University of Stuttgart
Germany

Tori for some locally integral group rings

Diagonalizability over a principal ideal domain

Master's Thesis

Fabian Hartkopf

May 2017

Contents

Preface

0.1 Introduction

Let R be a principal ideal domain. Let $K := \mathrm{frac}(R)$ be its field of fractions.

0.1.1 Diagonalization

0.1.1.1 Problem

Suppose given $A \in R^{n\times n}$. Then, in particular, $A \in K^{n\times n}$. By definition, A is K-diagonalizable if there exists $S \in \mathrm{GL}_n(K)$ such that $S^{-1}AS$ is diagonal. Furthermore, by definition, A is R-diagonalizable if there exists $T \in \mathrm{GL}_n(R)$ such that $T^{-1}AT$ is diagonal.

So if A is R-diagonalizable, then A is also K-diagonalizable.

Suppose A to be K-diagonalizable. We ask for conditions on A to be R-diagonalizable.

For instance, $\left(\begin{smallmatrix} 2 & -1 \\ 0 & 4 \end{smallmatrix}\right)$ is $\mathbb{Q}$-diagonalizable, but not $\mathbb{Z}$-diagonalizable, not even $\mathbb{Z}_{(2)}$-diagonalizable; cf. (2) in §0.1.1.3 below.

0.1.1.2 Diagonalizability over a principal ideal domain

Suppose $A \in R^{n\times n}$ to be K-diagonalizable. The intersections of eigenspaces and $R^{n\times 1}$ are R-submodules of $R^{n\times 1}$ which we call *eigenmodules of A*. Eigenmodules of A are finitely generated free R-modules.

After fixing R-linear bases of the eigenmodules of A we can define a matrix S that has in its columns these basis elements. Then we obtain the following; cf. Lemma 47.

$$A \text{ is } R\text{-diagonalizable} \iff \det(S) \text{ is a unit in } R$$

This allows us to state a characterization of R-diagonalizability that is independent of the choice of the bases of the eigenmodules; cf. Corollary 48.(1). Let $\sigma(A)$ be the set of eigenvalues of A. We have $\sigma(A) \subseteq R$; cf. Remark 37. Let $\mathrm{E}_A(\Lambda)$ be the eigenspace of A to the eigenvalue $\lambda \in \sigma(A)$. Then we have the following equivalence.

$$A \text{ is } R\text{-diagonalizable} \quad\iff\quad \bigoplus_{\lambda\in\sigma(A)} \left(\mathrm{E}_A(\lambda) \cap R^{n\times 1}\right) = R^{n\times 1} \tag{1}$$

In practice, we start with a K-linear basis of an eigenspace of A to find an R-linear basis of the corresponding eigenmodule of A using the elementary divisor theorem; cf. Lemma 49 and Algorithm 50.

Thus we can decide algorithmically whether A is R-diagonalizable.

0.1.1.3 Diagonalizability of linear combinations

Recall that K-linear combinations of commuting K-diagonalizable matrices are again K-diagonalizable.

We observe that R-linear combinations of commuting R-diagonalizable matrices are again R-diagonalizable; cf. Corollary 54.

Given commuting K-diagonalizable matrices, we ask which R-linear combinations of these are R-diagonalizable.

We consider the following two matrices in $\mathbb{Z}_{(2)}^{2\times 2}$ and their eigenmodules as submodules of $\mathbb{Z}_{(2)}^{2\times 1}$.

$$\begin{aligned}
A_1 &:= \begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix} \qquad \mathrm{E}_{A_1}(2) \cap \mathbb{Z}_{(2)}^{2\times 1} = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \qquad \mathrm{E}_{A_1}(4) \cap \mathbb{Z}_{(2)}^{2\times 1} = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle \\
A_2 &:= \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \qquad \mathrm{E}_{A_2}(0) \cap \mathbb{Z}_{(2)}^{2\times 1} = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \qquad \mathrm{E}_{A_2}(-2) \cap \mathbb{Z}_{(2)}^{2\times 1} = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle
\end{aligned} \tag{2}$$

Here we have $A_1 \cdot A_2 = A_2 \cdot A_1$. But both A_1 and A_2 are not $\mathbb{Z}_{(2)}$-diagonalizable since the direct sum of their respective eigenmodules is a proper submodule of $\mathbb{Z}_{(2)}^{2\times 1}$.

But we have $A_1 + A_2 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, which is a diagonal matrix, so in particular a $\mathbb{Z}_{(2)}$-diagonalizable matrix.

Suppose given a tuple $\Phi := (A_1, \ldots, A_k)$ of commuting K-diagonalizable matrices in $R^{n\times n}$. We define the *diagonalizability locus of* Φ as follows; cf. Definition 56.

$$\mathrm{C}_\Phi := \left\{ (\alpha_i)_{i\in[1,k]} \in R^{k\times 1} \;\middle|\; \sum_{i\in[1,k]} \alpha_i A_i \text{ is } R\text{-diagonalizable} \right\} \tag{3}$$

This is an R-submodule of $R^{k\times 1}$; cf. Lemma 57.

Suppose given $\alpha := (\alpha_i)_{i\in[1,k]} \in R^{k\times 1}$. We write $A_\alpha := \sum_{i\in[1,k]} \alpha_i A_i$. So A_α is R-diagonalizable if and only if $\alpha \in \mathrm{C}_\Phi$.

In order to determine C_Φ, we aim to find an R-linear basis of C_Φ. But testing all linear combinations of the matrices of Φ would lead to an infinite task. Our reduction to a finite test for the R-diagonalizability of A_α makes use of the fact that the eigenmodules of A_α are closely related to the eigenmodules of the matrices $A_1, \ldots, A_k$. We want to make use of (1) which requires the eigenvalues of A_α.

Since $A_1, \ldots, A_k$ are commuting K-diagonalizable matrices, we find $S \in \mathrm{GL}_n(K)$ such that all conjugates $S^{-1} A_i S$ are diagonal matrices. By multiplying with a common denominator, we can achieve that $S \in R^{n\times n}$. We denote the columns of S by $s_1, \ldots, s_n$ and the eigenvalues of A_i by $\lambda_{1,i}, \ldots, \lambda_{n,i}$, taken with multiplicities. So the following identities hold.

$$A_i s_j = \lambda_{j,i} s_j \quad \text{for } i \in [1,k] \text{ and } j \in [1,n]$$

We say that a tuple $\mu := (\mu_1, \mu_2, \ldots, \mu_k) \in R^{1\times k}$ is an *eigenvalue tuple* of Φ if there exists a non-zero $x \in R^{n\times 1}$ such that $A_i x = \mu_i x$ for $i \in [1,k]$; cf. Definition 52. For such an eigenvalue tuple μ we define the *simultaneous eigenmodule for* Φ as the following R-submodule $\mathrm{E}_\Phi(\mu)$ of $R^{n\times 1}$.

$$\mathrm{E}_\Phi(\mu) := \{ x \in R^n \mid A_i x = \mu_i x \text{ for } i \in [1,k] \}$$

So the tuples $(\lambda_{j,1}, \ldots, \lambda_{j,k})$, where $j \in [1,n]$, are the eigenvalue tuples of Φ, possibly with repetitions, and $s_j \in R^n$ is an element of the corresponding simultaneous eigenmodule $\mathrm{E}_\Phi((\lambda_{j,1}, \ldots, \lambda_{j,k}))$.

Note that $\mathrm{E}_\Phi(\mu) \subseteq R^{n\times 1}$ is an R-submodule whereas $\mathrm{E}_{A_i}(\mu_i) \subseteq K^{n\times 1}$ is a K-subspace.

Denoting by $\sigma(A_\alpha)$ the set of eigenvalues of A_α, we obtain the following relation between eigenvalue tuples of the A_i and the eigenvalues of A_α; cf. Remark 60.

$$\sigma(A_\alpha) = \left\{ \sum_{i\in[1,k]} \alpha_i \lambda_{j,i} \;\middle|\; j \in [1,\ell] \right\}.$$

Herein, different values of j may yield the same eigenvalue $\sum_{i\in[1,k]} \alpha_i \lambda_{j,i}$. To what extent this effect occurs depends on A. For example, if $\alpha = 0$, then A_α has only the eigenvalue 0.

We resume the example with the matrices A_1 and A_2 from (2).

Writing $\Phi := (A_1, A_2)$, we have the simultaneous eigenvalue tuples $(2,0)$ and $(4,-2)$ for Φ.

Let $\alpha = (1,-1)$. Then $A_\alpha = A_1 - A_2 = \left(\begin{smallmatrix} 2 & -2 \\ 0 & 6 \end{smallmatrix}\right)$ and hence $\sigma(A_\alpha) = \{2,6\}$. The matrix A_α has two eigenmodules each of which is of rank 1. More precisely, we have

$$\mathrm{E}_{A_\alpha}(2) \cap \mathbb{Z}_{(2)}^{2\times 1} = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle = \mathrm{E}_\Phi((2,0))$$

and

$$\mathrm{E}_{A_\alpha}(6) \cap \mathbb{Z}_{(2)}^{2\times 1} = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle = \mathrm{E}_\Phi((4,-2)).$$

So the eigenmodules of A_1, of A_2 and of $A_1 - A_2$ are the same.

Let $\tilde{\alpha} = (1,1)$. Then $A_{\tilde{\alpha}} = A_1 + A_2 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and hence $\sigma(A_{\tilde{\alpha}}) = \{2\}$. The matrix $A_{\tilde{\alpha}}$ has only one eigenmodule $\mathrm{E}_{A_{\tilde{\alpha}}}(2) \cap \mathbb{Z}_{(2)}^{2\times 1}$, viz. $\mathbb{Z}_{(2)}^{2\times 1}$. Note that

$$\mathrm{E}_{A_{\tilde{\alpha}}}(2) \cap \mathbb{Z}_{(2)}^{2\times 1} = \mathbb{Z}_{(2)}^{2\times 1} \supset {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle = \mathrm{E}_\Phi((2,0)) \oplus \mathrm{E}_\Phi((4,-2)).$$

So in a sense, the simultaneous eigenmodules for Φ, i.e. of A_1 and A_2, have fused to a single eigenmodule of $A_1 + A_2$.

We want to describe this behavior by using partitions of finite sets.

Denote by $\mathcal{P}_n$ the set of partitions of the set $\{1,2,\ldots,n\} = [1,n]$. So e.g. $(\{1,4\},\{2,5\},\{3\}) \in \mathcal{P}_5$.

We define three maps dependent on Φ and the number ℓ of different eigenvalue tuples for Φ; cf. Definitions 65, 77 and 78.

$$R^{k\times 1} \xrightarrow{\omega_\Phi} R^{\ell\times 1} \xrightarrow{\tau_\ell} \mathcal{P}_\ell \xrightarrow{\upsilon_\Phi} \mathrm{Sub}_R(N)$$

To this end, let $\Lambda_\Phi \in R^{\ell\times k}$ be the matrix containing the distinct eigenvalue tuples of Φ as rows; cf. Definition 77. The map ω_Φ sends $\alpha \in R^{k\times 1}$ to $\Lambda_\Phi\alpha \in R^{\ell\times 1}$, containing the eigenvalues of A_α. Then τ_ℓ maps $\beta \in R^{\ell\times 1}$ to the partition containing those subsets of $[1,\ell]$ where β, considered as a map from $[1,\ell]$ to R, is constant.

So under the map $\tau_\ell \circ \omega_\Phi$, the coefficient vector α is sent to the partition $P = (p_1,\ldots,p_u)$ in $\mathcal{P}_\ell$ such that j_1 and j_2 are in the same subset p_i of $[1,\ell]$ if and only if the simultaneous eigenmodules to row j_1 of Λ_Φ and to row j_2 of Λ_Φ are contained in the same eigenmodule of A_α. So in a sense, this map describes the fusion behavior of the simultaneous eigenmodules for Φ to eigenmodules of A_α.

Suppose given $P = (p_1,\ldots,p_u) \in \mathcal{P}_\ell$. Let $V_i \subseteq K^{n\times 1}$ be the sum of the simultaneous eigenspaces to the eigenvalue tuple in row j of Λ_Φ where j runs through p_i. Then we set

$$M_{\Phi,P} := \bigoplus_{i\in[1,u]} \left(V_i \cap R^{n\times 1}\right) \subseteq R^{n\times 1}.$$

The map υ_Φ sends the partition P to the R-module $M_{\Phi,P}$.

Consider the image of α under the map $\upsilon_\Phi \circ \tau_\ell \circ \omega_\Phi$. This is exactly the R-module occurring on the right hand side of (1) for $A = A_\alpha$. So if we want to decide whether A_α is R-diagonalizable, we may test whether the image α under the map $\upsilon_\Phi \circ \tau_\ell \circ \omega_\Phi$ equals $R^{n\times 1}$.

In other words, the preimage $(\upsilon_\Phi \circ \tau_\ell \circ \omega_\Phi)^{-1}(R^{n\times 1})$ equals the diagonalizability locus C_Φ of Φ.

Now we benefit from the fact that $\mathcal{P}_\ell$ is finite: We can determine algorithmically the preimage $\upsilon_\Phi^{-1}(R^{n\times 1}) \subseteq \mathcal{P}_\ell$. For every $P \in \upsilon_\Phi^{-1}(R^{n\times 1})$ we define a matrix $\mathrm{D}_{\Lambda_\Phi}^P$. This matrix is formed by row operations and row removals from Λ_Φ which depend on the partition P; cf. Definition 82.

Using these matrices, we obtain the following description of C_Φ; cf. Lemma 91.

$$\bigcup_{P\in\upsilon_\Phi^{-1}(R^{n\times 1})} \ker\left(\mathrm{D}_{\Lambda_\Phi}^P\right) = \mathrm{C}_\Phi \tag{4}$$

The advantage here is that, since $\mathcal{P}_\ell$ is finite, the preimage $\upsilon_\Phi^{-1}(R^{n\times 1})$ is finite and so we have a finite union. After determining the kernels of the matrices involved, we are able to establish an R-linear basis of C_Φ.

Moreover, we are able to reduce $\upsilon_\Phi^{-1}(R^{n\times 1})$ to the subset of the finest partitions in this preimage, using the fact that if a partition P is finer than a partition Q, then $M_{\Phi,P} \subseteq M_{\Phi,Q}$; cf. Lemma 68. This allows us to skip the calculation of $M_{\Phi,P}$ for certain partitions P when looping over $\mathcal{P}_\ell$, resulting in a speed improvement.

So in total, we establish an algorithm that calculates an R-linear basis of the diagonalizability locus C_Φ for a given $\Phi = (A_1, \ldots, A_k)$ as above. This algorithm is presented on the one hand as pseudocode in Algorithm 94, on the other hand as Magma code in §3.5.5 where it is part of the file "partalgo", cf. pages 69 – 73.

The theory presented here can be applied to commuting tuples of K-diagonalizable R-endomorphisms of a finitely generated free R-module. However, to use the implementation of the algorithm, one has to make the passage to describing matrices.

0.1.2 Tori

Let R be a principal ideal domain. Let $K := \mathrm{frac}(R)$ be its field of fractions.

In the theory of Lie algebras over $\mathbb{C}$, toral subalgebras are used to classify semisimple Lie algebras. We recall that toral subalgebras, also known as tori, consist of elements whose adjoint endomorphisms are semisimple. A maximal torus of a finite dimensional Lie algebra yields the root space decomposition of the Lie algebra and thus the root system.

Let $\mathfrak{g}$ be a Lie algebra over R and let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Lie subalgebra over R. We say that $\mathfrak{t}$ is a *rational torus of* $\mathfrak{g}$ if for $t \in \mathfrak{t}$, the adjoint endomorphism $\mathrm{ad}_\mathfrak{g}(t)$ is K-diagonalizable; cf. Definition 97. We say that $\mathfrak{t}$ is an *integral torus of* $\mathfrak{g}$ if for $t \in \mathfrak{t}$, the adjoint endomorphism $\mathrm{ad}_\mathfrak{g}(t)$ is R-diagonalizable; cf. Definition 97.

It follows that every integral torus is a rational torus.

We say that a rational torus $\mathfrak{t}$ of $\mathfrak{g}$ is a *maximal rational torus in* $\mathfrak{g}$ if for every rational torus $\mathfrak{t}'$ of $\mathfrak{g}$ such that $\mathfrak{t} \subseteq \mathfrak{t}' \subseteq \mathfrak{g}$, we have $\mathfrak{t} = \mathfrak{t}'$; cf. Definition 97. Similarly we define maximal integral tori of $\mathfrak{g}$; cf. Definition 97.

We will see that rational tori are abelian; cf. Lemma 107. Moreover, we will see that if a rational torus $\mathfrak{t}$ equals its centralizer in $\mathfrak{g}$, then $\mathfrak{t}$ is a maximal rational torus in $\mathfrak{g}$; cf. Lemma 112.

We can use a maximal rational torus $\mathfrak{t}$ of $\mathfrak{g}$ to find a decomposition of the $\mathfrak{t}$-module $\mathfrak{g}$ into indecomposables.

Let Γ be a direct product of matrix rings over R. Let Ω be a subalgebra of Γ such that $K \otimes_R \big(\Gamma/\Omega\big) = 0$.

Let $\Delta \subseteq \Gamma$ be the subalgebra consisting of those tuples that contain only diagonal matrices.

Then $\Omega \cap \Delta$ is a maximal commutative subalgebra of Ω; cf. Lemma 121. The commutator Lie algebra $\mathfrak{l}(\Omega \cap \Delta)$ is our standard example for a maximal rational torus in $\mathfrak{l}(\Omega)$; cf. Lemma 120.

However, maximal rational tori are in general not unique, not even unique up to conjugation; cf. Remark 125.

Suppose that $\mathfrak{t}$ is a rational torus in $\mathfrak{l}(\Omega)$. We define the *integral core of* $\mathfrak{t}$ *in* $\mathfrak{l}(\Omega)$ by

$$\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t}) := \big\{ t \in \mathfrak{t} \,\big|\, \mathrm{ad}_{\mathfrak{l}(\Omega)}(t) \text{ is } R\text{-diagonalizable} \big\}\,;$$

cf. Definition 130. This is an integral torus in $\mathfrak{l}(\Omega)$; cf. Lemma 131. In general, the integral core $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t})$ is contained properly in $\mathfrak{t}$. However, in general it is not a maximal integral torus in $\mathfrak{l}(\Omega)$; cf. Remark 152.(7). Any integral torus of $\mathfrak{l}(\Omega)$ that is contained in $\mathfrak{t}$ is also contained in $\mathrm{Cor}_{\mathfrak{l}(\Omega)}$.

To determine an integral core algorithmically, we make use of the algorithm introduced in §0.1.1.3. To that end, we choose an R-linear basis $(b_1, \ldots, b_k)$ of $\mathfrak{t}$. Then we can apply the algorithm to the tuple

$\Phi := (\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_1), \ldots, \mathrm{ad}_{\mathfrak{l}(\Omega)}(b_k))$. This gives us an R-linear basis of the diagonalizability locus of Φ which suffices to determine the integral core $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t})$ of $\mathfrak{t}$ in $\mathfrak{l}(\Omega)$.

We say that a torus $\mathfrak{t}$ in $\mathfrak{l}(\Omega)$ is a *primitive torus in* $\mathfrak{l}(\Omega)$ if it is a maximal rational torus in $\mathfrak{l}(\Omega)$ and if there exist idempotents $e_1, \ldots, e_n \in \mathfrak{t}$ that are primitive in Ω, such that $e_i e_j = 0$ for $i \neq j$ and $\sum_{i\in[1,n]} e_i = 1_\Omega$ and $e_i \Omega e_i$ is local for $i \in [1, n]$; cf. Definition 146.

Suppose given two primitive tori $\mathfrak{t}$ and $\mathfrak{t}'$ in $\mathfrak{l}(\Omega)$. We choose associated idempotents $e_i \in \mathfrak{t}$ $(i \in [1, m])$ and $e'_i \in \mathfrak{t}'$ $(i \in [1, n])$ as required in the definition of primitive tori. Then we have $m = n$ and there exists a unit u in Ω such that $\bigoplus_{i\in[1,n]} e_i\Omega e_i = u^{-1} \left(\bigoplus_{i\in[1,n]} e'_i\Omega e'_i\right) u$; cf. Lemma 149.

However, we also do not achieve uniqueness of primitive tori, not even uniqueness up to conjugation with units in Ω, as Remark 150 shows.

We illustrate this situation with the following diagram.

$$\begin{tikzcd}
\bigoplus_{i\in[1,n]} e_i\Omega e_i \arrow[rr, leftrightarrow, "\text{conjugate}", "\text{via } u \in \mathrm{U}(\Omega)"'] & & \bigoplus_{i\in[1,n]} e'_i\Omega e'_i \\
\mathfrak{t} \arrow[u, hook] & & \mathfrak{t}' \arrow[u, hook]
\end{tikzcd}$$

In particular, we consider the group rings $\mathbb{Z}_{(3)}\,\mathrm{S}_3$, $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ and $\mathbb{Z}_{(2)}\,\mathrm{S}_5$.

In §1 we will consider the group ring $\mathbb{Z}_{(3)}\,\mathrm{S}_3$. We use a Wedderburn embedding

$$\mathbb{Z}_{(3)}\,\mathrm{S}_3 \xrightarrow{\sim} \Omega \subseteq \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}^{2\times 2} \times \mathbb{Z}_{(3)}.$$

The standard torus $\mathfrak{l}(\Omega \cap \Delta)$ is both a maximal rational torus and a maximal integral torus in $\mathfrak{l}(\Omega)$. So in particular, the integral core $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega \cap \Delta))$ equals $\mathfrak{l}(\Omega \cap \Delta)$ here.

In §6 we will consider the group ring $\mathbb{Z}_{(2)}\,\mathrm{S}_4$. Using a Wedderburn embedding

$$\mathbb{Z}_{(2)}\,\mathrm{S}_4 \xrightarrow{\sim} \Omega \subseteq \mathbb{Z}_{(2)}^{2\times 2} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)},$$

we obtain the $\mathbb{Z}_{(2)}$-algebra Ω. As a $\mathbb{Z}_{(2)}$-module, Ω is of rank 24. The torus $\mathfrak{t} := \mathfrak{l}(\Omega \cap \Delta)$ is a maximal rational torus in $\mathfrak{l}(\Omega)$, but it is not an integral torus in $\mathfrak{l}(\Omega)$. The integral core $\mathrm{Cor}_{\mathfrak{l}(\Omega)}\,\mathfrak{t}$ is, considered as a $\mathbb{Z}_{(2)}$-module, of rank 7, whereas $\mathfrak{t}$ is of rank 10.

In §7 we will consider the group ring $\mathbb{Z}_{(2)}\,\mathrm{S}_5$. We use a Wedderburn embedding

$$\mathbb{Z}_{(2)}\,\mathrm{S}_5 \xrightarrow{\sim} \widetilde{\Omega} \subseteq \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{6\times 6}.$$

We consider a $\mathbb{Z}_{(2)}$-algebra

$$\Omega \subseteq \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{2\times 2} \times \mathbb{Z}_{(2)}^{2\times 2} \times \mathbb{Z}_{(2)}^{3\times 3}$$

that is Morita equivalent to $\widetilde{\Omega}$. As a $\mathbb{Z}_{(2)}$-module, Ω is of rank 21. The torus $\mathfrak{t} := \mathfrak{l}(\Omega \cap \Delta)$ is a maximal rational torus in $\mathfrak{l}(\Omega)$ of rank 13. Its integral core $\mathrm{Cor}_{\mathfrak{l}(\Omega)}\,\mathfrak{t}$ is, considered as a $\mathbb{Z}_{(2)}$-module, of rank 8.

In all three examples we observe that the integral core of the standard torus is generated by the center and by certain primitive idempotents; cf. Question 135.

Moreover, in each of these three examples we use the standard torus to find a decomposition of $\mathfrak{l}(\Omega)$ into indecomposable $\mathfrak{t}$-submodules of $\mathfrak{l}(\Omega)$. We compare such a decomposition to the Peirce decomposition of Ω. In the examples $\Omega \simeq \mathbb{Z}_{(3)}\,\mathrm{S}_3$ and $\Omega \simeq \mathbb{Z}_{(2)}\,\mathrm{S}_4$, the standard torus $\mathfrak{t}$ is a direct sum of such Peirce components. The components contained in $\mathfrak{t}$ decompose into $\mathfrak{t}$-submodules of rank 1, whereas the other Peirce components remain indecomposable. In the example of $\mathbb{Z}_{(2)}\,\mathrm{S}_5$, the standard torus $\mathfrak{t}$ is not a direct sum of such Peirce components. There exists one Peirce component of rank 8 that contains elements of $\mathfrak{t}$ and non-diagonal elements. This component decomposes into six indecomposable $\mathfrak{t}$-submodules, two of which are of rank 2. The other non-zero Peirce components remain indecomposable.

Since the standard torus $T := \Omega \cap \Delta$ is also a commutative subalgebra of Ω, we decompose the T-T-bimodule Ω into indecomposables. In the three examples under consideration, viz. those in §1, §6 and §7, the indecomposable summands coincide with the non-zero Peirce components. In general, however, this is not necessarily the case; cf. Remark 172.

0.2 Conventions

Let R be a commutative ring.

(1) Let X be a set. We write "for $x \in X$" if we mean "for all $x \in X$".

(2) We write $\mathbb{Z}$ for the set of integers. We denote by $\mathbb{N}$ the set of positive integers and we denote by $\mathbb{N}_0$ the set of non-negative integers. We denote by $\mathbb{F}_2$ the Galois field of two elements.

(3) Let $a, b \in \mathbb{Z}$. Then $[a, b]$ is defined as $\{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

(4) For a finite set X, we write $|X|$ for its cardinality.

(5) We write δ for Kronecker's delta. Suppose given elements x, y of the same set. Then $\delta_{x,y}$ is defined as follows.

$$\delta_{x,y} := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

(6) Suppose given $s \in \mathbb{N}$. We say that a tuple $P = (p_1, \ldots, p_k)$ of subsets of $[1, s]$ is a partition of $[1, s]$ if the following conditions hold.

- $\bigcup_{i \in [1,k]} p_i = [1, s]$
- $p_i \neq \emptyset$ for $i \in [1, k]$
- $p_i \cap p_j = \emptyset$ for $i, j \in [1, k]$ where $i \neq j$
- $\min p_i < \min p_j$ for $i, j \in [1, k]$ where $i < j$

We write $\mathcal{P}_s$ for the set of partitions of $[1, s]$.

(7) Let A be an abelian group, written additively. If unambiguous, we often write 0 for the zero element 0_A of A. We often write $A^\times := A \setminus \{0_A\}$.

(8) For a ring S, we denote by $\mathrm{U}(S)$ the group of units in S, i.e. the set of all invertible elements associated with the multiplication of S.

(9) Suppose that R is a discrete valuation ring with maximal ideal (π). Then we define the valuation function at π as follows.

$$\begin{aligned} \mathrm{v}_\pi \colon R^\times &\to \mathbb{N}_0 \\ x &\mapsto \max\left\{k \in \mathbb{N}_0 \,\middle|\, \exists\, y \in R \text{ such that } x = \pi^k y\right\} \end{aligned}$$

(10) Suppose given $a, b, s \in R$. We write $a \equiv_s b$ if there exists $r \in R$ such that $a - b = rs$, i.e. $a - b \in sR$

(11) Let $m \geq 0$. We denote by $R^{\times m}$ the direct product $\underbrace{R \times \ldots \times R}_{m \text{ times}}$.

(12) We often call the endomorphism R-algebra of an object simply endomorphism ring.

(13) Suppose given $m, n \in \mathbb{N}$. The R-module of $m \times n$-matrices over R is denoted by $R^{m \times n}$. We often identify $R = R^{1 \times 1}$.

When writing down a matrix, omitted matrix entries are supposed to be zero.

If unambiguous, we denote the i-th standard basis vector in $R^{m \times 1}$ by e_i for $i \in [1, m]$.

Suppose given $i \in [1, m]$ and $j \in [1, n]$. We define $E_{i,j} := (\delta_{(i,j),(k,l)})_{k,l} \in R^{m \times n}$. This is the matrix in $R^{m \times n}$ that has the entry 1 in the position (i, j) and the entry 0 in all other positions.

We say that

$$(E_{1,1}, E_{1,2}, \ldots, E_{1,n}, E_{2,1}, E_{2,2}, \ldots, E_{2,n}, \ldots, E_{m,1}, E_{m,2}, \ldots, E_{m,n})$$

is the standard basis of $R^{m \times n}$. We often denote it by $\mathcal{E}_{m,n}$.

(14) Suppose given finitely generated free R-modules M and N. Let $m := \operatorname{rk}_R M$ and $n = \operatorname{rk}_R N$. Suppose given an R-linear map $f\colon M \to N$. Let $\mathcal{B} = (b_1, \ldots, b_m)$ be an R-linear basis of M. Let $\mathcal{C} = (c_1, \ldots, c_n)$ be an R-linear basis of N. We say that a matrix $(a_{i,j})_{i \in [1,n],\, j \in [1,m]} \in R^{n \times m}$ is the describing matrix of f with respect to the bases $\mathcal{B}, \mathcal{C}$ if

$$f(b_j) = \sum_{i \in [1,n]} a_{i,j} c_i \quad \text{for } j \in [1,m].$$

In this case, we denote the matrix $(a_{i,j})_{i \in [1,n],\, j \in [1,m]}$ by $f_{\mathcal{C},\mathcal{B}} \in R^{n \times m}$.

If f is an endomorphism of M, then we simply call $f_{\mathcal{B},\mathcal{B}}$ the describing matrix of f with respect to the basis $\mathcal{B}$.

(15) Let $A = (a_{i,j})_{i \in [1,m],\, j \in [1,n]} \in R^{m \times n}$ be a matrix. We say that A is a *diagonal matrix* if $a_{i,j} = 0$ for $i \neq j$.

(16) By *ties* we mean congruences between matrix entries. For example, let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2 \times 2}$. The condition $a \equiv_2 d$ on this matrix is a tie. The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ fulfills the tie, but $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ does not.

(17) Suppose given an R-algebra A and $k \in \mathbb{N}$. We say that $\sum_{i \in [1,k]} e_i = 1_A$ is an *orthogonal decomposition (of 1_A) into idempotents in A* if the following conditions are satisfied.

- $0 \neq e_i \in A$ for $i \in [1,k]$
- $e_i e_j = 0$ for $i, j \in [1,k]$ and $i \neq j$
- $e_i^2 = e_i$ for $i \in [1,k]$

We say that $\sum_{i \in [1,k]} e_i = 1_A$ is an *orthogonal decomposition (of 1_A) into primitive idempotents in A* if $\sum_{i \in [1,k]} e_i = 1_A$ is an orthogonal decomposition (of 1_A) into idempotents and if for $i \in [1,k]$, there are no elements $e', e'' \in A \backslash \{0_A, e_i\}$ such that $e_i = e' + e''$ and $e'e'' = e''e' = 0$ and $(e')^2 = e'$ and $(e'')^2 = e''$.

Note that if $\sum_{i \in [1,k]} e_i = 1_A$ is an orthogonal decomposition into primitive idempotents in A, then $\sum_{i \in [1,k]} (u^{-1} e_i u) = 1_A$ also is an orthogonal decomposition into primitive idempotents in A for $u \in \mathrm{U}(A)$.

(18) Modules are supposed to be left modules.

(19) Let T be a commutative R-algebra. We often write $T^* := \operatorname{Hom}(T, R)$ for the R-module of R-linear maps $\varphi\colon T \to R$.

(20) Let A be an R-algebra and let B be an R-subalgebra of A. We often write $\mathrm{C}_A(B)$ for the centralizer of B in A, i.e.

$$\mathrm{C}_A(B) = \{x \in A \mid xy = yx \text{ for } y \in B\}.$$

(21) For an R-module M we write $\operatorname{Sub}_R(M)$ for the set of R-submodules of M.

(22) Let M be an R-module. Suppose given $x_i \in M$ for $i \in [1,s]$. We write ${}_R\langle x_1, \ldots, x_s \rangle$ for the R-submodule of M that is generated over R by $x_1, \ldots, x_s$.

We also write ${}_R\langle (x_1, \ldots, x_s) \rangle := {}_R\langle x_1, \ldots, x_s \rangle$.

(23) Suppose that R is a principal ideal domain. Denote by K its field of fractions. Suppose given R-modules M and N and an R-linear map $\varphi\colon M \to N$. Then the map

$$\begin{aligned} K \otimes_R \varphi\colon K \otimes_R M &\to K \otimes_R N \\ 1 \otimes_R m &\mapsto 1 \otimes_R \varphi(m) \end{aligned}$$

is a K-linear map. We often write $K\varphi := K \otimes_R \varphi$ and $KM := K \otimes_R M$ as well as $KN := K \otimes_R N$.

If M is a finitely generated torsion free module over R, the embedding $\iota_M\colon M \to KM$ that sends an element $m \in M$ to $\iota_M(m) := 1 \otimes m \in KM$ is an injective map. We identify M and $\iota_M(M)$ via ι_M.

(24) Suppose that R is a principal ideal domain. Denote by K its field of fractions. An R-algebra that is finitely generated free as an R-module is said to be an *R-order*.

Suppose given a finite dimensional K-algebra A. Suppose given an R-subalgebra $\Omega \subseteq A$. We say that an R-subalgebra $\Omega \subseteq A$ is an *R-order in A* if Ω is an R-order such that $K\Omega = A$.

For example, if G is a finite group and $A = K[G]$ is its group algebra over K, then $R[G]$ is an R-order in A.

We say that Γ is a *completely split R-order* if Γ is isomorphic to a finite direct product of matrix rings over R. In other words, there exist $k \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i \in [1,k]$ such that

$$\Gamma \simeq \prod_{i\in[1,k]} R^{n_i\times n_i}.$$

We identify $K\Gamma = \prod_{i\in[1,k]} K^{n_i\times n_i}$.

(25) Suppose that R is a principal ideal domain. Denote by K its field of fractions. Let M be a finitely generated free R-module. Let V be a vector space over K. Let $\varphi\colon M \to M$ be an R-linear map. Let $\psi\colon V \to V$ be a K-linear map.

We say that $m \in M$ is an eigenvector of φ to the eigenvalue $\lambda \in R$ if $1 \otimes_R m$ is an eigenvector of the K-linear map $K \otimes_R \varphi$ to the eigenvalue λ as defined by linear algebra.

We say that φ is diagonalizable as an R-linear endomorphism (or short: diagonalizable over R) if there exists an R-linear basis of M consisting of eigenvectors of φ.

We define the eigenmodule to the eigenvalue $\lambda \in R$ as the submodule of M that consists of all eigenvectors of φ to the eigenvalue λ and the zero element, i.e. all elements $x \in M$ satisfying $\varphi(x) = \lambda x$.

We denote the set of all eigenvalues of the R-endomorphism φ by $\sigma(\varphi)$. We denote the set of all eigenvalues of the K-endomorphism ψ by $\sigma(\psi)$.

We denote the eigenmodule to the eigenvalue λ of the R-linear map φ by $\mathrm{E}_\varphi(\lambda)$.

We denote the eigenspace to the eigenvalue μ of the K-linear map ψ by $\mathrm{E}_\psi(\mu)$.

Note that then we have $\mathrm{E}_\varphi(\lambda) = \mathrm{E}_{K\varphi}(\lambda) \cap M$ for $\lambda \in \sigma(\varphi)$.

(26) Suppose given a matrix $A \in R^{n\times n}$. Then eigenvalues and eigenvectors of A are to be calculated over $K := \operatorname{frac} R$, i.e. we consider the eigenvalues and eigenvectors of $A \in K^{n\times n}$.

For an eigenvalue λ of A, we denote the eigenspace of A to the eigenvalue λ by $\mathrm{E}_A(\lambda)$.

(27) Suppose given a field K and a vector space V over K. Suppose given $\Phi = (\varphi_1, \dots, \varphi_k)$ such that $\varphi_i \in \operatorname{End}_K(V)$ is diagonalizable for $i \in [1,k]$ and such that $\varphi_i\varphi_j = \varphi_j\varphi_i$ for $i,j \in [1,k]$. We say that $\lambda = (\lambda_i)_{i\in[1,k]} \in K^{1\times k}$ is an *eigenvalue tuple of* Φ if there exists $v \in V^\times$ such that $\varphi_i(v) = \lambda_i v$ for $i \in [1,k]$.

Suppose given an eigenvalue tuple $\lambda = (\lambda_i)_{i\in[1,k]}$ of Φ. Its *simultaneous eigenspace* $\mathrm{E}_\Phi(\lambda)$ is given by

$$\mathrm{E}_\Phi(\lambda) = \{v \in V \mid \varphi_i(v) = \lambda_i v \text{ for } i \in [1,k]\}\,.$$

Note that $\mathrm{E}_\Phi(\lambda) \neq 0$.

A *simultaneous eigenspace for* Φ is a simultaneous eigenspace for Φ for some eigenvalue tuple of Φ.

(28) Let A be an R-algebra. Let $\mathfrak{l}(A) := A$ as R-modules. Define the map

$$\begin{aligned} [-,=]\colon\quad \mathfrak{l}(A) \times \mathfrak{l}(A) &\longrightarrow \mathfrak{l}(A) \\ (x,y) &\longmapsto [x,y] := xy - yx \end{aligned}$$

which is a bilinear map. Then $\mathfrak{l}(A)$ together with the Lie bracket $[-,=]$ becomes a Lie algebra over R, called the *commutator Lie algebra of A*.

So we write A whenever we are in the context of associative algebras and we write $\mathfrak{l}(A)$ whenever we are in the context of Lie algebras.

(29) Let $\mathfrak{g}$ be a Lie algebra over R. Let M be an R-module. Let $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(M)$ be a morphism of Lie algebras. Then for $g \in \mathfrak{g}$, $m \in M$, we define $[g, m] := (\varphi(g))(m)$. We say that $M = (M, \varphi)$ is a $\mathfrak{g}$-Lie module. We also write $\mathrm{ad}_M(g) := \varphi(g) = \mathfrak{gl}(M) = \mathrm{End}_R(M)$ for $g \in \mathfrak{g}$.

Denoting by 0 the zero map $\mathfrak{g} \xrightarrow{0} \mathfrak{gl}(M)$, we call $M = (M, 0)$ a *trivial* $\mathfrak{g}$*-Lie module*. Then we have $[g, m] = 0$ for $g \in \mathfrak{g}$, $m \in M$.

0.3 List of Magma codes

We make use of the computer algebra system Magma; cf. [BCP97].

0.4 List of counterexamples

Remark 24	A finitely generated free module over an infinite principal ideal domain that can be written as a finite union of proper submodules. Cf. also Lemma 23.
Remark 36	An indecomposable object X in a preadditive category such that the endomorphism ring $\mathrm{End}(X)$ contains a non-trivial idempotent. Cf. also Lemma 35.
Remark 42	A discrete valuation ring R and a matrix $A \in R^{2\times 2}$ such that A is diagonalizable over $\mathrm{frac}(R)$, but A is not diagonalizable over R.
Remark 45	A discrete valuation ring R, finitely generated free R-modules $Y \subseteq X$ and an R-module endomorphism on X that is R-diagonalizable but that is not R-diagonalizable when restricted to Y. Cf. also Lemma 43 and Corollary 44.
Example 100	A Lie algebra $\mathfrak{g}$ over a principal ideal domain and an integral torus $\mathfrak{t} \subseteq \mathfrak{g}$ that is not pure in $\mathfrak{g}$.
Remark 101	A discrete valuation ring R, an R-algebra A and an integral torus $\mathfrak{t}$ in the Lie algebra $\mathfrak{l}(A)$ over R such that $\mathfrak{t}$ is not an R-subalgebra of A.
Remark 104	A discrete valuation ring R with field of fractions K, a Lie algebra $\mathfrak{g}$ over R and a maximal torus $\mathfrak{t} \subseteq K\mathfrak{g}$ such that $\mathfrak{t} \cap \mathfrak{g}$ is not an integral torus in $\mathfrak{g}$. Cf. also Lemma 103.
Remark 118	A discrete valuation ring R, a completely split R-order Γ and an element $x \in \mathfrak{l}(\Gamma)$ with an adjoint endomorphism $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)$ that is not diagonalizable over R but over $\mathrm{frac}\, R$.
Remark 125	A discrete valuation ring R, a completely split R-order Γ and two maximal rational tori in $\mathfrak{l}(\Gamma)$ that are not conjugate via a unit in Γ.
Remark 128	A discrete valuation ring R, a split R-order Ω, an element $x \in \mathfrak{l}(\Omega)$ and an orthogonal decomposition $1 = e + e'$ of 1_Ω into primitive idempotents in Ω such that $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ is diagonalizable over R but $\mathrm{ad}_{\mathfrak{l}(\Omega)}(exe')$ is not diagonalizable over $\mathrm{frac}\, R$.
Remark 150	A discrete valuation ring R, a split R-order Ω and two primitive tori of $\mathfrak{l}(\Omega)$ that are not conjugate via a unit in Ω.
Remark 152	A discrete valuation ring R and a split R-order Ω isomorphic to RG for a finite group G and R-subalgebras T and T_1 of Ω such that the following occur.

- T and T_1 are maximal commutative subalgebras of Ω that are not isomorphic as R-algebras.
- T and T_1 are not conjugate via a unit in $K\Omega$ but $\mathfrak{l}(T)$ and $\mathfrak{l}(T_1)$ are two maximal rational tori of $\mathfrak{l}(\Omega)$.
- $\mathfrak{l}(T_1) \subseteq \mathfrak{l}(\Omega)$ is a non-primitive maximal rational torus and $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$ is a primitive torus.
- A completely split R-overorder $\Gamma \supseteq \Omega$ with full diagonal Δ and $u \in \mathrm{U}(\Gamma)$ such that, writing $\mathring{\Omega} := u^{-1}\Omega u$, the lengths of the R-modules $\Delta/(\Omega \cap \Delta)$ and $\Delta/(\mathring{\Omega} \cap \Delta)$ are different.
- $\mathfrak{l}(T_1) \subseteq \mathfrak{l}(\Omega)$ is a maximal rational torus and 1_Ω is primitive in T_1, but not primitive in Ω.
- $\mathfrak{l}(T)$ and $\mathfrak{l}(T_1)$ are maximal rational tori in $\mathfrak{l}(\Omega)$ such that the integral cores of $\mathfrak{l}(T)$ and $\mathfrak{l}(T_1)$ in $\mathfrak{l}(\Omega)$ have, considered as R-modules, different ranks.
- A completely split R-overorder $\Gamma \supseteq \Omega$ with full diagonal Δ and $u \in \mathrm{U}(\Gamma)$ such that $\mathfrak{l}(\Omega \cap \Delta)$ is a maximal integral torus in $\mathfrak{l}(\Omega)$, but, writing $\mathring{\Omega} := u^{-1}\Omega u$, the integral core of the rational torus $\mathfrak{l}(\mathring{\Omega} \cap \Delta) \subseteq \mathfrak{l}(\mathring{\Omega})$ is not a maximal integral torus in $\mathfrak{l}(\mathring{\Omega})$.

Remark 169 — A discrete valuation ring R, a split R-order Ω, an element $x \in \Omega$ and a primitive idempotent $e \in \Omega$ such that $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ is R-diagonalizable, but $\mathrm{ad}_{\mathfrak{l}(\Omega)}(exe)$ is not R-diagonalizable. Moreover, an integral core C of a maximal rational torus in $\mathfrak{l}(\Omega)$ such that C is not an R-subalgebra of Ω. Cf. also Remark 101.

Remark 172 — A discrete valuation ring R, a split R-order Ω' in a completely split R-order Γ' such that, letting Δ' be the full diagonal in Γ' and $T' := \Omega' \cap \Delta'$, the following holds.
There exists an orthogonal decomposition of $1_{\Omega'} = e_1 + e_2$ into primitive idempotents in Ω' such that $e_1, e_2 \in T'$ and such that $e_1 \Omega' e_2$ is a decomposable T'-T'-bimodule.

0.5 List of open questions

Let R be a principal ideal domain.

Question 93 — Given a cd-tuple Φ on a finitely generated free R-module N; cf. Definition 51. We have the map υ_Φ; cf. Definition 65. Is there exactly one maximal element in the preimage $\upsilon_\Phi^{-1}(N)$?

Question 135 — Is the integral core of a maximal rational torus always generated by the primitive idempotents and central elements that are contained in the torus?

Question 145 — Suppose that Ω is a Wedderburn image of a group ring $\mathbb{Z}_{(p)} \mathrm{S}_n$. Denote by $\mathfrak{l}(T)$ the standard torus in $\mathfrak{l}(\Omega)$. Suppose given an orthogonal decomposition $1_\Omega = \sum_{i \in [1,\ell]} e_i$ into primitive idempotents in Ω where $e_i \in T$. Is $e_i \Omega e_j$ indecomposable as a T-T-bimodule for $i \neq j$?

Question 151 — Suppose that Ω is a split R-order. Is a maximal rational torus in $\mathfrak{l}(\Omega)$ always an R-subalgebra of Ω?

Question 173 — Suppose given an R-order Ω' and an orthogonal decomposition of $1_{\Omega'}$ into primitive idempotents by $\sum_{i \in [1,n]} e_i$. Suppose that $e_i \Omega' e_j \neq 0$.
Is $e_i \Omega' e_j$ indecomposable as a bimodule over the Peirce diagonal $\bigoplus_{i \in [1,n]} e_i \Omega' e_i$?

Chapter 1: Example $\mathbb{Z}_{(3)}\,\mathrm{S}_3$

1.1 Wedderburn: $\mathbb{Z}_{(3)}\,\mathrm{S}_3 \xrightarrow{\sim} \Omega$

As a first example, we consider $R := \mathbb{Z}_{(3)} = \left\{\frac{a}{b} \,\middle|\, a \in \mathbb{Z},\, b \in \mathbb{Z}^\times,\, b \not\equiv_3 0\right\}$ which is a discrete valuation ring with maximal ideal generated by 3, viz. $3\mathbb{Z}_{(3)} = \left\{\frac{a}{b} \in \mathbb{Z}_{(3)} \,\middle|\, a \in 3\mathbb{Z},\, b \in \mathbb{Z}^\times,\, b \not\equiv_3 0\right\}$. We often write $K := \mathbb{Q} = \mathrm{frac}(R)$ for the field of fractions of R. We consider the $\mathbb{Z}_{(3)}$-order

$$\mathbb{Z}_{(3)}\,\mathrm{S}_3 = \left\{ \sum_{\sigma \in \mathrm{S}_3} r_\sigma \sigma \,\middle|\, r_\sigma \in \mathbb{Z}_{(3)} \text{ for } \sigma \in \mathrm{S}_3 \right\}.$$

Since $R \subseteq \mathbb{Q}$, we can embed $R\,\mathrm{S}_3$ in $\mathbb{Q}\,\mathrm{S}_3$. By Maschke's theorem, the group algebra $\mathbb{Q}\,\mathrm{S}_3$ is semisimple. Then, by the Artin-Wedderburn theorem, there exists an isomorphism of $\mathbb{Q}$-algebras

$$\omega\colon \mathbb{Q}\,\mathrm{S}_3 \longrightarrow \prod_{i \in [1,k]} \mathbb{Q}^{n_i \times n_i}$$

where $k \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i \in [1,k]$ and all these integers are uniquely determined up to permutation of the n_i. In the case of $\mathbb{Q}\,\mathrm{S}_3$, we have $\mathbb{Q}\,\mathrm{S}_3 \simeq \mathbb{Q} \times \mathbb{Q}^{2\times 2} \times \mathbb{Q}$.

For the following, we denote by ϱ_1 the trivial representation of S_3 and we denote by ϱ_3 the sign representation of S_3. Moreover, we define ϱ_2 on generators of S_3 as follows; cf. [Kün01, §0.2].

$$\begin{aligned} \varrho_2 : \mathbb{Q}\,\mathrm{S}_3 &\longrightarrow \mathbb{Q}^{2\times 2} \\ (1,2) &\longmapsto \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix} \\ (1,2,3) &\longmapsto \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

This defines in fact a representation of S_3: Since S_3 is isomorphic to the group presented by generators s and t satisfying the relations $s^2 = 1$, $t^3 = 1$ and $(st)^2 = 1$ (via $s \mapsto (1,2)$ and $t \mapsto (1,2,3)$), it is sufficient to verify that the images of $(1,2)$ and $(1,2,3)$ satisfy these relations.

$$\begin{aligned} \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}^3 &= \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \left(\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} \right)^2 &= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This shows that ϱ_2 is in fact a representation of S_3, in particular it is a two-dimensional representation.

So the $\mathbb{Q}$-algebra isomorphism ω given by the Artin-Wedderburn theorem may take the form $(\varrho_1, \varrho_2, \varrho_3)$. We denote the restriction of ω to $R\,\mathrm{S}_3$ by ω^{r}. We obtain the following diagram.

$$\begin{tikzcd}
\sigma \arrow[rr, mapsto] & & (\varrho_1(\sigma), \varrho_2(\sigma), \varrho_3(\sigma)) \\
\mathbb{Q}\,\mathrm{S}_3 \arrow[rr, "\omega", "\sim"'] & & \mathbb{Q} \times \mathbb{Q}^{2\times 2} \times \mathbb{Q} \\
R\,\mathrm{S}_3 \arrow[u, hook] \arrow[rr, "\omega^{\mathrm{r}}"] \arrow[dr, "\sim"'] & & R \times R^{2\times 2} \times R =: \Gamma \arrow[u, hook] \\
& \omega^{\mathrm{r}}(R\,\mathrm{S}_3) =: \Omega \arrow[ur, hook] &
\end{tikzcd}$$

We want to verify that ω is in fact an isomorphism of $\mathbb{Q}$-algebras. Consider the following $\mathbb{Q}$-linear basis of the $\mathbb{Q}$-algebra $\mathbb{Q}\,\mathrm{S}_3$ and the images of these basis elements under ω.

σ	$\omega(\sigma)$	σ	$\omega(\sigma)$
id	$\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$	$(1,3)$	$\left(1, \begin{pmatrix} 1 & -3 \\ 0 & -1 \end{pmatrix}, -1\right)$
$(1,2)$	$\left(1, \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}, -1\right)$	$(2,3)$	$\left(1, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, -1\right)$
$(1,2,3)$	$\left(1, \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}, 1\right)$	$(1,3,2)$	$\left(1, \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}, 1\right)$

We define the following matrix that has in its rows the entries of all these images.

$$U := \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & -2 & 3 & -1 & 2 & -1 \\ 1 & -2 & 3 & -1 & 1 & 1 \\ 1 & 1 & -3 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & -3 & 1 & -2 & 1 \end{pmatrix}$$

We define the standard basis $\mathcal{E} := (\mathcal{E}_{1,1}, \mathcal{E}_{2,2}, \mathcal{E}_{1,1})$ of $\mathbb{Q} \times \mathbb{Q}^{2\times 2} \times \mathbb{Q} = K \otimes_R \Gamma$. Then we consider the determinant of the matrix U since this matrix is the describing matrix of ω with respect to the basis $\mathcal{E}$. We have

$$\det(U) = -54 \in \mathrm{U}(\mathbb{Q}).$$

This shows that ω is an isomorphism of $\mathbb{Q}$-algebras.

We invert this matrix as a matrix in $\mathbb{Q}^{6\times 6}$. Then the following multiple of U^{-1} is again a matrix in $R^{6\times 6}$.

$$6 \cdot U^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -4 & 2 & 2 & 2 & -4 \\ 0 & -2 & 2 & 0 & 2 & -2 \\ 0 & 6 & -6 & -6 & 0 & 6 \\ 2 & 4 & -4 & -2 & -2 & 2 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

The ties needed to describe Ω are obtained by the columns of this matrix. The factor 6 indicates that the ties are to be understood as ties modulo 6. Applying elementary column operations on $6 \cdot U^{-1}$, we obtain the following matrix.

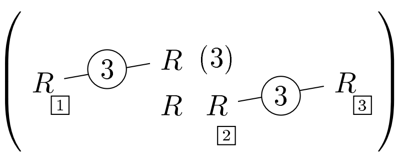
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 & 4 & 6 \end{pmatrix}$$

Since $2 \in \mathrm{U}(R)$ we get the following description of the image of $R\,\mathrm{S}_3$ under ω, i.e. of Ω.

$$\Omega = \omega(R\,\mathrm{S}_3) = \left\{ \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\right) \in R \times R^{2\times 2} \times R \;\middle|\; a \equiv_3 b,\, e \equiv_3 f,\, c \equiv_3 0 \right\}$$

Sometimes we use a more graphical way to illustrate R-algebras that can be described by ties. In this example, we have the following illustration of Ω.

$\Omega \quad =$



The number written in a box below a matrix is associated to the position of the respective matrix in the tuple.

