence, $\hat{\varepsilon} \circ f_1 : (A', m'_1) \to (\mathrm{Hom}^*_{R\mathrm{C}_n}(P, \tilde{R}), d_{\mathrm{Hom}^*_{R\mathrm{C}_n}(P,\tilde{R})})$ is an isomorphism of complexes. In particular, it is a quasi-isomorphism. Since $\hat{\varepsilon} : (A, m_1) \to (\mathrm{Hom}^*_{R\mathrm{C}_n}(P, \tilde{R}), d_{\mathrm{Hom}^*_{R\mathrm{C}_n}(P,\tilde{R})})$ is a quasi-isomorphism, we conclude that $f_1 : (A', m'_1) \to (A, m_1)$ is a quasi-isomorphism of complexes. $\square$

Remark 111. Suppose that the assumptions of Proposition 110 hold.
The A_∞-algebra $(A', (m'_k)_{k\geq 1})$ carries the structure of a minimal eA_∞-algebra, where the decomposition $A' = \oplus_{i,j\in\mathbb{Z}}(A')^{j,i}$ is given as follows. For $i' \in \mathbb{Z}_{\geq 0}$, we set $(A')^{0,2i'} := \langle\overline{\iota^{i'}}\rangle_R$ and $(A')^{1,2i'+1} := \langle\overline{\chi\iota^{i'}}\rangle_R$. We set all other $(A')^{j,i}$ to be zero. This way, we have $(A')^i = \oplus_{j\in\mathbb{Z}}(A')^{j,i}$ for $i \in \mathbb{Z}$.

We have $(A')^{0,-} = \langle\overline{\iota^{i'}} \mid i' \in \mathbb{Z}_{\geq 0}\rangle_R$ and $(A')^{1,-} = \langle\overline{\chi\iota^{i'}} \mid i' \in \mathbb{Z}_{\geq 0}\rangle_R$. For $j \in \mathbb{Z} \setminus \{0, 1\}$, we have $(A')^{j,-} = 0$. In particular, axiom (EA2) holds. Axiom (EA1) holds since $(A', (m'_k)_{k\geq 1})$ is an A_∞-algebra.

We need to verify (EA3$'$).

For $k \geq 1$ and $j_1, \ldots, j_k \in \mathbb{Z}$, we need to show that

$$m'_k((A')^{j_1,-} \otimes \ldots \otimes (A')^{j_k,-}) \subseteq \bigoplus_{j' \leq (j_1+\ldots+j_k)+(2k-3)} (A')^{j',-}.$$

From the definition of the m'_k, we have

$$\begin{aligned} m'_1((A')^{1,-}) &\subseteq (A')^{0,-} \\ m'_1((A')^{0,-}) &= 0. \end{aligned}$$

Hence $m'_1((A')^{j,-}) \subseteq (A')^{j-1,-}$ for $j \in \mathbb{Z}$.

We have

$$\begin{aligned} m'_2((A')^{0,-} \otimes (A')^{0,-}) &\subseteq (A')^{0,-} \\ m'_2((A')^{0,-} \otimes (A')^{1,-}) &\subseteq (A')^{1,-} \\ m'_2((A')^{1,-} \otimes (A')^{0,-}) &\subseteq (A')^{1,-} \\ m'_2((A')^{1,-} \otimes (A')^{1,-}) &\subseteq (A')^{0,-}. \end{aligned}$$

Hence $m'_2((A')^{j_1,-} \otimes (A')^{j_2,-}) \subseteq \oplus_{j' \leq j_1+j_2} (A')^{j',-} \subseteq \oplus_{j' \leq j_1+j_2+1} (A')^{j',-}$ for $j_1, j_2 \in \mathbb{Z}$.

For $k \geq 3$, we have

$$m'_k(\underbrace{(A')^{1,-} \otimes \ldots \otimes (A')^{1,-}}_{k \text{ factors}}) \subseteq (A')^{0,-}.$$

Note that given $(j_1, \ldots, j_k) \in \mathbb{Z}^k$, we have $m'_k((A')^{j_1,-} \otimes \ldots \otimes (A')^{j_k,-}) = 0$ if $(j_1, \ldots, j_k) \neq (1, \ldots, 1)$. Hence, we have $m'_k((A')^{j_1,-} \otimes \ldots \otimes (A')^{j_k,-}) \subseteq (A')^{(j_1+\ldots+j_k)-k} \subseteq \oplus_{j' \leq (j_1+\ldots+j_k)+(2k-3)} (A')^{j'}$.

So we have proven (EA3′).

Note that in this example, the bounds obtained are much stronger than the bounds required by (EA3′).

5.1. A simple solution

Lemma 112. *The sequences $(g_k)_{k\geq 1}$, $(h_k)_{k\geq 1}$, $(r_k)_{k\geq 1}$ given by*

$$\begin{aligned} g_k &:= \sum_{i \in [1,n-1]} \binom{n-i-1}{k} e^i \\ h_k &:= \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{else} \end{cases} \\ r_k &:= \binom{n}{k} \end{aligned}$$

satisfy the assumptions of Proposition 110.

Note that $g_k = 0$, $h_k = 0$ and $r_k = 0$ for $k > n$.

Proof. We have $g_k, h_k \in R\mathrm{C}_n$ and $r_k \in R$ for $k \geq 1$. Since $h_1 = 1$, $\varepsilon(h_1) = 1$ is a unit in R. It remains to check (67): For convenience, write $g_0 := \sum_{i=0}^{n-1} \binom{n-i-1}{0} e^i = \sum_{i=0}^{n-1} e^i = b$. By the definition of the h_k, we have

$$h_k b + \sum_{i \in [1,k-1]} g_i h_{k-i} = \begin{cases} g_0 & \text{if } k = 1 \\ 0 & \text{else} \end{cases} + \begin{cases} 0 & \text{if } k = 1 \\ g_{k-1} & \text{else} \end{cases} = g_{k-1}.$$

Thus for $k \geq 1$, we have

$$\begin{aligned}
& g_k a + h_k b + \sum_{i \in [1,k-1]} g_i h_{k-i} = g_k a + g_{k-1} \\
&= \sum_{i \in [0,n-1]} \binom{n-i-1}{k} e^i - \sum_{i \in [0,n-1]} \binom{n-i-1}{k} e^{i+1} + \sum_{i \in [0,n-1]} \binom{n-i-1}{k-1} e^i \\
&= \sum_{i \in [0,n-1]} \left(\binom{n-i-1}{k} + \binom{n-i-1}{k-1} \right) e^i - \sum_{i \in [0,n-1]} \binom{n-(i+1)}{k} e^{i+1} \\
&= \sum_{i \in [0,n-1]} \binom{n-i}{k} e^i - \sum_{i \in [1,n]} \binom{n-i}{k} e^i \\
&= \binom{n-0}{k} e^0 - \underbrace{\binom{n-n}{k}}_{=0} e^n = \binom{n}{k} = r_k.
\end{aligned}$$

□

In Definition/Remark 104 and Proposition 110, we have obtained a general framework for models of the A_∞-algebra A. In Remark 111, we established that models of that type are actually minimal in the sense of eA_∞-algebras. In Lemma 112, we obtained an explicit solution of the conditions of the framework. Combining all these, we obtain

Theorem 113 (Summary). *We have a minimal* eA_∞*-algebra* $(A' = \oplus_{i,j \in \mathbb{Z}} (A')^{j,i}, (m'_k)_{k \geq 1})$ *that is quasi-isomorphic to the dg-algebra* $A = \mathrm{Hom}^*_{R\mathrm{C}_n}(P, P)$*, where* P *is the projective resolution of the trivial* $R\mathrm{C}_n$*-module* R *given in* (63)*. The* eA_∞*-algebra* A' *is given as follows. For* $j \in \mathbb{Z}_{\geq 0}$*, the* R*-module* $(A')^{0,2j}$ *is free over the set* $\{\overline{\iota^j}\}$ *and the* R*-module* $(A')^{1,2j+1}$ *is free over the set* $\{\overline{\chi \iota^j}\}$*. All other* $(A')^{j,i}$ *are zero.*

$i =$						
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	
3	$\cdots$	0	$\langle\overline{\chi\iota^1}\rangle$	0	0	$\cdots$
2	$\cdots$	0	0	$\langle\overline{\iota^1}\rangle$	0	$\cdots$
1	$\cdots$	0	$\langle\overline{\chi\iota^0}\rangle$	0	0	$\cdots$
0	$\cdots$	0	0	$\langle\overline{\iota^0}\rangle$	0	$\cdots$
-1	$\cdots$	0	0	0	0	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$j =$	$\cdots$	2	1	0	-1	$\cdots$

We give the m_k *by giving them on the elements of the basis*

$$\mathfrak{B}^{\otimes k} = \{\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \mid \text{ all } a_i \in \{0,1\}, \text{ all } j_i \in \mathbb{Z}_{\geq 0}\} \subseteq (A')^{\otimes k}.$$

For elements $\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \in \mathfrak{B}^{\otimes k}$ *such that* $0 \in \{a_1, \ldots, a_k\}$, *we have*

$$\begin{aligned} m_2'(\overline{\chi^{a_1}\iota^{j_1}} \otimes \overline{\chi^{a_2}\iota^{j_2}}) &= \overline{\chi^{a_1+a_2}\iota^{j_1+j_2}} \qquad \text{(Note that } a_1 + a_2 \in \{0,1\}.\text{)} \\ m_k'(\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}}) &= 0 \text{ for } k \in \mathbb{Z}_{\geq 1} \setminus \{2\}. \end{aligned}$$

For elements $\overline{\chi^{a_1}\iota^{j_1}} \otimes \ldots \otimes \overline{\chi^{a_k}\iota^{j_k}} \in \mathfrak{B}^{\otimes k}$ *such that* $a_1 = \ldots = a_k = 1$, *we have*

$$m_k'(\overline{\chi\iota^{j_1}} \otimes \ldots \otimes \overline{\chi\iota^{j_k}}) = \binom{n}{k} \overline{\iota^{(j_1+\ldots+j_k)+1}} \text{ for } k \in \mathbb{Z}_{\geq 1}\,.$$

In particular, we have $m_k' = 0$ *for* $k > n$. *Concerning the differential, note that for* $j \in \mathbb{Z}_{\geq 0}$, *we have* $m_1'(\overline{\chi\iota^j}) = \overline{\iota^{j+1}}$ *and* $m_1'(\overline{\iota^j}) = 0$.

Remark 114 (Comparison with results of Madsen)**.** Let us examine the solution given in Lemma 112 in case $R = \mathbb{F}_p$ for a prime p and $n = p^c$ for some $c \in \mathbb{Z}_{\geq 1}$. By the binomial theorem, $r_k = \binom{p^c}{k} \in \mathbb{F}_p$ is the coefficient at X^k of the polynomial $(1+X)^{p^c} \in \mathbb{F}_p[X]$ for $k \geq 1$. But in $\mathbb{F}_p[X]$, we have $(1+X)^{p^c} = ((1+X)^p)^{p^{c-1}} = (1^p + X^p)^{p^{c-1}} = (1+X^p)^{p^{c-1}} = \ldots = 1 + X^{p^c}$. Thus we have

$$r_k = \begin{cases} 1 & \text{if } k = p^c \\ 0 & \text{else.} \end{cases}$$

In particular, we have $r_1 = 0$, so $m'_1 = 0$ in Proposition 110. This means that A' is a minimal model of A in the A_∞-sense.

Minimal models on the group cohomology of the cyclic group C_{p^c} over the field $\mathbb{F}_p$ were given by Madsen in [16, Appendix B Example 2.2] (actually, certain path algebras are considered, amongst them one isomorphic to the algebra $\mathbb{F}_p\mathrm{C}_{p^c}$) and adapted to the formulation of $\mathbb{F}_p\mathrm{C}_{p^c}$ as a group algebra by Vejdemo-Johansson in [23, Theorem 4.3.8]. Comparison of our minimal model and the minimal model given in [23, Theorem 4.3.8] yield that they are the same, so we have recovered this particular case.

5.2. A family of solutions

While experimenting with solutions of (67), where, say, $n = 3$ or $n = 5$ and where for $k \geq 2$, we have $r_k \in [0, n-1]$, I discovered that the resulting sequences for $(r_k)_{k\geq 1}$ had been described by Paul D. Hanna as coefficient series of powers of certain formal power series, cf. [7]. Further investigation showed that (67) is actually equivalent to the equation of formal powers series (71) and that taking powers of power series is the mechanism that governs the class of solutions of (67) that satisfy $h_k \in R$ for $k \geq 1$.

Suppose given sequences $(r_k)_{k\geq 1}$, $(g_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ such that $r_k \in R$ and $g_k, h_k \in R\mathrm{C}_n$ for $k \geq 1$. Let $g_0 := \sum_{i\in[0,n-1]} e^i = b \in R\mathrm{C}_n$ and $h_0 := 1 \in R\mathrm{C}_n$. Let $r_0 \in R$. We will discuss the choice of r_0 later.

Consider (67): The left hand side is

$$\begin{aligned} g_k a + h_k b + \sum_{i\in[1,k-1]} g_i h_{k-i} &= g_k - e g_k + h_k b + \sum_{i\in[1,k-1]} g_i h_{k-i} \\ &= g_k h_0 - e g_k + h_k g_0 + \sum_{i\in[1,k-1]} g_i h_{k-i} \\ &= - e g_k + \sum_{i\in[0,k]} g_i h_{k-i}\,. \end{aligned}$$

I.e. eq. (67) holds iff

$$-e g_k + \sum_{i\in[0,k]} g_i h_{k-i} = r_k \text{ for } k \in \mathbb{Z}_{\geq 1}\,. \tag{70}$$

Consider the formal power series $g := \sum_{i\geq 0} g_i X^i \in R\mathrm{C}_n[[X]]$, $h := \sum_{i\geq 0} h_i X^i \in R\mathrm{C}_n[[X]]$ and $r := \sum_{i\geq 0} r_i X^i$.

Consider the equation

$$-eg + gh = r. \tag{71}$$

By the multiplication rule for formal power series, the Cauchy product, we see that if (71) holds, then (70) and (67) hold.

So consider $(g_k)_{k\geq 1}$, $(h_k)_{k\geq 1}$ and $(r_k)_{k\geq 1}$ such that (67) holds. Recall that g_0 and h_0 are constants. Since (70) holds, the difference of the sides of (71) is

$$(-eg+gh)-r=(-eg_0+h_0g_0)-r_0=(1-e)g_0-r_0=ab-r_0=-r_0.$$

I.e. if (67) holds, then there is exactly one possible choice for r_0 such that (71) holds and that choice is $r_0 := 0$.

We have proven the

Proposition 115. *Let $g_0 := \sum_{i\in[0,n-1]} e^i = b \in R\mathrm{C}_n$ and $h_0 := 1 \in R\mathrm{C}_n$. Suppose given sequences $(r_k)_{k\geq 1}$, $(g_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ such that $r_k \in R$ and $g_k, h_k \in R\mathrm{C}_n$ for $k \geq 1$. The following are equivalent.*

(1) Condition (67) holds.

(2) There exists an $r_0 \in R$ such that for $g := \sum_{i\geq 0} g_iX^i \in R\mathrm{C}_n[[X]]$, $h := \sum_{i\geq 0} h_iX^i \in R\mathrm{C}_n[[X]]$ and $r := \sum_{i\geq 0} r_iX^i \in R[[X]]$, we have

$$-eg+gh=r.$$

In that case, we have $r_0 = 0$.

Lemma 116. *Suppose given $h \in R[[X]]$, $g \in R\mathrm{C}_n[[X]]$ and $r \in R[[X]]$. Then*

$$-eg+gh=r \tag{72}$$

if and only if there exists $\check{g} \in R[[X]]$ such that

$$\begin{aligned} g &= \sum_{i\in[0,n-1]} h^{n-1-i}\check{g}e^i \\ r &= (h^n-1)\check{g}. \end{aligned} \tag{73}$$

Furthermore, note that $\check{g}$ is the coefficient at e^{n-1} of g.

Proof. "$\Rightarrow$": Suppose given $h \in R[[X]]$, $g \in R\mathrm{C}_n[[X]]$ and $r \in R[[X]]$ such that (72) holds. We have $g = \sum_{i\in[0,n-1]} g^{(i)}e^i$ for some $g^{(i)} \in R[[X]]$, $i \in [0, n-1]$. Equation (72) becomes

$$\begin{aligned} r &= -eg+gh = -\sum_{i\in[0,n-1]} g^{(i)}e^{i+1} + \sum_{i\in[0,n-1]} hg^{(i)}e^i \\ &= hg^{(0)} - g^{(n-1)} + \sum_{i\in[1,n-1]} (hg^{(i)} - g^{(i-1)})e^i\,. \end{aligned} \tag{74}$$

Comparing coefficients of e^i for $i \in [1, n-1]$, we obtain

$$g^{(i-1)} = hg^{(i)}.$$

Applying this identity successively, we obtain

$$g^{(i)} = h^{n-1-i} g^{(n-1)} \text{ for } i \in [0, n-1].$$

In particular, we have $g^{(0)} = h^{n-1} g^{(n-1)}$. Thus comparing coefficients of e^0 in (74), we obtain

$$r = h^n g^{(n-1)} - g^{(n-1)} = (h^n - 1) g^{(n-1)}.$$

Thus choosing $\check{g} := g^{(n-1)}$, we have (73).

