On the Bisson-Tsemo model category of graphs

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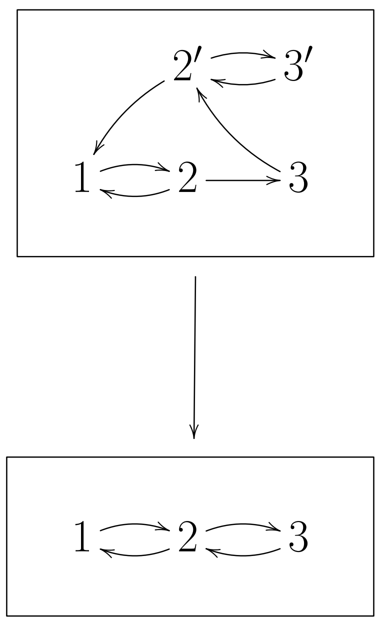

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Chapter 0

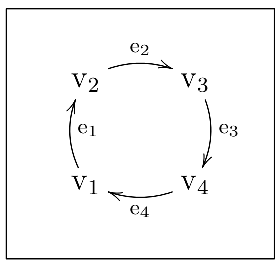
Introduction

0.1 Graphs

A graph G consists of a set of vertices V_G and of a set of edges E_G together with a source map $\mathrm{s}_G : \mathrm{E}_G \to \mathrm{V}_G$ and a target map $\mathrm{t}_G : \mathrm{E}_G \to \mathrm{V}_G$, mapping an edge to its source respectively to its target; cf. Definition 45.[1] Pictorially, we represent a graph by writing out the vertices of G and by drawing an arrow for each edge, pointing from its source to its target.

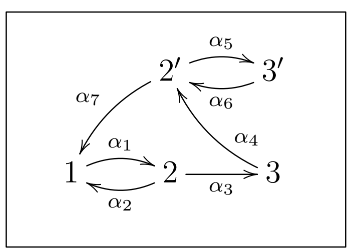
For example, we have the following cyclic graph.

C_4 :



Or, for example, we have the following graph.

G :



So e.g. the edge α_3 has source $\alpha_3\,\mathrm{s}_G = 2$ and target $\alpha_3\,\mathrm{t}_G = 3$.

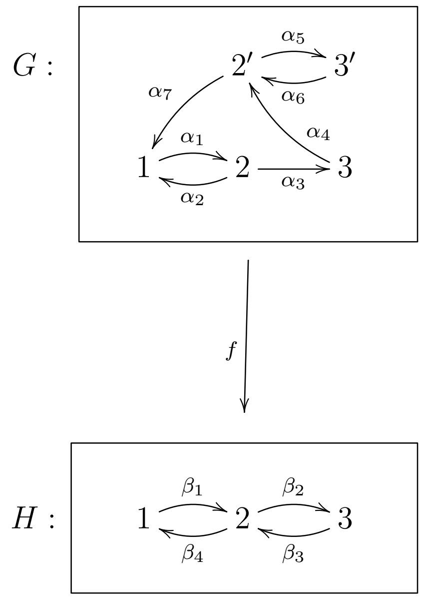
0.2 Graph morphisms

A graph morphism $f : G \to H$ between graphs G and H consists of a map $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ on the vertices of the graphs and a map $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ on the edges of the graphs such that

[1] In the literature, graphs in this sense are also called "directed graphs".

$\mathrm{E}_f\,\mathrm{s}_H = \mathrm{s}_G\,\mathrm{V}_f$ and $\mathrm{E}_f\,\mathrm{t}_H = \mathrm{t}_G\,\mathrm{V}_f$; cf. Definition 54.

For instance, we have the following graph morphism $f : G \to H$, mapping the vertices and the edges in a vertical way; cf. Example 215.



We have e.g. $\alpha_4\,\mathrm{E}_f\,\mathrm{t}_H = \beta_3\,\mathrm{t}_H = 2 = 2'\,\mathrm{V}_f = \alpha_4\,\mathrm{t}_G\,\mathrm{V}_f$.

The category of graphs and graph morphisms is denoted by Gph.

We denote the set of graph morphisms from G to H by $(G, H)_{\mathrm{Gph}}$; cf. Definition 64.

Given a graph morphism $f : G \to H$ and a graph K, we have the map

$$(K, f)_{\mathrm{Gph}} : (K, G)_{\mathrm{Gph}} \to (K, H)_{\mathrm{Gph}} : g \mapsto gf \ ;$$

cf. Definition 68.

0.3 A model category structure on Gph by Bisson and Tsemo

Bisson and Tsemo define data for a model category structure on the category Gph as follows.

A graph morphism $f : G \to H$ is called a quasiisomorphism if the map

$$(\mathrm{C}_k\,, f)_{\mathrm{Gph}} : (\mathrm{C}_k\,, G)_{\mathrm{Gph}} \to (\mathrm{C}_k\,, H)_{\mathrm{Gph}}$$

is bijective for $k \geqslant 1$. In other words, we require each graph morphism $\mathrm{C}_k \to H$ to have a unique lift $\mathrm{C}_k \to G$ along $G \xrightarrow{f} H$. Let $\mathrm{Qis} \subseteq \mathrm{Mor}(\mathrm{Gph})$ denote the subset of quasiisomorphisms; cf. Definition 115. A quasiisomorphism is written $G \overset{\approx}{\longrightarrow} H$. E.g. the example displayed in §0.2 is a quasiisomorphism.

For a graph G and a vertex $v \in \mathrm{V}_G$, we denote by $G(v, *) := \{e \in \mathrm{E}_G : e\,\mathrm{s}_G = v\}$ the set of edges with source v.

For a graph morphism $f : G \to H$ and a vertex $v \in \mathrm{V}_G$, we denote

$$\begin{array}{rcl}\mathrm{E}_{f,v} := \mathrm{E}_f \,|_{G(v,*)}^{H(v\,\mathrm{V}_f\,,*)} : G(v,*) & \to & H(v\,\mathrm{V}_f\,,*) \\ e & \mapsto & e\,\mathrm{E}_f\ .\end{array}$$

A graph morphism $f : G \to H$ is called a fibration if the map

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

is surjective for $v \in \mathrm{V}_G$; cf. Definition 127.(1). Let Fib $\subseteq$ Mor(Gph) denote the subset of fibrations. A fibration is written $G \longrightarrow\!\!\!\!\!\!\!+\;\; H$.

A graph morphism $f : G \to H$ is called an etale fibration if the map

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

is bijective for $v \in \mathrm{V}_G$; cf. Definition 127.(2). This notion will play a role in a sufficient condition on a morphism to be a quasiisomorphism; cf. §0.5.

A graph morphism $f : G \to H$ is called an acyclic cofibration if the properties (AcCofib 1–5) hold; cf. Definition 162. Pictorially, an acyclic cofibration is obtained as follows. Let G be a graph. Glue some trees at their roots to vertices of G to obtain the graph H. Then the inclusion morphism $\iota : G \to H$ is an acyclic cofibration. Every acyclic cofibration is essentially obtained in this way.

Let AcCofib $\subseteq$ Mor(Gph) denote the subset of acyclic cofibrations. An acyclic cofibration is written $G \longrightarrow\!\!\!\!\!\!\!\circ\;\; H$.

Let AcFib := Qis $\cap$ Fib $\subseteq$ Mor(Gph) denote the subset of acyclic fibrations. An acyclic fibration is written $G \longrightarrow\!\!\!\!\!\!\!\!\shortmid\!\shortmid\;\; H$. E.g. the example displayed in §0.2 is an acyclic fibration.

A graph morphism $f : G \to H$ is called a cofibration if it has the left lifting property with respect to AcFib. That is, $f : G \to H$ is a cofibration if, given a commutative quadrangle

$$\begin{tikzcd} G \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ H \arrow[r, "b"'] & Y \end{tikzcd}$$

in Gph, there exists a graph morphism $h : H \to X$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ H \arrow[r, "b"'] \arrow[ur, "h"'] & Y \end{tikzcd}$$

commutes.

Let Cofib $\subseteq$ Mor(Gph) denote the subset of cofibrations. A cofibration is written $G \longrightarrow\!\!\!\!\!\!\!\bullet\;\; H$.

Then Gph, together with Qis, Fib and Cofib, is a Quillen closed model category; cf. [**3,** Cor. 4.8], [**4,** Ch. I, §1, Def. 1; Ch. I, §5, Def. 1], Definition 198.(4), Proposition 204.

Moreover, for the set AcCofib, which is defined via (AcCofib 1–5) in Definition 162, we actually have the equality AcCofib = Cofib $\cap$ Qis; cf. Lemma 185.

0.4 Proof that Gph is a model category

Following Bisson and Tsemo [**3,** Cor. 4.8], we will show that Gph is a Quillen closed model category, using the data explained in §0.3; cf. Definition 198.(4).

Note that we have defined Cofib, Qis and AcCofib separately; cf. §0.3. We will show that Cofib ∩ Qis = AcCofib, but this will only be possible having factorization of morphisms in acyclic cofibrations and fibrations at our disposal; cf. Lemma 185.

First, we show that Qis, Fib and Cofib are closed under composition and under retracts.

Moreover, Qis, Fib and AcFib are shown to be stable under pullbacks, and Cofib and AcCofib are shown to be stable under pushouts.

In Assertion 251, we give an example that shows that Qis is not stable under pushouts along cofibrations.

Suppose given subsets $M, N \subseteq \mathrm{Mor}(\mathrm{Gph})$. We write $M \boxslash N$ if for every commutative quadrangle

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "m"'] & X' \arrow[d, "n"] \\ Y \arrow[r, "b"'] & Y' \end{tikzcd}$$

in Gph with $m \in M$ and $n \in N$, there exists a morphism $h : Y \to X'$ in Gph such that

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "m"'] & X' \arrow[d, "n"] \\ Y \arrow[r, "b"'] \arrow[ur, "h"] & Y' \end{tikzcd}$$

commutes.

By definition, we have Cofib $\boxslash$ AcFib.

To show that AcCofib $\boxslash$ Fib, a given acyclic cofibration is factored in successive inclusion morphisms, in each of which single-edge-trees are glued to the subgraph; cf. Lemma 174. Then to construct the required lift, the definition of a fibration can be used directly.

We have to show that each morphism $f : X \to Y$ can be factored into an acyclic cofibration followed by a fibration.

To this end, a resolution $p : F \to Y$ of Y is constructed, where F is a disjoint union of trees. This resolution is glued, via pushout, to the discrete subgraph $\dot{X}$ of X.

$$\begin{tikzcd} \dot{X} \arrow[r, "\circ" marking] \arrow[d] & F \arrow[d] \arrow[ddr, bend left, "p"] & \\ X \arrow[r, "\circ" marking] \arrow[drr, bend right, "f"'] & \tilde{F} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "|" marking] & \\ & & Y \end{tikzcd}$$

Using this factorization property, we are now able to show that we have in fact AcCofib = Cofib ∩ Qis.

We have to show that each morphism $f : X \to Y$ can be factored into a cofibration followed by an acyclic fibration.

Using a disjoint union $\tilde{X}$ of X with cyclic graphs, we factor f into a cofibration $X \xrightarrow{c} \tilde{X}$ followed by a graph morphism $\tilde{f} : \tilde{X} \to Y$ that becomes surjective under $(\mathrm{C}_k\,, -)_{\mathrm{Gph}}$ for $k \geqslant 1$.

$$\begin{tikzcd}
& \tilde{X} \arrow[dr, "\tilde{f}"] & \\
X \arrow[ur, "c"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Now we consider $\tilde{f} : \tilde{X} \to Y$. We iteratively glue cycles in the source graph using a pushout construction, where the definition of the sets M' and M ensure that cycles that map to the same cycle in Y are glued together.

$$\begin{tikzcd}
& \coprod_{M'} \mathrm{C}_n \arrow[r] \arrow[d] & \coprod_{M} \mathrm{C}_n \arrow[d] \arrow[ddr, bend left, "c"] & \\
& \tilde{X}_i \arrow[r, "g_{i,i+1}"] \arrow[ddrr, "\tilde{f}_i"'] & \tilde{X}_{i+1} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "\tilde{f}_{i+1}"] & \\
\tilde{X} \arrow[ur, "g_{0,i}"] \arrow[urr, "g_{0,i+1}"'] \arrow[rrr, "\tilde{f}"'] & & & Y
\end{tikzcd}$$

This yields a factorization over the direct limit $\tilde{X}_\infty$ as follows.

$$\begin{tikzcd}
& \tilde{X}_\infty \arrow[dr, "\tilde{f}_\infty", "\sim"'] & \\
\tilde{X} \arrow[ur, "g_{0,\infty}"] \arrow[rr, "\tilde{f}"'] & & Y
\end{tikzcd}$$

Using a factorization of $\tilde{f}_\infty$ into an acyclic cofibration $h :\ \tilde{X}_\infty \longrightarrow \tilde{\tilde{X}}$ followed by a fibration $\tilde{\tilde{f}} :\ \tilde{\tilde{X}} \longrightarrow Y$, which then is an acyclic fibration, we obtain

$$\begin{tikzcd}
& & \tilde{X}_\infty \arrow[r, "h"] \arrow[ddr, "\tilde{f}_\infty", "\sim"'] & \tilde{\tilde{X}} \arrow[dd, "\tilde{\tilde{f}}"] \\
& \tilde{X} \arrow[ur, "g_{0,\infty}"] \arrow[drr, "\tilde{f}"] & & \\
X \arrow[ur, "c"] \arrow[rrr, "f"'] & & & Y\,,
\end{tikzcd}$$

as required.

0.5 A sufficient condition for a graph morphism to be a quasiisomorphism

Suppose given graphs G and H.

Suppose given a graph morphism $f : G \to H$.

An edge e_H of H is called unitargeting with respect to f if we have

$$|\{\tilde{e}\,\mathrm{t}_G : \tilde{e} \in \mathrm{E}_G\,,\ \tilde{e}\,\mathrm{E}_f = e_H\}| = 1\ ,$$

that is, if its preimage in G has a unique target; cf. Definition 206.

Consider the following property.

(Uni) For $n \geqslant 1$ and each graph morphism $u : \mathrm{C}_n \to H$, there exists $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\mathrm{e}_i\,\mathrm{E}_u \in \mathrm{E}_H$ is unitargeting with respect to f.

Pictorially speaking, f satisfies (Uni) if every cycle in H contains at least a unitargeting edge. In practice, one removes all unitargeting edges from H to obtain a subgraph $\tilde{H}$, and one verifies that $(\mathrm{C}_n\,,\tilde{H})_{\mathrm{Gph}} = \emptyset$ for $n \geqslant 1$ in order to verify that (Uni) holds.

We show that if $f : G \to H$ is an etale fibration and satisfies (Uni), then it is a quasiisomorphism; cf. Proposition 210.(2)

E.g. the graph morphism $f : G \to H$ in §0.2 is a quasiisomorphism since it verifies this sufficient condition as follows.

First, $f : G \to H$ is an etale fibration; cf. Definition 127.(2). For instance, $G(2', *) = \{\alpha_7\,, \alpha_5\}$ maps bijectively to $H(2, *) = \{\beta_4\,, \beta_2\}$ via $\mathrm{E}_{f,2'}$ since $\alpha_7 \mapsto \beta_4$ and $\alpha_5 \mapsto \beta_2$.

Second, we show that f satisfies (Uni). The edge β_3 is unitargeting since $\alpha_4\,\mathrm{t}_G = \alpha_6\,\mathrm{t}_G = 2'$. The edge β_1 is unitargeting since α_1 is its only preimage. The edge β_4 is unitargeting since $\alpha_2\,\mathrm{t}_G = \alpha_7\,\mathrm{t}_G = 1$. But the edge β_2 is not unitargeting since $\alpha_3\,\mathrm{t}_G = 3 \neq 3' = \alpha_5\,\mathrm{t}_G$. Obtaining $\tilde{H}$ by removing the unitargeting edges in H there is just the edge β_2 left in $\tilde{H}$, i.e. $\mathrm{E}_{\tilde{H}} = \{\beta_2\}$. So there does not exist any graph morphism $\mathrm{C}_n \to \tilde{H}$ for $n \geqslant 1$.

Hence the graph morphism $f : G \to H$ in §0.2 is a quasiisomorphism.

There is a dual counterpart to this sufficient condition, obtainable as follows.

Given a graph G, we define the opposite graph G^{op} by letting $\mathrm{s}_{G^{\mathrm{op}}} := \mathrm{t}_G$ and $\mathrm{t}_{G^{\mathrm{op}}} := \mathrm{s}_G$.

Given a graph morphism $f : G \to H$ we define the opposite graph morphism

$$f^{\mathrm{op}} = (\mathrm{V}_{f^{\mathrm{op}}}\,, \mathrm{E}_{f^{\mathrm{op}}}) : G^{\mathrm{op}} \to H^{\mathrm{op}}$$

by $\mathrm{V}_{f^{\mathrm{op}}} := \mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and $\mathrm{E}_{f^{\mathrm{op}}} := \mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$.

Then $f^{\mathrm{op}} : G^{\mathrm{op}} \to H^{\mathrm{op}}$ is a quasiisomorphism if and only if $f : G \to H$ is a quasiisomorphism.

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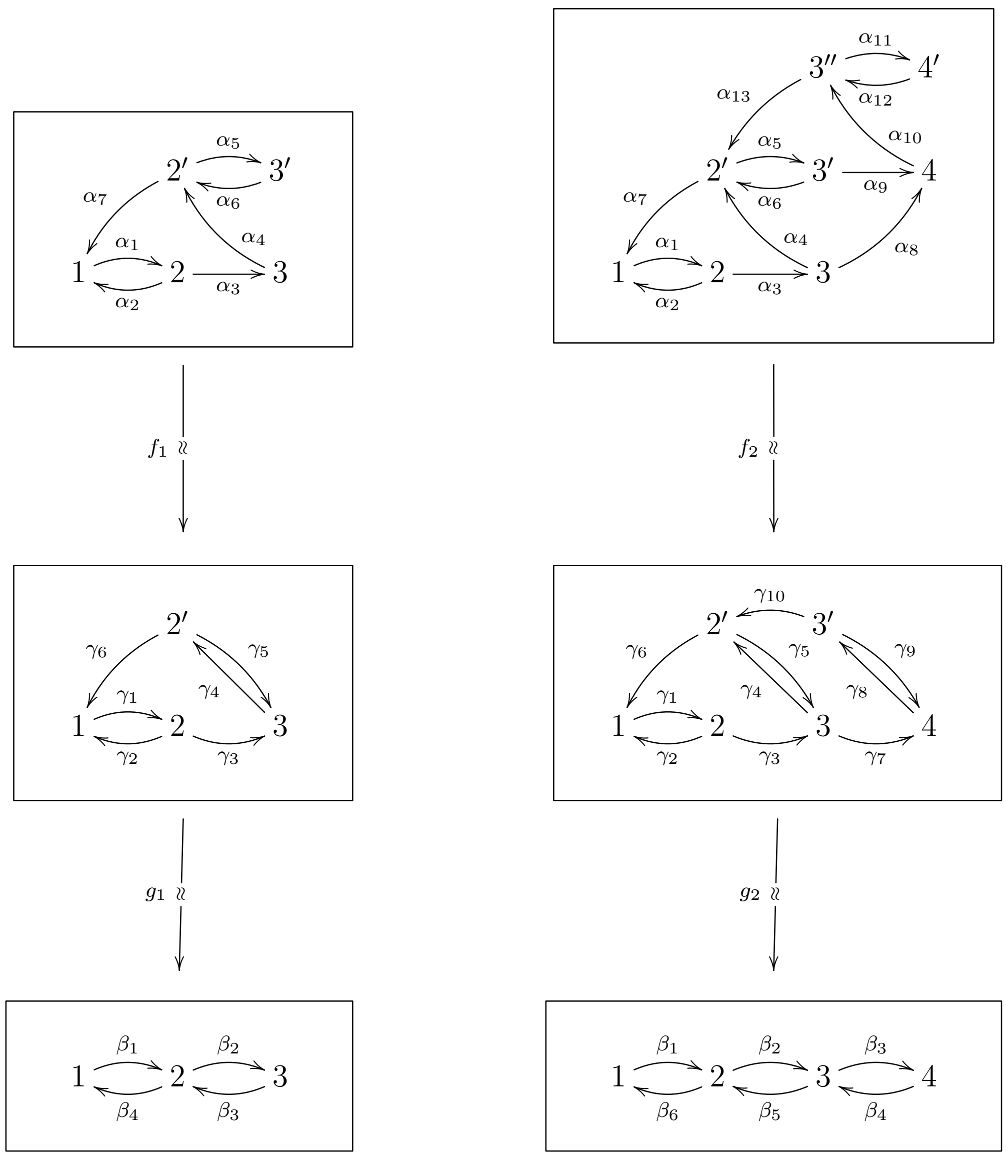
(2) Special thanks to Konrad Unger and Lukas Wiedmann who asked persistently if thinness is actually needed here. This led to a removal of this unnecessary condition.

0.6 Examples and counterexamples

0.6.1 Examples

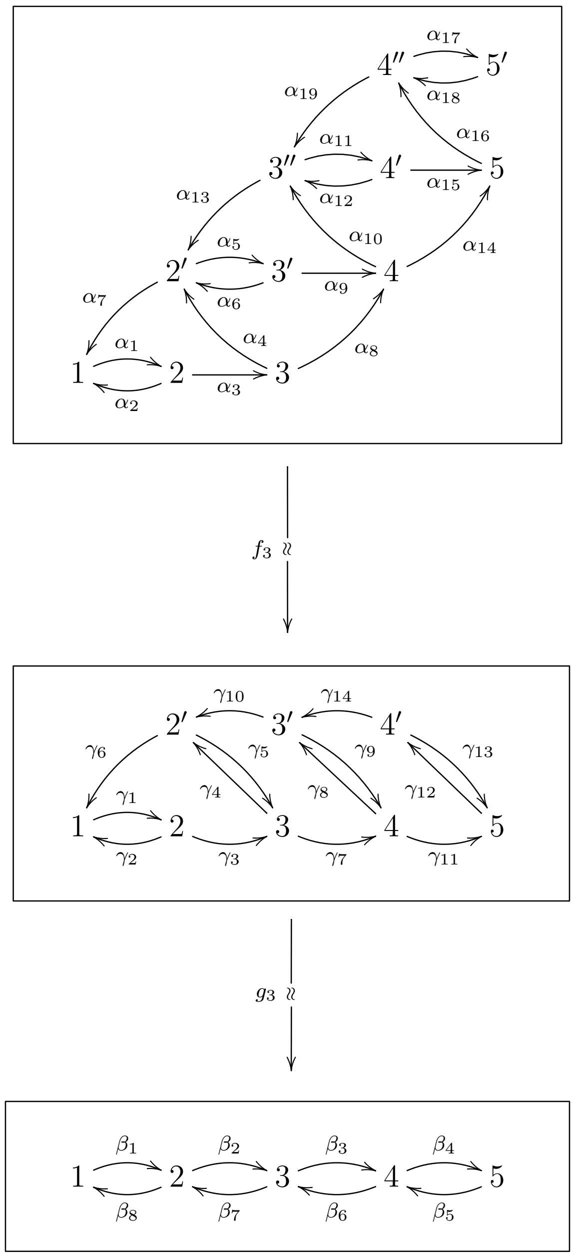
For sake of illustration, we show several examples for quasiisomorphisms obtained by the sufficient condition in Proposition 210, which we verify via Magma [**2**] in §9.1 using the functions given in §10.7.

We map the vertices and the edges in a vertical way.



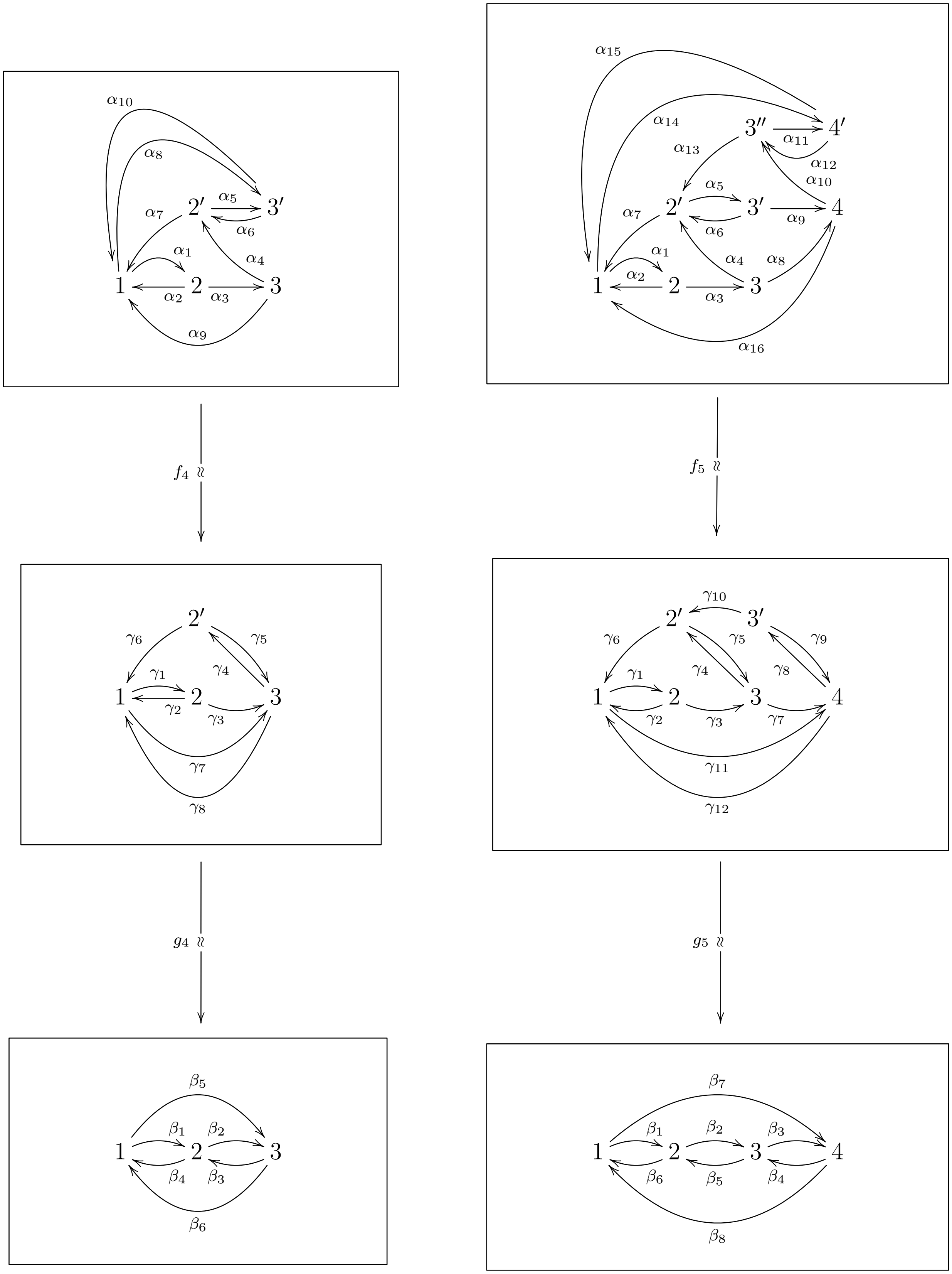
Here, $2\,\mathrm{V}_{f_1} = 2$ and $2'\,\mathrm{V}_{f_1} = 2'$. Moreover, $2\,\mathrm{V}_{f_2} = 2$, $2'\,\mathrm{V}_{f_2} = 2'$, $3\,\mathrm{V}_{f_2} = 3$, $3'\,\mathrm{V}_{f_2} = 3$, $3''\,\mathrm{V}_{f_2} = 3'$.

Varying the target graph and adjusting the other graphs accordingly, we get the following example.



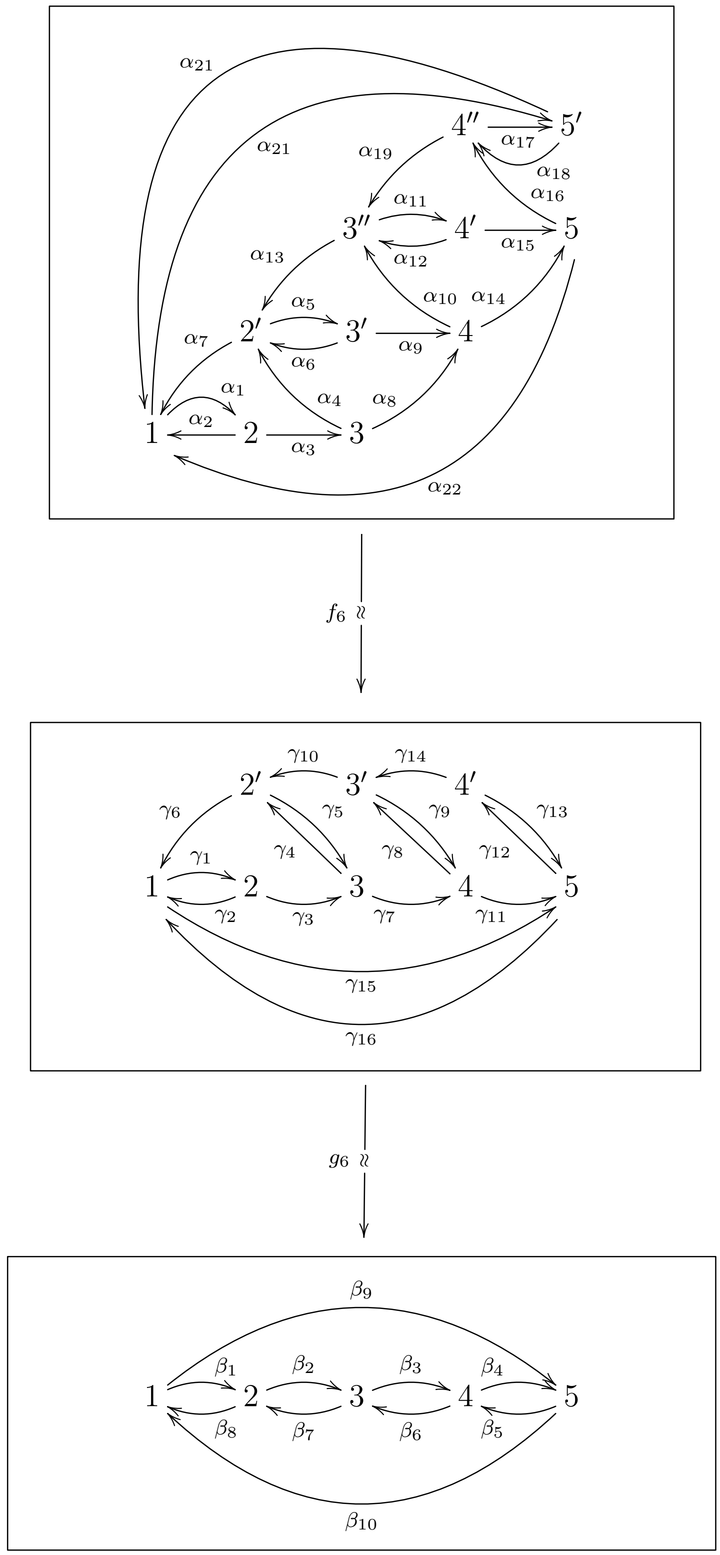
Here, $2\,\mathrm{V}_{f_3} = 2$, $2'\,\mathrm{V}_{f_3} = 2'$, $3\,\mathrm{V}_{f_3} = 3$, $3'\,\mathrm{V}_{f_3} = 3$, $3''\,\mathrm{V}_{f_3} = 3'$, $4\,\mathrm{V}_{f_3} = 4$, $4'\,\mathrm{V}_{f_3} = 4$, $4''\,\mathrm{V}_{f_3} = 4'$.

In the target graphs of the first two examples, we add two edges and adjust the other two graphs to get quasiisomorphisms as follows.



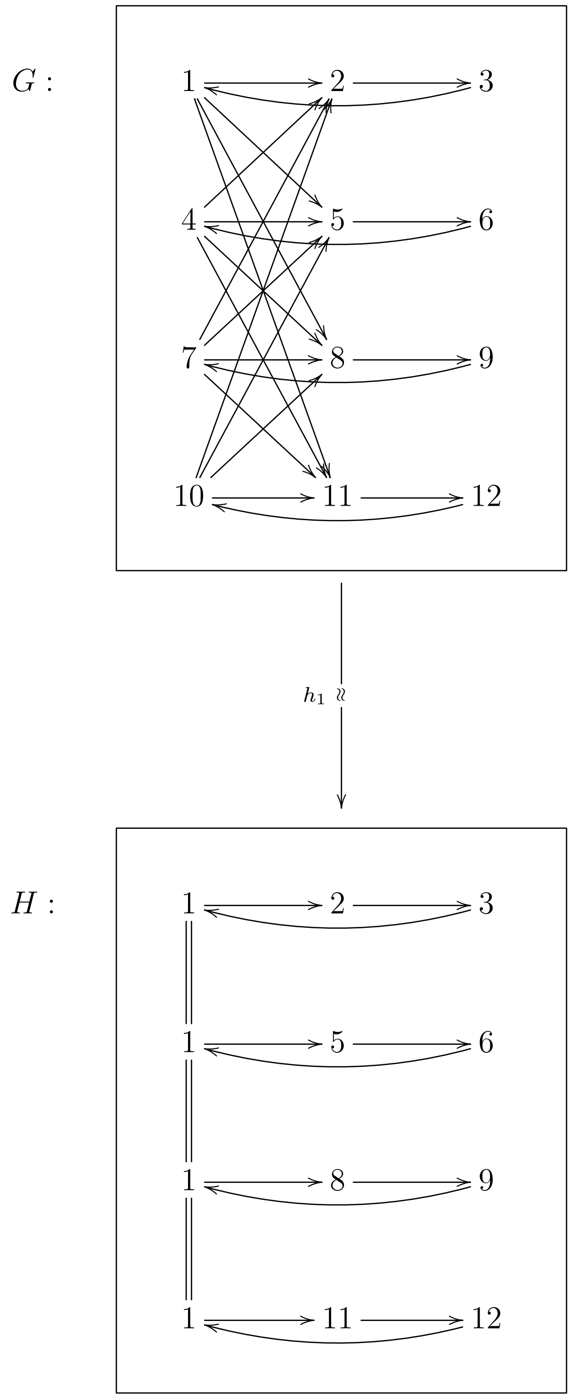
Here, $2\,\mathrm{V}_{f_4} = 2$, $2'\,\mathrm{V}_{f_4} = 2'$. Moreover $2\,\mathrm{V}_{f_5} = 2$, $2'\,\mathrm{V}_{f_5} = 2'$, $3\,\mathrm{V}_{f_5} = 3$, $3'\,\mathrm{V}_{f_5} = 3$, $3''\,\mathrm{V}_{f_5} = 3'$.

We can enlarge this example further to obtain the following quasiisomorphisms.



Here, $2\,\mathrm{V}_{f_6} = 2$, $2'\,\mathrm{V}_{f_6} = 2'$, $3\,\mathrm{V}_{f_6} = 3$, $3'\,\mathrm{V}_{f_6} = 3$, $3''\,\mathrm{V}_{f_6} = 3'$, $4\,\mathrm{V}_{f_6} = 4$, $4'\,\mathrm{V}_{f_6} = 4$, $4''\,\mathrm{V}_{f_6} = 4'$.

In the graph G in the following example we need to have edges from the vertices $1, 4, 7$ and 10 to the vertices $2, 5, 8$ and 11 to obtain an etale fibration. Since in addition the edges $2 \to 3$, $5 \to 6$, $8 \to 9$ and $11 \to 12$ of H are unitargeting, we conclude that h_1 is a quasiisomorphism; cf. §0.5.



We let

$$\begin{array}{llllll} 1\,\mathrm{V}_{h_1} := 1 & 2\,\mathrm{V}_{h_1} := 2 & 3\,\mathrm{V}_{h_1} := 3 & 4\,\mathrm{V}_{h_1} := 1 & 5\,\mathrm{V}_{h_1} := 5 & 6\,\mathrm{V}_{h_1} := 6 \\ 7\,\mathrm{V}_{h_1} := 1 & 8\,\mathrm{V}_{h_1} := 8 & 9\,\mathrm{V}_{h_1} := 9 & 10\,\mathrm{V}_{h_1} := 1 & 11\,\mathrm{V}_{h_1} := 11 & 12\,\mathrm{V}_{h_1} := 12\;. \end{array}$$

Here, the graph H has only a single vertex named 1, displayed four times for sake of clarity.

The following graph morphism between fibrant graphs is neither an acyclic fibration nor an acyclic cofibration but nevertheless a quasiisomorphism; cf. Example 226.

To prove this, we can not apply Proposition 210.

$$\begin{tikzcd}
 & 3 \arrow[dd, "\alpha_3"] & & 6 \arrow[dd, "\alpha_6"'] & \\
2 \arrow[ur, bend left, "\alpha_2"] & & & & 5 \arrow[ul, bend right, "\alpha_5"'] \\
 & 1 \arrow[ul, bend left, "\alpha_1"] & & 4 \arrow[ur, bend right, "\alpha_4"'] &
\end{tikzcd}$$

$$\Big\downarrow h_2 \ \wr$$

$$\begin{tikzcd}
 & 3 \arrow[dd, "\beta_3"] & & 6 \arrow[dd, "\beta_6"'] \arrow[ll, "\beta_8"] & \\
2 \arrow[ur, bend left, "\beta_2"] & & & & 5 \arrow[ul, bend right, "\beta_5"'] \\
 & 1 \arrow[ul, bend left, "\beta_1"] & & 4 \arrow[ur, bend right, "\beta_4"'] \arrow[ll, "\beta_7"] &
\end{tikzcd}$$

Here, $1\,\mathrm{V}_{h_2} = 1$, $2\,\mathrm{V}_{h_2} = 2$, $3\,\mathrm{V}_{h_2} = 3$, $4\,\mathrm{V}_{h_2} = 4$, $5\,\mathrm{V}_{h_2} = 5$, $6\,\mathrm{V}_{h_2} = 6$.

The following graph morphism is an acyclic fibration but not an etale fibration.

$$\begin{tikzcd}
1 \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & 2 \\
 & 3
\end{tikzcd}$$

$$\Big\downarrow h_3 \ \wr$$

$$\begin{tikzcd}
0 \arrow[r, "\beta_1"] & 1
\end{tikzcd}$$

So even for a fibration, the sufficient condition of Proposition 210 is not necessary for it to be a quasiisomorphism.

0.6.2 Counterexamples

We show by an example that the set of quasiisomorphisms is not stable under pushouts along cofibrations; cf. Assertion 251.

We show by an example that the pushout of two cofibrant graphs is not necessarily cofibrant; cf. Assertion 255.

We show by an example that the set of cofibrations is not stable under pullbacks; cf. Assertion 253.

We give an example of a pushout and an induced morphism as follows, in which the induced morphism f is not an acyclic cofibration, only a quasiisomorphism; cf. Assertion 254.

$$\begin{tikzcd}
X \arrow[r, "\circ" marking] \arrow[d, "\circ" marking] & Y \arrow[d, "\circ" marking] \arrow[ddr, bend left, "\circ" marking] & \\
X' \arrow[r, "\circ" marking] \arrow[drr, bend right, "\circ" marking] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "f", "\sim"'] & \\
 & & Z
\end{tikzcd}$$

We give an example $f : G \to H$ for which $(\mathrm{C}_1\,, f)_{\mathrm{Gph}}$ and $(\mathrm{C}_2\,, f)_{\mathrm{Gph}}$ are bijective, for which there exists no injective graph morphism $\mathrm{C}_k \to G$ or $\mathrm{C}_k \to H$ for $k \geqslant 3$, but which is not a quasiisomorphism; cf. Assertion 258.

Conventions

(1) Given a, $b \in \mathbb{Z}$, we write $[a, b] := \{\, z \in \mathbb{Z} \,:\, a \leqslant z \leqslant b \,\}$. In particular, $[a, b] = \emptyset$ if $a > b$.

(2) Given $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geqslant a} := \{\, z \in \mathbb{Z} \,:\, a \leqslant z \,\}$, etc.

(3) We set $\mathbb{N} := \mathbb{Z}_{\geqslant 1}$.

(4) We compose on the right. So given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, their composite is denoted by $X \xrightarrow{fg = f \cdot g} Z$ and maps $x \in X$ to $x(f \cdot g) = (xf)g$.

(5) Given a set X, "for $x \in X$" means "for all $x \in X$".

(6) Given a finite set X, we denote by $|X|$ its cardinality.

(7) Given sets X and Y, the set of maps from X to Y is denoted by Y^X.

(8) Given a set M, a relation $R \subseteq M \times M$, the equivalence relation $(\sim)$ generated by R and an element $m \in M$, we write $[m]_{(\sim)}$ for the equivalence class of m, i.e. the set of all elements $n \in M$ with $(m, n) \in R$.

(9) The symbol `%%` in a comment in Magma code refers to a function that is used in this line.

(10) The label **Reminder** indicates a summary of notations or definitions we remind of.

(11) The label **Assertion** indicates an assertion that we **falsify** by a counterexample.

(12) Whenever neccessary, we restrict the consideration to a given universe in the sense of Bourbaki [**1**].

Chapter 1

Preliminaries

1.1 Preliminaries on categories

Let $\mathcal{C}$ be a category.

1.1.1 The properties (2 of 6) and (2 of 3)

Definition 1 Suppose given a subset $Q \subseteq \mathrm{Mor}(\mathcal{C})$.

(1) We say that Q *satisfies* (2 of 6) if the following property holds.

Suppose given a commutative diagram

$$\begin{tikzcd} & X' \arrow[rr, "gh"] \arrow[dr, "g"] & & Y' \\ X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Y \arrow[ur, "h"'] & \end{tikzcd}$$

in $\mathcal{C}$.

Then the composites $X \xrightarrow{fg} Y$ and $X' \xrightarrow{gh} Y'$ are in Q if and only if f, g and h are in Q.

(2) We say that Q *satisfies* (2 of 3) if the following property holds.

Suppose given a commutative diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Z \end{tikzcd}$$

in $\mathcal{C}$.

Then (i, ii, iii) hold.

(i) If $f, g \in Q$ then $fg \in Q$.
(ii) If $f, fg \in Q$ then $g \in Q$.
(iii) If $g, fg \in Q$ then $f \in Q$.

Lemma 2 Suppose given a subset $Q \subseteq \mathrm{Mor}(\mathcal{C})$ such that $\mathrm{id}_X \in Q$ for $X \in \mathrm{Ob}(\mathcal{C})$. If Q satisfies (2 of 6), then Q satisfies (2 of 3).

Proof. Suppose given a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Z
\end{tikzcd}$$

in $\mathcal{C}$. We have to show that (i, ii, iii) from Definition 1 hold.

Ad (i). Suppose that $f, g \in Q$. We have to show that $fg \overset{!}{\in} Q$.

We have the following commutative diagram.

$$\begin{tikzcd}
& Y \arrow[rr, "g"] \arrow[dr, "g"] & & Z \\
X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Z \arrow[ur, "\mathrm{id}_Z"'] &
\end{tikzcd}$$

Since f, g, $\mathrm{id}_Z \in Q$, the morphism $fg : X \to Z$ is an element of Q by (2 of 6); cf. Definition 1.(1).

Ad (ii). Suppose that $f, fg \in Q$. We have to show that $g \overset{!}{\in} Q$.

We have the following commutative diagram.

$$\begin{tikzcd}
& X \arrow[rr, "fg"] \arrow[dr, "f"] & & Z \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "f"'] & & Y \arrow[ur, "g"'] &
\end{tikzcd}$$

Since f, $fg \in Q$, the morphism $g : Y \to Z$ is an element of Q by (2 of 6); cf. Definition 1.(1).

Ad (iii). Suppose that $g, fg \in Q$. We have to show that $f \overset{!}{\in} Q$.

We have the following commutative diagram.

$$\begin{tikzcd}
& Y \arrow[rr, "g"] \arrow[dr, "g"] & & Z \\
X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Z \arrow[ur, "\mathrm{id}_Z"'] &
\end{tikzcd}$$

Since $g, fg \in Q$, the morphism $f : X \to Y$ is an element of Q by (2 of 6); cf. Definition 1.(1). $\square$

Remark 3 The subset $\mathrm{Iso}(\mathcal{C}) \subseteq \mathrm{Mor}(\mathcal{C})$ satisfies (2 of 6).

Proof. Suppose given the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
 & X' \arrow[rr, "gh"] \arrow[dr, "g"] & & Y' \\
X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Y \arrow[ur, "h"'] &
\end{tikzcd}$$

Suppose that $f,\, g,\, h \in \mathrm{Iso}(\mathcal{C})$.

Then $fg,\, gh \in \mathrm{Iso}(\mathcal{C})$.

Conversely, suppose that $fg,\, gh \in \mathrm{Iso}(\mathcal{C})$.

Then $((fg)^{-1} \cdot f) \cdot g = \mathrm{id}_Y$ and $g \cdot (h \cdot (gh)^{-1}) = \mathrm{id}_{X'}$.

Thus $g \cdot ((fg)^{-1} \cdot f) = g \cdot ((fg)^{-1} \cdot f) \cdot g \cdot (h \cdot (gh)^{-1}) = g \cdot (h \cdot (gh)^{-1}) = \mathrm{id}_{X'}$.

Therefore, g is an isomorphism with $g^{-1} = (fg)^{-1} \cdot f$.

Hence $f = (fg) \cdot g^{-1}$ and $h = g^{-1} \cdot (gh)$ are isomorphisms. □

Remark 4 Suppose given a subset $Q \subseteq \mathrm{Mor}(\mathcal{C})$ such that $\mathrm{id}_X \in Q$ for $X \in \mathrm{Ob}(\mathcal{C})$.

Suppose that Q satisfies (2 of 6).

Then $\mathrm{Iso}(\mathcal{C}) \subseteq Q$.

Proof. Let $g : X \to Y$ be an isomorphism in $\mathcal{C}$.

We have the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
 & X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "g"] & & X \\
Y \arrow[ur, "g^{-1}"] \arrow[rr, "\mathrm{id}_Y"'] & & Y \arrow[ur, "g^{-1}"'] &
\end{tikzcd}$$

Since $\mathrm{id}_X, \mathrm{id}_Y \in Q$, we have $g \in Q$ by (2 of 6); cf. Definition 1.(1). □

1.1.2 Pushout and Pullback

Definition 5 Suppose given a quadrangle

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

in $\mathcal{C}$.

It is called a *pushout* if (Pushout 1–2) hold.

(Pushout 1) We have $g \cdot f' = f \cdot h$.

(Pushout 2) Suppose given an object $G \in \mathrm{Ob}(\mathcal{C})$ and morphisms $u : X' \to G$ and $v : Y \to G$ in $\mathrm{Mor}(\mathcal{C})$ such that $f \cdot v = g \cdot u$. Then there exists a unique morphism $w : Y' \to G$ in $\mathrm{Mor}(\mathcal{C})$ such that $f' \cdot w = u$ and $h \cdot w = v$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "v"] & \\ X' \arrow[r, "f'"] \arrow[drr, bend right, "u"'] & Y' \arrow[dr, dotted, "\exists! w"] \arrow[ul, phantom, "\lrcorner", very near start] & \\ & & G \end{tikzcd}$$

To indicate that this quadrangle is a pushout, we write

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} .$$

Then we also say that f' is a pushout of f.

Remark 6 Suppose given

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] & \\ X' \arrow[r, "f'"] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, shift right, "w"'] \arrow[dr, shift left, "\tilde{w}"] & \\ & & Z \end{tikzcd}$$

in $\mathcal{C}$ such that $f'w = f'\tilde{w}$ and $hw = h\tilde{w}$.

Then we have $w = \tilde{w}$.

Proof. Let $v := hw = h\tilde{w}$ and $u := f'w = f'\tilde{w}$.

We have $fv = fhw = gf'w = gu$.

Thus there exists exactly one morphism $w' : Y' \to Z$ such that $hw' = v$ and $f'w' = u$.

But we have $hw = v$ and $f'w = u$, and $h\tilde{w} = v$ and $f'\tilde{w} = u$.

So $w = w' = \tilde{w}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "v"] & \\
X' \arrow[r, "f'"] \arrow[drr, bend right, "u"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, shift right, "w"'] \arrow[dr, shift left, "\tilde{w}"] & \\
& & Z
\end{tikzcd}$$

□

Remark 7 Suppose given

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\tilde{h}"] \\
X' \arrow[r, "\tilde{f}'"'] & \tilde{Y}' \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

in $\mathcal{C}$.

Then there exists an isomorphism $\tilde{w} : \tilde{Y}' \xrightarrow{\sim} Y'$ such that $\tilde{f}'\tilde{w} = f'$ and $\tilde{h}\tilde{w} = h$.

Proof. Because Y' is a pushout there exists, by (Pushout 2), a unique morphism $w : Y' \to \tilde{Y}'$ such that $hw = \tilde{h}$ and $f'w = \tilde{f}'$.

Because $\tilde{Y}'$ is a pushout there exists, by (Pushout 2), a unique morphism $\tilde{w} : \tilde{Y}' \to Y'$ such that $\tilde{f}'\tilde{w} = f'$ and $\tilde{h}\tilde{w} = h$.

We have

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "\tilde{h}"] & \\
X' \arrow[r, "f'"] \arrow[drr, bend right, "\tilde{f}'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, shift right, "w"'] & \\
& & \tilde{Y}' \arrow[ul, shift right, "\tilde{w}"']
\end{tikzcd}$$

We have to show that $w\tilde{w} = \mathrm{id}_{Y'}$ and $\tilde{w}w = \mathrm{id}_{\tilde{Y}'}$.

We have $hw\tilde{w} = \tilde{h}\tilde{w} = h$ and $f'w\tilde{w} = \tilde{f}'\tilde{w} = f'$. So we have $w\tilde{w} = \mathrm{id}_{Y'}$; cf. Remark 6.

We have $\tilde{h}\tilde{w}w = hw = \tilde{h}$ and $\tilde{f}\tilde{w}w = f'w = \tilde{f}$. So we have $\tilde{w}w = \mathrm{id}_{\tilde{Y}'}$; cf. Remark 6. □

Remark 8 Suppose given

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[d, "w"'] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] \arrow[ul, phantom, "\lrcorner", very near start] & Z' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

in $\mathcal{C}$. Then

$$\begin{tikzcd} X \arrow[r, "fg"] \arrow[d, "u"'] & Z \arrow[d, "w"] \\ X' \arrow[r, "f'g'"'] & Z' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} .$$

Proof.

Ad (Pushout 1).

We have to show that $u(f'g') \overset{!}{=} (fg)w$.
We have $uf'g' = fvg' = fgw$; cf. Definition 5.

Ad (Pushout 2).

Suppose given an object $G \in \mathrm{Ob}(\mathcal{C})$ and morphisms $a : X' \to G$ and $b : Z \to G$ such that $u \cdot a = (fg) \cdot b$. We have to show that there exists exactly one morphism $d : Z' \to G$ such that $(f'g') \cdot d = a$ and $w \cdot d = b$. Because Y' is a pushout there exists exactly one morphism $c : Y' \to G$ such that $f' \cdot c = a$ and $v \cdot c = g \cdot b$. Because Z' is a pushout and because of $v \cdot c = g \cdot b$ there exists exactly one morphism $d : Z' \to G$ such that $g' \cdot d = c$ and $w \cdot d = b$. So there exists a morphism $d : Z' \to G$ such that $f'g' \cdot d = f'c = a$ and $w \cdot d = b$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[d, "w"'] \arrow[ddr, bend left, "b"] & \\ X' \arrow[r, "f'"] \arrow[drrr, bend right, "a"'] & Y' \arrow[r, "g'"] \arrow[ul, phantom, "\lrcorner", very near start] \arrow[drr, dotted, "\exists! c"'] & Z' \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dotted, "\exists! d"] & \\ & & & G \end{tikzcd}$$

It remains to show uniqueness.

Let $\tilde{d} : Z' \to G$ in $\mathrm{Mor}(\mathcal{C})$ be a morphism such that $a = f'g'\tilde{d}$ and such that $b = w\tilde{d}$.

We have to show that $d \overset{!}{=} \tilde{d}$.

Recall that $c : Y' \to G$ is the unique morphism with $f' \cdot c = a$ and $v \cdot c = g \cdot b$.

We have $f' \cdot (g'\tilde{d}) = a$ and $v \cdot (g'\tilde{d}) = gw\tilde{d} = gb$. Because of the uniqueness of the morphism c we get $g'\tilde{d} = c$.

Recall that $d : Z' \to G$ is the unique morphism with $g'd = c$ and $wd = b$.

But $g'\tilde{d} = c$ and $wd\tilde{} = b$. So $d = \tilde{d}$. $\square$

Definition 9 Suppose given a quadrangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in $\mathcal{C}$.

It is called a *pullback* if (Pullback 1–2) hold.

(Pullback 1) We have $f \cdot h = g \cdot f'$.

(Pullback 2) Suppose given an object $G \in \mathrm{Ob}(\mathcal{C})$ and morphisms $u : G \to Y$ and $v : G \to X'$ in $\mathrm{Mor}(\mathcal{C})$ such that $u \cdot h = v \cdot f'$. Then there exists a unique morphism $w : G \to X$ in $\mathrm{Mor}(\mathcal{C})$ such that $w \cdot f = u$ and $w \cdot g = v$.

$$\begin{tikzcd} G \arrow[dr, dotted, "\exists! w"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & X \arrow[r, "f"'] \arrow[d, "g"] & Y \arrow[d, "h"] \\ & X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

To indicate that this quadrangle is a pullback, we write

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd} .$$

Then we also say that f is a pullback of f'.

The following remarks on pullbacks have dual counterparts; cf. Remarks 6, 7 and 8. We carry out the proofs nonetheless.

Remark 10 Suppose given

$$\begin{tikzcd} Z \arrow[dr, shift right, "w"'] \arrow[dr, shift left, "\tilde{w}"] & & \\ & X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "h"] \\ & X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in $\mathcal{C}$ such that $wg = \tilde{w}g$ and $wf = \tilde{w}f$.

Then we have $w = \tilde{w}$.

Proof. Let $v := wg = \tilde{w}g$ and $u := wf = \tilde{w}f$.

We have $vf' = wgf' = wfh = uh$.

Thus there exists exactly one morphism $w' : Z \to X$ such that $w'g = v$ and $w'f = u$.

But we have $wg = v$ and $wf = u$, and $\tilde{w}g = v$ and $\tilde{w}f = u$.

So $w = w' = \tilde{w}$.

$$\begin{tikzcd}
Z \arrow[dr, shift right, "w"'] \arrow[dr, shift left, "\tilde{w}"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "h"] \\
& X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

□

Remark 11 Suppose given

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

and

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

in $\mathcal{C}$.

Then there exists an isomorphism $\tilde{w} : X \xrightarrow{\sim} \tilde{X}$ such that $\tilde{w}\tilde{f} = f$ and $\tilde{w}\tilde{g} = g$.

Proof. Because X is a pushout there exists, by (Pushout 2), a unique morphism $w : \tilde{X} \to X$ such that $wf = \tilde{f}$ and $wg = \tilde{g}$.

Because $\tilde{X}$ is a pushout there exists, by (Pushout 2), a unique morphism $\tilde{w} : X \to \tilde{X}$ such that $\tilde{w}\tilde{f} = f$ and $\tilde{w}\tilde{g} = g$.

We have

$$\begin{tikzcd}
\tilde{X} \arrow[dr, shift right, "w"'] \arrow[drr, bend left, "\tilde f"] \arrow[ddr, bend right, "\tilde g"'] & & \\
& X \arrow[r, "f"'] \arrow[d, "g"] \arrow[ul, shift right, "\tilde w"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "h"] \\
& X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

We have to show that $\tilde{w}w = \mathrm{id}_X$ and $w\tilde{w} = \mathrm{id}_{\tilde{X}}$.

We have $w\tilde{w}\tilde{f} = wf = \tilde{f}$ and $w\tilde{w}\tilde{g} = wg = \tilde{g}$. So we have $w\tilde{w} = \mathrm{id}_{\tilde{X}}$; cf. Remark 10.

We have $\tilde{w}wf = \tilde{w}\tilde{f} = f$ and $\tilde{w}wg = \tilde{w}\tilde{g} = g$. So we have $\tilde{w}w = \mathrm{id}_X$; cf. Remark 10. □

Remark 12 Suppose given

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "w"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'
\end{tikzcd}$$

in $\mathcal{C}$. Then

$$\begin{tikzcd}
X \arrow[r, "fg"] \arrow[d, "u"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "w"] \\
X' \arrow[r, "f'g'"'] & Z' .
\end{tikzcd}$$

Proof.

Ad (Pullback 1).

We have to show that $(fg)w \stackrel{!}{=} u(f'g')$.

We have $fgw = fvg' = uf'g'$; cf. Definition 9.

Ad (Pullback 2).

Suppose given an object $G \in \mathrm{Ob}(\mathcal{C})$ and morphisms $a : G \to Z$ and $b : G \to X'$ such that $a \cdot w = b \cdot (f'g')$. We have to show that there exists exactly one morphism $d : G \to X$ such that $d \cdot (fg) = a$ and $d \cdot u = b$. Because Y is a pullback there exist exactly one morphism $c : G \to Y$ such that $c \cdot g = a$ and $c \cdot v = b \cdot f'$. Because X is a pullback and because of $c \cdot v = b \cdot f'$ there exists exactly one morphism $d : G \to X$ such that $d \cdot u = b$ and $d \cdot f = c$. So there exists a morphism $d : G \to X$ such that $d \cdot u = b$ and $d \cdot fg = cg = a$.

$$\begin{tikzcd}
G \arrow[dr, dotted, "\exists! d"'] \arrow[drr, dotted, "\exists! c"] \arrow[drrr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & & \\
& X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[r, "g"] \arrow[d, "v"] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "w"] \\
& X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'
\end{tikzcd}$$

It remains to show uniqueness.

Let $\tilde{d} : G \to X$ in $\mathrm{Mor}(\mathcal{C})$ be a morphism such that $a = \tilde{d}fg$ and such that $b = \tilde{d}u$.

We have to show that $d \overset{!}{=} \tilde{d}$.

Recall that $c : G \to Y$ is the unique morphism with $c \cdot g = a$ and $c \cdot v = b \cdot f'$.

We have $(\tilde{d}f) \cdot g = a$ and $\tilde{d} \cdot u = b$. Because of the uniqueness of the morphism c we get $\tilde{d}f = c$.

Recall that $d : G \to X$ is the unique morphism with $df = c$ and $du = b$.

But $\tilde{d}f = c$ and $\tilde{d}u = b$. So $d = \tilde{d}$. $\square$

1.1.3 Lifting properties

Let $\mathcal{C}$ be a category.

Definition 13 Suppose given a set of morphisms $M \subseteq \mathrm{Mor}(\mathcal{C})$.

(1) For a morphism $f : X \to Y$ we define the *left lifting property* (LLP_M) as follows.

(LLP_M) For each morphism $f' : X' \to Y'$ in M and all morphisms $u : X \to X'$ and $v : Y \to Y'$ in $\mathrm{Mor}(\mathcal{C})$ with $fv = uf'$, there exists a morphism $h : Y \to X'$ such that $fh = u$ and $hf' = v$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "v"'] \arrow[ur, "h"] & Y'
\end{tikzcd}$$

The morphism f may or may not satisfy (LLP_M).

(2) Let the *left-liftable set* ${}^{\boxslash}M$ *of* M be defined as follows.

$${}^{\boxslash}M := \{ f \in \mathrm{Mor}(\mathcal{C}) : f \text{ satisfies } (\mathrm{LLP}_M) \}$$

For short, ${}^{\boxslash}M$ is also called M *left-lift*.

Definition 14 Suppose given a set of morphisms $M \subseteq \mathrm{Mor}(\mathcal{C})$.

(1) For a morphism $f : X \to Y$ we define the *right lifting property* (RLP_M) as follows.

(RLP_M) For each morphism $f' : X' \to Y'$ in M and all morphisms $u : X' \to X$ and $v : Y' \to Y$ in $\mathrm{Mor}(\mathcal{C})$ with $uf = f'v$, there exists a morphism $h : Y' \to X$ such that $f'h = u$ and $hf = v$.

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

The morphism f may or may not satisfy (RLP_M).

(2) Let the *right-liftable set* $M^{\boxslash}$ *of* M be defined as follows.

$$M^{\boxslash} := \{ f \in \mathrm{Mor}(\mathcal{C}) \, : \, f \text{ satisfies } (\mathrm{RLP}_M) \}$$

For short, the set $M^{\boxslash}$ is also called M *right-lift*.

Definition 15 Suppose given subsets $M, N \subseteq \mathrm{Mor}(\mathcal{C})$.

We write $M \boxslash N$ if $M \subseteq {}^{\boxslash}N$, or, equivalently, if $M^{\boxslash} \supseteq N$; cf. Definitions 13 and 14.

If $M \boxslash N$, we say M *lift* N.

Here, we use "lift" as a preposition.

Remark 16 Suppose given subsets $M \subseteq N \subseteq \mathrm{Mor}(\mathcal{C})$.

Then we have $M^{\boxslash} \supseteq N^{\boxslash}$ and we have ${}^{\boxslash}M \supseteq {}^{\boxslash}N$; cf. Definitions 13 and 14.

Remark 17 Suppose given a set of morphisms $M \subseteq \mathrm{Mor}(\mathcal{C})$.

Then we have $M \subseteq ({}^{\boxslash}M)^{\boxslash}$ and $M \subseteq {}^{\boxslash}(M^{\boxslash})$.

Proof.

Since we have ${}^{\boxslash}M \subseteq {}^{\boxslash}M$ we have ${}^{\boxslash}M \boxslash M$ and thus $M \subseteq ({}^{\boxslash}M)^{\boxslash}$; cf. Definition 15.

Since we have $M^{\boxslash} \subseteq M^{\boxslash}$ we have $M \boxslash M^{\boxslash}$ and thus $M \subseteq {}^{\boxslash}(M^{\boxslash})$; cf. Definition 15. ▫

Note that in general, we do **not** have $({}^{\boxslash}M)^{\boxslash} \subseteq M$. For instance, if $\mathcal{C}$ has an object X, then $\emptyset \not\supseteq \{\mathrm{id}_X\} \subseteq \mathrm{Mor}(\mathcal{C})^{\boxslash} = ({}^{\boxslash}\emptyset)^{\boxslash}$.

Note that in general, we do **not** have ${}^{\boxslash}(M^{\boxslash}) \subseteq M$. For instance, if $\mathcal{C}$ has an object X, then $\emptyset \not\supseteq \{\mathrm{id}_X\} \subseteq {}^{\boxslash}\mathrm{Mor}(\mathcal{C}) = {}^{\boxslash}(\emptyset^{\boxslash})$.

Remark 18 Suppose given a subset $M \subseteq \mathrm{Mor}(\mathcal{C})$.

We have $\mathrm{Iso}(\mathcal{C}) \subseteq {}^{\boxslash}M$ and $\mathrm{Iso}(\mathcal{C}) \subseteq M^{\boxslash}$.

Proof.

Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f", "\wr"'] & X' \arrow[d, "m"] \\ Y \arrow[r, "b"'] & Y' \end{tikzcd}$$

in $\mathcal{C}$ with $f \in \operatorname{Iso}(\mathcal{C})$ and $m \in M \subseteq \operatorname{Mor}(\mathcal{C})$.

We have the morphism $f^{-1}a : Y \to X'$ with $ff^{-1}a = a$ and with $f^{-1}am = f^{-1}fb = b$.

So we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f", "\wr"'] & X' \arrow[d, "m"] \\ Y \arrow[r, "b"'] \arrow[ur, "f^{-1}a"'] & Y' \end{tikzcd}$$

And so the isomorphism $f : X \to Y$ is in ${}^{\boxslash}M$.

So we have $\operatorname{Iso}(\mathcal{C}) \subseteq {}^{\boxslash}M$.

Now suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "m"'] & X' \arrow[d, "f", "\wr"'] \\ Y \arrow[r, "b"'] & Y' \end{tikzcd}$$

in $\mathcal{C}$ with $f \in \operatorname{Iso}(\mathcal{C})$ and $m \in M \subseteq \operatorname{Mor}(\mathcal{C})$.

We have the morphism $bf^{-1} : Y \to X'$ with $mbf^{-1} = aff^{-1} = a$ and with $bf^{-1}f = b$.

So we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "m"'] & X' \arrow[d, "f", "\wr"'] \\ Y \arrow[r, "b"'] \arrow[ur, "bf^{-1}"] & Y' \end{tikzcd}$$

And so the isomorphism $f : X \to Y$ is in $M^{\boxslash}$.

So we have $\operatorname{Iso}(\mathcal{C}) \subseteq M^{\boxslash}$. $\square$

Remark 19 Suppose given a subset $M \subseteq \operatorname{Mor}(\mathcal{C})$.

Suppose given morphisms $f : X \to Y$ and $g : Y \to Z$ in ${}^{\boxslash}M$.

Then the composite $fg : X \to Z$ is also in ${}^{\boxslash}M$.

Proof. Suppose given a commutative diagram as follows, where $m \in M$.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & X' \arrow[dd, "m"] \\
Y \arrow[d, "g"'] & \\
Z \arrow[r, "b"'] & Z'
\end{tikzcd}$$

We have to show that there exists a morphism $h : Z \to X'$ such that $(fg)h \overset{!}{=} a$ and $hm \overset{!}{=} b$.

Since $f(gb) = am$ and since the morphism $f : X \to Y$ is in ${}^{\boxslash}M$, there exists a morphism $k : Y \to X'$ such that $fk = a$ and $km = gb$.

Since $km = gb$ and since the morphism $g : Y \to Z$ is in ${}^{\boxslash}M$, there exists a morphism $h : Z \to X'$ such that $gh = k$ and $hm = b$.

So we have $(fg)h = fk = a$ and $hm = b$.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & X' \arrow[dd, "m"] \\
Y \arrow[ur, "k"] \arrow[d, "g"'] & \\
Z \arrow[uur, "h"'] \arrow[r, "b"'] & Z'
\end{tikzcd}$$

$\square$

Remark 20 Suppose given a subset $M \subseteq \mathrm{Mor}(\mathcal{C})$.

Suppose given morphisms $f : X \to Y$ and $g : Y \to Z$ in $M^{\boxslash}$.

Then the composite $fg : X \to Z$ is also in $M^{\boxslash}$.

Proof. Suppose given a commutative diagram as follows, where $m \in M$.

$$\begin{tikzcd}
X' \arrow[r, "a"] \arrow[dd, "m"'] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
Z' \arrow[r, "b"'] & Z
\end{tikzcd}$$

We have to show that there exists a morphism $h : Z' \to X$ such that $h(fg) \overset{!}{=} b$ and $mh \overset{!}{=} a$.

Since $(af)g = mb$ and since the morphism $g : Y \to Z$ is in $M^{\boxslash}$, there exists a morphism $k : Z' \to Y$ such that $kg = b$ and $mk = af$.

Since $mk = af$ and since the morphism $f : X \to Y$ is in $M^{\boxslash}$, there exists a morphism $h : Z' \to X$ such that $hf = k$ and $mh = a$.

So we have $h(fg) = kg = b$ and $mh = a$.

$$\begin{tikzcd}
X' \arrow[r, "a"] \arrow[dd, "m"'] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
Z' \arrow[r, "b"'] \arrow[uur, "h"] \arrow[ur, "k"'] & Z
\end{tikzcd}$$

$\square$

Remark 21 Suppose given a subset $M \subseteq \mathrm{Mor}(\mathcal{C})$.

We have ${}^{\boxslash}M \subseteq \mathrm{Mor}(\mathcal{C})$; cf. Definition 13.

The set ${}^{\boxslash}M$ is stable under pushouts; cf. Definition 82.

Proof. Suppose given a pushout

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
X' \arrow[r, "h'"'] \arrow[ur, phantom, "\lrcorner", very near end] & Y'
\end{tikzcd}$$

in $\mathcal{C}$ with $f : X \to X'$ in ${}^{\boxslash}M$.

We have to show that the morphism $g : Y \to Y'$ is in ${}^{\boxslash}M$, i.e. that the morphism g satisfies (LLP_M); cf. Definition 13.

Suppose given a commutative diagram as follows, where $m \in M$.

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[r, "p"] \arrow[d, "g"] & Z \arrow[d, "m"] \\
X' \arrow[r, "h'"'] \arrow[ur, phantom, "\lrcorner", very near end] & Y' \arrow[r, "q"'] & Z'
\end{tikzcd}$$

We have to show that there exists a morphism $\tilde{k} : Y' \to Z$ such that $g\tilde{k} = p$ and $\tilde{k}m = q$.

Because the morphism $f : X \to X'$ is in ${}^{\boxslash}M$ there exists a morphism $k : X' \to Z$ such that $fk = hp$ and $km = h'q$.

Because Y' is a pushout and $fk = hp$, there exists a morphism $\tilde{k} : Y' \to Z$ such that $h'\tilde{k} = k$ and $g\tilde{k} = p$.

So we have $g\tilde{k} = p$. It remains to show that $\tilde{k}m \overset{!}{=} q$.

We have $h'\tilde{k}m = km = h'q$ and $g\tilde{k}m = pm = gq$.

Cancelling h' and g simultaneously using Remark 6, we obtain $\tilde{k}m = q$.

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[r, "p"] \arrow[d, "g"] & Z \arrow[d, "m"] \\
X' \arrow[r, "h'"'] \arrow[ur, phantom, "\lrcorner", very near end] \arrow[urr, dotted, "k"] & Y' \arrow[r, "q"'] \arrow[ur, dotted, "\tilde{k}"'] & Z'
\end{tikzcd}$$

$\square$

Remark 22 Suppose given a subset $M \subseteq \operatorname{Mor}(\mathcal{C})$.

We have $M^{\boxslash} \subseteq \operatorname{Mor}(\mathcal{C})$; cf. Definition 13.

The set $M^{\boxslash}$ is stable under pullbacks; cf. Definition 96.

Proof. Suppose given a pullback

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "h"] \\
Y' \arrow[r, "f'"'] & Z'
\end{tikzcd}$$

in $\mathcal{C}$ with $h : Z \to Z'$ in $M^{\boxslash}$.

We have to show that the morphism $g : Y \to Y'$ is in $M^{\boxslash}$, i.e. that the morphism g satisfies (RLP_M); cf. Definition 14.

Suppose given a commutative diagram as follows, where $m \in M$.

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "m"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "h"] \\
X' \arrow[r, "q"'] & Y' \arrow[r, "f'"'] & Z'
\end{tikzcd}$$

We have to show that there exists a morphism $\tilde{k} : X' \to Z$ such that $m\tilde{k} = p$ and $\tilde{k}g = q$.

Because the morphism $h : Z \to Z'$ is in $M^{\boxslash}$ there exists a morphism $k : X' \to Z$ such that $mk = pf$ and $kh = qf'$.

Because Y is a pullback and $kh = pf'$, there exists a morphism $\tilde{k} : X' \to Y$ such that $\tilde{k}f = k$ and $\tilde{k}g = q$.

So we have $\tilde{k}g = q$. It remains to show that $m\tilde{k} \overset{!}{=} p$.

We have $m\tilde{k}f = mk = af$ and $m\tilde{k}g = mq = pg$.

Cancelling f and g simultaneously using Remark 10, we obtain $m\tilde{k} = p$.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[r, "f"] \arrow[d, "g"] & Z \arrow[d, "h"] \\
X' \arrow[r, "q"'] \arrow[ur, dotted, "k"] \arrow[urr, dotted, "\tilde{k}"] & Y' \arrow[r, "f'"'] & Z'
\end{tikzcd}$$

$\square$

1.1.4 Subsets of $\mathrm{Mor}(\mathcal{C})$ being closed under retracts

Definition 23 A subset of morphisms $M \subseteq \mathrm{Mor}(\mathcal{C})$ is called *closed under retracts* if the following property (C) holds.

(C) Suppose given a commutative diagram

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"'] \arrow[r, "f"] & H \arrow[u, "q"] \\
G' \arrow[u, "i"'] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

in $\mathcal{C}$ such that $f : G \to H$ in M.

Then the morphism $f' : G' \to H'$ is in M.

Remark 24 The subset of isomorphisms $\mathrm{Iso}(\mathcal{C}) \subseteq \mathrm{Mor}(\mathcal{C})$ is closed under retracts; cf. Definition 23.

Proof. Suppose given a commutative diagram in $\mathcal{C}$ as follows.

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"'] \arrow[r, "f", "\sim"'] & H \arrow[u, "q"] \\
G' \arrow[u, "i"'] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

We have to show that the morphism $f' : G' \to H'$ is an isomorphism.

We show that we have $f'^{-1} = jf^{-1}p : H' \to G'$.

We have $jf^{-1}p \cdot f' = jf^{-1}fq = jq = \mathrm{id}_{H'}$ and $f' \cdot jf^{-1}p = iff^{-1}p = ip = \mathrm{id}_{G'}$.

So the morphism $f' : G' \to H'$ is an isomorphism. $\square$

Remark 25 Suppose given a subset $M \subseteq \mathrm{Mor}(\mathcal{C})$.

The subset ${}^{\boxslash}M \subseteq \mathrm{Mor}(\mathcal{C})$ is closed under retracts; cf. Definition 23.

Proof. Suppose given a commutative diagram in $\mathcal{C}$ as follows, where $f \in {}^{\boxslash}M$.

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"'] \arrow[r, "f"] & H \arrow[u, "q"] \\
G' \arrow[u, "i"'] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

We have to show that the morphism $f' : G' \to H'$ is in ${}^{\boxslash}M$.

Suppose given a commutative diagram in $\mathcal{C}$ as follows, where $m \in M$.

$$\begin{tikzcd}
G'' \arrow[r, "m"] & H'' \\
G' \arrow[u, "u"] \arrow[r, "f'"] & H' \arrow[u, "v"'] \\
G \arrow[u, "p"'] \arrow[r, "f"] & H \arrow[u, "q"] \\
G' \arrow[u, "i"'] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

So we have to show that there exists a morphism $h : H' \to G''$ such that $hm = v$ and $f'h = u$; cf. Definition 13.

Since the morphism $f : G \to H$ is in ${}^{\boxslash}M$, there exists a morphism $k : H \to G''$ such that $km = qv$ and $fk = pu$.

We let $h := jk : H' \to G''$.

We have $hm = jkm = jqv = v$ and we have $f'h = f'jk = ifk = ipu = u$.

So the morphism $f' : G' \to H'$ is in ${}^{\boxslash}M$. ▫

Remark 26 Suppose given a subset $M \subseteq \mathrm{Mor}(\mathcal{C})$.

The subset $M^{\boxslash} \subseteq \mathrm{Mor}(\mathcal{C})$ is closed under retracts; cf. Definition 23.

Proof. Suppose given a commutative diagram in $\mathcal{C}$ as follows, where $f \in M^{\boxslash}$.

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"'] \arrow[r, "f"] & H \arrow[u, "q"] \\
G' \arrow[u, "i"'] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

We have to show that the morphism $f' : G' \to H'$ is in $M^{\boxslash}$.

Suppose given a commutative diagram in $\mathcal{C}$ as follows, where $m \in M$.

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"'] \arrow[r, "f"] & H \arrow[u, "q"] \\
G' \arrow[u, "i"'] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"] \arrow[uu, bend right, "\mathrm{id}_{H'}"'] \\
G'' \arrow[u, "u"] \arrow[r, "m"] & H'' \arrow[u, "v"']
\end{tikzcd}$$

We have to show that there exists a morphism $h : H'' \to G'$ such that $hf' = v$ and $mh = u$; cf. Definition 14.

Since the morphism $f : G \to H$ is in $M^{\boxslash}$, there exists a morphism $k : H'' \to G$ such that $kf = vj$ and $mk = ui$.

We let $h := kp : H'' \to G'$.

We have $hf' = kpf' = kfq = vjq = v$ and we have $mh = mkp = uip = u$.

So the morphism $f' : G' \to H'$ is in $M^{\boxslash}$. ▫

Definition 27 Suppose given subset $M, N \subseteq \mathrm{Mor}(\mathcal{C})$.

We write $M \mathbin{\hat{\boxslash}} N$ if the properties (C 1–3) hold.

(C 1) $M = {}^{\boxslash}N$.

(C 2) $M^{\boxslash} = N$.

(C 3) For a graph morphism $f : X \to Z$ there exist graph morphisms $m : X \to Y$ in M and $n : Y \to Z$ in N such that

$$\begin{tikzcd} & Y \arrow[dr, "n"] & \\ X \arrow[ur, "m"] \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes.

If $M \mathbin{\hat{\boxslash}} N$, we say M *closed-lift* N.

Here, we use "closed-lift" as a preposition.

Remark 28 Suppose given subsets $M, N \subseteq \mathrm{Mor}(\mathcal{C})$ such that $M \boxslash N$.

We have $M \mathbin{\hat{\boxslash}} N$ if and only if M and N are closed under retracts and (C 3) holds; cf. Definition 27.

Proof.

First, suppose that $M \mathbin{\hat{\boxslash}} N$.

Left-liftable sets are closed under retracts; cf. Remark 25.

Right-liftable sets are closed under retracts; cf. Remark 26.

Second, suppose that M and N are closed under retracts and that (C 3) holds.

We have to show that $M \overset{!}{=} {}^{\boxslash}N$ and that $M^{\boxslash} \overset{!}{=} N$.

Recall that $M \boxslash N$ is equivalent to $M \subseteq {}^{\boxslash}N$ and to $M^{\boxslash} \supseteq N$.

We have to show that $M \overset{!}{\supseteq} {}^{\boxslash}N$.

Suppose given a graph morphism $f : X \to Y$ in ${}^{\boxslash}N$. So $f \boxslash N$.

We have to show that $f \overset{!}{\in} M$.

Since (C 3) holds, we have a commutative diagram in Gph as follows, where $m \in M$ and $n \in N$.

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Z \arrow[d, "n"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd}$$

Since $f \boxslash N$, we have a commutative diagram in Gph as follows.

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Z \arrow[d, "n"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

We consider the following commutative diagram in Gph.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[r, "\mathrm{id}_X"'] \arrow[d, "m"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Z \arrow[r, "n"] & Y \end{tikzcd}$$

Since M is closed under retracts, we obtain $f \in M$.

We have to show that $M^{\boxslash} \overset{!}{\subseteq} N$.

Suppose given a graph morphism $f : X \to Y$ in $M^{\boxslash}$. So $M \boxslash f$.

We have to show that $f \overset{!}{\in} N$.

Since (C 3) holds, we have a commutative diagram in Gph as follows, where $m \in M$ and $n \in N$.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "m"'] & X \arrow[d, "f"] \\ Z \arrow[r, "n"'] & Y \end{tikzcd}$$

Since $M \boxslash f$, we have a commutative diagram in Gph as follows.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "m"'] & X \arrow[d, "f"] \\ Z \arrow[r, "n"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

We consider the following commutative diagram in Gph.

$$\begin{tikzcd} X \arrow[r, "m"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & Z \arrow[r, "h"'] \arrow[d, "n"] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_Y"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Y \arrow[r, "\mathrm{id}_Y"] & Y \end{tikzcd}$$

Since N is closed under retracts, we obtain $f \in N$. $\square$

1.2 Preliminaries on sets

1.2.1 Elementary constructions and properties

Remark 29 Suppose given sets $A \subseteq B \subseteq C$.

We have $C \setminus A = (C \setminus B) \,\dot{\cup}\, (B \setminus A)$.

Remark 30 Suppose given sets X, Y, Z.

Suppose given maps $f : X \to Y$, $g : Y \to Z$ with g injective.

We have $Z \setminus Xfg = (Z \setminus Yg) \,\dot{\cup}\, (Y \setminus Xf)g$.

Proof. We have to show that $Z \setminus Xfg \overset{!}{=} (Z \setminus Yg) \,\dot{\cup}\, (Yg \setminus Xfg)$, since $Xfg \subseteq Yg \subseteq Z$.

Because g is injective we have $Yg \setminus Xfg = (Y \setminus Xf)g$:

We show $Yg \setminus Xfg \overset{!}{=} (Y \setminus Xf)g$.

First, we show $Yg \setminus Xfg \overset{!}{\subseteq} (Y \setminus Xf)g$

Suppose given $z \in Yg \setminus Xfg$.

There exists $y \in Y$ such that $z = yg$. Then $y \notin Xf$ because of $z = yg \notin Xfg$.

So we have $z = yg \in (Y \setminus Xf)g$.

Second, we show $Yg \setminus Xfg \overset{!}{\supseteq} (Y \setminus Xf)g$.

Suppose given $z \in (Y \setminus Xf)g$.

There exists $y \in Y \setminus Xf$ such that $z = yg \in Yg$.

We *assume* that $z \in Xfg$. So we have $z = yg \in Xfg$.

There exists $x \in X$ such that $yg = xfg$. Because g is injective we have $y = xf$, which is a *contradiction.*

So we have $z \notin Xfg$. Hence $z \in Yg \setminus Xfg$. $\square$

Remark 31 Suppose given a surjective map $f : X \to Y$.

Then $f : X \to Y$ is an epimorphism in Set.

Proof. Suppose given maps: $u, u' : Y \to Z$ such that $fu = fu'$.

We have to show that $u \overset{!}{=} u'$.

Suppose given $y \in Y$.

Since $f : X \to Y$ is surjective there exists an element $x \in X$ such that $xf = y$.

Since $fu = fu'$ we have $yu = xfu = xfu' = yu'$.

So we conclude that $u = u'$. $\square$

Definition 32 Suppose given a set I.

Suppose given sets A_i for $i \in I$.

Then the *coproduct* or *disjoint union* $\coprod_{i\in I} A_i$ of the sets A_i for $i \in I$ is defined as follows.

$$\coprod_{i\in I} A_i := \bigcup_{i\in I} \{(i,a) : a \in A_i\}.$$

Suppose given $k \in I$. We let

$$\begin{array}{rrcl} \iota_k : & A_k & \to & \coprod_{i\in I} A_i \\ & a & \mapsto & (k,a) \end{array}$$

If $I = [1,n]$ for some $n \geqslant 1$, we often write $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n := \coprod_{i\in[1,n]} A_i$.

Example 33 Suppose given sets A and B.

Then we have the coproduct $A \sqcup B$ of A and B as follows; cf. Definition 32.

$$A \sqcup B = \{(1,a) : a \in A\} \cup \{(2,b) : b \in B\}$$

We have

$$\begin{array}{rrcl} \iota_1 : & A & \to & A \sqcup B \\ & a & \mapsto & (1,a) \end{array}$$

and

$$\begin{array}{rrcl} \iota_2 : & B & \to & A \sqcup B \\ & b & \mapsto & (2,b). \end{array}$$

Remark 34 Suppose given a set M.

Suppose given a relation $R \subseteq M \times M$.

Let $(\sim) \subseteq M \times M$ the equivalence relation generated by R.

Let

$$\begin{array}{rrcl} \rho : & M & \to & M/(\sim) \\ & m & \mapsto & [m]_{(\sim)} \ . \end{array}$$

Suppose given a set M.

Suppose given a map $f : M \to N$ such that $xf = yf$ for $(x,y) \in R$.

Then there exists a unique map $\bar{f} : M/(\sim) \to N$ such that $\rho\bar{f} = f$, i.e. we have the follwing commutative diagram.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho"'] & N \\ M/(\sim) \arrow[ur, dotted, "\exists!\bar{f}"'] & \end{tikzcd}$$

Remark 35 Let Q be the subset of bijective maps of Mor(Set). Then Q satisfies (2 of 6); cf. Definition 1.(1).

Proof. Suppose given sets X, X', Y and Y'.

Suppose given maps $f : X \to X'$, $g : X' \to Y$ and $h : Y \to Y'$.

We have the following commutative diagram in Set.

$$\begin{tikzcd} & X' \arrow[r, "gh"] \arrow[dr, "g"] & Y' \\ X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Y \arrow[u, "h"'] \end{tikzcd}$$

We have to show that the composites $fg : X \to Y$ and $gh : X' \to Y'$ are bijective if and only if f, g and h are bijective.

First, suppose that fg and gh are bijective.

Since fg is bijective, the map g is surjective.

Since gh is bijective, the map g is injective.

So g is bijective.

Hence $f = (fg) \cdot g^{-1}$ and $h = g^{-1} \cdot (gh)$ are bijective.

Now suppose that f, g and h are bijective.

Then the composites fg and gh are bijective. $\square$

1.2.2 Pushouts in Set

Construction 36 Suppose given the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X' & \end{tikzcd}$$

in Set.

We consider the coproduct $X' \sqcup Y$; cf. Definition 32.

On this set we have the relation $R := \{((1, xg), (2, xf)) : x \in X\} \subseteq (X' \sqcup Y) \times (X' \sqcup Y)$.

Let $(\sim)$ be the equivalence relation generated by R.

So we have the set of equivalence classes $Y' := {}^{(X' \sqcup Y)}\!/\!_{(\sim)}$.

Let

$$\begin{array}{rcl} Y & \xrightarrow{h} & Y' \\ y & \mapsto & [(2, y)]_{(\sim)} \end{array}$$

and

$$\begin{array}{rcl} X' & \xrightarrow{f'} & Y' \\ x' & \mapsto & [(1, x')]_{(\sim)} \, . \end{array}$$

Then, in Set, we have the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} .$$

Proof.

Ad (Pushout 1).

We have $xfh = [(2, xf)]_{(\sim)} = [(1, xg)]_{(\sim)} = xgf'$ for $x \in X$. So $fh = gf'$.

Ad (Pushout 2).

Suppose given a set Z together with maps $u : X' \to Z$ and $v : Y \to Z$ such that $fv = gu$.

We have to show that there exists exactly one map $w : Y' \to Z$ with $f'w = u$ and $hw = v$.

Existence.

We define the map

$$\begin{array}{rrcl} \hat{w}: & X \sqcup Y & \to & Z \\ & (1, x') & \mapsto & (1, x')\hat{w} := x'u \\ & (2, y) & \mapsto & (2, y)\hat{w} := yv\ . \end{array}$$

Suppose given $((1, xg), (2, xf)) \in R$, where $x \in X$. Then $(1, xg)\hat{w} = xgu = xfv = (2, xf)\hat{w}$.

Using $Y' = {}^{(X \sqcup Y)}\!\big/_{(\sim)}$, where $(\sim)$ is the equivalence relation generated by R, we obtain the map

$$\begin{array}{rccl} w: & Y' & \to & Z \\ & [(1, x')]_{(\sim)} & \mapsto & [(1, x')]_{(\sim)}w := x'u \\ & [(2, y)]_{(\sim)} & \mapsto & [(2, y)]_{(\sim)}w := yv\,. \end{array}$$

$$\begin{tikzcd} X \sqcup Y \arrow[r, "\hat{w}"] \arrow[d] & Z \\ Y' = {}^{(X \sqcup Y)}\!\big/_{(\sim)} \arrow[ur, dotted, "\exists ! w"'] & \end{tikzcd}$$

Then for $x \in X'$ we have $xf'w = [(1, x')]_{(\sim)}w = x'u$. So we have in fact $f'w = u$.

Then for $y \in Y$ we have $yhw = [(2, y)]_{(\sim)}w = yv$. So we have in fact $hw = v$.

Uniqueness.

Suppose given $\tilde{w} : Y' \to Z$ with $f'\tilde{w} = u$ and $h\tilde{w} = v$.

Suppose given $y' \in Y'$.

For $y' \in Y'$ we have $x' \in X$ such that $[(1, x')]_{(\sim)} = y'$ or we have $y \in Y$ such that $[(2, y)]_{(\sim)} = y'$.

If $y' = [(1, x')]_{(\sim)}$ with $x' \in X$, then we obtain $y'\tilde{w} = [(1, x')]_{(\sim)}\tilde{w} = x'f'\tilde{w} = x'u$.

If $y' = [(2, y)]_{(\sim)}$ with $y \in Y$, then we obtain $y'\tilde{w} = [(2, y)]_{(\sim)}\tilde{w} = yh\tilde{w} = yv$.

Since this holds for every such $\tilde{w}$, this shows uniqueness.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "v"] & \\
X' \arrow[r, "f'"] \arrow[drr, bend right, "u"'] & Y' \arrow[dr, dotted, "\exists! w"] \arrow[ul, phantom, "\lrcorner", very near start] & \\
& & Z
\end{tikzcd}
$$

Remark 37 Suppose given the pushout

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$

in Set.

Then we have $Y' = X'f' \cup Yh$.

Proof. It suffices to show that $Y' \overset{!}{\subseteq} X'f' \cup Yh$.

According to (Pushout 2) there exists a unique map $w : Y' \to X'f' \cup Yh$ such that $hw = h|^{X'f' \cup Yh}$ and $f'w = f|^{X'f' \cup Yh}$.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "h|^{X'f' \cup Yh}"] & \\
X' \arrow[r, "f'"] \arrow[drr, bend right, "f'|^{X'f' \cup Yh}"'] & Y' \arrow[dr, dotted, "\exists! w"] \arrow[ul, phantom, "\lrcorner", very near start] & \\
& & X'f' \cup Yh
\end{tikzcd}
$$

For $y' \in Y$ we calculate $y'w$.

If $y' \in X'f'$, then we write $y' = x'f'$ for some $x' \in X'$. We obtain $y'w = x'f'w = x'(f'|^{X'f' \cup Yh}) = x'f' = y'$.

If $y' \in Yh$, then we write $y' = yh$ for some $y \in Y$. We obtain $y'w = yhw = y(h||^{X'f' \cup Yh}) = yh = y'$.

So $y'w = y'$ in both cases.

In particular, $y' = y'w \in X'f' \cup Yh$.

So we have $Y' \subseteq X'f' \cup Yh$ and thus we have $Y' = X'f' \cup Yh$. $\square$

Remark 38 Suppose given

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \\ X' \end{tikzcd}$$

in Set with an injective map $f : X \to X'$.

Let $Y' := (X' \setminus Xf) \sqcup Y$.

Let

$$\begin{array}{rrcl} a' : & X' & \to & Y' \\ & x' & \mapsto & \begin{cases} (2, xa) & \text{if } x' = xf \in Xf \text{ for a unique } x \in X \\ (1, x') & \text{if } x' \in X' \setminus Xf\ . \end{cases} \end{array}$$

and we let

$$\begin{array}{rrcl} g : & Y & \to & (X' \setminus Xf) \sqcup Y \\ & y & \mapsto & (2, y) \end{array}$$

Then

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "a'"'] & Y' \end{tikzcd}$$

is a pushout in Set.

Proof.

Suppose given $x \in X$. We have $xfa' = (2, xa) = xag$. So we have $fa' = af'$.

Now we show that the commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "a'"'] & Y' \end{tikzcd}$$

is a pushout in Set.

Suppose given

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[ddr, bend left, "v"] & \\ X' \arrow[r, "a'"] \arrow[drr, bend right, "u"'] & Y' & \\ & & T \end{tikzcd}$$

in Set such that $av = fu$.

We have to show that there exists a unique map $w : Y' \to T$ such that $a'w = u$ and $gw = v$.

Uniqueness. This follows from $X'a' \cup Yg = Y'$.

We show the *existence* of a map $w : Y' \to T$ such that $a'w = u$ and $gw = v$.

We let

$$\begin{array}{rrcl} w: & Y' & \to & T \\ & (1, x') & \mapsto & x'u \\ & (2, y) & \mapsto & yv\ . \end{array}$$

So we have

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[ddr, bend left, "v"] & \\ X' \arrow[r, "a'"] \arrow[drr, bend right, "u"'] & Y' \arrow[dr, "w"] & \\ & & T\ . \end{tikzcd}$$

We have to show that $a'w \overset{!}{=} u$.

Suppose given $x' \in X'$.

We have $x'a'w = \left\{ \begin{array}{ll} (2, xa)w & \text{if } \exists_! x \in X : x' = xf \\ (1, x')w & \text{if } x' \in X' \setminus Xf \end{array} \right\} = \left\{ \begin{array}{ll} xav & \text{if } \exists_! x \in X : x' = xf \\ x'u & \text{if } x' \in X' \setminus Xf \end{array} \right\} \overset{fu=av}{=}$
$\left\{ \begin{array}{ll} xfu & \text{if } \exists_! x \in X : x' = xf \\ x'u & \text{if } x' \in X' \setminus Xf \end{array} \right\} = \left\{ \begin{array}{ll} x'u & \text{if } x' \in Xf \\ x'u & \text{if } x' \in X' \setminus Xf \end{array} \right\} = x'u.$

We have to show that $f'w \overset{!}{=} v$.

For $y \in Y$ we have $yf'w = (2, y)w = yv$. □

Remark 39 Suppose given

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "a'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

in Set.

If the map $f : X \to X'$ is injective, then the map $g : Y \to Y'$ is injective.

Proof.

Without loss of generality, the pushout is constructed in the way as in Remark 38; cf. Remark 7.

Then the map g is in fact injective. □

Remark 40 Suppose given

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "a'"'] & Y' \arrow[ul, phantom, "\lrcorner" very near start] \end{tikzcd}$$

in Set.

If the map $f : X \to X'$ is surjective, then the map $g : Y \to Y'$ is surjective.

Proof.

Without loss of generality, the pushout is constructed in the way as in Construction 36; cf. Remark 7.

So we have $Y' = {}^{(X' \sqcup Y)}\!\big/_{(\sim)}$ where $(\sim)$ is the equivalence relation generated by $\{((1, xf), (2, xa)) : x \in X\}$.

Suppose given $y' \in Y'$.

We have to show that there exists $y \in Y$ such that $yg = y'$.

Case: There exists $y \in Y$ such that $y' = [(2, y)]_{(\sim)}$.

Then we have $yg = [(2, y)]_{(\sim)} = y'$.

Case: There exists $x' \in X'$ such that $y' = [(1, x')]_{(\sim)}$.

Since $f : X \to X'$ is surjective there exists $x \in X$ such that $xf = x'$.

So we have $y' = [(1, x')]_{(\sim)} = [(1, xf)]_{(\sim)} \overset{\text{Def. } (\sim)}{=} [(2, xa)]_{(\sim)} \overset{\text{Def. } g}{=} (xa)g$. ▫

Without Construction 36 we can use $Y' = X'a' \cup Yg$ by Remark 37 to prove this Remark 40.

1.2.3 Pullbacks in Set

Construction 41 Suppose given the diagram

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in Set.

Define the set $X := \{(x', y) \in X' \times Y : x'f' = yh\}$.

Let

$$\begin{array}{rcl} X & \xrightarrow{g} & X' \\ (x', y) & \mapsto & x' \end{array}$$

and

$$\begin{array}{rcl} X & \xrightarrow{f} & Y \\ (x', y) & \mapsto & y \, . \end{array}$$

Then, in Set, we have the pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner" very near start] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd} .$$

Proof. We have $(x', y)fh = yh = x'f' = (x', y)gf'$ for $(x', y) \in X$. So $fh = gf'$. $\square$

Universal property.

Suppose given a set Z together with two maps $u : Z \to Y$ and $v : Z \to X'$ such that $uh = vf'$.

We have to show that there exists exactly one map $w : Z \to X$ with $wf = u$ and $wg = v$.

Existence.

We define the map

$$\begin{array}{rccl} w : & Z & \to & X \\ & z & \mapsto & zw := (zv,\, zu) \ . \end{array}$$

The element (zv, zu) is in fact contained in X since $zvf' = zuh$.

Then $wf = u$ since for $z \in Z$ we have $zwf = (zv, zu)f = zu$.

Moreover, $wg = v$ since for $z \in Z$ we have $zwg = (zv, zu)g = zv$.

Uniqueness.

Suppose given $\tilde{w} : Z \to X$ with $\tilde{w}g = v$ and $\tilde{w}f = u$.

Suppose given $z \in Z$. Write $z\tilde{w} =: (x', y) \in X$.

We obtain $x' = z\tilde{w}g = zv$ and $y = z\tilde{w}f = zu$.

So $z\tilde{w} = (zv, zu)$.

Since this holds for every such $\tilde{w}$, this shows uniqueness.

$$\begin{tikzcd} Z \arrow[dr, dotted, "\exists! w"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\ulcorner" very near start] & Y \arrow[d, "h"] \\ & X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

1.2.4 Colimit of a countable chain in Set

Suppose given

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} X_2 \xrightarrow{f_{2,3}} \cdots$$

in Set.

Let $f_{k,l} := f_{k,k+1} \cdot \ldots \cdot f_{l-1,l} : X_k \to X_l$ for $0 \leqslant k \leqslant l$. In particular, $f_{k,k} = \mathrm{id}_{X_k}$.

We define the relation $(\sim)$ on $\bigsqcup_{i \geqslant 0} X_i$ as follows.

For $(i, x_i), (j, x_j) \in \bigsqcup_{i \geqslant 0} X_i$ we let

$$(i, x_i) \sim (j, x_j) :\Leftrightarrow \text{ There exists } m \geqslant \max\{j, k\} \text{ such that } x_i f_{i,m} = x_j f_{j,m}\,.$$

The relation $(\sim)$ is an equivalence relation:

For $(j, x_j) \in \bigsqcup_{i \geqslant 0} X_i$ we have $(j, x_j) \sim (j, x_j)$, since $x_j f_{j,j} = x_j \,\mathrm{id}_{X_j} = x_j f_{j,j}$.

The relation $(\sim)$ is symmetric by definition.

Suppose given $(j, x_j), (k, x_k), (l, x_l) \in \bigsqcup_{i \geqslant 0} X_i$ such that $(j, x_j) \sim (k, x_k)$ and $(k, x_k) \sim (l, x_l)$.

So there exist $m \geqslant \max\{j, k\}$ and $n \geqslant \max\{k, l\}$ such that $x_j f_{j,m} = x_k f_{k,m}$ and $x_k f_{k,n} = x_l f_{l,n}$.

Let $p := \max\{m, n\}$.

Then we have $x_j f_{j,p} = x_j f_{j,m} f_{m,p} = x_k f_{k,m} f_{m,p} = x_k f_{k,p} = x_k f_{k,n} f_{n,p} = x_l f_{l,n} f_{n,p} = x f_{l,p}$.

So the relation $(\sim)$ is transitive.

Let $[j, x_j]$ be the equivalence class of (j, x_j) with respect to $(\sim)$.

Definition 42

(1) We define the *colimit* $X_\infty := \varinjlim_{i \geqslant 0} X_i := \bigsqcup_{i \geqslant 0} X_i \big/ (\sim) = \{[j, x_j] : j \geqslant 0, x_j \in X_j\}$.

(2) For $k \geqslant 0$, we let

$$\begin{array}{rccc} f_{k,\infty} : & X_k & \to & X_\infty \\ & x_k & \mapsto & [k, x_k]\,. \end{array}$$

Suppose given $0 \leqslant k \leqslant l$. Suppose given $x_k \in X_k$. We have $(k, x_k) \sim (l, x_k f_{k,l})$, since $x_k f_{k,l} = (x_k f_{k,l}) \,\mathrm{id}_{X_l} = (x_k f_{k,l}) f_{l,l}$.

So we have $x_k f_{k,l} f_{l,\infty} = [l, x_k f_{k,l}] = [k, x_k] = x_k f_{k,\infty}$.

So we have $f_{k,l} f_{l,\infty} = f_{k,\infty}$.

Lemma 43 (universal property) Suppose given maps $t_j : X_j \to T$ for $j \geqslant 0$ such that $f_{j,j+1}t_{j+1} = t_j$ for $j \geqslant 0$, i.e. $f_{j,k}t_k = t_j$ for $0 \leqslant j \leqslant k$.

$$\begin{tikzcd}
X_0 \arrow[r, "f_{0,1}"] \arrow[dr, "t_0"'] & X_1 \arrow[r, "f_{1,2}"] \arrow[d, "t_1"] & X_2 \arrow[r, "f_{2,3}"] \arrow[dl, "t_2"] & {} \\
& T & \cdots &
\end{tikzcd}$$

Then there exists a unique map $t_\infty : X_\infty \to T$ such that $f_{j,\infty}t_\infty = t_j$ for $j \geqslant 0$.

Proof.

Uniqueness.

Suppose given $t', t'' : X_\infty \to T$ such that $f_{j,\infty}t' = t_j$ and $f_{j,\infty}t'' = t_j$ for $j \geqslant 0$.

We have to show that $t' = t''$.

Let $[j, x_j] \in X_\infty$.

We have to show that $[j, x_j]t' \overset{!}{=} [j, x_j]t''$.

In fact, we have $[j, x_j]t' = x_j f_{j,\infty}t' = x_j t_j = x_j f_{j,\infty}t'' = [j, x_j]t''$.

Existence.

Let

$$\begin{array}{rccl} t_\infty : & X_\infty & \to & T \\ & [j, x_j] & \mapsto & x_j t_j \ . \end{array}$$

We show that this map is well-defined.

Suppose given $(j, x_j), (k, x_k) \in \bigsqcup_{i \geqslant 0} X_i$ such that $[j, x_j] = [k, x_k]$.

Then $x_j f_{j,m} = x_k f_{k,m}$ for some $m \geqslant \max\{j, k\}$.

We have to show that $x_j t_j \overset{!}{=} x_k t_k$.

In fact, we have $x_j t_j = x_j f_{j,m} t_m = x_k f_{k,m} t_m = x_k t_k$.

Suppose given $j \geqslant 0$. Suppose given $x_j \in X_j$. We have $x_j f_{j,\infty} t_\infty = [j, x_j]t_\infty \overset{\text{Def. } t_\infty}{=} x_j t_j$.

So we have $f_{j,\infty}t_\infty = t_j$.

$$\begin{tikzcd}
X_0 \arrow[r, "f_{0,1}"'] \arrow[dr, "t_0"'] \arrow[rrrr, bend left=50, "f_{0,\infty}"] & X_1 \arrow[r, "f_{1,2}"'] \arrow[d, "t_1"] \arrow[rrr, bend left=40, "f_{1,\infty}"] & X_2 \arrow[r, "f_{2,3}"'] \arrow[dl, "t_2"] \arrow[rr, bend left=40, "f_{2,\infty}"] & \cdots & X_\infty \arrow[dlll, bend left, "t_\infty"] \\
& T & \cdots & &
\end{tikzcd}$$

Definition 44 Suppose given a commutative diagram in Set as follows; cf. Definition 42.

$$
\begin{tikzcd}
X_0 \arrow[r, "f_{0,1}"'] \arrow[d, "u_0"'] \arrow[rrrrr, bend left=40, "f_{0,\infty}"] & X_1 \arrow[r, "f_{1,2}"'] \arrow[d, "u_1"] \arrow[rrrr, bend left=40, "f_{1,\infty}"] & X_2 \arrow[r, "f_{2,3}"'] \arrow[d, "u_2"] \arrow[rrr, bend left=40, "f_{2,\infty}"] & X_3 \arrow[r, "f_{3,4}"'] \arrow[d, "u_3"] \arrow[rr, bend left=40, "f_{3,\infty}"] & \cdots & X_\infty \\
Y_0 \arrow[r, "g_{0,1}"] \arrow[rrrrr, bend right=40, "g_{0,\infty}"'] & Y_1 \arrow[r, "g_{1,2}"] \arrow[rrrr, bend right=40, "g_{1,\infty}"'] & Y_2 \arrow[r, "g_{2,3}"] \arrow[rrr, bend right=40, "g_{2,\infty}"'] & Y_3 \arrow[r, "g_{3,4}"] \arrow[rr, bend right=40, "g_{3,\infty}"'] & \cdots & Y_\infty
\end{tikzcd}
$$

We have $f_{j,j+1}u_{j+1}g_{j+1,\infty} = u_j g_{j,j+1} g_{j+1,\infty} = u_j g_{j,\infty}$ for $j \geqslant 0$.

So because of the universal property there exists a unique map $u_\infty : X_\infty \to Y_\infty$ such that $f_{j,\infty}u_\infty = u_j g_{j,\infty}$ for $j \geqslant 0$; cf. Lemma 43.

We let $\varinjlim\limits_{i \geqslant 0} u_i := u_\infty$.

$$
\begin{tikzcd}
X_0 \arrow[r, "f_{0,1}"'] \arrow[d, "u_0"'] \arrow[rrrrr, bend left=40, "f_{0,\infty}"] & X_1 \arrow[r, "f_{1,2}"'] \arrow[d, "u_1"] \arrow[rrrr, bend left=40, "f_{1,\infty}"] & X_2 \arrow[r, "f_{2,3}"'] \arrow[d, "u_2"] \arrow[rrr, bend left=40, "f_{2,\infty}"] & X_3 \arrow[r, "f_{3,4}"'] \arrow[d, "u_3"] \arrow[rr, bend left=40, "f_{3,\infty}"] & \cdots & X_\infty \arrow[d, "u_\infty"] \\
Y_0 \arrow[r, "g_{0,1}"] \arrow[rrrrr, bend right=40, "g_{0,\infty}"'] & Y_1 \arrow[r, "g_{1,2}"] \arrow[rrrr, bend right=40, "g_{1,\infty}"'] & Y_2 \arrow[r, "g_{2,3}"] \arrow[rrr, bend right=40, "g_{2,\infty}"'] & Y_3 \arrow[r, "g_{3,4}"] \arrow[rr, bend right=40, "g_{3,\infty}"'] & \cdots & Y_\infty
\end{tikzcd}
$$

Chapter 2

Graphs

2.1 Definitions for graphs and graph morphisms

Definition 45 A *graph* $G = (\mathrm{V}_G, \mathrm{E}_G, \mathrm{s}_G, \mathrm{t}_G)$ is a tuple consisting of a *set of vertices* V_G and a *set of edges* E_G together with maps $\mathrm{s}_G : \mathrm{E}_G \to \mathrm{V}_G$, the *source map*, and $\mathrm{t}_G : \mathrm{E}_G \to \mathrm{V}_G$, the *target map*.