We choose an R-linear basis of Ω. Define $\mathcal{B} := (b_1, b_2, b_3, b_4, b_5, b_6)$ as follows.

$$b_1 := \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \quad b_2 := \left(3, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \quad b_3 := \left(0, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, 0\right),$$
$$b_4 := \left(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0\right), \quad b_5 := \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), \quad b_6 := \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right)$$

We find an orthogonal decomposition of 1_Ω into idempotents e and e' in Ω. Define

$$e := \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right) \text{ and } e' := \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\right).$$

Thus we obtain $e + e' = 1_\Omega$ and $ee' = e'e = 0$ and $e^2 = e$ and $e'^2 = e'$.

Using these two idempotents, we get a Peirce decomposition of Ω as follows.

$$\Omega = e\Omega e \oplus e'\Omega e' \oplus e\Omega e' \oplus e'\Omega e \tag{5}$$

Using the basis elements of Ω in the basis $\mathcal{B}$, this is the same as

$$\Omega = \underbrace{{}_R\left\langle \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \left(3, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)\right\rangle}_{e\Omega e} \oplus \underbrace{{}_R\left\langle \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right)\right\rangle}_{e'\Omega e'}$$
$$\oplus \underbrace{{}_R\left\langle \left(0, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, 0\right)\right\rangle}_{e\Omega e'} \oplus \underbrace{{}_R\left\langle \left(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0\right)\right\rangle}_{e'\Omega e}.$$

We want to show that e and e' are primitive idempotents in Ω. To see that e is primitive in Ω, it suffices to show that $e\Omega e$ is a local ring; cf. Remark 139.(2) below. Similarly, to see that e' is primitive in Ω, it suffices to show that $e'\Omega e'$ is a local ring.

We have the following isomorphism of R-algebras.

$$\begin{array}{ccc} e\Omega e & \xrightarrow{\sim} & \left\{(a, b) \in R^{\times 2} \,\middle|\, a \equiv_3 b\right\} \\ \left(a, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, 0\right) & \longmapsto & (a, b) \end{array}$$

The images of b_1 and of b_2 are $(1, 1)$ and $(3, 0)$ which is an R-linear basis of the right hand side. So we have in fact an isomorphism of R-algebras.

Applying Lemma 33 below, we conclude that $\left\{(a, b) \in R^{\times 2} \,\middle|\, a \equiv_3 b\right\}$ is local. Then also $e\Omega e$ is local and thus the idempotent e is primitive in Ω.

For $e'\Omega e'$, we have the following isomorphism of R-algebras.

$$\begin{array}{ccc} e'\Omega e' & \xrightarrow{\sim} & \left\{(a, b) \in R^{\times 2} \,\middle|\, a \equiv_3 b\right\} \\ \left(0, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, b\right) & \longmapsto & (a, b) \end{array}$$

Using similar arguments as for $e\Omega e$, we conclude that $e'\Omega e'$ is local. Thus e' is a primitive idempotent in Ω.

This shows that e and e' are primitive idempotents in Ω; cf. Remark 139.(2) below.

Hence $1_\Omega = e + e'$ is an orthogonal decomposition of 1_Ω into primitive idempotents in Ω.

1.2 The standard torus $\mathfrak{l}(T)$ in Ω

Keep the notation of §1.1.

As an R-algebra, we can intersect Ω with the R-subalgebra Δ of Γ where every matrix is a diagonal matrix. We denote this intersection by T.

$$T := \Omega \cap \Delta = \left\{ \left(a, \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}, f \right) \in R \times R^{2\times 2} \times R \,\middle|\, a \equiv_3 b,\ e \equiv_3 f \right\}$$

This is a commutative R-subalgebra of Ω.

We have the R-linear basis $\mathcal{B}_T := (b_1, b_2, b_5, b_6)$ of T. Note that

$$T = e\Omega e \oplus e'\Omega e' = Te \oplus Te'.$$

So $\Omega = T \oplus {}_R\langle b_3 \rangle \oplus {}_R\langle b_4 \rangle$.

We obtain the following illustration of T.

$$T = \left(\begin{array}{ccccc} R_{\boxed{1}} & \text{—}3\text{—} & \begin{array}{cc} R & 0 \\ 0 & R_{\boxed{2}} \end{array} & \text{—}3\text{—} & R_{\boxed{3}} \end{array} \right)$$

First we will show that T equals the centralizer $\mathrm{C}_\Omega(T)$ of T in Ω.

Since T is a commutative R-subalgebra of Ω, we have $T \subseteq \mathrm{C}_\Omega(T)$.

For the other direction $T \overset{!}{\supseteq} \mathrm{C}_\Omega(T)$, let $x \in \mathrm{C}_\Omega(T)$ and $y := \left(0, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \in T$. Then there exist $a, b, c, d, e, f \in R$ with $a \equiv_3 b$, $c \equiv_3 0$ and $e \equiv_3 f$ such that $x = \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f \right)$. We have $x \cdot y = y \cdot x$ leading to the condition $3 \left(\begin{smallmatrix} b & 0 \\ d & 0 \end{smallmatrix} \right) = 3 \left(\begin{smallmatrix} b & c \\ 0 & 0 \end{smallmatrix} \right)$. Since R is free of zero divisors, this is equivalent to $c = d = 0$, so we conclude that $T \supseteq \mathrm{C}_\Omega(T)$.

Altogether, we have shown that $T = \mathrm{C}_\Omega(T)$.

Let $\mathfrak{l}(\Omega) = \Omega$ as R-modules. We equip $\mathfrak{l}(\Omega)$ with the commutator Lie bracket.

$$\begin{array}{rccl} [-,=]\colon & \mathfrak{l}(\Omega) \times \mathfrak{l}(\Omega) & \longrightarrow & \mathfrak{l}(\Omega) \\ & (x, y) & \longmapsto & [x, y] := xy - yx \end{array}$$

Thus $\mathfrak{l}(\Omega)$ becomes a Lie algebra over R, the commutator Lie algebra of Ω. Likewise we have the Lie algebra $\mathfrak{l}(T)$ over R. We have $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$ as Lie algebras over R, so $\mathfrak{l}(\Omega)$ becomes an $\mathfrak{l}(T)$-Lie module.

From the theory of Lie algebras, we recall the adjoint morphism $\mathrm{ad}_{\mathfrak{l}(\Omega)}$; cf. [Kün15, Definition 8]. We use it now in our context of Lie algebras over R.

$$\begin{array}{rccl} \mathrm{ad}_{\mathfrak{l}(\Omega)}\colon & \mathfrak{l}(\Omega) & \longrightarrow & \mathrm{End}_R(\mathfrak{l}(\Omega)) \\ & x & \longmapsto & \mathrm{ad}_{\mathfrak{l}(\Omega)}(x)\colon \begin{array}{ccc} \mathfrak{l}(\Omega) & \to & \mathfrak{l}(\Omega) \\ y & \mapsto & [x, y] \end{array} \end{array}$$

We consider the describing matrices of $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_i)$ for the basis elements b_1, b_2, b_5 and b_6 of the basis $\mathcal{B}_T$ with respect to the basis $\mathcal{B}$.

$$(\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_1)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&-1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \qquad (\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_2)_{\mathcal{B},\mathcal{B}} = 0$$

$$(\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_5)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&-1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \qquad (\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_6)_{\mathcal{B},\mathcal{B}} = 0$$

Note that these matrices are all diagonal. But this is dependent on the choice of the basis $\mathcal{B}$. However, the property of being diagonalizable is independent of the choice of the basis. In particular, there exists a matrix $S \in \mathrm{GL}_6(R)$ such that $S^{-1} \cdot ((\mathrm{ad}_{\mathfrak{l}(\Omega)} b_i)_{\mathcal{B},\mathcal{B}}) \cdot S$ is a diagonal matrix for $i \in \{1, 2, 5, 6\}$, viz. $S = 1_{R^{6\times 6}}$.

Our aim is to establish a theory of maximal rational tori and maximal integral tori. Once established, $\mathfrak{l}(T)$ will be an example of both a maximal torus in $\mathfrak{l}(\Omega)$ and a maximal integral torus in $\mathfrak{l}(\Omega)$.

1.3 Decompositions of Ω

Keep the notation of §1.1 and §1.2.

We are now interested in decompositions of Ω into indecomposable submodules. We want to consider two possibilities of decomposing Ω.

On the one hand, we will decompose Ω as a T-T-bimodule. On the other hand, we will decompose $\mathfrak{l}(\Omega)$ as an $\mathfrak{l}(T)$-Lie module.

To see the indecomposability of the direct summands, we introduce the methods we will also use in the examples in §6 and §7 below, even though in this small example there might exist shorter ways.

1.3.1 A decomposition of Ω into T-T-bimodules

As a T-T-sub-bimodule of Ω, we can decompose T into the direct sum $T = Te \oplus Te'$. Since T is commutative, both Te and Te' are in fact T-T-sub-bimodules of Ω. Then we have the following decomposition of Ω.

$$\Omega = \underbrace{{}_R\langle b_1, b_2\rangle}_{Te} \oplus \underbrace{{}_R\langle b_5, b_6\rangle}_{Te'} \oplus \underbrace{{}_R\langle b_3\rangle}_{e\Omega e'} \oplus \underbrace{{}_R\langle b_4\rangle}_{e'\Omega e}$$

We will show in the following that this is a decomposition into indecomposable T-T-bimodules, i.e. we will show that Te and Te' are both indecomposable as T-T-bimodules.

Ad Te.

For a better distinction between the basis elements of Ω and the basis elements of the Peirce component Te, we write $x_1 := b_1$ and $x_2 := b_2$. So we have $Te = {}_R\langle x_1, x_2\rangle$.

To show the indecomposability of Te as a T-T-bimodule, it suffices to show that the T-T-endomorphism ring $\mathrm{End}_{T\text{-}T}(Te)$ is a local ring; cf. Lemma 35 below. This ring can be written as

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(Te) = \big\{h \in \mathrm{End}_R(Te) \mid\ & h(b_i x_j) = b_i h(x_j) \text{ for } i \in \{1,2,5,6\},\ j \in [1,2] \text{ and} \\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in \{1,2,5,6\},\ j \in [1,2]\big\}.\end{aligned}$$

We obtain the following diagram.

$$\begin{tikzcd} h \arrow[r, mapsto] & h_{(x_1,x_2),(x_1,x_2)} \\ \mathrm{End}_R(Te) \arrow[r, "\varphi_1", "\sim"'] & R^{2\times 2} \\ \mathrm{End}_{T\text{-}T}(Te) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_1(\mathrm{End}_{T\text{-}T}(Te)) =: E_1 \arrow[u, hook] \end{tikzcd}$$

Here the map $\varphi_1 \colon \mathrm{End}_R(Te) \to R^{2\times 2}$ is the ring isomorphism sending a map $h \in \mathrm{End}_R(Te)$ to its describing matrix in the ring of 2×2-matrices over R with respect to the R-linear basis (x_1, x_2) of Te. We can embed $\mathrm{End}_{T\text{-}T}(Te)$ into the endomorphism ring $\mathrm{End}_R(Te)$ and thus we can also apply φ_1 to $\mathrm{End}_{T\text{-}T}(Te)$. We denote the image $\varphi_1(\mathrm{End}_{T\text{-}T}(Te))$ by E_1. Since φ_1 is a ring morphism, E_1 is a subring of $R^{2\times 2}$.

So in order to show that $\operatorname{End}_{T\text{-}T}(Te)$ is a local ring, it suffices to show that E_1 is a local ring. To do so, we need a description of the elements in E_1.

For $i \in \{1,2,5,6\}$ we define $M_{(x_1,x_2),i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on Te from the left with respect to the basis (x_1,x_2). For $j \in \{1,2,5,6\}$ we define $M_{(x_1,x_2),j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on Te from the right with respect to the basis (x_1,x_2). Then we can describe E_1 as follows.

$$\begin{aligned}\operatorname{End}_{T\text{-}T}(Te) \simeq E_1 = \big\{ M \in R^{2\times 2} \mid M \cdot M_{(x_1,x_2),i,\mathrm{l}} = M_{(x_1,x_2),i,\mathrm{l}} \cdot M \text{ for } i \in \{1,2,5,6\} \text{ and} \\ M \cdot M_{(x_1,x_2),j,\mathrm{r}} = M_{(x_1,x_2),j,\mathrm{r}} \cdot M \text{ for } j \in \{1,2,5,6\} \,\big\}\end{aligned}$$

We determine the matrices $M_{(x_1,x_2),i,\mathrm{l}}$ and $M_{(x_1,x_2),i,\mathrm{r}}$ for $i \in \{1,2,5,6\}$.

i	$b_i \cdot x_1$	$b_i \cdot x_2$	$M_{(x_1,x_2),i,\mathrm{l}}$	$x_1 \cdot b_i$	$x_2 \cdot b_i$	$M_{(x_1,x_2),i,\mathrm{r}}$
1	x_1	x_2	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	x_1	x_2	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$
2	x_2	$3x_2$	$\left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right)$	x_2	$3x_2$	$\left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right)$
5	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
6	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$

In this example, we are in the case that we want to show the indecomposability of a submodule of a commutative R-algebra. Thus we have $M_{(x_1,x_2),i,\mathrm{l}} = M_{(x_1,x_2),i,\mathrm{r}}$ for $i \in \{1,2,5,6\}$.

Moreover, Te operates trivial on Te' since $T = Te \oplus Te'$ is commutative and we have $ee' = e'e = 0$.

Since $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ are central elements in the ring of 2×2-matrices, the description of E_1 shortens to

$$E_1 = \left\{ M \in R^{2\times 2} \,\middle|\, M \cdot \begin{pmatrix}0&0\\1&3\end{pmatrix} = \begin{pmatrix}0&0\\1&3\end{pmatrix} \cdot M \right\}.$$

Suppose given $M = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in R^{2\times 2}$ such that $M \cdot \left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right) \cdot M$. Then we obtain

$$\begin{aligned}\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) &\iff \left(\begin{smallmatrix}b&3b\\d&3d\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\a+3c&b+3d\end{smallmatrix}\right) \\ &\iff b = 0 \text{ and } a + 3c = d\end{aligned}$$

which is equivalent to $M = \left(\begin{smallmatrix}a&0\\c&a+3c\end{smallmatrix}\right)$, so $M \in {}_R\langle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right) \rangle$. This shows that

$$E_1 = {}_R\left\langle \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&3\end{pmatrix} \right\rangle.$$

It remains to show that E_1 is a local ring. We will determine the units in E_1. Then we will show that the sum of two non-units in E_1 always is a non-unit in E_1; cf. Lemma 28 and Remark 30 below. This will show that E_1 is a local ring.

Suppose given $x \in E_1$. Then there exist $a, c \in R$ such that $x = \left(\begin{smallmatrix}a&0\\c&a+3c\end{smallmatrix}\right)$. Now x is a unit in $R^{2\times 2}$ if and only if $\det x = a^2 + 3ac$ is a unit in R, i.e. if and only if a^2 is a unit in R. But this is the case if and only if a is a unit in R. So the units in E_1 are

$$\mathrm{U}(E_1) = \left\{ r \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) + s \left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right) \,\middle|\, r \in \mathrm{U}(R),\ s \in R \right\};$$

cf. Lemma 28.

We observe that the sum of two non-units in E_1 is again a non-unit in E_1 since R itself is a local ring; cf. Remark 30 below. Again by Remark 30, this shows that E_1 is a local ring which implies that $\operatorname{End}_{T\text{-}T}(Te)$ is also a local ring.

We conclude that Te is indecomposable as a T-T-sub-bimodule of Ω; cf. Lemma 35 below.

Alternatively, we could have argued that

$$\mathrm{End}_{T\text{-}T}(Te) \simeq \mathrm{End}_T(Te) \simeq eTe = Te \simeq \{(a,b) \in R \times R \mid a \equiv_3 b\}$$

where the latter ring is a local ring.

We can also show by direct calculation that E_1 does not contain a non-trivial idempotent; cf. Definition 31 below. This is also a sufficient condition for the indecomposability of Te as a T-T-sub-bimodule of Ω; cf. Lemma 35 below.

Assume that $M = \left(\begin{smallmatrix} a & 0 \\ c & a+3c \end{smallmatrix}\right)$ is a non-trivial idempotent in E_1. Then $M^2 = M$, so we have

$$\begin{pmatrix} a^2 & 0 \\ 2ac + 3c^2 & a^2 + 6ac + c^2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a + 3c \end{pmatrix}.$$

This leaves us two cases, either $a = 0$ or $a = 1$. If $a = 0$, then $\begin{pmatrix} 0 & 0 \\ 3c^2 & c^2 \end{pmatrix} = \left(\begin{smallmatrix} 0 & 0 \\ c & 3c \end{smallmatrix}\right)$, so $c = 0$ and hence $M = 0$. If $a = 1$, then $\begin{pmatrix} 1 & 0 \\ c(2+3c) & 1+6c+c^2 \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ c & 1+3c \end{smallmatrix}\right)$. But $2 + 3c \neq 0$ for $c \in \mathbb{Z}_{(3)}$, so the only solution is $c = 0$, hence $M = 1$. This is a *contradiction* to the non-triviality of M.

Ad Te'.

For a better distinction between the basis elements of Ω and the basis elements of the Peirce component Te', we write $x_3 := b_5$ and $x_4 := b_6$. So we have $Te' = {}_R\langle x_3, x_4\rangle$.

To show the indecomposability of Te' as a T-T-bimodule, it suffices to show that the T-T-endomorphism ring $\mathrm{End}_{T\text{-}T}(Te')$ is a local ring; cf. Lemma 35 below. This ring can be written as

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(Te') = \big\{h \in \mathrm{End}_R(Te') \mid\, & h(b_i x_j) = b_i h(x_j) \text{ for } i \in \{1,2,5,6\},\ j \in [3,4] \text{ and} \\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in \{1,2,5,6\},\ j \in [3,4]\big\}.\end{aligned}$$

We obtain the following diagram.

$$\begin{tikzcd} h \arrow[r, mapsto] & h_{(x_3,x_4),(x_3,x_4)} \\ \mathrm{End}_R(Te') \arrow[r, "\varphi_2", "\sim"'] & R^{2\times 2} \\ \mathrm{End}_{T\text{-}T}(Te') \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_2(\mathrm{End}_{T\text{-}T}(Te')) =: E_2 \arrow[u, hook] \end{tikzcd}$$

Here the map $\varphi_2\colon \mathrm{End}_R(Te') \to R^{2\times 2}$ is the ring isomorphism sending a map $h \in \mathrm{End}_R(Te')$ to its describing matrix in the ring of 2×2-matrices over R with respect to the R-linear basis (x_3, x_4) of Te'. We can embed $\mathrm{End}_{T\text{-}T}(Te')$ into the endomorphism ring $\mathrm{End}_R(Te')$ and thus we can also apply φ_2 to $\mathrm{End}_{T\text{-}T}(Te')$. We denote the image $\varphi_2(\mathrm{End}_{T\text{-}T}(Te'))$ by E_2. Since φ_2 is a ring morphism, E_2 is a subring of $R^{2\times 2}$.

So in order to show that $\mathrm{End}_{T\text{-}T}(Te')$ is a local ring, it suffices to show that E_2 is a local ring. To do so, we need a description of the elements in E_2.

For $i \in \{1,2,5,6\}$ we define $M_{(x_3,x_4),i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on Te' from the left with respect to the basis (x_3, x_4). For $j \in \{1,2,5,6\}$ we define $M_{(x_3,x_4),j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on Te' from the right with respect to the basis (x_3, x_4). Then we can describe E_2 as follows.

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(Te') \simeq E_2 = \big\{M \in R^{2\times 2} \mid\, & M \cdot M_{(x_3,x_4),i,\mathrm{l}} = M_{(x_3,x_4),i,\mathrm{l}} \cdot M \text{ for } i \in \{1,2,5,6\} \text{ and} \\ & M \cdot M_{(x_3,x_4),j,\mathrm{r}} = M_{(x_3,x_4),j,\mathrm{r}} \cdot M \text{ for } j \in \{1,2,5,6\}\,\big\}\end{aligned}$$

We determine the matrices $M_{(x_3,x_4),i,\mathrm{l}}$ and $M_{(x_3,x_4),i,\mathrm{r}}$ for $i \in \{1,2,5,6\}$.

i	$b_i \cdot x_3$	$b_i \cdot x_4$	$M_{(x_3,x_4),i,\mathrm{l}}$	$x_3 \cdot b_i$	$x_4 \cdot b_i$	$M_{(x_3,x_4),i,\mathrm{r}}$
1	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
2	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
5	x_3	x_4	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	x_3	x_4	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$
6	x_4	$3x_4$	$\left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right)$	x_4	$3x_4$	$\left(\begin{smallmatrix}0&0\\1&3\end{smallmatrix}\right)$

In this example, we are in the case that we want to show the indecomposability of a submodule of a commutative R-algebra. Thus we have $M_{(x_3,x_4),i,\mathrm{l}} = M_{(x_3,x_4),i,\mathrm{r}}$ for $i \in \{1,2,5,6\}$.

Moreover, Te' operates trivial on Te since $T = Te \oplus Te'$ is commutative and we have $ee' = e'e = 0$.

Since $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ are central elements in the ring of 2×2-matrices, the description of E_2 shortens to

$$E_2 = \left\{ M \in R^{2\times 2} \,\middle|\, M \cdot \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} \cdot M \right\}.$$

But this is the same as E_1. We have already seen that E_1 is a local ring, so E_2 is a local ring as well.

Using the same arguments as for $\mathrm{End}_{T\text{-}T}(Te)$ this shows that $\mathrm{End}_{T\text{-}T}(Te')$ is indecomposable as a T-T-sub-bimodule of Ω.

We summarize.

We get the following decomposition of Ω into indecomposable T-T-bimodules.

$$\Omega = \underbrace{{}_R\langle b_1, b_2\rangle}_{Te} \oplus \underbrace{{}_R\langle b_5, b_6\rangle}_{Te'} \oplus \underbrace{{}_R\langle b_3\rangle}_{e\Omega e'} \oplus \underbrace{{}_R\langle b_4\rangle}_{e'\Omega e}$$

But this is exactly the Peirce decomposition we found in §1.1; cf. equation (5).

There are two more endomorphism rings we can have a look at, viz. $\mathrm{End}_{T\text{-}T}(e\Omega e')$ and $\mathrm{End}_{T\text{-}T}(e'\Omega e)$. Both of them are isomorphic to R which is a local ring.

So for all direct summands in the decomposition, we have seen that their respective T-T-endomorphism ring is a local ring.

We will show one more thing: Each of the direct summands is not isomorphic as a T-T-bimodule to any of the other direct summands. To see this we will show that the annihilator in Ω of each summand is different from the annihilators of the other summands; cf. Lemma 26 below.

We have the following.

- $e(Te)e \neq 0$, but $e(Te')e = 0$, $e(e\Omega e')e = 0$, $e(e'\Omega e)e = 0$. This shows that Te is not isomorphic to any of the other direct summands.
- $e'(Te')e' \neq 0$, but $e'(e\Omega e')e' = 0$, $e'(e'\Omega e)e' = 0$. This shows that Te' is neither isomorphic to $e\Omega e'$ nor to $e'\Omega e$.
- $e(e\Omega e')e' \neq 0$, but $e(e'\Omega e)e' = 0$. This shows that $e\Omega e' \not\cong e'\Omega e$.

This shows that all direct summands of Ω in the Peirce decomposition are pairwise non-isomorphic as T-T-bimodules.

1.3.2 A decomposition of $\mathfrak{l}(\Omega)$ into $\mathfrak{l}(T)$-Lie modules

Now we want to decompose $\mathfrak{l}(\Omega)$ into a direct sum of indecomposable $\mathfrak{l}(T)$-Lie submodules. We abbreviate $T_i := {}_R\langle b_i \rangle$ for $i \in [1, 6]$. We have seen that T is commutative. We conclude that $\mathfrak{l}(T)$ is an abelian Lie algebra over R. Thus T is a trivial $\mathfrak{l}(T)$-Lie module. We could decompose T into submodules of rank 1, but this corresponds to the task of decomposing $R^{\oplus 4}$ into R-submodules of rank 1.

We see that T_3 and T_4 are in fact $\mathfrak{l}(T)$-Lie submodules because both are T-T-bimodules as well. In fact we have $[b_1, b_3] = b_3$, $[b_5, b_3] = -b_3$ and $[b_2, b_3] = [b_6, b_3] = 0$ and $[b_1, b_4] = -b_4$, $[b_5, b_4] = b_4$ and $[b_2, b_4] = [b_6, b_4] = 0$.

So we have a decomposition of $\mathfrak{l}(\Omega)$ into a direct sum of indecomposable $\mathfrak{l}(T)$-Lie submodules of rank 1 given by

$$\mathfrak{l}(\Omega) = \bigoplus_{i \in [1,6]} T_i.$$

If we keep T in the decomposition because it is a trivial $\mathfrak{l}(T)$-Lie module, then we get

$$\mathfrak{l}(\Omega) = T \oplus T_3 \oplus T_4.$$

Next we consider the $\mathfrak{l}(T)$-endomorphism rings $\mathrm{End}_{\mathfrak{l}(T)}(T_i)$ for $i \in [1, 6]$.

We know that $\mathrm{End}_R(T_i) = {}_R\langle \mathrm{id}_{T_i} \rangle$ for $i \in [1, 6]$. Since already $\mathrm{id}_{T_i} \in \mathrm{End}_{\mathfrak{l}(T)}(T_i) \subseteq \mathrm{End}_R(T_i)$, we conclude that $\mathrm{End}_R(T_i) = \mathrm{End}_{\mathfrak{l}(T)}(T_i) \simeq R$ for $i \in [1, 6]$.

This shows that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(T_i)$ is a local ring for $i \in [1, 6]$ since R itself is local.

In §1.3.1 we have shown that all summands in the T-T-bimodule-decomposition of Ω are pairwise non-isomorphic.

As $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$, we have $T_1 \simeq T_2 \simeq T_5 \simeq T_6$ as trivial $\mathfrak{l}(T)$-Lie modules of rank 1 over R, so all summands of T are isomorphic. The $\mathfrak{l}(T)$-Lie submodules T_3 and T_4 are not trivial $\mathfrak{l}(T)$-Lie modules.

Furthermore, we can show that T_3 is not isomorphic to T_4.

Assume that there exists an $\mathfrak{l}(T)$-linear isomorphism f between T_3 and T_4. Each element of T_3 is a multiple of b_3. This has to be sent to a multiple of b_4. So there exists $u \in \mathrm{U}(R)$ such that

$$\begin{aligned} f \colon T_3 &\xrightarrow{\sim} T_4 \\ \left(0, \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}, 0\right) &\longmapsto \left(0, \begin{pmatrix} 0 & 0 \\ ux & 0 \end{pmatrix}, 0\right) \end{aligned}$$

for $x \in R$.

Since f is $\mathfrak{l}(T)$-linear, we have $f[t, x \cdot b_3] = [t, f(x \cdot b_3)]$ for $t \in \mathfrak{l}(T)$ and $x \in R$.

We have

$$\begin{aligned} f([e, b_3]) &= f\left(\left[\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, 0\right)\right]\right) = f\left(\left(0, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, 0\right)\right) \\ &= \left(0, \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, 0\right) = u \cdot b_4 \\ [e, f(b_3)] &= \left[\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), f\left(\left(0, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, 0\right)\right)\right] = \left[\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, 0\right)\right] \\ &= \left(0, \begin{pmatrix} 0 & 0 \\ -u & 0 \end{pmatrix}, 0\right) = -u \cdot b_4. \end{aligned}$$

These two expressions are equal, thus $u \cdot b_4 = -u \cdot b_4$ and hence $u = -u$. So $u = 0$ which is a *contradiction* to $u \in \mathrm{U}(R)$.

This shows that T_3 and T_4 are not isomorphic as $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$.

1.4 Interlude: Roots and root paths

Let R be a principal ideal domain. Let Ω be an R-order. Let T be a commutative R-subalgebra of Ω.

Definition 1. Let $\chi \in T^* := \operatorname{Hom}(T, R)$ be an R-linear map.

If there exists an element $x \in \Omega^\times$ such that $(\operatorname{ad}_{\mathfrak{l}(\Omega)} t)(x) = \chi(t)x$ for $t \in T$, then χ is called a *root* of Ω. For a root $\chi \in T^*$ of Ω, we define the *root path* Ω_χ to the root χ of Ω as follows.

$$\Omega_\chi := \left\{x \in \Omega \,\middle|\, (\operatorname{ad}_{\mathfrak{l}(\Omega)} t)(x) = \chi(t)x \text{ for } t \in T\right\}$$

Let $\chi_\mathrm{l}, \chi_\mathrm{r} \in T^*$. If there exists an element $x \in \Omega^\times$ such that $tx = \chi_\mathrm{l}(t)x$ and $xt = \chi_\mathrm{r}(t)x$ for $t \in T$, we call the pair $(\chi_\mathrm{l},\, \chi_\mathrm{r}) \in T^* \times T^*$ a *biroot* of Ω. For a biroot $\chi' = (\chi_\mathrm{l},\, \chi_\mathrm{r}) \in T^* \times T^*$ of Ω, we define the *biroot path* $\Omega_{\chi'}$ to the biroot χ' of Ω as follows.

$$\Omega_{\chi'} := \{x \in \Omega \,|\, tx = \chi_\mathrm{l}(t)x \text{ for } t \in T \text{ and } xt = \chi_\mathrm{r}(t)x \text{ for } t \in T\}$$

Example 2. Let $\chi = 0 \in T^*$ be the zero homomorphism. Then Ω_0 is the set of all elements x in Ω where $tx - xt = 0$ for $t \in T$, i.e. $\Omega_0 = \mathrm{C}_\Omega(T)$, the centralizer of T in Ω.

Lemma 3. *Let χ be a root of Ω. Then the root path Ω_χ is an R-submodule of Ω.*

Proof. We have $0 \in \Omega_\chi$. Let $x, y \in \Omega_\chi$. Let $r, s \in R$. Suppose given $t \in \mathfrak{t}$. We calculate.

$$\begin{aligned}(\operatorname{ad}_{\mathfrak{l}(\Omega)} t)(rx + sy) &= [t, rx + sy] = [t, rx] + [t, sy] = r[t, x] + s[t, y] \\ &= r(\operatorname{ad}_{\mathfrak{l}(\Omega)} t)(x) + s(\operatorname{ad}_{\mathfrak{l}(\Omega)} t)(y) = r\chi(t)x + s\chi(t)y \\ &= \chi(t)rx + \chi(t)sy = \chi(t)(rx + sy)\end{aligned}$$

This shows that $rx + sy \in \Omega_\chi$. □

Lemma 4. *Let $\chi = (\chi_\mathrm{l},\, \chi_\mathrm{r})$ be a biroot of Ω. Then the biroot path Ω_χ is an R-submodule of Ω.*

Proof. We have $0 \in \Omega_\chi$. Let $x, y \in \Omega_\chi$. Let $r, s \in R$. Suppose given $t \in \mathfrak{t}$. We calculate.

$$\begin{aligned}t(rx + sy) &= rtx + sty = r\chi_\mathrm{l}(t)x + s\chi_\mathrm{l}(t)y \\ &= \chi_\mathrm{l}(t)(rx + sy) \\ (rx + sy)t &= rxt + syt = r\chi_\mathrm{r}(t)x + s\chi_\mathrm{r}(t)y \\ &= \chi_\mathrm{r}(t)(rx + sy)\end{aligned}$$

This shows that $rx + sy \in \Omega_\chi$. □

Lemma 5.

(1) *Let $\Omega = \bigoplus_{i \in [1,u]} T_i$ be a decomposition of Ω into a direct sum of T-T-sub-bimodules of Ω. Suppose given $p \in [1, u]$ such that T_p is an indecomposable summand of rank 1. Then there exists a unique biroot $\chi \in T^* \times T^*$ such that $T_p \subseteq \Omega_\chi$.*

(2) *Let $\mathfrak{l}(\Omega) = \bigoplus_{i \in [1,v]} S_i$ be a decomposition of $\mathfrak{l}(\Omega)$ into a direct sum of $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$. Suppose given $q \in [1, v]$ such that S_q is an indecomposable summand of rank 1 over R. Then there exists a unique root $\chi \in T^*$ such that $S_q \subseteq \Omega_\chi$.*

Proof. Ad (1). We find an element $x \in \Omega^\times$ such that $T_p = {}_R\langle x \rangle$.

For $t \in T$, the element tx is a multiple of x since $tx \in T_p$. Thus we can uniquely define a map $\chi_\mathrm{l} \in T^*$ such that $tx =: \chi_\mathrm{l}(t)x$ for $t \in T$. This map is independent of the choice of x.

We will show that χ_l is an R-linear map. Suppose given $\alpha, \alpha' \in R$ and $t, t' \in T$. We calculate.

$$\begin{aligned}\chi_\mathrm{l}(\alpha t + \alpha' t')x &= (\alpha t + \alpha' t')x = \alpha tx + \alpha' t' x \\ &= \alpha\chi_\mathrm{l}(t)x + \alpha'\chi_\mathrm{l}(t')x = \big(\alpha\chi_\mathrm{l}(t) + \alpha'\chi_\mathrm{l}(t')\big)\, x\end{aligned}$$

Since Ω is torsion free, we have $\chi_{\mathrm{l}}(\alpha t + \alpha' t') = (\alpha\chi_{\mathrm{l}}(t) + \alpha'\chi_{\mathrm{l}}(t'))$, so χ_{l} is an R-linear map.

Furthermore, the element xt is a multiple of x for $t \in T$, so we also can uniquely define a map $\chi_{\mathrm{r}} \in T^*$ such that $xt =: \chi_{\mathrm{r}}(t)x$ for $t \in T$. This map is independent of the choice of x. With similar arguments as for χ_{l} we conclude that χ_{r} is an R-linear map.

Taking both together, we obtain the biroot $\chi := (\chi_{\mathrm{l}}, \chi_{\mathrm{r}}) \in T^* \times T^*$ of Ω.

Suppose given $y \in T_p$. We find an element $r \in R$ such that $y = rx$. Suppose given $t \in T$. We calculate.

$$\begin{aligned} ty &= trx = rtx = r\chi_{\mathrm{l}}(t)x = \chi_{\mathrm{l}}(t)rx = \chi_{\mathrm{l}}(t)y \\ yt &= rxt = r\chi_{\mathrm{r}}(t)x = \chi_{\mathrm{r}}(t)rx = \chi_{\mathrm{r}}(t)y \end{aligned}$$

We conclude that $y \in \Omega_\chi$. This shows that $T_p \subseteq \Omega_\chi$.

Ad (2). We find an element $x \in \Omega^\times$ such that $S_q = {}_R\langle x\rangle$.

For $t \in T$, the element $[t,x]$ is a multiple of x since $[t,x] \in S_q$. Thus we can uniquely define a map $\chi \in T^*$ such that $(\mathrm{ad}_{\mathfrak{l}(\Omega)}\, t)(x) = [t,x] =: \chi(t)x$ for $t \in T$. This map is independent of the choice of x.

We will show that χ is an R-linear map. Suppose given $\alpha, \alpha' \in R$ and $t, t' \in T$. We calculate.

$$\begin{aligned} \chi(\alpha t + \alpha' t')x &= [\alpha t + \alpha' t', x] = \alpha[t,x] + \alpha'[t',x] \\ &= \alpha\chi(t)x + \alpha'\chi(t')x = \big(\alpha\chi(t) + \alpha'\chi(t')\big)\, x \end{aligned}$$

Since Ω is torsion free, we have $\chi(\alpha t + \alpha' t') = (\alpha\chi(t) + \alpha'\chi(t'))$, so χ is an R-linear map. This shows that χ is a root of Ω.

Suppose given $y \in S_q$. We find an element $r \in R$ such that $y = rx$. For $t \in T$, we have

$$(\mathrm{ad}_{\mathfrak{l}(\Omega)}\, t)(y) = [t,y] = [t,rx] = r[t,x] = r\chi(t)x = \chi(t)rx = \chi(t)y,$$

so $y \in \Omega_\chi$. This shows that $S_q \subseteq \Omega_\chi$. □

Example 6. As an example, we provide the biroots and roots of Ω of §1.1; cf. §1.5.1 and §1.5.2 below.

Lemma 7. *Let $(\chi_{\mathrm{l}}, \chi_{\mathrm{r}}) \in T^* \times T^*$ be a biroot of Ω. Then $\chi_{\mathrm{l}} - \chi_{\mathrm{r}}$ is a root of Ω and*

$$\Omega_{(\chi_{\mathrm{l}}, \chi_{\mathrm{r}})} \subseteq \Omega_{\chi_{\mathrm{l}} - \chi_{\mathrm{r}}}.$$

Proof. Suppose given $x \in \Omega^\times$ such that $tx = \chi_{\mathrm{l}}(t)x$ and $xt = \chi_{\mathrm{r}}(t)x$ for $t \in T$. Suppose given $t \in T$.

We have $(\mathrm{ad}_{\mathfrak{l}(\Omega)}\, t)(x) = [t,x] = tx - xt = \chi_{\mathrm{l}}(t)x - \chi_{\mathrm{r}}(t)x = ((\chi_{\mathrm{l}} - \chi_{\mathrm{r}})(t))x$. So $\chi_{\mathrm{l}} - \chi_{\mathrm{r}}$ is a root of Ω.

Ad $\subseteq$. Let $x \in \Omega_{(\chi_{\mathrm{l}}, \chi_{\mathrm{r}})}$. Then we have $tx = \chi_{\mathrm{l}}(t)x$ and $xt = \chi_{\mathrm{r}}(t)x$ for $t \in \mathfrak{t}$. Suppose given $t \in \mathfrak{t}$. Then $(\mathrm{ad}_{\mathfrak{l}(\Omega)}\, t)(x) = [t,x] = tx - xt = \chi_{\mathrm{l}}(t)x - \chi_{\mathrm{r}}(t)x = ((\chi_{\mathrm{l}} - \chi_{\mathrm{r}})(t))x$. So we have $x \in \Omega_{\chi_{\mathrm{l}} - \chi_{\mathrm{r}}}$. □

Example 8. Consider Ω of §1.1. We will give the roots and biroots of Ω; cf. §1.5.1 and §1.5.2 below. Note that for a biroot $(\chi_{\mathrm{l}}, \chi_{\mathrm{r}})$ of Ω in §1.5.1, the difference $\chi_{\mathrm{l}} - \chi_{\mathrm{r}}$ occurs as a root of Ω in §1.5.2.

Lemma 9. *Suppose that X and Y are two indecomposable summands of rank 1 in a decomposition of Ω into T-T-bimodules. Let $\chi_X = (\chi_{X,\mathrm{l}}, \chi_{X,\mathrm{r}})$ be the biroot of the summand X as constructed in Lemma 5.(1). Let $\chi_Y = (\chi_{Y,\mathrm{l}}, \chi_{Y,\mathrm{r}})$ be the biroot of the summand Y as constructed in Lemma 5.(1). Then we have the following equivalence:*

$$X \simeq Y \iff \chi_X = \chi_Y$$

Proof. Ad $\Longrightarrow$. Suppose given $x \in X$ and $y \in Y$ such that $X = {}_R\langle x\rangle$ and $Y = {}_R\langle y\rangle$. Suppose that $f\colon X \to Y$ is an isomorphism of R-modules. Since f is surjective, there exists an element $r \in R$ such that $y = r\cdot f(x)$. Moreover, we find an element $s \in R$ such that $f(x) = s\cdot y$. But now $y = r\cdot f(x) = r\cdot s\cdot y$ and since R is torsion free, we have $r \cdot s = 1$ and thus $r, s \in \mathrm{U}(R)$. We obtain $Y = {}_R\langle f(x)\rangle$.

So without loss of generality, we can assume that $y = f(x)$.

Suppose given $t \in T$. Then we have $tx = \chi_{X,\mathrm{l}}(t) \cdot x$; cf. Definition 1. Applying f on both sides amounts to $f(t \cdot x) = f(\chi_{X,\mathrm{l}}(t) \cdot x)$. But now

$$\begin{aligned} f(t \cdot x) &= t \cdot f(x) = t \cdot y = \chi_{Y,\mathrm{l}}(t) \cdot y \\ f(\chi_{X,\mathrm{l}}(t) \cdot x) &= \chi_{X,\mathrm{l}}(t) \cdot f(x) = \chi_{X,\mathrm{l}}(t) \cdot y \end{aligned}$$

and we conclude that $\chi_{Y,\mathrm{l}} = \chi_{X,\mathrm{l}}$.

Using similar arguments we obtain that $\chi_{Y,\mathrm{r}} = \chi_{X,\mathrm{r}}$.

We conclude that $\chi_X = \chi_Y$.

Ad $\Longleftarrow$. We write $\chi_X = \chi_Y =: (\chi_\mathrm{l}, \chi_\mathrm{r})$. Suppose given $x \in X$ and $y \in Y$ such that $X = {}_R\langle x \rangle$ and $Y = {}_R\langle y \rangle$. Define a map $f \colon X \to Y$ by $f(rx) := ry$ for $r \in R$. This map is well-defined and it is bijective. It is an R-linear map since (x) is an R-linear basis of X. It remains to show that f is a T-T-linear map. Suppose given $r, r' \in R$ and $t, t' \in T$. Then we have

$$\begin{aligned} f(t \cdot rx) &= r \cdot f(t \cdot x) = r \cdot f(\chi_\mathrm{l}(t) \cdot x) = r \cdot \chi_\mathrm{l}(t) \cdot f(x) = r \cdot \chi_\mathrm{l}(t) \cdot y \\ &= r \cdot ty = r \cdot tf(x) = t \cdot f(rx) \end{aligned}$$

and

$$\begin{aligned} f(r'x \cdot t') &= r' \cdot f(x \cdot t') = r' \cdot f(\chi_\mathrm{r}(t') \cdot x) = r' \cdot \chi_\mathrm{r}(t') \cdot f(x) = r' \cdot \chi_\mathrm{r}(t') \cdot y \\ &= r' \cdot yt' = r' \cdot f(x)t' = f(r'x) \cdot t'. \end{aligned}$$

So f is an isomorphism of T-T-bimodules. We conclude that $X \simeq Y$. □

1.5 Roots of Ω

Keep the notation of §1.1 and §1.2.

1.5.1 Biroots of Ω

In the decomposition of Ω into a direct sum of indecomposable T-T-bimodules we have found two direct summands of rank 1 over R, viz. $e\Omega e' = {}_R\langle b_3 \rangle$ and $e'\Omega e = {}_R\langle b_4 \rangle$. For each of these, we define two R-linear maps in T^* as follows.

$$\begin{aligned} &\chi_\mathrm{l}^3 \text{ such that } tb_3 = \chi_\mathrm{l}^3(t) b_3 \text{ for } t \in T \\ &\chi_\mathrm{r}^3 \text{ such that } b_3 t = \chi_\mathrm{r}^3(t) b_3 \text{ for } t \in T \end{aligned}$$

$$\begin{aligned} &\chi_\mathrm{l}^4 \text{ such that } tb_4 = \chi_\mathrm{l}^4(t) b_4 \text{ for } t \in T \\ &\chi_\mathrm{r}^4 \text{ such that } b_4 t = \chi_\mathrm{r}^4(t) b_4 \text{ for } t \in T \end{aligned}$$

This leads to the following table where we see the images of the basis elements of $\mathcal{B}_T$ under these maps.

i	b_i	$\chi_\mathrm{l}^3(b_i)$	$\chi_\mathrm{r}^3(b_i)$	$\chi_\mathrm{l}^4(b_i)$	$\chi_\mathrm{r}^4(b_i)$
1	$\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)$	1	0	0	1
2	$\left(3, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)$	0	0	0	0
5	$\left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$	0	1	1	0
6	$\left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right)$	0	0	0	0

(6)

We define the pairs $\chi^3 := (\chi_\mathrm{l}^3, \chi_\mathrm{r}^3) \in T^* \times T^*$ and $\chi^4 := (\chi_\mathrm{l}^4, \chi_\mathrm{r}^4) \in T^* \times T^*$. Both of these tuples are biroots of Ω; cf. Definition 1. Using (6) we see that $\chi^3 \neq \chi^4$. The biroot paths Ω_{χ^3} and Ω_{χ^4} are R-submodules of Ω; cf. Lemma 4. We have

$$\begin{aligned}\Omega_{\chi^3} &= \{x \in \Omega \,|\, tx = \chi_\mathrm{l}^3(t)x \text{ and } xt = \chi_\mathrm{r}^3(t)x \text{ for } t \in T\}\\ \Omega_{\chi^4} &= \{x \in \Omega \,|\, tx = \chi_\mathrm{l}^4(t)x \text{ and } xt = \chi_\mathrm{r}^4(t)x \text{ for } t \in T\}.\end{aligned}$$

In the following we will show that $\Omega_{\chi^3} \overset{!}{=} e\Omega e' = {}_R\langle b_3\rangle$ and that $\Omega_{\chi^4} \overset{!}{=} e'\Omega e = {}_R\langle b_4\rangle$.

In the descriptions of Ω_{χ^3} and of Ω_{χ^4} it suffices to consider the R-linear basis $\mathcal{B}_T$ of T.

$$\begin{aligned}\Omega_{\chi^3} &= \{x \in \Omega \,|\, tx = \chi_\mathrm{l}^3(t)x \text{ and } xt = \chi_\mathrm{r}^3(t)x \text{ for } t \in T\}\\ &= \{x \in \Omega \,|\, b_1x = x \text{ and } xb_5 = x \text{ and } b_ix = 0 \text{ for } i \in \{2,5,6\} \text{ and } xb_j = 0 \text{ for } j \in \{1,2,6\}\}\\ \Omega_{\chi^4} &= \{x \in \Omega \,|\, tx = \chi_\mathrm{l}^4(t)x \text{ and } xt = \chi_\mathrm{r}^4(t)x \text{ for } t \in T\}\\ &= \{x \in \Omega \,|\, b_5x = x \text{ and } xb_1 = x \text{ and } b_ix = 0 \text{ for } i \in \{1,2,6\} \text{ and } xb_j = 0 \text{ for } j \in \{2,5,6\}\}\end{aligned}$$

We want to determine the elements in Ω_{χ^3} and in Ω_{χ^4} via direct calculation. We are just considering the non-zero equations and we will see that this is already enough to determine Ω_{χ^3} and Ω_{χ^4}. Note that this has as a consequence that the following sequences of implications are not completely revertible.

Suppose given $y = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \in \Omega$. We have

$$\begin{aligned} y \in \Omega_{\chi^3} &\implies b_1y = y \text{ and } yb_5 = y\\ &\implies \left(y_1, \begin{pmatrix} y_2 & y_3 \\ 0 & 0 \end{pmatrix}, 0\right) = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \text{ and } \left(0, \begin{pmatrix} 0 & y_3 \\ 0 & y_5 \end{pmatrix}, y_6\right) = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right)\\ &\implies y = \left(0, \begin{pmatrix} 0 & y_3 \\ 0 & 0 \end{pmatrix}, 0\right) \implies y \in {}_R\langle b_3\rangle \end{aligned}$$

Additionally, we have that $b_ib_3 = 0$ for $i \in \{2,5,6\}$ and that $b_3b_j = 0$ for $j \in \{1,2,6\}$. This shows that

$$\Omega_{\chi^3} = e\Omega e'.$$

Using the same method, we go on with Ω_{χ^4}.

$$\begin{aligned} y \in \Omega_{\chi^4} &\implies b_5y = y \text{ and } yb_1 = y\\ &\implies \left(0, \begin{pmatrix} 0 & 0 \\ y_4 & y_5 \end{pmatrix}, y_6\right) = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \text{ and } \left(y_1, \begin{pmatrix} y_2 & 0 \\ y_4 & 0 \end{pmatrix}, 0\right) = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right)\\ &\implies y = \left(0, \begin{pmatrix} 0 & 0 \\ y_4 & 0 \end{pmatrix}, 0\right) \implies y \in {}_R\langle b_4\rangle \end{aligned}$$

Additionally, we have that $b_ib_4 = 0$ for $i \in \{1,2,6\}$ and that $b_4b_j = 0$ for $j \in \{2,5,6\}$. This shows that

$$\Omega_{\chi^4} = e'\Omega e.$$

1.5.2 Roots of Ω

In the decomposition of $\mathfrak{l}(\Omega)$ into a direct sum of indecomposable $\mathfrak{l}(T)$-Lie modules we have found two direct summands of rank 1 over R that are not contained in $\mathfrak{l}(T)$, viz. $e\Omega e' = {}_R\langle b_3\rangle$ and $e'\Omega e = {}_R\langle b_4\rangle$.

We have a look at the Lie operation of $\mathfrak{l}(T)$ on $e\Omega e'$ and $e'\Omega e$.

$$\begin{array}{rclcr@{\qquad}rclcr} \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_1\right)(b_3) &=& [b_1, b_3] &=& 1 \cdot b_3 & \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_1\right)(b_4) &=& [b_1, b_4] &=& -1 \cdot b_4\\ \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2\right)(b_3) &=& [b_2, b_3] &=& 0 \cdot b_3 & \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2\right)(b_4) &=& [b_2, b_4] &=& 0 \cdot b_4\\ \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_5\right)(b_3) &=& [b_5, b_3] &=& -1 \cdot b_3 & \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_5\right)(b_4) &=& [b_5, b_4] &=& 1 \cdot b_4\\ \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_6\right)(b_3) &=& [b_6, b_3] &=& 0 \cdot b_3 & \left(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_6\right)(b_4) &=& [b_6, b_4] &=& 0 \cdot b_4 \end{array}$$

For each of the summands $e\Omega e'$ and $e'\Omega e$ we define an R-linear map in T^* as follows.

$$\chi_3 \text{ such that } [t,b_3] = \chi_3(t)b_3 \text{ for } t \in T$$

$$\chi_4 \text{ such that } [t,b_4] = \chi_4(t)b_4 \text{ for } t \in T$$

The R-linear maps χ_3 resp. χ_4 describe the $\mathfrak{l}(T)$-Lie module structure on $e\Omega e'$ resp. on $e'\Omega e$. Furthermore, note that $\chi_3 = -\chi_4$. We give the images of the basis elements of $\mathcal{B}_T$ under χ_3 and χ_4.

$$\begin{array}{rcr} \chi_3\colon T & \to & R \\ b_1 & \mapsto & 1 \\ b_2 & \mapsto & 0 \\ b_5 & \mapsto & -1 \\ b_6 & \mapsto & 0 \end{array} \qquad \begin{array}{rcr} \chi_4\colon T & \to & R \\ b_1 & \mapsto & -1 \\ b_2 & \mapsto & 0 \\ b_5 & \mapsto & 1 \\ b_6 & \mapsto & 0 \end{array}$$

These maps are both roots of Ω; cf. Definition 1.

The root paths Ω_{χ_3} to the root χ_3 and the root path Ω_{χ_4} to the root χ_4 are R-submodules of Ω; cf. Lemma 3. We have

$$\begin{aligned} \Omega_{\chi_3} &= \{x \in \Omega \,|\, [t,x] = \chi_3(t)x \text{ for } t \in T\}\,, \\ \Omega_{\chi_4} &= \{x \in \Omega \,|\, [t,x] = \chi_4(t)x \text{ for } t \in T\}\,. \end{aligned}$$

In the following we will show that $\Omega_{\chi_3} \overset{!}{=} e\Omega e'$ and that $\Omega_{\chi_4} \overset{!}{=} e'\Omega e$.

In the descriptions of Ω_{χ_3} and of Ω_{χ_4} it suffices to consider the R-linear basis $\mathcal{B}_T$ of T.

$$\begin{aligned} \Omega_{\chi_3} &= \{x \in \Omega \,|\, [t,x] = \chi_3(t)x \text{ for } t \in T\} \\ &= \{x \in \Omega \,|\, [b_1,x] = x \text{ and } [b_5,x] = -x \text{ and } [b_i,x] = 0 \text{ for } i \in \{2,6\}\} \\ \Omega_{\chi_4} &= \{x \in \Omega \,|\, [t,x] = \chi_4(t)x \text{ for } t \in T\} \\ &= \{x \in \Omega \,|\, [b_1,x] = -x \text{ and } [b_5,x] = x \text{ and } [b_i,x] = 0 \text{ for } i \in \{2,6\}\} \end{aligned}$$

We want to determine the elements in Ω_{χ_3} and in Ω_{χ_4} via direct calculation. We are just considering the non-zero equations and we will see that this is already enough to determine Ω_{χ_3} and Ω_{χ_4}. Note that this has as a consequence that the following sequences of implications are not completely revertible.