"$\Leftarrow$": Suppose given $h \in R[[X]]$ and $\check{g} = \sum_{i \geq 0} \check{g}_i X^i \in R[[X]]$. Let r and g be given by (73). Then (72) holds since we have

$$\begin{aligned} -eg + hg &= -\sum_{i \in [0,n-1]} h^{n-(i+1)} \check{g} e^{i+1} + h \cdot \sum_{i \in [0,n-1]} h^{n-1-i} \check{g} e^i \\ &= -\sum_{i \in [1,n]} h^{n-i} \check{g} e^i + \sum_{i \in [0,n-1]} h^{n-i} \check{g} e^i = (h^n - 1)\check{g} = r. \end{aligned}$$

□

Proposition 117. *Let $h_0 := 1 \in R$, $\check{g}_0 := 1 \in R$.*

Suppose given $(h_i)_{i \geq 1}$, $(\check{g}_i)_{i \geq 1}$ such that $h_i, \check{g}_i \in R$ for $i \geq 1$ and such that h_1 is a unit in R. Write $\check{g} := \sum_{i \geq 0} \check{g}_i X^i \in R[[X]]$ and $h := \sum_{i \geq 0} h_i X^i \in R[[X]]$.

Consider the sequences $(r_i)_{i \geq 1}$ and $(g_i)_{i \geq 1}$ given by

$$\begin{aligned} \sum_{i \geq 0} g_i X^i &:= \sum_{i \in [0,n-1]} h^{n-1-i} \check{g} e^i \in R\mathrm{C}_n[[X]] \\ \sum_{i \geq 0} r_i X^i &:= (h^n - 1)\check{g} \in R[[X]]. \end{aligned} \tag{75}$$

Then the tuple of sequences $(g_i)_{i \geq 1}$, $(h_i)_{i \geq 1}$, $(r_k)_{i \geq 1}$ satisfies the assumptions of Proposition 110. Furthermore, all tuples of sequences $(g_i)_{i \geq 1}$, $(h_i)_{i \geq 1}$, $(r_k)_{i \geq 1}$ that satisfy the assumptions of Proposition 110 and that satisfy $h_i \in R$ for $i \geq 1$ can be obtained in this way.

Proof. Suppose given $(h_i)_{i \geq 1}$, $(\check{g}_i)_{i \geq 1}$ such that $h_i, \check{g}_i \in R$ for $i \geq 1$ and such that h_1 is a unit in R. Define the sequences $(r_i)_{i \geq 0}$ and $(g_i)_{i \geq 0}$ by (75).
Let $g := \sum_{i \geq 0} g_i X^i \in R\mathrm{C}_n[[X]]$ and $r := \sum_{i \geq 0} r_i X^i \in R[[X]]$.
Eq. (73) holds, so eq. (72) holds, cf. Lemma 116. Since $h_0 = 1$ and $\check{g}_0 = 1$, we have $g_0 = \sum_{i \in [0,n-1]} e^i$ by (75). Thus the assumptions of Proposition 115 hold. Thus by eq. (72) and by Proposition 115, condition (67) holds. Since $h_1 \overset{h_1 \in R}{=} \varepsilon(h_1)$ is a unit, the assumptions of Proposition 110 are satisfied. This proves the first assertion.

Now suppose given $(g_i)_{i \geq 1}$, $(h_i)_{i \geq 1}$, $(r_k)_{i \geq 1}$ such that the assumptions of Proposition 110 hold and that $h_i \in R$ for $i \geq 1$. In particular, (67) holds and $h_1 \overset{h_1 \in R}{=} \varepsilon(h_1)$ is a unit. Let

$g_0 := \sum_{i\in[0,n-1]} e^i = b \in R\mathrm{C}_n$. Let $r_0 := 0$. Let $g := \sum_{i\geq 0} g_i X^i$, $h := \sum_{i\geq 0} h_i X^i$ and $r := \sum_{i\geq 0} r_i X^i$. By Proposition 115 and by (67), eq. (71) holds. We have $g = \sum_{i\in[0,n-1]} g^{(i)} e^i$ for some $g^{(i)} \in R[[X]]$, $i \in [0, n-1]$. Note that since $g_0 = \sum_{i\in[0,n-1]} e^i$, the coefficient at X^0 of $g^{(n-1)}$ is $1 = \check{g}_0$. Hence $g^{(n-1)} = \sum_{i\geq 0} \check{g}_i X^i =: \check{g}$ for some $\check{g}_i \in R$, $i \geq 1$. By Lemma 116, we have $g = \sum_{i\in[0,n-1]} h^{n-1-i} \check{g} e^i$ and $r = (h^n - 1)\check{g}$. Thus $(g_i)_{i\geq 1}$ and $(r_i)_{i\geq 1}$ are of the form (75). $\square$

Remark 118. Concerning the assumptions of Proposition 117, a simple choice is setting $h_1 := 1$, $h_i := 0$ for $i \geq 2$, and $\check{g}_i := 0$ for $i \geq 1$. I.e. $h = 1 + X$ and $\check{g} = 1$. By the binomial theorem, this yields the solution given in Lemma 112.

6. The filt construction

In this section, we give explicit versions of key parts of Keller and Lefèvre-Hasegawa's "filt construction", cf. [11, Problem 2 and section 7.7]. For a comparison between this version of the filt construction and Keller and Lefèvre-Hasegawa's original version, see Remark 132.

Suppose given a commutative ground ring R.

6.1. Matrix versions of operators

Definition 119 (Matrix versions of operators)**.** Suppose given R-modules A and A'. Given $i, i' \in \mathbb{Z}_{\geq 0}$, let $A^{i\times i'}$ be the set of $i \times i'$-matrices with entries in A. Let $\mathrm{id}^{i\times i'} : A^{i\times i'} \to A^{i\times i'}$ be the identity map.

Suppose given $k \in \mathbb{Z}_{\geq 1}$. Suppose given an R-linear map $m : A^{\otimes k} \to A'$. Suppose given $i_0, \ldots, i_k \in \mathbb{Z}_{\geq 0}$. We define the R-linear map

$$\tilde{m} : A^{i_0\times i_1} \otimes \ldots \otimes A^{i_{k-1}\times i_k} \to (A')^{i_0\times i_k}$$

by

$$\begin{aligned}&\tilde{m}\left((a^1_{ij})_{i\in[1,i_0],j\in[1,i_1]} \otimes \ldots \otimes (a^k_{ij})_{i\in[1,i_{k-1}],j\in[1,i_k]}\right)\\ &:= \left(\sum_{(c_1,\ldots,c_{k-1})\in[1,i_1]\times\ldots\times[1,i_{k-1}]} m(a^1_{c_0c_1} \otimes \ldots \otimes a^k_{c_{k-1}c_k})\right)_{c_0\in[1,i_0],c_k\in[1,i_k]} .\end{aligned}$$

Note that if A and A' are graded modules and m is graded of degree k_m, then $\tilde{m}$ is also graded of degree k_m.

Abusing notation, we will often denote $\tilde{m}$ by m.

Example 120 (Matrix versions of operations resemble matrix multiplication)**.** Suppose given A, A', k, m as in Definition 119. We examine the case $k = 2$. Let $i_0 = i_1 = i_2 = 2$. Suppose given $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^{2\times 2} = A^{i_0\times i_1}$ and $T' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in A^{2\times 2} = A^{i_1\times i_2}$. We have

$$\tilde{m}(T \otimes T') = \begin{pmatrix} m(a \otimes a' + b \otimes c') & m(a \otimes b' + b \otimes d') \\ m(c \otimes a' + d \otimes c') & m(c \otimes b' + d \otimes d') \end{pmatrix} .$$

So $\tilde{m}$ is a tensor product variant of matrix multiplication followed by a matrix version of m.

6.2. Categories

Suppose given an A_∞-category $(\mathrm{Obj}\, A, A, (m_k)_{k\geq 1})$. Recall Definition/Remark 47. We have the corresponding pre-A_∞-triple $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, b)$.

Definition/Lemma 121. We define the A_∞-category $\operatorname{tw} A$ as follows. The set of objects $\operatorname{Obj} \operatorname{tw} A$ of the A_∞-category $\operatorname{tw} A$ consists of tuples $(l, (i_1, \dots, i_l), D)$, where

(1) $l \in \mathbb{Z}_{\geq 0}$,

(2) $(i_1, \dots, i_l) \subseteq \operatorname{Obj} A$ is a tuple of length l,

(3) $D = (D_{uv})_{u,v \in [1,l]} \in A^{l \times l}$ is a strictly lower triangular matrix such that for $u, v \in [1, l]$, we have $D_{uv} \in A^1(i_u, i_v)$ and

(4) we have

$$\sum_{k \geq 1} b_k((\omega^{-1}(D))^{\otimes k}) = 0\,. \tag{76}$$

Note that in (76), almost all summands are zero since D is strictly lower triangular. Hence, we understand the infinite sum in (76) as well as all other infinite sums in this section to be the sum of the non-zero summands.

Eq. (76) is called the generalized Maurer-Cartan equation, cf. e.g. [21, eq. (3.19)].

For $l \geq 0$ and a matrix $D \in A^{l \times l}$ where all entries are homogeneous of degree 1, we denote by $\llbracket D \rrbracket$ the R-linear map $\llbracket D \rrbracket : C(R) \to SA^{l^i \times l^i}, 1 \mapsto \omega^{-1}(D)$. Here $C(R)$ is the graded R-module which is R in degree 0 and 0 in all other degrees. Hence, $\llbracket D \rrbracket$ is a graded map of degree 0. Furthermore, given a graded R-module M, we identify $M \otimes C(R)$ and $C(R) \otimes M$ with M. Hence, given a graded map $g : M \to M'$ of degree k_g, we have e.g.

$$\begin{aligned} \llbracket D \rrbracket \otimes g : M &\longrightarrow A^{l_i \times l_i} \otimes M' \\ x &\longmapsto (\omega^{-1}(D^i)) \otimes g(x), \end{aligned}$$

which is a graded map of degree k_g.

Given objects $o = (l, (i_1, \dots, i_l), D)$, $o' = (l', (i'_1, \dots, i'_{l'}), D') \in \operatorname{Obj} \operatorname{tw} A$, the graded R-module $(\operatorname{tw} A)(o, o')$ of homomorphisms between o and o' consists of the matrices $E = (e_{uv})_{u \in [1,l], v \in [1,l']} \in A^{l \times l'}$ such that $e_{uv} \in A(i_u, i'_v)$ for $(u, v) \in [1, l] \times [1, l']$. Given $z \in \mathbb{Z}$, the z-th homogeneous component $(\operatorname{tw} A)(o, o')^z$ is given by the matrices E such that $e_{uv} \in A^z$ for $(u, v) \in [1, l] \times [1, l']$.

Suppose $k \geq 1$. Given objects $o_0 = (l^0, (i^0_1, \dots, i^0_{l^0}), D^0), \dots, o_k = (l^k, (i^k_1, \dots, i^k_{l^k}), D^k) \in \operatorname{Obj} \operatorname{tw} A$ and given $E^1 \in (\operatorname{tw} A)(o_0, o_1), \dots, E^k \in (\operatorname{tw} A)(o_{k-1}, o_k)$, we define

$$\begin{aligned} &m_k^{\mathrm{tw}}(E^1 \otimes \ldots \otimes E^k) \\ &:= \sum_{\text{all } j_x \geq 0} (-1)^{\frac{k(k-1)}{2}} \Big(\omega \circ b_{(j_0 + \ldots + j_k) + k} \circ \Big(\llbracket D^0 \rrbracket^{\otimes j_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1} \otimes \\ &\qquad \ldots \otimes \llbracket D^{k-1} \rrbracket^{\otimes j_{k-1}} \otimes \mathrm{id}^{l^{k-1} \times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j_k}\Big) \circ (\omega^{-1})^{\otimes k}\Big) (E^1 \otimes \ldots \otimes E^k). \end{aligned} \tag{77}$$

Given objects $o_0, \dots, o_{k-1}, o'_0, \dots, o'_{k-1} \in \operatorname{Obj} \operatorname{tw} A$ such that there exists $i \in [0, k-1]$ with $o'_i \neq o_{i+1}$, we set $m_k^{\mathrm{tw}}|_{(\operatorname{tw} A)(o_0, o'_0) \otimes \ldots \otimes (\operatorname{tw} A)(o_{k-1}, o'_{k-1})} := 0$.

Note that this definition of $\operatorname{tw} A$ is different from the definition given originally by Keller in [11, section 7.6]. For details, see Remark 132.

Proof. We need to show that $(\operatorname{Obj} \operatorname{tw} A, \operatorname{tw} A, (m_k^{\mathrm{tw}})_{k \geq 1})$ is an A_∞-category. By construction, $(\operatorname{Obj} \operatorname{tw} A, \operatorname{tw} A, (m_k^{\mathrm{tw}})_{k \geq 1})$ is a pre-A_∞-category. It is readily checked that $(\operatorname{Obj} \operatorname{tw} A, (m_k^{\mathrm{tw}})_{k \geq 1})$ is a pre-A_∞-algebra.

We have the corresponding pre-A_∞-triple $((m_k^{\mathrm{tw}})_{k \geq 1}, (b_k^{\mathrm{tw}})_{k \geq 1}, *)$. Suppose given $k \geq 1$. Suppose given $o_0 = (l^0, (i_1^0, \ldots, i_{l^0}^0), D^0), \ldots, o_k = (l^k, (i_1^k, \ldots, i_{l^k}^k), D^k) \in \operatorname{Obj} \operatorname{tw} A$. From the definition of the m_k^{tw} we obtain via the bar construction and (3) the definition of the b_k^{tw}: On $S(\operatorname{tw} A)(o_0, o_1) \otimes \ldots \otimes S(\operatorname{tw} A)(o_{k-1}, o_k)$, we have

$$\begin{aligned} b_k^{\mathrm{tw}} = \sum_{\text{all } j_x \geq 0} & b_{(j_0 + \ldots + j_k) + k} \circ (\llbracket D^0 \rrbracket^{\otimes j_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1} \otimes \\ & \ldots \otimes \llbracket D^{k-1} \rrbracket^{\otimes j_{k-1}} \otimes \mathrm{id}^{l^{k-1} \times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j_k}). \end{aligned} \tag{78}$$

Hence, we have on $S(\operatorname{tw} A)(o_0, o_1) \otimes \ldots \otimes S(\operatorname{tw} A)(o_{k-1}, o_k)$

$$\begin{aligned} & \sum_{\substack{k = r+s+t \\ r,t \geq 0, s \geq 1}} b_{r+1+t}^{\mathrm{tw}} \circ (1^{\otimes r} \otimes b_s^{\mathrm{tw}} \otimes 1^{\otimes t}) \\ = & \sum_{\substack{k = r+s+t \\ r,t \geq 0, s \geq 1 \\ \text{all } j_x, j'_x, j''_x \geq 0}} b_{\substack{j_0 + \ldots + j_r \\ + j''_0 + \ldots + j''_t \\ + r + 1 + t}} \circ \Big(\llbracket D^0 \rrbracket^{\otimes j_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1} \otimes \ldots \otimes \mathrm{id}^{l^{r-1} \times l^r} \otimes \llbracket D^r \rrbracket^{\otimes j_r} \otimes \mathrm{id}^{l^r \times l^{r+s}} \\ & \otimes \llbracket D^{r+s} \rrbracket^{\otimes j''_0} \otimes \mathrm{id}^{l^{r+s} \times l^{r+s+1}} \otimes \llbracket D^{r+s+1} \rrbracket^{\otimes j''_1} \otimes \ldots \otimes \mathrm{id}^{l^{r+s+t-1} \times l^{r+s+t}} \otimes \llbracket D^{r+s+t} \rrbracket^{\otimes j''_t} \Big) \\ & \circ \Big(1^{\otimes r} \otimes \Big(b_{(j'_0 + \ldots + j'_s) + s} \circ (\llbracket D^r \rrbracket^{\otimes j'_0} \otimes \mathrm{id}^{l^r \times l^{r+1}} \otimes \llbracket D^{r+1} \rrbracket^{\otimes j'_1} \otimes \\ & \ldots \otimes \mathrm{id}^{l^{r+s-1} \times l^{r+s}} \otimes \llbracket D^{r+s} \rrbracket^{\otimes j'_s}) \Big) \otimes 1^{\otimes t} \Big) \\ \overset{\substack{r' = r + j_0 + \ldots + j_r, \\ s' = s + j'_0 + \ldots + j'_s, \\ t' = t + j''_0 + \ldots + j''_t}}{\underset{(1)}{=}} & \sum_{\substack{r' + s' + t' = (j'''_0 + \ldots + j'''_k) + k, \\ \text{all } j'''_x \geq 0, \\ r', t' \geq 0, s' \geq 1, \\ b_{s'} \text{ has at least one} \\ \mathrm{id}^{x \times y} \text{ as argument}}} b_{r'+1+t'} \circ (1^{\otimes r'} \otimes b_{s'} \otimes 1^{\otimes t'}) \\ & \circ \Big(\llbracket D^0 \rrbracket^{\otimes j'''_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j'''_1} \ldots \otimes \mathrm{id}^{l^{k-1} \times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j'''_k} \Big) \\ \overset{(76)}{=} & \sum_{\substack{r' + s' + t' = (j'''_0 + \ldots + j'''_k) + k, \\ \text{all } j'''_x \geq 0, \\ r', t' \geq 0, s' \geq 1}} b_{r'+1+t'} \circ (1^{\otimes r'} \otimes b_{s'} \otimes 1^{\otimes t'}) \\ & \circ \Big(\llbracket D^0 \rrbracket^{\otimes j'''_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j'''_1} \ldots \otimes \mathrm{id}^{l^{k-1} \times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j'''_k} \Big) \overset{(12)}{=} 0. \end{aligned}$$