Remark 46 Suppose given a graph $G = (\mathrm{V}_G, \mathrm{E}_G, \mathrm{s}_G, \mathrm{t}_G)$. The elements of V_G are called *vertices* and the elements of E_G are called *edges*.

The element $e\,\mathrm{s}_G \in \mathrm{V}_G$ is called the *source* of $e \in \mathrm{E}_G$.

The element $e\,\mathrm{t}_G \in \mathrm{V}_G$ is called the *target* of $e \in \mathrm{E}_G$.

Pictorially, we represent G by writing out the vertices of G and by drawing an arrow for each edge e of G, pointing from its source $e\,\mathrm{s}_G$ to its target $e\,\mathrm{t}_G$.

Definition 47 Suppose given a graph $G = (\mathrm{V}_G\,, \mathrm{E}_G\,, \mathrm{s}_G\,, \mathrm{t}_G)$.

(1) A graph $G' = (\mathrm{V}_{G'}\,, \mathrm{E}_{G'}\,, \mathrm{s}_{G'}\,, \mathrm{t}_{G'})$ is called a *subgraph of* G if $\mathrm{V}_{G'} \subseteq \mathrm{V}_G$, $\mathrm{E}_{G'} \subseteq \mathrm{E}_G$, $\mathrm{s}_{G'} = \mathrm{s}_G|_{\mathrm{E}_{G'}}^{\mathrm{V}_{G'}}$, and $\mathrm{t}_{G'} = \mathrm{t}_G|_{\mathrm{E}_{G'}}^{\mathrm{V}_{G'}}$.

To indicate that G' is a subgraph of G, we write $G' \subseteq G$.

(2) Suppose given subgraphs $H \subseteq G$ and $K \subseteq G$. The *intersection* $H \cap K$ of the subgraphs H and K is the subgraph

$$H \cap K = (\mathrm{V}_{H\cap K}\,, \mathrm{E}_{H\cap K}\,, \mathrm{s}_{H\cap K}\,, \mathrm{t}_{H\cap K}) := (\mathrm{V}_H \cap \mathrm{V}_K\,, \mathrm{E}_H \cap \mathrm{E}_K\,, \mathrm{s}_G|_{\mathrm{E}_H\cap\mathrm{E}_K}^{\mathrm{V}_H\cap\mathrm{V}_K}\,, \mathrm{t}_G|_{\mathrm{E}_H\cap\mathrm{E}_K}^{\mathrm{V}_H\cap\mathrm{V}_K})$$

of G.

(3) A subgraph $G' \subseteq G$ is called a *full subgraph* if $\mathrm{E}_{G'} = \{e \in \mathrm{E}_G : e\,\mathrm{s}_G \in \mathrm{V}_{G'} \text{ and } e\,\mathrm{t}_G \in \mathrm{V}_{G'}\}$, i.e. if for $v', w' \in \mathrm{V}_{G'}$, each edge e of G having $e\,\mathrm{s}_G = v'$ and $e\,\mathrm{t}_G = w'$ is already an edge of G'.

(4) A graph $G = (\mathrm{V}_G\,, \mathrm{E}_G\,, \mathrm{s}_G\,, \mathrm{t}_G)$ is called *finite* if the sets V_G and E_G both are finite.

Remark 48 Suppose given a graph G.

(1) To define a subgraph G' of G, it suffices to give subsets $\mathrm{E}_{G'} \subseteq \mathrm{E}_G$ and $\mathrm{V}_{G'} \subseteq \mathrm{V}_G$ such that $\mathrm{E}_{G'}\, \mathrm{s}_G \subseteq \mathrm{V}_{G'}$ and $\mathrm{E}_{G'}\, \mathrm{t}_G \subseteq \mathrm{V}_{G'}$.

(2) To define a full subgraph G' of G, it suffices to give a subset $\mathrm{V}_{G'} \subseteq \mathrm{V}_G$.

Example 49 The graph G having $\mathrm{V}_G = \{1, 2, 3, 4\}$ and $\mathrm{E}_G = \{\alpha_1\, , \alpha_2\, , \alpha_3\, , \alpha_4\}$ with

$$\begin{aligned} \alpha_1\, \mathrm{s}_G &= 1, & \alpha_1\, \mathrm{t}_G &= 2, \\ \alpha_2\, \mathrm{s}_G &= 1, & \alpha_2\, \mathrm{t}_G &= 2, \\ \alpha_3\, \mathrm{s}_G &= 3, & \alpha_3\, \mathrm{t}_G &= 2, \\ \alpha_4\, \mathrm{s}_G &= 1, & \alpha_4\, \mathrm{t}_G &= 1, \end{aligned}$$

is represented the following way.

$$G : \quad \boxed{\begin{tikzcd} 1 \arrow[loop left, "\alpha_4"] \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2 & 3 \arrow[l, "\alpha_3"'] & 4 \end{tikzcd}}$$

Note that the graph H having $\mathrm{V}_H = \{1, 2, 3\}$ and $\mathrm{E}_H = \{\alpha_1\, , \alpha_2\, , \alpha_3\}$ with

$$\begin{aligned} \alpha_1\, \mathrm{s}_H &= 1, & \alpha_3\, \mathrm{t}_H &= 2, \\ \alpha_2\, \mathrm{s}_H &= 1, & \alpha_3\, \mathrm{t}_H &= 2, \\ \alpha_3\, \mathrm{s}_H &= 3, & \alpha_3\, \mathrm{t}_H &= 2, \end{aligned}$$

which is represented as

$$H : \quad \boxed{\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2 & 3 \arrow[l, "\alpha_3"'] \end{tikzcd}}$$

is a full subgraph of G; cf. Definition 47.

Moreover, the graph K having $\mathrm{V}_K = \{1, 2\}$ and $\mathrm{E}_K = \{\alpha_1\, , \alpha_4\}$ with

$$\begin{aligned} \alpha_1\, \mathrm{s}_K &= 1, & \alpha_1\, \mathrm{t}_K &= 2, \\ \alpha_4\, \mathrm{s}_K &= 1, & \alpha_4\, \mathrm{t}_K &= 1, \end{aligned}$$

which is represented as

$$K : \quad \boxed{\begin{tikzcd} 1 \arrow[loop left, "\alpha_4"] \arrow[r, bend left, "\alpha_1"] & 2\,, \end{tikzcd}}$$

is a subgraph of G, but not a full subgraph.

As intersection, we obtain $H \cap K$ having $\mathrm{V}_{H\cap K} = \{1, 2\}$ and $\mathrm{E}_{H\cap K} = \{\alpha_1\}$ with

$$\alpha_1 \, \mathrm{s}_{H\cap K} = 1, \quad \alpha_3 \, \mathrm{t}_{H\cap K} = 2,$$

which is represented as

$$H \cap K : \quad \boxed{\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \, . \end{tikzcd}}$$

Remark 50 Suppose given a graph G.

Suppose given full subgraphs $G' \subseteq G$ and $G'' \subseteq G$ with $\mathrm{V}_{G'} = \mathrm{V}_{G''}$.

Then $G' = G''$.

Proof. We have to show that $G' \overset{!}{=} G''$. It remains to show that $\mathrm{E}_{G'} \overset{!}{=} \mathrm{E}_{G''}$ as subsets of E_G.

Write $V := \mathrm{V}_{G'} = \mathrm{V}_{G''}$. We have $\mathrm{E}_{G'} = \{e \in \mathrm{E}_G : e\,\mathrm{s}_G \in V \text{ and } e\,\mathrm{t}_G \in V\} = \mathrm{E}_{G''}$. □

Notation 51 Suppose given a graph G.

Suppose given vertices $v, w \in \mathrm{V}_G$.

Suppose given subsets $V, W \subseteq \mathrm{V}_G$.

(1) We denote $G(v, w) := \{e \in \mathrm{E}_G : e\,\mathrm{s}_G = v,\ e\,\mathrm{t}_G = w\} \subseteq \mathrm{E}_G$.
Write $G(v, *) := \{e \in \mathrm{E}_G : e\,\mathrm{s}_G = v\} \subseteq \mathrm{E}_G$.
Write $G(*, w) := \{e \in \mathrm{E}_G : e\,\mathrm{t}_G = w\} \subseteq \mathrm{E}_G$.

(2) We denote $G(V, W) := \{e \in \mathrm{E}_G : e\,\mathrm{s}_G \in V,\ e\,\mathrm{t}_G \in W\} \subseteq \mathrm{E}_G$.
Write $G(V, w) := G(V, \{w\})$ and $G(v, W) := G(\{v\}, W)$.
Write $G(V, *) := \{e \in \mathrm{E}_G : e\,\mathrm{s}_G \in V\} \subseteq \mathrm{E}_G$.
Write $G(*, W) := \{e \in \mathrm{E}_G : e\,\mathrm{t}_G \in W\} \subseteq \mathrm{E}_G$.
In particular, $G(v, *) = G(\{v\}, *)$ and $G(*, w) = G(*, \{w\})$.

Definition 52 Let $n \in \mathbb{N}$. We will define a graph C_n.

For $i \in \mathbb{Z}$ we often abbreviate $i := i + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$.

To denote vertices in C_n, we use symbols v_i, for $i \in \mathbb{Z}/n\mathbb{Z}$. To denote edges in C_n, we use symbols e_i, for $i \in \mathbb{Z}/n\mathbb{Z}$.

The *cyclic graph* $\mathrm{C}_n = (\mathrm{V}_{\mathrm{C}_n}, \mathrm{E}_{\mathrm{C}_n}, \mathrm{s}_{\mathrm{C}_n}, \mathrm{t}_{\mathrm{C}_n})$ is the graph with $\mathrm{V}_{\mathrm{C}_n} := \{\mathrm{v}_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$,

$\mathrm{E}_{\mathrm{C}_n} := \{\mathrm{e}_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ and with

$$\mathrm{e}_i \, \mathrm{s}_{\mathrm{C}_n} = \mathrm{v}_i, \quad \mathrm{e}_i \, \mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_{i+1} \text{ for } i \in \mathbb{Z}/n\mathbb{Z}\,.$$

Example 53 For example the cyclic graph C_4 is represented the following way.

$$\mathrm{C}_4 : \boxed{\begin{tikzcd} \mathrm{v}_2 \arrow[r, bend left, "\mathrm{e}_2"] & \mathrm{v}_3 \arrow[d, bend left, "\mathrm{e}_3"] \\ \mathrm{v}_1 \arrow[u, bend left, "\mathrm{e}_1"'] & \mathrm{v}_4 \arrow[l, bend left, "\mathrm{e}_4"] \end{tikzcd}}$$

Note that $\mathrm{v}_4 = \mathrm{v}_0$ and $\mathrm{e}_4 = \mathrm{e}_0$.

Definition 54 Suppose given graphs $G = (\mathrm{V}_G, \mathrm{E}_G, \mathrm{s}_G, \mathrm{t}_G)$ and $H = (\mathrm{V}_H, \mathrm{E}_H, \mathrm{s}_H, \mathrm{t}_H)$.

A *graph morphism* $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$ between G and H is a tuple consisting of a map on the vertices of the graphs $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and a map on the edges of the graphs $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ such that (Morph 1–2) hold.

(Morph 1) We have $\mathrm{E}_f\, \mathrm{s}_H = \mathrm{s}_G\, \mathrm{V}_f$.

(Morph 2) We have $\mathrm{E}_f\, \mathrm{t}_H = \mathrm{t}_G\, \mathrm{V}_f$.

The graph morphism $f = (\mathrm{V}_f\,, \mathrm{E}_f)$ is called *injective* if V_f and E_f are injective.

The graph morphism $f = (\mathrm{V}_f\,, \mathrm{E}_f)$ is called *surjective* if V_f and E_f are surjective.

The graph morphism $f = (\mathrm{V}_f\,, \mathrm{E}_f)$ is called *bijective* if V_f and E_f are bijective.

Definition 55 Suppose given a graph $G = (\mathrm{V}_G, \mathrm{E}_G, \mathrm{s}_G, \mathrm{t}_G)$.

The graph morphism $\mathrm{id}_G := (\mathrm{V}_{\mathrm{id}_G}, \mathrm{E}_{\mathrm{id}_G}) : G \to G$ with

$$\begin{array}{rccc} \mathrm{V}_{\mathrm{id}_G} := \mathrm{id}_{\mathrm{V}_G} : & \mathrm{V}_G & \to & \mathrm{V}_G \\ & v & \mapsto & v \end{array}$$

for $v \in \mathrm{V}_G$ and with

$$\begin{array}{rccc} \mathrm{E}_{\mathrm{id}_G} := \mathrm{id}_{\mathrm{E}_G} : & \mathrm{E}_G & \to & \mathrm{E}_G \\ & e & \mapsto & e \end{array}$$

for $e \in \mathrm{E}_G$ is called the *identity on G*.

Note that the identity id_G is bijective and thus a graph isomorphism.

Definition 56 Suppose given $n \in \mathbb{N}$ and $k, l \in \mathbb{N}$ with $k \leqslant l$. We will define a graph D_n , as well as graph morphisms $r_n : \mathrm{D}_n \to \mathrm{C}_n$ and $\iota_{k,l} : \mathrm{D}_k \to \mathrm{D}_l$.

(1) To denote vertices in D_n , we use symbols $\hat{\mathrm{v}}_i$, for $i \in [0, n]$. To denote edges in D_n , we use symbols $\hat{\mathrm{e}}_i$, for $i \in [0, n-1]$.

The *direct graph* $\mathrm{D}_n = (\mathrm{V}_{\mathrm{D}_n}, \mathrm{E}_{\mathrm{D}_n}, \mathrm{s}_{\mathrm{D}_n}, \mathrm{t}_{\mathrm{D}_n})$ is the graph with $\mathrm{V}_{\mathrm{D}_n} := \{\hat{\mathrm{v}}_i \mid i \in [0, n]\}$, $\mathrm{E}_{\mathrm{D}_n} := \{\hat{\mathrm{e}}_i \mid i \in [0, n-1]\}$ and with

$$\hat{\mathrm{e}}_i\, \mathrm{s}_{\mathrm{D}_n} = \hat{\mathrm{v}}_i \text{ for } i \in [0, n-1], \quad \hat{\mathrm{e}}_i\, \mathrm{t}_{\mathrm{D}_n} = \hat{\mathrm{v}}_{i+1} \text{ for } i \in [0, n-1] .$$

(2) We have the graph morphism

$$\begin{array}{rccl} r_n : & \mathrm{D}_n & \to & \mathrm{C}_n \\ \mathrm{V}_{r_n} : & \hat{\mathrm{v}}_i & \to & \mathrm{v}_{i+n\mathbb{Z}} = \mathrm{v}_i \;\text{ for } i \in [0,n] \\ \mathrm{E}_{r_n} : & \hat{\mathrm{e}}_i & \to & \mathrm{e}_{i+n\mathbb{Z}} = \mathrm{e}_i \;\text{ for } i \in [0,n-1]\ . \end{array}$$

To verify that r_n is in fact a graph morphism, we have to show that $\hat{\mathrm{e}}_i \, \mathrm{s}_{\mathrm{D}_n} \, \mathrm{V}_{r_n} = \hat{\mathrm{e}}_i \, \mathrm{E}_{r_n} \, \mathrm{s}_{\mathrm{C}_n}$ and that $\hat{\mathrm{e}}_i \, \mathrm{t}_{\mathrm{D}_n} \, \mathrm{V}_{r_n} = \hat{\mathrm{e}}_i \, \mathrm{E}_{r_n} \, \mathrm{t}_{\mathrm{C}_n}$ for $i \in [0, n-1]$.

We have $\hat{\mathrm{e}}_i \, \mathrm{s}_{\mathrm{D}_n} \, \mathrm{V}_{r_n} = \hat{\mathrm{v}}_i \, \mathrm{V}_{r_n} = \mathrm{v}_{i+n\mathbb{Z}} = \mathrm{e}_{i+n\mathbb{Z}} \, \mathrm{s}_{\mathrm{C}_n} = \hat{\mathrm{e}}_i \, \mathrm{E}_{r_n} \, \mathrm{s}_{\mathrm{C}_n}$ for $i \in [0, n-1]$ and we have $\hat{\mathrm{e}}_i \, \mathrm{t}_{\mathrm{D}_n} \, \mathrm{V}_{r_n} = \hat{\mathrm{v}}_{i+1} \, \mathrm{V}_{r_n} = \mathrm{v}_{i+1+n\mathbb{Z}} = \mathrm{e}_{i+n\mathbb{Z}} \, \mathrm{t}_{\mathrm{C}_n} = \hat{\mathrm{e}}_i \, \mathrm{E}_{r_n} \, \mathrm{t}_{\mathrm{C}_n}$ for $i \in [0, n-1]$.

So $r_n : \mathrm{D}_n \to \mathrm{C}_n$ is in fact a graph morphism. ▫

(3) We have the graph morphism

$$\begin{array}{rcccl} \iota_{k,l} : & \mathrm{D}_k & \to & \mathrm{D}_l & \\ \mathrm{V}_{\iota_{k,l}} : & \hat{\mathrm{v}}_i & \to & \hat{\mathrm{v}}_i & \text{for } i \in [0,k] \\ \mathrm{E}_{\iota_{k,l}} : & \hat{\mathrm{e}}_i & \to & \hat{\mathrm{e}}_i & \text{for } i \in [0,k-1]. \end{array}$$

We often identify the direct graph D_k with the subgraph $\mathrm{D}_k \, \iota_{k,l} \subseteq \mathrm{D}_l$ of the direct graph D_l .

We often abbreviate $\iota_n := \iota_{0,n} : \mathrm{D}_0 \to \mathrm{D}_n$, where $\mathrm{V}_{\iota_n} : \hat{\mathrm{v}}_0 \mapsto \hat{\mathrm{v}}_0$.

Example 57 For example the direct graph D_1 is represented the following way.

$$\mathrm{D}_1 : \quad \boxed{\hat{\mathrm{v}}_0 \xrightarrow{\hat{\mathrm{e}}_0} \hat{\mathrm{v}}_1}$$

Example 58 For example the direct graph D_3 is represented the following way.

$$\mathrm{D}_3 : \quad \boxed{\hat{\mathrm{v}}_0 \xrightarrow{\hat{\mathrm{e}}_0} \hat{\mathrm{v}}_1 \xrightarrow{\hat{\mathrm{e}}_1} \hat{\mathrm{v}}_2 \xrightarrow{\hat{\mathrm{e}}_2} \hat{\mathrm{v}}_3}$$

Remark 59 Suppose given a graph G.

Suppose given an edge $e \in \mathrm{E}_G$.

Then there exists a unique graph morphism $f : \mathrm{D}_1 \to G$ such that $\hat{\mathrm{e}}_0 \, \mathrm{E}_f = e$.

Proof.

Existence. Let $\hat{\mathrm{e}}_0 \, \mathrm{E}_f := e$, $\hat{\mathrm{v}}_0 \, \mathrm{V}_f := e \, \mathrm{s}_G$ and $\hat{\mathrm{v}}_1 \, \mathrm{V}_f := e \, \mathrm{t}_G$.

We have $\hat{\mathrm{e}}_0 \, \mathrm{s}_{\mathrm{D}_1} \, \mathrm{V}_f = \hat{\mathrm{v}}_0 \, \mathrm{V}_f = e \, \mathrm{s}_G = \hat{\mathrm{e}}_0 \, \mathrm{E}_f \, \mathrm{s}_G$ and $\hat{\mathrm{e}}_0 \, \mathrm{t}_{\mathrm{D}_1} \, \mathrm{V}_f = \hat{\mathrm{v}}_0 \, \mathrm{V}_f = e \, \mathrm{t}_G = \hat{\mathrm{e}}_0 \, \mathrm{E}_f \, \mathrm{t}_G$.

So f is in fact a graph morphism.

Uniqueness. Given $f : \mathrm{D}_1 \to G$ such that $\hat{\mathrm{e}}_0 \, \mathrm{E}_f = e$, we necessarily have $\hat{\mathrm{v}}_0 \, \mathrm{V}_f = \hat{\mathrm{e}}_0 \, \mathrm{s}_{\mathrm{D}_1} \, \mathrm{V}_f = \hat{\mathrm{e}}_0 \, \mathrm{E}_f \, \mathrm{s}_G = e \, \mathrm{s}_G$ and $\hat{\mathrm{v}}_1 \, \mathrm{V}_f = \hat{\mathrm{e}}_0 \, \mathrm{t}_{\mathrm{D}_1} \, \mathrm{V}_f = \hat{\mathrm{e}}_0 \, \mathrm{E}_f \, \mathrm{t}_G = e \, \mathrm{t}_G$. ▫

Definition 60 Suppose given graphs $X = (\mathrm{V}_X, \mathrm{E}_X, \mathrm{s}_X, \mathrm{t}_X)$, $Y = (\mathrm{V}_Y, \mathrm{E}_Y, \mathrm{s}_Y, \mathrm{t}_Y)$ and $Z = (\mathrm{V}_Z, \mathrm{E}_Z, \mathrm{s}_Z, \mathrm{t}_Z)$.

Suppose given graph morphisms $f = (\mathrm{V}_f\,, \mathrm{E}_f) : X \to Y$ and $g = (\mathrm{V}_g, \mathrm{E}_g) : Y \to Z$.

Let $f \cdot g := (\mathrm{V}_f \cdot \mathrm{V}_g, \mathrm{E}_f \cdot \mathrm{E}_g)$. We often write $fg := f \cdot g$.

Then $f \cdot g$ is also a graph morphism, called the *composite* of the graph morphisms f and g.

Proof. We need to show that fg is a graph morphism.

We have to show that (Morph 1) and (Morph 2) hold for fg.

So we have to show that $E_{fg}\,\mathrm{s}_Z \overset{!}{=} \mathrm{s}_X\,\mathrm{V}_{fg}$ and that $\mathrm{E}_{fg}\,\mathrm{t}_Z \overset{!}{=} \mathrm{t}_X\,\mathrm{V}_{fg}$.

We have $\mathrm{E}_{fg}\,\mathrm{s}_Z = \mathrm{E}_f \cdot \mathrm{E}_g \cdot \mathrm{s}_Z \overset{\text{(Morph 1) for } g}{=} \mathrm{E}_f \cdot \mathrm{s}_Y \cdot \mathrm{V}_g \overset{\text{(Morph 1) for } f}{=} \mathrm{s}_X \cdot \mathrm{V}_f \cdot \mathrm{V}_g = \mathrm{s}_X \cdot \mathrm{V}_{fg}$.

We have $\mathrm{E}_{fg}\,\mathrm{t}_Z = \mathrm{E}_f \cdot \mathrm{E}_g \cdot \mathrm{t}_Z \overset{\text{(Morph 2) for } g}{=} \mathrm{E}_f \cdot \mathrm{t}_Y \cdot \mathrm{V}_g \overset{\text{(Morph 2) for } f}{=} \mathrm{t}_X \cdot \mathrm{V}_f \cdot \mathrm{V}_g = \mathrm{t}_X \cdot \mathrm{V}_{fg}$. ▫

So

$$\begin{aligned} \mathrm{V}_{f\cdot g} &= \mathrm{V}_f \cdot \mathrm{V}_g \\ \mathrm{E}_{f\cdot g} &= \mathrm{E}_f \cdot \mathrm{E}_g \ . \end{aligned}$$

Definition 61 Suppose given graphs G and H. Suppose given a graph morphism $f : G \to H$.

The *image* Gf of the graph morphism f is the graph $(\mathrm{V}_{Gf}, \mathrm{E}_{Gf}, \mathrm{s}_{Gf}, \mathrm{t}_{Gf})$ consisting of the set of vertices $\mathrm{V}_{Gf} := \mathrm{V}_G\,\mathrm{V}_f \subseteq \mathrm{V}_H$, the set of edges $\mathrm{E}_{Gf} := \mathrm{E}_G\,\mathrm{E}_f \subseteq \mathrm{E}_H$, the source map $\mathrm{s}_{Gf} := \mathrm{s}_H|_{\mathrm{E}_{Gf}}^{\mathrm{V}_{Gf}}$ and the target map $\mathrm{t}_{Gf} := \mathrm{t}_H|_{\mathrm{E}_{Gf}}^{\mathrm{V}_{Gf}}$.

Note that the image Gf of the graph morphism f is a subgraph of H; cf. Definitions 47 and 54.

Example 62 Let G be the graph having $\mathrm{V}_G = \{1, 2, 3\}$ and $\mathrm{E}_G = \{\alpha_1\,, \alpha_2\}$ with

$$\begin{aligned} \alpha_1\,\mathrm{s}_G &= 1, \quad \alpha_1\,\mathrm{t}_G = 2, \\ \alpha_2\,\mathrm{s}_G &= 3, \quad \alpha_2\,\mathrm{t}_G = 2, \end{aligned}$$

Let H be the graph having $\mathrm{V}_H = \{1, 2, 3\}$ and $\mathrm{E}_H = \{\beta_1\,, \beta_2\}$ with

$$\begin{aligned} \beta_1\,\mathrm{s}_H &= 1, \quad \beta_1\,\mathrm{t}_H = 2, \\ \beta_2\,\mathrm{s}_H &= 2, \quad \beta_2\,\mathrm{t}_H = 3, \end{aligned}$$

Now we have the following situation

$$G : \quad \boxed{\; 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \;}$$

$$H : \quad \boxed{\; 1 \xrightarrow{\beta_1} 2 \xrightarrow{\beta_2} 3 \;}$$

Let $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$ be the graph morphism with

$$1\,\mathrm{V}_f = 1, \quad 2\,\mathrm{V}_f = 2, \quad 3\,\mathrm{V}_f = 1,$$

and with

$$\begin{aligned} \alpha_1\,\mathrm{E}_f &= \beta_1\,, \\ \alpha_2\,\mathrm{E}_f &= \beta_1\,, \end{aligned}$$

Then the image Gf of the graph morphism f is the graph having $\mathrm{V}_{Gf} = \{1, 2\}$ and $\mathrm{E}_{Gf} = \{\beta_1\}$ with

$$\beta_1\,\mathrm{s}_{Gf} = 1, \quad \beta_1\,\mathrm{t}_{Gf} = 2,$$

The image Gf is represented as

$$Gf : \boxed{\quad 1 \xrightarrow{\beta_1} 2 \quad}$$

Definition 63 Suppose given a graph morphism $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$.

Suppose given subgraphs $G' \subseteq G$ and $H' \subseteq H$ such that $G'f \subseteq H'$.

Let the *restriction* of f to G' and H' be defined as

$$f|_{G'}^{H'} := (\mathrm{V}_f\,|_{\mathrm{V}_{G'}}^{\mathrm{V}_{H'}}, \mathrm{E}_f\,|_{\mathrm{E}_{G'}}^{\mathrm{E}_{H'}})$$

Then $f|_{G'}^{H'} : G' \to H'$ is a graph morphism.

We have $\mathrm{V}_{f|_{G'}^{H'}} = \mathrm{V}_f\,|_{\mathrm{V}_{G'}}^{\mathrm{V}_{H'}}$ and $\mathrm{E}_{f|_{G'}^{H'}} = \mathrm{E}_f\,|_{\mathrm{E}_{G'}}^{\mathrm{E}_{H'}}$.

If $H' = H$, then we also write $f|_{G'} := f|_{G'}^{H} : G' \to H$.

If $G' = G$, then we also write $f|^{H'} := f|_{G}^{H'} : G \to H'$.

Note that, in particular, we can restrict the graph morphism f to the subgraph $Gf \subseteq H$ to obtain the surjective graph morphism $f|^{Gf} : G \to Gf$.

Moreover, we obtain the *inclusion morphism* $\mathrm{id}_G\,|_{G'} : G' \to G$.

Definition 64 The *category of graphs* Gph consists of the set of objects

$$\mathrm{Ob}(\mathrm{Gph}) := \{G : \ G \text{ is a graph}\}$$

and the set of morphisms

$$\mathrm{Mor}(\mathrm{Gph}) := \{f : \ f \text{ is a graph morphism}\}.$$

Cf. Definitions 45 and 54.

The category Gph has the identity morphisms introduced in Definition 55 and carries the composition introduced in Definition 60.

Suppose given graphs $G, H \in \mathrm{Ob}(\mathrm{Gph})$. By

$$(G, H)_{\mathrm{Gph}} := \{G \xrightarrow{f} H : \ f \text{ is a graph morphism}\} \subseteq \mathrm{Mor}(\mathrm{Gph})$$

we denote the set of graph morphisms from G to H. We often abbreviate $(G, H) := (G, H)_{\mathrm{Gph}}$.

Remark 65 Suppose given graphs $G, H \in \mathrm{Ob}(\mathrm{Gph})$. A graph morphism $f : G \to H$ is an isomorphism if and only if it is bijective, i.e. if the map $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and the map $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ both are bijective.

Proof. Suppose given a bijective graph morphism $f = (\mathrm{V}_f, \mathrm{E}_f) : G \to H$.

Then the map $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ is bijective and so there exists its inverse $\mathrm{V}_f^{-1} : \mathrm{V}_H \to \mathrm{V}_G$.

Moreover, the map $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ is bijective and so there exists its inverse $\mathrm{E}_f^{-1} : \mathrm{E}_H \to \mathrm{E}_G$.

Therefore $(\mathrm{V}_f^{-1}, \mathrm{E}_f^{-1}) : H \to G$ is a graph morphism with $ff^{-1} = \mathrm{id}_G$ and $f^{-1}f = \mathrm{id}_H$, since we have $\mathrm{E}_f^{-1}\mathrm{s}_G = \mathrm{E}_f^{-1}\mathrm{s}_G\mathrm{V}_f\mathrm{V}_f^{-1} = \mathrm{E}_f^{-1}\mathrm{E}_f\mathrm{s}_H\mathrm{V}_f^{-1} = \mathrm{s}_H\mathrm{V}_f^{-1}$ and $\mathrm{E}_f^{-1}\mathrm{t}_G = \mathrm{E}_f^{-1}\mathrm{t}_G\mathrm{V}_f\mathrm{V}_f^{-1} = \mathrm{E}_f^{-1}\mathrm{E}_f\mathrm{t}_H\mathrm{V}_f^{-1} = \mathrm{t}_H\mathrm{V}_f^{-1}$.

Furthermore, $(\mathrm{V}_f, \mathrm{E}_f)(\mathrm{V}_f^{-1}, \mathrm{E}_f^{-1}) = (\mathrm{V}_f\mathrm{V}_f^{-1}, \mathrm{E}_f\mathrm{E}_f^{-1}) = (\mathrm{id}_{\mathrm{V}_G}, \mathrm{id}_{\mathrm{E}_G}) = \mathrm{id}_G$ and $(\mathrm{V}_f^{-1}, \mathrm{E}_f^{-1})(\mathrm{V}_f, \mathrm{E}_f) = (\mathrm{V}_f^{-1}\mathrm{V}_f, \mathrm{E}_f^{-1}\mathrm{E}_f) = (\mathrm{id}_{\mathrm{V}_H}, \mathrm{id}_{\mathrm{E}_H}) = \mathrm{id}_H$. Thus $f^{-1} = (\mathrm{V}_{f^{-1}}, \mathrm{E}_{f^{-1}}) = (\mathrm{V}_f^{-1}, \mathrm{E}_f^{-1})$. So f is an isomorphism.

Conversely, if f is an isomorphism, then there exists a graph morphism $g : H \to G$ such that $fg = \mathrm{id}_G$ and $gf = \mathrm{id}_H$. Hence there exist maps $\mathrm{V}_g : \mathrm{V}_H \to \mathrm{V}_G$ and $\mathrm{E}_g : \mathrm{E}_H \to \mathrm{E}_G$ such that $\mathrm{V}_f\mathrm{V}_g = \mathrm{id}_{\mathrm{V}_G}$ and $\mathrm{V}_g\mathrm{V}_f = \mathrm{id}_{\mathrm{V}_H}$ and such that $\mathrm{E}_f\mathrm{E}_g = \mathrm{id}_{\mathrm{E}_G}$ and $\mathrm{E}_g\mathrm{E}_f = \mathrm{id}_{\mathrm{E}_H}$.

So both maps V_f and E_f are bijective. ▫

Remark 66 Suppose given an injective graph morphism $f : X \to Y$.

Then the restriction $f|^{Xf} : X \to Xf$ is a graph isomorphism.

Proof. We have to show that the graph morphism $f|^{Xf}$ is bijective; cf. Remark 65.

The graph morphism f is injective by supposition. Hence the graph morphism $f|^{Xf}$ is injective.

By construction, the graph morphism $f|^{Xf}$ is surjective.

So the graph morphism $f|^{Xf}$ is bijective. ▫

Definition 67 Suppose given $n \in \mathbb{N}$.

Suppose given $s \in \mathbb{Z}$.

We define the graph automorphism

$$\begin{array}{rcl} \mathrm{a}_s : \mathrm{C}_n & \xrightarrow{\sim} & \mathrm{C}_n \\ \mathrm{e}_i & \mapsto & \mathrm{e}_{i+s} \\ \mathrm{v}_i & \mapsto & \mathrm{v}_{i+s} \ . \end{array}$$

In fact, for $i \in \mathbb{Z}/n\mathbb{Z}$ we have

$(\mathrm{e}_i\,\mathrm{E}_{\mathrm{a}_s})\,\mathrm{s}_{\mathrm{C}_n} = \mathrm{e}_{i+s}\,\mathrm{s}_{\mathrm{C}_n} = \mathrm{v}_{i+s} = \mathrm{v}_i\,\mathrm{V}_{\mathrm{a}_s} = (\mathrm{e}_i\,\mathrm{s}_{\mathrm{C}_n})\,\mathrm{V}_{\mathrm{a}_s}$ and

$(\mathrm{e}_i\,\mathrm{E}_{\mathrm{a}_s})\,\mathrm{t}_{\mathrm{C}_n} = \mathrm{e}_{i+s}\,\mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_{i+s+1} = \mathrm{v}_{i+1}\,\mathrm{V}_{\mathrm{a}_s} = (\mathrm{e}_i\,\mathrm{t}_{\mathrm{C}_n})\,\mathrm{V}_{\mathrm{a}_s}$.

So a_s is a graph morphism; cf. Definition 54.

We have $\mathrm{a}_s \cdot \mathrm{a}_t = \mathrm{a}_{s+t}$ for $s, t \in \mathbb{Z}$.

We have $\mathrm{a}_0 = \mathrm{id}_{\mathrm{C}_n}$; cf. Definition 55.

In particular, we have $\mathrm{a}_s \cdot \mathrm{a}_{-s} = \mathrm{a}_0 = \mathrm{id}_{\mathrm{C}_n}$ and $\mathrm{a}_{-s} \cdot \mathrm{a}_s = \mathrm{a}_0 = \mathrm{id}_{\mathrm{C}_n}$, so that $\mathrm{a}_{-s} = \mathrm{a}_s^{-1}$ for $s \in \mathbb{Z}$.

So a_s is a graph automorphism.

Definition 68 Suppose given graphs G, H, X, Y.

Suppose given a graph morphism $f : G \to H$.

We have the map

$$\begin{array}{rcl} (X,G)_{\mathrm{Gph}} & \xrightarrow{(X,f)_{\mathrm{Gph}}} & (X,H)_{\mathrm{Gph}} \\ g & \mapsto & g(X,f)_{\mathrm{Gph}} := gf \ . \end{array}$$

We have the map

$$\begin{array}{rcl} (H,Y)_{\mathrm{Gph}} & \xrightarrow{(f,Y)_{\mathrm{Gph}}} & (G,Y)_{\mathrm{Gph}} \\ g & \mapsto & g(f,Y)_{\mathrm{Gph}} := fg \ . \end{array}$$

Remark 69 Suppose given a graphs X, Y and graph morphisms $f : G \to H, g : H \to K$ and their composite $fg : G \to K$.

We have

(1) $(X, fg)_{\mathrm{Gph}} = (X, f)_{\mathrm{Gph}} \cdot (X, g)_{\mathrm{Gph}}$
$(X, \mathrm{id}_G)_{\mathrm{Gph}} = \mathrm{id}_{(X,G)_{\mathrm{Gph}}}$

(2) $(fg, Y)_{\mathrm{Gph}} = (g, Y)_{\mathrm{Gph}} \cdot (f, Y)_{\mathrm{Gph}}$
$(\mathrm{id}_K, Y)_{\mathrm{Gph}} = \mathrm{id}_{(K,Y)_{\mathrm{Gph}}}$.

Proof.

Ad (1).

Suppose given a graph morphism $u : X \to G$.

We have $u(X, fg)_{\mathrm{Gph}} = u(fg) = (uf)g = (u(X, f)_{\mathrm{Gph}}) \cdot g = u\left((X, f)_{\mathrm{Gph}} \cdot (X, g)_{\mathrm{Gph}}\right)$.

We have $u(X, \mathrm{id}_G)_{\mathrm{Gph}} = u\,\mathrm{id}_G = u$.

Ad (2).

Suppose given a graph morphism $v : K \to Y$.

We have $v(fg, Y)_{\mathrm{Gph}} = (fg)v = f(gv) = (gv)(f, Y)_{\mathrm{Gph}} = v\left((g, Y)_{\mathrm{Gph}} \cdot (f, Y)_{\mathrm{Gph}}\right)$.

We have $v(\mathrm{id}_K, Y)_{\mathrm{Gph}} = \mathrm{id}_K\, v = v$. □

Remark 70

(1) The cyclic graph C_1 is the terminal object in the category Gph.

Let τ_X be the unique graph morphism $\tau_X : X \to \mathrm{C}_1$.

We have $v\,\mathrm{V}_{\tau_X} = \mathrm{v}_0$ for $v \in \mathrm{V}_X$ and we have $e\,\mathrm{E}_{\tau_X} = \mathrm{e}_0$ for $e \in \mathrm{E}_X$.

(2) The *empty graph* $\emptyset := (\mathrm{V}_\emptyset, \mathrm{E}_\emptyset)$ with $\mathrm{V}_\emptyset := \emptyset$ and $\mathrm{E}_\emptyset := \emptyset$ is the initial object in the category Gph.

Let ι_X be the unique graph morphism $\iota_X : \emptyset \to X$.

Definition 71 Suppose given a graph G.

We will define the *discrete subgraph* $\dot{G} \subseteq G$.

We let $\mathrm{V}_{\dot{G}} := \mathrm{V}_G$ and $\mathrm{E}_{\dot{G}} := \emptyset$.

We have the maps $\mathrm{s}_{\dot{G}} : \emptyset \to \mathrm{V}_{\dot{G}}$ and $\mathrm{t}_{\dot{G}} : \emptyset \to \mathrm{V}_{\dot{G}}$.

We have the inclusion morphism o_G from $\dot{G}$ to G as follows.

$$
\begin{array}{rccc}
 & \dot{G} & \xrightarrow{\mathrm{o}_G} & G \\
\mathrm{V}_{\mathrm{o}_G}: & \mathrm{V}_G & \to & \mathrm{V}_G \\
 & x & \mapsto & x \\
\mathrm{E}_{\mathrm{o}_G}: & \emptyset & \to & \mathrm{E}_G
\end{array}
$$

.

Remark 72 Suppose given a surjective graph morphism $f : G \to H$.

Then $f : G \to H$ is an epimorphism in Gph.

Proof. Suppose given graph morphisms $u, u' : H \to K$ such that $fu = fu'$.

We have to show that $u \overset{!}{=} u'$. I.e. we have to show that $\mathrm{V}_u \overset{!}{=} \mathrm{V}_{u'}$ and $\mathrm{E}_u \overset{!}{=} \mathrm{E}_{u'}$.

We have $\mathrm{V}_f\,\mathrm{V}_u = \mathrm{V}_{fu} = \mathrm{V}_{fu'} = \mathrm{V}_f\,\mathrm{V}_u$.

Since V_f is surjective, it is epimorphic; cf. Remark 31. So $\mathrm{V}_u = \mathrm{V}_{u'}$.

We have $\mathrm{E}_f\,\mathrm{E}_u = \mathrm{E}_{fu} = \mathrm{E}_{fu'} = \mathrm{E}_f\,\mathrm{E}_u$.

Since E_f is surjective, it is epimorphic; cf. Remark 31. So $\mathrm{E}_u = \mathrm{E}_{u'}$.

So we conclude $u = u'$. $\square$

2.2 Thin graphs

Definition 73 A graph G is called *thin* if for $v, v' \in \mathrm{V}_G$ we have

$$|\{e \in \mathrm{E}_G : e\,\mathrm{s}_G = v,\ e\,\mathrm{t}_G = v'\}| \leqslant 1.$$

I.e. between two vertices there is at most one edge.

Equivalently, a graph G is thin if the map

$$\begin{array}{ccc} \mathrm{E}_G & \xrightarrow{(\mathrm{s}_G\,,\mathrm{t}_G)} & \mathrm{V}_G \times \mathrm{V}_G \\ e & \mapsto & (e\,\mathrm{s}_G\,, e\,\mathrm{t}_G) \end{array}$$

is injective.

Example 74 The cyclic graph C_n is thin for $n \in \mathbb{N}$; cf. Definitions 52 and 73.

For $i, j \in \mathbb{Z}/n\mathbb{Z}$ we have

$$|\{\mathrm{e}_k \in \mathrm{E}_{\mathrm{C}_n} \,:\, \mathrm{e}_k\,\mathrm{s}_{\mathrm{C}_n} = \mathrm{v}_i\,,\ \mathrm{e}_k\,\mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_j\}| = |\{\mathrm{e}_i \in \mathrm{E}_{\mathrm{C}_n} \,:\, \mathrm{e}_i\,\mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_j\}| = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{if } j \neq i+1. \end{cases}$$

Example 75 The direct graph D_n is thin for $n \in \mathbb{N}$; cf. Definitions 56 and 73.

In fact, for $i, j \in [0, n]$ we have $|\{\hat{\mathrm{e}}_k \in \mathrm{E}_{\mathrm{D}_n} \,:\, \hat{\mathrm{e}}_k\,\mathrm{s}_{\mathrm{D}_n} = \hat{\mathrm{v}}_i\,,\ \hat{\mathrm{e}}_k\,\mathrm{t}_{\mathrm{D}_n} = \hat{\mathrm{v}}_j\}| = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{if } j \neq i+1. \end{cases}$

Remark 76 Suppose given a thin graph G. Suppose given a subgraph $G' \subseteq G$.

Then G' is thin.

Proof. Suppose given vertices $v, v' \in \mathrm{V}_{G'} \subseteq \mathrm{V}_G$.

We have

$$\begin{aligned} & |\{e' \in \mathrm{E}_{G'} \,:\, e'\,\mathrm{s}_{G'} = v,\ e'\,\mathrm{t}_{G'} = v'\}| \overset{\text{Def. 47.(1)}}{=} |\{e' \in \mathrm{E}_{G'} \,:\, e'\,\mathrm{s}_G = v,\ e'\,\mathrm{t}_G = v'\}| \\ \overset{\mathrm{E}_{G'} \subseteq \mathrm{E}_G}{\leqslant} \ & |\{e \in \mathrm{E}_G \,:\, e\,\mathrm{s}_G = v,\ e\,\mathrm{t}_G = v'\}| \leqslant 1\,. \end{aligned}$$

□

Remark 77 Suppose given a graph X. Suppose given a thin graph Y.

(1) Suppose given graph morphisms $f, g : X \to Y$ with $\mathrm{V}_f = \mathrm{V}_g$.

Then $f = g$.

In other words, the map

$$\begin{array}{rcl} (X, Y)_{\mathrm{Gph}} & \to & (\mathrm{V}_X, \mathrm{V}_Y)_{\mathrm{Set}} \\ f & \mapsto & \mathrm{V}_f\ . \end{array}$$

is injective.

(2) A map $u : \mathrm{V}_X \to \mathrm{V}_Y$ is called *monotone* if for each edge $e \in \mathrm{E}_X$ there exists an edge $\tilde{e} \in \mathrm{E}_Y$ such that $\tilde{e}\,\mathrm{s}_Y = (e\,\mathrm{s}_X)u$ and $\tilde{e}\,\mathrm{t}_Y = (e\,\mathrm{t}_X)u$.

We have a bijective map

$$\begin{array}{rcl} (X, Y)_{\mathrm{Gph}} & \to & \{u \in (\mathrm{V}_X, \mathrm{V}_Y)_{\mathrm{Set}} : u \text{ is monotone}\} \\ f & \mapsto & \mathrm{V}_f\ . \end{array}$$

Proof.

Ad (1). We have to show that $\mathrm{E}_f \overset{!}{=} \mathrm{E}_g$.

The graph Y is thin. So we have $|\{e \in \mathrm{E}_Y : e\,\mathrm{s}_Y = v,\, e\,\mathrm{t}_Y = v'\}| \leqslant 1$ for $v, v' \in \mathrm{V}_Y$.

We have $\mathrm{E}_f\,\mathrm{t}_Y = \mathrm{t}_X\,\mathrm{V}_f = \mathrm{t}_X\,\mathrm{V}_g = \mathrm{E}_g\,\mathrm{t}_Y$ as well as $\mathrm{E}_f\,\mathrm{s}_Y = \mathrm{s}_X\,\mathrm{V}_f = \mathrm{s}_X\,\mathrm{V}_g = \mathrm{E}_g\,\mathrm{s}_Y$.

Suppose given an edge $e_X \in \mathrm{E}_X$.

We have $v_1 := e_X\,\mathrm{E}_f\,\mathrm{s}_Y = e_X\,\mathrm{E}_g\,\mathrm{s}_Y \in \mathrm{V}_Y$ and $v_2 := e_X\,\mathrm{E}_f\,\mathrm{t}_Y = e_X\,\mathrm{E}_g\,\mathrm{t}_Y \in \mathrm{V}_Y$.

We have $1 \geqslant |\{e \in \mathrm{E}_Y : e\,\mathrm{s}_Y = v_1,\, e\,\mathrm{t}_Y = v_2\}| \geqslant |\{e_X\,\mathrm{E}_f\,, e_X\,\mathrm{E}_g\}|$ because Y is thin.

So we have $e_X\,\mathrm{E}_f = e_X\,\mathrm{E}_g$.

Therefore, $\mathrm{E}_f = \mathrm{E}_g$.

Ad (2). By (1), the map in question is injective. It remains to show that it is surjective.

Suppose given a monotone map $u : \mathrm{V}_X \to \mathrm{V}_Y$.

We let $\mathrm{V}_f := u$.

For $e \in \mathrm{E}_X$, we let $e\,\mathrm{E}_f := \tilde{e}$, where $\tilde{e}$ is the unique edge in E_Y having $\tilde{e}\,\mathrm{s}_Y = (e\,\mathrm{s}_X)u$ and $\tilde{e}\,\mathrm{t}_Y = (e\,\mathrm{t}_X)u$.

Then $e\,\mathrm{E}_f\,\mathrm{s}_Y = \tilde{e}\,\mathrm{s}_Y = (e\,\mathrm{s}_X)u = e\,\mathrm{s}_X\,\mathrm{V}_f$ and $e\,\mathrm{E}_f\,\mathrm{t}_Y = \tilde{e}\,\mathrm{t}_Y = (e\,\mathrm{t}_X)u = e\,\mathrm{t}_X\,\mathrm{V}_f$.

So $f = (\mathrm{V}_f\,, \mathrm{E}_f)$ is a graph morphism with $\mathrm{V}_f = u$. ▫

Example 78 Suppose given a graph X.

Suppose given $n \in \mathbb{N}$.

Suppose given graph morphisms $f, g : X \to \mathrm{C}_n$ with $\mathrm{V}_f = \mathrm{V}_g$.

Then $f = g$, since the cyclic graph C_n is thin; cf. Remarks 74 and 77.(1).

Remark 79 Suppose given a thin graph Y.

Suppose given $n \in \mathbb{N}$.

Suppose given a map $u : \mathrm{V}_{\mathrm{C}_n} \to \mathrm{V}_Y$ such that $Y(\mathrm{v}_i\,u, \mathrm{v}_{i+1}\,u) \neq \emptyset$ for $i \in \mathbb{Z}/n\mathbb{Z}$.

Then there exists a unique graph morphism $f : \mathrm{C}_n \to Y$ such that $\mathrm{V}_f = u$.

Proof.

Uniqueness. This follows by Remark 77.(1).

Existence. Write $Y(\mathrm{v}_i\,u, \mathrm{v}_{i+1}\,u) =: \{\alpha_i\}$ for $i \in \mathbb{Z}/n\mathbb{Z}$.

Let $\mathrm{V}_f := u$.

Let

$$\begin{array}{rccc} \mathrm{E}_f : & \mathrm{E}_{\mathrm{C}_n} & \to & \mathrm{E}_Y \\ & \mathrm{e}_i & \mapsto & \alpha_i \end{array}$$

Then $f := (\mathrm{V}_f\,, \mathrm{E}_f) : \mathrm{C}_n \to Y$ is a graph morphism, since $\mathrm{e}_i\,\mathrm{E}_f\,\mathrm{s}_Y = \alpha_i\,\mathrm{s}_Y = \mathrm{v}_i\,u = \mathrm{e}_i\,\mathrm{s}_{\mathrm{C}_n}\,\mathrm{V}_f$ and $\mathrm{e}_i\,\mathrm{E}_f\,\mathrm{t}_Y = \alpha_i\,\mathrm{t}_Y = \mathrm{v}_{i+1}\,u = \mathrm{e}_i\,\mathrm{t}_{\mathrm{C}_n}\mathrm{V}_f$ for $i \in \mathbb{Z}/n\mathbb{Z}$. Cf. also Remark 77.(2). ▫

Remark 80 Suppose given a thin graph G.

Suppose given a graph morphism $f : G \to H$ such that the map V_f is injective.

Then the graph morphism $f : G \to H$ is injective, i.e. both maps V_f and E_f are injective.

Proof. Suppose given edges e and $\tilde{e}$ in E_G such that $e\,\mathrm{E}_f = \tilde{e}\,\mathrm{E}_f$.

We have to show that $e \overset{!}{=} \tilde{e}$.

We have $e\,\mathrm{s}_G\,\mathrm{V}_f = e\,\mathrm{E}_f\,\mathrm{s}_H = \tilde{e}\,\mathrm{E}_f\,\mathrm{s}_H = \tilde{e}\,\mathrm{s}_G\,\mathrm{V}_f$.

Since the map $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ is injective, we have $e\,\mathrm{s}_G = \tilde{e}\,\mathrm{s}_G$.

Likewise we have $e\,\mathrm{t}_G = \tilde{e}\,\mathrm{t}_G$.

Since the graph G is thin, we conclude that $e = \tilde{e}$. $\square$

Example 81 We consider the following graph morphism.

$$G : \quad \boxed{\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2 \end{tikzcd}}$$

$$\Big\downarrow f$$

$$H : \quad \boxed{\begin{tikzcd} 1 \arrow[r, "\beta_1"] & 2 \end{tikzcd}}$$

Here, $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$ is the graph morphism mapping the vertices and the edges in a vertical way. I.e.

$$1\,\mathrm{V}_f = 1, \quad 2\,\mathrm{V}_f = 2$$

and

$$\alpha_1\,\mathrm{E}_f = \beta_1\,, \quad \alpha_2\,\mathrm{E}_f = \beta_1\,.$$

The map V_f is injective, while the map E_f is not injective. Note that the graph G is **not** thin.

2.3 Pushout and pullback of graphs

2.3.1 Pushout of graphs

Reminder 82 Suppose given a quadrangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in Gph. It is called a pushout if (Pushout 1–2) hold; cf. Definition 5.

(Pushout 1) We have $g \cdot f' = f \cdot h$.

(Pushout 2) Suppose given a graph G and graph morphisms $u : X' \to G$ and $v : Y \to G$ such that $f \cdot v = g \cdot u$. Then there exists a unique graph morphism $w : Y' \to G$ such that $f' \cdot w = u$ and $h \cdot w = v$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"'] \arrow[ddr, bend left, "v"] & \\ X' \arrow[r, "f'"] \arrow[drr, bend right, "u"'] & Y' \arrow[dr, dotted, "\exists ! w"] & \\ & & G \end{tikzcd}$$

To indicate that this quadrangle is a pushout, we write

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} .$$

Then we also say that f' is a pushout of f.

Construction 83 Suppose given a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X' & \end{tikzcd}$$

in Gph.

We aim to construct a pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner", very near end] & Y'
\end{tikzcd} .$$

In particular, we have to construct maps $\mathrm{E}_h : \mathrm{E}_Y \to \mathrm{E}_{Y'}$ and $\mathrm{E}_{f'} : \mathrm{E}_{X'} \to \mathrm{E}_{Y'}$ and $\mathrm{V}_h : \mathrm{V}_Y \to \mathrm{V}_{Y'}$ and $\mathrm{V}_{f'} : \mathrm{V}_{X'} \to \mathrm{V}_{Y'}$.

We form pushouts in Set as follows.

$$\begin{tikzcd}
\mathrm{E}_X \arrow[r, "\mathrm{E}_f"] \arrow[d, "\mathrm{E}_g"'] & \mathrm{E}_Y \arrow[d, "\mathrm{E}_h"] \\
\mathrm{E}_{X'} \arrow[r, "\mathrm{E}_{f'}"'] \arrow[ur, phantom, "\lrcorner", very near end] & \mathrm{E}_{Y'}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{V}_X \arrow[r, "\mathrm{V}_f"] \arrow[d, "\mathrm{V}_g"'] & \mathrm{V}_Y \arrow[d, "\mathrm{V}_h"] \\
\mathrm{V}_{X'} \arrow[r, "\mathrm{V}_{f'}"'] \arrow[ur, phantom, "\lrcorner", very near end] & \mathrm{V}_{Y'}
\end{tikzcd}$$

For instance, we can use Construction 36 to achieve this.

Because of the universal property of the pushout $\mathrm{E}_{Y'}$, the map $\mathrm{t}_{Y'}$ is uniquely existent with respect to $\mathrm{t}_{X'}\,\mathrm{V}_{f'} = \mathrm{E}_{f'}\,\mathrm{t}_{Y'}$ and $\mathrm{t}_Y\,\mathrm{V}_h = \mathrm{E}_h\,\mathrm{t}_{Y'}$.

Because of the universal property of the pushout $\mathrm{E}_{Y'}$, the map $\mathrm{s}_{Y'}$ is uniquely existent with respect to $\mathrm{s}_{X'}\,\mathrm{V}_{f'} = \mathrm{E}_{f'}\,\mathrm{s}_{Y'}$ and $\mathrm{s}_Y\,\mathrm{V}_h = \mathrm{E}_h\,\mathrm{s}_{Y'}$.

$$\begin{tikzcd}
& \mathrm{E}_X \arrow[rr, "\mathrm{E}_f"] \arrow[dl, "\mathrm{E}_g"'] \arrow[dd, shift right, "\mathrm{t}_X"'] \arrow[dd, shift left, "\mathrm{s}_X"] & & \mathrm{E}_Y \arrow[dl, "\mathrm{E}_h"'] \arrow[dd, shift right, "\mathrm{t}_Y"'] \arrow[dd, shift left, "\mathrm{s}_Y"] \\
\mathrm{E}_{X'} \arrow[rr, "\mathrm{E}_{f'}", near end] \arrow[dd, shift right, "\mathrm{t}_{X'}"'] \arrow[dd, shift left, "\mathrm{s}_{X'}"] & & \mathrm{E}_{Y'} \arrow[dd, shift right, "\mathrm{t}_{Y'}"', near end] \arrow[dd, shift left, "\mathrm{s}_{Y'}", near end] & \\
& \mathrm{V}_X \arrow[rr, "\mathrm{V}_f", near end] \arrow[dl, "\mathrm{V}_g"] & & \mathrm{V}_Y \arrow[dl, "\mathrm{V}_h"] \\
\mathrm{V}_{X'} \arrow[rr, "\mathrm{V}_{f'}"'] & & \mathrm{V}_{Y'} &
\end{tikzcd}$$

And so we have the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner", very near end] & Y'
\end{tikzcd}$$

in Gph.

Proof. We have $\mathrm{s}_{X'}\,\mathrm{V}_{f'} = \mathrm{E}_{f'}\,\mathrm{s}_{Y'}$ and $\mathrm{t}_{X'}\,\mathrm{V}_{f'} = \mathrm{E}_{f'}\,\mathrm{t}_{Y'}$.

So $f' = (\mathrm{V}_{f'}, \mathrm{E}_{f'})$ is a graph morphism from $X' = (\mathrm{V}_{X'}, \mathrm{E}_{X'}, \mathrm{s}_{X'}, \mathrm{t}_{X'})$ to $Y' = (\mathrm{V}_{Y'}, \mathrm{E}_{Y'}, \mathrm{s}_{Y'}, \mathrm{t}_{Y'})$.

We have $\mathrm{s}_Y\,\mathrm{V}_h = \mathrm{E}_h\,\mathrm{s}_{Y'}$ and $\mathrm{t}_Y\,\mathrm{V}_h = \mathrm{E}_h\,\mathrm{t}_{Y'}$.

So $h = (\mathrm{V}_h, \mathrm{E}_h)$ is a graph morphism from $Y = (\mathrm{V}_Y, \mathrm{E}_Y, \mathrm{s}_Y, \mathrm{t}_Y)$ to $Y' = (\mathrm{V}_{Y'}, \mathrm{E}_{Y'}, \mathrm{s}_{Y'}, \mathrm{t}_{Y'})$.

We have $gf' = (\mathrm{V}_g, \mathrm{E}_g) \cdot (\mathrm{V}_{f'}\,\mathrm{E}_{f'}) = (\mathrm{V}_g \cdot \mathrm{V}_{f'}, \mathrm{E}_g \cdot \mathrm{E}_{f'}) = (\mathrm{V}_{gf'}, \mathrm{E}_{gf'}) = (\mathrm{V}_{fh}, \mathrm{E}_{fh}) = (\mathrm{V}_f \cdot \mathrm{V}_h, \mathrm{E}_f \cdot \mathrm{E}_h) = (\mathrm{V}_f\,, \mathrm{E}_f) \cdot (\mathrm{V}_h\,\mathrm{E}_h) = fh$. ▫

Universal property.

Suppose given a graph Z together with graph morphisms $v : X' \to Z$ and $u : Y \to Z$ such that $fu = gv$.

Since $\mathrm{E}_{Y'}$ is constructed as a pushout, we obtain the unique map $\mathrm{E}_w : \mathrm{E}_{Y'} \to \mathrm{E}_Z$ such that $\mathrm{E}_h\,\mathrm{E}_w = \mathrm{E}_u$ and $\mathrm{E}_{f'}\,\mathrm{E}_w = \mathrm{E}_v$.

Since $\mathrm{V}_{Y'}$ is constructed as a pushout, we obtain the unique map $\mathrm{V}_w : \mathrm{V}_{Y'} \to \mathrm{V}_Z$ such that $\mathrm{V}_h\,\mathrm{V}_w = \mathrm{V}_u$ and $\mathrm{V}_{f'}\,\mathrm{V}_w = \mathrm{V}_v$.

It remains to show that the pair of maps $w := (\mathrm{V}_w, \mathrm{E}_w)$ is in fact a graph morphism.

We have to show that the map w is a graph morphism.

So we have to show that $\mathrm{s}_{Y'}\,\mathrm{V}_w \overset{!}{=} \mathrm{E}_w\,\mathrm{s}_Z$ and that $\mathrm{t}_{Y'}\,\mathrm{V}_w \overset{!}{=} \mathrm{E}_w\,\mathrm{t}_Z$.

We have $\mathrm{E}_{f'}(\mathrm{s}_{Y'}\,\mathrm{V}_w) = \mathrm{s}_{X'}\,\mathrm{V}_{f'}\,\mathrm{V}_w = \mathrm{s}_{X'}\,\mathrm{V}_v = \mathrm{E}_v\,\mathrm{s}_Z = \mathrm{E}_{f'}(\mathrm{E}_w\,\mathrm{s}_Z)$.

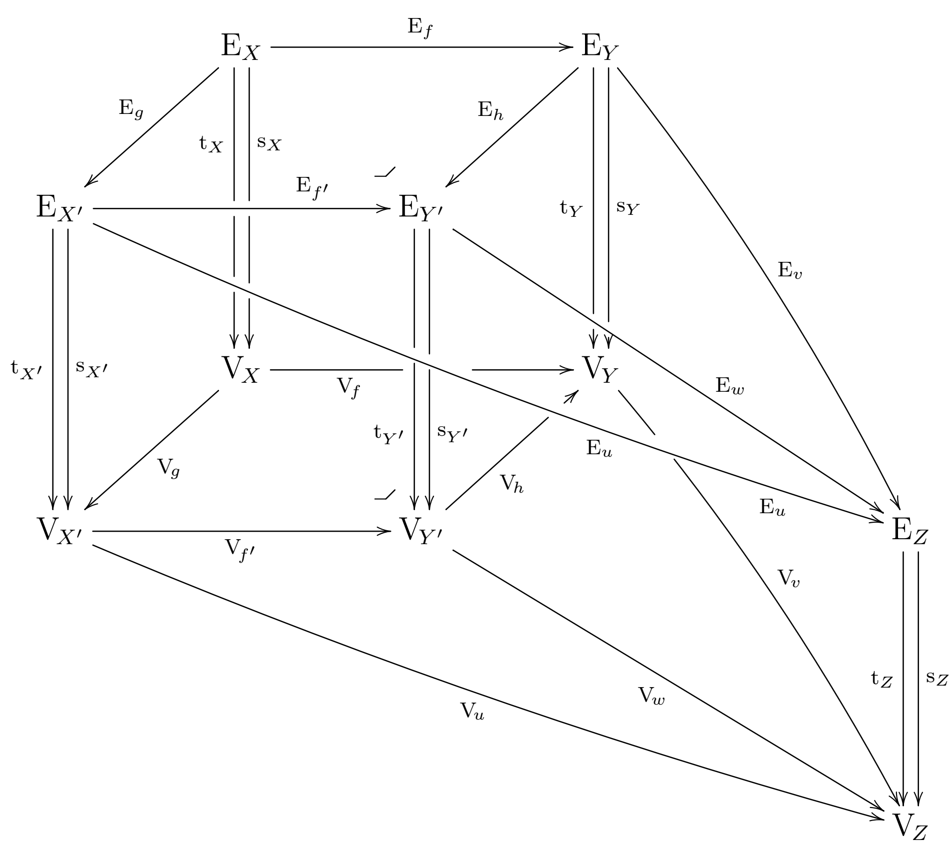
We have $\mathrm{E}_h(\mathrm{s}_{Y'}\,\mathrm{V}_w) = \mathrm{s}_Y\,\mathrm{V}_h\,\mathrm{V}_w = \mathrm{s}_Y\,\mathrm{V}_u = \mathrm{E}_u\,\mathrm{s}_Z = \mathrm{E}_h(\mathrm{E}_w\,\mathrm{s}_Z)$.

This shows that $\mathrm{s}_{Y'}\,\mathrm{V}_w = \mathrm{E}_w\,\mathrm{s}_Z$ by Remark 6.

We have $\mathrm{E}_{f'}(\mathrm{t}_{Y'}\,\mathrm{V}_w) = \mathrm{t}_{X'}\,\mathrm{V}_{f'}\,\mathrm{V}_w = \mathrm{t}_{X'}\,\mathrm{V}_v = \mathrm{E}_v\,\mathrm{t}_Z = \mathrm{E}_{f'}(\mathrm{E}_w\,\mathrm{t}_Z)$.

We have $\mathrm{E}_h(\mathrm{t}_{Y'}\,\mathrm{V}_w) = \mathrm{t}_Y\,\mathrm{V}_h\,\mathrm{V}_w = \mathrm{t}_Y\,\mathrm{V}_u = \mathrm{E}_u\,\mathrm{t}_Z = \mathrm{E}_h(\mathrm{E}_w\,\mathrm{t}_Z)$.

This shows that $\mathrm{t}_{Y'}\,\mathrm{V}_w = \mathrm{E}_w\,\mathrm{t}_Z$ by Remark 6.



So we have obtained the following diagram in Gph.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "v"] & \\
X' \arrow[r, "f'"] \arrow[drr, bend right, "u"'] & Y' \arrow[dr, "w"] & \\
& & Z
\end{tikzcd}$$

Remark 84 Suppose given

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & Y \\
X' &
\end{tikzcd}$$

in Gph with an injective graph morphism $f : X \to X'$.

We let $\mathrm{V}_{Y'} := (\mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}) \sqcup \mathrm{V}_Y$ and $\mathrm{E}_{Y'} := (\mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}) \sqcup \mathrm{E}_Y$.

We let

$$\begin{array}{rrcl}
\mathrm{V}_{a'} : & \mathrm{V}_{X'} & \to & \mathrm{V}_{Y'} = (\mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}) \sqcup \mathrm{V}_Y \\
& v_{X'} & \mapsto & \begin{cases} (2, v_X \, \mathrm{V}_a) & \text{if } v_{X'} = v_X \, \mathrm{V}_f \in \mathrm{V}_{Xf} \text{ for a unique } v_X \in \mathrm{V}_X \\ (1, v_{X'}) & \text{if } v_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf} \end{cases}
\end{array}$$

and

$$\begin{array}{rrcl}
\mathrm{E}_{a'} : & \mathrm{E}_{X'} & \to & \mathrm{E}_{Y'} = (\mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}) \sqcup \mathrm{E}_Y \\
& e_{X'} & \mapsto & \begin{cases} (2, e_X \, \mathrm{E}_a) & \text{if } e_{X'} = e_X \, \mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ (1, e_{X'}) & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \ . \end{cases}
\end{array}$$

We let

$$\begin{array}{rrcl}
\mathrm{V}_g : & \mathrm{V}_Y & \to & \mathrm{V}_{Y'} = (\mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}) \sqcup \mathrm{V}_Y \\
& v_Y & \mapsto & (2, v_Y)
\end{array}$$

and

$$\begin{array}{rrcl}
\mathrm{E}_g : & \mathrm{E}_Y & \to & \mathrm{E}_{Y'} = (\mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}) \sqcup \mathrm{E}_Y \\
& e_Y & \mapsto & (2, e_Y)
\end{array}$$

We have the pushouts

$$\begin{tikzcd}
\mathrm{V}_X \arrow[r, "\mathrm{V}_a"] \arrow[d, "\mathrm{V}_f"'] & \mathrm{V}_Y \arrow[d, "\mathrm{V}_g"] \\
\mathrm{V}_{X'} \arrow[r, "\mathrm{V}_{a'}"'] & \mathrm{V}_{Y'}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{E}_X \arrow[r, "\mathrm{E}_a"] \arrow[d, "\mathrm{E}_f"'] & \mathrm{E}_Y \arrow[d, "\mathrm{E}_g"] \\
\mathrm{E}_{X'} \arrow[r, "\mathrm{E}_{a'}"'] & \mathrm{E}_{Y'} \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

in Set; cf. Remark 38.

We will construct the maps $s_{Y'}$ and $\mathrm{t}_{Y'}$.

Let

$$\begin{array}{rrcl}
\mathrm{s}_{Y'}: & \mathrm{E}_{Y'} = (\mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}) \sqcup \mathrm{E}_Y & \to & (\mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}) \sqcup \mathrm{V}_Y = \mathrm{V}_{Y'} \\
& (1, e_{X'}) & \mapsto & e_{X'}\,\mathrm{s}_{X'}\,\mathrm{V}_{a'} = \begin{cases} (2, v_X\,\mathrm{V}_a) & \text{if } e_{X'}\,\mathrm{s}_{X'} = v_X\,\mathrm{V}_f \in \mathrm{V}_{Xf} \\ & \text{for a unique } v_X \in \mathrm{V}_X \\ (1, e_{X'}\,\mathrm{s}_{X'}) & \text{if } e_{X'}\,\mathrm{s}_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}\ . \end{cases} \\
& (2, e_Y) & \mapsto & (2, e_Y\,\mathrm{s}_Y) = e_Y\,\mathrm{s}_Y\,\mathrm{V}_g
\end{array}$$

Let

$$\begin{array}{rrcl}
\mathrm{t}_{Y'}: & \mathrm{E}_{Y'} = (\mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}) \sqcup \mathrm{E}_Y & \to & (\mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}) \sqcup \mathrm{V}_Y = \mathrm{V}_{Y'} \\
& (1, e_{X'}) & \mapsto & e_{X'}\,\mathrm{t}_{X'}\,\mathrm{V}_{a'} = \begin{cases} (2, v_X\,\mathrm{V}_a) & \text{if } e_{X'}\,\mathrm{t}_{X'} = v_X\,\mathrm{V}_f \in \mathrm{V}_{Xf} \\ & \text{for a unique } v_X \in \mathrm{V}_X \\ (1, e_{X'}\,\mathrm{t}_{X'}) & \text{if } e_{X'}\,\mathrm{t}_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}\ . \end{cases} \\
& (2, e_Y) & \mapsto & (2, e_Y\,\mathrm{t}_Y) = e_Y\,\mathrm{t}_Y\,\mathrm{V}_g
\end{array}$$

In order to show that we have the pushout

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
X' \arrow[r, "a'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

it suffices to show that $\mathrm{E}_{a'}\,\mathrm{s}_{Y'} \overset{!}{=} \mathrm{s}_{X'}\,\mathrm{V}_{a'}$ and $\mathrm{E}_g\,\mathrm{s}_{Y'} \overset{!}{=} \mathrm{s}_Y\,\mathrm{V}_g$, and that $\mathrm{E}_{a'}\,\mathrm{t}_{Y'} \overset{!}{=} \mathrm{t}_{X'}\,\mathrm{V}_{a'}$ and $\mathrm{E}_g\,\mathrm{t}_{Y'} \overset{!}{=} \mathrm{t}_Y\,\mathrm{V}_g$; cf. Construction 83.

We show that $\mathrm{E}_{a'}\,\mathrm{s}_{Y'} \overset{!}{=} \mathrm{s}_{X'}\,\mathrm{V}_{a'}$ and $\mathrm{E}_g\,\mathrm{s}_{Y'} \overset{!}{=} \mathrm{s}_Y\,\mathrm{V}_g$.

For $e_{X'} \in \mathrm{E}_{X'}$ we have $e_{X'}\,\mathrm{E}_{a'} = \begin{cases} (2, e_X\,\mathrm{E}_a) & \text{if } e_{X'} = e_X\,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ (1, e_{X'}) & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}\ . \end{cases}$

So we have

$$e_{X'}\,\mathrm{E}_{a'}\,\mathrm{s}_{Y'} = \begin{cases} (2, e_X\,\mathrm{E}_a)\,\mathrm{s}_{Y'} & \text{if } e_{X'} = e_X\,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ (1, e_{X'})\,\mathrm{s}_{Y'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$$

$$= \begin{cases} (2, e_X\,\mathrm{E}_a\,\mathrm{s}_Y) & \text{if } e_{X'} = e_X\,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'}\,\mathrm{s}_{X'}\,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$$

$= \begin{cases} (2, e_X \,\mathrm{s}_X \,\mathrm{V}_a) & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'} \,\mathrm{s}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} e_X \,\mathrm{s}_X \,\mathrm{V}_f \,\mathrm{V}_{a'} & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'} \,\mathrm{s}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} e_X \,\mathrm{E}_f \,\mathrm{s}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'} \,\mathrm{s}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} e_{X'} \,\mathrm{s}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{Xf} \\ e_{X'} \,\mathrm{s}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= e_{X'} \,\mathrm{s}_{X'} \,\mathrm{V}_{a'}$.

For $e_Y \in \mathrm{E}_Y$ we have $e_Y \,\mathrm{E}_g \,\mathrm{s}_{Y'} = (2, e_Y) \,\mathrm{s}_{Y'} = (2, e_Y \,\mathrm{s}_Y) = e_Y \,\mathrm{s}_Y \,\mathrm{V}_g$.

We show that $\mathrm{E}_{a'} \,\mathrm{t}_{Y'} \overset{!}{=} \mathrm{t}_{X'} \,\mathrm{V}_{a'}$ and $\mathrm{E}_g \,\mathrm{t}_{Y'} \overset{!}{=} \mathrm{t}_Y \,\mathrm{V}_g$.

We have

$e_{X'} \,\mathrm{E}_{a'} \,\mathrm{t}_{Y'} = \begin{cases} (2, e_X \,\mathrm{E}_a) \,\mathrm{t}_{Y'} & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ (1, e_{X'}) \,\mathrm{t}_{Y'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} (2, e_X \,\mathrm{E}_a \,\mathrm{t}_Y) & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} (2, e_X \,\mathrm{t}_X \,\mathrm{V}_a) & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} e_X \,\mathrm{t}_X \,\mathrm{V}_f \,\mathrm{V}_{a'} & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} e_X \,\mathrm{E}_f \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} = e_X \,\mathrm{E}_f \in \mathrm{E}_{Xf} \text{ for a unique } e_X \in \mathrm{E}_X \\ e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= \begin{cases} e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{Xf} \\ e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} & \text{if } e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf} \end{cases}$
$= e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'}$.

For $e_Y \in \mathrm{E}_Y$ we have $e_Y \,\mathrm{E}_g \,\mathrm{t}_{Y'} = (2, e_Y) \,\mathrm{t}_{Y'} = (2, e_Y \,\mathrm{t}_Y) = e_Y \,\mathrm{t}_Y \,\mathrm{V}_g$.

So we have the pushout

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "a'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

□

2.3.2 Coproducts

Definition 85 Suppose given graphs X and Y.

We will define the *coproduct* $X \sqcup Y$ of X and Y.

Note that the graph morphism $\iota_X : \emptyset \to X$ is injective.

By Construction 83 and Remark 84, we may form the pushout

$$\begin{tikzcd} \emptyset \arrow[r, "\iota_Y"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_2"] \\ X \arrow[r, "\iota_1"'] \arrow[ur, phantom, "\lrcorner"] & X \sqcup Y \end{tikzcd}$$

with $\mathrm{V}_{X\sqcup Y} = (\mathrm{V}_X \setminus \mathrm{V}_{\emptyset\iota_X}) \sqcup \mathrm{V}_Y = \mathrm{V}_X \sqcup \mathrm{V}_Y$ and $\mathrm{E}_{X\sqcup Y} = (\mathrm{E}_X \setminus \mathrm{E}_{\emptyset\iota_X}) \sqcup \mathrm{E}_Y = \mathrm{E}_X \sqcup \mathrm{E}_Y$ and with

$$\begin{array}{rrcl} \mathrm{s}_{X\sqcup Y}: & \mathrm{E}_{X\sqcup Y} & \to & \mathrm{V}_{X\sqcup Y} \\ & (1, e_X) & \mapsto & (1, e_X\,\mathrm{s}_X) \\ & (2, e_Y) & \mapsto & (2, e_Y\,\mathrm{s}_Y) \end{array}$$

and

$$\begin{array}{rrcl} \mathrm{t}_{X\sqcup Y}: & \mathrm{E}_{X\sqcup Y} & \to & \mathrm{V}_{X\sqcup Y} \\ & (1, e_X) & \mapsto & (1, e_X\,\mathrm{t}_X) \\ & (2, e_Y) & \mapsto & (2, e_Y\,\mathrm{t}_Y)\;. \end{array}$$

The graph $X \sqcup Y$ is called the *coproduct* of X and Y.

And we have

$$\begin{array}{rrcl} \iota_1: & X & \to & X \sqcup Y \\ \mathrm{V}_{\iota_1}: & \mathrm{V}_X & \to & \mathrm{V}_{X\sqcup Y} = \mathrm{V}_X \sqcup \mathrm{V}_Y \\ & v_X & \mapsto & (1, v_X) \\ \mathrm{E}_{\iota_1}: & \mathrm{E}_X & \to & \mathrm{E}_{X\sqcup Y} = \mathrm{E}_X \sqcup \mathrm{E}_Y \\ & e_X & \mapsto & (1, e_X) \end{array}$$

and

$$\begin{array}{rrcl} \iota_2: & Y & \to & X \sqcup Y \\ \mathrm{V}_{\iota_2}: & \mathrm{V}_Y & \to & \mathrm{V}_{X\sqcup Y} = \mathrm{V}_X \sqcup \mathrm{V}_Y \\ & v_Y & \mapsto & (2, v_Y) \\ \mathrm{E}_{\iota_2}: & \mathrm{E}_Y & \to & \mathrm{E}_{X\sqcup Y} = \mathrm{E}_X \sqcup \mathrm{E}_Y \\ & e_Y & \mapsto & (2, e_Y) \end{array}$$

Given a graph morphism $g : X \sqcup Y \to G$, we also write $g|_X := \iota_1 g$ and $g|_Y := \iota_2 g$.

The following definition is a way to express the universal property of the coproduct.

Definition 86 Suppose given graph morphisms $a : X \to G$ and $b : Y \to G$.

We have the following commutative diagram in Gph; cf. Remminder 82.

$$\begin{tikzcd} \emptyset \arrow[r, "\iota_X"] \arrow[d, "\iota_Y"'] & X \arrow[d, "\iota_1"] \arrow[ddr, bend left, "a"] & \\ Y \arrow[r, "\iota_2"] \arrow[ur, phantom, "\lrcorner"] \arrow[drr, bend right, "b"'] & X \sqcup Y \arrow[dr, "\exists!\binom{a}{b}"] & \\ & & G \end{tikzcd}$$

Since $\iota_1 \binom{a}{b} = a$ and since $\iota_2 \binom{a}{b} = b$, the graph morphism $\binom{a}{b} : X \sqcup Y \to G$ maps as follows.

$$\begin{array}{rrcl}
\mathrm{V}_{\binom{a}{b}} : & \mathrm{V}_{X\sqcup Y} & \to & \mathrm{V}_G \\
& (1, v_X) = v_X \, \mathrm{V}_{\iota_1} & \mapsto & v_X \, \mathrm{V}_a \\
& (2, v_Y) = v_Y \, \mathrm{V}_{\iota_2} & \mapsto & v_Y \, \mathrm{V}_b
\end{array}$$

$$\begin{array}{rrcl}
\mathrm{E}_{\binom{a}{b}} : & \mathrm{E}_{X\sqcup Y} & \to & \mathrm{E}_G \\
& (1, e_X) = e_X \, \mathrm{E}_{\iota_1} & \mapsto & e_X \, \mathrm{E}_a \\
& (2, e_Y) = e_Y \, \mathrm{E}_{\iota_2} & \mapsto & e_Y \, \mathrm{E}_b
\end{array}$$

□

Definition 87 Suppose given graph morphisms $f : X \to X'$ and $g : Y \to Y'$.

We have the graph morphisms $\iota_1 : X \to X \sqcup Y$, $\iota_2 : Y \to X \sqcup Y$, $\iota'_1 : X' \to X' \sqcup Y'$ and $\iota'_2 : Y' \to X' \sqcup Y'$; cf. Definition 85.

Then we have the graph morphism $f \sqcup g := \begin{pmatrix} f\iota'_1 \\ g\iota'_2 \end{pmatrix} : X \sqcup Y \to X' \sqcup Y'$; cf. Definition 86.

So we have $\iota_1(f \sqcup g) = f\iota'_1$ and $\iota_2(f \sqcup g) = g\iota'_2$.

We have

$$\begin{array}{rcl}
\mathrm{V}_{f\sqcup g} : \quad \mathrm{V}_{X\sqcup Y} & \to & \mathrm{V}_{X'\sqcup Y'} \\
(1, v_X) & \mapsto & v_X \, \mathrm{V}_{\iota_1} \, \mathrm{V}_{f\sqcup g} = v_X \, \mathrm{V}_f \, \mathrm{V}_{\iota'_1} = (1, v_X \, \mathrm{V}_f) \\
(2, v_Y) & \mapsto & v_Y \, \mathrm{V}_{\iota_2} \, \mathrm{V}_{f\sqcup g} = v_Y \, \mathrm{V}_g \, \mathrm{V}_{\iota'_2} = (2, v_Y \, \mathrm{V}_g)
\end{array}$$

$$\begin{array}{rcl}
\mathrm{E}_{f\sqcup g} : \quad \mathrm{E}_{X\sqcup Y} & \to & \mathrm{E}_{X'\sqcup Y'} \\
(1, e_X) & \mapsto & e_X \, \mathrm{E}_{\iota_1} \, \mathrm{E}_{f\sqcup g} = e_X \, \mathrm{E}_f \, \mathrm{E}_{\iota'_1} = (1, e_X \, \mathrm{E}_f) \\
(2, e_Y) & \mapsto & e_Y \, \mathrm{E}_{\iota_2} \, \mathrm{E}_{f\sqcup g} = e_Y \, \mathrm{E}_g \, \mathrm{E}_{\iota'_2} = (2, e_Y \, \mathrm{E}_g)
\end{array}$$

$$\begin{tikzcd}
& \emptyset \arrow[rr] \arrow[dl] \arrow[dd, "\mathrm{id}_\emptyset"] \arrow[dr, phantom, "\ulcorner"] & & Y \arrow[dl, "\iota_2"'] \arrow[dd, "g"] \\
X \arrow[rr, "\iota_1"] \arrow[dd, "f"'] & & X \sqcup Y \arrow[dd, "f\sqcup g"] & \\
& \emptyset \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\ulcorner"] & & Y' \arrow[dl, "\iota'_2"] \\
X' \arrow[rr, "\iota'_1"'] & & X' \sqcup Y' &
\end{tikzcd}$$

Remark 88 Suppose graph morphisms $a : X \to Y$, $a' : X' \to Y$ and $f : Y \to Z$.

We have the graph morphism $\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) : X \sqcup X' \to Y$; cf. Definition 86.

We have $\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) f = \left(\begin{smallmatrix} af \\ a'f \end{smallmatrix}\right)$.

Proof. Recall the graph morphisms $\iota_1 : X \to X \sqcup X'$ and $\iota_2 : X' \to X \sqcup X'$ from Definition 86.

Since we have $\iota_1 \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) f = af = \iota_1 \left(\begin{smallmatrix} af \\ af \end{smallmatrix}\right)$ and since we have $\iota_2 \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) f = a'f = \iota_2 \left(\begin{smallmatrix} af \\ af \end{smallmatrix}\right)$, we have $\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) f = \left(\begin{smallmatrix} af \\ a'f \end{smallmatrix}\right)$.

□

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d, "\iota_1"] \arrow[ddr, bend left, "a"] & & \\
X' \arrow[r, "\iota_2"] \arrow[drr, bend right, "a'"'] & X \sqcup X' \arrow[dr, "\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right)"] \arrow[ddrr, bend right, "\left(\begin{smallmatrix} af \\ a'f \end{smallmatrix}\right)"'] & & \\
& & Y \arrow[dr, "f"] & \\
& & & Z
\end{tikzcd}
$$

Remark 89 Suppose given graph morphisms $X \xrightarrow{f} X' \xrightarrow{a'} G$ and $Y \xrightarrow{g} Y' \xrightarrow{b'} G$.

We have $(f \sqcup g) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = \left(\begin{smallmatrix} fa' \\ gb' \end{smallmatrix}\right)$.

Proof. Recall the graph morphisms $\iota_1 : X \to X \sqcup Y$, $\iota_1' : X' \to X' \sqcup Y'$, $\iota_2 : Y \to X \sqcup Y$ and $\iota_2' : Y' \to X' \sqcup Y'$ from Definition 86.

We have $f \sqcup g = \left(\begin{smallmatrix} f\iota_1' \\ \\ g\iota_2' \end{smallmatrix}\right)$; cf. Definition 87.

So we have to show that $(f \sqcup g) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = \left(\begin{smallmatrix} f\iota_1' \\ g\iota_2' \end{smallmatrix}\right) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) \overset{!}{=} \left(\begin{smallmatrix} fa' \\ gb' \end{smallmatrix}\right)$.

It suffices to show that $\iota_1(f \sqcup g) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) \overset{!}{=} \iota_1 \left(\begin{smallmatrix} fa' \\ gb' \end{smallmatrix}\right)$ and that $\iota_2(f \sqcup g) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) \overset{!}{=} \iota_2 \left(\begin{smallmatrix} fa' \\ gb' \end{smallmatrix}\right)$.

We have $\iota_1(f \sqcup g) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = \iota_1 \left(\begin{smallmatrix} f\iota_1' \\ \\ g\iota_2' \end{smallmatrix}\right) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = f\iota_1' \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = fa' = \iota_1 \left(\begin{smallmatrix} fa' \\ gb' \end{smallmatrix}\right)$.

We have $\iota_2(f \sqcup g) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = \iota_2 \left(\begin{smallmatrix} f\iota_1' \\ \\ g\iota_2' \end{smallmatrix}\right) \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = g\iota_2' \left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right) = gb' = \iota_2 \left(\begin{smallmatrix} fa' \\ gb' \end{smallmatrix}\right)$. □

Now we generalizes from the coproduct of two graphs to coproducts indexed with a set.

Definition 90 Suppose given a set I.

Suppose given graphs A_i for $i \in I$.

We define the *coproduct* $\coprod_{i \in I} A_i$ of the graphs A_i as follows.

We let $\mathrm{V}_{\coprod_{i\in I} A_i} := \coprod_{i\in I} \mathrm{V}_{A_i}$ and $\mathrm{E}_{\coprod_{i\in I} A_i} := \coprod_{i\in I} \mathrm{E}_{A_i}$; cf. Definition 32.

Additionally we let

$$\begin{array}{rrcl} \mathrm{s}_{\coprod_{i\in I} A_i} : & \mathrm{E}_{\coprod_{i\in I} A_i} & \to & \mathrm{V}_{\coprod_{i\in I} A_i} \\ & (i,e) & \mapsto & (i, e\,\mathrm{s}_{A_i}) \text{ for } i \in I \end{array}$$

and

$$\begin{array}{rrcl} \mathrm{t}_{\coprod_{i\in I} A_i} : & \mathrm{E}_{\coprod_{i\in I} A_i} & \to & \mathrm{V}_{\coprod_{i\in I} A_i} \\ & (i,e) & \mapsto & (i, e\,\mathrm{t}_{A_i}) \text{ for } i \in I. \end{array}$$

We have the inclusion graph morphism $\iota_k : A_k \to \coprod_{i\in I} A_i$ for $k \in I$ as follows.

$$\begin{array}{rrcl} \iota_k : & A_k & \to & \coprod_{i\in I} A_i \\ \mathrm{V}_{\iota_k} : & \mathrm{V}_{A_k} & \to & \mathrm{V}_{\coprod_{i\in I} A_i} \\ & v_k & \to & (k, v_k) \\ \mathrm{E}_{\iota_k} : & \mathrm{E}_{A_k} & \to & \mathrm{E}_{\coprod_{i\in I} A_i} \\ & e_k & \to & (k, e_k) \end{array}$$

Suppose given an edge $e_k \in A_k$.

We have $e_k\,\mathrm{E}_{\iota_k}\,\mathrm{s}_{\coprod_{i\in I} A_i} = (k, e_k)\,\mathrm{s}_{\coprod_{i\in I} A_i} = (k, e_k\,\mathrm{s}_{A_k}) = e_k\,\mathrm{s}_{A_k}\,\mathrm{V}_{\iota_k}$.

We have $e_k\,\mathrm{E}_{\iota_k}\,\mathrm{t}_{\coprod_{i\in I} A_i} = (k, e_k)\,\mathrm{t}_{\coprod_{i\in I} A_i} = (k, e_k\,\mathrm{t}_{A_k}) = e_k\,\mathrm{t}_{A_k}\,\mathrm{V}_{\iota_k}$.

So the pair of maps $\iota_k = (\mathrm{V}_{\iota_k}, E_{\iota_k})$ is in fact a graph morphism since we have $\mathrm{E}_{\iota_k}\,\mathrm{s}_{\coprod_{i\in I} A_i} = \mathrm{s}_{A_k}\,\mathrm{V}_{\iota_k}$ and $\mathrm{E}_{\iota_k}\,\mathrm{t}_{\coprod_{i\in I} A_i} = \mathrm{t}_{A_k}\,\mathrm{V}_{\iota_k}$.

Definition 91 Let I be a set.

Suppose given graph morphisms $f_i : X_i \to Y$ for $i \in I$.

We have the graph morphism $f =: (f_i)_{i\in I} : \coprod_{i\in I} X_i \to Y$ with

$$\begin{array}{ll} (i, v_{X_i})\,\mathrm{V}_f := v_{X_i}\,\mathrm{V}_{f_i} & \text{for } i \in I \text{ and } v_{X_i} \in \mathrm{V}_{X_i} \\ (i, e_{X_i})\,\mathrm{E}_f := e_{X_i}\,\mathrm{E}_{f_i} & \text{for } i \in I \text{ and } e_{X_i} \in \mathrm{E}_{X_i}\,; \end{array}$$

cf. also Definition 86.

We often abbreviate $(f_i)_i := (f_i)_{i\in I}$.

The pair of maps $(\mathrm{V}_f, \mathrm{E}_f)$ is in fact a graph morphism, since $(i, e_{X_i})\,\mathrm{E}_f\,\mathrm{s}_Y = e_{X_i}\,\mathrm{E}_{f_i}\,\mathrm{s}_Y = e_{X_i}\,\mathrm{s}_{X_i}\,\mathrm{V}_{f_i} = (i, e_{X_i}\,\mathrm{s}_{X_i})\,\mathrm{V}_f = (i, e_{X_i})\,\mathrm{s}_{\coprod_{i\in I} X_i}\,\mathrm{V}_f$ and $(i, e_{X_i})\,\mathrm{E}_f\,\mathrm{t}_Y = e_{X_i}\,\mathrm{E}_{f_i}\,\mathrm{t}_Y = e_{X_i}\,\mathrm{t}_{X_i}\,\mathrm{V}_{f_i} = (i, e_{X_i}\,\mathrm{t}_{X_i})\,\mathrm{V}_f = (i, e_{X_i})\,\mathrm{t}_{\coprod_{i\in I} X_i}\,\mathrm{V}_f$.

Note that $\iota_j f = \iota_j (f_i)_{i\in I} = f_j$ for $j \in I$ and that f is unique with this property; cf. Definition 90.

Remark 92 Suppose given a set I.

Suppose graph morphisms $a_i : X_i \to Y$ for $i \in I$ and $f : Y \to Z$.

We have the graph morphism $(a_i)_i : \coprod_{i\in I} X_i \to Y$; cf. Definition 91.

We have $(a_i)_i \cdot f = (a_i \cdot f)_i$.

Proof. Recall the graph morphism $\iota_k : X_k \to \coprod_{i\in I} X_i$ for $k \in I$ from Definition 91.

Since we have $\iota_k \cdot (a_i)_i \cdot f = a_k \cdot f = \iota_k \cdot (a_i \cdot f)_i$ for $k \in I$, we have $(a_i)_i \cdot f = (a_i \cdot f)_i$.

Definition 93 Suppose given a set I.

Suppose given graph morphisms $f_i : X_i \to Y_i$.

Recall the graph morphisms $\iota_k : X_k \to \coprod_{i\in I} X_i$ and $\iota_k : Y_k \to \coprod_{i\in I} Y_i$ for $k \in I$, from Definition 91.

We define the graph morphism $\coprod_{i\in I} f_i := (f_i \cdot \iota_i)_{i\in I} : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$.

Then we have $\iota_k \cdot \coprod_{i\in I} f_i = \iota_k \cdot (f_i \cdot \iota_i)_{i\in I} = f_k \cdot \iota_k$ for $k \in I$.

Remark 94 Suppose given a set I.

Suppose given graph morphisms $X_i \xrightarrow{f_i} Y_i \xrightarrow{a_i} G$ for $i \in I$.

We have $\left(\coprod_{i\in I} f_i\right) \cdot (a_i)_i = (f_i \cdot a_i)_i$.

Proof. Recall the graph morphisms $\iota_k : Y_k \to \coprod_{i\in I} Y_i$ for $k \in I$ from Definition 91.

We have $\left(\coprod_j f_j\right) \cdot (a_i)_i \overset{\text{Def. 93}}{=} (f_j \cdot \iota_j)_j \cdot (a_i)_i \overset{\text{Rem. 92}}{=} \left(f_j \cdot \iota_j \cdot (a_i)_i\right)_j \overset{\text{Def. 91}}{=} (f_j \cdot a_j)_j$. ▫

Remark 95 Suppose given a set I.

Suppose given a graph morphism $f : G \to H$ and graphs X_i for $i \in I$.

We have the following commutative diagram

$$\begin{tikzcd}
\left(\coprod_{i\in I} X_i, G\right)_{\mathrm{Gph}} \arrow[d, "(\coprod_{i\in I} X_i, f)_{\mathrm{Gph}}"'] & \prod_{i\in I}(X_i, G)_{\mathrm{Gph}} \arrow[l, "\sim", "a"'] \arrow[d, "\prod_{i\in I}(X_i, f)_{\mathrm{Gph}}"] \\
\left(\coprod_{i\in I} X_i, H\right)_{\mathrm{Gph}} & \prod_{i\in I}(X_i, H)_{\mathrm{Gph}} \arrow[l, "b", "\sim"']
\end{tikzcd}$$

in Set with

$$\begin{array}{rrcl} a: & \prod_{i\in I}(X_i, G)_{\mathrm{Gph}} & \xrightarrow{\sim} & \left(\coprod_{i\in I} X_i, G\right)_{\mathrm{Gph}} \\ & (g_i)_{i\in I} & \mapsto & (g_i)_{i\in I} \end{array}$$

and

$$\begin{array}{rrcl} b: & \prod_{i\in I}(X_i, H)_{\mathrm{Gph}} & \xrightarrow{\sim} & \left(\coprod_{i\in I} X_i, H\right)_{\mathrm{Gph}} \\ & (h_i)_{i\in I} & \mapsto & (h_i)_{i\in I}\,. \end{array}$$

In particular, $(\coprod_{i\in I} X_i, f)_{\mathrm{Gph}}$ is bijective if and only if $\prod_{i\in I}(X_i, f)_{\mathrm{Gph}}$ is bijective.

Proof. Because of the universal property of the coproduct, the maps a and b are bijective and we have

$$\begin{array}{rrcl} a^{-1}: & \left(\coprod_{i\in I} X_i, G\right)_{\mathrm{Gph}} & \xrightarrow{\sim} & \prod_{i\in I}(X_i, G)_{\mathrm{Gph}} \\ & g & \mapsto & (\iota_i g)_i \end{array}$$

and

$$b^{-1}: \begin{array}{rcl} \left(\coprod_{i\in I} X_i, H\right)_{\mathrm{Gph}} & \xrightarrow{\sim} & \prod_{i\in I}(X_i, H)_{\mathrm{Gph}} \\ h & \mapsto & (\iota_i h)_i\ . \end{array}$$

So we have

$$\begin{array}{ccccccc} (\coprod_{i\in I} X_i\,, G)_{\mathrm{Gph}} & \xrightarrow{a^{-1}} & \prod_{i\in I}(X_i\,, G)_{\mathrm{Gph}} & \xrightarrow{\prod_{i\in I}(X_i\,,f)_{\mathrm{Gph}}} & \prod_{i\in I}(X_i\,, H)_{\mathrm{Gph}} & \xrightarrow{b} & (\coprod_{i\in I} X_i\,, H)_{\mathrm{Gph}} \\ g & \mapsto & (\iota_i g)_i & \mapsto & (\iota_i g f)_i & \mapsto & gf\ . \end{array}$$

So we have $\left(\coprod_{i\in I} X_i, f\right)_{\mathrm{Gph}} = a^{-1}\cdot \prod_{i\in I}(X_i\,, f)_{\mathrm{Gph}}\cdot b$. $\square$

2.3.3 Pullback of graphs

Reminder 96 Suppose given a quadrangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in Gph. It is called a pullback if (Pullback 1–2) hold; cf. Definition 9.

(Pullback 1) We have $f\cdot h = g\cdot f'$.

(Pullback 2) Suppose given a graph G and graph morphisms $u : G \to Y$ and $v : G \to X'$ such that $u\cdot h = v\cdot f'$. Then there exists a unique graph morphism $w : G \to X$ such that $w\cdot f = u$ and $w\cdot g = v$.

$$\begin{tikzcd} G \arrow[dr, dotted, "\exists! w"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & X \arrow[r, "f"'] \arrow[d, "g"] & Y \arrow[d, "h"] \\ & X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

To indicate that this quadrange is a pullback, we write

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y'\ . \end{tikzcd}$$

Then we also say that f is a pullback of f'.

Construction 97 Suppose given a diagram

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in Gph.

We aim to construct a pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

In particular, we have to construct maps $\mathrm{E}_g : \mathrm{E}_X \to \mathrm{E}_{X'}$ and $\mathrm{E}_f : \mathrm{E}_X \to \mathrm{E}_Y$ and $\mathrm{V}_g : \mathrm{V}_X \to \mathrm{V}_{X'}$ and $\mathrm{V}_f : \mathrm{V}_X \to \mathrm{V}_Y$.

We construct the pullbacks

$$\begin{tikzcd} \mathrm{E}_X \arrow[r, "\mathrm{E}_f"] \arrow[d, "\mathrm{E}_g"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{E}_Y \arrow[d, "\mathrm{E}_h"] \\ \mathrm{E}_{X'} \arrow[r, "\mathrm{E}_{f'}"'] & \mathrm{E}_{Y'} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{V}_X \arrow[r, "\mathrm{V}_f"] \arrow[d, "\mathrm{V}_g"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{V}_Y \arrow[d, "\mathrm{V}_h"] \\ \mathrm{V}_{X'} \arrow[r, "\mathrm{V}_{f'}"'] & \mathrm{V}_{Y'} \end{tikzcd}$$

using Construction 41.

Because of the universal property of the pullback E_X , the map t_X is uniquely existent with respect to $\mathrm{t}_X \, \mathrm{V}_g = \mathrm{E}_g \, \mathrm{t}_{X'}$ and $\mathrm{t}_X \, \mathrm{V}_f = \mathrm{E}_f \, \mathrm{t}_Y$.

Because of the universal property of the pullback E_X , the map s_X is uniquely existent with respect to $\mathrm{s}_X \, \mathrm{V}_g = \mathrm{E}_g \, \mathrm{s}_{X'}$ and $\mathrm{s}_X \, \mathrm{V}_f = \mathrm{E}_f \, \mathrm{s}_Y$.

$$\begin{tikzcd} & \mathrm{E}_X \arrow[rr, "\mathrm{E}_f"] \arrow[dl, "\mathrm{E}_g"'] \arrow[dd, shift right, "\mathrm{t}_X"'] \arrow[dd, shift left, "\mathrm{s}_X"] & & \mathrm{E}_Y \arrow[dl, "\mathrm{E}_h"'] \arrow[dd, shift right, "\mathrm{t}_Y"'] \arrow[dd, shift left, "\mathrm{s}_Y"] \\ \mathrm{E}_{X'} \arrow[rr, "\mathrm{E}_{f'}"] \arrow[dd, shift right, "\mathrm{t}_{X'}"'] \arrow[dd, shift left, "\mathrm{s}_{X'}"] & & \mathrm{E}_{Y'} \arrow[dd, shift right, "\mathrm{t}_{Y'}"'] \arrow[dd, shift left, "\mathrm{s}_{Y'}"] & \\ & \mathrm{V}_X \arrow[rr, "\mathrm{V}_f"] \arrow[dl, "\mathrm{V}_g"] & & \mathrm{V}_Y \\ \mathrm{V}_{X'} \arrow[rr, "\mathrm{V}_{f'}"'] & & \mathrm{V}_{Y'} \arrow[ur, "\mathrm{V}_h"'] & \end{tikzcd}$$

And so we have the pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in Gph.

Proof. We have $\mathrm{s}_X \, \mathrm{V}_g = \mathrm{E}_g \, \mathrm{s}_{X'}$ and $\mathrm{t}_X \, \mathrm{V}_g = \mathrm{E}_g \, \mathrm{t}_{X'}$.

So $g = (\mathrm{V}_g, \mathrm{E}_g)$ is a graph morphism from $X = (\mathrm{V}_X, \mathrm{E}_X, \mathrm{s}_X, \mathrm{t}_X)$ to $X' = (\mathrm{V}_{X'}, \mathrm{E}_{X'}, \mathrm{s}_{X'}, \mathrm{t}_{X'})$.

We have $\mathrm{s}_X \, \mathrm{V}_f = \mathrm{E}_f \, \mathrm{s}_Y$ and $\mathrm{t}_X \, \mathrm{V}_f = \mathrm{E}_f \, \mathrm{t}_Y$.

So $f = (\mathrm{V}_f \, , \mathrm{E}_f)$ is a graph morphism from $X = (\mathrm{V}_X, \mathrm{E}_X, \mathrm{s}_X, \mathrm{t}_X)$ to $Y = (\mathrm{V}_Y, \mathrm{E}_Y, \mathrm{s}_Y, \mathrm{t}_Y)$.

We have $gf' = (\mathrm{V}_g, \mathrm{E}_g) \cdot (\mathrm{V}_{f'} \, \mathrm{E}_{f'}) = (\mathrm{V}_g \cdot \mathrm{V}_{f'}, \mathrm{E}_g \cdot \mathrm{E}_{f'}) = (\mathrm{V}_{gf'}, \mathrm{E}_{gf'}) = (\mathrm{V}_{fh}, \mathrm{E}_{fh}) = (\mathrm{V}_f \cdot \mathrm{V}_h, \mathrm{E}_f \cdot \mathrm{E}_h) = (\mathrm{V}_f \, , \mathrm{E}_f) \cdot (\mathrm{V}_h \, \mathrm{E}_h) = fh$. □

Universal property.

Suppose given a graph Z together with graph morphisms $u : Z \to Y$ and $v : Z \to X'$ such that $uh = vf'$.

Since E_X is constructed as a pullback, we obtain the unique map $\mathrm{E}_w : \mathrm{E}_Z \to \mathrm{E}_X$ such that $\mathrm{E}_w \, \mathrm{E}_f = \mathrm{E}_u$ and $\mathrm{E}_w \, \mathrm{E}_g = \mathrm{E}_v$.

Since V_X is constructed as a pullback, we obtain the unique map $\mathrm{V}_w : \mathrm{V}_Z \to \mathrm{V}_X$ such that $\mathrm{V}_w \, \mathrm{V}_f = \mathrm{V}_u$ and $\mathrm{V}_w \, \mathrm{V}_g = \mathrm{V}_v$.

It remains to show that the pair of maps $w := (\mathrm{V}_w, \mathrm{E}_w)$ is in fact a graph morphism.

We have to show that the map w is a graph morphism.

So we have to show that $\mathrm{s}_Z \, \mathrm{V}_w \overset{!}{=} \mathrm{E}_w \, \mathrm{s}_X$ and that $\mathrm{t}_Z \, \mathrm{V}_w \overset{!}{=} \mathrm{E}_w \, \mathrm{t}_X$.

We have $(\mathrm{s}_Z \, \mathrm{V}_w) \, \mathrm{V}_g = \mathrm{s}_Z \, \mathrm{V}_v = \mathrm{E}_v \, \mathrm{s}_{X'} = \mathrm{E}_w \, \mathrm{E}_g \, \mathrm{s}_{X'} = (\mathrm{E}_w \, \mathrm{s}_X) \, \mathrm{V}_g$.

We have $(s_Z \, \mathrm{V}_w) \, \mathrm{V}_f = \mathrm{s}_Z \, \mathrm{V}_u = \mathrm{E}_u \, \mathrm{s}_Y = \mathrm{E}_w \, \mathrm{E}_f \, \mathrm{s}_Y = (\mathrm{E}_w \, \mathrm{s}_X) \, \mathrm{V}_f$.

This shows that $\mathrm{s}_Z \, \mathrm{V}_w = \mathrm{E}_w \, \mathrm{s}_X$ by Remark 10.

We have $(\mathrm{t}_Z \, \mathrm{V}_w) \, \mathrm{V}_g = \mathrm{t}_Z \, \mathrm{V}_v = \mathrm{E}_v \, \mathrm{t}_{X'} = \mathrm{E}_w \, \mathrm{E}_g \, \mathrm{t}_{X'} = (\mathrm{E}_w \, \mathrm{t}_X) \, \mathrm{V}_g$.

We have $(\mathrm{t}_Z \, \mathrm{V}_w) \, \mathrm{V}_f = \mathrm{t}_Z \, \mathrm{V}_u = \mathrm{E}_u \, \mathrm{t}_Y = \mathrm{E}_w \, \mathrm{E}_f \, \mathrm{t}_Y = (\mathrm{E}_w \, \mathrm{t}_X) \, \mathrm{V}_f$.

This shows that $\mathrm{t}_Z \, \mathrm{V}_w = \mathrm{E}_w \, \mathrm{t}_X$ by Remark 10.

$$
\begin{tikzcd}
\mathrm{E}_Z \arrow[d, shift right, "\mathrm{t}_Z"'] \arrow[d, shift left, "\mathrm{s}_Z"] \arrow[drrr, "\mathrm{E}_w"] \arrow[drrrrr, "\mathrm{E}_u"] \arrow[ddrr, "\mathrm{E}_v"] & & & & & \\
\mathrm{V}_Z \arrow[ddrrrrr, "\mathrm{V}_u"] \arrow[ddrrr, "\mathrm{V}_w"] \arrow[dddrr, "\mathrm{V}_v"] & & & \mathrm{E}_X \arrow[rr, "\mathrm{E}_f"] \arrow[dl, "\mathrm{E}_g"'] \arrow[dd, shift right, "\mathrm{t}_X"'] \arrow[dd, shift left, "\mathrm{s}_X"] \arrow[dr, phantom, "\ulcorner", very near start] & & \mathrm{E}_Y \arrow[dl, "\mathrm{E}_h"'] \arrow[dd, shift right, "\mathrm{t}_Y"'] \arrow[dd, shift left, "\mathrm{s}_Y"] \\
& & \mathrm{E}_{X'} \arrow[rr, "\mathrm{E}_{f'}"] \arrow[dd, shift right, "\mathrm{t}_{X'}"'] \arrow[dd, shift left, "\mathrm{s}_{X'}"] & & \mathrm{E}_{Y'} \arrow[dd, shift right, "\mathrm{t}_{Y'}"'] \arrow[dd, shift left, "\mathrm{s}_{Y'}"] & \\
& & & \mathrm{V}_X \arrow[rr, "\mathrm{V}_f"] \arrow[dl, "\mathrm{V}_g"'] \arrow[dr, phantom, "\ulcorner", very near start] & & \mathrm{V}_Y \\
& & \mathrm{V}_{X'} \arrow[rr, "\mathrm{V}_{f'}"'] & & \mathrm{V}_{Y'} \arrow[ur, "\mathrm{V}_h"'] &
\end{tikzcd}
$$

So we have the following commutative diagram in Gph.

$$
\begin{tikzcd}
Z \arrow[dr, "w"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& X \arrow[r, "f"'] \arrow[d, "g"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "h"] \\
& X' \arrow[r, "f'"'] & Y'
\end{tikzcd}
$$

2.4 Colimit of a countable chain in Gph

Definition 98 Suppose given

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} X_2 \xrightarrow{f_{2,3}} \cdots$$

in Gph.