Suppose given $y = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \in \Omega$. We have

$$\begin{aligned} y \in \Omega_{\chi_3} &\implies [b_1,y] = y \text{ and } [b_5,y] = -y \\ &\implies \left(0, \begin{pmatrix} 0 & y_3 \\ -y_4 & 0 \end{pmatrix}, 0\right) = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \\ &\qquad\quad \text{and } \left(0, \begin{pmatrix} 0 & -y_3 \\ y_4 & 0 \end{pmatrix}, 0\right) = -\left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \\ &\implies y = \left(0, \begin{pmatrix} 0 & y_3 \\ 0 & 0 \end{pmatrix}, 0\right) \implies y \in {}_R\langle b_3\rangle \end{aligned}$$

Additionally, we have that $[b_i, b_3] = 0$ for $i \in \{2,6\}$. This shows that

$$\Omega_{\chi_3} = e\Omega e'.$$

Using the same method, we go on with Ω_{χ_4}.

$$\begin{aligned} y \in \Omega_{\chi^4} &\implies [b_1,y] = -y \text{ and } [b_5,y] = y \\ &\implies \left(0, \begin{pmatrix} 0 & y_3 \\ -y_4 & 0 \end{pmatrix}, 0\right) = -\left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \\ &\qquad\quad \text{and } \left(0, \begin{pmatrix} 0 & -y_3 \\ y_4 & 0 \end{pmatrix}, 0\right) = \left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix}, y_6\right) \\ &\implies y = \left(0, \begin{pmatrix} 0 & 0 \\ y_4 & 0 \end{pmatrix}, 0\right) \implies y \in {}_R\langle b_4\rangle \end{aligned}$$

Additionally, we have that $[b_i, b_4] = 0$ for $i \in \{2, 6\}$. This shows that

$$\Omega_{\chi_4} = e'\Omega e.$$

1.6 Summary

When comparing the two decompositions we have found in §1.3.1 and §1.3.2, we recognize that outside of T, both decompositions are the same. The only difference occurs in the decomposition of the commutative subalgebra T of Ω.

In this example, we have found a decomposition of Ω into indecomposable T-T-sub-bimodules two of which are of rank 1 over R. Both these modules occur again as biroot paths of biroots of Ω.

In the case of a complex semisimple Lie algebra $\mathfrak{g}$, the root space decomposition of $\mathfrak{g}$ relative to a maximal toral subalgebra $\mathfrak{t}$ consists of summands each of which is of dimension 1 except for $\mathfrak{t}$ itself. Our example $\mathbb{Z}_{(3)}\,\mathrm{S}_3$ shows an analogous behavior. However, in another example below we will find indecomposable summands of rank greater than one; cf. §6.5.1 and §6.5.2. So in order to pursue this analogy, we consider decompositions into indecomposables instead of mere (bi)root paths, as T-T-bimodules and as $\mathfrak{l}(T)$-Lie modules, respectively.

1.7 Magma

The following two codes are used for calculations with $\Omega \simeq \mathbb{Z}_{(3)}\,\mathrm{S}_3$ in Magma. However, note that initialization files such as “pre” and “definitions” are required; cf. Magma Codes 3 and 4.

Magma Code 1: z3s3Init1

```
// global definitions
Sizes := [1,2,1];
nb := #Sizes; // number of blocks
nt := 3; // number of ties needed to describe Omega
rt := &+Sizes; // rank of torus
rl := &+[Sizes[i]^2 : i in [1..nb]]; // rank of Omega
prime := 3; // R is Z localized at the prime number 3
e := 3; // ties that describe Omega are given mod e

RM := RMatrixSpace(Z,rl,rt);
RMQ := RMatrixSpace(Q,rl,rt);
RV := RMatrixSpace(Z,rl,1);
RQV := VectorSpace(Q,rl);
RM2 := RMatrixSpace(Z,nt,rl);
RMB := RMatrixSpace(Z,rl,rl);
RMBQ := KMatrixSpace(Q,rl,rl);
RMVQ := KMatrixSpace(Q,rl,1);

Ties_Omega :=   // Ties mod e that describe Omega,
                //given in the rows of this matrix
    RM2!Matrix([
        [ 1, -1, 0, 0, 0,  0],
        [ 0,  0, 0, 0, 1, -1],
        [ 0,  0, 1, 0, 0,  0]
]);
```

Magma Code 2: z3s3Init2

```
// R-linear basis of Omega
b := [];
```

```
// e Omega e
b[1] := CoerceGamma([1,1,0,0,0,0]);
b[2] := CoerceGamma([3,0,0,0,0,0]);
// e Omega e'
b[3] := CoerceGamma([0,0,3,0,0,0]);
// e' Omega e
b[4] := CoerceGamma([0,0,0,1,0,0]);
// e' Omega e'
b[5] := CoerceGamma([0,0,0,0,1,1]);
b[6] := CoerceGamma([0,0,0,0,0,3]);

// describing matrices of the adjoint endomorphisms of the elements
// of b with respect to the basis Basis_Omega which is defined in
// the file "definitions"
A := [RMBQ!admatrix(x) : x in b];
```

Chapter 2: Preliminaries

Let R be a principal ideal domain and $K = \operatorname{frac} R$ its field of fractions, which is not necessarily algebraically closed. Note that the special case $K = R$ is not excluded. When tensoring over R, we often abbreviate $\otimes_R$ by $\otimes$.

2.1 Preliminaries on modules

Let N be a finitely generated free R-module. Let $M \subseteq N$ be an R-submodule.

Remark 10. *Suppose given $k \in \mathbb{N}$. Suppose given R-submodules $M_i \subseteq N$ and $N_i \subseteq N$ such that $M_i \subseteq N_i$ for $i \in [1,k]$. Then the following implication holds.*

$$\bigoplus_{i\in[1,k]} M_i = \bigoplus_{i\in[1,k]} N_i \implies M_i = N_i \text{ for } i \in [1,k]$$

Proof. On the one hand, we have

$$\left(\bigoplus_{i\in[1,k]} N_i\right) \Big/ \left(\bigoplus_{i\in[1,k]} M_i\right) = 0.$$

On the other hand, we have

$$\left(\bigoplus_{i\in[1,k]} N_i\right) \Big/ \left(\bigoplus_{i\in[1,k]} M_i\right) = \bigoplus_{i\in[1,k]} N_i/M_i.$$

We conclude that $N_i/M_i = 0$ for $i \in [1,k]$ and thus $M_i = N_i$ for $i \in [1,k]$. □

Remark 11. *Let $\varphi\colon N \to N$ be an R-linear bijective map such that $\varphi(M) \subseteq M$.*

Suppose that N/M is finite as a set.

Then $\varphi|_M^M$ is bijective.

Proof. The map $\varphi|_M^M$ is injective as it is a restriction of a bijective map.

We have the following diagram.

$$\begin{tikzcd} N \arrow[r, "\varphi", "\sim"'] & N \\ M \arrow[u, hook] \arrow[r, "\varphi|_M^M"] & M \arrow[u, hook] \end{tikzcd}$$

Since φ is an isomorphism, we have $N/M \simeq \varphi(N)/\varphi(M)$. But $\varphi(N) = N$, so $N/M \simeq N/\varphi(M)$.

In particular, we have

$$\left|N/M\right| = \left|N/\varphi(M)\right| = \left|N/M\right| \cdot \left|M/\varphi(M)\right|.$$

We conclude that $\left|M/\varphi(M)\right| = 1$. That is, $\varphi|_M^M$ is surjective. □

Definition 12. Define $\operatorname{cl}_N(M) := \{x \in N \mid \exists r \in R^\times : rx \in M\}$ as the *pure closure* of M in N. We call an R-submodule M of N *pure in N*, if $M = \operatorname{cl}_N(M)$. In other words, an R-submodule $M \subseteq N$ is pure in N if for $r \in R^\times$ and $x \in N$ such that $rx \in M$, we have $x \in M$.

Note that $\mathrm{cl}_N(M)$ is an R-submodule of N: We have $0 \in \mathrm{cl}_N(M)$. For $x, y \in \mathrm{cl}_N(M)$, we choose $r, s \in R^\times$ such that $rx \in M$ and $sy \in M$. Then for $a, b \in R$, we have $rs(ax+by) = as(rx)+rb(sy) \in M$ since $as, rb \in R$ and $rx, sy \in M$. So $ax + by \in \mathrm{cl}_N(M)$ since $rs \in R^\times$.

Moreover, note that $M \subseteq \mathrm{cl}_N(M)$ as R-modules.

Example 13.

(1) Suppose M to be a direct summand of N. Then M is a pure submodule of N.

To see that, we write $N = M \oplus M'$. Let $x \in N$ and $r \in R^\times$ such that $rx \in M$. We have unique $y \in M$, $y' \in M'$ with $x = y + y'$. Now $ry + ry' = rx \in M$ and we get $ry' = 0$. Since N is finitely generated free and thus is torsion-free, we obtain that $y' = 0$. Thus $x = y \in M$.

(2) More generally, let X be a torsion-free R-module and let $f : N \to X$ be an R-linear map. Then $\ker f$ is pure in N.

To see that, suppose given $r \in R^\times$ and $n \in N$ such that $rn \in \ker f$. Then $0 = f(rn) = rf(n)$. Since X is torsion-free, we conclude that $n \in \ker f$.

(3) In the case that R is a field, N is a vector space. Then each subspace of N is pure in N.

Remark 14. *Suppose given R-submodules $M_1, M_2 \subseteq M$. Then the following implication holds.*

$$M_1 \subseteq M_2 \implies \mathrm{cl}_N(M_1) \subseteq \mathrm{cl}_N(M_2)$$

Proof. Suppose given $x \in \mathrm{cl}_M(M_1)$. Then there exists $r \in R^\times$ such that $rx \in M_1$. Hence $rx \in M_2$ since $M_1 \subseteq M_2$. We conclude that $x \in \mathrm{cl}_M(M_2)$. $\square$

Remark 15. *Suppose given finitely generated free R-modules M_1, M_2, M_3 such that $M_1 \subseteq M_2 \subseteq M_3$. If $M_1 \subseteq M_3$ is pure, then $M_1 \subseteq M_2$ is pure.*

Proof. Suppose given $r \in R^\times$ and $m \in M_2$ such that $rm \in M_1$. Then $m \in M_3$ since $M_2 \subseteq M_3$. So we get $rm \in M_3$ and since $M_1 \subseteq M_3$ is pure, we conclude that $m \in M_1$. $\square$

Lemma 16. *We write $m := \mathrm{rk}_R M$ and $n := \mathrm{rk}_R N$. Now R is a principal ideal domain, so by the elementary divisor theorem, we can choose R-linear bases $\mathcal{B} = (b_1, \ldots, b_m)$ of M and $\mathcal{C} = (c_1, \ldots, c_n)$ of N such that $b_i = d_i c_i$ for $i \in [1, m]$ and for certain $d_i \in R^\times$.*

Then we have the following equivalence.

$$M \text{ is a pure submodule of } N \iff d_1, \ldots, d_m \in \mathrm{U}(R)$$

Proof. Ad $\Longleftarrow$. Let $x \in N$ and $r \in R^\times$ with $rx \in M$. We want to show that $x \overset{!}{\in} M$.

There exist $a_i \in R$ for $i \in [1, m]$ such that $rx = \sum_{i\in[1,m]} a_i b_i$. We have $b_i = d_i c_i$ for $i \in [1, m]$, so $rx = \sum_{i\in[1,m]} a_i d_i c_i$. A priori we have $a_i' \in R$ for $i \in [1, n]$ such that $x = \sum_{i\in[1,n]} a_i' c_i$ and we can compare the coefficients: We get $ra_i' = a_i d_i$ for $i \in [1, m]$ and $ra_j' = 0$, implying $a_j' = 0$, for $j \in [m + 1, n]$. Thus the sum $x = \sum_{i\in[1,n]} a_i' c_i$ shortens to $x = \sum_{i\in[1,m]} a_i' c_i = \sum_{i\in[1,m]} a_i' d_i^{-1} b_i$. By assumption, $a_i' d_i^{-1} \in R$ for $i \in [1, m]$, so $x \in M$.

Ad $\Longrightarrow$. Suppose given $i \in [1, m]$. Since $d_i \in R^\times$ and $d_i c_i = b_i$, we have $c_i \in \mathrm{cl}_N(M) = M$ by assumption. So there exist $a_{i,j} \in R$ for $j \in [1, m]$ such that $c_i = \sum_{j\in[1,m]} a_{i,j} b_j$. Multiplying by d_i, we get $b_i = d_i c_i = \sum_{j\in[1,m]} d_i a_{i,j} b_j$. Comparing coefficients, we have $1 = d_i a_{i,i}$. This shows $d_i \in \mathrm{U}(R)$. $\square$

Corollary 17. *Keep the setting of Lemma 16. If M is a pure submodule of N, one can choose an R-linear basis $\mathcal{B}$ of M such that the elementary divisors $d_1, \ldots, d_m$ are all 1.*

That means that the describing matrix of the inclusion map $\iota\colon M \to N$ with respect to the basis $\mathcal{B}$ takes the following form.

$$\iota_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in R^{n\times m}$$

Lemma 18. *Suppose given $m, n \in \mathbb{N}$ with $m \leq n$. Suppose given $A, B \in R^{n\times m}$ and $X \in R^{m\times m}$ satisfying the following conditions.*

- $\det(X) \neq 0$.
- *All elementary divisors of B are units in R.*
- $BX = A$.

Writing A' for the R-module generated by the columns of A and B' for the R-module generated by the columns of B, we have

$$\mathrm{cl}_{R^n}(A') = B'.$$

Proof. The situation can be illustrated as follows.

$$\begin{tikzcd} R^m \arrow[rr, "A(-)"] \arrow[dr, "X(-)"'] & & R^n \\ & R^m \arrow[ur, "B(-)"'] & \end{tikzcd}$$

Ad $\subseteq$. First we show that $A' \subseteq B'$. Suppose given $a' \in A'$. Thus we find y in R^m such that $Ay = a'$. Thus $a' = BXa'$, hence $a' \in B'$.

We obtain that $\mathrm{cl}_{R^n}(A') \subseteq \mathrm{cl}_{R^n}(B')$; cf. Remark 14.

B' is pure in R^n since all elementary divisors of B are units in R; cf. Lemma 16. So we obtain

$$\mathrm{cl}_{R^n}(A') \subseteq \mathrm{cl}_{R^n}(B') = B'.$$

Ad $\supseteq$. Suppose given $b' \in B'$. We find $y \in R^m$ such that $By = b'$. Then in K^m, we calculate.

$$b' = By = AX^{-1}y = \frac{1}{\det(X)} A \underbrace{(X^{-1}\det(X))y}_{\in R^m}$$

So $\det(X)b' = A(X^{-1}\det(X))y \in A'$ and hence $b' \in \mathrm{cl}_{R^n}(A')$. $\square$

Remark 19. *Suppose that $X \subseteq KN$ is a K-subspace. The intersection $X \cap N \subseteq N$ is again a finitely generated free R-module. Then the following assertions hold.*

(1) *The intersection $X \cap N$ is a pure R-submodule of N.*

(2) *We have $\mathrm{rk}_R(X \cap N) = \dim_K(X)$.*

Proof. Ad (1). Suppose given $n \in N$ and $r \in R^\times$ such that $rn \in X \cap N$. We have to show that $n \overset{!}{\in} X \cap N$. It suffices to show that $n \overset{!}{\in} X$. But X is a K-vector space. Thus we can write $n = \frac{1}{r}rn \in X$ since $\frac{1}{r} \in K$. This shows that $n \in X$.

Ad (2). Ad $\geq$. We write $k := \dim_K(X)$. Choose a K-linear basis $(x_1, \dots, x_k)$ of X such that $x_i \in N$ for $i \in [1,k]$. We have ${}_R\langle x_1, \dots, x_k\rangle \subseteq X \cap N$. The K-linear basis $(x_1, \dots, x_k)$ of X is linearly independent over R as well, so we have $\mathrm{rk}_R(X \cap N) \geq k = \dim_K(X)$.

Ad $\leq$. We write $l := \mathrm{rk}_R(X \cap N)$. Choose an R-linear basis $(y_1, \dots, y_l)$ of the R-module $X \cap N$. We have ${}_K\langle y_1, \dots, y_l\rangle \subseteq X$. The basis $(y_1, \dots, y_l)$ is linearly independent over K as well, so we have $\mathrm{rk}_R(X \cap N) = l \leq \dim_K(X)$. $\square$

Corollary 20. *Suppose that $X \subseteq KN$ is a K-subspace. Then we have*

$$K(X \cap N) = X.$$

Example 21. Suppose given an R-linear map $\varphi\colon N \to N$. Suppose given an eigenspace X of the K-linear map $K\varphi$. Then $X \cap N$ is an eigenmodule of φ. Now Remark 19.(1) shows that $X \cap N$ is pure in N.

This shows that eigenmodules of R-endomorphisms on N are pure submodules of N.

Remark 22. *As R-modules, we have the following equation.*

$$\mathrm{cl}_N(M) = KM \cap N$$

Proof. Ad $\subseteq$. Let $x \in \mathrm{cl}_N(M)$. Then $x \in N$ and there exists $r \in R^\times$ such that $rx \in M$. So there exists $m \in M$ such that $rx = m$. But then $x = \frac{1}{r}(1 \otimes m) = \frac{1}{r} \otimes m = \frac{1}{r}m \in KM$.

Ad $\supseteq$. Let $x \in KM \cap N$. Then $x \in N$ and x is a K-linear combination of elements of M. Hence there exists an element $r \in R^\times$ such that $rx \in M$ and hence $x \in \mathrm{cl}_N(M)$. $\square$

Lemma 23. *Suppose R to be infinite. Let $m \geq 0$. Suppose $U_1, \dots, U_m \subset N$ to be pure submodules of N. Then $\bigcup_{i\in[1,m]} U_i \subset N$.*

Proof. The proof given here follows the proof of Question 42 of [Kün15].

Without loss of generality, we may assume that $U_j \not\subseteq \bigcup_{i\in[1,m]\setminus\{j\}} U_i$ for $j \in [1,m]$, since in the other case, we could omit U_j leaving the union to be the same.

Without loss of generality, we have $m \geq 2$.

It suffices to show that $\bigcup_{i\in[1,m]} U_i$ is not an R-submodule of N.

Assume that $\bigcup_{i\in[1,m]} U_i$ is an R-submodule of N.

Choose $\alpha_k \in R^\times$ for $k \in [1, m-1]$ such that $|\{\alpha_k \,|\, k \in [1, m-1]\}| = m-1$. This is possible since $|R| \geq m$. Choose $u_1 \in U_1 \setminus \bigcup_{i\in[1,m]\setminus\{1\}} U_i$. Choose $u_2 \in U_2 \setminus \bigcup_{i\in[1,m]\setminus\{2\}} U_i$.

Suppose given $k \in [1, m-1]$. Then $u_1 + \alpha_k u_2 \in \bigcup_{i\in[1,m]} U_i$. But $u_1 + \alpha_k u_2 \notin U_1$, since in this case, we would have $\alpha_k u_2 \in U_1$. Since U_1 is pure in N, this would imply that $u_2 \in U_1$. Also we have $u_1 + \alpha_k u_2 \notin U_2$, since in this case, we would have $u_1 \in U_2$. So $u_1 + \alpha_k u_2 \in \bigcup_{i\in[3,m]} U_i$.

Since $u_2 \neq 0$, we have $|\,\{u_1 + \alpha_k u_2 \,|\, k \in [1, m-1]\}\,| = m-1$. So there exist $s, t \in [1, m-1]$ and $j \in [3, m]$ such that $s \neq t$ and $u_1 + \alpha_s u_2 \in U_j$ and $u_1 + \alpha_t u_2 \in U_j$. The difference $(\alpha_s - \alpha_t)u_2$ has to be in U_j, so $u_2 \in U_j$. But this is a *contradiction* to the choice of u_2. $\square$

Remark 24. *There exists an infinite principal ideal domain R and a finitely generated free R-module N that can be written as a finite union of proper submodules of N.*

This shows that the assertion of Lemma 23 does not hold if we omit the condition on the U_i to be pure submodules of N.

Proof. Consider the module $N = \mathbb{Z} \oplus \mathbb{Z}$ over the principal ideal domain $\mathbb{Z}$.

Let $U_1 := (2) \oplus \mathbb{Z}$, let $U_2 := \mathbb{Z} \oplus (2)$ and let $U_3 := {}_{\mathbb{Z}}\langle(1,1),(0,2)\rangle$. Then $U_i \subset N$ are proper $\mathbb{Z}$-submodules for $i \in [1,3]$. Moreover, we have $U_1 \cup U_2 \cup U_3 = \mathbb{Z} \oplus \mathbb{Z} = N$. $\square$

Lemma 25. *Suppose given a commutative R-algebra T. Suppose given an idempotent $e \in T$. Consider T as a T-T-bimodule and as a T-module, respectively. Then*

$$\mathrm{End}_{T\text{-}T}(Te) = \mathrm{End}_T(Te).$$

Moreover, as R-algebras, we have

$$Te \simeq \mathrm{End}_T(Te).$$

Proof. The first claim follows from the commutativity of T.

For the second claim, define

$$\begin{array}{rcl} \mu\colon Te & \longrightarrow & \mathrm{End}_T(Te) \\ te & \longmapsto & \mu(te)\colon x \mapsto xte \end{array} \quad \text{and} \quad \begin{array}{rcl} \varepsilon\colon \mathrm{End}_T(Te) & \longrightarrow & Te \\ \varphi & \longmapsto & \varphi(e). \end{array}$$

We will show that μ is an R-algebra morphism.

We have $\mu(e)\colon x \mapsto xe = x$ which is the identity map in $\mathrm{End}_T(T)$.

Suppose given $t, t' \in T$, $\alpha, \alpha' \in R$. Then $\mu(\alpha te + \alpha' t'e)$ is the map that sends an element $x \in Te$ to $x(\alpha te + \alpha' t'e) = \alpha xte + \alpha' xt'e$. But this is the same map as the sum $\alpha\mu(te) + \alpha'\mu(t'e)$ which sends $x \in Te$ to $(\alpha\mu(te) + \alpha'\mu(t'e))(x) = \alpha xte + \alpha' xt'e$.

Suppose given $t, t' \in T$. Then $\mu(tet'e)$ is the map that sends $x \in Te$ to $xtet'e = xtt'e$. But this is the same as the map $\mu(te) \circ \mu(t'e)$ which sends $x \in Te$ to $\mu(te)(\mu(t'e)(x)) = \mu(te)(xt'e) = xt'ete = xtt'e$.

This shows that μ is an R-algebra morphism.

We will show that $\varepsilon \circ \mu \overset{!}{=} \mathrm{id}_{Te}$.

For $t \in T$, we obtain

$$\varepsilon(\mu(te)) = \mu(te)(e) = ete = te.$$

We will show that $\mu \circ \varepsilon \overset{!}{=} \mathrm{id}_{\mathrm{End}_T(Te)}$.

For $\varphi \in \mathrm{End}_T(Te)$ and for $x \in Te$, we obtain

$$\mu(\varepsilon(\varphi))(x) = x\varepsilon(\varphi) = x\varphi(e) = \varphi(xe) = \varphi(x)$$

and thus $\varphi = \mu(\varepsilon(\varphi)) = (\mu \circ \varepsilon)(\varphi)$.

This shows that μ and ε are R-algebra isomorphisms and that $\varepsilon = \mu^{-1}$. □

2.2 Nonisomorphic modules

Let T be a commutative R-algebra.

Lemma 26. *Suppose given T-T-bimodules X and Y. If there exists a pair $(s, t) \in T \times T$ such that $sXt = 0$ and $sYt \neq 0$, then $X \not\simeq Y$ as T-T-bimodules.*

Proof. Assume that there exists a T-T-isomorphism $f\colon X \xrightarrow{\sim} Y$. Choose $y \in Y$ such that $syt \neq 0$. Then we obtain

$$0 \neq syt = sf(f^{-1}(y))t = f(s\underbrace{f^{-1}(y)}_{\in X}t) = 0$$

which is a *contradiction.* □

Lemma 27. *Suppose given $\mathfrak{l}(T)$-Lie modules X and Y. If there exists an element $t \in \mathfrak{l}(T)$ such that $[t, X] = 0$ and $[t, Y] \neq 0$, then $X \not\simeq Y$ as $\mathfrak{l}(T)$-Lie modules.*

Proof. Assume that there exists an $\mathfrak{l}(T)$-isomorphism $f\colon X \xrightarrow{\sim} Y$. Choose $y \in Y$ such that $[t, y] \neq 0$. Then we obtain

$$0 \neq [t, y] = [t, f(f^{-1}(y))] = f([t, \underbrace{f^{-1}(y)}_{\in X}]) = 0$$

which is a *contradiction.* □

2.3 Preliminaries on local rings

Lemma 28. *Suppose given $k \in \mathbb{N}$ and a subset $\mathcal{S} \subseteq R^{k\times k}$. Define*

$$E := \left\{ M \in R^{k\times k} \,\middle|\, M \cdot X = X \cdot M \text{ for } X \in \mathcal{S} \right\}.$$

Then the following implication holds.

$$\Big(M \in E \text{ and } M \in \mathrm{U}(R^{k\times k}) \Big) \implies M \in \mathrm{U}(E)$$

Proof. We have to show that if $M \in E$ and M is invertible in $R^{k\times k}$, then $M^{-1} \in E$.

Suppose given a matrix $X \in \mathcal{S}$. We have $X = XMM^{-1} = (MX)M^{-1}$ since $M \in E$. Also we have $MM^{-1}X = X$. Hence $MM^{-1}X = MXM^{-1}$. We multiply by M^{-1} from the left and thus we obtain $M^{-1}X = XM^{-1}$.

This shows that $M^{-1} \in E$. □

Definition 29. Let A be a ring. A is called a *local ring*, if $A \setminus \mathrm{U}(A)$ is an ideal of A.

Remark 30. *A is a local ring if and only if $0_A \neq 1_A$ and for any $a_1, a_2 \in A$ with $a_1 + a_2 \in \mathrm{U}(A)$, it follows that $a_1 \in \mathrm{U}(A)$ or $a_2 \in \mathrm{U}(A)$.*

Proof. A proof is given in [Mül13, Remark 192]. □

Definition 31. Suppose given a ring A. We say that $e \in A$ is a *non-trivial idempotent in A* if $e^2 = e$ and $e \notin \{0_A, 1_A\}$.

Remark 32. *Suppose that A is a local ring. Then A does not contain a non-trivial idempotent.*

Proof. Assume that $e \in A$ is a non-trivial idempotent, i.e. $e^2 = e$ and $e \notin \{0_A, 1_A\}$. Then we have $e \cdot (1 - e) = e - e^2 = 0$. This shows that $e \notin \mathrm{U}(A)$ and that $1 - e \notin \mathrm{U}(A)$. Since A is a local ring, we have the maximal ideal $\mathcal{I} := A \setminus \mathrm{U}(A)$ of A; cf. Definition 29. We have $\mathcal{I} \neq A$ since $1 \in \mathrm{U}(A)$. By definition of $\mathcal{I}$ we have $e \in \mathcal{I}$ and $(1 - e) \in \mathcal{I}$. But $e + (1 - e) = 1 \in \mathcal{I}$, a *contradiction*. □

Lemma 33. *Suppose that R is a discrete valuation ring with maximal ideal (π). Suppose given an R-subalgebra $\Theta \subseteq R^{\times m} =: \Gamma$ for a certain $m \in \mathbb{N}$ such that Γ/Θ has finite length. Define the following R-subalgebra of Γ.*

$$\widetilde{\Gamma} := \{(a_i)_i \in \Gamma \,|\, a_i \equiv_\pi a_j \text{ for } i, j \in [1, m]\} \subseteq \Gamma$$

If $\Theta \subseteq \widetilde{\Gamma}$, then Θ is a local ring.

Proof. Since the factor R-module Γ/Θ is of finite length, Θ is stable; cf. [Mül13, Definition 207 and Remark 208].

For $j \in [1, m]$ we define $\varepsilon_j := (a_i)_{i\in[1,m]} \in K\Theta = K\Gamma = K^{\times m}$ by $a_i = 1$ if $i = j$ and $a_i = 0$ for $i \in [1, m] \setminus \{j\}$. Then $1_{K\Theta} = \sum_{j\in[1,m]} \varepsilon_j$ is an orthogonal decomposition of $1_{K\Theta}$ into central idempotents in $K\Theta$. So the Jacobson radical $\mathrm{Jac}(\Theta)$ of Θ can be written as

$$\mathrm{Jac}(\Theta) = \Theta \cap \bigoplus_{j\in[1,m]} \mathrm{Jac}(\varepsilon_j \Theta);$$

cf. [Mül13, Proposition 222]. Note that for $j \in [1, m]$, we have $\varepsilon_j \Gamma = {}_R\langle \varepsilon_j \rangle \subseteq \varepsilon_j \Theta \subseteq \varepsilon_j \Gamma$ and hence $\varepsilon_j \Theta = \varepsilon_j \Gamma \simeq R$ as R-algebras. So $\mathrm{Jac}(\varepsilon_j \Theta) = \pi \varepsilon_j \Theta$ for $j \in [1, m]$, viz. $\mathrm{Jac}(\varepsilon_1 \Theta) = (\pi) \times 0 \times \ldots \times 0$, $\ldots$, $\mathrm{Jac}(\varepsilon_m \Theta) = 0 \times \ldots \times 0 \times (\pi)$. We conclude that

$$\mathrm{Jac}(\Theta) = \Theta \cap (\pi)^{\times m} = \Theta \cap ((\pi) \times \ldots \times (\pi)).$$

Claim 1. We have $\widetilde{\Gamma}/((\pi)^{\times m}) \simeq R/(\pi)$.

Define the R-algebra morphism ψ as follows.

$$\begin{array}{rccc} \psi\colon & \widetilde{\Gamma}/((\pi)^{\times m}) & \to & R/(\pi) \\ & (a_i)_i + (\pi)^{\times m} & \mapsto & (a_1) + (\pi) \end{array}$$

We will show that ψ is an R-algebra isomorphism.

Suppose given $a + (\pi) \in R + (\pi)$. Then $\psi((a)_i + (\pi)^{\times m}) = a + (\pi)$. This shows that ψ is surjective.

Suppose given $(a_i)_i + (\pi)^{\times m} \in \ker\psi$, i.e. $\psi((a_i)_i + (\pi)^{\times m}) = 0$. But then $a_1 \equiv_\pi 0$, so $a_i \equiv_\pi 0$ for $i \in [1, m]$ since $(a_i)_i \in \widetilde{\Gamma}$. Hence $(a_i)_i \in (\pi)^{\times m}$. This shows that ψ is injective.

So ψ is an isomorphism.

This *proves Claim 1.*

Claim 2. We have $\Theta/\operatorname{Jac}(\Theta) \simeq R/(\pi)$ *as* R*-algebras.*

Denote by ι the inclusion map of $\Theta/(\Theta \cap (\pi)^{\times m})$ into $\widetilde{\Gamma}/(\widetilde{\Gamma} \cap (\pi)^{\times m}) = \widetilde{\Gamma}/((\pi)^{\times m})$. Define the R-algebra morphism $\alpha\colon R/(\pi) \to \Theta/\operatorname{Jac}(\Theta)$ by $\alpha(r + (\pi)) := (r)_i + (\Theta \cap (\pi)^{\times m})$.

We have the following R-algebra morphisms.

$$\begin{array}{ccccccccc} R/(\pi) & \xrightarrow{\alpha} & \Theta/\operatorname{Jac}(\Theta) & = & \Theta/(\Theta \cap (\pi)^{\times m}) & \xrightarrow{\iota} & \widetilde{\Gamma}/((\pi)^{\times m}) & \xrightarrow{\psi} & R/(\pi) \\ r + (\pi) & \mapsto & (r)_i + \operatorname{Jac}(\Theta) & & (a_i)_i + (\Theta \cap (\pi)^{\times m}) & \mapsto & (a_i)_i + ((\pi)^{\times m}) & \mapsto & a_1 + (\pi) \end{array}$$

We know that ψ is bijective by Claim 1. Moreover, we know that ι is injective and that $\psi \circ \iota \circ \alpha$ is the identity map, in particular it is bijective. So we conclude that $\iota \circ \alpha = \psi^{-1}$ is bijective. This entails that ι is surjective. But since ι is injective by definition, ι is bijective. So also $\alpha = \iota^{-1} \circ \psi^{-1}$ is bijective.

This *proves Claim 2.*

But $R/(\pi)$ is isomorphic to a field, so $\Theta/\operatorname{Jac}(\Theta)$ is isomorphic to a field by Claim 2. Then Θ is a local ring; cf. [Mül13, Remark 192]. ☐

Definition 34. Suppose given a preadditive category $\mathcal{A}$. Suppose given $X \in \operatorname{Ob}\mathcal{A}$. We say that X is *indecomposable* if $X \not\simeq 0$ and if for $Y, Z \in \operatorname{Ob}\mathcal{A}$ the implication $X \simeq Y \oplus Z \implies (X \simeq 0 \text{ or } Z \simeq 0)$ holds; cf. [Ste12, Convention (12)].

Lemma 35. *Suppose given a preadditive category* $\mathcal{A}$ *having a zero-object* 0. *Suppose given* $X \in \operatorname{Ob}\mathcal{A}$. *We have the endomorphism ring* $\operatorname{End}(X) = ({}_{\mathcal{A}}(X, X), +, \circ)$. *Consider the following assertions.*

(1) *The ring* $\operatorname{End}(X)$ *is a local ring.*

(2) *The ring* $\operatorname{End}(X)$ *has no non-trivial idempotent and* $X \not\simeq 0$.

(3) X *is indecomposable.*

We have $(1) \implies (2)$. *We have* $(2) \implies (3)$.

Proof. Ad $(1) \implies (2)$. If $\operatorname{End}(X)$ is a local ring, then $\operatorname{End}(X)$ does not contain non-trivial idempotents; cf. Remark 32. Moreover, in $\operatorname{End}(X)$ we have $1 \neq 0$; cf. Remark 30. So in X there are at least two different objects, in particular, we have $X \not\simeq 0$.

Ad $(2) \implies (3)$. We will prove this by contraposition. Suppose that $X \not\simeq 0$. Suppose that there exist $Y, Z \in \operatorname{Ob}\mathcal{A}$ such that $Y \not\simeq 0$ and $Z \not\simeq 0$ and $X \simeq Y \oplus Z$. Let $\pi_1\colon Y \oplus Z \to Y$ and $\pi_2\colon Y \oplus Z \to Z$ be the projection morphisms. Let $\iota_1\colon Y \to Y \oplus Z$ and $\iota_2\colon Z \to Y \oplus Z$ be the inclusion morphisms; cf. [Ste12, Conventions, (12)].

Choose an isomorphism $\varphi\colon X \to Y \oplus Z$. So we are in the situation as described by the following diagram.

$$\begin{tikzcd}
 & & Y \arrow[dl, bend left, "\iota_1"] \\
X \arrow[r, "\varphi", "\sim"'] & Y \oplus Z \arrow[ur, bend left, "\pi_1"] \arrow[dr, bend right, "\pi_2"'] & \\
 & & Z \arrow[ul, bend right, "\iota_2"']
\end{tikzcd}$$

We define $e := \varphi^{-1} \circ \iota_1 \circ \pi_1 \circ \varphi \in \operatorname{End}(X)$. Then $e^2 = e$ is an idempotent in $\operatorname{End}(X)$. Moreover, we have $1 - e = \varphi^{-1} \circ \iota_2 \circ \pi_2 \circ \varphi \in \operatorname{End}(X)$. We have to show that $e \notin \{0, 1\}$, i.e. we have to show that $e \overset{!}{\neq} 0$ and that $1 - e \overset{!}{\neq} 0$. It suffices to show that $e \neq 0$ because of symmetric reasons.

It suffices to show that $\iota_1 \circ \pi_1 \neq 0$ because φ is an isomorphism.

Assume that $\iota_1 \circ \pi_1 = 0$, then $\mathrm{id}_Y = \mathrm{id}_Y \circ \mathrm{id}_Y = \pi_1 \circ (\iota_1 \circ \pi_1) \circ \iota_1 = 0$, so we would have $Y \simeq 0$, a *contradiction.* □

Remark 36. *There exists an indecomposable object X in a preadditive category such that the endomorphism ring* $\operatorname{End}(X)$ *contains a non-trivial idempotent.*

This shows that in Lemma 35 we cannot deduce assertion (2) from assertion (3).

Proof. Let $R := \mathbb{C} \times \mathbb{C}$. Let $\mathcal{A}$ be the preadditive category with one object R and with the endomorphism ring $\operatorname{End}_{\mathcal{A}}(R) = R$. Then $\operatorname{End}_{\mathcal{A}}(R)$ contains the idempotent $e = (1, 0) \notin = \{0_R, 1_R\}$.

Assume that R is decomposable, i.e. we find objects $Y, Z \in \operatorname{Ob} \mathcal{A}$ such that $R \simeq Y \oplus Z$. Necessarily this means that $Y = Z = R$.

But then $R \simeq R \oplus R$. Since in $\mathcal{A}$, there is just one object R, we have $R = R \oplus R$.

Now on the one hand, we have $\operatorname{End}_{\mathcal{A}}(R \oplus R) \simeq (\operatorname{End}_{\mathcal{A}}(R))^{2\times 2} \simeq (\mathbb{C} \times \mathbb{C})^{2\times 2} \simeq \mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2}$. This is a non-commutative ring. On the other hand, we have $\operatorname{End}_{\mathcal{A}}(R) = R = \mathbb{C} \times \mathbb{C}$. This is a commutative ring.

But if $R = R \oplus R$, then the two endomorphism rings have to be the same.

This is a *contradiction.* □

Chapter 3: Diagonalizability

Let R be a principal ideal domain. Let $K = \operatorname{frac} R$ be its field of fractions. When tensoring over R, we often abbreviate $\otimes_R$ by $\otimes$.

3.1 On R-diagonalizability of K-diagonalizable endomorphisms

Let N be a finitely generated free R-module. We write $n := \operatorname{rk}_R(N)$. Suppose given an R-linear map $\varphi\colon N \to N$.

Remark 37. *Suppose given an eigenvalue $\lambda \in K$ of φ. Then $\lambda \in R$.*

In particular, we have $\sigma(\varphi) = \sigma(K\varphi)$.

Proof. Choose an R-linear basis $\mathcal{B}$ of N. Let $A = \varphi_{\mathcal{B},\mathcal{B}} \in R^{n\times n}$ be the describing matrix of φ with respect to the basis $\mathcal{B}$. Then λ is a zero of the characteristic polynomial of A. Since all entries in A are elements of R, the coefficients of this polynomial also are in R. So λ is contained in the integral closure of R in K. But R is integrally closed, hence $\lambda \in R$. □

Definition 38. We say that φ is *diagonalizable over K* (or short: *K-diagonalizable*) if the K-linear map $K \otimes \varphi\colon K \otimes N \to K \otimes N$ is diagonalizable (as defined by linear algebra).

Definition 39. We say that φ is *diagonalizable over R* (or short: *R-diagonalizable*) if there exists an R-linear basis of N that consists of eigenvectors of φ.

We say that a matrix $A \in R^{n\times n}$ is *diagonalizable over R* (or short: *R-diagonalizable*) if the R-endomorphism $R^n \to R^n\colon x \mapsto Ax$ is R-diagonalizable. So $A \in R^{n\times n}$ is diagonalizable over R if and only if there exists $S \in \mathrm{GL}_n(R)$ such that $S^{-1}AS \in R^{n\times n}$ is a diagonal matrix.

Remark 40. Note that if R is a field, then $R = K$. In this case, the property of being R-diagonalizable and the property of being K-diagonalizable both coincide with the property of being diagonalizable in the sense of linear algebra.

Remark 41. *If φ is diagonalizable over R, then φ is diagonalizable over K.*

Proof. Suppose that φ is diagonalizable over R. Then there exists an R-linear basis $\mathcal{B} = (b_i)_{i\in[1,n]}$ of N consisting of eigenvectors of φ. So $(1 \otimes b_i)_{i\in[1,n]}$ is a K-linear basis of $K \otimes N$ and these basis elements are eigenvectors of $K \otimes \varphi$, so $K \otimes \varphi$ is diagonalizable. By Definition 38, this means that φ is diagonalizable over K. □

Remark 42. *There exists a discrete valuation ring R and a matrix $A \in R^{2\times 2}$ such that A is diagonalizable over* $\operatorname{frac}(R)$*, but A is not diagonalizable over R.*

This shows that in general, there is a difference between K-diagonalizability and R-diagonalizability when $R \neq K$; cf. Definitions 38 and 39.

Proof. Let $R := \mathbb{Z}_{(3)}$ and $A := \left(\begin{smallmatrix}3&0\\1&0\end{smallmatrix}\right) \in R^{2\times 2}$. We define $S := \left(\begin{smallmatrix}3&0\\1&1\end{smallmatrix}\right) \in R^{2\times 2}$. We have the inverse $S^{-1} = \begin{pmatrix}\frac{1}{3}&0\\-\frac{1}{3}&1\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}) = \mathrm{GL}_2(\operatorname{frac}(R))$. Then $S^{-1}AS = \left(\begin{smallmatrix}3&0\\0&0\end{smallmatrix}\right)$ is diagonal. Note that $S \notin \mathrm{GL}_2(R)$.

Assume that there exists $T \in \mathrm{GL}_2(R)$ such that $T^{-1}AT$ is a diagonal matrix. Without loss of generality, we have $T^{-1}AT = \left(\begin{smallmatrix}3&0\\0&0\end{smallmatrix}\right)$. So $AT = T\left(\begin{smallmatrix}3&0\\0&0\end{smallmatrix}\right)$.

Writing $T = \left(\begin{smallmatrix} u & v \\ r & s \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$, this amounts to

$$\begin{pmatrix} 3u & 3v \\ u & v \end{pmatrix} = \begin{pmatrix} 3u & 0 \\ 3r & 0 \end{pmatrix}.$$

So $v = 0$ and $u = 3r$ whence $T = \left(\begin{smallmatrix} 3r & 0 \\ r & s \end{smallmatrix}\right)$. But then $\det(T) = 3rs \notin \mathrm{U}(R)$ which is a *contradiction*. $\square$

Lemma 43. *Suppose M to be a pure R-submodule of N such that $\varphi(M) \subseteq M$. If φ is diagonalizable over R, then the restricted map $\varphi|_M^M$ is diagonalizable over R, too.*

The proof is a variation of an idea of the user "Zorn" on math.stackexchange.com; cf. [Zor11].

Proof. Denote by $\lambda_1, \ldots, \lambda_l \in R$ the distinct eigenvalues of φ. We have the eigenmodules

$$\mathrm{E}_\varphi(\lambda_i) = \{x \in N \mid \varphi(x) = \lambda_i x\} \subseteq N$$

of φ to the eigenvalues λ_i for $i \in [1, l]$. Then $\mathrm{E}_\varphi(\lambda_i)$ is a non-zero submodule of N for $i \in [1, l]$. We write $g_i := \mathrm{rk}_R(\mathrm{E}_\varphi(\lambda_i))$.

Choose an R-linear basis

$$(x_{1,1}, \ldots, x_{1,g_1}, \ldots, x_{l,1}, \ldots, x_{l,g_l})$$

of N consisting of eigenvectors $x_{i,j}$ of φ to the eigenvalues λ_i , i.e. $\varphi(x_{i,j}) = \lambda_i x_{i,j}$ for $i \in [1, l]$, $j \in [1, g_i]$.

Claim 1. For $i \in [1, l]$, we have

$$\mathrm{E}_\varphi(\lambda_i) = {}_R\langle x_{i,1}, \ldots, x_{i,g_i}\rangle$$

and thus

$$N = \bigoplus_{i\in[1,l]} \mathrm{E}_\varphi(\lambda_i).$$

We show the first statement. It suffices to show $\overset{!}{\subseteq}$.

Suppose given $k \in [1, l]$ and $x \in \mathrm{E}_\varphi(\lambda_k)$. Then $\varphi(x) = \lambda_k x$. There exist certain $c_{i,j} \in R$ such that $x = \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} c_{i,j} x_{i,j}$. Now we can write $\varphi(x)$ in two ways.

$$\begin{aligned} \varphi(x) &= \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} c_{i,j}\varphi(x_{i,j}) = \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} c_{i,j}\lambda_i x_{i,j} \\ \varphi(x) &= \lambda_k x = \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} c_{i,j}\lambda_k x_{i,j} \end{aligned}$$

By subtracting both equations, we have $0 = \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} c_{i,j}(\lambda_i - \lambda_k)x_{i,j}$. Now $\lambda_i - \lambda_k = 0$ if and only if $i = k$, so we conclude that $c_{i,j} = 0$ for $i \in [1, l]\backslash \{k\}$ and for $j \in [1, g_i]$. This shows $\subseteq$.

For the second statement it suffices to show that the sum is direct. Suppose that we may write $x = \sum_{i\in[1,l]} y_i = \sum_{i\in[1,l]} \tilde{y}_i$ with $y_i, \tilde{y}_i \in \mathrm{E}_\varphi(\lambda_i)$ for $i \in [1, l]$. So there exist certain $c_{i,j}, \tilde{c}_{i,j} \in R$ for $i \in [1, l]$, $j \in [1, g_i]$ such that $y_i = \sum_{j\in[1,g_i]} c_{i,j}x_{i,j}$ and $\tilde{y}_i = \sum_{j\in[1,g_i]} \tilde{c}_{i,j}x_{i,j}$ for $i \in [1, l]$. This yields $x = \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} c_{i,j}x_{i,j} = \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} \tilde{c}_{i,j}x_{i,j}$. Comparing coefficients we get $c_{i,j} = \tilde{c}_{i,j}$ for $i \in [1, l]$, $j \in [1, g_i]$ and thus $y_i = \tilde{y}_i$ for $i \in [1, l]$. This *proves Claim 1.*

Claim 2. Suppose given $k \in \mathbb{N}_0$. Suppose given eigenvectors $y_j \in N$ of φ for $j \in [1, k]$ to pairwise distinct eigenvalues μ_j of φ, i.e. $y_j \neq 0$ and $\varphi(y_j) = \mu_j y_j$ where $\mu_j \neq \mu_{j'}$ for $j, j' \in [1, k]$ and $j \neq j'$. If $y_1 + y_2 + \ldots + y_k \in M$, then $y_i \in M$ for $i \in [1, k]$.

Proceed by induction on $k \geq 0$. Write $y := y_1 + y_2 + \ldots + y_k$. Then

$$\varphi(y) - \mu_k y = (\mu_1 - \mu_k)y_1 + \ldots + (\mu_{k-1} - \mu_k)y_{k-1} \in M.$$

By induction we have $(\mu_i - \mu_k)y_i \in M$ for $i \in [1, k-1]$ and thus $y_i \in M$ for $i \in [1, k-1]$ since M is a pure R-submodule of N. So we also get $y - (y_1 + \ldots + y_{k-1}) = y_k \in M$. This *proves Claim 2.*

Claim 3. We have

$$M = \bigoplus_{i\in[1,l]} (\mathrm{E}_\varphi(\lambda_i) \cap M).$$

It follows from Claim 1 that the sum is direct.

It suffices to show $\subseteq$. Suppose given $x \in M$. Then we have unique coefficients $c_{i,j} \in R$ such that $x = \sum_{i\in[1,l]} \sum_{j\in[1,g_i]} c_{i,j}x_{i,j}$. Define $y_i := \sum_{j\in[1,g_i]} c_{i,j}x_{i,j}$ for $i \in [1, l]$. If $y_i \neq 0$, then y_i is an eigenvector to the eigenvalue λ_i. So now $x = \sum_{i\in[1,m],y_i\neq 0} y_i$ is a sum as in Claim 2. Thus we have $y_i \in M$ for $i \in [1, l]$. Hence $x \in \bigoplus_{i\in[1,l]}(\mathrm{E}_\varphi(\lambda_i) \cap M)$. This *proves Claim 3.*

Now R is a principal ideal domain, so we can choose R-linear bases of $\mathrm{E}_\varphi(\lambda_i) \cap M$ for $i \in [1, l]$ and concatenate these to a basis of M, using Claim 3. We conclude that M has a basis consisting of eigenvectors of φ and thus the restricted map $\varphi|_M^M$ is diagonalizable over R. □

Corollary 44. *Suppose M to be a submodule of N such that $\varphi(M) \subseteq M$. If φ is diagonalizable over K, then the restricted map $\varphi|_M^M$ is diagonalizable over K, too.*

Proof. If φ is diagonalizable over K, then $K\varphi$ is a diagonalizable K-linear map. Now we can apply Lemma 43 on the map $K\varphi\colon KN \to KN$ restricted to the subspace KM of KN. We obtain that $(K\varphi)|_{KM}^{KM}$ is diagonalizable. But this map is the same as $K(\varphi|_M^M)$ and thus $\varphi|_M^M$ is a K-diagonalizable map. □

Remark 45. *There exist a discrete valuation ring R, two finitely generated R-modules X and Y and an R-linear map $\psi\colon X \to X$ such that Y is an R-submodule of X, the map ψ is diagonalizable over R but the restricted map $\psi|_Y^Y$ is not diagonalizable over R.*

This shows that the assertion of Lemma 43 does not hold if we omit the condition on the X and Y to be pure submodules of N.

Proof. Suppose that $R = \mathbb{Z}_{(3)}$. Define $X := R^2$ and $A := \left(\begin{smallmatrix} 3 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Let $\mathcal{E} := \left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$ be the standard basis of X. Define $\psi\colon X \to X$ as the R-endomorphism on X given by $x \mapsto Ax$, so $A = \psi_{\mathcal{E},\mathcal{E}}$.

Now we restrict ψ to $Y := {}_R\langle\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -3 \end{smallmatrix}\right)\rangle \subseteq X$. So $\mathcal{F} := \left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -3 \end{smallmatrix}\right)\right)$ is an R-linear basis of Y. Note that $\psi(Y) \subseteq Y$.

Then the describing matrix of $\psi|_Y^Y$ with respect to the basis $\mathcal{F}$ is $B := (\psi|_Y^Y)_{\mathcal{F},\mathcal{F}} = \left(\begin{smallmatrix} 3 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

We have shown that the matrix B is not diagonalizable over $\mathbb{Z}_{(3)}$ in Remark 42. So $\psi|_Y^Y$ is not diagonalizable over R. □

Corollary 46. *Suppose given two R-submodules N_1, N_2 of N such that $N = N_1 \oplus N_2$ and such that $\varphi(N_1) \subseteq N_1$ and $\varphi(N_2) \subseteq N_2$. Then we have the following equivalence:*

$$\varphi \textit{ is diagonalizable over } R \iff \varphi|_{N_1}^{N_1} \textit{ and } \varphi|_{N_2}^{N_2} \textit{ are diagonalizable over } R$$

Proof.

Ad $\Longrightarrow$. N_1 and N_2 are pure submodules of N; cf. Example 13.(1). Then we can apply Lemma 43.

Ad $\Longleftarrow$. Choose an R-linear basis $\mathcal{B}_1$ of N_1 consisting of eigenvectors of $\varphi|_{N_1}^{N_1}$. Choose an R-linear basis $\mathcal{B}_2$ of N_2 consisting of eigenvectors of $\varphi|_{N_2}^{N_2}$. Then we can concatenate $\mathcal{B}_1$ and $\mathcal{B}_2$ to an R-linear basis $\mathcal{B}$ of N. Thus $\mathcal{B}$ consists of eigenvectors of φ. □

3.2 On R-diagonalizability of K-diagonalizable matrices

Let $n \in \mathbb{N}$. Let $A \in R^{n\times n}$ be a matrix that is diagonalizable over K. We write $\ell := |\sigma(A)|$. We denote the distinct eigenvalues of A by λ_i where $i \in [1, \ell]$. For $i \in [1, \ell]$, we write $g_i := \dim_K(\mathrm{E}_A(\lambda_i))$.

There exists $Q \in \mathrm{GL}_n(K)$ such that $Q^{-1}AQ$ is a diagonal matrix. Without loss of generality, we may assume that $Q \in R^{n\times n}$.

We establish a method to decide whether there is a matrix $\tilde{Q} \in \mathrm{GL}_n(R)$ such that $\tilde{Q}^{-1}A\tilde{Q}$ is diagonal.

Lemma 47. *For $i \in [1, \ell]$, we choose an R-linear basis $\mathcal{B}_i = (v_{i,1}, \ldots, v_{i,g_i})$ of $\mathrm{E}_A(\lambda_i) \cap R^{n\times 1}$; cf. Remark 19. We define the matrix*

$$V := \begin{pmatrix} v_{1,1} & \cdots & v_{1,g_1} & \cdots & v_{\ell,1} & \cdots & v_{\ell,g_\ell} \end{pmatrix}$$

that contains the basis elements of the $\mathcal{B}_i$ in its columns, ordered by eigenvalues. We have the following equivalence.

$$A \text{ is diagonalizable over } R \iff \det(V) \in \mathrm{U}(R)$$

Proof. Ad $\Longleftarrow$. Since $\det(V) \in \mathrm{U}(R)$, we have $V^{-1} \in R^{n\times n}$. The fact that $V^{-1}AV$ is diagonal follows from the theory of vector spaces over fields. Thus A is diagonalizable over R; cf. Definition 39.