Since $(\operatorname{Obj}\operatorname{tw} A, \operatorname{tw} A, (m_k^{\mathrm{tw}})_{k\geq 1})$ is a pre-A_∞-category, (12)[k] holds for $k \geq 1$. Thus by Theorem 48, the tuple $(\operatorname{tw} A, (m_k^{\mathrm{tw}})_{k\geq 1})$ is an A_∞-algebra. So $(\operatorname{Obj}\operatorname{tw} A, \operatorname{tw} A, (m_k^{\mathrm{tw}})_{k\geq 1})$ is an A_∞-category. □

6.3. Functors

Suppose given A_∞-categories $(\operatorname{Obj} A, A, (m_k)_{k\geq 1})$, $(\operatorname{Obj}\check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ and an A_∞-functor $(f_{\mathrm{Obj}}, (f_k)_{k\geq 1})$ from A to $\check{A}$. We have corresponding triples $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, b)$, $((\check{m}_k)_{k\geq 1}, (\check{b}_k)_{k\geq 1}, \check{b})$ and $((f_k)_{k\geq 1}, (F_k)_{k\geq 1}, F)$.

Definition/Lemma 122. We define the A_∞-functor $\operatorname{tw} f = (f_{\mathrm{Obj}}^{\mathrm{tw}}, (f_k^{\mathrm{tw}})_{k\geq 1})$ from $\operatorname{tw} A$ to $\operatorname{tw}\check{A}$ as follows.

Given an object $o = (l, (i_1, \ldots, i_l), D) \in \operatorname{Obj}(\operatorname{tw} A)$, we define

$$f_{\mathrm{Obj}}^{\mathrm{tw}}(o) := (l, (f_{\mathrm{Obj}}(i_1), \ldots, f_{\mathrm{Obj}}(i_l)), \textstyle\sum_{k\geq 1}\check{\omega}(F_k((\omega^{-1}(D))^{\otimes k})).$$

Note that as declared in Definition/Lemma 121, infinite sums are the sums of their non-zero summands.

Suppose $k \geq 1$. Given objects $o_0 = (l^0, (i_1^0, \ldots, i_{l^0}^0), D^0), \ldots, o_k = (l^k, (i_1^k, \ldots, i_{l^k}^k), D^k) \in \operatorname{Obj}\operatorname{tw} A$ and given $E^1 \in (\operatorname{tw} A)(o_0, o_1), \ldots, E^k \in (\operatorname{tw} A)(o_{k-1}, o_k)$, we define

$$\begin{aligned} & f_k^{\mathrm{tw}}(E^1 \otimes \ldots \otimes E^k) \\ := & \sum_{\text{all } j_x \geq 0} (-1)^{\frac{k(k-1)}{2}} \Big(\check{\omega} \circ F_{(j_0+\ldots+j_k)+k} \circ \Big([\![D^0]\!]^{\otimes j_0} \otimes \mathrm{id}^{l^0\times l^1} \otimes [\![D^1]\!]^{\otimes j_1} \otimes \\ & \ldots \otimes [\![D^{k-1}]\!]^{\otimes j_{k-1}} \otimes \mathrm{id}^{l^{k-1}\times l^k} \otimes [\![D^k]\!]^{\otimes j_k}\Big) \circ (\omega^{-1})^{\otimes k}\Big)\,(E^1 \otimes \ldots \otimes E^k). \end{aligned}$$

Given objects $o_0, \ldots, o_{k-1}, o'_0, \ldots, o'_{k-1} \in \operatorname{Obj}\operatorname{tw} A$ such that there exists $i \in [0, k-1]$ with $o'_i \neq o_{i+1}$, we set $f_k^{\mathrm{tw}}|_{(\operatorname{tw} A)(o_0, o'_0)\otimes\ldots\otimes(\operatorname{tw} A)(o_{k-1}, o'_{k-1})} := 0$.

Proof. We need to prove that $\operatorname{tw} f$ is an A_∞-functor. First, we need to prove that $\operatorname{tw} f$ is a pre-A_∞-functor. The only non-immediate part is to prove that the images of $f_{\mathrm{Obj}}^{\mathrm{tw}}$ are elements of $\operatorname{Obj}\operatorname{tw}\check{A}$. So suppose given $o = (l, (i_1, \ldots, i_l), D) \in \operatorname{Obj}\operatorname{tw} A$. We have $f_{\mathrm{Obj}}^{\mathrm{tw}}(o) := (l, (f_{\mathrm{Obj}}(i_1), \ldots, f_{\mathrm{Obj}}(i_l)), \check{D})$, where $\check{D} = \sum_{k\geq 1}\check{\omega}(F_k((\omega^{-1}(D))^{\otimes k}))$. Conditions (1) - (3) of Definition/Lemma 121 are readily checked. Condition (4) is proven by the following.

$$\begin{aligned} \sum_{k\geq 1} \check{b}_k((\check{\omega}^{-1}(\check{D}))^{\otimes k}) &= \sum_{k\geq 1}\sum_{j_1,\ldots,j_k\geq 1} \check{b}_k(F_{j_1}((\omega^{-1}(D))^{\otimes j_1}) \otimes \ldots \otimes F_{j_k}((\omega^{-1}(D))^{\otimes j_k})) \\ &\overset{(1)}{=} \sum_{k\geq 1}\sum_{j_1,\ldots,j_k\geq 1} (\check{b}_k \circ (F_{j_1} \otimes \ldots \otimes F_{j_k}))((\omega^{-1}(D))^{\otimes j_1+\ldots+j_k}) \\ &\overset{\text{L.50}}{=} \sum_{r,t\geq 0, s\geq 1} (F_{r+1+t} \circ (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}))((\omega^{-1}(D))^{\otimes r+s+t}) \end{aligned}$$

$$\overset{(1)}{=} \sum_{r,t\geq 0, s\geq 1} F_{r+1+t}((\omega^{-1}(D))^{\otimes r} \otimes b_s((\omega^{-1}(D))^{\otimes s}) \otimes (\omega^{-1}(D))^{\otimes t})$$

$$\overset{(76)}{=} 0$$

Thus $\operatorname{tw} f$ is a pre-A_∞-functor. It is readily checked that $(f_k^{\mathrm{tw}})_{k\geq 1}$ is a pre-A_∞-morphism from A to $\hat{A}$. We have the corresponding triples $((m_k^{\mathrm{tw}})_{k\geq 0}, (b_k^{\mathrm{tw}})_{k\geq 1}, *)$, $((\check{m}_k^{\mathrm{tw}})_{k\geq 0}, (\check{b}_k^{\mathrm{tw}})_{k\geq 1}, *)$ and $((f_k^{\mathrm{tw}})_{k\geq 0}, (F_k^{\mathrm{tw}})_{k\geq 1}, *)$. We obtain $(b_k^{\mathrm{tw}})_{k\geq 1}$ and $(\check{b}_k^{\mathrm{tw}})_{k\geq 1}$ from (78). Suppose given $k \geq 1$. Suppose given $o_0 = (l^0, (i_1^0, \ldots, i_{l^0}^0), D^0), \ldots, o_k = (l^k, (i_1^k, \ldots, i_{l^k}^k), D^k) \in \operatorname{Obj}(\operatorname{tw} A)$. From the definition of the f_k^{tw} we obtain via the bar construction and (3) the F_k^{tw}: On $S(\operatorname{tw} A)(o_0, o_1) \otimes \ldots \otimes S(\operatorname{tw} A)(o_{k-1}, o_k)$, we have

$$\begin{aligned} F_k^{\mathrm{tw}} = \sum_{\text{all } j_x \geq 0} & F_{(j_0+\ldots+j_k)+k} \circ (\llbracket D^0 \rrbracket^{\otimes j_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1} \otimes \\ & \ldots \otimes \llbracket D^{k-1} \rrbracket^{\otimes j_{k-1}} \otimes \mathrm{id}^{l^{k-1} \times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j_k}). \end{aligned} \tag{79}$$

Note that for $j \in [0, k]$, we have $f_{\mathrm{Obj}}^{\mathrm{tw}}(o_j) = (l^j, (f_{\mathrm{Obj}}(i_1^j), \ldots, f_{\mathrm{Obj}}(i_{l^j}^j)), \check{D}^j)$, where $\check{D}^j = \sum_{i\geq 1} \check{\omega}(F_i((\omega^{-1}(D^j))^{\otimes i}))$. Hence for $j \in [0, k]$, we have

$$\llbracket \check{D}^j \rrbracket = \sum_{i\geq 1} F_i \circ \llbracket D^j \rrbracket^{\otimes i}. \tag{80}$$

On $S(\operatorname{tw} A)(o_0, o_1) \otimes \ldots \otimes S(\operatorname{tw} A)(o_{k-1}, o_k)$, we have

$$\begin{aligned} & \sum_{\substack{k=r+s+t \\ r,t\geq 0, s\geq 1}} F_{r+1+t}^{\mathrm{tw}} \circ (1^{\otimes r} \otimes b_s^{\mathrm{tw}} \otimes 1^{\otimes t}) \\ = & \sum_{\substack{k=r+s+t \\ r,t\geq 0, s\geq 1 \\ \text{all } j_x, j'_x, j''_x \geq 0}} F_{\substack{j_0+\ldots+j_r \\ +j''_0+\ldots+j''_t \\ +r+1+t}} \circ \Big(\llbracket D^0 \rrbracket^{\otimes j_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1} \otimes \ldots \otimes \mathrm{id}^{l^{r-1} \times l^r} \otimes \llbracket D^r \rrbracket^{\otimes j_r} \otimes \mathrm{id}^{l^r \times l^{r+s}} \\ & \otimes \llbracket D^{r+s} \rrbracket^{\otimes j''_0} \otimes \mathrm{id}^{l^{r+s} \times l^{r+s+1}} \otimes \llbracket D^{r+s+1} \rrbracket^{\otimes j''_1} \otimes \ldots \otimes \mathrm{id}^{l^{r+s+t-1} \times l^{r+s+t}} \otimes \llbracket D^{r+s+t} \rrbracket^{\otimes j''_t} \Big) \\ & \circ \Big(1^{\otimes r} \otimes \Big(b_{(j'_0+\ldots+j'_s)+s} \circ (\llbracket D^r \rrbracket^{\otimes j'_0} \otimes \mathrm{id}^{l^r \times l^{r+1}} \otimes \llbracket D^{r+1} \rrbracket^{\otimes j'_1} \otimes \\ & \ldots \otimes \mathrm{id}^{l^{r+s-1} \times l^{r+s}} \otimes \llbracket D^{r+s} \rrbracket^{\otimes j'_s}) \Big) \otimes 1^{\otimes t} \Big) \\ \overset{\substack{r'=r+j_0+\ldots+j_r, \\ s'=s+j'_0+\ldots+j'_s, \\ t'=t+j''_0+\ldots+j''_t}}{\underset{(1)}{=}} & \sum_{\substack{r'+s'+t'=(j'''_0+\ldots+j'''_k)+k, \\ \text{all } j'''_x \geq 0, \\ r',t'\geq 0, s'\geq 1, \\ b_{s'} \text{ has at least one} \\ \mathrm{id}^{x\times y} \text{ as argument}}} F_{r'+1+t'} \circ (1^{\otimes r'} \otimes b_{s'} \otimes 1^{\otimes t'}) \\ & \circ \Big(\llbracket D^0 \rrbracket^{\otimes j'''_0} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j'''_1} \otimes \ldots \otimes \mathrm{id}^{l^{k-1} \times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j'''_k} \Big) \end{aligned}$$

$$\overset{(76)}{=} \sum_{\substack{r'+s'+t'=(j_0'''+\ldots+j_k''')+k,\\ \text{all } j_x'''\geq 0,\\ r',t'\geq 0, s'\geq 1}} F_{r'+1+t'} \circ (1^{\otimes r'} \otimes b_{s'} \otimes 1^{\otimes t'})$$

$$\circ \left(\llbracket D^0 \rrbracket^{\otimes j_0'''} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1'''} \otimes \ldots \otimes \mathrm{id}^{l^{k-1}\times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j_k'''} \right)$$

$$\overset{\text{L.50}}{=} \sum_{\substack{r\geq 1,\\ j_1+\ldots+j_r=(j_0'''+\ldots+j_k''')+k,\\ \text{all } j_x\geq 1, \text{all } j_x'''\geq 0}} \check{b}_r \circ (F_{j_1} \otimes \ldots \otimes F_{j_r})$$

$$\circ \left(\llbracket D^0 \rrbracket^{\otimes j_0'''} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1'''} \otimes \ldots \otimes \mathrm{id}^{l^{k-1}\times l^k} \otimes \llbracket D^k \rrbracket^{\otimes j_k'''} \right)$$

$$\overset{(1)}{=} \sum_{\substack{r'\geq 1,\\ j_1''+\ldots+j_{r'}''=(j_0''''+\ldots+j_k'''')+k,\\ \text{all } j_x''\geq 1, \text{ all } j_x', j_x''''\geq 0,\\ \text{each } F_{j_x''} \text{ has at least one}\\ \mathrm{id}^{x\times y} \text{ as argument}}} \check{b}_{r'+j_0'+\ldots+j_{r'}'}$$

$$\circ \left(\left[\textstyle\sum_{n\geq 1} F_n \circ \llbracket D^0 \rrbracket^{\otimes n}\right]^{\otimes j_0'} \otimes F_{j_1''} \otimes \left[\textstyle\sum_{n\geq 1} F_n \circ \llbracket D^* \rrbracket^{\otimes n}\right]^{\otimes j_1'} \otimes \right.$$

$$\left. \ldots \otimes F_{j_{r'}''} \otimes \left[\textstyle\sum_{n\geq 1} F_n \circ \llbracket D^k \rrbracket^{\otimes n}\right]^{\otimes j_{r'}'} \right)$$

$$\circ \left(\llbracket D^0 \rrbracket^{\otimes j_0''''} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1''''} \otimes \ldots \otimes \mathrm{id}^{l^{k-1}\times l^k} \otimes (\omega^{-1}(D^k))^{\otimes j_k''''} \right)$$

$$\overset{\substack{r=r',\\(80)}}{=} \sum_{\substack{r\geq 1,\\ j_1''+\ldots+j_r''=(j_0''''+\ldots+j_k'''')+k,\\ \text{all } j_x''\geq 1, \text{ all } j_x', j_x''''\geq 0,\\ \text{each } F_{j_x''} \text{ has at least one}\\ \mathrm{id}^{x\times y} \text{ as argument}}} \check{b}_{r+j_0'+\ldots+j_r'}$$

$$\circ \left(\llbracket \check{D}^0 \rrbracket^{\otimes j_0'} \otimes F_{j_1''} \otimes \llbracket \check{D}^* \rrbracket^{\otimes j_1'} \otimes \ldots \otimes F_{j_r''} \otimes \llbracket \check{D}^k \rrbracket^{\otimes j_r'} \right)$$

$$\circ \left(\llbracket D^0 \rrbracket^{\otimes j_0''''} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1''''} \otimes \ldots \otimes \mathrm{id}^{l^{k-1}\times l^k} \otimes (\omega^{-1}(D^k))^{\otimes j_k''''} \right)$$

$$\overset{(1)}{=} \sum_{\substack{r\geq 1,\\ j_1''+\ldots+j_r''=(j_0''''+\ldots+j_k'''')+k,\\ \text{all } j_x''\geq 1, \text{ all } j_x''''\geq 0,\\ \text{each } F_{j_x''} \text{ has at least one}\\ \mathrm{id}^{x\times y} \text{ as argument}}} \check{b}_r^{\mathrm{tw}} \circ (F_{j_1''} \otimes \ldots \otimes F_{j_r''})$$

$$\circ \left(\llbracket D^0 \rrbracket^{\otimes j_0''''} \otimes \mathrm{id}^{l^0 \times l^1} \otimes \llbracket D^1 \rrbracket^{\otimes j_1''''} \otimes \ldots \otimes \mathrm{id}^{l^{k-1}\times l^k} \otimes (\omega^{-1}(D^k))^{\otimes j_k''''} \right)$$

$$\overset{(1)}{=} \sum_{\substack{r\geq 1, \text{ all } j_x\geq 1\\ j_1+\ldots+j_r=k}} \check{b}_r^{\mathrm{tw}} \circ (F_{j_1}^{\mathrm{tw}} \otimes \ldots \otimes F_{j_r}^{\mathrm{tw}}).$$

Here, the index $*$ is the number of arguments of $F_{j_1''}$ of type $\mathrm{id}^{x\times y}$. Hence, since $(\mathrm{Obj}\,\mathrm{tw}\,A, \mathrm{tw}\,A, (m_k^{\mathrm{tw}})_{k\geq 1})$ and $(\mathrm{Obj}\,\mathrm{tw}\,\check{A}, \mathrm{tw}\,\check{A}, (\check{m}_k^{\mathrm{tw}})_{k\geq 1})$ are pre-A$_\infty$-categories and since $(f_{\mathrm{Obj}}^{\mathrm{tw}}, (f_k^{\mathrm{tw}})_{k\geq 1})$ is a pre-A$_\infty$-functor from $\mathrm{tw}\,A$ to $\mathrm{tw}\,\check{A}$, eq. (14)[k] holds for $k \geq 1$.