We define

$$\mathrm{V}_{X_\infty} := \varinjlim_{i \geqslant 0} \mathrm{V}_{X_i} = \{[j, v_j] : j \geqslant 0,\ v_j \in \mathrm{V}_{X_j}\}$$

and

$$\mathrm{E}_{X_\infty} := \varinjlim_{i \geqslant 0} \mathrm{E}_{X_i} = \{[j, e_j] : j \geqslant 0,\ e_j \in \mathrm{E}_{X_j}\}\ ;$$

cf. Definition 42.

Note that $[j, v_j] = [k, v_j\, \mathrm{V}_{f_{j,k}}]$ for $0 \leqslant j \leqslant k$ and $v_j \in \mathrm{V}_{X_j}$.

Note that $[j, e_j] = [k, e_j\, \mathrm{E}_{f_{j,k}}]$ for $0 \leqslant j \leqslant k$ and $e_j \in \mathrm{E}_{X_j}$.

For $k \geqslant 0$, Definition 42.(2) gives the maps

$$\begin{array}{rrcl} \mathrm{V}_{f_{k,\infty}} : & \mathrm{V}_{X_k} & \mapsto & \mathrm{V}_{X_\infty} \\ & v_k & \mapsto & [k, v_k] \\ \mathrm{E}_{f_{k,\infty}} : & \mathrm{E}_{X_k} & \mapsto & \mathrm{E}_{X_\infty} \\ & e_k & \mapsto & [k, e_k]\ . \end{array}$$

We define $\mathrm{s}_{X_\infty} := \varinjlim_{i \geqslant 0} \mathrm{s}_{X_i}$ and $\mathrm{t}_{X_\infty} := \varinjlim_{i \geqslant 0} \mathrm{t}_{X_i}$, which is possible since $\mathrm{s}_{X_i}\, \mathrm{V}_{f_{i,i+1}} = \mathrm{E}_{f_{i,i+1}}\, \mathrm{s}_{X_{i+1}}$ and $\mathrm{t}_{X_i}\, \mathrm{V}_{f_{i,i+1}} = \mathrm{E}_{f_{i,i+1}}\, \mathrm{t}_{X_{i+1}}$ for $i \geqslant 0$; cf. Definition 44.

We have

$$\begin{array}{rrcl} \mathrm{s}_{X_\infty} : & \mathrm{E}_{X_\infty} & \to & \mathrm{V}_{X_\infty} \\ & [k, e_k] & \mapsto & [k, e_k\, \mathrm{s}_{X_k}] \end{array}$$

and

$$\begin{array}{rrcl} \mathrm{t}_{X_\infty} : & \mathrm{E}_{X_\infty} & \to & \mathrm{V}_{X_\infty} \\ & [k, e_k] & \mapsto & [k, e_k\, \mathrm{t}_{X_k}]\ . \end{array}$$

In fact, we have $[k, e_k]\, \mathrm{s}_{X_\infty} = e_k\, \mathrm{E}_{f_{k,\infty}}\, \mathrm{s}_{X_\infty} \overset{\text{Def. 44}}{=} e_k\, \mathrm{s}_{X_k}\, \mathrm{V}_{f_{k,\infty}} = [k, e_k\, \mathrm{s}_{X_k}]$ and $[k, e_k]\, \mathrm{t}_{X_\infty} = e_k\, \mathrm{E}_{f_{k,\infty}}\, \mathrm{t}_{X_\infty} \overset{\text{Def. 44}}{=} e_k\, \mathrm{t}_{X_k}\, \mathrm{V}_{f_{k,\infty}} = [k, e_k\, \mathrm{t}_{X_k}]$.

This defines the graph $X_\infty := (\mathrm{V}_{X_\infty}, \mathrm{E}_{X_\infty}, \mathrm{s}_{X_\infty}, \mathrm{t}_{X_\infty})$.

For $j \geqslant 0$ we have $\mathrm{E}_{f_{j,\infty}}\, \mathrm{s}_{X_\infty} = \mathrm{s}_{X_j}\, \mathrm{V}_{f_{j,\infty}}$ and $\mathrm{E}_{f_{j,\infty}}\, \mathrm{t}_{X_\infty} = \mathrm{t}_{X_j}\, \mathrm{V}_{f_{j,\infty}}$; cf. Definition 44.

So the pair of maps $f_{j,\infty} = (\mathrm{V}_{f_{j,\infty}}, \mathrm{E}_{f_{j,\infty}}) : X_j \to X_\infty$ is a graph morphism.

$$\begin{tikzcd}
\mathrm{V}_{X_0} \arrow[r, "\mathrm{V}_{f_{0,1}}"'] \arrow[rrrr, bend left, "\mathrm{V}_{f_{0,\infty}}"] & \mathrm{V}_{X_1} \arrow[r, "\mathrm{V}_{f_{1,2}}"'] \arrow[rrr, bend left, "\mathrm{V}_{f_{1,\infty}}"] & \mathrm{V}_{X_2} \arrow[r] \arrow[rr, bend left, "\mathrm{V}_{f_{2,\infty}}"] & \cdots & \mathrm{V}_{X_\infty} \\
\mathrm{E}_{X_0} \arrow[u, shift right, "\mathrm{s}_{X_0}"] \arrow[u, shift left, "\mathrm{t}_{X_0}"'] \arrow[r, "\mathrm{E}_{f_{0,1}}"] \arrow[rrrr, bend right, "\mathrm{E}_{f_{0,\infty}}"'] & \mathrm{E}_{X_1} \arrow[u, shift right, "\mathrm{s}_{X_1}"] \arrow[u, shift left, "\mathrm{t}_{X_1}"'] \arrow[r, "\mathrm{E}_{f_{1,2}}"] \arrow[rrr, bend right, "\mathrm{E}_{f_{1,\infty}}"'] & \mathrm{E}_{X_2} \arrow[u, shift right, "\mathrm{s}_{X_2}"] \arrow[u, shift left, "\mathrm{t}_{X_2}"'] \arrow[r] \arrow[rr, bend right, "\mathrm{E}_{f_{2,\infty}}"'] & \cdots & \mathrm{E}_{X_\infty} \arrow[u, shift right, "\mathrm{s}_{X_\infty}"] \arrow[u, shift left, "\mathrm{t}_{X_\infty}"']
\end{tikzcd}$$

We have $f_{i,i+1} f_{i+1,\infty} = f_{i,\infty}$; cf. Definition 42.(2), applied to vertices and edges.

Lemma 99 (universal property) Suppose given graph morphisms $t_j : X_j \to T$ for $j \geqslant 0$ such that $f_{j,j+1} t_{j+1} = t_j$ for $j \geqslant 0$, i.e. $f_{j,k} t_k = t_j$ for $0 \leqslant j \leqslant k$.

$$\begin{tikzcd} X_0 \arrow[r, "f_{0,1}"] \arrow[dr, "t_0"'] & X_1 \arrow[r, "f_{1,2}"] \arrow[d, "t_1"] & X_2 \arrow[r, "f_{2,3}"] \arrow[dl, "t_2"] & \ldots \\ & T & \ldots & \end{tikzcd}$$

Then there exists a unique graph morphism $t_\infty : X_\infty \to T$ such that $f_{j,\infty} t_\infty = t_j$ for $j \geqslant 0$.

So we may define $\varinjlim_{i \geqslant 0} X_i := X_\infty$.

Proof.

Uniqueness.

Suppose given $t', t'' : X_\infty \to T$ such that $f_{j,\infty} t' = t_j$ and $f_{j,\infty} t'' = t_j$ for $j \geqslant 0$.

We have to show that $t' = t''$.

Suppose given $[j, v_j] \in \mathrm{V}_{X_\infty}$. We have to show that $[j, v_j]\, \mathrm{V}_{t'} \overset{!}{=} [j, v_j]\, \mathrm{V}_{t''}$.

In fact, we have $[j, v_j]\, \mathrm{V}_{t'} = v_j\, \mathrm{V}_{f_{j,\infty}}\, \mathrm{V}_{t'} = v_j\, \mathrm{V}_{t_j} = v_j\, \mathrm{V}_{f_{j,\infty}}\, \mathrm{V}_{t''} = [j, v_j]\, \mathrm{V}_{t''}$.

Suppose given $[j, e_j] \in \mathrm{E}_{X_\infty}$. We have to show that $[j, e_j]\, \mathrm{E}_{t'} \overset{!}{=} [j, e_j]\, \mathrm{E}_{t''}$.

In fact, we have $[j, e_j]\, \mathrm{E}_{t'} = e_j\, \mathrm{E}_{f_{j,\infty}}\, \mathrm{E}_{t'} = e_j\, \mathrm{E}_{t_j} = e_j\, \mathrm{E}_{f_{j,\infty}}\, \mathrm{E}_{t''} = [j, e_j]\, \mathrm{E}_{t''}$.

Existence.

We have the following commutative diagram.

$$\begin{tikzcd} \mathrm{V}_{X_0} \arrow[r, "\mathrm{V}_{f_{0,1}}"] \arrow[dr, "\mathrm{V}_{t_0}"'] & \mathrm{V}_{X_1} \arrow[r, "\mathrm{V}_{f_{1,2}}"] \arrow[d, "\mathrm{V}_{t_1}"] & \mathrm{V}_{X_2} \arrow[r, "\mathrm{V}_{f_{2,3}}"] \arrow[dl, "\mathrm{V}_{t_2}"] & \ldots \\ & \mathrm{V}_T & \ldots & \end{tikzcd}$$

So because of the universal property of V_{X_∞} there exists a unique map $\mathrm{V}_{t_\infty} : \mathrm{V}_{X_\infty} \to \mathrm{V}_T$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{V}_{X_0} \arrow[r, "\mathrm{V}_{f_{0,1}}"'] \arrow[rrrr, bend left, "\mathrm{V}_{f_{0,\infty}}"] \arrow[dr, "\mathrm{V}_{t_0}"'] & \mathrm{V}_{X_1} \arrow[r, "\mathrm{V}_{f_{1,2}}"'] \arrow[rrr, bend left, "\mathrm{V}_{f_{1,\infty}}"] \arrow[d, "\mathrm{V}_{t_1}"] & \mathrm{V}_{X_2} \arrow[r, "\mathrm{V}_{f_{2,3}}"'] \arrow[rr, bend left, "\mathrm{V}_{f_{2,\infty}}"] \arrow[dl, "\mathrm{V}_{t_2}"] & \ldots & \mathrm{V}_{X_\infty} \arrow[dlll, bend left, "\mathrm{V}_{t_\infty}"] \\ & \mathrm{V}_T & \ldots & & \end{tikzcd}$$

We have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{E}_{X_0} \arrow[r, "\mathrm{E}_{f_{0,1}}"] \arrow[dr, "\mathrm{E}_{t_0}"'] & \mathrm{E}_{X_1} \arrow[r, "\mathrm{E}_{f_{1,2}}"] \arrow[d, "\mathrm{E}_{t_1}"] & \mathrm{E}_{X_2} \arrow[r, "\mathrm{E}_{f_{2,3}}"] \arrow[dl, "\mathrm{E}_{t_2}\ \cdots"] & \ldots \\
& \mathrm{E}_T & &
\end{tikzcd}$$

So because of the universal property of E_{X_∞} there exists a unique map $\mathrm{E}_{t_\infty} : \mathrm{E}_{X_\infty} \to \mathrm{E}_T$ such that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{E}_{X_0} \arrow[r, "\mathrm{E}_{f_{0,1}}"'] \arrow[rrrr, bend left, "\mathrm{E}_{f_{0,\infty}}"] \arrow[dr, "\mathrm{E}_{t_0}"'] & \mathrm{E}_{X_1} \arrow[r, "\mathrm{E}_{f_{1,2}}"'] \arrow[rrr, bend left, "\mathrm{E}_{f_{1,\infty}}"] \arrow[d, "\mathrm{E}_{t_1}"] & \mathrm{E}_{X_2} \arrow[r, "\mathrm{E}_{f_{2,3}}"'] \arrow[rr, bend left, "\mathrm{E}_{f_{2,\infty}}"] \arrow[dl, "\mathrm{E}_{t_2}\ \cdots"] & \cdots & \mathrm{E}_{X_\infty} \arrow[dlll, bend left, "\mathrm{E}_{t_\infty}"] \\
& \mathrm{E}_T & & &
\end{tikzcd}$$

We let

$$\begin{array}{rccl}
t_\infty : & X_\infty & \to & T \\
\mathrm{V}_{t_\infty} : & \mathrm{V}_{X_\infty} & \mapsto & \mathrm{V}_T \\
& [j, v_j] & \mapsto & v_j \, \mathrm{V}_{t_j} \\
\mathrm{E}_{t_\infty} : & \mathrm{E}_{X_\infty} & \mapsto & \mathrm{E}_T \\
& [j, e_j] & \mapsto & e_j \, \mathrm{E}_{t_j}
\end{array}$$

We show that these maps are well-defined.

Suppose given $(j, v_j), (k, v_k) \in \bigsqcup_{i \geqslant 0} \mathrm{V}_{X_i}$ such that $[j, v_j] = [k, v_k]$.

Then $v_j \, \mathrm{V}_{f_{j,m}} = v_k \, \mathrm{V}_{f_{k,m}}$ for some $m \geqslant \max\{j, k\}$.

We have to show that $v_j \, \mathrm{V}_{t_j} \overset{!}{=} v_k \, \mathrm{V}_{t_k}$.

In fact, we have $v_j \, \mathrm{V}_{t_j} = v_j \, \mathrm{V}_{f_{j,m} t_m} = v_k \, \mathrm{V}_{f_{k,m} t_m} = v_k \, \mathrm{V}_{t_k}$.

Suppose given $v_j \in \mathrm{V}_{X_j}$. For $j \geqslant 0$ we have $v_j \, \mathrm{V}_{f_{j,\infty}} \, \mathrm{V}_{t_\infty} = [j, v_j] \, \mathrm{V}_{t_\infty} \overset{\text{Def. } \mathrm{V}_{t_\infty}}{=} v_j \, \mathrm{V}_{t_j}$.

So we have $\mathrm{V}_{f_{j,\infty}} \, \mathrm{V}_{t_\infty} = \mathrm{V}_{t_j}$.

Suppose given $(j, e_j), (k, e_k) \in \bigsqcup_{i \geqslant 0} \mathrm{E}_{X_i}$ such that $[j, e_j] = [k, e_k]$.

Then $e_j \, \mathrm{E}_{f_{j,m}} = e_k \, \mathrm{E}_{f_{k,m}}$ for some $m \geqslant \max\{j, k\}$.

We have to show that $e_j \, \mathrm{E}_{t_j} \overset{!}{=} e_k \, \mathrm{E}_{t_k}$.

In fact, we have $e_j \, \mathrm{E}_{t_j} = e_j \, \mathrm{E}_{f_{j,m} t_m} = e_k \, \mathrm{E}_{f_{k,m} t_m} = e_k \, \mathrm{E}_{t_k}$.

Suppose given $e_j \in \mathrm{E}_{X_j}$. For $j \geqslant 0$ we have $e_j \, \mathrm{E}_{f_{j,\infty}} \, \mathrm{E}_{t_\infty} = [j, e_j] \, \mathrm{E}_{t_\infty} \overset{\text{Def. } \mathrm{E}_{t_\infty}}{=} e_j \, \mathrm{E}_{t_j}$.

So we have $\mathrm{E}_{f_{j,\infty}}\,\mathrm{E}_{t_\infty} = \mathrm{E}_{t_j}$.

To show that $t_\infty := (\mathrm{V}_{t_\infty}, \mathrm{E}_{t_\infty})$ is the unique graph morphism such that

$$\begin{tikzcd}
X_0 \arrow[r, "f_{0,1}"'] \arrow[rrrr, bend left, "f_{0,\infty}"] \arrow[dr, "t_0"'] & X_1 \arrow[r, "f_{1,2}"'] \arrow[rrr, bend left, "f_{0,\infty}"] \arrow[d, "t_1"] & X_2 \arrow[r, "f_{2,3}"'] \arrow[rr, bend left, "f_{0,\infty}"] \arrow[dl, "t_2"] & \cdots & X_\infty \arrow[dlll, bend left, "t_\infty"] \\
& T & \cdots & &
\end{tikzcd}$$

is commutative, it remains to show that t_∞ actually is a graph morphism.

We have to show that $\mathrm{E}_{t_\infty}\,\mathrm{s}_T \overset{!}{=} \mathrm{s}_{X_\infty}\,\mathrm{V}_{t_\infty}$ and that $\mathrm{E}_{t_\infty}\,\mathrm{t}_T \overset{!}{=} \mathrm{t}_{X_\infty}\,\mathrm{V}_{t_\infty}$.

Suppose given an edge $e_\infty = [j, e_j] \in \mathrm{E}_{X_\infty}$ with $j \geqslant 0$ and $e_j \in \mathrm{E}_{X_j}$.

We have $e_\infty\,\mathrm{E}_{t_\infty}\,\mathrm{s}_T = [j, e_j]\,\mathrm{E}_{t_\infty}\,\mathrm{s}_T = e_j\,\mathrm{E}_{t_j}\,\mathrm{s}_T = e_j\,\mathrm{s}_{X_j}\,\mathrm{V}_{t_j} = [j, e_j\,\mathrm{s}_{X_j}]\,\mathrm{V}_{t_\infty} = [j, e_j]\,\mathrm{s}_{X_\infty}\,\mathrm{V}_{t_\infty} = e_\infty\,\mathrm{s}_{X_\infty}\,\mathrm{V}_{t_\infty}$.

And we have $e_\infty\,\mathrm{E}_{t_\infty}\,\mathrm{t}_T = [j, e_j]\,\mathrm{E}_{t_\infty}\,\mathrm{t}_T = e_j\,\mathrm{E}_{t_j}\,\mathrm{t}_T = e_j\,\mathrm{t}_{X_j}\,\mathrm{V}_{t_j} = [j, e_j\,\mathrm{t}_{X_j}]\,\mathrm{V}_{t_\infty} = [j, e_j]\,\mathrm{t}_{X_\infty}\,\mathrm{V}_{t_\infty} = e_\infty\,\mathrm{t}_{X_\infty}\,\mathrm{V}_{t_\infty}$. ▫

Remark 100 Suppose given graph morphisms w', $w'' : X_\infty \to T$ such that $f_{k,\infty}w' = f_{k,\infty}w''$ for $k \geqslant 0$.

Then $w' = w''$.

We say that for $k \geqslant 0$ the graph morphisms $f_{k,\infty}$ are *collectively epimorphic*.

Proof. Both graph morphisms w' and w'' are induced by $(f_{k,\infty}w')_{k\geqslant 0} = (f_{k,\infty}w'')_{k\geqslant 0}$.

By the uniqueness in the universal property, it follows that $w' = w''$; cf. Lemma 99. ▫

Lemma 101 Suppose given a finite graph G, i.e. the sets V_G and E_G are finite.

Suppose given

$$\begin{tikzcd}
X_0 \arrow[r, "f_{0,1}"'] \arrow[rrrr, bend left, "f_{0,\infty}"] & X_1 \arrow[r, "f_{1,2}"'] \arrow[rrr, bend left, "f_{1,\infty}"] & X_2 \arrow[r, "f_{2,3}"'] \arrow[rr, bend left, "f_{2,\infty}"] & \cdots & X_\infty
\end{tikzcd}$$

(1) Suppose given a graph morphism $w : G \to X_\infty$. Then there exist $m \geqslant 0$ and a graph morphism $\hat{w} : G \to X_m$ such that $\hat{w} f_{m,\infty} = w$.

$$\begin{tikzcd}
X_m \arrow[r, "f_{m,\infty}"] & X_\infty \\
& G \arrow[ul, "\hat{w}"] \arrow[u, "w"']
\end{tikzcd}$$

(2) Suppose given graph morphisms $w' : G \to X_\infty$ and $w'' : G \to X_\infty$. Then there exist $m \geqslant 0$ and $\hat{w}' : G \to X_m$ and $\hat{w}'' : G \to X_m$ such that we have $\hat{w}' f_{m,\infty} = w'$ and $\hat{w}'' f_{m,\infty} = w''$.

$$\begin{tikzcd} X_m \arrow[r, "f_{m,\infty}"] & X_\infty \\ & G \arrow[ul, shift left, "\hat{w}''"'] \arrow[ul, shift right, "\hat{w}'"] \arrow[u, shift left, "w'"] \arrow[u, shift right, "w''"'] \end{tikzcd}$$

Proof.

Ad (1). We will construct $\hat{w}$.

Step 1. We choose $j \geqslant 0$ such that $e\,\mathrm{E}_w \in \mathrm{E}_{X_j f_{j,\infty}}$ and $v\,\mathrm{V}_w \in \mathrm{V}_{X_j f_{j,\infty}}$ for $e \in \mathrm{E}_G$, $v \in \mathrm{V}_G$.

This is possible since the graph G is finite; cf. Definition 98.

Step 2. For $e \in \mathrm{E}_G$ we choose $e' \in \mathrm{E}_{X_j}$ such that $[j, e'] = e\,\mathrm{E}_w$.

Note that $[j, e'] = e'\,\mathrm{E}_{f_{j,\infty}}$.

For $v \in \mathrm{V}_G$ we choose $v' \in \mathrm{V}_{X_j}$ such that $[j, v'] = v\,\mathrm{V}_w$.

Note that $[j, v'] = v'\,\mathrm{V}_{f_{j,\infty}}$.

Step 3. Let $e \in \mathrm{E}_G$.

We have $[j, e'\,\mathrm{s}_{X_j}] \overset{\text{Def. 98}}{=} [j, e']\,\mathrm{s}_{X_\infty} = e\,\mathrm{E}_w\,\mathrm{s}_{X_\infty} = e\,\mathrm{s}_G\,\mathrm{V}_w = [j, (e\,\mathrm{s}_G)'].$

So there exists $m_{e,1} \geqslant j$ such that $e'\,\mathrm{s}_{X_j}\,\mathrm{V}_{f_{j,m_{e,1}}} = (e\,\mathrm{s}_G)'\,\mathrm{V}_{f_{j,m_{e,1}}}$.

We have $[j, e'\,\mathrm{t}_{X_j}] \overset{\text{Def. 98}}{=} [j, e']\,\mathrm{t}_{X_\infty} = e\,\mathrm{E}_w\,\mathrm{t}_{X_\infty} = e\,\mathrm{t}_G\,\mathrm{V}_w = [j, (e\,\mathrm{t}_G)'].$

So there exists $m_{e,2} \geqslant j$ such that $e'\,\mathrm{t}_{X_j}\,\mathrm{V}_{f_{j,m_{e,2}}} = (e\,\mathrm{t}_G)'\,\mathrm{V}_{f_{j,m_{e,2}}}$.

Step 4. Let $m := \max(\{m_{e,1} : e \in \mathrm{E}_G\} \cup \{m_{e,2} : e \in \mathrm{E}_G\})$. This is existent since the set E_G is finite.

Then $e'\,\mathrm{s}_{X_j}\,\mathrm{V}_{f_{j,m}} = (e\,\mathrm{s}_G)'\,\mathrm{V}_{f_{j,m}}$ and $e'\,\mathrm{t}_{X_j}\,\mathrm{V}_{f_{j,m}} = (e\,\mathrm{t}_G)'\,\mathrm{V}_{f_{j,m}}$ for $e \in \mathrm{E}_G$.

This follows by application of $\mathrm{V}_{f_{m_{e,1},m}}$ respectively $\mathrm{V}_{f_{m_{e,2},m}}$ to the equations in Step 3.

Step 5. We have the pair of maps

$$\begin{array}{rccl} \hat{w}: & G & \to & X_m \\ \mathrm{E}_{\hat{w}}: & e & \mapsto & e'\,\mathrm{E}_{f_{j,m}} \\ \mathrm{V}_{\hat{w}}: & v & \mapsto & v'\,\mathrm{V}_{f_{j,m}} \;. \end{array}$$

Step 6. The pair of maps $\hat{w} = (\mathrm{V}_{\hat{w}}, \mathrm{E}_{\hat{w}}) : G \to X_m$ is a graph morphism.

Suppose given $e \in \mathrm{E}_G$.

We have to show that $e\,\mathrm{E}_{\hat{w}}\,\mathrm{s}_{X_m} \overset{!}{=} e\,\mathrm{s}_G\,\mathrm{V}_{\hat{w}}$ and that $e\,\mathrm{E}_{\hat{w}}\,\mathrm{t}_{X_m} \overset{!}{=} e\,\mathrm{t}_G\,\mathrm{V}_{\hat{w}}$.

We have $e\,\mathrm{E}_{\hat{w}}\,\mathrm{s}_{X_m} \overset{\text{Step 5}}{=} e'\,\mathrm{E}_{f_{j,m}}\,\mathrm{s}_{X_m} = e'\,\mathrm{s}_{X_j}\,\mathrm{V}_{f_{j,m}} \overset{\text{Step 4}}{=} (e\,\mathrm{s}_G)'\,\mathrm{V}_{f_{j,m}} \overset{\text{Step 5}}{=} e\,\mathrm{s}_G\,\mathrm{V}_{\hat{w}}$.

We have $e\,\mathrm{E}_{\hat{w}}\,\mathrm{t}_{X_m} \overset{\text{Step 5}}{=} e'\,\mathrm{E}_{f_{j,m}}\,\mathrm{t}_{X_m} = e'\,\mathrm{t}_{X_j}\,\mathrm{V}_{f_{j,m}} \overset{\text{Step 4.}}{=} (e\,\mathrm{t}_G)'\,\mathrm{V}_{f_{j,m}} \overset{\text{Step 5}}{=} e\,\mathrm{t}_G\,\mathrm{V}_{\hat{w}}$.

Step 7. We have to show that $\hat{w} \cdot f_{m,\infty} \overset{!}{=} w$.

Suppose given $e \in \mathrm{E}_G$.

We have $e\,\mathrm{E}_{\hat{w} f_{m,\infty}} = e\,\mathrm{E}_{\hat{w}}\,\mathrm{E}_{f_{m,\infty}} \overset{\text{Step 5}}{=} e'\,\mathrm{E}_{f_{j,m}}\,\mathrm{E}_{f_{m,\infty}} = e'\,\mathrm{E}_{f_{j,\infty}} = [j, e'] \overset{\text{Step 2}}{=} e\,\mathrm{E}_w$.

Suppose given $v \in \mathrm{V}_G$.

We have $v\,\mathrm{V}_{\hat{w} f_{m,\infty}} = v\,\mathrm{V}_{\hat{w}}\,\mathrm{V}_{f_{m,\infty}} \overset{\text{Step 5}}{=} v'\,\mathrm{V}_{f_{j,m}}\,\mathrm{V}_{f_{m,\infty}} = v'\,\mathrm{V}_{f_{j,\infty}} = [j, v'] \overset{\text{Step 2}}{=} v\,\mathrm{V}_w$. ▫

Ad (2). By (1) there exist $m', m'' \geqslant 0$ and graph morphisms $\tilde{w}' : G \to X_{m'}$ and $\tilde{w}'' : G \to X_{m''}$ such that $\tilde{w}' f_{m',\infty} = w'$ and $\tilde{w}'' f_{m'',\infty} = w''$.

Let $m := \max\{m', m''\}$.

Let $\hat{w}' := \tilde{w}' f_{m',m} : G \to X_m$ and let $\hat{w}'' := \tilde{w}'' f_{m'',m} : G \to X_m$.

Then we have $\hat{w}' f_{m,\infty} = \tilde{w}' f_{m',m} f_{m,\infty} = \tilde{w}' f_{m',\infty} = w'$.

And we have $\hat{w}'' f_{m,\infty} = \tilde{w}'' f_{m'',m} f_{m,\infty} = \tilde{w}'' f_{m'',\infty} = w''$.

Lemma 102 Suppose given $A \subseteq \mathrm{Mor}(\mathrm{Gph})$ and

$$\begin{tikzcd}
X_0 \arrow[r, "g_{0,1}"'] \arrow[rrrr, bend left=50, "g_{0,\infty}"] & X_1 \arrow[r, "g_{1,2}"'] \arrow[rrr, bend left=45, "g_{1,\infty}"] & X_2 \arrow[r, "g_{2,3}"'] \arrow[rr, bend left=45, "g_{2,\infty}"] & \cdots & X_\infty := \varinjlim_{i \geqslant 0} X_i
\end{tikzcd}$$

in Gph such that $g_{i,i+1} \in {}^{\boxslash}A$ for $i \geqslant 0$.

Then the graph morphism $g_{0,\infty} : X_0 \to X_\infty$ is in ${}^{\boxslash}A$.

Proof. Suppose given

$$\begin{tikzcd}
X_0 \arrow[r, "a"] \arrow[d, "g_{0,\infty}"'] & Y \arrow[d, "h"] \\
X_\infty \arrow[r, "b"'] & Z ,
\end{tikzcd}$$

where $h \in A$.

We have to show that $g_{0,\infty} \boxslash h$, i.e. that there exists a graph morphism $c : X_\infty \to Y$ such that $g_{0,\infty} c = a$ and $ch = b$.

Let $c_0 := a$.

Since the graph morphism $g_{0,1} : X_0 \to X_1$ is in ${}^{\boxslash}A$ and since we have $g_{0,1} \cdot g_{1,\infty} b = g_{0,\infty} b = ah = c_0 h$, we may choose a graph morphism $c_1 : X_1 \to Y$ such that $g_{0,1} c_1 = c_0 = a$ and $c_1 h = g_{1,\infty} b$.

Since the graph morphism $g_{1,2} : X_1 \to X_2$ is in ${}^{\boxslash}A$ and since we have $g_{1,2} \cdot g_{2,\infty} b = g_{1,\infty} b = c_1 h$, we may choose a graph morphism $c_2 : X_2 \to Y$ such that $g_{1,2} c_2 = c_1$ and $c_2 h = g_{2,\infty} b$.

Since the graph morphism $g_{2,3} : X_2 \to X_3$ is in ${}^{\boxslash}A$ and since we have $g_{2,3} \cdot g_{3,\infty} b = g_{2,\infty} b = c_2 h$, we may choose a graph morphism $c_3 : X_3 \to Y$ such that $g_{2,3} c_3 = c_2$ and $c_3 h = g_{3,\infty} b$.

Etc.

In a recursive way we may choose a graph morphism $c_k : X_k \to Y$ such that $g_{k-1,k}c_k = c_{k-1}$ and $c_k h = g_{k,\infty} b$ for $k \geqslant 0$.

We obtain $g_{0,k}c_k = g_{0,1} \cdot g_{1,2} \cdot \ldots \cdot g_{k-1,k}c_k = c_0$ for $k \geqslant 0$.

Because of the universal property of the colimit in Gph there exists a unique graph morphism $c : X_\infty \to Y$ such that $g_{k,\infty}c = c_k$ for $k \geqslant 0$; cf. Lemma 99.

$$\begin{tikzcd}
X_0 \arrow[rr, "a=c_0"] \arrow[d, "g_{0,1}"] \arrow[dddd, bend right=50, "g_{0,\infty}"'] & & Y \arrow[dddd, "h"] \\
X_1 \arrow[urr, "c_1"] \arrow[d, "g_{1,2}"] \arrow[ddd, bend right=40, "g_{1,\infty}"'] & & \\
X_2 \arrow[uurr, "c_2"] \arrow[d, "g_{2,3}"] \arrow[dd, bend right=20, "g_{2,\infty}"'] & & \\
\vdots & & \\
X_\infty \arrow[uuuurr, "c"] \arrow[rr, "b"'] & & Z
\end{tikzcd}$$

So we have

$$\begin{tikzcd}
X_0 \arrow[r, "a=c_0"] \arrow[d, "g_{0,\infty}"'] & Y \arrow[d, "h"] \\
X_\infty \arrow[ur, "c"] \arrow[r, "b"'] & Z
\end{tikzcd}$$

such that $g_{0,\infty}c = c_0 = a$.

We have to show that $ch \overset{!}{=} b$.

So we have $g_{k,\infty}ch = c_k h = g_{k,\infty}b$ for $k \geqslant 0$.

Now we deduce $ch = b$ by Remark 100. $\square$

Definition 103 Suppose given a countable chain of subgraphs $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y$ of Y.

Let $\mathrm{V}_{Y_\infty} := \bigcup_{i \geqslant 0} \mathrm{V}_{Y_i} \subseteq \mathrm{V}_Y$ and $\mathrm{E}_{Y_\infty} := \bigcup_{i \geqslant 0} \mathrm{E}_{Y_i} \subseteq \mathrm{E}_Y$.

Let $\mathrm{s}_{Y_\infty} := \mathrm{s}_Y |_{\mathrm{E}_{Y_\infty}}^{\mathrm{V}_{Y_\infty}}$ and $\mathrm{t}_{Y_\infty} = \mathrm{t}_Y |_{\mathrm{E}_{Y_\infty}}^{\mathrm{V}_{Y_\infty}}$.

Then the graph $Y_\infty := \bigcup_{i \geqslant 0} Y_i := (\mathrm{V}_{Y_\infty}, \mathrm{E}_{Y_\infty}, \mathrm{s}_{Y_\infty}, \mathrm{t}_{Y_\infty})$ is a subgraph of Y; cf. Definition 47.(1).

In fact, suppose given an edge $e \in \mathrm{E}_{Y_\infty} = \bigcup_{i \geqslant 0} \mathrm{E}_{Y_i}$. There exists $i \geqslant 0$ such that $e \in \mathrm{E}_{Y_i}$.

We have $e\,\mathrm{s}_Y = e\,\mathrm{s}_{Y_i} \in \mathrm{V}_{Y_i} \subseteq \mathrm{V}_{Y_\infty}$ and $e\,\mathrm{t}_Y = e\,\mathrm{t}_{Y_i} \in \mathrm{V}_{Y_i} \subseteq \mathrm{V}_{Y_\infty}$. $\square$

Remark 104 Suppose given a graph Y.

Suppose given a countable chain of subgraphs $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y$ such that $Y_i \subseteq Y$ is a full subgraph for $i \in \mathbb{Z}_{\geqslant 0}$.

Then $\bigcup_{i\in\mathbb{Z}_{\geqslant 0}} Y_i \subseteq Y$ is a full subgraph; cf. Definition 103.

Proof. Suppose given vertices $v, w \in \mathrm{V}_{\bigcup_{i\in\mathbb{Z}_{\geqslant 0}} Y_i} = \bigcup_{i\in\mathbb{Z}_{\geqslant 0}} \mathrm{V}_{Y_i}$.

Suppose given an edge $e \in Y(v, w)$.

We have to show that $e \overset{!}{\in} \mathrm{E}_{\bigcup_{i\in\mathbb{Z}_{\geqslant 0}} Y_i} = \bigcup_{i\in\mathbb{Z}_{\geqslant 0}} \mathrm{E}_{Y_i}$.

We can choose $j \in \mathbb{Z}_{\geqslant 0}$ such that $v \in Y_j$ and $w \in Y_j$.

The subgraph $Y_j \subseteq Y$ is a full subgraph.

So we have $e \in \mathrm{E}_{Y_j} \subseteq \bigcup_{i\in\mathbb{Z}_{\geqslant 0}} \mathrm{E}_{Y_i}$. □

Remark 105 Suppose given a countable chain of subgraphs $X = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y$ of Y.

The graph $Y_\infty = \bigcup_{i\geqslant 0} Y_i \subseteq Y$ is a subgraph of Y; cf. Definition 103.

Then the graph Y_∞ is a colimit of the countable chain of subgraphs $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y$; cf. Definition 42.

Proof. First, we abbreviate $\kappa_{i,j} := \mathrm{id}_{Y_j}|_{Y_i} : Y_i \to Y_j$ for $0 \leqslant i \leqslant j$.

Suppose given $i \geqslant 0$. Since the graph Y_i is a subgraph of $Y_\infty = \bigcup_{i\geqslant 0} Y_i$, we have the graph morphisms $\kappa_{i,\infty} := \mathrm{id}_{Y_\infty}|_{Y_i} : Y_i \to Y_\infty$.

We have

$$\begin{tikzcd}
Y_0 \arrow[r, "\kappa_{0,1}"'] \arrow[rrrr, bend left, "\kappa_{0,\infty}"] & Y_1 \arrow[r, "\kappa_{1,2}"'] \arrow[rrr, bend left, "\kappa_{1,\infty}"] & Y_2 \arrow[r, "\kappa_{2,3}"'] \arrow[rr, bend left, "\kappa_{2,\infty}"] & \cdots & Y_\infty
\end{tikzcd}$$

We show that the graph Y_∞ is a colimit of the chain of subgraphs $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y$ with respect to $\kappa_{i,\infty} : Y_i \to Y_\infty$ for $i \geqslant 0$.

Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd}
Y_0 \arrow[r, "\kappa_{0,1}"'] \arrow[rrrr, bend left, "\kappa_{0,\infty}"] \arrow[drrrr, bend right, "f_0"'] & Y_1 \arrow[r, "\kappa_{1,2}"'] \arrow[rrr, bend left, "\kappa_{1,\infty}"] \arrow[drrr, bend right, "f_1"'] & Y_2 \arrow[r, "\kappa_{2,3}"'] \arrow[rr, bend left, "\kappa_{2,\infty}"] \arrow[drr, "f_2"'] & \cdots & Y_\infty \\
 & & & & Z
\end{tikzcd}$$

We have to show that there exists a unique graph morphism $f_\infty : Y_\infty \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
Y_0 \arrow[r, "\kappa_{0,1}"'] \arrow[rrrr, bend left, "\kappa_{0,\infty}"] \arrow[drrrr, bend right, "f_0"'] & Y_1 \arrow[r, "\kappa_{1,2}"'] \arrow[rrr, bend left, "\kappa_{1,\infty}"] \arrow[drrr, bend right, "f_1"'] & Y_2 \arrow[r, "\kappa_{2,3}"'] \arrow[rr, bend left, "\kappa_{2,\infty}"] \arrow[drr, "f_2"'] & \cdots & Y_\infty \arrow[d, "f_\infty"] \\
& & & & Z
\end{tikzcd}$$

Uniqueness. This follows by $\mathrm{V}_{Y_\infty} = \bigcup_{i \geqslant 0} \mathrm{V}_{Y_i}$ and $\mathrm{E}_{Y_\infty} = \bigcup_{i \geqslant 0} \mathrm{E}_{Y_i}$.

Existence. We let

$$\begin{array}{rrcl} \mathrm{V}_{f_\infty} : & \mathrm{V}_{Y_\infty} = \bigcup_{i \geqslant 0} \mathrm{V}_{Y_i} & \to & \mathrm{V}_Z \\ & v & \mapsto & v \, \mathrm{V}_{f_i} \text{ , if } v \in \mathrm{V}_{Y_i} \ . \end{array}$$

We have to show that this map is well-defined.

Suppose given $v \in \mathrm{V}_{Y_\infty}$ and $v \in \mathrm{V}_{Y_i}$ and $v \in \mathrm{V}_{Y_j}$, where $i, j \geqslant 0$.

We show that we have $v \, \mathrm{V}_{f_i} \stackrel{!}{=} v \, \mathrm{V}_{f_j}$.

Without loss of generality, we have $i \leqslant j$ and thus $v \, \mathrm{V}_{f_i} = v \, \mathrm{V}_{\kappa_{i,j} f_j} = v \, \mathrm{V}_{\kappa_{i,j}} \, \mathrm{V}_{f_j} = v \, \mathrm{V}_{f_j}$.

We let

$$\begin{array}{rrcl} \mathrm{E}_{f_\infty} : & \mathrm{E}_{Y_\infty} = \bigcup_{i \geqslant 0} \mathrm{E}_{Y_i} & \to & \mathrm{E}_Z \\ & v & \mapsto & v \, \mathrm{E}_{f_i} \text{ , if } v \in \mathrm{E}_{Y_i} \ . \end{array}$$

We have to show that this map is well-defined.

Suppose given $e \in \mathrm{E}_{Y_\infty}$ and $e \in \mathrm{E}_{Y_i}$ and $e \in \mathrm{E}_{Y_j}$, where $i, j \geqslant 0$.

We show that we have $e \, \mathrm{E}_{f_i} \stackrel{!}{=} e \, \mathrm{E}_{f_j}$.

Without loss of generality, we have $i \leqslant j$ and thus $e \, \mathrm{E}_{f_i} = e \, \mathrm{E}_{\kappa_{i,j} f_j} = e \, \mathrm{E}_{\kappa_{i,j}} \, \mathrm{E}_{f_j} = e \, \mathrm{E}_{f_j}$.

We show that the pair of maps $f_\infty := (\mathrm{V}_{f_\infty}, \mathrm{E}_{f_\infty})$ is a graph morphism.

Suppose given an edge $e \in \mathrm{E}_{Y_\infty}$.

We may choose $i \geqslant 0$ such that $e \in \mathrm{E}_{Y_i}$.

We have to show that $e \, \mathrm{s}_{Y_\infty} \, \mathrm{V}_{f_\infty} \stackrel{!}{=} e \, \mathrm{E}_{f_\infty} \, \mathrm{s}_Z$.

We have $e \, \mathrm{s}_{Y_\infty} \, \mathrm{V}_{f_\infty} \stackrel{\text{Def. 103}}{=} e \, \mathrm{s}_{Y_i} \, \mathrm{V}_{f_\infty} \stackrel{\text{Def. } \mathrm{V}_{f_\infty}}{=} e \, \mathrm{s}_{Y_i} \, \mathrm{V}_{f_i} = e \, \mathrm{E}_{f_i} \, \mathrm{s}_Z \stackrel{\text{Def. } \mathrm{E}_{f_\infty}}{=} e \, \mathrm{E}_{f_\infty} \, \mathrm{s}_Z$.

We have to show that $e \, \mathrm{t}_{Y_\infty} \, \mathrm{V}_{f_\infty} \stackrel{!}{=} e \, \mathrm{E}_{f_\infty} \, \mathrm{t}_Z$.

We have $e \, \mathrm{t}_{Y_\infty} \, \mathrm{V}_{f_\infty} \stackrel{\text{Def. 103}}{=} e \, \mathrm{t}_{Y_i} \, \mathrm{V}_{f_\infty} \stackrel{\text{Def. } \mathrm{V}_{f_\infty}}{=} e \, \mathrm{t}_{Y_i} \, \mathrm{V}_{f_i} = e \, \mathrm{E}_{f_i} \, \mathrm{t}_Z \stackrel{\text{Def. } \mathrm{E}_{f_\infty}}{=} e \, \mathrm{E}_{f_\infty} \, \mathrm{t}_Z$.

We now show that we have $\kappa_{i,\infty} f_\infty \stackrel{!}{=} f_i$ for $i \geqslant 0$.

Suppose given $v \in \mathrm{V}_{Y_i}$. We have $v \, \mathrm{V}_{\kappa_{i,\infty}} \, \mathrm{V}_{f_\infty} = v \, \mathrm{V}_{f_\infty} \stackrel{\text{Def. } f_\infty}{=} v \, \mathrm{V}_{f_i}$.

Suppose given $e \in \mathrm{E}_{Y_i}$. We have $e \, \mathrm{E}_{\kappa_{i,\infty}} \, \mathrm{E}_{f_\infty} = e \, \mathrm{E}_{f_\infty} \stackrel{\text{Def. } f_\infty}{=} e \, \mathrm{E}_{f_i}$. $\square$

Lemma 106 Suppose given $A \subseteq \operatorname{Mor}(\mathrm{Gph})$.

Suppose given a countable chain of subgraphs $X = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y$ such that $Y = \bigcup_{i \geqslant 0} Y_i$.

Suppose that the inclusion morphism $\mathrm{id}_{Y_{i+1}}|_{Y_i} : Y_i \to Y_{i+1}$ is in ${}^{\boxslash}A$ for $i \in \mathbb{Z}_{\geqslant 0}$.

Then the inclusion morphism $\mathrm{id}_Y|_X : X \to Y$ is a morphism in ${}^{\boxslash}A$.

Proof. The graph Y is the colimit of countable chain of subgraphs $X = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y$ with respect to the inclusion morphisms; cf. Remark 105.

Then by Lemma 102 the inclusion morphism $\mathrm{id}_Y|_X : X \to Y$ is a morphism in ${}^{\boxslash}A$. ▫

2.5 Tree graphs

Definition 107 A graph morphism $p : \mathrm{D}_n \to X$ for some $n \geqslant 0$ is called *path* in X.

Given a path $p : \mathrm{D}_n \to X$, we let $p\,\mathrm{s} := \hat{\mathrm{v}}_0\,\mathrm{V}_p$ and $p\,\mathrm{t} := \hat{\mathrm{v}}_n\,\mathrm{V}_p$.

We say p is a path from $p\,\mathrm{s}$ to $p\,\mathrm{t}$ with *length* n. We write $\operatorname{length}(p) := n$.

We often write $p = (\hat{\mathrm{v}}_0\,\mathrm{V}_p\,; \hat{\mathrm{e}}_0\,\mathrm{E}_p\,, \ldots, \hat{\mathrm{e}}_{n-1}\,\mathrm{E}_p)$.

Conversely, given $v \in \mathrm{V}_X$, $m \geqslant 0$ and $e_0, \ldots, e_{m-1} \in \mathrm{E}_X$, then $(v; e_0, \ldots, e_{m-1})$ is a path in X if $e_0\,\mathrm{s}_X = v$ and $e_i\,\mathrm{t}_X = e_{i+1}\,\mathrm{s}_X$ for $i \in [0, m-2]$.

A graph morphism $p : \mathrm{D}_0 \to X$ with $\hat{\mathrm{v}}_0\,\mathrm{V}_p = x \in \mathrm{V}_X$ is called the *empty path at* x, also written $p = (x;\,)$.

Suppose given a path $p =: (v; e_0, \ldots, e_{n-1})$ in X. We have $p\,\mathrm{s} = v$. If $n \geqslant 1$ then we have $p\,\mathrm{t} = e_{n-1}\,\mathrm{t}_X$, if $n = 0$ then we have $p\,\mathrm{t} = v$.

For a path $p = (v; e_0, \ldots, e_{n-1})$ in X and an edge $e \in \mathrm{E}_X$ with $e\,\mathrm{s}_X = p\,\mathrm{t}$, we let

$$p \cdot e := (v; e_0, \ldots, e_{n-1}, e)\ .$$

Definition 108 A graph G is called a *tree* if there exists $r \in \mathrm{V}_G$ such that the following properties (Tree 1–3) hold.

(Tree 1) We have $|\{e \in \mathrm{E}_G : (e)\,\mathrm{t}_G = v\}| = 1$ for $v \in \mathrm{V}_G \setminus \{r\}$.

(Tree 2) We have $(e)\,\mathrm{t}_G \neq r$ for $e \in \mathrm{E}_G$.

(Tree 3) There exists a path from r to v for $v \in \mathrm{V}_G$.

The vertex $r \in \mathrm{V}_G$ is unique with these properties. We call this vertex the *root* r of G.

Proof. Suppose given two vertices $r, \tilde{r} \in \mathrm{V}_G$ with $r \neq \tilde{r}$ satisfying (Tree 1–3).

Then by (Tree 1) we have $|\{e \in \mathrm{E}_G : (e)\,\mathrm{t}_G = \tilde{r}\}| = 1$ since $\tilde{r} \in \mathrm{V}_G \setminus \{r\}$, in *contradiction* to (Tree 2). ▫

Definition 109 Suppose given a graph X and a vertex $x \in \mathrm{V}_X$.

We will define the graph $\mathrm{Tree}(x, X)$.

We let
$$\mathrm{V}_{\mathrm{Tree}(x,X)} := \{\alpha : \alpha \text{ is a path in } X \text{ with } \alpha\,\mathrm{s} = x\} .$$

We let
$$\mathrm{E}_{\mathrm{Tree}(x,X)} := \{(\alpha, e, \beta) : \alpha, \beta \in \mathrm{V}_{\mathrm{Tree}(x,X)} ,\ e \in \mathrm{E}_X \text{ such that } \alpha\,\mathrm{t} = e\,\mathrm{s}_X \text{ and } \alpha \cdot e = \beta\} .$$

Note that for $(\alpha, e, \beta) \in \mathrm{E}_{\mathrm{Tree}(x,X)}$ we have $e\,\mathrm{t}_X = (\alpha \cdot e)\,\mathrm{t} = \beta\,\mathrm{t}$.

For $(\alpha, e, \beta) \in \mathrm{E}_{\mathrm{Tree}(x,X)}$, we let $(\alpha, e, \beta)\,\mathrm{s}_{\mathrm{Tree}(x,X)} := \alpha$ and $(\alpha, e, \beta)\,\mathrm{t}_{\mathrm{Tree}(x,X)} := \beta$.

Remark 110 Suppose given a graph X. Suppose given a vertex $x \in \mathrm{V}_X$.

We have
$$\mathrm{V}_{\mathrm{Tree}(x,X)} = \{(x; e_0 , \ldots , e_{n-1}) : n \geqslant 0,\ (x; e_0 , \ldots , e_{n-1}) \text{ path in } X\}$$

and
$$\mathrm{E}_{\mathrm{Tree}(x,X)} = \{((x; e_0 , \ldots , e_{n-1}), e_n , (x; e_0 , \ldots , e_n)) : n \geqslant 0,\ (x; e_0 , \ldots , e_n) \text{ path in } X\} .$$

Proof.

Ad $\mathrm{V}_{\mathrm{Tree}(x,X)}$.

Suppose given $n \geqslant 1$. Suppose given a path $\alpha = (x; e_0 , \ldots , e_{n-1})$ in X. Then we have $\alpha\,\mathrm{s} = x$; cf. Definition 109.

Conversely, suppose given a path α in X with $\alpha\,\mathrm{s} = x$. Then we have $\mathrm{length}(\alpha) \geqslant 0$ and we write $\alpha = (\alpha\,\mathrm{s}; \hat{\mathrm{e}}_0\,\mathrm{E}_\alpha , \ldots , \hat{\mathrm{e}}_{\mathrm{length}(\alpha)-1}\,\mathrm{E}_\alpha) =: (x; e_0 , \ldots , e_{\mathrm{length}(\alpha)-1})$, which is a path in X.

Ad $\mathrm{E}_{\mathrm{Tree}(x;X)}$.

Suppose given $(\alpha, e, \beta) \in \mathrm{E}_{\mathrm{Tree}(x,X)}$. So α and β are paths in X and we have an edge $e \in \mathrm{E}_X$ such that $\alpha\,\mathrm{s} = x$ and $\alpha\,\mathrm{t} = e\,\mathrm{s}_X$ and $\alpha \cdot e = \beta$.

We write $n := \mathrm{length}(\alpha) \geqslant 0$. Then $\mathrm{length}(\beta) = \mathrm{length}(\alpha \cdot e) = \mathrm{length}(\alpha) + 1 = n + 1$. We write $\alpha = (\alpha\,\mathrm{s}; \hat{\mathrm{e}}_0\,\mathrm{E}_\alpha , \ldots , \hat{\mathrm{e}}_{n-1}\,\mathrm{E}_\alpha) =: (x; e_0 , \ldots , e_{n-1})$. We write $e_n := e$. So we have $\beta = \alpha \cdot e = \alpha \cdot e_n = (x; e_0 , \ldots , e_{n-1} , e_n)$.

So we have $(\alpha, e, \beta) = ((x; e_0 , \ldots , e_{n-1}), e, (x; e_0 , \ldots , e_{n-1} , e_n))$.

Conversely, suppose given $n \geqslant 0$ and $((x; e_0 , \ldots , e_{n-1}), e_n , (x; e_0 , \ldots , e_n))$ such that $(x; e_0 , \ldots , e_n) =: \beta$ is a path in X.

Then we have the graph morphism $\beta : \mathrm{D}_{n+1} \to X$ with $(\hat{\mathrm{v}}_0\,\mathrm{V}_\beta ; \hat{\mathrm{e}}_0\,\mathrm{E}_\beta , \ldots , \hat{\mathrm{e}}_n\,\mathrm{E}_\beta) = (x; e_0 , \ldots , e_n)$. We let $\alpha : \mathrm{D}_n \to X$ be the graph morphism with $\hat{\mathrm{v}}_i\,\mathrm{V}_\alpha := \hat{\mathrm{v}}_i\,\mathrm{V}_\beta$ for $i \in [0, n]$ and with $\hat{\mathrm{e}}_i\,\mathrm{E}_\alpha := \hat{\mathrm{e}}_i\,\mathrm{E}_\beta$ for $i \in [0, n-1]$. So we have $\alpha = \beta|_{\mathrm{D}_n} : \mathrm{D}_n \to X$.

We obtain $\alpha = (\hat{\mathrm{v}}_0\,\mathrm{V}_\alpha ; \hat{\mathrm{e}}_0\,\mathrm{E}_\alpha , \ldots , \hat{\mathrm{e}}_{n-1}\,\mathrm{E}_\alpha) = (\hat{\mathrm{v}}_0\,\mathrm{V}_\beta ; \hat{\mathrm{e}}_0\,\mathrm{E}_\beta , \ldots , \hat{\mathrm{e}}_{n-1}\,\mathrm{E}_\beta) = (x; e_0 , \ldots , e_{n-1})$.

So we have $((x; e_0 , \ldots , e_{n-1}), e_n , (x; e_0 , \ldots , e_n)) = (\alpha, e_n , \beta)$, where α and β are paths in X and the edge $e_n = \hat{\mathrm{e}}_n\,\mathrm{E}_\beta$ is in E_X. Furthermore, we have $\alpha\,\mathrm{s} = x$ and $\alpha\,\mathrm{t} = \hat{\mathrm{v}}_n\,\mathrm{V}_\alpha = \hat{\mathrm{v}}_n\,\mathrm{V}_\beta = \hat{\mathrm{e}}_n\,\mathrm{s}_{\mathrm{D}_{n+1}}\,\mathrm{V}_\beta = \hat{\mathrm{e}}_n\,\mathrm{E}_\beta\,\mathrm{s}_X = e_n\,\mathrm{s}_X$.

And we have $\alpha \cdot e_n = (x; e_0 , \ldots , e_{n-1}) \cdot e_n = (x; e_0 , \ldots , e_{n-1}, e_n) = \beta$. $\square$

Example 111 We consider the following graph.

$$X : \quad \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[rr, "c"'] & & 3 \end{tikzcd}$$

We have the graph

$$\mathrm{Tree}(1, X) : \quad \begin{tikzcd} & (1;a) \arrow[dr, "((1;a),b,(1;a,b))"] & \\ (1;\,) \arrow[ur, "((1;),a,(1;a))"] \arrow[dr, "((1;),c,(1;c))"'] & & (1;a,b)\,. \\ & (1;c) & \end{tikzcd}$$

In the following Remark 112 we will show that for every graph G and every vertex $x \in \mathrm{V}_G$, the graph $\mathrm{Tree}(x, G)$ actually is a tree.

Remark 112 Suppose given a graph G and a vertex $x \in \mathrm{V}_G$.

Then the graph $\mathrm{Tree}(x, G)$ is a tree with root $r := (x;\,) \in \mathrm{V}_{\mathrm{Tree}(x,G)}$.

Proof.

We have to show that the properties (Tree 1–3) hold for $\mathrm{Tree}(x, G)$; cf. Definitions 108 and 109.

Ad (Tree 1).

Suppose given a vertex $v \in \mathrm{V}_{\mathrm{Tree}(x,G)} \backslash \{r\}$.

We have to show that $|\{\epsilon \in \mathrm{E}_{\mathrm{Tree}(x,G)} \,:\, (\epsilon)\,\mathrm{t}_{\mathrm{Tree}(x,G)} = v\}| \overset{!}{=} 1$.

The vertex $v \in \mathrm{V}_{\mathrm{Tree}(x,G)}$ is a path $v : \mathrm{D}_n \to G$ in G with $v\,\mathrm{s} = x$ and $n \geqslant 1$ since $v \neq r$.

We write $v = (x; e_0\,, \ldots, e_n)$ with $e_i \in \mathrm{E}_G$ for $i \in [1, n]$.

For an edge $\epsilon = (\alpha, e, \beta) \in \mathrm{E}_{\mathrm{Tree}(x,G)}$, we have $(\alpha, e, \beta)\,\mathrm{t}_{\mathrm{Tree}(x,G)} = \beta$.

So $(\epsilon)\,\mathrm{t}_{\mathrm{Tree}(x,G)} = v$ if and only if $\epsilon = (\alpha, e, v)$, where $\alpha : \mathrm{D}_{n-1} \to G$ is a path with $\alpha\,\mathrm{t} = e\,\mathrm{s}_G$ and $\alpha \cdot e = v$. So it has to be $e = e_n$ and $\alpha = (x; e_1\,, \ldots, e_{n-1})$; cf. Remark 110.

We obtain

$$\{e \in \mathrm{E}_{\mathrm{Tree}(x,G)} \,:\, e\,\mathrm{t}_{\mathrm{Tree}(x,G)} = v\} = \{((x; e_0\,, \ldots, e_{n-1}), e_n\,, (x; e_0\,, \ldots, e_n))\}\,,$$

which contains a single element.

Ad (Tree 2).

We have to show that $(\epsilon)\,\mathrm{t}_{\mathrm{Tree}(x,G)} \overset{!}{\neq} r$ for $\epsilon \in \mathrm{E}_{\mathrm{Tree}(x,G)}$.

Suppose given an edge $(\alpha, e, \beta) \in \mathrm{E}_{\mathrm{Tree}(x,G)}$.

We have $(\alpha, e, \beta)\, \mathrm{t}_{\mathrm{Tree}(x,G)} = \beta$. Since $\alpha \cdot e = \beta$ we have $\mathrm{length}(\beta) = \mathrm{length}(\alpha \cdot e) = \mathrm{length}(\alpha) + 1 \geqslant 1$, whereas $\mathrm{length}(r) = 0$. So we conclude that $(\alpha, e, \beta)\, \mathrm{t}_{\mathrm{Tree}(x,G)} = \beta \neq r$.

Ad (Tree 3).

We have to show that there exists a path from $(x;)$ to v for $v \in \mathrm{V}_{\mathrm{Tree}(x,G)}$.

The vertex $v \in \mathrm{V}_{\mathrm{Tree}(x,G)}$ is a path $v : \mathrm{D}_n \to G$ in G from $v\,\mathrm{s} = x$, where $n := \mathrm{length}(v) \geqslant 0$.

We write $v = (x; e_0, \dots, e_{n-1})$ with $e_i \in \mathrm{E}_G$ for $i \in [1, n-1]$.

The following graph morphism $p : \mathrm{D}_n \to \mathrm{Tree}(x, G)$ will be a path in $\mathrm{Tree}(x, G)$ from $p\,\mathrm{s} = \hat{\mathrm{v}}_0\, \mathrm{V}_p = (x;)$ to $p\,\mathrm{t} = \hat{\mathrm{v}}_n\, \mathrm{V}_p = v = (x; e_0, \dots, e_{n-1})$.

Using Remark 110, we let

$$\begin{array}{rrcl} p: & \mathrm{D}_n & \to & \mathrm{Tree}(x,G) \\ \mathrm{V}_p: & \mathrm{V}_{\mathrm{D}_n} & \to & \mathrm{V}_{\mathrm{Tree}(x,G)} \\ & \hat{\mathrm{v}}_i & \mapsto & (x; e_0, \dots, e_{i-1}) \text{ for } i \in [0,n] \\ \mathrm{E}_p: & \mathrm{E}_{\mathrm{D}_n} & \to & \mathrm{E}_{\mathrm{Tree}(x,G)} \\ & \hat{\mathrm{e}}_i & \mapsto & ((x; e_0, \dots, e_{i-1}), e_i, (x; e_0, \dots, e_i)) \text{ for } i \in [0, n-1]. \end{array}$$

We have to show that $\mathrm{E}_p\, \mathrm{s}_{\mathrm{Tree}(x,G)} \overset{!}{=} \mathrm{s}_{\mathrm{D}_n}\, \mathrm{V}_p$ and that $\mathrm{E}_p\, \mathrm{t}_{\mathrm{Tree}(x,G)} \overset{!}{=} \mathrm{t}_{\mathrm{D}_n}\, \mathrm{V}_p$.

Suppose given an edge $\hat{\mathrm{e}}_i \in \mathrm{E}_{\mathrm{D}_n}$, where $i \in [0, n-1]$.

We have $\hat{\mathrm{e}}_i\, \mathrm{E}_p\, \mathrm{s}_{\mathrm{Tree}(x,G)} = ((x; e_0, \dots, e_{i-1}), e_i, (x; e_0, \dots, e_i))\, \mathrm{s}_{\mathrm{Tree}(x,G)} = (x; e_0, \dots, e_{i-1}) = \hat{\mathrm{v}}_i\, \mathrm{V}_p = \hat{\mathrm{e}}_i\, \mathrm{s}_{\mathrm{D}_n}\, \mathrm{V}_p$.

And we have $\hat{\mathrm{e}}_i\, \mathrm{E}_p\, \mathrm{t}_{\mathrm{Tree}(x,G)} = ((x; e_0, \dots, e_{i-1}), e_i, (x; e_0, \dots, e_i))\, \mathrm{t}_{\mathrm{Tree}(x,G)} = (x; e_0, \dots, e_i) = \hat{\mathrm{v}}_{i+1}\, \mathrm{V}_p = \hat{\mathrm{e}}_i\, \mathrm{t}_{\mathrm{D}_n}\, \mathrm{V}_p$.

Finally, $p\,\mathrm{s} = \hat{\mathrm{v}}_0\, \mathrm{V}_p = (x;) = r$ and $p\,\mathrm{t} = \hat{\mathrm{v}}_n\, \mathrm{V}_p = (x; e_0, \dots, e_{n-1}) = v$.

$$(x;) \xrightarrow{((x;), e_0, (x; e_0))} (x; e_0) \xrightarrow{((x; e_0), e_1, (x; e_0, e_1))} (x; e_0, e_1) \xrightarrow{((x; e_0, e_1), e_2, (x; e_0, e_1, e_2))} \cdots \qquad (x; e_1, \dots, e_n)$$

□

Definition 113 Suppose given a graph G and a vertex $x \in \mathrm{V}_G$.

We define the *projection morphism* $\mathrm{p}_x : \mathrm{Tree}(x, G) \to G$ at x.

For a path $\alpha : \mathrm{D}_n \to G$ in $\mathrm{V}_{\mathrm{Tree}(x,G)}$ we let $\alpha\, \mathrm{V}_{\mathrm{p}_x} := \hat{\mathrm{v}}_{\mathrm{length}(\alpha)}\, \mathrm{V}_\alpha$.

For an edge $(\alpha, e, \beta) \in \mathrm{E}_{\mathrm{Tree}(x,G)}$ we let $(\alpha, e, \beta)\, \mathrm{E}_{\mathrm{p}_x} := e \in \mathrm{E}_G$.

To verify that

$$\begin{array}{rrcl} \mathrm{p}_x: & \mathrm{Tree}(x,G) & \to & G \\ \mathrm{V}_{\mathrm{p}_x}: & \mathrm{V}_{\mathrm{Tree}(x,G)} & \to & \mathrm{V}_G \\ & \alpha & \mapsto & \hat{\mathrm{v}}_{\mathrm{length}(\alpha)}\, \mathrm{V}_\alpha \\ \mathrm{E}_{\mathrm{p}_x}: & \mathrm{E}_{\mathrm{Tree}(x,G)} & \to & \mathrm{E}_G \\ & (\alpha, e, \beta) & \mapsto & e \end{array}$$

is a graph morphism, we have to show that $\mathrm{E}_{\mathrm{p}_x}\,\mathrm{s}_G \overset{!}{=} \mathrm{s}_{\mathrm{Tree}(x,G)}\,\mathrm{V}_{\mathrm{p}_x}$ and that $\mathrm{E}_{\mathrm{p}_x}\,\mathrm{t}_G \overset{!}{=} \mathrm{t}_{\mathrm{Tree}(x,G)}\,\mathrm{V}_{\mathrm{p}_x}$.

So suppose given an edge $(\alpha, e, \beta) \in \mathrm{E}_{\mathrm{Tree}(x,G)}$.

Note that $e \in \mathrm{E}_G$ with $e\,\mathrm{s}_G = \alpha\,\mathrm{t} = \hat{\mathrm{v}}_{\mathrm{length}(\alpha)}\,\mathrm{V}_\alpha$ and with $e\,\mathrm{t}_G = \beta\,\mathrm{t} = \hat{\mathrm{v}}_{\mathrm{length}(\beta)}\,\mathrm{V}_\beta$.

We have

$$(\alpha, e, \beta)\,\mathrm{E}_{\mathrm{p}_x}\,\mathrm{s}_G = e\,\mathrm{s}_G = \alpha\,\mathrm{t} = \hat{\mathrm{v}}_{\mathrm{length}(\alpha)}\,\mathrm{V}_\alpha = \alpha\,\mathrm{V}_{\mathrm{p}_x} = (\alpha, e, \beta)\,\mathrm{s}_{\mathrm{Tree}(x,G)}\,\mathrm{V}_{\mathrm{p}_x} \ .$$

And we have

$$(\alpha, e, \beta)\,\mathrm{E}_{\mathrm{p}_x}\,\mathrm{t}_G = e\,\mathrm{t}_G = \beta\,\mathrm{t} = \hat{\mathrm{v}}_{\mathrm{length}(\beta)}\,\mathrm{V}_\beta = \beta\,\mathrm{V}_{\mathrm{p}_x} = (\alpha, e, \beta)\,\mathrm{t}_{\mathrm{Tree}(x,G)}\,\mathrm{V}_{\mathrm{p}_x} \ .$$

So p_x is in fact a graph morphism.

Given $n \geqslant 1$ and $\alpha = (x; e_0\,, \dots, e_{n-1}) \in \mathrm{V}_{\mathrm{Tree}(x;G)}$, we have $\alpha\,\mathrm{V}_{\mathrm{p}_x} = e_{n-1}\,\mathrm{t}_G$.

Moreover, we have $(x;\,)\,\mathrm{V}_{\mathrm{p}_x} = x$.

Given $(\alpha, e, \beta) = (x; (e_0\,, \dots, e_{n-1}), e_n, (e_0\,, \dots, e_n)) \in \mathrm{E}_{\mathrm{Tree}(x,G)}$, we have

$$(\alpha, e, \beta)\,\mathrm{E}_{\mathrm{p}_x} = (x; (e_0\,, \dots, e_{n-1}), e_n\,, (e_0\,, \dots, e_n))\,\mathrm{E}_{\mathrm{p}_x} = e_n \ .$$

Remark 114 Suppose given a graph G and a vertex $x \in \mathrm{V}_G$.

We recall the projection morphism $\mathrm{p}_x : \mathrm{Tree}(x, G) \to G$; cf. Definition 113.

We have the bijection

$$\mathrm{E}_{\mathrm{p}_x}|^{G((x;)\,\mathrm{V}_{\mathrm{p}_x},*)}_{\mathrm{Tree}(x,G)((x;),*)} : \mathrm{Tree}(x, G)((x;\,), *) \to G((x;\,)\,\mathrm{V}_{\mathrm{p}_x}, *) = G(x, *) \ .$$

Proof. Note that $(x;\,)$ is the empty path in $x \in \mathrm{V}_G$ and thus a vertex in $\mathrm{Tree}(x, G)$. So the set $\mathrm{Tree}(x, G)((x;\,), *)$ consists of all edges in $\mathrm{Tree}(x, G)$ which have the empty path as source. So

$$\mathrm{Tree}(x; G)((x;\,), *) = \{((x;\,), e, (x; e)) :\ e \in G(x, *) = G((x;\,)\,\mathrm{V}_{\mathrm{p}_x}, *)\} \ .$$

The claimed bijection ensues. □

Chapter 3

Properties of graph morphisms

We shall show in §6, Proposition 204, that Gph, equipped with a set of fibrations, a set of cofibrations and a set of quasiisomorphisms is a model category in the sense of Definition 198. To this end, we shall introduce these sets, already employing the language and symbolism of model categories before §6.

3.1 Quasiisomorphisms

Definition 115 Suppose given graphs G and H.

A graph morphism $f : G \to H$ is called a *quasiisomorphism* if the map $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$ is bijective for $n \geqslant 1$.

To indicate that f is a quasiisomorphism, we often write $G \overset{f}{\underset{\approx}{-\!\!\!\rightarrow}} H$.

By $\mathrm{Qis}(\mathrm{Gph}) \subseteq \mathrm{Mor}(\mathrm{Gph})$ we denote the set of quasiisomorphisms in the category Gph.

We often write $\mathrm{Qis} := \mathrm{Qis}(\mathrm{Gph})$.

So $f : G \to H$ is a quasiisomorphism if and only if for $n \geqslant 1$ and for each graph morphism $h : \mathrm{C}_n \to H$ there exists a unique graph morphism $g : \mathrm{C}_n \to G$ such that $gf = h$.

$$\begin{tikzcd} & G \arrow[d, "f"] \\ \mathrm{C}_n \arrow[ur, "\exists! g"] \arrow[r, "h"'] & H \end{tikzcd}$$

Remark 116 We have $\mathrm{Iso}(\mathrm{Gph}) \subseteq \mathrm{Qis}(\mathrm{Gph})$.

Proof. Suppose given a graph isomorphism $f : G \to H$.

We have to show that the graph isomorphism $f : G \to H$ is a quasiisomorphism, i.e. that the map $(\mathrm{C}_n\,, f)_{\mathrm{Gph}} : (\mathrm{C}_n\,, G)_{\mathrm{Gph}} \to (\mathrm{C}_n\,, H)_{\mathrm{Gph}}$ is bijective for $n \geqslant 1$; cf. Definition 115.

Suppose given $n \geqslant 1$. Then $\mathrm{C}_n\,, f)_{\mathrm{Gph}} \cdot (\mathrm{C}_n\,, f^{-1})_{\mathrm{Gph}} \overset{\text{Rem. 69.(1)}}{=} (\mathrm{C}_n\,, f \cdot f^{-1})_{\mathrm{Gph}} = (\mathrm{C}_n\,, \mathrm{id}_H)_{\mathrm{Gph}} \overset{\text{Rem. 69.(1)}}{=} \mathrm{id}_{(\mathrm{C}_n\,,H)_{\mathrm{Gph}}}$.

Moreover $(\mathrm{C}_n\,, f^{-1})_{\mathrm{Gph}} \cdot (\mathrm{C}_n\,, f)_{\mathrm{Gph}} \overset{\text{Rem. 69}}{=} (\mathrm{C}_n\,, f^{-1} \cdot f)_{\mathrm{Gph}} = (\mathrm{C}_n\,, \mathrm{id}_G)_{\mathrm{Gph}} \overset{textRem.\ 69}{=} \mathrm{id}_{(\mathrm{C}_n\,,G)_{\mathrm{Gph}}}$.

So $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$ is bijective.

So the map $(\mathrm{C}_n\,, f)_{\mathrm{Gph}} : (\mathrm{C}_n\,, G)_{\mathrm{Gph}} \to (\mathrm{C}_n\,, H)_{\mathrm{Gph}}$ is bijective for $n \geqslant 1$ and thus the graph isomorphism $f : G \to H$ is a quasiisomorphism. ▫

Remark 117 In Gph, the subset of quasiisomorphisms Qis $\subseteq$ Mor is closed under retracts; cf. Definition 23.

Proof. Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"] \arrow[r, "f", "\approx"'] & H \arrow[u, "q"'] \\
G' \arrow[u, "i"] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"'] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

We have to show that the graph morphism $f' : G' \to H'$ is a quasiisomorphism; cf. Definition 115.

Suppose given

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"] \arrow[r, "f", "\approx"'] & H \arrow[u, "q"'] \\
G' \arrow[u, "i"] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"'] \arrow[uu, bend right, "\mathrm{id}_{H'}"'] \\
 & \mathrm{C}_n \arrow[u, "u"']
\end{tikzcd}$$

We have to show that there exists a unique graph morphism $\tilde{v} : \mathrm{C}_n \to G'$ such that $\tilde{v}f' = u$; cf. Definition 115.

Existence. Because the graph morphism $f : G \to H$ is a quasiisomorphism there exists a unique graph morphism $v : \mathrm{C}_n \to G$ such that $vf = uj : \mathrm{C}_n \to H$.

So we have the graph morphism $\tilde{v} := vp : \mathrm{C}_n \to G$ satisfying $\tilde{v}f' = vpf' = vfq = ujq = u$.

Uniqueness. Suppose given

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"] \arrow[r, "f", "\approx"'] & H \arrow[u, "q"'] \\
G' \arrow[u, "i"] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"'] \arrow[uu, bend right, "\mathrm{id}_{H'}"'] \\
 & \mathrm{C}_n \arrow[u, "u"'] \arrow[ul, shift left, "\tilde{v}"'] \arrow[ul, shift right, "v"]
\end{tikzcd}$$

such that $vf' = \tilde{v}f' = u$.

We have to show that $v \stackrel{!}{=} \tilde{v}$.

We have $vif = vf'j = uj = \tilde{v}f'j = \tilde{v}if$

Since the graph morphism $f : G \to H$ is a quasiisomorphism, we have $vi = \tilde{v}i$; cf. Definition 115.

So we have $v = vip = \tilde{v}ip = \tilde{v}$ and this shows uniqueness. ▫

Example 118 The unique graph morphism $\iota_{\mathrm{D}_0} : \emptyset \to \mathrm{D}_0$ is a quasiisomorphism; cf. Definitions 70 and 115:

Suppose given $n \in \mathbb{N}$. There does not exist a graph morphism from C_n to $\emptyset$ and there does not exist a graph morphism from C_n to D_0, because $\mathrm{E}_{\mathrm{C}_n} \neq \emptyset$, but $\mathrm{E}_\emptyset = \emptyset$ and $\mathrm{E}_{\mathrm{D}_0} = \emptyset$. So

$$(\mathrm{C}_n\,, \iota_{\mathrm{D}_0}) : \quad (\mathrm{C}_n\,, \emptyset)_{\mathrm{Gph}} \quad \to \quad (\mathrm{C}_n\,, \mathrm{D}_0)_{\mathrm{Gph}}$$

is bijective.

Lemma 119 Suppose given a graph X such that $(\mathrm{C}_n\,, X) = \emptyset$ for $n \in \mathbb{X}$.

Then the graph morphism $\iota_X : \emptyset \to X$ is a quasiisomorphism.

Proof. For $n \in \mathbb{N}$ we have $(\mathrm{C}_n\,, X) = \emptyset = (\mathrm{C}_n\,, \emptyset)$. ▫

Example 120 We consider the following graph.

$$X : \quad \boxed{\begin{tikzcd} & 3 \arrow[dr, bend left, "\alpha_2"] & \\ 1 \arrow[ur, bend left, "\alpha_3"] \arrow[rr, "\alpha_1"'] & & 2 \end{tikzcd}}$$

Then $(\mathrm{C}_n\,, X) = \emptyset$ for $n \in \mathbb{N}$. So the graph morphism $\iota_X : \emptyset \to X$ is a quasiisomorphism; cf. Lemma 119.

Example 121 Suppose given a graph morphism $f : X \to Y$ with $(\mathrm{C}_n\,, X) = \emptyset = (\mathrm{C}_n\,, Y)$ for $n \in \mathbb{N}$. Then the graph morphism $f : X \to Y$ is a quasiisomorphism; cf. Definition 115.

E.g. for $k, l \in \mathbb{N}$ with $k \leqslant l$, we may consider the graph morphism $\iota_{k,l} : \mathrm{D}_k \to \mathrm{D}_l$; cf. Definition 56. Then $\iota_{k,l}$ is a quasiisomorphism, since $(\mathrm{C}_n\,, \mathrm{D}_m) = \emptyset$ for $n, m \in \mathbb{N}$.

The following remark is called (2 of 6).

Remark 122 The subset $\mathrm{Qis} \subseteq \mathrm{Mor}(\mathrm{Gph})$ satisfies (2 of 6).

Proof. Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd} & X' \arrow[rr, "gh"] \arrow[dr, "g"] & & Y' \\ X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Y \arrow[ur, "h"'] & \end{tikzcd}$$

We have to show that the composites $fg : X \to Y$ and $gh : X' \to Y'$ are quasiisomorphisms if and only if f, g and h are quasiisomorphisms.

Applying $(\mathrm{C}_n\,, -)_{\mathrm{Gph}}$ to the given commutative diagram yields the following commutative diagram in Set; cf. Remark 69.

$$\begin{tikzcd} & (\mathrm{C}_n\,, X')_{\mathrm{Gph}} \arrow[rr, "(\mathrm{C}_n\,,gh)_{\mathrm{Gph}}"] \arrow[dr, "(\mathrm{C}_n\,,g)_{\mathrm{Gph}}"] & & (\mathrm{C}_n\,, Y')_{\mathrm{Gph}} \\ (\mathrm{C}_n\,, X)_{\mathrm{Gph}} \arrow[ur, "(\mathrm{C}_n\,,f)_{\mathrm{Gph}}"] \arrow[rr, "(\mathrm{C}_n\,,fg)_{\mathrm{Gph}}"'] & & (\mathrm{C}_n\,, Y)_{\mathrm{Gph}} \arrow[ur, "(\mathrm{C}_n\,,h)_{\mathrm{Gph}}"'] & \end{tikzcd}$$

In this situation we have (2 of 6); cf. Remark 35.

The morphisms f, g and h are quasiisomorphisms if and only if the maps $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$, $(\mathrm{C}_n\,, g)_{\mathrm{Gph}}$ and $(\mathrm{C}_n\,, h)_{\mathrm{Gph}}$ are bijective for $n \in \mathbb{N}$. This holds if and only if the maps $(\mathrm{C}_n\,, f)_{\mathrm{Gph}} \cdot (\mathrm{C}_n\,, g)_{\mathrm{Gph}} = (\mathrm{C}_n\,, fg)_{\mathrm{Gph}}$ and $(\mathrm{C}_n\,, g)_{\mathrm{Gph}} \cdot (\mathrm{C}_n\,, h)_{\mathrm{Gph}} = (\mathrm{C}_n\,, gh)_{\mathrm{Gph}}$ are bijective for $n \in \mathbb{N}$; cf. Remark 35. This holds if and only if fg and gh are quasiisomorphisms; cf. Definition 115. ▫

Remark 123 The subset Qis $\subseteq$ Mor(Gph) satisfies (2 of 3).

Explicitly, this means the following.

Suppose given graph morphisms $f : X \to Y$ and $g : Y \to Z$.

Note the graph morphism $fg : X \to Z$.

We have the following commutative triangle.

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "fg"'] & & Z \end{tikzcd}$$

The composite $fg : X \to Z$ is a quasiisomorphism if f and g are quasiisomorphisms.

The graph morphism g is a quasiisomorphism if f and fg are quasiisomorphisms.

The graph morphism f is a quasiisomorphism if g and fg are quasiisomorphisms.

Proof. Since Qis satisfies (2 of 6) and contains all identities in Gph, we conclude that Qis satisfies (2 of 3) by Lemma 2; cf. Remark 116. ▫

Remark 124 Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd} & Y \arrow[dr, "b"] & \\ X \arrow[ur, "a"] \arrow[rr, "f"'] & & Z \end{tikzcd}$$

Suppose given $k \geqslant 1$.

Suppose that the map $(\mathrm{C}_k\,, f)_{\mathrm{Gph}}$ is surjective.

Then the map $(\mathrm{C}_k, b)_{\mathrm{Gph}}$ is surjective.

Proof. Since $(\mathrm{C}_k\,, a)_{\mathrm{Gph}} \cdot (\mathrm{C}_k\,, b)_{\mathrm{Gph}} = (\mathrm{C}_k\,, f)_{\mathrm{Gph}}$ is surjective, so is $(\mathrm{C}_k\,, b)_{\mathrm{Gph}}$. □

Remark 125 In Gph, a pullback of a quasiisomorphism is a quasiisomorphism; cf. Definition 115.

Proof. Suppose given a pullback in Gph as follows.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[d, "h", "\wr"'] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

We have to show that the graph morphism $g : X \to X'$ is a quasiisomorphism.

Suppose given $n \geqslant 1$ and a graph morphism $c' : \mathrm{C}_n \to X'$.

We have to show that there exists a unique graph morphism $c : \mathrm{C}_n \to X'$ with $cg = c'$.

First, we remark that there exists a unique graph morphism $w : \mathrm{C}_n \to Y$ such that $wh = c'f'$ because h is a quasiisomorphism.

Existence.

Because X is a pullback, we may choose a graph morphism $c : \mathrm{C}_n \to X$ such that $cf = w$ and $cg = c'$.

Uniqueness.

Now we have to show that c is unique with respect to $cg = c'$.

Suppose given a graph morphism $\tilde{c} : \mathrm{C}_n \to X$ such that $\tilde{c}g = c'$.

We have to show that $c \overset{!}{=} \tilde{c}$.

We have $\tilde{c}fh = \tilde{c}gf' = c'f'$ and thus $\tilde{c}f = w$.

So we have $cf = \tilde{c}f$.

And we have $cg = \tilde{c}g$.

So we have $c = \tilde{c}$; cf. Remark 10.

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[d, "h", "\wr"'] \\ \mathrm{C}_n \arrow[ur, dotted, "\exists!\, c"] \arrow[urr, dotted, "\exists!\, w"'] \arrow[r, "c'"'] & X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

□

Question 126 Suppose given a graph morphism $f : G \to H$ with G and H finite.

(1) Is it possible to give an algorithm to calculate a number $\zeta_f \in \mathbb{Z}_{\geqslant 1}$ such that

$$\min\{n \in \mathbb{N} : (\mathrm{C}_n\,, f) \text{ is not bijective}\} \leqslant \zeta_f$$

if f is not a quasiisomorphism?

An affirmative answer would allow to algorithmically decide whether f is a quasiisomorphism; for then f would be a quasiisomorphism if and only if $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$ is bijective for $n \in [1, \zeta_f]$.

If f is not a quasiisomorphism, then the left hand side

$$\min\{n \in \mathbb{N} : (\mathrm{C}_n\,, f) \text{ is not bijective}\}$$

seems to be difficult to calculate.

We can only give an algorithm that verifies a sufficient condition for f to be an acyclic fibration, in particular a quasiisomorphism; cf. Proposition 210. The function `SuffCond` is given in §10.6.

(2) Experiments indicate that $\zeta_f = \max\{|\,\mathrm{E}_G\,|, |\,\mathrm{E}_H\,|\}$ could be a possible choice in (1).

3.2 Fibrations and fibrant graphs

Definition 127 Suppose given graphs G and H.

(1) A graph morphism $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$ is called a *fibration* if the map

$$\mathrm{E}_{f,v} := \mathrm{E}_f\,|_{G(v,*)}^{H(v\,\mathrm{V}_f\,,*)} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

is surjective for $v \in \mathrm{V}_G$.

To indicate that f is a fibration, we often write $G \overset{f}{\twoheadrightarrow} H$.

By $\mathrm{Fib(Gph)} \subseteq \mathrm{Mor(Gph)}$ we denote the set of fibrations in the category Gph.

We often write $\mathrm{Fib} := \mathrm{Fib(Gph)}$.

(2) A graph morphism $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$ is called an *etale fibration* if the map

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

is bijective for $v \in \mathrm{V}_G$.

By $\mathrm{EtaleFib(Gph)} \subseteq \mathrm{Mor(Gph)}$ we denote the set of etale fibrations in the category Gph.

The Assertion 249 below shows that being surjective is not a sufficient condition for a graph morphism to be a fibration.

Example 128 Suppose given a graph H.

The morphism $\iota_H : \emptyset \to H$ is a fibration; cf. Definitions 70 and 127.(1).

We have to show that the map

$$\mathrm{E}_{h,v} = \mathrm{E}_h\,|_{\emptyset(v,*)}^{H(v\,\mathrm{V}_h,*)} : \emptyset(v,*) \to H(v\,\mathrm{V}_h,*)$$

is surjective for $v \in \mathrm{V}_\emptyset$.

The set $\mathrm{V}_\emptyset = \emptyset$ does not contain any element, so there is nothing to show and the condition for h to be a fibration is satisfied. ▫

Remark 129 We have $\mathrm{Iso}(\mathrm{Gph}) \subseteq \mathrm{EtaleFib}(\mathrm{Gph}) \subseteq \mathrm{Fib}(\mathrm{Gph})$.

Proof.

Ad $\mathrm{EtaleFib}(\mathrm{Gph}) \overset{!}{\subseteq} \mathrm{Fib}(\mathrm{Gph})$. Suppose given an etale fibration $f : G \to H$ in $\mathrm{EtaleFib}(\mathrm{Gph})$.

Then the map

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

is bijective and thus surjective for $v \in \mathrm{V}_G$; cf. Definition 127. So the graph morphism $f : G \to H$ is a fibration and thus in Fib.

Ad $\mathrm{Iso}(\mathrm{Gph}) \overset{!}{\subseteq} \mathrm{EtaleFib}(\mathrm{Gph})$. Suppose given an isomorphism $f : G \to H$ in $\mathrm{Iso}(\mathrm{Gph})$.

We have to show that the isomorphism $f : G \to H$ is an etale fibration.

Therefor we have to show that the map

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

is bijective for $v \in \mathrm{V}_G$.

Because the graph morphism $f : G \to H$ is an isomorphism, the map $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ is bijective.

Suppose given a vertex $v \in \mathrm{V}_G$.

Since we have $\mathrm{E}_{f,v} = \mathrm{E}_f\,|_{G(v,*)}^{H(v\,\mathrm{V}_f\,,*)}$, the map $\mathrm{E}_{f,v}$ is injective because the map E_f is injective.

We have to show that the map $\mathrm{E}_{f,v}$ is surjective.

Suppose given an edge $e \in H(v\,\mathrm{V}_f\,,*) \subseteq \mathrm{E}_H$.

We have $e\,\mathrm{s}_H = v\,\mathrm{V}_f$.

We have to find an edge $e_G \in G(v,*) \subseteq \mathrm{E}_G$ such that $e_G\,\mathrm{E}_{f,v} = e_G\,\mathrm{E}_f \overset{!}{=} e$.

Let $e_G := e\,\mathrm{E}_{f^{-1}}$. Then $e_G\,\mathrm{E}_f = e\,\mathrm{E}_{f^{-1}}\,\mathrm{E}_f = e$.

Moreover, $e_G \in G(v,*)$ since $e_G\,\mathrm{s}_G = e\,\mathrm{E}_{f^{-1}}\,\mathrm{s}_G = e\,\mathrm{s}_H\,\mathrm{V}_{f^{-1}} = v\,\mathrm{V}_f\,\mathrm{V}_{f^{-1}} = v$.

So the map $\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$ is bijective. ▫

Remark 130 In Gph, the subset of fibrations $\mathrm{Fib} \subseteq \mathrm{Mor}$ is closed under retracts; cf. Definition 23.

Proof. Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"] \arrow[r, "f", "+" marking] & H \arrow[u, "q"'] \\
G' \arrow[u, "i"] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"'] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

We have to show that the graph morphism $f' : G' \to H'$ is a fibration; cf. Definition 127.(1).

Since the graph morphism $f : G \to H$ is a fibration, given a vertex $v \in \mathrm{V}_G$, the map $\mathrm{E}_{f,v} : G(v, *) \to H(v\,\mathrm{V}_f\,, *)$ is surjective; cf. Definition 127.(1).

Suppose given a vertex $v' \in \mathrm{V}_{G'}$.

We have to show that the map $\mathrm{E}_{f',v'} : G'(v', *) \to H'(v'\,\mathrm{V}_{f'}, *)$ is surjective.

Suppose given an edge $e' \in H'(v'\,\mathrm{V}_{f'}, *)$.

We have to show that there exists an edge $\hat{e}' \in G'(v', *)$ such that $\hat{e}'\,\mathrm{E}_{f',v'} = \hat{e}'\,\mathrm{E}_{f'} \overset{!}{=} e'$.

Since $e'\,\mathrm{E}_j\,\mathrm{s}_H = e\,\mathrm{s}_{H'}\,\mathrm{V}_j = v\,\mathrm{V}_{f'}\,\mathrm{V}_j = v\,\mathrm{V}_{f'j}$ we have $e'\,\mathrm{E}_j \in H(v'\,\mathrm{V}_{f'j}, *)$.

Let $v := v'\,\mathrm{V}_i$. We have $v\,\mathrm{V}_f = v'\,\mathrm{V}_i\,\mathrm{V}_f = v'\,\mathrm{V}_{if} = v'\,\mathrm{V}_{f'j}$. So we have $e'\,\mathrm{E}_j \in H(v\,\mathrm{V}_f\,, *)$.

Since the graph morphism $f : G \to H$ is a fibration, there exists an edge $\hat{e} \in G(v, *)$ such that $\hat{e}\,\mathrm{E}_f = e'\,\mathrm{E}_j$.

Let $\hat{e}' := \hat{e}\,\mathrm{E}_p$. Then we have $\hat{e}'\,\mathrm{s}_{G'} = \hat{e}\,\mathrm{E}_p\,\mathrm{s}_{G'} = \hat{e}\,\mathrm{s}_G\,\mathrm{V}_p = v\,\mathrm{V}_p = v'\,\mathrm{V}_i\,\mathrm{V}_p = v'\,\mathrm{V}_{ip} = v'\,\mathrm{V}_{\mathrm{id}_{G'}} = v'$. So we have $\hat{e}' \in G'(v', *)$.

And we have $\hat{e}'\,\mathrm{E}_{f'} = \hat{e}\,\mathrm{E}_p\,\mathrm{E}_{f'} = \hat{e}\,\mathrm{E}_{pf'} = \hat{e}\,\mathrm{E}_{fq} = \hat{e}\,\mathrm{E}_f\,\mathrm{E}_q = e'\,\mathrm{E}_j\,\mathrm{E}_q = e'\,\mathrm{E}_{jq} = e'\,\mathrm{E}_{\mathrm{id}_{H'}} = e'$.

So the graph morphism $f' : G' \to H'$ is a fibration. □

Once Lemma 192 below is known, which states that we have $\mathrm{Fib} = \mathrm{AcCofib}^{\boxslash}$, we can also argue by Remark 26 to obtain the statement of Remark 130.

Remark 131 Suppose given fibrations $f :\ X \xrightarrow{+} Y$ and $g :\ Y \xrightarrow{+} Z$.

Then the composite $fg : X \to Z$ is also a fibration.

Proof.

The map $\mathrm{E}_{f,v} : X(v, *) \to Y(v\,\mathrm{V}_f\,, *)$ is surjective for $v \in \mathrm{V}_X$.

The map $\mathrm{E}_{g,v\,\mathrm{V}_f} : Y(v\,\mathrm{V}_f\,, *) \to Z(v\,\mathrm{V}_f\,\mathrm{V}_g, *)$ is surjective for $v \in \mathrm{V}_X$.

We have to show that the map $\mathrm{E}_{fg,v} : X(v, *) \to Z(v\,\mathrm{V}_{fg}, *)$ is surjective for $v \in \mathrm{V}_X$.

We *claim* that $\mathrm{E}_{f,v} \cdot \mathrm{E}_{g,v\,\mathrm{V}_f} \overset{!}{=} \mathrm{E}_{fg,v}$.

Suppose given $e \in X(v, *)$, i.e. $e \in \mathrm{E}_X$ with $e\,\mathrm{s}_X = v$.

Then $e\,\mathrm{E}_{f,v} \cdot \mathrm{E}_{g,v\,\mathrm{V}_f} = e\,\mathrm{E}_f\,\mathrm{E}_g = e\,\mathrm{E}_{fg} = e\,\mathrm{E}_{fg,v}$.

This proves the *claim*.

Now since $\mathrm{E}_{f,v}$ and $\mathrm{E}_{g,v\,\mathrm{V}_f}$ are surjective, so is $\mathrm{E}_{fg,v}$. $\square$

Once Lemma 192 below is known, which states that we have $\mathrm{Fib} = \mathrm{AcCofib}^{\boxslash}$, we can also argue by Remark 20 to obtain the statement of Remark 131.

Remark 132 Suppose given etale fibrations $f : X \to Y$ and $g : Y \to Z$.

Then the composite $fg : X \to Z$ is also an etale fibration.

Proof. The map $\mathrm{E}_{f,v} : X(v,*) \to Y(v\,\mathrm{V}_f\,,*)$ is bijective for $v \in \mathrm{V}_X$.

The map $\mathrm{E}_{g,v\,\mathrm{V}_f} : Y(v\,\mathrm{V}_f\,,*) \to Z(v\,\mathrm{V}_f\,\mathrm{V}_g,*)$ is bijective for $v \in \mathrm{V}_X$.

We have to show that the map $\mathrm{E}_{fg,v} : X(v,*) \to Z(v\,\mathrm{V}_{fg},*)$ is bijective for $v \in \mathrm{V}_X$.

We have $\mathrm{E}_{f,v} \cdot \mathrm{E}_{g,v\,\mathrm{V}_f} = \mathrm{E}_{fg,v}$; cf. proof of Remark 131.

Now since $\mathrm{E}_{f,v}$ and $\mathrm{E}_{g,v\,\mathrm{V}_f}$ are bijective, so is $\mathrm{E}_{fg,v}$. $\square$

Remark 133 In Gph, a pullback of a fibration is a fibration; cf. Definition 127.(1).

Proof. Suppose given a pullback in Gph as follows.

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f", tail] \\ Y' \arrow[r, "h"'] & Y \end{tikzcd}$$

We have to show that the graph morphism $f' : X' \to Y'$ is a fibration.

Suppose given $v_{X'} \in \mathrm{V}_{X'}$.

We write $v_{Y'} := v_{X'}\,\mathrm{V}_{f'} \in \mathrm{V}_{Y'}$ and $v_X := v_{X'}\,\mathrm{V}_g \in \mathrm{V}_X$ and

$$v_Y := v_{Y'}\,\mathrm{V}_h = v_{X'}\,\mathrm{V}_{f'}\,\mathrm{V}_h = v_{X'}\,\mathrm{V}_{f'h} \overset{f'h=gf}{=} v_{X'}\,\mathrm{V}_{gf} = v_{X'}\,\mathrm{V}_g\,\mathrm{V}_f = v_X\,\mathrm{V}_f \in \mathrm{V}_Y \ .$$

We have to show that the map

$$\mathrm{E}_{f',v_{X'}} := \mathrm{E}_{f'}\,|_{X'(v_{X'},*)}^{Y'(v_{Y'},*)} : X'(v_{X'},*) \to Y'(v_{Y'},*)$$

is surjective.

Suppose given an edge $e_{Y'} \in Y'(v_{Y'},*) \subseteq \mathrm{E}_{Y'}$.

We have to show that there exists an edge $e_{X'} \in X'(v_{X'}\,,*) \subseteq \mathrm{E}_{X'}$ such that $e_{X'}\,\mathrm{E}_{f'} = e_{Y'}$.

We have $v_{Y'} = e_{Y'}\,\mathrm{s}_{Y'}$. We write $w_{Y'} := e_{Y'}\,\mathrm{t}_{Y'}$.

We write $e_Y := e_{Y'}\,\mathrm{E}_h \in Y(v_{Y'}\,\mathrm{V}_h\,,*) = Y(v_Y,*)$.

We write $w_Y := w_{Y'}\,\mathrm{V}_h = e_{Y'}\,\mathrm{t}_{Y'}\,\mathrm{V}_h = e_{Y'}\,\mathrm{E}_h\,\mathrm{t}_Y = e_Y\,\mathrm{t}_Y$.

We have $v_Y = v_{Y'} \mathrm{V}_h = e_{Y'} \mathrm{s}_{Y'} \mathrm{V}_h = e_{Y'} \mathrm{E}_h \mathrm{s}_Y = e_Y \mathrm{s}_Y$.

Because the graph morphism $f : X \longrightarrow\!\!\!\!\!\!\!+\; Y$ is a fibration, the map $\mathrm{E}_{f,v_X} = \mathrm{E}_f\big|_{X(v_X,*)}^{Y(v_Y,*)}$ is surjective.

So there exists an edge $e_X \in X(v_X, *) \subseteq \mathrm{E}_X$ such that $e_X \mathrm{E}_f = e_Y$.

Note that $e_X \mathrm{s}_X = v_X$. We write $w_X := e_X \mathrm{t}_X$.

Note that $w_X \mathrm{V}_f = e_X \mathrm{t}_X \mathrm{V}_f = e_X \mathrm{E}_f \mathrm{t}_Y = e_Y \mathrm{t}_Y = w_Y$.

Recall that we have to show that there exists an edge $e_{X'} \in X'(v_{X'}, *) \subseteq \mathrm{E}_{X'}$, i.e. $e_{X'} \mathrm{s}_{X'} = v_{X'}$, such that $e_{X'} \mathrm{E}_{f'} = e_{Y'}$.

We consider the direct graph D_1 with $\mathrm{V}_{\mathrm{D}_1} = \{\hat{\mathrm{v}}_0 , \hat{\mathrm{v}}_1\}$ and $\mathrm{E}_{\mathrm{D}_1} = \{\hat{\mathrm{e}}_0\}$ and $\hat{\mathrm{e}}_0 \mathrm{s}_{\mathrm{D}_1} = \hat{\mathrm{v}}_0$ and $\hat{\mathrm{e}}_0 \mathrm{t}_{\mathrm{D}_1} = \hat{\mathrm{v}}_1$; cf. Definition 56.

We define the graph morphism $q : \mathrm{D}_1 \to X$ by $\hat{\mathrm{v}}_0 \mathrm{V}_q := v_X$, $\hat{\mathrm{v}}_1 \mathrm{V}_q := w_X$ and $\hat{\mathrm{e}}_0 \mathrm{E}_q := e_X$.

Note that $\hat{\mathrm{e}}_0 \mathrm{E}_q \mathrm{s}_X = e_X \mathrm{s}_X = v_X = \hat{\mathrm{v}}_0 \mathrm{V}_q = \hat{\mathrm{e}}_0 \mathrm{s}_{\mathrm{D}_1} \mathrm{V}_q$ and that $\hat{\mathrm{e}}_0 \mathrm{E}_q \mathrm{t}_X = e_X \mathrm{t}_X = w_X = \hat{\mathrm{v}}_1 \mathrm{V}_q = \hat{\mathrm{e}}_0 \mathrm{t}_{\mathrm{D}_1} \mathrm{V}_q$. So $q : \mathrm{D}_1 \to X$ is in fact a graph morphism.

We define the graph morphism $p : \mathrm{D}_1 \to Y'$ with $\hat{\mathrm{v}}_0 \mathrm{V}_p := v_{Y'}$, $\hat{\mathrm{v}}_1 \mathrm{V}_p := w_{Y'}$ and $\hat{\mathrm{e}}_0 \mathrm{E}_p := e_{Y'}$.

Note that $\hat{\mathrm{e}}_0 \mathrm{E}_p \mathrm{s}_{Y'} = e_{Y'} \mathrm{s}_{Y'} = v_{Y'} = \hat{\mathrm{v}}_0 \mathrm{V}_p = \hat{\mathrm{e}}_0 \mathrm{s}_{\mathrm{D}_1} \mathrm{V}_p$ and that $\hat{\mathrm{e}}_0 \mathrm{E}_p \mathrm{t}_{Y'} = e_{Y'} \mathrm{t}_{Y'} = w_{Y'} = \hat{\mathrm{v}}_1 \mathrm{V}_p = \hat{\mathrm{e}}_0 \mathrm{t}_{\mathrm{D}_1} \mathrm{V}_p$. So $p : \mathrm{D}_1 \to Y'$ is in fact a graph morphism.

We have $\hat{\mathrm{v}}_0 \mathrm{V}_{qf} = v_X \mathrm{V}_f = v_Y = v_{Y'} \mathrm{V}_h = \hat{\mathrm{v}}_0 \mathrm{V}_{ph}$.

We have $\hat{\mathrm{v}}_1 \mathrm{V}_{qf} = w_X \mathrm{V}_f = w_Y = w_{Y'} \mathrm{V}_h = \hat{\mathrm{v}}_1 \mathrm{V}_{ph}$.

We have $\hat{\mathrm{e}}_0 \mathrm{E}_{qf} = e_X \mathrm{E}_f = e_Y = e_{Y'} \mathrm{E}_h = \hat{\mathrm{e}}_0 \mathrm{E}_{ph}$.

So we have $qf = ph$.

Because X' is a pullback there exists a unique graph morphism $r : \mathrm{D}_1 \to X'$ such that $rg = q$ and $rf' = p$.

We let $e_{X'} := \hat{\mathrm{e}}_0 \mathrm{E}_r \in \mathrm{E}_{X'}$. We have $e_{X'} \mathrm{E}_{f'} = \hat{\mathrm{e}}_0 \mathrm{E}_r \mathrm{E}_{f'} = \hat{\mathrm{e}}_0 \mathrm{E}_{rf'} = \hat{\mathrm{e}}_0 \mathrm{E}_p = e_{Y'}$.

We will show that $e_{X'} \overset{!}{\in} X'(v_{X'}, *)$, i.e. that $e_{X'} \mathrm{s}_{X'} = \hat{\mathrm{e}}_0 \mathrm{E}_r \mathrm{s}_{X'} = \hat{\mathrm{e}}_0 \mathrm{s}_{\mathrm{D}_1} \mathrm{V}_r = \hat{\mathrm{v}}_0 \mathrm{V}_r \overset{!}{=} v_{X'}$.

We consider the subgraph $\mathrm{D}_0 \subseteq \mathrm{D}_1$ with $\mathrm{V}_{\mathrm{D}_0} = \{\hat{\mathrm{v}}_0\}$ and $\mathrm{E}_{\mathrm{D}_0} = \emptyset$; cf. Definition 56.(1).

We consider the graph morphism $\iota_1 : \mathrm{D}_0 \to \mathrm{D}_1$ with $\hat{\mathrm{v}}_0 \mathrm{V}_{\iota_1} = \hat{\mathrm{v}}_0$; cf. Definition 56.(3).

We write $p' := \iota_1 p = \iota_1 r f'$ and $q' := \iota_1 q = \iota_1 r g$.

We have $p'h = \iota_1 r f' h = \iota_1 r g f = q' f$.

So there exists a unique graph morphism $r' : \mathrm{D}_0 \to X'$ such that $r'f' = p'$ and $r'g = q'$.

We define the graph morphism $\tilde{r} : \mathrm{D}_0 \to X$ by $\hat{\mathrm{v}}_0 \mathrm{V}_{\tilde{r}} := v_{X'}$.

We show that we have $\tilde{r} f' \overset{!}{=} p'$ and $\tilde{r} g \overset{!}{=} q'$.

We have $\hat{\mathrm{v}}_0 \mathrm{V}_{\tilde{r}f'} = v_{X'} \mathrm{V}_{f'} = v_{Y'} = \hat{\mathrm{v}}_0 \mathrm{V}_p = \hat{\mathrm{v}}_0 \mathrm{V}_{\iota_1 p} = \hat{\mathrm{v}}_0 \mathrm{V}_{p'}$. So we have $\tilde{r} f' = p$.

We have $\hat{\mathrm{v}}_0 \mathrm{V}_{\tilde{r}g} = v_{X'} \mathrm{V}_g = v_X = \hat{\mathrm{v}}_0 \mathrm{V}_q = \hat{\mathrm{v}}_0 \mathrm{V}_{\iota_1 q} = \hat{\mathrm{v}}_0 \mathrm{V}_{q'}$. So we have $\tilde{r} g = q'$.

Because of the uniqueness of r', we obtain $r' = \tilde{r}$.

We consider the graph morphism $\iota_1 r : \mathrm{D}_0 \to X$. We have $(\iota_1 r) f' = p'$ and $(\iota_1 r) g = q'$.

Because of the uniqueness of r', we obtain $r' = \iota_1 r$.

Altogether, we have $\iota_1 r = \tilde{r}$. In particular $\hat{\mathrm{v}}_0\,\mathrm{V}_r = \hat{\mathrm{v}}_0\,\mathrm{V}_{\iota_1 r} = \hat{\mathrm{v}}_0\,\mathrm{V}_{\tilde{r}} = v_{X'}$.

So the graph morphism $f' : X' \to Y'$ is a fibration.

$$
\begin{tikzcd}
\mathrm{D}_0 \arrow[dr, "\iota_1"] \arrow[ddrrr, bend left, "q'"] \arrow[dddrr, bend right, "p'"'] & & & \\
& \mathrm{D}_1 \arrow[dr, "r"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& & X' \arrow[r, "g"'] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "f"] \\
& & Y' \arrow[r, "h"'] & Y
\end{tikzcd}
$$

□

Once Lemma 192 below is known, which states that we have $\mathrm{Fib} = \mathrm{AcCofib}^{\boxslash}$, we can also argue by Remark 22 to obtain the statement of Remark 133.

Remark 134 Suppose given a thin graph Y and an etale fibration $X \xrightarrow{f} Y$; cf. Definitions 73 and 127.(2).

Then the graph X is thin.

Proof. We *assume* that the graph X is not thin.

Then we may choose two edges $e_1 \neq e_2$ in E_X such that $e_1\,\mathrm{s}_X = e_2\,\mathrm{s}_X$ and $e_1\,\mathrm{t}_X = e_2\,\mathrm{t}_X$.

Let $v_1 := e_1\,\mathrm{s}_X$. Let $v_2 := e_1\,\mathrm{t}_X$.