Ad $\Longrightarrow$. Suppose that A is diagonalizable over R. Then there exists a matrix $T \in \mathrm{GL}_n(R)$ such that $T^{-1}AT$ is diagonal; cf. Definition 39.

We find a permutation matrix $P \in \mathrm{GL}_n(R)$ that reorders the columns of T such that the j-th column of TP and the j-th column of V are (considered as elements of $R^{n\times 1}$) eigenvectors of A to the same eigenvalue for $j \in [1, n]$. We write $T' := TP$. Then T' takes the following form.

$$T' = \begin{pmatrix} c_{1,1} & \cdots & c_{1,g_1} & \cdots & c_{\ell,1} & \cdots & c_{\ell,g_\ell} \end{pmatrix}$$

where $Ac_{i,j} = \lambda_i c_{i,j}$ for $i \in [1, \ell]$, $j \in [1, g_i]$. Since T and thus T' is invertible in R, the columns of T' form an R-linear basis $\mathcal{C} = (c_{1,1}, \ldots, c_{1,g_1}, \ldots, c_{\ell,1}, \ldots, c_{\ell,g_\ell})$ of $R^{n\times 1}$.

For $i \in [1, \ell]$, we write $\mathcal{C}_i := (c_{i,1}, \ldots, c_{i,g_i})$ for the elements of $\mathcal{C}$ that are eigenvectors of A to the eigenvalue λ_i.

Claim. For $i \in [1, \ell]$, the tuple $\mathcal{C}_i$ forms an R-linear basis of $R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)$.

For $i \in [1, \ell]$, we write $X_i := {}_R\langle c_{i,1}, \ldots, c_{i,g_i}\rangle$.

$\mathcal{C}_i$ is linearly independent over R for $i \in [1, \ell]$ since all columns of T' are linearly independent over R. Moreover, we have $X_i \subseteq R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)$ definition of X_i.

It remains to show that $X_i \overset{!}{=} R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)$ for $i \in [1, \ell]$.

We have

$$\bigoplus_{i\in[1,\ell]} X_i \subseteq \bigoplus_{i\in[1,\ell]} (R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)) \subseteq R^{n\times 1}.$$

But we know that $\bigoplus_{i\in[1,\ell]} X_i = \mathrm{im}\,(T) = R^{n\times 1}$. So we conclude that

$$\bigoplus_{i\in[1,\ell]} X_i = \bigoplus_{i\in[1,\ell]} (R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)).$$

Now we can apply Remark 10 and we obtain that $X_i = R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)$ for $i \in [1, \ell]$.

This *proves the Claim.*

Suppose given $i \in [1, \ell]$. The columns $(v_{i,1}, \ldots, v_{i,g_i})$ of V form an R-linear basis of $R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)$. By the Claim, the tuple $\mathcal{C}_i = (c_{i,1}, \ldots, c_{i,g_i})$ also is an R-linear basis of $R^{n\times 1} \cap \mathrm{E}_A(\lambda_i)$. So we find a matrix $D_i \in \mathrm{GL}_{g_i}(R)$ such that, as matrices, we obtain

$$\begin{pmatrix} c_{i,1} & \cdots & c_{i,g_i} \end{pmatrix} D_i = \begin{pmatrix} v_{i,1} & \cdots & v_{i,g_i} \end{pmatrix}.$$

We define the block diagonal matrix $D := \mathrm{diag}(D_1, \ldots, D_\ell) \in \mathrm{GL}_n(R)$. Thus we obtain

$$V = T'D = TPD.$$

Since $\det(P)$ and $\det(D)$ are units in R, we have $\det(V) \in \mathrm{U}(R) \iff \det(T) \in \mathrm{U}(R)$. But T is invertible in R, hence $\det(V) \in \mathrm{U}(R)$. □

Corollary 48. *Recall that $A \in R^{n\times n}$ is diagonalizable over K.*

(1) *We have the following equivalence.*

$$A \text{ is diagonalizable over } R \iff \bigoplus_{\lambda\in\sigma(A)} \left(\mathrm{E}_A(\lambda) \cap R^{n\times 1}\right) = R^{n\times 1}$$

(2) *Let N be a finitely generated free R-module. Let $\varphi \in \mathrm{End}_R(N)$. Suppose that φ is diagonalizable over K. Then we have the following equivalence.*

$$\varphi \text{ is diagonalizable over } R \iff \bigoplus_{\lambda\in\sigma(\varphi)} \mathrm{E}_\varphi(\lambda) = N$$

Proof. Ad (1). The equality on the right hand side is equivalent to the columns of the matrix V in Lemma 47 being an R-linear basis of $R^{n\times 1}$, whence Corollary 48 follows from Lemma 47.

Ad (2). This follows from (1) by passage to describing matrices. □

In the following, we want to find a method which allows us to construct R-linear bases of the eigenmodules as required in Corollary 48 with matrix operations.

Lemma 49. *Suppose given $i \in [1, \ell]$. Let $(w_1, \ldots, w_{g_i})$ be a K-linear basis of $\mathrm{E}_A(\lambda_i)$ that is contained in $R^{n\times 1}$. Define $W := \begin{pmatrix} w_1 & \cdots & w_{g_i}\end{pmatrix} \in R^{n\times g_i}$. By the elementary divisor theorem, we find matrices $S \in \mathrm{GL}_n(R)$ and $T \in \mathrm{GL}_{g_i}(R)$ such that*

$$SWT =: D = (d_{i,j})_{i\in[1,n],\, j\in[1,g_i]} \in R^{n\times g_i} \tag{7}$$

is a diagonal matrix. This can e.g. be achieved using the Smith normal form of W.

Then the first g_i columns of S^{-1} form an R-linear basis of $\mathrm{E}_A(\lambda_i) \cap R^{n\times 1}$.

Note that W itself does not necessarily contain an R-linear basis of $\mathrm{E}_A(\lambda_i) \cap R^{n\times 1}$ in its columns. We only know that $\mathrm{E}_A(\lambda_i) \cap R^{n\times 1}$ is the pure closure of ${}_R\langle w_1, \ldots, w_{g_i}\rangle$ in $R^{n\times 1}$.

Proof. We define the following matrices.

$$D' = \begin{pmatrix} d_{1,1} & & \\ & \ddots & \\ & & d_{g_i,g_i}\end{pmatrix} \in R^{g_i\times g_i} \quad \text{and} \quad U = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \in R^{n\times g_i}$$

Note that D' has only non-zero elements on its diagonal. We have

$$UD' = D. \tag{8}$$

We obtain the following commutative diagram.

$$\begin{tikzcd}
R^{g_i\times 1} \arrow[rr, "W(-)"] \arrow[dr, "D'T^{-1}(-)"] \arrow[dd, shift right, "T^{-1}(-)"'] & & R^{n\times 1} \arrow[dd, shift right, "S(-)"'] \\
& R^{g_i\times 1} \arrow[ur, "S^{-1}U(-)"] \arrow[dd, equal] & \\
R^{g_i\times 1} \arrow[rr, "D(-)"] \arrow[dr, "D'(-)"'] \arrow[uu, shift right, "T(-)"'] & & R^{n\times 1} \arrow[uu, shift right, "S^{-1}(-)"'] \\
& R^{g_i\times 1} \arrow[ur, "U(-)"'] &
\end{tikzcd}$$

The map $S^{-1}U$ is injective because U is injective and S is an isomorphism. The map $D'T^{-1}$ is injective because D' is injective and T is an isomorphism.

The matrix $D'T^{-1}$ is invertible in $\mathrm{GL}_{g_i}(K)$. Moreover, all elementary divisors of $S^{-1}U$ are units in R by the definition of U and since $S \in \mathrm{GL}_n(R)$. We have the product $S^{-1}U \cdot D'T^{-1} = W$, so we can apply Lemma 18 to obtain that the R-module generated by the columns of $S^{-1}U$ is the same as the pure closure of the R-module generated by the columns of W which is $\mathrm{E}_A(\lambda_i) \cap R^{n\times 1}$. So the columns of $S^{-1}U$ generate $\mathrm{E}_A(\lambda_i) \cap R^{n\times 1}$ as an R-module.

But the columns of the matrix product $S^{-1}U$ are the first g_i columns of S^{-1} followed by zero columns. Hence an R-linear basis of $\mathrm{E}_A(\lambda_i) \cap R^{n\times 1}$ is given by the first g_i columns of S^{-1}.

This completes the proof. $\square$

Algorithm 50. *Using Lemma 49, we obtain the following algorithm, written in pseudocode, that constructs an R-linear basis for every eigenmodule of A. This allows us to decide whether A is diagonalizable over R; cf. Lemma 47 and Corollary 48.(1).*

More precisely, the algorithm yields a matrix V that has in its columns an R-linear basis for every eigenmodule of the K-diagonalizable matrix A. Then A is diagonalizable over R if and only if $\det(V)$ *is a unit in R.*

for $i \in [1, \ell]$ **do**
  Choose a K-linear basis $(w_1, \ldots, w_{g_i})$ of $\mathrm{E}_A(\lambda_i)$ that is contained in $R^{n\times 1}$.
  Define the matrix $W_i := \begin{pmatrix} w_1 & \cdots & w_{g_i} \end{pmatrix}$.
  Find matrices $S_i \in \mathrm{GL}_n(R)$ and $T_i \in \mathrm{GL}_{g_i}(R)$ such that $S_i W_i T_i$ is diagonal.
  Let $\mathcal{C}_i$ be the set of the first g_i columns of S^{-1}.
end for
Set $\mathcal{B} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_\ell$.
Write the elements of $\mathcal{B}$ in the columns of a matrix V.
if $\det(V) \in \mathrm{U}(R)$ **then**
  print A is diagonalizable over R.
else
  print A is not diagonalizable over R.
end if
return V

Note that $\mathcal{B}$ is an R-linear basis of $\bigoplus_{\lambda\in\sigma(A)} \left(\mathrm{E}_A(\lambda) \cap R^{n\times 1}\right)$.

3.3 Commuting tuples of K-diagonalizable endomorphisms

Let N be a finitely generated free R-module. We write $n := \mathrm{rk}_R(N)$. Let $k \in \mathbb{N}$. Let $\Phi = (\varphi_1, \ldots, \varphi_k)$ be a tuple of R-linear endomorphisms of N.

Definition 51. We say that Φ is a *cd-tuple on N* if the following conditions are fulfilled.

- $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for $i, j \in [1, k]$.
- φ_i is K-diagonalizable for $i \in [1, k]$.

Here “cd” stands for “commuting and diagonalizable”.

Suppose given $\alpha := (\alpha_j)_{j\in[1,k]} \in R^{k\times 1}$. Then we define

$$\varphi_{\Phi,\alpha} := \sum_{i\in[1,k]} \alpha_i \varphi_i.$$

If Φ is a cd-tuple, we often write $K\Phi := (K\varphi_1, \ldots, K\varphi_k)$.

Note that the property of being a cd-tuple does not depend on the order of the endomorphisms. Moreover, every tuple that arises from a cd-tuple by omitting some of the endomorphisms is again a cd-tuple. Also note that all eigenvalues of the endomorphisms of Φ are elements of R; cf. Remark 37.

Definition 52. Suppose that $\Phi = (\varphi_1, \ldots, \varphi_k)$ is a cd-tuple on N. We say that $\lambda = (\lambda_i)_{i\in[1,k]} \in R^{1\times k}$ is an *eigenvalue tuple of* Φ if there exists a non-zero element $x \in N$ such that $\varphi_i(x) = \lambda_i x$ for $i \in [1, k]$.

Suppose given an eigenvalue tuple $\lambda = (\lambda_i)_{i\in[1,k]}$ of Φ. Its *simultaneous eigenmodule* $\mathrm{E}_\Phi(\lambda)$ is given by

$$\mathrm{E}_\Phi(\lambda) = \{x \in N \mid \varphi_i(x) = \lambda_i x \text{ for } i \in [1, k]\}.$$

A *simultaneous eigenmodule for* Φ is a simultaneous eigenmodule for Φ for some eigenvalue tuple of Φ.

3.4 On simultaneous R-diagonalizability of R-diagonalizable endomorphisms

Let N be a finitely generated free R-module. We write $n := \mathrm{rk}_R(N)$. Let $\Phi = (\varphi_1, \ldots, \varphi_k)$ be a cd-tuple on N. Suppose that φ_i is diagonalizable over R for $i \in [1, k]$.

Lemma 53. *There exists an R-linear basis $\mathcal{B} = (b_j)_{j\in[1,n]}$ of N such that $\varphi_i(b_j) \in {}_R\langle b_j\rangle$ for $i \in [1, k]$, $j \in [1, n]$.*

In this sense, the maps $\varphi_1, \ldots, \varphi_k$ are simultaneously diagonalizable over R.

Proof. First we need to restrict endomorphisms to eigenmodules.

Claim. Suppose given $i \in [1, k]$. Suppose given an eigenvalue $\lambda \in R$ of φ_i. Let $E := \mathrm{E}_{\varphi_i}(\lambda)$. Then $\varphi_j(E) \subseteq E$ for $j \in [1, k]$.

For $x \in E$, we have $\varphi_i(\varphi_j(x)) = \varphi_j(\varphi_i(x)) = \varphi_j(\lambda x) = \lambda\varphi_j(x)$, so $\varphi_j(x) \in E$. This *proves the Claim.*

Suppose given $i \in [1, k]$. Suppose given R-submodules X, Y of N such that $N = X \oplus Y$ and $\varphi_i(X) \subseteq X$ and $\varphi_i(Y) \subseteq Y$. Then $\varphi_i|_X^X$ is diagonalizable over R; cf. Corollary 46.

We proceed by induction on k.

For $k = 1$, we have that φ_1 is R-diagonalizable by assumption. So there exists an R-linear basis of N that consists of eigenvectors of φ_1; cf. Definition 39.

Suppose given $k \geq 2$. Suppose that the statement holds for each cd-tuple on N of length $k - 1$. Denote the distinct eigenvalues of φ_1 by $\lambda_1, \ldots, \lambda_l \in R$. Denote the corresponding eigenmodules by $E_i := \mathrm{E}_{\varphi_1}(\lambda_i)$ for $i \in [1, l]$. Then $N = \bigoplus_{i\in[1,l]} E_i$ is a decomposition of N into pure R-submodules of N; cf. Example 13.(1) and Corollary 48.(2). By the Claim, we have $\varphi_j(E_i) \subseteq E_i$ for $j \in [1,k]$, $i \in [1,l]$. By Lemma 43, the maps $\varphi_j|_{E_i}^{E_i}$ are diagonalizable over R for $j \in [2,k]$, $i \in [1,l]$.

For $i \in [2,k]$, choose an R-linear basis $\mathcal{B}_i := (b_{i,j})_{j\in[1,\mathrm{rk}_R E_i]}$ of E_i such that $\varphi_r(b_{i,j}) \in {}_R\langle b_{i,j}\rangle$ for $r \in [2,k]$, $j \in [1, \mathrm{rk}_R E_i]$ which is possible by induction.

Moreover, $\varphi_1(b_{i,j}) \in {}_R\langle b_{i,j}\rangle$ for $i \in [1,k]$, $j \in [1, \mathrm{rk}_R E_i]$ since $b_{i,j} \in E_i$ which is an eigenmodule of φ_1.

So $\mathcal{B} := (b_{i,j})_{i\in[1,k],\, j\in[1,\mathrm{rk}_R E_i]}$ is an R-linear basis of N. Thus we obtain a basis that fulfills the required properties, completing the proof. □

Corollary 54. *Suppose given $\alpha = (\alpha_j)_{j\in[1,k]} \in R^{k\times 1}$. The R-endomorphism $\varphi_{\Phi,\alpha} = \sum_{j\in[1,k]} \alpha_j\varphi_j$ is diagonalizable over R.*

Proof. Choose an R-linear basis $\mathcal{B} = (b_i)_{i\in[1,n]}$ of N such that $\varphi_j(b_i) \in {}_R\langle b_i\rangle$ for $i \in [1,n]$, $j \in [1,k]$; cf. Lemma 53.

Suppose given $i \in [1,n]$. We denote the eigenvalue of φ_j to the eigenvector b_i by $\lambda_j \in R$ for $j \in [1,k]$, i.e. $\varphi_j(b_i) = \lambda_j b_i$ for $j \in [1,k]$.

We obtain

$$\varphi_{\Phi,\alpha}(b_i) = \left(\sum_{j\in[1,k]} \alpha_j\varphi_j\right)(b_i) = \sum_{j\in[1,k]} \alpha_j(\varphi_j(b_i)) = \sum_{j\in[1,k]} \alpha_j\lambda_j b_i.$$

This shows that b_i is an eigenvector of $\varphi_{\Phi,\alpha}$ to the eigenvalue $\sum_{j\in[1,k]} \alpha_j\lambda_j \in R$.

We conclude that $\mathcal{B}$ is an R-linear basis of N consisting of eigenvectors of $\varphi_{\Phi,\alpha}$. Thus $\varphi_{\Phi,\alpha}$ is R-diagonalizable, completing the proof. □

Corollary 55. *Denote the ℓ distinct eigenvalue tuples of Φ by $\lambda_j \in R^{1\times k}$ for $j \in [1,\ell]$.*

We obtain the following decomposition of N.

$$N = \bigoplus_{j\in[1,\ell]} \mathrm{E}_\Phi(\lambda_j)$$

Proof. By Lemma 53, there exist an R-linear basis $\mathcal{B} = (x_1, \ldots, x_n)$ of N and $\mu_t = (\mu_{t,i})_{i\in[1,k]} \in R^{k\times 1}$ for $t \in [1,n]$ such that

$$\varphi_i(x_t) = \mu_{t,i}x_t \quad \text{for } i \in [1,k],\ t \in [1,n].$$

Let $\Lambda := \{\mu_t \mid t \in [1,n]\}$. Write $\ell := |\Lambda|$. Write $\Lambda = \{\lambda_1, \ldots, \lambda_\ell\}$ so that $\lambda_u \neq \lambda_v$ for $u, v \in [1,\ell]$ with $u \neq v$. Write $\lambda_j =: (\lambda_{j,i})_{i\in[1,k]} \in R^{k\times 1}$ for $j \in [1,\ell]$. After reordering if necessary we may suppose that

$$\mathcal{B} = (b_{1,1}, \ldots, b_{1,g_1}, \ldots, b_{\ell,1}, \ldots, b_{\ell,g_\ell})$$

with certain $g_j \in \mathbb{N}$ for $j \in [1,\ell]$ such that

$$\varphi_i(b_{j,s}) = \lambda_{j,i}b_{j,s} \quad \text{for } i \in [1,k],\ j \in [1,\ell],\ s \in [1,g_j].$$

We write $\mathrm{E}'_\Phi(\lambda_j) := {}_R\langle b_{j,s} \mid s \in [1,g_j]\rangle$. Then we obtain that

$$\mathrm{E}'_\Phi(\lambda_j) \subseteq \mathrm{E}_\Phi(\lambda_j) \text{ for } j \in [1,\ell]. \tag{9}$$

Moreover, we have

$$\bigoplus_{j\in[1,\ell]} \mathrm{E}'_\Phi(\lambda_j) = N. \tag{10}$$

Claim. We have the direct sum $\bigoplus_{j\in[1,\ell]} \mathrm{E}_\Phi(\lambda_j) \subseteq N$.

Assume that the sum is not direct. Then there exists a tuple $y = (y_j)_{j\in[1,\ell]}$ where $y_j \in \mathrm{E}_\Phi(\lambda_j)$ for $j \in [1,\ell]$ satisfying the following conditions.

(C1) We have $\sum_{j\in[1,\ell]} y_j = 0$.

(C2) We have $\{j \in [1,\ell] \,|\, y_j \neq 0\} \neq \emptyset$.

(C3) There is no tuple $z = (z_j)_{j\in[1,\ell]}$ where $z_j \in \mathrm{E}_\Phi(\lambda_j)$ for $j \in [1,\ell]$ such that

$$|\{j \in [1,\ell] \,|\, z_j \neq 0\}| < |\{j \in [1,\ell] \,|\, y_j \neq 0\}| .$$

Choose $u, v \in [1,\ell]$ such that $y_u \neq 0$, $y_v \neq 0$ and $u \neq v$. This is possible because of (C1) and (C2).

Suppose given $i \in [1,k]$. We calculate.

$$\begin{aligned}
\sum_{j\in[1,\ell]} (\lambda_{u,i} - \lambda_{j,i}) y_j &= \lambda_{u,i} \sum_{j\in[1,\ell]} y_j - \sum_{j\in[1,\ell]} \lambda_{j,i} y_j \\
&= \lambda_{u,i} \sum_{j\in[1,\ell]} y_j - \varphi_i \left(\sum_{j\in[1,\ell]} y_j \right) \\
&\overset{\text{(C1)}}{=} 0
\end{aligned}$$

We have the following inclusion for $i \in [1,k]$.

$$\{j \in [1,\ell] \,|\, (\lambda_{u,i} - \lambda_{j,i}) y_j \neq 0\} \subset \{j \in [1,\ell] \,|\, y_j \neq 0\}$$

This is a proper inclusion since u is contained in the right hand side but it is not contained in the left hand side.

Using (C3) we conclude that $(\lambda_{u,i} - \lambda_{j,i}) y_j = 0$ for $i \in [1,k]$, $j \in [1,\ell]$. Note that this includes that $(\lambda_{u,i} - \lambda_{v,i}) y_v = 0$ for $i \in [1,k]$. But $y_v \neq 0$ by choice, so $\lambda_{u,i} = \lambda_{v,i}$ for $i \in [1,k]$ since N is torsion free. But this amounts to $\lambda_u = \lambda_v$ which is *a contradiction*.

This *proves the Claim*.

Using the Claim, we obtain the following chain of inclusions.

$$N \overset{(10)}{=} \bigoplus_{j\in[1,\ell]} \mathrm{E}'_\Phi(\lambda_j) \overset{(9)}{\subseteq} \bigoplus_{j\in[1,\ell]} \mathrm{E}_\Phi(\lambda_j) \subseteq N$$

So we obtain that

$$\bigoplus_{j\in[1,\ell]} \mathrm{E}'_\Phi(\lambda_j) = \bigoplus_{j\in[1,\ell]} \mathrm{E}_\Phi(\lambda_j). \tag{11}$$

By (9) and (11) we can apply Remark 10 to obtain that $\mathrm{E}'_\Phi(\lambda_j) = \mathrm{E}_\Phi(\lambda_j)$ for $j \in [1,k]$. □

3.5 On R-diagonalizability of linear combinations of K-diagonalizable endomorphisms

Let N be a finitely generated free R-module. We write $n := \mathrm{rk}_R(N)$. Let $k \in \mathbb{N}$. Let $\Phi = (\varphi_1, \ldots, \varphi_k)$ be a cd-tuple on N.

Denote the ℓ distinct eigenvalue tuples of Φ by $\lambda_j = (\lambda_{j,i})_{i\in[1,k]} \in R^{1\times k}$ for $j \in [1,\ell]$. Denote the simultaneous eigenmodules for Φ by $E_j := \mathrm{E}_\Phi(\lambda_j)$ for $j \in [1,\ell]$.

We write $g_j := \mathrm{rk}_R(E_j) = \dim_K(\mathrm{E}_{K\Phi}(\lambda_j))$; cf. Remark 19.(2). Choose a K-linear basis

$$\mathcal{B} := (b_{1,1}, \ldots, b_{1,g_1}, \ldots, b_{\ell,1}, \ldots, b_{\ell,g_\ell})$$

of KN such that

$$(K\varphi_i)(b_{j,s}) = \lambda_{j,i} b_{j,s} \quad \text{for } i \in [1,k], j \in [1,\ell], s \in [1,g_j];$$

cf. Corollary 53, note that K is a principal ideal domain.

If all φ_i are diagonalizable over R, we know that $\varphi_{\Phi,\alpha}$ is diagonalizable over R for $\alpha \in R^{k\times 1}$; cf. Corollary 54, Definition 51.

Now we want to investigate the question for which $\alpha \in R^{k\times 1}$ the map $\varphi_{\Phi,\alpha}$ is diagonalizable over R. We have the trivial solution $\alpha_i = 0$ for $i \in [1,k]$ because the zero map is indeed diagonalizable over R. But in the general case we just know that $\varphi_{\Phi,\alpha}$ is diagonalizable over K for $\alpha \in R^{k\times 1}$.

If we wanted to test all possible linear combinations $\varphi_{\Phi,\alpha}$ for $\alpha \in R^{k\times 1}$, this would lead to an infinite task (provided that R is infinite).

In the following we will develop an algorithm that gives an R-linear basis of an R-module consisting of all solutions to this problem. Thus we know all R-linear combinations of endomorphisms of Φ that are diagonalizable over R.

In particular, this also allows us to decide whether there is a non-zero solution at all.

3.5.1 Setup and examples

Definition 56. We define the *diagonalizability locus of* Φ as

$$\mathrm{C}_\Phi := \left\{ \alpha = (\alpha_i)_{i\in[1,k]} \in R^{k\times 1} \,\middle|\, \varphi_{\Phi,\alpha} \text{ is diagonalizable over } R \right\}.$$

Lemma 57. *C_Φ is a pure R-submodule of $R^{k\times 1}$.*

Proof. We have $\mathrm{C}_\Phi \subseteq R^{k\times 1}$.

We have $0_{R^{k\times 1}} \in \mathrm{C}_\Phi$.

C_Φ *is closed under addition:* Suppose given $\alpha := (\alpha_i)_{i\in[1,k]} \in \mathrm{C}_\Phi$ and $\beta := (\beta_i)_{i\in[1,k]} \in \mathrm{C}_\Phi$. Then $\varphi_{\Phi,\alpha}$ and $\varphi_{\Phi,\beta}$ are diagonalizable over R.

We have to show that $\alpha + \beta \overset{!}{\in} \mathrm{C}_\Phi$. It suffices to show that $\varphi_{\Phi,\alpha+\beta}$ is diagonalizable over R.

Suppose given $x \in N$.

$$\begin{aligned}
(\varphi_{\Phi,\alpha} \circ \varphi_{\Phi,\beta})(x) &= \varphi_{\Phi,\alpha}\left(\sum_{j\in[1,k]} \beta_j \varphi_j(x)\right) = \sum_{i\in[1,k]} \alpha_i \varphi_i \left(\sum_{j\in[1,k]} \beta_j \varphi_j(x)\right) \\
&= \sum_{i\in[1,k]} \alpha_i \left(\sum_{j\in[1,k]} \beta_j \varphi_i(\varphi_j(x))\right) = \sum_{j\in[1,k]} \beta_j \left(\sum_{i\in[1,k]} \alpha_i \varphi_i(\varphi_j(x))\right) \\
&= \sum_{j\in[1,k]} \beta_j \left(\sum_{i\in[1,k]} \alpha_i \varphi_j(\varphi_i(x))\right) = \sum_{j\in[1,k]} \beta_j \left(\varphi_j\left(\sum_{i\in[1,k]} \alpha_i \varphi_i(x)\right)\right) \\
&= \sum_{j\in[1,k]} \beta_j \left(\varphi_j\left(\varphi_{\Phi,\alpha}(x)\right)\right) = (\varphi_{\Phi,\beta} \circ \varphi_{\Phi,\alpha})(x)
\end{aligned}$$

This shows that $\varphi_{\Phi,\alpha} \circ \varphi_{\Phi,\beta} = \varphi_{\Phi,\beta} \circ \varphi_{\Phi,\alpha}$. Both $\varphi_{\Phi,\alpha}$ and $\varphi_{\Phi,\beta}$ are R-diagonalizable, so $(\varphi_{\Phi,\alpha}, \varphi_{\Phi,\beta})$ is a cd-tuple on N.

Now $\varphi_{\Phi,\alpha} + \varphi_{\Phi,\beta}$ is diagonalizable over R by Corollary 54. But this is the same as $\varphi_{\Phi,\alpha+\beta}$.

C_Φ *is closed under scalar multiplication:* Suppose given $\alpha := (\alpha_i)_{i\in[1,k]} \in \mathrm{C}_\Phi$ and $r \in R$. Then $\varphi_{\Phi,\alpha}$ is diagonalizable over R. We have to show that $r\alpha \overset{!}{\in} \mathrm{C}_\Phi$. It suffices to show that $\varphi_{\Phi,r\alpha}$ is diagonalizable over R.

But we have $\varphi_{\Phi,r\alpha} = r\varphi_{\Phi,\alpha}$ and this map is diagonalizable over R by Corollary 54.

C_Φ *is pure in* $R^{k\times 1}$*:* Suppose given $\alpha := (\alpha_i)_{i\in[1,k]} \in R^{k\times 1}$ and $r \in R^\times$ such that $r\alpha \in \mathrm{C}_\Phi$. Then $\varphi_{\Phi,r\alpha} = r\varphi_{\Phi,\alpha}$ is R-diagonalizable, i.e. there exists an R-linear basis of N that consists of eigenvectors of $r\varphi_{\Phi,\alpha}$. But each of these basis elements is again an eigenvector of $\varphi_{\Phi,\alpha}$. So $\varphi_{\Phi,\alpha}$ is R-diagonalizable and thus $\alpha \in \mathrm{C}_\Phi$. □

Remark 58. *Suppose given $j \in [1, k]$. Suppose that φ_i is diagonalizable over R for $i \in [1, j]$. Suppose given $\alpha = (\alpha_i)_{i\in[1,k]} \in R^{k\times 1}$. Then we have the following equivalence.*

$$\sum_{i\in[j+1,k]} \alpha_i\varphi_i \textit{ is diagonalizable over } R \Longleftrightarrow \sum_{i\in[1,k]} \alpha_i\varphi_i \textit{ is diagonalizable over } R$$

Proof. We write $\psi := \sum_{i\in[1,j]} \alpha_i\varphi_i$. This is an R-endomorphism that is diagonalizable over R; cf. Corollary 54. Thus we have $\left(\sum_{i\in[j+1,k]} \alpha_i\varphi_i\right) + \psi = \sum_{i\in[1,k]} \alpha_i\varphi_i$.

Ad $\Longrightarrow$. If $\sum_{i\in[j+1,k]} \alpha_i\varphi_i$ is diagonalizable over R, then $\left(\sum_{i\in[j+1,k]} \alpha_i\varphi_i\right) + \psi$ also is diagonalizable over R as it is a sum of two commuting R-diagonalizable R-endomorphisms; cf. Corollary 54.

Ad $\Longleftarrow$. If $\sum_{i\in[1,k]} \alpha_i\varphi_i$ is diagonalizable over R, then $\left(\sum_{i\in[1,k]} \alpha_i\varphi_i\right) + (-\psi)$ also is diagonalizable over R as it is a sum of two commuting R-diagonalizable R-endomorphisms; cf. Corollary 54. □

Remark 59. Suppose given $j \in [1, k]$. Suppose that φ_i is diagonalizable over R for $i \in [1, j]$. Define $\Phi' := (\varphi_{j+1}, \ldots, \varphi_k)$ which is also a cd-tuple on N. Suppose given $\alpha = (\alpha_i)_{i\in[1,k]} \in R^{k\times 1}$. Then we have the following equivalences.

$$\begin{aligned}
\alpha \in \mathrm{C}_\Phi \quad &\Longleftrightarrow \quad \sum_{i\in[1,k]} \alpha_i\varphi_i \text{ is diagonalizable over } R \\
&\overset{\text{Rem. 58}}{\Longleftrightarrow} \quad \sum_{i\in[j+1,k]} \alpha_i\varphi_i \text{ is diagonalizable over } R \\
&\Longleftrightarrow \quad (\alpha_i)_{i\in[j+1,k]} \in \mathrm{C}_{\Phi'}
\end{aligned}$$

So $\mathrm{C}_\Phi = \left\{(\alpha_i)_{i\in[1,k]} \in R^{k\times 1} \,\middle|\, (\alpha_i)_{i\in[j+1,k]} \in R^{(k-j)\times 1} \in \mathrm{C}_{\Phi'}\right\}$.

Recall that in Corollary 48, we have used eigenspaces to decide whether an R-linear endomorphism is diagonalizable over R. Therefore we need to know the eigenvalues of linear combinations of the endomorphisms in Φ. This gives reason to the following Remark.

Remark 60. *Suppose given $\alpha = (\alpha_i)_{i\in[1,k]} \in R^{k\times 1}$. Recall that $K\varphi_{\Phi,\alpha}$ is a K-endomorphism of KN. Then*

$$\sigma(K\varphi_{\Phi,\alpha}) = \left\{ \sum_{i\in[1,k]} \alpha_i\lambda_{j,i} \,\middle|\, j \in [1,\ell] \right\}.$$

In particular, given $\mu \in \sigma(K\varphi_{\Phi,\alpha})$, we have

$$\begin{aligned}
\mathrm{E}_{\varphi_{\Phi,\alpha}}(\mu) &= {}_K\langle b_{j,s} : s \in [1, g_j],\ j \in [1,\ell] \textit{ with } \sum_{i\in[1,k]} \alpha_i\lambda_{j,i} = \mu\rangle \cap N \\
&= \left(\bigoplus_{\substack{j\in[1,\ell] \\ \sum_{i\in[1,k]} \alpha_i\lambda_{j,i} = \mu}} \mathrm{E}_{K\Phi}(\lambda_j) \right) \cap N.
\end{aligned}$$

Proof. Recall the K-linear basis $\mathcal{B} = (b_{1,1}, \ldots, b_{1,g_1}, \ldots, b_{\ell,1}, \ldots, b_{\ell,g_\ell})$ of KN.

For $j \in [1, \ell]$, $s \in [1, g_j]$, we obtain

$$\begin{aligned}(K\varphi_{\Phi,\alpha})(b_{j,s}) &= \left(\sum_{i\in[1,k]} \alpha_i (K\varphi_i)\right)(b_{j,s}) = \sum_{i\in[1,k]} \left(\alpha_i \left(K\varphi_i\right)(b_{j,s})\right) \\ &= \sum_{i\in[1,k]} \left(\alpha_i \left(\lambda_{j,i} b_{j,s}\right)\right) = \left(\sum_{i\in[1,k]} \alpha_i \lambda_{j,i}\right)(b_{j,s}).\end{aligned}$$

□

Example 61. Suppose that $R = \mathbb{Z}$ and $N = \mathbb{Z}^{2\times 1}$. We have the standard basis $\mathcal{E} := \mathcal{E}_{2,1}$ of $\mathbb{Z}^{2\times 2}$. Let $\tilde{\varphi}_1, \tilde{\varphi}_2$ be $\mathbb{Z}$-linear maps on N such that $(\tilde{\varphi}_1)_{\mathcal{E},\mathcal{E}} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ and $(\tilde{\varphi}_2)_{\mathcal{E},\mathcal{E}} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$. Let τ be a $\mathbb{Z}$-linear map on N such that $\tau_{\mathcal{E},\mathcal{E}} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$. Note that $\tau_{\mathcal{E},\mathcal{E}} \notin \mathrm{GL}_2(\mathbb{Z})$.

As restricted $\mathbb{Q}$-endomorphisms, we define the $\mathbb{Z}$-endomorphisms $\varphi_1 := \big((\mathbb{Q}\tau^{-1})(\mathbb{Q}\tilde{\varphi}_1)(\mathbb{Q}\tau)\big)\,|_N^N$ and $\varphi_2 := \big((\mathbb{Q}\tau^{-1})(\mathbb{Q}\tilde{\varphi}_2)(\mathbb{Q}\tau)\big)\,|_N^N$. Then $(\varphi_1)_{\mathcal{E},\mathcal{E}} = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ and $(\varphi_2)_{\mathcal{E},\mathcal{E}} = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$.

We have $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$ and both φ_1 and φ_2 are diagonalizable over $\mathbb{Z}$ by construction. So $\Phi := (\varphi_1, \varphi_2)$ is a cd-tuple on $\mathbb{Z}^{2\times 1}$.

The matrix $(\varphi_1)_{\mathcal{E},\mathcal{E}}$ has eigenvalues 1 and -1, the corresponding eigenvectors are $b_{1,1} := \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ and $b_{2,1} := \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, respectively. So we obtain the corresponding eigenspaces are $\mathrm{E}_{(\varphi_1)_{\mathcal{E},\mathcal{E}}}(1) = {}_{\mathbb{Q}}\langle \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right) \rangle$ and $\mathrm{E}_{(\varphi_1)_{\mathcal{E},\mathcal{E}}}(-1) = {}_{\mathbb{Q}}\langle \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \rangle$.

We have

$$(\mathrm{E}_{(\varphi_1)_{\mathcal{E},\mathcal{E}}}(1) \cap \mathbb{Z}^2) \oplus (\mathrm{E}_{(\varphi_1)_{\mathcal{E},\mathcal{E}}}(-1) \cap \mathbb{Z}^2) = {}_{\mathbb{Z}}\langle \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \rangle \neq \mathbb{Z}^2.$$

Therefore the matrices $(\varphi_1)_{\mathcal{E},\mathcal{E}}$ and $(\varphi_2)_{\mathcal{E},\mathcal{E}}$ are not diagonalizable over $\mathbb{Z}$; cf. Corollary 48. So φ_1 and φ_2 are not diagonalizable over $\mathbb{Z}$.

But φ_1 and φ_2 are diagonalizable over $\mathbb{Q}$ by construction.

For $\alpha := \left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right) \in \mathbb{Z}^2$, the $\mathbb{Z}$-linear combination

$$(\varphi_{\Phi,\alpha})_{\mathcal{E},\mathcal{E}} := \alpha_1 (\varphi_1)_{\mathcal{E},\mathcal{E}} + \alpha_2 (\varphi_2)_{\mathcal{E},\mathcal{E}} = \left(\begin{smallmatrix} 0 & -\alpha_1-\alpha_2 \\ -\alpha_1-\alpha_2 & 0 \end{smallmatrix}\right)$$

is diagonalizable if $\alpha_1 = -\alpha_2$ since the zero matrix is diagonalizable over $\mathbb{Z}$. Let now $\alpha_1 = 1$ and $\alpha_2 = -1$. In this case, we have $\mathrm{E}_{\varphi_{\Phi,\alpha}}(1) = \mathbb{Z}^2$ and $\mathrm{E}_{\varphi_{\Phi,\alpha}}(1) \not\subseteq \mathrm{E}_{\varphi_1}(1)$ as well as $\mathrm{E}_{\varphi_{\Phi,\alpha}}(1) \not\subseteq \mathrm{E}_{\varphi_1}(-1)$. So in a sense, the simultaneous eigenmodules $\mathrm{E}_\Phi((1,1))$ and $\mathrm{E}_\Phi((-1,-1))$ for Φ fuse to the eigenmodule $\mathrm{E}_{\varphi_{\Phi,\alpha}}(1)$ of $\varphi_{\Phi,\alpha}$.

In this small example, we will have a closer look at the eigenvalues and the eigenvectors $b_{1,1}$ and $b_{2,1}$. We have $(\varphi_1)_{\mathcal{E},\mathcal{E}} \cdot b_{1,1} = 1 \cdot b_{1,1}$ and $(\varphi_1)_{\mathcal{E},\mathcal{E}} \cdot b_{2,1} = -1 \cdot b_{2,1}$. Denote the eigenvalues by $\lambda_{1,1} := 1$ and $\lambda_{2,1} := -1$. We have $(\varphi_2)_{\mathcal{E},\mathcal{E}} \cdot b_{1,1} = 1 \cdot b_{1,1}$ and $(\varphi_2)_{\mathcal{E},\mathcal{E}} \cdot b_{2,1} = -1 \cdot b_{2,1}$. Denote the eigenvalues by $\lambda_{1,2} := 1$ and $\lambda_{2,2} := -1$.

Note that $\sigma(\varphi_{\Phi,\alpha}) = \sigma(K\varphi_{\Phi,\alpha})$; cf. Remark 37. Let $\nu \in \sigma(\varphi_{\Phi,\alpha})$. Then we have the eigenmodule $\mathrm{E}_{\varphi_{\Phi,\alpha}}(\nu) = {}_{\mathbb{Z}}\langle b_{j,1} : j \in [1,2] \text{ and } \alpha_1\lambda_{j,1} + \alpha_2\lambda_{j,2} = \nu \rangle$. In order that a fusion of eigenmodules can happen, we need $\alpha_1, \alpha_2 \in R$ such that

$$\alpha_1\lambda_{1,1} + \alpha_2\lambda_{1,2} = \alpha_1\lambda_{2,1} + \alpha_2\lambda_{2,2}$$

since we have

$$\begin{aligned}(\varphi_{\Phi,\alpha})_{\mathcal{E},\mathcal{E}}(b_{1,1}) &= (\alpha_1(\varphi_1)_{\mathcal{E},\mathcal{E}} + \alpha_2(\varphi_2)_{\mathcal{E},\mathcal{E}})(b_{1,1}) = (\alpha_1\lambda_{1,1} + \alpha_2\lambda_{1,2})b_{1,1}\,, \\ (\varphi_{\Phi,\alpha})_{\mathcal{E},\mathcal{E}}(b_{2,1}) &= (\alpha_1(\varphi_1)_{\mathcal{E},\mathcal{E}} + \alpha_2(\varphi_2)_{\mathcal{E},\mathcal{E}})(b_{2,1}) = (\alpha_1\lambda_{2,1} + \alpha_2\lambda_{2,2})b_{2,1}\,.\end{aligned}$$

Example 62. Suppose given a cd-tuple $\Phi = (\varphi_1, \varphi_2)$ on R^2 such that φ_1 and φ_2 are not diagonalizable over R. Suppose that $\lambda_1 = (\lambda_{1,1}, \lambda_{1,2}) = (1, -1)$ and $\lambda_2 = (\lambda_{2,1}, \lambda_{2,2}) = (-1, 2)$ are the eigenvalue tuples of Φ. Note that this entails that each eigenmodule of φ_1 resp. of φ_2 is an R-module of rank 1. Recall that we have the simultaneous eigenmodules $E_1 = \mathrm{E}_\Phi(\lambda_1)$ and $E_2 = \mathrm{E}_\Phi(\lambda_2)$ for Φ.

We want to find out if there exists $\alpha = \binom{\alpha_1}{\alpha_2} \in R^{2\times 1}$ such that $\varphi_{\Phi,\alpha}$ is diagonalizable over R.

Let $\alpha_1 = 5$ and $\alpha_2 = 7$. Then the eigenvalues of $\varphi_{\Phi,\alpha}$ are $\alpha_1\lambda_{1,1} + \alpha_2\lambda_{1,2} = 5\cdot 1 + 7\cdot(-1) = -2$ and $\alpha_1\lambda_{2,1} + \alpha_2\lambda_{2,2} = 5\cdot(-1) + 7\cdot 2 = 9$; cf. Remark 60. Thus $\varphi_{\Phi,\alpha}$ also has two non-zero eigenmodules, so each of these has to be of rank 1. Hence every eigenmodule of $\varphi_{\Phi,\alpha}$ is contained in an eigenmodule of φ_1. So by Corollary 48.(2), $\varphi_{\Phi,\alpha}$ is not R-diagonalizable.

It also can happen that $\varphi_{\Phi,\alpha}$ has only one eigenmodule which is of rank 2. This would be the case if, in a sense, E_1 and E_2 fuse to a single eigenmodule of $\varphi_{\Phi,\alpha}$. But this can only happen if $\alpha = \binom{\alpha_1}{\alpha_2}$ is chosen such that

$$\begin{aligned} & \alpha_1\cdot 1 + \alpha_2\cdot(-1) = \alpha_1\cdot(-1) + \alpha_2\cdot 2 \\ \iff \quad & 2\cdot\alpha_1 = 3\cdot\alpha_2. \end{aligned}$$

But we have $\left\{\binom{\alpha_1}{\alpha_2} \in R^{2\times 1} \,\middle|\, 2\alpha_1 = 3\alpha_2\right\} = {}_R\langle\binom{3}{2}\rangle$. So once we know that $3\varphi_1 + 2\varphi_2$ is diagonalizable over R, we know that $\mathrm{C}_\Phi = {}_R\langle\binom{3}{2}\rangle$.

Note that the eigenmodule of $3\varphi_1 + 2\varphi_2$ to the eigenvalue 1 is of rank 2 and it is pure in R^2, so it equals R^2. Hence $3\varphi_1 + 2\varphi_2 = \mathrm{id}$.

Example 63. Let $R = \mathbb{Z}$. Let $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ be a cd-tuple on R^4. Let $(b_{1,1}, b_{2,1}, b_{2,2}, b_{3,1})$ be an R-linear basis of R^4 such that $\varphi_i(b_{j,s}) = \lambda_{j,i} b_{j,s}$ for $i \in [1,3]$, $j \in [1,3]$, $s \in [1, g_j]$ with $g_1 = g_3 = 1$, $g_2 = 2$ and eigenvalues $\lambda_{j,i}$ given as follows.

$$\begin{array}{lll} \lambda_{1,1} = 1 & \lambda_{2,1} = 1 & \lambda_{3,1} = 0 \\ \lambda_{1,2} = 1 & \lambda_{2,2} = 2 & \lambda_{3,2} = -1 \\ \lambda_{1,3} = 3 & \lambda_{2,3} = -1 & \lambda_{3,3} = 4 \end{array}$$

For $j \in [1,3]$, the eigenvalue tuple $\lambda_j := (\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3})$ has a simultaneous eigenmodule E_j for Φ. These simultaneous eigenmodules are

$$E_1 = \mathrm{E}_\Phi(\lambda_1) = {}_R\langle b_{1,1}\rangle, \quad E_2 = \mathrm{E}_\Phi(\lambda_2) = {}_R\langle b_{2,1}, b_{2,2}\rangle, \quad E_3 = \mathrm{E}_\Phi(\lambda_3) = {}_R\langle b_{3,1}\rangle.$$

For some choices of the coefficients $\alpha = \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} \in R^{3\times 1}$, two or three of the simultaneous eigenmodules for Φ fuse to a single eigenmodule of $\varphi_{\Phi,\alpha}$. We assign a partition of the finite set $\{1,2,3\}$ to the tuple α, describing which of the simultaneous eigenmodules fuse. If P is a partition of the set $\{1,2,3\}$, we will have $\{i,j\} \subseteq p$ for some p in the tuple P if and only if $E_i \oplus E_j \subseteq E$ as R-modules for an eigenmodule E of $\varphi_{\Phi,\alpha}$.

For certain tuples $\alpha = \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}$, we obtain the following.

α_1	α_2	α_3	set of eigenvalues of $\varphi_{\Phi,\alpha}$	eigenmodules of $\varphi_{\Phi,\alpha}$	corresponding partition of $\{1,2,3\}$
1	1	1	$\{5,2,3\}$	E_1, E_2, E_3	$(\{1\},\{2\},\{3\})$
1	4	1	$\{8,0\}$	$\mathrm{cl}_N(E_1\oplus E_2), E_3$	$(\{1,2\},\{3\})$
8	-1	1	$\{10,5\}$	$E_1, \mathrm{cl}_N(E_2\oplus E_3)$	$(\{1\},\{2,3\})$
-1	1	1	$\{3,0\}$	$\mathrm{cl}_N(E_1\oplus E_3), E_2$	$(\{1,3\},\{2\})$
-7	4	1	$\{0\}$	$\mathrm{cl}_N(E_1\oplus E_2\oplus E_3)$	$(\{1,2,3\})$

In this example, all partitions of $\{1,2,3\}$ have occurred. In general, this will not necessarily be the case.

Example 64. Suppose given $\alpha := (\alpha_i)_{i\in[1,k]} \in R^{k\times 1}$. Suppose that φ_1 is not diagonalizable over R. Consider $\varphi := \varphi_{\Phi,\alpha}$. Suppose that for $\mu \in \sigma(\varphi)$, the eigenmodule $\mathrm{E}_\varphi(\mu)$ of φ to the eigenvalue μ is contained in an eigenmodule $\mathrm{E}_{\varphi_1}(\lambda)$ of φ_1 for some $\lambda \in \sigma(\varphi_1)$.

Since φ_1 is not diagonalizable over R, we have a proper inclusion $\bigoplus_{\lambda\in\sigma(\varphi_1)} \mathrm{E}_{\varphi_1}(\lambda) \subset N$.

Taking both together, we get

$$\bigoplus_{\mu\in\sigma(\varphi)} \mathrm{E}_{\varphi}(\mu) \subseteq \bigoplus_{\lambda\in\sigma(\varphi_1)} \mathrm{E}_{\varphi_1}(\lambda) \subset N$$

and thus φ cannot be diagonalizable over R; cf. Corollary 48.(2).

3.5.2 On partitions of finite sets

We will collect some facts on partitions of finite sets as far as it is necessary to understand the Partitions Algorithm; cf. Algorithm 94 below. We will explain how we handle partitions in Magma and give some examples.

Definition 65. Recall that ℓ is the number of distinct eigenvalue tuples of Φ. Suppose given a partition $P = (p_1, \ldots, p_u)$ of the set $[1, \ell]$. We define the following R-submodule of N.

$$\mathrm{M}_{\Phi,P} := \bigoplus_{i\in[1,u]} \left(\left(\bigoplus_{j\in p_i} \mathrm{E}_{K\Phi}(\lambda_j) \right) \cap N \right) \subseteq N$$

We define the following map.

$$\begin{aligned} \upsilon_\Phi \colon \mathcal{P}_\ell &\to \mathrm{Sub}_R(N) \\ P &\mapsto \mathrm{M}_{\Phi,P} \end{aligned}$$

Remark 66. Let $P := ([1,\ell]) = (\{1,2,\ldots,\ell\}) \in \mathcal{P}_\ell$. Then $\upsilon_\Phi(P) = N$. In particular, the preimage $\upsilon_\Phi^{-1}(N)$ is not empty.

Definition 67. Let $s \in \mathbb{N}$. Let $P = (p_1, \ldots, p_u)$ and $Q = (q_1, \ldots, q_v)$ be partitions of $[1, s]$.

We say that P is *finer than* Q if for $i \in [1, u]$ there exists $j \in [1, v]$ such that $p_i \subseteq q_j$. If P is finer than Q, we write $P \succeq Q$. If $P \succeq Q$ and $P \neq Q$, we write $P \succ Q$.

We say that P is *coarser than* Q if Q is finer than P. If P is coarser than Q, we write $P \preceq Q$. If $P \preceq Q$ and $P \neq Q$, we write $P \prec Q$.

The relation $\preceq$ is a *partial order* on the set $\mathcal{P}_s$ of partitions of the set $[1, s]$. So for every subset $L \subseteq \mathcal{P}_s$ we can define $\min(L)$ and $\max(L)$ as follows.

$$\begin{aligned} \min(L) &:= \{P \in L \mid \{Q \in L \mid Q \preceq P\} = \{P\}\} \\ \max(L) &:= \{P \in L \mid \{Q \in L \mid Q \succeq P\} = \{P\}\} \end{aligned}$$

Lemma 68. *Suppose given $s \in \mathbb{N}$. Suppose given $P, Q \in \mathcal{P}_s$. Then*

$$P \succeq Q \implies \mathrm{M}_{\Phi,P} \subseteq \mathrm{M}_{\Phi,Q}\,.$$

Proof. We write $P = (p_1, \ldots, p_u)$ and $Q = (q_1, \ldots, q_v)$ where $u, v \in \mathbb{N}$. We calculate.

$$\begin{aligned} \mathrm{M}_{\Phi,P} &= \bigoplus_{i\in[1,u]} \left(\left(\bigoplus_{j\in p_i} \mathrm{E}_{K\Phi}(\lambda_j) \right) \cap N \right) \\ &= \bigoplus_{i\in[1,v]} \bigoplus_{\substack{j\in[1,u] \\ p_j\subseteq q_i}} \left(\left(\bigoplus_{s\in p_j} \mathrm{E}_{K\Phi}(\lambda_s) \right) \cap N \right) \\ &\subseteq \bigoplus_{i\in[1,v]} \left(\left(\bigoplus_{j\in q_i} \mathrm{E}_{K\Phi}(\lambda_j) \right) \cap N \right) \\ &= \mathrm{M}_{\Phi,Q} \end{aligned}$$

$\square$

Definition 69. Suppose given $n \in \mathbb{N}$. We define

$$\mathcal{I}_1 := \{(1)\}.$$

For $n \geq 2$, we recursively define

$$\mathcal{I}_n := \bigcup_{(i_1,\ldots,i_{n-1}) \in \mathcal{I}_{n-1}} \left\{ (i_1, \ldots, i_{n-1}, i_n) \,\middle|\, i_n \in [1, (\max_{j \in [1,n-1]} i_j) + 1] \right\}.$$

Note that this is a disjoint union and that $\mathcal{I}_n$ is a finite set for $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, the set $\mathcal{I}_n$ equipped with the lexicographic order $(\leq_{\text{lex}})$ is a linearly ordered set.