Thus Lemma 50 shows that $\operatorname{tw} f$ is an A_∞-functor.

$\square$

Lemma 123. *Suppose given a commutative diagram of complexes as follows.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[d, "c"] \\ \check{A} \arrow[r] & \check{B} \arrow[r] & \check{C} \end{tikzcd} \tag{81}$$

If the rows are short exact and if a and c are quasi-isomorphisms, then b is a quasi-isomorphism.

Proof. Each row of (81) is a short exact sequence of complexes and hence gives rise to a long exact sequence, cf. e.g. [26, Theorem 1.3.1]. By naturality of these long exact sequences and since the vertical arrows of (81) give a morphism of short exact sequences of complexes, we obtain a morphism between the long exact sequences as follows:

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^{j-1}C \arrow[r, "\partial"] \arrow[d, "\mathrm{H}^{j-1}c"] & \mathrm{H}^{j}A \arrow[r] \arrow[d, "\mathrm{H}^{j}a"] & \mathrm{H}^{j}B \arrow[r] \arrow[d, "\mathrm{H}^{j}b"] & \mathrm{H}^{j}C \arrow[r, "\partial"] \arrow[d, "\mathrm{H}^{j}c"] & \mathrm{H}^{j+1}A \arrow[r] \arrow[d, "\mathrm{H}^{j+1}a"] & \cdots \\ \cdots \arrow[r] & \mathrm{H}^{j-1}\check{C} \arrow[r, "\partial"] & \mathrm{H}^{j}\check{A} \arrow[r] & \mathrm{H}^{j}\check{B} \arrow[r] & \mathrm{H}^{j}\check{C} \arrow[r, "\partial"] & \mathrm{H}^{j+1}\check{A} \arrow[r] & \cdots \end{tikzcd} \tag{82}$$

Suppose given $j \in \mathbb{Z}$. Since a and c are quasi-isomorphisms, the maps $\mathrm{H}^{j-1}c$, $\mathrm{H}^{j}a$, $\mathrm{H}^{j}c$ and $\mathrm{H}^{j+1}a$ are isomorphisms. So application of the five lemma to (82) yields that $\mathrm{H}^{j}b$ is an isomorphism for $j \in \mathbb{Z}$. Hence, b is a quasi-isomorphism. $\square$

Recall that we have A_∞-categories $(\operatorname{Obj} A, A, (m_k)_{k\geq 1})$, $(\operatorname{Obj} \check{A}, \check{A}, (\check{m}_k)_{k\geq 1})$ and an A_∞-functor $(f_{\mathrm{Obj}}, (f_k)_{k\geq 1})$ from A to $\check{A}$. Recall that we have corresponding triples $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, b)$, $((\check{m}_k)_{k\geq 1}, (\check{b}_k)_{k\geq 1}, \check{b})$ and $((f_k)_{k\geq 1}, (F_k)_{k\geq 1}, F)$.

Proposition 124. *If f is a local quasi-isomorphism (recall Definition 32), then* $\operatorname{tw} f$ *is also a local quasi-isomorphism.*

Proof. Suppose given objects $o = (l, (i_1, \ldots, i_l), D), o' = (l', (i'_1, \ldots, i'_{l'}), D') \in \operatorname{Obj}(\operatorname{tw} A)$. Let $\check{o} := f_{\mathrm{Obj}}(o) =: (l, (\check{i}_1, \ldots, \check{i}_l), \check{D})$ and $\check{o}' := f_{\mathrm{Obj}}(o') =: (l', (\check{i}'_1, \ldots, \check{i}'_{l'}), \check{D}')$. We need to show that the complex morphism $f_1^{\mathrm{tw}} : ((\operatorname{tw} A)(o, o'), m_1^{\mathrm{tw}}) \to ((\operatorname{tw} \check{A})(\check{o}, \check{o}'), \check{m}_1^{\mathrm{tw}})$ is a quasi-isomorphism. Recall that $(\operatorname{tw} A)(o, o')$ (and similarly $(\operatorname{tw} \check{A})(\check{o}, \check{o}')$) consists of matrices $(e_{uv})_{u\in[1,l], v\in[1,l']} \in A^{l\times l'}$ such that $e_{uv} \in A(i_u, i'_v)$ for $u \in [1, l]$, $v \in [1, l']$.

Write $\lambda := l \cdot l'$. We arrange the elements of $[1, l] \times [1, l']$ into a finite sequence $(u_1, v_1), \ldots, (u_\lambda, v_\lambda)$ such that for $k, k' \in [1, \lambda]$, we have $k \leq k'$ whenever $u_k \geq u_{k'}$ and $v_k \leq v_{k'}$. I.e. when using the (u_j, v_j) as indices of entries of $l \times l'$-matrices, indices that are further to the right and upwards in the matrix appear later in the sequence. For $k \in [0, \lambda]$, we define the graded submodules

$$V_k := \{(e_{uv})_{u\in[1,l], v\in[1,l']} \in (\operatorname{tw} A)(o, o') \mid e_{u_j v_j} = 0 \text{ for } j > k\} \subseteq (\operatorname{tw} A)(o, o')$$

$$\check{V}_k := \{(e_{uv})_{u\in[1,l],v\in[1,l']} \in (\operatorname{tw}\check{A})(\check{o},\check{o}') \mid e_{u_jv_j} = 0 \text{ for } j > k\} \subseteq (\operatorname{tw}\check{A})(\check{o},\check{o}').$$

Note that for $k \in [0,\lambda]$, we have

$$\begin{aligned} V_k &\simeq \oplus_{j\in[1,k]} A(i_{u_j}, i'_{v_j}) \\ \check{V}_k &\simeq \oplus_{j\in[1,k]} \check{A}(\check{i}_{u_j}, \check{i}'_{v_j}). \end{aligned} \tag{83}$$

Note that $0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_\lambda = (\operatorname{tw}A)(o,o')$ and $0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq \check{V}_\lambda = (\operatorname{tw}\check{A})(\check{o},\check{o}')$.

On $(\operatorname{tw}A)(o,o')$, we have

$$\begin{aligned} m_1^{\mathrm{tw}} &= \sum_{j_0,j_1\geq 0} (-1)^0 \omega \circ b_{j_0+j_1+1} \circ (⟦D⟧^{\otimes j_0} \otimes \mathrm{id}^{l\times l'} \otimes ⟦D'⟧^{\otimes j_1}) \circ \omega^{-1} \\ f_1^{\mathrm{tw}} &= \sum_{j_0,j_1\geq 0} (-1)^0 \check{\omega} \circ F_{j_0+j_1+1} \circ (⟦D⟧^{\otimes j_0} \otimes \mathrm{id}^{l\times l'} \otimes ⟦D'⟧^{\otimes j_1}) \circ \omega^{-1}. \end{aligned}$$

Similarly, we have on $(\operatorname{tw}\check{A})(\check{o},\check{o}')$

$$\check{m}_1^{\mathrm{tw}} = \sum_{j_0,j_1\geq 0} (-1)^0 \check{\omega} \circ \check{b}_{j_0+j_1+1} \circ (⟦\check{D}⟧^{\otimes j_0} \otimes \mathrm{id}^{l\times l'} \otimes ⟦\check{D}'⟧^{\otimes j_1}) \circ \check{\omega}^{-1}.$$

Hence since $D, D', \check{D}$ and $\check{D}'$ are strictly lower triangular and by the ordering of the (u_j, v_j), we have for $k \in [1,\lambda]$ and $E \in V_k$, $\check{E} \in \check{V}_k$

$$\begin{aligned} m_1^{\mathrm{tw}}(E) &\in (\omega \circ b_{0+0+1} \circ \mathrm{id}^{l\times l'} \circ \omega^{-1})(E) + V_{k-1} = m_1(E) + V_{k-1} \\ \check{m}_1^{\mathrm{tw}}(\check{E}) &\in (\check{\omega} \circ \check{b}_{0+0+1} \circ \mathrm{id}^{l\times l'} \circ \check{\omega}^{-1})(\check{E}) + \check{V}_{k-1} = \check{m}_1(\check{E}) + \check{V}_{k-1} \\ \check{f}_1^{\mathrm{tw}}(E) &\in (\check{\omega} \circ F_{0+0+1} \circ \mathrm{id}^{l\times l'} \circ \omega^{-1})(E) + \check{V}_{k-1} = f_1(E) + \check{V}_{k-1}. \end{aligned} \tag{84}$$

In particular, we have $m_1^{\mathrm{tw}}(V_k) \subseteq V_k$, $f_1^{\mathrm{tw}}(V_k) \subseteq \check{V}_k$ and $\check{m}_1(\check{V}_k) \subseteq \check{V}_k$ for $k \in [0, ll']$.

Thus f_1^{tw} restricts for $k \in [0,\lambda]$ to a complex morphism from $(V_k, m_1^{\mathrm{tw}}|_{V_k}^{V_k})$ to $(\check{V}_k, \check{m}_1^{\mathrm{tw}}|_{\check{V}_k}^{\check{V}_k})$. We prove by induction on $k \in [0,\lambda]$ that

$$f_1^{\mathrm{tw}}|_{V_k}^{\check{V}_k} : (V_k, m_1^{\mathrm{tw}}|_{V_k}^{V_k}) \to (\check{V}_k, \check{m}_1^{\mathrm{tw}}|_{\check{V}_k}^{\check{V}_k}) \tag{85}$$

is a quasi-isomorphism. The initial step $k = 0$ follows from $V_0 = 0$ and $\check{V}_0 = 0$.
For the induction step suppose given a $k \in [0, \lambda-1]$ such that (85) is a quasi-isomorphism.

Consider the following diagram.

$$\begin{tikzcd} (V_k, m_1^{\mathrm{tw}}|_{V_k}^{V_k}) \arrow[r, "\subseteq"] \arrow[d, "f_1^{\mathrm{tw}}|_{V_k}^{\check{V}_k}"] & (V_{k+1}, m_1^{\mathrm{tw}}|_{V_{k+1}}^{V_{k+1}}) \arrow[r, "p"] \arrow[d, "f_1^{\mathrm{tw}}|_{V_{k+1}}^{\check{V}_{k+1}}"] & (A(i_{u_{k+1}}, i'_{v_{k+1}}), m_1) \arrow[d, "f_1|_{A(i_{u_{k+1}}, i'_{v_{k+1}})}^{\check{A}(\check{i}_{u_{k+1}}, \check{i}'_{v_{k+1}})}"] \\ (\check{V}_k, \check{m}_1^{\mathrm{tw}}|_{\check{V}_k}^{\check{V}_k}) \arrow[r, "\subseteq"] & (\check{V}_{k+1}, \check{m}_1^{\mathrm{tw}}|_{\check{V}_{k+1}}^{\check{V}_{k+1}}) \arrow[r, "\check{p}"] & (\check{A}(\check{i}_{u_{k+1}}, \check{i}'_{v_{k+1}}), \check{m}_1) \end{tikzcd} \tag{86}$$

Here, p and $\check{p}$ are the maps $p : V_{k+1} \to A(i_{u_{k+1}}, i'_{v_{k+1}}), (e_{uv})_{u\in[1,l],v\in[1,l']} \mapsto e_{u_{k+1}v_{k+1}}$ and $\check{p} : \check{V}_{k+1} \to \check{A}(\check{i}_{u_{k+1}}, \check{i}'_{v_{k+1}}), (e_{uv})_{u\in[1,l],v\in[1,l']} \mapsto e_{u_{k+1}v_{k+1}}$ (These are effectively the residue class maps $V_{k+1} \to V_{k+1}/V_k$ and $\check{V}_{k+1} \to \check{V}_{k+1}/\check{V}_k$, cf. (83)). The maps denoted by $\subseteq$ are the inclusion maps. By (84), all maps in (86) are complex morphisms and (86) is commutative. By construction, each row of (86) is a short exact sequence of complexes. Consider the vertical morphisms in (86): By the induction hypothesis, the left morphism is a quasi-isomorphism. Since f is a local quasi-isomorphism, the right morphism is a quasi-isomorphism. So by Lemma 123, the morphism in the middle is a quasi-isomorphism. This proves the induction step. Thus for $k \in [0, \lambda]$, the morphism (85) is a quasi-isomorphism. For $k = \lambda$ we obtain in particular that

$$f_1^{\mathrm{tw}} : ((\mathrm{tw}\, A)(o, o'), m_1^{\mathrm{tw}}) \to ((\mathrm{tw}\, \check{A})(\check{o}, \check{o}'), \check{m}_1^{\mathrm{tw}})$$

is a quasi-isomorphism which completes the proof. □

6.4. $\mathrm{H}^0\, \mathrm{tw}\, \mathrm{Hom}^*(\cdot, \cdot)$

Suppose given an R-algebra B. Suppose given a set I and suppose given complexes $(C^{(i)}, d^{(i)})$ over B for $i \in I$.

We define the A_∞-category $(I, A, (m_k)_{k\geq 1})$ as in Example 31 and Lemma 33.

Definition/Remark 125. For an object $o = (l, (i_1, \ldots, i_l), D = (D_{uv})_{u,v\in[1,l]}) \in \mathrm{Obj}\, \mathrm{tw}\, A$, we define the complex (C^o, d^o) over B as follows.

- We set $C^o := \oplus_{j\in[1,l]} C^{(i_j)}$. For $j \in [1, l]$, let $\iota_j^{C^o} : C^{(i_j)} \to C^o$ and $\pi_j^{C^o} : C^o \to C^{(i_j)}$ be the canonical inclusions and projections of the direct sum C^o.
- We set $d^o : C^o \to C^o$, $d^o := \sum_{j\in[1,l]} \iota_j^{C^o} \circ d^{(i_j)} \circ \pi_j^{C^o} + \sum_{j,j'\in[1,l]} \iota_j^{C^o} \circ D_{jj'} \circ \pi_{j'}^{C^o}$.