Since the graph morphism $X \xrightarrow{f} Y$ is an etale fibration, the map

$$\mathrm{E}_{f,v_1} : X(v_1\,,*) \to Y(v_1\,\mathrm{V}_f\,,*)$$

is bijective.

Since $v_1 = e_1\,\mathrm{s}_X = e_2\,\mathrm{s}_X$ we have $e_1,\ e_2 \in X(v_1\,,*) = X(e_1\,\mathrm{s}_X,*) = X(e_2\,\mathrm{s}_X,*)$.

We have $e_1\,\mathrm{E}_{f,v_1},\ e_2\,\mathrm{E}_{f,v_1} \in Y(v_1\,\mathrm{V}_f\,,*)$. Since the map E_{f,v_1} is bijective, we have $e_1\,\mathrm{E}_f = e_1\,\mathrm{E}_{f,v_1} \neq e_2\,\mathrm{E}_{f,v_1} = e_2\,\mathrm{E}_f$.

Because $f : X \to Y$ is a graph morphism we have $e_1\,\mathrm{E}_f\,\mathrm{t}_Y = e_1\,\mathrm{t}_X\,\mathrm{V}_f = v_2\,\mathrm{V}_f = e_2\,\mathrm{t}_X\,\mathrm{V}_f = e_2\,\mathrm{E}_f\,\mathrm{t}_Y$.

So we have $e_1\,\mathrm{E}_f \neq e_2\,\mathrm{E}_f$ in $Y(v_1\,\mathrm{V}_f\,,v_2\,\mathrm{V}_f)$, which is a *contradiction* to the fact that the graph Y is thin. □

Definition 135 Suppose given a graph X. The graph X is called *fibrant* if the graph morphism $\tau_X : X \to \mathrm{C}_1$ is a fibration; cf. Remark 70 and Definition 127.(1).

Remark 136 Suppose given a graph X.

The graph X is fibrant if and only if $X(v,*) \neq \emptyset$ for $v \in \mathrm{V}_X$.

Proof. The graph X is fibrant if and only if the graph morphism $\tau_X : X \to \mathrm{C}_1$ is a fibration, i.e. if and only if the map

$$\mathrm{E}_{\tau_X,v} := \mathrm{E}_{\tau_X}|_{X(v,*)}^{\mathrm{C}_1(v\,\mathrm{V}_{\tau_X},*)} : X(v,*) \to \mathrm{C}_1(v\,\mathrm{V}_{\tau_X},*)$$

is surjective for $v \in \mathrm{V}_X$; cf. Definitions 127.(1) and 135.

For $v \in \mathrm{V}_X$ we have $v\,\mathrm{V}_{\tau_X} = \mathrm{v}_1$ and so $\mathrm{C}_1(v\,\mathrm{V}_{\tau_X}\,,*) = \mathrm{C}_1(\mathrm{v}_1\,,*) = \{\mathrm{e}_1\}$.

So $\mathrm{E}_{\tau_X,v}$ is surjective for $v \in \mathrm{V}_X$ if and only if $X(v,*) \neq \emptyset$ for $v \in \mathrm{V}_X$. ▫

Note that the empty graph $\emptyset$ is fibrant; cf. Remark 70.(2).

Example 137 Suppose given $n \in \mathbb{N}$.

The cyclic graph C_n is fibrant; cf. Definitions 52 and 135 and Remark 136.

The direct graph D_n is **not** fibrant, since $e\,\mathrm{t}_{\mathrm{D}_n} \neq \hat{\mathrm{v}}_n$ for $e \in \mathrm{E}_{\mathrm{D}_n}$.

3.3 Acyclic fibrations

Recall the notion of quasiisomorphisms from Definition 115 and of fibrations from Definition 127.(1).

Definition 138 A fibration that is a quasiisomorphism is called an *acyclic fibration*; cf. Definitions 115 and 127.(1).

To indicate that a graph morphism $G \xrightarrow{f} H$ is an acyclic fibration, we often write $G \overset{f}{\twoheadrightarrow} H$.

By $\mathrm{AcFib} := \mathrm{AcFib}(\mathrm{Gph}) := \mathrm{Fib} \cap \mathrm{Qis} \subseteq \mathrm{Mor}(\mathrm{Gph})$ we denote the set of acyclic fibrations in the category Gph.

Remark 139 We have $\mathrm{Iso}(\mathrm{Gph}) \subseteq \mathrm{AcFib}(\mathrm{Gph})$.

Proof. In Gph, we have $\mathrm{Iso} \subseteq \mathrm{Fib} \cap \mathrm{Qis} \overset{\text{Def. 138}}{=} \mathrm{AcFib}$ since $\mathrm{Iso} \subseteq \mathrm{Fib}$ by Remark 129 and $\mathrm{Iso} \subseteq \mathrm{Qis}$ by Remark 116. ▫

Example 140 We consider the graph morphism $\iota_{\mathrm{D}_0} : \emptyset \to \mathrm{D}_0$.

Recall from Example 128 that ι_{D_0} is a fibration.

Recall from Example 118 that ι_{D_0} is a quasiisomorphism.

So $\iota_{\mathrm{D}_0} : \emptyset \to \mathrm{D}_0$ is an acyclic fibration.

Remark 141 In Gph, the subset of acyclic fibrations $\mathrm{AcFib} \subseteq \mathrm{Mor}$ is closed under retracts; cf. Definition 23.

Proof. Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd}
G' \arrow[r, "f'"] & H' \\
G \arrow[u, "p"] \arrow[r, "f"] & H \arrow[u, "q"'] \\
G' \arrow[u, "i"] \arrow[r, "f'"] \arrow[uu, bend left, "\mathrm{id}_{G'}"] & H' \arrow[u, "j"'] \arrow[uu, bend right, "\mathrm{id}_{H'}"']
\end{tikzcd}$$

We have to show that the graph morphism $f' : G' \to H'$ is an acyclic fibration; cf. Definition 138.

Since the subset of quasiisomorphisms $\mathrm{Qis} \subset \mathrm{Mor}(\mathrm{Gph})$ is closed under retracts the graph morphism $f' : G' \to H'$ is a quasiisomorphism; cf. Remark 117.

Since the subset of fibrations $\mathrm{Fib} \subset \mathrm{Mor}(\mathrm{Gph})$ is closed under retracts the graph morphism $f' : G' \to H'$ is a fibration; cf. Remark 130.

So the subset of acyclic fibrations $\mathrm{AcFib} \subseteq \mathrm{Mor}(\mathrm{Gph})$ is closed under retracts, since the graph morphism $f' : G' \to H'$ is an acyclic fibration; cf. Definition 23. □

Once Lemma 193 below is known, which states that we have $\mathrm{AcFib} = \mathrm{Cofib}^{\boxslash}$, we can also argue by Remark 26 to obtain the statement of Remark 141.

Remark 142 Suppose given acyclic fibrations $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$.

Then the composite $fg : X \to Z$ is also an acyclic fibration.

Proof. Since f and g are fibrations, so is fg; cf. Definition 127.(1) and Remark 131.

Since f and g are quasiisomorphisms, so is fg; cf. Definition 115 and Remark 122.

So the graph morphism $fg : X \to Z$ is a fibration and a quasiisomorphism. Hence fg is an acyclic fibration. □

Once Lemma 193 below is known, which states that we have $\mathrm{AcFib} = \mathrm{Cofib}^{\boxslash}$ we can also argue by Remark 20 to obtain the statement of Remark 142.

Remark 143 In Gph, a pullback of an acyclic fibration is an acyclic fibration; cf. Definition 138.

Proof. Recall that $\mathrm{AcFib} = \mathrm{Fib} \cap \mathrm{Qis}$. I.e. a morphism is an acyclic fibration if and only if it is a fibration and a quasiisomorphism.

A pullback of a fibration is a fibration; cf. Remark 133.

A pullback of a quasiisomorphism is a quasiisomorphism; cf. Remark 125.

So a pullback of an acyclic fibration is an acyclic fibration. □

Once Lemma 193 below is known, which states that we have $\mathrm{AcFib} = \mathrm{Cofib}^{\boxslash}$, we can also argue by Remark 22 to obtain the statement of Remark 143.

3.4 Cofibrations and cofibrant graphs

Definition 144 A graph morphism $f : X \to Y$ is called a *cofibration* if it satisfies $(\mathrm{LLP}_{\mathrm{AcFib}})$; cf. Definitions 13 and 138.

To indicate that f is a cofibration, we often write $X \overset{f}{\bullet\!\!\longrightarrow} Y$.

By

$$\mathrm{Cofib}(\mathrm{Gph}) := {}^{\boxslash}\mathrm{AcFib}(\mathrm{Gph}) \subseteq \mathrm{Mor}(\mathrm{Gph})$$

we denote the set of cofibrations in the category Gph.

We often write $\mathrm{Cofib} := \mathrm{Cofib}(\mathrm{Gph})$.

Remark 145 We have $\mathrm{Iso}(\mathrm{Gph}) \subseteq \mathrm{Cofib}(\mathrm{Gph})$.

Proof. By definition we have $\mathrm{Cofib} = {}^{\boxslash}\mathrm{AcFib}$.

We have $\mathrm{Iso} \subseteq {}^{\boxslash}\mathrm{AcFib} = \mathrm{Cofib}$; cf. Remark 18. ▫

Remark 146 In Gph, the subset of cofibrations $\mathrm{Cofib} \subseteq \mathrm{Mor}$ is closed under retracts; cf. Definition 23.

Proof. Since, by definition, $\mathrm{Cofib} = {}^{\boxslash}\mathrm{AcFib}$ we can argue by Remark 25. ▫

For an example of a cofibration we refer to Remark 155, which makes use of Remark 151.

Remark 147 Suppose given cofibrations $f : X \bullet\!\!\longrightarrow Y$ and $g : Y \bullet\!\!\longrightarrow Z$.

Then the composite $fg : X \to Z$ is also a cofibration.

Proof. Since $\mathrm{Cofib} = {}^{\boxslash}\mathrm{AcFib}$, this follows by Remark 19.

Remark 148 In Gph, a pushout of a cofibration is a cofibration.

Proof. Suppose given a pushout in Gph as follows.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"', "\bullet" marking] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

We have to show that the graph morphism $h : Y \to Y'$ is a cofibration; cf. Definition 144.

We have to show that the graph morphism h satisfies $(\mathrm{LLP}_{\mathrm{AcFib}})$; cf. Definitions 144, 13 and 138.

Suppose given the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"', "\bullet" marking] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[r, "u"] \arrow[d, "h"] & Z \arrow[d, "k", "\dashv" marking] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "v"'] & Z'
\end{tikzcd} .$$

We have to show that there exists a graph morphism $\tilde{w} : Y' \to Z$ such that $h\tilde{w} = u$ and $\tilde{w}k = v$.

Because the graph morphism $X \xrightarrow{g} X'$ is a cofibration there exists a graph morphism $w : X' \to Z$ such that $gw = fu$ and $wk = f'v$.

Because Y' is a pushout and $gw = fu$ there exists a graph morphism $\tilde{w} : Y' \to Z$ such that $f'\tilde{w} = w$ and $h\tilde{w} = u$.

It remains to show that $\tilde{w}k \overset{!}{=} v$.

We have $f'\tilde{w}k = wk = f'v$ and $h\tilde{w}k = uk = hv$.

Cancelling f' and h simultaneously using Remark 6, we obtain $\tilde{w}k = v$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[r, "u"] \arrow[d, "h"] & Z \arrow[d, "k"] \\
X' \arrow[r, "f'"'] \arrow[urr, dotted, "w"] & Y' \arrow[r, "v"'] \arrow[ur, dotted, "\tilde{w}"'] \arrow[ul, phantom, "\lrcorner"] & Z'
\end{tikzcd}$$

□

Using Definition 144 where we have $\mathrm{Cofib(Gph)} := {}^{\boxslash}\mathrm{AcFib(Gph)}$ and Remark 21 the set Cofib(Gph) is stable under pushouts.

Lemma 149 Suppose given a set I and cofibrations $X_i \xrightarrow{g_i} Y_i$ for $i \in I$.

Then the graph morphism $\coprod_{i\in I} g_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is a cofibration; cf. Definition 93.

Proof. Suppose given a commutative quadrangle

$$\begin{tikzcd}
\coprod_{i\in I} X_i \arrow[r, "(a_i)_i"] \arrow[d, "\coprod_{i\in I} g_i"'] & G \arrow[d, "f"] \\
\coprod_{i\in I} Y_i \arrow[r, "(b_i)_i"] & H
\end{tikzcd}$$

in Gph with $a_i : X_i \to G$ and $b_i : Y_i \to H$ for $i \in I$.

We have to show that there exists a graph morphism $(h_i)_i : \coprod_{i\in I} Y_i \to G$ such that $(\coprod_{i\in I} g_i) \cdot (h_i)_i = (a_i)_i$ and $(h_i)_i \cdot f = (b_i)_i$.

We have $(g_i \cdot b_i)_i \overset{\text{Rem. 94}}{=} (\coprod_{i\in I} g_i) \cdot (b_i)_i = (a_i)_i \cdot f \overset{\text{Rem. 92}}{=} (a_i \cdot f)_i : \coprod_{i\in I} X_i \to H$.

So we have $g_i \cdot b_i = a_i \cdot f$ for $i \in I$.

$$\begin{tikzcd}
X_i \arrow[r, "a_i"] \arrow[d, "g_i"'] & G \arrow[d, "f"] \\
Y_i \arrow[r, "b_i"'] & H
\end{tikzcd}$$

Since the graph morphism $g_i : X_i \to Y_i$ is a cofibration, there exists a graph morphism $h_i : Y_i \to G$ such that $g_i \cdot h_i = a_i$ and $h_i \cdot f = b_i$, for each $i \in I$.

$$\begin{tikzcd}
X_i \arrow[r, "a_i"] \arrow[d, "g_i"'] & G \arrow[d, "f"] \\
Y_i \arrow[r, "b_i"'] \arrow[ur, "h_i"] & H
\end{tikzcd}$$

So we have the graph morphism $(h_i)_i : \coprod_{i\in I} Y_i \to G$, where $h_i : Y_i \to G$.

We have $(\coprod_{i\in I} g_i) \cdot (h_i)_i \overset{\text{Rem. 94}}{=} (g_i \cdot h_i)_i = (a_i)_i$ and $(h_i)_i \cdot f \overset{\text{Rem. 92}}{=} (h_i \cdot f)_i = (b_i)_i$.

Therefore, the following diagram commutes.

$$\begin{tikzcd}
\coprod_{i\in I} X_i \arrow[r, "(a_i)_i"] \arrow[d, "\coprod_{i\in I} g_i"'] & G \arrow[d, "f"] \\
\coprod_{i\in I} Y_i \arrow[r, "(b_i)_i"] \arrow[ur, "(h_i)_i"] & H
\end{tikzcd}$$

So the graph morphism $\coprod_{i\in I} g_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is a cofibration. ▫

Definition 150 Suppose given a graph X. The graph X is called *cofibrant* if the graph morphism $\iota_X : \emptyset \to X$ is a cofibration; cf. Remark 70 and Definition 144.

Remark 151 Suppose given $n \in \mathbb{N}$.

The cyclic graph C_n is cofibrant; cf. Definitions 52 and 150.

Proof. Suppose given

$$\begin{tikzcd}
\emptyset \arrow[r, "u"] \arrow[d, "\iota_{\mathrm{C}_n}"'] & X \arrow[d, "f'"] \\
\mathrm{C}_n \arrow[r, "v"'] & Y
\end{tikzcd}$$

in Gph such that $uf' = \iota_{\mathrm{C}_n} v$.

Since the graph morphism $X \xrightarrow{f'} Y$ is an acyclic fibration, f' is in particular a quasiisomorphism; cf. Definition 138. Since $v \in (\mathrm{C}_n\,, Y) \simeq (\mathrm{C}_n\,, X)$, there exists a graph morphism $g : \mathrm{C}_n \to X$ such that $gf' = v$.

We have $\iota_{\mathrm{C}_n} g = \iota_X = u$.

So the graph morphism $\iota_{\mathrm{C}_n} : \emptyset \to \mathrm{C}_n$ is a cofibration and so the cyclic graph C_n is cofibrant. ▫

Remark 152 Suppose given a set I.

Suppose given cofibrant graphs X_i for $i \in I$.

Then the coproduct $\coprod_{i\in I} X_i$ is cofibrant.

Proof. Since the graphs X_i are cofibrant, the graph morphisms $\iota_{X_i} : \emptyset \to X_i$ are cofibrations for $i \in I$.

Since $\coprod_{i\in I} \emptyset = \emptyset$, the graph morphism $\iota_{\coprod_{i\in I} Y_i} : \emptyset \to \coprod_{i\in I} Y_i$ is a cofibration; cf. Lemma 149.

So the coproduct $\coprod_{i\in I} Y_i$ is cofibrant. ▫

Example 153 The cyclic graph C_2 is cofibrant; cf. Remark 151.

Thus the coproduct $\mathrm{C}_2 \sqcup \mathrm{C}_2$ is cofibrant; cf. Remark 152.

Definition 154 Suppose given the graph X.

We have the coproduct $X \sqcup X$; cf. Definition 85.

We define the *diagonal* graph morphism $d_X := \left(\begin{smallmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{smallmatrix}\right) : X \sqcup X \to X$; cf. Definition 86.

In detail, we have

$$\begin{array}{rccccccl} & X & \sqcup & X & \xrightarrow{d_X = \left(\begin{smallmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{smallmatrix}\right)} & X & \\ \mathrm{V}_{d_X} : & (1 & , & v) & \mapsto & v & \text{for } v \in \mathrm{V}_X \\ & (2 & , & v) & \mapsto & v & \text{for } v \in \mathrm{V}_X \\ \mathrm{E}_{d_X} : & (1 & , & e) & \mapsto & e & \text{for } e \in \mathrm{E}_X \\ & (2 & , & e) & \mapsto & e & \text{for } e \in \mathrm{E}_X\ . \end{array}$$

Remark 155 Suppose given $n \in \mathbb{N}$.

The diagonal graph morphism $d_{\mathrm{C}_n} : \mathrm{C}_n \sqcup \mathrm{C}_n \to \mathrm{C}_n$ is a cofibration, i.e. d_{C_n} satisfies $(\mathrm{LLP}_{\mathrm{AcFib}})$, cf. Definitions 144, 138, 154.

Proof. In this proof we abbreviate $d := d_{\mathrm{C}_n}$.

Suppose given the following commutative quadrangle in Gph.

$$\begin{tikzcd} \mathrm{C}_n \sqcup \mathrm{C}_n \arrow[r, "\left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} \mathrm{id}_{\mathrm{C}_n} \\ \mathrm{id}_{\mathrm{C}_n} \end{smallmatrix}\right) = d"'] & X \arrow[d, "f"] \\ \mathrm{C}_n \arrow[r, "a"'] & Y \end{tikzcd}$$

In particular, we have $b_1 f = a$ and $b_2 f = a$.

Since f is a quasiisomorphism, the map $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$ is bijective and so we have $b_1 = b_2 =: \hat{a}$.

Thus $d\hat{a} = \left(\begin{smallmatrix} \mathrm{id}_{\mathrm{C}_n} \\ \mathrm{id}_{\mathrm{C}_n} \end{smallmatrix}\right) \hat{a} = \left(\begin{smallmatrix} \hat{a} \\ \hat{a} \end{smallmatrix}\right) = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$.

Moreover, we have $\hat{a} f = b_1 f = a$.

$$\begin{tikzcd} \mathrm{C}_n \sqcup \mathrm{C}_n \arrow[r, "\left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)"] \arrow[d, "d"'] & X \arrow[d, "f"] \\ \mathrm{C}_n \arrow[r, "a"'] \arrow[ur, "\hat{a}"] & Y \end{tikzcd}$$

So the diagonal graph morphism $d_{\mathrm{C}_n} : \mathrm{C}_n \sqcup \mathrm{C}_n \to \mathrm{C}_n$ is shown to be a cofibration.

The following lemma generalizes Remark 155.

Lemma 156 Suppose given $n \in \mathbb{N}$.

Suppose given sets M' and M, and a map $\mu : M' \to M$ in Mor(Set).

Suppose given a map $\nu : M \to \mathbb{N}$.

We have the coproducts $\coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu}$ and $\coprod_{m \in M} \mathrm{C}_{m\nu}$; cf. Definition 90.

We define the graph morphism $j : \coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu} \to \coprod_{m \in M} \mathrm{C}_{m\nu}$ as follows.

$$\begin{array}{rrcl} j: & \coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu} & \to & \coprod_{m \in M} \mathrm{C}_{m\nu} \\ \mathrm{V}_j : & (m', \mathrm{v}_i) & \mapsto & (m'\mu, \mathrm{v}_i) \text{ for } i \in \mathbb{Z}/m'\mu\nu\mathbb{Z} \\ \mathrm{E}_j : & (m', \mathrm{e}_i) & \mapsto & (m'\mu, \mathrm{e}_i) \text{ for } i \in \mathbb{Z}/m'\mu\nu\mathbb{Z} \end{array}$$

Then $j = (\mathrm{V}_j, \mathrm{E}_j) : \coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu} \to \coprod_{m \in M} \mathrm{C}_{m\nu}$ is a cofibration; cf. Definition 144.

Proof. Since for an edge $(m', \mathrm{e}_i) \in \mathrm{E}_{\coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu}}$ we have

$$\begin{array}{rclcl} (m', \mathrm{e}_i)\, \mathrm{E}_j\, \mathrm{s}_{\coprod_{m \in M} \mathrm{C}_{m\nu}} & = & (m'\mu, \mathrm{e}_i)\, \mathrm{s}_{\coprod_{m \in M} \mathrm{C}_{m\nu}} & = & (m'\mu, \mathrm{e}_i\, \mathrm{s}_{\mathrm{C}_{m'\mu\nu}}) \\ & = & (m', \mathrm{e}_i\, \mathrm{s}_{\mathrm{C}_{m'\mu\nu}})\, \mathrm{V}_j & = & (m', \mathrm{e}_i)\, \mathrm{s}_{\coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu}}\, \mathrm{V}_j \end{array}$$

and

$$\begin{array}{rclcl} (m', \mathrm{e}_i)\, \mathrm{E}_j\, \mathrm{t}_{\coprod_{m \in M} \mathrm{C}_{m\nu}} & = & (m'\mu, \mathrm{e}_i)\, \mathrm{t}_{\coprod_{m \in M} \mathrm{C}_{m\nu}} & = & (m'\mu, \mathrm{e}_i\, \mathrm{t}_{\mathrm{C}_{m'\mu\nu}}) \\ & = & (m', \mathrm{e}_i\, \mathrm{t}_{\mathrm{C}_{m'\mu\nu}})\, \mathrm{V}_j & = & (m', \mathrm{e}_i)\, \mathrm{t}_{\coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu}}\, \mathrm{V}_j\ , \end{array}$$

the tuple $(\mathrm{V}_j\, , \mathrm{E}_j) = j : \coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu} \to \coprod_{m \in M} \mathrm{C}_{m\nu}$ is in fact a graph morphism.

Suppose given

$$\begin{tikzcd} \coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu} \arrow[r, "b"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \coprod_{m \in M} \mathrm{C}_{m\nu} \arrow[r, "a"'] & Y \end{tikzcd}$$

in Gph such that $ja = bf$.

Given $m' \in M'$, we have the graph morphism

$$\begin{array}{rrcl} \iota_{m'} : & \mathrm{C}_{m'\mu\nu} & \to & \coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu} \\ \mathrm{V}_{\iota_{m'}} : & \mathrm{v}_i & \mapsto & (m', \mathrm{v}_i) \\ \mathrm{E}_{\iota_{m'}} : & \mathrm{e}_i & \mapsto & (m', \mathrm{e}_i)\ . \end{array}$$

Since the graph morphism $f : X \to Y$ is an acyclic fibration, in particular a quasiisomorphism, the map $\prod_{m \in M}(\mathrm{C}_{m\nu}\, , f)_{\mathrm{Gph}} : \prod_{m \in M}(\mathrm{C}_{m\nu}\, , X)_{\mathrm{Gph}} \to \prod_{m \in M}(\mathrm{C}_{m\nu}\, , Y)_{\mathrm{Gph}}$ is bijective.

Hence the map $\left(\coprod_{m \in M} \mathrm{C}_{m\nu}\, , f\right)_{\mathrm{Gph}} : \left(\coprod_{m \in M} \mathrm{C}_{m\nu}\, , X\right)_{\mathrm{Gph}} \to \left(\coprod_{m \in M} \mathrm{C}_{m\nu}\, , Y\right)_{\mathrm{Gph}}$ is bijective, cf. Remark 95.

So there exists a unique graph morphism $\hat{a} : \coprod_{m \in M} \mathrm{C}_{m\nu} \to X$ such that $\hat{a} f = a$.

We have

$$\begin{tikzcd} & \coprod_{m' \in M'} \mathrm{C}_{m'\mu\nu} \arrow[r, "b"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \mathrm{C}_{m'\mu\nu} \arrow[ur, "\iota_{m'}"] & \coprod_{m \in M} \mathrm{C}_{m\nu} \arrow[r, "a"'] \arrow[ur, "\hat{a}"] & Y \end{tikzcd}$$

We have to show that $j\hat{a} \overset{!}{=} b$.

It suffices to show that $\iota_{m'} j \hat{a} \overset{!}{=} \iota_{m'} b$ for $m' \in M'$.

Since the graph morphism $f : X \to Y$ is a quasiisomorphism and thus $(\mathrm{C}_{m'\mu\nu}, f)_{\mathrm{Gph}}$ is bijective, it suffices to show that $\iota_{m'} j \hat{a} f \stackrel{!}{=} \iota_{m'} b f$ for $m' \in M'$.

In fact, we have $\iota_{m'} j \hat{a} f = \iota_{m'} j a = \iota_{m'} b f$. ▫

Remark 157 Suppose given a graph X and a cofibrant graph Y, i.e. we have $\emptyset \xrightarrow{\iota_Y} Y$.

Then we have the cofibration $\iota_1 : X \to X \sqcup Y$; cf. Definition 85.

Proof. By Definition 85 and Remark 148 we have

$$\begin{tikzcd} \emptyset \arrow[r, "\iota_Y"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "\iota_1"'] \arrow[ur, phantom, "\lrcorner" very near end] & X \sqcup Y \end{tikzcd} .$$

▫

Using a result obtained **below** in Proposition 210, we show that gluing cofibrant graphs via pushout does not yield a cofibrant graph in general; cf. Assertion 255.

The following Example 158 also shows that fibrations are not necessarily surjective.

Example 158 Suppose given $n \geqslant 0$.

We consider the graph morphism $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$; cf. Remark 70.(2), Definition 56.

(1) The graph morphism $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$ is an acyclic fibration; cf. Definition 138.

(2) The graph morphism $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$ is not a cofibration; cf. Definition 144.
That is, D_n is not cofibrant.

Proof.

Ad (1). Suppose given $k \in \mathbb{N}$.

We have $(\mathrm{C}_k, \mathrm{D}_n)_{\mathrm{Gph}} = \emptyset = (\mathrm{C}_k, \emptyset)_{\mathrm{Gph}}$. So the graph morphism ι_{D_n} is a quasiisomorphism; cf. Definition 115.

Since $\mathrm{V}_\emptyset = \emptyset$, the graph morphism ι_{D_n} is a fibration; cf. Definition 127.(1).

So the graph morphism $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$ is an acyclic fibration.

Ad (2). We have the commutative quadrangle

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "\iota_{\mathrm{D}_n}"'] & \emptyset \arrow[d, "\iota_{\mathrm{D}_n}"'] \\ \mathrm{D}_n \arrow[r, "\mathrm{id}_{\mathrm{D}_n}"'] & \mathrm{D}_n \end{tikzcd}$$

in Gph; cf. (1).

But there does not exist a graph morphism $g : \mathrm{D}_n \to \emptyset$.

So the acyclic fibration $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$ is not a cofibration and thus the graph D_n is not cofibrant; cf. Definition 150. ▫

Once Lemma 185 below is known, which states that we have $\mathrm{AcCofib} = \mathrm{Cofib} \cap \mathrm{Qis}$, we can also argue as follows.

The graph morphism $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$ is a quasiisomorphism; cf. Definition 138.

The graph morphism $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$ is not an acyclic cofibration since (AcCofib 5) is not satisfied.

Since $\mathrm{AcCofib} \overset{\text{Lemma 185}}{=} \mathrm{Cofib} \cap \mathrm{Qis}$, the quasiisomorphism $\iota_{\mathrm{D}_n} : \emptyset \to \mathrm{D}_n$ is not a cofibration.

Example 159 Suppose given $n \in \mathbb{N}$.

We consider the graph morphism $f : \mathrm{D}_0 \to \mathrm{C}_n$ with $\hat{\mathrm{v}}_0 \, \mathrm{V}_f := \mathrm{v}_1$.

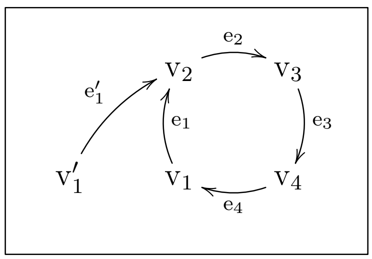
Then the graph morphism $f : \mathrm{D}_0 \to \mathrm{C}_n$ is not a cofibration; cf. Definition 144.

Proof. Consider the graph X with $\mathrm{V}_X := \mathrm{V}_{\mathrm{C}_n} \cup \{\mathrm{v}'_1\}$ and with $\mathrm{E}_X := \mathrm{E}_{\mathrm{C}_n} \cup \{\mathrm{e}'_1\}$ with $\mathrm{e}'_1 \, \mathrm{s}_X := \mathrm{v}'_1$ and with $\mathrm{e}'_1 \, \mathrm{t}_X := \mathrm{v}_2$.

We have the acyclic fibration $g : X \xrightarrow{\;\;} \mathrm{C}_n$ with $g|_{\mathrm{C}_n} := \mathrm{id}_{\mathrm{C}_n}$ and with $\mathrm{e}'_1 \, \mathrm{E}_g := \mathrm{e}_1$ and with $\mathrm{v}'_1 \, \mathrm{V}_g := \mathrm{v}_1$; cf. Definition 138.

E.g. for $n = 4$ we have

X :



So we have $\mathrm{v}_1 \, \mathrm{V}_g = \mathrm{v}'_1 \, \mathrm{V}_g = \mathrm{v}_1$.

Since the graph morphism $g : X \xrightarrow{\;\;} \mathrm{C}_n$ is an acyclic fibration, there exists a unique graph morphism $k : \mathrm{C}_n \to X$ such that $kg = \mathrm{id}_{\mathrm{C}_n}$; cf. Definition 138.

We have $k|^{\mathrm{C}_n} = \mathrm{id}_{\mathrm{C}_n}$.

We have the commutative diagram

$$\begin{tikzcd} \mathrm{D}_0 \arrow[d, "f"'] & X \arrow[d, "g"] \\ \mathrm{C}_n \arrow[r, "\mathrm{id}_{\mathrm{C}_n}"'] \arrow[ur, "\exists !\, k"] & \mathrm{C}_n \end{tikzcd}$$

in Gph.

We will find a graph morphism $p : \mathrm{D}_0 \to X$ such that $f \, \mathrm{id}_{\mathrm{C}_n} = pg$ and such that $fk \overset{!}{\neq} p$.

We have $\mathrm{v}_1 \, \mathrm{V}_g = \mathrm{v}'_1 \, \mathrm{V}_g$ and we have $\mathrm{v}_1 \, \mathrm{V}_k = \mathrm{v}_1$.

So we let $p : \mathrm{D}_0 \to X$ be the graph morphism with $\hat{\mathrm{v}}_0 \, \mathrm{V}_p := \mathrm{v}'_1$.

Now we have $f \, \mathrm{id}_{\mathrm{C}_n} = pg$ and we have $kg = \mathrm{id}_{\mathrm{C}_n}$. But we have $fk \neq p$ since $\hat{\mathrm{v}}_0 \, \mathrm{V}_f \, \mathrm{V}_k = \mathrm{v}_1 \, \mathrm{V}_k = \mathrm{v}_1 \neq \mathrm{v}'_1 = \hat{\mathrm{v}}_0 \, \mathrm{V}_p$.

So the graph morphism $f : \mathrm{D}_0 \to \mathrm{C}_n$ is not a cofibration. ▫

3.5 Bifibrant graphs

Definition 160 A graph X is called *bifibrant* if X is fibrant and cofibrant; cf. Definitions 135 and 150.

Example 161 Suppose given $n \in \mathbb{N}$.

The cyclic graph C_n is bifibrant; cf. Example 137 and Remark 151.

3.6 Acyclic cofibrations

Definition 162 An *acyclic cofibration* is a graph morphism $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$ that satisfies (AcCofib 1–5).

(AcCofib 1) The map $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ is injective.

(AcCofib 2) The map $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ is injective.

(AcCofib 3) We have $|\{e \in \mathrm{E}_H : (e)\,\mathrm{t}_H = v_H\}| = 1$ for $v_H \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$.

(AcCofib 4) We have $(e)\,\mathrm{t}_H \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ for $e \in \mathrm{E}_H \setminus \mathrm{E}_{Gf}$.

(AcCofib 5) For $v_H \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ there exist $n \geqslant 1$ and $e_i \in \mathrm{E}_H$ for $i \in [1, n]$ such that $(e_1)\,\mathrm{s}_H \in \mathrm{V}_{Gf}$, such that $(e_i)\,\mathrm{t}_H = (e_{i+1})\,\mathrm{s}_H$ for $i \in [1, n-1]$ and such that $(e_n)\,\mathrm{t}_H = v_H$.

To indicate that f is an acyclic cofibration, we often write $G \xrightarrow{f} H$.

By AcCofib(Gph) $\subseteq$ Mor(Gph) we denote the set of acyclic cofibrations in the category Gph.

We often write AcCofib := AcCofib(Gph).

Remark 163 Bisson and Tsemo [**3,** Def. 3.2] call the acyclic cofibrations *whiskerings*.

Remark 164 Condition (AcCofib 5) in Definition 162 is equivalent to the following condition (AcCofib 5′).

(AcCofib 5′) For $v_H \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$, there exist $n \geqslant 1$ and a graph morphism $p : \mathrm{D}_n \to H$ with $\hat{\mathrm{v}}_0\,\mathrm{V}_p \in \mathrm{V}_{Gf}$ and $\hat{\mathrm{v}}_n\,\mathrm{V}_p = v_H$.

Here, p is a path from a vertex in V_{Gf} to v_H ; cf. Definition 107.

Remark 165 We have Iso(Gph) $\subseteq$ AcCofib(Gph).

Proof. Suppose given a graph isomorphism $f : G \xrightarrow{\sim} H$.

We have to show that $f : G \xrightarrow{\sim} H$ is an acyclic cofibration; cf. Definition 162.

Ad (AcCofib 1–2). The maps $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ are injective; cf. Definition 55.

Ad (AcCofib 3). We have $\mathrm{V}_H \setminus \mathrm{V}_{Gf} = \mathrm{V}_H \setminus \mathrm{V}_H = \emptyset$.

Ad (AcCofib 4). We have $\mathrm{E}_H \setminus \mathrm{E}_{Gf} = \mathrm{E}_H \setminus \mathrm{E}_H = \emptyset$.

Ad (AcCofib 5). We have $\mathrm{V}_H \setminus \mathrm{V}_{Gf} = \mathrm{V}_H \setminus \mathrm{V}_H = \emptyset$. ▫

Remark 166 Suppose given $0 \leqslant i \leqslant k$ in $\mathbb{Z}_{\geqslant 0}$.

The graph morphism $\iota_{i,k} : \mathrm{D}_i \to \mathrm{D}_k$ is an acyclic cofibration; cf. Definition 56.(3).

Proof.

Ad (AcCofib 1–2). We have $\iota_{i,k} = \mathrm{id}_{\mathrm{D}_k}|_{\mathrm{D}_i}$ and so the maps $\mathrm{V}_{\iota_{i,k}} : \mathrm{V}_{\mathrm{D}_i} \to \mathrm{V}_{\mathrm{D}_k}$ and $\mathrm{E}_{\iota_{i,k}} : \mathrm{E}_{\mathrm{D}_i} \to \mathrm{E}_{\mathrm{D}_k}$ are injective.

Ad (AcCofib 3). We have $\mathrm{V}_{\mathrm{D}_k} \setminus \mathrm{V}_{\mathrm{D}_i \, \iota_{i,k}} = \mathrm{V}_{\mathrm{D}_k} \setminus \mathrm{V}_{\mathrm{D}_i} = \{\hat{\mathrm{v}}_j : j \in [i+1, k]\}$.

So suppose given $j \in [i+1, k]$.

We have $|\{e \in \mathrm{E}_{\mathrm{D}_k} : (e)\,\mathrm{t}_{\mathrm{D}_k} = \hat{\mathrm{v}}_j\}| = |\{\hat{\mathrm{e}}_{j-1}\}| = 1$.

Ad (AcCofib 4). We have $\mathrm{E}_{\mathrm{D}_k} \setminus \mathrm{E}_{\mathrm{D}_i \, \iota_{i,k}} = \mathrm{E}_{\mathrm{D}_k} \setminus \mathrm{E}_{\mathrm{D}_i} = \{\hat{\mathrm{e}}_j : j \in [i, k-1]\}$.

So suppose given $j \in [i, k-1]$.

We have $(\hat{\mathrm{e}}_j)\,\mathrm{t}_{\mathrm{D}_k} = \hat{\mathrm{v}}_{j+1} \in \mathrm{V}_{\mathrm{D}_k} \setminus \mathrm{V}_{\mathrm{D}_i}$.

Ad (AcCofib 5). We have $\mathrm{V}_{\mathrm{D}_k} \setminus \mathrm{V}_{\mathrm{D}_i \, \iota_{i,k}} = \mathrm{V}_{\mathrm{D}_k} \setminus \mathrm{V}_{\mathrm{D}_i} = \{\hat{\mathrm{v}}_j : j \in [i+1, k]\}$.

So suppose given $j \in [i+1, k]$.

We let $n := j - i$. Let $e_u := \hat{\mathrm{e}}_{i+u-1}$ for $u \in [1, n]$. We consider the edges $\{\hat{\mathrm{e}}_u\}_{u \in [i, j-1]}$ We have $e_1\,\mathrm{s}_{\mathrm{D}_k} = \hat{\mathrm{e}}_i\,\mathrm{s}_{\mathrm{D}_k} = \hat{\mathrm{v}}_i \in \mathrm{V}_{\mathrm{D}_i}$. We have $e_u\,\mathrm{t}_{\mathrm{D}_k} = \hat{\mathrm{e}}_{i+u-1}\,\mathrm{t}_{\mathrm{D}_k} = \hat{\mathrm{v}}_{i+u} = \hat{\mathrm{e}}_{i+u}\,\mathrm{s}_{\mathrm{D}_k} = e_{u+1}\,\mathrm{s}_{\mathrm{D}_k}$ for $u \in [1, n-1]$. And we have $e_n\,\mathrm{t}_{\mathrm{D}_k} = \hat{\mathrm{e}}_{j-1}\,\mathrm{t}_{\mathrm{D}_k} = \hat{\mathrm{v}}_j$. ▫

Example 167 We consider the graphs

$$G : \quad \boxed{\begin{tikzcd} 1 \arrow[r, "\alpha_1"] & 2 \arrow[r, bend left, "\alpha_2"] \arrow[r, bend right, "\alpha_3"'] & 3 \end{tikzcd}}$$

and

$$H : \quad \boxed{\begin{tikzcd} 1 \arrow[r, "\beta_1"] & 2 \arrow[r, bend left, "\beta_2"] \arrow[r, bend right, "\beta_3"'] \arrow[d, "\beta_7"'] & 3 \arrow[r, "\beta_4"] & 4 \arrow[r, "\beta_5"] \arrow[d, "\beta_6"'] & 5 \\ & 7 & & 6 & \end{tikzcd}}$$

Let $f = (\mathrm{V}_f, \mathrm{E}_f) : G \to H$ be the graph morphism with

$$1\,\mathrm{V}_f = 1, \quad 2\,\mathrm{V}_f = 2, \quad 3\,\mathrm{V}_f = 3,$$

and with

$$\alpha_1\,\mathrm{E}_f = \beta_1\,, \quad \alpha_2\,\mathrm{E}_f = \beta_2\,, \quad \alpha_3\,\mathrm{E}_f = \beta_3\,.$$

Then the graph morphism f is an acyclic cofibration; cf. Definition 162.

Proof. We use this opportunity to illustrate the properties (AcCofib 1–5) from Definition 162.

Ad (AcCofib 1, 2).

The maps V_f and E_f both are injective.

Ad (AcCofib 3).

We have to show that $|\{e \in \mathrm{E}_H : (e)\,\mathrm{t}_H = v_H\}| = 1$ for $v_H \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$.

We have $\mathrm{V}_H \setminus \mathrm{V}_{Gf} = \{1,2,3,4,5,6,7\} \setminus \{1,2,3\} = \{4,5,6,7\}$.

We have $|\{e \in \mathrm{E}_H : (e)\,\mathrm{t}_H = 4\}| = |\{\beta_4\}| = 1$ and $|\{e \in \mathrm{E}_H : (e)\,\mathrm{t}_H = 5\}| = |\{\beta_5\}| = 1$ and $|\{e \in \mathrm{E}_H : (e)\,\mathrm{t}_H = 6\}| = |\{\beta_6\}| = 1$ and $|\{e \in \mathrm{E}_H : (e)\,\mathrm{t}_H = 7\}| = |\{\beta_7\}| = 1$.

So (AcCofib 3) holds for f.

Ad (AcCofib 4).

We have to show that $(e)\,\mathrm{t}_H \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ for $e \in \mathrm{E}_H \setminus \mathrm{E}_{Gf}$.

We have $\mathrm{E}_H \setminus \mathrm{E}_{Gf} = \{\beta_1\,, \beta_2\,, \beta_3\,, \beta_4\,, \beta_5\,, \beta_6\,, \beta_7\} \setminus \{\beta_1\,, \beta_2\,, \beta_3\} = \{\beta_4\,, \beta_5\,, \beta_6\,, \beta_7\}$.

We have $\mathrm{V}_H \setminus \mathrm{V}_{Gf} = \{4,5,6,7\}$.

We have $\beta_4\,\mathrm{t}_H = 4$, $\beta_5\,\mathrm{t}_H = 5$, $\beta_6\,\mathrm{t}_H = 6$, $\beta_7\,\mathrm{t}_H = 7$, which are elements in $\mathrm{V}_H \setminus \mathrm{V}_{Gf} = \{4,5,6,7\}$, so (AcCofib 4) holds for f.

Ad (AcCofib 5).

We have $\mathrm{V}_H \setminus \mathrm{V}_{Gf} = \{4,5,6,7\}$.

We have $\mathrm{V}_{Gf} = \{1,2,3\}$.

For the vertex $4 \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ we may choose $n := 1$ and $e_1 := \beta_4$. Then $e_1\,\mathrm{s}_H = \beta_4\,\mathrm{s}_H = 3 \in \mathrm{V}_{Gf}$ and $e_1\,\mathrm{t}_H = \beta_4\,\mathrm{t}_H = 4$.

For the vertex $5 \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ we may choose $n := 2$ and $e_1 := \beta_4$ and $e_2 := \beta_5 \in \mathrm{E}_H$. Then $e_1\,\mathrm{s}_H = \beta_4\,\mathrm{s}_H = 3 \in \mathrm{V}_{Gf}$ and $e_1\,\mathrm{t}_H = \beta_4\,\mathrm{t}_H = 4 = \beta_5\,\mathrm{s}_H = e_2\,\mathrm{s}_H$ and $e_2\,\mathrm{t}_H = \beta_5\,\mathrm{t}_H = 5$.

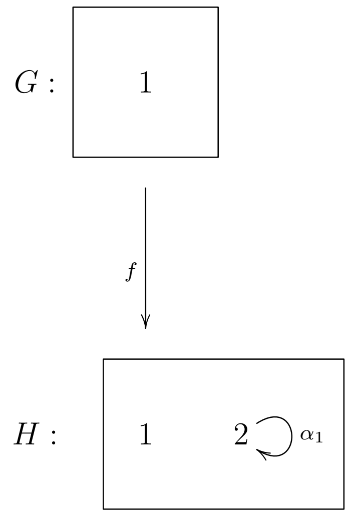
For the vertex $6 \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ we may choose $n := 2$ and $e_1 := \beta_4$ and $e_2 := \beta_6 \in \mathrm{E}_H$. Then $e_1\,\mathrm{s}_H = \beta_4\,\mathrm{s}_H = 3 \in \mathrm{V}_{Gf}$ and $e_1\,\mathrm{t}_H = \beta_4\,\mathrm{t}_H = 4 = \beta_6\,\mathrm{s}_H = e_2\,\mathrm{s}_H$ and $e_2\,\mathrm{t}_H = \beta_6\,\mathrm{t}_H = 6$.

For the vertex $7 \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ we may choose $n := 1$ and $e_1 := \beta_7$. Then $e_1\,\mathrm{s}_H = \beta_7\,\mathrm{s}_H = 2 \in \mathrm{V}_{Gf}$ and $e_1\,\mathrm{t}_H = \beta_7\,\mathrm{t}_H = 7$. ◻

Alternatively, via Magma [**2**] we may proceed as follows, using the functions given in §10 below.

```
G := <[1,2,3],[<1,1,2>,<2,2,3>,<2,3,3>]>;
H := <[1,2,3,4,5,6,7],[<1,1,2>,<2,2,3>,<2,3,3>,<3,4,4>,<4,5,5>,<4,6,6>,<2,7,7>]>;
f := <[<1,1>,<2,2>,<3,3>],[<<1,1,2>,<1,1,2>>,<<2,2,3>,<2,2,3>>,<<2,3,3>,<2,3,3>>]>;
> AcCofib1to4(f,G,H);
true
> AcCofib5(f,G,H);
true
> IsAcCofib(f,G,H);
true
```

Example 168 We consider the following graph morphism.



We let $1\,\mathrm{V}_f := 1$.

Then the graph morphism f satisfies (AcCofib 1–4), but **not** (AcCofib 5).

Hence the graph morphism f is **not** an acyclic cofibration.

Proof. We show that (AcCofib 5) is not satisfied.

We *assume* that (AcCofib 5) is satisfied.

So for $v_H \in \mathrm{V}_H \setminus \mathrm{V}_{Gf}$ there exist $n \geqslant 1$ and $e_i \in \mathrm{E}_H$ for $i \in [1, n]$ such that $e_1\,\mathrm{s}_H \in \mathrm{V}_{Gf}$, such that $e_i\,\mathrm{t}_H = e_{i+1}\,\mathrm{s}_H$ for $i \in [1, n-1]$ and such that $e_n\,\mathrm{t}_H = v_H$.

Then $e_1\,\mathrm{s}_H = 1$. But there is no edge in H with this property. *Contradiction.*

Alternatively, via Magma we may proceed as follows, using the functions given in §10 below.

```
G := <[1],[]>;
H := <[1,2],[<2,1,2>]>;
f := <[<1,1>],[]>;
> IsGraphMorphism(f,G,H);
true
> AcCofib1to4(f,G,H);
true
> AcCofib5(f,G,H);
false
> IsAcCofib(f,G,H);
false
```

$\square$

Remark 169 Suppose given acyclic cofibrations $f : X \mathrel{-\!\!\circ\!\!\rightarrowtail} Y$ and $g : Y \mathrel{-\!\!\circ\!\!\rightarrowtail} Z$.

Then the composite fg is also an acyclic cofibration.

Proof.

We have to show that (AcCofib 1–5) hold for the graph morphism fg.

Ad (AcCofib 1).

The composite of injective maps is injective.

The composite of the injective maps V_f and V_g yields the injective map $\mathrm{V}_{fg} = \mathrm{V}_f \cdot \mathrm{V}_g$ and so (AcCofib 1) holds for fg.

Ad (AcCofib 2).

The composite of injective maps is injective.

The composite of the injective maps E_f and E_g yields the injective map $\mathrm{E}_{fg} = \mathrm{E}_f \cdot \mathrm{E}_g$ and so (AcCofib 2) holds for fg.

Ad (AcCofib 3).

We have $|\{e_Y \in \mathrm{E}_Y : (e_Y)\,\mathrm{t}_Y = v_Y\}| = 1$ for $v_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

We have $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| = 1$ for $v_Z \in \mathrm{V}_Z \setminus \mathrm{V}_{Yg}$.

We have to show that $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \overset{!}{=} 1$ for $v_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$.

We have $\mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg} = \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_f\,\mathrm{V}_g \overset{\text{Rem. 30}}{=} (\mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g) \,\dot{\cup}\, (\mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f)\,\mathrm{V}_g$, since $\mathrm{V}_{fg} = \mathrm{V}_f\,\mathrm{V}_g$.

Suppose given $v_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$. We have to show that $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \overset{!}{=} 1$.

We consider two cases.

Case 1: $v_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g$. We have to show that $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \overset{!}{=} 1$. This follows by g being an acyclic cofibration.

Case 2: $v_Z \in (\mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f)\,\mathrm{V}_g$. We have to show that $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \overset{!}{=} 1$. There exists $v_Y \in \mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f$ with $(v_Y)\,\mathrm{V}_g = v_Z$. We obtain $|\{e_Y \in \mathrm{E}_Y : (e_Y)\,\mathrm{t}_Y = v_Y\}| = 1$ by f being an acyclic cofibration.

We obtain $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \overset{!}{=} 1$:

First, we show that $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \overset{!}{\geqslant} 1$.

Suppose given $e_Y \in \mathrm{E}_Y$ with $(e_Y)\,\mathrm{t}_Y = v_Y$. Then $(e_Y\,\mathrm{E}_g)\,\mathrm{t}_Z = (e_Y\,\mathrm{t}_Y)\,\mathrm{V}_g = v_Y\,\mathrm{V}_g = v_Z$. So we obtain $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \geqslant 1$.

Second, we show that $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| \overset{!}{\leqslant} 1$.

Suppose given $e_Z, e'_Z \in \mathrm{E}_Z$ with $(e_Z)\,\mathrm{t}_Z = v_Z = (e'_Z)\,\mathrm{t}_Z$. Since $v_Z \in (\mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f)\,\mathrm{V}_g$, we may write $v_Z = v_Y\,\mathrm{V}_g$ with $v_Y \notin (\mathrm{V}_X)\,\mathrm{V}_f$.

We have to show that $e_Z \overset{!}{=} e'_Z$.

We *claim* $e_Z \in (\mathrm{E}_Y)\,\mathrm{E}_g$.

We *assume* $e_Z \notin (\mathrm{E}_Y)\,\mathrm{E}_g$. With (AcCofib 4) for g we get
$(\mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f)\,\mathrm{V}_g \ni v_Z = (e_Z)\,\mathrm{t}_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g$, which is a *contradiction*.

This proves the *claim*.

So we have $e_Z \in (\mathrm{E}_Y)\,\mathrm{E}_g$. Likewise we have $e'_Z \in (\mathrm{E}_Y)\,\mathrm{E}_g$.

There exist $e_Y, e'_y \in \mathrm{E}_Y$ such that $e_Z = e_Y\,\mathrm{E}_g$ and $e'_Z = e'_Y\,\mathrm{E}_g$.

We have to show that $e_Y \overset{!}{=} e'_Y$.

But we have $(e_Y\,\mathrm{t}_Y)\,\mathrm{V}_g = (e_Y\,\mathrm{E}_g)\,\mathrm{t}_Z = (e_Z)\,\mathrm{t}_Z = v_Z = (e'_Z)\,\mathrm{t}_Z = (e'_Y\,\mathrm{E}_g)\,\mathrm{t}_Z = (e'_Y\,\mathrm{t}_Y)\,\mathrm{V}_g$.

Moreover, $v_Z = (v_Y)\,\mathrm{V}_g$.

Because V_g is injective we get $e_Y\,\mathrm{t}_Y = e'_Y\,\mathrm{t}_Y = v_Y$.

And so $e_Y, e'_Y \in \{\tilde{e}_Y \in \mathrm{E}_Y : (\tilde{e}_Y)\,\mathrm{t}_Y = v_Y\}$ which contains at most one element by (AcCofib 3) for f. So $e_Y = e'_Y$.

This proves $|\{e_Z \in \mathrm{E}_Z : (e_Z)\,\mathrm{t}_Z = v_Z\}| = 1$.

So (AcCofib 3) holds for fg.

Ad (AcCofib 4).

We have $(e_Y)\,\mathrm{t}_Y \in \mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f$ for $e_Y \in \mathrm{E}_Y \setminus (\mathrm{E}_X)\,\mathrm{E}_f$ by (AcCofib 4) for f.

We have $(e_Z)\,\mathrm{t}_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g$ for $e_Z \in \mathrm{E}_Z \setminus (E_Y)\,\mathrm{E}_g$ by (AcCofib 4) for g.

We have $\mathrm{E}_Z \setminus (\mathrm{E}_X)\,\mathrm{E}_{fg} = \mathrm{E}_Z \setminus (\mathrm{E}_X)\,\mathrm{E}_f\,\mathrm{E}_g \overset{\text{Rem. 30}}{=} (\mathrm{E}_Z \setminus (\mathrm{E}_Y)\,\mathrm{E}_g) \,\dot\cup\, (\mathrm{E}_Y \setminus (\mathrm{E}_X)\,\mathrm{E}_f)\,\mathrm{E}_g$, since $\mathrm{E}_{fg} = \mathrm{E}_f\,\mathrm{E}_g$.

Suppose given $e_Z \in \mathrm{E}_Z \setminus (\mathrm{E}_X)\,\mathrm{E}_{fg}$. We have to show that $(e_Z)\,\mathrm{t}_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$.

We consider two cases.

Case 1: $e_Z \in \mathrm{E}_Z \setminus (\mathrm{E}_Y)\,\mathrm{E}_g$. We have to show that $(e_Z)\,\mathrm{t}_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$. We obtain:

We have $(e_Z)\,\mathrm{t}_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g \subseteq \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_f\,\mathrm{V}_g = \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$.

Case 2: $e_Z \in (\mathrm{E}_Y \setminus (\mathrm{E}_X)\,\mathrm{E}_f)\,\mathrm{E}_g$. We have to show that $(e_Z)\,\mathrm{t}_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$. There exists $e_Y \in \mathrm{E}_Y \setminus (\mathrm{E}_X)\,\mathrm{E}_f$ such that $(e_Y)\,\mathrm{E}_g = e_Z$. We have $(e_Y)\,\mathrm{t}_Y \in \mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f$. So we get

$$(e_Z)\,\mathrm{t}_Z = (e_Y)\,\mathrm{E}_g\,\mathrm{t}_Z = (e_Y)\,\mathrm{t}_Y\,\mathrm{V}_g \in (\mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f)\,\mathrm{V}_g \overset{\text{Rem. 30}}{\subseteq} \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}\,.$$

So (AcCofib 4) holds for fg.

Ad (AcCofib 5).

Suppose given $v_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$. We have so show that there exist $n \geqslant 1$ and $e_i \in \mathrm{E}_Z$ for $i \in [1, n]$ such that $e_1\,\mathrm{s}_Z \in (\mathrm{V}_X)\,\mathrm{V}_{fg}$, such that $e_i\,\mathrm{t}_Z = e_{i+1}\,\mathrm{s}_Z$ for $i \in [1, n-1]$ and such that $e_n\,\mathrm{t}_Z = v_Z$.

We have $(\mathrm{V}_X)\,\mathrm{V}_{fg} \subseteq (\mathrm{V}_Y)\,\mathrm{V}_g \subseteq \mathrm{V}_Z$.

We have $\mathrm{V}_Z \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg} = (\mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g) \,\dot\cup\, ((\mathrm{V}_Y\,\mathrm{V}_g) \setminus (\mathrm{V}_X\,\mathrm{V}_{fg})) = (\mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g) \,\dot\cup\, (\mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f)\,\mathrm{V}_g$; cf. Remark 30.

Case $v_Z \in (\mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f)\,\mathrm{V}_g$.

There exists a unique vertex $v_Y \in \mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f$ such that $v_Y\,\mathrm{V}_g = v_Z$.

The graph morphism $f : X \to Y$ is an acyclic cofibration. So we may choose $n \geqslant 1$ and $\tilde{e}_i \in \mathrm{E}_Y$ for $i \in [1,n]$ such that $\tilde{e}_1 \,\mathrm{s}_Y \in (\mathrm{V}_X)\,\mathrm{V}_f$, such that $\tilde{e}_i \,\mathrm{t}_Y = \tilde{e}_{i+1} \,\mathrm{s}_Y$ for $i \in [1,n-1]$ and such that $\tilde{e}_n \,\mathrm{t}_Y = v_Y$.

So $(\tilde{e}_1 \,\mathrm{E}_g)\,\mathrm{s}_Z = (\tilde{e}_1 \,\mathrm{s}_Y)\,\mathrm{V}_g \in (\mathrm{V}_X)\,\mathrm{V}_{fg}$. Moreover, $(\tilde{e}_i \,\mathrm{E}_g)\,\mathrm{t}_Z = (\tilde{e}_i \,\mathrm{t}_Y)\,\mathrm{V}_g = (\tilde{e}_{i+1} \,\mathrm{s}_Y)\,\mathrm{V}_g = (\tilde{e}_{i+1} \,\mathrm{E}_g)\,\mathrm{s}_Z$ for $i \in [1,n-1]$. Finally, $(\tilde{e}_n \,\mathrm{E}_g)\,\mathrm{t}_Z = (\tilde{e}_n \,\mathrm{t}_Y)\,\mathrm{V}_g = v_Y \,\mathrm{V}_g = v_Z$.

So we may take $e_i := \tilde{e}_i \,\mathrm{E}_g$ for $i \in [1,n]$.

Case $v_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g$.

The graph morphism $g : X \to Y$ is an acyclic cofibration. So we may choose $k \geqslant 1$ and $e'_i \in \mathrm{E}_Z$ for $i \in [1,k]$ such that $e'_1 \,\mathrm{s}_Z \in (\mathrm{V}_Y)\,\mathrm{V}_g$, such that $e'_i \,\mathrm{t}_Z = e'_{i+1} \,\mathrm{s}_Z$ for $i \in [1,k-1]$ and such that $e'_k \,\mathrm{t}_Z = v_Z$.

There exists a unique vertex $v_Y \in \mathrm{V}_Y$ such that $v_Y \,\mathrm{V}_g = e'_1 \,\mathrm{s}_Z$.

Subcase 1: $v_Y \in (\mathrm{V}_X)\,\mathrm{V}_f$.

We map $n := k$, and $e_i := e'_i$ for $i \in [1,n]$.

Then, in particular, $e_1 \,\mathrm{s}_Z = e'_1 \,\mathrm{s}_Z = v_Y \,\mathrm{V}_g \in (\mathrm{V}_X)\,\mathrm{V}_{fg}$.

Subcase 2: $v_Y \in \mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_{fg}$.

Because $f : X \to Y$ is an acyclic cofibration and because of $v_Y \in \mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f$, we may choose $m \geqslant 1$ and $e''_i \in \mathrm{E}_Y$ for $i \in [1,m]$ such that $e''_1 \,\mathrm{s}_Y \in (\mathrm{V}_X)\,\mathrm{V}_f$, such that $e''_i \,\mathrm{t}_Y = e''_{i+1} \,\mathrm{s}_Y$ for $i \in [1,m-1]$ and such that $e''_m \,\mathrm{t}_Y = v_Y$.

Let $n := m+k$. For $i \in [1,n]$ let $e_i := \begin{cases} e''_i \,\mathrm{E}_g & \text{if } i \in [1,m] \\ e'_{i-m} & \text{if } i \in [m+1,m+k]. \end{cases}$

Then $e_1 \,\mathrm{s}_Z = e''_1 \,\mathrm{E}_g \,\mathrm{s}_Z = e''_1 \,\mathrm{s}_Y \,\mathrm{V}_g \in (\mathrm{V}_X)\,\mathrm{V}_f \,\mathrm{V}_g = (\mathrm{V}_X)\,\mathrm{V}_{fg}$.

For $i \in [1,m-1]$ we have $e_i \,\mathrm{t}_Z = e''_i \,\mathrm{E}_g \,\mathrm{t}_Z = e''_i \,\mathrm{t}_Y \,\mathrm{V}_g = e''_{i+1} \,\mathrm{s}_Y \,\mathrm{V}_g = e''_{i+1} \,\mathrm{E}_g \,\mathrm{s}_Z = e_{i+1} \,\mathrm{s}_Z$.

For $i = m$, we have $e_m \,\mathrm{t}_Z = e''_m \,\mathrm{E}_g \,\mathrm{t}_Z = e''_m \,\mathrm{t}_Y \,\mathrm{V}_g = v_Y \,\mathrm{V}_g = e'_1 \,\mathrm{s}_Z = e_{m+1} \,\mathrm{s}_Z$.

For $i \in [m+1,n-1]$ we have $e_i \,\mathrm{t}_Z = e'_{i-m} \,\mathrm{t}_Z = e'_{i+1-m} \,\mathrm{s}_Z = e_{i+1} \,\mathrm{s}_Z$.

Finally, we have $e_n \,\mathrm{t}_Z = e_{m+k} \,\mathrm{t}_Z = e'_k \,\mathrm{t}_Z = v_Z$.

So we may choose $m \geqslant 1$ and $\tilde{e}_i \,\mathrm{E}_g \in \mathrm{E}_Y \,\mathrm{E}_g \subseteq \mathrm{E}_Z$ for $i \in [1,m]$ such that $(\tilde{e}_1 \,\mathrm{E}_g)\,\mathrm{s}_Z = (\tilde{e}_1 \,\mathrm{s}_Y)\,\mathrm{V}_g \in (\mathrm{V}_X)\,\mathrm{V}_{fg}$, such that $(\tilde{e}_i \,\mathrm{E}_g)\,\mathrm{t}_Z = (\tilde{e}_i \,\mathrm{t}_Y)\,\mathrm{V}_g = (\tilde{e}_{i+1} \,\mathrm{s}_Y)\,\mathrm{V}_g = (\tilde{e}_{i+1})E_g)\,\mathrm{s}_Z$ for $i \in [1,m-1]$ and such that $(\tilde{e}_m \,\mathrm{E}_g)\,\mathrm{t}_Z = (\tilde{e}_m \,\mathrm{t}_Y)\,\mathrm{V}_g = v_Y \,\mathrm{V}_g = v_Z$.

So for $v_Z \in \mathrm{V}_Z \setminus (\mathrm{V}_Y)\,\mathrm{V}_g$ we may choose $n := m+k \geqslant 1$ and $\tilde{e}_i \,\mathrm{E}_g \in \mathrm{E}_Y \,\mathrm{E}_g \subseteq \mathrm{E}_Z$ for $i \in [1,m]$ such that $(\tilde{e}_1 \,\mathrm{E}_g)\,\mathrm{s}_Z = (\tilde{e}_1 \,\mathrm{s}_Y)\,\mathrm{V}_g \in (\mathrm{V}_X)\,\mathrm{V}_{fg}$, such that $(\tilde{e}_i \,\mathrm{E}_g)\,\mathrm{t}_Z = (\tilde{e}_i \,\mathrm{t}_Y)\,\mathrm{V}_g = (\tilde{e}_{i+1} \,\mathrm{s}_Y)\,\mathrm{V}_g = (\tilde{e}_{i+1})E_g)\,\mathrm{s}_Z$ for $i \in [1,m-1]$ and such that $(\tilde{e}_m \,\mathrm{E}_g)\,\mathrm{t}_Z = (\tilde{e}_m \,\mathrm{t}_Y)\,\mathrm{V}_g = v_Y \,\mathrm{V}_g = v_Z$.

And we may choose $e_i \in \mathrm{E}_Z$ for $i \in [m+1,m+k]$ such that $e_{m+1} \,\mathrm{s}_Z \in \mathrm{V}_Y \,\mathrm{V}_g$, such that $e_i \,\mathrm{t}_Z = e_{i+1} \,\mathrm{s}_Z$ for $i \in [m+1,m+k-1]$ and such that $e_{m+k} \,\mathrm{t}_Z = v_Z$.

So we may choose $n := k+m \geqslant 1$ and $e_i \in \mathrm{E}_Z$ for $i \in [1,n]$ such that $e_1 \,\mathrm{s}_Z \in (\mathrm{V}_X)\,\mathrm{V}_{fg}$, such that $e_i \,\mathrm{t}_Z = e_{i+1} \,\mathrm{s}_Z$ for $i \in [1,n-1]$ and such that $e_n \,\mathrm{t}_Z = v_Z$.

So (AcCofib 5) holds for fg and so the graph morphism $fg : X \to Z$ is an acyclic cofibration. ▫

Once Lemma 191 below is known, which states that we have $\mathrm{AcCofib} = {}^{\boxslash}\mathrm{Fib}$, we can also argue by Remark 19 to obtain the statement of Remark 169.

Remark 170 Suppose given an acyclic cofibration $f : X \multimap Y$.

Then the image $Xf \subseteq Y$ is a full subgraph of Y; cf. Definition 47.

Proof. We have to show that we have $\mathrm{E}_{Xf} = \{e_Y \in \mathrm{E}_Y : e_Y\,\mathrm{s}_Y \in \mathrm{V}_{Xf} \text{ and } e_Y\,\mathrm{t}_Y \in \mathrm{V}_{Xf}\}$, i.e. we have to show that we have $e \in \mathrm{E}_{Xf}$ for $e \in \mathrm{E}_Y$, $v, w \in \mathrm{V}_{Xf}$ with $e\,\mathrm{s}_Y = v$, $e\,\mathrm{t}_Y = w$.

Suppose given $e \in \mathrm{E}_Y$, $v, w \in \mathrm{V}_{Xf}$ with $e\,\mathrm{s}_Y = v$, $e\,\mathrm{t}_Y = w$.

We have to show that $e \in \mathrm{E}_{Xf} \subseteq \mathrm{E}_Y$.

We have $(e)\,\mathrm{t}_Y \in \mathrm{V}_Y \setminus (\mathrm{V}_X)\,\mathrm{V}_f$ for $e \in \mathrm{E}_Y \setminus (\mathrm{E}_X)\,\mathrm{E}_f$; cf. Definition 162 (AcCofib 4).

So we have $e \in (\mathrm{E}_X)\,\mathrm{E}_f$ for $(e)\,\mathrm{t}_Y \in (\mathrm{V}_X)\,\mathrm{V}_f$.

We have $e\,\mathrm{t}_Y = w \in \mathrm{V}_{Xf} = \mathrm{V}_X\,\mathrm{V}_f$.

So we have $e \in (\mathrm{E}_X)\,\mathrm{E}_f = \mathrm{E}_{Xf}$. □

Remark 171 In Gph, a pushout of an acyclic cofibration is an acyclic cofibration; cf. Definition 162.

Proof. Suppose given a pushout in Gph as follows.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"', circle] & Y \arrow[d, "g"] \\ X' \arrow[r, "a'"'] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

We use the alternative construction for the pushout in Gph from Remark 84; cf. Remark 38.

We may use this construction of the pushout to prove that g is an acyclic cofibration, since pushouts are unique up to isomorphism; cf. Remarks 7, 165 and 169.

We have the pushouts

$$\begin{tikzcd} \mathrm{V}_X \arrow[r, "\mathrm{V}_a"] \arrow[d, "\mathrm{V}_f"'] & \mathrm{V}_Y \arrow[d, "\mathrm{V}_g"] \\ \mathrm{V}_{X'} \arrow[r, "\mathrm{V}_{a'}"'] & \mathrm{V}_{Y'} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{E}_X \arrow[r, "\mathrm{E}_a"] \arrow[d, "\mathrm{E}_f"'] & \mathrm{E}_Y \arrow[d, "\mathrm{E}_g"] \\ \mathrm{E}_{X'} \arrow[r, "\mathrm{E}_{a'}"'] & \mathrm{E}_{Y'} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

in Gph; cf. Remarks 38 and 84.

Ad (AcCofib 1, 2).

Since the graph morphism $f : X \to X'$ is an acyclic cofibration, the maps $\mathrm{V}_f : \mathrm{V}_X \to \mathrm{V}_{X'}$ and $\mathrm{E}_f : \mathrm{E}_X \to \mathrm{E}_{X'}$ are injective; cf. (AcCofib 1, 2).

So the maps $\mathrm{V}_g : \mathrm{V}_Y \to \mathrm{V}_{Y'}$ and $\mathrm{E}_g : \mathrm{E}_Y \to \mathrm{E}_{Y'}$ are injective; cf. Remark 39.

So (AcCofib 1) and (AcCofib 2) hold for the graph morphism $g : Y \to Y'$.

Ad (AcCofib 3). Suppose given a vertex $v_{Y'} \in \mathrm{V}_{Y'} \setminus \mathrm{V}_{Yg}$.

So $v_{Y'} = (1, v_{X'}) \in \mathrm{V}_{Y'} = (\mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}) \sqcup \mathrm{V}_Y$ with $v_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}$.

We have to show that $|\{e_{Y'} \in \mathrm{E}_{Y'} : e_{Y'} \,\mathrm{t}_{Y'} = v_{Y'}\}| \overset{!}{=} 1$.

First, we show that $|\{e_{Y'} \in \mathrm{E}_{Y'} : e_{Y'} \,\mathrm{t}_{Y'} = v_{Y'}\}| \overset{!}{\geqslant} 1$.

The graph morphism $f : X \to X'$ is an acyclic cofibration and we have $v_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}$.

Because of (AcCofib 3) for f there exists a unique edge $e_{X'} \in \mathrm{E}_{X'}$ such that $e_{X'} \,\mathrm{t}_{X'} = v_{X'}$.

Assume that $e_{X'} \in \mathrm{E}_{Xf}$. Then $e_{X'} = e_X \,\mathrm{E}_f$ for some $e_X \in \mathrm{E}_X$.

Hence $v_{X'} = e_{X'} \,\mathrm{t}_{X'} = e_X \,\mathrm{E}_f \,\mathrm{t}_{X'} = (e_X \,\mathrm{t}_X)\,\mathrm{V}_f \in \mathrm{V}_{Xf}$. *Contradiction.*

So $e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}$. Thus we have $(1, e_{X'}) \in \mathrm{E}_{Y'} = (\mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}) \sqcup \mathrm{E}_Y$.

We obtain $(1, e_{X'})\,\mathrm{t}_{Y'} = e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} = v_{X'} \,\mathrm{V}_{a'} = (1, v_{X'}) = v_{Y'}$, because $v_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}$.

Second, we show that $|\{e_{Y'} \in \mathrm{E}_{Y'} : e_{Y'} \,\mathrm{t}_{Y'} = v_{Y'}\}| \overset{!}{\leqslant} 1$.

Suppose given $e_{Y'}, \tilde{e}_{Y'} \in \mathrm{E}_{Y'}$ such that $e_{Y'} \,\mathrm{t}_{Y'} = v_{Y'} = \tilde{e}_{Y'} \,\mathrm{t}_{Y'}$.

We have to show that $e_{Y'} \overset{!}{=} \tilde{e}_{Y'}$.

We *assume* that $e_{Y'} \in \mathrm{E}_{Yg}$. Then there exists an edge $e_Y \in \mathrm{E}_Y$ such that $e_{Y'} = e_Y \,\mathrm{E}_g$.

So we have $v_{Y'} = e_{Y'} \,\mathrm{t}_{Y'} = e_Y \,\mathrm{E}_g \,\mathrm{t}_{Y'} = (e_Y \,\mathrm{t}_Y)\,\mathrm{V}_g \in \mathrm{V}_{Yg}$, which is a *contradiction.*

So we have $e_{Y'} \in \mathrm{E}_{Y'} \setminus \mathrm{E}_{Yg}$. Thus there exists an edge $e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}$ such that $e_{Y'} = (1, e_{X'})$.

We *assume* that $\tilde{e}_{Y'} \in \mathrm{E}_{Yg}$. Then there exists an edge $\tilde{e}_Y \in \mathrm{E}_Y$ such that $\tilde{e}_{Y'} = \tilde{e}_Y \,\mathrm{E}_g$.

So we have $v_{Y'} = \tilde{e}_{Y'} \,\mathrm{t}_{Y'} = \tilde{e}_Y \,\mathrm{E}_g \,\mathrm{t}_{Y'} = (\tilde{e}_Y \,\mathrm{t}_Y)\,\mathrm{V}_g \in \mathrm{V}_{Yg}$, which is a *contradiction.*

So we have $\tilde{e}_{Y'} \in \mathrm{E}_{Y'} \setminus \mathrm{E}_{Yg}$. Thus there exists an edge $\tilde{e}_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}$ such that $\tilde{e}_{Y'} = (1, \tilde{e}_{X'})$.

Now we have

$$\begin{aligned} v_{Y'} &= e_{Y'} \,\mathrm{t}_{Y'} = (1, e_{X'})\,\mathrm{t}_{Y'} = e_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} \\ &= \begin{cases} (2, v_X \,\mathrm{V}_a) & \text{if } e_{X'} \,\mathrm{t}_{X'} = v_X \,\mathrm{V}_f \in \mathrm{V}_{Xf} \text{ for a unique } v_X \in \mathrm{V}_X \\ (1, e_{X'} \,\mathrm{t}_{X'}) & \text{if } e_{X'} \,\mathrm{t}_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf} \;. \end{cases} \end{aligned}$$

We *assume* that there exists a vertex $v_X \in \mathrm{V}_X$ such that $e_{X'} \,\mathrm{t}_{X'} = v_X \,\mathrm{V}_f$.

Then we have $v_{Y'} = (2, v_X \,\mathrm{V}_a) \in \mathrm{V}_{Yg}$. But we have $v_{Y'} \in \mathrm{V}_Y \setminus \mathrm{V}_{Yg}$, which is a *contradiction.*

So we have $v_{Y'} = (1, e_{X'} \,\mathrm{t}_{X'})$.

Now we have

$$\begin{aligned} v_{Y'} &= \tilde{e}_{Y'} \,\mathrm{t}_{Y'} = (1, \tilde{e}_{X'})\,\mathrm{t}_{Y'} = \tilde{e}_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{a'} \\ &= \begin{cases} (2, v_X \,\mathrm{V}_a) & \text{if } \tilde{e}_{X'} \,\mathrm{t}_{X'} = v_X \,\mathrm{V}_f \in \mathrm{V}_{Xf} \text{ for a unique } v_X \in \mathrm{V}_X \\ (1, \tilde{e}_{X'} \,\mathrm{t}_{X'}) & \text{if } \tilde{e}_{X'} \,\mathrm{t}_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf} \;. \end{cases} \end{aligned}$$

We *assume* that there exists a vertex $v_X \in \mathrm{V}_X$ such that $\tilde{e}_{X'}\,\mathrm{t}_{X'} = v_X\,\mathrm{V}_f$.

Then we have $v_{Y'} = (2, v_X\,\mathrm{V}_a) \in \mathrm{V}_{Yg}$. But we have $v_{Y'} \in \mathrm{V}_Y \setminus \mathrm{V}_{Yg}$, which is a *contradiction.*

So we have $v_{Y'} = (1, \tilde{e}_{X'}\,\mathrm{t}_{X'})$.

Recall that $v_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}$ and $v_{Y'} = (1, v_{X'})$.

Therefore we have $e_{X'}\,\mathrm{t}_{X'} = \tilde{e}_{X'}\,\mathrm{t}_{X'} = v_{X'}$.

Because of (AcCofib 3) for f we have $e_{X'} = \tilde{e}_{X'}$.

So we have $e_{Y'} = (1, e_{X'}) = (1, \tilde{e}_{X'}) = \tilde{e}_{Y'}$.

So (AcCofib 3) holds for the graph morphism $g : Y \to Y'$.

Ad (AcCofib 4). Suppose given an edge $e_{Y'} \in \mathrm{E}_{Y'} \setminus \mathrm{E}_{Yg}$.

We have to show that $e_{Y'}\,\mathrm{t}_{Y'} \overset{!}{\in} \mathrm{V}_{Y'} \setminus \mathrm{V}_{Yg}$.

We have $e_{Y'} \notin \mathrm{E}_{Yg}$. So there exists an edge $e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}$ such that $e_{Y'} = (1, e_{X'})$.

So we have

$$\begin{aligned} e_{Y'}\,\mathrm{t}_{Y'} &= (1, e_{X'})\,\mathrm{t}_{Y'} = e_{X'}\,\mathrm{t}_{X'}\,\mathrm{V}_{a'} \\ &= \begin{cases} (2, v_X\,\mathrm{V}_a) & \text{if } e_{X'}\,\mathrm{t}_{X'} = v_X\,\mathrm{V}_f \in \mathrm{V}_{Xf} \text{ for a unique } v_X \in \mathrm{V}_X \\ (1, e_{X'}\,\mathrm{t}_{X'}) & \text{if } e_{X'}\,\mathrm{t}_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf} \quad . \end{cases} \end{aligned}$$

We have $e_{X'} \in \mathrm{E}_{X'} \setminus \mathrm{E}_{Xf}$. Because of (AcCofib 4) for f we conclude that $e_{X'}\,\mathrm{t}_{X'} \notin \mathrm{V}_{Xf}$.

So we have $e_{Y'}\,\mathrm{t}_{Y'} = (1, e_{X'}\,\mathrm{t}_{X'}) \notin \mathrm{V}_{Yg}$.

Hence (AcCofib 4) holds for the graph morphism $g : Y \to Y'$.

Ad (AcCofib 5). Suppose given $v_{Y'} \in \mathrm{V}_{Y'} \setminus \mathrm{V}_{Yg}$, where $\mathrm{V}_{Yg} = \{(2, v_Y) : v_Y \in \mathrm{V}_Y\}$.

We have to show that there exist $n \geqslant 1$ and $e_{Y',i} \in \mathrm{E}_{Y'}$ for $i \in [1, n]$ such that $e_{Y',1}\,\mathrm{s}_{Y'} \in \mathrm{V}_{Yg}$, such that $e_{Y',i}\,\mathrm{t}_{Y'} = e_{Y',i+1}\,\mathrm{s}_{Y'}$ for $i \in [1, n-1]$ and such that $e_{Y',n}\,\mathrm{t}_{Y'} = v_{Y'}$.

Since $v_{Y'} \in \mathrm{V}_{Y'} \setminus \mathrm{V}_{Yg}$, there exists a unique vertex $v_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}$ such that $v_{Y'} = (1, v_{X'})$.

Now (AcCofib 5) holds for the graph morphism $f : X \to X'$.

So we may choose $n \geqslant 1$ and edges $e_{X',i} \in \mathrm{E}_{X'}$ for $i \in [1, n]$ such that $e_{X',1}\,\mathrm{s}_{X'} \in \mathrm{V}_{Xf}$, such that $e_{X',i}\,\mathrm{t}_{X'} = e_{X',i+1}\,\mathrm{s}_{X'}$ for $i \in [1, n-1]$ and such that $e_{X',n}\,\mathrm{t}_{X'} = v_{X'}$.

Without loss of generality, we have $e_{X',i}\,\mathrm{t}_{X'} \notin \mathrm{V}_{Xf}$ for $i \in [1, n]$. So we have $e_{X',i} \notin \mathrm{E}_{Xf}$ for $i \in [1, n]$.

We let $e_{Y',i} := (1, e_{X',i})$ for $i \in [1, n]$.

We have $e_{Y',1}\,\mathrm{s}_{Y'} = (1, e_{X',1})\,\mathrm{s}_{Y'} = e_{X',1}\,\mathrm{s}_{X'}\,\mathrm{V}_{a'}$.

Now there exists a unique vertex $v_X \in \mathrm{V}_X$ such that $e_{X',1}\,\mathrm{s}_{X'} = v_X\,\mathrm{V}_f$.

So we have $e_{X',1}\,\mathrm{s}_{X'}\,\mathrm{V}_{a'} = (2, v_X\,\mathrm{V}_a) = v_X\,\mathrm{V}_a\,\mathrm{V}_g \in \mathrm{V}_{Yg}$.

Suppose given $i \in [1, n-1]$. Then we have

$$e_{Y',i}\,\mathrm{t}_{Y'} = (1, e_{X',i})\,\mathrm{t}_{Y'} = e_{X',i}\,\mathrm{t}_{X'}\,\mathrm{V}_{a'} = e_{X',i+1}\,\mathrm{s}_{X'}\,\mathrm{V}_{a'} = (1, e_{X',i+1})\,\mathrm{s}_{Y'} = e_{Y',i+1}\,\mathrm{s}_{Y'} \quad .$$

And we have

$$e_{Y',n}\,\mathrm{t}_{Y'} \;\; = (1, e_{X',n})\,\mathrm{t}_{Y'} \;\; = e_{X',n}\,\mathrm{t}_{X'}\,\mathrm{V}_{a'} \;\; = v_{X'}\,\mathrm{V}_{a'} = (1, v_{X'}) = v_{Y'}\ ,$$

because $v_{X'} \in \mathrm{V}_{X'} \setminus \mathrm{V}_{Xf}$.

So (AcCofib 5) holds for the graph morphism $g : Y \to Y'$.

So the graph morphism $g : Y \to Y'$ is an acyclic cofibration; cf. Definition 162.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
X' \arrow[r, "a'"'] \arrow[ur, phantom, "\lrcorner" near end] & Y'
\end{tikzcd}$$

$\square$

Once Lemma 191 below is known, which states that we have $\mathrm{AcCofib} = {}^{\boxslash}\mathrm{Fib}$, we can also argue by Remark 21 to obtain the statement of Remark 171.

Remark 172 In Gph, the subset of acyclic cofibrations $\mathrm{AcCofib} \subseteq \mathrm{Mor}$ is closed under retracts; cf. Definition 23.

Proof. Suppose given the following commutative diagram in Gph.

$$\begin{tikzcd}
X' \arrow[r, "a"'] \arrow[rr, bend left, "\mathrm{id}_{X'}"] \arrow[d, "f'"] & X \arrow[r, "a'"'] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y' \arrow[r, "b"] \arrow[rr, bend right, "\mathrm{id}_{Y'}"'] & Y \arrow[r, "b'"] & Y'
\end{tikzcd}$$

We have to show that the graph morphism $f' : X' \to Y'$ is an acyclic cofibration.

Ad (AcCofib 1).

We have to show that the map $\mathrm{V}_{f'} : \mathrm{V}_{X'} \to \mathrm{V}_{Y'}$ is injective.

We have $\mathrm{V}_{f'} \cdot \mathrm{V}_b = \mathrm{V}_a \cdot \mathrm{V}_f$. So it suffices to show that the map $\mathrm{V}_a \cdot \mathrm{V}_f$ is injective.

The map V_f is injective since $f \in \mathrm{AcCofib}$.

The map V_a is injective since $\mathrm{V}_a \cdot \mathrm{V}_{a'} = \mathrm{V}_{\mathrm{id}_{X'}}$.

Ad (AcCofib 2).

We have to show that the map $\mathrm{E}_{f'} : \mathrm{E}_{X'} \to \mathrm{E}_{Y'}$ is injective.

We have $\mathrm{E}_{f'} \cdot \mathrm{E}_b = \mathrm{E}_a \cdot \mathrm{E}_f$. So it suffices to show that the map $\mathrm{E}_a \cdot \mathrm{E}_f$ is injective.

The map E_f is injective since $f \in \mathrm{AcCofib}$.

The map E_a is injective since $\mathrm{E}_a \cdot \mathrm{E}_{a'} = \mathrm{E}_{\mathrm{id}_{X'}}$.

Ad (AcCofib 3).

Suppose given a vertex $v_{Y'} \in \mathrm{V}_{Y'} \setminus \mathrm{V}_{X'f'}$.

We have to show that we have $|\{e \in \mathrm{E}_{Y'} : (e)\,\mathrm{t}_{Y'} = v_{Y'}\}| \overset{!}{=} 1$.

We show that $v_{Y'}\,\mathrm{V}_b \overset{!}{\notin} \mathrm{V}_{Xf}$.

We *assume* that $v_{Y'}\,\mathrm{V}_b \in \mathrm{V}_{Xf}$. Then there exists a unique vertex $v_X \in \mathrm{V}_X$ such that $v_{Y'}\,\mathrm{V}_b = v_X\,\mathrm{V}_f$.

Then we have $v_{Y'} = v_{Y'}\,\mathrm{V}_b\,\mathrm{V}_{b'} = v_X\,\mathrm{V}_f\,\mathrm{V}_{b'} = (v_X\,\mathrm{V}_{a'})\,\mathrm{V}_{f'} \in \mathrm{V}_{X'f'}$, which is a *contradiction.*

First, we show that $|\{e \in \mathrm{E}_{Y'} : e\,\mathrm{t}_{Y'} = v_{Y'}\}| \overset{!}{\geqslant} 1$.

Because the graph morphism $f : X \to Y$ is an acyclic cofibration, there exists a unique edge $e_Y \in \mathrm{E}_Y$ such that $e_Y\,\mathrm{t}_Y = v_{Y'}\,\mathrm{V}_b$. We have $(e_Y\,\mathrm{E}_{b'})\,\mathrm{t}_{Y'} = e_Y\,\mathrm{t}_Y\,\mathrm{V}_{b'} = v_{Y'}\,\mathrm{V}_b\,\mathrm{V}_{b'} = v_{Y'}$.

Second, we show that $|\{e \in \mathrm{E}_{Y'} : e\,\mathrm{t}_{Y'} = v_{Y'}\}| \overset{!}{\leqslant} 1$.

Suppose given $e_{Y'}, \tilde{e}_{Y'} \in \mathrm{E}_{Y'}$ such that $e_{Y'}\,\mathrm{t}_{Y'} = \tilde{e}_{Y'}\,\mathrm{t}_{Y'} = v_{Y'}$. We have to show that $e_{Y'} \overset{!}{=} \tilde{e}_{Y'}$.

We have $(e_{Y'}\,\mathrm{E}_b)\,\mathrm{t}_Y = e_{Y'}\,\mathrm{t}_{Y'}\,\mathrm{V}_b = v_{Y'}\,\mathrm{V}_b = \tilde{e}_{Y'}\,\mathrm{t}_{Y'}\,\mathrm{V}_b = (\tilde{e}_{Y'}\,\mathrm{E}_b)\,\mathrm{t}_Y$.

Because the graph morphism $f : X \to Y$ is an acyclic cofibration and because we have $v_{Y'}\,\mathrm{V}_b \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$ we conclude that $e_{Y'}\,\mathrm{E}_b = \tilde{e}_{Y'}\,\mathrm{E}_b$.

So we have $e_{Y'} = e_{Y'}\,\mathrm{E}_b\,\mathrm{E}_{b'} = \tilde{e}_{Y'}\,\mathrm{E}_b\,\mathrm{E}_{b'} = \tilde{e}_{Y'}$.

Ad (AcCofib 4).

Suppose given an edge $e_{Y'} \in \mathrm{E}_{Y'} \setminus \mathrm{E}_{X'f'}$. We have to show that $e_{Y'}\,\mathrm{t}_{Y'} \overset{!}{\notin} \mathrm{V}_{X'f'}$.

We show that $e_{Y'}\,\mathrm{E}_b \overset{!}{\in} \mathrm{E}_Y \setminus \mathrm{E}_{Xf}$.

We *assume* that $e_{Y'}\,\mathrm{E}_b \in \mathrm{E}_{Xf}$. Then there exists a unique edge $e_X \in \mathrm{E}_X$ such that $e_{Y'}\,\mathrm{E}_b = e_X\,\mathrm{E}_f$.

Then we have $e_{Y'} = e_{Y'}\,\mathrm{E}_b\,\mathrm{E}_{b'} = e_X\,\mathrm{E}_f\,\mathrm{E}_{b'} = (e_X\,\mathrm{E}_{a'})\,\mathrm{E}_{f'} \in \mathrm{E}_{X'f'}$, which is a *contradiction.*

We *assume* that $e_{Y'}\,\mathrm{t}_{Y'} \in \mathrm{V}_{X'f'}$. Then there exists a unique vertex $v_{X'} \in \mathrm{V}_{X'}$ such that we have $e_{Y'}\,\mathrm{t}_{Y'} = v_{X'}\,\mathrm{V}_{f'}$. So we have $(e_{Y'}\,\mathrm{E}_b)\,\mathrm{t}_Y = e_{Y'}\,\mathrm{t}_{Y'}\,\mathrm{V}_b = v_{X'}\,\mathrm{V}_{f'}\,\mathrm{V}_b = (v_{X'}\,\mathrm{V}_a)\,\mathrm{V}_f \in \mathrm{V}_{Xf}$.

But we have $e_{Y'}\,\mathrm{E}_b \in \mathrm{E}_Y \setminus \mathrm{E}_{Xf}$. Since the graph morphism $f : X \to Y$ is an acyclic cofibration, we conclude that $e_{Y'}\,\mathrm{E}_b\,\mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$ which is a *contradiction.*

Ad (AcCofib 5).

Suppose given a vertex $v_{Y'} \in \mathrm{V}_{Y'} \setminus \mathrm{V}_{X'f'}$.

We have to show that there exist $n \geqslant 1$ and $e_i \in \mathrm{E}_{Y'}$ for $i \in [1, n]$ such that $e_1\,\mathrm{s}_{Y'} \in \mathrm{V}_{X'f}$, such that $e_i\,\mathrm{t}_{Y'} = e_{i+1}\,\mathrm{s}_{Y'}$ for $i \in [1, n-1]$ and such that $e_n\,\mathrm{t}_{Y'} = v_{Y'}$.

We show that $v_{Y'}\,\mathrm{V}_b \overset{!}{\notin} \mathrm{V}_{Xf}$.

We *assume* that $v_{Y'}\,\mathrm{V}_b \in \mathrm{V}_{Xf}$. Then there exists a unique vertex $v_X \in \mathrm{V}_X$ such that $v_{Y'}\,\mathrm{V}_b = v_X\,\mathrm{V}_f$.

Then we have $v_{Y'} = v_{Y'}\,\mathrm{V}_b\,\mathrm{V}_{b'} = v_X\,\mathrm{V}_f\,\mathrm{V}_{b'} = (v_X\,\mathrm{V}_{a'})\,\mathrm{V}_{f'} \in \mathrm{V}_{X'f'}$, which is a *contradiction.*

Now (AcCofib 5) holds for the acyclic cofibration $f : X \to Y$. So we may choose $n \geqslant 1$ and edges $e_{Y,i} \in \mathrm{E}_Y$ for $i \in [1, n]$ such that $e_{Y,1}\,\mathrm{s}_Y \in \mathrm{V}_{Xf}$ and $e_{Y,i}\,\mathrm{t}_Y = e_{Y,i+1}\,\mathrm{s}_Y$ for $i \in [1, n-1]$ and such that $e_{Y,n}\,\mathrm{t}_Y = v_{Y'}\,\mathrm{V}_b$.

We let $e_i := e_{Y,i}\,\mathrm{E}_{b'}$ for $i \in [1,n]$.

We will show that the following statements (i, ii, iii) hold.

(i) $e_1\,\mathrm{s}_{Y'} \overset{!}{\in} \mathrm{V}_{X'f'}$

(ii) $e_i\,\mathrm{t}_{Y'} \overset{!}{=} e_{i+1}\,\mathrm{s}_{Y'}$ for $i \in [1, n-1]$.

(iii) $e_n\,\mathrm{t}_{Y'} \overset{!}{=} v_{Y'}$

Ad (i). There exists a unique vertex $v_X \in \mathrm{V}_X$ such that $e_{Y,1}\,\mathrm{s}_Y = v_X\,\mathrm{V}_f$.

Then we have $e_1\,\mathrm{s}_{Y'} = (e_{Y,1}\,\mathrm{E}_{b'})\,\mathrm{s}_{Y'} = e_{Y,1}\,\mathrm{s}_Y\,\mathrm{V}_{b'} = v_X\,\mathrm{V}_f\,\mathrm{V}_{b'} = (v_X\,\mathrm{V}_{a'})\,\mathrm{V}_{f'} \in \mathrm{V}_{X'f'}$.

Ad (ii). Suppose given $i \in [1, n-1]$.

We have $e_{Y,i}\,\mathrm{t}_Y = e_{Y,i+1}\,\mathrm{s}_Y$. So we have $e_i\,\mathrm{t}_{Y'} = (e_{Y,i}\,\mathrm{E}_{b'})\,\mathrm{t}_{Y'} = e_{Y,i}\,\mathrm{t}_Y\,\mathrm{V}_{b'} = e_{Y,i+1}\,\mathrm{s}_Y\,\mathrm{V}_{b'} = (e_{Y,i+1}\,\mathrm{E}_{b'})\,\mathrm{s}_{Y'} = e_{i+1}\,\mathrm{s}_{Y'}$.

Ad (iii). We have $e_{Y,n}\,\mathrm{t}_Y = v_{Y'}\,\mathrm{V}_b$.

So we have $e_n\,\mathrm{t}_{Y'} = (e_{Y,n}\,\mathrm{E}_{b'})\,\mathrm{t}_{Y'} = e_{Y,n}\,\mathrm{t}_Y\,\mathrm{V}_{b'} = v_{Y'}\,\mathrm{V}_b\,\mathrm{V}_{b'} = v_{Y'}$.

Altogether, the graph morphism $f' : X' \to Y'$ is an acyclic cofibration, i.e. f' is in AcCofib. $\square$

Once Lemma 191 below is known, which states that we have AcCofib $= {}^{\boxslash}$Fib, we can also argue by Remark 25 to obtain the statement of Remark 172.

However, we need Remark 172 to prove Lemma 185, which we need to prove Lemma 191.

Lemma 173 Suppose given a graph Y.

Suppose given a full subgraph $X \subseteq Y$ such that the following properties (1, 2) hold.

(1) We have $e_Y\,\mathrm{s}_Y \in \mathrm{V}_X$ for $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$.

(2) For $v_Y \in \mathrm{V}_Y \setminus \mathrm{V}_X$, there exists a unique $e_Y \in \mathrm{E}_Y$ such that $e_Y\,\mathrm{t}_Y = v_Y$.

Then the inclusion morphism $\iota := \mathrm{id}_Y|_X : X \to Y$ is in ${}^{\boxslash}$Fib.

Proof. Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "\iota"'] & X' \arrow[d, "f'", tail] \\ Y \arrow[r, "b"'] & Y' \end{tikzcd}$$

We have to show that there exists a graph morphism $h : Y \to X'$ such that $\iota h = a$ and $hf' = b$.

Let $v_X\,\mathrm{V}_h := v_X\,\mathrm{V}_a$ for $v_X \in \mathrm{V}_X$ and $e_X\,\mathrm{E}_h := e_X\,\mathrm{E}_a$ for $e_X \in \mathrm{E}_X$.

Now suppose given an edge $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$.

We now consider the edge $e_Y\,\mathrm{E}_b \in \mathrm{E}_{Y'}$.

We abbreviate $v_{X'} := e_Y \, \mathrm{s}_Y \, \mathrm{V}_a \in \mathrm{V}_{X'}$; cf. property (1).

Since $\iota b = af'$, we have $e_Y \, \mathrm{E}_b \, \mathrm{s}_{Y'} = e_Y \, \mathrm{s}_Y \, \mathrm{V}_b \overset{e_Y \, \mathrm{s}_Y \in \mathrm{V}_X}{=} e_Y \, \mathrm{s}_Y \, \mathrm{V}_\iota \, \mathrm{V}_b = e_Y \, \mathrm{s}_Y \, \mathrm{V}_a \, \mathrm{V}_{f'} = v_{X'} \, \mathrm{V}_{f'}$.

Since the graph morphism $f' : X' \to Y'$ is a fibration, the map

$$\mathrm{E}_{f', v_{X'}} = \mathrm{E}_{f'} |_{X'(v_{X'}, *)}^{Y'(v_{X'} \, \mathrm{V}_{f'}, *)} : X'(v_{X'}, *) \to Y'(v_{X'} \, \mathrm{V}_{f'}, *)$$

is surjective; cf. Definition 127.(1).

We have $v_{X'} \, \mathrm{V}_{f'} = e_Y \, \mathrm{E}_b \, \mathrm{s}_{Y'}$, and therefore $e_Y \, \mathrm{E}_b \in Y'(v_{X'} \, \mathrm{V}_{f'}, *)$.

So we may choose an edge $(e_Y)_{X'} \in X'(v_{X'}, *)$ such that $(e_Y)_{X'} \, \mathrm{E}_{f'} = e_Y \, \mathrm{E}_b$.

So for the edge $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$, we let $e_Y \, \mathrm{E}_h := (e_Y)_{X'} \in \mathrm{E}_{X'}$.

Finally, suppose given a vertex $v_Y \in \mathrm{V}_Y \setminus \mathrm{V}_X$.

By (2), there exists a unique edge $e_{v_Y} \in \mathrm{E}_Y$ such that $e_{v_Y} \, \mathrm{t}_Y = v_Y$.

Note that $e_{v_Y} \in \mathrm{E}_Y \setminus \mathrm{E}_X$ since $e_{v_Y} \, \mathrm{t}_Y = v_Y \in \mathrm{V}_Y \setminus \mathrm{V}_X$.

We let $v_Y \, \mathrm{V}_h := (e_{v_Y})_{X'} \, \mathrm{t}_{X'} \in \mathrm{V}_{X'}$.

Before we show that $h : Y \to X'$ is a graph morphism we make a remark.

Suppose given an edge $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$.

We *assume* that $e_Y \, \mathrm{t}_Y \in \mathrm{V}_X$.

We have $e_Y \, \mathrm{s}_Y \in \mathrm{V}_X$; cf. property (1).

Since $X \subseteq Y$ is a full subgraph, we have $e_Y \in \mathrm{E}_X$. *Contradiction.*

So for the edge $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$ we have $e_Y \, \mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_X$.

In particular, we have $e_Y = e_{e_Y \, \mathrm{t}_Y}$. This finishes the remark.

We have to show that $h : Y \to X'$ is a graph morphism.

First, we have to show that $\mathrm{E}_h \, \mathrm{s}_{X'} \overset{!}{=} \mathrm{s}_Y \, \mathrm{V}_h$.

Suppose given an edge $e_Y \in \mathrm{E}_Y$. We have to show that $e_Y \, \mathrm{E}_h \, \mathrm{s}_{X'} \overset{!}{=} e_Y \, \mathrm{s}_Y \, \mathrm{V}_h$.

Case $e_Y \in \mathrm{E}_X$.

We have $e_Y \, \mathrm{E}_h \, \mathrm{s}_{X'} = e_Y \, \mathrm{E}_a \, \mathrm{s}_{X'} = e_Y \, \mathrm{s}_X \, \mathrm{V}_a \overset{e_Y \, \mathrm{s}_X \in \mathrm{V}_X}{=} e_Y \, \mathrm{s}_X \, \mathrm{V}_h = e_Y \, \mathrm{s}_Y \, \mathrm{V}_h$.

Case $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$.

We have $e_Y \, \mathrm{E}_h \, \mathrm{s}_{X'} = (e_Y)_{X'} \, \mathrm{s}_{X'} \overset{(e_Y)_{X'} \in X'(v_{X'}, *)}{=} v_{X'} \overset{\text{Def. } v_{X'}}{=} e_Y \, \mathrm{s}_Y \, \mathrm{V}_a \overset{(1)}{=} e_Y \, \mathrm{s}_Y \, \mathrm{V}_h$.

Second, we have to show that $\mathrm{E}_h \, \mathrm{t}_{X'} \overset{!}{=} \mathrm{t}_Y \, \mathrm{V}_h$.

Suppose given an edge $e_Y \in \mathrm{E}_Y$. We have to show that $e_Y \, \mathrm{E}_h \, \mathrm{t}_{X'} \overset{!}{=} e_Y \, \mathrm{t}_Y \, \mathrm{V}_h$.

Case $e_Y \in \mathrm{E}_X$.

We have $e_Y \, \mathrm{E}_h \, \mathrm{t}_{X'} = e_Y \, \mathrm{E}_a \, \mathrm{t}_{X'} = e_Y \, \mathrm{t}_X \, \mathrm{V}_a \overset{e_Y \, \mathrm{t}_X \in \mathrm{V}_X}{=} e_Y \, \mathrm{t}_X \, \mathrm{V}_h = e_Y \, \mathrm{t}_Y \, \mathrm{V}_h$.

Case $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$.

Now we need the remark made above.

Writing $v_Y := e_Y \,\mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_X$, we obtain

$$e_Y \,\mathrm{t}_Y \,\mathrm{V}_h = v_Y \,\mathrm{V}_h = (e_{v_Y})_{X'} \,\mathrm{t}_{X'} = e_{v_Y} \,\mathrm{E}_h \,\mathrm{t}_{X'} = e_Y \,\mathrm{E}_h \,\mathrm{t}_{X'} \ .$$

So $h : Y \to X'$ is in fact a graph morphism.

We now show that $a \overset{!}{=} \iota h$.

Therefor we have to show that $\mathrm{V}_a \overset{!}{=} \mathrm{V}_\iota \,\mathrm{V}_h$ and $\mathrm{E}_a \overset{!}{=} \mathrm{E}_\iota \,\mathrm{E}_h$.

Suppose given a vertex $v_X \in \mathrm{V}_X$. We have $v_X \,\mathrm{V}_\iota \,\mathrm{V}_h = v_X \,\mathrm{V}_h = v_X \,\mathrm{V}_a$.

Suppose given an edge $e_X \in \mathrm{E}_X$. We have $e_X \,\mathrm{E}_\iota \,\mathrm{E}_h = e_X \,\mathrm{E}_h = e_X \,\mathrm{E}_a$.

We now show that $hf' \overset{!}{=} b$.

First, we have to show that $\mathrm{V}_h \,\mathrm{V}_{f'} \overset{!}{=} \mathrm{V}_b$.

Suppose given a vertex $v_Y \in \mathrm{V}_Y$. We have to show that $v_Y \,\mathrm{V}_h \,\mathrm{V}_{f'} \overset{!}{=} v_Y \,\mathrm{V}_b$.

Case $v_Y \in \mathrm{V}_X$.

We have $v_Y \,\mathrm{V}_h \,\mathrm{V}_{f'} = v_Y \,\mathrm{V}_a \,\mathrm{V}_{f'} = v_Y \,\mathrm{V}_\iota \,\mathrm{V}_b = v_Y \,\mathrm{V}_b$.

Case $v_Y \in \mathrm{V}_Y \setminus \mathrm{V}_X$.

We have $v_Y \,\mathrm{V}_h \,\mathrm{V}_{f'} = (e_{v_Y})_{X'} \,\mathrm{t}_{X'} \,\mathrm{V}_{f'} = (e_{v_Y})_{X'} \,\mathrm{E}_{f'} \,\mathrm{t}_{Y'} \overset{\text{choice of } (e_{v_Y})_{X'}}{=} e_{v_Y} \,\mathrm{E}_b \,\mathrm{t}_{Y'} = e_{v_Y} \,\mathrm{t}_Y \,\mathrm{V}_b = v_Y \,\mathrm{V}_b$.

Second, we have to show that $\mathrm{E}_h \,\mathrm{E}_{f'} \overset{!}{=} \mathrm{E}_b$.

Suppose given $e_Y \in \mathrm{E}_Y$. We have to show that $e_Y \,\mathrm{E}_h \,\mathrm{E}_{f'} \overset{!}{=} e_Y \,\mathrm{E}_b$.

Case $e_Y \in \mathrm{E}_X$.

We have $e_Y \,\mathrm{E}_h \,\mathrm{E}_{f'} = e_Y \,\mathrm{E}_a \,\mathrm{E}_{f'} = e_Y \,\mathrm{E}_\iota \,\mathrm{E}_b = e_Y \,\mathrm{E}_b$.

Case $e_Y \in \mathrm{E}_Y \setminus \mathrm{E}_X$.

We have $e_Y \,\mathrm{E}_h \,\mathrm{E}_{f'} = (e_Y)_{X'} \,\mathrm{E}_{f'} \overset{\text{choice of } (e_Y)_{X'}}{=} e_Y \,\mathrm{E}_b$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "\iota"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "b"'] \arrow[ur, "h"] & Y' \end{tikzcd}$$

□

Lemma 174 We have $\mathrm{AcCofib} \subseteq {}^{\boxslash}\mathrm{Fib}$.

Proof. Suppose given an acyclic cofibration $f : X \to Y$.

We have to show that f satisfies ($\mathrm{LLP}_{\mathrm{Fib}}$); cf. Definition 13.

Suppose given a fibration $g : X' \to Y'$.

Suppose given graph morphisms $u : X \to X'$ and $r : Y \to Y'$ such that $ug = fr$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "r"'] & Y' \end{tikzcd}$$

We have to show that there exists a graph morphism $h : Y \to X'$ such that $fh = u$ and $hg = r$.

Since (AcCofib 1, 2) hold for the acyclic cofibration $f : X \multimap\!\!\!\rightarrow Y$, the graph morphism $f|^{Xf} : X \to Xf \subseteq Y$ is bijective, i.e. a graph isomorphism; cf. Remark 66. Thus $f|^{Xf} \in \mathrm{Iso} \subseteq {}^{\boxslash}\mathrm{Fib}$; cf. Remark 18.

Moreover, the graph Xf is a full subgraph in Y; cf. Remark 170.

We shall define full subgraphs $Y_n \subseteq Y$ recursively.

Let $Y_0 := Xf$.

For $n \in \mathbb{Z}_{\geqslant 1}$, let Y_n be the full subgraph of Y with

$$\mathrm{V}_{Y_n} := \mathrm{V}_{Y_{n-1}} \cup \{v \in \mathrm{V}_Y : \exists e \in Y(\mathrm{V}_{Y_{n-1}}, v)\} ;$$

cf. Definition 47.(3) and Notation 51.(2).

We now have a countable chain of subgraphs

$$Xf = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y .$$

Note that we have the full subgraph $Y_k \subseteq Y_{k+1}$ for $k \in \mathbb{Z}_{\geqslant 0}$.