Example 70. We have

$$\begin{aligned}
\mathcal{I}_1 = \; & \big\{(1)\big\}, \\
\mathcal{I}_2 = \; & \big\{(1,1), (1,2)\big\}, \\
\mathcal{I}_3 = \; & \big\{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (1,2,3)\big\} \text{ and} \\
\mathcal{I}_4 = \; & \big\{(1,1,1,1), (1,1,1,2), (1,1,2,1), (1,1,2,2), (1,1,2,3), \\
& (1,2,1,1), (1,2,1,2), (1,2,1,3), (1,2,2,1), (1,2,2,2), \\
& (1,2,2,3), (1,2,3,1), (1,2,3,2), (1,2,3,3), (1,2,3,4) \; \big\}.
\end{aligned}$$

We have written all elements ordered linearly with respect to $(\leq_{\text{lex}})$, starting with the smallest element. For example, we have $(1,1,1,1) \leq_{\text{lex}} (1,2,1,3) \leq_{\text{lex}} (1,2,3,4)$.

Remark 71. *Suppose given $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in \mathcal{I}_n$. Then*

$$\{i_1, \ldots, i_n\} = [1, \max_{j \in [1,n]} i_j].$$

Proof. We proceed by induction on n.

Let $n = 1$. Then $(i_1) = 1$ and $\{i_1\} = \{1\} = [1,1]$.

Let $n \geq 2$. Suppose that the statement holds for $n-1$. Suppose given $(i_1, \ldots, i_n) \in \mathcal{I}_n$. Write $m := \max_{j \in [1,n]} i_j$. Write $m' := \max_{j \in [1,n-1]} i_j$. Then $(i_1, \ldots, i_{n-1}) \in \mathcal{I}_{n-1}$ and $i_n \in [1, m'+1]$; cf. Definition 69.

By induction we have $\{i_1, \ldots, i_{n-1}\} = [1, m']$. So we obtain $\{i_1, \ldots, i_n\} = [1, m'] \cup \{i_n\}$.

We proceed by case distinction.

If $i_n = m' + 1$, then $\{i_1, \ldots, i_n\} = [1, m'+1] = [1, m]$.

If $i_n \in [1, m']$, then $\{i_1, \ldots, i_n\} = [1, m'] = [1, m]$.

This shows that the statement holds for $n \in \mathbb{N}$. □

Remark 72. *For $n \in \mathbb{N}$, we define the following map.*

$$\begin{aligned}
\gamma_n \colon \quad & \mathcal{I}_n \to \mathcal{P}_n \\
& (i_1, \ldots, i_n) \mapsto \Big(\{k \in [1,n] \mid i_k = s\} : s \in [1, \max_{j \in [1,n]} i_j] \Big)
\end{aligned}$$

Furthermore, we define

$$\begin{aligned}
\gamma'_n \colon \quad & \mathcal{P}_n \to \mathcal{I}_n \\
& (p_1, \ldots, p_u) \mapsto (i_1, \ldots, i_n)
\end{aligned}$$

where $j \in p_{i_j}$ for $j \in [1,n]$.

Then γ_n is bijective. Moreover, we have $\gamma_n^{-1} = \gamma'_n$.

We identify $\mathcal{I}_n$ and $\mathcal{P}_n$ along γ_n. In particular, we have $P \succeq Q$ if and only if $\gamma_n(P) \succeq \gamma_n(Q)$ for $P, Q \in \mathcal{I}_n$.

Proof. Suppose given $n \in \mathbb{N}$.

γ_n is well-defined.

Suppose given $(i_1, \ldots, i_n) \in \mathcal{I}_n$. We have to show that $\gamma_n((i_1, \ldots, i_n)) \overset{!}{\in} \mathcal{P}_n$. Let $m := \max_{j \in [1,n]} i_j$. We write $p_j := \{k \in [1,n] \mid i_k = j\}$ for $j \in [1,m]$. Since the index s in the definition of $\gamma_n((i_1, \ldots, i_n))$ increases from 1 to m, we obtain $\gamma_n((i_1, \ldots, i_n)) = (p_1, \ldots, p_m)$.

It follows that $p_i \neq \emptyset$ for $i \in [1,m]$; cf. Remark 71. Moreover, we have $p_i \cap p_j = \emptyset$ if $i \neq j$ for $i, j \in [1,m]$ by definition of the sets p_i. We also obtain that $\bigcup_{i \in [1,m]} p_i = [1,n]$.

Assume that there exist $j_1, j_2 \in [1,m]$ such that $j_1 < j_2$ and $m_1 := \min p_{j_1} > \min p_{j_2} =: m_2$. Then $i_{m_1} = j_1$ and $i_{m_2} = j_2$. Consider the tuple $(i_1, \ldots, i_{m_2})$. Recall the recursive definition of the $\mathcal{I}_{n'}$ for $n' \in \mathbb{N}$; cf. Definition 69. Then $(i_1, \ldots, i_{m_2}) = (i_1, \ldots, i_{m_2-1}, j_2) \in \mathcal{I}_{m_2}$. This implies that the number j_1 which is smaller than j_2 is an entry of this tuple; cf. Remark 71. So there exists $m_3 < m_2$ such that $i_{m_3} = j_1$. Hence $m_3 \in p_{j_1}$. But now $\min p_{j_1} > \min p_{j_2} = m_2 > m_3 \in p_{j_1}$, a *contradiction*.

Thus we have $\min p_{j_1} < \min p_{j_2}$ if and only if $j_1 < j_2$ for $j_1, j_2 \in [1,m]$. This shows that the entries of $(p_1, \ldots, p_m)$ are ordered by their smallest element.

So we have shown that $\gamma_n((i_1, \ldots, i_n)) \in \mathcal{P}_n$.

γ'_n is well-defined.

For $n = 1$ there is nothing to show, so suppose that $n \geq 2$.

Suppose given $(p_1, \ldots, p_u) \in \mathcal{P}_n$. We have to show that $\gamma'_n((p_1, \ldots, p_u)) \overset{!}{\in} \mathcal{I}_n$.

We write $\gamma'_n((p_1, \ldots, p_u)) =: (i_1, \ldots, i_n)$. We have $i_1 = 1$ since $1 \in p_1$. Moreover, $i_j \in [1,u]$ for $j \in [1,n]$, in particular $i_j \geq 1$ for $j \in [1,n]$. We have to show that

$$i_j \overset{!}{\leq} \left(\max_{r \in [1,j-1]} i_r \right) + 1 \text{ for } j \in [2,n].$$

Assume that there exists $j \in [2,n]$ such that $i_j > \left(\max_{r \in [1,j-1]} i_r\right) + 1$. Then $i_j - 1 \notin \{i_r \mid r \in [1, j-1]\}$. This entails that $r \notin p_{i_j - 1}$ for $r \in [1, j-1]$. Moreover, $i_j - 1 \neq i_j$, so $j \notin p_{i_j - 1}$. So we conclude that $\min p_{i_j - 1} \notin ([1, j-1] \cup \{j\}) = [1,j]$.

But since $\min p_{i_j} \leq j$ by definition of γ'_n we obtain $\min p_{i_j} \leq j < \min p_{i_j - 1}$, a *contradiction* to $(p_1, \ldots, p_u) \in \mathcal{P}_n$.

This shows that γ'_n is well-defined.

It remains to show that $\gamma'_n \circ \gamma_n \overset{!}{=} \mathrm{id}_{\mathcal{I}_n}$ and that $\gamma_n \circ \gamma'_n \overset{!}{=} \mathrm{id}_{\mathcal{P}_n}$.

Ad $\gamma'_n \circ \gamma_n \overset{!}{=} \mathrm{id}_{\mathcal{I}_n}$.

Suppose given $(i_1, \ldots, i_n) \in \mathcal{I}_n$. We write $m := \max_{j \in [1,n]} i_j$. Then

$$\begin{aligned}(\gamma'_n \circ \gamma_n)(i_1, \ldots, i_n) &= \gamma'_n(\underbrace{\{r \in [1,n] \mid i_r = j\}}_{=: p_j} : j \in [1,m]) \\ &= \gamma'_n(p_1, \ldots, p_m) =: (\tilde{i}_1, \ldots, \tilde{i}_n).\end{aligned}$$

Here we have $j \in p_{\tilde{i}_j}$ for $j \in [1,n]$.

Suppose given $r \in [1,n]$. We have $r \in p_{i_r}$ by definition of γ_n. Now $r \in p_{\tilde{i}_r}$ by definition of $(\tilde{i}_1, \ldots, \tilde{i}_n)$. Since partitions are decompositions into disjoint subsets, we have $\tilde{i}_r = i_r$.

We conclude that $(\tilde{i}_1, \ldots, \tilde{i}_n) = (i_1, \ldots, i_n)$.

This shows that $\gamma'_n \circ \gamma_n = \mathrm{id}_{\mathcal{I}_n}$.

Ad $\gamma_n \circ \gamma'_n \stackrel{!}{=} \mathrm{id}_{\mathcal{P}_n}$.

Suppose given $(p_1, \ldots, p_u) \in \mathcal{P}_n$. Let $(i_1, \ldots, i_n) := \gamma'_n(p_1, \ldots, p_u)$. Then we have $i_j = s$ if and only if $j \in p_s$ for $j \in [1, n]$ and $s \in [1, u]$.

Let $(\tilde{p}_1, \ldots, \tilde{p}_{\tilde{u}}) := \gamma_n(i_1, \ldots, i_n)$ where we write $\tilde{p}_s := \{j \in [1, n] \mid i_j = s\}$ for $s \in [1, \max_{j \in [1,n]} i_j]$. Note that $u = \max_{j \in [1,n]} i_j = \tilde{u}$, so $\tilde{u} = u$. Then we have $j \in \tilde{p}_s$ if and only if $i_j = s$ for $j \in [1, n]$ and $s \in [1, u]$.

We conclude that $j \in p_s$ if and only if $j \in \tilde{p}_s$ for $j \in [1, n]$ and $s \in [1, u]$. Hence we obtain $(p_1, \ldots, p_u) = (\tilde{p}_1, \ldots, \tilde{p}_u) = (\gamma_n \circ \gamma'_n)\left((p_1, \ldots, p_u)\right)$.

This shows that $\gamma_n \circ \gamma'_n = \mathrm{id}_{\mathcal{P}_n}$. $\square$

Remark 73. Using Remark 72, we have a bijection γ_n between the linearly ordered set $\mathcal{I}_n$ and $\mathcal{P}_n$ for $n \in \mathbb{N}$. This allows us to loop easily over all partitions of the set $[1, n]$ in our algorithm.

In particular, we obtain that $|\mathcal{I}_n| = |\mathcal{P}_n|$ for $n \in \mathbb{N}$. The cardinality of $\mathcal{P}_n$ is called *Bell number* B_n.

Bell numbers can be calculated as sums of Stirling numbers of the second kind or recursively using binomial coefficients.

The first few Bell numbers are given by

$$\begin{array}{lll} B_1 = 1 & B_4 = 15 & B_7 = 877 \\ B_2 = 2 & B_5 = 52 & B_8 = 4140 \\ B_3 = 5 & B_6 = 203 & B_9 = 21147. \end{array}$$

For further information on this topic, we refer to [Gro07, Chapter 5].

Definition 74. Suppose given $P \in \mathcal{P}_n$. Write $\gamma_n^{-1}(P) =: (i_1, \ldots, i_n)$. We define the following map.

$$\begin{aligned} \pi_P \colon [1, n] &\to \mathbb{N} \\ j &\mapsto \pi_P(j) := i_j \end{aligned}$$

Example 75. Let $n := 4$. Let $P := (\{1, 3\}, \{2\}, \{4\}) \in \mathcal{P}_4$. Then $\gamma_4^{-1}(P) = (1, 2, 1, 3) \in \mathcal{I}_4$.

We obtain the following.

$$\begin{aligned} \pi_P(1) &= 1 \\ \pi_P(2) &= 2 \\ \pi_P(3) &= 1 \\ \pi_P(4) &= 3 \end{aligned}$$

So the map π_P maps $i \in [1, n]$ to the number of the set in P in which i is contained.

Remark 76. Suppose given $P, Q \in \mathcal{P}_n$. Then we have

$$P \prec Q \implies \gamma_n^{-1}(P) <_{\mathrm{lex}} \gamma_n^{-1}(Q).$$

Proof. Write $\gamma_n^{-1}(P) =: (i_1, \ldots, i_n)$ and $\gamma_n^{-1}(Q) =: (j_1, \ldots, j_n)$.

We will show that $\gamma_n^{-1}(P) >_{\mathrm{lex}} \gamma_n^{-1}(Q)$ implies $P \nprec Q$.

Suppose that $\gamma_n^{-1}(P) >_{\mathrm{lex}} \gamma_n^{-1}(Q)$. *Assume* that $P \prec Q$. Define

$$y := \min\{s \in [1, n] \mid i_s > j_s\}.$$

In particular, we have $i_y > j_y$. Since $\gamma_n^{-1}(P) >_{\mathrm{lex}} \gamma_n^{-1}(Q)$, we have $i_t = j_t$ for $t \in [1, y-1]$, i.e. $(i_1, \ldots, i_{y-1}) = (j_1, \ldots, j_{y-1})$.

Let

$$x := \min\left(\pi_Q^{-1}(j_y)\right).$$

This is the minimum of the subset of $[1,n]$ that occurs in the partition Q and that contains the element y. We have $j_x = j_y$ since in the partition Q, the elements x and y are in the same subset of $[1,n]$. Moreover, as sets we have $\pi_Q^{-1}(j_x) = \pi_Q^{-1}(j_y)$.

Claim 1. We have $x < y$.

We have $x \le y$ by definition of x. *Assume* that $x = y$. Then $x = \min\left(\pi_Q^{-1}(j_x)\right)$ and thus we get $j_x \notin \{j_s \mid s \in [1, x-1]\}$. Note that the latter set is an interval starting from 1; cf. Remark 71. Moreover, it coincides with the set $\{i_s \mid s \in [1, x-1]\}$. Since $i_x > j_x$, we obtain that $i_x \notin \{i_s \mid s \in [1, x-1]\}$. By construction of $\mathcal{I}_n$, we have

$$i_x = (\max_{s \in [1,x-1]} i_s) + 1 = (\max_{s \in [1,x-1]} j_s) + 1 = j_x$$

which is a *contradiction*. This *proves Claim 1.*

Claim 2. We have $\pi_P^{-1}(i_x) \cap \pi_Q^{-1}(j_x) \neq \emptyset$.

The element x is contained in both sets, so it is also in their intersection. This *proves Claim 2.*

Claim 3. We have $\pi_P^{-1}(i_x) \not\supseteq \pi_Q^{-1}(j_x)$.

We have $y \in \pi_Q^{-1}(j_y) = \pi_Q^{-1}(j_x)$ since $j_x = j_y$.

Recall that $j_y < i_y$. Note that $i_x = j_x$ by Claim 1. So we have $i_x = j_x = j_y < i_y$. In particular, $i_y \neq i_x$ which implies that $y \notin \pi_P^{-1}(i_x)$.

This *proves Claim 3.*

By assumption, P is coarser than Q, so $\pi_P^{-1}(i_x)$ is a disjoint union of sets of the form $\pi_Q^{-1}(j_s)$ for certain $s \in [1,n]$. One of these sets has to be $\pi_Q^{-1}(j_x)$ by Claim 2. Thus we have $\pi_P^{-1}(i_x) \supseteq \pi_Q^{-1}(j_x)$ which is a *contradiction* to Claim 3.

This completes the proof. $\square$

3.5.3 A description of the diagonalizability locus

Definition 77. Recall that $\lambda_j = (\lambda_{j,i})_{i \in [1,k]} \in R^{1 \times k}$ for $j \in [1,\ell]$ are the distinct eigenvalue tuples of Φ. We define the matrix

$$\Lambda_\Phi := (\lambda_{j,i})_{j \in [1,\ell], i \in [1,k]} \in R^{\ell \times k}.$$

The multiplication on $R^{k \times 1}$ with this matrix from the left defines a map.

$$\begin{aligned} \omega_\Phi \colon R^{k \times 1} &\to R^{\ell \times 1} \\ \alpha &\mapsto \Lambda_\Phi \cdot \alpha \end{aligned}$$

Note that for $\alpha \in R^{k \times 1}$, the image $\omega_\Phi(\alpha) = \Lambda_\Phi \cdot \alpha$ has the eigenvalues of $\varphi_{\Phi,\alpha}$ as entries; cf. Remark 60 and Convention (25).

Definition 78. We define the map

$$\begin{aligned} \tau_\ell \colon R^{\ell \times 1} &\to \mathcal{P}_\ell \\ \beta &\mapsto \tau_\ell(\beta) \end{aligned}$$

where for $\beta \in R^{\ell \times 1}$, we define

$$\tau_\ell(\beta) := (T \subseteq [1,\ell] : \text{there exists } r \in R \text{ such that } T = \{i \in [1,\ell] \mid \beta_i = r\} \neq \emptyset)$$

such that the entries of $\tau_\ell(\beta)$ are ordered by their smallest element.

That is, $\tau_\ell = (p_1, \ldots, p_s)$, where for $j \in [1,s]$, we have

$$p_j := \left\{ t \in [1,\ell] \,\middle|\, \beta_t = \beta_m \text{ where } m := \min\left\{ i \in [1,\ell] \,\middle|\, \beta_i \notin \{\beta_u \mid u \in \bigcup_{k \in [1,j-1]} p_k\} \right\} \right\}.$$

Corollary 79. *We can compose the maps defined in Definitions 65, 77 and 78 as follows.*

$$\begin{array}{ccccccc} R^{k\times 1} & \xrightarrow{\omega_\Phi} & R^{\ell\times 1} & \xrightarrow{\tau_\ell} & \mathcal{P}_\ell & \xrightarrow{\upsilon_\Phi} & \mathrm{Sub}_R(N) \\ \alpha & \longmapsto & \Lambda_\Phi\cdot\alpha & \longmapsto & \tau_\ell(\Lambda_\Phi\cdot\alpha) & \longmapsto & \mathrm{M}_{\Phi,\tau_\ell(\Lambda_\Phi\cdot\alpha)} \end{array}$$

Then we have the following equivalence for $\alpha \in R^{k\times 1}$.

$$\varphi_{\Phi,\alpha} \text{ is diagonalizable over } R \iff (\upsilon_\Phi\circ\tau_\ell\circ\omega_\Phi)(\alpha) = N$$

Proof. Note that $(\upsilon_\Phi\circ\tau_\ell\circ\omega_\Phi)(\alpha) = \mathrm{M}_{\Phi,\tau_\ell(\Lambda_\Phi\cdot\alpha)}$ for $\alpha\in R^{k\times 1}$.

If $\alpha = 0$, then $\varphi_{\Phi,\alpha}$ is the zero map which is R-diagonalizable. Moreover, we have $(\upsilon_\Phi\circ\tau_\ell\circ\omega_\Phi)(\alpha) = \upsilon_\Phi(([1,\ell])) = N$.

Suppose given $\alpha\in R^{k\times 1}\setminus\{0\}$. Note that $\Lambda_\Phi\cdot\alpha$ has the eigenvalues of $\varphi_{\Phi,\alpha}$ as entries; cf. Definition 77. We write $P := (p_1,\ldots,p_u) := \tau_\ell(\Lambda_\Phi\cdot\alpha)$

Recall that for $j\in[1,\ell]$, the tuple λ_j is a simultaneous eigenvalue tuple of Φ. We obtain the following.

$$\begin{array}{cl} & \varphi_{\Phi,\alpha} \text{ is diagonalizable over } R \\ \overset{\text{Cor. 48.(2)}}{\iff} & \bigoplus\limits_{\mu\in\sigma(\varphi_{\Phi,\alpha})} \mathrm{E}_{\varphi_{\Phi,\alpha}}(\mu) = N \\ \overset{\text{Rem. 60}}{\iff} & \bigoplus\limits_{\mu\in\sigma(\varphi_{\Phi,\alpha})} \left(\Bigg(\bigoplus\limits_{\substack{j\in[1,\ell] \\ \sum_{i\in[1,k]}\alpha_i\lambda_{j,i}=\mu}} \mathrm{E}_{K\Phi}(\lambda_j)\Bigg)\cap N\right) = N \\ \overset{\text{Def. 78}}{\iff} & \bigoplus\limits_{i\in[1,u]}\left(\left(\bigoplus\limits_{j\in p_i}\mathrm{E}_{K\Phi}(\lambda_j)\right)\cap N\right) = N \\ \overset{\text{Def. 65}}{\iff} & \mathrm{M}_{\Phi,P} = N \\ \iff & (\upsilon_\Phi\circ\tau_\ell\circ\omega_\Phi)(\alpha) = N \end{array}$$

□

Remark 80. Using Corollary 79, we want to find all $\alpha\in R^{k\times 1}$ such that $\varphi_{\Phi,\alpha}$ is R-diagonalizable. More precisely, we aim to find an R-linear basis of the diagonalizability locus C_Φ of Φ; cf. Definition 56. Note that $\mathcal{P}_\ell$ is a finite set, so we can write an algorithm to determine the preimage $\upsilon_\Phi^{-1}(N)$ of N under υ_Φ .

Later we will use $\upsilon_\Phi^{-1}(N)$ to determine an R-linear basis of C_Φ.

Lemma 81. *The preimage $\upsilon_\Phi^{-1}(N)$ of N under υ_Φ is a lower set of the poset $\mathcal{P}_\ell$.*

Proof. Suppose given $P\in\upsilon_\Phi^{-1}(N)$. Then for $Q\in\mathcal{P}_\ell$ with $Q\preceq P$ we have $\mathrm{M}_{\Phi,Q}\supseteq\mathrm{M}_{\Phi,P}$; cf. Lemma 68. Since $P\in\upsilon_\Phi^{-1}(N)$, we have $\mathrm{M}_{\Phi,P} = N$. Thus we get the following chain of inclusions.

$$N\supseteq\mathrm{M}_{\Phi,Q}\supseteq\mathrm{M}_{\Phi,P} = N$$

We conclude that $\mathrm{M}_{\Phi,Q} = N$. So $Q\in\upsilon_\Phi^{-1}(N)$. □

Definition 82. Suppose given $P = (p_1,\ldots,p_u)\in\mathcal{P}_\ell$. For $s\in[1,u]$, we write $p_s =: \{p_{s,1},\ldots,p_{s,|p_s|}\}$ such that $p_{s,t} < p_{s,t+1}$ for $t\in[1,|p_s|-1]$. This means that for $s\in[1,u]$, we sort the elements in p_s in ascending order. We write $m_s := \max(p_s) = p_{s,|p_s|}$ for $s\in[1,u]$.

We define the following matrix.

$$\mathrm{D}^P_{\Lambda_\Phi} := \left(\begin{array}{cccc} \lambda_{p_{1,1},1}-\lambda_{m_1,1} & \lambda_{p_{1,1},2}-\lambda_{m_1,2} & \cdots & \lambda_{p_{1,1},k}-\lambda_{m_1,k} \\ \lambda_{p_{1,2},1}-\lambda_{m_1,1} & \lambda_{p_{1,2},2}-\lambda_{m_1,2} & \cdots & \lambda_{p_{1,2},k}-\lambda_{m_1,k} \\ \vdots & \vdots & & \vdots \\ \lambda_{p_{1,|p_1|-1},1}-\lambda_{m_1,1} & \lambda_{p_{1,|p_1|-1},2}-\lambda_{m_1,2} & \cdots & \lambda_{p_{1,|p_1|-1},k}-\lambda_{m_1,k} \\ \hline \lambda_{p_{2,1},1}-\lambda_{m_2,1} & \lambda_{p_{2,1},2}-\lambda_{m_2,2} & \cdots & \lambda_{p_{2,1},k}-\lambda_{m_2,k} \\ \vdots & \vdots & & \vdots \\ \lambda_{p_{2,|p_2|-1},1}-\lambda_{m_2,1} & \lambda_{p_{2,|p_2|-1},2}-\lambda_{m_2,2} & \cdots & \lambda_{p_{2,|p_2|-1},k}-\lambda_{m_2,k} \\ \hline \vdots & \vdots & & \vdots \\ \hline \lambda_{p_{u,1},1}-\lambda_{m_u,1} & \lambda_{p_{u,1},2}-\lambda_{m_u,2} & \cdots & \lambda_{p_{u,1},k}-\lambda_{m_u,k} \\ \vdots & \vdots & & \vdots \\ \lambda_{p_{u,|p_u|-1},1}-\lambda_{m_u,1} & \lambda_{p_{u,|p_u|-1},2}-\lambda_{m_u,2} & \cdots & \lambda_{p_{u,|p_u|-1},k}-\lambda_{m_u,k} \end{array}\right) \in R^{(\ell-u)\times k}$$

We call this matrix the *difference matrix of* Λ_Φ *with respect to (the partition)* P.

Remark 83. Keep the notation of Definition 82. We denote the j-th row of the matrix Λ_Φ by $\lambda_{j,\bullet} \in R^{1\times k}$. Then we have

$$\mathrm{D}^P_{\Lambda_\Phi} = \left(\begin{array}{c} \lambda_{p_{1,1},\bullet}-\lambda_{p_{1,|p_1|},\bullet} \\ \lambda_{p_{1,2},\bullet}-\lambda_{p_{1,|p_1|},\bullet} \\ \vdots \\ \lambda_{p_{1,|p_1|-1},\bullet}-\lambda_{p_{1,|p_1|},\bullet} \\ \hline \vdots \\ \hline \lambda_{p_{u,1},\bullet}-\lambda_{p_{u,|p_u|},\bullet} \\ \vdots \\ \lambda_{p_{u,|p_u|-1},\bullet}-\lambda_{p_{u,|p_u|},\bullet} \end{array}\right) = \left(\begin{array}{c} \lambda_{p_{1,1},\bullet}-\lambda_{m_1,\bullet} \\ \lambda_{p_{1,2},\bullet}-\lambda_{m_1,\bullet} \\ \vdots \\ \lambda_{p_{1,|p_1|-1},\bullet}-\lambda_{m_1,\bullet} \\ \hline \vdots \\ \hline \lambda_{p_{u,1},\bullet}-\lambda_{m_u,\bullet} \\ \vdots \\ \lambda_{p_{u,|p_u|-1},\bullet}-\lambda_{m_u,\bullet} \end{array}\right).$$

For each $s \in [1,u]$, we obtain a block that has in its rows the differences of the row on position $p_{s,t}$ and the row on position $p_{s,|p_s|}$ of Λ_Φ for $t \in [1, |p_s|-1]$.

Example 84. We will give examples of difference matrices as defined in Definition 82.

Let $R = \mathbb{Z}$. Suppose given Φ such that

$$\Lambda_\Phi := \begin{pmatrix} 1 & -2 & 0 & 2 \\ 2 & -1 & -1 & 0 \\ 4 & 2 & 2 & 1 \\ -1 & -1 & 0 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

So $k = 4$ and $\ell = 6$.

Let $P_1 := (p_1, p_2, p_3) := (\{1,3,4\}, \{2,5\}, \{6\})$. So $u = 3$. We obtain the following difference matrix of Λ_Φ with respect to P_1.

$$\mathrm{D}^{P_1}_{\Lambda_\Phi} = \begin{pmatrix} 1-(-1) & -2-(-1) & 0-0 & 2-(-1) \\ 4-(-1) & 2-(-1) & 2-0 & 1-(-1) \\ 2-3 & -1-1 & -1-3 & 0-(-2) \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 5 & 3 & 2 & 2 \\ -1 & -2 & -4 & 2 \end{pmatrix} \in R^{3\times 4} = R^{(6-3)\times k}.$$

Note that the matrix block corresponding to $p_3 = \{6\}$ has zero rows since $|p_3| = 1$.
Let $P_2 := (\{1,2,3,4,5,6\})$. So $u = 1$. We obtain

$$\mathrm{D}_{\Lambda_\Phi}^{P_2} = \begin{pmatrix} 1-0 & -2-0 & 0-2 & 2-2 \\ 2-0 & -1-0 & -1-2 & 0-2 \\ 4-0 & 2-0 & 2-2 & 1-2 \\ -1-0 & -1-0 & 0-2 & -1-2 \\ 3-0 & 1-0 & 3-2 & -2-2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 & 0 \\ 2 & -1 & -3 & -2 \\ 4 & 2 & 0 & -1 \\ -1 & -1 & -2 & -3 \\ 3 & 1 & 1 & -4 \end{pmatrix} \in R^{5\times 4} = R^{(6-1)\times k}.$$

Let $P_3 := (\{1\},\{2\},\{3\},\{4\},\{5\},\{6\})$. So $u = 6$. Then $\mathrm{D}_{\Lambda_\Phi}^{P_3} \in R^{0\times 4} = R^{6-6\times 4}$.

Lemma 85. *Suppose given $\alpha \in R^{k\times 1}$. Suppose given $Q \in \mathcal{P}_\ell$ such that $(\tau_\ell \circ \omega_\Phi)(\alpha) \preceq Q$. Then*

$$\mathrm{D}_{\Lambda_\Phi}^{Q} \cdot \alpha = 0.$$

Proof. We write $P := (p_1,\ldots,p_u) := (\tau_\ell \circ \omega_\Phi)(\alpha)$. We write $Q = (q_1,\ldots,q_v)$. For the sets in the tuple P, we write $p_i =: \{p_{i,1},\ldots,p_{i,|p_i|}\}$ such that $p_{i,t} < p_{i,t+1}$ for $t \in [1,|p_i|-1]$. Similarly, we write $q_j =: \left\{q_{j,1},\ldots,q_{j,|q_j|}\right\}$ such that $q_{j,t} < q_{j,t+1}$ for $t \in [1,|q_j|-1]$.
Suppose given $t \in [1,v]$. It suffices to show that we have

$$\sum_{i\in[1,k]} \alpha_i \left(\lambda_{q_{t,j},i} - \lambda_{q_{t,|q_t|},i}\right) \overset{!}{=} 0 \qquad \text{for } j \in [1,|q_t|-1]. \tag{12}$$

There exists $s \in [1,u]$ such that $q_t \subseteq p_s$. But since $(p_1,\ldots,p_u) = (\tau_\ell \circ \omega_\Phi)(\alpha)$, we have

$$\sum_{i\in[1,k]} \alpha_i \lambda_{j_1,i} = \sum_{i\in[1,k]} \alpha_i \lambda_{j_2,i} \qquad \text{for } j_1, j_2 \in p_s.$$

This implies that

$$\sum_{i\in[1,k]} \alpha_i \left(\lambda_{j_1,i} - \lambda_{j_2,i}\right) = 0 \qquad \text{for } j_1, j_2 \in p_s.$$

So we have the following equation.

$$\sum_{i\in[1,k]} \alpha_i \left(\lambda_{p_{s,y_1},i} - \lambda_{p_{s,y_2},i}\right) = 0 \qquad \text{for } y_1, y_2 \in [1,|p_s|] \tag{13}$$

Suppose given $j \in [1,|q_t|-1]$.
Then there exists $y_1 \in [1,|p_s|]$ such that $q_{t,j} = p_{s,y_1}$. Moreover, there exists $y_2 \in [1,|p_s|]$ such that $q_{t,|q_t|} = p_{s,y_2}$.
Thus by equation (13) we obtain

$$\sum_{i\in[1,k]} \alpha_i \left(\lambda_{q_{t,j},i} - \lambda_{q_{t,|q_t|},i}\right) = 0$$

as required in equation (12). □

Lemma 86. *Suppose given $Q \in \mathcal{P}_\ell$. Suppose given $\alpha \in \ker \mathrm{D}_{\Lambda_\Phi}^{Q} \subseteq R^{k\times 1}$. Then $(\tau_\ell \circ \omega_\Phi)(\alpha) \preceq Q$.*

Proof. We write $\alpha = (\alpha_i)_{i\in[1,k]}$. We write $Q = (q_1,\ldots,q_u)$. We write $P := (p_1,\ldots,p_v) := (\tau_\ell \circ \omega_\Phi)(\alpha)$. Since $\alpha \in \ker \mathrm{D}_{\Lambda_\Phi}^{Q} \subseteq R^{k\times 1}$, we have the following equation.

$$\sum_{i\in[1,k]} \alpha_i (\lambda_{q_{s,j},i} - \lambda_{q_{s,|q_s|},i}) = 0 \qquad \text{for } j \in [1,|q_s|-1],\ s \in [1,u]$$

We write $\omega_\Phi(\alpha) =: \beta = (\beta_i)_{i\in[1,\ell]} \in R^{\ell\times 1}$. By the previous equation we have the identities

$$\beta_{q_{s,j}} = \sum_{i\in[1,k]} \alpha_i(\lambda_{q_{s,j},i}) = \sum_{i\in[1,k]} \alpha_i(\lambda_{q_{s,|q_s|},i}) = \beta_{q_{s,|q_s|}} \qquad \text{for } j \in [1, |q_s|-1], \text{ for } s \in [1,u]$$

and thus we obtain

$$|\{\beta_j \mid j \in q_s\}| = 1 \qquad \text{for } s \in [1,u].$$

This shows that for $s \in [1,u]$, there exists $t \in [1,v]$ such that $q_s \subseteq p_t$ and hence $(\tau_\ell \circ \omega_\Phi)(\alpha) \preceq Q$. $\square$

Note that we do not know whether there exist $j_1, j_2 \in [1,u]$, $i \in [1,v]$ such that $j_1 \neq j_2$ and $q_{j_1} \cup q_{j_2} \subseteq p_i$. Thus we cannot conclude that $P = Q$.

Corollary 87. *Suppose given $Q \in \mathcal{P}_\ell$ and $\alpha \in R^{k\times 1}$. We have the following equivalence.*

$$\mathrm{D}^{Q}_{\Lambda_\Phi} \cdot \alpha = 0 \Longleftrightarrow (\tau_\ell \circ \omega_\Phi)(\alpha) \preceq Q$$

In particular, we have $\mathrm{D}^{(\tau_\ell\circ\omega_\Phi)(\alpha)}_{\Lambda_\Phi} \cdot \alpha = 0$.

Proof. This is a combination of Lemma 85 and Lemma 86. $\square$

Corollary 88. *Suppose given $P \in \mathcal{P}_\ell$. Then the following assertions hold.*

(1) *If $\mathrm{rk}(\mathrm{D}^{P}_{\Lambda_\Phi}) = k$, then $P \notin (\tau_\ell \circ \omega_\Phi)(R^{k\times 1} \setminus \{0\})$.*

(2) *If $\mathrm{rk}(\mathrm{D}^{P}_{\Lambda_\Phi}) < k$, then there exists $Q \in (\tau_\ell \circ \omega_\Phi)(R^{k\times 1} \setminus \{0\})$ such that $P \succeq Q$.*

Proof. Ad (1). Suppose that $P \in (\tau_\ell \circ \omega_\Phi)(R^{k\times 1} \setminus \{0\})$. Then there exists a non-zero $\alpha \in R^{k\times 1}$ such that $\mathrm{D}^{(\tau_\ell\circ\omega_\Phi)(\alpha)}_{\Lambda_\Phi} \cdot \alpha = 0$; cf. Corollary 87. Hence, in particular, $\mathrm{rk}(\mathrm{D}^{P}_{\Lambda_\Phi})$ is smaller than k.

Ad (2). If $\mathrm{rk}(\mathrm{D}^{P}_{\Lambda_\Phi}) < k$, then $\ker \mathrm{D}^{P}_{\Lambda_\Phi} \neq 0$. Choose $\alpha \in \ker(\mathrm{D}^{P}_{\Lambda_\Phi}) \setminus \{0\}$. Let $Q := (\tau_\ell \circ \omega_\Phi)(\alpha) \in (\tau_\ell \circ \omega_\Phi)(R^{k\times 1} \setminus \{0\})$. Then $P \succeq Q$; cf. Corollary 87. $\square$

Corollary 89. *We have the following inclusion of sets containing minimal elements with respect to $\preceq$.*

$$\min \left\{ P \in \mathcal{P}_\ell \,\middle|\, \mathrm{rk}(\mathrm{D}^{P}_{\Lambda_\Phi}) < k \right\} \subseteq \min \left\{ (\tau_\ell \circ \omega_\Phi)(R^{k\times 1} \setminus \{0\}) \right\}$$

Proof. We have $\left\{ (\tau_\ell \circ \omega_\Phi)(R^{k\times 1} \setminus \{0\}) \right\} \subseteq \left\{ P \in \mathcal{P}_\ell \,\middle|\, \mathrm{rk}(\mathrm{D}^{P}_{\Lambda_\Phi}) < k \right\}$ by Corollary 88.(1). Then we can apply Corollary 88.(2). $\square$

Lemma 90. *Let $s \in \mathbb{N}$. Suppose given $P, Q \in \mathcal{P}_s$. Then*

$$P \preceq Q \implies \ker \mathrm{D}^{P}_{\Lambda_\Phi} \subseteq \ker \mathrm{D}^{Q}_{\Lambda_\Phi} .$$

Proof. We write $P =: (p_1, \ldots, p_u)$ and $Q =: (q_1, \ldots, q_v)$. For $i \in [1,u]$ there exists $j \in [1,v]$ such that $q_i \subseteq p_j$ since $P \preceq Q$. Suppose given $\alpha \in \ker \mathrm{D}^{P}_{\Lambda_\Phi}$. We have to show that $\mathrm{D}^{Q}_{\Lambda_\Phi} \cdot \alpha \overset{!}{=} 0$.

Suppose given $s \in [1,v]$. By definition of $\mathrm{D}^{Q}_{\Lambda_\Phi}$ it suffices to show that

$$\sum_{i\in[1,k]} \alpha_i \left(\lambda_{x,i} - \lambda_{y,i}\right) \overset{!}{=} 0 \qquad \text{for } x, y \in q_s.$$

There exists $p_t \in [1,u]$ such that $q_s \subseteq p_t$. But we have

$$\sum_{i\in[1,k]} \alpha_i \left(\lambda_{x,i} - \lambda_{y,i}\right) = 0 \qquad \text{for } x, y \in p_t$$

since $\alpha \in \ker \mathrm{D}^{P}_{\Lambda_\Phi}$. In particular, this equation holds for $x, y \in q_s$.

This completes the proof. $\square$

Lemma 91. *We have the following description of* C_Φ*.*

$$\bigcup_{P \in \upsilon_\Phi^{-1}(N)} \ker\left(\mathrm{D}^P_{\Lambda_\Phi}\right) = \mathrm{C}_\Phi$$

Proof. Ad $\subseteq$. Suppose given $P \in \upsilon_\Phi^{-1}(N)$. It suffices to show that $\ker\left(\mathrm{D}^P_{\Lambda_\Phi}\right) \subseteq \mathrm{C}_\Phi$. Suppose given $\alpha \in \ker\left(\mathrm{D}^P_{\Lambda_\Phi}\right)$, so $\mathrm{D}^P_{\Lambda_\Phi} \cdot \alpha = 0$. Then $(\tau_\ell \circ \omega_\Phi)(\alpha) \preceq P$; cf. Lemma 86. This implies that $(\upsilon_\Phi \circ \tau_\ell \circ \omega_\Phi)(\alpha) \supseteq \upsilon_\Phi(P) = N$; cf. Lemma 68. So $(\upsilon_\Phi \circ \tau_\ell \circ \omega_\Phi)(\alpha) = N$. By Corollary 79 the map $\varphi_{\Phi,\alpha}$ is R-diagonalizable. We conclude that $\alpha \in \mathrm{C}_\Phi$.

Ad $\supseteq$. Suppose given $\alpha \in \mathrm{C}_\Phi$. Then $\varphi_{\Phi,\alpha}$ is R-diagonalizable; cf. Definition 56. By Corollary 79 we obtain that $(\upsilon_\Phi \circ \tau_\ell \circ \omega_\Phi)(\alpha) = N$. In particular, we have $P := (\tau_\ell \circ \omega_\Phi)(\alpha) \in \upsilon_\Phi^{-1}(N)$. Applying Corollary 87 shows that $\alpha \in \ker\left(\mathrm{D}^P_{\Lambda_\Phi}\right)$, completing the proof. $\square$

Theorem 92.

(1) *We have*

$$\bigcup_{P \in \max(\upsilon_\Phi^{-1}(N))} \ker\left(\mathrm{D}^P_{\Lambda_\Phi}\right) = \mathrm{C}_\Phi\,.$$

(2) *Suppose R to be infinite. There exists $P_0 \in \max(\upsilon_\Phi^{-1}(N))$ such that*

$$\mathrm{C}_\Phi = \ker\left(\mathrm{D}^{P_0}_{\Lambda_\Phi}\right).$$

Proof. Ad (1). This is a combination of Lemma 90 and Lemma 91.

Ad (2). The diagonalizability locus C_Φ of Φ is pure in $R^{k\times 1}$; cf. Lemma 57. Suppose given a partition $P \in \max(\upsilon_\Phi^{-1}(N))$. Then $\ker(\mathrm{D}^P_{\Lambda_\Phi})$ is pure in $R^{k\times 1}$; cf. Example 13.(2). Thus we obtain that $\ker(\mathrm{D}^P_{\Lambda_\Phi})$ is pure in C_Φ; cf. Remark 15.

By Lemma 23 we obtain that there exists $P_0 \in \max(\upsilon_\Phi^{-1}(N))$ such that $\mathrm{C}_\Phi = \ker\left(\mathrm{D}^{P_0}_{\Lambda_\Phi}\right)$. $\square$

Question 93. Recall that Φ is a cd-tuple on the finitely generated free R-module N; cf. Definition 51. We have the map υ_Φ; cf. Definition 65.

We ask whether $|\max(\upsilon_\Phi^{-1}(N))| = 1$.

3.5.4 Algorithm

Here we use pseudocode to describe an algorithm to determine an R-linear basis of the diagonalizability locus for a given cd-tuple Φ consisting of matrices. That is, we still use mathematical expressions in the procedure. In §3.5.5 below, we provide the Magma code of our implementation.

Algorithm 94 (Partitions Algorithm)**.** *Suppose that $\Phi = (\varphi_1, \ldots, \varphi_k)$ is a cd-tuple on R^n where $\varphi_i \in R^{n\times n}$ for $i \in [1,k]$. The following algorithm written in pseudocode returns an R-linear basis of the R-module* C_Φ*.*

$s_0 := (i \in [1,k] : \varphi_i$ is not R-diagonalizable$)$.
$\Phi_0 := (\varphi_i : i$ in $s_0)$.
$s_1 := (i \in [1,k] : \varphi_i$ is R-diagonalizable$)$.
$\Phi_1 := (\varphi_i : i$ in $s_1)$.
Define k_0 to be the length of s_0.
if Φ_0 is empty **then**
  print All elements of Φ are R-diagonalizable.
  $\mathcal{D} := (\mathrm{e}_i \in R^k : i \in [1,k])$.
else

Choose $S \in \mathrm{GL}_n(K) \cap R^{n\times n}$ such that $S\varphi S^{-1}$ is diagonal for $\varphi \in \Phi_0$.
$D_i := S\varphi_i S^{-1}$ for i in s_0.
Define $\widetilde{\Lambda}_{\Phi_0} := (\widetilde{\lambda}_{j,i})_{j\in[1,n],\, i\in[1,k]} \in R^{n\times k}$ such that $\widetilde{\lambda}_{j,i} = (D_i)_{j,j}$.
$\Lambda_{\Phi_0} := \mathrm{RemoveDuplicateRows}(\widetilde{\Lambda}_{\Phi_0})$.
$\ell_0 := \mathrm{NumberOfRows}(\Lambda_{\Phi_0})$.
$P := (1 : i \in [1, \ell_0]) \in \mathcal{I}_{\ell_0}$.
$L_{\mathrm{eff}} := [\,]$.
$L_{\mathrm{bad}} := [\,]$.
while $P \neq (1, 2, ..., \ell_0)$ **do**
  if $\exists\, Q \in L_{\mathrm{bad}} : P \succeq Q$ **then**
    Skip P.
  else
    if $\mathrm{rk}(\mathrm{D}^P_{\Lambda_{\Phi_0}}) < k_0$ **then**
      if $\mathrm{M}_{\Phi_0,P} = R^n$ **then**
        Remove all elements from L_{eff} that are coarser than P.
        Append P to L_{eff}.
      else
        Append P to L_{bad}.
      end if
    end if
  end if
  $P := \min_{\leq_{\mathrm{lex}}} \{Q \in \mathcal{I}_{\ell_0} \,|\, P \leq_{\mathrm{lex}} Q \text{ and } P \neq Q\}$.
end while
Choose an R-linear basis $\mathcal{B}'$ of ${}_R\langle \ker(\mathrm{D}^P_{\Lambda_{\Phi_0}}) : P \text{ in } L_{\mathrm{eff}}\rangle$.
Let $s_0 =: (i_1, \ldots, i_{k_0})$. Define the R-linear map $\xi\colon R^{k_0} \to R^k$ by $\mathrm{e}_j \mapsto \mathrm{e}_{i_j}$ for $j \in [1, k_0]$.
$\mathcal{B} := (\xi(b') \in R^{k_0} : b' \text{ in } \mathcal{B}')$.
$\mathcal{C} := (\mathrm{e}_i \in R^k : i \text{ in } s_1)$.
Define $\mathcal{D}$ as the concatenation of $\mathcal{B}$ and $\mathcal{C}$.
end if
return $\mathcal{D}$.

Proof. We have to show that

$$\mathrm{C}_\Phi \overset{!}{=} {}_R\langle \mathcal{D}\rangle.$$

Claim 1. If $\mathcal{B}'$ is an R-linear basis of C_{Φ_0}, then $\mathcal{D}$ is an R-linear basis of C_Φ.

Note that both $\mathcal{B}$ and $\mathcal{C}$ are linearly independent. Moreover, the non-zero entries of elements of $\mathcal{B}$ only are in positions where the elements of $\mathcal{C}$ have zeros and vice versa. So $\mathcal{D}$ is linearly independent.

It remains to show that $\mathrm{C}_\Phi \overset{!}{=} {}_R\langle \mathcal{D}\rangle$. But since $(\mathrm{e}_i \in R^{k-k_0} : i \in [1, k-k_0])$ is an R-linear basis of C_{Φ_1}, the tuple $\mathcal{D}$ is a generating set of C_Φ as we see using Remark 59 after reordering the elements of Φ.

This *proves Claim 1.*

By Claim 1, it suffices to show that

$$\mathrm{C}_{\Phi_0} \overset{!}{=} {}_R\langle \mathcal{B}'\rangle = {}_R\langle \ker(\mathrm{D}^P_{\Lambda_{\Phi_0}}) : P \text{ in } L_{\mathrm{eff}}\rangle.$$

Note that L_{eff} only depends on Φ_0.

So for the rest of this proof we may assume that Φ_1 is the empty tuple and thus $\Phi = \Phi_0$. Hence the map ξ is the identity map on R^k, the tuple $\mathcal{C}$ is empty and $\mathcal{B} = \mathcal{B}' = \mathcal{D}$. Then we also have $k = k_0$. Furthermore, let ℓ be the number of rows of the matrix Λ_Φ. Then also $\ell = \ell_0$.

So we need to show that
$$\mathrm{C}_\Phi \overset{!}{=} {}_R\langle \mathcal{B}\rangle = {}_R\langle \ker(\mathrm{D}^P_{\Lambda_\Phi}) : P \text{ in } L_{\text{eff}}\rangle.$$

Ad $\supseteq$. It suffices to show that for P in the list L_{eff}, we have $\ker(\mathrm{D}^P_{\Lambda_\Phi}) \subseteq \mathrm{C}_\Phi$. So suppose given $P \in \mathcal{I}_\ell$ that is an element of L_{eff}. Suppose given $\alpha = (\alpha_i)_{i\in[1,k]} \in \ker(\mathrm{D}^P_{\Lambda_\Phi})$. We have to show that the R-endomorphism $\sum_{i\in[1,k]} \alpha_i\varphi_i$ is R-diagonalizable.

By construction of L_{eff} we have $\mathrm{M}_{\Phi,P} = R^n$. Moreover, we know that $(\tau_\ell \circ \omega_\Phi)(\alpha) \preceq P$; cf. Lemma 86.

> This means that, in a sense, if one defines a partition $(\tau_\ell \circ \omega_\Phi)(\alpha)$ that indicates how the simultaneous eigenmodules of the endomorphisms of Φ have fused to the eigenmodules of $\varphi_{\Phi,\alpha}$, this partition is coarser than or equal to P. So in a sense, one can say that $\varphi_{\Phi,\alpha}$ shows "at least the fusion behavior P".

But from $(\tau_\ell \circ \omega_\Phi)(\alpha) \preceq P$ we conclude that $\mathrm{M}_{\Phi,(\tau_\ell\circ\omega_\Phi)(\alpha)} \supseteq \mathrm{M}_{\Phi,P}$; cf. Lemma 68. We already have $\mathrm{M}_{\Phi,P} = R^n$, so we conclude that $\mathrm{M}_{\Phi,(\tau_\ell\circ\omega_\Phi)(\alpha)} = R^n$.

This shows that $\varphi_{\Phi,\alpha}$ is R-diagonalizable; cf. Corollary 79.

Ad $\subseteq$. Suppose given $\alpha = (\alpha_i)_{i\in[1,k]} \in \mathrm{C}_\Phi \setminus \{0\}$. Then $\sum_{i\in[1,k]} \alpha_i\varphi_i$ is R-diagonalizable.

We write
$$P := (\tau_\ell \circ \omega_\Phi)(\alpha)$$
which we view as an element of $\mathcal{I}_\ell$; cf. Remark 72.

Consider the while-loop that iterates over $\mathcal{I}_\ell$, ordered lexicographically. At some point, the loop reaches the partition P.

Claim 2. P is not skipped in the first if-statement in the while-loop under consideration

Assume that P is skipped in the first if-statement. Then there exists $Q \in L_{\text{bad}}$ such that $P \succeq Q$. Since $Q \in L_{\text{bad}}$, we have $\mathrm{M}_{\Phi,Q} \subset R^n$. Since $\varphi_{\Phi,\alpha}$ is R-diagonalizable, we have $\mathrm{M}_{\Phi,P} = R^n$; cf. Corollary 79. Using Lemma 68, we obtain $R^n = \mathrm{M}_{\Phi,P} \subseteq \mathrm{M}_{\Phi,Q} \subset R^n$, a *contradiction*.

This *proves Claim 2.*

Since $P \in (\tau_\ell \circ \omega_\Phi)(R^{k\times 1} \setminus \{0\})$, we have $\mathrm{rk}(\mathrm{D}^P_{\Lambda_\Phi}) < k$; cf. Corollary 88.(1).

Since $\varphi_{\Phi,\alpha}$ is R-diagonalizable, we have $\mathrm{M}_{\Phi,P} = R^n$; cf. Corollary 79. We conclude that P is appended to L_{eff}.

We proceed by case distinction.

Case 1. At the end of the algorithm, P is still an element of L_{eff}.

We have $\alpha \in \ker(\mathrm{D}^P_{\Lambda_\Phi})$; cf. Corollary 87.

Case 2. At the end of the algorithm, P is no longer an element of L_{eff}.

The only way to remove an element of the list L_{eff} is to replace it with a finer partition. So at the end of the algorithm there exist $s \geq 1$ and a chain $Q_s \succeq Q_{s-1} \succeq \ldots \succeq Q_1 \succeq P$ where $Q_i \in \mathcal{I}_\ell$ for $i \in [1,s]$ such that Q_s is a member of L_{eff}.

But then $\alpha \in \ker \mathrm{D}^P_{\Lambda_\Phi} \subseteq \ker \mathrm{D}^{Q_s}_{\Lambda_\Phi}$; cf. Corollary 90. □

3.5.5 Magma Code

In the following we provide the Magma codes including certain definitions and functions and our implementation of the Partitions Algorithm; cf. Algorithm 94. The file "partalgo" requires initialization files such as "pre", "z3s3Init1" and "definitions"; cf. Magma Codes 3, 1, 4 and 5.

The files provided here are used in the examples $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ and $\mathbb{Z}_{(2)}\,\mathrm{S}_5$; cf. §6.4 and §7.4.