Proof. We need to prove that (C^o, d^o) is actually a complex for $o \in \mathrm{Obj}\, \mathrm{tw}\, A$. That is, we need to prove $(d^o)^2 = 0$. We have the corresponding pre-A_∞-triple $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, *)$. Recall that for $k \geq 3$, we have $m_k = 0$ hence $b_k = 0$. By (76), we have

$$\begin{aligned} 0 &= \sum_{k\geq 1} b_k((\omega^{-1}(D))^{\otimes k}) = b_1(\omega^{-1}(D)) + b_2((\omega^{-1}(D)) \otimes (\omega^{-1}(D))) \\ &\overset{(1)}{=} (b_1 \circ \omega^{-1})(D) - (b_2 \circ (\omega^{-1})^{\otimes 2})(D^{\otimes 2}) \\ &= (\omega^{-1} \circ m_1 \circ \omega \circ \omega^{-1})(D) - (\omega^{-1} \circ m_2 \circ \omega^{\otimes 2} \circ (\omega^{-1})^{\otimes 2})(D^{\otimes 2}) \\ &\overset{(3)}{=} (\omega^{-1} \circ m_1)(D) + (\omega^{-1} \circ m_2)(D^{\otimes 2}) \,. \end{aligned} \tag{87}$$

In particular, we have $0 = m_1(D) + m_2(D \otimes D)$. Breaking this equation of matrices down into components, we obtain $m_1(D_{jj''}) + \sum_{j'\in[1,l]} m_2(D_{jj'} \otimes D_{j'j''}) = 0$ for $j, j'' \in [1, l]$. Thus we have

$$(d^o)^2 = \left(\sum_{j\in[1,l]} \iota_j^{C^o} \circ d^{(i_j)} \circ \pi_j^{C^o} + \sum_{j,j'\in[1,l]} \iota_j^{C^o} \circ D_{jj'} \circ \pi_{j'}^{C^o} \right)^2$$

$$
\begin{aligned}
&= \sum_{j\in[1,l]} \iota_j^{C^o} \circ (d^{(i_j)})^2 \circ \pi_j^{C^o} + \sum_{j,j''\in[1,l]} \iota_j^{C^o} \circ (d^{(i_j)} \circ D_{jj''} + D_{jj''} \circ d^{(i_{j''})}) \circ \pi_{j''}^{C^o} \\
&\quad + \sum_{j,j',j''\in[1,l]} \iota_j^{C^o} \circ D_{jj'} \circ D_{j'j''} \circ \pi_{j''}^{C^o} \\
&= 0 + \sum_{j,j''\in[1,l]} \iota_j^{C^o} \circ d_{\mathrm{Hom}_B^*(C^{(i_{j''})},C^{(i_j)})}(D_{jj''}) \circ \pi_{j''}^{C^o} + \sum_{j,j',j''\in[1,l]} \iota_j^{C^o} \circ m_2(D_{jj'} \otimes D_{j'j''}) \circ \pi_{j''}^{C^o} \\
&= \sum_{j,j''\in[1,l]} \iota_j^{C^o} \circ \left(m_1(D_{jj''}) + \sum_{j'\in[1,l]} m_2(D_{jj'} \otimes D_{j'j''}) \right) \circ \pi_{j''}^{C^o} = 0.
\end{aligned}
$$

Thus d^o is a differential. $\square$

Definition/Lemma 126. For objects $o = (l, (i_1, \ldots, i_l), D), o' = (l', (i'_1, \ldots, i'_{l'}), D') \in \mathrm{Obj}\,\mathrm{tw}\, A$, we define the map

$$
\begin{aligned}
T_{oo'} : (\mathrm{tw}\, A)(o, o') &\longrightarrow \mathrm{Hom}_B^*(C^{o'}, C^o) \\
(e_{uv})_{u\in[1,l],v\in[1,l']} &\longmapsto \sum_{u\in[1,l],v\in[1,l']} \iota_u^{C^o} \circ e_{uv} \circ \pi_v^{C^{o'}} .
\end{aligned}
$$

For $o = (l, (i_1, \ldots, i_l), D) \in \mathrm{Obj}\,\mathrm{tw}\, A$, we define the diagonal matrix $d_0^o := (d_{uv})_{u,v\in[1,l]} \in A^1(o, o)$ given by $d_{uu} := d^{(i_u)}$ for $u \in [1, l]$ and $d_{uv} := 0$ for $u, v \in [1, l]$ with $u \neq v$.

(a) For $o, o' \in \mathrm{Obj}\,\mathrm{tw}\, A$, the map $T_{oo'}$ is bijective.

(b) For $o = (l, (i_1, \ldots, i_l), D) \in \mathrm{Obj}\,\mathrm{tw}\, A$, we have $d^o = T_{oo}(d_0^o + D)$.

(c) For $o, o', o'' \in \mathrm{Obj}\,\mathrm{tw}\, A$, have $m_2^{\mathrm{tw}}|_{(\mathrm{tw}\, A)(o,o')\otimes(\mathrm{tw}\, A)(o',o'')} = m_2|_{(\mathrm{tw}\, A)(o,o')\otimes(\mathrm{tw}\, A)(o',o'')}$.

(d) For $o, o', o'' \in \mathrm{Obj}\,\mathrm{tw}\, A$, $E \in (\mathrm{tw}\, A)(o, o')$ and $E' \in (\mathrm{tw}\, A)(o', o'')$, we have

$$T_{oo''}(m_2(E \otimes E')) = T_{oo'}(E) \circ T_{o'o''}(E').$$

(e) For $o, o' \in \mathrm{Obj}\,\mathrm{tw}\, A$ and $E \in (\mathrm{tw}\, A)(o, o')$, we have

$$T_{oo'}(m_1^{\mathrm{tw}}(E)) = d_{\mathrm{Hom}_B^*(C^{o'},C^o)}(T_{oo'}(E)).$$

Thus $T_{oo'} : ((\mathrm{tw}\, A)(o, o'), m_1^{\mathrm{tw}}) \to (\mathrm{Hom}_B^*(C^{o'}, C^o), d_{\mathrm{Hom}_B^*(C^{o'},C^o)})$ is by (a) an isomorphism of complexes.

(f) We have $m_k^{\mathrm{tw}} = 0$ for $k \geq 3$.

(g) $\mathrm{H}^0(\mathrm{tw}\, A)$ has identities. I.e. it is a category.

(h) We obtain a functor $Q : \mathrm{H}^0(\mathrm{tw}\, A) \to B\text{-}\mathtt{Mod}$ by setting

- For $o \in \mathrm{Obj}\,\mathrm{tw}\, A = \mathrm{Obj}\,\mathrm{H}^0(\mathrm{tw}\, A)$, we set $Q(o) := \mathrm{H}_0(C^o, d^o)$.
- For $o, o' \in \mathrm{Obj}\,\mathrm{tw}A = \mathrm{Obj}\,\mathrm{H}^0(\mathrm{tw}A)$ and $\bar{f} = f + \mathrm{B}^0(\mathrm{tw}A)(o, o') \in \mathrm{H}^0(\mathrm{tw}\, A)(o, o')$ for some $f \in \mathrm{Z}^0(\mathrm{tw}\, A)(o, o')$, we set

  $$Q_{oo'}(\bar{f}) := \mathrm{H}_0(T_{oo'}(f)) \in \mathrm{Hom}_B(\mathrm{H}_0(C^{o'}, d^{o'}), \mathrm{H}_0(C^o, d^o)) = \mathrm{Hom}_B(Q(o'), Q(o))$$

  (note the reversal of o and o').

Proof. We have the corresponding pre-A_∞-triple $((m_k)_{k\geq 1},(b_k)_{k\geq 1},*)$.
(a) and (b) hold by construction.
(c): Suppose $o=(l,(i_1,\dots,i_l),D), o'=(l',(i'_1,\dots,i'_{l'}),D'), o''=(l'',(i''_1,\dots,i''_{l''}),D'')\in \operatorname{Obj}\operatorname{tw}A$. Since $m_k=0$ for $k\geq 3$, we have $b_k=0$ for $k\geq 3$. Thus in the definition of m_2 (cf. (77)), all summands with $j_0+\ldots+j_k\neq 0$ are zero. Hence, we have on $(\operatorname{tw}A)(o,o')\otimes(\operatorname{tw}A)(o',o'')$

$$\begin{aligned}m_2^{\mathrm{tw}} =&(-1)^{\frac{2(2-1)}{2}}\omega\circ b_2\circ(\mathrm{id}^{l\times l'}\otimes\mathrm{id}^{l'\times l''})\circ(\omega^{-1})^{\otimes 2}=-\omega\circ b_2\circ(\omega^{-1})^{\otimes 2}\\ =&-\omega\circ\omega^{-1}\circ m_2\circ\omega^{\otimes 2}\circ(\omega^{-1})^{\otimes 2}\overset{(3)}{=}m_2.\end{aligned}$$

(d): Suppose $o=(l,(i_1,\dots,i_l),D), o'=(l',(i'_1,\dots,i'_{l'}),D'), o''=(l'',(i''_1,\dots,i''_{l''}),D'')\in \operatorname{Obj}\operatorname{tw}A$, $E=(e_{uv})_{u\in[1,l],v\in[1,l']}\in(\operatorname{tw}A)(o,o')$ and $E'=(e'_{uv})_{u\in[1,l'],v\in[1,l'']}\in(\operatorname{tw}A)(o',o'')$. We have

$$\begin{aligned}T_{oo'}(E)\circ T_{o'o''}(E') &=\left(\sum_{j\in[1,l],j'\in[1,l']}\iota_j^{C^o}\circ e_{jj'}\circ\pi_{j'}^{C^{o'}}\right)\circ\left(\sum_{j'\in[1,l'],j''\in[1,l'']}\iota_{j'}^{C^{o'}}\circ e'_{j'j''}\circ\pi_{j''}^{C^{o''}}\right)\\ &=\sum_{j\in[1,l],j'\in[1,l'],j''\in[1,l'']}\iota_j^{C^o}\circ e_{jj'}\circ e'_{j'j''}\circ\pi_{j''}^{C^{o''}}\\ &=\sum_{\substack{j\in[1,l],\\ j''\in[1,l'']}}\iota_j^{C^o}\circ\left(\sum_{j'\in[1,l']}m_2(e_{jj'}\otimes e'_{j'j''})\right)\circ\pi_{j''}^{C^{o''}}=T_{oo''}(m_2(E\otimes E')).\end{aligned}$$

(e): Suppose given $o=(l,(i_1,\dots,i_l),D), o'=(l',(i'_1,\dots,i'_{l'}),D')\in\operatorname{Obj}\operatorname{tw}A$. Suppose given a homogeneous element $E=(e_{uv})_{u\in[1,l],v\in[1,l']}\in(\operatorname{tw}A)(o,o')^{\mathsf{k}_E}$ for some $\mathsf{k}_E\in\mathbb{Z}$. Since $b_k=0$ for $k\geq 3$, we have

$$\begin{aligned}m_1^{\mathrm{tw}}(E)&\overset{(77)}{=}(-1)^0\left(\omega\circ\left(b_1\circ\mathrm{id}^{l\times l'}+b_2\circ([\![D]\!]\otimes\mathrm{id}^{l\times l'}+\mathrm{id}^{l\times l'}\otimes[\![D']\!])\right)\circ(\omega^{-1})^{\otimes 1}\right)(E)\\ &\overset{(1)}{=}\omega\left(b_1(\omega^{-1}(E))+b_2\left(\omega^{-1}(D)\otimes\omega^{-1}(E)+\omega^{-1}(E)\otimes\omega^{-1}(D')\right)\right)\\ &=\omega\left((\omega^{-1}\circ m_1\circ\omega)(\omega^{-1}(E))\right.\\ &\qquad\left.+(\omega^{-1}\circ m_2\circ\omega^{\otimes 2})\left(\omega^{-1}(D)\otimes\omega^{-1}(E)+\omega^{-1}(E)\otimes\omega^{-1}(D')\right)\right)\\ &=m_1(E)+(m_2\circ\omega^{\otimes 2})(\omega^{-1}(D)\otimes\omega^{-1}(E)+\omega^{-1}(E)\otimes\omega^{-1}(D'))\\ &\overset{(1)}{=}m_1(E)+m_2(\omega(\omega^{-1}(D))\otimes\omega(\omega^{-1}(E))+(-1)^{\mathsf{k}_E-1}\omega(\omega^{-1}(E))\otimes\omega(\omega^{-1}(D')))\\ &=m_1(E)+m_2(D\otimes E-(-1)^{\mathsf{k}_E}E\otimes D').\end{aligned}$$

Thus we have

$$\begin{aligned}&T_{oo'}(m_1^{\mathrm{tw}}(E))\\ =&T_{oo'}\left((m_1(E)+m_2\circ(D\otimes E-(-1)^{\mathsf{k}_E}E\otimes D')\right)\\ \overset{(d)}{=}&\left(\sum_{j\in[1,l],j'\in[1,l']}\iota_j^{C^o}\circ m_1(e_{jj'})\circ\pi_{j'}^{C^{o'}}\right)+T_{oo}(D)\circ T_{oo'}(E)-(-1)^{\mathsf{k}_E}T_{oo'}(E)\circ T_{o'o'}(D')\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in[1,l],j'\in[1,l']} \iota_j^{C^o} \circ d_{\mathrm{Hom}^*_B(C^{(j')},C^{(j)})}(e_{jj'}) \circ \pi_{j'}^{C^{o'}} \\
&\quad + T_{oo}(D) \circ T_{oo'}(E) - (-1)^{\mathsf{k}_E} T_{oo'}(E) \circ T_{o'o'}(D') \\
&= \sum_{j\in[1,l],j'\in[1,l']} \iota_j^{C^o} \circ (d^{(j)} \circ e_{jj'} - (-1)^{\mathsf{k}_E} e_{jj'} \circ d^{(j')}) \circ \pi_{j'}^{C^{o'}} \\
&\quad + T_{oo}(D) \circ T_{oo'}(E) - (-1)^{\mathsf{k}_E} T_{oo'}(E) \circ T_{o'o'}(D') \\
&= T_{oo}(d_0^o) \circ T_{oo'}(E) - (-1)^{\mathsf{k}_E} T_{oo'}(E) \circ T_{o'o'}(d_0^{o'}) \\
&\quad + T_{oo}(D) \circ T_{oo'}(E) - (-1)^{\mathsf{k}_E} T_{oo'}(E) \circ T_{o'o'}(D') \\
&= T_{oo}(D + d_0^o) \circ T_{oo'}(E) - (-1)^{\mathsf{k}_E} T_{oo'}(E) \circ T_{o'o'}(D' + d_0^{o'}) \\
&\overset{(b)}{=} d^o \circ T_{oo'}(E) - (-1)^{\mathsf{k}_E} T_{oo'}(E) \circ d^{o'} \\
&= d_{\mathrm{Hom}^*_B(C^{o'},C^o)}(T_{oo'}(E)).
\end{aligned}
$$

(f): For $k \geq 3$, we have $m_k = 0$, hence $b_k = 0$. Thus for $k \geq 3$, all summands in (77) are zero.

(g): By Definition/Remark 37, $\mathrm{H}^0 \operatorname{tw} A$ is a semicategory. Suppose given $o = (l, (i_1, \ldots, i_l), D) \in \operatorname{Obj} \operatorname{tw} A$. Consider the diagonal matrix $Z = (Z_{uv})_{u,v\in[1,l]} \in (\operatorname{tw} A)(o,o)^0$ given by $Z_{uu} := \mathrm{id}_{C^{i_u}} \in A(i_u, i_u)$ for $u \in [1,l]$ and $Z_{uv} := 0$ for $u, v \in [1,l]$ with $u \neq v$. We have $T_{oo}(Z) = \mathrm{id}_{C^o}$, so $T_{oo}(Z)$ is in particular a complex morphism. Hence $0 = d_{\mathrm{Hom}^*_B(C^o,C^o)}(T_{oo}(Z)) \overset{(e)}{=} T_{oo}(m_1^{\mathrm{tw}}(Z))$. Since T_{oo} is injective, we obtain $m_1^{\mathrm{tw}}(Z) = 0$. Hence, Z represents the homology class $\bar{Z} := Z + \mathrm{B}^0(\operatorname{tw} A)(o,o) \in \mathrm{H}^0(\operatorname{tw} A)(o,o)$.

Suppose given $o' \in \operatorname{Obj} \operatorname{tw} A$. For all $a \in \mathrm{Z}^0(\operatorname{tw} A)(o,o')$ and $b \in \mathrm{Z}^0(\operatorname{tw} A)(o',o)$, assertion (c) implies

$$
\begin{aligned}
m_2^{\mathrm{tw}}(Z \otimes a) &= m_2(Z \otimes a) = a \\
m_2^{\mathrm{tw}}(b \otimes Z) &= m_2(b \otimes Z) = b.
\end{aligned}
$$

Hence, we have $\bar{Z} \cdot \bar{a} = \bar{a}$ and $\bar{b} \cdot \bar{Z} = \bar{b}$ for all $\bar{a} \in \mathrm{H}^0(\operatorname{tw} A)(o,o')$ and all $\bar{b} \in \mathrm{H}^0(\operatorname{tw} A)(o',o)$. Thus $\bar{Z}$ is the identity of the object o in $\mathrm{H}^0(\operatorname{tw} A)$.

(h): We need to show the following.

(i) The map $Q : \operatorname{Obj} \mathrm{H}^0(\operatorname{tw} A) \to \operatorname{Obj} B\text{-}\mathtt{Mod}$ is well-defined.