Suppose given a vertex $v_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

Since (AcCofib 5) holds for the acyclic cofibration $f : X \multimap\!\!\!\rightarrow Y$, we may choose $n \geqslant 1$ and edges $e_i \in \mathrm{E}_Y$ for $i \in [1, n]$ such that $e_1\,\mathrm{s}_Y \in \mathrm{V}_{Xf}$, such that $e_i\,\mathrm{t}_Y = e_{i+1}\,\mathrm{s}_Y$ for $i \in [1, n-1]$ and such that $e_n\,\mathrm{t}_Y = v_Y$.

We have $e_1\,\mathrm{s}_Y \in \mathrm{V}_{Xf} = \mathrm{V}_{Y_0}$.

So we have $e_1\,\mathrm{t}_Y \in \mathrm{V}_{Y_1}$.

Since $e_2\,\mathrm{s}_Y = e_1\,\mathrm{t}_Y \in \mathrm{V}_{Y_1}$, we have $e_2\,\mathrm{t}_Y \in \mathrm{V}_{Y_2}$.

Since $e_3\,\mathrm{s}_Y = e_2\,\mathrm{t}_Y \in \mathrm{V}_{Y_2}$, we have $e_3\,\mathrm{t}_Y \in \mathrm{V}_{Y_3}$.

Etc.

So we deduce that $v_Y = e_n\,\mathrm{t}_Y \in \mathrm{V}_{Y_n} \subseteq \bigcup_{k\in\mathbb{Z}_{\geqslant 0}} \mathrm{V}_{Y_k}$.

So we deduce that $\bigcup_{k\in\mathbb{Z}_{\geqslant 0}} \mathrm{V}_{Y_k} = \mathrm{V}_Y$.

Since the graph $\bigcup_{k\in\mathbb{Z}_{\geqslant 0}} Y_k$ is a full subgraph of Y, we have $\bigcup_{k\in\mathbb{Z}_{\geqslant 0}} Y_k = Y$; cf. Remarks 104 and 50.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. The graph Y_k is a full subgraph of Y_{k+1} . In order to apply Lemma 173, we will show that the following properties (1, 2) hold.

(1) We have $e_{Y_{k+1}}\,\mathrm{s}_{Y_{k+1}} \in \mathrm{V}_{Y_k}$ for $e_{Y_{k+1}} \in \mathrm{E}_{Y_{k+1}} \setminus \mathrm{E}_{Y_k}$.

(2) For $v_{Y_{k+1}} \in \mathrm{V}_{Y_{k+1}} \setminus \mathrm{V}_{Y_k}$, there exists a unique $e_{Y_{k+1}} \in \mathrm{E}_{Y_{k+1}}$ such that $e_{Y_{k+1}}\,\mathrm{t}_{Y_{k+1}} = v_{Y_{k+1}}$.

Ad (1). Suppose given an edge $e \in \mathrm{E}_{Y_{k+1}} \setminus \mathrm{E}_{Y_k}$. We have to show that $e\,\mathrm{s}_Y \overset{!}{\in} \mathrm{V}_{Y_k}$.

We have $e\,\mathrm{t}_Y \in \mathrm{V}_{Y_{k+1}}$.

There exists a unique minimal $l \in [0, k+1]$ such that $e\,\mathrm{t}_Y \in \mathrm{V}_{Y_l}$.

We have $e \in \mathrm{E}_{Y_{k+1}} \setminus \mathrm{E}_{Y_k} \subseteq \mathrm{E}_Y \setminus \mathrm{E}_{Xf}$.

Since (AcCofib 4) holds for the acyclic cofibration $f :\ X \multimap\!\!\!\rightarrow Y$, we have $e\,\mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf} = \mathrm{V}_Y \setminus \mathrm{V}_{Y_0}$.

So we deduce that $l \geqslant 1$.

By minimality of l, we have $e\,\mathrm{t}_Y \in \mathrm{V}_{Y_l} \setminus \mathrm{V}_{Y_{l-1}}$.

By construction of V_{Y_l} , we may choose an edge $\tilde{e} \in \mathrm{E}_Y$ such that $\tilde{e}\,\mathrm{s}_Y \in \mathrm{V}_{Y_{l-1}}$ and $\tilde{e}\,\mathrm{t}_Y = e\,\mathrm{t}_Y$.

Note that we have $\tilde{e}\,\mathrm{t}_Y = e\,\mathrm{t}_Y \in \mathrm{V}_{Y_l} \setminus \mathrm{V}_{Y_{l-1}} \subseteq \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

Since (AcCofib 3) holds for the acyclic cofibration $f :\ X \multimap\!\!\!\rightarrow Y$, we infer $e = \tilde{e}$.

So we have $e\,\mathrm{s}_Y = \tilde{e}\,\mathrm{s}_Y \in \mathrm{V}_{Y_{l-1}} \subseteq \mathrm{V}_{Y_k}$.

Ad (2). Suppose given a vertex $v \in \mathrm{V}_{Y_{k+1}} \setminus \mathrm{V}_{Y_k}$. We have to show that there exists a unique edge $e \in \mathrm{E}_{Y_{k+1}}$ such that $e\,\mathrm{t}_Y \overset{!}{=} v$.

By definition of $\mathrm{V}_{Y_{k+1}}$, we may choose an edge $e \in \mathrm{E}_Y$ such that $e\,\mathrm{s}_Y \in \mathrm{V}_{Y_k}$ and $e\,\mathrm{t}_Y = v$.

Since $Y_{k+1} \subseteq Y$ is a full subgraph, we have $e \in \mathrm{E}_{Y_{k+1}}$.

Uniqueness follows since $v \in \mathrm{V}_{Y_{k+1}} \setminus \mathrm{V}_{Y_k} \subseteq \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$ by (AcCofib 3) for the acyclic cofibration $f :\ X \multimap\!\!\!\rightarrow Y$.

So the inclusion morphism $\mathrm{id}_{Y_{k+1}}|_{Y_k} : Y_k \to Y_{k+1}$ is in ${}^{\boxslash}\mathrm{Fib}$; cf. Lemma 173.

Since the inclusion morphisms $\mathrm{id}_{Y_{k+1}}|_{Y_k} : Y_k \to Y_{k+1}$ are in ${}^{\boxslash}\mathrm{Fib}$ for $k \geqslant 0$ and since $\bigcup_{k \in \mathbb{Z}_{\geqslant 0}} Y_k = Y$, the graph morphism $\mathrm{id}_Y|_{Y_0} : Y_0 \to Y$ is in ${}^{\boxslash}\mathrm{Fib}$; cf. Lemma 106.

Altogether, $f = f|^{Xf} \cdot \mathrm{id}_Y|_{Y_0}$ is in ${}^{\boxslash}\mathrm{Fib}$; cf. Remark 19. $\square$

The following lemma is due to Bisson and Tsemo [**3,** Prop. 4.5].

Lemma 175 In Gph, we have $\mathrm{AcCofib} \subseteq \mathrm{Cofib} \cap \mathrm{Qis}$; cf. Definitions 115, 144 and 162.

Proof. We *claim* that $\mathrm{AcCofib} \overset{!}{\subseteq} \mathrm{Qis}$.

Suppose given an acyclic cofibration $f :\ X \multimap\!\!\!\rightarrow Y$. We have to show that f is a quasiisomorphism.

Suppose given $n \geqslant 1$ and a graph morphism $u : \mathrm{C}_n \to Y$. We have to show that $\mathrm{C}_n\,u \overset{!}{\subseteq} Xf$; cf. Remark 66.

$$\begin{tikzcd}
 & X \arrow[d, "f|^{Xf}", "\wr"'] \\
 & Xf \arrow[d, "\mathrm{id}_Y|_{Xf}"] \\
\mathrm{C}_n \arrow[r, "u"] & Y
\end{tikzcd}$$

We *assume* that $\mathrm{C}_n\,u \not\subseteq Xf$. So $\mathrm{V}_{\mathrm{C}_n}\,\mathrm{V}_u \not\subseteq \mathrm{V}_{Xf}$ or $\mathrm{E}_{\mathrm{C}_n}\,\mathrm{E}_u \not\subseteq \mathrm{E}_{Xf}$.

By (AcCofib 4), it is impossible to have $\mathrm{V}_{\mathrm{C}_n} \mathrm{V}_u \subseteq \mathrm{V}_{Xf}$ and $\mathrm{E}_{\mathrm{C}_n} \mathrm{E}_u \not\subseteq \mathrm{E}_{Xf}$.

So $\mathrm{V}_{\mathrm{C}_n} \mathrm{V}_u \not\subseteq \mathrm{V}_{Xf}$.

Hence, we may choose $k \in \mathbb{Z}/n\mathbb{Z}$ with $\mathrm{v}_k \mathrm{V}_u \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

Because of (AcCofib 5) we may choose $m \geqslant 1$ and $e_i \in \mathrm{E}_Y$ for $i \in [1, m]$ such that $e_1 \mathrm{s}_Y \in \mathrm{V}_{Xf}$, such that $e_i \mathrm{t}_Y = e_{i+1} \mathrm{s}_Y$ for $i \in [1, m-1]$ and such that $e_m \mathrm{t}_Y = \mathrm{v}_k \mathrm{V}_u$.

Without loss of generality, we have $e_i \mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$ for $i \in [1, m]$.

And we have $\mathrm{e}_{k-1} \mathrm{E}_u \mathrm{t}_Y = \mathrm{e}_{k-1} \mathrm{t}_{\mathrm{C}_n} \mathrm{V}_u = \mathrm{v}_k \mathrm{V}_u = e_m \mathrm{t}_Y$.

Because of (AcCofib 3) we have $e_m = \mathrm{e}_{k-1} \mathrm{E}_u$.

If $m \geqslant 2$, then we have $e_{m-1} \mathrm{t}_Y = e_m \mathrm{s}_Y = \mathrm{e}_{k-1} \mathrm{E}_u \mathrm{s}_Y = \mathrm{e}_{k-1} \mathrm{s}_{\mathrm{C}_n} \mathrm{V}_u = \mathrm{v}_{k-1} \mathrm{V}_u = \mathrm{e}_{k-2} \mathrm{t}_{\mathrm{C}_n} \mathrm{V}_u = \mathrm{e}_{k-2} \mathrm{E}_u \mathrm{t}_Y$, in $\mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

Because of (AcCofib 3), we obtain $e_{m-1} = \mathrm{e}_{k-2} \mathrm{E}_u$.

If $m \geqslant 3$, then, by the same argument, we obtain $e_{m-2} = e_{k-3} \mathrm{E}_u$.

If $m \geqslant 4$, then, by the same argument, we obtain $e_{m-3} = \mathrm{e}_{k-4} \mathrm{E}_u$.

Etc.

Continuing this procedure, we obtain $e_1 = \mathrm{e}_{k-m} \mathrm{E}_u$.

So we have $\mathrm{V}_{Xf} \ni e_1 \mathrm{s}_Y = \mathrm{e}_{k-m} \mathrm{E}_u \mathrm{s}_Y = \mathrm{e}_{k-m} \mathrm{s}_{\mathrm{C}_n} \mathrm{V}_u = \mathrm{v}_{k-m} \mathrm{V}_u$.

We have $\mathrm{e}_k \mathrm{E}_u \mathrm{s}_Y = \mathrm{e}_k \mathrm{s}_{\mathrm{C}_n} \mathrm{V}_u = \mathrm{v}_k \mathrm{V}_u \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

So we have $\mathrm{e}_k \mathrm{E}_u \in \mathrm{E}_Y \setminus \mathrm{E}_{Xf}$. Because of (AcCofib 4) we have $\mathrm{e}_{k+1} \mathrm{E}_u \mathrm{s}_Y = \mathrm{e}_{k+1} \mathrm{s}_{\mathrm{C}_n} \mathrm{V}_u = \mathrm{v}_{k+1} \mathrm{V}_u = \mathrm{e}_k \mathrm{t}_{\mathrm{C}_n} \mathrm{V}_u = \mathrm{e}_k \mathrm{E}_u \mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

So $\mathrm{e}_{k+1} \mathrm{E}_u \in \mathrm{E}_Y \setminus \mathrm{E}_{Xf}$. By the same argument, we conclude that $\mathrm{e}_{k+2} \mathrm{E}_u \mathrm{s}_Y = \mathrm{v}_{k+2} \mathrm{V}_u = \mathrm{e}_{k+1} \mathrm{E}_u \mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

So $\mathrm{e}_{k+2} \mathrm{E}_u \in \mathrm{E}_Y \setminus \mathrm{E}_{Xf}$. By the same argument, we conclude that $\mathrm{e}_{k+3} \mathrm{E}_u \mathrm{s}_Y = \mathrm{v}_{k+3} \mathrm{V}_u = \mathrm{e}_{k+2} \mathrm{E}_u \mathrm{t}_Y \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$.

Etc.

Continuing this procedure, we obtain $\mathrm{v}_{k-m} \mathrm{V}_u = \mathrm{v}_{k+l} \mathrm{V}_u \in \mathrm{V}_Y \setminus \mathrm{V}_{Xf}$, where $l \geqslant 1$ is such that $k + l = k - m \in \mathbb{Z}/n\mathbb{Z}$.

But $\mathrm{v}_{k-m} \mathrm{V}_u \in \mathrm{V}_{Xf}$. This *contradiction* proves the *claim*.

$$\begin{tikzcd}
 & & X \arrow[d, "\wr"] \\
 & & Xf \arrow[d, "\mathrm{id}_Y|_{Xf}"] \\
\mathrm{C}_n \arrow[r, "u"'] \arrow[rruu] & \mathrm{C}_n u \arrow[r] \arrow[ru] & Y
\end{tikzcd}$$

Moreover, we have $\mathrm{AcCofib} \overset{\text{Lemma 174}}{\subseteq} {}^{\boxslash}\mathrm{Fib} \subseteq {}^{\boxslash}\mathrm{AcFib} \overset{\text{Def. 144}}{=} \mathrm{Cofib}$.

Altogether, we have $\mathrm{AcCofib} \subseteq \mathrm{Cofib} \cap \mathrm{Qis}$. $\square$

In Lemma 175 we have shown the inclusion $\mathrm{AcCofib} \subseteq \mathrm{Cofib} \cap \mathrm{Qis}$.

In Lemma 185 below, we will show that the inclusion $\mathrm{AcCofib} \overset{!}{\supseteq} \mathrm{Cofib} \cap \mathrm{Qis}$ also holds, using the factorization provided by Lemma 184.

Remark 176 In Gph, a pullback of an acyclic cofibration is a quasiisomorphism.

Proof. We have $\mathrm{AcCofib} \subseteq \mathrm{Qis}$; cf. Lemma 175.

So an acyclic cofibration is a quasiisomorphism.

A pullback of a quasiisomorphism is a quasiisomorphism; cf. Remark 125.

So a pullback of an acyclic cofibration is a quasiisomorphism. ▫

Example 177 Suppose given $n \in \mathbb{N}$.

The graph morphism $\iota_{\mathrm{C}_n} : \emptyset \to \mathrm{C}_n$ is not an acyclic cofibration.

Proof. Since $\mathrm{V}_{\emptyset \iota_{\mathrm{C}_n}} = \emptyset$ and $\mathrm{V}_{\mathrm{C}_n} \neq \emptyset$, the graph morphism ι_{C_n} does not satisfy (AcCofib 5).

Alternatively, since $(\mathrm{C}_n\,, \emptyset) = \emptyset \neq (\mathrm{C}_n\,, \mathrm{C}_n)$, the graph morphism ι_{C_n} is not a quasiisomorphism, and thus not an acyclic cofibration; cf. Lemma 175. ▫

Remark 178 A graph G is a tree if and only if there exists an acyclic cofibration $c : \mathrm{D}_0 \to G$.

Proof.

Suppose given an acyclic cofibration $c : \mathrm{D}_0 \to G$.

We have to show that the graph G is a tree, i.e. that the properties (Tree 1–3) hold; cf. Definition 108.

We let $r := \hat{\mathrm{v}}_0 \, \mathrm{V}_c$. So we have $\{r\} = \mathrm{V}_{\mathrm{D}_0 \, c}$.

Ad (Tree 1). Because of (AcCofib 3) we have $|\{e \in \mathrm{E}_G : e \, \mathrm{t}_G = v_G\}| = 1$ for $v_G \in \mathrm{V}_G \setminus \mathrm{V}_{\mathrm{D}_0 \, c} = \mathrm{V}_G \setminus \{r\}$.

Ad (Tree 2). Because of (AcCofib 4) we have $e \, \mathrm{t}_G \in \mathrm{V}_G \setminus \mathrm{V}_{\mathrm{D}_0 \, c} = \mathrm{V}_G \setminus \{r\}$ for $e \in \mathrm{E}_G \setminus \mathrm{E}_{\mathrm{D}_0 \, c} = \mathrm{E}_G \setminus \emptyset = \mathrm{E}_G$. So we have $e \, \mathrm{t}_G \neq r$ for $e \in \mathrm{E}_G$ and thus (Tree 2).

Ad (Tree 3). Suppose given $v \in \mathrm{V}_G$.

We have to show that there exists a path from r to v, i.e. $n \geqslant 0$ and a graph morphism $p : \mathrm{D}_n \to G$ such that $\hat{\mathrm{v}}_0 \, \mathrm{V}_p = r$ and $\hat{\mathrm{v}}_n \, \mathrm{V}_p = v$.

If $v = r$, then we may take $n = 0$ and $p = c$.

So suppose that $v \in \mathrm{V}_G \setminus \{r\}$.

Because of (AcCofib 5) we may choose $n \geqslant 1$ and $e_i \in \mathrm{E}_G$ for $i \in [1, n]$ such that $e_1 \, \mathrm{s}_G \in \{r\}$, such that $e_i \, \mathrm{t}_G = e_{i+1} \, \mathrm{s}_G$ for $i \in [1, n-1]$ and such that $e_n \, \mathrm{t}_G = v$.

So for $v \in \mathrm{V}_G$ we let $\hat{\mathrm{e}}_i \, \mathrm{E}_p := e_{i+1}$ for $i \in [0, n-1]$ and $\hat{\mathrm{v}}_i \, \mathrm{V}_p := e_{i+1} \, \mathrm{s}_G$ for $i \in [0, n-1]$ and $\hat{\mathrm{v}}_n \, \mathrm{V}_p := e_n \, \mathrm{t}_G$.

Then $p : \mathrm{D}_n \to G$ is a graph morphism because we have $\hat{\mathrm{e}}_i \, \mathrm{E}_p \, \mathrm{s}_G = e_{i+1} \, \mathrm{s}_G = \hat{\mathrm{v}}_i \, \mathrm{V}_p = \hat{\mathrm{e}}_i \, \mathrm{s}_{\mathrm{D}_n} \, \mathrm{V}_p$ for $i \in [0, n-1]$ and since we have $\hat{\mathrm{e}}_i \, \mathrm{E}_p \, \mathrm{t}_G = e_{i+1} \, \mathrm{t}_G = e_{i+2} \, \mathrm{s}_G = \hat{\mathrm{v}}_{i+1} \, \mathrm{V}_p = \hat{\mathrm{e}}_i \, \mathrm{t}_{\mathrm{D}_n} \, \mathrm{V}_p$ for $i \in [0, n-2]$ and $\hat{\mathrm{e}}_{n-1} \, \mathrm{E}_p \, \mathrm{t}_G = e_n \, \mathrm{t}_G = \hat{\mathrm{v}}_n \, \mathrm{V}_p = \hat{\mathrm{e}}_n \, \mathrm{t}_{\mathrm{D}_n} \, \mathrm{V}_p$.

Suppose given a tree G.

We have to show that there exists an acyclic cofibration $c : \mathrm{D}_0 \to G$.

We let $\hat{\mathrm{v}}_0 \, \mathrm{V}_c := r$.

We have to show that the graph morphism c is an acyclic cofibration; cf. Definition 162.

Ad (AcCofib 1, 2). The graph morphism $c : \mathrm{D}_0 \to G$ is injective; i.e. the maps V_c and E_c are injective.

Ad (AcCofib 3). Because of (Tree 1) we have $|\{e \in \mathrm{E}_G : (e)\,\mathrm{t}_G = v\}| = 1$ for $v \in \mathrm{V}_G \setminus \mathrm{V}_{\mathrm{D}_0\, c} = \mathrm{V}_G \setminus \{r\}$.

Ad (AcCofib 4). Because of (Tree 2) we have $(e)\,\mathrm{t}_G \neq r$ for $e \in \mathrm{E}_G$ and thus $(e)\,\mathrm{t}_G \in \mathrm{V}_G \setminus \{r\}$ for $e \in \mathrm{E}_G = \mathrm{E}_G \setminus \mathrm{E}_{\mathrm{D}_0\, c}$.

Ad (AcCofib 5). Suppose given $v \in \mathrm{V}_G \setminus \mathrm{V}_{\mathrm{D}_0\, c}$. Because of (Tree 3) we have a path from r to v, i.e. we have a graph morphism $p : \mathrm{D}_n \to G$ such that $\hat{\mathrm{v}}_0 \, \mathrm{V}_p = r$ and $\hat{\mathrm{v}}_n \, \mathrm{V}_p = v$.

We let $e_i := \hat{\mathrm{e}}_{i-1} \, \mathrm{E}_p$ for $i \in [1, n]$. So we have $e_1 \, \mathrm{s}_G = \hat{\mathrm{e}}_0 \, \mathrm{E}_p \, \mathrm{s}_G = \hat{\mathrm{e}}_0 \, \mathrm{s}_G \, \mathrm{V}_p = \hat{\mathrm{v}}_0 \, \mathrm{V}_p = r \in \mathrm{V}_{\mathrm{D}_n\, c}$. And we have $e_i \, \mathrm{t}_G = \hat{\mathrm{e}}_{i-1} \, \mathrm{E}_p \, \mathrm{t}_G = \hat{\mathrm{e}}_{i-1} \, \mathrm{t}_G \, \mathrm{V}_p = \hat{\mathrm{e}}_i \, \mathrm{s}_G \, \mathrm{V}_p = \hat{\mathrm{e}}_i \, \mathrm{E}_p \, \mathrm{s}_G = e_{i+1} \, \mathrm{s}_G$ for $i \in [1, n-1]$ and $e_n \, \mathrm{t}_G = \hat{\mathrm{e}}_{n-1} \, \mathrm{E}_p \, \mathrm{t}_G = \hat{\mathrm{e}}_{n-1} \, \mathrm{t}_G \, \mathrm{V}_p = \hat{\mathrm{v}}_n \, \mathrm{V}_p = v$. ▫

Example 179 Let $n \in \mathbb{N}$.

Since the graph morphism $\iota_{0,n} : \mathrm{D}_0 \to \mathrm{D}_n$ is an acyclic cofibration, the graph D_n is a tree; cf. Definition 108, Remark 166 and Remark 178.

This can also be verified directly. The graph D_n has root $r = \hat{\mathrm{v}}_0$. Then (Tree 1) and (Tree 2) hold since we have $\hat{\mathrm{e}}_{k-1} \, \mathrm{t}_{\mathrm{D}_n} = \hat{\mathrm{v}}_k$ for $k \in [1, n]$. Then (Tree 3) holds since we have the paths $\iota_{k,n} : \mathrm{D}_k \to \mathrm{D}_n$ for $k \in [0, n]$.

Now we can give another proof of Remark 112.

Remark 180 Suppose given a graph G and a vertex $x \in \mathrm{V}_G$.

Then the graph $\mathrm{Tree}(x, G)$ is a tree with root $r := (x;) \in \mathrm{V}_{\mathrm{Tree}(x,G)}$.

Proof. We have to show that there exists an acyclic cofibration $c : \mathrm{D}_0 \to \mathrm{Tree}(x, G)$.

Let $\hat{\mathrm{v}}_0 \, \mathrm{V}_c := (x;) \in \mathrm{V}_{\mathrm{Tree}(x,G)}$.

We will show that the graph morphism $c : \mathrm{D}_0 \to \mathrm{Tree}(x, G)$ with $\hat{\mathrm{v}}_0 \, \mathrm{V}_c := (x;)$ is an acyclic cofibration.

Ad (AcCofib 1, 2). The maps V_c and E_c are injective.

Ad (AcCofib 3). We have to show that $|\{e \in \mathrm{E}_{\mathrm{Tree}(x,G)} : e \, \mathrm{t}_{\mathrm{Tree}(x,G)} = v\}| \overset{!}{=} 1$ for $v \in \mathrm{V}_{\mathrm{Tree}(x,G)} \setminus \{(x;)\}$.

For a path $(x; e_1, \ldots, e_n) \in \mathrm{V}_{\mathrm{Tree}(x,G)}$ we have $\{e \in \mathrm{E}_{\mathrm{Tree}(x,G)} : e \, \mathrm{t}_{\mathrm{Tree}(x,G)} = (x; e_1, \ldots, e_n)\} = \{((x; e_1 \ldots, e_{n-1}), e_n, (x; e_1, \cdots, e_n))\}$.

Ad (AcCofib 4). Suppose given $e \in \mathrm{E}_{\mathrm{Tree}(x,G)} \setminus \mathrm{E}_{\mathrm{D}_0} = \mathrm{E}_{\mathrm{Tree}(x,G)} \setminus \emptyset$.

We have $e = (\gamma, e_n, \gamma \cdot e_n)$ for a path γ and an edge $e_n \in \mathrm{E}_G$.

Then we have $\gamma \cdot e_n \neq (x;)$.

Ad (AcCofib 5) Suppose given $(x; e_1, \ldots, e_n) \in \mathrm{V}_{\mathrm{Tree}(x,G)}$.

Then there exists

$$(x;) \xrightarrow{((x;),e_1,(x;e_1))} (x;e_1) \xrightarrow{((x;e_1),e_2,(x;e_1\,e_2))} (x;e_1,e_2) \xrightarrow{((x;e_1,e_2),e_3,(x;e_1,e_2,e_3))} \cdots \qquad \cdots \longrightarrow (x;e_1,\ldots,e_n)\,.$$

□

3.7 Summary of some notations

Reminder 181 Suppose given a graph morphism $f : G \to H$.

To indicate that f is a quasiisomorphism, we often write $G \overset{f}{\underset{\approx}{\dashrightarrow}} H$; cf. Definition 115.

To indicate that f is a fibration, we often write $G \overset{f}{\longrightarrow\!\!\!\!\!\!+\;\;} H$; cf. Definition 127.(1).

To indicate that f is an acyclic fibration, we often write $G \xrightarrow{f} H$; cf. Definition 138.

To indicate that f is a cofibration, we often write $G \xrightarrow{f} H$; cf. Definition 144.

To indicate that f is an acyclic cofibration, we often write $G \xrightarrow{f} H$; cf. Definition 162.

Reminder 182

(1) We denote $\mathrm{Qis} = \{f \in \mathrm{Mor(Gph)} : f \text{ is a quasiisomorphism}\}$; cf. Definition 115.

(2) We denote $\mathrm{Fib} = \{f \in \mathrm{Mor(Gph)} : f \text{ is a fibration}\}$; cf. Definition 127.(1).

We denote $\mathrm{AcFib} = \mathrm{Fib} \cap \mathrm{Qis} = \{f \in \mathrm{Mor(Gph)} : f \text{ is an acyclic fibration}\}$; cf. Definition 138.

(3) We denote $\mathrm{Cofib} = \{f \in \mathrm{Mor(Gph)} : f \text{ is a cofibration}\}$; cf. Definition 144.

We denote $\mathrm{AcCofib} = \{f \in \mathrm{Mor(Gph)} : f \text{ is an acyclic cofibration}\}$; cf. Definition 162 and Lemma 175.

Chapter 4

Factorization of graph morphisms

Remark 183 Suppose given a graph morphism $f : X \to Y$.

Consider the graph $F := \coprod_{x \in \mathrm{V}_X} \mathrm{Tree}(x\,\mathrm{V}_f\,, Y)$; cf. Definitions 109 and 90.

Consider the discrete subgraph $\dot{X} \subseteq X$; cf. Definition 71.

We define the graph morphism

$$\begin{array}{rrcl} a: & \dot{X} & \to & F = \coprod_{x \in \mathrm{V}_X} \mathrm{Tree}(x\,\mathrm{V}_f\,, Y) \\ \mathrm{V}_a: & \mathrm{V}_X = \mathrm{V}_{\dot{X}} \ni x & \mapsto & (x, (x\,\mathrm{V}_f\,;)) \\ \mathrm{E}_a: & \mathrm{E}_{\dot{X}} = \emptyset\,. & & \end{array}$$

Recall that $(x\,\mathrm{V}_f\,;)$ is an empty path at $x\,\mathrm{V}_f$; cf. Definition 107.

Then the graph morphism $a : \dot{X} \to F$ is an acyclic cofibration.

Cf. also Remark 178.

Proof.

Note that for $x \in \mathrm{V}_X$, the graph $\mathrm{Tree}(x\,\mathrm{V}_f\,, Y)$ is a tree with root $r := (x\,\mathrm{V}_f\,;)$; cf. Remark 112 and Definition 108.

We have to show that (AcCofib 1–5) hold for $a : \dot{X} \to F$; cf. Definition 162.

Ad (AcCofib 1, 2): The maps V_a and E_a are injective.

Ad (AcCofib 3): Suppose given $v_F \in \mathrm{V}_F \setminus \mathrm{V}_{\dot{X}a}$.

There exists a unique vertex $x \in X$, a unique integer $n \geqslant 1$ and a unique path $p : \mathrm{D}_n \to Y$ in Y from $x\,\mathrm{V}_f$ such that $v_F = (x, p)$. Note that $\hat{\mathrm{v}}_0\,\mathrm{V}_p = x\,\mathrm{V}_f$.

So the path p is not the root of the graph $\mathrm{Tree}(x\,\mathrm{V}_f\,, Y)$; i.e. $p \neq r$; cf. Definition 108.

So because of (Tree 1) we have $|\{e \in \mathrm{E}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)} : (e)\,\mathrm{t}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)} = p\}| = 1$ since $p \in \mathrm{V}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)} \setminus \{r\}$.

Let $e \in \mathrm{E}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)}$ be this edge with $e\,\mathrm{t}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)} = p$.

So we have $(x, e)\,\mathrm{t}_F \overset{\text{Def. 90}}{=} (x, e\,\mathrm{t}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)}) = (x, p)$.

Conversely, suppose given an edge $(\tilde{x}, \tilde{e}) \in \mathrm{E}_F$ such that $(\tilde{x}, \tilde{e})\,\mathrm{t}_F = (x, p)$. Then we have $(x, p) = (\tilde{x}, \tilde{e})\,\mathrm{t}_F = (\tilde{x}, \tilde{e}\,\mathrm{t}_{\mathrm{Tree}(\tilde{x}\,\mathrm{V}_f\,,Y)})$.

So we have $x = \tilde{x}$ and $p = \tilde{e}\,\mathrm{t}_{\mathrm{Tree}(\tilde{x}\,\mathrm{V}_f\,,Y)} = \tilde{e}\,\mathrm{t}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)}$.

Because of (Tree 1) we deduce that $e = \tilde{e}$.

Ad (AcCofib 4).

Suppose given $e_F \in \mathrm{E}_F \setminus \mathrm{E}_{\dot{X}_a} = \mathrm{E}_F$.

There exist a unique vertex $x \in \mathrm{V}_X$ and a unique edge $e \in \mathrm{E}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)}$ such that $e_F = (x, e)$.

So we have $e_F\,\mathrm{t}_F = (x, e)\,\mathrm{t}_F = (x, e\,\mathrm{t}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)})$.

Because of (Tree 2) we have $e\,\mathrm{t}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)} \neq (x\,\mathrm{V}_f\,;)$.

So we have $e_F\,\mathrm{t}_F \in \mathrm{V}_F \setminus \mathrm{V}_{\dot{X}a}$.

Ad (AcCofib 5).

Suppose given $v_F \in \mathrm{V}_F \setminus \mathrm{V}_{\dot{X}a}$.

Then there exist a vertex $p \in \mathrm{V}_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)}$ such that $v_F = (x, p)$ and $p \neq (x\,\mathrm{V}_f\,;) =: r$.

Because of (Tree 3) we may choose $n \geqslant 0$ and a path $\alpha : \mathrm{D}_n \to \mathrm{Tree}(x\,\mathrm{V}_f\,, Y)$ from $\hat{\mathrm{v}}_0\,\mathrm{V}_\alpha = r = (x\,\mathrm{V}_f\,;)$ to $\hat{\mathrm{v}}_n\,\mathrm{V}_\alpha = p$.

$$\begin{tikzcd} \mathrm{Tree}(x\,\mathrm{V}_f\,,Y) \arrow[r, "\iota_x"] & F \\ \mathrm{D}_n \arrow[u, "\alpha"] \arrow[ur, "\alpha\cdot\iota_x"'] & \end{tikzcd}$$

Then $\alpha \cdot \iota_x$ is a path from $\hat{\mathrm{v}}_0(\alpha \cdot \iota_x) = r\iota_x = (x, r) = (x, x\,\mathrm{V}_f\,;) = x\,\mathrm{V}_a$
to $\hat{\mathrm{v}}_n(\alpha \cdot \iota_x) = p\iota_x = (x, p) = v_F$; cf. Remark 164.

So the graph morphism $a : \dot{X} \to F$ is an acyclic cofibration. ▫

The following lemma is due to Bisson and Tsemo [**3,** Prop. 3.5].

Lemma 184 Suppose given a graph morphism $f : X \to Y$.

Then there exists a commutative triangle in Gph as follows.

$$\begin{tikzcd} & G \arrow[dr, "q"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Proof. We consider the discrete subgraph $\dot{X} \subseteq X$ and the inclusion morphism $\mathrm{o}_X : \dot{X} \to X$; cf. Definition 71.

Consider the graph $F := \coprod_{x\in\mathrm{V}_X} \mathrm{Tree}(x\,\mathrm{V}_f\,, Y)$.

We define the graph morphism

$$\begin{array}{rrcl} a : & \dot{X} & \to & F = \coprod_{x\in\mathrm{V}_X} \mathrm{Tree}(x\,\mathrm{V}_f\,,Y) \\ \mathrm{V}_a : & \mathrm{V}_X = \mathrm{V}_{\dot{X}} \ni x & \mapsto & (x, (x\,\mathrm{V}_f\,;)) \\ \mathrm{E}_a : & \mathrm{E}_{\dot{X}} = \emptyset\,. & & \end{array}$$

Recall that $(x\,\mathrm{V}_f\,;\,)$ is an empty path at $x\,\mathrm{V}_f$.

Note that the graph morphism $a : \dot{X} \to F$ is an acyclic cofibration; cf. Remark 183; cf. also Remark 178.

We now form the pushout as in Construction 83; cf. Remark 171.

$$\begin{tikzcd} \dot{X} \arrow[r, "a"] \arrow[d, "\mathrm{o}_X"'] & F \arrow[d, "h"] \\ X \arrow[r, "g"'] & \tilde{F} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

We have the graph morphism $\mathrm{p}_{x\,\mathrm{V}_f} : \mathrm{Tree}(x\,\mathrm{V}_f\,, Y) \to Y$ for $x \in \mathrm{V}_X$; cf. Definition 113.

So because of the universal property of the coproduct we have the graph morphism

$$p := (\mathrm{p}_{x\,\mathrm{V}_f})_{x\in \mathrm{V}_X} : F = \coprod_{x\in \mathrm{V}_X} \mathrm{Tree}(x\,\mathrm{V}_f\,, Y) \to Y$$

with $(x, v_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)})\,\mathrm{V}_p = v_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)}\,\mathrm{V}_{\mathrm{p}_{x\,\mathrm{V}_f}}$ and $(x, e_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)})\,\mathrm{E}_p = e_{\mathrm{Tree}(x\,\mathrm{V}_f\,,Y)}\,\mathrm{E}_{\mathrm{p}_{x\,\mathrm{V}_f}}$; cf. Definitions 90 and 91.

We have the following commutative diagram.

$$\begin{tikzcd} \dot{X} \arrow[r, "a"] \arrow[d, "\mathrm{o}_X"'] & F \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

In fact, suppose given $x \in \mathrm{V}_{\dot{X}} = \mathrm{V}_X$.

We have $(x\,\mathrm{V}_a)\,\mathrm{V}_p = (x, (x\,\mathrm{V}_f\,;\,))\,\mathrm{V}_p = (x\,\mathrm{V}_f\,;\,)\,\mathrm{V}_{\mathrm{p}_{x\,\mathrm{V}_f}} \overset{\text{Def. 113}}{=} x\,\mathrm{V}_f = x\,\mathrm{V}_{\mathrm{o}_X}\,\mathrm{V}_f$.

Because $\tilde{F}$ is a pushout, there exists a unique graph morphism $q : \tilde{F} \to Y$ such that $hq = p$ and $gq = f$. We *claim* that the graph morphism $q : \tilde{F} \to Y$ is a fibration.

$$\begin{tikzcd} \dot{X} \arrow[r, "a"] \arrow[d, "\mathrm{o}_X"'] & F \arrow[d, "h"'] \arrow[ddr, bend left, "p"] & \\ X \arrow[r, "g"] \arrow[drr, bend right, "f"'] & \tilde{F} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "q"] & \\ & & Y \end{tikzcd}$$

Suppose given a vertex $z \in \mathrm{V}_{\tilde{F}} = \mathrm{V}_{Xg} \cup \mathrm{V}_{Fh}$; cf. Remark 37.

We consider two cases.

Case 1: $z \in \mathrm{V}_{Xg}$.

We choose a vertex $x \in \mathrm{V}_X$ such that $z = x\,\mathrm{V}_g$.

We have to show that the map $\mathrm{E}_q\,|_{\tilde{F}(z,*)}^{Y(z\,\mathrm{V}_q\,,*)} : \tilde{F}(z,*) \to Y(z\,\mathrm{V}_q\,,*)$ is surjective.

So suppose given an edge $e \in Y(z\,\mathrm{V}_q\,,*) \subseteq \mathrm{E}_Y$. We have $e\,\mathrm{s}_Y = z\,\mathrm{V}_q = x\,\mathrm{V}_g\,\mathrm{V}_q = x\,\mathrm{V}_f$. We write $y := e\,\mathrm{t}_Y \in \mathrm{V}_Y$. So for the edge $e \in \mathrm{E}_Y$ we have the path $(x\,\mathrm{V}_f\,;e)$ in Y of length

1 from $x\,\mathrm{V}_f = e\,\mathrm{s}_Y$ to $y = e\,\mathrm{t}_Y$. This path $(x\,\mathrm{V}_f\,; e)$ is a vertex in $\mathrm{Tree}(x\,\mathrm{V}_f\,, Y)$. So we have the vertex $(x, (x\,\mathrm{V}_f\,; e)) \in \mathrm{V}_F$. We have the vertex $(x, (x\,\mathrm{V}_f\,;)) \in \mathrm{V}_F$. We have the edge $e_T := (x, ((x\,\mathrm{V}_f\,;), e, (x\,\mathrm{V}_f\,; e))) \in \mathrm{E}_F$; cf. Definition 109. So $e_T\,\mathrm{s}_F = (x, (x\,\mathrm{V}_f\,;))$ and $e_T\,\mathrm{t}_F = (x, (x\,\mathrm{V}_f\,; e))$; cf. Definition 90.

We have $e_T\,\mathrm{E}_h \in \mathrm{E}_{\tilde{F}}$, where its source vertex is $e_T\,\mathrm{E}_h\,\mathrm{s}_{\tilde{F}} = e_T\,\mathrm{s}_F\,\mathrm{V}_h = (x, (x\,\mathrm{V}_f\,;))\,\mathrm{V}_h = x\,\mathrm{V}_a\,\mathrm{V}_h = x\,\mathrm{V}_{\mathrm{o}_X}\,\mathrm{V}_g = x\,\mathrm{V}_g = z$. So the edge $e_T\,\mathrm{E}_h \in \mathrm{E}_{\tilde{F}}$ has source z and thus $e_T\,\mathrm{E}_h \in \tilde{F}(z, *)$.

We show that $(e_T\,\mathrm{E}_h)\,\mathrm{E}_q \overset{!}{=} e \in Y(z\,\mathrm{V}_q\,, *)$.

We have $\mathrm{E}_h\,\mathrm{E}_q = \mathrm{E}_p$. So we have

$$e_T\,\mathrm{E}_h\,\mathrm{E}_q = e_T\,\mathrm{E}_p = (x, ((x\,\mathrm{V}_f\,;), e, (x\,\mathrm{V}_f\,; e)))\,\mathrm{E}_p \overset{\text{Def. 91}}{=} ((x\,\mathrm{V}_f\,;), e, (x\,\mathrm{V}_f\,; e))\,\mathrm{E}_{\mathrm{p}_{x\,\mathrm{V}_f}} \overset{\text{Def. 113}}{=} e\,.$$

Case 2: $z \in \mathrm{V}_{Fh}$.

We may choose a vertex in V_F, consisting of a vertex $x \in \mathrm{V}_X$ and a vertex v in $\mathrm{Tree}(x\,\mathrm{V}_f\,, Y)$, i.e. a path $v = (x\,\mathrm{V}_f\,; e_0\,, \ldots, e_{n-1})$ in Y, such that $z = (x, v)\,\mathrm{V}_h$.

We write $y_n := v\,\mathrm{V}_{\mathrm{p}_{x\,\mathrm{V}_f}} \in \mathrm{V}_Y$; cf. Definition 113.

We have to show that the map $\mathrm{E}_q\big|_{\tilde{F}(z,*)}^{Y(z\,\mathrm{V}_q\,,*)} : \tilde{F}(z, *) \to Y(z\,\mathrm{V}_q\,, *)$ is surjective.

We have $z\,\mathrm{V}_q = (x, v)\,\mathrm{V}_h\,\mathrm{V}_q \overset{\mathrm{V}_h\,\mathrm{V}_q = \mathrm{V}_p}{=} (x, v)\,\mathrm{V}_p \overset{\text{Def. 91}}{=} v\,\mathrm{V}_{\mathrm{p}_{x\,\mathrm{V}_f}} = y_n$.

So suppose given an edge $e \in Y(z\,\mathrm{V}_q\,, *) = Y(y_n, *)$. Note that $e\,\mathrm{s}_Y = y_n$.

We have to find an edge e_F in $\tilde{F}(z, *)$ such that $e_F\,\mathrm{E}_q \overset{!}{=} e$.

We first consider the edge $\tau := (x, ((x\,\mathrm{V}_f\,; e_0\,, \ldots, e_{n-1}), e, (x\,\mathrm{V}_f\,; e_0\,, \ldots, e_{n-1}, e))) \in \mathrm{E}_F$ which has the source vertex $(x, v) = (x, (x\,\mathrm{V}_f\,; e_0\,, \ldots, e_{n-1}))$ and the target vertex $(x, (x\,\mathrm{V}_f\,; e_0\,, \ldots, e_{n-1}, e))$.

We have $\tau\,\mathrm{E}_h\,\mathrm{s}_{\tilde{F}} = \tau\,\mathrm{s}_F\,\mathrm{V}_h = (x, v)\,\mathrm{V}_h = z$. So we have $\tau\,\mathrm{E}_h \in \tilde{F}(z, *)$.

We now show that $\tau\,\mathrm{E}_h\,\mathrm{E}_q \overset{!}{=} e$.

We have $hq = p$ and thus we have $\mathrm{E}_h\,\mathrm{E}_q = \mathrm{E}_p$.

So we have $\tau\,\mathrm{E}_h\,\mathrm{E}_q = \tau\,\mathrm{E}_p \overset{\text{Def. 91}}{=} ((x\,\mathrm{V}_f\,; e_0\,, \ldots ; e_{n-1}), e, (x\,\mathrm{V}_f\,; e_0\,, \ldots, e_{n-1}, e))\,\mathrm{E}_{p_{x\,\mathrm{V}_f}} \overset{\text{Def. 113}}{=} e$.

So the graph morphism $q : \tilde{F} \to Y$ is a fibration.

$$\begin{tikzcd}
\dot{X} \arrow[r, "a"] \arrow[d, "\mathrm{o}_X"'] & F \arrow[d, "h"] \arrow[ddr, bend left, "p"] & \\
X \arrow[r, "g"] \arrow[drr, bend right, "f"'] & \tilde{F} \arrow[dr, "q"] & \\
& & Y
\end{tikzcd}$$

$\square$

Lemma 185 In Gph, we have AcCofib = Cofib $\cap$ Qis.

Proof. We show that AcCofib $\overset{!}{\supseteq}$ Cofib $\cap$ Qis.

Suppose given a quasiisomorphism $f : X \to Y$ that is a cofibration, i.e. that satisfies ($\mathrm{LLP}_{\mathrm{AcFib}}$); cf. Definition 144.

We have to show that f is an acyclic cofibration, i.e. that (AcCofib 1–5) hold for the graph morphism f; cf. Definition 162.

By Lemma 184 we may choose an acyclic cofibration $w : X \to Z$ and a fibration $p : Z \to Y$ such that $f = wp$.

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "w"] \arrow[rr, "f", "\approx"'] & & Y \end{tikzcd}$$

Because the graph morphism $f : X \xrightarrow{\approx} Y$ is a quasiisomorphism and because the graph morphism $w : X \longrightarrow Z$ is an acyclic cofibration and thus a quasiisomorphism, we may conclude by (2 of 3) that the fibration $p : Z \to Y$ is a quasiisomorphism and thus an acyclic fibration; cf. Remark 123 and Definition 138.

So we consider the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "w"] \arrow[d, "f"', "\wr"] & Z \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd}$$

Because the graph morphism $f : X \to Y$ is a cofibration and thus in ${}^{\boxslash}\mathrm{AcFib}$, there exists a graph morphism $h : Y \to Z$ such that $fh = w$ and such that $hp = \mathrm{id}_Y$.

$$\begin{tikzcd} X \arrow[r, "w"] \arrow[d, "f"', "\wr"] & Z \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, "h"'] & Y \end{tikzcd}$$

We consider the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[r, "\mathrm{id}_X"'] \arrow[d, "w"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

Since the set of acyclic cofibrations AcCofib is closed under retracts, the graph morphism $f : X \to Y$ is an acyclic cofibration; cf. Remark 172.

So we have AcCofib $\supseteq$ Cofib $\cap$ Qis.

We have AcCofib $\subseteq$ Cofib $\cap$ Qis; cf. Lemma 175.

Altogether, we have AcCofib $=$ Cofib $\cap$ Qis. $\square$

Lemma 186 Suppose given a graph morphism $f : X \to Y$.

Then there exist a commutative triangle in Gph as follows, where $(\mathrm{C}_n\,, f')_{\mathrm{Gph}}$ is surjective for $n \geqslant 1$.

$$\begin{tikzcd} & \tilde{X} \arrow[dr, "f'"] & \\ X \arrow[ur, "g'"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Proof. For $n \geqslant 1$, we let $M_n := (\mathrm{C}_n\,, Y)_{\mathrm{Gph}} \setminus \mathrm{Im}(\mathrm{C}_n\,, g)_{\mathrm{Gph}}\,.$

Let $M := \coprod_{n \geqslant 1} M_n = \{(n,u) \,:\, n \geqslant 1,\, u \in M_n\}$.

Let $C := \coprod_{(n,u)\in M} \mathrm{C}_n\,.$

We have $\mathrm{V}_C = \{((n,u),\mathrm{v}_i) \,:\, n \geqslant 1,\, u \in M_n\,,\, i \in \mathbb{Z}/n\mathbb{Z}\}$.

And we have $\mathrm{E}_C = \{((n,u),\mathrm{e}_i) \,:\, n \geqslant 1,\, u \in M_n\,,\, i \in \mathbb{Z}/n\mathbb{Z}\}$.

We write

$$(n,u,\mathrm{v}_i) := ((n,u),\mathrm{v}_i)$$

and

$$(n,u,\mathrm{e}_i) := ((n,u),\mathrm{e}_i)\ .$$

Using Definition 91, we have the graph morphism

$$\begin{array}{rrcl} h = (u)_{(n,u)\in M} : & C & \to & Y \\ & (n,u,\mathrm{v}_i) & \mapsto & \mathrm{v}_i\,\mathrm{V}_u \;\text{ for } i \in \mathbb{Z}/n\mathbb{Z} \\ & (n,u,\mathrm{e}_i) & \mapsto & \mathrm{e}_i\,\mathrm{E}_u \;\text{ for } i \in \mathbb{Z}/n\mathbb{Z}\ . \end{array}$$

Let $X' := X \sqcup C$; cf. Definition 85. Then

$$\mathrm{V}_{X'} = \mathrm{V}_X \sqcup \mathrm{V}_C = \{(1,v_X) \,:\, v_X \in \mathrm{V}_X\} \cup \{(2,v_C) \,:\, v_C \in \mathrm{V}_C\}$$

and

$$\mathrm{E}_{X'} = \mathrm{E}_X \sqcup \mathrm{E}_C = \{(1,e_X) \,:\, e_X \in \mathrm{E}_X\} \cup \{(2,e_C) \,:\, e_C \in \mathrm{E}_C\}\ .$$

Moreover,

$$\begin{array}{rrcl} \mathrm{s}_{X'} : & \mathrm{E}_{X'} & \to & \mathrm{V}_{X'} \\ & (1,e_X) & \mapsto & (1,e_X\,\mathrm{s}_X) \\ & (2,e_C) & \mapsto & (2,e_C\,\mathrm{s}_C) \end{array}$$

and

$$\begin{array}{rrcl} \mathrm{t}_{X'} : & \mathrm{E}_{X'} & \to & \mathrm{V}_{X'} \\ & (1,e_X) & \mapsto & (1,e_X\,\mathrm{t}_X) \\ & (2,e_C) & \mapsto & (2,e_C\,\mathrm{t}_C)\ . \end{array}$$

We have the graph morphism $f' := \binom{g}{h} : X \sqcup C \to Y$ satisfying $\iota_1 f = g$ and $\iota_2 f = h$; cf. Definition 86.

Writing $g' := \iota_1 : X \to X \sqcup C$, we have $g' f = g$.

Claim 1. The graph morphism $g' : X \to X' = X \sqcup C$ is a cofibration; cf. Definition 144.

The cyclic graph C_n is cofibrant for $n \geqslant 1$; cf. Remark 151. So the graph $C = \coprod_{(n,u)\in M} \mathrm{C}_n$ is cofibrant, i.e. the graph morphism $\iota_C : \emptyset \to C$ is a cofibration; cf. Definitions 70, 150 and Remark 152.

So the graph morphism $g' : X \to X \sqcup C$ is a cofibration as a pushout of a cofibration; cf. Remark 157.

This proves *Claim 1*.

Claim 2. Given $n \geqslant 1$. The map $(\mathrm{C}_n\,, f')_{\mathrm{Gph}} : (\mathrm{C}_n\,, X \sqcup C)_{\mathrm{Gph}} \to (\mathrm{C}_n\,, Y)_{\mathrm{Gph}}$ is surjective.

Suppose given a graph morphism $u : \mathrm{C}_n \to Y$ in $(\mathrm{C}_n\,, Y)_{\mathrm{Gph}}$.

Case 1: $u \in \mathrm{Im}(\mathrm{C}_n\,, g)_{\mathrm{Gph}}$.

Then we may choose a graph morphism $\tilde{u} : \mathrm{C}_n \to X$ in $(\mathrm{C}_n\,, X)_{\mathrm{Gph}}$ such that $u = \tilde{u} g$.

$$\begin{tikzcd}
X \sqcup C \arrow[r, "f'=\binom{g}{h}"] & Y \\
X \arrow[u, "g'"] \arrow[ur, "g"] & \\
\mathrm{C}_n \arrow[u, "\tilde{u}"] \arrow[uur, "u"'] &
\end{tikzcd}$$

So we have $\tilde{u} g' \in (\mathrm{C}_n\,, X \sqcup C)_{\mathrm{Gph}}$ and $(\tilde{u} g')(\mathrm{C}_n\,, f')_{\mathrm{Gph}} = \tilde{u} g' f' = \tilde{u} g = u$.

Case 2: $u \in (\mathrm{C}_n\,, Y)_{\mathrm{Gph}} \setminus \mathrm{Im}(\mathrm{C}_n\,, g)_{\mathrm{Gph}}$.

So we have $u \in M_n$, i.e. $(n, u) \in M$.

We have the graph morphism $\iota_{(n,u)} : \mathrm{C}_n \to C$; cf. Definition 90.

We have $\iota_{(n,u)} h = \iota_{(n,u)} (u)_{(n,u)\in M} \overset{\text{Def. 91}}{=} u$.

Consider the graph morphism $\iota_2 : C \to X \sqcup C$.

We have $\iota_{(n,u)} \cdot \iota_2 \in (\mathrm{C}_n\,, X \sqcup C)_{\mathrm{Gph}}$ and $(\iota_{(n,u)} \cdot \iota_2)(\mathrm{C}_n\,, f')_{\mathrm{Gph}} = \iota_{(n,u)} \cdot \iota_2 \cdot f' = \iota_{(n,u)} \cdot h = u$.

$$\begin{tikzcd}
X \sqcup C \arrow[r, "f'=\binom{g}{h}"] & Y \\
C \arrow[u, "\iota_2"] \arrow[ur, "h"] & \\
\mathrm{C}_n \arrow[u, "\iota_{n,u}"] \arrow[uur, "u"'] &
\end{tikzcd}$$

This proves *Claim 2*.

So we have the commutative triangle

$$\begin{tikzcd} & X \sqcup C \arrow[dr, "f'"] & \\ X \arrow[ur, "g'"] \arrow[rr, "g"'] & & Y \end{tikzcd} ,$$

where $(\mathrm{C}_n\, , f')_{\mathrm{Gph}}$ is surjective for $n \geqslant 1$. ▫

The following lemma is an iterated version of an argument of Bisson and Tsemo [**3,** Prop. 4.6].

Lemma 187 Suppose given a graph morphism $f : X \to Y$ such that $(\mathrm{C}_n\, , f)_{\mathrm{Gph}}$ is surjective for $n \geqslant 1$.

Then there exists a commutative triangle in Gph as follows.

$$\begin{tikzcd} & G \arrow[dr, "h", "\wr"'] & \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Proof. We have the graph morphism $f_0 := f : X \to Y$.

Let $X_0 := X$.

Let $g_{0,0} := \mathrm{id}_X : X \to X_0$.

Then $g_{0,0} \cdot f_0 = f$.

For $i \geqslant 0$, we shall recursively construct a commutative diagram in Gph as follows for suitably defined sets M_i and M and j_i resulting from Lemma 156.

$$\begin{tikzcd} & \coprod_{(n,p)\in M_i} \mathrm{C}_n \arrow[r, "j_i"] \arrow[d, "a_i"'] & \coprod_{(n,u)\in M} \mathrm{C}_n \arrow[d, "b_i"] \arrow[ddr, bend left, "c"] & \\ & X_i \arrow[r, "g_{i,i+1}"] \arrow[drr, "f_i"] & X_{i+1} \arrow[dr, "f_{i+1}"] \arrow[ul, phantom, "\lrcorner", very near start] & \\ X \arrow[ur, "g_{0,i}"] \arrow[urr, "g_{0,i+1}"'] \arrow[rrr, "f"'] & & & Y \end{tikzcd} .$$

Suppose that the commutative triangle

$$\begin{tikzcd} & X_k \arrow[dr, "f_k"] & \\ X \arrow[ur, "g_{0,k}"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

is constructed for a given $k \geqslant 0$.

Let $M := \{(n, u) : n \geqslant 1,\ \mathrm{C}_n \xrightarrow{u} Y\ \}$.

Then we have the graph $\coprod_M \mathrm{C}_n := \coprod_{(n,u)\in M} \mathrm{C}_n$ with

$$\begin{aligned} \mathrm{V}_{\coprod_M \mathrm{C}_n} &= \{(n,u,\mathrm{v}_i) : n \geqslant 1,\ \mathrm{C}_n \xrightarrow{u} Y\ ,\ i \in \mathbb{Z}/n\mathbb{Z}\} \\ \mathrm{E}_{\coprod_M \mathrm{C}_n} &= \{(n,u,\mathrm{e}_i) : n \geqslant 1,\ \mathrm{C}_n \xrightarrow{u} Y\ ,\ i \in \mathbb{Z}/n\mathbb{Z}\}\ . \end{aligned}$$

Let $M_k := \{(n,p) : n \geqslant 1,\ \mathrm{C}_n \xrightarrow{p} X_k\ \}$.

Then we have the graph $\coprod_{M_k} \mathrm{C}_n := \coprod_{(n,p)\in M_k} \mathrm{C}_n$ with

$$\begin{aligned} \mathrm{V}_{\coprod_{M_k} \mathrm{C}_n} &= \{(n,p,\mathrm{v}_i) : n \geqslant 1,\ \mathrm{C}_n \xrightarrow{p} X_k\ ,\ i \in \mathbb{Z}/n\mathbb{Z}\} \\ \mathrm{E}_{\coprod_{M_k} \mathrm{C}_n} &= \{(n,p,\mathrm{e}_i) : n \geqslant 1,\ \mathrm{C}_n \xrightarrow{p} X_k\ ,\ i \in \mathbb{Z}/n\mathbb{Z}\}\ . \end{aligned}$$

Using the maps $\mu : M_k \to M : (n,p) \mapsto (n, pf_k)$ and $\nu : M \to \mathbb{N} : (n,u) \mapsto n$, Lemma 156 gives the cofibration

$$\begin{array}{rccc} j_k : & \coprod_{M_k} \mathrm{C}_n & \to & \coprod_M \mathrm{C}_n \\ \mathrm{V}_{j_k} : & (n,p,\mathrm{v}_i) & \mapsto & (n,pf_k,\mathrm{v}_i) \\ \mathrm{E}_{j_k} : & (n,p,\mathrm{e}_i) & \mapsto & (n,pf_k,\mathrm{e}_i)\ . \end{array}$$

Using Definition 91, we obtain the following graph morphisms.

$$\begin{array}{rccl} a_k := (p)_{(n,p)} & \coprod_{M_k} \mathrm{C}_n & \to & X_k \\ \mathrm{V}_{a_k} : & (n,p,\mathrm{v}_i) & \mapsto & \mathrm{v}_i\,\mathrm{V}_p \ \text{ for }\ \mathrm{v}_i \in \mathrm{V}_{\mathrm{C}_n} \\ \mathrm{E}_{a_k} : & (n,p,\mathrm{e}_i) & \mapsto & \mathrm{e}_i\,\mathrm{E}_p \ \text{ for }\ \mathrm{e}_i \in \mathrm{E}_{\mathrm{C}_n} \end{array}$$

$$\begin{array}{rccl} c := (u)_{(n,u)\in M} : & \coprod_M \mathrm{C}_n & \to & Y \\ \mathrm{V}_c : & (n,u,\mathrm{v}_i) & \mapsto & \mathrm{v}_i\,\mathrm{V}_u \ \text{ for }\ \mathrm{v}_i \in \mathrm{V}_{\mathrm{C}_n} \\ \mathrm{E}_c : & (n,u,\mathrm{e}_i) & \mapsto & \mathrm{e}_i\,\mathrm{E}_u \ \text{ for }\ \mathrm{e}_i \in \mathrm{E}_{\mathrm{C}_n} \end{array}$$

We note that since $(\mathrm{C}_n\,,f_k)_{\mathrm{Gph}}$ is surjective, the graph morphism j_k is surjective.

We show that we have $j_k c \overset{!}{=} a_k f_k$.

Suppose given $(n,p,\mathrm{v}_i) \in \mathrm{V}_{\coprod_{M_i} \mathrm{C}_n}$.

We have $(n,p,\mathrm{v}_i)\,\mathrm{V}_{j_k}\,\mathrm{V}_c = (n,pf_k,\mathrm{v}_i)\,\mathrm{V}_c = \mathrm{v}_i\,\mathrm{V}_{pf_k} = \mathrm{v}_i\,\mathrm{V}_p\,\mathrm{V}_{f_k} = (n,p,\mathrm{v}_i)\,\mathrm{V}_{a_k}\,\mathrm{V}_{f_k}$.

Suppose given $(n,p,\mathrm{e}_i) \in \mathrm{E}_{\coprod_{M_i} \mathrm{C}_n}$.

We have $(n,p,\mathrm{e}_i)\,\mathrm{E}_{j_k}\,\mathrm{E}_c = (n,pf_k,\mathrm{e}_i)\,\mathrm{E}_c = \mathrm{e}_i\,\mathrm{E}_{pf_k} = \mathrm{e}_i\,\mathrm{E}_p\,\mathrm{E}_{f_k} = (n,p,\mathrm{e}_i)\,\mathrm{E}_{a_k}\,\mathrm{E}_{f_k}$.

So we have $j_k c = a_k f_k$.

Forming the pushout, we obtain the following commutative diagram.

$$\begin{tikzcd} & \coprod_{(n,p)\in M_i} \mathrm{C}_n \arrow[r, "j_i"] \arrow[d, "a_i"'] & \coprod_{(n,u)\in M} \mathrm{C}_n \arrow[d, "b_i"] \arrow[ddr, bend left, "c"] & \\ & X_i \arrow[r, "g_{i,i+1}"] \arrow[ur, phantom, "\lrcorner"] \arrow[drr, "f_i"'] & X_{i+1} \arrow[dr, "f_{i+1}"] & \\ X \arrow[ur, "g_{0,i}"] \arrow[urr, "g_{0,i+1}"'] \arrow[rrr, "f"'] & & & Y\ . \end{tikzcd}$$

In this diagram, the graph morphism $g_{k,k+1} : X_k \to X_{k+1}$ is a cofibration; cf. Remark 148.

With respect to the morphisms $X_i \xrightarrow{g_{i,i+1}} X_{i+1}$ for $i \geqslant 0$, we let $X_\infty := \varinjlim_{i \geqslant 0} X_i$; cf. Definition 98 and Lemma 99.

We have graph morphisms $g_{i,\infty} : X_i \to X_\infty$ such that $g_{i,i+1} \cdot g_{i+1,\infty} = g_{i,\infty}$ for $i \geqslant 0$; cf. Definition 98.

Since $g_{i,i+1}$ is a cofibration for $i \geqslant 0$, we conclude that $g_{0,\infty} : X_0 \to X_\infty$ is a cofibration; cf. Lemma 102.

There exists a unique graph morphism $f_\infty : X_\infty \to Y$ such that $g_{i,\infty} \cdot f_\infty = f_i$ for $i \geqslant 0$; cf. Lemma 99.

$$\begin{tikzcd}
X = X_0 \arrow[r, "g_{0,1}"'] \arrow[d, "f_0"'] \arrow[rrrrr, bend left=40, "g_{0,\infty}"] & X_1 \arrow[r, "g_{1,2}"'] \arrow[d, "f_1"] \arrow[rrrr, bend left=35, "g_{1,\infty}"] & X_2 \arrow[r, "g_{2,3}"'] \arrow[d, "f_2"] \arrow[rrr, bend left=30, "g_{2,\infty}"] & X_3 \arrow[d, "f_3"] \arrow[rr, bend left=25, "g_{3,\infty}"] & \cdots & \varinjlim_{i \geqslant 0} X_i =: X_\infty \arrow[d, "\exists ! f_\infty"] \\
Y \arrow[r, "\mathrm{id}_Y"] \arrow[rrrrr, bend right=40, "\mathrm{id}_Y"'] & Y \arrow[r, "\mathrm{id}_Y"] \arrow[rrrr, bend right=35, "\mathrm{id}_Y"'] & Y \arrow[r, "\mathrm{id}_Y"] \arrow[rrr, bend right=30, "\mathrm{id}_Y"'] & Y \arrow[rr, bend right=25, "\mathrm{id}_Y"'] & \cdots & Y
\end{tikzcd}$$

By the surjectivity of $(\mathrm{C}_n\,, f)_{\mathrm{Gph}} = (\mathrm{C}_n\,, f_0)_{\mathrm{Gph}} = (\mathrm{C}_n\,, g_{0,\infty})_{\mathrm{Gph}} \cdot (\mathrm{C}_n\,, f_\infty)_{\mathrm{Gph}}$, we conclude that the map $(\mathrm{C}_n\,, f_\infty)_{\mathrm{Gph}}$ is surjective for $n \geqslant 0$.

We *claim* that the graph morphism $f_\infty : X_\infty \to Y$ is a quasiisomorphism.

Suppose given $n \geqslant 1$. It suffices to show that the map $(\mathrm{C}_n\,, f_\infty)_{\mathrm{Gph}}$ is injective.

Suppose given graph morphisms $w', w'' : \mathrm{C}_n \to X_\infty$ such that $w' f_\infty = w'' f_\infty$.

We have to show that $w' \overset{!}{=} w''$.

We write $w := w' f_\infty = w'' f_\infty$.

There exist $k \geqslant 0$ and graph morphisms $\hat{w}', \hat{w}'' : \mathrm{C}_n \to X_k$ such that $\hat{w}' g_{k,\infty} = w'$ and such that $\hat{w}'' g_{k,\infty} = w''$; cf. Lemma 101.(2).

We have $\hat{w}' f_k = \hat{w}' g_{k,\infty} f_\infty = w' f_\infty = w = w'' f_\infty = \hat{w}'' g_{k,\infty} f_\infty = \hat{w}'' f_k$.

$$\begin{tikzcd}
X_k \arrow[r, "g_{k,\infty}"] & X_\infty \arrow[d, "f_\infty"] \\
\mathrm{C}_n \arrow[u, shift left, "\hat{w}'"] \arrow[u, shift right, "\hat{w}''"'] \arrow[ur, shift left, "w'"] \arrow[ur, shift right, "w''"'] \arrow[r, "w"'] & Y
\end{tikzcd}$$

It now suffices to show that $\hat{w}' g_{k,k+1} \overset{!}{=} \hat{w}'' g_{k,k+1}$.

Because then we have $w' = \hat{w}' g_{k,\infty} = \hat{w}' g_{k,k+1} g_{k+1,\infty} = \hat{w}'' g_{k,k+1} g_{k+1,\infty} = \hat{w}'' g_{k,\infty} = w''$.

We have $(n, \hat{w}''), (n, \hat{w}') \in M_k$.

We show that $\hat{w}' g_{k,k+1} \overset{!}{=} \hat{w}'' g_{k,k+1}$.

So suppose given $\mathrm{v}_i \in \mathrm{V}_{\mathrm{C}_n}$, where $i \in \mathbb{Z}/n\mathbb{Z}$.

We have to show that $\mathrm{v}_i \, \mathrm{V}_{\hat{w}' g_{k,k+1}} \overset{!}{=} \mathrm{v}_i \, \mathrm{V}_{\hat{w}'' g_{k,k+1}}$.

We have $\mathrm{v}_i \, \mathrm{V}_{\hat{w}' g_{k,k+1}} = \mathrm{v}_i \, \mathrm{V}_{\hat{w}'} \, \mathrm{V}_{g_{k,k+1}} \overset{\text{Def. } a_k}{=} (n, \hat{w}', \mathrm{v}_i) \, \mathrm{V}_{a_k} \, \mathrm{V}_{g_{k,k+1}} = (n, \hat{w}', \mathrm{v}_i) \, \mathrm{V}_{a_k g_{k,k+1}} = (n, \hat{w}', \mathrm{v}_i) \, \mathrm{V}_{j_k b_k} = (n, \hat{w}', \mathrm{v}_i) \, \mathrm{V}_{j_k} \, \mathrm{V}_{b_k} \overset{\text{Def. } j_k}{=} (n, w, \mathrm{v}_i) \, \mathrm{V}_{b_k} \overset{\text{Def. } j_k}{=} (n, \hat{w}'', \mathrm{v}_i) \, \mathrm{V}_{j_k} \, \mathrm{V}_{b_k} = (n, \hat{w}'', \mathrm{v}_i) \, \mathrm{V}_{j_k b_k} = (n, \hat{w}'', \mathrm{v}_i) \, \mathrm{V}_{a_k g_{k,k+1}} = (n, \hat{w}'', \mathrm{v}_i) \, \mathrm{V}_{a_k} \, \mathrm{V}_{g_{k,k+1}} \overset{\text{Def. } a_k}{=} \mathrm{v}_i \, \mathrm{V}_{\hat{w}''} \, \mathrm{V}_{g_{k,k+1}} = \mathrm{v}_i \, \mathrm{V}_{\hat{w}'' g_{k,k+1}}$.

Now suppose given an edge $\mathrm{e}_i \in \mathrm{E}_{\mathrm{C}_n}$.

We have to show that $\mathrm{e}_i \, \mathrm{E}_{\hat{w}' g_{k,k+1}} \overset{!}{=} \mathrm{e}_i \, \mathrm{E}_{\hat{w}'' g_{k,k+1}}$.

We have $\mathrm{e}_i \, \mathrm{E}_{\hat{w}' g_{k,k+1}} = \mathrm{e}_i \, \mathrm{E}_{\hat{w}'} \, \mathrm{E}_{g_{k,k+1}} \overset{\text{Def. } a_k}{=} (n, \hat{w}', \mathrm{e}_i) \, \mathrm{E}_{a_k} \, \mathrm{E}_{g_{k,k+1}} = (n, \hat{w}', \mathrm{e}_i) \, \mathrm{E}_{a_k g_{k,k+1}} = (n, \hat{w}', \mathrm{e}_i) \, \mathrm{E}_{j_k b_k} = (n, \hat{w}', \mathrm{e}_i) \, \mathrm{E}_{j_k} \, \mathrm{E}_{b_k} \overset{\text{Def. } j_k}{=} (n, w, \mathrm{e}_i) \, \mathrm{E}_{b_k} \overset{\text{Def. } j_k}{=} (n, \hat{w}'', \mathrm{e}_i) \, \mathrm{E}_{j_k} \, \mathrm{E}_{b_k} = (n, \hat{w}'', \mathrm{e}_i) \, \mathrm{E}_{j_k b_k} = (n, \hat{w}'', \mathrm{e}_i) \, \mathrm{E}_{a_k g_{k,k+1}} = (n, \hat{w}'', \mathrm{e}_i) \, \mathrm{E}_{a_k} \, \mathrm{E}_{g_{k,k+1}} \overset{\text{Def. } a_k}{=} \mathrm{e}_i \, \mathrm{E}_{\hat{w}''} \, \mathrm{E}_{g_{k,k+1}} = \mathrm{e}_i \, \mathrm{E}_{\hat{w}'' g_{k,k+1}}$.

So we have $\hat{w}' g_{k,k+1} = \hat{w}'' g_{k,k+1}$.

Altogether, we have obtained a commutative triangle

$$\begin{tikzcd} & X_\infty \arrow[dr, "f_\infty", "\wr"'] & \\ X = X_0 \arrow[ur, "g_{0,\infty}"] \arrow[rr, "f = f_0"'] & & Y \end{tikzcd}$$

as was to be shown. $\square$

Lemma 188 Suppose given a graph morphism $g : X \to Y$.

Then there exists a commutative triangle in Gph as follows.

$$\begin{tikzcd} & G \arrow[dr, "f"] & \\ X \arrow[ur, "d"] \arrow[rr, "g"'] & & Y \end{tikzcd}$$

Proof.

Step 1. There exists a commutative triangle in Gph as follows, where $(\mathrm{C}_n \, , \tilde{f})_{\mathrm{Gph}}$ is surjective for $n \geqslant 1$; cf. Lemma 186.

$$\begin{tikzcd} & \tilde{X} \arrow[dr, "\tilde{f}"] & \\ X \arrow[ur, "c"] \arrow[rr, "g"'] & & Y \end{tikzcd}$$

Step 2. By Lemma 187 there exists a commutative triangle in Gph as follows.

$$\begin{tikzcd} & \tilde{\tilde{X}} \arrow[dr, "\tilde{\tilde{f}}", "\wr"'] & \\ \tilde{X} \arrow[ur, "\tilde{c}"] \arrow[rr, "\tilde{f}"'] & & Y \end{tikzcd}$$

Step 3. By Lemma 184 there exists a commutative triangle in Gph as follows.

$$\begin{tikzcd}
& G \arrow[dr, "f"] & \\
\tilde{\tilde{X}} \arrow[ur, "\tilde{\tilde{c}}"] \arrow[rr, "\approx", "\tilde{\tilde{f}}"'] & & Y
\end{tikzcd}$$

Because Qis satisfies (2 of 3) and because AcCofib $\subseteq$ Qis, the fibration $f : G \longrightarrow Y$ is a quasiisomorphism and thus an acyclic fibration; cf. Lemma 175 and Definition 138.

In conclusion we have the following commutative diagram in Gph.

$$\begin{tikzcd}
& & \tilde{\tilde{X}} \arrow[r, "\tilde{\tilde{c}}"] \arrow[ddr, "\tilde{\tilde{f}}"', "\wr"] & G \arrow[dd, "f"] \\
& \tilde{X} \arrow[ur, "\tilde{c}"] \arrow[drr, "\tilde{f}"'] & & \\
X \arrow[ur, "c"] \arrow[rrr, "g"'] & & & Y
\end{tikzcd}$$

The acyclic cofibration $\tilde{\tilde{c}} : \tilde{\tilde{X}} \longrightarrow G$ is in particular a cofibration; cf. Lemma 185.

Since the composite of cofibrations is a cofibration, the graph morphism $d := c\tilde{c}\tilde{\tilde{c}} : X \to G$ is a cofibration; cf. Remark 147. $\square$

Chapter 5

Subsets of $\mathrm{Mor}(\mathrm{Gph})$ and their lifting sets

Lemma 189 In Gph, we have $\mathrm{AcCofib}^{\boxslash} \subseteq \mathrm{Fib}$.

Proof. Suppose given a graph morphism $f : X \to Y$ in $\mathrm{AcCofib}^{\boxslash}$; cf. Definitions 162 and 14.

We have to show that the graph morphism $f : X \to Y$ is in Fib, i.e. that f is a fibration.

Suppose given a vertex $x \in \mathrm{V}_X$.

We have to show that the map $\mathrm{E}_{f,x} = \mathrm{E}_{f|_{X(x,*)}^{Y(x\,\mathrm{V}_f\,,*)}} : X(x,*) \to Y(x\,\mathrm{V}_f\,,*)$ is surjective; cf. Definition 127.(1).

Suppose given an edge $e \in Y(x\,\mathrm{V}_f\,,*)$, i.e. $e \in \mathrm{E}_Y$ with $e\,\mathrm{s}_Y = x\,\mathrm{V}_f$.

We write $y := e\,\mathrm{t}_Y \in \mathrm{V}_Y$.

We have to find an edge $\tilde{e} \in X(x,*)$ such that $\tilde{e}\,\mathrm{E}_f \overset{!}{=} e$.

Let $a : \mathrm{D}_0 \to X$ be defined by $\hat{\mathrm{v}}_0\,\mathrm{V}_a := x$. So $a = (x;)$; cf. Definition 107.

Let $b : \mathrm{D}_1 \to X$ be defined by $\hat{\mathrm{e}}_0\,\mathrm{E}_b := e$, $\hat{\mathrm{v}}_0\,\mathrm{V}_b := e\,\mathrm{s}_Y = x\,\mathrm{V}_f$ and $\hat{\mathrm{v}}_1\,\mathrm{V}_b := e\,\mathrm{t}_Y = y$. So $b = (x\,\mathrm{V}_f\,; e)$; cf. Definition 107.

In fact, $b : \mathrm{D}_1 \to Y$ is a graph morphism since $\hat{\mathrm{e}}_0\,\mathrm{s}_{\mathrm{D}_1}\,\mathrm{V}_b = \hat{\mathrm{v}}_0\,\mathrm{V}_b = e\,\mathrm{s}_Y = \hat{\mathrm{e}}_0\,\mathrm{E}_b\,\mathrm{s}_Y$ and since $\hat{\mathrm{e}}_0\,\mathrm{t}_{\mathrm{D}_1}\,\mathrm{V}_b = \hat{\mathrm{v}}_1\,\mathrm{V}_b = e\,\mathrm{t}_Y = \hat{\mathrm{e}}_0\,\mathrm{E}_b\,\mathrm{t}_Y$.

Note that the graph morphism $\iota_{0,1} : \mathrm{D}_0 \to \mathrm{D}_1$ is an acyclic cofibration; cf. Remark 166.

We have the following commutative diagram

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_{0,1}"'] & X \arrow[d, "f"] \\ \mathrm{D}_1 \arrow[r, "b"] & Y \end{tikzcd}$$

since $\hat{\mathrm{v}}_0\,\mathrm{V}_a\,\mathrm{V}_f \overset{\text{Def. } a}{=} x\,\mathrm{V}_f \overset{\text{Def. } b}{=} \hat{\mathrm{v}}_0\,\mathrm{V}_b = \hat{\mathrm{v}}_0\,\mathrm{V}_{\iota_{0,1}}\,\mathrm{V}_b$.

We have $\hat{\mathrm{e}}_0\,\mathrm{s}_{\mathrm{D}_1} = \hat{\mathrm{v}}_0 = \hat{\mathrm{v}}_0\,\mathrm{V}_{\iota_{0,1}}$.

Because the graph morphism $f : X \to Y$ is in $\mathrm{AcCofib}^{\boxslash}$ there exists a graph morphism

$h : \mathrm{D}_1 \to X$ such that $\iota_{0,1} h = a$ and $hf = b$.

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_{0,1}"'] & X \arrow[d, "f"] \\ \mathrm{D}_1 \arrow[r, "b"] \arrow[ur, "h"] & Y \end{tikzcd}$$

Let $\tilde{e} := \hat{\mathrm{e}}_0 \, \mathrm{E}_h \in \mathrm{E}_X$.

Then we have $\tilde{e}\, \mathrm{s}_X = \hat{\mathrm{e}}_0 \, \mathrm{E}_h \, \mathrm{s}_X = \hat{\mathrm{e}}_0 \, \mathrm{s}_{\mathrm{D}_1} \, \mathrm{V}_h = \hat{\mathrm{v}}_0 \, \mathrm{V}_h = \hat{\mathrm{v}}_0 \, \mathrm{V}_{\iota_{0,1}} \, \mathrm{V}_h = \hat{\mathrm{v}}_0 \, \mathrm{V}_a \overset{\text{Def. } a}{=} x$.

So we have $\tilde{e} \in X(x, *)$.

Furthermore, $\tilde{e}\, \mathrm{E}_f = \hat{\mathrm{e}}_0 \, \mathrm{E}_h \, \mathrm{E}_f = \hat{\mathrm{e}}_0 \, \mathrm{E}_b \overset{\text{Def. } b}{=} e$. ▫

Lemma 190 In Gph, we have ${}^{\boxslash}\mathrm{Fib} \subseteq \mathrm{AcCofib}$.

Proof. Suppose given a graph morphism $f : X \to Y$ in ${}^{\boxslash}\mathrm{Fib}$, i.e. such that $f \boxslash \mathrm{Fib}$.

We have to show that $f \overset{!}{\in} \mathrm{AcCofib}$, i.e. that $f : X \to Y$ is an acyclic cofibration.

There exist an acyclic cofibration $X \xrightarrow{a} Z$ and a fibration $Z \xrightarrow{b} Y$ such that $f = ab$; cf. Lemma 184.

We consider the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "b"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd}$$

Because $f \boxslash \mathrm{Fib}$ there exists a graph morphism $h : Y \to Z$ such that $fh = a$ and $hb = \mathrm{id}_Y$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "b"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

We consider the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[r, "\mathrm{id}_X"'] \arrow[d, "a"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Z \arrow[r, "b"] & Y \end{tikzcd}$$

Because the set of acyclic cofibrations AcCofib is closed under retracts, the graph morphism $f : X \to Y$ is an acyclic cofibration; cf. Remark 172 ▫

Lemma 191 In Gph, we have $\mathrm{AcCofib} = {}^{\boxslash}\mathrm{Fib}$.

Proof.

We have $\mathrm{AcCofib} \subseteq {}^{\boxslash}\mathrm{Fib}$; cf. Lemma 174.

We have $\mathrm{AcCofib} \supseteq {}^{\boxslash}\mathrm{Fib}$; cf. Lemma 190. ▫

Lemma 192 In Gph, we have $\mathrm{AcCofib}^{\boxslash} = \mathrm{Fib}$.

Proof.

We have $\mathrm{AcCofib}^{\boxslash} \subseteq \mathrm{Fib}$; cf. Lemma 189.

We have $\mathrm{AcCofib}^{\boxslash} \supseteq \mathrm{Fib}$, i.e. $\mathrm{AcCofib} \boxslash \mathrm{Fib}$, i.e. $\mathrm{AcCofib} \subseteq {}^{\boxslash}\mathrm{Fib}$; cf. Lemma 174. ▫

Lemma 193 In Gph, we have $\mathrm{Cofib}^{\boxslash} = \mathrm{AcFib}$.

Proof.

We have $\mathrm{Cofib} \subseteq {}^{\boxslash}\mathrm{AcFib}$, i.e. $\mathrm{Cofib} \boxslash \mathrm{AcFib}$, i.e. $\mathrm{Cofib}^{\boxslash} \supseteq \mathrm{AcFib}$; cf. Definition 144.

We have to show that $\mathrm{Cofib}^{\boxslash} \overset{!}{\subseteq} \mathrm{AcFib}$.

So suppose given a graph morphism $f : X \to Y$ in $\mathrm{Cofib}^{\boxslash}$.

We have to show that the graph morphism $f : X \to Y$ is an acyclic fibration; i.e. a fibration that is a quasiisomorphism.

Since $\mathrm{AcCofib} \subseteq \mathrm{Cofib}$ we have $\mathrm{Cofib}^{\boxslash} \subseteq \mathrm{AcCofib}^{\boxslash}$; cf. Lemma 175.

So we have $f \in \mathrm{Cofib}^{\boxslash} \subseteq \mathrm{AcCofib}^{\boxslash} = \mathrm{Fib}$; cf. Lemma 192.

So the graph morphism $f : X \to Y$ is a fibration.

We now have to show that the fibration $f : X \to Y$ is a quasiisomorphism.

Let $n \geqslant 1$.

The graph morphism $\iota_{\mathrm{C}_n} : \emptyset \to \mathrm{C}_n$ is a cofibration; cf. Remark 151.

Suppose given a graph morphism $p : \mathrm{C}_n \to Y$.

We have to show that there exists a unique graph morphism $q : \mathrm{C}_n \to X$ such that $qf = p$.

Existence. We have the commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "\iota_{\mathrm{C}_n}"'] & X \arrow[d, "f"] \\ \mathrm{C}_n \arrow[r, "p"'] & Y \end{tikzcd} ;$$

cf. Remark 151.

Since the fibration $f : X \to Y$ is in $\mathrm{Cofib}^{\boxslash}$ there exists a graph morphism $q : \mathrm{C}_n \to X$ such that $qf = p$.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "\iota_{\mathrm{C}_n}"'] & X \arrow[d, "f"] \\ \mathrm{C}_n \arrow[r, "p"'] \arrow[ur, "q"] & Y \end{tikzcd}$$

Uniqueness.

Suppose given graph morphisms $q, \tilde{q} : \mathrm{C}_n \to X$ such that $qf = \tilde{q}f = p$.

We have to show that $q \overset{!}{=} \tilde{q}$.

We consider the cofibration $d_{\mathrm{C}_n} : \mathrm{C}_n \sqcup \mathrm{C}_n \to \mathrm{C}_n$; cf. Remark 155.

We consider the following commutative diagram.

$$\begin{tikzcd}
 & \mathrm{C}_n \sqcup \mathrm{C}_n \arrow[r, "\left(\begin{smallmatrix} q \\ \tilde{q} \end{smallmatrix}\right)"] \arrow[d, "d_{\mathrm{C}_n}"'] & X \arrow[d, "f"] \\
\mathrm{C}_n \arrow[ur, "\iota_1"] \arrow[r, "\mathrm{id}_{\mathrm{C}_n}"'] & \mathrm{C}_n \arrow[r, "p"'] & Y
\end{tikzcd}$$

To show that the diagram is commutative we have to show that $d_{\mathrm{C}_n} \overset{!}{=} \left(\begin{smallmatrix} p \\ p \end{smallmatrix}\right) f$.

It suffices to show that $p = \iota_1 d_{\mathrm{C}_n} p \overset{!}{=} \iota_1 \left(\begin{smallmatrix} q \\ \tilde{q} \end{smallmatrix}\right) f = qf$ and that $p = \iota_2 d_{\mathrm{C}_n} p \overset{!}{=} \iota_2 \left(\begin{smallmatrix} q \\ \tilde{q} \end{smallmatrix}\right) f = \tilde{q}f$. This holds since $qf = p = \tilde{q}f$.

Since the graph morphism $f : X \to Y$ is in $\mathrm{Cofib}^{\boxslash}$, there exists a graph morphism $q' : \mathrm{C}_n \to X$ such that $q'f = p$ and $d_{\mathrm{C}_n} q' = \left(\begin{smallmatrix} q \\ \tilde{q} \end{smallmatrix}\right)$.

So we have $q' = \mathrm{id}_{\mathrm{C}_n} q' = \iota_1 d_{\mathrm{C}_n} q' = \iota_1 \left(\begin{smallmatrix} q \\ \tilde{q} \end{smallmatrix}\right) = q$ and $q' = \mathrm{id}_{\mathrm{C}_n} q' = \iota_2 d_{\mathrm{C}_n} q' = \iota_2 \left(\begin{smallmatrix} q \\ \tilde{q} \end{smallmatrix}\right) = \tilde{q}$.

So we have $q = q' = \tilde{q}$. □

Using Lemma 188, we can alternatively prove Lemma 193 as follows.

Proof. We now have to show that the fibration $f : X \to Y$ is a quasiisomorphism.

We have the factorization $f = ab$ with $X \xrightarrow{a} G$ and $G \xrightarrow{b} Y$; cf. Lemma 188 below.

We consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "a"'] & X \arrow[d, "f"] \\
G \arrow[r, "b"'] & Y
\end{tikzcd}$$

Because $f \in \mathrm{Cofib}^{\boxslash}$ there exists a graph morphism $k : G \to X$ such that $ak = \mathrm{id}_X$ and $kf = b$.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "a"'] & X \arrow[d, "f"] \\
G \arrow[r, "b"'] \arrow[ur, "k"] & Y
\end{tikzcd}$$

We consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "a"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & G \arrow[r, "k"'] \arrow[d, "b"] & X \arrow[d, "f"] \\
Y \arrow[r, "\mathrm{id}_Y"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Y \arrow[r, "\mathrm{id}_Y"] & Y
\end{tikzcd}$$

Since the set of quasiisomorphisms $\mathrm{Qis} \subseteq \mathrm{Mor}(\mathrm{Gph})$ is closed under retracts the graph morphism $f : X \to Y$ is a quasiisomorphism; cf. Remark 117

So the graph morphism $f : X \to Y$ is an acyclic fibration.

> Since the set of acyclic fibrations $\mathrm{AcFib} \subseteq \mathrm{Mor}(\mathrm{Gph})$ is closed under retracts the graph morphism $f : X \to Y$ is an acyclic fibration; cf. Remark 141

Hence we have $\mathrm{Cofib}^{\boxslash} \subseteq \mathrm{AcFib}$ and so we conclude $\mathrm{Cofib}^{\boxslash} = \mathrm{AcFib}$. $\square$

Remark 194 We have $\mathrm{AcCofib}^{\boxslash} \cap \mathrm{AcFib}^{\boxslash} = \mathrm{Qis}^{\boxslash}$.

Proof.

Ad $\mathrm{Qis}^{\boxslash} \subseteq \mathrm{AcCofib}^{\boxslash} \cap \mathrm{AcFib}^{\boxslash}$.

Since $\mathrm{AcCofib} \subseteq \mathrm{Qis}$, we have $\mathrm{Qis}^{\boxslash} \subseteq \mathrm{AcCofib}^{\boxslash}$; cf. Lemma 175.

Since $\mathrm{AcFib} \subseteq \mathrm{Qis}$, we have $\mathrm{Qis}^{\boxslash} \subseteq \mathrm{AcFib}^{\boxslash}$; cf. Definition 138.

So we have $\mathrm{Qis}^{\boxslash} \subseteq \mathrm{AcCofib}^{\boxslash} \cap \mathrm{AcFib}^{\boxslash}$.

Ad $\mathrm{AcCofib}^{\boxslash} \cap \mathrm{AcFib}^{\boxslash} \subseteq \mathrm{Qis}^{\boxslash}$.

Suppose given a graph morphism $f : X \to Y$ in $\mathrm{AcCofib}^{\boxslash} \cap \mathrm{AcFib}^{\boxslash}$.

We have to show that the graph morphism f is in $\mathrm{Qis}^{\boxslash}$.

Suppose given a commutative diagram in Gph as follows.

$$\begin{tikzcd}
G \arrow[r, "a"] \arrow[d, "q \wr"'] & X \arrow[d, "f"] \\
H \arrow[r, "b"'] & Y
\end{tikzcd}$$

We show that there exists a graph morphism $h : H \to X$ such that $qh = a$ and $hf = b$.

By Lemma 184 and since Qis satisfies (2 of 3), we have a commutative triangle as follows.

$$\begin{tikzcd}
 & Z \arrow[dr, "d"] & \\
G \arrow[ur, "c"] \arrow[rr, "\approx"', "q"'] & & H
\end{tikzcd}$$

So we have the following commutative diagram.

$$\begin{tikzcd}
G \arrow[r, "a"] \arrow[d, "c"'] & X \arrow[dd, "f"] \\
Z \arrow[d, "d"'] & \\
H \arrow[r, "b"'] & Y
\end{tikzcd}$$

Since f is in $\mathrm{AcCofib}^{\boxslash}$, there exists a graph morphism $g : Z \to X$ such that $cg = a$ and $gf = db$.

$$\begin{tikzcd}
G \arrow[r, "a"] \arrow[d, "c"'] & X \arrow[dd, "f"] \\
Z \arrow[d, "d"'] \arrow[ur, "g"] & \\
H \arrow[r, "b"'] & Y
\end{tikzcd}$$

Since f is in $\mathrm{AcFib}^{\boxslash}$ and since $gf = db$, there exists a graph morphism $h : H \to X$ such that $dh = g$ and $hf = b$.

$$\begin{tikzcd}
G \arrow[r, "a"] \arrow[d, "c"'] & X \arrow[dd, "f"] \\
Z \arrow[d, "d"'] \arrow[ur, "g"] & \\
H \arrow[r, "b"'] \arrow[uur, "h"] & Y
\end{tikzcd}$$

So we have $cdh = cg = a$ and $hf = b$.

So f is in $\mathrm{Qis}^{\boxslash}$. $\square$

Remark 195 In Gph, the following statements (1–4) hold.

(1) We have $\mathrm{AcCofib} = {}^{\boxslash}\mathrm{Fib}$; cf. Lemma 191.

(2) We have $\mathrm{AcCofib}^{\boxslash} = \mathrm{Fib}$; cf. Lemma 192.

(3) We have $\mathrm{Cofib} = {}^{\boxslash}\mathrm{AcFib}$; cf. Definition 144.

(4) We have $\mathrm{Cofib}^{\boxslash} = \mathrm{AcFib}$; cf. Lemma 193.

Remark 196 In Gph, the following statements (1–4) hold.

(1) We have $({}^{\boxslash}\mathrm{Fib})^{\boxslash} = \mathrm{AcCofib}^{\boxslash} = \mathrm{Fib}$; cf. Remark 195.

(2) We have $({}^{\boxslash}\mathrm{AcFib})^{\boxslash} = \mathrm{Cofib}^{\boxslash} = \mathrm{AcFib}$; cf. Remark 195.

(3) We have ${}^{\boxslash}(\mathrm{Cofib}^{\boxslash}) = {}^{\boxslash}\mathrm{AcFib} = \mathrm{Cofib}$; cf. Remark 195.

(4) We have ${}^{\boxslash}(\mathrm{AcCofib}^{\boxslash}) = {}^{\boxslash}\mathrm{Fib} = \mathrm{AcCofib}$; cf. Remark 195.

Remark 197 In Gph, we have

$$\mathrm{Iso} = \mathrm{Fib} \cap {}^{\boxslash}\mathrm{Fib} \overset{\text{Lemma 191}}{=} \mathrm{Fib} \cap \mathrm{AcCofib} \overset{\text{Lemma 185}}{=} \mathrm{Fib} \cap \mathrm{Cofib} \cap \mathrm{Qis} \ .$$

Proof. We show that $\mathrm{Fib} \cap \mathrm{AcCofib} \overset{!}{=} \mathrm{Iso}$.

Ad $\mathrm{Iso} \overset{!}{\subseteq} \mathrm{Fib} \cap \mathrm{AcCofib}$.

We have $\mathrm{Iso} \subseteq \mathrm{Fib}$; cf. Remark 129.

We have $\mathrm{Iso} \subseteq \mathrm{AcCofib}$; cf. Remark 165.

Ad $\mathrm{Fib} \cap \mathrm{AcCofib} \overset{!}{\subseteq} \mathrm{Iso}$.

Suppose given a fibration $f : G \to H$ that is an acyclic cofibration.

We consider the following commutative diagram.

$$\begin{tikzcd}
G \arrow[r, "\mathrm{id}_G"] \arrow[d, "f"'] & G \arrow[d, "f"] \\
H \arrow[r, "\mathrm{id}_H"'] & H
\end{tikzcd}$$

We have $\mathrm{AcCofib} = {}^{\boxslash}\mathrm{Fib}$ and $\mathrm{AcCofib}^{\boxslash} = \mathrm{Fib}$; cf. Lemmas 191 and 192.

So there exists a graph morphism $h : H \to G$ such that the following diagram is commutative.

$$\begin{tikzcd}
G \arrow[r, "\mathrm{id}_G"] \arrow[d, "f"'] & G \arrow[d, "f"] \\
H \arrow[r, "\mathrm{id}_H"'] \arrow[ur, "h"] & H
\end{tikzcd}$$

Thus f is an isomorphism. $\square$

Chapter 6

Gph is a model category

In Gph, we have defined subsets Qis(Gph), Fib(Gph) and Cofib(Gph) of Mor(Gph) in Definitions 115, 127.(1) and 144.

We want to show that the category Gph, together with these subsets, is a model category in the sense of Definition 198 below.

We will use notations introduced in Definitions 5, 9, 14, 13 and 15.

Definition 198 Suppose given a category $\mathcal{M}$ having an initial object ¡ and a terminal object ! . Suppose given subsets $\mathrm{Fib}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ of *fibrations*, $\mathrm{Cofib}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ of *cofibrations* and $\mathrm{Qis}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ of *quasiisomorphisms*. We consider the axioms $\mathrm{A_{Fib}}$, $\mathrm{A_{Cofib}}$, $\mathrm{A_{Qis}}$, $\mathrm{A_{Lift}}$, $\mathrm{A_{Fact}}$, $\mathrm{M_{Fib}}$ and $\mathrm{M_{Cofib}}$ below.

(1) The category $\mathcal{M}$ together with $\mathrm{Fib}(\mathcal{M})$, $\mathrm{Cofib}(\mathcal{M})$ and $\mathrm{Qis}(\mathcal{M})$ is called a *basic model category* if ($\mathrm{A_{Fib}}$ 1, 2), ($\mathrm{A_{Cofib}}$ 1, 2), ($\mathrm{A_{Qis}}$ 1, 2), $\mathrm{A_{Lift}}$, $\mathrm{A_{Fact}}$, ($\mathrm{M_{Fib}}$ 1, 2) and ($\mathrm{M_{Cofib}}$ 1, 2) hold.

(2) The category $\mathcal{M}$ together with $\mathrm{Fib}(\mathcal{M})$, $\mathrm{Cofib}(\mathcal{M})$ and $\mathrm{Qis}(\mathcal{M})$ is called a *proper basic model category* if ($\mathrm{A_{Fib}}$ 1, 2), ($\mathrm{A_{Cofib}}$ 1, 2), ($\mathrm{A_{Qis}}$ 1, 2), $\mathrm{A_{Lift}}$, $\mathrm{A_{Fact}}$, $\mathrm{M_{Fib}}$ and $\mathrm{M_{Cofib}}$ hold.

(3) The category $\mathcal{M}$ together with $\mathrm{Fib}(\mathcal{M})$, $\mathrm{Cofib}(\mathcal{M})$ and $\mathrm{Qis}(\mathcal{M})$ is called a *Quillen model category* if $\mathrm{M_{PO}}$, $\mathrm{M_{PB}}$, ($\mathrm{A_{Fib}}$ 1, 2), ($\mathrm{A_{Cofib}}$ 1, 2), ($\mathrm{A_{Qis}}$ 1, 2), $\mathrm{A_{Lift}}$, $\mathrm{A_{Fact}}$, ($\mathrm{M_{Fib}}$ 1, 2) and $\mathrm{M_{Cofib}}$ (1, 2) hold.

(4) The category $\mathcal{M}$ together with $\mathrm{Fib}(\mathcal{M})$, $\mathrm{Cofib}(\mathcal{M})$ and $\mathrm{Qis}(\mathcal{M})$ is called a *Quillen closed model category* if $\mathrm{M_{PO}}$, $\mathrm{M_{PB}}$, $\mathrm{A_{Fib}}$, $\mathrm{A_{Cofib}}$, $\mathrm{A_{Qis}}$, $\mathrm{A_{Lift}}$, $\mathrm{A_{Fact}}$, ($\mathrm{M_{Fib}}$ 1, 2) and ($\mathrm{M_{Cofib}}$ 1, 2) hold.

We define the subset $\mathrm{AcFib}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ of *acyclic fibrations* to be

$$\mathrm{AcFib}(\mathcal{M}) := \mathrm{Fib}(\mathcal{M}) \cap \mathrm{Qis}(\mathcal{M}).$$

We define the subset $\mathrm{AcCofib}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ of *acyclic cofibrations* to be

$$\mathrm{AcCofib}(\mathcal{M}) := \mathrm{Cofib}(\mathcal{M}) \cap \mathrm{Qis}(\mathcal{M}).$$

Suppose given a morphism $f : G \to H$ in $\mathrm{Mor}(\mathcal{M})$.

To indicate that f is a quasiisomorphism, we often write $\begin{tikzcd} G \arrow[r, "f", "\approx" marking] & H \end{tikzcd}$.

To indicate that f is a fibration, we often write $\begin{tikzcd} G \arrow[r, "f", "|" marking] & H \end{tikzcd}$.

To indicate that f is an acyclic fibration, we often write $\begin{tikzcd} G \arrow[r, "f", "\square" marking] & H \end{tikzcd}$.

To indicate that f is a cofibration, we often write $\begin{tikzcd} G \arrow[r, "f", "\bullet" marking] & H \end{tikzcd}$.

To indicate that f is an acyclic cofibration, we often write $\begin{tikzcd} G \arrow[r, "f", "\circ" marking] & H \end{tikzcd}$.

Suppose given $G \in \mathrm{Ob}(\mathcal{M})$. Then G is called *fibrant*, if $\begin{tikzcd} G \arrow[r, "|" marking] & ! \end{tikzcd}$. Moreover G is called *cofibrant*, if $\begin{tikzcd} \text{¡} \arrow[r, "\bullet" marking] & G \end{tikzcd}$.

We consider the following properties.

(M_{PO}) Suppose given the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \\ X' & \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pushout in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] \arrow[ur, phantom, "\lrcorner" near end] & Y' \end{tikzcd}$$

(M_{PB}) Suppose given the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pullback in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner" very near start] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

- $\mathrm{A}_{\mathrm{Fib}} := (\mathrm{A}_{\mathrm{Fib}}\ 1) \wedge (\mathrm{A}_{\mathrm{Fib}}\ 2) \wedge (\mathrm{A}_{\mathrm{Fib}}\ 3)$ where:

($\mathrm{A}_{\mathrm{Fib}}$ 1) We have $\mathrm{Iso}(\mathcal{M}) \subseteq \mathrm{Fib}(\mathcal{M})$.

($\mathrm{A}_{\mathrm{Fib}}$ 2) Suppose given $\begin{tikzcd} X \arrow[r, "f", "|" marking] & Y \end{tikzcd}$ and $\begin{tikzcd} Y \arrow[r, "g", "|" marking] & Z \end{tikzcd}$ in $\mathrm{Mor}(\mathcal{M})$. Then we have $\begin{tikzcd} X \arrow[r, "fg", "|" marking] & Z \end{tikzcd}$.

($\mathrm{A}_{\mathrm{Fib}}$ 3) The subset $\mathrm{Fib}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ is closed under retracts; cf. Definition 23.

- $\mathrm{A}_{\mathrm{Cofib}} := (\mathrm{A}_{\mathrm{Cofib}}\ 1) \wedge (\mathrm{A}_{\mathrm{Cofib}}\ 2) \wedge (\mathrm{A}_{\mathrm{Cofib}}\ 3)$, where:

($\mathrm{A}_{\mathrm{Cofib}}$ 1) We have $\mathrm{Iso}(\mathcal{M}) \subseteq \mathrm{Cofib}(\mathcal{M})$.

($\mathrm{A_{Cofib}}$ 2) Suppose given $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathrm{Mor}(\mathcal{M})$. Then we have $X \xrightarrow{fg} Z$.

($\mathrm{A_{Cofib}}$ 3) The subset $\mathrm{Cofib}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ is closed under retracts; cf. Definition 23.

- $\mathrm{A_{Qis}} := (\mathrm{A_{Qis}}\ 1) \wedge (\mathrm{A_{Qis}}\ 2) \wedge (\mathrm{A_{Qis}}\ 3)$, where:

($\mathrm{A_{Qis}}$ 1) We have $\mathrm{Iso}(\mathcal{M}) \subseteq \mathrm{Qis}(\mathcal{M})$.

($\mathrm{A_{Qis}}$ 2) The set $\mathrm{Qis}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ satisfies (2 of 3).

($\mathrm{A_{Qis}}$ 3) The subset $\mathrm{Qis}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ is closed under retracts; cf. Definition 23.

- $\mathrm{A_{Lift}} := (\mathrm{A_{Lift}}\ 1) \wedge (\mathrm{A_{Lift}}\ 2)$, where:

($\mathrm{A_{Lift}}$ 1) We have $\mathrm{AcCofib}(\mathcal{M}) \boxslash \mathrm{Fib}(\mathcal{M})$; cf. Definition 15.

($\mathrm{A_{Lift}}$ 2) We have $\mathrm{Cofib}(\mathcal{M}) \boxslash \mathrm{AcFib}(\mathcal{M})$; cf. Definition 15.

- $\hat{\mathrm{A}}_{\mathrm{Lift}} := (\hat{\mathrm{A}}_{\mathrm{Lift}}\ 1) \wedge (\hat{\mathrm{A}}_{\mathrm{Lift}}\ 2)$, where:

($\hat{\mathrm{A}}_{\mathrm{Lift}}$ 1) We have $\mathrm{AcCofib}(\mathcal{M}) \mathbin{\hat{\boxslash}} \mathrm{Fib}(\mathcal{M})$; cf. Definition 27.

($\hat{\mathrm{A}}_{\mathrm{Lift}}$ 2) We have $\mathrm{Cofib}(\mathcal{M}) \mathbin{\hat{\boxslash}} \mathrm{AcFib}(\mathcal{M})$; cf. Definition 27.

- $\mathrm{A_{Fact}} := (\mathrm{A_{Fact}}\ 1) \wedge (\mathrm{A_{Fact}}\ 2)$, where:

($\mathrm{A_{Fact}}$ 1) For $X \xrightarrow{f} Y$ in $\mathcal{M}$ there exists a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd} & U \arrow[dr, "h"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

($\mathrm{A_{Fact}}$ 2) For $X \xrightarrow{f} Y$ in $\mathcal{M}$ there exists a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd} & U \arrow[dr, "h"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

- $\mathrm{M_{Fib}} := (\mathrm{M_{Fib}}\ 1) \wedge (\mathrm{M_{Fib}}\ 2) \wedge (\mathrm{M_{Fib}}\ 3)$, where:

($\mathrm{M_{Fib}}$ 1) Suppose given the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pullback in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

($\mathrm{M}_{\mathrm{Fib}}$ 2) Suppose given the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pullback in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

($\mathrm{M}_{\mathrm{Fib}}$ 3) Suppose given the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X' \arrow[r, "u'", "\approx"'] & Y' \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pullback in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u", "\approx"'] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'", "\approx"'] & Y' \end{tikzcd}$$

- $\mathrm{M}_{\mathrm{Cofib}} := (\mathrm{M}_{\mathrm{Cofib}}\ 1) \wedge (\mathrm{M}_{\mathrm{Cofib}}\ 2) \wedge (\mathrm{M}_{\mathrm{Cofib}}\ 3)$, where:

($\mathrm{M}_{\mathrm{Cofib}}$ 1) Suppose given the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \\ X' & \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pushout in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

($\mathrm{M}_{\mathrm{Cofib}}$ 2) Suppose given the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \\ X' & \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pushout in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \arrow[ul, phantom, "\lrcorner" very near start] \end{tikzcd}$$

($\mathrm{M}_{\mathrm{Cofib}}$ 3) Suppose given the diagram

$$\begin{tikzcd} X \arrow[r, "u", "\approx"', dashed] \arrow[d, "f"'] & Y \\ X' & \end{tikzcd}$$

in $\mathcal{M}$. Then there exists a pushout in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u", "\approx"', dashed] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'", "\approx"', dashed] & Y' \arrow[ul, phantom, "\lrcorner" very near start] \end{tikzcd}$$

Remark 199

(1) We have $\hat{\mathrm{A}}_{\mathrm{Lift}} \wedge (\mathrm{A}_{\mathrm{Qis}}\,3)$ if and only if $\mathrm{A}_{\mathrm{Lift}} \wedge (\mathrm{A}_{\mathrm{Fib}}\,3) \wedge (\mathrm{A}_{\mathrm{Cofib}}\,3) \wedge (\mathrm{A}_{\mathrm{Qis}}\,3) \wedge \mathrm{A}_{\mathrm{Fact}}$.

(2) The category $\mathcal{M}$ together with $\mathrm{Fib}(\mathcal{M})$, $\mathrm{Cofib}(\mathcal{M})$ and $\mathrm{Qis}(\mathcal{M})$ is a Quillen closed model category if and only if M_{PO}, M_{PB}, $(\mathrm{A}_{\mathrm{Fib}}\,1,2)$, $(\mathrm{A}_{\mathrm{Cofib}}\,1,2)$, $\mathrm{A}_{\mathrm{Qis}}$, $\hat{\mathrm{A}}_{\mathrm{Lift}}$, $(\mathrm{M}_{\mathrm{Fib}}\,1,2)$ and $(\mathrm{M}_{\mathrm{Cofib}}\,1,2)$ hold.