Magma Code 3: pre

```
Z := Integers();
Q := Rationals();

kernel := function(A)
// INPUT: integer matrix A
// OUTPUT: integer matrix B that has in its columns a Z-linear basis
// of ker(A)
   n := NumberOfColumns(A);
   D,S,T := SmithForm(A);
   l := #[x : x in Diagonal(D) | x ne 0];
   return SubmatrixRange(T,1,l+1,n,n);
end function;

sim_diag := function(L)
// INPUT: list L of commuting Q-diagonalizable square matrices over Q
// of the same size
// OUTPUT: matrix S such that S*L[i]*S^-1 is diagonal for i in [1..#L]
   size := NumberOfRows(L[1]);
   blocks := [1, size+1];
   J,S := JordanForm(L[1]);
   D := Diagonal(S*L[1]*S^-1);
   for j in [i : i in [2..size] | D[i-1] ne D[i] and not i in blocks] do
      Include(~blocks,j);
   end for;
   Sort(~blocks);
   for i in [2 .. #L] do
      Matrix := S*L[i]*S^-1;
      T := <>;
      for i in [1 .. #blocks - 1] do
         J,U := JordanForm(SubmatrixRange(Matrix,blocks[i],blocks[i],
            blocks[i+1]-1,blocks[i+1]-1));
         Append(~T,U);
      end for;
      S := DiagonalJoin(T)*S;
      D := Diagonal(S*L[i]*S^-1);
      for j in [i : i in [2..size] | D[i-1] ne D[i] and not i in blocks]
         do
         Include(~blocks,j);
      end for;
      Sort(~blocks);
   end for;
   return S;
end function;

intmatrix := function(A)
// INPUT: matrix A
// OUTPUT: integer matrix A where every column is multiplied by its lcm
   noc := NumberOfColumns(A);
   nor := NumberOfRows(A);
   for i in [1..noc] do
      lcm := LCM([Denominator(A[j,i]) : j in [1..nor]]);
      if lcm ne 1 then
```

```
         MultiplyColumn(~A,lcm,i);
      end if;
   end for;
   return RMatrixSpace(Z,nor,noc)!A;
end function;
```

Magma Code 4: definitions

```
Gamma := CartesianProduct([MatrixRing(Z,i) : i in Sizes]);
QGamma := CartesianProduct([MatrixRing(Q,i) : i in Sizes]);
sp := [&+([0] cat [Sizes[i]^2 : i in [1..j-1]]) + 1 : j in [1..nb]];
// starting positions of the blocks
ep := [&+[Sizes[i]^2 : i in [1..j]] : j in [1..nb]];
// ending positions of the blocks

CoerceGamma := function(v);
// INPUT: tuple v with rl entries in R
// OUTPUT: element of Gamma that has the entries of v in its matrices.
// matrices are filled row-wise
   return Gamma!<MatrixRing(Z,Sizes[j])![v[i] : i in [sp[j]..ep[j]]] : j
      in [1..nb]>;
end function;

ProdTup := function(x);
// INPUT: nonempty list x of tuples of the same format
// OUTPUT: product of all elements in x
   return < &*[x[j][i] : j in [1..#x]] : i in [1..#(x[1])] >;
end function;

AddTup := function(x);
// INPUT: nonempty list x of tuples of the same format
// OUTPUT: sum of all elements in x
   return &+[x[j] : j in [1..#x]];
end function;

SubTup := function(x,y);
// INPUT: tuples x,y of the same format
// OUTPUT: difference x-y
   return < x[i] - y[i] : i in [1..#x] >;
end function;

InvTup := function(x);
// INPUT: tuple of invertible rational matrices
// OUTPUT: multiplicative inverse of x
   return < x[i]^-1 : i in [1..#x] >;
end function;

ConjTup := function(x,s);
// INPUT: tuple x of rational matrices and tuple s of invertible rational
// matrices
// OUTPUT: conjugate tuple inv(s)*x*s
   return ProdTup([InvTup(s),x,s]);
end function;
```

```
LieTup := function(x,y);
// INPUT: tuples x,y of the same format
// OUTPUT: Lie bracket [x,y] = xy - yx as tuple
   return SubTup(ProdTup([x,y]),ProdTup([y,x]));
end function;

ScalMultTup := function(x,c)
// INPUT: tuple x of matrices and scalar value c
// OUTPUT: tuple c*x
   return < x[i] * Parent(x[i])!c : i in [1..#x]>;
end function;

Ties_to_Basis := function(Ties,e);
// INPUT: a matrix Ties and a divisor e
// OUTPUT: calculates a matrix that has in its columns an R-linear basis
// of the module generated by the ties modulo e in the rows of Ties
   D,S,T := SmithForm(Ties);
   k := Rank(D);
   return T * DiagonalMatrix([Z!(e/Gcd(D[i,i],e)) : i in [1..k]] cat [1 :
        i in [k+1..rl]]);
end function;

Basis_Omega := Ties_to_Basis(Ties_Omega,e);
invBasis_Omega := (RMBQ!Basis_Omega)^-1;
Omeganull := CoerceGamma([0: i in [1..rl]]);

admatrix := function(x)
// INPUT: x an element of Gamma
// OUTPUT: matrix ad(x) with respect to the basis Basis_Omega
   pre_ad := RMBQ!0;
   for j in [1..rl] do
      v_tup := LieTup(x,CoerceGamma([Basis_Omega[i,j] : i in [1..rl]]));
      v_vec := &cat[ElementToSequence(v_tup[i]):i in [1..#Sizes]];
      for i in [1..rl] do
         pre_ad[i,j] := v_vec[i];
      end for;
   end for;
   return RMB!((RMBQ!Basis_Omega)^-1*pre_ad);
end function;

rdiag := function(ad,p)
// INPUT: ad a square Integer Matrix
//   p an integer prime number or 0
// OUTPUT: boolean if ad is diagonalizable over Z localized at p
//   (over Q if p=0)
//   if no then the reason is printed
   nor := NumberOfRows(ad);
   eigval_set := Eigenvalues(Transpose(ad));
   if not &+([0] cat [Dimension(Eigenspace(Transpose(ad),x[1])) : x in
      eigval_set]) eq nor then
      print "Not␣diagonalizable␣over␣Q";
      return false;
   end if;
```

```
   if p eq 0 then
      return true;
   end if;
   // matrix over ground ring is required for SmithForm
   R := MatrixRing(Z,nor)!0;
   i := 1;
   for x in eigval_set do
      Espace := Eigenspace(Transpose(ad),x[1]);
      D,S,T := SmithForm(Transpose(intmatrix(BasisMatrix(Espace))));
      invS := S^-1;
      for j in [1..x[2]] do
         for k in [1..nor] do
            R[k,i] := invS[k,j];
         end for;
         i := i+1;
      end for;
   end for;
   if Valuation(Determinant(R),p) eq 0 then
      return true;
   else
      // print "Valuation at", p, "equals ", Valuation(Determinant(R),p);
      return false;
   end if;
end function;
```

Magma Code 5: partalgo

```
ispartition := function(p);
// INPUT: integer sequence p of length n
// OUTPUT: boolean whether p can be interpreted as a partition of [1..n]
   return (p[1] eq 1) and &and[p[i] in [1 .. (Max([p[j] : j in [1,i-1]])
      +1)] : i in [2..#p]];
end function;

increasepartition := function(p);
// INPUT: integer sequence p of length n, interpreted as a
//  partition of [1..n]
// OUTPUT: integer sequence p of length n, interpreted as a
//  partition of [1..n] which is input partition increased by 1.
// Maximal partition is returned unchanged.
   s := #p;
   active := true;
   while s ge 1 and active do
      if p[s] lt Max([0] cat [p[i] : i in [1..s-1]])+1 then
         p[s] +:= 1;
         active := false;
      else
         p[s] := 1;
         s +:= -1;
      end if;
   end while;
   return p;
end function;
```

```
diffmatrix := function(A,p);
// INPUT: integer matrix A of size r x k, partition p of [1..r]
// OUTPUT: difference matrix of A with respect to the partition p
   k := NumberOfColumns(A);
   max := Max(p);
   C := RMatrixSpace(Z, 1, k)!0;
   occ := [[i : i in [1 .. #p] | p[i] eq j] : j in [1 .. max]];
   for i in [1 .. max] do
      if #occ[i] gt 1 then
         B := Submatrix(A, occ[i], [1 .. k]);
         for j in [1 .. #occ[i] - 1] do
            AddRow(~B, -1, #occ[i], j);
         end for;
         C := VerticalJoin(C,RemoveRow(B,#occ[i]));
      end if;
   end for;
   return RemoveRow(C,1);
end function;

isfiner := function(p,q);
// INPUT: integer sequences p, q of the same length n, interpreted
// as partitions of [1..n]
// OUTPUT: boolean whether p is a finer partition than q
   maxp := Max(p);
   maxq := Max(q);
   if maxp le maxq then
      return p eq q;
   end if;
   n := #p;
   return &and[#{q[t] : t in [1..n] | p[t] eq s} eq 1 : s in [1..maxp]];
end function;

Partalgo := function(Phi);
// INPUT: List Phi of commuting rational square matrices of the same
// sizes each of which is diagonalizable over Q
// OUTPUT: matrix that has in its rows an Z_(p)-linear basis of C_Phi
   k0 := #Phi;

   // Check whether the requirements on Phi are fulfilled or not
   if #SequenceToSet([Parent(Phi[i]) : i in [1..k0]]) ne 1 then
      print "Error:␣Elements␣in␣the␣given␣list␣do␣not␣have␣the␣same␣
         parents.";
      return false;
   end if;
   if NumberOfRows(Phi[1]) ne NumberOfColumns(Phi[1]) then
      print "Error:␣Elements␣are␣no␣square␣matrices.";
      return false;
   end if;
   JF := [JordanForm(Phi[i]) : i in [1..k0]];
   if not &and[JF[i] eq DiagonalMatrix(Q,#Diagonal(JF[i]),Diagonal(JF[i])
      ) : i in [1..k0]] then
      print "Error:␣Not␣all␣matrices␣are␣diagonalizable␣over␣Q.";
      return false;
```

```
end if;
if not &and[Phi[i] * Phi[j] eq Phi[j] * Phi[i] : i in [1..k0], j in
   [1..k0] | i gt j] then
   print "Error:␣Not␣all␣pairs␣of␣matrices␣are␣commuting.";
   return false;
end if;

rdiagendos := [];
for i in [1..#Phi] do
   if rdiag(Phi[i],prime) then
      Include(~rdiagendos,i);
   end if;
end for;
nonrdiagendos := [i : i in [1..k0] | not i in rdiagendos];

if #nonrdiagendos eq 0 then
   print "All␣given␣matrices␣are␣diagonalizable␣over␣Z␣localized␣at",
      prime,".";
   print "The␣diagonalizability␣locus␣is␣spanned␣by␣the␣columns␣of␣the
      ␣following␣matrix.";
   K2 := RMatrixSpace(Z,k0,k0)!0;
   for i in [1 .. k0] do
      K2[i,rdiagendos[i]] := 1;
   end for;
   print K2;
   return true;
end if;

Phi := [Phi[i] : i in nonrdiagendos];
k := #Phi;
m := NumberOfRows(Phi[1]);
S := sim_diag(Phi);
invS := S^-1;
Evalues := [Diagonal(S*M*S^-1) : M in Phi];

tupEvalues := [[Evalues[i][j] : i in [1..#Phi]] : j in [1..m]];
matEvalues := Matrix(Z, #tupEvalues, #tupEvalues[1], [<i,j,tupEvalues[
   i][j]> : i in [1..#tupEvalues], j in [1..#tupEvalues[1]]]);

// in the rows of S^-1 we have a K-linear basis of the simultaneous
// eigenspaces of the matrices.
// we need to bring each basis vector into Z^m
invS := RMatrixSpace(Z,m,m)!intmatrix(invS);

// Remove duplicate rows of matEvalues, store the result in A
A := Submatrix(matEvalues, [i : i in [1..NumberOfRows(matEvalues)] | &
   and[matEvalues[i] ne matEvalues[j] : j in [1..i-1]]],[1..
   NumberOfColumns(matEvalues)]);

l := NumberOfRows(A);
Belllist := [1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570,
   4213597, 27644437, 190899322, 1382958545, 10480142147];
```

```
if l le 16 then
   print "l␣=␣",l,",␣there␣are", Belllist[l],"partitions␣to␣check.";
end if;

// start with the coarsest partition
partition := [1 : i in [1..l]];
running := partition[l] ne l;

Listeff := [];
Listbad := [];

while running do
   skip_partition := false;
   ip := 1;
   while not (skip_partition or (ip eq (#Listbad + 1))) do
      skip_partition := isfiner(partition,Listbad[ip]);
      ip +:= 1;
   end while;
   if not skip_partition then
      if Rank(diffmatrix(A,partition)) lt k then
      // if Rank = k then we can skip the partition
         occ := [[i : i in [1 .. #partition] | partition[i] eq j]:j in
              [1 .. Max(partition)]];
         cols := <>;
         for tup in occ do
            colindices := [s : s in [1..rl] | &or[matEvalues[s] eq A[i
               ] : i in tup]];
            D,U,T := SmithForm(Submatrix(invS,[1..rl],colindices));
            Append(~cols,ColumnSubmatrix(U^-1,#colindices));
         end for;

         if Valuation(Determinant(HorizontalJoin(cols)),prime) eq 0
            then
            for P in Listeff do
               if isfiner(partition,P) then
                  Exclude(~Listeff,P);
               end if;
            end for;
            Include(~Listeff, partition);
         else
            Include(~Listbad, partition);
         end if;
      end if;
   end if;

   running := partition[l] ne l;
   // we are done if the last entry is l
   if running then
      partition := increasepartition(partition);
   end if;
end while;
print "List␣of␣finest␣partitions␣contains", #Listeff, "element(s).";
```

```
  if #Listeff ge 1 or #rdiagendos ge 1 then
    basis_list := RMatrixSpace(Z,k0,0)!0;

    for partition in Listeff do
      B := Transpose(KernelMatrix(Transpose(diffmatrix(A,partition))))
        ;

      B1 := RMatrixSpace(Z,k0,NumberOfColumns(B))!0;
      for i in [1..k] do
        for j in [1 .. NumberOfColumns(B)] do
          B1[nonrdiagendos[i]][j] := B[i][j];
        end for;
      end for;
      basis_list := HorizontalJoin(basis_list,B1);
    end for;

    B2 := RMatrixSpace(Z,k0,k0-k)!0;
    for i in [1 .. #rdiagendos] do
      B2[rdiagendos[i],i] := 1;
    end for;
    if #Listeff eq 0 then
      print "There␣is␣no␣non-trivial␣linear␣combination␣of␣the␣given␣
        matrices␣that␣is␣Z_(p)-diagonalizable.";
    else
      print "Partitions␣in␣L_eff:";
      print Listeff;
    end if;
    // The diagonalizability locus is spanned by the
    // columns of the following matrix.
    B := HorizontalJoin(basis_list,B2);
    D,S,T := SmithForm(B);
    print "A␣Z_(p)-linear␣basis␣of␣the␣diagonalizability␣locus␣is␣given
      ␣by␣the␣columns␣of␣the␣following␣matrix.";
    return ColumnSubmatrixRange(S^-1*D,1,Rank(D));
  else
    print "There␣is␣only␣the␣zero␣solution.";
    print "C_Phi␣is␣the␣zero␣module";
    return 0;
  end if;
end function;
```

Chapter 4: Tori

Let R be a principal ideal domain with the maximal ideal (π). Let $K = \operatorname{frac} R$ be the field of fractions of R. We often abbreviate $\otimes_R$ by $\otimes$.

4.1 Maximal tori of Lie algebras over R

Let $\mathfrak{g}$ be a Lie algebra over R that is finitely generated free as an R-module. Let $n = \operatorname{rk}_R \mathfrak{g}$.

Remark 95. *Recall that*

$$K\mathfrak{g} = K \otimes \mathfrak{g} = \left\{ \sum_{i\in[1,k]} c_i \otimes g_i \;\middle|\; k \in \mathbb{N}_0,\ c_i \in K,\ g_j \in \mathfrak{g} \text{ for } i,j \in [1,k] \right\}.$$

We can shorten this description to

$$K\mathfrak{g} = \left\{ \frac{1}{s} \otimes g \;\middle|\; s \in R^\times,\ g \in \mathfrak{g} \right\}.$$

Moreover, for $s, s' \in R^\times$ and $g, g' \in \mathfrak{g}$, we have in $K\mathfrak{g}$, that

$$\frac{1}{s} \otimes g = \frac{1}{s'} \otimes g' \iff s'g = sg'.$$

Proof. It suffices to show $\subseteq$. Suppose given an element $g = \sum_{i\in[1,k]} c_i \otimes g_i \in K \otimes \mathfrak{g}$. Then we can write $g = \sum_{i\in[1,k]} \frac{r_i}{s_i} \otimes g_i$, where $r_i \in R$, $s_i \in R^\times$ for $i \in [1,k]$. Define s as a common denominator of these fractions (e.g. $s := s_1 s_2 \cdots s_k$). Then $g = \sum_{i\in[1,k]} \frac{r_i'}{s} \otimes g_i$ for certain $r_i' \in R$. So we get $g = \frac{1}{s} \otimes \sum_{i\in[1,k]} r_i' g_i$ where $\sum_{i\in[1,k]} r_i' g_i \in \mathfrak{g}$.
Thus we have

$$K\mathfrak{g} = \left\{ \frac{1}{s} \otimes g \;\middle|\; s \in R^\times,\ g \in \mathfrak{g} \right\}.$$

For the second claim, we have to show two implications.
Ad $\Longleftarrow$. We have $\frac{1}{s} \otimes g = \frac{1}{s}\frac{1}{s'} \otimes s'g = \frac{1}{s}\frac{1}{s'} \otimes sg' = \frac{1}{s'} \otimes g'$.
Ad $\Longrightarrow$. Suppose that $\frac{1}{s} \otimes g = \frac{1}{s'} \otimes g'$ in $K\mathfrak{g}$. We multiply by ss' from the left and we obtain $ss'(\frac{1}{s} \otimes g) = ss'(\frac{1}{s'} \otimes g')$. The left side calculates to $ss'(\frac{1}{s} \otimes g) = s'(1 \otimes g) = 1 \otimes s'g$ and for the right side, we have $ss'(\frac{1}{s'} \otimes g') = s(1 \otimes g') = 1 \otimes sg'$. Thus we get $1 \otimes s'g = 1 \otimes sg'$. Since the embedding $\mathfrak{g} \to K\mathfrak{g}$ that maps an element $g \in \mathfrak{g}$ to $1 \otimes g$ is injective, we conclude that $s'g = sg'$. $\square$

Definition 96. Recall that $K\mathfrak{g} = K \otimes \mathfrak{g}$ is a K-vector space. We define a multiplication on $K\mathfrak{g}$ by

$$\begin{aligned} [-,=] \colon K\mathfrak{g} \times K\mathfrak{g} &\to K\mathfrak{g} \\ \left(\frac{1}{s} \otimes g, \frac{1}{t} \otimes h\right) &\mapsto \frac{1}{st} \otimes [g,h] =: [s \otimes g, t \otimes h] \end{aligned}$$

This operation is well-defined:
Suppose given $\frac{1}{s} \otimes g = \frac{1}{s'} \otimes g'$ and $\frac{1}{t} \otimes h = \frac{1}{t'} \otimes h'$ with $s, s', t, t' \in R^\times$ and $g, g', h, h' \in \mathfrak{g}$. Then $sg' = s'g$ and $th' = t'h$; cf. Remark 95. We have to show that

$$\left[\frac{1}{s} \otimes g, \frac{1}{t} \otimes h\right] = \frac{1}{st} \otimes [g,h] \overset{!}{=} \frac{1}{s't'} \otimes [g',h'] = \left[\frac{1}{s'} \otimes g', \frac{1}{t'} \otimes h'\right],$$

i.e. we have to show that

$$s't'[g,h] \stackrel{!}{=} st[g',h']$$

But $s't'[g,h] = [s'g, t'h] = [sg', th'] = st[g',h']$.

Since $\mathfrak{g}$ is a Lie algebra over R, this operation satisfies $[\frac{1}{s} \otimes g, \frac{1}{s} \otimes g] = \frac{1}{s^2}[g,g] = 0$ for $s \in R^\times$ and $g \in \mathfrak{g}$. Similarly, it satisfies the Jacobi identity.

Thus $K\mathfrak{g}$ becomes a Lie algebra over K.

Definition 97 (cf. [Kün01, Definition 82])**.** A Lie subalgebra $\mathfrak{t} \subseteq K\mathfrak{g}$ is said to be a *torus* in $K\mathfrak{g}$ if the adjoint endomorphism $\mathrm{ad}_{K\mathfrak{g}}\, t \in \mathfrak{gl}(K\mathfrak{g})$ is diagonalizable over K for $t \in \mathfrak{t}$.

A torus $\mathfrak{t} \subseteq K\mathfrak{g}$ is said to be a *maximal torus* if $\mathfrak{t} \subseteq \mathfrak{t}' \subseteq K\mathfrak{g}$ and $\mathfrak{t}'$ is a torus in $K\mathfrak{g}$ implies that $\mathfrak{t} = \mathfrak{t}'$.

Since $0 \subseteq K\mathfrak{g}$ is a torus, there always exists a maximal torus for dimensional reasons.

We define a torus of the Lie algebra $\mathfrak{g}$ over R similar to Definition 97.

Definition 98. A Lie subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is said to be a *rational torus* in $\mathfrak{g}$ if the adjoint endomorphism

$$\begin{aligned} \mathrm{ad}_{\mathfrak{g}}\, t \colon \mathfrak{g} &\longrightarrow \mathfrak{g} \\ y &\longmapsto [t,y] \end{aligned}$$

is diagonalizable over K for $t \in \mathfrak{t}$; cf. Definition 38.

A rational torus $\mathfrak{t} \subseteq \mathfrak{g}$ is said to be a *maximal rational torus* if for all rational tori $\mathfrak{t}' \subseteq \mathfrak{g}$ we have that $\mathfrak{t} \subseteq \mathfrak{t}' \subseteq \mathfrak{g}$ implies that $\mathfrak{t} = \mathfrak{t}'$.

A Lie subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is said to be an *integral torus (over R)* in $\mathfrak{g}$ if the adjoint endomorphism

$$\begin{aligned} \mathrm{ad}_{\mathfrak{g}}\, t \colon \mathfrak{g} &\longrightarrow \mathfrak{g} \\ y &\longmapsto [t,y] \end{aligned}$$

is diagonalizable over R for $t \in \mathfrak{t}$; cf. Definition 39.

An integral torus $\mathfrak{t} \subseteq \mathfrak{g}$ is said to be a *maximal integral torus* if for all integral tori $\mathfrak{t}' \subseteq \mathfrak{g}$ we have that $\mathfrak{t} \subseteq \mathfrak{t}' \subseteq \mathfrak{g}$ implies that $\mathfrak{t} = \mathfrak{t}'$.

Remark 99. The following remarks are immediate consequences of Definition 98.

(1) Every integral torus in $\mathfrak{g}$ is a rational torus in $\mathfrak{g}$.

(2) Since R is a Noetherian ring, $\mathfrak{g}$ is Noetherian as an R-module. Thus it satisfies the ascending chain condition on its submodules. We conclude that in $\mathfrak{g}$, every rational torus is contained in a maximal rational torus and each integral torus is contained in a maximal integral torus.

(3) The zero Lie algebra $0 \subseteq \mathfrak{g}$ is an integral torus in $\mathfrak{g}$. Thus there always exists a maximal integral torus in $\mathfrak{g}$ contained in a maximal rational torus in $\mathfrak{g}$.

(4) We recall the definition of pure submodules and the pure closure of a submodule; cf. Definition 12. For a rational torus $\mathfrak{t}$ in $\mathfrak{g}$, the pure closure $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{t})$ of $\mathfrak{t}$ in $\mathfrak{g}$ is given by

$$\mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}) = \left\{ g \in \mathfrak{g} \,\middle|\, \exists r \in R^\times : rg \in \mathfrak{t} \right\}.$$

Moreover, $\mathfrak{t}$ is pure in $\mathfrak{g}$ if $\mathfrak{t} = \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t})$.

Example 100. Let $\mathfrak{g} := \mathfrak{gl}_2(\mathbb{Z})$ and $\mathfrak{t} := {}_{\mathbb{Z}}\langle \left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 7 \end{smallmatrix}\right) \rangle \subseteq \mathfrak{g}$. We write $\mathcal{E}$ for the standard basis $\mathcal{E}_{2,2} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right)$ of $\mathfrak{gl}_2(\mathbb{Z})$.

Suppose given $t \in \mathfrak{t}$. Then there exist $x, y \in R$ such that $t = \left(\begin{smallmatrix} 2x & 0 \\ 0 & 7y \end{smallmatrix}\right)$. We get

$$\mathrm{ad}_{\mathfrak{g}}(t)_{\mathcal{E},\mathcal{E}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2x-7y & 0 & 0 \\ 0 & 0 & 7y-2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is diagonal, in particular it is diagonalizable over R.

This shows that $\mathfrak{t}$ is an integral torus in $\mathfrak{g}$.

For the pure closure of $\mathfrak{t}$ in $\mathfrak{g}$, we get

$$\begin{aligned} \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{g} \;\middle|\; \exists r \in \mathbb{Z}^\times : r \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in {}_{\mathbb{Z}}\langle \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 7 \end{pmatrix} \rangle \right\} \\ &= {}_{\mathbb{Z}}\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle. \end{aligned}$$

So we have proper inclusions $\mathfrak{t} \subset \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}) \subset \mathfrak{g}$. This shows that $\mathfrak{t}$ is not pure in $\mathfrak{g}$.

Remark 101. *There exist a discrete valuation ring R, an R-algebra A and an integral torus $\mathfrak{t}$ in the Lie algebra $\mathfrak{l}(A)$ over R such that $\mathfrak{t}$ is not an R-subalgebra of A.*

Proof. Suppose that $R := \mathbb{Z}_{(3)}$. Consider the R-algebra $A := R^{3\times 3}$. We have the commutator Lie algebra $\mathfrak{l}(R^{3\times 3}) = \mathfrak{gl}_3(R) =: \mathfrak{g}$.

Define the elements $t_1 := \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} \in \mathfrak{g}$ and $t_2 := \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \in \mathfrak{g}$. We have the finitely generated free R-module $\mathfrak{t} := {}_R\langle t_1, t_2 \rangle$ in $\mathfrak{g}$. Then $\mathfrak{t}$ is an abelian Lie subalgebra of $\mathfrak{g}$.

Let $\mathcal{E} := \mathcal{E}_{3,3}$ be the standard basis of $\mathfrak{g}$.

Suppose given $i, j \in [1, 3]$. Then $E_{i,j}$ is an element of $\mathcal{E}$. The product $t_1 E_{i,j}$ is either $E_{i,j}$ or 0 since t_1 is a diagonal matrix with 0 and 1 on its diagonal. Similarly we have $E_{i,j} t_1 \in \{E_{i,j}, 0\}$. We conclude that $[t_1, E_{i,j}] \in \{0, E_{i,j}, -E_{i,j}\}$.

Similarly we obtain that $[t_2, E_{i,j}] \in \{0, E_{i,j}, -E_{i,j}\}$.

Therefore the matrices $(\mathrm{ad}_{\mathfrak{g}}(t_1))_{\mathcal{E},\mathcal{E}}$ and $(\mathrm{ad}_{\mathfrak{g}}(t_2))_{\mathcal{E},\mathcal{E}}$ are diagonal. In particular, the R-linear maps $\mathrm{ad}_{\mathfrak{g}}(t_1)$ and $\mathrm{ad}_{\mathfrak{g}}(t_2)$ are diagonalizable over R.

Now $[t_1, t_2] = 0$. Since $\mathrm{ad}_{\mathfrak{g}}$ is a morphism of Lie algebras over R, also $\mathrm{ad}_{\mathfrak{g}}(t_1)$ and $\mathrm{ad}_{\mathfrak{g}}(t_2)$ commute.

Suppose given $r_1, r_2 \in R$. Applying Corollary 54, we get that $r_1 \,\mathrm{ad}_{\mathfrak{g}}(t_1) + r_2 \,\mathrm{ad}_{\mathfrak{g}}(t_2)$ is R-diagonalizable. But this map is the same as $\mathrm{ad}_{\mathfrak{g}}(r_1 t_1 + r_2 t_2)$.

We conclude that $\mathfrak{t}$ is an integral torus in $\mathfrak{g}$.

But $\mathfrak{t}$ is not an R-subalgebra of $R^{3\times 3}$ since $t_1 t_2 \notin \mathfrak{t}$. Moreover, $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \notin \mathfrak{t}$. □

Remark 102. *Suppose given a Lie subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. Then $\mathfrak{t}$ is a rational torus in $\mathfrak{g}$ if and only if $K\mathfrak{t}$ is a torus in $K\mathfrak{g}$.*

Proof. Ad $\Longrightarrow$. Suppose that $\mathfrak{t}$ is a rational torus in $\mathfrak{g}$. The subspace $K\mathfrak{t}$ is a Lie subalgebra of $K\mathfrak{g}$; cf. Definition 96. We have to show that the map $\mathrm{ad}_{K\mathfrak{g}}(\frac{1}{s} \otimes t)$ is diagonalizable over K for $t \in \mathfrak{t}$ and $s \in R^\times$; cf. Remark 95.

Suppose given $t \in \mathfrak{t}$ and $s \in R^\times$. By assumption, the map $\mathrm{ad}_{\mathfrak{g}} t \colon \mathfrak{g} \to \mathfrak{g}$ is diagonalizable over K for $t \in \mathfrak{t}$. Therefore the K-linear map $K \,\mathrm{ad}_{\mathfrak{g}} t \colon K\mathfrak{g} \to K\mathfrak{g}$ is diagonalizable; cf. Definition 38. Thus $\mathrm{ad}_{K\mathfrak{g}}(\frac{1}{s} \otimes t) = \frac{1}{s} \mathrm{ad}_{K\mathfrak{g}}(1 \otimes t) = \frac{1}{s} K \,\mathrm{ad}_{\mathfrak{g}} t$ is diagonalizable.

Ad $\Longleftarrow$. Suppose that $K\mathfrak{t}$ is a torus in $K\mathfrak{g}$. Then for $t \in \mathfrak{t}$, the map $\mathrm{ad}_{K\mathfrak{g}}(1 \otimes t) \colon K\mathfrak{g} \to K\mathfrak{g}$ is diagonalizable. But this map is the same as $K \,\mathrm{ad}_{\mathfrak{g}} t$. We conclude that $\mathrm{ad}_{\mathfrak{g}} t$ also is diagonalizable over K for $t \in \mathfrak{t}$.

Thus $\mathfrak{t} \subseteq \mathfrak{g}$ is a rational torus. □

Lemma 103. *Suppose given a torus $\mathfrak{t} \subseteq K\mathfrak{g}$. Then $\mathfrak{t} \cap \mathfrak{g}$ is a rational torus in $\mathfrak{g}$.*

Proof. Consider $K\mathfrak{g}$ as a Lie algebra over R. Recall that we identify $\mathfrak{g}$ as a Lie subalgebra of $K\mathfrak{g}$ via $\mathfrak{g} \to K\mathfrak{g}$, $g \mapsto 1 \otimes g$. Since $\mathfrak{t}$ and $\mathfrak{g}$ are Lie subalgebras of $K\mathfrak{g}$, the intersection $\mathfrak{t} \cap \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$. Suppose given $g \in \mathfrak{t} \cap \mathfrak{g}$. Then $g \in \mathfrak{t}$. Since $\mathfrak{t}$ is a torus in $K\mathfrak{g}$, we have that $\mathrm{ad}_{K\mathfrak{g}} g = K \,\mathrm{ad}_{\mathfrak{g}} g$ is diagonalizable. This shows that $\mathfrak{t} \cap \mathfrak{g} \subseteq \mathfrak{g}$ is a rational torus. □

Remark 104. *There exists a discrete valuation ring R with field of fractions K, a Lie algebra $\mathfrak{g}$ over R and a maximal torus $\mathfrak{t} \subseteq K\mathfrak{g}$ such that $\mathfrak{t} \cap \mathfrak{g}$ is not an integral torus in $\mathfrak{g}$.*

This shows that Lemma 103 does not hold when replacing "rational torus" by "integral torus".

Proof. Consider $R = \mathbb{Z}_{(3)}$ and $K = \mathrm{frac}(R) = \mathbb{Q}$. We have the Lie algebra $\mathfrak{g} := \mathfrak{gl}_2(R)$ over R. Define $\tilde{\mathfrak{t}} := {}_K\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \rangle \subseteq \mathfrak{gl}_2(K) = K\mathfrak{g}$. This is a maximal torus in $K\mathfrak{g}$. Let $\gamma := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ and consider the conjugate Lie subalgebra $\mathfrak{t} := \gamma\tilde{\mathfrak{t}}\gamma^{-1}$ of $K\mathfrak{g}$.

$$\mathfrak{t} = {}_K\langle \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \rangle = {}_K\langle \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \rangle$$

Then $\mathfrak{t}$ is also a maximal torus in $K\mathfrak{g}$ since conjugation with γ is a Lie algebra automorphism. Intersecting with $\mathfrak{g} = \mathfrak{gl}_2(R)$, we get

$$\begin{aligned} & q_1 \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix} + q_2 \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \in \mathfrak{g} \\ \iff & q_1 \in R,\ q_2 \in R,\ \frac{1}{3}q_1 - \frac{1}{3}q_2 \in R \\ \iff & q_1, q_2 \in R \text{ and } q_1 \equiv_3 q_2 \end{aligned}$$

and thus

$$\mathfrak{t} \cap \mathfrak{g} = {}_R\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \rangle.$$

Using the standard basis $\mathcal{E} := \mathcal{E}_{2,2} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right)$ of $\mathfrak{gl}_2(R)$, we get

$$\mathrm{ad}_{\mathfrak{g}} \left(\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \right)_{\mathcal{E},\mathcal{E}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & -3 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} =: A$$

Assume that A is diagonalizable over R, i.e. there exists $S \in \mathrm{GL}_4(R)$ such that $S^{-1}AS$ is a diagonal matrix. Then also over K, the columns of S are eigenvectors of A. The following four vectors form a K-linear basis of $K^{4\times 1}$ since these are linearly independent eigenvectors of A.

$$v_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 3 \\ -1 \\ 9 \\ -3 \end{pmatrix}$$

We have $Av_1 = Av_2 = 0$, $Av_3 = -3v_3$ and $Av_4 = 3v_4$. By assumption there exist $c_1, \ldots, c_6 \in R$ such that $(c_1v_1 + c_2v_2, c_3v_1 + c_4v_2, c_5v_3, c_6v_4)$ is an R-linear basis of $R^{4\times 1}$. In particular, there is an R-linear combination of these vectors that equals e_3. This is a *contradiction.*

So A is not diagonalizable over R. Note that we also can apply Corollary 48.(1) here.

Thus $\mathfrak{t} \cap \mathfrak{g}$ is not an integral torus in $\mathfrak{g}$. □

Lemma 105. *Suppose given a maximal rational torus $\mathfrak{t}$ in $\mathfrak{g}$. Then $K\mathfrak{t}$ is a maximal torus in $K\mathfrak{g}$.*

Proof. $K\mathfrak{t}$ is a torus in $K\mathfrak{g}$; cf. Remark 102.

Assume that $K\mathfrak{t} \subseteq K\mathfrak{g}$ is not a maximal torus. Then there exists a maximal torus $\mathfrak{t}' \subseteq K\mathfrak{g}$ such that $K\mathfrak{t} \subset \mathfrak{t}' \subseteq K\mathfrak{g}$. By intersecting $\mathfrak{t}'$ and $\mathfrak{g}$, we get again a Lie subalgebra of $\mathfrak{g}$. This intersection is a rational torus in $\mathfrak{g}$; cf. Lemma 103.

But then, as vector spaces, we have $\dim_K \mathfrak{t}' > \dim_K K\mathfrak{t}$, but

$$\mathrm{rk}_R(\mathfrak{t}' \cap \mathfrak{g}) = \dim_K \mathfrak{t}' > \dim_K K\mathfrak{t} = \mathrm{rk}_R\, \mathfrak{t};$$

cf. Remark 19.(2). Thus we get
$$\mathfrak{t} \subset \mathfrak{t}' \cap \mathfrak{g} \subseteq \mathfrak{g}.$$
This is a *contradiction* to the maximality of $\mathfrak{t}$ as a rational torus in $\mathfrak{g}$. □

Lemma 106. *Suppose given a maximal torus* $\mathfrak{t} \subseteq K\mathfrak{g}$. *Then* $\mathfrak{t} \cap \mathfrak{g}$ *is a maximal rational torus in* $\mathfrak{g}$.

Proof. The intersection $\mathfrak{t} \cap \mathfrak{g}$ is a rational torus in $\mathfrak{g}$; cf. Lemma 103.

It remains to show the maximality of $\mathfrak{t} \cap \mathfrak{g}$ as a rational torus in $\mathfrak{g}$.

Assume that $\mathfrak{t} \cap \mathfrak{g}$ is not a maximal rational torus in $\mathfrak{g}$. Then there exists a maximal rational torus $\mathfrak{t}'$ in $\mathfrak{g}$ such that
$$\mathfrak{t} \cap \mathfrak{g} \subset \mathfrak{t}' \subseteq \mathfrak{g}.$$
We have $K(\mathfrak{t} \cap \mathfrak{g}) \subset K\mathfrak{t}'$:

Assume that $K(\mathfrak{t} \cap \mathfrak{g}) = K\mathfrak{t}'$. Choose $x \in \mathfrak{t}' \backslash (\mathfrak{t} \cap \mathfrak{g})$. Then we find $y \in \mathfrak{t} \cap \mathfrak{g}$ and $r \in R^\times$ such that $x = \frac{1}{r} \otimes y$, i.e. $rx = y$. So $rx \in \mathfrak{t} \cap \mathfrak{g}$. Since $\mathfrak{t} \cap \mathfrak{g} \subseteq \mathfrak{g}$ is pure by Remark 19.(1), we have $x \in \mathfrak{t} \cap \mathfrak{g}$, a *contradiction*.

We know that $K\mathfrak{t}'$ is a maximal torus in $K\mathfrak{g}$; cf. Lemma 105.

We have $K(\mathfrak{t} \cap \mathfrak{g}) = \mathfrak{t}$; cf. Corollary 20.

So we have the chain of inclusions
$$\mathfrak{t} = K(\mathfrak{t} \cap \mathfrak{g}) \subset K\mathfrak{t}' \underset{\max}{\subseteq} K\mathfrak{g}.$$
But this is a *contradiction* to the maximality of $\mathfrak{t}$ in $K\mathfrak{g}$. □

Remark 107. *Let* $\mathfrak{t}$ *be a rational torus in* $\mathfrak{g}$. *Then* $\mathfrak{t}$ *is abelian.*

This proof follows the idea of [Hum72, §8.1, Lemma].

Proof. Define $\tilde{\mathfrak{g}} := K\mathfrak{g}$ and $\tilde{\mathfrak{t}} := K\mathfrak{t}$ as Lie algebras over K. Firstly, we show that $\tilde{\mathfrak{t}}$ is abelian. Suppose given $\tilde{t} \in \tilde{\mathfrak{t}}$. Since $\tilde{\mathfrak{t}} \subseteq \tilde{\mathfrak{g}}$ is a torus, the endomorphism $\mathrm{ad}_{\tilde{\mathfrak{g}}} \tilde{t}$ is diagonalizable over K; cf. Remark 102. By Lemma 43, the restricted map $\mathrm{ad}_{\tilde{\mathfrak{g}}} \tilde{t}|_{\tilde{\mathfrak{t}}}^{\tilde{\mathfrak{t}}} = \mathrm{ad}_{\tilde{\mathfrak{t}}} \tilde{t}$ is diagonalizable over K, too.

We want to show that $\mathrm{ad}_{\tilde{\mathfrak{t}}} \tilde{t} \overset{!}{=} 0$.

Assume that $\mathrm{ad}_{\tilde{\mathfrak{t}}} \tilde{t} \neq 0$. Then $\mathrm{ad}_{\tilde{\mathfrak{t}}} \tilde{t}$ has an eigenvalue $\lambda \in K^\times$ with corresponding eigenvector $u \in \tilde{\mathfrak{t}}^\times$, i.e. $(\mathrm{ad}_{\tilde{\mathfrak{t}}} \tilde{t})(u) = \lambda u$. Since $u \in \tilde{\mathfrak{t}}$, the map $\mathrm{ad}_{\tilde{\mathfrak{t}}} u$ is also diagonalizable over K, so there exists a K-linear basis $(b_1, \dots, b_k)$ of $\tilde{\mathfrak{t}}$ such that $(\mathrm{ad}_{\tilde{\mathfrak{t}}} u)(b_i) = \mu_i b_i$ for certain $\mu_i \in K$, where $i \in [1, k]$. Write $\tilde{t} = \sum_{i \in [1,k]} \nu_i b_i$ in this basis with $\nu_i \in K$. Then
$$0 \neq \lambda u = [\tilde{t}, u] = -[u, \tilde{t}\,] = - \sum_{i \in [1,k]} \nu_i [u, b_i] = - \sum_{i \in [1,k]} \nu_i \mu_i b_i$$
so that we find $j \in [1, k]$ such that $\nu_j \neq 0$ and $\mu_j \neq 0$. But then
$$0 = [u, \lambda u] = - \sum_{i \in [1,k]} \nu_i \mu_i [u, b_i] = - \sum_{i \in [1,k]} \nu_i \mu_i^2 b_i$$
which is *impossible* since $\nu_j \mu_j^2 \neq 0$ and $(b_1, \dots, b_k)$ is linearly independent.

So $\mathrm{ad}_{\tilde{\mathfrak{t}}} \tilde{t} = 0$ for $\tilde{t} \in \tilde{\mathfrak{t}}$, showing that $\tilde{\mathfrak{t}}$ is abelian.

Now given $t, t' \in \mathfrak{t}$, we get that the embedding $\mathfrak{t} \to \tilde{\mathfrak{t}}$ maps $[t, t'] \in \mathfrak{t}$ to $1 \otimes [t, t'] = [1 \otimes t, 1 \otimes t'] = 0$ since $\tilde{\mathfrak{t}}$ is abelian. The injectivity of the embedding gives us $[t, t'] = 0$. Thus $\mathfrak{t}$ is abelian, completing the proof. □

Lemma 108. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be an abelian Lie subalgebra over R. Let $(t_1, \dots, t_k)$ be an R-linear generating tuple of $\mathfrak{t}$.*

(1) *Suppose that $\mathrm{ad}_{\mathfrak{g}}(t_i)$ is diagonalizable over R for $i \in [1,k]$. Then $\mathrm{ad}_{\mathfrak{g}}(t)$ is diagonalizable over R for $t \in \mathfrak{t}$.*

(2) *Suppose that $\mathrm{ad}_{\mathfrak{g}}(t_i)$ is diagonalizable over K for $i \in [1,k]$. Then $\mathrm{ad}_{\mathfrak{g}}(t)$ is diagonalizable over K for $t \in \mathfrak{t}$.*

Proof. Ad (1). Suppose given $t, t' \in \mathfrak{t}$ such that $\mathrm{ad}_{\mathfrak{g}}(t)$ and $\mathrm{ad}_{\mathfrak{g}}(t')$ are R-diagonalizable. Suppose given $r \in R$. It suffices to show that the map $\mathrm{ad}_{\mathfrak{g}}(rt + t')$ is R-diagonalizable. This map is the same as $r\,\mathrm{ad}_{\mathfrak{g}}(t) + \mathrm{ad}_{\mathfrak{g}}(t')$.

Now $[t, t'] = 0$ since $\mathfrak{t}$ is abelian. Since $\mathrm{ad}_{\mathfrak{g}}$ is a morphism of Lie algebras over R, also $\mathrm{ad}_{\mathfrak{g}}(t)$ and $\mathrm{ad}_{\mathfrak{g}}(t')$ commute.

So we conclude that $r\,\mathrm{ad}_{\mathfrak{g}}(t) + \mathrm{ad}_{\mathfrak{g}}(t')$ is R-diagonalizable; cf. Corollary 54.

Ad (2). Suppose given $t, t' \in \mathfrak{t}$ such that $\mathrm{ad}_{\mathfrak{g}}(t)$ and $\mathrm{ad}_{\mathfrak{g}}(t')$ are K-diagonalizable. Suppose given $r \in R$. It suffices to show that the map $\mathrm{ad}_{\mathfrak{g}}(rt + t')$ is K-diagonalizable. This map is the same as $r\,\mathrm{ad}_{\mathfrak{g}}(t) + \mathrm{ad}_{\mathfrak{g}}(t')$.

Now $[t, t'] = 0$ since $\mathfrak{t}$ is abelian. Since $\mathrm{ad}_{\mathfrak{g}}$ is a morphism of Lie algebras over R, also $\mathrm{ad}_{\mathfrak{g}}(t)$ and $\mathrm{ad}_{\mathfrak{g}}(t')$ commute. Hence $K\,\mathrm{ad}_{\mathfrak{g}}(t)$ and $K\,\mathrm{ad}_{\mathfrak{g}}(t')$ commute.

So we conclude that $K\,(r\,\mathrm{ad}_{\mathfrak{g}}(t) + \mathrm{ad}_{\mathfrak{g}}(t')) = r\,(K\,(\mathrm{ad}_{\mathfrak{g}}(t))) + K\,(\mathrm{ad}_{\mathfrak{g}}(t'))$ is K-diagonalizable; cf. Corollary 54. □

Remark 109. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Lie subalgebra. Then $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}) \subseteq \mathfrak{g}$ is a Lie subalgebra.

Proof. We have $0 \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t})$.

Let $x, y \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t})$ and $a, b \in R$. Then there exist $r, s \in R^{\times}$ such that $rx \in \mathfrak{t}$ and $sy \in \mathfrak{t}$. Note that $rs \in R^{\times}$. Then $ax + by \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t})$ since $rs(ax + by) = sa(rx) + rb(sy) \in \mathfrak{t}$. Furthermore, we have $[x, y] \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t})$ since $rs[x, y] = [rx, sy] \in \mathfrak{t}$ and $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{g}$. □

Lemma 110.

(1) *Let $\mathfrak{t}_0$ be a rational torus in $\mathfrak{g}$. Then $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_0)$ is a rational torus in $\mathfrak{g}$.*

(2) *Let $\mathfrak{t}_1$ be an integral torus in $\mathfrak{g}$. Then $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_1)$ is an integral torus in $\mathfrak{g}$.*

Proof. Ad (1). $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_0)$ is a Lie subalgebra of $\mathfrak{g}$; cf. Remark 109. It remains to show that for $x \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_0)$, the R-linear map $\mathrm{ad}_{\mathfrak{g}}(x)$ is diagonalizable over K.

Suppose given $x \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_0)$. Choose $r \in R^{\times}$ such that $rx \in \mathfrak{t}_0$, then $\mathrm{ad}_{\mathfrak{g}}(rx)$ is diagonalizable over K. So there exists a K-linear basis $\mathcal{B}$ of $K\mathfrak{g}$ such that $(K \otimes \mathrm{ad}_{\mathfrak{g}}(rx))_{\mathcal{B},\mathcal{B}} \in K^{n \times n}$ is a diagonal matrix. But $(K \otimes \mathrm{ad}_{\mathfrak{g}}(rx))_{\mathcal{B},\mathcal{B}} = r((K \otimes \mathrm{ad}_{\mathfrak{g}}(x))_{\mathcal{B},\mathcal{B}})$, so the matrix $(K \otimes \mathrm{ad}_{\mathfrak{g}}(x))_{\mathcal{B},\mathcal{B}}$ is diagonal, i.e. the map $\mathrm{ad}_{\mathfrak{g}}(x)$ is diagonalizable over K.

Ad (2). $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_1)$ is a Lie subalgebra of $\mathfrak{g}$; cf. Remark 109. It remains to show that for $x \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_1)$, the R-linear map $\mathrm{ad}_{\mathfrak{g}}(x)$ is diagonalizable over R.

Suppose given $x \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{t}_1)$. Choose $r \in R^{\times}$ such that $rx \in \mathfrak{t}_1$, then $\mathrm{ad}_{\mathfrak{g}}(rx)$ is diagonalizable over R. So there exists an R-linear basis $\mathcal{B}$ of $\mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(rx)_{\mathcal{B},\mathcal{B}} \in R^{n \times n}$ is a diagonal matrix. By linearity of this map, we have $\mathrm{ad}_{\mathfrak{g}}(rx)_{\mathcal{B},\mathcal{B}} = r(\mathrm{ad}_{\mathfrak{g}}(x)_{\mathcal{B},\mathcal{B}})$. Since R is free of zero divisors, $\mathrm{ad}_{\mathfrak{g}}(x)_{\mathcal{B},\mathcal{B}}$ is a diagonal matrix with entries in R, i.e. the map $\mathrm{ad}_{\mathfrak{g}}(x)$ is diagonalizable over R. □

Corollary 111.

(1) *A maximal rational torus in* $\mathfrak{g}$ *is a pure* R*-submodule of* $\mathfrak{g}$ *and thus a pure rational torus.*

(2) *A maximal integral torus in* $\mathfrak{g}$ *is a pure* R*-submodule of* $\mathfrak{g}$ *and thus a pure integral torus.*

Proof. This follows from Lemma 110. □

Lemma 112. *Let* $\mathfrak{t}$ *be a rational torus in* $\mathfrak{g}$*. If* $\mathfrak{t} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$*, then* $\mathfrak{t}$ *is a maximal rational torus in* $\mathfrak{g}$*. If* $\mathfrak{t} \subseteq \mathfrak{g}$ *is a maximal rational torus and an integral torus, then* $\mathfrak{t}$ *is a maximal integral torus in* $\mathfrak{g}$*.*

Proof. Assume that $\mathfrak{t} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$ and $\mathfrak{t}$ is not a maximal rational torus in $\mathfrak{g}$. Then we find a rational torus $\mathfrak{t}' \subseteq \mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{t}' \subseteq \mathfrak{g}$. By Remark 107, $\mathfrak{t}'$ is abelian, so $\mathfrak{t} \subset \mathfrak{t}' \subseteq \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}') \subseteq \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$, hence $\mathfrak{t} \neq \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$, a *contradiction.* □

4.2 Maximal tori of split orders over R

4.2.1 Definitions for R-orders

Definition 113. Suppose given an R-order Γ. We say that Γ is a *completely split R-order* if Γ is isomorphic to a finite direct product of matrix rings over R. In other words, there exist $k \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i \in [1, k]$ such that

$$\Gamma \simeq \prod_{i \in [1,k]} R^{n_i \times n_i}.$$

Definition 114. Suppose given an R-order Ω. We say that Ω is a *split R-order* if there exist a completely split R-order Γ and an injective R-algebra morphism $\iota\colon \Omega \to \Gamma$ such that

$$K \otimes_R \big(\Gamma/\iota(\Omega)\big) = 0.$$

Example 115. Suppose given a completely split R-order Γ. Let Ω be an R-subalgebra of Γ. If $K \otimes_R (\Gamma/\Omega) = 0$, then Ω is a split R-order.

Remark 116. Let Ω be an R-order such that there exists a K-algebra isomorphism

$$\varphi\colon K\Omega \xrightarrow{\sim} \prod_{i \in [1,k]} K^{n_i \times n_i}$$

for certain $k \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i \in [1, k]$. Then Ω is a split R-order.

Proof. We write $\tilde{\varphi} := \varphi|_{\Omega}^{\varphi(\Omega)}$. Then we have $\tilde{\varphi}(\Omega) \subseteq \prod_{i \in [1,k]} K^{n_i \times n_i}$ where the right hand side is a separable K-algebra. So there exists a maximal R-order Γ such that $\tilde{\varphi}(\Omega) \subseteq \Gamma \subseteq \prod_{i \in [1,k]} K^{n_i \times n_i}$; cf. [CR81, Theorem 26.5].

Then $\Gamma = \prod_{i \in [1,k]} \Gamma_i$ where Γ_i is a maximal order in $K^{n_i \times n_i}$ for $i \in [1, k]$; cf. [CR81, Theorem 26.20(i)]. Thus Γ is a completely split R-order.

For $i \in [1, k]$, Γ_i is conjugate to $R^{n_i \times n_i}$; cf. [CR81, Exercise 26.10].

We define $\tilde{\iota}\colon \tilde{\varphi}(\Omega) \to \Gamma$ to be the embedding map. We define $\iota\colon \Omega \to \Gamma$ to be the composite map $\tilde{\iota} \circ \tilde{\varphi}$. Then ι is an injective R-algebra morphism.