(ii) For $o, o' \in \operatorname{Obj} \mathrm{H}^0(\operatorname{tw} A) = \operatorname{Obj} \operatorname{tw} A$, the map

$$
Q_{oo'} : \mathrm{H}^0(\operatorname{tw} A)(o,o') \to \mathrm{Hom}_B(Q(o'), Q(o))
$$

is well-defined.

(iii) For $o \in \operatorname{Obj} \mathrm{H}^0(\operatorname{tw} A) = \operatorname{Obj} \operatorname{tw} A$, we have $Q_{oo}(\mathrm{id}_o) = \mathrm{id}_{Q(o)}$.

(iv) For $o, o', o'' \in \operatorname{Obj} \mathrm{H}^0(\operatorname{tw} A) = \operatorname{Obj} \operatorname{tw} A$ and $f \in \mathrm{H}^0(\operatorname{tw} A)(o,o')$, $g \in \mathrm{H}^0(\operatorname{tw} A)(o',o'')$, we have $Q_{oo'}(f) \circ Q_{o'o''}(g) = Q_{oo''}(f \cdot g)$.

Assertion (i) is immediate from the definition of Q.
Assertion (ii) is proven as follows.
Suppose given $o, o' \in \operatorname{Obj} \mathrm{H}^0(\operatorname{tw} A) = \operatorname{Obj} \operatorname{tw} A$. Suppose given $\bar{f} \in \mathrm{H}^0(\operatorname{tw} A)(o, o')$. Suppose given $f \in \mathrm{Z}^0(\operatorname{tw} A)(o, o')$ such that $\bar{f} = f + \mathrm{B}^0(\operatorname{tw} A)(o, o')$. In particular, we have $m_1^{\mathrm{tw}}(f) = 0$. By (e), the map $T_{oo'}(f) \in \operatorname{Hom}_B^0(C^{o'}, C^o)$ is a complex morphism. Thus $\mathrm{H}_0 T_{oo'}(f) : \mathrm{H}_0(C^{o'}, d^{o'}) \to \mathrm{H}_0(C^o, d^o)$ exists. Suppose given $f' \in \mathrm{Z}^0(\operatorname{tw} A)(o, o')$ such that $\bar{f} = f' + \mathrm{B}^0(\operatorname{tw} A)(o, o')$. I.e. $f - f' \in \mathrm{B}^0(\operatorname{tw} A)(o, o')$, which by (e) implies that the complex morphisms $T_{oo'}(f)$ and $T_{oo'}(f')$ are homotopy equivalent. Therefore, $T_{oo'}(f)$ and $T_{oo'}(f')$ induce the same maps in homology. In particular, we have $\mathrm{H}_0 T_{oo'}(f) = \mathrm{H}_0 T_{oo'}(f')$. This proves that $Q_{oo'}$ is well-defined.
To prove assertion (iii), suppose given $o \in \operatorname{Obj} \mathrm{H}^0(\operatorname{tw} A) = \operatorname{Obj} \operatorname{tw} A$. By the proof of (g), the identity $\mathrm{id}_o \in \mathrm{H}^0(\operatorname{tw} A)(o, o)$ is represented by $Z \in \mathrm{Z}^0(\operatorname{tw} A)(o, o)$, for which we have $T_{oo}(Z) = \mathrm{id}_{C^o}$. Hence $Q_{oo}(\mathrm{id}_o) = \mathrm{H}_0 \, \mathrm{id}_{C^o} = \mathrm{id}_{\mathrm{H}_0(C^o, d^o)} = \mathrm{id}_{Q(o)}$.
Assertion (iv) follows from (c), (d) and the functoriality of taking the 0-th homology. □

Definition 127. We call a complex $C = (\cdots \to C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \to \cdots)$ over B a *pr-complex* if

- $C_k = 0$ for $k < 0$,
- all C_k are projective over B and
- $\mathrm{H}_k C = 0$ for $k > 0$.

The notation pr-complex is motivated by the fact each projective resolution of a B-module is a pr-complex. Furthermore, we have

Remark 128. Suppose given a pr-complex $C = (\cdots \to C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \to \cdots)$. Then C is a projective resolution of $\mathrm{H}_0 C = \mathrm{Z}_0 C / \mathrm{B}_0 C = C_0 / (\operatorname{im} d_1)$ with the augmentation $\varepsilon : C_0 \to C_0/(\operatorname{im} d_1) = \mathrm{H}_0 C$ defined as the residue class map.

Lemma 129. *Suppose given complexes (C, d) and (C', d') over B. For $e \in \operatorname{Hom}_B^1(C, C')$, let $d_e := \iota \circ d \circ \pi + \iota' \circ d' \circ \pi' + \iota' \circ e \circ \pi \in \operatorname{Hom}_B^1(C \oplus C', C \oplus C')$, where $\iota : C \to C \oplus C'$, $\iota' : C' \to C \oplus C'$ are the canonical inclusions and $\pi : C \oplus C' \to C$, $\pi' : C \oplus C' \to C'$ are the canonical projections.*

(a) We have $e \in \mathrm{Z}^1 \operatorname{Hom}_B^*(C, C') \Leftrightarrow d_e^2 = 0$. I.e. $(C \oplus C', d_e)$ is a complex if and only if $e \in \mathrm{Z}^1 \operatorname{Hom}_B^*(C, C')$.

(b) For $e, e' \in \mathrm{Z}^1 \operatorname{Hom}_B^*(C, C')$ with $e - e' \in \mathrm{B}^1 \operatorname{Hom}_B^*(C, C')$, the complexes $(C \oplus C', d_e)$ and $(C \oplus C', d_{e'})$ are isomorphic.

Proof. (a): We have

$$\begin{aligned} d_e^2 &= (\iota \circ d \circ \pi + \iota' \circ d' \circ \pi' + \iota' \circ e \circ \pi)^2 \\ &= \iota \circ d^2 \circ \pi + \iota' \circ (d' \circ e + e \circ d) \circ \pi + \iota' \circ d'^{\,2} \circ \pi' \\ &= \iota' \circ d_{\operatorname{Hom}_B^*(C, C')}(e) \circ \pi. \end{aligned}$$

Hence $d_e^2 = 0 \Leftrightarrow d_{\mathrm{Hom}^*_B(C,C')}(e) = 0 \Leftrightarrow e \in \mathrm{Z}^1 \mathrm{Hom}^*_B(C, C')$.

(b): Suppose that $e - e' = d_{\mathrm{Hom}^*_B(C,C')}(h) = d' \circ h - h \circ d$ for some $h \in \mathrm{Hom}^0_B(C, C')$. Consider the morphism $f := \iota \circ \pi + \iota' \circ \pi' + \iota' \circ h \circ \pi \in \mathrm{Hom}^0_B(C \oplus C', C \oplus C')$. We have

$$\begin{aligned} f \circ d_e - d_{e'} \circ f &= (\iota \circ \pi + \iota' \circ \pi' + \iota' \circ h \circ \pi) \circ (\iota \circ d \circ \pi + \iota' \circ d' \circ \pi' + \iota' \circ e \circ \pi) \\ &\quad - (\iota \circ d \circ \pi + \iota' \circ d' \circ \pi' + \iota' \circ e' \circ \pi) \circ (\iota \circ \pi + \iota' \circ \pi' + \iota' \circ h \circ \pi) \\ &= \iota \circ (d - d) \circ \pi + \iota' \circ (d' - d') \circ \pi' + \iota' \circ (e + h \circ d - d' \circ h - e') \circ \pi = 0. \end{aligned}$$

Hence $f : (C \oplus C', d_e) \to (C \oplus C', d_{e'})$ is a complex morphism. Since f is inverted by $f^{-1} = \iota \circ \pi + \iota' \circ \pi' + \iota' \circ (-h) \circ \pi \in \mathrm{Hom}^0_B(C \oplus C', C \oplus C')$, it is an isomorphism of complexes. $\square$

Lemma 130. *Suppose that $(C^{(i)}, d^{(i)})$ is a pr-complex for $i \in I$.*

Let `filt` *be the full subcategory of B-*`Mod` *that consists of modules that have a finite filtration such that each subquotient is isomorphic to some $\mathrm{H}_0(C^{(i)}, d^{(i)})$ for some $i \in I$.*

(a) For $o \in \mathrm{Obj}\,\mathrm{tw}\,A$, the complex (C^o, d^o) is a pr-complex.

(b) For each $o \in \mathrm{Obj}\,\mathrm{tw}\,A$, the B-module $\mathrm{H}_0(C^o, d^o)$ is in Obj `filt`.

(c) Suppose given an A_∞-category $(\mathrm{Obj}\,A', A', (m'_k)_{k\geq 1})$ and a local quasi-isomorphism of A_∞-categories $(f_{\mathrm{Obj}}, (f_k)_{k\geq 1}) : (\mathrm{Obj}\,A', A', (m'_k)_{k\geq 1}) \to (I, A, (m_k)_{k\geq 1})$. Suppose that for each $i \in I$, there exists $o' \in \mathrm{Obj}\,A'$ such that $\mathrm{H}_0(C^{(i)}, d^{(i)}) \simeq \mathrm{H}_0(C^{(f_{\mathrm{Obj}}(o'))}, d^{(f_{\mathrm{Obj}}(o'))})$. Then for each B-module M in Obj `filt`*, there exists $o' \in \mathrm{Obj}\,\mathrm{tw}\,A'$ such that $\mathrm{H}_0(C^o, d^o) \simeq M$ for $o := f^{\mathrm{tw}}_{\mathrm{Obj}}(o')$.*

(d) Suppose given a B-module M in Obj `filt`*. Then there exists $o \in \mathrm{Obj}\,\mathrm{tw}\,A$ such that $\mathrm{H}_0(C^o, d^o) \simeq M$.*

(e) The functor Q defined in Definition/Lemma 126(h) from the category $\mathrm{H}^0(\mathrm{tw}\,A)$ to B-`Mod` *is fully faithful.*

(f) The functor Q defined in Definition/Lemma 126(h) from the category $\mathrm{H}^0(\mathrm{tw}\,A)$ to `filt` *(cf. (b)) is fully faithful and dense. I.e. it is an equivalence of categories from $\mathrm{H}^0(\mathrm{tw}\,A)$ to* `filt`.

Proof. We have the corresponding pre-A_∞-triple $((m_k)_{k\geq 1}, (b_k)_{k\geq 1}, *)$.
(a): Suppose given $o = (l, (i_1, \ldots, i_l), D) \in \mathrm{Obj}\,\mathrm{tw}\,A$. We need to show that (C^o, d^o) is a pr-complex. The first two properties given in Definition 127 follow from $C^o = \oplus_{j\in[1,l]} C^{(i_j)}$ and the fact that the $(C^{(i)}, d^{(i)})$ are pr-complexes. We prove the third property by induction on $l \geq 0$.
The assertion holds for $l = 0$ since then C^o is the zero complex, hence $\mathrm{H}_k(C^o, d^o) = 0$ for $k \in \mathbb{Z}$.
For the induction step assume that the assertion holds for an $l \geq 0$. Suppose given $o = (l+1, (i_1, \ldots, i_{l+1}), D = (D_{uv})_{u,v\in[1,l+1]}) \in \mathrm{Obj}\,\mathrm{tw}\,A$. Let $\check{o} := (1, (i_{l+1}), \check{D} := (0)) \in \mathrm{Obj}\,\mathrm{tw}\,A$. Let $\hat{o} := (l, (i_1, \ldots, i_l), \hat{D} := (D_{uv})_{u,v\in[1,l]})$. We prove $\hat{o} \in \mathrm{Obj}\,\mathrm{tw}\,A$: Properties

(1), (2) and (3) in Definition/Lemma 121 follow from the respective properties of o. Note that since $o \in \operatorname{Obj} \operatorname{tw} A$, we have $0 = \sum_{k\geq 1} b_k((\omega^{-1}(D))^{\otimes k}) =: \bar{D} = (\bar{D}_{uv})_{u,v\in[1,l+1]}$. Since D can be described as a block matrix of the form

$$D = \left(\begin{array}{c|c} \hat{D} & 0 \\ \hline \lambda & 0 = \check{D} \end{array}\right), \tag{88}$$

for some $1 \times l$-matrix $\lambda \in A^{1\times l}$, we have

$$(\bar{D}_{uv})_{u\in[1,l],v\in[1,l]} = \sum_{k\geq 1} b_k((\omega^{-1}(\hat{D}))^{\otimes k}).$$

So since $\bar{D} = 0$, we have $\sum_{k\geq 1} b_k((\omega^{-1}(\hat{D}))^{\otimes k}) = 0$. This completes the proof that $\hat{o} \in \operatorname{Obj} \operatorname{tw} A$.

Note that $(C^{\check{o}}, d^{\check{o}}) = (C^{(i_{l+1})}, d^{(i_{l+1})})$. We identify $C^o = \oplus_{j\in[1,l+1]} C^{(i_j)} = \left(\oplus_{j\in[1,l]} C^{(i_j)}\right) \oplus C^{(i_{j+1})}$ with $C^{\hat{o}} \oplus C^{\check{o}} = \left(\oplus_{j\in[1,l]} C^{(i_j)}\right) \oplus C^{(i_{j+1})}$. Let $\iota_{\hat{o}} : C^{\hat{o}} \to C^o$ and $\iota_{\check{o}} : C^{\check{o}} \to C^o$ be the inclusion maps. Let $\pi_{\hat{o}} : C^o \to C^{\hat{o}}$ and $\pi_{\check{o}} : C^o \to C^{\check{o}}$ be the projection maps.

By (88) and Definition/Lemma 126(b), we have

$$\begin{aligned} d^o &= T_{oo}(d_0^o + D) \\ &= \iota_{\hat{o}} \circ T_{\hat{o}\hat{o}}(d_0^{\hat{o}}) \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ T_{\check{o}\check{o}}(d_0^{\check{o}}) \circ \pi_{\check{o}} \\ &\quad + \iota_{\hat{o}} \circ T_{\hat{o}\hat{o}}(D^{\hat{o}}) \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ T_{\check{o}\check{o}}(\check{D}) \circ \pi_{\check{o}} + \iota_{\check{o}} \circ T_{\check{o}\hat{o}}(\lambda) \circ \pi_{\hat{o}} \\ &= \iota_{\hat{o}} \circ d^{\hat{o}} \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ d^{\check{o}} \circ \pi_{\check{o}} + \iota_{\check{o}} \circ T_{\check{o}\hat{o}}(\lambda) \circ \pi_{\hat{o}}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \pi_{\hat{o}} \circ d^o &= d^{\hat{o}} \circ \pi_{\hat{o}} \\ d^o \circ \iota_{\check{o}} &= \iota_{\check{o}} \circ d^{\check{o}}. \end{aligned}$$

So $\pi_{\hat{o}}$ and $\iota_{\check{o}}$ are morphisms of complexes. Hence, we obtain the short exact sequence of complexes

$$C^{\check{o}} \xrightarrow{\iota_{\check{o}}} C^o \xrightarrow{\pi_{\hat{o}}} C^{\hat{o}}. \tag{89}$$

Since $(C^{\check{o}}, d^{\check{o}}) = (C^{(i_{l+1})}, d^{(i_{l+1})})$ and since $(C^{(i_{l+1})}, d^{(i_{l+1})})$ is a pr-complex, we have $\mathrm{H}_k(C^{\check{o}}, d^{\check{o}}) = 0$ for $k > 0$. By the induction hypothesis, we have $\mathrm{H}_k(C^{\hat{o}}, d^{\hat{o}}) = 0$ for $k > 0$. Thus considering the long exact sequence arising from (89) (cf. e.g. [26, Theorem 1.3.1]), we obtain $\mathrm{H}_k(C^o, d^o) = 0$ for $k > 0$. Thus the assertion holds for $l + 1$, which completes the induction step.