Proof.

Ad (1).

Ad $\Rightarrow$.

We have $\mathrm{AcCofib}(\mathcal{M}) \,\hat{\boxslash}\, \mathrm{Fib}(\mathcal{M})$ and $\mathrm{Cofib}(\mathcal{M}) \,\hat{\boxslash}\, \mathrm{AcFib}(\mathcal{M})$ and the subset $\mathrm{Qis}(\mathcal{M}) \subseteq \mathrm{Mor}(\mathcal{M})$ is closed under retracts.

Because of (C 3) we have $\mathrm{A}_{\mathrm{Fact}}$; cf. Definition 27.

Since $\mathrm{AcCofib}(\mathcal{M}) \,\hat{\boxslash}\, \mathrm{Fib}(\mathcal{M})$, we have $\mathrm{AcCofib}(\mathcal{M}) \boxslash \mathrm{Fib}(\mathcal{M})$ and $(\mathrm{A}_{\mathrm{Fib}}\,3)$; cf. Remark 28.

Since $\mathrm{Cofib}(\mathcal{M}) \,\hat{\boxslash}\, \mathrm{AcFib}(\mathcal{M})$, we have $\mathrm{Cofib}(\mathcal{M}) \boxslash \mathrm{AcFib}(\mathcal{M})$ and $(\mathrm{A}_{\mathrm{Cofib}}\,3)$; cf. Remark 28.

So because of $\mathrm{AcCofib}(\mathcal{M}) \boxslash \mathrm{Fib}(\mathcal{M})$ and $\mathrm{Cofib}(\mathcal{M}) \boxslash \mathrm{AcFib}(\mathcal{M})$ we have $\mathrm{A}_{\mathrm{Lift}}$.

Ad $\Leftarrow$.

The subsets $\mathrm{Fib}(\mathcal{M})$, $\mathrm{Cofib}(\mathcal{M})$ and $\mathrm{Qis}(\mathcal{M})$ in $\mathrm{Mor}(\mathcal{M})$ are closed under retracts by $(\mathrm{A}_{\mathrm{Fib}}\,3)$, $(\mathrm{A}_{\mathrm{Cofib}}\,3)$ and $(\mathrm{A}_{\mathrm{Qis}}\,3)$.

Hence the subsets $\mathrm{AcFib}(\mathcal{M})$ and $\mathrm{AcCofib}(\mathcal{M})$ in $\mathrm{Mor}(\mathcal{M})$ are closed under retracts.

Because of ($\mathrm{A}_{\mathrm{Fact}}\,1$), ($\mathrm{A}_{\mathrm{Lift}}\,1$), ($\mathrm{A}_{\mathrm{Fib}}\,3$) and $\mathrm{AcCofib}(\mathcal{M})$ being closed under retracts we have ($\hat{\mathrm{A}}_{\mathrm{Lift}}\;1$) by Remark 28.

Because of ($\mathrm{A}_{\mathrm{Fact}}\,2$), ($\mathrm{A}_{\mathrm{Lift}}\,2$), $\mathrm{AcFib}(\mathcal{M})$ being closed under retracts and ($\mathrm{A}_{\mathrm{Cofib}}\,3$) we have ($\hat{\mathrm{A}}_{\mathrm{Lift}}\;2$) by Remark 28. $\square$

Ad (2).

This follows using (1). $\square$

Reminder 200 We recall the meaning of $\mathrm{AcCofib}(\mathcal{M}) \boxslash \mathrm{Fib}(\mathcal{M})$; cf. $\mathrm{A}_{\mathrm{Lift}}$, Definition 15.

Suppose given an acyclic cofibration $X \xrightarrow{f} X'$ in $\mathrm{AcCofib}(\mathcal{M})$ and a fibration $Y \xrightarrow{g} Y'$ in $\mathrm{Fib}(\mathcal{M})$. Suppose given a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"'] & Y' \end{tikzcd}$$

Then there exists a $h : X' \to Y$ in $\mathrm{Mor}(\mathcal{M})$ such that $fh = u$ and such that $hg = u'$, i.e. we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"'] \arrow[ur, dotted, "\exists h"] & Y' \end{tikzcd}$$

The morphism h is also called a *lift* of this quadrangle.

Reminder 201 We recall the meaning of $\mathrm{Cofib}(\mathcal{M}) \boxslash \mathrm{AcFib}(\mathcal{M})$; cf. $\mathrm{A}_{\mathrm{Lift}}$, Definition 15.

Suppose given a cofibration $X \xrightarrow{f} X'$ in $\mathrm{Cofib}(\mathcal{M})$ and an acyclic fibration $Y \xrightarrow{g} Y'$ in $\mathrm{AcFib}(\mathcal{M})$. Suppose given a commutative diagram as follows.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"'] & Y' \end{tikzcd}$$

Then there exists $h : X' \to Y$ in $\mathrm{Mor}(\mathcal{M})$ such that $fh = u$ and such that $hg = u'$, i.e. we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"'] \arrow[ur, dotted, "\exists h"] & Y' \end{tikzcd}$$

The morphism h is also called a *lift* of this quadrangle.

Remark 202

(1) D. Quillen introduced Quillen model categories under the name *model categories* in [**4,** §I.1].

He introduced Quillen closed model categories under the name *closed model categories* in [**4,** §I.5]; cf. [**4,** §I.5, Prop. 2].

(2) M. Ritter used basic proper model categories under the name *model categories* in [**6,** Def. 172].

(3) The conditions ($\mathrm{M}_{\mathrm{Fib}}$ 3) and ($\mathrm{M}_{\mathrm{Cofib}}$ 3) were introduced in [**5,** §1, Def. 1.2].

Reminder 203 We defined the subsets Fib(Gph), Qis(Gph) $\subseteq$ Mor(Gph) directly; cf. Definitions 127 and 115.

We defined AcFib(Gph) = Fib(Gph) $\cap$ Qis(Gph); cf. Definition 138.

We defined Cofib(Gph) = ${}^{\boxslash}$AcFib(Gph); cf. Definition 144.

The following proposition is due to Bisson and Tsemo [**3,** Cor. 4.8].

Proposition 204 The category Gph together with Fib(Gph), Cofib(Gph) and Qis(Gph), is a Quillen closed model category; cf. Definition 198.(4), Reminder 203.

In addition, Gph satisfies ($\mathrm{M}_{\mathrm{Fib}}$ 3).

Moreover, for AcCofib(Gph), as defined by (AcCofib 1–5) in Definition 162, we have the equality AcCofib(Gph) = Cofib(Gph) $\cap$ Qis(Gph).

Finally, note that Gph satisfies $\hat{\mathrm{A}}_{\mathrm{Lift}}$; cf. Remark 199.

Proof. We have AcCofib = Cofib $\cap$ Qis; cf. Lemma 185.

We have M_{PO} ; cf. Construction 83.

We have M_{PB} ; cf. Construction 97.

We have ($\mathrm{A}_{\mathrm{Fib}}$ 1); cf. Remark 129.

We have ($\mathrm{A}_{\mathrm{Fib}}$ 2); cf. Remark 131.

We have ($\mathrm{A}_{\mathrm{Fib}}$ 3); cf. Remark 130.

We have ($\mathrm{A}_{\mathrm{Cofib}}$ 1); cf. Remark 145.

We have ($\mathrm{A}_{\mathrm{Cofib}}$ 2); cf. Remark 147.

We have ($\mathrm{A}_{\mathrm{Cofib}}$ 3); cf. Remark 146.

We have ($\mathrm{A}_{\mathrm{Qis}}$ 1); cf. Remark 116.

We have ($\mathrm{A}_{\mathrm{Qis}}$ 2); cf. Remark 123.

We have ($\mathrm{A}_{\mathrm{Qis}}$ 3); cf. Remark 117.

We have ($\mathrm{A}_{\mathrm{Lift}}$ 1), i.e. AcCofib(Gph) $\boxslash$ Fib(Gph); cf. Remark 174.

We have ($\mathrm{A}_{\mathrm{Lift}}$ 2), i.e. Cofib(Gph) $\boxslash$ AcFib(Gph); cf. Definition 144.

We have ($\mathrm{A}_{\mathrm{Fact}}\,1$); cf. Lemma 184.

We have ($\mathrm{A}_{\mathrm{Fact}}\,2$); cf. Lemma 188.

We have ($\mathrm{M}_{\mathrm{Fib}}\,1$); cf. Remark 133.

We have ($\mathrm{M}_{\mathrm{Fib}}\,2$); cf. Remark 143.

We have ($\mathrm{M}_{\mathrm{Fib}}\,3$); cf. Remarks 125 and 133.

We have ($\mathrm{M}_{\mathrm{Cofib}}\,1$); cf. Remark 148.

We have ($\mathrm{M}_{\mathrm{Cofib}}\,2$); cf. Remark 171. □

Remark 205 In Gph, the axiom ($\mathrm{M}_{\mathrm{Cofib}}\ 3$) does **not** hold.

In other words, there exists a quasiisomorphism in Gph, whose pushout along a cofibration is not a quasiisomorphism.

Counterexample.

Consider the following thin graph.

$$Y:\ \boxed{\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] & 3 \arrow[l, "\alpha_5"'] \arrow[r, bend left, "\alpha_3"] & 4 \arrow[l, bend left, "\alpha_4"] \end{tikzcd}}$$

We consider the cofibration $d_{\mathrm{C}_2}:\ \mathrm{C}_2 \sqcup \mathrm{C}_2 \xrightarrow{\ \bullet\ } \mathrm{C}_2$; cf. Definition 154.

We consider the quasiisomorphism $f : \mathrm{C}_2 \sqcup \mathrm{C}_2 \to Y$ with

$$\begin{aligned} (1, \mathrm{v}_0)\,\mathrm{V}_f &:= 1, \quad (1, \mathrm{v}_1)\,\mathrm{V}_f := 2, \\ (2, \mathrm{v}_0)\,\mathrm{V}_f &:= 3, \quad (2, \mathrm{v}_1)\,\mathrm{V}_f := 4 \end{aligned}$$

and with

$$\begin{aligned} (1, \mathrm{e}_0)\,\mathrm{E}_f &:= \alpha_1\,, \quad (1, \mathrm{e}_1)\,\mathrm{E}_f := \alpha_2\,, \\ (2, \mathrm{e}_0)\,\mathrm{E}_f &:= \alpha_3\,, \quad (2, \mathrm{e}_1)\,\mathrm{E}_f := \alpha_4\ . \end{aligned}$$

Constructing the pushout using Magma, we obtain

$$\begin{tikzcd} \mathrm{C}_2 \sqcup \mathrm{C}_2 \arrow[r, "f", "\approx"'] \arrow[d, "d_{\mathrm{C}_2}"', "\bullet" marking] & Y \arrow[d, "h"] \\ \mathrm{C}_2 \arrow[r, "f'"] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

with

$$Y':\ \boxed{\begin{tikzcd} 1 \arrow[r, bend left=15, "\beta_1"] \arrow[r, bend left=80, "\beta_3"] & 2 \arrow[l, bend left=15, "\beta_2"] \end{tikzcd}}$$

$$1\,\mathrm{V}_h = 1, \quad 2\,\mathrm{V}_h = 2, \quad 3\,\mathrm{V}_h = 1, \quad 4\,\mathrm{V}_h = 2,$$

$$\alpha_1\,\mathrm{E}_h = \beta_1\,,\quad \alpha_2\,\mathrm{E}_h = \beta_2\,,\quad \alpha_3\,\mathrm{E}_h = \beta_1\,,$$
$$\alpha_4\,\mathrm{E}_h = \beta_2\,,\quad \alpha_5\,\mathrm{E}_h = \beta_3$$

and

$$\mathrm{v}_0\,\mathrm{V}_{f'} = 1,\quad \mathrm{e}_0\,\mathrm{E}_{f'} = \beta_1\,,$$
$$\mathrm{v}_1\,\mathrm{V}_{f'} = 2,\quad \mathrm{e}_1\,\mathrm{E}_{f'} = \beta_2\;.$$

But the resulting graph morphism $f' : \mathrm{C}_2 \to Y'$ is **not** a quasiisomorphism because $|(\mathrm{C}_2\,,\mathrm{C}_2)_{\mathrm{Gph}}| = 2 \neq 4 = |(\mathrm{C}_2\,,Y')_{\mathrm{Gph}}|$.

Via Magma [**2**] we proceed as follows, using the functions given in §10 below.

```
C2C2 := DCN(2)[1];
C2 := DCN(2)[2];
d := DCN(2)[3];
Y := <[1,2,3,4],[<1,1,2>,<2,2,1>,<3,3,4>,<4,4,3>,<3,5,2>]>;
f := IsSubgraph(C2C2,Y)[1];
PO := PushoutGraphs(C2C2,Y,C2,f,d);
Yp := PO[1]; // "Y prime" = Y'
fp := PO[2]; // "f prime" = f'
h := PO[3];
> C2;
<[ 1, 2 ], [ <1, 1, 2>, <2, 2, 1> ]>
> C2C2;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 1>, <3, 3, 4>, <4, 4, 3> ]>
> d;
<[ <1, 1>, <2, 2>, <3, 1>, <4, 2> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 2, 1>>, <<3, 3, 4>, <1, 1, 2>>, <<4, 4, 3>, <2, 2, 1>> ]>
> Y;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 1>, <3, 3, 4>, <4, 4, 3>, <3, 5, 2> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 2, 1>>, <<3, 3, 4>, <3, 3, 4>>, <<4, 4, 3>, <4, 4, 3>> ]>
> Yp;
<[ 1, 2 ], [ <1, 1, 2>, <2, 2, 1>, <1, 3, 2> ]>
> fp;
<[ <1, 1>, <2, 2> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2, 1>> ]>
> h;
<[ <1, 1>, <2, 2>, <3, 1>, <4, 2> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 2, 1>>, <<3, 3, 4>, <1, 1, 2>>, <<4, 4, 3>, <2, 2, 1>>,
 <<3, 5, 2>, <1, 3, 2>> ]>
> IsQis_Bound(fp,C2,Yp,2);
false
> #ListOfnCycles(C2,2);
2
> #ListOfnCycles(Yp,2);
4
```

Note that the calculated graph morphism $h : Y \to Y'$ is a cofibration since cofibrations are stable under pushouts; cf. Remark 148.

In the similar situation of the pointed category of graphs, D. Vicinsky has constructed an example that shows that quasiisomorphisms are not stable under pushouts along cofibrations; cf. [**7,** Prop. 5.8].

Chapter 7

A sufficient condition for a graph morphism to be a quasiisomorphism

Definition 206 Suppose given a graph morphism $f : G \to H$. An edge $e_H \in \mathrm{E}_H$ is called *unitargeting* with respect to f, if

$$|\{\tilde{e}\,\mathrm{t}_G : \tilde{e} \in \mathrm{E}_G\,,\ \tilde{e}\,\mathrm{E}_f = e_H\}| = 1.$$

Example 207 We consider the following graph morphism.

$$G : \quad \boxed{\begin{tikzcd} 1 \arrow[dr, "\alpha_1"] & \\ 2 \arrow[r, "\alpha_2"] & 4 \\ 3 \arrow[ur, "\alpha_3"'] & \end{tikzcd}}$$

$$\Big\downarrow f$$

$$H : \quad \boxed{\begin{tikzcd} 1 \arrow[r, "\beta"] & 2 \end{tikzcd}}$$

Here, $f : G \to H$ is the graph morphism mapping the vertices and the edges in a vertical way. Then the edge $\beta \in \mathrm{E}_H$ is unitargeting with respect to f, since we have

$$|\{\tilde{e}\,\mathrm{t}_G :\ \tilde{e} \in \mathrm{E}_G\,,\ \tilde{e}\,\mathrm{E}_f = \beta\}| = |\{\alpha_1\,\mathrm{t}_G\,,\ \alpha_2\,\mathrm{t}_G\,,\ \alpha_3\,\mathrm{t}_G\}| = |\{4\}| = 1\ .$$

Lemma 208 Suppose given graphs G and H and an etale fibration $f : G \to H$; cf. Definition 127.(2).

Recall that f being an etale fibration means that the map

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

is bijective for $v \in \mathrm{V}_G$.

Suppose given a commutative diagram in Gph as follows, where $\iota_n = \iota_{0,n} : \mathrm{D}_0 \to \mathrm{D}_n$; cf. Definition 56.(3).

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_n"'] & G \arrow[d, "f"] \\
\mathrm{D}_n \arrow[r, "y"] & H
\end{tikzcd}$$

Then there exists a unique graph morphism $\hat{y} : \mathrm{D}_n \to G$ such that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_n"'] & G \arrow[d, "f"] \\
\mathrm{D}_n \arrow[r, "y"'] \arrow[ur, "\hat{y}"] & H
\end{tikzcd}$$

Proof. We have to show that there exists a graph morphism $\hat{y} : \mathrm{D}_n \to G$ such that $\iota_n\hat{y} \overset{!}{=} a$ and such that $\hat{y}f \overset{!}{=} y$.

Existence.

We *claim* that for $k \in [0,n]$ there exists $\hat{y}_k : \mathrm{D}_k \to G$ such that $\iota_{0,k}\hat{y}_k = a$ and $\hat{y}_k f = \iota_{k,n}y$; cf. Definition 56.(3).

We carry out an *induction* for $k \in [0,n]$:

If $k = 0$, we let $\hat{y}_0 := a : \mathrm{D}_0 \to G$. We have $\iota_{0,0}\hat{y}_0 = \mathrm{id}_{\mathrm{D}_0} a = a$ and $\hat{y}_0 f = \iota_{0,n}y$.

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\mathrm{id}_{\mathrm{D}_0}"'] & G \arrow[d, "f"] \\
\mathrm{D}_0 \arrow[r, "\iota_{0,n}y"'] \arrow[ur, "\hat{y}_0=a"] & H
\end{tikzcd}$$

Induction step: Suppose given $k \in [0,n-1]$ and $\hat{y}_k : \mathrm{D}_k \to G$ such that $\iota_{0,k}\hat{y}_k = a$ and $\hat{y}_k f = \iota_{k,n}y$.

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_{0,k}"'] & G \arrow[dd, "f"] \\
\mathrm{D}_k \arrow[ur, "\hat{y}_k"] \arrow[d, "\iota_{k,k+1}"'] & \\
\mathrm{D}_{k+1} \arrow[r, "\iota_{k+1,n}y"'] & H
\end{tikzcd}$$

We show that there exists $\hat{y}_{k+1} : \mathrm{D}_{k+1} \to G$ such that $\iota_{0,k+1}\hat{y}_{k+1} = a$ and $\hat{y}_{k+1} f = \iota_{k+1,n} y$.

We have $\hat{\mathrm{v}}_k \, \mathrm{V}_{\hat{y}_k} \in \mathrm{V}_G$ and $\hat{\mathrm{v}}_k \, \mathrm{V}_{\hat{y}_k} \, \mathrm{V}_f = \hat{\mathrm{v}}_k \, \mathrm{V}_y \in \mathrm{V}_H$.

Since the map $\mathrm{E}_{f, \hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}_k}} : G(\hat{\mathrm{v}}_k \, \mathrm{V}_{\hat{y}_k}, *) \to H(\hat{\mathrm{v}}_k \, \mathrm{V}_y, *)$ is bijective and since $\hat{\mathrm{e}}_k \, \mathrm{E}_y \, \mathrm{s}_H = \hat{\mathrm{e}}_k \, \mathrm{s}_{\mathrm{D}_n} \, \mathrm{V}_y = \hat{\mathrm{v}}_k \, \mathrm{V}_y$, there exists a unique edge $\tilde{e} \in G(\hat{\mathrm{v}}_k \, \mathrm{V}_{\hat{y}_k}, *) \subseteq \mathrm{E}_G$ such that $\tilde{e} \, \mathrm{E}_f = \hat{\mathrm{e}}_k \, \mathrm{E}_y \in H(\hat{\mathrm{v}}_k \, \mathrm{V}_y \, , *)$.

So let $\hat{y}_{k+1} : \mathrm{D}_{k+1} \to G$ with $\hat{\mathrm{v}}_i \, \mathrm{V}_{\hat{y}_{k+1}} = \hat{\mathrm{v}}_i \, \mathrm{V}_{\hat{y}_k}$ for $i \in [0, k]$ and $\hat{\mathrm{v}}_{k+1} \, \mathrm{V}_{\hat{y}_{k+1}} = \tilde{e} \, \mathrm{t}_G$, as well as $\hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_{k+1}} := \hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_k}$ for $i \in [0, k-1]$ and $\hat{\mathrm{e}}_k \, \mathrm{E}_{\hat{y}_{k+1}} := \tilde{e}$.

We have to show that $\hat{y}_{k+1}$ is in fact a graph morphism.

We have to show that $\hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_{k+1}} \, \mathrm{s}_G \overset{!}{=} \hat{\mathrm{e}}_i \, \mathrm{s}_{\mathrm{D}_{k+1}} \, \mathrm{V}_{\hat{y}_{k+1}}$ and that $\hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_{k+1}} \, \mathrm{t}_G \overset{!}{=} \hat{\mathrm{e}}_i \, \mathrm{t}_{\mathrm{D}_{k+1}} \, \mathrm{V}_{\hat{y}_{k+1}}$ in V_G for $i \in [0, k]$.

If $i \in [0, k-1]$, then we have $\hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_{k+1}} \, \mathrm{s}_G = \hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_k} \, \mathrm{s}_G = \hat{\mathrm{e}}_i \, \mathrm{s}_{\mathrm{D}_k} \, \mathrm{V}_{\hat{y}_k} = \hat{\mathrm{v}}_i \, \mathrm{V}_{\hat{y}_k} = \hat{\mathrm{v}}_i \, \mathrm{V}_{\hat{y}_{k+1}} = (\hat{\mathrm{e}}_i \, \mathrm{s}_{\mathrm{D}_{k+1}}) \, \mathrm{V}_{\hat{y}_{k+1}}$.

If $i \in [0, k-1]$, then we have $\hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_{k+1}} \, \mathrm{t}_G = \hat{\mathrm{e}}_i \, \mathrm{E}_{\hat{y}_k} \, \mathrm{t}_G = \hat{\mathrm{e}}_i \, \mathrm{t}_{\mathrm{D}_k} \, \mathrm{V}_{\hat{y}_k} = \hat{\mathrm{v}}_{i+1} \, \mathrm{V}_{\hat{y}_k} = \hat{\mathrm{v}}_{i+1} \, \mathrm{V}_{\hat{y}_{k+1}} = (\hat{\mathrm{e}}_i \, \mathrm{t}_{\mathrm{D}_{k+1}}) \, \mathrm{V}_{\hat{y}_{k+1}}$.

If $i = k$, then we have $\hat{\mathrm{e}}_k \, \mathrm{E}_{\hat{y}_{k+1}} \, \mathrm{s}_G = \tilde{e} \, \mathrm{s}_G = \hat{\mathrm{v}}_k \, \mathrm{V}_{\hat{y}_k} = \hat{\mathrm{v}}_k \, \mathrm{V}_{\hat{y}_{k+1}} = \hat{\mathrm{e}}_k \, \mathrm{s}_{\mathrm{D}_{k+1}} \, \mathrm{V}_{\hat{y}_{k+1}}$.

If $i = k$, then we have $\hat{\mathrm{e}}_k \, \mathrm{E}_{\hat{y}_{k+1}} \, \mathrm{t}_G = \tilde{e} \, \mathrm{t}_G = \hat{\mathrm{v}}_{k+1} \, \mathrm{V}_{\hat{y}_{k+1}} = (\hat{\mathrm{e}}_k \, \mathrm{t}_{\mathrm{D}_{k+1}}) \, \mathrm{V}_{\hat{y}_{k+1}}$.

So $\hat{y}_{k+1}$ is in fact a graph morphism.

We have to show that $\iota_{0,k+1}\hat{y}_{k+1} \overset{!}{=} a$.

In fact, we have $\hat{\mathrm{v}}_0 \, \mathrm{V}_{\iota_{0,k+1}} \, \mathrm{V}_{\hat{y}_{k+1}} = \hat{\mathrm{v}}_0 \, \mathrm{V}_{\hat{y}_{k+1}} = \hat{\mathrm{v}}_0 \, \mathrm{V}_{\hat{y}_k} = \hat{\mathrm{v}}_0 \, \mathrm{V}_{\iota_{0,k}} \, \mathrm{V}_{\hat{y}_k} = \hat{\mathrm{v}}_0 \, \mathrm{V}_a$.

We have to show that $\hat{y}_{k+1} f \overset{!}{=} \iota_{k+1,n} y$.

Since the direct graph D_{k+1} is thin, it suffices to show that $\mathrm{V}_{\hat{y}_{k+1} f} \overset{!}{=} \mathrm{V}_{\iota_{k+1,n} y}$.

Suppose given $i \in [0, k+1]$.

If $i \in [0, k]$, then we have $\hat{\mathrm{v}}_i \, \mathrm{V}_{\hat{y}_{k+1} f} = \hat{\mathrm{v}}_i \, \mathrm{V}_{\hat{y}_{k+1}} \, \mathrm{V}_f = \hat{\mathrm{v}}_i \, \mathrm{V}_{\hat{y}_k} \, \mathrm{V}_f = \hat{\mathrm{v}}_i \, \mathrm{V}_{\iota_{k,n} y} = \hat{\mathrm{v}}_i \, \mathrm{V}_{\iota_{k,n}} \, \mathrm{V}_y = \hat{\mathrm{v}}_i \, \mathrm{V}_y = \hat{\mathrm{v}}_i \, \mathrm{V}_{\iota_{k+1,n} y}$.

If $i = k+1$ then we have $\hat{\mathrm{v}}_{k+1} \, \mathrm{V}_{\hat{y}_{k+1} f} = \hat{\mathrm{v}}_{k+1} \, \mathrm{V}_{\hat{y}_{k+1}} \, \mathrm{V}_f = \tilde{e} \, \mathrm{t}_G \, \mathrm{V}_f = \tilde{e} \, \mathrm{E}_f \, \mathrm{t}_H = \hat{\mathrm{e}}_k \, \mathrm{E}_y \, \mathrm{t}_H = \hat{\mathrm{e}}_k \, \mathrm{t}_{\mathrm{D}_n} \, \mathrm{V}_y = \hat{\mathrm{v}}_{k+1} \, \mathrm{V}_y = \hat{\mathrm{v}}_{k+1} \, \mathrm{V}_{\iota_{k+1,n} y}$.

$$
\begin{tikzcd}
\mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_{0,k}"'] & G \arrow[ddd, "f"] \\
\mathrm{D}_k \arrow[ur, "\hat{y}_k"] \arrow[d, "\iota_{k,k+1}"'] & \\
\mathrm{D}_{k+1} \arrow[uur, "\hat{y}_{k+1}"'] \arrow[d, "\iota_{k+1,n}"'] & \\
\mathrm{D}_n \arrow[r, "y"] & H
\end{tikzcd}
$$

So we have shown in an inductive way, that there exists a graph morphism $\hat{y} := \hat{y}_n : \mathrm{D}_n \to G$ such that $\iota_{0,n}\hat{y} = a$ and $\hat{y} f = y$.

Uniqueness. Suppose given commutative diagrams as follows.

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_{0,n}"'] & G \arrow[d, "f"] \\ \mathrm{D}_n \arrow[r, "y"] \arrow[ur, "\hat{y}"] & H \end{tikzcd} \qquad \begin{tikzcd} \mathrm{D}_0 \arrow[r, "a"] \arrow[d, "\iota_{0,n}"'] & G \arrow[d, "f"] \\ \mathrm{D}_n \arrow[r, "y"] \arrow[ur, "\hat{y}'"] & H \end{tikzcd}$$

We have to show that $\hat{y} \overset{!}{=} \hat{y}'$.

Since the direct graph D_n is thin, it suffices to show that $\mathrm{V}_{\hat{y}} \overset{!}{=} \mathrm{V}_{\hat{y}'}$.

We want to show that $\hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}} \overset{!}{=} \hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}'}$ for $k \in [0, n]$.

We proceed by induction on k.

If $k = 0$ then we have $\hat{\mathrm{v}}_0 \mathrm{V}_{\hat{y}} \overset{\text{Def. } \iota_{0,n}}{=} \hat{\mathrm{v}}_0 \mathrm{V}_{\iota_{0,n}} \mathrm{V}_{\hat{y}} \overset{\iota_{0,n}\hat{y}=a}{=} \hat{\mathrm{v}}_0 \mathrm{V}_a \overset{\iota_{0,n}\hat{y}'=a}{=} \hat{\mathrm{v}}_0 \mathrm{V}_{\iota_{0,n}} \mathrm{V}_{\hat{y}'} \overset{\text{Def. } \iota_{0,n}}{=} \hat{\mathrm{v}}_0 \mathrm{V}_{\hat{y}'}$.

Induction step: Let $k \in [0, n-1]$. We suppose that $\hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}} = \hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}'}$.

We have to show that $\hat{\mathrm{v}}_{k+1} \mathrm{V}_{\hat{y}} \overset{!}{=} \hat{\mathrm{v}}_{k+1} \mathrm{V}_{\hat{y}'}$.

Let $v_G := \hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}} = \hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}'}$. Since the graph morphism $f : G \to H$ is an etale fibration, the map $\mathrm{E}_{f,v_G} : G(v_G, *) \to H(v_G \mathrm{V}_f , *)$ is bijective.

We have $v_G \mathrm{V}_f = \hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}} \mathrm{V}_f = \hat{\mathrm{v}}_k \mathrm{V}_y$.

We have $\hat{\mathrm{e}}_k \mathrm{E}_y \in H(v_G \mathrm{V}_f , *)$ since $(\hat{\mathrm{e}}_k \mathrm{E}_y) \mathrm{s}_H = \hat{\mathrm{e}}_k \mathrm{s}_{\mathrm{D}_n} \mathrm{V}_y = \hat{\mathrm{v}}_k \mathrm{V}_y = v_G \mathrm{V}_f$.

We have $\hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}} \in G(v_G, *)$ because $(\hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}}) \mathrm{s}_G = \hat{\mathrm{e}}_k \mathrm{s}_{\mathrm{D}_n} \mathrm{V}_{\hat{y}} = \hat{\mathrm{v}}_k \mathrm{V}_{\hat{y}} = v_G$.

Moreover, $(\hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}}) \mathrm{E}_{f,v_G} = \hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}} \mathrm{E}_f = (\hat{\mathrm{e}}_k \mathrm{E}_y)$.

Likewise, we have $\hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}'} \in G(v_G, *)$ and $(\hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}'}) \mathrm{E}_{f,v_G} = (\hat{\mathrm{e}}_k \mathrm{E}_y)$.

Since the map E_{f,v_G} is bijective, we have $\hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}} = \hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}'}$ and so $\hat{\mathrm{v}}_{k+1} \mathrm{V}_{\hat{y}} = \hat{\mathrm{e}}_k \mathrm{t}_{\mathrm{D}_n} \mathrm{V}_{\hat{y}} = \hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}} \mathrm{t}_G = \hat{\mathrm{e}}_k \mathrm{E}_{\hat{y}'} \mathrm{t}_G = \hat{\mathrm{e}}_k \mathrm{t}_G \mathrm{V}_{\hat{y}'} = \hat{\mathrm{v}}_{k+1} \mathrm{V}_{\hat{y}'}$. ▫

Lemma 209 Suppose given graphs G and H and an etale fibration $f : G \to H$; cf. Definition 127.(2).

Recall that f being an etale fibration means that the map

$$\mathrm{E}_{f,v} := \mathrm{E}_f \big|_{G(v,*)}^{H(v \mathrm{V}_f ,*)} : G(v, *) \to H(v \mathrm{V}_f , *)$$

is bijective for $v \in \mathrm{V}_G$.

Suppose given graph morphisms $\hat{u} : \mathrm{C}_n \to G$ and $\hat{u}' : \mathrm{C}_n \to G$ such that $\hat{u}f = \hat{u}'f$ and $\mathrm{v}_0 \mathrm{V}_{\hat{u}} = \mathrm{v}_0 \mathrm{V}_{\hat{u}'}$.

Then $\hat{u} = \hat{u}'$.

Proof. Let $u := \hat{u}f = \hat{u}'f : \mathrm{C}_n \to H$.

Consider the graph morphism $a : \mathrm{D}_0 \to G$ with $\hat{\mathrm{v}}_0 \mathrm{V}_a = \mathrm{v}_0 \mathrm{V}_{\hat{u}} = \mathrm{v}_0 \mathrm{V}_{\hat{u}'}$, the graph morphism $r_n : \mathrm{D}_n \to \mathrm{C}_n$ with $\hat{\mathrm{v}}_i \mathrm{V}_r = \mathrm{v}_i$ for $i \in [0, n]$ and the graph morphism $\iota_n = \iota_{0,n} : \mathrm{D}_0 \to \mathrm{D}_n$; cf.

Definition 56.(3). We have the following commutative diagrams.

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[rr, "a"] \arrow[d, "\iota_n"'] & & G \arrow[d, "f"] \\
\mathrm{D}_n \arrow[r, "r_n"'] & \mathrm{C}_n \arrow[ur, "\hat{u}"] \arrow[r, "u"'] & H
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[rr, "a"] \arrow[d, "\iota_n"'] & & G \arrow[d, "f"] \\
\mathrm{D}_n \arrow[r, "r_n"'] & \mathrm{C}_n \arrow[ur, "\hat{u}'"] \arrow[r, "u"'] & H
\end{tikzcd}$$

Then we have $(r_n\hat{u})f = r_n u$ and $(r_n\hat{u}')f = r_n u$, and we have $\iota_n(r_n\hat{u}) = a$ and $\iota_n(r_n\hat{u}') = a$.

Because of Lemma 208 there exists a unique graph morphism $\hat{y} : \mathrm{D}_n \to G$ such that $\iota_n\hat{y} = a$ and $\hat{y}f = r_n u$. Therefore, we have $r_n\hat{u} = \hat{y} = r_n\hat{u}'$.

Since r_n is surjective, we conclude that $\hat{u} = \hat{u}'$; cf. Remark 72. ▫

Proposition 210 Suppose given graphs G and H.

Suppose given an etale fibration $f : G \to H$ that satisfies the following property (Uni).

(Uni) For $n \geqslant 1$ and each graph morphism $u : \mathrm{C}_n \to H$, there exists $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\mathrm{e}_i\,\mathrm{E}_u \in \mathrm{E}_H$ is unitargeting with respect to f.

Then the graph morphism $f : G \to H$ is a quasiisomorphism; cf. Definition 115.

Note that altogether, f is an acyclic fibration; cf. Definition 138.

Proof. Suppose given $n \geqslant 1$. Suppose given a graph morphism $u_0 : \mathrm{C}_n \to H$.

We have to show that there exists a unique graph morphism $\hat{u}_0 : \mathrm{C}_n \to G$ such that $\hat{u}_0 f = u$.

Since (Uni) is satisfied, we may choose $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\mathrm{e}_i\,\mathrm{E}_{u_0} \in \mathrm{E}_H$ is unitargeting.

Let $u := \mathrm{a}_{i+1}\,u_0$; cf. Definition 67. Then $u_0 = a_{-i-1}u$.

So the edge $\mathrm{e}_i\,\mathrm{E}_{u_0} = \mathrm{e}_i\,\mathrm{a}_{-i-1}\,\mathrm{E}_u = \mathrm{e}_{-1}\,\mathrm{E}_u \in \mathrm{E}_H$ is unitargeting.

We have

$$\begin{array}{rl}
 & \{\hat{u}_0 \in (\mathrm{C}_n\,, G) : \hat{u}_0 f = u_0\} \\
= & \{\mathrm{a}_{-i-1}\,\hat{u} : \hat{u} \in (\mathrm{C}_n\,, G),\ \mathrm{a}_{-i-1}\,\hat{u}f = u_0 = \mathrm{a}_{-i-1}\,u\} \\
= & \{\mathrm{a}_{-i-1}\,\hat{u} : \hat{u} \in (\mathrm{C}_n\,, G)_{\mathrm{Gph}}\,,\ \hat{u}f = u\} \\
= & (\{\hat{u} \in (\mathrm{C}_n\,, G) : \hat{u}f = u\})(\mathrm{a}_{-i-1}\,, G)\ .
\end{array}$$

So we have to show that there exists a unique graph morphism $\hat{u} : \mathrm{C}_n \to G$ such that $\hat{u}f = u$.

$$\begin{tikzcd}
 & G \arrow[d, "f"] \\
\mathrm{C}_n \arrow[ur, dotted, "\hat{u}"] \arrow[r, "\tilde{u}"'] & H
\end{tikzcd}$$

Uniqueness.

Suppose given graph morphisms $\hat{u} : \mathrm{C}_n \to G$ and $\hat{u}' : \mathrm{C}_n \to G$ such that $\hat{u} f = u = \hat{u}' f$.

We have to show that $\hat{u} \overset{!}{=} \hat{u}'$.

We have $(\mathrm{e}_{-1}\,\mathrm{E}_{\hat{u}})\,\mathrm{E}_f = \mathrm{e}_{-1}\,\mathrm{E}_u = (\mathrm{e}_{-1}\,\mathrm{E}_{\hat{u}'})\,\mathrm{E}_f$.

Since $\mathrm{e}_{-1}\,\mathrm{E}_u$ is unitargeting, we have $\mathrm{e}_{-1}\,\mathrm{E}_{\hat{u}}\,\mathrm{t}_G = \mathrm{e}_{-1}\,\mathrm{E}_{\hat{u}'}\,\mathrm{t}_G$.

So we have $\mathrm{v}_0\,\mathrm{V}_{\hat{u}} = \mathrm{e}_{-1}\,\mathrm{t}_{\mathrm{C}_n}\,\mathrm{V}_{\hat{u}} = \mathrm{e}_{-1}\,\mathrm{E}_{\hat{u}}\,\mathrm{t}_G = \mathrm{e}_{-1}\,\mathrm{E}_{\hat{u}'}\,\mathrm{t}_G = \mathrm{e}_{-1}\,\mathrm{t}_{\mathrm{C}_n}\,\mathrm{V}_{\hat{u}'} = \mathrm{v}_0\,\mathrm{V}_{\hat{u}'}$.

By Lemma 209 we obtain $\hat{u} = \hat{u}'$.

Existence.

Let $\{v_G\} := \{\hat{e}\,\mathrm{t}_G : \hat{e} \in \mathrm{E}_G\,,\ \hat{e}\,\mathrm{E}_f = \mathrm{e}_{-1}\,\mathrm{E}_u\}$, using that $\mathrm{e}_{-1}\,\mathrm{E}_u$ is unitargeting.

Consider the graph morphism $a : \mathrm{D}_0 \to G$ with $\hat{\mathrm{v}}_0\,\mathrm{V}_a := v_G$, the graph morphism $r_n : \mathrm{D}_n \to \mathrm{C}_n$ with $\hat{\mathrm{v}}_i\,\mathrm{V}_r = \mathrm{v}_i$ for $i \in [0, n]$ and the graph morphism $\iota_n = \iota_{0,n} : \mathrm{D}_0 \to \mathrm{D}_n$; cf. Definition 56.(3). We have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[rr, "a"] \arrow[d, "\iota_n"'] & & G \arrow[d, "f"] \\
\mathrm{D}_n \arrow[r, "r_n"'] & \mathrm{C}_n \arrow[r, "u"'] & H
\end{tikzcd}$$

We show that it commutes, i.e. that $af \overset{!}{=} \iota_n r_n u$.

We choose $\hat{e} \in \mathrm{E}_G$ with $\hat{e}\,\mathrm{E}_f = \mathrm{e}_{-1}\,\mathrm{E}_u$. Then $\hat{\mathrm{v}}_0\,\mathrm{V}_{af} = v_G\,\mathrm{V}_f = \hat{e}\,\mathrm{t}_G\,\mathrm{V}_f = \hat{e}\,\mathrm{E}_f\,\mathrm{t}_H = \mathrm{e}_{-1}\,\mathrm{E}_u\,\mathrm{t}_H = \mathrm{e}_{-1}\,\mathrm{t}_{\mathrm{C}_n}\,\mathrm{V}_u = \mathrm{v}_0\,\mathrm{V}_u = \hat{\mathrm{v}}_0\,\mathrm{V}_{ru} = \hat{\mathrm{v}}_0\,\mathrm{V}_{\iota_n r u}$.

So the diagram is in fact commutative.

By Lemma 208 there exists a graph morphism $\hat{y} : \mathrm{D}_n \to G$ such that

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[rr, "a"] \arrow[d, "\iota_n"'] & & G \arrow[d, "f"] \\
\mathrm{D}_n \arrow[r, "r_n"'] \arrow[urr, "\hat{y}"] & \mathrm{C}_n \arrow[r, "u"'] & H
\end{tikzcd}$$

commutes.

Now we show that there exists a graph morphism $\hat{u} : \mathrm{C}_n \to G$ such that $r_n \hat{u} \overset{!}{=} \hat{y}$.

Therefor we have to show that $\hat{\mathrm{v}}_n\,\mathrm{V}_{\hat{y}} \overset{!}{=} \hat{\mathrm{v}}_0\,\mathrm{V}_{\hat{y}}$.

We have $\hat{\mathrm{e}}_{n-1}\,\mathrm{E}_{\hat{y}} \in \mathrm{E}_G$.

We have $(\hat{\mathrm{e}}_{n-1}\,\mathrm{E}_{\hat{y}})\,\mathrm{E}_f = \hat{\mathrm{e}}_{n-1}\,\mathrm{E}_{r_n}\,\mathrm{E}_u = \mathrm{e}_{n-1}\,\mathrm{E}_u = \mathrm{e}_{-1}\,\mathrm{E}_u$.

So we have $\hat{\mathrm{e}}_{n-1}\,\mathrm{E}_{\hat{y}}\,\mathrm{t}_G = v_G$.

So we have $\hat{\mathrm{v}}_n\,\mathrm{V}_{\hat{y}} = \hat{\mathrm{e}}_{n-1}\,\mathrm{t}_{\mathrm{D}_n}\,\mathrm{V}_{\hat{y}} = \hat{\mathrm{e}}_{n-1}\,\mathrm{E}_{\hat{y}}\,\mathrm{t}_G = v_G = \hat{\mathrm{v}}_0\,\mathrm{V}_a = \hat{\mathrm{v}}_0\,\mathrm{V}_{\iota_n}\,\mathrm{V}_{\hat{y}} = \hat{\mathrm{v}}_0\,\mathrm{V}_{\hat{y}}$.

So we have $\hat{\mathrm{v}}_n \, \mathrm{V}_{\hat{y}} = \hat{\mathrm{v}}_0 \, \mathrm{V}_{\hat{y}}$.

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[rr, "a"] \arrow[d, "\iota_n"'] & & G \arrow[d, "f"] \\
\mathrm{D}_n \arrow[urr, "\hat{y}"] \arrow[r, "r_n"'] & \mathrm{C}_n \arrow[ur, "\hat{u}"'] \arrow[r, "u"'] & H
\end{tikzcd}$$

Now we have $r_n \hat{u} f = \hat{y} f = r_n u$.

Since r_n is surjective, we conclude that $\hat{u} f = u$; cf. Remark 72. □

Chapter 8

Duality

Definition 211

(1) Suppose given a graph G.

We define the *opposite* graph $G^{\mathrm{op}} := (\mathrm{V}_{G^{\mathrm{op}}}, \mathrm{E}_{G^{\mathrm{op}}}, \mathrm{s}_{G^{\mathrm{op}}}, \mathrm{t}_{G^{\mathrm{op}}})$ by $\mathrm{V}_{G^{\mathrm{op}}} := \mathrm{V}_G$, $\mathrm{E}_{G^{\mathrm{op}}} := \mathrm{E}_G$, $\mathrm{s}_{G^{\mathrm{op}}} := \mathrm{t}_G$ and $\mathrm{t}_{G^{\mathrm{op}}} := \mathrm{s}_G$.

(2) Suppose given a graph morphism $f : G \to H$.

We define the *opposite* graph morphism $f^{\mathrm{op}} = (\mathrm{V}_{f^{\mathrm{op}}}, \mathrm{E}_{f^{\mathrm{op}}}) : G^{\mathrm{op}} \to H^{\mathrm{op}}$ with $\mathrm{V}_{f^{\mathrm{op}}} := \mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and $\mathrm{E}_{f^{\mathrm{op}}} := \mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$.

In fact, we have $\mathrm{E}_{f^{\mathrm{op}}}\,\mathrm{s}_{H^{\mathrm{op}}} = \mathrm{E}_f\,\mathrm{t}_H = \mathrm{t}_G\,\mathrm{V}_f = \mathrm{s}_{G^{\mathrm{op}}}\,\mathrm{V}_{f^{\mathrm{op}}}$ and $\mathrm{E}_{f^{\mathrm{op}}}\,\mathrm{t}_{H^{\mathrm{op}}} = \mathrm{E}_f\,\mathrm{s}_H = \mathrm{s}_G\,\mathrm{V}_f = \mathrm{t}_{G^{\mathrm{op}}}\,\mathrm{V}_{f^{\mathrm{op}}}$.

Remark 212 Suppose given graph morphisms $f : X \to Y$ and $g : Y \to Z$.

(1) We have $(f^{\mathrm{op}})^{\mathrm{op}} = f$.

(2) We have $(fg)^{\mathrm{op}} = f^{\mathrm{op}} g^{\mathrm{op}}$.

Proof.

Ad (1). We have $(X^{\mathrm{op}})^{\mathrm{op}} = X$ and $(Y^{\mathrm{op}})^{\mathrm{op}} = Y$; cf. Definition 211.(1).

We have $\mathrm{V}_{(f^{\mathrm{op}})^{\mathrm{op}}} = \mathrm{V}_{f^{\mathrm{op}}} = \mathrm{V}_f$ and $\mathrm{E}_{(f^{\mathrm{op}})^{\mathrm{op}}} = \mathrm{E}_{f^{\mathrm{op}}} = \mathrm{E}_f$; cf. Definition 211.(1).

Ad (2). We have $\mathrm{V}_{(fg)^{\mathrm{op}}} = \mathrm{V}_{fg} = \mathrm{V}_f\,\mathrm{V}_g = \mathrm{V}_{f^{\mathrm{op}}}\,\mathrm{V}_{g^{\mathrm{op}}} = \mathrm{V}_{f^{\mathrm{op}}g^{\mathrm{op}}}$ and $\mathrm{E}_{(fg)^{\mathrm{op}}} = \mathrm{E}_{fg} = \mathrm{E}_f\,\mathrm{E}_g = \mathrm{E}_{f^{\mathrm{op}}}\,\mathrm{E}_{g^{\mathrm{op}}} = \mathrm{E}_{f^{\mathrm{op}}g^{\mathrm{op}}}$; cf. Definition 211. □

Remark 213 Suppose given $n \in \mathbb{N}$.

Wel define the graph isomorphism $\zeta_n : \mathrm{C}_n \xrightarrow{\sim} \mathrm{C}_n^{\mathrm{op}}$ as follows.

We let

$$\begin{array}{rrcl}
\zeta_n : & \mathrm{C}_n & \xrightarrow{\sim} & \mathrm{C}_n^{\mathrm{op}} \\
\mathrm{V}_{\zeta_n} : & \mathrm{V}_{\mathrm{C}_n} & \to & \mathrm{V}_{\mathrm{C}_n^{\mathrm{op}}} \\
 & \mathrm{v}_k & \mapsto & \mathrm{v}_{-k} \text{ for } k \in \mathbb{Z}/n\mathbb{Z} \\
\mathrm{E}_{\zeta_n} : & \mathrm{E}_{\mathrm{C}_n} & \to & \mathrm{E}_{\mathrm{C}_n^{\mathrm{op}}} \\
 & \mathrm{e}_k & \mapsto & \mathrm{e}_{-1-k} \text{ for } k \in \mathbb{Z}/n\mathbb{Z}
\end{array}$$

Suppose given an edge $\mathrm{e}_k \in \mathrm{E}_{\mathrm{C}_n}$.

We have $\mathrm{e}_k\,\mathrm{E}_{\zeta_n}\,\mathrm{s}_{\mathrm{C}_n^{\mathrm{op}}} = \mathrm{e}_{-1-k}\,\mathrm{s}_{\mathrm{C}_n^{\mathrm{op}}} = \mathrm{e}_{-1-k}\,\mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_{-k} = \mathrm{v}_k\,\mathrm{V}_{\zeta_n} = \mathrm{e}_k\,\mathrm{s}_{\mathrm{C}_n}\,\mathrm{V}_{\zeta_n}$.

We have $\mathrm{e}_k\,\mathrm{E}_{\zeta_n}\,\mathrm{t}_{\mathrm{C}_n^{\mathrm{op}}} = \mathrm{e}_{-1-k}\,\mathrm{t}_{\mathrm{C}_n^{\mathrm{op}}} = \mathrm{e}_{-1-k}\,\mathrm{s}_{\mathrm{C}_n} = \mathrm{v}_{-1-k} = \mathrm{v}_{k+1}\,\mathrm{V}_{\zeta_n} = \mathrm{e}_k\,\mathrm{t}_{\mathrm{C}_n}\,\mathrm{V}_{\zeta_n}$.

So $\zeta_n : \mathrm{C}_n \xrightarrow{\sim} \mathrm{C}_n^{\mathrm{op}}$ is in fact a graph isomorphism.

We have $(\mathrm{C}_n^{\mathrm{op}})^{\mathrm{op}} = \mathrm{C}_n$; cf. Remark 212.

We have the graph isomorphism $\zeta_n^{\mathrm{op}} := (\mathrm{V}_{\zeta_n}, \mathrm{E}_{\zeta_n}) : \mathrm{C}_n^{\mathrm{op}} \to \mathrm{C}_n$; cf. Definition 211.

We show that we have $\zeta_n\zeta_n^{\mathrm{op}} \overset{!}{=} \mathrm{id}_{\mathrm{C}_n}$ and $\zeta_n^{\mathrm{op}}\zeta_n \overset{!}{=} \mathrm{id}_{\mathrm{C}_n^{\mathrm{op}}}$.

Suppose given a vertex $\mathrm{v}_k \in \mathrm{V}_{\mathrm{C}_n} = \mathrm{V}_{\mathrm{C}_n^{\mathrm{op}}}$.

We have $\mathrm{v}_k\,\mathrm{V}_{\zeta_n\zeta_n^{\mathrm{op}}} = \mathrm{v}_k\,\mathrm{V}_{\zeta_n}\,\mathrm{V}_{\zeta_n^{\mathrm{op}}} = \mathrm{v}_{-k}\,\mathrm{V}_{\zeta_n^{\mathrm{op}}} = \mathrm{v}_{-k}\,\mathrm{V}_{\zeta_n} = \mathrm{v}_{-(-k)} = \mathrm{v}_k = \mathrm{v}_k\,\mathrm{V}_{\mathrm{id}_{\mathrm{C}_n}}$.

We have $\mathrm{v}_k\,\mathrm{V}_{\zeta_n^{\mathrm{op}}\zeta_n} = \mathrm{v}_k\,\mathrm{V}_{\zeta_n^{\mathrm{op}}}\,\mathrm{V}_{\zeta_n} = \mathrm{v}_k\,\mathrm{V}_{\zeta_n}\,\mathrm{V}_{\zeta_n} = \mathrm{v}_{-k}\,\mathrm{V}_{\zeta_n} = \mathrm{v}_{-(-k)} = \mathrm{v}_k = \mathrm{v}_k\,\mathrm{V}_{\mathrm{id}_{\mathrm{C}_n^{\mathrm{op}}}}$.

Suppose given an edge $\mathrm{e}_k \in \mathrm{E}_{\mathrm{C}_n} = \mathrm{E}_{\mathrm{C}_n^{\mathrm{op}}}$.

We have $\mathrm{e}_k\,\mathrm{E}_{\zeta_n\zeta_n^{\mathrm{op}}} = \mathrm{e}_k\,\mathrm{E}_{\zeta_n}\,\mathrm{E}_{\zeta_n^{\mathrm{op}}} = \mathrm{e}_{-1-k}\,\mathrm{E}_{\zeta_n^{\mathrm{op}}} = \mathrm{e}_{-1-k}\,\mathrm{E}_{\zeta_n} = \mathrm{e}_{-1-(-1-k)} = \mathrm{e}_k = \mathrm{e}_k\,\mathrm{E}_{\mathrm{id}_{\mathrm{C}_n}}$.

We have $\mathrm{e}_k\,\mathrm{E}_{\zeta_n^{\mathrm{op}}\zeta_n} = \mathrm{e}_k\,\mathrm{E}_{\zeta_n^{\mathrm{op}}}\,\mathrm{E}_{\zeta_n} = \mathrm{e}_k\,\mathrm{E}_{\zeta_n}\,\mathrm{E}_{\zeta_n} = \mathrm{e}_{-1-k}\,\mathrm{E}_{\zeta_n} = \mathrm{e}_{-1-(-1-k)} = \mathrm{e}_k = \mathrm{e}_k\,\mathrm{E}_{\mathrm{id}_{\mathrm{C}_n^{\mathrm{op}}}}$.

So we have $\zeta_n^{-1} = \zeta_n^{\mathrm{op}}$. □

Remark 214 Suppose given a graph morphism $f : G \to H$.

Then the graph morphism $f^{\mathrm{op}} : G^{\mathrm{op}} \to H^{\mathrm{op}}$ is a quasiisomorphism if and only if the graph morphism $f : G \to H$ is a quasiisomorphism.

Proof. Suppose given $k \geqslant 1$.

We will show that the map $(\mathrm{C}_k\,, f)_{\mathrm{Gph}}$ is bijective if and only if the map $(\mathrm{C}_k^{\mathrm{op}}\,, f^{\mathrm{op}})$ is bijective, which is the case if and only if the map $(\mathrm{C}_k\,, f^{\mathrm{op}})$ is bijective.

We consider the following commutative diagram.

$$\begin{tikzcd}
g \arrow[d, mapsto] & (\mathrm{C}_k\,, G)_{\mathrm{Gph}} \arrow[r, "(\mathrm{C}_k\,,f)_{\mathrm{Gph}}"] \arrow[d, "\sigma", "\wr"'] & (\mathrm{C}_k\,, H)_{\mathrm{Gph}} \arrow[d, "\tau"', "\wr"] & g \arrow[d, mapsto] \\
g^{\mathrm{op}} & (\mathrm{C}_k^{\mathrm{op}}\,, G^{\mathrm{op}})_{\mathrm{Gph}} \arrow[r, "(\mathrm{C}_k^{\mathrm{op}}\,,f^{\mathrm{op}})_{\mathrm{Gph}}"] \arrow[d, "(\zeta_k\,,G^{\mathrm{op}})_{\mathrm{Gph}}", "\wr"'] & (\mathrm{C}_k^{\mathrm{op}}\,, H^{\mathrm{op}})_{\mathrm{Gph}} \arrow[d, "(\zeta_k\,,H^{\mathrm{op}})_{\mathrm{Gph}}", "\wr"'] & g^{\mathrm{op}} \\
 & (\mathrm{C}_k\,, G^{\mathrm{op}})_{\mathrm{Gph}} \arrow[r, "(\mathrm{C}_k\,,f^{\mathrm{op}})_{\mathrm{Gph}}"'] & (\mathrm{C}_k\,, H^{\mathrm{op}})_{\mathrm{Gph}} &
\end{tikzcd}$$

□

Suppose given a graph morphism $u \in (\mathrm{C}_k\,,G)_{\mathrm{Gph}}$.

We have $u^{\mathrm{op}}(\mathrm{C}_k^{\mathrm{op}}\,,f^{\mathrm{op}})_{\mathrm{Gph}} = u^{\mathrm{op}}f^{\mathrm{op}} \overset{\text{Rem. 212}}{=} (uf)^{\mathrm{op}} = (u(\mathrm{C}_k\,,f)_{\mathrm{Gph}})^{\mathrm{op}}$.

So the "upper quadrangle" commutes.

Suppose given a graph morphism $r \in (\mathrm{C}_k^{\mathrm{op}}\,,G^{\mathrm{op}})_{\mathrm{Gph}}$.

We have $r(\zeta_n\,,G^{\mathrm{op}})_{\mathrm{Gph}}(\mathrm{C}_k\,,f^{\mathrm{op}})_{\mathrm{Gph}} = \zeta_n r(\mathrm{C}_k\,,f^{\mathrm{op}})_{\mathrm{Gph}} = \zeta_n r f^{\mathrm{op}} = r f^{\mathrm{op}}(\zeta_n\,,H^{\mathrm{op}})_{\mathrm{Gph}} = r(\mathrm{C}_k^{\mathrm{op}}\,,f^{\mathrm{op}})_{\mathrm{Gph}}(\zeta_n\,,H^{\mathrm{op}})_{\mathrm{Gph}}$.

So the "lower quadrangle" commutes.

The maps σ and τ are bijective because of Remark 212.

The maps $(\zeta_n\,,G^{\mathrm{op}})_{\mathrm{Gph}}$ and $(\zeta_n\,,H^{\mathrm{op}})_{\mathrm{Gph}}$ are bijective because of Remark 213.

The graph morphism $f : G \to H$ is a quasiisomorphism if and only if the map $(\mathrm{C}_k\,,f)_{\mathrm{Gph}}$ is bijective for $k \geqslant 1$.

By the commutative diagram above, $(\mathrm{C}_k\,,f)_{\mathrm{Gph}}$ is bijective if and only if $(\mathrm{C}_k\,,f^{\mathrm{op}})_{\mathrm{Gph}}$ is bijective.

Hence f is a quasiisomorphism if and only if f^{op} is a quasiisomorphism. □

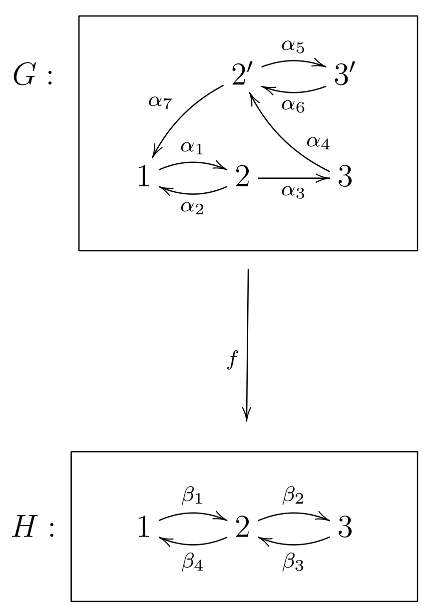
Chapter 9

Some examples and counterexamples

The functions that are used in this section to calculate graphs and graph morphisms are given in §10. They can also be obtained using the electronic appendix, cf. §A.

9.1 Some examples for quasiisomorphisms

Example 215 We consider the following graph morphism.



Here, $f = (\mathrm{V}_f\,, \mathrm{E}_f) : G \to H$ is the graph morphism mapping the vertices and the edges in a vertical way. So we let

$$\begin{aligned} 1\,\mathrm{V}_f &:= 1, \quad 2\,\mathrm{V}_f := 2, \quad 3\,\mathrm{V}_f := 3, \\ & \quad\; 2'\,\mathrm{V}_f := 2, \quad 3'\,\mathrm{V}_f := 3 \end{aligned}$$

and

$$\begin{aligned}
\alpha_1\,\mathrm{E}_f &:= \beta_1\ ,\\
\alpha_2\,\mathrm{E}_f &:= \beta_4\ ,\\
\alpha_3\,\mathrm{E}_f &:= \beta_2\ ,\\
\alpha_4\,\mathrm{E}_f &:= \beta_3\ ,\\
\alpha_5\,\mathrm{E}_f &:= \beta_2\ ,\\
\alpha_6\,\mathrm{E}_f &:= \beta_3\ ,\\
\alpha_7\,\mathrm{E}_f &:= \beta_4\ .
\end{aligned}$$

Then the graph morphism f is an acyclic fibration.

Proof. Via Magma [**2**] we will calculate that the graph morphism f is an etale fibration that satisfies (Uni), using the function `SuffCond`.

This will show that f is a quasiisomorphism; cf. Proposition 210.

The necessary Magma functions can be found in §10.7, §10.5 and in §10.6; cf. also §A.

The graph G can be obtained with the function `trygraph` by setting `n := 3`.

The graph H can be obtained with the function `c2chain` by setting `n := 3`.

The graph morphism f can be obtained with the function `tryacyclic` by setting `n := 3`.

So letting

```
G := trygraph(3);
H := c2chain(3);
f := tryacyclic(3);
> G;
<[ 1, 2, 3, 4, 5 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 5>,
                      <5, 5, 4>, <3, 6, 4>, <4, 7, 1> ]>
> H;
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 4, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 5>, <2, 2, 3>>,
 <<5, 5, 4>, <3, 3, 2>>, <<3, 6, 4>, <3, 3, 2>>, <<4, 7, 1>, <2, 4, 1>> ]>
```

we get

```
> IsGraphMorphism(f,G,H);
true
> SuffCond(f,G,H);
<true, true>
```

Alternative proof without Proposition 210.

Via Magma we calculate that the graph morphism f is in fact a fibration, using the function `IsFibration` given in §10 below.

```
> IsFibration(f,G,H);
true
```

We will now give a pedestrian proof that f is a quasiisomorphism by classifying all morphisms from C_n to G respectively to H.

It might be useful as a blueprint for a quasiisomorphy verification in a case where we cannot apply Proposition 210, or its dual statement using Remark 214.

Let $k \in \mathbb{N}$.

We remark that $(\mathrm{C}_{2k-1}\,, H) = \emptyset$ and thus $(\mathrm{C}_{2k-1}\,, G) = \emptyset$.

To prove that f is acyclic, we have to show that the map $(\mathrm{C}_{2k}\,, f) : (\mathrm{C}_{2k}\,, G) \to (\mathrm{C}_{2k}\,, H)$ is bijective.

We have to show that for each graph morphism $u : \mathrm{C}_{2k} \to H$ there exists a unique graph morphism $v : \mathrm{C}_{2k} \to G$ such that $vf = u$.

$$\begin{tikzcd} & G \arrow[d, "f"] \\ \mathrm{C}_{2k} \arrow[ur, "\exists! v"] \arrow[r, "u"'] & H \end{tikzcd}$$

We first show that it suffices to show this for each graph morphism $u : \mathrm{C}_{2k} \to H$ such that $\mathrm{v}_0 \,\mathrm{V}_u = 2$.

Write $(\mathrm{C}_{2k}\,, H)_2 := \{u \in (\mathrm{C}_{2k}\,, H) : u \text{ graph morphism}, \mathrm{v}_0 \,\mathrm{V}_u = 2\}$.

Write $(\mathrm{C}_{2k}\,, G)_{2,2'} := \{u \in (\mathrm{C}_{2k}\,, G) : u \text{ graph morphism}, \mathrm{v}_0 \,\mathrm{V}_u \in \{2, 2'\}\}$.

The map $(\mathrm{C}_{2k}\,, f) : (\mathrm{C}_{2k}\,, G) \to (\mathrm{C}_{2k}\,, H)$ restricts to the map

$$(\mathrm{C}_{2k}\,, f)_2 := (\mathrm{C}_{2k}\,, f)|_{(\mathrm{C}_{2k}\,,G)_{2,2'}}^{(\mathrm{C}_{2k}\,,H)_2} : (\mathrm{C}_{2k}\,, G)_{2,2'} \to (\mathrm{C}_{2k}\,, H)_2\ .$$

Recall that the vertices of the graph C_{2k} are written v_i for $i \in \mathbb{Z}/2k\mathbb{Z}$.

Recall that the edges of the graph C_{2k} are written e_i for $i \in \mathbb{Z}/2k\mathbb{Z}$.

Claim. If $(\mathrm{C}_{2k}\,, f)_2$ is bijective, then $(\mathrm{C}_{2k}\,, f)$ is bijective.

Suppose given $u : \mathrm{C}_{2k} \to H$. We have to show that there exists a unique graph morphism $v : \mathrm{C}_{2k} \to G$ such that $vf = u$.

$$\begin{tikzcd} & G \arrow[d, "f"] \\ \mathrm{C}_{2k} \arrow[ur, "\exists! v"] \arrow[r, "u"'] & H \end{tikzcd}$$

A consideration of H shows that there exists $s \in \mathbb{Z}/2k\mathbb{Z}$ such that $\mathrm{v}_s \,\mathrm{V}_u = 2$.

Let $\tilde{u} := \mathrm{a}_s \cdot u : \mathrm{C}_{2k} \to H$.

We have $\mathrm{v}_0 \,\mathrm{V}_{\tilde{u}} = \mathrm{v}_0 \,\mathrm{V}_{\mathrm{a}_s} \,\mathrm{V}_u = \mathrm{v}_s \,\mathrm{V}_u = 2$.

Since $(\mathrm{C}_{2k}\,, f)_2$ is bijective, there exists a unique graph morphism $\tilde{v} \in (\mathrm{C}_{2k}\,, G)_{2,2'}$ such that $\tilde{v}f = \tilde{u}$.

$$\begin{tikzcd} & G \arrow[d, "f"] \\ \mathrm{C}_{2k} \arrow[ur, "\exists! \tilde{v}"] \arrow[r, "\tilde{u}"'] & H \end{tikzcd}$$

To prove *existence*, we let $v := \mathrm{a}_s^{-1} \cdot \tilde{v}$.

We obtain $vf = \mathrm{a}_s^{-1}\, \tilde{v}f = \mathrm{a}_s^{-1}\, \tilde{u} = u$.

Uniqueness. Suppose given $v, v' \in (\mathrm{C}_{2k}\,, G)$ such that $vf = u = v'f$.

We have to show that $v \overset{!}{=} v'$.

We have $\mathrm{a}_s \, vf = \mathrm{a}_s \, u = \tilde{u}$ and $\mathrm{a}_s \, v'f = \mathrm{a}_s \, u = \tilde{u}$.

Note that $\mathrm{V}_f^{-1}(\{2\}) = \{2, 2'\}$.

In particular, $\mathrm{v}_0\,\mathrm{V}_{\tilde{u}} = 2$ implies $\mathrm{v}_0\,\mathrm{V}_{\mathrm{a}_s v} \in \{2, 2'\}$ and $\mathrm{v}_0\,\mathrm{V}_{\mathrm{a}_s v'} \in \{2, 2'\}$, so $\mathrm{a}_s v, \mathrm{a}_s v' \in (\mathrm{C}_{2k}, G)_{2,2'}$.

Since $(\mathrm{C}_{2k}, f)_2$ is bijective, we conclude that $\mathrm{a}_s v = \mathrm{a}_s v'$. Thus $v = v'$.

This proves the *claim*.

Note that the map

$$\begin{array}{rcl} \mathbb{Z}/k\mathbb{Z} & \to & \mathbb{Z}/2k\mathbb{Z} \\ i + k\mathbb{Z} & \mapsto & 2i + 2k\mathbb{Z} \end{array}$$

exists.

Note that for a graph morphism $r : \mathrm{C}_{2k} \to H$ in $(\mathrm{C}_{2k}, H)_2$ we have $\mathrm{v}_{2i}\,\mathrm{V}_r \in \{2\}$, $\mathrm{v}_{2i+1}\,\mathrm{V}_r \in \{1, 3\}$, $\mathrm{e}_{2i}\,\mathrm{E}_r \in \{\beta_2, \beta_4\}$ and $\mathrm{e}_{2i+1}\,\mathrm{E}_r \in \{\beta_1, \beta_3\}$ for $i \in \mathbb{Z}/k\mathbb{Z}$.

Note that for a graph morphism $r : \mathrm{C}_{2k} \to G$ in $(\mathrm{C}_{2k}, G)_{2,2'}$ we have $\mathrm{v}_{2i}\,\mathrm{V}_r \in \{2, 2'\}$, $\mathrm{v}_{2i+1}\,\mathrm{V}_r \in \{1, 3, 3'\}$, $\mathrm{e}_{2i}\,\mathrm{E}_r \in \{\alpha_2, \alpha_3, \alpha_5, \alpha_7\}$ and $\mathrm{e}_{2i+1}\,\mathrm{E}_r \in \{\alpha_1, \alpha_4, \alpha_6\}$ for $i \in \mathbb{Z}/k\mathbb{Z}$.

We will introduce bijective maps $\gamma : (\mathrm{C}_{2k}, G)_{2,2'} \to \{0,1\}^{\mathbb{Z}/k\mathbb{Z}}$ and $\delta : (\mathrm{C}_{2k}, H)_2 \to \{0,1\}^{\mathbb{Z}/k\mathbb{Z}}$. In order to show that $(\mathrm{C}_{2k}, f)_2$ is bijective, we will show that $\gamma = (\mathrm{C}_{2k}, f)_2 \cdot \delta$.

We establish the bijection $\delta : (\mathrm{C}_{2k}, H)_2 \to \{0,1\}^{\mathbb{Z}/k\mathbb{Z}}$.

Let

$$\begin{array}{rrcl} \delta : & (\mathrm{C}_{2k}, H)_2 & \to & \{0,1\}^{\mathbb{Z}/k\mathbb{Z}} \\ & u & \mapsto & u\delta : i \mapsto \begin{cases} 0 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_u = 1 \\ 1 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_u = 3\,. \end{cases} \end{array}$$

Let

$$\begin{array}{rccc} \tilde{\delta} : & \{0,1\}^{\mathbb{Z}/k\mathbb{Z}} & \to & (\mathrm{C}_{2k}, H)_2 \\ & \varphi & \mapsto & \varphi\tilde{\delta} \end{array}$$

be defined as follows.

Because the graph H is thin it suffices to give $\mathrm{V}_{\varphi\tilde{\delta}}$; cf. Remark 77.(1).

Let

$$\begin{array}{rccl} \mathrm{V}_{\varphi\tilde{\delta}} : & \mathrm{V}_{\mathrm{C}_{2k}} & \to & \mathrm{V}_H = \{1, 2, 3\} \\ & \mathrm{v}_{2i} & \mapsto & 2, \text{ for } i \in \mathbb{Z}/k\mathbb{Z} \\ & \mathrm{v}_{2i+1} & \mapsto & \left\{ \begin{array}{ll} 1 & \text{if } i\varphi = 0 \\ 3 & \text{if } i\varphi = 1 \end{array} \right\} \text{ for } i \in \mathbb{Z}/k\mathbb{Z}\,. \end{array}$$

To show that the graph morphism $\varphi\tilde{\delta}$ is well-defined, we show that there is an edge in E_H from $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\delta}}$ to $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\delta}}$ for $j \in \mathbb{Z}/2k\mathbb{Z}$; cf. Remark 79.

We consider two cases:

Case 1: $j = 2i$ for some $i \in \mathbb{Z}/k\mathbb{Z}$.

Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2i}\,\mathrm{V}_{\varphi\tilde{\delta}} = 2$ and $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\delta}} \in \{1, 3\}$.

Subcase: $i\varphi = 0$. Then $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = 1$. And we have $\beta_4 \in H(2, 1)$.

Subcase: $i\varphi = 1$. Then $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = 3$. And we have $\beta_2 \in H(2, 3)$.

Case 2: $j = 2i + 1$ for some $i \in \mathbb{Z}/k\mathbb{Z}$.

Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\delta}} \in \{1, 3\}$ and $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2i+2}\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2(i+1)}\,\mathrm{V}_{\varphi\tilde{\delta}} = 2$.

Subcase: $i\varphi = 0$. Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = 1$. And we have $\beta_1 \in H(1, 2)$.

Subcase: $i\varphi = 1$. Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\delta}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = 3$. And we have $\beta_3 \in H(3, 2)$.

We have to show that $\delta \cdot \tilde{\delta} \overset{!}{=} \mathrm{id}_{(\mathrm{C}_{2k}, H)_2}$ and that $\tilde{\delta} \cdot \delta \overset{!}{=} \mathrm{id}_{\{0,1\}^{\mathbb{Z}/k\mathbb{Z}}}$.

Ad $\delta \cdot \tilde{\delta} \overset{!}{=} \mathrm{id}_{(\mathrm{C}_{2k}\,, H)_2}$. We have to show that $(u)\delta\tilde{\delta} \overset{!}{=} u$ for $u \in (\mathrm{C}_{2k}\,, H)_2$.

We have to show that $\mathrm{V}_{u\delta\tilde{\delta}} \overset{!}{=} \mathrm{V}_u$; cf. Remark 77.(1).

We have $\mathrm{v}_{2i}\,\mathrm{V}_{(u\delta)\tilde{\delta}} = 2 = \mathrm{v}_{2i}\,\mathrm{V}_u$ for $i \in \mathbb{Z}/k\mathbb{Z}$.

It remains to show that $\mathrm{v}_{2i+1}\,\mathrm{V}_{(u\delta)\tilde{\delta}} \overset{!}{=} \mathrm{v}_{2i+1}\,\mathrm{V}_u$ in $\{1, 3\}$ for $i \in \mathbb{Z}/k\mathbb{Z}$.

So we have to show that $1 \overset{!}{=} \mathrm{v}_{2i+1}\,\mathrm{V}_u$ if $i(u\delta) = 0$ and that $3 \overset{!}{=} \mathrm{v}_{2i+1}\,\mathrm{V}_u$ if $i(u\delta) = 1$. This holds by definition of δ.

Ad $\tilde{\delta} \cdot \delta \overset{!}{=} \mathrm{id}_{\{0,1\}^{\mathbb{Z}/k\mathbb{Z}}}$. We have to show that $\varphi\delta^{-1}\delta \overset{!}{=} \varphi$ for $\varphi \in \{0, 1\}^{\mathbb{Z}/k\mathbb{Z}}$. We obtain

$$i((\varphi\tilde{\delta})\delta) \overset{\text{Def. } \delta}{=} \left\{ \begin{array}{ll} 0 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = 1 \\ 1 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\delta}} = 3 \end{array} \right\} \overset{\text{Def. } \tilde{\delta}}{=} \left\{ \begin{array}{ll} 0 & \text{if } i\varphi = 0 \\ 1 & \text{if } i\varphi = 1 \end{array} \right\} = i\varphi$$

for $i \in \mathbb{Z}/k\mathbb{Z}$.

So we have $\tilde{\delta} = \delta^{-1}$.

We now establish the bijection $\gamma : (\mathrm{C}_{2k}\,, G)_{2,2'} \to \{0, 1\}^{\mathbb{Z}/k\mathbb{Z}}$.

Let

$$\begin{array}{rccl} \gamma : & (\mathrm{C}_{2k}\,, G)_{2,2'} & \to & \{0, 1\}^{\mathbb{Z}/k\mathbb{Z}} \\ & v & \mapsto & v\gamma : i \mapsto \left\{ \begin{array}{ll} 0 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_v = 1 \\ 1 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_v \in \{3, 3'\}. \end{array} \right. \end{array}$$

Let

$$\begin{array}{rccc} \tilde{\gamma} : & \{0, 1\}^{\mathbb{Z}/k\mathbb{Z}} & \to & (\mathrm{C}_{2k}\,, G)_{2,2'} \\ & \varphi & \mapsto & \varphi\tilde{\gamma} \end{array}$$

be defined as follows.

Because the graph G is thin it suffices to give $\mathrm{V}_{\varphi\tilde{\gamma}}$; cf. Remark 77.(1).

Let

$$\begin{array}{rccl} \mathrm{V}_{\varphi\tilde{\gamma}} : & \mathrm{V}_{\mathrm{C}_{2k}} & \to & \mathrm{V}_G = \{1, 2, 2', 3, 3'\} \\ & \mathrm{v}_{2i+1} & \mapsto & \left\{ \begin{array}{ll} 1 & \text{if } i\varphi = 0 \\ 3 & \text{if } i\varphi = 1 \text{ and } \mathrm{v}_{2i}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 2 \\ 3' & \text{if } i\varphi = 1 \text{ and } \mathrm{v}_{2i}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 2' \end{array} \right\} \text{ for } i \in \mathbb{Z}/k\mathbb{Z} \\ & \mathrm{v}_{2i+2} & \mapsto & \left\{ \begin{array}{ll} 2 & \text{if } i\varphi = 0 \\ 2' & \text{if } i\varphi = 1 \end{array} \right\} \text{ for } i \in \mathbb{Z}/k\mathbb{Z} \,. \end{array}$$

To show that the graph morphism $\varphi\tilde{\gamma}$ is well-defined, we show that there is an edge in E_G from $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\gamma}}$ to $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}}$ for $j \in \mathbb{Z}/2k\mathbb{Z}$; cf. Remark 79.

We consider two cases:

Case 1: $j = 2i + 2$ for some $i \in \mathbb{Z}/k\mathbb{Z}$.

Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+2}\,\mathrm{V}_{\varphi\tilde{\gamma}} \in \{2, 2'\}$ and $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+3}\,\mathrm{V}_{\varphi\tilde{\gamma}} \in \{1, 3, 3'\}$.

Subcase: $i\varphi = 0$. Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+2}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 2$.

Subsubcase: $(i + 1)\varphi = 0$. Then $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2(i+1)+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+3}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 1$. And we have $\alpha_2 \in G(2, 1)$.

Subsubcase: $(i + 1)\varphi = 1$. Then $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2(i+1)+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+3}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 3$. And we have $\alpha_3 \in G(2, 3)$.

Subcase: $i\varphi = 1$. Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+2}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 2'$.

Subsubcase: $(i + 1)\varphi = 0$. Then $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2(i+1)+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+3}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 1$. And we have $\alpha_7 \in G(2', 1)$.

Subsubcase: $(i+1)\varphi = 1$. Then $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2(i+1)+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+3}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 3'$. And we have $\alpha_5 \in G(2', 3')$.

Case 2: $j = 2i+1$ for some $i \in \mathbb{Z}/k\mathbb{Z}$.

Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} \in \{1, 3, 3'\}$ and $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+2}\,\mathrm{V}_{\varphi\tilde{\gamma}} \in \{2, 2'\}$.

Subcase: $i\varphi = 0$. Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 1$ and $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+2}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 2$. And we have $\alpha_1 \in G(1, 2)$.

Subcase: $i\varphi = 1$. Then $\mathrm{v}_j\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} \in \{3, 3'\}$ and $\mathrm{v}_{j+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = \mathrm{v}_{2i+2}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 2'$. And we have $\alpha_4 \in G(3, 2')$ and $\alpha_6 \in G(3', 2')$.

We have to show that $\gamma \cdot \tilde{\gamma} \overset{!}{=} \mathrm{id}_{(\mathrm{C}_{2k}\,, G)_{2,2'}}$ and that $\tilde{\gamma} \cdot \gamma \overset{!}{=} \mathrm{id}_{\{0,1\}^{\mathbb{Z}/k\mathbb{Z}}}$.

Ad $\gamma \cdot \tilde{\gamma} \overset{!}{=} \mathrm{id}_{(\mathrm{C}_{2k}\,, G)_{2,2'}}$. We have to show that $(v)\gamma\tilde{\gamma} \overset{!}{=} v$ for $v \in (\mathrm{C}_{2k}\,, G)_{2,2'}$.

We have to show that $\mathrm{V}_{v\gamma\tilde{\gamma}} \overset{!}{=} \mathrm{V}_v$; cf. Remark 77.(1).

We have to show that $\mathrm{v}_{2i+2}\,\mathrm{V}_{(v\gamma)\tilde{\gamma}} \overset{!}{=} \mathrm{v}_{2i+2}\,\mathrm{V}_v$ in $\{2, 2'\}$ for $i \in \mathbb{Z}/k\mathbb{Z}$.

So we have to show that $2 \overset{!}{=} \mathrm{v}_{2i+2}\,\mathrm{V}_v$ if $i(v\gamma) = 0$ and that $2' \overset{!}{=} \mathrm{v}_{2i+2}\,\mathrm{V}_f$ if $i(v\gamma) = 1$.

If $i(v\gamma) = 0$ then we have $\mathrm{v}_{2i+1}\,\mathrm{V}_v = 1$ by definition of γ and consequently $\mathrm{v}_{2i+2}\,\mathrm{V}_v = 2$.

If $i(v\gamma) = 1$ then we have $\mathrm{v}_{2i+1}\,\mathrm{V}_v \in \{3, 3'\}$ by definition of γ and consequently $\mathrm{v}_{2i+2}\,\mathrm{V}_v = 2'$. This holds by definition of γ.

We have to show that $\mathrm{v}_{2i+1}\,\mathrm{V}_{(v\gamma)\tilde{\gamma}} \overset{!}{=} \mathrm{v}_{2i+1}\,\mathrm{V}_v$ in $\{1, 3, 3'\}$ for $i \in \mathbb{Z}/k\mathbb{Z}$.

So we have to show that $1 \overset{!}{=} \mathrm{v}_{2i+1}\,\mathrm{V}_f$ if $i(f\gamma) = 0$ and that $3 \overset{!}{=} \mathrm{v}_{2i+1}\,\mathrm{V}_v$ if $i(v\gamma) = 1$ and $\mathrm{v}_{2i}\,\mathrm{V}_{(v\gamma)\tilde{\gamma}} = 2$ and that $3' \overset{!}{=} \mathrm{v}_{2i+1}\,\mathrm{V}_v$ if $i(v\gamma) = 1$ and $\mathrm{v}_{2i}\,\mathrm{V}_{(v\gamma)\tilde{\gamma}} = 2'$.

If $i(v\gamma) = 0$ then we have $\mathrm{v}_{2i+1}\,\mathrm{V}_v = 1$ by definition of γ.

If $i(v\gamma) = 1$ then, by definition of γ, we have $\mathrm{v}_{2i+1}\,\mathrm{V}_v \in \{3, 3'\}$.

If $i(v\gamma) = 1$ and $\mathrm{v}_{2i}\,\mathrm{V}_{(v\gamma)\tilde{\gamma}} = 2$ then, by definition of $\tilde{\gamma}$, we have $(i-1)(v\gamma) = 0$. Then, by definition of γ, we have $\mathrm{v}_{2(i-1)+1}\,\mathrm{V}_v = \mathrm{v}_{2i-1}\,\mathrm{V}_v = 1$. By the structure of G we obtain $\mathrm{v}_{2i+1}\,\mathrm{V}_v = 3$.

If $i(v\gamma) = 1$ and $\mathrm{v}_{2i}\,\mathrm{V}_{(v\gamma)\tilde{\gamma}} = 2'$ then, by definition of $\tilde{\gamma}$, we have $(i-1)(v\gamma) = 1$. Then, by definition of γ, we have $\mathrm{v}_{2(i-1)+1}\,\mathrm{V}_v = \mathrm{v}_{2i-1}\,\mathrm{V}_v \in \{3, 3'\}$. By the structure of G we obtain $\mathrm{v}_{2i+1}\,\mathrm{V}_v = 3'$.

Ad $\tilde{\gamma} \cdot \gamma \overset{!}{=} \mathrm{id}_{\{0,1\}^{\mathbb{Z}/k\mathbb{Z}}}$. We have to show that $\varphi\tilde{\gamma}\gamma \overset{!}{=} \varphi$ for $\varphi \in \{0,1\}^{\mathbb{Z}/k\mathbb{Z}}$. We obtain

$$i((\varphi\tilde{\gamma})\gamma) \overset{\text{Def. } \gamma}{=} \left\{ \begin{array}{ll} 0 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} = 1 \\ 1 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_{\varphi\tilde{\gamma}} \in \{3, 3'\} \end{array} \right\} \overset{\text{Def. } \tilde{\gamma}}{=} \left\{ \begin{array}{ll} 0 & \text{if } i\varphi = 0 \\ 1 & \text{if } i\varphi = 1 \end{array} \right\} = i\varphi$$

for $i \in \mathbb{Z}/k\mathbb{Z}$.

We have to show commutativity of the following diagram.

$$\begin{tikzcd}
(\mathrm{C}_{2k}\,, G)_{2,2'} \arrow[r, "\gamma"] \arrow[d, "(\mathrm{C}_{2k}\,, f)"'] & \{0,1\}^{\mathbb{Z}/k\mathbb{Z}} \arrow[d, "\mathrm{id}"] \\
(\mathrm{C}_{2k}\,, H)_2 \arrow[r, "\delta"'] & \{0,1\}^{\mathbb{Z}/k\mathbb{Z}}
\end{tikzcd}$$

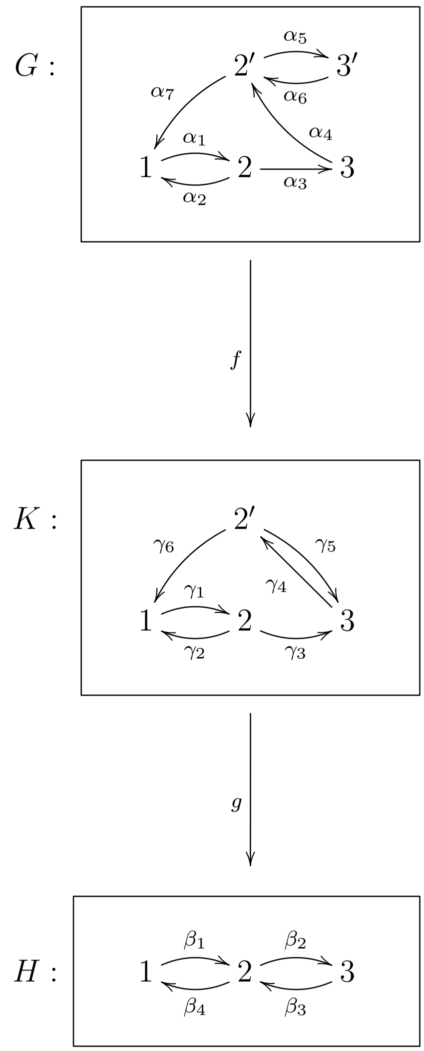
We have to show that $v\gamma \overset{!}{=} (v \cdot f)\delta$ for $v \in (\mathrm{C}_{2k}\,, G)_{2,2'}$.

For $i \in \mathbb{Z}/k\mathbb{Z}$, we have

$$i((v \cdot f)\delta) = \left\{ \begin{array}{ll} 0 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_{v \cdot f} = 1 \\ 1 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_{v \cdot f} = 3 \end{array} \right\} \overset{\text{pre-images}}{=} \left\{ \begin{array}{ll} 0 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_v = 1 \\ 1 & \text{if } \mathrm{v}_{2i+1}\,\mathrm{V}_v \in \{3, 3'\} \end{array} \right\} = i(v\gamma).$$

□

Example 216 We consider the following graph morphisms, where G and H are as in Example 215.



Here, the graph morphisms $f : G \to K$ and $g : K \to H$ map the vertices and the edges in a vertical way, where $2'\,\mathrm{V}_f = 2'$, $2\,\mathrm{V}_f = 2$.

We will verify that the graph morphisms f, g and fg are etale fibrations that satisfy (Uni).

This will show that f, g and fg are quasiisomorphisms; cf. Proposition 210.

Via Magma [**2**] we may proceed as follows, using the functions given in §10 below.

The graph G can be obtained with the function `trygraph` by setting `n := 3`.

The graph K can be obtained with the function `idtrygraph` by setting `n := 3`.

The graph H can be obtained with the function `c2chain` by setting `n := 3`.

The graph morphism f can be obtained with the function `tryfactorization` by setting `n := 3`.

The graph morphism g can be obtained with the function `idtryacyclic` by setting `n := 3`.

The graph morphism fg can be obtained with the function `tryacyclic` by setting `n := 3`.

So letting

```
G := trygraph(3);
K := idtrygraph(3);
H := c2chain(3);
f := tryfactorization(3);
g  := idtryacyclic(3);
fg := tryacyclic(3);
> G;
<[ 1, 2, 3, 4, 5 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 5>, <5, 5, 4>,
                      <3, 6, 4>, <4, 7, 1> ]>
> K;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 3>, <3, 5, 4>,
                   <4, 7, 1> ]>
> H;
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 2, 1>>, <<2, 3, 3>, <2, 3, 3>>, <<4, 4, 5>, <4, 4, 3>>,
 <<5, 5, 4>, <3, 5, 4>>, <<3, 6, 4>, <3, 5, 4>>, <<4, 7, 1>, <4, 7, 1>> ]>
> g;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 4, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 3>, <2, 2, 3>>,
 <<3, 5, 4>, <3, 3, 2>>, <<4, 7, 1>, <2, 4, 1>> ]>
> fg;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 4, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 5>, <2, 2, 3>>,
 <<5, 5, 4>, <3, 3, 2>>, <<3, 6, 4>, <3, 3, 2>>, <<4, 7, 1>, <2, 4, 1>> ]>
```

we get

```
> IsGraphMorphism(f,G,K);
true
> SuffCond(f,G,K);
<true, true>
> IsGraphMorphism(g,K,H);
true
> SuffCond(g,K,H);
<true, true>
> IsEqual(fg,ComposeGraphMorphisms(f,g));
true
> IsGraphMorphism(fg,G,H);
true
> SuffCond(fg,G,H);
<true, true>
```

Varying the input `n`, we obtain further quasiisomorphisms; cf. Examples 218 and 219 below. This seems to hold independently of $n \geqslant 3$.

Example 217 We consider the thin graph H from Example 215.

$$H : \quad \boxed{\begin{tikzcd} 1 \arrow[r, bend left, "\beta_1"] & 2 \arrow[l, bend left, "\beta_4"] \arrow[r, bend left, "\beta_2"] & 3 \arrow[l, bend left, "\beta_3"] \end{tikzcd}}$$

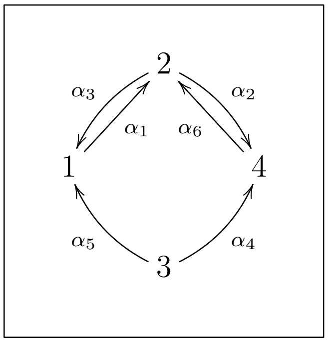
We want to find graphs G and graph morphisms $f : G \to H$ such that $\mathrm{V}_G = \{1, 2, 3, 4\}$ and $1\,\mathrm{V}_f := 1$, $2\,\mathrm{V}_f := 2$, $3\,\mathrm{V}_f := 2$ and $4\,\mathrm{V}_f := 3$ and such that f is an etale fibration that satisfies (Uni).

Via Magma we proceed as follows.

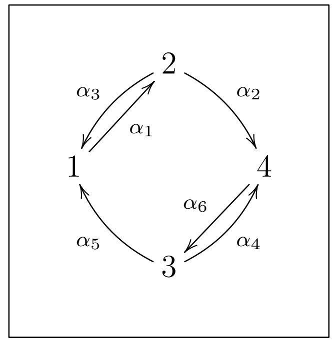
```
L := EFU(c2chain(3),[1,2,1]);
> L;
[
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
     <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
     <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 2>, <3, 3, 2>> ]>,
    <[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
     <4, 6, 2> ]>, <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>,
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ],[ <<1, 1, 2>, <1, 1, 2>>,
     <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
     <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 3>, <3, 3, 2>> ]>,
    <[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
     <4, 6, 3> ]>, <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>,
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ], [ <<1, 1, 3>, <1, 1, 2>>,
     <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
     <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 2>, <3, 3, 2>> ]>,
    <[ 1, 2, 3, 4 ], [ <1, 1, 3>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
     <4, 6, 2> ]>, <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>,
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ], [ <<1, 1, 3>, <1, 1, 2>>,
     <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
     <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 3>, <3, 3, 2>> ]>,
    <[ 1, 2, 3, 4 ], [ <1, 1, 3>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
     <4, 6, 3> ]>, <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>
]


> [x[2] : x in L];
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
                   <4, 6, 2> ]>
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
                   <4, 6, 3> ]>
<[ 1, 2, 3, 4 ], [ <1, 1, 3>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
                   <4, 6, 2> ]>
<[ 1, 2, 3, 4 ], [ <1, 1, 3>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>,
                   <4, 6, 3> ]>
```

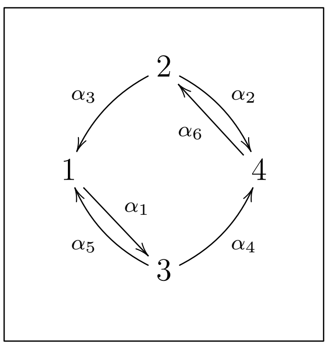
G_1 :



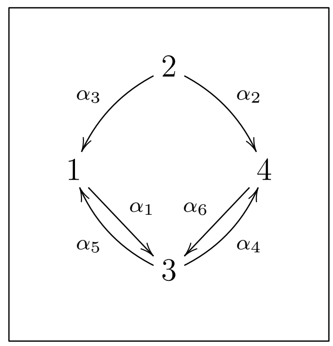
G_2 :



G_3 :



G_4 :

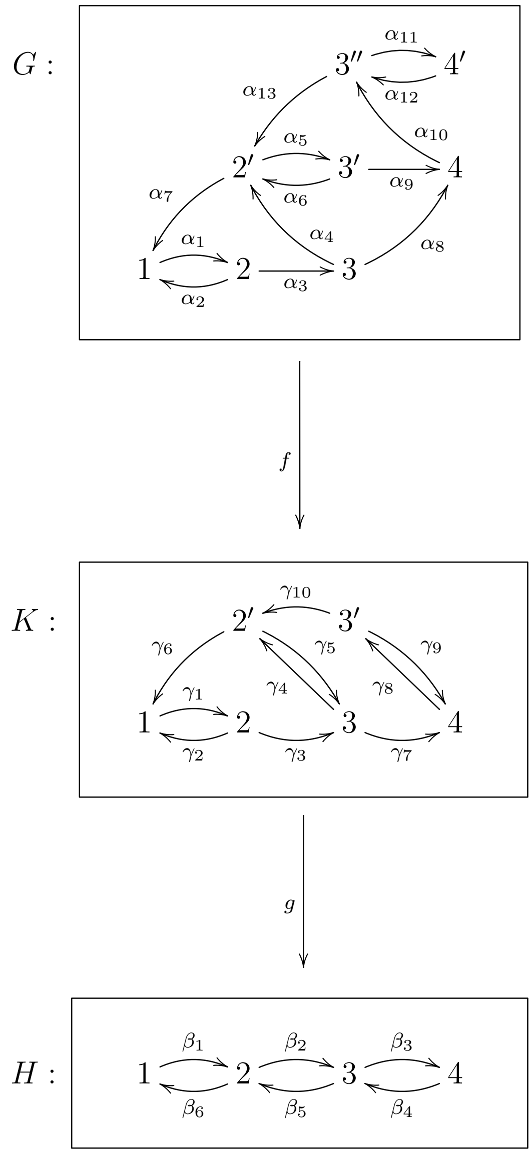


Note that $G_1 \simeq G_4$ and $G_2 \simeq G_3$, with isomorphisms respecting the morphisms to H.

Since our sufficient condition is satisfied the graph morphisms are quasiisomorphisms; cf. Proposition 210.

Cf. also $g : K \to H$ in Example 216, where K is isomorphic to G_3.

Example 218 We consider the following graph morphisms.



Here, the graph morphisms $f : G \to K$ and $g : K \to H$ map the vertices and the edges in a vertical way, where $2'\,\mathrm{V}_f = 2'$, $2\,\mathrm{V}_f = 2$, $3''\,\mathrm{V}_f = 3'$, $3'\,\mathrm{V}_f = 3$ and $3\,\mathrm{V}_f = 3$.

We will verify that the graph morphisms f, g and fg are etale fibrations that satisfy (Uni).

This will show that f, g and fg are quasiisomorphisms; cf. Proposition 210.

Via Magma we may proceed as follows, using the functions given in §10 below.

The graph G can be obtained with the function `trygraph` by setting `n := 4`.

The graph K can be obtained with the function `idtrygraph` by setting `n := 4`.

The graph H can be obtained with the function `c2chain` by setting `n := 4`.

The graph morphism f can be obtained with the function `tryfactorization` by setting `n := 4`.

The graph morphism g can be obtained with the function `idtryacyclic` by setting `n := 4`.

The graph morphism fg can be obtained with the function `tryacyclic` by setting `n := 4`.

So letting

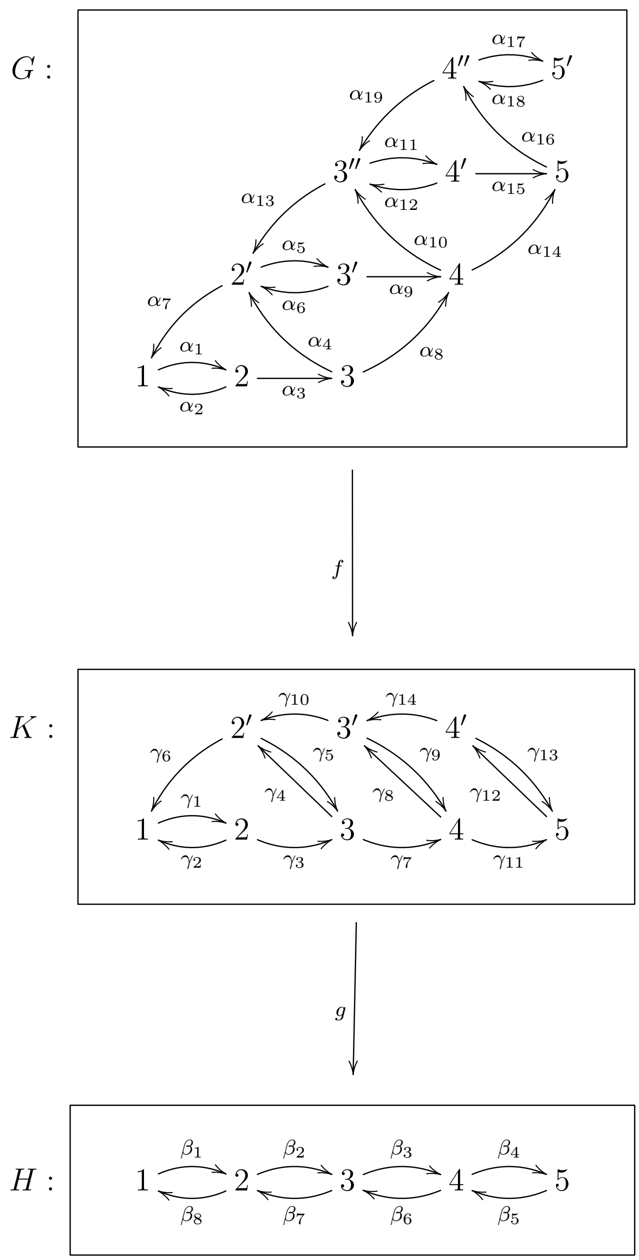
```
G := trygraph(4);
K := idtrygraph(4);
H := c2chain(4);
f := tryfactorization(4);
g  := idtryacyclic(4);
fg := tryacyclic(4);
> G;
<[ 1, 2, 3, 4, 5, 6, 7, 8 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 5>,
 <5, 5, 4>, <5, 6, 6>, <7, 7, 8>, <8, 8, 7>, <3, 9, 4>, <4, 10, 1>,
 <3, 11, 6>, <6, 12, 7>, <7, 13, 4> ]>
> K;
<[ 1, 2, 3, 4, 6, 7 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 3>,
 <3, 5, 4>, <3, 6, 6>, <7, 7, 6>, <6, 8, 7>, <4, 10, 1>, <7, 13, 4> ]>
> H;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 4>,
                   <4, 4, 3>, <3, 5, 2>, <2, 6, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 3>, <6, 6>, <7, 7>, <8, 6> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2, 1>>, <<2, 3, 3>, <2, 3, 3>>,
   <<4, 4, 5>, <4, 4, 3>>, <<5, 5, 4>, <3, 5, 4>>, <<5, 6, 6>, <3, 6, 6>>,
   <<7, 7, 8>, <7, 7, 6>>, <<8, 8, 7>, <6, 8, 7>>, <<3, 9, 4>, <3, 5, 4>>,
   <<4, 10, 1>, <4, 10, 1>>, <<3, 11, 6>, <3, 6, 6>>,
  <<6, 12, 7>, <6, 8, 7>>, <<7, 13, 4>, <7, 13, 4>> ]>
> g;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <6, 4>, <7, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 6, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 3>, <2, 2, 3>>,
 <<3, 5, 4>, <3, 5, 2>>, <<3, 6, 6>, <3, 3, 4>>, <<7, 7, 6>, <3, 3, 4>>,
 <<6, 8, 7>, <4, 4, 3>>, <<4, 10, 1>, <2, 6, 1>>, <<7, 13, 4>, <3, 5, 2>> ]>
> fg;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3>, <6, 4>, <7, 3>, <8, 4> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 6, 1>>, <<2, 3, 3>, <2, 2, 3>>,
   <<4, 4, 5>, <2, 2, 3>>, <<5, 5, 4>, <3, 5, 2>>, <<5, 6, 6>, <3, 3, 4>>,
   <<7, 7, 8>, <3, 3, 4>>, <<8, 8, 7>, <4, 4, 3>>, <<3, 9, 4>, <3, 5, 2>>,
   <<4, 10, 1>, <2, 6, 1>>, <<3, 11, 6>, <3, 3, 4>>, <<6, 12, 7>,
   <4, 4, 3>>, <<7, 13, 4>, <3, 5, 2>> ]>
```

we get

```
> IsGraphMorphism(f,G,K);
true
> SuffCond(f,G,K);
<true, true>
> IsGraphMorphism(g,K,H);
true
```

```
> SuffCond(g,K,H);
<true, true>
> IsEqual(fg,ComposeGraphMorphisms(f,g));
true
> IsGraphMorphism(fg,G,H);
true
> SuffCond(fg,G,H);
<true, true>
```

Example 219 We consider the following graph morphisms.



Here, the graph morphisms $f : G \to K$ and $g : K \to H$ map the vertices and the edges in a vertical way, where $2\,\mathrm{V}_f = 2$, $2'\,\mathrm{V}_f = 2'$, $3\,\mathrm{V}_f = 3$, $3'\,\mathrm{V}_f = 3$, $3''\,\mathrm{V}_f = 3'$, $4\,\mathrm{V}_f = 4$, $4'\,\mathrm{V}_f = 4$ and $4''\,\mathrm{V}_f = 4'$.

We will verify that the graph morphisms f, g and fg are etale fibrations that satisfy (Uni).

This will show that f, g and fg are quasiisomorphisms; cf. Proposition 210.

Via Magma we may proceed as follows, using the functions given in §10 below.

The graph G can be obtained with the function `trygraph` by setting `n := 5`.

The graph K can be obtained with the function `idtrygraph` by setting `n := 5`.

The graph H can be obtained with the function `c2chain` by setting `n := 5`.

The graph morphism f can be obtained with the function `tryfactorization` by setting `n := 5`.

The graph morphism g can be obtained with the function `idtryacyclic` by setting `n := 5`.

The graph morphism fg can be obtained with the function `tryacyclic` by setting `n := 5`.

So letting

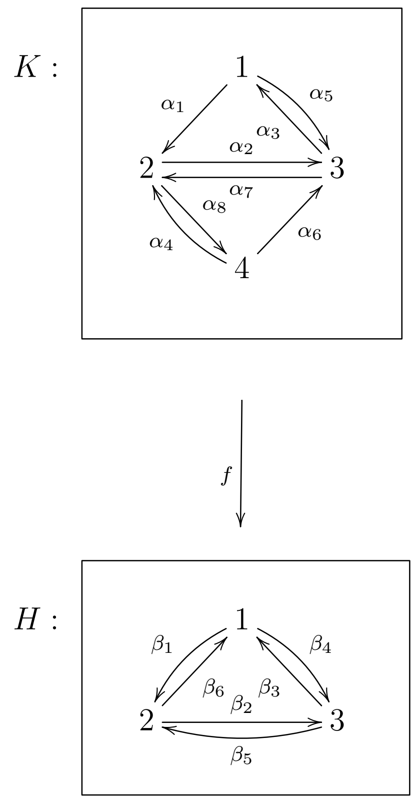
```
G := trygraph(5);
K := idtrygraph(5);
H := c2chain(5);
f := tryfactorization(5);
g  := idtryacyclic(5);
fg := tryacyclic(5);
> G;
<[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>,
 <4, 4, 5>, <5, 5, 4>, <5, 6, 6>, <7, 7, 8>, <8, 8, 7>, <8, 9, 9>,
 <10, 10, 11>, <11, 11, 10>, <3, 12, 4>, <4, 13, 1>, <3, 14, 6>,
 <6, 15, 7>, <7, 16, 4>, <6, 17, 9>, <9, 18, 10>, <10, 19, 7> ]>
> K;
<[ 1, 2, 3, 4, 6, 7, 9, 10 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 3>,
 <3, 5, 4>, <3, 6, 6>, <7, 7, 6>, <6, 8, 7>, <6, 9, 9>, <10, 10, 9>,
 <9, 11, 10>, <4, 13, 1>, <7, 16, 4>, <10, 19, 7> ]>
> H;
<[ 1, 2, 3, 4, 5 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 4>, <4, 4, 5>, <5, 5, 4>,
 <4, 6, 3>, <3, 7, 2>, <2, 8, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 3>, <6, 6>, <7, 7>, <8, 6>, <9, 9>,
 <10, 10>, <11, 9> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2, 1>>,
 <<2, 3, 3>, <2, 3, 3>>, <<4, 4, 5>, <4, 4, 3>>, <<5, 5, 4>, <3, 5, 4>>,
 <<5, 6, 6>, <3, 6, 6>>, <<7, 7, 8>, <7, 7, 6>>, <<8, 8, 7>, <6, 8, 7>>,
 <<8, 9, 9>, <6, 9, 9>>, <<10, 10, 11>, <10, 10, 9>>,
 <<11, 11, 10>, <9, 11, 10>>, <<3, 12, 4>, <3, 5, 4>>,
 <<4, 13, 1>, <4, 13, 1>>, <<3, 14, 6>, <3, 6, 6>>,
 <<6, 15, 7>, <6, 8, 7>>, <<7, 16, 4>, <7, 16, 4>>,
 <<6, 17, 9>, <6, 9, 9>>, <<9, 18, 10>, <9, 11, 10>>,
 <<10, 19, 7>, <10, 19, 7>> ]>
```

```
> g;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <6, 4>, <7, 3>, <9, 5>, <10, 4> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 8, 1>>, <<2, 3, 3>, <2, 2, 3>>,
   <<4, 4, 3>, <2, 2, 3>>, <<3, 5, 4>, <3, 7, 2>>, <<3, 6, 6>, <3, 3, 4>>,
   <<7, 7, 6>, <3, 3, 4>>, <<6, 8, 7>, <4, 6, 3>>, <<6, 9, 9>, <4, 4, 5>>,
   <<10, 10, 9>, <4, 4, 5>>, <<9, 11, 10>, <5, 5, 4>>,
   <<4, 13, 1>, <2, 8, 1>>, <<7, 16, 4>, <3, 7, 2>>,
   <<10, 19, 7>, <4, 6, 3>> ]>
> fg;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3>, <6, 4>, <7, 3>, <8, 4>, <9, 5>,
 <10, 4>, <11, 5> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 8, 1>>,
 <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 5>, <2, 2, 3>>, <<5, 5, 4>, <3, 7, 2>>,
 <<5, 6, 6>, <3, 3, 4>>, <<7, 7, 8>, <3, 3, 4>>, <<8, 8, 7>, <4, 6, 3>>,
 <<8, 9, 9>, <4, 4, 5>>, <<10, 10, 11>, <4, 4, 5>>,
 <<11, 11, 10>, <5, 5, 4>>, <<3, 12, 4>, <3, 7, 2>>,
 <<4, 13, 1>, <2, 8, 1>>, <<3, 14, 6>, <3, 3, 4>>,
 <<6, 15, 7>, <4, 6, 3>>, <<7, 16, 4>, <3, 7, 2>>,
 <<6, 17, 9>, <4, 4, 5>>, <<9, 18, 10>, <5, 5, 4>>,
 <<10, 19, 7>, <4, 6, 3>> ]>
```

we get

```
> IsGraphMorphism(f,G,K);
true
> SuffCond(f,G,K);
<true, true>
> IsGraphMorphism(g,K,H);
true
> SuffCond(g,K,H);
<true, true>
> IsEqual(fg,ComposeGraphMorphisms(f,g));
true
> IsGraphMorphism(fg,G,H);
true
> SuffCond(fg,G,H);
<true, true>
```

Example 220 We consider the following graph morphism.