We can illustrate this situation as follows.

$$\begin{tikzcd} \Omega \arrow[r, "\sim", "\tilde{\varphi}"'] \arrow[rr, bend right, "\iota"'] & \tilde{\varphi}(\Omega) \arrow[r, hook, "\tilde{\iota}"] & \Gamma \arrow[r, hook] & \prod_{i \in [1,k]} K^{n_i \times n_i} \end{tikzcd}$$

Now $\mathrm{rk}_R(\Omega) = \mathrm{rk}_R(\iota(\Omega)) = \mathrm{rk}_R(\Gamma) = \sum_{i \in [1,k]} n_i^2$. We conclude that $K \otimes_R \big(\Gamma/\iota(\Omega)\big) = 0$. □

Definition 117. Let Ω be a split R-order. Then there exists a completely split R-order Γ and an injective R-algebra morphism $\iota\colon \Omega \to \Gamma$ such that $K\iota$ is an isomorphism. There exist $k \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i \in [1,k]$ and an R-algebra isomorphism ψ

$$\psi\colon \Gamma \xrightarrow{\sim} \prod_{i\in[1,k]} R^{n_i\times n_i}.$$

So we get a K-algebra isomorphism

$$\varphi := (K\psi)\circ(K\iota)\colon K\Omega \xrightarrow{\sim} \prod_{i\in[1,k]} K^{n_i\times n_i}.$$

Since ι is an R-algebra morphism and since Ω is an R-algebra, $\iota(\Omega)$ is an R-subalgebra of Γ. We define

$$\Delta := \left\{ (x_1,\dots,x_k) \in \prod_{i\in[1,k]} R^{n_i\times n_i} \;\middle|\; x_i \text{ is a diagonal matrix for } i \in [1,k] \right\}.$$

We say that Δ is the *full diagonal* in $\prod_{i\in[1,k]} R^{n_i\times n_i}$. This is in fact a subalgebra of Γ since $0 \in \Delta$ and $1 \in \Delta$ and sums and products of diagonal matrices are again diagonal.

The intersection $\varphi(\Omega)\cap\Delta = \psi(\iota(\Omega))\cap\Delta$ is the *full diagonal of $\psi(\iota(\Omega))$ (with respect to ι and ψ).*

This will often be used in the case where ψ is the identity and ι is an embedding of a subalgebra. In this case, the full diagonal of Ω in Γ is given by $\Omega \cap \Delta$.

4.2.2 Tori of split R-orders

Let $k \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i \in [1,k]$. Define the completely split R-order $\Gamma := \prod_{i\in[1,k]} R^{n_i\times n_i}$. Let Ω be an R-subalgebra of Γ such that $K \otimes_R (\Gamma/\Omega) = 0$. So Ω is a split R-order. Let Δ be the full diagonal in Γ.

Recall that $\mathfrak{l}(\Omega)$ is the commutator Lie algebra of Ω. Similarly, we have the commutator Lie algebra $\mathfrak{l}(\Gamma)$ of Γ.

First we want to illustrate that even in the special case of completely split R-orders, there is a difference between diagonalizability over R and diagonalizability over K.

Remark 118. *There exist a discrete valuation ring R, a completely split R-order Γ and $x \in \Gamma$ such that $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)$ is diagonalizable over $\mathrm{frac}(R)$ but $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)$ is not diagonalizable over R.*

Proof. Let $R = \mathbb{Z}_{(2)}$ and $K = \mathrm{frac}(R) = \mathbb{Q}$. Let Γ be the completely split $\mathbb{Z}_{(2)}$-order $\mathbb{Z}_{(2)}^{2\times 2}$. Let $\mathcal{E} := \mathcal{E}_{2,2}$ be the standard basis of Γ. Let $x := \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \in \Gamma$.

We have

$$A := \mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)_{\mathcal{E},\mathcal{E}} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

and $\sigma(A) = \{0, 2, -2\}$. The eigenspaces of A are given by

$$\mathrm{E}_A(0) = {}_{\mathbb{Q}}\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \rangle,\quad \mathrm{E}_A(2) = {}_{\mathbb{Q}}\langle \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \rangle,\quad \mathrm{E}_A(-2) = {}_{\mathbb{Q}}\langle \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} \rangle.$$

We have the following intersections.

$$\mathrm{E}_A(0)\cap\mathbb{Z}_{(2)}^4 = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \rangle,\quad \mathrm{E}_A(2)\cap\mathbb{Z}_{(2)}^4 = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \rangle,\quad \mathrm{E}_A(-2)\cap\mathbb{Z}_{(2)}^4 = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} \rangle$$

But

$$\bigoplus_{\lambda\in\sigma(A)} \left(\mathrm{E}_A(\lambda)\cap\mathbb{Z}_{(2)}^4\right) = {}_{\mathbb{Z}_{(2)}}\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\4\end{pmatrix}\rangle \subset \mathbb{Z}_{(2)}^4,$$

so A is not diagonalizable over $\mathbb{Z}_{(2)}$. But the matrix

$$S := \begin{pmatrix} 1 & 0 & 1 & 1\\ 0 & 1 & -1 & 1\\ 0 & 1 & 1 & -1\\ 1 & 0 & -1 & -1 \end{pmatrix}$$

is invertible as a matrix in $\mathbb{Q}^{4\times 4}$ and we have the diagonal matrix

$$S^{-1}\cdot A\cdot S = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & -2 \end{pmatrix},$$

showing that A is diagonalizable over $\mathbb{Q}$. □

Lemma 119. *For $x\in\mathfrak{l}(\Omega)$, we have the following equivalence.*

$$\mathrm{ad}_{\mathfrak{l}(\Omega)}(x) \textit{ is diagonalizable over } K \iff \mathrm{ad}_{\mathfrak{l}(\Gamma)}(x) \textit{ is diagonalizable over } K$$

Proof. By Definition 38, the R-linear map $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ is K-diagonalizable if and only if the K-linear map $K\otimes\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)\colon K\otimes\mathfrak{l}(\Omega)\to K\otimes\mathfrak{l}(\Omega)$ is diagonalizable.

In the same way the R-linear map $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)$ is K-diagonalizable if and only if the K-linear map $K\otimes\mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)\colon K\otimes\mathfrak{l}(\Gamma)\to K\otimes\mathfrak{l}(\Gamma)$ is diagonalizable.

We have the embedding $\iota\colon\Omega\to\Gamma$ which is an injective R-algebra morphism. The following diagram commutes.

$$\begin{tikzcd} K\otimes\mathfrak{l}(\Gamma) \arrow[r, "K\otimes\mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)"] & K\otimes\mathfrak{l}(\Gamma) \\ K\otimes\mathfrak{l}(\Omega) \arrow[u, "K\otimes\iota", "\wr"'] \arrow[r, "K\otimes\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)"] & K\otimes\mathfrak{l}(\Omega) \arrow[u, "\wr", "K\otimes\iota"'] \end{tikzcd}$$

So we have shown that $K\otimes\mathrm{ad}_{\mathfrak{l}(\Gamma)}(x)$ is K-diagonalizable if and only if $K\otimes\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ is K-diagonalizable.

This completes the proof. □

Lemma 120. *The full diagonal $\mathfrak{l}(\Omega\cap\Delta)$ of $\mathfrak{l}(\Omega)$ in $\mathfrak{l}(\Gamma)$ is a maximal rational torus in $\mathfrak{l}(\Omega)$.*

Proof. We divide the proof into three steps.

Step 1. The full diagonal $\mathfrak{l}(\Omega\cap\Delta)$ is a Lie subalgebra over R of $\mathfrak{l}(\Omega)$.

Both $\mathfrak{l}(\Delta)$ and $\mathfrak{l}(\Omega)$ are Lie subalgebras over R of $\mathfrak{l}(\Gamma)$. Thus $\mathfrak{l}(\Omega\cap\Delta)$ also is a Lie subalgebra over R of $\mathfrak{l}(\Gamma)$. Since $\mathfrak{l}(\Omega\cap\Delta)\subseteq\mathfrak{l}(\Omega)\subseteq\mathfrak{l}(\Gamma)$, we have that $\mathfrak{l}(\Omega\cap\Delta)$ is a Lie subalgebra over R of $\mathfrak{l}(\Omega)$.

Step 2. Suppose given $x\in\mathfrak{l}(\Omega\cap\Delta)$. Then $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ is diagonalizable over K.

Define $q := \min\{r\in\mathbb{N}\,|\,\pi^r\Gamma\subseteq\Omega\}$. This number q exists since Γ/Ω is a torsion-R-module and thus there exists $\tilde{q}\in\mathbb{N}$ such that $\pi^{\tilde{q}}(\Gamma/\Omega)=0$ and hence $\pi^{\tilde{q}}\Gamma\subseteq\Omega$.

We define the following elements of Γ.

$$\eta_{i;j,l} := \Big(\big(\quad\big),\ldots,\big(\quad\big),\underbrace{E_{j,l}}_{\text{position } i},\big(\quad\big),\ldots,\big(\quad\big)\Big) \text{ for } i\in[1,k],\ j,l\in[1,n_i]$$

Recall that the matrices without entries are to be understood as zero matrices.

We have $\pi^q \eta_{i;j,l} \in \Omega$ for $i \in [1,k]$, $j,l \in [1,n_i]$. We have $\pi^q \eta_{i;j,j} \in \Delta$ for $i \in [1,k]$, $j \in [1,n_i]$.

Then for $i \in [1,k]$, $j \in [1,n_i]$, we have $\pi^q \eta_{i;j,j} \in \Omega \cap \Delta$.

Since for the K-diagonalizability of an element, a constant factor in $K^\times$ does not play a role, we have

$$\mathrm{ad}_{\mathfrak{l}(\Omega)}(x) \text{ is diagonalizable over } K \iff \mathrm{ad}_{\mathfrak{l}(\Omega)}(\pi^q \cdot x) \text{ is diagonalizable over } K.$$

Now $\pi^q \cdot x$ is a linear combination of the $\pi^q \cdot \eta_{i;j,j}$. Thus it suffices to show that $\mathrm{ad}_{\mathfrak{l}(\Omega)}(\pi^q \cdot \eta_{i;j,j})$ is K-diagonalizable for $i \in [1,k]$, $j \in [1,n_i]$; cf. Lemma 108.(2). By Lemma 119, it suffices to show that $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(\pi^q \cdot \eta_{i;j,j})$ is K-diagonalizable for $i \in [1,k]$, $j \in [1,n_i]$. This is the case if and only if $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(\eta_{i;j,j})$ is diagonalizable over K for $i \in [1,k]$, $j \in [1,n_i]$.

Denote by

$$\mathcal{E} := (\eta_{1;1,1}, \ldots, \eta_{1;1,n_1}, \eta_{1;2,1}, \ldots, \eta_{1;n_1,n_1}, \ldots, \eta_{k;n_k,1}, \ldots, \eta_{k;n_k,n_k})$$

the standard basis of Γ.

Then it suffices to show that $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(\eta_{i;j,j})_{\mathcal{E},\mathcal{E}}$ is a diagonal matrix for $i \in [1,k]$, $j \in [1,n_i]$.

But given $i \in [1,k]$, $j \in [1,n_i]$, we have that

$$\left(\mathrm{ad}_{\mathfrak{l}(\Gamma)}(\eta_{i;j,j})\right)(\eta_{r;s,t}) = \delta_{i,r} \cdot (\delta_{j,s} - \delta_{j,t}) \cdot \eta_{r;s,t} \text{ for } r \in [1,k],\ s,t \in [1,n_r].$$

This shows that $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(\eta_{i;j,j})_{\mathcal{E},\mathcal{E}}$ is a diagonal matrix for $i \in [1,k]$, $j \in [1,n_i]$.

Step 3. The full diagonal of $\mathfrak{l}(\Omega)$ in $\mathfrak{l}(\Gamma)$ is a maximal rational torus in $\mathfrak{l}(\Omega)$.

By Lemma 112, it suffices to show that $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega \cap \Delta)) = \mathfrak{l}(\Omega \cap \Delta)$.

Ad $\supseteq$. We have shown that $\mathfrak{l}(\Omega \cap \Delta)$ is a rational torus in step 1 and step 2. So it is abelian and thus $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega \cap \Delta)) \supseteq \mathfrak{l}(\Omega \cap \Delta)$; cf. Remark 107.

Ad $\subseteq$. We have the standard basis $\mathcal{E}$ of $\mathfrak{l}(\Gamma)$. We have $\{\pi^q \eta_{i;j,j} \,|\, i \in [1,k],\ j \in [1,n_i]\} \subseteq \mathfrak{l}(\Omega \cap \Delta)$. Suppose given $x \in \mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega \cap \Delta))$. Then $xy = yx$ for $y \in \{\pi^q \eta_{i;j,j} \,|\, i \in [1,k],\ j \in [1,n_i]\}$.

Suppose given $r \in [1,k]$. Then $x \cdot (\pi^q \eta_{r;j,j}) = (\pi^q \eta_{r;j,j}) \cdot x$ for $j \in [1,n_r]$. Let the r-th component of x be given by the matrix $(x_{s,t})_{s,t \in [1,n_r]}$.

Let L_1 be the r-th component of $x \cdot (\pi^q \eta_{r;j,j})$. Then for $u,v \in [1,n_r]$ and $j \in [1,n_r]$, we have

$$(L_1)_{u,v} = x_{u,v} \cdot \delta_{v,j} \cdot \pi^q.$$

Let L_2 be the r-th component of $(\pi^q \eta_{r;j,j}) \cdot x$. Then for $u,v \in [1,n_r]$ and $j \in [1,n_r]$ we have

$$(L_2)_{u,v} = x_{u,v} \cdot \delta_{u,j} \cdot \pi^q.$$

Now it follows from $L_1 = L_2$ that $x_{u,v} \cdot \delta_{v,j} = x_{u,v} \cdot \delta_{u,j}$ for $u,v \in [1,n_r]$ and $j \in [1,n_r]$.

Consider the j-th row of these matrices, i.e. let $u = j$. Then $x_{u,v} = 0$ or $j = v = u$. This means that there can be at most one non-zero entry in the j-th row and this entry is on the position (j,j). Since this holds for $j \in [1,n_i]$, the r-th component of x is a diagonal matrix.

This holds for $r \in [1,k]$, so every component of x is a diagonal matrix.

This shows that $x \in \mathfrak{l}(\Delta)$ and since $x \in \mathfrak{l}(\Omega)$ by assumption, we have $x \in \mathfrak{l}(\Omega \cap \Delta)$. □

Lemma 121. *The intersection $\Omega \cap \Delta$ is a maximal commutative R-subalgebra of Ω.*

Proof. In the case that Γ is a direct product of copies of R, then $\Omega \cap \Delta = \Omega$ is a commutative algebra itself. So suppose that there exists $i \in [1,k]$ with $n_i > 1$.

Assume that there exists a commutative R-subalgebra C of Ω such that $\Omega \cap \Delta \subset C$. Then, in particular, we have $C \not\subseteq \Delta$. This entails that $KC \not\subseteq K\Delta$. From $\Omega \cap \Delta \subset C \subseteq \Omega$, we conclude that

$$K(\Omega \cap \Delta) = K\Delta \subset KC \subseteq K\Omega \subseteq K\Gamma.$$

Note that the inclusion $K\Delta \subset KC$ is a proper inclusion since $K\Delta = KC$ would be a contradiction to $KC \not\subseteq K\Delta$.

Since C is commutative, also KC is commutative. Choose an element $x \in KC \backslash K\Delta$. Then there exist $i_0 \in [1,k]$ and $j_1, j_2 \in [1, n_{i_0}]$ such that $j_1 \neq j_2$ and in the i_0-th component of x, there is an entry $z \in K^\times$ on position (j_1, j_2).

Using the notation as in the proof of Lemma 120, we have the K-linear basis

$$(\eta_{1;1,1}, \eta_{1;1,2}, \ldots, \eta_{1;1,n_1}, \eta_{1;2,1}, \ldots, \eta_{1;n_1,n_1}, \ldots, \eta_{k;n_k,1}, \ldots, \eta_{k;n_k,n_k})$$

of $K\Gamma$. Moreover, we have the K-linear basis

$$(\eta_{1;1,1}, \eta_{1;2,2}, \ldots, \eta_{1;n_1,n_1}, \eta_{2;1,1}, \ldots, \eta_{2;n_2,n_2}, \ldots, \eta_{k;1,1}, \ldots, \eta_{k;n_k,n_k})$$

of $K\Delta$. So we can write

$$\eta_{i_0;j_1,j_2} = \underbrace{z^{-1}}_{\in K^\times} \cdot \underbrace{\eta_{i_0;j_1,j_1}}_{\in K\Delta \subseteq KC} \cdot \underbrace{x}_{\in KC} \cdot \underbrace{\eta_{i_0;j_2,j_2}}_{\in K\Delta \subseteq KC} .$$

We conclude that $\eta_{i_0;j_1,j_2} \in KC$.

But then we have

$$\begin{aligned} \eta_{i_0;j_1,j_1} \cdot \eta_{i_0;j_1,j_2} &= \eta_{i_0;j_1,j_2} \neq 0 \\ \eta_{i_0;j_1,j_2} \cdot \eta_{i_0;j_1,j_1} &= 0. \end{aligned}$$

This shows that KC is not commutative, a *contradiction.* □

Remark 122. Let T be an R-subalgebra of Ω. Then as R-submodules of Ω, we have

$$\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) = \mathrm{C}_\Omega(T).$$

Proof. We have

$$\begin{aligned} \mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) &= \{x \in \mathfrak{l}(\Omega) \,|\, [x,t] = 0 \text{ for } t \in \mathfrak{l}(T)\} \\ &= \{x \in \Omega \,|\, xt - tx = 0 \text{ for } t \in T\} \\ &= \{x \in \Omega \,|\, xt = tx \text{ for } t \in T\} \\ &= \mathrm{C}_\Omega(T). \end{aligned}$$

□

Lemma 123. *Suppose given $x \in \Omega$. Suppose given $u \in \mathrm{U}(K\Omega)$. We write $\tilde{\Omega} := u^{-1}\Omega u \subseteq K\Omega$.*

(1) *The map $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ is diagonalizable over R if and only if $\mathrm{ad}_{\mathfrak{l}(\tilde{\Omega})}(u^{-1}xu)$ is diagonalizable over R.*

(2) *The map $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ is diagonalizable over K if and only if $\mathrm{ad}_{\mathfrak{l}(\tilde{\Omega})}(u^{-1}xu)$ is diagonalizable over K.*

Proof. We have the conjugation isomorphism of Lie algebras

$$\begin{aligned} \varphi \colon \mathfrak{l}(\Omega) &\longrightarrow \mathfrak{l}(\tilde{\Omega}) \\ y &\longmapsto u^{-1}yu. \end{aligned}$$

We have the R-linear maps

$$\begin{aligned} \mathrm{ad}_{\mathfrak{l}(\Omega)}(x) \colon \mathfrak{l}(\Omega) &\longrightarrow \mathfrak{l}(\Omega) \\ y &\longmapsto xy - yx \end{aligned}$$

and

$$\begin{aligned}\mathrm{ad}_{\mathfrak{l}(\tilde{\Omega})}(u^{-1}xu)\colon \mathfrak{l}(\tilde{\Omega}) &\longrightarrow \mathfrak{l}(\tilde{\Omega})\\ y &\longmapsto u^{-1}xuy - yu^{-1}xu.\end{aligned}$$

We verify the commutativity of the following diagram.

$$\begin{tikzcd}
\mathfrak{l}(\tilde{\Omega}) \arrow[r, "\mathrm{ad}_{\mathfrak{l}(\tilde{\Omega})}(u^{-1}xu)"] & \mathfrak{l}(\tilde{\Omega}) \\
\mathfrak{l}(\Omega) \arrow[u, "\varphi", "\wr"'] \arrow[r, "\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)"] & \mathfrak{l}(\Omega) \arrow[u, "\varphi", "\wr"']
\end{tikzcd}$$

For $y \in \mathfrak{l}(\Omega)$, we have

$$\begin{aligned}(\mathrm{ad}_{\mathfrak{l}(\tilde{\Omega})}(u^{-1}xu) \circ \varphi)(y) &= (\mathrm{ad}_{\mathfrak{l}(\tilde{\Omega})}(u^{-1}xu))(u^{-1}yu)\\ &= u^{-1}xuu^{-1}yu - u^{-1}yuu^{-1}xu\\ &= u^{-1}(xy-yx)u\\ &= \varphi(xy-yx) = \left(\varphi \circ \mathrm{ad}_{\mathfrak{l}(\Omega)}(x)\right)(y).\end{aligned}$$

This shows (1). After tensoring with K, this shows (2). □

Corollary 124. *Suppose given $u \in \mathrm{U}(K\Omega)$. We write $\tilde{\Omega} := u^{-1}\Omega u \subseteq K\Omega$.*
Suppose given a commutative subalgebra $T \subseteq \Omega$. We write $\tilde{T} := u^{-1}Tu \subseteq \tilde{\Omega}$.

(1) *$\mathfrak{l}(T)$ is an integral torus in $\mathfrak{l}(\Omega)$ if and only if $\mathfrak{l}(\tilde{T})$ is an integral torus in $\mathfrak{l}(\tilde{\Omega})$.*

(2) *$\mathfrak{l}(T)$ is a rational torus in $\mathfrak{l}(\Omega)$ if and only if $\mathfrak{l}(\tilde{T})$ is a rational torus in $\mathfrak{l}(\tilde{\Omega})$.*

Proof. This follows from Lemma 123. □

Remark 125. *There exist a discrete valuation ring R, a completely split R-order Γ and two maximal rational tori in $\mathfrak{l}(\Gamma)$ one of which is an integral torus in $\mathfrak{l}(\Gamma)$ and the other is not. In particular, they are not conjugate via a unit in Γ.*

Proof. Let $R = \mathbb{Z}_{(2)}$ and $K = \mathrm{frac}(R) = \mathbb{Q}$. Let Γ be the completely split $\mathbb{Z}_{(2)}$-order $\mathbb{Z}_{(2)}^{2\times 2}$. We define

$$T := {}_R\left\langle \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix} \right\rangle.$$

This is a commutative R-subalgebra of Γ.

We have seen that $\mathrm{ad}_{\mathfrak{l}(\Gamma)}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ is not $\mathbb{Z}_{(2)}$-diagonalizable, but $\mathbb{Q}$-diagonalizable; cf. Example 118. So $\mathfrak{l}(T)$ is not ans integral torus in $\mathfrak{l}(\Gamma)$. In Γ, the matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ is a central element, thus $\mathrm{ad}_{\mathfrak{l}(\Gamma)}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = 0$. This shows that $\mathfrak{l}(T)$ is a rational torus in $\mathfrak{l}(\Gamma)$; cf. Lemma 108.(2).

We use the commutativity of rational tori to show that $\mathfrak{l}(T)$ is a maximal rational torus. Suppose given $M = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \mathfrak{l}(\Gamma)$ such that $[M, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)] = 0$. Then

$$\begin{pmatrix}0&0\\0&0\end{pmatrix} = \left[\begin{pmatrix}a&b\\c&d\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right] = \begin{pmatrix}b-c & a-d\\ d-a & c-b\end{pmatrix}$$

and we conclude that $b = c$ and $a = d$. So $M = \left(\begin{smallmatrix}a&b\\b&a\end{smallmatrix}\right) \in \mathfrak{l}(T)$. This shows that $\mathfrak{l}(T) = \mathfrak{c}_{\mathfrak{l}(\Gamma)}(\mathfrak{l}(T))$, so $\mathfrak{l}(T)$ is a maximal rational torus in $\mathfrak{l}(\Gamma)$; cf. Lemma 112.

Furthermore, we have the full diagonal $\Delta = \mathbb{Z}_{(2)}\langle \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\rangle$ in Γ. The describing matrices of $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right))$ and $\mathrm{ad}_{\mathfrak{l}(\Gamma)}(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right))$ with respect to the standard basis $\mathcal{E}_{2,2}$ of Γ are diagonal, so $\mathfrak{l}(\Delta) \subseteq \mathfrak{l}(\Gamma)$ is an integral torus; cf. Lemma 108.(1). But $\mathfrak{l}(\Delta)$ is also a maximal rational torus in $\mathfrak{l}(\Gamma)$; cf. Lemma 120. So $\mathfrak{l}(\Delta)$ is a maximal integral torus in $\mathfrak{l}(\Gamma)$; cf. Lemma 112.

Assume that there exists $u_0 \in \mathrm{GL}_2(\mathbb{Z}_{(2)}) = \mathrm{U}(\Gamma)$ such that $u_0^{-1}Tu_0 = \Delta$. Note that $u_0^{-1}\Gamma u_0 = \Gamma$. Since $\mathfrak{l}(\Delta)$ is an integral torus in $\mathfrak{l}(\Gamma)$, its conjugate $u_0\Delta u_0^{-1} = T$ also is an integral torus in $\mathfrak{l}(\Gamma)$; cf. Corollary 124.(1). But this is a *contradiction*.

Thus we have found two maximal rational tori $\mathfrak{l}(T)$ and $\mathfrak{l}(\Delta)$ in $\mathfrak{l}(\Gamma)$ that are not conjugate via a unit in Γ. Moreover, $\mathfrak{l}(\Delta)$ is an integral torus in $\mathfrak{l}(\Gamma)$ whereas $\mathfrak{l}(T)$ is not. □

Lemma 126. *Suppose given an orthogonal decomposition $1_\Omega = e_1 + \ldots + e_n$ of 1_Ω into primitive idempotents in Ω. Then $\mathrm{ad}_{\mathfrak{l}(\Omega)}(e_i)$ is diagonalizable over R for $i \in [1, n]$.*

Proof. For $i, j \in [1, n]$ we choose an R-linear basis $\mathcal{B}_{i,j}$ of the Peirce component $e_i\Omega e_j$ of Ω. Denote the R-linear basis of $\mathfrak{l}(\Omega)$ that we obtain by concatenating these bases by $\mathcal{B}$.

Suppose given $k \in [1, n]$. We consider the describing matrix of $\mathrm{ad}_{\mathfrak{l}(\Omega)}(e_k)$ with respect to the basis $\mathcal{B}$.

Suppose given $i, j \in [1, n]$ and an element x of $\mathcal{B}_{i,j}$. In particular, we have $x = e_ixe_j$.

Then $e_kx - xe_k = e_k(e_ixe_j) - (e_ixe_j)e_k = \delta_{i,k}x - \delta_{j,k}x \in \{0, x, -x\}$.

But this implies that $\left(\mathrm{ad}_{\mathfrak{l}(\Omega)}(e_k)\right)_{\mathcal{B},\mathcal{B}}$ is a diagonal matrix with entries in $\{0, +1, -1\}$ on its diagonal. In particular, $\mathrm{ad}_{\mathfrak{l}(\Omega)}(e_k)$ is diagonalizable over R. □

Lemma 127. *Suppose given an orthogonal decomposition $1_\Omega = e_1 + \ldots + e_n$ of 1_Ω into primitive idempotents in Ω such that $e_i \in \Delta$ for $i \in [1, n]$. Then for $x \in \mathfrak{l}(\Omega \cap \Delta)$, the following equivalence holds.*

$$\mathrm{ad}_{\mathfrak{l}(\Omega)}(x) \text{ is diagonalizable over } R \iff \left(\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)\right)\Big|_{e_i\Omega e_j}^{e_i\Omega e_j} \text{ is diagonalizable over } R \text{ for } i, j \in [1, n]$$

Here we have $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)\big|_{e_i\Omega e_i}^{e_i\Omega e_i} = \mathrm{ad}_{\mathfrak{l}(e_i\Omega e_i)}(e_ixe_i)$ for $i \in [1, n]$.

Proof. We have the Peirce decompositions $\Omega = \bigoplus_{i,j\in[1,n]} e_i\Omega e_j$ and $\Omega \cap \Delta = \bigoplus_{i\in[1,n]} e_i(\Omega \cap \Delta)e_i$.

Suppose given an element $x \in \mathfrak{l}(\Omega \cap \Delta)$. Then $x = \sum_{i\in[1,n]} e_ixe_i$. Suppose given $j, l \in [1, n]$ and $y \in e_j\Omega e_l$. We calculate.

$$\begin{aligned}
[x, y] &= \left[\left(\sum_{i\in[1,n]} e_ixe_i\right), e_jye_l\right] \\
&= \left(\sum_{i\in[1,n]} (e_ixe_i) \cdot (e_jye_l)\right) - \left(\sum_{i\in[1,n]} (e_jye_l) \cdot (e_ixe_i)\right) \\
&= \left(\sum_{i\in[1,n]} e_ix\delta_{i,j}e_iye_l\right) - \left(\sum_{i\in[1,n]} e_jy\delta_{l,i}e_ixe_i\right) \\
&= (e_jxe_jye_l) - (e_jye_lxe_l) \\
&= e_j(xe_jy - ye_lx)e_l = e_j(xy - yx)e_l = e_j[x, y]e_l \in e_j\Omega e_l
\end{aligned}$$

This shows that $(\mathrm{ad}_{\mathfrak{l}(\Omega)}(x))(e_j\Omega e_l) \subseteq e_j\Omega e_l$ for $j, l \in [1, n]$. Applying Corollary 46 iteratively shows the equivalence.

Moreover, for $i \in [1, n]$ and $y \in e_i\Omega e_i$, by the calculation above we have

$$\left(\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)\big|_{e_i\Omega e_i}^{e_i\Omega e_i}\right)(y) = [x, y] = e_i(xe_iy - ye_ix)e_i = [e_ixe_i, y] = \left(\mathrm{ad}_{\mathfrak{l}(e_i\Omega e_i)}(e_ixe_i)\right)(y).$$

□

Remark 128. *There exist a discrete valuation ring R, a split R-order Ω, an element $x \in \mathfrak{l}(\Omega)$ and an orthogonal decomposition $1 = e + e'$ of 1_Ω into primitive idempotents in Ω such that $\operatorname{ad}_{\mathfrak{l}(\Omega)}(x)$ is diagonalizable over R but $\operatorname{ad}_{\mathfrak{l}(\Omega)}(exe')$ is not diagonalizable over $\operatorname{frac} R$ and thus in particular it is not diagonalizable over R.*

This shows that we cannot replace the maps $\left(\operatorname{ad}_{\mathfrak{l}(\Omega)}(x)\right)\big|_{e_i\Omega e_j}^{e_i\Omega e_j}$ in the assertion of Lemma 127 by the maps $\operatorname{ad}_{\mathfrak{l}(\Omega)}(e_i x e_j)$.

Proof. Consider the setting of §1 with the $\mathbb{Z}_{(3)}$-order $\Omega \simeq \mathbb{Z}_{(3)}\,\mathrm{S}_3$. We have the Peirce decomposition $\Omega = e\Omega e \oplus e\Omega e' \oplus e'\Omega e \oplus e'\Omega e'$ with

$$e = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \quad e' = \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$$

since $1 = e + e'$ is an orthogonal decomposition of 1_Ω into primitive idempotents in Ω.

Let $x := \left(1, \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, 0\right) \in \Omega$. Then $\operatorname{ad}_{\mathfrak{l}(\Omega)}(x)$ is diagonalizable over R.

In fact, we have

$$(\operatorname{ad}_{\mathfrak{l}(\Omega)}(x))_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let

$$S := \begin{pmatrix} -2 & 0 & 2 & -6 & 2 & 0 \\ 1 & 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_6(\mathbb{Z}_{(3)}).$$

Then $S \cdot (\operatorname{ad}_{\mathfrak{l}(\Omega)} x)_{\mathcal{B},\mathcal{B}} \cdot S^{-1}$ is a diagonal matrix.

But we have

$$exe' = b_3 = \left(0, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, 0\right)$$

and $\operatorname{ad}_{\mathfrak{l}(\Omega)}(b_3)$ is not $\mathbb{Q}$-diagonalizable:

We have $b_3 = exe' \in e\Omega e'$ and

$$A := \left(\operatorname{ad}_{\mathfrak{l}(\Omega)}(b_3)\right)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We have $A^3 = 0 \in R^{6\times 6}$, hence A is nilpotent.

Assume that A is diagonalizable over $\mathbb{Q}$. Then there exists $T \in \mathrm{GL}_n(\mathbb{Q})$ such that $T^{-1}AT = D$ where the matrix D is diagonal and nilpotent. So D is the zero matrix, thus $0 = TDT^{-1} = A$ which is a *contradiction*.

Alternatively we can use Magma to see that $\operatorname{ad}_{\mathfrak{l}(\Omega)}(b_3)$ is not $\mathbb{Q}$-diagonalizable.

Magma Code 6: z3s3Example

```
load pre;
load z3s3Init1;
load definitions;
load z3s3Init2;

rdiag(admatrix(b[3]),3);

Not diagonalizable over Q
false
```

$\square$

For a further counterexample in this context, see Remark 169 below.

4.2.3 The integral core

Keep the notation of §4.2.2.

Lemma 129. *Suppose given an orthogonal decomposition $1_\Omega = \sum_{i\in[1,n]} e_i$ of 1_Ω into primitive idempotents in Ω, where $e_i \in \Omega \cap \Delta$ for $i \in [1,n]$. Let $\mathrm{Z}(\Omega) := \mathrm{C}_\Omega(\Omega)$ be the center of Ω. Define*

$$\mathfrak{t}_0 := {}_R\langle e_1, \ldots, e_n, \mathrm{Z}(\Omega)\rangle$$

as an R-submodule of $\Omega \cap \Delta$. So $\mathfrak{t}_0$ is a Lie subalgebra of $\mathfrak{l}(\Omega \cap \Delta)$.

Then $\mathfrak{t}_0$ is an integral torus in $\mathfrak{l}(\Omega)$.

Proof. Since $\mathfrak{t}_0$ is a Lie subalgebra of $\mathfrak{l}(\Omega \cap \Delta)$, it is also a Lie subalgebra of $\mathfrak{l}(\Omega)$.

It remains to show that the maps $\mathrm{ad}_{\mathfrak{l}(\Omega)}(e_i)$ and $\mathrm{ad}_{\mathfrak{l}(\Omega)}(c)$ are diagonalizable over R for $i \in [1,n]$ and for $c \in \mathrm{Z}(\Omega)$; cf. Lemma 108.(1). By Corollary 126, we know that $\mathrm{ad}_{\mathfrak{l}(\Omega)}(e_i)$ is R-diagonalizable for $i \in [1,n]$. Suppose given $c \in \mathrm{Z}(\Omega)$. Then $cx = xc$ for $x \in \Omega$, so $[c,x] = 0$ for $x \in \Omega$. Thus we have $\mathrm{ad}_{\mathfrak{l}(\Omega)}(c) = 0$. In particular, the map $\mathrm{ad}_{\mathfrak{l}(\Omega)}(c)$ is diagonalizable over R. $\square$

Definition 130. Let $\mathfrak{t}$ be a rational torus in $\mathfrak{l}(\Omega)$. We define

$$\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t}) := \left\{ t \in \mathfrak{t} \,\middle|\, \mathrm{ad}_{\mathfrak{l}(\Omega)}(t) \text{ is diagonalizable over } R \right\}$$

as the *integral core of* $\mathfrak{t}$ *in* $\mathfrak{l}(\Omega)$.

Lemma 131. *Let $\mathfrak{t}$ be a rational torus in $\mathfrak{l}(\Omega)$. Then $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t})$ is an integral torus in $\mathfrak{l}(\Omega)$.*

Proof. We have $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t}) \subseteq \mathfrak{t} \subseteq \mathfrak{l}(\Omega)$ by definition of the integral core; cf. Definition 130.

The rational torus $\mathfrak{t}$ is abelian by Lemma 107, whence it suffices to show that for $t, t' \in \mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t})$ and $r \in R$, we have $(rt + t') \in \mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t})$. In fact, we can write $\mathrm{ad}_{\mathfrak{l}(\Omega)}(rt + t') = r\,\mathrm{ad}_{\mathfrak{l}(\Omega)}(t) + \mathrm{ad}_{\mathfrak{l}(\Omega)}(t')$ which is R-diagonalizable since $[\mathrm{ad}_{\mathfrak{l}(\Omega)}(t), \mathrm{ad}_{\mathfrak{l}(\Omega)}(t')] = \mathrm{ad}_{\mathfrak{l}(\Omega)}([t,t']) = 0$; cf. Corollary 54. Thus we have $rt + t' \in \mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{t})$. $\square$

Corollary 132. *Let $\mathfrak{t}$ be a rational torus in $\mathfrak{l}(\Omega)$. Let $\mathfrak{t}'$ be an integral torus in $\mathfrak{l}(\Omega)$ such that $\mathfrak{t}' \subseteq \mathfrak{t}$. Then $\mathfrak{t}' \subseteq \mathrm{Cor}_{\mathfrak{t}}(\mathfrak{t})$.*

Example 133. In §1.2, where $\Omega \simeq \mathbb{Z}_{(3)}\,\mathrm{S}_3$, we find that the integral core of the full diagonal $\mathfrak{l}(T)$ of $\mathfrak{l}(\Omega)$ in $\mathfrak{l}(\Gamma)$ equals $\mathfrak{l}(T)$.

In §6.4 below, where $\Omega \simeq \mathbb{Z}_{(2)}\,\mathrm{S}_4$, we find that the integral core of the full diagonal $\mathfrak{l}(T)$ of $\mathfrak{l}(\Omega)$ in $\mathfrak{l}(\Gamma)$ is a proper Lie subalgebra of $\mathfrak{l}(T)$. In §7.4 below, where Ω is Morita-equivalent to $\mathbb{Z}_{(2)}\,\mathrm{S}_5$, we also have a proper inclusion.

Remark 134. For an integral torus $\mathfrak{t} \subseteq \mathfrak{l}(\Omega)$ satisfying $\mathfrak{t} \subseteq \mathfrak{l}(T)$, we have $\mathfrak{t} \subseteq \mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$.

However, in general the integral core $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ is not a maximal integral torus in $\mathfrak{l}(\Omega)$; cf. Remark 152.(7) below.

Moreover, in general the integral core $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ is not an (associative) R-subalgebra of Ω; cf. Remark 169 below.

Question 135. Suppose that T is the full diagonal $\Omega \cap \Delta$ of Ω in Γ.

Suppose given an orthogonal decomposition $1_\Omega = \sum_{i\in[1,n]} e_i$ of 1_Ω into primitive idempotents in Ω, where $e_i \in T$ for $i \in [1, n]$. Let $\mathrm{Z}(\Omega) := \mathrm{C}_\Omega(\Omega)$ be the center of Ω.

Define $\mathfrak{t}_0 := {}_R\langle e_1, \ldots, e_n, \mathrm{Z}(\Omega)\rangle$ as an R-submodule of T. Then $\mathfrak{t}_0$ is an integral torus in $\mathfrak{l}(\Omega)$; cf. Lemma 129.

Then $\mathfrak{t}_0$ is contained in $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ since $\mathfrak{t}_0 \subseteq \mathfrak{l}(T)$ and $\mathfrak{t}_0$ is an integral torus in $\mathfrak{l}(\Omega)$; cf. Lemma 129.

We ask if we have equality here.

$$\mathfrak{t}_0 \stackrel{?}{=} \mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$$

4.3 Decompositions of R-orders

Let Ω be an R-order. Let $T \subseteq \Omega$ be a commutative R-subalgebra.

Definition 136. We write T as a direct sum of non-zero ideals of T.

$$T = \bigoplus_{j\in[1,k]} I_j$$

This is also a decomposition of T into T-submodules of T and it is as well a decomposition of T into T-T-sub-bimodules of T.

For $j \in [1, k]$, we define the T-linear maps

$$T \xrightarrow{\pi_j} I_j \xrightarrow{\iota_j} T$$

where π_j is the projection morphism and ι_j is the inclusion morphism. For $j \in [1, k]$, we define

$$(\iota_j \circ \pi_j)(1) =: e_j$$

and thus the composed map $\iota_j \circ \pi_j \colon T \to T$ is given by multiplication by e_j.

$$\begin{aligned} \iota_j \circ \pi_j \colon T &\longrightarrow T \\ 1 &\longmapsto e_j \\ t \cdot 1 &\longmapsto t \cdot e_j \end{aligned}$$

So on the one hand, we have $\mathrm{im}\,(\iota_j \circ \pi_j) = I_j$ because $\iota_j(I_j) = I_j$. On the other hand, we can write the image of $\iota_j \circ \pi_j$ as Te_j. This shows that $I_j = Te_j$.

Remark 137. For $j \in [1, k]$, let I_j and e_j be defined as in Definition 136. We have an orthogonal decomposition of 1_T into idempotents as follows.

$$1 = \sum_{j\in[1,k]} e_j$$

Proof. We have $1 = \mathrm{id}(1) = \left(\sum_{j\in[1,k]} \iota_j \circ \pi_j\right)(1) = \sum_{j\in[1,k]} e_j$.

For $j \in [1, k]$ we have

$$\begin{aligned} e_j^2 = e_j \cdot e_j \cdot 1 &= (\iota_j \circ \pi_j)((\iota_j \circ \pi_j)(1)) \\ &= (\iota_j \circ \underbrace{(\pi_j \circ \iota_j)}_{=\mathrm{id}_{I_j}} \circ \pi_j)(1) \\ &= (\iota_j \circ \pi_j)(1) = e_j. \end{aligned}$$

For $j, j' \in [1, k]$ and $j \neq j'$, we have

$$\begin{aligned} e_j e_{j'} &= e_j e_{j'} \cdot 1 = (\iota_j \circ \pi_j)\left(\iota_{j'} \circ \pi_{j'}(1)\right) \\ &= \Big(\iota_j \circ \underbrace{(\pi_j \circ \iota_{j'})}_{=0} \circ \pi_{j'}\Big)(1) = 0. \end{aligned}$$

$\square$

Remark 138. Suppose given an idempotent $e \in T$. Then the following equivalence holds.

$$Te \text{ is indecomposable as a } T\text{-}T\text{-bimodule} \iff e \text{ is primitive in } T$$

Proof. Ad $\Longleftarrow$. *Assume* that Te is decomposable as a T-T-bimodule. Then we find T-T-sub-bimodules $T' \neq 0$ and $T'' \neq 0$ of Te such that $Te = T' \oplus T''$.

We define the projection maps π', π'' and the inclusion maps ι', ι'' as follows.

$$\begin{tikzcd} Te \arrow[rr, "\iota' \circ \pi'"] \arrow[dr, "\pi'"'] & & Te \\ & T' \arrow[ur, "\iota'"'] & \end{tikzcd} \qquad \begin{tikzcd} Te \arrow[rr, "\iota'' \circ \pi''"] \arrow[dr, "\pi''"'] & & Te \\ & T'' \arrow[ur, "\iota''"'] & \end{tikzcd}$$

We define $(\iota' \circ \pi')(e) =: e'$ and $(\iota'' \circ \pi'')(e) =: e''$.

We have $ee' = e'$: Using the map $\iota' \circ \pi'$, we get $(\iota' \circ \pi')(e) = e'$ and $(\iota' \circ \pi')(ee) = ee'$. But since $e = e^2$, we have $ee' = e'$.

Likewise we get $ee'' = e''$.

Furthermore, we can write the identity map on Te as $\mathrm{id}_{Te} = (\iota' \circ \pi') + (\iota'' \circ \pi'')$. Applying this map to e, we obtain the decomposition $e = ((\iota' \circ \pi') + (\iota'' \circ \pi''))(e) = e' + e''$.

Now

$$\begin{aligned} e'^{\,2} &= e'^{\,2} e = (\iota' \circ \pi')(\iota' \circ \pi'(e)) = (\iota' \circ \underbrace{(\pi' \circ \iota')}_{=\mathrm{id}_{Te}} \circ \pi')(e) \\ &= (\iota' \circ \pi')(e) = e' \end{aligned}$$

and similarly we get $e''^{\,2} = e''$. Moreover, we have

$$e'e'' = e'(e - e') = e'e - e'^{\,2} = e' - e' = 0.$$

So $e = e' + e''$ is an orthogonal decomposition of e into idempotents.

Since $Te' = \mathrm{im}\,(\iota' \circ \pi') = \mathrm{im}\,(\pi') = T'$ and $T' \neq 0$, we conclude that $e' \neq 0$. Similarly, we have $Te'' = \mathrm{im}\,(\iota'' \circ \pi'') = \mathrm{im}\,(\pi'') = T''$ and $T'' \neq 0$, so we conclude that $e'' \neq 0$.

This is a *contradiction* to the primitivity of e in T.

Ad $\Longrightarrow$. *Assume* that e is not primitive in T.

We find an orthogonal decomposition of e into idempotents $e = e' + e''$ with $e', e'' \in T^{\times}$.

We have $e' = e'e' + e'e'' = e'(e' + e'') = e'e \in Te$, so $Te' \subseteq Te$.

Similarly we conclude that $Te'' \subseteq Te$.

Suppose given $x \in Te' \cap Te''$. From $x \in Te'$, we conclude that $x = xe'$. From $x \in Te''$, we conclude that $x = xe''$. Now we calculate $x = xe' = (xe'')e' = x(e'e'') = 0$.

This shows that $Te' \cap Te'' = 0$.

Suppose given $x \in Te$. Then we have $x = xe = x(e' + e'') = xe' + xe''$.

This shows that $Te' + Te'' = Te$.

We conclude that T is decomposable as T-T-bimodule, a *contradiction*. $\square$

Remark 139. *Suppose given an idempotent $e \in \Omega$. Then the following assertions hold.*

(1) *The idempotent e is primitive in Ω if and only if e is primitive in $e\Omega e$.*

(2) *If $e\Omega e$ is local, then the idempotent e is primitive in Ω.*

Proof. Ad (1).

Ad $\Longrightarrow$. Suppose that e is not primitive in $e\Omega e$. Then we find non-zero idempotents $e', e'' \in e\Omega e$ such that $e' + e'' = e$ and $e'e'' = 0 = e''e'$. In particular, we have $e', e'' \in \Omega$, so e is not primitive in Ω.

Ad $\Longleftarrow$. Suppose that e is not primitive in Ω. Choose non-zero idempotents $e', e'' \in \Omega$ such that $e' + e'' = e$ and $e'e'' = 0 = e''e'$. We have

$$ee'e = (e' + e'')e'(e' + e'') = e'e'e' + e'\underbrace{e'e''}_{=0} + \underbrace{e''e'}_{=0}e' + \underbrace{e''e'}_{=0}e'' = e'.$$

This shows that $e' \in e\Omega e$. Similarly we can show that $e'' \in e\Omega e$. So $e = e' + e''$ is a non-trivial orthogonal decomposition of e into idempotents in $e\Omega e$. Hence e is not primitive in $e\Omega e$.

Ad (2). The idempotent e is primitive in $e\Omega e$; cf. Remark 32. Hence e is primitive in Ω by (1). $\square$

Lemma 140. *Suppose that $1_\Omega = \sum_{i\in[1,k]} e_i$ is an orthogonal decomposition of 1_Ω into idempotents in Ω such that $e_i \in T$ for $i \in [1,k]$. Suppose given $l \in [1,k]$. Suppose that T is a maximal commutative subalgebra of Ω and that $e_l\Omega e_l$ is commutative.*

Then the following assertions hold.

(1) *We have $e_l\Omega e_l = Te_l$. We have $\mathrm{End}_{T\text{-}T}(e_l\Omega e_l) \simeq e_l\Omega e_l$ as R-algebras.*

(2) *Consider the T-T-sub-bimodule Te_l of Ω.*

 If $e_l\Omega e_l$ is local, then Te_l is indecomposable as a T-T-bimodule.

Proof. We write $f := e_l$.

Ad (1). We have $T = \bigoplus_{i\in[1,k]} Te_i$. Let $\widetilde{T} := \left(\bigoplus_{i\in[1,k]\setminus\{l\}} Te_i\right) \oplus e_l\Omega e_l$. This is a commutative subalgebra of Ω. Note that $Tf = Te_l = e_lTe_l \subseteq e_l\Omega e_l = f\Omega f$. Since T is a maximal commutative subalgebra of Ω and $\widetilde{T}$ is a commutative subalgebra that contains T, we conclude that $T = \widetilde{T}$ and hence $f\Omega f = Tf$.

By Lemma 25, the R-algebra $\mathrm{End}_{T\text{-}T}(Tf)$ is isomorphic to Tf.

But this is the same as to say that $\mathrm{End}_{T\text{-}T}(f\Omega f) \simeq f\Omega f$ as R-algebras.

Ad (2). By (1), the endomorphism ring $\mathrm{End}_{T\text{-}T}(f\Omega f)$ is isomorphic to $f\Omega f$. Since $f\Omega f$ is local, the endomorphism ring $\mathrm{End}_{T\text{-}T}(f\Omega f)$ also is local. Again by (1), we obtain that $\mathrm{End}_{T\text{-}T}(Tf)$ is local. Then we can apply Lemma 35 to conclude that Tf is indecomposable as a T-T-bimodule. $\square$

Lemma 141. *Suppose that $T_0 \subseteq \mathfrak{l}(\Omega)$ is a summand in a decomposition of $\mathfrak{l}(\Omega)$ into a direct sum of $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$. If T_0 is a T-T-sub-bimodule of Ω and T_0 is indecomposable as an $\mathfrak{l}(T)$-Lie module, then T_0 is indecomposable as a T-T-bimodule.*

Proof. We will prove this by contraposition. Suppose that T_0 is decomposable as a T-T-bimodule. Choose non-zero T-T-sub-bimodules $T_1, T_2 \subseteq T_0$ such that $T_0 = T_1 \oplus T_2$. We have $tT_1t' \subseteq T_1$ and $tT_2t' \subseteq T_2$ for $t, t' \in T$.

Suppose given $t \in T$. Since $1 \in T$, we have $tT_1 \subseteq T_1$ and $T_1t \subseteq T_1$, so $[t, T_1] \subseteq T_1$. Likewise we have $[t, T_2] \subseteq T_2$.

This shows that T_1, T_2 are $\mathfrak{l}(T)$-Lie submodules of T_0, so $T_0 = T_1 \oplus T_2$ is also a decomposition of T_0 into $\mathfrak{l}(T)$-Lie submodules. $\square$

Example 142.

(1) The implication of Lemma 141 can be illustrated with the example of $\Omega \simeq \mathbb{Z}_{(3)}\,\mathrm{S}_3$ in §1.

The example of $\Omega \simeq \mathbb{Z}_{(3)}\,\mathrm{S}_3$ also shows that it can happen that a decomposition of $\mathfrak{l}(\Omega)$ into $\mathfrak{l}(T)$-Lie submodules contains a summand such as ${}_R\langle b_1\rangle$ that is not a T-T-sub-bimodule of Ω, so we cannot apply Lemma 141 here; cf. §1.3.1 and §1.3.2.

Furthermore, in the example of $\Omega \simeq \mathbb{Z}_{(3)}\,\mathrm{S}_3$ the reverse implication of Lemma 141 does hold for all summands that are not contained in T:

All summands in the decomposition of $\mathfrak{l}(\Omega)$ into indecomposable $\mathfrak{l}(T)$-Lie submodules that have trivial intersection with T are in fact T-T-bimodules and as such, they are also indecomposable; cf. also Lemma 143.(2) below.

(2) Consider the example $\mathbb{Z}_{(2)}\,\mathrm{S}_5$ in §7 below. In the decomposition of $\mathfrak{l}(\Omega)$ into indecomposable $\mathfrak{l}(T)$-Lie submodules, there exist summands such as $T_7 = {}_{\mathbb{Z}_{(2)}}\langle b_7, c\rangle$ that are not contained in T, but have non-trivial intersection with T; cf. §7.5.2.

Lemma 143. *Suppose that $1_\Omega = \sum_{l\in[1,k]} e_i$ is an orthogonal decomposition of 1_Ω into primitive idempotents in Ω such that $e_l \in T$ for $l \in [1,k]$.*

Suppose given $i, j \in [1,k]$ such that $i \neq j$. Then the following assertions hold.

(1) *$e_i\Omega e_j$ is a T-T-sub-bimodule of Ω.*

(2) *$e_i\Omega e_j$ is indecomposable as T-T-bimodule if and only if $e_i\Omega e_j$ is indecomposable as $\mathfrak{l}(T)$-Lie module.*

(3) *We have $\mathrm{End}_{T\text{-}T}(e_i\Omega e_j) = \mathrm{End}_{\mathfrak{l}(T)}(e_i\Omega e_j)$ as subrings of $\mathrm{End}_R(e_i\Omega e_j)$.*

Proof. Recall that T is commutative. We have $e_sTe_t = T(e_se_t) = 0$ for $s, t \in [1,k]$ with $s \neq t$. We conclude that $T = \bigoplus_{l\in[1,k]} e_lTe_l \subseteq \bigoplus_{l\in[1,k]} e_l\Omega e_l$.

Ad (1). Note that $Te_l = e_lTe_l = e_lT$ for $l \in [1,k]$.

Thus we have

$$Te_i\Omega e_j = e_iT\Omega e_j \subseteq e_i\Omega e_j$$

and also

$$e_i\Omega e_jT = e_i\Omega Te_j \subseteq e_i\Omega e_j.$$

So $e_i\Omega e_j$ is a T-T-sub-bimodule of Ω.

Ad (2). Ad $\Longleftarrow$. This is Lemma 141, using (1).

Ad $\Longrightarrow$. We will prove this by contraposition. Suppose given a decomposition $e_i\Omega e_j = M_1 \oplus M_2$ into non-zero $\mathfrak{l}(T)$-Lie submodules. It suffices to show that M_1 and M_2 are T-T-sub-bimodules of $e_i\Omega e_j$. Because of symmetric reasons it suffices to show that M_1 is a T-T-sub-bimodule of $e_i\Omega e_j$.