(b): We need to prove the assertion for all $o = (l, (i_1, \ldots, i_l), D) \in \operatorname{Obj} \operatorname{tw} A$. We show the assertion by induction on $l \geq 0$. The initial step $l = 0$ holds since then $C^o = 0$, hence $\mathrm{H}^0(C^o, d^o) = 0$ which implies the assertion.
For the induction step, assume that the assertion holds for some $l \geq 0$. Suppose

given $o = (l+1, (i_1, \dots, i_{l+1}), D) \in \operatorname{Obj} \operatorname{tw} A$. In the same way as in the proof of (a), we obtain $\hat{o} := (l, (i_1, \dots, i_l), \hat{D} = (D_{uv})_{u,v \in [1,l]}) \in \operatorname{Obj} \operatorname{tw} A$, $\check{o} := (1, (i_{l+1}), \check{D} := (0)) \in \operatorname{Obj} \operatorname{tw} A$ and the short exact sequence of complexes (89). Since (a) implies $\mathrm{H}_k(C^o, d^o) = \mathrm{H}_k(C^{\hat{o}}, d^{\hat{o}}) = \mathrm{H}_k(C^{\check{o}}, d^{\check{o}}) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$, the only part of the long exact sequence arising from (89) (cf. e.g. [26, Theorem 1.3.1]) that may be non-zero is the short exact sequence

$$\mathrm{H}_0(C^{\check{o}}, d^{\check{o}}) \xrightarrow{\mathrm{H}_0 \iota_{\check{o}}} \mathrm{H}_0(C^o, d^o) \xrightarrow{\mathrm{H}_0 \pi_{\hat{o}}} \mathrm{H}_0(C^{\hat{o}}, d^{\hat{o}}).$$

By the induction hypothesis, $\mathrm{H}_0(C^{\hat{o}}, d^{\hat{o}})$ has a finite filtration such that each subquotient is isomorphic to $\mathrm{H}_0(C^{(i)}, d^{(i)})$ for some $i \in I$. Since $\mathrm{H}_0(C^{\check{o}}, d^{\check{o}}) = \mathrm{H}_0(C^{(i_{l+1})}, d^{(i_{l+1})})$, this also holds for $\mathrm{H}_0(C^o, d^o)$. This completes the induction step.

(c): We have the corresponding triples $((m'_k)_{k \ge 1}, (b'_k)_{k \ge 1}, *)$ and $((f_k)_{k \ge 1}, (F_k)_{k \ge 1}, *)$. Suppose given a B-module N and a filtration $N = N_0 \geqslant \ldots \geqslant N_l = 0$ such that for $j \in [0, l-1]$, the module N_j / N_{j+1} is isomorphic to $\mathrm{H}_0(C^{(i)}, d^{(i)})$ for an $i \in I$. We will prove that there exists $o' = (l, (i'_1, \dots, i'_l), D') \in \operatorname{Obj} \operatorname{tw} A'$ such that for $o := f^{\mathrm{tw}}_{\mathrm{Obj}}(o') \in \operatorname{Obj} \operatorname{tw} A$, we have $N \simeq \mathrm{H}_0(C^o, d^o)$. We prove this by induction on the filtration length $l \geq 0$.
For the initial step, let $l = 0$. Thus $N = 0$ and $N = \mathrm{H}_0(C^o, d^o)$ for $o := (0, (), ()) \in \operatorname{Obj} \operatorname{tw} A$ (the third entry is the 0×0-matrix, hence it has no entries). For $o' := (0, (), ()) \in \operatorname{Obj} \operatorname{tw} A'$, we have $o = f^{\mathrm{tw}}_{\mathrm{Obj}}(o')$. This proves the initial step $l = 0$.
For the induction step, assume that for a $l \geq 0$, the assertion holds for modules with filtration length $\leq l$. Suppose given a module N and a filtration $N = N_0 \geqslant \ldots \geqslant N_{l+1} = 0$ such that for $j \in [0, l]$ the module N_j / N_{j+1} is isomorphic to $\mathrm{H}_0(C^{(i)}, d^{(i)})$ for an $i \in I$. By the induction hypothesis, there exists $\hat{o}' = (l, (i'_1, \dots, i'_l), \hat{D}') \in \operatorname{Obj} \operatorname{tw} A'$ such that for $f^{\mathrm{tw}}_{\mathrm{Obj}}(\hat{o}') =: (l, (i_1, \dots, i_l), \hat{D}) =: \hat{o} \in \operatorname{Obj} \operatorname{tw} A$, we have $N/N_l \simeq \mathrm{H}_0(C^{\hat{o}}, d^{\hat{o}})$. The assumptions yield an $i'_{l+1} \in \operatorname{Obj} A'$ such that for $i_{l+1} := f_{\mathrm{Obj}}(i'_{l+1})$, we have $N_l = N_l / N_{l+1} \simeq \mathrm{H}_0(C^{(i_{l+1})}, d^{(i_{l+1})})$. Thus setting $\check{o}' := (1, (i'_{l+1}), (0) =: \check{D}') \in \operatorname{Obj} \operatorname{tw} A'$ and $\check{o} := f^{\mathrm{tw}}_{\mathrm{Obj}}(\check{o}') = (1, (i_{l+1}), (0) =: \check{D}) \in \operatorname{Obj} \operatorname{tw} A$, we have $N_l \simeq \mathrm{H}_0(C^{\check{o}}, d^{\check{o}})$.

By Remark 128, there exist $\hat{\varepsilon} : C_0^{\hat{o}} \to N/N_l$ and $\check{\varepsilon} : C_0^{\check{o}} \to N_l$ such that $(C^{\hat{o}}, d^{\hat{o}})$ is a projective resolution of N/N_l with augmentation $\hat{\varepsilon}$ and $(C^{\check{o}}, d^{\check{o}})$ is a projective resolution of N_l with augmentation $\check{\varepsilon}$. Consider the following diagram.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & C_2^{\check{o}} \arrow[r] & C_1^{\check{o}} \arrow[r] & C_0^{\check{o}} \arrow[r, "\check{\varepsilon}"] & N_l \arrow[r] \arrow[d, "\iota"] & 0 \\
 & & & & N \arrow[d, "\pi"] & \\
\cdots \arrow[r] & C_2^{\hat{o}} \arrow[r] & C_1^{\hat{o}} \arrow[r] & C_0^{\hat{o}} \arrow[r, "\hat{\varepsilon}"] & N/N_l \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

Here, $\iota : N_l \to N$ is the inclusion map and $\pi : N \to N/N_l$ is the residue class map. The column is a short exact sequence. The rows are the augmented projective resolutions obtained from $C^{\check{o}}$ and $C^{\hat{o}}$. Let $C := C^{\hat{o}} \oplus C^{\check{o}}$. Let $\iota_{\hat{o}} : C^{\hat{o}} \to C$ and $\iota_{\check{o}} : C^{\check{o}} \to C$ be the inclusion maps. Let $\pi_{\hat{o}} : C \to C^{\hat{o}}$ and $\pi_{\check{o}} : C \to C^{\check{o}}$ be the projection maps. By the horseshoe lemma (cf. e.g. [26, Lemma 2.2.8]), there exists a differential ${}_0d$ on C such that

$$C^{\check{o}} \xrightarrow{\iota_{\check{o}}} C \xrightarrow{\pi_{\hat{o}}} C^{\hat{o}}$$

is a short exact sequence of complexes and such that $\mathrm{H}_0(C, {}_0d) \simeq N$. We have

$$\begin{aligned}
\pi_{\hat{o}} \circ {}_0d \circ \iota_{\hat{o}} &= d^{\hat{o}} \circ \pi_{\hat{o}} \circ \iota_{\hat{o}} = d^{\hat{o}} \\
\pi_{\check{o}} \circ {}_0d \circ \iota_{\check{o}} &= \pi_{\check{o}} \circ \iota_{\check{o}} \circ d^{\check{o}} = d^{\check{o}} \\
\pi_{\hat{o}} \circ {}_0d \circ \iota_{\check{o}} &= d^{\hat{o}} \circ \pi_{\hat{o}} \circ \iota_{\check{o}} = 0.
\end{aligned}$$

Thus we obtain the well-known result (cf. e.g. [26, Exercise 2.2.4]) that ${}_0d$ is of the form

$${}_0d = \iota_{\hat{o}} \circ d^{\hat{o}} \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ {}_0\lambda \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ d^{\check{o}} \circ \pi_{\check{o}}$$

for some ${}_0\lambda \in \mathrm{Hom}^1_B(C^{\hat{o}}, C^{\check{o}})$. By Lemma 129(a), we have ${}_0\lambda \in \mathrm{Z}^1 \mathrm{Hom}^*_B(C^{\hat{o}}, C^{\check{o}})$. By Proposition 124, the complex morphism $f_1^{\mathrm{tw}} : ((\mathrm{tw}\, A')(\check{o}', \hat{o}'), {m'_1}^{\mathrm{tw}}) \to ((\mathrm{tw}\, A)(\check{o}, \hat{o}), m_1^{\mathrm{tw}})$ is a quasi-isomorphism. By Definition/Lemma 126(e), the map $T_{\check{o}\hat{o}} : ((\mathrm{tw}\, A)(\check{o}, \hat{o}), m_1^{\mathrm{tw}}) \to (\mathrm{Hom}^*_B(C^{\hat{o}}, C^{\check{o}}), d_{\mathrm{Hom}^*_B(C^{\hat{o}}, C^{\check{o}})})$ is an isomorphism of complexes. Thus there exists a cycle $\tau \in \mathrm{Z}^1((\mathrm{tw}\, A')(\check{o}', \hat{o}'), {m'_1}^{\mathrm{tw}})$ such that for $\lambda := (T_{\check{o}\hat{o}}(f_1^{\mathrm{tw}}(\tau))) \in \mathrm{Z}^1 \mathrm{Hom}^*_B(C^{\hat{o}}, C^{\check{o}})$, we have $\lambda - {}_0\lambda \in \mathrm{B}^1 \mathrm{Hom}^*_B(C^{\hat{o}}, C^{\check{o}})$. Let

$$d := \iota_{\hat{o}} \circ d^{\hat{o}} \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ \lambda \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ d^{\check{o}} \circ \pi_{\check{o}}, \tag{90}$$

which is by Lemma 129(a) a differential on C. Since $\lambda - {}_0\lambda \in \mathrm{B}^1 \mathrm{Hom}^*_B(C^{\hat{o}}, C^{\check{o}})$, Lemma 129(b) implies that the complexes (C, d) and $(C, {}_0d)$ are isomorphic. In particular, we have $\mathrm{H}_0(C, d) \simeq \mathrm{H}_0(C, {}_0d) \simeq N$. Thus it suffices to show that $(C, d) = (C^{f^{\mathrm{tw}}_{\mathrm{Obj}}(o')}, d^{f^{\mathrm{tw}}_{\mathrm{Obj}}(o')})$ for an $o' = (l+1, (i'_1, \ldots, i'_{l+1}), D') \in \mathrm{Obj}\,\mathrm{tw}\, A'$. We define the matrix $D' = (D'_{uv})_{u,v \in [1,l+1]} \in (A')^{(n+1)\times(n+1)}$ as follows. Note that $\tau \in (\mathrm{tw}\, A')(\check{o}', \hat{o}')^1$ is an $1 \times l$-matrix. Let

$$\begin{aligned}
(D'_{uv})_{u\in[1,l],v\in[1,l]} &:= \hat{D}' & \qquad (D'_{uv})_{u\in[1,l],v=l+1} &:= 0 \in (A')^{l\times 1} \\
(D'_{uv})_{u=l+1,v\in[1,l]} &:= \tau & \qquad (D'_{uv})_{u=l+1,v=l+1} &:= \check{D}' = 0\,.
\end{aligned}$$

I.e. as a block matrix, we have

$$D' = \left(\begin{array}{c|c} \hat{D}' & 0 \\ \hline \tau & \check{D}' = 0 \end{array}\right).$$

Let $o' := (l+1, (i'_1, \ldots, i'_{l+1}), D')$. We show $o' \in \mathrm{Obj}\,\mathrm{tw}\, A'$: Properties (1), (2) and (3) in Definition/Lemma 121 hold by construction of o'. Let $\bar{D} = (\bar{D}_{uv})_{u\in[1,l+1],v\in[1,l+1]} := \sum_{k\geq 1} b'_k(((\omega')^{-1}(D'))^{\otimes k})$. Since D' is strictly lower diagonal, we have

$$(\bar{D}_{uv})_{u\in[1,l+1],v=l+1} = 0 \in (A')^{(l+1)\times 1}$$

$$
\begin{aligned}
(\bar{D}_{uv})_{u\in[1,l],v\in[1,l]} &= \sum_{k\geq 1} b'_k(((\omega')^{-1}(\hat{D}'))^{\otimes k}) \overset{\hat{o}'\in \operatorname{Obj}\operatorname{tw} A'}{=} 0 \\
(\bar{D}_{uv})_{u=l+1,v\in[1,l]} &= \sum_{k,k'\geq 0} b'_{k+1+k'}(((\omega')^{-1}(\check{D}'))^{\otimes k} \otimes (\omega')^{-1}(\tau) \otimes ((\omega')^{-1}(\hat{D}'))^{\otimes k'}) \\
&\overset{(1)}{=} \sum_{k,k'\geq 0} (b'_{k+1+k'} \circ (\llbracket \check{D}' \rrbracket^{\otimes k} \otimes \mathrm{id}^{1\otimes l} \otimes \llbracket \hat{D}' \rrbracket^{\otimes k'}))((\omega')^{-1}(\tau)) \\
&= (-1)^{\frac{1(1-1)}{2}} (\omega')^{-1}(m_1'^{\operatorname{tw}}|_{(\operatorname{tw} A')(\check{o}',\hat{o}')}(\tau)) \overset{*}{=} 0,
\end{aligned}
$$

where at the step marked by "$*$", we use $\tau \in \mathrm{Z}^1((\operatorname{tw} A')(\check{o}',\hat{o}'), m_1'^{\operatorname{tw}})$. Hence $\bar{D} = 0$ which proves property (4) in Definition/Lemma 121. Thus $o' \in \operatorname{Obj}\operatorname{tw} A'$.

Let $o = f^{\operatorname{tw}}_{\operatorname{Obj}}(o') \in \operatorname{Obj}\operatorname{tw} A$. By the choice of the i_j for $j \in [1, l+1]$, we have $o = (l + 1, (i_1, \ldots, i_{l+1}), D)$ for the matrix $D = (D_{uv})_{u,v\in[1,l+1]} = \sum_{k\geq 1} \omega(F_k(((\omega')^{-1}(D'))^{\otimes k})) \in A^{(l+1)\times(l+1)}$.

Since D' is a strictly lower diagonal matrix, we have

$$
\begin{aligned}
(D_{uv})_{u\in[1,l+1],v=l+1} &= 0 \in A^{(l+1)\times 1} \\
(D_{uv})_{u\in[1,l],v\in[1,l]} &= \sum_{k\geq 1} \omega(F_k(((\omega')^{-1}(\hat{D}'))^{\otimes k})) \overset{\hat{o}=f^{\operatorname{tw}}_{\operatorname{Obj}}(\hat{o}')}{=} \hat{D} \\
(D_{uv})_{u=l+1,v\in[1,l]} &= \sum_{k,k'\geq 0} \omega(F_{k+1+k'}(((\omega')^{-1}(\check{D}'))^{\otimes k} \otimes (\omega')^{-1}(\tau) \otimes ((\omega')^{-1}(\hat{D}'))^{\otimes k'})) \\
&\overset{(1)}{=} \sum_{k,k'\geq 0} (\omega \circ F_{k+1+k'} \circ (\llbracket \check{D}' \rrbracket^{\otimes k} \otimes \mathrm{id}^{1\otimes l} \otimes \llbracket \hat{D}' \rrbracket^{\otimes k'}) \circ (\omega')^{-1})(\tau) \\
&= (-1)^{\frac{1(1-1)}{2}} f_1^{\operatorname{tw}}|_{(\operatorname{tw} A)(\check{o},\hat{o})}(\tau) = f_1^{\operatorname{tw}}(\tau).
\end{aligned} \tag{91}
$$

We identify $C^o = \oplus_{j\in[1,l+1]} C^{(i_j)} = (\oplus_{j\in[1,l]} C^{(i_j)}) \oplus C^{i_{l+1}}$ with $C^{\hat{o}} \oplus C^{\check{o}} = C$. In particular, we have $C^o = C$. By Definition/Lemma 126(b), we have

$$
\begin{aligned}
d^o &= T_{oo}(d_0^o + D) \\
&\overset{(91)}{=} \iota_{\hat{o}} \circ T_{\hat{o}\hat{o}}(d_0^{\hat{o}}) \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ T_{\check{o}\check{o}}(d_0^{\check{o}}) \circ \pi_{\check{o}} + \iota_{\hat{o}} \circ T_{\hat{o}\hat{o}}(\hat{D}) \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ T_{\check{o}\check{o}}(\check{D}) \circ \pi_{\check{o}} \\
&\quad + \iota_{\check{o}} \circ T_{\check{o}\hat{o}}(f_1^{\operatorname{tw}}(\tau)) \circ \pi_{\hat{o}} \\
&\overset{\text{D./L.126}(b)}{=} \iota_{\hat{o}} \circ d^{\hat{o}} \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ d^{\check{o}} \circ \pi_{\check{o}} + \iota_{\check{o}} \circ T_{\check{o}\hat{o}}(f_1^{\operatorname{tw}}(\tau)) \circ \pi_{\hat{o}} \\
&= \iota_{\hat{o}} \circ d^{\hat{o}} \circ \pi_{\hat{o}} + \iota_{\check{o}} \circ d^{\check{o}} \circ \pi_{\check{o}} + \iota_{\check{o}} \circ \lambda \circ \pi_{\hat{o}} \\
&\overset{(90)}{=} d.
\end{aligned}
$$

Hence, we have $(C, d) = (C^o, d^o) = (C^{f^{\operatorname{tw}}_{\operatorname{Obj}}(o')}, d^{f^{\operatorname{tw}}_{\operatorname{Obj}}(o')})$. In particular, we have $N \simeq \mathrm{H}_0(C, d) = \mathrm{H}_0(C^{f^{\operatorname{tw}}_{\operatorname{Obj}}(o')}, d^{f^{\operatorname{tw}}_{\operatorname{Obj}}(o')})$ for $o' := (l + 1, (i'_1, \ldots, i'_{l+1}), D') \in \operatorname{Obj}\operatorname{tw} A'$. This completes the induction step.