Here, the graph morphism $f : K \to H$ maps the vertices and the edges in a vertical way, i.e. we let

$$1\,\mathrm{V}_f := 1, \quad 2\,\mathrm{V}_f := 2, \quad 3\,\mathrm{V}_f := 3, \quad 4\,\mathrm{V}_f := 1$$

and

$$\begin{aligned}
\alpha_1\,\mathrm{E}_f &:= \beta_1\ , \\
\alpha_2\,\mathrm{E}_f &:= \beta_2\ , \\
\alpha_3\,\mathrm{E}_f &:= \beta_3\ , \\
\alpha_4\,\mathrm{E}_f &:= \beta_1\ , \\
\alpha_5\,\mathrm{E}_f &:= \beta_4\ , \\
\alpha_6\,\mathrm{E}_f &:= \beta_4\ , \\
\alpha_7\,\mathrm{E}_f &:= \beta_5\ , \\
\alpha_8\,\mathrm{E}_f &:= \beta_6\ .
\end{aligned}$$

Then the graph morphism f is an acyclic fibration.

Via Magma we may proceed as follows, using the functions given in §10 below.

The graph K can be obtained with the function `idTrygraph` by setting `n := 3`.

The graph H can be obtained with the function `Doublecyclic` by setting `n := 3`.

The graph morphism f can be obtained with the function `idTryacyclic` by setting `n := 3`.

So letting

```
K := idTrygraph(3);
H := Doublecyclic(3);
f := idTryacyclic(3);
> K;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 3>, <3, 5, 4>,
 <4, 7, 1>, <1, 8, 3>, <3, 9, 1> ]>
> H;
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>,
                <2, 4, 1>, <1, 5, 3>, <3, 6, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 4, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 3>, <2, 2, 3>>,
 <<3, 5, 4>, <3, 3, 2>>, <<4, 7, 1>, <2, 4, 1>>, <<1, 8, 3>, <1, 5, 3>>,
 <<3, 9, 1>, <3, 6, 1>> ]>
```

we get

```
> IsGraphMorphism(f,K,H);
true
```

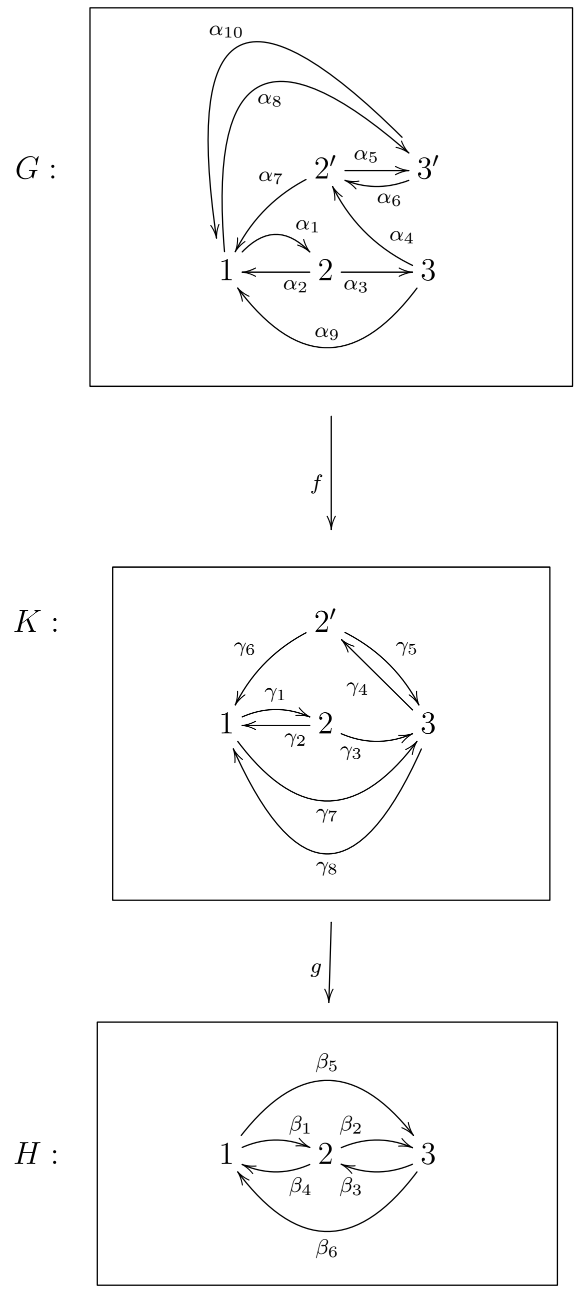
Every edge of H is unitargeting with respect to f.

We obtain that f is an acyclic fibration, in particular a quasiisomorphism, since it satisfies the sufficient condition of Proposition 210:

```
> SuffCond(f,K,H);
<true, true>
```

We extend Example 220 to a commutative trianlge consisting of three quasiisomorphisms in Example 221.

Example 221 We consider the following graph morphisms.



Here, the graph morphisms $f : G \to K$ and $g : K \to H$ map the vertices and the edges in a vertical way, where $2' \, \mathrm{V}_f = 2'$ and $2 \, \mathrm{V}_f = 2$.

We will verify that the graph morphisms f, g and fg are etale fibrations that satisfy (Uni).

This will show that f, g and fg are quasiisomorphisms; cf. Proposition 210.

Via Magma we may proceed as follows, using the functions given in §10 below.

The graph G can be obtained with the function `Trygraph` by setting `n := 3`.

The graph K can be obtained with the function `idTrygraph` by setting `n := 3`.

The graph H can be obtained with the function `Doublecyclic` by setting `n := 3`.

The graph morphism f can be obtained with the function `Tryfactorization` by setting `n := 3`.

The graph morphism g can be obtained with the function `idTryacyclic` by setting `n := 3`.

The graph morphism fg can be obtained with the function `Tryacyclic` by setting `n := 3`.

So letting

```
G := Trygraph(3);
K := idTrygraph(3);
H := Doublecyclic(3);
f := Tryfactorization(3);
g := idTryacyclic(3);
fg := Tryacyclic(3);
> G;
<[ 1, 2, 3, 4, 5 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 5>, <5, 5, 4>,
                      <3, 6, 4>, <4, 7, 1>, <1, 8, 5>, <5, 9, 1>, <3, 10, 1> ]>
> K;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 3>, <3, 5, 4>,
                   <4, 7, 1>, <1, 8, 3>, <3, 9, 1> ]>
> H;
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1>, <1, 5, 3>,
                <3, 6, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 2, 1>>, <<2, 3, 3>, <2, 3, 3>>, <<4, 4, 5>, <4, 4, 3>>,
 <<5, 5, 4>, <3, 5, 4>>, <<3, 6, 4>, <3, 5, 4>>, <<4, 7, 1>, <4, 7, 1>>,
 <<1, 8, 5>, <1, 8, 3>>, <<5, 9, 1>, <3, 9, 1>>, <<3, 10, 1>, <3, 9, 1>> ]>
> g;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 4, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 3>, <2, 2, 3>>,
 <<3, 5, 4>, <3, 3, 2>>, <<4, 7, 1>, <2, 4, 1>>, <<1, 8, 3>, <1, 5, 3>>,
 <<3, 9, 1>, <3, 6, 1>> ]>
> fg;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 4, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 5>, <2, 2, 3>>,
 <<5, 5, 4>, <3, 3, 2>>, <<3, 6, 4>, <3, 3, 2>>, <<4, 7, 1>, <2, 4, 1>>,
 <<1, 8, 5>, <1, 5, 3>>, <<5, 9, 1>, <3, 6, 1>>, <<3, 10, 1>, <3, 6, 1>> ]>
```

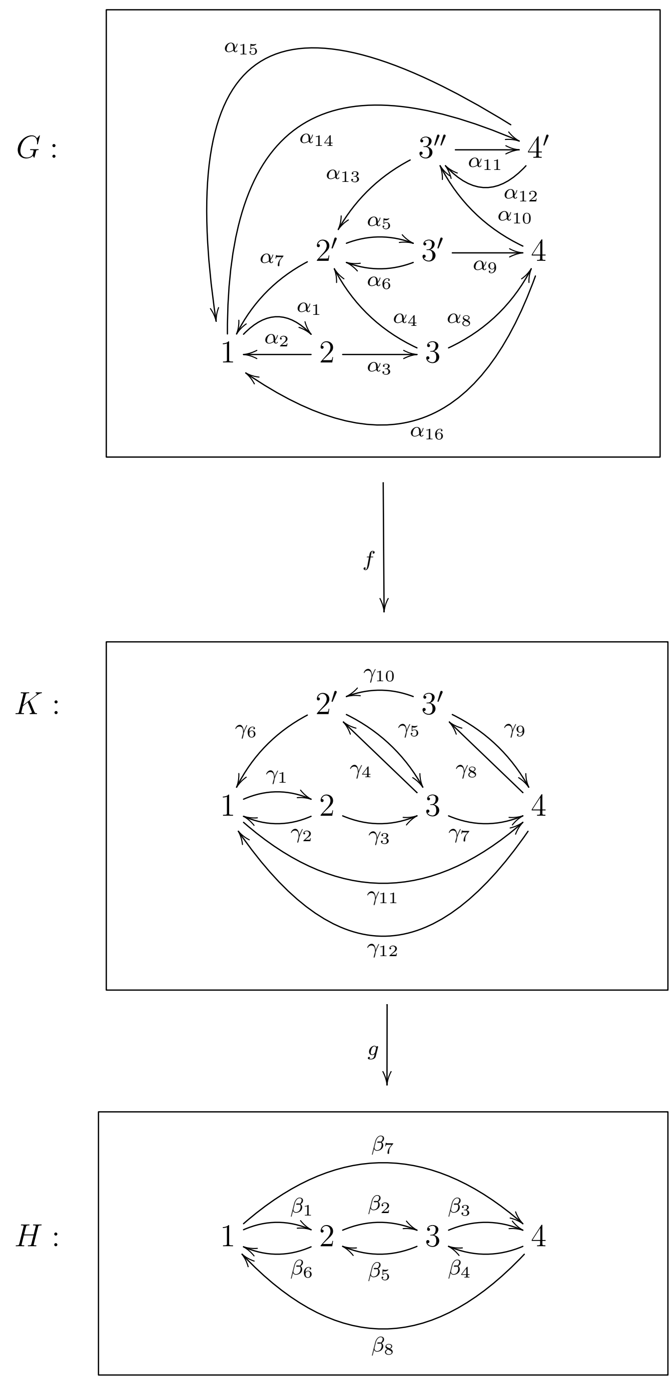
we get

```
> IsGraphMorphism(f,G,K);
true
> SuffCond(f,G,K);
<true, true>
> IsGraphMorphism(g,K,H);
true
> SuffCond(g,K,H);
<true, true>
```

```
> IsEqual(fg,ComposeGraphMorphisms(f,g));
true
> IsGraphMorphism(fg,G,H);
true
> SuffCond(fg,G,H);
<true, true>
```

Varying the input `n`, we obtain further quasiisomorphisms; cf. Examples 222 and 223 below. This seems to hold independently of $n \geqslant 3$.

Example 222 We consider the following graph morphisms.



Here, the graph morphisms $f : G \to K$ and $g : K \to H$ map the vertices and the edges in a vertical way, where $2\,\mathrm{V}_f = 2$, $2'\,\mathrm{V}_f = 2'$, $3\,\mathrm{V}_f = 3$, $3'\,\mathrm{V}_f = 3$ and $3''\,\mathrm{V}_f = 3'$.

We will verify that the graph morphisms f, g and fg are etale fibrations that satisfy (Uni).

This will show that f, g and fg are quasiisomorphisms; cf. Proposition 210.

Via Magma we may proceed as follows, using the functions given in §10 below.

The graph G can be obtained with the function `Trygraph` by setting `n := 4`.

The graph K can be obtained with the function `idTrygraph` by setting `n := 4`.

The graph H can be obtained with the function `Doublecyclic` by setting `n := 4`.

The graph morphism f can be obtained with the function `Tryfactorization` by setting `n := 4`.

The graph morphism g can be obtained with the function `idTryacyclic` by setting `n := 4`.

The graph morphism fg can be obtained with the function `Tryacyclic` by setting `n := 4`.

So letting

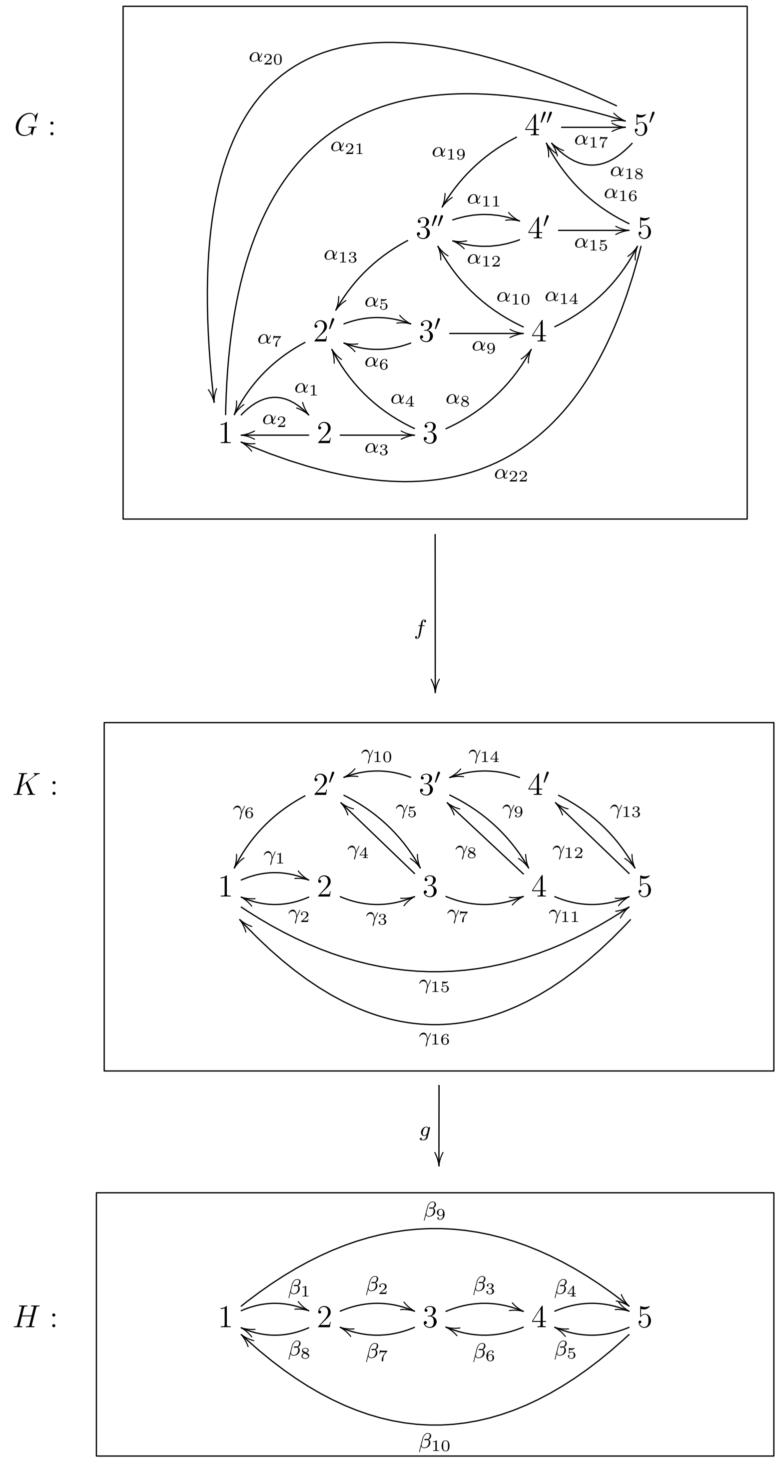
```
G := Trygraph(4);
K := idTrygraph(4);
H := Doublecyclic(4);
f := Tryfactorization(4);
g := idTryacyclic(4);
fg := Tryacyclic(4);
> G;
<[ 1, 2, 3, 4, 5, 6, 7, 8 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 5>,
 <5, 5, 4>, <5, 6, 6>, <7, 7, 8>, <8, 8, 7>, <3, 9, 4>, <4, 10, 1>, <3, 11, 6>,
 <6, 12, 7>, <7, 13, 4>, <1, 14, 8>, <8, 15, 1>, <6, 16, 1> ]>
> K;
<[ 1, 2, 3, 4, 6, 7 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 3>, <3, 5, 4>,
 <3, 6, 6>, <7, 7, 6>, <6, 8, 7>, <4, 10, 1>, <7, 13, 4>, <1, 14, 6>, <6, 15, 1> ]>
> H;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 4>, <4, 4, 3>,
                   <3, 5, 2>, <2, 6, 1>, <1, 7, 4>, <4, 8, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 3>, <6, 6>, <7, 7>, <8, 6> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2, 1>>, <<2, 3, 3>, <2, 3, 3>>,
 <<4, 4, 5>, <4, 4, 3>>, <<5, 5, 4>, <3, 5, 4>>, <<5, 6, 6>, <3, 6, 6>>,
 <<7, 7, 8>, <7, 7, 6>>, <<8, 8, 7>, <6, 8, 7>>, <<3, 9, 4>, <3, 5, 4>>,
 <<4, 10, 1>, <4, 10, 1>>, <<3, 11, 6>, <3, 6, 6>>, <<6, 12, 7>, <6, 8, 7>>,
 <<7, 13, 4>, <7, 13, 4>>, <<1, 14, 8>, <1, 14, 6>>, <<8, 15, 1>, <6, 15, 1>>,
 <<6, 16, 1>, <6, 15, 1>> ]>
```

```
> g;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <6, 4>, <7, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 6, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 3>, <2, 2, 3>>,
 <<3, 5, 4>, <3, 5, 2>>, <<3, 6, 6>, <3, 3, 4>>, <<7, 7, 6>, <3, 3, 4>>,
 <<6, 8, 7>, <4, 4, 3>>, <<4, 10, 1>, <2, 6, 1>>, <<7, 13, 4>, <3, 5, 2>>,
 <<1, 14, 6>, <1, 7, 4>>, <<6, 15, 1>, <4, 8, 1>> ]>
> fg;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3>, <6, 4>, <7, 3>, <8, 4> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 6, 1>>, <<2, 3, 3>, <2, 2, 3>>,
 <<4, 4, 5>, <2, 2, 3>>, <<5, 5, 4>, <3, 5, 2>>, <<5, 6, 6>, <3, 3, 4>>,
 <<7, 7, 8>, <3, 3, 4>>, <<8, 8, 7>, <4, 4, 3>>, <<3, 9, 4>, <3, 5, 2>>,
 <<4, 10, 1>, <2, 6, 1>>, <<3, 11, 6>, <3, 3, 4>>, <<6, 12, 7>, <4, 4, 3>>,
 <<7, 13, 4>, <3, 5, 2>>, <<1, 14, 8>, <1, 7, 4>>, <<8, 15, 1>, <4, 8, 1>>,
 <<6, 16, 1>, <4, 8, 1>> ]>
```

we get

```
> IsGraphMorphism(f,G,K);
true
> SuffCond(f,G,K);
<true, true>
> IsGraphMorphism(g,K,H);
true
> SuffCond(g,K,H);
<true, true>
> IsEqual(fg,ComposeGraphMorphisms(f,g));
true
> IsGraphMorphism(fg,G,H);
true
> SuffCond(fg,G,H);
<true, true>
```

Example 223 We consider the following graph morphisms.



Here, the graph morphisms $f : G \to K$ and $g : K \to H$ map the vertices and the edges in a vertical way, where $2\,\mathrm{V}_f = 2$, $2'\,\mathrm{V}_f = 2'$, $3\,\mathrm{V}_f = 3$, $3'\,\mathrm{V}_f = 3$, $3''\,\mathrm{V}_f = 3'$, $4\,\mathrm{V}_f = 4$, $4'\,\mathrm{V}_f = 4$ and $4''\,\mathrm{V}_f = 4'$.

We will verify that the graph morphisms f, g and fg are etale fibrations that satisfy (Uni).

This will show that f, g and fg are quasiisomorphisms; cf. Proposition 210.

Via Magma we may proceed as follows, using the functions given in §10 below.

The graph G can be obtained with the function `Trygraph` by setting `n := 5`.

The graph K can be obtained with the function `idTrygraph` by setting `n := 5`.

The graph H can be obtained with the function `Doublecyclic` by setting `n := 5`.

The graph morphism f can be obtained with the function `Tryfactorization` by setting `n := 5`.

The graph morphism g can be obtained with the function `idTryacyclic` by setting `n := 5`.

The graph morphism fg can be obtained with the function `Tryacyclic` by setting `n := 5`.

So letting

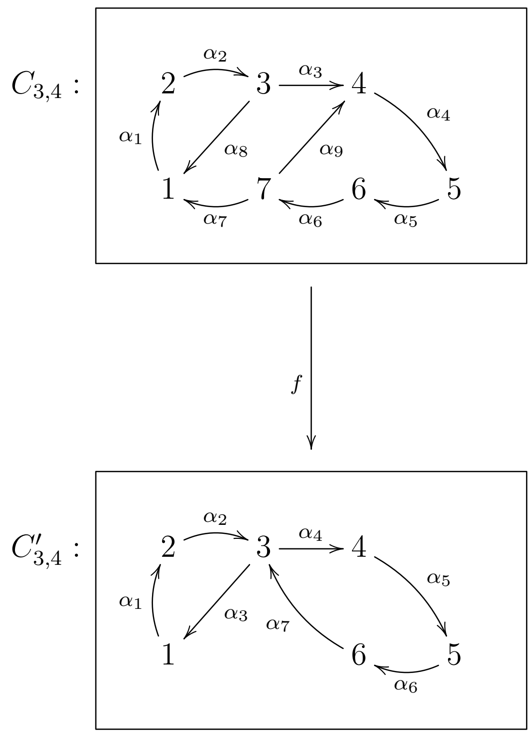
```
G := Trygraph(5);
K := idTrygraph(5);
H := Doublecyclic(5);
f := Tryfactorization(5);
g := idTryacyclic(5);
fg := Tryacyclic(5);
> G;
<[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>,
 <4, 4, 5>, <5, 5, 4>, <5, 6, 6>, <7, 7, 8>, <8, 8, 7>, <8, 9, 9>, <10, 10, 11>,
 <11, 11, 10>, <3, 12, 4>, <4, 13, 1>, <3, 14, 6>, <6, 15, 7>, <7, 16, 4>,
 <6, 17, 9>, <9, 18, 10>, <10, 19, 7>, <1, 20, 11>, <11, 21, 1>, <9, 22, 1> ]>
> K;
<[ 1, 2, 3, 4, 6, 7, 9, 10 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 3>,
 <3, 5, 4>, <3, 6, 6>, <7, 7, 6>, <6, 8, 7>, <6, 9, 9>, <10, 10, 9>,
 <9, 11, 10>, <4, 13, 1>, <7, 16, 4>, <10, 19, 7>, <1, 20, 9>, <9, 21, 1> ]>
> H;
<[ 1, 2, 3, 4, 5 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 4>, <4, 4, 5>, <5, 5, 4>,
                      <4, 6, 3>, <3, 7, 2>, <2, 8, 1>, <1, 9, 5>, <5, 10, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 3>, <6, 6>, <7, 7>, <8, 6>, <9, 9>,
 <10, 10>, <11, 9> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2, 1>>,
 <<2, 3, 3>, <2, 3, 3>>, <<4, 4, 5>, <4, 4, 3>>, <<5, 5, 4>, <3, 5, 4>>,
 <<5, 6, 6>, <3, 6, 6>>, <<7, 7, 8>, <7, 7, 6>>, <<8, 8, 7>, <6, 8, 7>>,
 <<8, 9, 9>, <6, 9, 9>>, <<10, 10, 11>, <10, 10, 9>>,
 <<11, 11, 10>, <9, 11, 10>>, <<3, 12, 4>, <3, 5, 4>>,
 <<4, 13, 1>, <4, 13, 1>>, <<3, 14, 6>, <3, 6, 6>>,
 <<6, 15, 7>, <6, 8, 7>>, <<7, 16, 4>, <7, 16, 4>>,
 <<6, 17, 9>, <6, 9, 9>>, <<9, 18, 10>, <9, 11, 10>>,
 <<10, 19, 7>, <10, 19, 7>>, <<1, 20, 11>, <1, 20, 9>>,
 <<11, 21, 1>, <9, 21, 1>>, <<9, 22, 1>, <9, 21, 1>> ]>
```

```
> g;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <6, 4>, <7, 3>, <9, 5>, <10, 4> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 8, 1>>, <<2, 3, 3>, <2, 2, 3>>,
 <<4, 4, 3>, <2, 2, 3>>, <<3, 5, 4>, <3, 7, 2>>, <<3, 6, 6>, <3, 3, 4>>,
 <<7, 7, 6>, <3, 3, 4>>, <<6, 8, 7>, <4, 6, 3>>, <<6, 9, 9>, <4, 4, 5>>,
 <<10, 10, 9>, <4, 4, 5>>, <<9, 11, 10>, <5, 5, 4>>,
 <<4, 13, 1>, <2, 8, 1>>, <<7, 16, 4>, <3, 7, 2>>,
 <<10, 19, 7>, <4, 6, 3>>, <<1, 20, 9>, <1, 9, 5>>,
 <<9, 21, 1>, <5, 10, 1>> ]>
> fg;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3>, <6, 4>, <7, 3>, <8, 4>, <9, 5>,
 <10, 4>, <11, 5> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 8, 1>>,
 <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 5>, <2, 2, 3>>, <<5, 5, 4>, <3, 7, 2>>,
 <<5, 6, 6>, <3, 3, 4>>, <<7, 7, 8>, <3, 3, 4>>, <<8, 8, 7>, <4, 6, 3>>,
 <<8, 9, 9>, <4, 4, 5>>, <<10, 10, 11>, <4, 4, 5>>, <<11, 11, 10>, <5, 5, 4>>,
 <<3, 12, 4>, <3, 7, 2>>, <<4, 13, 1>, <2, 8, 1>>,
 <<3, 14, 6>, <3, 3, 4>>, <<6, 15, 7>, <4, 6, 3>>,
 <<7, 16, 4>, <3, 7, 2>>, <<6, 17, 9>, <4, 4, 5>>,
 <<9, 18, 10>, <5, 5, 4>>, <<10, 19, 7>, <4, 6, 3>>,
 <<1, 20, 11>, <1, 9, 5>>, <<11, 21, 1>, <5, 10, 1>>,
 <<9, 22, 1>, <5, 10, 1>> ]>
```

we get

```
> IsGraphMorphism(f,G,K);
true
> SuffCond(f,G,K);
<true, true>
> IsGraphMorphism(g,K,H);
true
> SuffCond(g,K,H);
<true, true>
> IsEqual(fg,ComposeGraphMorphisms(f,g));
true
> IsGraphMorphism(fg,G,H);
true
> SuffCond(fg,G,H);
<true, true>
```

Example 224 We consider the following graph morphism.



Here, the graph morphism $f : C_{3,4} \to C'_{3,4}$ maps the vertices and the edges in a vertical way, where $1\,\mathrm{V}_f = 1$, $2\,\mathrm{V}_f = 2$, $4\,\mathrm{V}_f = 4$ and $6\,\mathrm{V}_f = 6$.

We will verify that the graph morphism f is an etale fibration that satisfies (Uni).

This will show that f is a quasiisomorphism; cf. Proposition 210.

Via Magma we may proceed as follows, using the functions given in §10 below.

The graph $C_{3,4}$ can be obtained with the function `CnCm` by setting `n := 3` and `m := 4`.

The graph $C'_{3,4}$ can be obtained with the function `cncm` by setting `n := 3` and `m := 4`.

The graph morphism f can be obtained with the function `cncmqis` by setting `n := 3` and `m := 4`.

So letting

```
C34 := CnCm(3,4);
Cp34 := cncm(3,4); // "C prime" = C’
f := cncmqis(3,4);
> C34;
<[ 1, 2, 3, 4, 5, 6, 7 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 4>, <4, 4, 5>,
 <5, 5, 6>, <6, 6, 7>, <7, 7, 1>, <3, 8, 1>, <7, 9, 4> ]>
> Cp34;
<[ 1, 2, 3, 4, 5, 6 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 1>, <3, 4, 4>, <4, 5, 5>,
<5, 6, 6>, <6, 7, 3> ]>
```

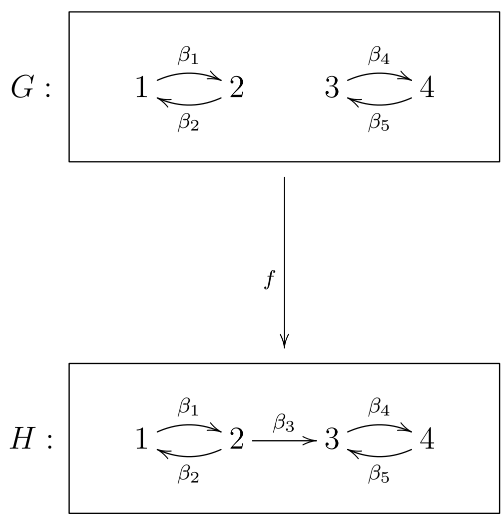
```
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 5>, <6, 6>, <7, 3> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 3>, <2, 2, 3>>, <<3, 3, 4>, <3, 4, 4>>,
 <<4, 4, 5>, <4, 5, 5>>, <<5, 5, 6>, <5, 6, 6>>, <<6, 6, 7>, <6, 7, 3>>,
 <<7, 7, 1>, <3, 3, 1>>, <<3, 8, 1>, <3, 3, 1>>, <<7, 9, 4>, <3, 4, 4>> ]>
```

we get

```
> IsGraphMorphism(f,C34,Cp34);
true
> SuffCond(f,C34,Cp34);
<true, true>
```

Varying the input `n` and `m`, it is possible to construct further quasiisomorphisms. This seems to hold independently of $n \geqslant 2$.

Example 225 We consider the following inclusion morphism f of the subgraph G into the graph H.



Then the graph morphism f is a quasiisomorphism.

Via Magma, we calculated that the graph morphism f satisfies (Uni) as follows.

```
G := <[1,2,3,4],[<1,1,2>,<2,2,1>,<3,4,4>,<4,5,3>]>;
H := <[1,2,3,4],[<1,1,2>,<2,2,1>,<2,3,3>,<3,4,4>,<4,5,3>]>;
f := IsSubgraph(G,H)[1];
> IsGraphMorphism(f,G,H);
true
> Uni(f,G,H);
true
> IsEtaleFibration(f,G,H);
false
```

But we can not apply Proposition 210 because f is not an etale fibration.

Proof. Suppose given the following diagram in Gph.

$$\begin{tikzcd} & G \arrow[d, "f"] \\ \mathrm{C}_n \arrow[r, "p"] & H \end{tikzcd}$$

We have to show that there exists a unique graph morphism $q : \mathrm{C}_n \to G$ such that $qf = p$.

Uniqueness. The graph morphism $q : \mathrm{C}_n \to G$ is unique since the graph morphism $f : G \to H$ is injective, whence $(C_n\,, f)_{\mathrm{Gph}}$ is injective.

Existence. In order to be able to let $q := p|^G$, we have to show that $\mathrm{C}_n\, p \subseteq G$. I.e. we have to show that $\beta_3 \notin \mathrm{E}_{\mathrm{C}_n}\, \mathrm{E}_p$.

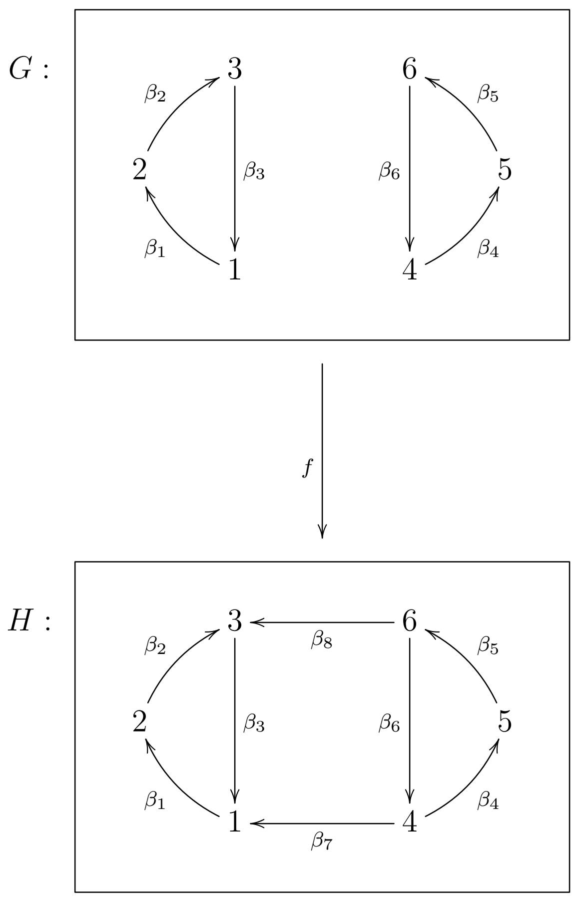
We *assume* that $\beta_3 \in \mathrm{E}_{\mathrm{C}_n}\, \mathrm{E}_p$.

So there exists an edge $\mathrm{e}_j \in \mathrm{E}_{\mathrm{C}_n}$ such that $\mathrm{e}_j\, \mathrm{E}_p = \beta_3$.

Thus $\mathrm{e}_{j+1}\, \mathrm{E}_p = \beta_4$. Thus $\mathrm{e}_{j+2}\, \mathrm{E}_p = \beta_5$. Thus $\mathrm{e}_{j+3}\, \mathrm{E}_p = \beta_4$. Etc.

We deduce that $\mathrm{e}_i\, \mathrm{E}_p \in \{\beta_4\,, \beta_5\}$ for $i \in \mathbb{Z}/n\mathbb{Z}$, *contradicting* $\mathrm{e}_j\, \mathrm{E}_p = \beta_3$. ▫

Example 226 We consider the following inclusion morphism f of the subgraph G into the graph H.



Then the graph morphism f is a quasiisomorphism.

Both G and H are fibrant.

The graph morphism f is not a fibration.

Via Magma we calculated that the graph morphism f satisfies (Uni), using the functions given in §10 below.

But we can not apply Proposition 210 because f is not an etale fibration.

With the function `CNCN` we calculated the graphs `G` and `H` and the graph morphism `f`.

So letting

```
G := CNCN(3)[1];
H := CNCN(3)[2];
f := CNCN(3)[3];
```

we get

```
> IsSubgraph(G,H)[2];
true
> IsGraphMorphism(f,G,H);
true
> Uni(f,G,H);
true
> IsEtaleFibration(f,G,H);
false
```

Proof. Suppose given the following diagram in Gph.

$$\begin{tikzcd} & G \arrow[d, "f"] \\ \mathrm{C}_n \arrow[r, "p"] & H \end{tikzcd}$$

We have to show that there exists a unique graph morphism $q : \mathrm{C}_n \to G$ such that $qf = p$.

Uniqueness. The graph morphism $q : \mathrm{C}_n \to G$ is unique since the graph morphism $f : G \to H$ is injective, whence $(C_n\,, f)_{\mathrm{Gph}}$ is injective

Existence. In order to be able to let $q := p|^G$, we have to show that $\mathrm{C}_n\, p \subseteq G$. I.e. we have to show that $\beta_7\,, \beta_8 \notin \mathrm{E}_{\mathrm{C}_n}\,\mathrm{E}_p\,$.

We show that $\beta_7 \notin \mathrm{C}_n\, p\,$, the proof for β_8 being similar.

We *assume* that $\beta_7 \in \mathrm{E}_{\mathrm{C}_n}\,\mathrm{E}_p\,$.

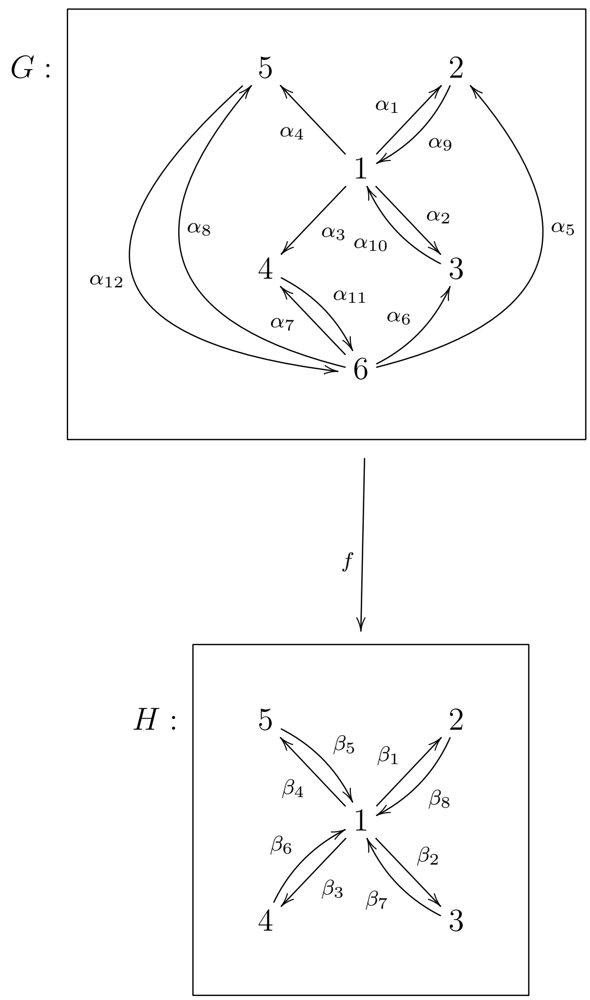
So there exists an edge $\mathrm{e}_j \in \mathrm{E}_{\mathrm{C}_n}$ such that $\mathrm{e}_j\,\mathrm{E}_p = \beta_7$.

Thus $\mathrm{e}_{j+1}\,\mathrm{E}_p = \beta_1\,$. Thus $\mathrm{e}_{j+2}\,\mathrm{E}_p = \beta_2\,$. Thus $\mathrm{e}_{j+3}\,\mathrm{E}_p = \beta_3\,$. Thus $\mathrm{e}_{j+4}\,\mathrm{E}_p = \beta_1\,$. Etc.

We deduce that $\mathrm{e}_i\,\mathrm{E}_p \in \{\beta_1\,, \beta_2\,, \beta_3\}$ for $i \in \mathbb{Z}/n\mathbb{Z}$, *contradicting* $\mathrm{e}_j\,\mathrm{E}_p = \beta_7\,$. □

Varying the input `n`, it is possible to construct further quasiisomorphisms.

Example 227 We consider the following graphs.



We consider the graph morphism $f : G \to H$ with

$$
\begin{aligned}
&1\,\mathrm{V}_f := 1, \quad 2\,\mathrm{V}_f := 2, \quad 3\,\mathrm{V}_f := 3, \\
&4\,\mathrm{V}_f := 4, \quad 5\,\mathrm{V}_f := 5, \quad 6\,\mathrm{V}_f := 1;
\end{aligned}
$$

and with the corresponding map on the edges.

The graph morphism $f : G \to H$ is an acyclic fibration.

Proof. Via Magma we will calculate that the graph morphism f is an etale fibration that satisfies (Uni), using the functions given in §10 below.

This will show that f is a quasiisomorphism; cf. Proposition 210.

The graph morphism `f` together with the graphs `G` and `H` can be obtained with Magma with the function `Exflower` by setting `n := 4` and `list := [2,3]`.

So letting

```
G := Exflower(4,[2,3])[1];
H := Exflower(4,[2,3])[2];
f := Exflower(4,[2,3])[3];
> G;
<[ 1, 2, 3, 4, 5, 6 ], [ <1, 1, 2>, <1, 2, 3>, <1, 3, 4>, <1, 4, 5>, <6, 5, 2>,
 <6, 6, 3>, <6, 7, 4>, <6, 8, 5>, <2, 9, 1>, <3, 10, 1>, <4, 11, 6>,
 <5, 12, 6> ]>
> H;
<[ 1, 2, 3, 4, 5 ], [ <1, 1, 2>, <1, 2, 3>, <1, 3, 4>, <1, 4, 5>, <5, 5, 1>,
 <4, 6, 1>, <3, 7, 1>, <2, 8, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 5>, <6, 1> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<1, 2, 3>, <1, 2, 3>>, <<1, 3, 4>, <1, 3, 4>>, <<1, 4, 5>, <1, 4, 5>>,
 <<6, 5, 2>, <1, 1, 2>>, <<6, 6, 3>, <1, 2, 3>>, <<6, 7, 4>, <1, 3, 4>>,
 <<6, 8, 5>, <1, 4, 5>>, <<2, 9, 1>, <2, 8, 1>>, <<3, 10, 1>, <3, 7, 1>>,
 <<4, 11, 6>, <4, 6, 1>>, <<5, 12, 6>, <5, 5, 1>> ]>
```

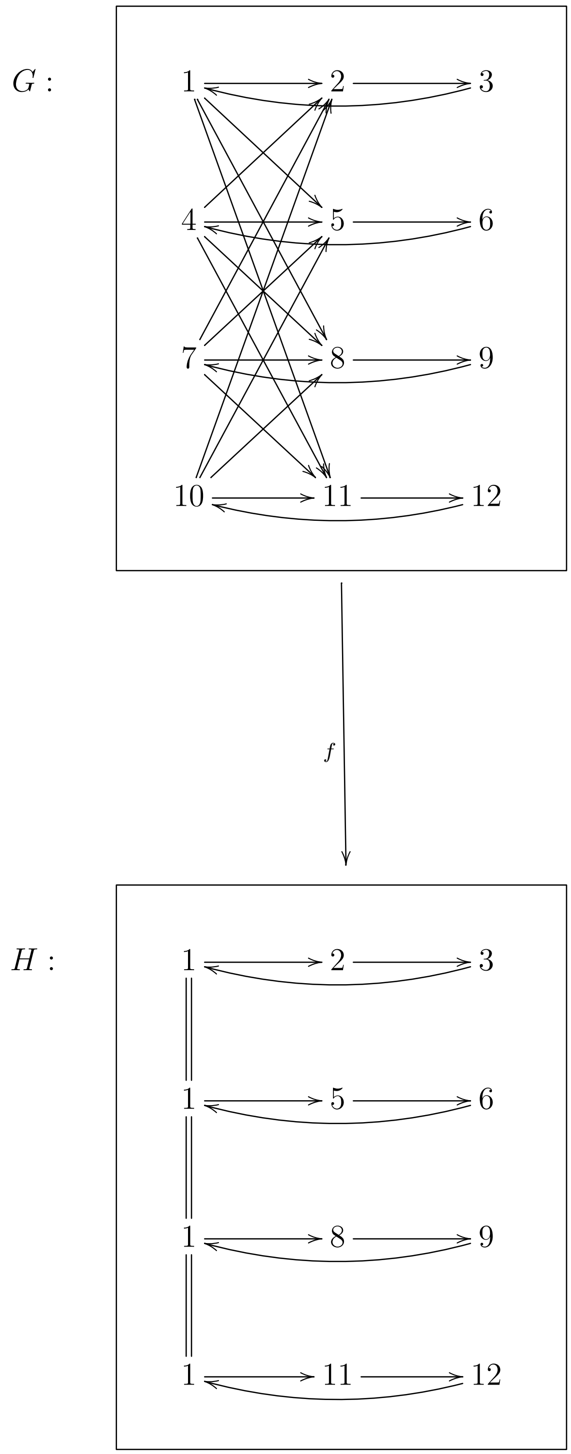
we get

```
> IsGraphMorphism(f,G,H);
true
> SuffCond(f,G,H);
<true, true>
```

By varying the input `n` and `list` it is possible to construct further quasiisomorphisms using `Exflower(n,list)`.

The "lower" graph consists of `4` respectively `n` cyclic graphs C_2 glued together at one vertex such that the form of a flower is visible.

Example 228 We consider the graph morphism



where we let

$$\begin{array}{llllll} 1\,\mathrm{V}_f := 1 & 2\,\mathrm{V}_f := 2 & 3\,\mathrm{V}_f := 3 & 4\,\mathrm{V}_f := 1 & 5\,\mathrm{V}_f := 5 & 6\,\mathrm{V}_f := 6 \\ 7\,\mathrm{V}_f := 1 & 8\,\mathrm{V}_f := 8 & 9\,\mathrm{V}_f := 9 & 10\,\mathrm{V}_f := 1 & 11\,\mathrm{V}_f := 11 & 12\,\mathrm{V}_f := 12\ . \end{array}$$

Here, the graph H has only a single vertex named 1, displayed four times for sake of clarity.

The graph morphism $f : G \to H$ is an acyclic fibration.

Proof. Via Magma we will calculate that the graph morphism f is an etale fibration that satisfies (Uni), using the functions given in §10 below.

This will show that f is a quasiisomorphism; cf. Proposition 210.

The graph morphism `f` together with the graphs `G` and `H` can be obtained with Magma with the function `Exflower2` by setting `n := 4` and `k := 3`.

So letting

```
G := Exflower2(4,3)[1];
H := Exflower2(4,3)[2];
f := Exflower2(4,3)[3];
> G;
<[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 1>,
 <4, 4, 5>, <5, 5, 6>, <6, 6, 4>, <7, 7, 8>, <8, 8, 9>, <9, 9, 7>,
 <10, 10, 11>, <11, 11, 12>, <12, 12, 10>, <1, 13, 5>, <1, 14, 8>, <1, 15, 11>,
 <4, 16, 2>, <4, 17, 8>, <4, 18, 11>, <7, 19, 2>, <7, 20, 5>, <7, 21, 11>,
 <10, 22, 2>, <10, 23, 5>, <10, 24, 8> ]>
> H;
<[ 1, 2, 3, 4, 5, 6, 7, 8, 9 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 1>, <1, 4, 4>,
 <4, 5, 5>, <5, 6, 1>, <1, 7, 6>, <6, 8, 7>, <7, 9, 1>, <1, 10, 8>, <8, 11, 9>,
 <9, 12, 1> ]>
> f;
<[ <1, 1>, <4, 1>, <7, 1>, <10, 1>, <2, 2>, <3, 3>, <5, 4>, <6, 5>, <8, 6>,
 <9, 7>, <11, 8>, <12, 9> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 3>, <2, 2, 3>>,
 <<3, 3, 1>, <3, 3, 1>>, <<4, 4, 5>, <1, 4, 4>>, <<5, 5, 6>, <4, 5, 5>>,
 <<6, 6, 4>, <5, 6, 1>>, <<7, 7, 8>, <1, 7, 6>>, <<8, 8, 9>, <6, 8, 7>>,
 <<9, 9, 7>, <7, 9, 1>>, <<10, 10, 11>, <1, 10, 8>>, <<11, 11, 12>, <8, 11, 9>>,
 <<12, 12, 10>, <9, 12, 1>>, <<1, 13, 5>, <1, 4, 4>>, <<1, 14, 8>, <1, 7, 6>>,
 <<1, 15, 11>, <1, 10, 8>>, <<4, 16, 2>, <1, 1, 2>>, <<4, 17, 8>, <1, 7, 6>>,
 <<4, 18, 11>, <1, 10, 8>>, <<7, 19, 2>, <1, 1, 2>>, <<7, 20, 5>, <1, 4, 4>>,
 <<7, 21, 11>, <1, 10, 8>>, <<10, 22, 2>, <1, 1, 2>>, <<10, 23, 5>, <1, 4, 4>>,
 <<10, 24, 8>, <1, 7, 6>> ]>
```

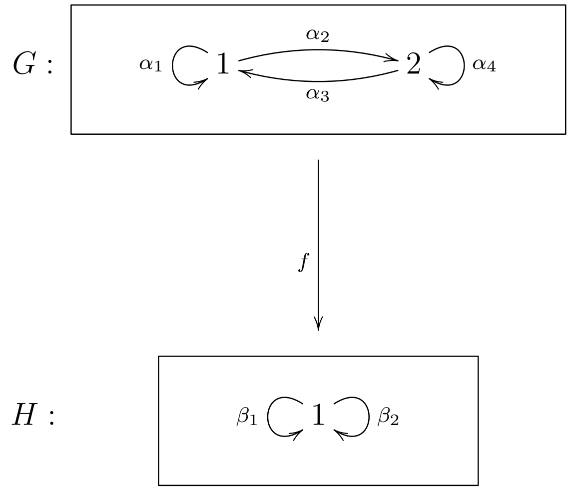
we get

```
> SuffCond(f,G,H);
<true, true>
```

Note that we need to have edges from the vertices $1, 4, 7$ and 10 to the vertices $2, 5, 8$ and 11 to obtain an etale fibration.

Via Magma, we calculated that we obtain quasiisomorphisms with `Exflower2` for input values `n` in $[1, 30]$ and `k` in $[2, 30]$.

Example 229 We consider the graph morphism



where

$$
\begin{aligned}
1\,\mathrm{V}_f &= 1, & 2\,\mathrm{V}_f &= 1,\\
\alpha_1\,\mathrm{E}_f &= \beta_1\,, & \alpha_3\,\mathrm{E}_f &= \beta_1\,,\\
\alpha_2\,\mathrm{E}_f &= \beta_2\,, & \alpha_4\,\mathrm{E}_f &= \beta_2\,.
\end{aligned}
$$

Via Magma we will calculate that the graph morphism f is an etale fibrations that satisfies (Uni), using the functions given in §10 below.

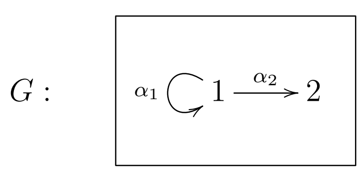
Letting

```
G := <[1,2],[<1,1,1>,<1,2,2>,<2,3,1>,<2,4,2>]>;
H := <[1],[<1,1,1>,<1,2,1>]>;
f := <[<1,1>,<2,1>],[<<1,1,1>,<1,1,1>>,<<1,2,2>,<1,2,1>>,<<2,3,1>,<1,1,1>>,
      <<2,4,2>,<1,2,1>>]>;
```

we get

```
> SuffCond(f,G,H);
<true, true>
```

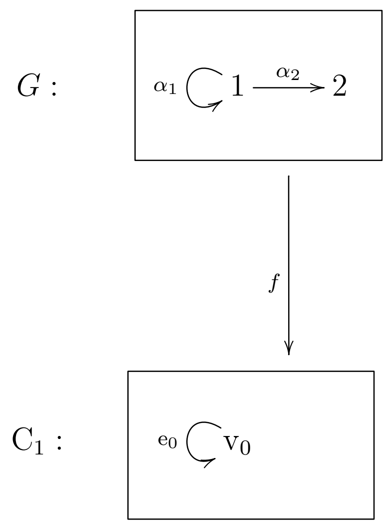
Example 230 We consider the graph



and the cyclic graph C_1 ; cf. Definition 52.

We consider the graph morphism $f : G \to \mathrm{C}_1$ with

$$
\begin{aligned}
1\,\mathrm{V}_f &= \mathrm{v}_0\,, & \alpha_1\,\mathrm{E}_f &= \mathrm{e}_0\,,\\
2\,\mathrm{V}_f &= \mathrm{v}_0\,, & \alpha_2\,\mathrm{E}_f &= \mathrm{e}_0\;.
\end{aligned}
$$



We will calculate that the graph G and the cyclic graph C_1 are thin and that the graph morphism $f : G \to \mathrm{C}_1$ does **not** satisfy (Uni) and f is **not** a fibration; cf. Definitions 52 and 127.(1) and Proposition 210. But the graph morphism $f : G \to \mathrm{C}_1$ is a quasiisomorphism; cf. Definition 115.

Via Magma we may proceed as follows, using the functions given in §10 below.

```
G := <[1,2],[<1,1,1>,<1,2,2>]>;
> IsThin(G);
true
C1 := C(1);
> IsThin(C1);
true
f := ListGraphMorphisms(G,C1)[1];
> f;
<[ <1, 1>, <2, 1> ], [ <<1, 1, 1>, <1, 1, 1>>, <<1, 2, 2>, <1, 1, 1>> ]>
> Uni(f,G,C1);
false
> IsFibration(f,G,C1);
false
```

9.2 Some examples of graph morphisms related to the sufficient condition of Proposition 210

Example 231 We consider the graph morphism $f : \emptyset \to \mathrm{C}_1$.

Via Magma, we calculated that the empty graph $\emptyset$ and the cyclic graph $\mathrm{C}_1 =: H$ are thin and that the graph morphism $f : \emptyset \to \mathrm{C}_1$ is an etale fibration; cf. Definitions 70, 52 and 127.(2). But the graph morphism f does **not** satisfy (Uni); cf. Proposition 210. Moreover, f is **not** a quasiisomorphism, since $(\mathrm{C}_1\,, \emptyset)_{\mathrm{Gph}} = \emptyset \neq (\mathrm{C}_1\,, \mathrm{C}_1)_{\mathrm{Gph}}$.

```
G := <[],[]>;
H := C(1);
f := <[],[]>;
> IsThin(G);
true
> IsThin(H);
true
> IsEtaleFibration(f,G,H);
true
> Uni(f,G,H);
false
> IsQis_Bound(f,G,H,1);
false
```

Example 232 We consider the graph morphism $f : \mathrm{D}_1 \to \mathrm{C}_1$.

Via Magma, we calculated that the direct graph D_1 and the cyclic graph C_1 are thin and that the graph morphism $f : \mathrm{D}_1 \to \mathrm{C}_1$ satisfies (Uni); cf. Definitions 56 and 52 and Proposition 210. But the graph morphism f is **not** a fibration; cf. Definition 127.(1). Moreover, f is **not** a quasiisomorphism, since $(\mathrm{C}_1\,,\mathrm{D}_1)_{\mathrm{Gph}} = \emptyset \neq (\mathrm{C}_1\,,\mathrm{C}_1)_{\mathrm{Gph}}$.

```
G := D(1);
> IsThin(G);
true
H := C(1);
> IsThin(H);
true
> f := ListGraphMorphisms(G,H)[1];
> f;
<[ <0, 1>, <1, 1> ], [ <<0, 0, 1>, <1, 1, 1>> ]>
> Uni(f,G,H);
true
> IsFibration(f,G,H);
false
```

Example 233 We consider the graphs

$$G : \quad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_4"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\alpha_3"] \end{tikzcd}$$

and C_2 .

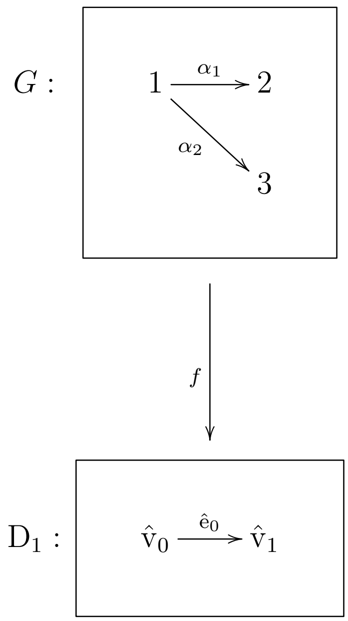
Let $f : G \to \mathrm{C}_2$ be the graph morphism with

$$\begin{array}{llll} 1\,\mathrm{V}_f = \mathrm{v}_1\,, & 2\,\mathrm{V}_f = \mathrm{v}_2\,, & 3\,\mathrm{V}_f = \mathrm{v}_1\,, & \\ \alpha_1\,\mathrm{E}_f = \mathrm{e}_1\,, & \alpha_2\,\mathrm{E}_f = \mathrm{e}_2\,, & \alpha_3\,\mathrm{E}_f = \mathrm{e}_1\,, & \alpha_4\,\mathrm{E}_f = \mathrm{e}_2\ . \end{array}$$

Via Magma, we calculated that the graph G and the cyclic graph C_2 are thin and that the graph morphism $f : G \to \mathrm{C}_2$ satisfies (Uni); cf. Definition 52 and Proposition 210. Moreover, f is a fibration; cf. Definition 127.(1). But the graph morphism f is **not** an etale fibration; cf. Definition 127.(2). And f is **not** a quasiisomorphism since $|(\mathrm{C}_2\,, G)_{\mathrm{Gph}}| = 4 \neq 2 = |(\mathrm{C}_2\,, \mathrm{C}_2)_{\mathrm{Gph}}|$; cf. Definition 115.

```
G := c2chain(3);
> G;
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>
> IsThin(G);
true
H := C(2);
> H;
<[ 1, 2 ], [ <1, 1, 2>, <2, 2, 1> ]>
> IsThin(H);
true
> f := ListGraphMorphisms(G,H)[2];
> f;
<[ <1, 1>, <2, 2>, <3, 1> ],
  [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 3>, <2, 2, 1>>,
    <<3, 3, 2>, <1, 1, 2>>, <<2, 4, 1>, <2, 2, 1>> ]>
> Uni(f,G,H);
true
> IsFibration(f,G,H);
true
> IsEtaleFibration(f,G,H);
false
> IsQis_Bound(f,G,H,2);
false
```

Example 234 We consider the following graph morphism, mapping the vertices and the edges in a vertical way.



Via Magma, we calculated that the graph morphism $f : G \to \mathrm{D}_1$ satisfies (Uni) and that f is a fibration; cf. Proposition 210 and Definition 127.(1). But the graph morphism f is not an etale fibration; cf. Definition 127.(2). However, f is a quasiisomorphism, since $(\mathrm{C}_n\,, G)_{\mathrm{Gph}} = \emptyset = (\mathrm{C}_n\,, \mathrm{D}_1)_{\mathrm{Gph}}$ for $n \in \mathbb{N}$.

```
G := <[0,1,2],[<0,1,1>,<0,2,2>]>;
H := D(1);
f := ListGraphMorphisms(G,H)[1];
> f;
<[ <0, 0>, <1, 1>, <2, 1> ], [ <<0, 1, 1>, <0, 0, 1>>, <<0, 2, 2>, <0, 0, 1>> ]>
> IsFibration(f,G,H);
true
> Uni(f,G,H);
true
> IsEtaleFibration(f,G,H);
false
```

So the converse to Proposition 210 does not hold, even for fibrations satisfying (Uni).

Example 235 Suppose given $n \in \mathbb{N}$.

We consider the graph morphism $f := \tau_{\mathrm{C}_1 \sqcup \mathrm{C}_n} : \mathrm{C}_1 \sqcup \mathrm{C}_n \to \mathrm{C}_1$.

The graph morphism f is an etale fibration. The edge e_0 of the target C_1 is not unitargeting. In fact, the edges mapping to e_0 have $n+1 \geqslant 2$ distinct targets.

The graph morphism f can be calculated via Magma with the function `exampleforbadbound2` given in §10.8.

E.g., setting `n := 3`, we get

```
> exampleforbadbound2(3);
<<[ 1, 2, 3, 4 ], [ <1, 1, 1>, <2, 2, 3>, <3, 3, 4>, <4, 4, 2> ]>,
 <[ 1 ], [ <1, 1, 1> ]>, <[ <1, 1>, <2, 1>, <3, 1>, <4, 1> ],
  [ <<1, 1, 1>, <1, 1, 1>>, <<2, 2, 3>, <1, 1, 1>>, <<3, 3, 4>,
    <1, 1, 1>>, <<4, 4, 2>, <1, 1, 1>> ]>>
```

The map $(\mathrm{C}_k\,, f)_{\mathrm{Gph}}$ is bijective for $1 \leqslant k < n$ since there are unique graph morphisms $\mathrm{C}_k \to \mathrm{C}_1 \sqcup \mathrm{C}_n$ and $\mathrm{C}_k \to \mathrm{C}_1$.

But the map $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$ is not bijective since we may have $p, q : \mathrm{C}_n \to \mathrm{C}_1 \sqcup \mathrm{C}_n$ with $\mathrm{C}_n\, p = \mathrm{C}_1$ and $\mathrm{C}_n\, q = \mathrm{C}_n$.

In particular, f is not a quasiisomorphism.

E.g., we get

```
f := exampleforbadbound2(3)[3];
G := exampleforbadbound2(3)[1];
> IsQis_Bound(f,G,C(1),2);
true
> IsQis_Bound(f,G,C(1),3);
false
```

and

```
f := exampleforbadbound2(10)[3];
G := exampleforbadbound2(10)[1];
> IsQis_Bound(f,G,C(1),9);
true
> IsQis_Bound(f,G,C(1),10);
false
```

9.3 Some inequalities of subsets of $\mathrm{Mor}(\mathrm{Gph})$

We consider subsets of $\mathrm{Mor}(\mathrm{Gph})$.

Remark 236 We have $\mathrm{AcFib} \subsetneq \mathrm{Fib}$.

Proof. We consider the graph morphism $\iota_1 : \mathrm{D}_0 \to \mathrm{D}_0 \sqcup \mathrm{C}_1$.

Note that the map $\mathrm{E}_{\iota_1\,,\hat{\mathrm{v}}_0} : \emptyset \to \emptyset$ is surjective. Thus the graph morphism $\iota_1 : \mathrm{D}_0 \to \mathrm{D}_0 \sqcup \mathrm{C}_1$ is a fibration; cf. Definition 127.(1). But ι_1 is not a quasiisomorphism; cf. Definitions 115 and 127.(1).

So $\iota_1 \in \mathrm{Fib} \setminus \mathrm{AcFib}$. $\square$

Remark 237 We have $\mathrm{AcFib} \subsetneq \mathrm{Qis}$.

Proof. Let

$$P : \quad \begin{tikzcd} 1 \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & 2 \\ & 3 \end{tikzcd}$$

We consider the acyclic cofibration $g : \mathrm{D}_1 \to P$ mapping $\hat{\mathrm{e}}_1$ to α_2 ; cf. Assertion 254 below.

The graph morphism $g : \mathrm{D}_1 \to P$ is an acyclic cofibration and thus a quasiisomorphism; cf. Lemma 175. But g is not a fibration and thus not an acyclic fibration; cf. Definitions 115, 127.(1), 138.

```
> P := <[1,2,3],[<1,1,2>,<1,2,3>]>;
> g := VtoE(D(1),P,[<0,1>,<1,3>]);
> IsAcCofib(g,D(1),P);
true
```

So $g \in \mathrm{Qis} \setminus \mathrm{AcFib}$. ▫

Remark 238 We have $\mathrm{AcCofib} \subsetneq \mathrm{Cofib}$.

Proof. We consider the cofibration $\iota_{\mathrm{C}_1} : \emptyset \to \mathrm{C}_1$; cf. Remark 151.

The graph morphism $\iota_{\mathrm{C}_1} : \emptyset \to \mathrm{C}_1$ is not a quasiisomorphism; cf. Definition 115.

So $\iota_{\mathrm{C}_1} \in \mathrm{Cofib} \setminus \mathrm{AcCofib}$. ▫

Remark 239 We have $\mathrm{AcCofib} \subsetneq \mathrm{Qis}$.

Proof. We consider the acyclic fibration $\iota_{\mathrm{D}_0} : \emptyset \to \mathrm{D}_0$; cf. Example 158.(1).

The graph morphism $\iota_{\mathrm{D}_0} : \emptyset \to \mathrm{D}_0$ is an acyclic fibration and thus a quasiisomorphism.

But ι_{D_0} is not a cofibration; cf. Example 158.(2). Thus it is not an acyclic cofibration; cf. Lemma 175.

Alternatively, the graph morphism $\iota_{\mathrm{D}_0} : \emptyset \to \mathrm{D}_0$ is not an acyclic cofibration because it does not satisfy (AcCofib 3) since $\mathrm{V}_{\mathrm{D}_0} \setminus \mathrm{V}_\emptyset \neq \emptyset$, but $\mathrm{E}_{\mathrm{D}_0} = \emptyset$.

So $\iota_{\mathrm{D}_0} \in \mathrm{Qis} \setminus \mathrm{AcCofib}$. ▫

Remark 240 We have ${}^{\boxslash}\mathrm{Qis} \subsetneq {}^{\boxslash}\mathrm{AcFib} \overset{\text{Def. 144}}{=} \mathrm{Cofib}$.

In fact, we have $\mathrm{AcCofib} \setminus {}^{\boxslash}\mathrm{Qis} \neq \emptyset$.

Proof. We consider the graph morphism $\iota_{0,1} : \mathrm{D}_0 \to \mathrm{D}_1$; cf. Definition 56. It is an acyclic cofibration; cf. Remark 166. Thus it is a cofibration and a quasiisomorphism; cf. Lemma 175.

We consider the following commutative quadrangle.

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "\mathrm{id}_{\mathrm{D}_0}"] \arrow[d, "\iota_{0,1}"'] & \mathrm{D}_0 \arrow[d, "\wr\, \iota_{0,1}"] \\ \mathrm{D}_1 \arrow[r, "\mathrm{id}_{\mathrm{D}_1}"'] & \mathrm{D}_1 \end{tikzcd}$$

Since $\mathrm{E}_{\mathrm{D}_1} = \{\hat{\mathrm{e}}_0\}$ and $\mathrm{E}_{\mathrm{D}_0} = \emptyset$, there does not exist a graph morphism $c : \mathrm{D}_1 \to \mathrm{D}_0$.

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "\mathrm{id}_{\mathrm{D}_0}"] \arrow[d, "\iota_{0,1}"'] & \mathrm{D}_0 \arrow[d, "\wr\, \iota_{0,1}"] \\ \mathrm{D}_1 \arrow[r, "\mathrm{id}_{\mathrm{D}_1}"'] \arrow[ur, dotted, "\nexists"] & \mathrm{D}_1 \end{tikzcd}$$

So the graph morphism $\iota_{0,1} : \mathrm{D}_0 \to \mathrm{D}_1$ is in $\mathrm{AcCofib} \setminus {}^{\boxslash}\mathrm{Qis}$. $\square$

Note that we also have ${}^{\boxslash}\mathrm{Fib} \subsetneq {}^{\boxslash}\mathrm{AcFib}$ since

$$ {}^{\boxslash}\mathrm{Fib} \overset{\text{Lemma 191}}{=} \mathrm{AcCofib} \overset{\text{Remark 238}}{\subsetneq} \mathrm{Cofib} \overset{\text{Definition 144}}{=} {}^{\boxslash}\mathrm{AcFib} \,. $$

Remark 241 We have ${}^{\boxslash}\mathrm{Qis} \setminus \mathrm{AcCofib} \overset{\text{Lemma 191}}{=} {}^{\boxslash}\mathrm{Qis} \setminus {}^{\boxslash}\mathrm{Fib} \neq \emptyset$.

Proof. We consider the graph morphism $f := \iota_{\mathrm{C}_2} : \emptyset \to \mathrm{C}_2$.

First, we show that f is in ${}^{\boxslash}\mathrm{Qis}$.

Suppose given a commutative diagram as follows.

$$\begin{tikzcd} \emptyset \arrow[r, "p"] \arrow[d, "f"'] & G \arrow[d, "\wr\, g"] \\ \mathrm{C}_2 \arrow[r, "q"'] & H \end{tikzcd}$$

Since the graph morphism $g : G \to H$ is a quasiisomorphism, there exists a graph morphism $h : \mathrm{C}_2 \to G$ such that $hg = q$.

Thus we have the following commutative diagram.

$$\begin{tikzcd} \emptyset \arrow[r, "p"] \arrow[d, "f"'] & G \arrow[d, "\wr\, g"] \\ \mathrm{C}_2 \arrow[r, "q"'] \arrow[ur, "h"] & H \end{tikzcd}$$

So the graph morphism $f : \emptyset \to \mathrm{C}_2$ is in ${}^{\boxslash}\mathrm{Qis}$.

In particular, f is a cofibration; cf. Remark 195.(3) and Definition 138, or Remark 151.

Secondly, we show that f is not in ${}^{\boxslash}\mathrm{Fib}$.

We consider the following commutative diagram.

$$\begin{tikzcd} \emptyset \arrow[r, "p"] \arrow[d, "f"'] & \mathrm{C}_4 \arrow[d, "\tau_{\mathrm{C}_4}"] \\ \mathrm{C}_2 \arrow[r, "\tau_{\mathrm{C}_2}"'] & \mathrm{C}_1 \end{tikzcd}$$

Note that the graph morphism $\tau_{\mathrm{C}_4} : \mathrm{C}_4 \to \mathrm{C}_1$ is a fibration; cf. Example 137.

But there does not exist a graph morphism $h : \mathrm{C}_2 \to \mathrm{C}_4$ at all.

So f is not in ${}^{\boxslash}\mathrm{Fib}$.

Alternatively, using ${}^{\boxslash}\mathrm{Fib} \overset{\text{Lemma 191}}{=} \mathrm{AcCofib}$, f is not an acyclic cofibration since it does not satisfy (AcCofib 5).

So $f \in {}^{\boxslash}\mathrm{Qis} \setminus {}^{\boxslash}\mathrm{Fib}$. $\square$

Remark 242 We have ${}^{\boxslash}\mathrm{Fib} \setminus {}^{\boxslash}\mathrm{Qis} \overset{\text{Lemma 191}}{=} \mathrm{AcCofib} \setminus {}^{\boxslash}\mathrm{Qis} \neq \emptyset$.

Proof. We consider the acyclic cofibration $f := \iota_{0,1} : \mathrm{D}_0 \to \mathrm{D}_1$; cf. Remark 166.

We consider the following commutative diagram.

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "\mathrm{id}_{\mathrm{D}_0}"] \arrow[d, "f"'] & \mathrm{D}_0 \arrow[d, "\wr\, f"] \\ \mathrm{D}_1 \arrow[r, "\mathrm{id}_{\mathrm{D}_1}"'] & \mathrm{D}_1 \end{tikzcd}$$

There does not exist a graph morphism $h : \mathrm{D}_1 \to \mathrm{D}_0$.

So f is not in ${}^{\boxslash}\mathrm{Qis}$.

So $f \in \mathrm{AcCofib} \setminus {}^{\boxslash}\mathrm{Qis}$. $\square$

Remark 243 We have $\mathrm{Qis}^{\boxslash} \subsetneq \mathrm{AcCofib}^{\boxslash} \overset{\text{Lemma 192}}{=} \mathrm{Fib}$.

In fact, we have $\mathrm{AcFib} \setminus \mathrm{Qis}^{\boxslash} \neq \emptyset$.

Proof. We consider the graph morphism $f := \iota_1 : \mathrm{D}_0 \to \mathrm{D}_0 \sqcup \mathrm{D}_0$, where $\hat{\mathrm{v}}_0\, \mathrm{V}_f = (1, \hat{\mathrm{v}}_0)$.

The graph morphism $f : \mathrm{D}_0 \to \mathrm{D}_0 \sqcup \mathrm{D}_0$ is a quasiisomorphism since $(\mathrm{C}_n\, , \mathrm{D}_0)_{\mathrm{Gph}} = \emptyset = (\mathrm{C}_n\, , \mathrm{D}_0 \sqcup \mathrm{D}_0)_{\mathrm{Gph}}$.

The map $\mathrm{E}_{f,\hat{\mathrm{v}}_0} : \emptyset \to \emptyset$ is surjective. So the graph morphism $f : \mathrm{D}_0 \to D_0 \sqcup \mathrm{D}_0$ is a fibration.

So the graph morphism $f : \mathrm{D}_0 \to D_0 \sqcup \mathrm{D}_0$ is an acyclic fibration.

We consider the following commutative quadrangle.

$$\begin{tikzcd} \mathrm{D}_0 \arrow[rr, "\mathrm{id}_{\mathrm{D}_0}"] \arrow[d, "f\, \wr"'] & & \mathrm{D}_0 \arrow[d, "f"] \\ \mathrm{D}_0 \sqcup \mathrm{D}_0 \arrow[rr, "\mathrm{id}_{\mathrm{D}_0 \sqcup \mathrm{D}_0}"'] & & \mathrm{D}_0 \sqcup \mathrm{D}_0 \end{tikzcd}$$

There exists the unique graph morphism $c : \mathrm{D}_0 \sqcup \mathrm{D}_0 \to \mathrm{D}_0$, having $(1, \hat{\mathrm{v}}_0)\,\mathrm{V}_c = (2, \hat{\mathrm{v}}_0)\,\mathrm{V}_c = \hat{\mathrm{v}}_0$.
But we do not have $cf = \mathrm{id}_{\mathrm{D}_0 \sqcup \mathrm{D}_0}$ since $(2, \hat{\mathrm{v}}_0)\,\mathrm{V}_{cf} = (1, \hat{\mathrm{v}}_0) \neq (2, \hat{\mathrm{v}}_0) = (2, \hat{\mathrm{v}}_0)\,\mathrm{V}_{\mathrm{id}_{\mathrm{D}_0 \sqcup \mathrm{D}_0}}$.

$$\begin{tikzcd}
\mathrm{D}_0 \arrow[r, "\mathrm{id}_{\mathrm{D}_0}"] \arrow[d, "f\,\wr"'] & \mathrm{D}_0 \arrow[d, "f"] \\
\mathrm{D}_0 \sqcup \mathrm{D}_0 \arrow[r, "\mathrm{id}_{\mathrm{D}_0 \sqcup \mathrm{D}_0}"'] \arrow[ur, dotted, "\nexists"] & \mathrm{D}_0 \sqcup \mathrm{D}_0
\end{tikzcd}$$

So the graph morphism $f : \mathrm{D}_0 \to \mathrm{D}_0 \sqcup \mathrm{D}_0$ is in $\mathrm{AcFib} \setminus \mathrm{Qis}^{\boxslash}$. ▫

Note that we also have $\mathrm{Cofib}^{\boxslash} \subsetneq \mathrm{AcCofib}^{\boxslash}$ since

$$\mathrm{Cofib}^{\boxslash} \overset{\text{Lemma 193}}{=} \mathrm{AcFib} \overset{\text{Remark 236}}{\subsetneq} \mathrm{Fib} \overset{\text{Lemma 192}}{=} \mathrm{AcCofib}^{\boxslash}.$$

Remark 244 We have $\mathrm{Cofib}^{\boxslash} \setminus \mathrm{Qis}^{\boxslash} \overset{\text{Lemma 193}}{=} \mathrm{AcFib} \setminus \mathrm{Qis}^{\boxslash} \neq \emptyset$.

Proof. We consider the acyclic fibration $f : \emptyset \to \mathrm{D}_0$; cf. Example 158.(1).

We consider the following commutative diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "f\,\wr"'] & \emptyset \arrow[d, "f"] \\
\mathrm{D}_0 \arrow[r, "\mathrm{id}_{\mathrm{D}_0}"'] & \mathrm{D}_0
\end{tikzcd}$$

There does not exist a graph morphism $h : \mathrm{D}_0 \to \emptyset$.

So the graph morphism f is not in $\mathrm{Qis}^{\boxslash}$.

So $f \in \mathrm{Cofib}^{\boxslash} \setminus \mathrm{Qis}^{\boxslash}$. ▫

9.4 Counterexamples for model categories

Recall that Gph is a Quillen closed model category; cf. Proposition 204.

Recall that the label **Assertion** indicates an assertion that we **falsify** by a counterexample.

Some of the following assertions are dual to each other, leading to further possibilities to falsify them.

9.4.1 Elementary counterexamples

Assertion 245 *In a Quillen closed model category, each acyclic fibration is a retraction.*

This assertion is **false**.

Counterexample 1 in Gph.

We consider the acyclic fibration $f : \emptyset \to \mathrm{D}_0$; cf. Example 158.

But f is not a retraction since there does not exist a graph morphism $g : \mathrm{D}_0 \to \emptyset$.

Counterexample 2 in Gph.

Consider the graph morphism $f : G \to H$ as in Example 215.

The acyclic fibration f is not a retraction.

We *assume* that f is a retraction.

Let $g : H \to G$ be a corresponding coretraction, i.e. $g \cdot f = \mathrm{id}_H$.

So $1\,\mathrm{V}_g = 1$. Thus $\beta_1\,\mathrm{E}_g = \alpha_1$. So $2\,\mathrm{V}_g = 2$. Thus $\beta_2\,\mathrm{E}_g = \alpha_3$. So $3\,\mathrm{V}_g = 3$. Thus $\beta_3\,\mathrm{E}_g = \alpha_4$. So $2\,\mathrm{V}_g = 2'$.

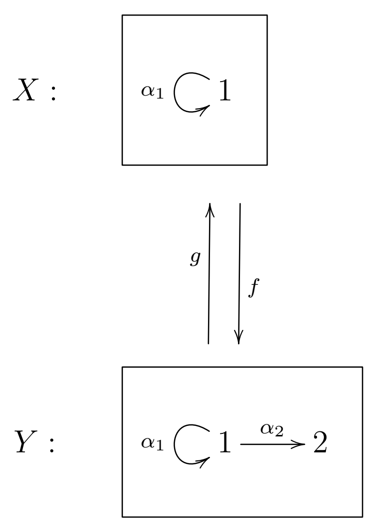
Contradiction. ▫

Assertion 246 *The subset of fibrations in a Quillen closed model category satisfies (2 of 3).*

This assertion is **false**.

Counterexample in Gph.

Consider the following graphs.



We consider the unique graph morphisms $f : X \to Y$ and $g := \tau_Y : Y \to X$, as well as their composite $fg = \mathrm{id}_X : X \to X$.

The graph morphisms id_X and g are acyclic fibrations.

But f is not a fibration; cf. Definition 127.

```
X := C(1);
Y := <[1,2],[<1,1,1>,<1,2,2>]>;
f := VtoE(X,Y,[<1,1>]);
> f;
<[ <1, 1> ], [ <<1, 1, 1>, <1, 1, 1>> ]>
> IsFibration(f,X,Y);
false
```

▫

Assertion 247 *In a Quillen closed model category, each acyclic cofibration is a coretraction.*

This assertion is **false**.

Counterexample in Gph.

We consider the acyclic cofibration $\iota_{0,1} : \mathrm{D}_0 \to \mathrm{D}_1$; cf. Remark 166.

But $\iota_{0,1}$ is not a coretraction since there does not exist a graph morphism $g : \mathrm{D}_1 \to \mathrm{D}_0$. ▫

Assertion 248

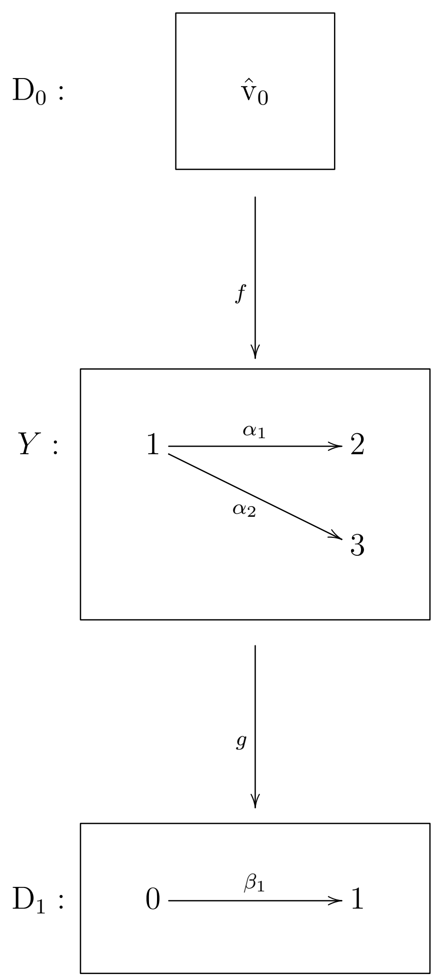
(1) *The subset of acyclic cofibrations in a Quillen closed model category satisfies (2 of 3).*

(2) *The subset of cofibrations in a Quillen closed model category satisfies (2 of 3).*

The assertions (1) and (2) are **false**.

Counterexample in Gph.

We consider the following graph morphisms.



We consider the graph morphisms $f : \mathrm{D}_0 \to Y$, $g : Y \to \mathrm{D}_1$ and $fg = \iota_1 : \mathrm{D}_0 \to \mathrm{D}_1$, where $\hat{\mathrm{v}}_0 \, \mathrm{V}_f := 1$ and g mapping the vertices and the edges in a vertical way.

The graph morphisms f and fg are acyclic cofibrations; cf. Remark 166.

But g is not an acyclic cofibration since it does not satisfy e.g. (AcCofib 2).

```
Y := <[1,2,3],[<1,1,2>,<1,2,3>]>;
f := VtoE(D(0),Y,[<0,1>]);
g := ListGraphMorphisms(Y,D(1))[1];
> IsAcCofib(f,D(0),Y);
true
> IsAcCofib(ComposeGraphMorphisms(f,g),D(0),D(1));
true
> IsAcCofib(g,Y,D(1));
false
```

Since g is a quasiisomorphism, we infer that g is not a cofibration; cf. Lemma 185. $\square$

Assertion 249 Consider the following assertions (1–6) in a Quillen closed model category.

(1) *Each quasiisomorphism that is a retraction is a fibration.*

(2) *Each quasiisomorphism that is a coretraction is a cofibration.*

(3) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "g"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

Then $f : Y \to Z$ is a fibration.

(4) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "g"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

Then $f : Y \to Z$ is an acyclic fibration.

(5) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "g"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

Then $g : X \to Y$ is a cofibration.

(6) *Suppose given a commutative diagram as follows.*

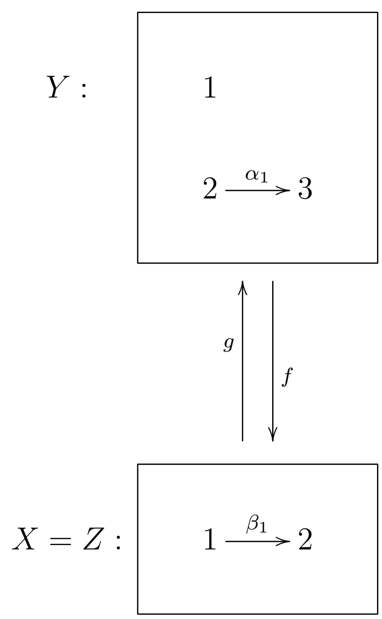
$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "g"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

Then $g : X \to Y$ is an acyclic cofibration.

The assertions (1–6) are **false**.

Counterexample in Gph.

We consider the following graph morphisms.



Here, the graph morphisms $f : Y \to Z$ and $g : X \to Y$ map the vertices and the edges in a vertical way, where $1\,\mathrm{V}_g := 2$, $2\,\mathrm{V}_g := 3$ and $\beta_1\,\mathrm{E}_g := \alpha_1$.

So we have $gf = \mathrm{id}_X$.

So the graph morphism $f : Z \to Y$ is a retraction and the graph morphism $g : X \to Z$ is a coretraction.

Since $(\mathrm{C}_k\,, X)_{\mathrm{Gph}} = (\mathrm{C}_k\,, Y)_{\mathrm{Gph}} = \emptyset$ for $k \geqslant 1$, the graph morphisms f and g are quasiisomorphisms.

```
X := D(1);
Z := D(1);
Y := <[1,2,3],[<2,1,3>]>;
g := VtoE(X,Y,[<0,2>,<1,3>]);
f := VtoE(Y,Z,[<1,0>,<2,0>,<3,1>]);
```

Ad (1).

The graph morphism $f : Y \to Z$ is a retraction and a quasiisomorphism but not a fibration.

```
> IsFibration(f,Y,Z);
false
```

Ad (2).

The graph morphism $g : X \to Z$ is a coretraction and a quasiisomorphism but not a cofibration.

```
> IsAcCofib(g,X,Z);
false
```

The graph morphism g is not an acyclic cofibration since g does not satisfy (AcCofib 3, 5).

Since g is a quasiisomorphism and since $\mathrm{AcCofib} = \mathrm{Cofib} \cap \mathrm{Qis}$, g is not a cofibration; cf. Lemma 185.

Alternative proof for (2). We have $\mathrm{Cofib} = {}^{\boxslash}\mathrm{AcFib}$; cf. Definition 144. The graph morphism g is an acyclic fibration since it is a quasiisomorphism and a fibration; cf. Definition 138.

```
> IsFibration(g,X,Y);
true
```

We consider the following commutative diagram.

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "g"'] & X \arrow[d, "g"] \\
Y \arrow[r, "\mathrm{id}_Y"'] & Y
\end{tikzcd}
$$

But there does not exist a graph morphism $h : Z \to X$ such that $gh = \mathrm{id}_X$ and $hg = \mathrm{id}_X$ since g is not a graph isomorphism.

So g is not a cofibration.

Ad (3).

We have $gf = \mathrm{id}_X : \ X \longrightarrow Z$, but f is not a fibration.

Ad (4).

We have $gf = \mathrm{id}_X : \ X \longrightarrow Z$, but f is not an acyclic fibration since it is not a fibration.

Ad (5).

We have $gf = \mathrm{id}_X : \ X \longrightarrow Z$, but g is not a cofibration.

Ad (6).

We have $gf = \mathrm{id}_X : \ X \longrightarrow Z$, but g is not an acyclic cofibration. $\square$

Remark 250 Using $\mathrm{Gph}^{\mathrm{op}}$, we can falsify the assertions dual to Assertions 245 – 249.

9.4.2 Counterexamples for pushouts and pullbacks

The following assertions take place in a Quillen closed model category.

Assertion 251 *In a Quillen closed model category, a pushout of a quasiisomorphism along a cofibration is a quasiisomorphism.*

This assertion is **false**.

Counterexample in Gph.

A counterexample is given in Remark 205. ▫

Assertion 252

Consider the following assertions (1–4) in a Quillen closed model category.

(1) *A pushout of a fibration is a fibration.*

(2) *A pushout of an acyclic fibration is a fibration.*

(3) *A pushout of an acyclic fibration is a quasiisomorphism.*

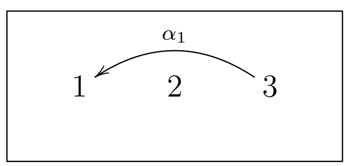
(4) *A pushout of an etale fibration is a fibration.*

The assertions (1), (2), (3) and (4) are **false**.

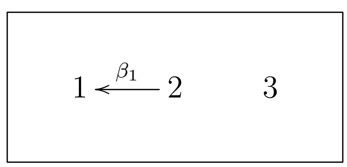
Counterexample in Gph.

We consider the following graphs.

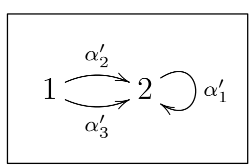
X :



Y :



X' :



Let $f = (\mathrm{V}_f\,, \mathrm{E}_f) : X \to Y$ be the graph morphism with

$$1\,\mathrm{V}_f = 1, \quad 2\,\mathrm{V}_f = 1, \quad 3\,\mathrm{V}_f = 2$$

and with

$$\alpha_1\,\mathrm{E}_f = \beta_1\ .$$

Then the graph morphism f is an etale fibration as we calculated with Magma [**2**].

```
X := <[1,2,3],[<3,1,1>]>;
Y := <[1,2,3],[<2,1,1>]>;
f := <[<1,1>,<2,1>,<3,2>],[<<3,1,1>,<2,1,1>>]>;
> IsFibration(f,X,Y);
true
> IsEtaleFibration(f,X,Y);
true
```

Moreover, f is a quasiisomorphism since $(\mathrm{C}_n\,, X)_{\mathrm{Gph}} = \emptyset = (\mathrm{C}_n\,, Y)_{\mathrm{Gph}}$ for $n \in \mathbb{N}$.

In conclusion, f is an acyclic fibration.

Let $g = (\mathrm{V}_g, \mathrm{E}_g) : X \to X'$ be the graph morphism with

$$1\,\mathrm{V}_g = 2, \quad 2\,\mathrm{V}_g = 1, \quad 3\,\mathrm{V}_g = 2$$

and with

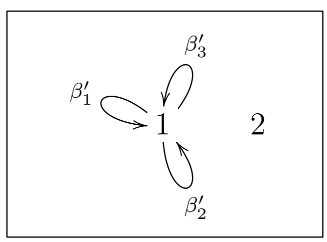
$$\alpha_1\,\mathrm{E}_g = \alpha'_1\,.$$

We calculated a pushout with Magma.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner"] & Y' \end{tikzcd}$$

We obtained the pushout Y'.

Y' :



We have

$$\begin{aligned} &1\,\mathrm{V}_{f'} = 1, \quad 2\,\mathrm{V}_{f'} = 1, \\ &\alpha'_1\,\mathrm{E}_{f'} = \beta'_1\,, \quad \alpha'_2\,\mathrm{E}_{f'} = \beta'_2\,, \quad \alpha'_3\,\mathrm{E}_{f'} = \beta'_3 \end{aligned}$$

and

$$\begin{aligned} &1\,\mathrm{V}_h = 1, \quad 2\,\mathrm{V}_h = 1, \quad 3\,\mathrm{V}_h = 2, \\ &\beta_1\,\mathrm{E}_h = \beta'_1\,. \end{aligned}$$

The graph morphism f is an etale fibration but the graph morphism f' is not.

```
Xp := <[1,2],[<2,1,2>,<1,2,2>,<1,3,2>]>; // "X prime" = X'
g := ListGraphMorphisms(X,Xp)[1];
> g;
<[ <1, 2>, <2, 1>, <3, 2> ], [ <<3, 1, 1>, <2, 1, 2>> ]>
> IsGraphMorphism(g,X,Xp);
true
```

```
PO := PushoutGraphs(X,Y,Xp,f,g);
> PO;
<<[ 1, 2 ], [ <1, 1, 1>, <1, 2, 1>, <1, 3, 1> ]>,
<[ <1, 1>, <2, 1> ], [ <<2, 1, 2>, <1, 1, 1>>, <<1, 2, 2>, <1, 2, 1>>,
                       <<1, 3, 2>, <1, 3, 1>> ]>,
<[ <1, 1>, <2, 1>, <3, 2> ], [ <<2, 1, 1>, <1, 1, 1>> ]>>
Yp := PO[1]; // "Y prime" = Y'
fp := PO[2]; // "f prime" = f'
h := PO[3];
> IsFibration(fp,Xp,Yp);
false
> IsQis_Bound(fp,Xp,Yp,1);
false
```

Here, the pushout $f' : X' \to Y'$ of the acyclic fibration $f : X \to Y$ is neither a fibration nor a quasiisomorphism, as we calculated with Magma. We can also directly see that

$$|\{e \in \mathrm{E}_{X'} : e\,\mathrm{s}_{X'} = 2\}| = 1 \neq 3 = |\{e \in \mathrm{E}_{Y'} : e\,\mathrm{s}_{Y'} = 2\,\mathrm{V}_{f'} = 1\}|$$

and that $|(\mathrm{C}_1\,, X')_{\mathrm{Gph}}| = 1 \neq 3 = |(\mathrm{C}_1\,, Y')_{\mathrm{Gph}}|$. ▫

Assertion 253 *In a Quillen closed model category, a pullback of an acyclic cofibration is a cofibration.*

This assertion is **false**.

Counterexample in Gph*.*

We consider the following graphs.

$$X : \quad \boxed{\; 1 \;}$$

$$Y' : \quad \boxed{\; 1 \xrightarrow{\alpha} 2 \qquad 3 \;}$$

$$Y : \quad \boxed{\; 1 \xrightarrow{\beta} 2 \;}$$

Let $f = (\mathrm{V}_f\,, \mathrm{E}_f) : X \to Y$ be the graph morphism with

$$1\,\mathrm{V}_f = 1.$$

Then the graph morphism f is an acyclic cofibration, as we calculated with Magma [**2**].

Let $q = (\mathrm{V}_q\,, \mathrm{E}_q) : Y' \to Y$ be the graph morphism with

$$\begin{aligned} 1\,\mathrm{V}_q &= 1, \\ 2\,\mathrm{V}_q &= 2, \\ 3\,\mathrm{V}_q &= 2, \end{aligned}$$

and with

$$\alpha\,\mathrm{E}_q = \beta\,.$$

Let $f' : X' \to Y'$ be the pullback of $f : X \to Y$ along q.

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"'] & Y\,. \end{tikzcd}$$

Via Magma, we obtain $\mathrm{V}_{X'} = \{1\}$ and $\mathrm{E}_{X'} = \emptyset$ and

$$1\,\mathrm{V}_{f'} = 1,$$

and

$$1\,\mathrm{V}_p = 1.$$

$$X' : \boxed{\quad 1 \quad}$$

The graph morphism f is an acyclic cofibration, but the graph morphism f' is not, as we calculated with Magma:

```
X := <[1],[]>;
Y := <[1,2],[<1,1,2>]>;
Yp := <[1,2,3],[<1,1,2>]>; // "Y prime" = Y'
f := <[<1,1>],[]>;
q := <[<1,1>,<2,2>,<3,2>],[<<1,1,2>,<1,1,2>>]>;
> IsGraphMorphism(f,X,Y);
true
> IsAcCofib(f,X,Y);
true
> IsGraphMorphism(q,Yp,Y);
true
PB := PullbackGraphs(X,Y,Yp,f,q);
> PB;
<<[ 1 ], []>, <[ <1, 1> ], []>, <[ <1, 1> ], []>>;
Xp := PB[1]; // "X prime" = X'
fp := PB[2]; // "f prime" = f'
q := PB[3];
```

```
> IsAcCofib(fp,Xp,Yp);
false
> AcCofib1to4(fp,Xp,Yp);
false
> AcCofib5(fp,Xp,Yp);
false
```

We can see that the vertex $3 \in \mathrm{V}_{Y'} \setminus \mathrm{V}_{X'f'}$ does not have an edge $e \in \mathrm{E}_{Y'}$ such that $e\,\mathrm{t}_{Y'} = 3$. So the graph morphism $f' : X' \to Y'$ does not satisfy (AcCofib 3). Hence it is not an acyclic cofibration.

The pullback $f' : X' \to Y'$ of the acyclic cofibration $f : X \to Y$ is **not** a cofibration since first, it is not an acyclic cofibration, and second, it is a quasiisomorphism as $(\mathrm{C}_n\,, Y')_{\mathrm{Gph}} = \emptyset = (\mathrm{C}_n\,, X')_{\mathrm{Gph}}$, for $n \in \mathbb{N}$, or by Remark 125; cf. Lemma 185.

Alternatively, to show that f' is not a cofibration, we can consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"'] & \mathrm{C}_1 \arrow[d, "\iota_1"] \\ Y' \arrow[r, "r"'] & \mathrm{C}_1 \sqcup \mathrm{D}_0\ , \end{tikzcd}$$

where $1\,\mathrm{V}_p := \mathrm{v}_0$, $1\,\mathrm{V}_r = (1, \mathrm{v}_0)$, $2\,\mathrm{V}_r = (1, \mathrm{v}_0)$ and $3\,\mathrm{V}_r := (2, \hat{\mathrm{v}}_0)$.

The graph morphism ι_1 is in fact a quasiisomorphism since $(\mathrm{C}_k\,, \iota_1)_{\mathrm{Gph}}$ is injective and $|(\mathrm{C}_k\,, \mathrm{C}_1)_{\mathrm{Gph}}| = 1 = |(\mathrm{C}_k\,, \mathrm{C}_1 \sqcup \mathrm{D}_0)_{\mathrm{Gph}}|$.

Moreover, ι_1 is a fibration, as we will verify with Magma. So ι_1 is an acyclic fibration.

The unique graph morphism $h : Y' \to \mathrm{C}_1$ maps the vertex 3 to v_0 .

So we have $3\,\mathrm{V}_r = (2, \hat{\mathrm{v}}_0) \neq (1, \mathrm{v}_0) = \mathrm{v}_0\,\mathrm{V}_{\iota_1} = 3\,\mathrm{V}_{h\iota_1}$.

Thus $h\iota_1 \neq r$.

So f' does not have the left lifting property with respect to the acyclic fibration ι_1 .

So $f' : X' \to Y'$ is not a cofibration; cf. Definition 144. ▫

```
Xp := <[1],[]>; // "X prime" = X'
p := VtoE(Xp,C(1),[<1,1>]);
> p;
<[ <1, 1> ], []>
Yp := <[1,2,3],[<1,1,2>]>; // "Y prime" = Y'
fp := <[<1,1>],[]>; // "f prime" = f'
C1D0 := <[0,1],[<0,1,0>]>;
r := <[<1,0>,<2,0>,<3,1>],[<<1,1,2>,<0,1,0>>]>;
iota_1 := VtoE(C(1),C1D0,[<1,0>]);
> iota_1;
<[ <1, 0> ], [ <<1, 1, 1>, <0, 1, 0>> ]>
> IsFibration(iota_1,C(1),C1D0);
true
```

```
> Lift(Xp,C(1),Yp,C1D0,p,r,fp,iota_1);
false
```

Note that in Gph, a pullback of an acyclic cofibration is a quasiisomorphism; cf. Remark 176.

Assertion 254 Consider the following assertions (1–2) in a Quillen closed model category.

(1) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "q"] & \\ X' \arrow[r, "f'"] \arrow[drr, bend right, "p"'] \arrow[ur, phantom, "\lrcorner"] & Y' \arrow[dr, "w"] & \\ & & Z \end{tikzcd}$$

Then the graph morphism $w : Y' \to Z$ is an acyclic cofibration.

(2) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "q"] & \\ X' \arrow[r, "f'"] \arrow[drr, bend right, "p"'] \arrow[ur, phantom, "\lrcorner"] & Y' \arrow[dr, "w"] & \\ & & Z \end{tikzcd}$$

Then the graph morphism $w : Y' \to Z$ is a cofibration.

The assertions (1) and (2) are **false**.

Ad (1). *Counterexample in* Gph.

We consider

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "\iota_{0,1}"] \arrow[d, "\iota_{0,1}"'] & \mathrm{D}_1 \arrow[d, "g_1"] \\ \mathrm{D}_1 \arrow[r, "g_2"'] \arrow[ur, phantom, "\lrcorner"] & P \end{tikzcd} \;;$$

where $\hat{\mathrm{e}}_0\,\mathrm{E}_{g_1} := e_1$ and $\hat{\mathrm{e}}_0\,\mathrm{E}_{g_2} := e_2$; cf. Remarks 166 and 171.

Magma gives

$$P : \boxed{\begin{tikzcd} 1 \arrow[r, "e_1"] \arrow[dr, "e_2"'] & 2 \\ & 3 \end{tikzcd}}$$

We now consider

$$
\begin{tikzcd}
\mathrm{D}_0 \arrow[r, "\iota_{0,1}"] \arrow[d, "\iota_{0,1}"'] & \mathrm{D}_1 \arrow[d, "g_1"'] \arrow[ddr, bend left, "\mathrm{id}_{\mathrm{D}_1}"] & \\
\mathrm{D}_1 \arrow[r, "g_2"] \arrow[drr, bend right, "\mathrm{id}_{\mathrm{D}_1}"'] & P \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "h"] & \\
 & & \mathrm{D}_1
\end{tikzcd}
$$

We have

$$
\begin{array}{rccl}
h: & P & \to & \mathrm{D}_1 \\
\mathrm{V}_h: & 1 & \mapsto & \hat{\mathrm{v}}_0 \\
 & 2 & \mapsto & \hat{\mathrm{v}}_1 \\
 & 3 & \mapsto & \hat{\mathrm{v}}_1 \\
\mathrm{E}_h: & e_1 & \mapsto & \hat{\mathrm{e}}_0 \\
 & e_2 & \mapsto & \hat{\mathrm{e}}_0
\end{array}
$$

Note that the graph morphism $\mathrm{id}_{\mathrm{D}_1} : \mathrm{D}_1 \to \mathrm{D}_1$ is an acyclic cofibration; cf. Remark 165.

But the graph morphism $h : P \to \mathrm{D}_1$ is **not** an acyclic cofibration, since neither V_h nor E_h are injective; cf. Definition 162.

```
D0 := D(0);
D1 := D(1);
f := VtoE(D0,D1,[<0,0>]); // iota_{0,1}
> D0;
<[ 0 ], []>
> D1;
<[ 0, 1 ], [ <0, 0, 1> ]>
> f;
<[ <0, 0> ], []>
PO := PushoutGraphs(D0,D1,D1,f,f);
> PO[1];
<[ 1, 2, 3 ], [ <1, 1, 2>, <1, 2, 3> ]>
> PO[2];
<[ <0, 1>, <1, 2> ], [ <<0, 0, 1>, <1, 1, 2>> ]>
> PO[3];
<[ <0, 1>, <1, 3> ], [ <<0, 0, 1>, <1, 2, 3>> ]>
> IsAcCofib(Identity(D1),D1,D1);
true
h := VtoE(PO[1],D1,[<1,0>,<2,1>,<3,1>]);
> h;
<[ <1, 0>, <2, 1>, <3, 1> ], [ <<1, 1, 2>, <0, 0, 1>>, <<1, 2, 3>, <0, 0, 1>> ]>
> IsEqual(ComposeGraphMorphisms(PO[2],h),Identity(D1));
true
```

```
> IsEqual(ComposeGraphMorphisms(PO[3],h),Identity(D1));
true
> IsAcCofib(h,PO[1],D1);
false
> IsFibration(h,PO[1],D1);
true
```

Note that h is a quasiisomorphism by (2 of 3).

Thus h is not a cofibration.

By the way, h is an acyclic fibration as it is a quasiisomorphism and a fibration.

Ad (2). *Counterexample in* Gph.

We consider

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "\iota_{0,1}"] \arrow[d, "\iota_{0,1}"'] & \mathrm{D}_1 \arrow[d, "g_1"] \\ \mathrm{D}_1 \arrow[r, "g_2"'] & P \end{tikzcd}$$

where $\hat{\mathrm{e}}_0\,\mathrm{E}_{g_1} := e_1$ and $\hat{\mathrm{e}}_0\,\mathrm{E}_{g_2} := e_2$ as above; cf. Lemma 185 and Remark 148.

We now consider

$$\begin{tikzcd} \mathrm{D}_0 \arrow[r, "\iota_{0,1}"] \arrow[d, "\iota_{0,1}"'] & \mathrm{D}_1 \arrow[d, "g_1"'] \arrow[ddr, bend left, "\mathrm{id}_{\mathrm{D}_1}"] & \\ \mathrm{D}_1 \arrow[r, "g_2"] \arrow[drr, bend right, "\mathrm{id}_{\mathrm{D}_1}"'] & P \arrow[dr, "h"] & \\ & & \mathrm{D}_1 \end{tikzcd}$$

as above.

Note that the graph morphism $\mathrm{id}_{\mathrm{D}_1} : \mathrm{D}_1 \to \mathrm{D}_1$ is a cofibration; cf. Remark 145.

But we will show that the graph morphism $h : P \to \mathrm{D}_1$ is not a cofibration.

Since $(\mathrm{C}_n\,, P) = \emptyset = (\mathrm{C}_n\,, D_1)$ the graph morphism $h : P \to \mathrm{D}_1$ is a quasiisomorphism; cf. Definition 115.

But the graph morphism h is not an acyclic cofibration since it does not satisfy (AcCofib 1) since $2\,\mathrm{V}_h = \hat{\mathrm{v}}_1 = 3\,\mathrm{V}_h$.

Since we have AcCofib $=$ Cofib$\cap$Qis the quasiisomorphism h is not a cofibration; cf. Lemma 175. ▫

Assertion 255 *Suppose given a pushout as follows.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

If Y and X' are cofibrant, then Y' is cofibrant.

This assertion is **false**.

Counterexample in Gph.

We consider the following graph.

$$
H : \quad \boxed{\begin{tikzcd} 1 \arrow[r, bend left, "\beta_1"] & 2 \arrow[l, bend left, "\beta_4"] \arrow[r, bend left, "\beta_2"] & 3 \arrow[l, bend left, "\beta_3"] \end{tikzcd}}
$$

The graph H is not cofibrant; cf. Definition 150.

Assume that H is cofibrant. Consider $f : G \to H$ from Example 215. Then we obtain a commutative diagram as follows

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & G \arrow[d, "f"] \\
H \arrow[r, "\mathrm{id}_H"'] \arrow[ur] & H .
\end{tikzcd}
$$

But $f : G \to H$ is not a retraction; cf. Counterexample 2 to Assertion 245.