(d): This follows from (c) by setting $(\operatorname{Obj} A', A', (m'_k)_{k\geq 1}) := (I, A, (m_k)_{k\geq 1})$ and by setting $(f_{\operatorname{Obj}}, (f_k)_{k\geq 1}) := (\mathrm{id}_I, \mathrm{strict}_\infty(\mathrm{id}_A))$. That is, we set the A_∞-functor f to be the identity on A.

(e): Suppose given $o, o' \in \operatorname{Obj} \mathrm{H}^0(\operatorname{tw} A) = \operatorname{Obj} \operatorname{tw} A$. We have proven in Definition/Lemma 126(e) that $T_{oo'} : ((\operatorname{tw} A)(o, o'), m_1^{\mathrm{tw}}) \to (\mathrm{Hom}^*_B(C^{o'}, C^o), d_{\mathrm{Hom}^*_B(C^{o'}, C^o)})$ is an isomorphism of complexes. Recall that (C^o, d^o) resp. $(C^{o'}, d^{o'})$ are projective resolutions of $\mathrm{H}_0(C^o, d^o) = Q(o)$ resp. $\mathrm{H}_0(C^{o'}, d^{o'}) = Q(o')$. Suppose given $f \in \mathrm{Hom}_B(Q(o'), Q(o))$. By the comparison theorem (cf. e.g. [26, Comparison Theorem 2.2.6]), there exists $f_Z \in \mathrm{Z}^0(\mathrm{Hom}^*_B(C^{o'}, C^o), d_{\mathrm{Hom}^*_B(C^{o'}, C^o)})$ such that $f = \mathrm{H}_0 f_Z$. Together with the fact that $T_{oo'}$ is an isomorphism of complexes, this shows surjectivity of $Q_{oo'}$. Uniqueness up to chain homotopy equivalence in the comparison theorem shows that f_Z is unique up to elements of $\mathrm{B}^0(\mathrm{Hom}^*_B(C^{o'}, C^o), d_{\mathrm{Hom}^*_B(C^{o'}, C^o)})$. Once more since $T_{oo'}$ is an isomorphism of complexes, this shows injectivity of $Q_{oo'}$. Note[7].

(f): By (e), the functor Q is fully faithful. By (d), it is dense. $\square$

In the case of the ground ring being a field, a variant of the following theorem was given by Keller, cf. [11, section 7.7]. Our constructions and methods are somewhat different, cf. Remark 132.

Recall that R is a commutative ring. Recall that B is an R-algebra. Recall that I is a set.

Theorem 131 (The filt construction)**.** *Suppose given B-modules M_i for $i \in I$. Suppose that for $i \in I$, the complex $(C^{(i)}, d^{(i)})$ is a projective resolution of M_i. Let* `filt` *be the full subcategory of B-*`Mod` *consisting of the modules that have a finite filtration such that each subquotient is isomorphic to M_i for an $i \in I$. Recall that the A_∞-category $(I, A, (m_k)_{k\geq 1})$ is defined by Example 31 and Lemma 33.*

Suppose given an A_∞-category $(\operatorname{Obj} A', A', (m'_k)_{k\geq 1})$ and a local quasi-isomorphism of A_∞-categories $f = (f_{\operatorname{Obj}}, (f_k)_{k\geq 1}) : (\operatorname{Obj} A', A', (m'_k)_{k\geq 1}) \to (I, A, (m_k)_{k\geq 1})$ such that f_{Obj} is surjective.

We have the equivalence of categories $Q : \mathrm{H}^0 \operatorname{tw} A \to \mathtt{filt}$ given by Lemma 130(f).

Then the semicategory $\mathrm{H}^0 \operatorname{tw} A'$ is a category and $Q \circ \mathrm{H}^0(\operatorname{tw} f) : \mathrm{H}^0 \operatorname{tw} A' \to \mathtt{filt}$ is an equivalence of categories.

Note that such f and A' may e.g. be obtained via Theorems 98 and 99.

[7] Basically, we have proven here $\mathrm{H}^0(\operatorname{tw} A)(o, o') \cong \mathrm{H}^0(\mathrm{Hom}^*_B(C^{o'}, C^o)) = \mathrm{Ext}^0_B(Q(o'), Q(o)) \cong \mathrm{Hom}_B(Q(o'), Q(o))$. Note in particular the connection of $\mathrm{H}^0(\operatorname{tw} A)$ to Ext^0. Analogous to the category $\mathrm{H}^0 \operatorname{tw} A$, one may define the category $\mathrm{H}^* \operatorname{tw} A$. Instead of $\mathrm{H}^0(\operatorname{tw} A)(o, o') = \mathrm{Ext}^0(Q(o'), Q(o)) \cong \mathrm{Hom}_B(Q(o'), Q(o))$, we then obtain $\mathrm{H}^*(\operatorname{tw} A)(o, o') = \mathrm{Ext}^*(Q(o'), Q(o))$. So instead of an equivalence to the category `filt`, we obtain an equivalence to the category Ext^* `filt`. The category Ext^* `filt` has the same objects as `filt`, but the morphisms are given by the elements of the Ext^*-spaces and composition of morphisms is given by the Yoneda product.

Proof. By Proposition 124, the morphism of A_∞-categories $\operatorname{tw} f$ is a local quasi-isomorphism. By Lemma 39, the semifunctor $H^0(\operatorname{tw} f) : H^0(\operatorname{tw} A') \to H^0(\operatorname{tw} A)$ is fully faithful. Recall that $H^0(\operatorname{tw} A)$ is a category by Definition/Lemma 126(g), so Lemma 36 implies that $H^0(\operatorname{tw} A')$ is a category and that $H^0(\operatorname{tw} f) : H^0(\operatorname{tw} A') \to H^0(\operatorname{tw} A)$ is a fully faithful functor. Thus $Q \circ H^0(\operatorname{tw} f) : H^0(\operatorname{tw} A') \to \mathtt{filt}$ is a fully faithful functor. The map f_{Obj} is surjective, so Lemma 130(c) yields that $Q \circ H^0(\operatorname{tw} f)$ is dense. Thus $Q \circ H^0(\operatorname{tw} f) : H^0(\operatorname{tw} A') \to \mathtt{filt}$ is an equivalence of categories. □

Remark 132 (Comparison with Keller and Lefèvre-Hasegawa's original version of the filt construction)**.** We compare our version of the filt construction with the version given by Keller in [11].

- Keller's version of the filt construction uses A_∞-modules as intermediary step. Given an A_∞-category A, he defines the derived category $D_\infty A$ of A_∞-modules over A. Then a factorisation of a Yoneda functor is used to obtain $\operatorname{tw} A$. Our version uses a direct and explicit approach as detailed in the proof of Theorem 131.
- Keller uses a field as a ground ring. In particular, he may therefore use Kadeishvili's minimality theorem to obtain A' and f. Our version is designed to work over arbitrary ground rings, so it was necessary to generalize Kadeishvili's minimality theorem to obtain A' and f, cf. section 4.
- Our definition of $\operatorname{tw} A$ is slightly different from Keller's version. Compared to our version, Keller's version of $\operatorname{tw} A$ has more objects by allowing the objects of A to be formally shifted in their degrees. Keller calls this 'closure under shifts', cf. [11, section 7.6]. The construction which is called filt(A) in Keller's notation (cf. [11, section 7.7]) is called $H^0 \operatorname{tw} A$ in our notation.

A. Appendix

A.1. The principle of dependent choice and the Countable Axiom of Choice

In the following, we briefly discuss the Principle of Dependent Choice, the Countable Axiom of Choice and their relationship with the Axiom of Choice. Furthermore, we refine the Principle of Dependent Choice to a version which we use e.g. in Kadeishvili's algorithm and its variants.

The Principle of Dependent Choice and the Countable Axiom of Choice are strictly weaker[8] than the Axiom of Choice. They formalize the common concept of "constructing a sequence by successively constructing its elements". We will usually use them without further comment.

Principle of Dependent Choice. Suppose given a binary relation ρ over a nonempty set A such that for every $a \in A$ there exists $b \in A$ such that $a \,\rho\, b$. Then for each $a_0 \in A$, there is a sequence $a_1, a_2, \dots$ in A such that

$$a_n \,\rho\, \alpha_{n+1} \quad \text{for all } n \in \mathbb{Z}_{\geq 0}.$$

Countable Axiom of Choice. Suppose given a countably infinite set I. Suppose given nonempty sets M_i for $i \in I$. Then there is a function $f : I \to \cup_{i \in I} M_i$ such that $f(i) \in M_i$ for $i \in I$.

We have the well-known assertions of

Lemma 133. *(i) The Axiom of Choice implies the Principle of Dependent Choice.*

(ii) The Principle of Dependent Choice implies the Countable Axiom of Choice.

Proof. Ad (i): Suppose given ρ, A and a_0 as in the Principle of Dependent Choice. For $a \in A$, let $A_a := \{b \in A \mid a \,\rho\, b\}$. We have $A_a \neq \emptyset$ for $a \in A$. By the Axiom of Choice, there is a function $f : A \to A$ such that $f(a) \in A_a$ for all $a \in A$. I.e. $a \,\rho\, f(a)$ for all $a \in A$. For $i \in \mathbb{Z}_{\geq 1}$, set $a_i := f^i(a_0)$. This way, we have $a_i \,\rho\, a_{i+1}$ for $i \in \mathbb{Z}_{\geq 0}$.

Ad (ii): Suppose given a countably infinite set I. Suppose given nonempty sets A_i for $i \in I$. Since I is countably infinite, I is as a set isomorphic to $\mathbb{Z}_{\geq 0}$. So we may assume $I = \mathbb{Z}_{\geq 0}$. For $i \in \mathbb{Z}_{\geq 0}$, let $A'_i := \{(a, i) \mid a \in A_i\}$. The sets of tuples A'_i, $i \in \mathbb{Z}_{\geq 0}$ are pairwise disjoint. Let $A' := \bigcup_{i \in \mathbb{Z}_{\geq 0}} A'_i$. We define on $A' \neq \emptyset$ the relation ρ by

$$a \,\rho\, b \quad \Leftrightarrow \quad \exists j \in \mathbb{Z}_{\geq 0} : a \in A'_j, b \in A'_{j+1}.$$

[8] For strictness of the implications given in Lemma 133, confer [8]. The Axiom of Choice is Form 1. The Countable Axiom of Choice is Form 8. The Principle of Dependent Choice is Form 43 S. The unprovability of the converse of the implications of Lemma 133 is referenced on p. 321 resp. p. 330

A' and ρ satisfy the assumptions of the Principle of Dependent Choice. Choose $a'_0 \in A'_0$. The Principle of Dependent Choice yields a sequence $a'_1, a'_2, \ldots$, such that $a'_i \,\rho\, a'_{i+1}$ for $i \in \mathbb{Z}_{\geq 0}$.

We prove by induction on $i \in \mathbb{Z}_{\geq 0}$ that $a'_i \in A'_i$. The initial step is $a'_0 \in A'_0$, which holds by choice of a'_0. So suppose $i \in \mathbb{Z}_{\geq 0}$ and $a'_i \in A'_i$. We have $a'_i \,\rho\, a'_{i+1}$. I.e. $a'_i \in A'_j$, $a'_{i+1} \in A'_{j+1}$ for some $j \in \mathbb{Z}_{\geq 0}$. Since the A'_k, $k \in \mathbb{Z}_{\geq 0}$ are pairwise disjoint, we have $j = i$. Hence $a'_{i+1} \in A'_{j+1} = A'_{i+1}$, which completes the induction step.

Thus we have $a'_i \in A'_i$ for $i \in \mathbb{Z}_{\geq 0}$. I.e. for $i \in \mathbb{Z}_{\geq 0}$, we have $a'_i = (a_i, i)$ for some $a_i \in A_i$. Hence the function $f : \mathbb{Z}_{\geq 0} \to \cup_{i \in \mathbb{Z}_{\geq 0}} A_i$, $f(i) := a_i$ is a choice function. $\square$

We will derive from the Principle of Dependent Choice the following

Lemma 134 (Successive construction of an infinite sequence)**.** *Suppose given sets M_k for $k \in \mathbb{Z}_{\geq 1}$. Suppose given binary functions $c_k : M_1 \times \ldots \times M_k \to \{0, 1\}$ for $k \in \mathbb{Z}_{\geq 1}$. For $n \in \mathbb{Z}_{\geq 0}$, we call a tuple $(y_1, \ldots, y_n) \in M_1 \times \ldots \times M_n$ of length n* admissible *if for $k \in [1, n]$, we have $c_k(y_1, \ldots, y_k) = 1$. Note that the empty tuple $()$ is admissible.*

Suppose that for admissible tuples $(y_1, \ldots, y_n) \in M_1 \times \ldots \times M_n$ of length $n \in \mathbb{Z}_{\geq 0}$, there exists an element $y_{n+1} \in M_{n+1}$ such that $(y_1, \ldots, y_{n+1})$ is admissible.

Suppose given $N \in \mathbb{Z}_{\geq 0}$ and an admissible tuple $(x_1, \ldots, x_N)$ of length N.

Then there exists a sequence $(x_n)_{n \in \mathbb{Z}_{\geq N+1}} \in \prod_{n \in \mathbb{Z}_{\geq N+1}} M_n$ such that $c_k(x_1, \ldots, x_k) = 1$ for $k \in \mathbb{Z}_{\geq 1}$.

In applications, $c_k(x_1, \ldots, x_k)$ is defined to equal 1 if the tuple $(x_1, \ldots, x_k)$ has certain desired properties and to equal 0 otherwise. To apply Lemma 134 one needs to show that given $(x_1, \ldots, x_k)$ having the desired properties, there is x_{k+1} such that $(x_1, \ldots, x_{k+1})$ has the desired properties. Therefore constructions invoking Lemma 134 will often be called "successive" constructions.

Proof. Denote by $A \subseteq \cup_{k \geq 0}(M_1 \times \ldots \times M_k)$ the set of admissible tuples of length ≥ 0. Note that the empty tuple $()$ is an element of A. We define the relation ρ on A by

$$(y_1, \ldots, y_i) \,\rho\, (y'_1, \ldots, y'_j) \quad :\Leftrightarrow \quad i + 1 = j \text{ and } (y_1, \ldots, y_i) = (y'_1, \ldots, y'_i).$$

For $i \in [0, N]$, let $t_i := (x_1, \ldots, x_i) \in A$. Thus $t_i \,\rho\, t_{i+1}$ for $i \in [0, N-1]$.

The assumptions of Lemma 134 ensure that for each $t \in A$, there is a $t' \in A$ such that $t \,\rho\, t'$. Thus the Principle of Dependent Choice asserts the existence of a sequence $(t_n)_{n \in \mathbb{Z}_{\geq N+1}}$ with $t_n \in A$ for $n \in \mathbb{Z}_{\geq N+1}$ such that $t_i \,\rho\, t_{i+1}$ for $i \in \mathbb{Z}_{\geq N}$. Hence $t_i \,\rho\, t_{i+1}$ for $i \in \mathbb{Z}_{\geq 0}$.

For $i \in \mathbb{Z}_{\geq 0}$, the tuple t_i has the form $(x_1^{(i)}, \ldots, x_i^{(i)})$. For $i \in [1, N]$, we have $x_i = x_i^{(i)}$. For $i \in \mathbb{Z}_{\geq N+1}$, we set $x_i := x_i^{(i)} \in M_i$. Since $t_i \,\rho\, t_{i+1}$ for $i \in \mathbb{Z}_{\geq 0}$, we have $t_i = (x_1, \ldots, x_i)$ for $i \in \mathbb{Z}_{\geq 0}$. Thus since t_i is admissible, we have $c_k(x_1, \ldots, x_i) = 1$ for $i \in \mathbb{Z}_{\geq 0}$. I.e. the sequence $(x_n)_{n \in \mathbb{Z}_{\geq N+1}} \in \prod_{n \in \mathbb{Z}_{\geq N+1}} M_n$ satisfies the assertions of Lemma 134 $\square$

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