However, Magma yields a pushout as follows.

$$
\begin{tikzcd}
\mathrm{D}_0 \arrow[r] \arrow[d] & \mathrm{C}_2 \arrow[d] \\
\mathrm{C}_2 \arrow[r] & H \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$

$\square$

Note that C_2 is cofibrant; cf. Remark 151.

```
D0 := D(0);
C := C(2);
f := VtoE(D0,C,[<0,1>]);
> D0;
<[ 0 ], []>
> C;
<[ 1, 2 ], [ <1, 1, 2>, <2, 2, 1> ]>
> f;
<[ <0, 1> ], []>
PO := PushoutGraphs(D0,C,C,f,f);
> PO[1];
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 1>, <1, 3, 3>, <3, 4, 1> ]>
> PO[2];
<[ <1, 1>, <2, 2> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2, 1>> ]>
> PO[3];
<[ <1, 1>, <2, 3> ], [ <<1, 1, 2>, <1, 3, 3>>, <<2, 2, 1>, <3, 4, 1>> ]>
```

We obtain that `PO[1]` $\simeq H$.

Remark 256 Using $\mathrm{Gph}^{\mathrm{op}}$, we can falsify the assertions dual to Assertions 251 – 255.

9.5 Counterexamples in Gph

Assertion 257

(1) *Suppose given an acyclic fibration* $f:\ G \twoheadrightarrow H$.

Suppose given $n \in \mathbb{Z}_{\geqslant 1}$. *Suppose given graph morphisms* $\mathrm{C}_n \xrightarrow{p} G$ *and* $\mathrm{C}_n \xrightarrow{q} G$.

If $C_n p \cap \mathrm{C}_n\, q = \emptyset$, *then* $\mathrm{C}_n\, pf \cap \mathrm{C}_n\, qf = \emptyset$.

(2) *Suppose given an acyclic fibration* $f:\ G \twoheadrightarrow H$.

Suppose given $n \in \mathbb{Z}_{\geqslant 1}$. *Suppose given graph morphisms* $\mathrm{C}_n \xrightarrow{p} G$, $\mathrm{C}_n \xrightarrow{q} G$ *and* $\mathrm{C}_n \xrightarrow{r} G$.

If $\mathrm{C}_n\, p$, $\mathrm{C}_n\, q$ *and* $\mathrm{C}_n\, r$ *are pairwise disjoint, so are* $\mathrm{C}_n\, pf$, $\mathrm{C}_n\, qf$ *and* $\mathrm{C}_n\, rf$.

The assertions (1, 2) are **false**.

Counterexamples.

Ad (1). We consider the graph morphism $g : K \to H$ from Example 216.

We let $n := 2$

We consider the graph morphisms $p, q : \mathrm{C}_2 \to K$, where $\mathrm{v}_0\, \mathrm{V}_p := 1$, $\mathrm{v}_1\, \mathrm{V}_p := 2$, $\mathrm{v}_0\, \mathrm{V}_q := 2'$ and $\mathrm{v}_1\, \mathrm{V}_q := 3$.

Then we have $\mathrm{C}_2\, p \cap \mathrm{C}_2\, q = \emptyset$, but $\mathrm{C}_2\, pf \cap \mathrm{C}_2\, qf$ consists the vertex 2.

Ad (2). We consider the graph morphism $g : K \to H$ from Example 218.

We let $n := 2$

We consider the graph morphisms $p, q, r : \mathrm{C}_2 \to K$ where $\mathrm{v}_0\, \mathrm{V}_p := 1$, $\mathrm{v}_1\, \mathrm{V}_p := 2$, $\mathrm{v}_0\, \mathrm{V}_q := 2'$, $\mathrm{v}_1\, \mathrm{V}_q := 3$, $\mathrm{v}_0\, \mathrm{V}_r := 3'$ and $\mathrm{v}_1\, \mathrm{V}_r := 4$.

Then we have $\mathrm{C}_2\, p \cap \mathrm{C}_2\, q = \emptyset$, but $\mathrm{C}_2\, pf \cap \mathrm{C}_2\, qf$ consists the vertex 2.

And we have $\mathrm{C}_2\, q \cap \mathrm{C}_2\, r = \emptyset$, but $\mathrm{C}_2\, qf \cap \mathrm{C}_2\, rf$ consists the vertex 3. □

Assertion 258

(1) *Suppose given a graph morphism* $f : G \to H$.

If for each $n \geqslant 1$ *and each graph morphism* $u : \mathrm{C}_n \to H$ *such that* E_u *is injective there exists a unique graph morphism* $\hat{u} : \mathrm{C}_n \to G$ *with* $\hat{u} f = u$, *then* f *is a quasiisomorphism.*

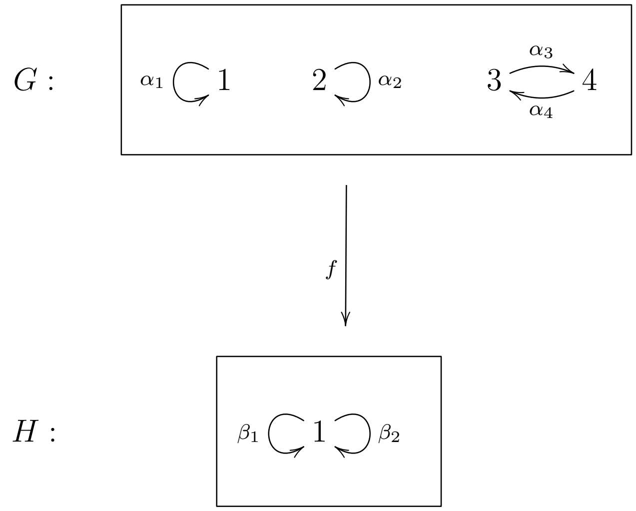
(2) *Suppose given* $m \in \mathbb{N}$ *such that there does not exist an injective graph morphism* $g :\ \mathrm{C}_n \to\ G$ *for* $n > m$ *and such that there does not exist an injective graph morphism* $h : \mathrm{C}_n \to H$ *for* $n > m$ *and such that the map* $(\mathrm{C}_n\, , f)_{\mathrm{Gph}}$ *is bijective for* $n \in [1, m]$.

Then the graph morphism $f : G \to H$ *is a quasiisomorphism.*

The assertions (1, 2) are **false**.

Counterexample.

We consider the following graph morphism.



Here, $f : G \to H$ is the graph morphism with

$$\begin{array}{cccc} 1\,\mathrm{V}_f = 1 & 2\,\mathrm{V}_f = 1 & 3\,\mathrm{V}_f = 1 & 4\,\mathrm{V}_f = 1 \\ \alpha_1\,\mathrm{E}_f = \beta_1 & \alpha_2\,\mathrm{E}_f = \beta_2 & \alpha_3\,\mathrm{E}_f = \beta_1 & \alpha_4\,\mathrm{E}_f = \beta_2 \end{array}$$

Ad (1).

There are two graph morphisms $p, q : \mathrm{C}_1 \to H$ with $\mathrm{E}_p\,, \mathrm{E}_q$ injective, having $\mathrm{e}_0\,\mathrm{E}_p = \beta_1$ and $\mathrm{e}_0\,\mathrm{E}_q = \beta_2\,$.

For each of these injective graph morphisms there exists a unique graph morphism $\tilde{p}, \tilde{q} : \mathrm{C}_1 \to G$ such that $\tilde{p}f = p$ and $\tilde{q}f = q$.

There are two graph morphisms $p, q : \mathrm{C}_2 \to H$ with E_p and E_q injective. In particular, we have $\mathrm{e}_0\,\mathrm{E}_p = \beta_1$ and $\mathrm{e}_0\,\mathrm{E}_q = \beta_2\,$.

For each of these injective graph morphisms exists a unique graph morphism $\tilde{p}, \tilde{q} : \mathrm{C}_2 \to G$ such that $\tilde{p}f = p$ and $\tilde{q}f = q$.

But the graph morphism $f : G \to H$ is not a quasiisomorphism.

Let $g : \mathrm{C}_3 \to G$ be the graph morphism with

$$\begin{array}{ccc} \mathrm{v}_0\,\mathrm{V}_g = 1, & \mathrm{v}_1\,\mathrm{V}_g = 1, & \mathrm{v}_2\,\mathrm{V}_g = 1, \\ \mathrm{e}_0\,\mathrm{E}_g = \beta_1\,, & \mathrm{e}_1\,\mathrm{E}_g = \beta_2\,, & \mathrm{e}_2\,\mathrm{E}_f = \beta_1\,. \end{array}$$

There does not exist a graph morphism $\tilde{g} : \mathrm{C}_3 \to G$ such that $\tilde{g}f = g$.

So the graph morphism $f : G \to H$ is not a quasiisomorphism.

Ad (2).

We have $m = 2$ since there does not exist an injective graph morphism $g : \mathrm{C}_n \to G$ for $n > 2$ and there does not exist an injective graph morphism $h : \mathrm{C}_n \to H$ for $n > 2$. Furthermore, $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$ is bijective for $n \in [1, 2]$, but $(\mathrm{C}_3\,, f)_{\mathrm{Gph}}$ is not bijective since it is not surjective. ▫

Assertion 259 *Suppose given*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in Gph.

(1) *If* $(\mathrm{C}_n\,,f)_{\mathrm{Gph}}$ *is surjective for* $n \geqslant 1$*, then* $(\mathrm{C}_n\,,f')_{\mathrm{Gph}}$ *is surjective for* $n \geqslant 1$*.*

(2) *If* f *is a retraction, then* f' *is a retraction.*

The assertions (1) and (2) are **false**.

Counterexample.

Let $X := \mathrm{C}_2 \sqcup \mathrm{C}_2$. Let $Y := \mathrm{C}_2$. Let $f := d_{\mathrm{C}_2} : \mathrm{C}_2 \sqcup \mathrm{C}_2 \to \mathrm{C}_2$.

Then f is a retraction. In particular, $(\mathrm{C}_n\,,f)_{\mathrm{Gph}}$ is surjective for $n \geqslant 1$.

Let

$$X' : \boxed{\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] \arrow[r, "\alpha_3"] & 3 \arrow[r, "\alpha_4"] & 4 \arrow[r, bend left, "\alpha_5"] & 5 \arrow[l, bend left, "\alpha_6"] \end{tikzcd}}$$

Let $g : \mathrm{C}_2 \sqcup \mathrm{C}_2 \to X$ be defined by

$$\begin{aligned} (1, \mathrm{v}_0)\,\mathrm{V}_g &:= 1, \quad (1, \mathrm{v}_1)\,\mathrm{V}_g := 2, \\ (2, \mathrm{v}_0)\,\mathrm{V}_g &:= 5, \quad (2, \mathrm{v}_1)\,\mathrm{V}_g := 4. \end{aligned}$$

This defines g since $\mathrm{C}_2 \sqcup \mathrm{C}_2$ is thin; cf. Remark 77.

We form the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d, "h"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

via Magma.

```
X := DUC([2,2]);
Y := C(2);
Xp := <[1,2,3,4,5],[<1,1,2>,<2,2,1>,<2,3,3>,<3,4,4>,<4,5,5>,<5,6,4>]>;
      // "X prime" = X'
f := VtoE(X,Y,[<1,1>,<2,2>,<3,1>,<4,2>]);
g := VtoE(X,Xp,[<1,1>,<2,2>,<3,5>,<4,4>]);
PO := PushoutGraphs(X,Y,Xp,f,g);
Yp := PO[1]; // "Y prime" = Y'
fp := PO[2]; // "f prime" = f'
h := PO[3];
```

We have obtained

$$Y' : \quad \begin{tikzcd} 1 \arrow[r, bend left, "\beta_1"] & 2 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\beta_3"] & 3 \arrow[l, bend left, "\beta_4"] \end{tikzcd}$$

We verify that $(\mathrm{C}_n\,, f')_{\mathrm{Gph}}$ is not surjective.

```
listXp := ListGraphMorphisms(C(2),Xp);
listYp := Sort(ListGraphMorphisms(C(2),Yp));
listXpfp := RedSeq([ComposeGraphMorphisms(k,fp) : k in listXp]);
           // set of images
> #listYp;
4
> #listXpfp;
2
```

In particular, f' is not a retraction.

We verify this again independently as follows.

We search for a coretraction to f'; i.e. a graph morphism $c : Y' \to X'$ such that $cf \cdot f' = \mathrm{id}_{Y'}$.

```
> Identity(Yp);
<[ <1, 1>, <2, 2>, <3, 3> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2, 1>>,
                             <<2, 3, 3>, <2, 3, 3>>, <<3, 4, 2>, <3, 4, 2>> ]>
list := ListGraphMorphisms(Yp,Xp);
> #list;
4

Cf := [ComposeGraphMorphisms(l,fp) : l in list];
> Cf;
[
    <[ <1, 1>, <2, 2>, <3, 1> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2,
    1>>, <<2, 3, 3>, <2, 2, 1>>, <<3, 4, 2>, <1, 1, 2>> ]>,
    <[ <1, 2>, <2, 1>, <3, 2> ], [ <<1, 1, 2>, <2, 2, 1>>, <<2, 2, 1>, <1, 1,
    2>>, <<2, 3, 3>, <1, 1, 2>>, <<3, 4, 2>, <2, 2, 1>> ]>,
    <[ <1, 2>, <2, 1>, <3, 2> ], [ <<1, 1, 2>, <2, 2, 1>>, <<2, 2, 1>, <1, 1,
    2>>, <<2, 3, 3>, <1, 1, 2>>, <<3, 4, 2>, <2, 2, 1>> ]>,
    <[ <1, 1>, <2, 2>, <3, 1> ], [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 1>, <2, 2,
    1>>, <<2, 3, 3>, <2, 2, 1>>, <<3, 4, 2>, <1, 1, 2>> ]>
]

Cfid := [l : l in list | IsEqual(ComposeGraphMorphisms(l,fp),Identity(Yp))];
> #Cfid;
0
```

So $f' : X' \to Y'$ is not a retraction since there does not exist a coretraction. $\square$

Assertion 260 *Suppose given an acyclic fibration $f : G \mathrel{-\!\!\!\square\!\!\!\rightarrow} H$.*

Suppose given $n \in \mathbb{N}$ and an injective graph morphism $\iota : \mathrm{C}_n \to G$.

Then the graph morphism $\iota \cdot f$ is injective.

This assertion is **false**.

Counterexample.

We consider the graphs G, H and the acyclic fibration f as in Example 215. Consider the injective graph morphism $\iota : \mathrm{C}_4 \to G$ given by

$$\begin{aligned} \mathrm{v}_1 \, \mathrm{V}_\iota &= 1, \quad \mathrm{v}_2 \, \mathrm{V}_\iota = 2, \\ \mathrm{v}_3 \, \mathrm{V}_\iota &= 3, \quad \mathrm{v}_4 \, \mathrm{V}_\iota = 2', \end{aligned}$$

and by

$$\begin{aligned} \mathrm{e}_1 \, \mathrm{E}_\iota &= \alpha_1 \, , \quad \mathrm{e}_2 \, \mathrm{E}_\iota = \alpha_3 \, , \\ \mathrm{e}_3 \, \mathrm{E}_\iota &= \alpha_4 \, , \quad \mathrm{e}_4 \, \mathrm{E}_\iota = \alpha_7 \, . \end{aligned}$$

Then $(\mathrm{v}_2) \, \mathrm{V}_{\iota f} = 2 = (\mathrm{v}_4) \, \mathrm{V}_{\iota f}$. So the graph morphism $\iota \cdot f$ is not injective. ▫

Assertion 261 *Suppose given $n \in \mathbb{N}$ and a graph morphism $f : \mathrm{C}_n \to G$. Then there exists $k \in \mathbb{N}$ and a graph morphism $g : \mathrm{C}_k \to G$ such that E_g is injective and $\mathrm{C}_n \, f = \mathrm{C}_k \, g$.*

This assertion is **false**.

Counterexample.

We consider the following graph.

$$G : \quad \boxed{\begin{tikzcd} 1 \arrow[r, "\alpha_3"] \arrow[rr, bend left, "\alpha_1"] & 3 \arrow[r, "\alpha_4"] & 2 \arrow[ll, bend left, "\alpha_2"] \end{tikzcd}}$$

We consider the graph morphism $f : \mathrm{C}_5 \to G$ with

$$\begin{aligned} &\mathrm{v}_0 \, \mathrm{V}_f := 1, \quad \mathrm{v}_1 \, \mathrm{V}_f := 2, \quad \mathrm{v}_2 \, \mathrm{V}_f := 1, \quad \mathrm{v}_3 \, \mathrm{V}_f := 3, \quad \mathrm{v}_4 \, \mathrm{V}_f := 2, \\ &\mathrm{e}_0 \, \mathrm{E}_f := \alpha_1 \, , \quad \mathrm{e}_1 \, \mathrm{E}_f := \alpha_2 \, , \quad \mathrm{e}_2 \, \mathrm{E}_f := \alpha_3 \, , \quad \mathrm{e}_3 \, \mathrm{E}_f := \alpha_4 \, , \quad \mathrm{e}_4 \, \mathrm{E}_f := \alpha_2 \, . \end{aligned}$$

Assume that there exists $k \in \mathbb{N}$ and an injective graph morphism $g : \mathrm{C}_k \to G$ such that $\mathrm{C}_n \, f = \mathrm{C}_k \, g$.

Since $\mathrm{C}_n \, f = G$, we have $k = |\, \mathrm{E}_G \,| = 4$.

Without loss of generality, we have $\mathrm{e}_0 \, \mathrm{E}_g = \alpha_1$.

Hence $\mathrm{e}_1 \, \mathrm{E}_g = \alpha_2$.

Hence $\mathrm{e}_2 \, \mathrm{E}_g = \alpha_3$, using injectivity of E_g .

Hence $\mathrm{e}_3 \, \mathrm{E}_g = \alpha_4$.

Hence $\mathrm{e}_0 \, \mathrm{E}_g = \mathrm{e}_4 \, \mathrm{E}_g = \alpha_2$.

Contradiction. ▫

Chapter 10

Algorithmic treatment of graphs

We use Magma [**2**] to codify finite graphs and graph morphisms in order to perform calculations.

10.1 Implementation of graphs

In the following functions we calculate graphs and graph morphisms as introduced in Definition 45 and 54.

In our implementation, a graph is a tuple consisting of its list of vertices as first entry and its list of edges as second entry.

An edge is a triple consisting of its source vertex as first entry, its name as second entry and its target vertex as third entry.

For instance, in Example 49 we consider the graph

$$G: \quad \boxed{\begin{tikzcd} 1 \arrow[loop left, "\alpha_4"] \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2 & 3 \arrow[l, "\alpha_3"'] & 4 \end{tikzcd}}$$

It has the following codification.

```
G := <[1,2,3,4],[<1,1,2>,<1,2,2>,<3,3,2>,<1,4,1>]>;
> G;
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <1, 2, 2>, <3, 3, 2>, <1, 4, 1> ]>
```

Note that the edges are codified by their indices, so e.g. $1 \xrightarrow{\alpha_2} 2$ is codified as `<1,2,3>`, and $1 \xrightarrow{\alpha_4} 1$ is codified as `<1,4,1>`.

Here,

```
> G[1];
[ 1, 2, 3, 4 ]
```

is the list of the four vertices in the graph G and

```
> G[2];
[ <1, 1, 2>, <1, 2, 2>, <3, 3, 2>, <1, 4, 1> ]
```

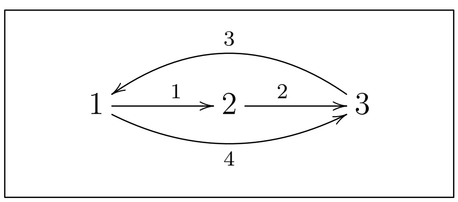
is the list of the four edges in the graph G.

The following function `SetGraphExample` gives us a random graph with exactly `e` edges and `v` vertices.

```
SetGraphExample := function(v,e) // v: number of vertices, e: number of edges
 edges := [];
 for j in [1..e] do
  edges cat:= [<Random([1..v]),j,Random([1..v])>];
 end for;
 return <[i : i in [1..v]],edges>;
end function;
```

For example the following graph has been obtained by this function.

```
> SetGraphExample(3,4);
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 1>, <1, 4, 3> ]>
```



Given `n` in $\mathbb{N}$, with the function `CyclicGraph` we calculate the cyclic graph C_n .

```
CyclicGraph := function(n) // returns cyclic graph with n edges
 return <[i : i in [1..n]],[<i,i,i+1> : i in [1..n-1]] cat [<n,n,1>]>;
end function;
```

For example, we get

```
> CyclicGraph(1);
<[ 1 ], [ <1, 1, 1> ]>
> CyclicGraph(2);
<[ 1, 2 ], [ <1, 1, 2>, <2, 2, 1> ]>
> CyclicGraph(3);
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 1> ]>
> CyclicGraph(4);
<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 4>, <4, 4, 1> ]>
```

$$\mathtt{C}_4 : \quad \begin{tikzcd} \mathrm{v}_2 \arrow[r, bend left, "\mathrm{e}_2"] & \mathrm{v}_3 \arrow[d, bend left, "\mathrm{e}_3"] \\ \mathrm{v}_1 \arrow[u, bend left, "\mathrm{e}_1"] & \mathrm{v}_4 \arrow[l, bend left, "\mathrm{e}_4"] \end{tikzcd}$$

Note that we have $\mathrm{v}_4 = \mathrm{v}_0$ and $\mathrm{e}_4 = \mathrm{e}_0$.

For further use, we abbreviate

```
C := CyclicGraph; // %%
```

Given `n` in $\mathbb{N}$, with the function `DirectGraph` we calculate the direct graph D_n .

```
DirectGraph := function(n)
 return <[i : i in [0..n]],[<i,i,i+1> : i in [0..n-1]]>;
end function;
```

For example, we get

```
> DirectGraph(0);
<[ 0 ], []>
> DirectGraph(1);
<[ 0, 1 ], [ <0, 0, 1> ]>
> DirectGraph(2);
<[ 0, 1, 2 ], [ <0, 0, 1>, <1, 1, 2> ]>
> DirectGraph(3);
<[ 0, 1, 2, 3 ], [ <0, 0, 1>, <1, 1, 2>, <2, 2, 3> ]>
```

$$\mathtt{D}_3 : \quad \hat{\mathrm{v}}_0 \xrightarrow{\hat{\mathrm{e}}_0} \hat{\mathrm{v}}_1 \xrightarrow{\hat{\mathrm{e}}_1} \hat{\mathrm{v}}_2 \xrightarrow{\hat{\mathrm{e}}_2} \hat{\mathrm{v}}_3$$

For further use, we abbreviate

```
D := DirectGraph; // %%
```

With the function `IsThin` we can test if a given graph `G` is thin.

```
IsThin := function(G)
 E := {<e[1],e[3]> : e in G[2]};
 return #E eq #G[2];
end function;
```

For example, given the graph `G` from above we get

```
> IsThin(G);
false
> IsThin(C(4));
true
```

10.2 Implementation of graph morphisms

In our implementation, a graph morphism $f : G \to H$ is a tuple consisting of the map V_f as first entry and the map E_f as second entry.

Such a map on vertices is a list of tuples with each vertex $v_G \in \mathrm{V}_G$ of the graph G as first entry and its image $v_G \, \mathrm{V}_f \in \mathrm{V}_H$ as second entry.

Such a map on edges is a list of tuples with each edge $e_G \in \mathrm{E}_G$ of the graph G as first entry and its image $e_G \, \mathrm{E}_f \in \mathrm{E}_H$ as second entry.

Given graphs `G` and `H` as shown below, for example `f` is a graph morphism from `G` to `H`.

```
G := <[ 1, 2, 3, 4 ], [ <1, 1, 1>, <1, 2, 2>, <3, 3, 2> ]>;
H := <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 2>, <3, 3, 2> ]>;
f := <[ <1, 2>, <2, 2>, <3, 1>, <4, 1> ],
 [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]>;
```

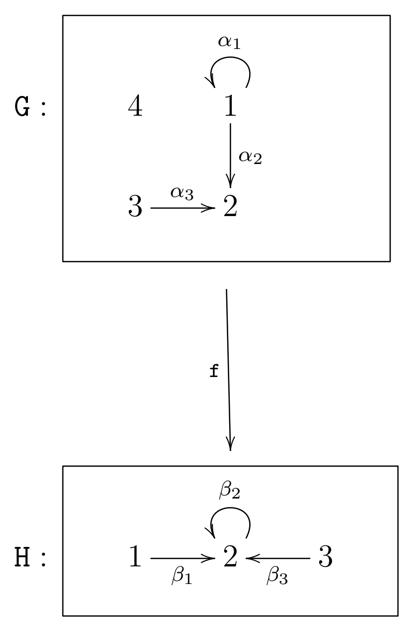
We have the map on the vertices

```
> f[1];
[ <1, 2>, <2, 2>, <3, 1>, <4, 1> ]
```

and the map on the edges

```
> f[2];
[ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]
```

We have



and the graph morphism `f: G` $\to$ `H` with

$$
\begin{array}{llll}
1\,\mathrm{V}_{\mathtt{f}} = 2 & 2\,\mathrm{V}_{\mathtt{f}} = 2 & 3\,\mathrm{V}_{\mathtt{f}} = 1 & 4\,\mathrm{V}_{\mathtt{f}} = 1 \\
\alpha_1\,\mathrm{E}_{\mathtt{f}} = \beta_2 & \alpha_2\,\mathrm{E}_{\mathtt{f}} = \beta_2 & \alpha_3\,\mathrm{E}_{\mathtt{f}} = \beta_1\,. &
\end{array}
$$

With the function `IsGraphMorphism` we can test if a given pair of maps `m` from a given graph `G` to a given graph `H` is a graph morphism.

```
IsGraphMorphism := function(m,G,H) // m = <v,e>
 v := m[1];
 e := m[2];
 // test, if v is a map:
 if not SequenceToMultiset([x[1] : x in v]) eq SequenceToMultiset(G[1]) then
  return false;
 end if;
 if not SequenceToSet([x[2] : x in v]) subset SequenceToSet(H[1]) then
  return false;
 end if;
 // test, if e is a map:
 if not SequenceToMultiset([x[1] : x in e]) eq SequenceToMultiset(G[2]) then
  return false;
 end if;
 if not SequenceToSet([x[2] : x in e]) subset SequenceToSet(H[2]) then
  return false;
 end if;
 // test, if v and e are compatible concerning source and target:
 for x in e do
  if not &and[<x[1][1],x[2][1]> in v, <x[1][3],x[2][3]> in v] then
   return false;
  end if;
 end for;
 return true;
end function;
```

We can confirm that the graph morphism `f` above actually is a graph morphism.

```
> IsGraphMorphism(f,G,H);
true
```

Moreover, we give an example of a pair of maps `g`

```
g := <[<1,2>,<2,2>,<3,2>,<4,1>],
      [<<1,1,1>,<2,2,2>>,<<1,2,2>,<2,2,2>>,<<3,3,2>,<1,1,2>>]>;
```

for which

```
> IsGraphMorphism(g,G,H);
false
```

Using the following functions we calculate the list of all graph morphisms from a given graph `G` to a given graph `H`.

```
IsRightUnique := function(u) // u: relation, e.g. u := [<1,3>,<1,4>,<2,3>],
                             // or u := {<1,3>,<1,4>,<2,3>}
 right_unique := true;
 left_elements := {x[1] : x in u};
 for y in left_elements do
  if #{x[2] : x in u | x[1] eq y} ge 2 then
   right_unique := false;
   break y;
  end if;
 end for;
 return right_unique;
end function;

CompletionsToMaps := function(D,C,u)
 // D/C: list of elements in the domain/codomain,
 // u: right unique relation
 // e.g. D := [1,2,3,4]; C := [1,2,3,4,5]; u := {<1,3>, <3,5>};
 to_be_mapped := [x : x in D | not x in {y[1] : y in u}];
 list := [u];
 for i in to_be_mapped do
  list_new := [];
  for v in list do
   for j in C do
    list_new cat:= [v join {<i,j>}];
   end for;
  end for;
  list := list_new;
 end for;
 return list;
end function;

RelationOnVerticesFromPartialMapOnEdges := function(x);
 // x: partial map on edges from graph G to graph H (G, H not required as data)
 return {<z[1][1],z[2][1]> : z in x} join {<z[1][3],z[2][3]> : z in x};
end function;
```

For further use, we abbreviate

```
RVPME := RelationOnVerticesFromPartialMapOnEdges; // %%

ListGraphMorphisms := function(G,H)
 list := [[]];
 for z in G[2] do
  list_new := [];
  for w in H[2] do
   for x in list do
    x_test := x cat [<z,w>];
```

```
    if IsRightUnique(RVPME(x_test)) then // %%
     list_new cat:= [x_test];
    end if;
   end for;
  end for;
  list := list_new;
 end for;
 list_mor := [];
 for y in list do
  list_completions_to_maps_on_vertices := [Sort(SetToSequence(x)) : x in
    CompletionsToMaps(G[1],H[1],RVPME(y))]; // %%
  list_mor cat:= [<x,y> : x in list_completions_to_maps_on_vertices];
 end for;
 return list_mor;
end function;
```

For example, we can list all graph morphisms between the graphs `G` and `H` given above.

```
> ListGraphMorphisms(G,H);
[
  <[ <1, 2>, <2, 2>, <3, 1>, <4, 1> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 1>, <4, 2> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 1>, <4, 3> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 2>, <4, 1> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <2, 2, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 2>, <4, 2> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <2, 2, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 2>, <4, 3> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <2, 2, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 3>, <4, 1> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <3, 3, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 3>, <4, 2> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <3, 3, 2>> ]>,
  <[ <1, 2>, <2, 2>, <3, 3>, <4, 3> ],
   [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <3, 3, 2>> ]>
]
> ListGraphMorphisms(H,G);
[
    <[ <1, 1>, <2, 1>, <3, 1> ], [ <<1, 1, 2>, <1, 1, 1>>,
       <<2, 2, 2>, <1, 1, 1>>, <<3, 3, 2>, <1, 1, 1>> ]>
]
```

Note that the graph morphism `f: G` $\to$ `H` from above

```
> f;
<[ <1, 2>, <2, 2>, <3, 1>, <4, 1> ],
 [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]>
```

is the first entry in `ListGraphMorphisms(G,H)`.

Given a graph G, we can get the identity id_G with the following function `Identity`.

```
Identity := function(G);
 return <[<x,x> : x in G[1]], [<x,x> : x in G[2]]>;
end function;
```

For example, for `G` as above we get

```
> G;
<[ 1, 2, 3, 4 ], [ <1, 1, 1>, <1, 2, 2>, <3, 3, 2> ]>
> Identity(G);
<[ <1, 1>, <2, 2>, <3, 3>, <4, 4> ],
 [ <<1, 1, 1>, <1, 1, 1>>, <<1, 2, 2>, <1, 2, 2>>, <<3, 3, 2>, <3, 3, 2>> ]>
```

With the function `Is_Injective` we can test if a given graph morphism `f` from a given graph `G` to a given graph `H` is injective.

With the function `Is_Surjective` we can test if it is surjective, with the function `Is_Bijective` we can test if it is bijective.

```
Is_Injective := function(m,G,H)
 // m: G -> H: <morphism_on_vertices,morphism_on_edges>
 return #SequenceToSet([u[2] : u in m[1]]) eq #m[1] and
        #SequenceToSet([u[2] : u in m[2]]) eq #m[2];
end function;

Is_Surjective := function(m,G,H)
 // m: G -> H: <morphism_on_vertices,morphism_on_edges>
 return #SequenceToSet([u[2] : u in m[1]]) eq #H[1] and
        #SequenceToSet([u[2] : u in m[2]]) eq #H[2];
end function;

Is_Bijective := function(f,G,H)
 return Is_Injective(f,G,H) and Is_Surjective(f,G,H); // %%
end function;
```

For example, given `G`, `H` and `f` as above, we get

```
> Is_Injective(f,G,H);
false
> Is_Surjective(f,G,H);
false
> Is_Bijective(f,G,H);
false
```

With the function `ComposeGraphMorphisms` we can compose given graph morphisms $G \xrightarrow{p} H \xrightarrow{q} I$.

```
ComposeGraphMorphisms := function(p,q)
 v := []; // v for vertices
 for x in p[1] do
  v cat:= [<x[1],y[2]> : y in q[1] | x[2] eq y[1]];
 end for;
 e := []; // e for edges
 for x in p[2] do
  e cat:= [<x[1],y[2]> : y in q[2] | x[2] eq y[1]];
 end for;
 return <v,e>;
end function;
```

For example, let

```
p := ListGraphMorphisms(G,H)[8];
q := ListGraphMorphisms(H,H)[1];
```

Then we get

```
> ComposeGraphMorphisms(p,q);
<[ <1, 2>, <2, 2>, <3, 1>, <4, 2> ],
 [ <<1, 1, 1>, <2, 2, 2>>, <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]>
```

The result equals `ListGraphMorphisms(G,H)[2]`.

With the function `IsIsomorphic` we can test if given graphs `G` and `H` are isomorphic.

In case of isomorphic graphs, we additionally give a graph isomorphism as output.

```
IsIsomorphic := function(G,H) // G, H: graphs
 if #G[1] ne #H[1] or #G[2] ne #H[2] then
  return <0,false>;
 end if;
 list := ListGraphMorphisms(G,H); // %%
 for m in list do
  if Is_Bijective(m,G,H) then // %%
   return <m,true>;
  end if;
 end for;
 return <0,false>;
end function;
```

For example:

```
> IsIsomorphic(G,H);
<0, false>
```

With the function `IsSubgraph` we can test if a given graph `G` is a subgraph of a given graph `H`.

If so, we additionally give the inclusion morphism as output.

```
IsSubgraph := function(G,H) // returns true if G is a subgraph of H
 for v in G[1] do
  if not v in H[1] then
   return <0,false>;
  end if;
 end for;
 for e in G[2] do
  if not e in H[2] then
   return <0,false>;
  end if;
 end for;
 m := <[<v,v> : v in G[1]],[<e,e> : e in G[2]]>;
 return <m,true>;
end function;
```

A test whether `G` is a full subgraph of `H`:

```
IsFullSubgraph := function(G,H);
 if not IsSubgraph(G,H)[2] then // %%
  return false;
 end if;
 return not &or[e[1] in G[1] and e[3] in G[1] and not e in G[2] : e in H[2]];
end function;
```

For example, for

```
G1 := <[ 1, 2 ], [ <1, 1, 1>, <1, 2, 2> ]>;
H1 := <[ 1, 2, 3 ], [ <1, 1, 1>, <1, 2, 2>, <2, 3, 1>, <2, 4, 2>, <3, 5, 2> ]>;
```

we get

```
> IsSubgraph(G1,H1);
<<[ <1, 1>, <2, 2> ], [ <<1, 1, 1>, <1, 1, 1>>, <<1, 2, 2>, <1, 2, 2>> ]>, true>
> IsFullSubgraph(G1,H1);
false
```

With the function `VtoE` we can complete a given map on vertices from a given graph `G` to a given thin graph `H` to a graph morphism.

If such a graph morphism does not exist, the function prints this out and returns `<0,0>`.

```
VtoE := function(G,H,Vf) // H thin, Vf = [<1,2>,<2,5>,<3,1>,<4,2>] map on
                         // vertices
```

```
 for e in G[2] do
  if #[h : h in H[2] | h[1] eq [v[2] : v in Vf | v[1] eq e[1]][1] and
     h[3] eq [v[2] : v in Vf | v[1] eq e[3]][1]] eq 0 then
   print "graph morphism does not exist";
   return <0,0>;
  end if;
 end for;
 Ef := [<e,[h : h in H[2] | h[1] eq [v[2] : v in Vf | v[1] eq e[1]][1] and
        h[3] eq [v[2] : v in Vf | v[1] eq e[3]][1]][1]> : e in G[2]];
 return <Vf,Ef>;
end function;
```

E.g. the graph morphism `f` from above can be obtained as

```
> VtoE(G,H,f[1]);
```

With `ListGraphMorphisms_partial` we get the list of graph morphisms `f: G` $\to$ `H` that obey given partial mapping rules, without having to calculate the whole list of graph morphisms from `G` to `H`.

This is useful for the search for quasiisomorphisms.

```
ListGraphMorphisms_partial := function(f,G,H)
 // returns all graph morphisms that obey given partial mapping rule f
 vertices_partial := f[1];
 v1 := [n[1] : n in f[1]];
 edges_partial := f[2];
 e1 := [n[1] : n in f[2]];
 list := [[]];
 for z in G[2] do
  list_new := [];
  if z in e1 then
   H_edges := [n[2] : n in f[2] | n[1] eq z];
  else
   H_edges := H[2];
  end if;
  source := [v : v in H[1] | #[e : e in H[2] | e[1] eq v] ge 1];
  target := [v : v in H[1] | #[e : e in H[2] | e[3] eq v] ge 1];
  if z[1] in v1 then
   source := [n[2] : n in f[1] | n[1] eq z[1]];
  end if;
  if z[3] in v1 then
   target := [n[2] : n in f[1] | n[1] eq z[3]];
  end if;
  H_edges := [h : h in H_edges | h[1] in source and h[3] in target];
  for w in H_edges do
   for x in list do
```

```
    x_test := x cat [<z,w>];
    if IsRightUnique(RVPME(x_test)) then // %%
     list_new cat:= [x_test];
    end if;
   end for;
  end for;
  list := list_new;
 end for;
 list_mor := [];
 for y in list do
  list_completions_to_maps_on_vertices := [Sort(SetToSequence(x)) : x in
    CompletionsToMaps(G[1],H[1],RVPME(y)
    join SequenceToSet(f[1]))]; // %%
  list_mor cat:= [<x,y> : x in list_completions_to_maps_on_vertices];
 end for;
 return list_mor;
end function;
```

For example, for

```
G := <[ 1, 2, 3, 4 ], [ <1, 1, 1>, <1, 2, 2>, <3, 3, 2> ]>;
H := <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 2>, <3, 3, 2> ]>;
```

as above, we get

```
> ListGraphMorphisms_partial(<[<4,1>],[]>,G,H);
[
    <[ <1, 2>, <2, 2>, <3, 1>, <4, 1> ], [ <<1, 1, 1>, <2, 2, 2>>,
    <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <1, 1, 2>> ]>,
    <[ <1, 2>, <2, 2>, <3, 2>, <4, 1> ], [ <<1, 1, 1>, <2, 2, 2>>,
    <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <2, 2, 2>> ]>,
    <[ <1, 2>, <2, 2>, <3, 3>, <4, 1> ], [ <<1, 1, 1>, <2, 2, 2>>,
    <<1, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <3, 3, 2>> ]>
]
```

The function `DisjointUnionCycles` produces a graph isomorphic to a disjoint union of cycles. E.g. for `list := [2,4,5]`, it returns a graph isomorphic to $\mathrm{C}_2 \sqcup \mathrm{C}_4 \sqcup \mathrm{C}_5$.

```
DisjointUnionCycles := function(list) // e.g. list := [2,4,5]
 G1 := [i : i in [1..&+list]]; // vertices
 lists := [[u : u in [1..list[1]]]] cat [[u : u in
  [&+[list[k] : k in [1..i-1]]+1..&+[list[k] : k in [1..i]]]] : i in [2..#list]];
 edges := &cat[[<t[i],t[i+1]> : i in [1..#t-1]] cat [<t[#t],t[1]>] : t in lists];
 edges_numbered := [<edges[i][1],i,edges[i][2]> : i in [1..#edges]];
 G := <G1,edges_numbered>;
 return G;
end function;
```

We abbreviate as follows.

```
DUC := DisjointUnionCycles; // %%
```

E.g. we get

```
> DUC([2,4,5]);
<[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 ], [ <1, 1, 2>, <2, 2, 1>, <3, 3, 4>,
 <4, 4, 5>, <5, 5, 6>, <6, 6, 3>, <7, 7, 8>,
 <8, 8, 9>, <9, 9, 10>, <10, 10, 11>, <11, 11, 7> ]>
```

With the function `DCN` we calculate the diagonal graph morphism $d_{\mathrm{C}_n} : \mathrm{C}_n \sqcup \mathrm{C}_n \to \mathrm{C}_n$ for a given `n`.

```
DCN := function(n)
 G := DUC([n,n]); // %%
 H := C(n); // %%
 f := VtoE(G,H,Sort([<i,i mod n> : i in [1..2*n-1] | not i eq n] cat
      [<n,n>,<2*n,n>]));
 return <G,H,f>;
end function;
```

E.g. we get

```
> DCN(3)[3];
<[ <1, 1>, <2, 2>, <3, 3>, <4, 1>, <5, 2>, <6, 3> ],
 [ <<1, 1, 2>, <1, 1, 2>>, <<2, 2, 3>, <2, 2, 3>>, <<3, 3, 1>, <3, 3, 1>>,
   <<4, 4, 5>, <1, 1, 2>>, <<5, 5, 6>, <2, 2, 3>>, <<6, 6, 4>, <3, 3, 1>> ]>
```

10.3 Calculating a pushout and a pullback of graphs

10.3.1 Calculating a pushout of graphs

The following functions return the equivalence relation that is generated by a given relation `R` on `M`. Here a relation on `M` is a list of pairs with both entries in `M`.

```
RedSeq := function(S); // S: sequence, to be reduced and sorted
 return Sort(SetToSequence(SequenceToSet(S)));
end function;

Equivclasses := function(R,M) // R: relation on set M
 Rinv := [<r[2],r[1]> : r in R];
 Diag := [<m,m> : m in M];
 RR := [r : r in R cat Rinv | not r[1] eq r[2]];
 equivclasses := [];
```

```
 Mtodo := [m : m in M];
 while not #Mtodo eq 0 do
  k := Mtodo[1];
  kclassold := [];
  kclassnew := [k];
  while not #kclassnew eq #kclassold do
   kclassold := kclassnew;
   kclassnew cat:= [j[2] : j in RR | j[1] in kclassold];
   kclassnew := RedSeq(kclassnew); // %%
  end while;
  equivclasses cat:= [kclassnew];
  Mtodo := [u : u in Mtodo | not u in kclassnew];
 end while;
 return equivclasses;
end function;

Equivrelation := function(R,M) // R: relation on set M
 equivclasses := Equivclasses(R,M); // %%
 return Sort(&cat[[<k,l> : k, l in x] : x in equivclasses]);
end function;
```

With the following functions we calculate a pushout in Set.

```
DisjointUnionSets := function(X,Y); // X, Y lists
 return [<1,x> : x in X] cat [<2,y> : y in Y];
end function;


PushoutSets := function(X,Y,X2,f,g); // f : X -> Y, g : X -> X2 maps
 M := DisjointUnionSets(X2,Y); // %%
 R := [ [<<1, g_elt[2]>, <2, f_elt[2]>> : g_elt in g, f_elt in f |
          g_elt[1] eq x and f_elt[1] eq x][1] : x in X];
 equiv := Equivclasses(R,M); // %%
 u := [ [<x2,t> : t in equiv | <1,x2> in t][1] : x2 in X2];
 v := [ [< y,t> : t in equiv | <2, y> in t][1] :  y in  Y];
 return <equiv, u, v>; // pushout, u : X2 -> pushout, v : Y -> pushout
end function;
```

With the following function we calculate a pushout in Gph.

```
PushoutGraphs := function(X,Y,X2,f,g); // f : X -> Y, g : X -> X2
 // graph morphisms, returns the pushout
 vertices := PushoutSets(X[1],Y[1],X2[1],f[1],g[1]); // %%
 edges := PushoutSets(X[2],Y[2],X2[2],f[2],g[2]); // %%
 N := [i : i in [1..#vertices[1]]];
 E := [i : i in [1..#edges[1]]]; // edges without source and target
 EE := [ <Index(vertices[1], [n : n in vertices[1] | <edges[1][e][1][1],
```

```
       edges[1][e][1][2][1]> in n][1]),e, Index(vertices[1], [n :
      n in vertices[1] | <edges[1][e][1][1],edges[1][e][1][2][3]> in n][1])> :
      e in E];
      // edges with source and target
 PP := <N,EE>;
 uN := [<x2, Index(vertices[1], [n[2] : n in vertices[2] | x2 eq n[1]][1]) > :
        x2 in X2[1]]; // <p,p@v>
 uE := [<x2, Index(edges[1], [e[2] : e in edges[2] | x2 eq e[1]][1]) > :
        x2 in X2[2]]; // <p,p@v>, second entry is number of edge
 uEE := [ <x[1], <[ee[1] : ee in EE | ee[2] eq x[2]][1],x[2],[ee[3] : ee in EE |
      ee[2] eq x[2]][1]>> : x in uE]; // <p,p@u>
      // second entry with source and target
 u := <uN,uEE>;
 vN := [<y, Index(vertices[1], [n[2] : n in vertices[3] | y eq n[1]][1]) > :
        y in Y[1]]; // <p,p@v>
 vE := [<y, Index(edges[1], [e[2] : e in edges[3] | y eq e[1]][1]) > :
        y in Y[2]]; // <p,p@v>, , second entry is number of edge
 vEE := [ <x[1], <[ee[1] : ee in EE | ee[2] eq x[2]][1],x[2],[ee[3] : ee in EE |
          ee[2] eq x[2]][1]>> : x in vE]; // <p,p@u>
          // second entry with source and target
 v := <vN,vEE>;
 return <PP,u,v>; // pushout, u : X2 -> pushout, v : Y -> pushout
end function;
```

E.g. we get

```
> PushoutGraphs(G,H,H,f,f);
<<[ 1, 2, 3, 4 ], [ <1, 1, 2>, <2, 2, 2>, <3, 3, 2>, <4, 4, 2> ]>,
   <[ <1, 1>, <2, 2>, <3, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
   <<2, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <3, 3, 2>> ]>,
   <[ <1, 1>, <2, 2>, <3, 4> ], [ <<1, 1, 2>, <1, 1, 2>>,
   <<2, 2, 2>, <2, 2, 2>>, <<3, 3, 2>, <4, 4, 2>> ]>>
```

10.3.2 Calculating a pullback of graphs

With the following function we calculate a pullback in Set.

```
PullbackSets := function(X,Y,Y2,f,g) // f: X -> Y, g: Y2 -> Y, maps between sets
 // given as a list of tuples, e.g. f = [<x_1,y_1>,...,<x_n,y_n>]
 P := Sort([<x,y2> : x in X, y2 in Y2 | #[<s,t> : s in f, t in g | x eq s[1] and
      y2 eq t[1] and s[2] eq t[2]] eq 1]);
 return P;
end function;
```

With the following function we calculate a pullback in Gph.

```
PullbackGraphs := function(X,Y,Y2,f,g)
 // f: X -> Y, g: Y2 -> Y graph morphisms
 vertices := PullbackSets(X[1],Y[1],Y2[1],f[1],g[1]); // %%
 edges := PullbackSets(X[2],Y[2],Y2[2],f[2],g[2]); // %%
 N := [i : i in [1..#vertices]];
 E := [i : i in [1..#edges]]; // edges without source and target
 EE := [<Index(vertices,<edges[e][1][1],edges[e][2][1]>),e,
         Index(vertices,<edges[e][1][3],edges[e][2][3]>)> : e in E];
         // edges with source and target
 PP := <N, EE>;
 vE := [<ee,edges[ee[2]][1]> : ee in EE]; // <p,p@v>
 uE := [<ee,edges[ee[2]][2]> : ee in EE]; // <p,p@u>
 vN := [<n,vertices[n][1]> : n in N]; // <p,p@v>
 uN := [<n,vertices[n][2]> : n in N]; // <p,p@u>
 v := <vN,vE>;
 u := <uN,uE>;
 return <PP,u,v>; // pullback, u: pullback -> Y2, v: pullback -> X
end function;
```

E.g. we get

```
> PullbackGraphs(G,H,G,f,f);
<<[ 1, 2, 3, 4, 5, 6, 7, 8 ], [ <1, 1, 1>, <1, 2, 2>, <1, 3, 3>, <1, 4, 4>,
  <5, 5, 4> ]>,
 <[ <1, 1>, <2, 2>, <3, 1>, <4, 2>, <5, 3>, <6, 4>, <7, 3>, <8, 4> ],
  [ <<1, 1, 1>, <1, 1, 1>>, <<1, 2, 2>, <1, 2, 2>>, <<1, 3, 3>, <1, 1, 1>>,
    <<1, 4, 4>, <1, 2, 2>>, <<5, 5, 4>, <3, 3, 2>> ]>,
 <[ <1, 1>, <2, 1>, <3, 2>, <4, 2>, <5, 3>, <6, 3>, <7, 4>, <8, 4> ],
  [ <<1, 1, 1>, <1, 1, 1>>, <<1, 2, 2>, <1, 1, 1>>, <<1, 3, 3>, <1, 2, 2>>,
   <<1, 4, 4>, <1, 2, 2>>, <<5, 5, 4>, <3, 3, 2>> ]>>
```

10.3.3 Calculating induced morphisms of pushouts and pullbacks of graphs

With the following functions we calculate the induced morphism of a given pushout respectively pullback.

```
InducedMorphismSetsPO := function(X,X2,Y,Y2,u,u2,v,v2)
 // u: X -> X2, u2: X2 -> Y2, v: X -> Y, v2: Y -> Y2
 T := <Y2,u2,v2>;
 P := PushoutSets(X,Y,X2,v,u); // %%
 x2 := [<i,[t[2] : t in T[2] | t[1] in [r[1] : r in P[2] | r[2] eq i]]> :
        i in P[1]];
 y := [<i,[t[2] : t in T[3] | t[1] in [r[1] : r in P[3] | r[2] eq i]]> :
```

```
      i in P[1]];
 c := [<r[1],r[2][1]> : r in x2 | not #r[2] eq 0] cat [<r[1],r[2][1]> :
      r in y | not #r[2] eq 0];
 return RedSeq(c); // %%
end function;

InducedMorphismGraphsPO := function(X,X2,Y,Y2,u,u2,v,v2)
 // u: X -> X2, u2: X2 -> Y2, v: X -> Y, v2: Y -> Y2
 T := <Y2,u2,v2>;
 P := PushoutGraphs(X,Y,X2,v,u); // %%
 x2_vertices := [<i,[t[2] : t in T[2][1] | t[1] in [r[1] : r in P[2][1] |
                 r[2] eq i]]> : i in P[1][1]];
 x2_edges := [<i,[t[2] : t in T[2][2] | t[1] in [r[1] : r in P[2][2] |
              r[2] eq i]]> : i in P[1][2]];
 y_vertices := [<i,[t[2] : t in T[3][1] | t[1] in [r[1] : r in P[3][1] |
                r[2] eq i]]> : i in P[1][1]];
 y_edges := [<i,[t[2] : t in T[3][2] | t[1] in [r[1] : r in P[3][2] |
             r[2] eq i]]> : i in P[1][2]];
 c_vertices := [<r[1],r[2][1]> : r in x2_vertices | not #r[2] eq 0]
               cat [<r[1],r[2][1]> : r in y_vertices | not #r[2] eq 0];
 c_edges := [<r[1],r[2][1]> : r in x2_edges | not #r[2] eq 0]
            cat [<r[1],r[2][1]> : r in y_edges | not #r[2] eq 0];
 c := <RedSeq(c_vertices),RedSeq(c_edges)>; // %%
 return c;
end function;
```

This allows us to decide whether a given commutative quadrangle is a pushout:

```
IsPushoutGraphs := function(X,X2,Y,Y2,u,u2,v,v2)
 // u: X -> X2, u2: X2 -> Y2, v: X -> Y, v2: Y -> Y2
 p := InducedMorphismGraphsPO(X,X2,Y,Y2,u,u2,v,v2); // %%
 return Is_Bijective(p,PushoutGraphs(X,Y,X2,v,u)[1],Y2); // %%
end function;
```

With the following functions we calculate the induced morphism of a given pullback.

```
PullbackSets_num := function(X,Y,Y2,f,g) // f: X -> Y, g: Y2 -> Y
                                         // maps between sets
  // given as a list of tuples, e.g. f = [<x_1,y_1>,...,<x_n,y_n>]
 P := Sort([<x,y2> : x in X, y2 in Y2 | #[<s,t> : s in f, t in g |
           x eq s[1] and y2 eq t[1] and s[2] eq t[2]] eq 1]);
 PP := [i : i in [1..#P]];
 v := [<i,P[i][1]> : i in PP]; // <p,p@v>
 u := [<i,P[i][2]> : i in PP]; // <p,p@u>
 num := [<i,P[i]> : i in [1..#P]];
 return <PP,u,v,num>; // pullback PP, u: PP -> Y2, v: PP -> X
end function;
```

```
PullbackGraphs_num := function(X,Y,Y2,f,g) // f: X -> Y, g: Y2 -> Y
                                           // graph morphisms
 vertices := PullbackSets(X[1],Y[1],Y2[1],f[1],g[1]); // %%
 edges := PullbackSets(X[2],Y[2],Y2[2],f[2],g[2]); // %%
 N := [i : i in [1..#vertices]];
 E := [i : i in [1..#edges]]; // edges without source and target
 EE := [<Index(vertices,<edges[e][1][1],edges[e][2][1]>),e,
   Index(vertices,<edges[e][1][3],edges[e][2][3]>)> : e in E];
   // edges with source and target
 PP := <N, EE>;
 vE := [<ee,edges[ee[2]][1]> : ee in EE]; // <p,p@v>
 uE := [<ee,edges[ee[2]][2]> : ee in EE]; // <p,p@u>
 vN := [<n,vertices[n][1]> : n in N]; // <p,p@v>
 uN := [<n,vertices[n][2]> : n in N]; // <p,p@u>
 v := <vN,vE>;
 u := <uN,uE>;
 numvertices := [<i,vertices[i]> : i in [1..#vertices]];
 numedges := [<i,edges[i]> : i in [1..#edges]];
 num := <numvertices,numedges>;
 return <PP,u,v,num>; // pullback PP, u: PP -> Y2, v: PP -> X
end function;

InducedMorphismSetsPB := function(X,X2,Y,Y2,u,u2,v,v2)
 // u: X -> X2, u2: X2 -> Y2, v: X -> Y, v2: Y -> Y2
 T := <X,v,u>;
 P := PullbackSets_num(Y,Y2,X2,v2,u2); // %%
 c_uncode := [<t,<[r[2] : r in T[2] | r[1] eq t][1],[r[2] : r in T[3] |
             r[1] eq t][1]>> : t in T[1]];
 c := [<r[1],Index([p[2] : p in P[4]],r[2])> : r in c_uncode];
 return c;
end function;

InducedMorphismGraphsPB := function(X,X2,Y,Y2,u,u2,v,v2)
 // u: X -> X2, u2: X2 -> Y2, v: X -> Y, v2: Y -> Y2
 T := <X,v,u>;
 P := PullbackGraphs_num(Y,Y2,X2,v2,u2); // %%
 c_uncode_vertices := [<t,<[r[2] : r in T[2][1] | r[1] eq t][1],[r[2] :
                       r in T[3][1] | r[1] eq t][1]>> : t in T[1][1]];
 c_uncode_edges := [<t,<[r[2] : r in T[2][2] | r[1] eq t][1],[r[2] :
                    r in T[3][2] | r[1] eq t][1]>> : t in T[1][2]];
 c_vertices := [<r[1],Index([p[2] : p in P[4][1]],r[2])> :
                r in c_uncode_vertices];
 c_edges := [<r[1],Index([p[2] : p in P[4][2]],r[2])> : r in c_uncode_edges];
 c := <c_vertices,c_edges>;
 return c;
end function;
```

This allows us to decide whether a given commutative quadrangle is a pullback:

```
IsPullbackGraphs := function(X,X2,Y,Y2,u,u2,v,v2)
 // u: X -> X2, u2: X2 -> Y2, v: X -> Y, v2: Y -> Y2
 p := InducedMorphismGraphsPB(X,X2,Y,Y2,u,u2,v,v2); // %%
 return Is_Bijective(p,X,PullbackGraphs_num(Y,Y2,X2,v2,u2)[1]); // %%
end function;
```

10.4 Calculating tree graphs

With the following functions we calculate the graph $\mathrm{Tree}(x, X)$ at a given vertex $x \in \mathrm{V}_X$ of a given graph X if $\mathrm{Tree}(x, X)$ is finite.

The function `CyclesFromVertex` returns `true` if cycles in a given graph `G` exist that contain a given vertex `x`.

```
CyclesFromVertex := function(x,G)  // G graph, x vertex in G
 if #[a : a in G[2] | a[1] eq x] eq 0 then
  return false;
 end if;
 S := {};   // S: set of vertices to achieve
 Snew := {x};
 while not #Snew eq #S do
  S := Snew;
  Snew join:= {a[3] : a in G[2] | a[1] in S};
  if x in {a[3] : a in G[2] | a[1] in S} then
   return true;
  end if;
 end while;
 return false;
end function;
```

For example, for the graph

```
G2 := <[ 1, 2, 3 ], [ <2, 1, 2>, <3, 2, 2>, <2, 3, 3>, <1, 4, 2> ]>;
```

the function yields

```
> CyclesFromVertex(1,G2);
false
> CyclesFromVertex(2,G2);
true
> CyclesFromVertex(3,G2);
true
```

Given a graph `G` and a vertex `r` of `G`, the function `VerticesToAchieve` returns the list of vertices v in `G` for which there exists a path from `r` to v.

```
VerticesToAchieve := function(G,r) // G: graph, r vertex in G
 if #[a : a in G[2] | a[1] eq r] eq 0 then
  return [];
 end if;
 S := {};   // S: set of vertices to achieve
 Snew := {r};
 while not #Snew eq #S do
  S := Snew;
  Snew join:= {a[3] : a in G[2] | a[1] in S};
 end while;
 return Sort(SetToSequence(Snew));
end function;
```

E.g. we get

```
> VerticesToAchieve(G2,1);
[ 1, 2, 3 ]
> VerticesToAchieve(G2,2);
[ 2, 3 ]
> VerticesToAchieve(G2,3);
[ 2, 3 ]
```

The following function `CyclesInPathFromx` returns `true` if some path from a given vertex `x` in a given graph `G` can be restricted to a graph morphism to a cyclic graph C_n for some n.

```
CyclesInPathFromx := function(x,G)
 return &or[CyclesFromVertex(v,G) : v in VerticesToAchieve(G,x)];
end function;
```

E.g. we get

```
> CyclesInPathFromx(1,G2);
true
```

The function `Paths` returns the list of paths in `G` starting in `x` if there are finitely many. Such a path is given as a list of edges.

```
Paths := function(x,G) // G graph, x vertex in G[1],
 if not CyclesInPathFromx(x,G) then // %%
  Listofpaths := [[[]]];
  if not #[[[e] : e in G[2] | e[1] eq x]] eq 0 then
   Listofpaths cat:= [[[e] : e in G[2] | e[1] eq x]];
  else
```

```
   return &cat(Listofpaths);
  end if;
  newpaths := [<p,[e : e in G[2] | e[1] eq p[#p][3]]> :
               p in Listofpaths[#Listofpaths] | not #[e : e in G[2] |
               e[1] eq p[#p][3]] eq 0];
  while not #newpaths eq 0 do
   Listofpaths cat:= [&cat[&cat[[n[1] cat [n[2][i]]] : i in [1..#n[2]]] :
                      n in newpaths]];
   newpaths := [<p,[e : e in G[2] | e[1] eq p[#p][3]]> :
               p in Listofpaths[#Listofpaths] | not #[e : e in G[2] |
               e[1] eq p[#p][3]] eq 0];
  end while;
  return Listofpaths;
 end if;
 return "infinite";
end function;
```

The function `TreeOfPaths` returns the graph $\mathrm{Tree}(x, G)$ in case it is finite, for a given graph `G` and one of its vertices `x`.

```
TreeOfPaths := function(x,G) // G graph, x vertex in G
 if not CyclesInPathFromx(x,G) then // %%
  P := Paths(x,G); // %%
  V := &cat(P);
  E := [<[],<[],p[1],p>,p> : p in P[2]];
  E cat:= [<[p[i] : i in [1..#p-1]],<[p[i] : i in [1..#p-1]],p[#p],p>,p> :
          p in V | #p ge 2];
  return <V,E>;
 end if;
 return "infinite";
end function;
```

For example, for

```
G3 := <[ 1, 2, 3 ], [ <1, "a", 2>, <2, "b", 3>, <1, "c", 3> ]>;
C(3);
```

we get

```
> Paths(1,G3);
[
    [
        []
    ],
    [
        [ <1, "a", 2> ],
        [ <1, "c", 3> ]
```

```
    ],
    [
        [ <1, "a", 2>, <2, "b", 3> ]
    ]
]
> Paths(1,C(3));
infinite
> TreeOfPaths(1,G3);
<[
    [],
    [ <1, "a", 2> ],
    [ <1, "c", 3> ],
    [ <1, "a", 2>, <2, "b", 3> ]
], [
    <[], <[], <1, "a", 2>, [ <1, "a", 2> ]>, [ <1, "a", 2> ]>,
    <[], <[], <1, "c", 3>, [ <1, "c", 3> ]>, [ <1, "c", 3> ]>,
    <[ <1, "a", 2> ],
     <[ <1, "a", 2> ], <2, "b", 3>, [ <1, "a", 2>, <2, "b", 3> ]>,
      [ <1, "a", 2>, <2, "b", 3> ]>]>
> TreeOfPaths(1,C(3));
infinite
```

With the function `IsTreeDef` we can calculate whether a given graph `G` is a tree using Definition 108.

```
IsTreeDef := function(G)
 for r in G[1] do // searching root
  if #[v : v in [g : g in G[1] | not g eq r] | not #[e : e in G[2] |
       e[3] eq v] eq 1] eq 0 then // (Tree 1)
   if #[e : e in G[2] | e[3] eq r] eq 0 then // (Tree 2)
    if &and[v in VerticesToAchieve(G,r) : v in G[1]] then // (Tree 3) // %%
     return true;
    end if;
   end if;
  end if;
 end for;
 return false;
end function;
```

Cf. the function `IsTree` in §10.5 below.

10.5 Testing properties of graph morphisms

With the following functions we test properties of graph morphisms as discussed in §3.

```
ListOfnCycles := function(G,n) // G: graph
 return ListGraphMorphisms(C(n),G); // %%
end function;
```

The function `Cnf_Bij` returns `true` if $(\mathrm{C}_n\,, f)_{\mathrm{Gph}}$ is bijective for a given graph morphism `f: G` $\to$ `H` and for a given number `n`.

```
Cnf_Bij := function(f,G,H,n) // G, H: graphs, f: G -> H: graph morphism
 if not #ListOfnCycles(G,n) eq #ListOfnCycles(H,n) then // %%
  return false;
 end if;
 if SequenceToSet([ComposeGraphMorphisms(ListOfnCycles(G,n)[i],f) : i in
     [1..#ListOfnCycles(G,n)]]) eq SequenceToSet(ListOfnCycles(H,n)) then
  return true; // %%
 end if;
 return false;
end function;
```

Given a graph morphism `f: G -> H` and an upper bound `ub`, the function `IsQis_Bound` returns `true` if the map $(\mathrm{C}_k\,, f)_{\mathrm{Gph}}$ is bijective for $k \leqslant$ `ub`.

```
IsQis_Bound := function(f,G,H,ub) // ub: upper bound
 i := 1;
 while Cnf_Bij(f,G,H,i) and i le ub do // %%
  i := i+1;
 end while;
 return i eq ub+1;
end function;
```

With the function `IsFibration` we can calculate whether a given graph morphism `f: G -> H` is a fibration or not; cf. Definition 127.(1).

```
IsFibration := function(f,G,H) // G, H: graphs, f: G -> H graph morphism
 for x in G[1] do
  y := [a[2] : a in f[1] | a[1] eq x][1];
  for b in [h : h in H[2] | h[1] eq y] do
   if #[0 : a in G[2] | <a,b> in f[2] and a[1] eq x] eq 0 then
    return false;
   end if;
  end for;
 end for;
 return true;
end function;
```

With the function `IsEtaleFibration` we can calculate whether a given graph morphism `f: G -> H` is an etale fibration or not; cf. Definition 127.(2).

```
IsEtaleFibration := function(f,G,H) // G, H: graphs, f: G -> H graph morphism
 for x in G[1] do
  y := [a[2] : a in f[1] | a[1] eq x][1];
  if not #[e : e in G[2] | e[1] eq x] eq #[h : h in H[2] | h[1] eq y] then
   return false;
  elif not Sort([a[2] : a in f[2] | a[1][1] eq x])
           eq Sort([h : h in H[2] | h[1] eq y]) then
    return false;
  end if;
 end for;
 return true;
end function;
```

For example, given

```
G := <[1,2,3],[<1,1,2>,<1,2,3>]>;
f := VtoE(G,D(1),[<1,0>,<2,1>,<3,1>]);
```

we get

```
> IsFibration(f,G,D(1));
true
> IsEtaleFibration(f,G,D(1));
false
```

With the function `IsFibrant` we can calculate whether a given graph `X` is fibrant or not; cf. Definition 135.

```
IsFibrant := function(X)
 return &and[not #[e : e in X[2] | e[1] eq v] eq 0: v in X[1]];
end function;
```

For example, we get

```
> IsFibrant(C(2));
true
> IsFibrant(D(2));
false
```

With the functions `AcCofib1to4` and `AcCofib5` we can check if a given graph morphism `f: G -> H` satisfies the properties (AcCofib 1–4) respectively (AcCofib 5).

The function `IsAcCofib` checks all properties (AcCofib 1–5), i.e. it decides whether `f` is an acyclic cofibration.

```
AcCofib1to4 := function(f,G,H) // f: G -> H graph morphism
 if #[0 : x in H[1] | #[0 : a in G[1] | <a,x> in f[1]] ge 2] ge 1 then
    // (AcCofib 1)
  return false;
 end if;
 if #[0 : x in H[2] | #[0 : a in G[2] | <a,x> in f[2]] ge 2] ge 1 then
    // (AcCofib 2)
  return false;
 end if;
 HH1 := [x : x in H[1] | #[0 : a in G[1] | <a,x> in f[1]] eq 0];
 HH2 := [x : x in H[2] | #[0 : a in G[2] | <a,x> in f[2]] eq 0];
 if #[0 : x in HH1 | not #[a : a in H[2] | a[3] eq x] eq 1] ge 1 then
    // (AcCofib 3)
  return false;
 end if;
 if #[0 : x in HH2 | not x[3] in HH1] ge 1 then
    // (AcCofib 4)
  return false;
 end if;
 return true;
end function;

AcCofib5 := function(f,G,H)
 HH1 := [x : x in H[1] | #[0 : a in G[1] | <a,x> in f[1]] eq 0];
 max := #H[2];
 L := [];
 for i in [1..max] do
  L cat:= [ListGraphMorphisms(D(i),H)]; // %%
 end for;
 for v in HH1 do
  list := [];
  for i in [1..max] do
   list cat:= [l : l in L[i] | not l[1][1][2] in HH1 and l[1][#l[1]][2] eq v];
  end for;
  if #list eq 0 then
   return false;
  end if;
 end for;
 return true;
end function;

IsAcCofib := function(f,G,H)
 return AcCofib1to4(f,G,H) and AcCofib5(f,G,H);
end function;
```

For example, for `f := VtoE(D(0),D(2),[<0,0>]);`

we get

```
> IsAcCofib(f,D(0),D(2));
true
```

For example, for

```
G := <[],[]>;
f := <[],[]>;
```

we get

```
> AcCofib1to4(f,G,C(1));
true
> AcCofib5(f,G,C(1)); // %%
false
```

With the function `IsTree` we can calculate whether a given graph `G` is a tree using Remark 178.

```
IsTree := function(G)
 for x in G[1] do
  if IsAcCofib(VtoE(D(0),G,[<0,x>]),D(0),G) then // %%
   return true;
  end if;
 end for;
 return false;
end function;
```

10.6 Testing the sufficient condition of Proposition 210 for graph morphisms

The function `Unitargeting` returns the unitargeting edges in `H` with respect to `f`.

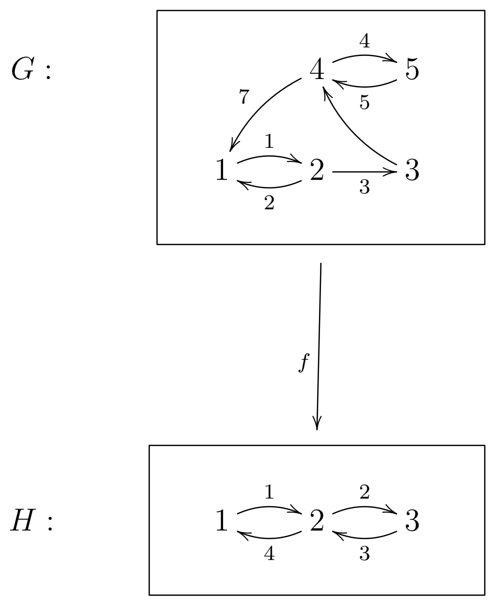
```
Unitargeting := function(f,G,H)
 return [e : e in H[2] | #RedSeq([ee[1][3] : ee in f[2] | ee[2] eq e]) eq 1];
  // %%
end function;
```

For example, using functions from §10.7, for

```
G := trygraph(3);
H := c2chain(3);
f := tryacyclic(3);
```

we get

```
> G;
<[ 1, 2, 3, 4, 5 ], [ <1, 1, 2>, <2, 2, 1>, <2, 3, 3>, <4, 4, 5>, <5, 5, 4>,
                      <3, 6, 4>, <4, 7, 1> ]>
> H;
<[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>
> f;
<[ <1, 1>, <2, 2>, <3, 3>, <4, 2>, <5, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
 <<2, 2, 1>, <2, 4, 1>>, <<2, 3, 3>, <2, 2, 3>>, <<4, 4, 5>, <2, 2, 3>>,
 <<5, 5, 4>, <3, 3, 2>>, <<3, 6, 4>, <3, 3, 2>>, <<4, 7, 1>, <2, 4, 1>> ]>
> Unitargeting(f,G,H);
[ <1, 1, 2>, <3, 3, 2>, <2, 4, 1> ]
```



The function `Uni` tests if the property (Uni) holds for the given graph morphism `f: G -> H`.

```
Uni := function(f,G,H)
 U := Unitargeting(f,G,H); // %%
 HH := <H[1],[e : e in H[2] | not e in U]>; // H without unitargeting edges
 n := Minimum([#HH[1],#HH[2]]);
 return &and[#ListGraphMorphisms(C(i),HH) eq 0 : i in [1..n]]; // %%
end function;
```

With the function `SuffCond` we test if our sufficient condition for a morphism to be a quasi-isomorphism holds; cf. Proposition 210.

```
SuffCond := function(f,G,H)
 return <IsEtaleFibration(f,G,H),Uni(f,G,H)>; // %%
end function;
```

For example, we get

```
> Uni(f,G,H);
true
> SuffCond(f,G,H);
<true, true>
```

For

```
G := <[ 1, 2, 3 ], [ <2, 1, 2>, <3, 2, 1>, <3, 3, 1> ]>;
H := <[ 1, 2, 3 ], [ <2, 1, 2>, <3, 2, 1>, <1, 3, 3> ]>;
L := ListGraphMorphisms(G,H);
M := ListGraphMorphisms(H,G);
```

we get

```
> M;
[
    <[ <1, 2>, <2, 2>, <3, 2> ], [ <<2, 1, 2>, <2, 1, 2>>,
       <<3, 2, 1>, <2, 1, 2>>, <<1, 3, 3>, <2, 1, 2>> ]>
]
> L;
[
    <[ <1, 2>, <2, 2>, <3, 2> ], [ <<2, 1, 2>, <2, 1, 2>>,
       <<3, 2, 1>, <2, 1, 2>>, <<3, 3, 1>, <2, 1, 2>> ]>,
    <[ <1, 1>, <2, 2>, <3, 3> ], [ <<2, 1, 2>, <2, 1, 2>>,
       <<3, 2, 1>, <3, 2, 1>>, <<3, 3, 1>, <3, 2, 1>> ]>,
    <[ <1, 3>, <2, 2>, <3, 1> ], [ <<2, 1, 2>, <2, 1, 2>>,
       <<3, 2, 1>, <1, 3, 3>>, <<3, 3, 1>, <1, 3, 3>> ]>
]
> SuffCond(M[1],H,G);
<true, false>
> SuffCond(L[1],G,H);
<false, false>
> SuffCond(L[2],G,H);
<false, true>
> SuffCond(L[3],G,H);
<false, true>
```

10.7 Functions to calculate examples in §9.1

With the following functions we calculated the examples mentioned in §9.1.

```
c2chain := function(n)
 edges := [<i,i,i+1> : i in [1..n-1]] cat [<i+1,2*n-1-i,i> : i in [1..n-1]];
 edges_tosort := Sort([<e[2],e[1],e[3]> : e in edges]);
 edges := [<e[2],e[1],e[3]> : e in edges_tosort];
 return <[i : i in [1..n]],edges>;
end function;

trygraph := function(n) // n geq 3
 edges := [<3*n-5,3*n-5,3*n-4>,<3*n-4,3*n-4,3*n-5>,<3*n-5,6*n-11,3*n-8>,
           <3*n-6,6*n-12,3*n-5>];
 edges cat:= &cat[[<3*k-2,3*k-2,3*k-1>,<3*k-1,3*k-1,3*k-2>,<3*k-1,3*k,3*k>] :
                 k in [1..n-2]]; // innerhalb der Stufen
 edges cat:= &cat[[<3*k+1,3*n-5+3*k,3*k-2>,<3*k,3*n-6+3*k,3*k+1>,
                   <3*k,3*n-4+3*k,3*k+3>] : k in [1..n-3]];
 edges_tosort := Sort([<e[2],e[1],e[3]> : e in edges]);
 edges := [<e[2],e[1],e[3]> : e in edges_tosort];
 G := <[i : i in [1..3*n-4]],edges>;
 return G;
end function;

tryacyclic := function(n) // trygraph -> c2chain
 vertices := [<1,1>,<2,2>,<4,2>,<3*n-6,n>,<3*n-4,n>];
 vertices cat:= &cat[[<3*k-6,k>,<3*k-4,k>,<3*k-2,k>] : k in [3..n-1]];
 if n eq 3 then
  edges := [<<1,1,2>,<1,1,2>>,<<2,2,1>,<2,2*n-2,1>>,<<2,3,3>,<2,2,3>>,
            <<4,4,5>,<2,2,3>>,<<5,5,4>,<3,2*n-3,2>>,
            <<3,3*n-3,4>,<3,2*n-3,2>>,<<4,3*n-2,1>,<2,2*n-2,1>>];
 else
  edges := [<<1,1,2>,<1,1,2>>,<<2,2,1>,<2,2*n-2,1>>,<<2,3,3>,<2,2,3>>,
            <<4,4,5>,<2,2,3>>,<<5,5,4>,<3,2*n-3,2>>,
            <<3,3*n-3,4>,<3,2*n-3,2>>,<<4,3*n-2,1>,<2,2*n-2,1>>,
            <<7,3*n+1,4>,<3,2*n-3,2>>,<<3*n-7,3*n-6,3*n-6>,<n-1,n-1,n>>,
            <<3*n-5,3*n-5,3*n-4>,<n-1,n-1,n>>,
            <<3*n-4,3*n-4,3*n-5>,<n,n,n-1>>,
            <<3*n-9,6*n-13,3*n-6>,<n-1,n-1,n>>,
            <<3*n-6,6*n-12,3*n-5>,<n,n,n-1>>];
  edges cat:= &cat[[<<3*k-4,3*k-3,3*k-3>,<k,k,k+1>>,
   <<3*k-2,3*k-2,3*k-1>,<k,k,k+1>>,<<3*k-1,3*k-1,3*k-2>,<k+1,2*n-1-k,k>>,
   <<3*k-6,3*n-10+3*k,3*k-3>,<k,k,k+1>>,
   <<3*k-3,3*n-9+3*k,3*k-2>,<k+1,2*n-1-k,k>>,
   <<3*k+1,3*n-5+3*k,3*k-2>,<k+1,2*n-1-k,k>>] : k in [3..n-2]];
 end if;
 edges_tosort := Sort([<<e[1][2],e[1][1],e[1][3]>,e[2]> : e in edges]);
 edges := [<<e[1][2],e[1][1],e[1][3]>,e[2]> : e in edges_tosort];
 return <Sort(vertices),edges>;
end function;
```

```
glue_vertices := function(G,list) // e.g. list := [[2,3],[1,4,5]] list of
 // sublists of vertices to glue
 list := [RedSeq(l) : l in list | #RedSeq(l) ge 2]; // %%
 vertices_to_glue := RedSeq(&cat(list)); // %%
 vertices_left_over := [n : n in G[1] | not n in vertices_to_glue];
 vertices := Sort([l[1] : l in list] cat vertices_left_over);
 edges_1 := [e : e in G[2] | not e[1] in vertices_to_glue
             and not e[3] in vertices_to_glue];
 edges := [];
 for e in G[2] do
  if not e[1] in vertices_to_glue and not e[3] in vertices_to_glue then
   edges cat:= [e];
  else if not e[1] in vertices_to_glue then
   edges cat:= [<e[1],e[2],[l[1] : l in list | e[3] in l][1]>];
  else
   edges cat:= [<[l[1] : l in list | e[1] in l][1],e[2],e[3]>];
  end if;
 end if;
 end for;
 edges_named := [];
 for e in edges do
  if e[1] in &cat(list) then
   e1_new := [l : l in list | e[1] in l][1][1];
  else
   e1_new := e[1];
  end if;
  if e[3] in &cat(list) then
   e3_new := [l : l in list | e[3] in l][1][1];
  else
   e3_new := e[3];
  end if;
  edges_named cat:= [<e1_new,e[2],e3_new>];
 end for;
 return <vertices,edges_named>;
end function;

glue_vertices_including_edges := function(G,list)
 // e.g. list := [[2,3],[1,4,5]] list of sublists of vertices to glue
 G := glue_vertices(G,list); // %%
 edges := [];
 for g in G[2] do
  if #[e : e in edges | e[1] eq g[1] and e[3] eq g[3]] eq 0 then
   edges cat:= [g];
  end if;
 end for;
 return <G[1],edges>;
end function;
```

For example, for

```
G := <[1,2,3],[<1,1,2>]>;
H := <[1,2,3],[<1,1,2>,<1,2,2>,<2,3,3>,<2,4,3>]>;
```

we get

```
> glue_vertices(G,[[1,2]]);
<[ 1, 3 ], [ <1, 1, 1> ]>
> glue_vertices(H,[[1,2]]);
<[ 1, 3 ], [ <1, 1, 1>, <1, 2, 1>, <1, 3, 3>, <1, 4, 3> ]>
> glue_vertices_including_edges(H,[[1,2]]);
<[ 1, 3 ], [ <1, 1, 1>, <1, 3, 3> ]>
```

```
try_id_vertices := function(n) // n ge 3
 return [[3*i,3*i+2] : i in [1..n-2]];
end function;

idtrygraph := function(n)
 return glue_vertices_including_edges(trygraph(n),try_id_vertices(n));
  // %%
end function;

tryfactorization := function(n) // trygraph --> idtrygraph
 T := try_id_vertices(n); // %%
 G := trygraph(n); // %%
 vertices := Sort([<t[2],t[1]> : t in T] cat [<i,i> : i in [1..3*n-4] |
                  not i in [t[2] : t in T]]);
 edges_to_map := trygraph(n)[2];
 edges_images := idtrygraph(n)[2];
 edges := [];
 for e in edges_to_map do
  if e in edges_images then
   im_e := e;
  else
   im_e1 := [n[2] : n in vertices | n[1] eq e[1]][1];
   im_e3 := [n[2] : n in vertices | n[1] eq e[3]][1];
   im_e := [edge : edge in edges_images |
            edge[1] eq im_e1 and edge[3] eq im_e3][1];
  end if;
  edges cat:= [<e,im_e>];
 end for;
 return <vertices,edges>;
end function;
```

```
idtryacyclic := function(n) // idtrygraph -> c2chain
 T := try_id_vertices(n); // %%
 T2 := [t[2] : t in try_id_vertices(n)]; // %%
 G :=  idtrygraph(n); // %%
 f := tryacyclic(n); // %%
 vertices := [n : n in f[1] | n[1] in G[1]];
 edges := [];
 for e in f[2] do
  if e[1] in G[2] then
   edges cat:= [e];
  else
   e1 := e[1][1];
   e3 := e[1][3];
   if e[1][1] in T2 then
    e1 := [t[1] : t in T | t[2] eq e[1][1]][1];
   end if;
   if e[1][3] in T2 then
    e3 := [t[1] : t in T | t[2] eq e[1][3]][1];
   end if;
   edges cat:= [<<e1,e[1][2],e3>,e[2]>];
  end if;
 end for;
 edges2 := [];
 for e in edges do
  if not <e[1][1],e[1][3]> in [<e[1][1],e[1][3]> : e in edges2] then
   edges2 cat:= [e];
  end if;
 end for;
 return <vertices,edges2>;
end function;

Doublecyclic := function(n)
 C := c2chain(n); // %%
 return <C[1],C[2] cat [<1,#C[2]+1,#C[1]>] cat [<#C[1],#C[2]+2,1>]>;
end function;

Trygraph := function(n)
 T := trygraph(n); // %%
 return <T[1],T[2] cat [<1,#T[2]+1,#T[1]>, <#T[1],#T[2]+2,1>, <#T[1]-2,#T[2]+3,1>]>;
end function;

Tryacyclic := function(n) // Trygraph -> Doublecyclic
 T := Trygraph(n); // %%
 t := #T[2];
 D := Doublecyclic(n); // %%
 d := #D[2];
 f := tryacyclic(n); // %%
```

```
 return <f[1],f[2] cat [<T[2][t-2],D[2][d-1]>,
 <T[2][t-1],D[2][d]>, <T[2][t],D[2][d]>]>;
end function;


idTrygraph := function(n)
 T := idtrygraph(n); // %%
 t := [r[2] : r in T[2]];
 return <T[1],T[2] cat [<1,t[#t]+1,T[1][#T[1]-1]>]
         cat [<T[1][#T[1]-1],t[#t]+2,1>]>;
end function;


Tryfactorization := function(n) // Trygraph -> idTrygraph
 T := trygraph(n); // %%
 TT := idtrygraph(n); // %%
 t := [r[2] : r in TT[2]];
 f := tryfactorization(n); // %%
 return <f[1],f[2] cat [<<1,#T[2]+1,#T[1]>,<1,t[#t]+1,TT[1][#TT[1]-1]>>]
         cat [<<#T[1],#T[2]+2,1>,<TT[1][#TT[1]-1],t[#t]+2,1>>]
         cat [<<#T[1]-2,#T[2]+3,1>,<TT[1][#TT[1]-1],t[#t]+2,1>>]>;
end function;


idTryacyclic := function(n) // idTrygraph -> Doublecyclic
 f := idtryacyclic(n); // %%
 T := idtrygraph(n); // %%
 t := [r[2] : r in T[2]];
 C := c2chain(n); // %%
 return <f[1],f[2] cat [<<1,t[#t]+1,T[1][#T[1]-1]>,<1,#C[2]+1,#C[1]>>]
        cat [<<T[1][#T[1]-1],t[#t]+2,1>,<#C[1],#C[2]+2,1>>]>;
end function;


cncm := function(n,m) // cn glued to cm at vertex n
 V := [i : i in [1..n+m-1]];
 E := [<i,i,i+1> : i in [1..n-1]] cat [<n,n,1>] cat [<n+i,n+i+1,n+i+1> :
       i in [0..m-2]] cat [<n+m-1,n+m,n>];
 return <V,E>;
end function;


CnCm := function(n,m)
 V := [i : i in [1..n+m]];
 E := [<i,i,i+1> : i in [1..n+m-1]] cat [<n+m,n+m,1>]
      cat [<n,n+m+1,1>,<n+m,n+m+2,n+1>];
 return <V,E>;
end function;
```

```
cncmqis := function(n,m)
 return VtoE(CnCm(n,m),cncm(n,m),[<i,i> : i in [1..n+m-1]] cat [<n+m,n>]);
  // %%
end function;

CNCN := function(n)
 G := DUC([n,n]); // %%
 H := <G[1],G[2] cat [<n+1,2*n+1,1>,<2*n,2*n+2,n>]>;
 f := VtoE(G,H,[<i,i> : i in [1..2*n]]); // %%
 return <G,H,f>;
end function;

c2graph := function(list) // list e.g. [<1,3>,<2,4>,<6,6>] list of tuples
                          // of vertices
 n := Maximum([x[1] : x in list] cat [x[2] : x in list]);
 edges := [];
 for x in list do
  i := Index(list,x);
  edges cat:= [<x[1],i,x[2]>,<x[2],2*#list-i+1,x[1]>];
 end for;
 edges_tosort := Sort([<e[2],e[1],e[3]> : e in edges]);
 edges := [<e[2],e[1],e[3]> : e in edges_tosort];
 return <[i : i in [1..n]],edges>;
end function;

Exflower := function(n,list)  // list := [2,3,5] contains vertices
 // that are connected with "upper" vertex 1 in G
 edges_G := Sort(&cat[[<1,i>,<n+2,i>] : i in [2..n+1]]);
 if #list eq 0 then // alle unten
  edges_G cat:= [<i,n+2> : i in [2..n+1]];
  edges_G := [<edges_G[i][1],i,edges_G[i][2]> : i in [1..#edges_G]];
 else
  edges_G cat:= [<i,1> : i in list];
  edges_G cat:= [<i,n+2> : i in [2..n+1] | not i in list];
  edges_G := [<edges_G[i][1],i,edges_G[i][2]> : i in [1..#edges_G]];
 end if;
 flowerG := <[i : i in [1..n+2]],edges_G>;
 flowerH := c2graph([<1,i> : i in [2..n+1]]); // %%
 flowerf := VtoE(flowerG,flowerH,[<i,i> : i in [1..n+1]] cat [<n+2,1>]);
  // %%
 return <flowerG,flowerH,flowerf>;
end function;

Exflower2 := function(n,k)
 // n: number of "petals", k: size of "petals"
 G := DUC([k : i in [1..n]]); // %%
 VG := G[1];
```

```
 EG := G[2];
 V := [1+i*k : i in [0..n-1]];
 EG2 := &cat[[<v,vv> : vv in [vv+1 : vv in [vv : vv in V | not vv eq v]]] :
        v in V];
 EG2 := [<e[1],#EG+Index(EG2,e),e[2]> : e in EG2];
 flowerG := <VG,EG cat EG2>;
 VH := [i : i in [1..n*(k-1)+1]];
 EH := [<1,1+i*k,2+i*(k-1)> : i in [0..n-1]];
 if not k eq 2 then
  for i in [1..n] do
   EH cat:= [<1+(i-1)*(k-1)+l,1+(i-1)*k+l,2+(i-1)*(k-1)+l> : l in [1..k-2]];
  end for;
 end if;
 EH cat:= [<k+i*(k-1),k*(i+1),1> : i in [0..n-1]];
 EH_sort := [<e[2],e[1],e[3]> : e in Sort([<e[2],e[1],e[3]> : e in EH])];
 flowerH := <VH,EH_sort>; // %%
 VtoE_vertices := [<v,1> : v in V] cat &cat[[<1+(i-1)*k+l,2+(i-1)*k+l-i> :
                  l in [1..k-1]] : i in [1..n]];
 flowerf := VtoE(flowerG,flowerH,VtoE_vertices); // %%
 return <flowerG,flowerH,flowerf>;
end function;
```

10.8 Functions to calculate more examples

```
exampleforbadbound := function(n) // (c_2n,f) first not to be bijective
 V := [i : i in [1..2*n+1]];
 E := [<i,i+1,i+1> : i in [3..2*n]];
 E cat:= [<1,1,2>, <2,2,1>, <1,3,3>, <2*n+1,2*n+2,1>];
 X := <V,E>;
 f := ListGraphMorphisms(X,C(2))[2]; // %%
 return <X,f>;
end function;

exampleforbadbound2 := function(n) // (c_n,f) first not to be bijective
 G := DUC([1,n]); // %%
 f := <[<v,1> : v in G[1]],[<e,<1,1,1>> : e in G[2]]>;
 return <G,C(1),f>;
end function;

exampleforbadbound3 := function(n) // n even, n ge 4
 G := DUC([2,n-2,n]); // %%
 H := <[i : i in [1..n-1]],[<i,i,i+1> : i in [1..n-2]]
       cat [<n-1,n-1,n-2>,<n-2,n,1>]>;
 u_vertices := [<1,n-1>,<2,n-2>] cat [<i,i-2> : i in [3..n]]
  cat [<i,i-n> : i in [n+1..2*n-1]] cat [<2*n,n-2>];
```

```
 u_edges := [<c,[h : h in H[2] | h[1] eq [u[2] : u in u_vertices |
   u[1] eq c[1]][1] and h[3] eq [u[2] : u in u_vertices |
   u[1] eq c[3]][1]][1]> : c in G[2]];
 u := <u_vertices,u_edges>;
 return <G,H,u>;
end function;

DTB := function(n) // decimal to binary
 A := [];
 if n eq 0 then
  return [0];
 end if;
 while n gt 0 do
  A cat:= [n mod 2];
  n := Integers()!((n-(n mod 2))/2);
 end while;
 return A;
end function;

DTO := function(list) // decimal to other, e.g. list := [2,4,3]
 W := [[i : i in [1..list[1]]]];
 for i in [2..#list] do
  W cat:= [[u : u in [W[#W][#W[#W]]+1..W[#W][#W[#W]]+list[i]]]];
 end for;
 n := #list;
 A := [];
 for i in [0..&*list-1] do
  j := i;
  AA := [];
  for l in [list[#list-i] : i in [0..#list-1]] do
   AA cat:= [j mod l];
   j := Integers()!((j-(j mod l))/l);
  end for;
  A cat:= [[AA[#AA-i] : i in [0..#AA-1]]];
 end for;
 return A;
end function;
```

With the function `Thins` we list all thin graphs that have `n` vertices.

```
Thins := function(n)
 PV := [<i,j> : i,j in [1..n]]; // pairs of vertices for one edge
 LG := []; // list of thin graphs
 for i in [0..2^(n^2)-1] do
  A := DTB(i); // %%
  E := [PV[i] : i in [1..#A] | A[i] eq 1];
  E := [<e[1],Index(E,e),e[2]> : e in E];
```

```
  LG cat:= [<[i : i in [1..n]],E>];
 end for;
 return LG;
end function;
```

With the function `Thins2` we list all thin graphs that have `n` vertices and that do not have an edge that has the same vertex as source as as target.

```
Thins2 := function(n)
 PV := [<i,j> : i,j in [1..n]]; // pairs of vertices for one edge
 LG := []; // list of thin graphs
 for i in [0..2^(n^2)-1] do
  A := DTB(i); // %%
  E := [PV[i] : i in [1..#A] | A[i] eq 1];
  E := [e : e in E | not e[1] eq e[2]];
  E := [<e[1],Index(E,e),e[2]> : e in E];
  LG cat:= [<[i : i in [1..n]],E>];
 end for;
 return RedSeq(LG); // %%
end function;
```

For a graph `H` and a list `F` of prescribed cardinalities, the function `EFU` gives the list of etale fibrations `f: G -> H` satisfying (Uni) such that the cardinalities of the fibres of the vertices of `H` under V_f are listed in `F`.

```
EFU := function(H,F) // returns all etale fibrations f: G -> H that
 // satisfy (Uni)
 // F: sizes of fibres in list, e.g. F := [1,2,1]
 n := #H[1];
 VG := [i : i in [1..&+F]];
 W := [[i : i in [1..F[1]]]];
 for i in [2..#F] do
  W cat:= [[u : u in [W[#W][#W[#W]]+1..W[#W][#W[#W]]+F[i]]]];
 end for;
 Vf := [<i,Index(W,[a : a in W | i in a][1])> : i in [1..&+F]];
 EG_all := [[<i,j> : i in W[e[1]], j in W[e[3]]] : e in H[2]];
 Ef_all_sort := &cat([[<[e : e in EG_all[i] | e[1] eq v],H[2][i]> :
  i in [1..#EG_all]]: v in VG]);
 Ef_all_sort := [e : e in Ef_all_sort | not #e[1] eq 0];
 Ef_poss := [[<Ef_all_sort[i][1][d[i]+1],Ef_all_sort[i][2]> : i in [1..#d]] :
  d in DTO([#e[1] : e in Ef_all_sort])]; // %%
 Ef_poss_numbered := [[<<Ef_poss[i][j][1][1],j,Ef_poss[i][j][1][2]>,
  Ef_poss[i][j][2]> : j in [1..#Ef_poss[i]]] : i in [1..#Ef_poss]];
 EF := [<<Vf,Ef_poss_numbered[i]>,<VG,[Ef_poss_numbered[i][j][1] : j in
   [1..#Ef_poss_numbered[i]]]>,H> : i in [1..#Ef_poss_numbered]];
      // <f,G,H> all possibilities that f is an etale fibration
 return [f : f in EF | &and[SuffCond(f[1],f[2],f[3])[i] : i in [1,2]]];
end function;
```

For example, given

```
H := c2chain(3);
F := [1,2,1];
```

we get

```
> EFU(H,F);
[
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
      <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
      <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 2>, <3, 3, 2>> ]>, <[ 1, 2, 3, 4 ],
      [ <1, 1, 2>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>, <4, 6, 2> ]>,
      <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>,
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ], [ <<1, 1, 2>, <1, 1, 2>>,
      <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
      <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 3>, <3, 3, 2>> ]>, <[ 1, 2, 3, 4 ],
      [ <1, 1, 2>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>, <4, 6, 3> ]>,
      <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>,
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ], [ <<1, 1, 3>, <1, 1, 2>>,
      <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
      <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 2>, <3, 3, 2>> ]>, <[ 1, 2, 3, 4 ],
      [ <1, 1, 3>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>, <4, 6, 2> ]>,
      <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>,
    <<[ <1, 1>, <2, 2>, <3, 2>, <4, 3> ], [ <<1, 1, 3>, <1, 1, 2>>,
      <<2, 2, 4>, <2, 2, 3>>, <<2, 3, 1>, <2, 4, 1>>, <<3, 4, 4>, <2, 2, 3>>,
      <<3, 5, 1>, <2, 4, 1>>, <<4, 6, 3>, <3, 3, 2>> ]>, <[ 1, 2, 3, 4 ],
      [ <1, 1, 3>, <2, 2, 4>, <2, 3, 1>, <3, 4, 4>, <3, 5, 1>, <4, 6, 3> ]>,
      <[ 1, 2, 3 ], [ <1, 1, 2>, <2, 2, 3>, <3, 3, 2>, <2, 4, 1> ]>>
]
```

10.9 More useful functions

```
IsEqual := function(f,g) // returns true if graph morphisms
  // f, g: G -> H are equal
  // returns true if graphs f, g are equal
 return (SequenceToSet(f[1]) eq SequenceToSet(g[1]))
        and (SequenceToSet(f[2]) eq SequenceToSet(g[2]));
end function;
```

```
IsoRepresentatives := function(T,n); // returns representatives of isoclasses
 TT := [T[1]];
 for i in [1..n] do
  if not &or[IsIsomorphic(T[i],t)[2] : t in TT] then // %%
   TT cat:= [T[i]];
  end if;
 end for;
 return TT;
end function;

Is_Comm_Quad_Graphs := function(X,Y,X2,Y2,f,f2,g,h)
  // f: X -> Y, f2: X2 -> Y2, g: X -> X2, h: Y -> Y2
 if IsEqual(ComposeGraphMorphisms(f,h),ComposeGraphMorphisms(g,f2)) then // %%
  return true;
 end if;
 return false;
end function;
```

The function `Lift` returns a lift of a given commutative quadrangle if existent.

```
Lift := function(X,Y,X2,Y2,f,f2,g,h) // f: X -> Y, f2: X2 -> Y2,
  // g: X -> X2, h: Y -> Y2
 L := ListGraphMorphisms(X2,Y); // %%
 L_comm := [l : l in L | IsEqual(ComposeGraphMorphisms(g,l),f) and
            IsEqual(ComposeGraphMorphisms(l,h),f2)]; // %%
 if not #L_comm eq 0 then
  return L_comm[1];
 end if;
 return false;
end function;

remove_edges := function(G,list)
 return <G[1],[e : e in G[2] | not e in list]>;
end function;

remove_vertices := function(G,list)
 return <[n : n in G[1] | not n in list],[e : e in G[2] |
         not e[1] in list and not e[2] in list]>;
end function;

graph_op := function(G); // functor, reverses the direction of arrows in graph
  // exchanges source and target of edges in graph
 return <G[1],[<e[3],e[2],e[1]> : e in G[2]]>;
end function;
```

```
connection := function(G) // returns the list of connected components
 list := [];
 G_done := {};
 while not #G_done eq #G[1] do
  S := {};   // S: set of vertices to achieve
  r := [g : g in G[1] | not g in G_done][1];
  Snew := {r};
  while not #Snew eq #S do
   S := Snew;
   Snew join:= {a[1] : a in G[2] | a[3] in S} join {a[3] : a in G[2] | a[1] in S};
  end while;
  G_done join:= Snew;
  Snew := Sort(SetToSequence(Snew));
  edges := Sort([a : a in G[2] | a[1] in Snew or a[3] in Snew]);
  list cat:= [<Snew,edges>];
 end while;
 return list;
end function;
```

Appendix A

Explanation for electronic appendix

On the memory stick attached there is a file called `electronic_appendix.txt`. It contains all functions in Magma code that are mentioned in this master thesis. It can be loaded into Magma with

```
load "electronic_appendix.txt";
```

References

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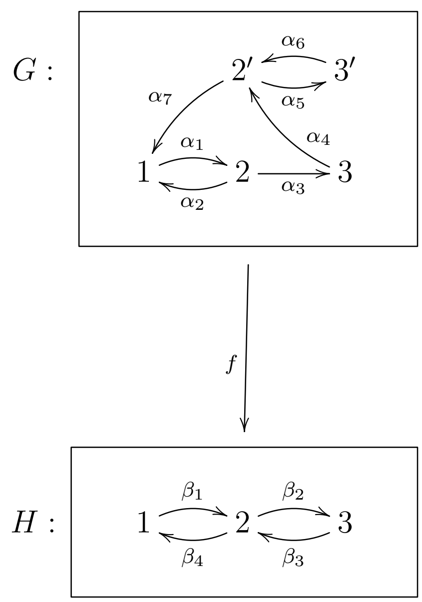
Zusammenfassung

Graphen. Ein Graph G besteht aus einer Menge von Knoten V_G und einer Menge von Kanten E_G, zusammen mit einer Startabbildung $\mathrm{s}_G : \mathrm{E}_G \to \mathrm{V}_G$ und einer Zielabbildung $\mathrm{t}_G : \mathrm{E}_G \to \mathrm{V}_G$, welche einer Kante ihren jeweiligen Start- bzw. Zielknoten zuordnen.

Graphmorphismen. Ein Graphmorphismus $f : G \to H$ zwischen Graphen G und H besteht aus einer Abbildung $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ auf den Knoten und einer Abbildung $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ auf den Kanten derart, dass $\mathrm{E}_f\, \mathrm{s}_H = \mathrm{s}_G\, \mathrm{V}_f$ und $\mathrm{E}_f\, \mathrm{t}_H = \mathrm{t}_G\, \mathrm{V}_f$ ist.

Zum Beispiel bildet der folgende Graphmorphismus $f : G \to H$ die Knoten und Kanten vertikal ab.



Die Kategorie der Graphen bezeichnen wir mit Gph. Die Menge der Graphmorphismen von G nach H bezeichnen wir mit $(G, H)_{\mathrm{Gph}}$. Zu einem Graphmorphismus $f : G \to H$ und einem Graphen K haben wir die Abbildung

$$(K, f)_{\mathrm{Gph}} : (K, G)_{\mathrm{Gph}} \to (K, H)_{\mathrm{Gph}} : g \mapsto gf\ .$$

Eine Modellkategorienstruktur auf Gph. Bisson und Tsemo definieren Daten für eine Modellkategorienstruktur auf Gph wie folgt.

Ein Graphmorphismus $f : G \to H$ ist ein Quasiisomorphismus, wenn

$$(\mathrm{C}_k\, , f)_{\mathrm{Gph}} : (\mathrm{C}_k\, , G)_{\mathrm{Gph}} \to (\mathrm{C}_k\, , H)_{\mathrm{Gph}}$$

bijektiv ist für $k \geqslant 1$. Wir bezeichnen die Menge der Quasiisomorphismen mit $\mathrm{Qis} \subseteq \mathrm{Mor}(\mathrm{Gph})$. Für einen Knoten $v \in \mathrm{V}_G$ bezeichnen wir mit $G(v, *) := \{e \in \mathrm{E}_G : e\, \mathrm{s}_G = v\}$ die Menge der Kanten mit Start v. Wir haben die Abbildung

$$\begin{array}{rcl} \mathrm{E}_{f,v} := \mathrm{E}_f\, |_{G(v,*)}^{H(v\, \mathrm{V}_f\, ,*)} : G(v, *) & \to & H(v\, \mathrm{V}_f\, , *) \\ e & \mapsto & e\, \mathrm{E}_f\ . \end{array}$$

Ein Graphmorphismus $f : G \to H$ heißt eine Faserung, wenn die Abbildung

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

surjektiv ist für $v \in \mathrm{V}_G$. Wir bezeichnen die Menge der Faserungen mit $\mathrm{Fib} \subseteq \mathrm{Mor}(\mathrm{Gph})$.

Ein Graphmorphismus $f : G \to H$ heißt eine etale Faserung, wenn die Abbildung

$$\mathrm{E}_{f,v} : G(v,*) \to H(v\,\mathrm{V}_f\,,*)$$

bijektiv ist für $v \in \mathrm{V}_G$. Dies wird für ein hinreichendes Kriterium für Quasiisomorphismen benötigt.

Ein Graphmorphismus $f : G \to H$ heißt eine azyklische Kofaserung, wenn die Eigenschaften (AcCofib 1–5) erfüllt sind; cf. Definition 162. Anschaulich erhalten wir eine azyklische Kofaserung $f : G \to H$, wenn der Graph H durch Ankleben von Bäumen an den Graphen G entsteht. Wir bezeichnen die Menge der azyklischen Kofaserungen mit $\mathrm{AcCofib} \subseteq \mathrm{Mor}(\mathrm{Gph})$.

Die Menge der azyklischen Faserungen bezeichnen wir mit $\mathrm{AcFib} := \mathrm{Qis} \cap \mathrm{Fib} \subseteq \mathrm{Mor}(\mathrm{Gph})$. Wir schreiben $\begin{tikzcd} G \arrow[r, "\square" marking] & H \end{tikzcd}$ für eine azyklische Faserung.

Ein Graphmorphismus $f : G \to H$ heißt eine Kofaserung, wenn wir zu einem kommutativen Viereck

$$\begin{tikzcd}
G \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "g", "\square" marking] \\
H \arrow[r, "b"'] & Y
\end{tikzcd},$$

immer einen Lift $h : H \to X$ so finden, dass zwei kommutative Dreiecke entstehen. Die Menge der Kofaserungen bezeichnen wir mit $\mathrm{Cofib} \subseteq \mathrm{Mor}(\mathrm{Gph})$.

Nun wird Gph zusammen mit Qis, Fib und Cofib zu einer geschlossenen Quillen-Modellkategorie; cf. [**3,** Cor. 4.8].

Zusätzlich gilt $\mathrm{AcCofib} = \mathrm{Cofib} \cap \mathrm{Qis}$.

Beweis, dass Gph **eine Modellkategorie ist**. Zum Beweis folgen wir Bisson und Tsemo [**3**]. Beim Nachweis der Faktorisierung eines gegebenen Morphismus in eine Kofaserung, gefolgt von einer azyklischen Faserung verwenden wir eine iterierte Pushout-Konstruktion.

Ein hinreichendes Kriterium für Quasiisomorphismen. Sei $f : G \to H$ ein Graphmorphismus. Eine Kante $e \in \mathrm{E}_H$ heißt einzielig, wenn das Urbild der Kante e unter E_f einen eindeutigen Zielknoten besitzt.

Wir betrachten folgende Eigenschaft.

(Uni) Für $n \geqslant 1$ und jeden Graphmorphismus $u : \mathrm{C}_n \to H$ gibt es ein $i \in \mathbb{Z}/n\mathbb{Z}$ so, dass $\mathrm{e}_i\,\mathrm{E}_u \in \mathrm{E}_H$ einzielig ist.

Nach Entfernung der einzieligen Kanten aus H darf es also keinen Zykel mehr darin geben, damit (Uni) erfüllt ist.

Wenn eine etale Faserung $f : G \to H$ die Eigenschaft (Uni) erfüllt, ist sie ein Quasiisomorphismus.

Z.B. ist der Graphmorphismus $f : G \to H$ von oben ein Quasiisomorphismus, wie damit überprüft werden kann.

Beispiele und Gegenbeispiele. Wir geben eine Reihe von Beispielen für Quasiisomorphismen, berechnet mit Magma [**2**] mithilfe unseres hinreichenden Kriteriums.

Wir zeigen, dass Quasiisomorphismen nicht stabil sind unter Pushouts entlang Kofaserungen.

Wir zeigen, dass Kofaserungen nicht stabil sind unter Pullbacks.

Wir geben ein Beispiel $f : G \to H$ so, dass $(\mathrm{C}_1\,, f)_{\mathrm{Gph}}$ und $(\mathrm{C}_2\,, f)_{\mathrm{Gph}}$ bijektiv sind und es keinen injektiven Graphmorphismus $\mathrm{C}_k \to G$ oder $\mathrm{C}_k \to H$ für $k \geqslant 3$ gibt, bei welchem jedoch f kein Quasiisomorphismus ist.

Versicherung

Hiermit versichere ich,

1. dass ich meine Arbeit selbstständig verfasst habe,
2. dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,
3. dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und
4. dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

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