Suppose given $u, v \in [1,k]$ and $t \in e_uTe_u$ and $t' \in e_vTe_v$. It suffices to show that $tM_1t' \subseteq M_1$. Recall that $M_1 \subseteq e_i\Omega e_j$, so $tM_1t' = 0$ if $u \neq i$ or $v \neq j$.

But if $u = i$ and $v = j$, then $M_1t = 0$ and $t'M_1 = 0$ because of $i \neq j$. So we have $tM_1 = [t, M_1]$ and we have $M_1t' = [M_1, t']$. Now M_1 and M_2 are $\mathfrak{l}(T)$-Lie submodules of $e_i\Omega e_j$, so $[t, M_1] \subseteq M_1$ and $[M_1, t'] \subseteq M_1$.

We obtain that $(tM_1)t' \subseteq M_1t' \subseteq M_1$, completing the proof.

Ad (3). Suppose given $x \in e_i\Omega e_j$ and $t \in e_lTe_l = Te_l$ for some $l \in [1,k]$. We obtain the following.

$$[t, x] = \begin{cases} 0 & \text{if } l \notin \{i, j\} \\ tx & \text{if } l = i \\ -xt & \text{if } l = j \end{cases}$$

Suppose given $h \in \mathrm{End}_R(e_i\Omega e_j)$. Applying this map, we obtain the following.

$$h([t,x]) = \begin{cases} 0 & \text{if } l \notin \{i,j\} \\ h(tx) & \text{if } l = i \\ h(-xt) & \text{if } l = j \end{cases} \qquad \text{and} \qquad [t,h(x)] = \begin{cases} 0 & \text{if } l \notin \{i,j\} \\ th(x) & \text{if } l = i \\ -h(x)t & \text{if } l = j \end{cases} \tag{14}$$

Using this equation, we calculate.

$$\begin{aligned}
\mathrm{End}_{T\text{-}T}(e_i\Omega e_j) &= \big\{h \in \mathrm{End}_R(e_i\Omega e_j) \mid h(tx) = th(x) \text{ for } x \in e_i\Omega e_j \text{ and } t \in T \text{ and} \\
&\qquad\qquad h(xt') = h(x)t' \text{ for } x \in e_i\Omega e_j \text{ and } t' \in T\big\}. \\
&= \big\{h \in \mathrm{End}_R(e_i\Omega e_j) \mid h(tx) = th(x) \text{ for } x \in e_i\Omega e_j \text{ and } t \in Te_l \text{ for } l \in [1,k] \text{ and} \\
&\qquad\qquad h(xt') = h(x)t' \text{ for } x \in e_i\Omega e_j \text{ and } t' \in Te_l \text{ for } l \in [1,k]\big\}. \\
&= \big\{h \in \mathrm{End}_R(e_i\Omega e_j) \mid h(tx) = th(x) \text{ for } x \in e_i\Omega e_j \text{ and } t \in Te_i \text{ and} \\
&\qquad\qquad h(xt') = h(x)t' \text{ for } x \in e_i\Omega e_j \text{ and } t' \in Te_j\big\} \\
&\overset{(14)}{=} \big\{h \in \mathrm{End}_R(e_i\Omega e_j) \mid h([t,x]) = [t,h(x)] \text{ for } x \in e_i\Omega e_j \text{ and } t \in Te_l \text{ for } l \in [1,k]\big\} \\
&= \big\{h \in \mathrm{End}_R(e_i\Omega e_j) \mid h([t,x]) = [t,h(x)] \text{ for } x \in e_i\Omega e_j \text{ and } t \in T\big\} \\
&= \mathrm{End}_{\mathfrak{l}(T)}(e_i\Omega e_j)
\end{aligned}$$

□

Lemma 144. *Suppose that $1_\Omega = \sum_{i\in[1,k]} e_i$ is an orthogonal decomposition of 1_Ω into idempotents in T. Suppose given $i,j,i',j' \in [1,k]$ such that $e_i\Omega e_j \neq 0$.*

Then $e_i\Omega e_j$ and $e_{i'}\Omega e_{j'}$ are isomorphic as T-T-bimodules if and only if $(i,j) = (i',j')$.

Proof. If $(i,j) = (i',j')$, then $e_i\Omega e_j = e_{i'}\Omega e_{j'}$ and we obtain the identity isomorphism.

If $(i,j) \neq (i',j')$, then $e_ie_{i'} = 0$ or $e_{j'}e_j = 0$ and thus $e_i(e_{i'}\Omega e_{j'})e_j = 0$. We apply Lemma 26 and we conclude that $e_i\Omega e_j$ and $e_{i'}\Omega e_{j'}$ are not isomorphic as T-T-bimodules. □

Question 145. Consider the group ring $\mathbb{Z}_{(p)}\,\mathrm{S}_n$ for some $n \geq 1$ and some prime p dividing $n!$. Suppose given a Wedderburn embedding $\omega\colon \mathbb{Z}_{(p)}\,\mathrm{S}_n \to \prod_{i\in[1,l]} \mathbb{Z}_{(p)}^{n_i\times n_i} =: \Gamma$ such that its image Ω admits an orthogonal decomposition of 1_Ω into primitive idempotents in Ω by $1 = \sum_{i\in[1,k]} e_i$ where e_i is contained in the full diagonal of Ω in Γ for $i \in [1,k]$.

Writing $T := \Omega \cap \Delta$, we ask whether $e_i\Omega e_j$ is indecomposable as a T-T-bimodule for $i,j \in [1,k]$ with $i \neq j$.

However, cf. Remark 172 below.

4.4 Primitive tori

Let Γ be a completely split R-order. Let Ω be a split R-order in Γ. Let Δ be the full diagonal in Γ.

Definition 146. Suppose given a maximal rational torus $\mathfrak{t}$ in $\mathfrak{l}(\Omega)$. We say that $\mathfrak{t}$ is a *primitive torus in* $\mathfrak{l}(\Omega)$ if there exists an orthogonal decomposition $1_\Omega = \sum_{i\in[1,k]} e_i$ into primitive idempotents in Ω such that $e_i \in \mathfrak{t}$ for $i \in [1,k]$ and such that $e_i\Omega e_i$ is local for $i \in [1,k]$.

Example 147. The rational tori considered in our examples in §1.2, in §6.3 and in §7.3 are primitive.

Remark 148. Suppose given an idempotent $e \in \Delta$ that is primitive in Ω. Then e is primitive in $\Omega \cap \Delta$.

However, in general, the reverse implication does not hold; cf. Remark 152.(5).

Lemma 149. *Suppose given two primitive tori $\mathfrak{t}$ and $\mathfrak{t}'$ of $\mathfrak{l}(\Omega)$. Choose an orthogonal decomposition $1_\Omega = \sum_{i\in[1,m]} e_i$ of 1_Ω into primitive idempotents in Ω such that $e_i \in \mathfrak{t}$ and such that $e_i\Omega e_i$ is local for $i \in [1,n]$. Choose an orthogonal decomposition of 1_Ω into primitive idempotents $1 = \sum_{i\in[1,n]} e_i'$ in Ω such that $e_i \in \mathfrak{t}'$.*

Then $m = n$ and there exists $u \in \mathrm{U}(\Omega)$ such that

$$u^{-1}\left(\bigoplus_{i\in[1,n]} e_i'\Omega e_i'\right)u = \bigoplus_{i\in[1,n]} e_i\Omega e_i.$$

We illustrate this situation with the following diagram.

$$\begin{tikzcd}
\bigoplus_{i\in[1,n]} e_i\Omega e_i \arrow[rr, leftrightarrow, "\text{conjugate}", "\text{via } u\in\mathrm{U}(\Omega)"'] & & \bigoplus_{i\in[1,n]} e_i'\Omega e_i' \\
\mathfrak{t} \arrow[u, hook] & & \mathfrak{t}' \arrow[u, hook]
\end{tikzcd}$$

Note that it might happen that the vertical inclusions are strict; cf. §7. In Remark 150 we see two primitive tori in a split R-order Ω that are not conjugate via a unit in Ω.

Proof. By Lemma 176, we have $m = n$ and there exist $u \in \mathrm{U}(\Omega)$ and $\sigma \in \mathrm{S}_n$ such that $u^{-1}e_i'u = e_{\sigma(i)}$ for $i \in [1,n]$. Now conjugation with $u \in \mathrm{U}(\Omega)$ is an R-algebra automorphism of Ω. So we have

$$u^{-1}\left(\bigoplus_{i\in[1,n]} e_i'\Omega e_i'\right)u = \bigoplus_{i\in[1,n]} \underbrace{u^{-1}e_i'u}_{e_{\sigma(i)}}\,\Omega\,\underbrace{u^{-1}e_i'u}_{e_{\sigma(i)}}$$

which equals $\bigoplus_{i\in[1,n]} e_i\Omega e_i$. $\square$

Remark 150. *There exist a discrete valuation ring R, a split R-order Ω and two primitive tori of $\mathfrak{l}(\Omega)$ that are not conjugate via a unit in Ω.*

Proof. Let $R = \mathbb{Z}_{(2)}$ and let

$$\Omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2} \,\middle|\, a \equiv_2 d \text{ and } c \equiv_2 0 \right\}.$$

Then Ω is a split R-order in $\Gamma := R^{2\times 2}$. Let $u := \left(\begin{smallmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{U}(K\Omega)$. Note that $u \notin \Omega$ and, in particular, $u \notin \mathrm{U}(\Omega)$. We define $t := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and $t' := u^{-1}tu = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. Furthermore, we define

$$T := {}_R\langle \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle = {}_R\langle t, 1\rangle$$

and

$$T' := {}_R\langle \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle = {}_R\langle t', 1\rangle.$$

Then T is the full diagonal of Ω in Γ, so $\mathfrak{l}(T)$ is a maximal rational torus in $\mathfrak{l}(\Omega)$; cf. Lemma 120. Moreover, the describing matrix of $\mathrm{ad}_{\mathfrak{l}(\Omega)} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ with respect to the R-linear basis

$$\mathcal{B} := \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right) \right)$$

of Ω is a diagonal matrix. So $\mathfrak{l}(T)$ is also an integral torus in $\mathfrak{l}(\Omega)$; cf. Lemma 108.(1).

We determine the matrix $\left(\mathrm{ad}_{\mathfrak{l}(\Omega)}(\left(\begin{smallmatrix}0&1\\0&2\end{smallmatrix}\right))\right)_{\mathcal{B},\mathcal{B}}$.

$$A := \left(\mathrm{ad}_{\mathfrak{l}(\Omega)}(\left(\begin{smallmatrix}0&1\\0&2\end{smallmatrix}\right))\right)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The eigenspaces of A are the following.

$$\mathrm{E}_A(0) = {}_{\mathbb{Q}}\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \rangle, \qquad \mathrm{E}_A(2) = {}_{\mathbb{Q}}\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \rangle, \qquad \mathrm{E}_A(-2) = {}_{\mathbb{Q}}\langle \begin{pmatrix}2\\-2\\-1\\2\end{pmatrix} \rangle$$

These four vectors form a $\mathbb{Q}$-linear basis of $\mathbb{Q}^{4\times 1}$, hence $\mathfrak{l}(T')$ is a rational torus in $\mathfrak{l}(\Omega)$. Intersecting the eigenspaces with $\mathbb{Z}_{(2)}^{4\times 1}$, we obtain the eigenmodules of A. But e.g. the element e_4 is not contained in an eigenmodule of A. So we conclude that A is not R-diagonalizable; cf. Corollary 48.(1). Thus $\mathfrak{l}(T')$ is not an integral torus in $\mathfrak{l}(\Omega)$.

Suppose given $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Omega$ such that $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) t' = t' \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix}0&a+2b\\0&c+2d\end{smallmatrix}\right) = \left(\begin{smallmatrix}c&d\\2c&2d\end{smallmatrix}\right)$, so $c = 0$ and $a + 2b = d$. We conclude that $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in {}_R\langle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&2\end{smallmatrix}\right) \rangle = T'$.

This shows that $\mathfrak{l}(T')$ is a maximal rational torus in $\mathfrak{l}(\Omega)$.

Suppose given $A \in \Omega$. There exist $a, b, c, d \in R$ such that $A = \left(\begin{smallmatrix}a&b\\2c&a+2d\end{smallmatrix}\right)$. The determinant of A is $a^2 + 2ad - 2bc$. We conclude that $\det(A) \in \mathrm{U}(R)$ if and only if $a \in \mathrm{U}(R)$. Moreover, if $\det(A) \in \mathrm{U}(R)$, then $(\det A)^{-1} \left(\begin{smallmatrix}a+2d&-b\\-2c&a\end{smallmatrix}\right) \in \Omega$.

This shows that $A = \left(\begin{smallmatrix}a&b\\2c&a+2d\end{smallmatrix}\right) \in \mathrm{U}(\Omega)$ if and only if $a \in \mathrm{U}(R)$.

Suppose given $A, A' \in \Omega \setminus \mathrm{U}(\Omega)$. Then there exist $a, b, c, d, a', b', c', d' \in R$ such that $A = \left(\begin{smallmatrix}a&b\\2c&a+2d\end{smallmatrix}\right)$ and $A' = \left(\begin{smallmatrix}a'&b'\\2c'&a'+2d'\end{smallmatrix}\right)$. We conclude that $a, a' \notin \mathrm{U}(R)$. Since R is local, we conclude that $a + a' \notin \mathrm{U}(R)$. But then $\det(A + A') = (a + a')(a + 2d + a' + 2d') - (b + b')(2c + 2c') \equiv_2 (a + a')^2 \notin \mathrm{U}(R)$.

This shows that Ω is local.

So $1_\Omega = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ is a primitive idempotent in Ω and it is contained in T and in T'.

Thus $\mathfrak{l}(T)$ and $\mathfrak{l}(T')$ are primitive tori in $\mathfrak{l}(\Omega)$

Suppose given $\tilde{t} \in T$. Then we find $u, v \in R$ such that $\tilde{t} = \left(\begin{smallmatrix}u&0\\0&u+2v\end{smallmatrix}\right)$.

Assume that there exists $w \in \mathrm{U}(\Omega)$ such that $w^{-1} t' w = \tilde{t}$. We find $a, b, c, d \in R$ such that we have $w = \left(\begin{smallmatrix}a&b\\2c&a+2d\end{smallmatrix}\right) \in \mathrm{U}(\Omega)$. Then $a \equiv_2 1$. We have

$$\begin{aligned}
\begin{pmatrix} a & b \\ 2c & a+2d \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ 2c & a+2d \end{pmatrix} &= \begin{pmatrix} u & 0 \\ 0 & u+2v \end{pmatrix} \\
\iff \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ 2c & a+2d \end{pmatrix} &= \begin{pmatrix} a & b \\ 2c & a+2d \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u+2v \end{pmatrix} \\
\iff \begin{pmatrix} 2c & a+2d \\ 4c & 2a+4d \end{pmatrix} &= \begin{pmatrix} au & bu+2bv \\ 2cu & au+2av+2du+4dv \end{pmatrix}
\end{aligned}$$

We conclude that $au \equiv_2 0$ from position (1,1). Since $a \equiv_2 1$, we have $u \equiv_2 0$. The equation on position (1,2) is $a + 2d = bu + 2v$. But $a + 2d \equiv_2 a \equiv_2 1$ and $bu + 2v \equiv_2 bu \equiv_2 0$ since $u \equiv_2 0$. This is a *contradiction*.

This shows that there is no $u \in \mathrm{U}(\Omega)$ such that $u^{-1} T' u = T$. We conclude that primitive tori are not unique, not even unique up to conjugation with units in Ω. □

Question 151. We ask whether every maximal rational torus in $\mathfrak{l}(\Omega)$ is a subalgebra of Ω.

4.5 A counterexample

Remark 152. *There exist a discrete valuation ring R and a split R-order Ω isomorphic to RG for a finite group G such that the assertions* (1) – (7) *hold.*

(1) *There exist two maximal commutative R-subalgebras T and T_1 of Ω that are not isomorphic as R-algebras.*

(2) *There exist two maximal commutative R-subalgebras T and T_1 of Ω such that $\mathfrak{l}(T)$ and $\mathfrak{l}(T_1)$ are maximal rational tori of $\mathfrak{l}(\Omega)$, but T and T_1 are not conjugate via a unit in $K\Omega$.*

(3) *There exist two maximal commutative R-subalgebras T and T_1 of Ω such that $\mathfrak{l}(T_1) \subseteq \mathfrak{l}(\Omega)$ is a non-primitive maximal rational torus and $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$ is a primitive torus.*

(4) *There exist a completely split R-overorder $\Gamma \supseteq \Omega$ with full diagonal Δ and $u \in \mathrm{U}(\Gamma)$ such that, writing $\mathring{\Omega} := u^{-1}\Omega u$, the lengths of the R-modules $\Delta/(\Omega \cap \Delta)$ and $\Delta/(\mathring{\Omega} \cap \Delta)$ are different.*

(5) *There exists a maximal commutative R-subalgebra T_1 of Ω such that $\mathfrak{l}(T_1) \subseteq \mathfrak{l}(\Omega)$ is a maximal rational torus and 1_Ω is primitive in T_1, but not primitive in Ω.*

(6) *There exist two maximal commutative R-subalgebras T and T_1 of Ω such that $\mathfrak{l}(T)$ and $\mathfrak{l}(T_1)$ are maximal rational tori of $\mathfrak{l}(\Omega)$ and such that the integral cores of $\mathfrak{l}(T)$ and of $\mathfrak{l}(T_1)$ in $\mathfrak{l}(\Omega)$ have, considered as R-modules, different ranks.*

(7) *There exists a completely split R-overorder $\Gamma \supseteq \Omega$ and $u \in \mathrm{U}(\Gamma)$ such that the full diagonal of $\mathfrak{l}(\Omega)$ in $\mathfrak{l}(\Gamma)$ is a maximal integral torus in $\mathfrak{l}(\Omega)$, but, writing $\mathring{\Omega} := u^{-1}\Omega u$, the integral core of the full diagonal of $\mathfrak{l}(\mathring{\Omega})$ in $\mathfrak{l}(\Gamma)$ is not a maximal integral torus in $\mathfrak{l}(\mathring{\Omega})$.*

Proof. Let $R := \mathbb{Z}_{(3)}$ and $\Gamma := R \times R^{2\times 2} \times R$.

We have seen in §1 that $\mathbb{Q}\,\mathrm{S}_3$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}^{2\times 2} \times \mathbb{Q}$. By the Artin-Wedderburn-Theorem, we have uniqueness of the skew fields and of the sizes of the matrix rings over these skew fields. But we can choose different isomorphisms.

We will use the following two Wedderburn isomorphisms in this example.

$$\begin{array}{rccc} \omega: & \mathbb{Q}\,\mathrm{S}_3 & \longrightarrow & \mathbb{Q} \times \mathbb{Q}^{2\times 2} \times \mathbb{Q} \\ & (1,2) & \longmapsto & \left(1, \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}, -1\right) \\ & (1,2,3) & \longmapsto & \left(1, \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}, 1\right) \end{array} \qquad \begin{array}{rccc} \mathring{\omega}: & \mathbb{Q}\,\mathrm{S}_3 & \longrightarrow & \mathbb{Q} \times \mathbb{Q}^{2\times 2} \times \mathbb{Q} \\ & (1,2) & \longmapsto & \left(1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -1\right) \\ & (1,2,3) & \longmapsto & \left(1, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, 1\right) \end{array}$$

Recall that ω is the morphism we used in §1 where we also showed that it is an isomorphism of $\mathbb{Q}$-algebras. The morphism $\mathring{\omega}$ is obtained by conjugating ω with $u := \left(1, \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right), 1\right) \in \mathrm{U}(\Gamma)$ from the right. So also $\mathring{\omega}$ is an isomorphism of $\mathbb{Q}$-algebras.

Using the notation of §1.1, we have

$$\Omega = \omega(R\,\mathrm{S}_3) = \left\{ \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\right) \in R \times R^{2\times 2} \times R \;\middle|\; a \equiv_3 b,\ e \equiv_3 f,\ c \equiv_3 0 \right\}.$$

We determine the images of the elements of S_3 under $\mathring{\omega}$. Since S_3 is an R-linear basis of $\mathbb{Q}\,\mathrm{S}_3$, we can thus determine the ties needed to describe the image of $R\,\mathrm{S}_3$ under $\mathring{\omega}$. Each element $\left(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f\right) \in \mathring{\omega}(\mathrm{S}_3)$ yields a row (a, b, c, d, e, f) in the matrix $U_{\mathring{\omega}} \in \mathbb{Q}^{6\times 6}$ with entries in R. Thus we obtain

$$U_{\mathring{\omega}} := \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \quad \text{and } 6 \cdot U_{\mathring{\omega}}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & -2 & 0 & 2 & -2 \\ 0 & 2 & -2 & 2 & 0 & -2 \\ 0 & 2 & 0 & -2 & -2 & 2 \\ 2 & 0 & 2 & -2 & -2 & 0 \\ 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

The ties modulo 6 needed to describe $\mathring{\Omega}$ are given by the columns of $6 \cdot U_{\mathring{\omega}}^{-1}$. Applying elementary column operations on $6 \cdot U_{\mathring{\omega}}^{-1}$ we obtain the following matrix. From its columns we get another set of ties that also describe $\mathring{\Omega}$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 4 & 6 & 0 \\ 3 & 4 & 0 & 2 & 0 & 6 \end{pmatrix}$$

Since $2 \in \mathrm{U}(R)$, this results in the following.

$$\mathring{\Omega} := \mathring{\omega}(R\,\mathrm{S}_3) = \left\{ \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f \right) \in R \times R^{2\times 2} \times R \;\middle|\; a \equiv_3 c + e,\, b \equiv_3 c + f,\, e \equiv_3 d + f \right\}$$

We have the full diagonal $\Delta := \left\{ (a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f) \in \Gamma \;\middle|\; c = 0,\, d = 0 \right\}$ in Γ. Then we define the following intersections.

$$T := \Omega \cap \Delta = \left\{ \left(a, \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}, f \right) \in R \times R^{2\times 2} \times R \;\middle|\; a \equiv_3 b,\, e \equiv_3 f \right\}$$
$$\mathring{T} := \mathring{\Omega} \cap \Delta = \left\{ \left(a, \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}, f \right) \in R \times R^{2\times 2} \times R \;\middle|\; a \equiv_3 e,\, b \equiv_3 f,\, e \equiv_3 f \right\}$$

By Lemma 120 we know that $\mathfrak{l}(T)$ is a maximal integral torus in $\mathfrak{l}(\Omega)$ and that $\mathfrak{l}(\mathring{T})$ is a maximal rational torus in $\mathfrak{l}(\mathring{\Omega})$. Moreover, in §1.1 we have found idempotents $e, e' \in T$ that are primitive in Ω and such that $1_\Omega = e + e'$ and $e\Omega e$ and $e'\Omega e'$ are local. So $\mathfrak{l}(T)$ is a primitive torus in $\mathfrak{l}(\Omega)$.

We choose R-linear bases $\mathcal{B}_T$ of T and $\mathcal{B}_{\mathring{T}}$ of $\mathring{T}$.

$$\mathcal{B}_T := \left(\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \left(3, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right) \right) \tag{15}$$
$$\mathcal{B}_{\mathring{T}} := \left(\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), \left(0, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right) \right) \tag{16}$$

Note that $\Delta/(\Omega \cap \Delta)$ is an R-module of length 2 and $\Delta/(\mathring{\Omega} \cap \Delta)$ is an R-module of length 3.

This shows (4).

By Lemma 33, $\mathring{T}$ is a local ring. In particular, in $\mathring{T}$ there are only the idempotents $(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0)$ and $(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1)$.

But in T, we have the non-trivial idempotents $(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0)$ and $(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1)$ and thus T is not a local ring.

This shows that $T \not\simeq \mathring{T}$ as R-algebras.

We conclude that $\mathring{T}$ cannot be obtained by conjugating T with an element in $\mathrm{U}(K\Gamma)$ and in particular not with an element in $\mathrm{U}(\Gamma)$ or in $\mathrm{U}(\Omega)$.

An R-linear basis $\mathcal{C}$ of $\mathring{\Omega}$ is given as follows.

$$\mathcal{C} = \Bigg(\left(1, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, 0\right), \left(1, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, 0\right), \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right),$$
$$\left(0, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right) \Bigg)$$

We consider the rational torus $\mathfrak{l}(\mathring{T}) \subseteq \mathfrak{l}(\mathring{\Omega})$. We calculate.

$$\left(\mathrm{ad}_{\mathfrak{l}(\mathring{\Omega})}\left(\left(0, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)\right)\right)_{\mathcal{C},\mathcal{C}} = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(\mathrm{ad}_{\mathfrak{l}(\mathring{\Omega})}\left(\left(0, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, 0\right)\right)\right)_{\mathcal{C},\mathcal{C}} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

These matrices are not diagonalizable over R as we see e.g. using Magma.

Magma Code 7: counterex

```
load pre;
load z3s3Init1;
load definitions;

Basis_Omegacirc := Matrix([
   [1,1,1,0,0,0],
   [0,1,1,3,0,0],
   [0,1,0,0,0,0],
   [1,0,0,0,0,0],
   [1,0,1,0,3,0],
   [0,0,1,0,0,3]
]);

admatrix := function(x)
// INPUT: x an element of Gamma
// OUTPUT: matrix ad(x) with respect to the basis Basis_Omegacirc
   pre_ad := RMBQ!0;
   for j in [1..rl] do
      v_tup := LieTup(x,CoerceGamma([Basis_Omegacirc[i,j] : i in [1..rl
         ]]));
      v_vec := &cat[ElementToSequence(v_tup[i]):i in [1..#Sizes]];
      for i in [1..rl] do
         pre_ad[i,j] := v_vec[i];
      end for;
   end for;
   return RMB!((RMBQ!Basis_Omegacirc)^-1*pre_ad);
end function;

rdiag(admatrix(CoerceGamma([0,3,0,0,0,0])),3);
rdiag(admatrix(CoerceGamma([0,0,0,0,3,0])),3);
```

This shows that $\mathfrak{l}(\mathring{T})$ is not an integral torus of $\mathring{\Omega}$.

But since $(0, \left(\begin{smallmatrix} 3 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0) + (0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 3 \end{smallmatrix}\right), 0) = (0, \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), 0)$ is central in $\mathring{\Omega}$, its adjoint endomorphism is zero, in particular it is diagonalizable over R. So $(0, \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), 0)$ is an element of the integral core $\mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T}))$. We conclude that

$$\mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T})) = {}_R\langle \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), \left(0, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right) \rangle. \tag{17}$$

Recall that $\mathring{\Omega}$ is obtained by conjugating Ω with $u = (1, \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right), 1)$ from the right, so $\mathring{\Omega} = u^{-1}\Omega u$. We define the conjugate R-algebra

$$T_1 := u\mathring{T}u^{-1} \subseteq u\mathring{\Omega}u^{-1} = \Omega.$$

Then $\mathfrak{l}(T_1)$ is a rational torus in $\mathfrak{l}(\Omega)$; cf. Corollary 124.(2).

An R-linear basis of T_1 can be obtained by conjugating the elements of $\mathcal{B}_{\mathring{T}}$ with u from the left. This yields the following.

$$T_1 = u\mathring{T}u^{-1} = {}_R\langle \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), \left(0, \begin{pmatrix} 0 & 6 \\ 0 & 3 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix}, 0\right), \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3\right) \rangle$$

Note that $T_1 \not\subseteq \Delta$. We show the maximality of the torus $\mathfrak{l}(T_1) \subseteq \mathfrak{l}(\Omega)$ by direct calculation.

Suppose given $t := \left(a + 3b, \begin{pmatrix} a & 3c \\ d & e \end{pmatrix}, e + 3f\right) \in \Omega$ where $a, b, c, d, e, f \in R$ such that t commutes with $\left(0, \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix}, 0\right) \in T_1$. Then, in particular, we have

$$\begin{pmatrix} 3a & -6a \\ 3d & -6d \end{pmatrix} = \begin{pmatrix} a & 3c \\ d & e \end{pmatrix} \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 3c \\ d & e \end{pmatrix} = \begin{pmatrix} 3a - 6d & 9c - 6e \\ 0 & 0 \end{pmatrix}.$$

We conclude that $d = 0$ and $-6a = 9c - 6e$. But this implies that $t \in T_1$.

This shows that $\mathfrak{l}(T_1) \subseteq \mathfrak{l}(\Omega)$ is a maximal rational torus.

Moreover, $\mathfrak{l}(T_1)$ is not a primitive torus in $\mathfrak{l}(\Omega)$ since $1_\Omega = (1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1)$ is not primitive in Ω but 1_Ω is primitive in T_1 since T_1 is conjugate to $\mathring{T}$ which is local.

This shows (3).

Recall that $u\mathring{\Omega}u^{-1} = \Omega$. In Ω, the idempotents $e := \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)$ and $e' := \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$ are primitive. Its sum is 1_Ω and we have $ee' = e'e = 0$, so $1_\Omega = e + e'$ is an orthogonal decomposition of 1_Ω into primitive idempotents in Ω. In particular, 1_Ω is not primitive in Ω.

But we have seen that $\mathring{T}$ is local, so its conjugate T_1 also is local. We conclude that $1_{\mathring{\Omega}}$ is primitive in $\mathring{T}_1$.

This shows (5).

Note that the following products are in T_1.

$$\left(0, \begin{pmatrix} 0 & 6 \\ 0 & 3 \end{pmatrix}, 0\right)^2 = 3 \cdot \left(0, \begin{pmatrix} 0 & 6 \\ 0 & 3 \end{pmatrix}, 0\right)$$
$$\left(0, \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix}, 0\right)^2 = 3 \cdot \left(0, \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix}, 0\right)$$
$$\left(0, \begin{pmatrix} 0 & 6 \\ 0 & 3 \end{pmatrix}, 0\right) \cdot \left(0, \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix}, 0\right) = 0$$

Moreover, we have $1_\Omega \in T_1$. This shows that T_1 is an R-subalgebra of Ω. We conclude that T_1 is a maximal commutative R-subalgebra of Ω.

Note that $T_1 \neq T$. So we have found two maximal rational tori $\mathfrak{l}(T_1)$ and $\mathfrak{l}(T)$ in Ω. Moreover, both T and T_1 are maximal commutative R-subalgebras of Ω. As R-algebras, they are not isomorphic since T is not local while T_1 as conjugate of the local R-algebra $\mathring{T}$ is local.

This shows (1) and (2).

We determine the integral core of $\mathfrak{l}(T_1)$ in $\mathfrak{l}(\Omega)$. This is obtained by conjugating $\mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T}))$ with u from the right. But we observe that $\mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T}))$ consists of elements that are central in $K\Gamma$. So we conclude that

$$\mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T})) = \mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T_1)).$$

In particular, as an R-module, this is of rank 3 whereas $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ as an R-module is of rank 4. This shows (6).

We consider $c := (1, \left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right), 0) \in \mathring{\Omega}$. Note that $c \notin \Delta$. We have

$$(\mathrm{ad}_{\mathfrak{l}(\mathring{\Omega})}(c))_{\mathcal{C},\mathcal{C}} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

This is a matrix that is diagonalizable over $\mathbb{Z}_{(2)}$.

Recall that $\mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T}))$ is central in $\mathring{\Omega}$; cf. equation (17). Denote by T_2 the R-algebra generated by $\mathring{T}$ and c. Then $\mathfrak{l}(T_2)$ is an integral torus in $\mathfrak{l}(\mathring{\Omega})$.

So the integral core $\mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T}))$ is maximal as an integral torus in $\mathring{T}$. But it is not maximal as an integral torus in $\mathfrak{l}(\mathring{\Omega})$ since $\mathfrak{l}(T_2) \supset \mathrm{Cor}_{\mathfrak{l}(\mathring{\Omega})}(\mathfrak{l}(\mathring{T}))$ is a proper inclusion and T_2 is an integral torus in $\mathfrak{l}(\mathring{\Omega})$. This shows (7). $\square$

Chapter 5: Certain local $\mathbb{Z}_{(2)}$-algebras

Let $R := \mathbb{Z}_{(2)}$.

In the following we will collect some $\mathbb{Z}_{(2)}$-algebras. Some of these algebras are given as matrix algebras, others are given by descriptions using ties.

We will show that the $\mathbb{Z}_{(2)}$-algebras under consideration are local $\mathbb{Z}_{(2)}$-algebras. For this purpose we will use two different approaches:

The first one will use the definition of a local ring and verify the required properties ad hoc.

The second one will use further arguments on Jacobson radicals and factor algebras. However, the Jacobson radicals are just used in the proof of Lemma 33. We will make use of this Lemma without determining Jacobson radicals here.

Moreover, for some of these $\mathbb{Z}_{(2)}$-algebras we will give descriptions as polynomial factor rings.

The local $\mathbb{Z}_{(2)}$-algebras covered here will occur in the context of $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ and of $\mathbb{Z}_{(2)}\,\mathrm{S}_5$; cf. §6 and §7 below.

Recall the following characterizations of the units in R.

Remark 153. For $r \in R^\times$, the following assertions are equivalent.

(1) $r \in \mathrm{U}(R)$

(2) $\mathrm{v}_2(r) = 0$

(3) $r \not\equiv_2 0$

(4) $r \in R\backslash 2R$

Lemma 154. *Suppose given R-algebras A and B such that $A \subseteq B$ is a subalgebra. Suppose that A is generated R-linearly by $x_1, \ldots, x_n \in A$. Then we have the following equality.*

$$\mathrm{C}_B(A) = \{b \in B \mid bx_i = x_i b \text{ for } i \in [1, n]\}$$

Proof. Ad $\subseteq$. Suppose given $b \in \mathrm{C}_B(A)$. Then $ba = ab$ for $a \in A$, in particular $bx_i = x_i b$ for $i \in [1, n]$.

Ad $\supseteq$. Suppose given $b \in B$ such that $bx_i = x_i b$ for $i \in [1, n]$. Suppose given $a \in A$. There exist $r_1, \ldots, r_n \in R$ such that $a = \sum_{i\in[1,n]} r_i x_i$. We obtain

$$\begin{aligned} ab &= \left(\sum_{i\in[1,n]} r_i x_i\right) b = \sum_{i\in[1,n]} (r_i x_i b) \\ &= \sum_{i\in[1,n]} (r_i(x_i b)) = \sum_{i\in[1,n]} (r_i(bx_i)) \\ &= \sum_{i\in[1,n]} (b(r_i x_i)) = b\left(\sum_{i\in[1,n]} r_i x_i\right) = ba, \end{aligned}$$

showing that $b \in \mathrm{C}_B(A)$. $\square$

5.1 The $\mathbb{Z}_{(2)}$-algebra L_1

We define the R-subalgebra A_1 of $R^{2\times 2}$ as follows.

$$A_1 := \left\{ \begin{pmatrix} a & 0 \\ b & a+2b \end{pmatrix} \,\middle|\, a, b \in R \right\} = {}_R\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\rangle \tag{18}$$

To determine $\mathrm{C}_{R^{2\times 2}}(A_1)$ it suffices to consider the R-linear generators of A_1; cf. Lemma 154. Note that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a central element in $R^{2\times 2}$. It suffices to consider the R-linear generators of A_1 that are not central in $R^{2\times 2}$.

Suppose given a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in R^{2\times 2}$. Then we have the following equivalences.

$$\begin{aligned} M \in \mathrm{C}_{R^{2\times 2}}(A_1) \quad &\Longleftrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &\Longleftrightarrow \quad \begin{pmatrix} b & 2b \\ d & 2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a+2c & b+2d \end{pmatrix} \\ &\Longleftrightarrow \quad b = 0 \text{ and } a + 2c = d \end{aligned}$$

Hence we obtain the following description of $\mathrm{C}_{R^{2\times 2}}(A_1)$.

$$L_1 := \mathrm{C}_{R^{2\times 2}}(A_1) = {}_R\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}}_{=:S_{1,2}} \right\rangle \tag{19}$$

Note that in this case, we have $A_1 = L_1$. In general, this will not necessarily be the case.

Moreover, we want to set L_1 in relation to a factor ring of the polynomial ring $R[X]$ in one indeterminate X. We calculate.

$$\begin{aligned} S_{1,2}^2 = \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} &= 2S_{1,2} \\ \Longrightarrow \; S_{1,2}^2 - 2S_{1,2} &= 0 \end{aligned}$$

Denoting by $\mathcal{I}_1 := (X^2 - 2X)$ the ideal generated by $X^2 - 2X$ in $R[X]$, we get

$$\begin{aligned} \psi_1 \colon \quad R[X]/\mathcal{I}_1 \quad &\xrightarrow{\sim} \quad L_1 \\ X + \mathcal{I}_1 \quad &\longmapsto \quad \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

The map ψ_1 is surjective since L_1 as a module over R is generated by $S_{1,2}^0$ and $S_{1,2}^1$. Since L_1 is free of rank 2 over R like $R[X]/\mathcal{I}_1$ is, we see that ψ_1 is an isomorphism.

5.1.1 L_1 is local: ad hoc method

Remark 155. *The units in L_1 are given as follows.*

$$\mathrm{U}(L_1) = \left\{ \begin{pmatrix} a & 0 \\ b & a+2b \end{pmatrix} \in R^{2\times 2} \,\middle|\, a \in \mathrm{U}(R),\ b \in R \right\}$$

Proof. Ad $\subseteq$. Suppose given $M \in \mathrm{U}(L_1)$. Then there exist $a, b \in R$ such that $M = \left(\begin{smallmatrix} a & 0 \\ b & a+2b \end{smallmatrix}\right)$. Moreover, we have $\det(M) = a^2 + 2ab$. So $\det(M) \equiv_2 a^2$. Since $\det(M) \in \mathrm{U}(R)$, this entails that a^2 is a unit in R. But if a^2 is a unit in R, then a is also a unit in R.

This shows that M is an element of the right hand side.

Ad $\supseteq$. Suppose given $M := \left(\begin{smallmatrix} a & 0 \\ b & a+2b \end{smallmatrix}\right) \in R^{2\times 2}$ where $a \in \mathrm{U}(R)$. Then $\det(M) = a^2 + 2ab \not\equiv_2 0$ and so $\det(M) \in \mathrm{U}(R)$. Now, by definition, L_1 is the centralizer of a matrix algebra, viz. of A_1. So we can apply Lemma 28 and we obtain that $M \in \mathrm{U}(L_1)$. $\square$

Remark 156. *L_1 is a local ring.*

Proof. Suppose given $M_1, M_2 \in L_1$ such that $M_1 \notin \mathrm{U}(L_1)$ and $M_2 \notin \mathrm{U}(L_1)$. It suffices to show that $M_1 + M_2 \overset{!}{\notin} \mathrm{U}(L_1)$; cf. Remark 30.

There exist $a_1, b_1, a_2, b_2 \in R$ such that $M_1 = \left(\begin{smallmatrix} a_1 & 0 \\ b_1 & a_1+2b_1 \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} a_2 & 0 \\ b_2 & a_2+2b_2 \end{smallmatrix}\right)$. Moreover, we have $a_1 \notin \mathrm{U}(R)$ and $a_2 \notin \mathrm{U}(R)$; cf. Remark 155. Consider the sum

$$M_1 + M_2 = \begin{pmatrix} a_1 + a_2 & 0 \\ b_1 + b_2 & a_1 + 2b_1 + a_2 + 2b_2 \end{pmatrix} \in L_1.$$

Since R is a local ring and both a_1 and a_2 are non-units in R, we conclude that $a_1 + a_2$ also is a non-unit in R; cf. Remark 30.

This shows that $M_1 + M_2 \notin \mathrm{U}(L_1)$; cf. Remark 155.

This completes the proof that L_1 is a local ring. $\square$

5.1.2 L_1 is local: using the radical

Define the R-algebra morphism

$$\begin{array}{rccc} \mu_1 \colon & L_1 & \longrightarrow & R \times R \\ & \begin{pmatrix} a & 0 \\ b & a+2b \end{pmatrix} & \longmapsto & (a, a+2b). \end{array}$$

This is the R-algebra morphism that maps a matrix in L_1 to the tuple of its diagonal entries. Note that μ_1 is an injective R-algebra morphism.

Its image in $R^{\times 2}$ can be described by ties. We get the ties needed to describe the image by inverting the matrix that contains in its rows the entries of the images of the R-linear generators of L_1; cf. equation (19). So we define the matrix U_1 that has the entries of the elements $(1,1)$ and $(0,2)$ as rows and we invert this matrix.

$$U_1 := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad 2 \cdot U_1^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

The ties are given by the columns of U_1^{-1}. The factor 2 indicates that the ties are to be read modulo 2. An element $(r_1, r_2) \in R \times R$ fulfills the ties if the following identities hold.

$$\begin{array}{rrcl} 2r_1 & & \equiv_2 & 0 \\ -r_1 & +r_2 & \equiv_2 & 0 \end{array}$$

The first tie can be skipped since it is always fulfilled. We obtain the following description of $\mu_1(L_1)$.

$$\mu_1(L_1) = \{(a, b) \in R \times R \,|\, a \equiv_2 b\}$$

Since μ_1 is injective, this results in the following illustration.

$$\begin{array}{ccc} L_1 & \simeq & \left(R_{\boxed{1}} \;\text{———}\; 2 \;\text{———}\; R_{\boxed{2}} \right) \\ \begin{pmatrix} a & 0 \\ b & a+2b \end{pmatrix} & \mapsto & (a, a+2b) \end{array}$$

We can apply Lemma 33 on the $\mathbb{Z}_{(2)}$-algebra $\mu_1(L_1)$ which shows that $\mu_1(L_1)$ is local. We conclude that L_1 is local.

5.2 The $\mathbb{Z}_{(2)}$-algebra L_2

We define the R-subalgebra A_2 of $R^{3\times 3}$ as follows.

$$\begin{aligned} A_2 &:= \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a+2b & 0 \\ c & b-2c & a-2b+8c \end{pmatrix} \,\middle|\, a,b,c \in R \right\} \\ &= {}_R\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 8 \end{pmatrix} \right\rangle \end{aligned} \tag{20}$$

To determine $\mathrm{C}_{R^{3\times 3}}(A_2)$ it suffices to consider the R-linear generators of A_2; cf. Lemma 154. Note that $\left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right)$ is a central element in $R^{3\times 3}$. It suffices to consider the R-linear generators of A_2 that are not central in $R^{3\times 3}$.

Suppose given a matrix $M = \left(\begin{smallmatrix} a&b&c\\d&e&f\\g&h&i \end{smallmatrix}\right) \in R^{3\times 3}$. Then we have the following equivalences.

$$\begin{aligned} M \in \mathrm{C}_{R^{3\times 3}}(A_2) \iff & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ and} \\ & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 8 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \\ \iff & \begin{pmatrix} b & 2b+c & -2c \\ e & 2e+f & -2f \\ h & 2h+i & -2i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a+2d & b+2e & c+2f \\ d-2g & e-2h & f-2i \end{pmatrix} \text{ and} \\ & \begin{pmatrix} c & -2c & 8c \\ f & -2f & 8f \\ i & -2i & 8i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a-2d+8g & b-2e+8h & c-2f+8i \end{pmatrix} \\ \iff & b = c = f = 0 \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ e & 2e & 0 \\ h & 2h+i & -2i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a+2d & 2e & 0 \\ d-2g & e-2h & -2i \end{pmatrix} \text{ and} \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & -2i & 8i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a-2d+8g & b-2e+8h & 8i \end{pmatrix} \end{aligned}$$

Now we have to solve a system of linear equations and we get that

$$\begin{aligned} M \in \mathrm{C}_{R^{3\times 3}}(A_2) \iff\ & b = c = f = 0 \text{ and} \\ & e = a+2d, \\ & h = d-2g, \\ & 2h+i = e-2h, \\ & i = a-2d+8g, \\ & -2i = -2e+8h \\ \iff\ & b = c = f = 0 \text{ and} \\ & a = -2d+e, \\ & d = 2g+h, \\ & e = 4h+i, \\ & i = -4d+e+8g \end{aligned}$$

In the last term we can skip the condition on i. We can write the entries a, d and e in dependency of the entries g, h and i.

$$\begin{aligned} M \in \mathrm{C}_{R^{3\times 3}}(A_2) \iff & \; b = c = f = 0 \text{ and} \\ & \begin{array}{rcl} a &=& -4g+2h+i, \\ d &=& 2g+h, \\ e &=& 4h+i \end{array} \\ \iff & \; M = \begin{pmatrix} -4g+2h+i & 0 & 0 \\ 2g+h & 4h+i & 0 \\ g & h & i \end{pmatrix} \end{aligned}$$

Hence we obtain the following description of $\mathrm{C}_{R^{3\times 3}}(A_2)$.

$$L_2 := \mathrm{C}_{R^{3\times 3}}(A_2) = {}_R\left\langle \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=:S_{2,1}}, \underbrace{\begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{=:S_{2,2}}, \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{=:S_{2,3}} \right\rangle \tag{21}$$

In fact we have

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix} &= -2 \cdot S_{2,1} + S_{2,3} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 8 \end{pmatrix} &= 8 \cdot S_{2,1} + S_{2,2} - 2 \cdot S_{2,3} \end{aligned}$$

and

$$\begin{aligned} S_{2,2} &= -4 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 8 \end{pmatrix} \\ S_{2,3} &= 2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix}. \end{aligned}$$

This shows that $A_2 = L_2$.

We obtain the following multiplication table of L_2.

$S_{2,i} \cdot S_{2,j}$	$j=1$	$j=2$	$j=3$
$i=1$	$S_{2,1}$	$S_{2,2}$	$S_{2,3}$
$i=2$	$S_{2,2}$	$-4S_{2,2}$	$2S_{2,2}$
$i=3$	$S_{2,3}$	$2S_{2,2}$	$S_{2,2}+4S_{2,3}$

(22)

In equation (21) we have seen R-linear generators of L_2. But we have

$$S_{2,2} = \begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 - 4 \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix} = S_{2,3}^2 - 4S_{2,3}.$$

So as an R-algebra, L_2 is generated by $\left(\begin{smallmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right) = S_{2,3}$.

Moreover, we want to set L_2 in relation to a factor ring of the polynomial ring $R[X]$ in one indeterminate X. Using (22), we obtain the following.

$$\begin{aligned} S_{2,3}^2 &= S_{2,2} + 4S_{2,3} \\ &\implies S_{2,2} = S_{2,3}^2 - 4S_{2,3} \\ S_{2,3}^3 &= S_{2,3} \cdot (S_{2,2} + 4S_{2,3}) = 6S_{2,2} + 16S_{2,3} = 6S_{2,3}^2 - 8S_{2,3} \\ &\implies S_{2,3}^3 - 6S_{2,3}^2 + 8S_{2,3} = 0 \end{aligned}$$

Denoting by $\mathcal{I}_2 := (X^3 - 6X^2 + 8X)$ the ideal generated by $X^3 - 6X^2 + 8X$ in $R[X]$, we get

$$\begin{array}{rccc} \psi_2\colon & R[X]/\mathcal{I}_2 & \stackrel{\sim}{\longrightarrow} & L_2 \\ & X + \mathcal{I}_2 & \longmapsto & \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

The map ψ_2 is surjective since L_2 as a module over R is generated by $S^0_{2,3}$, $S^1_{2,3}$ and $S^2_{2,3}$. Since L_2 is free of rank 3 over R like $R[X]/\mathcal{I}_2$ is, we see that ψ_2 is an isomorphism.

5.2.1 L_2 is local: ad hoc method

Remark 157. *The units in L_2 are given as follows.*

$$\mathrm{U}(L_2) = \left\{ \begin{pmatrix} a - 4b + 2c & 0 & 0 \\ 2b + c & a + 4c & 0 \\ b & c & a \end{pmatrix} \in R^{3\times 3} \,\middle|\, a \in \mathrm{U}(R),\ b, c \in R \right\}$$

Proof. Ad $\subseteq$. Suppose given $M \in \mathrm{U}(L_2)$. Then there exist $a, b, c \in R$ such that $M = \left(\begin{smallmatrix} a-4b+2c & 0 & 0 \\ 2b+c & a+4c & 0 \\ b & c & a \end{smallmatrix}\right)$. Moreover, we have $\det(M) = a^3 - 4a^2b + 6a^2c - 16abc + 8ac^2$. So $\det(M) \equiv_2 a^3$. Since $\det(M) \in \mathrm{U}(R)$, this entails that a^3 is a unit in R. But if a^3 is a unit in R, then a is also a unit in R.

This shows that M is an element of the right hand side.

Ad $\supseteq$. Suppose given a matrix $M := \left(\begin{smallmatrix} a-4b+2c & 0 & 0 \\ 2b+c & a+4c & 0 \\ b & c & a \end{smallmatrix}\right) \in R^{3\times 3}$ such that $a \in \mathrm{U}(R)$. Then we obtain $\det(M) = a^3 - 4a^2b + 6a^2c - 16abc + 8ac^2 \not\equiv_2 0$ and so $\det(M) \in \mathrm{U}(R)$. Now, by definition, L_2 is the centralizer of the matrix algebra A_2. So we can apply Lemma 28 and we obtain that $M \in \mathrm{U}(L_2)$. $\square$

Remark 158. *L_2 is a local ring.*

Proof. Suppose given $M_1, M_2 \in L_2$ such that $M_1 \notin \mathrm{U}(L_2)$ and $M_2 \notin \mathrm{U}(L_2)$. It suffices to show that $M_1 + M_2 \stackrel{!}{\notin} \mathrm{U}(L_2)$; cf. Remark 30.

There exist $a_1, b_1, c_1, a_2, b_2, c_2 \in R$ such that

$$M_1 = \begin{pmatrix} a_1 - 4b_1 + 2c_1 & 0 & 0 \\ 2b_1 + c_1 & a_1 + 4c_1 & 0 \\ b_1 & c_1 & a_1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} a_2 - 4b_2 + 2c_2 & 0 & 0 \\ 2b_2 + c_2 & a_2 + 4c_2 & 0 \\ b_2 & c_2 & a_2 \end{pmatrix}.$$

Moreover, we have $a_1 \notin \mathrm{U}(R)$ and $a_2 \notin \mathrm{U}(R)$; cf. Remark 157. Consider the sum

$$M_1 + M_2 = \begin{pmatrix} a_1 - 4b_1 + 2c_1 + a_2 - 4b_2 + 2c_2 & 0 & 0 \\ 2b_1 + c_1 + 2b_2 + c_2 & a_1 + 4c_1 + a_2 + 4c_2 & 0 \\ b_1 + b_2 & c_1 + c_2 & a_1 + a_2 \end{pmatrix} \in L_2.$$

Since R is a local ring and both a_1 and a_2 are non-units in R, we conclude that $a_1 + a_2$ also is a non-unit in R; cf. Remark 30.

This shows that $M_1 + M_2 \notin \mathrm{U}(L_2)$; cf. Remark 157.

This completes the proof that L_2 is a local ring. $\square$

5.2.2 L_2 is local: using the radical

Define the R-algebra morphism

$$\begin{array}{rccc} \mu_2 \colon & L_2 & \longrightarrow & R^{\times 3} \\ & \begin{pmatrix} a-4b+2c & 0 & 0 \\ 2b+c & a+4c & 0 \\ b & c & a \end{pmatrix} & \longmapsto & (a-4b+2c, a+4c, a). \end{array}$$

This is the R-algebra morphism that maps a matrix in L_2 to the tuple of its diagonal entries. Note that μ_2 is an injective R-algebra morphism.

Its image in $R^{\times 3}$ can be described by ties. We get the ties needed to describe the image by inverting the matrix that contains in its rows the entries of the images of the R-linear generators of L_2; cf. equation (21). So we define the matrix U_2 that has the entries of the elements $(1,1,1)$, $(-4,0,0)$ and $(2,4,0)$ as rows and we invert this matrix.

$$U_2 := \begin{pmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 2 & 4 & 0 \end{pmatrix} \qquad 8 \cdot U_2^{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & 2 \\ 8 & 1 & -2 \end{pmatrix}$$

The ties are given by the columns of U_2^{-1}. The factor 8 indicates that the ties are to be read modulo 8. An element $(r_1, r_2, r_3) \in R^{\times 3}$ fulfills the ties if the following identities hold.

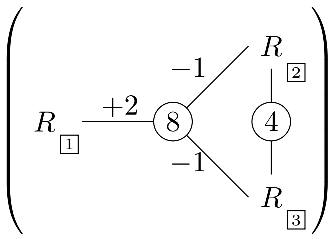
$$\begin{array}{rrrcl} & & 8r_3 & \equiv_8 & 0 \\ -2r_1 & +r_2 & +r_3 & \equiv_8 & 0 \\ & 2r_2 & -2r_3 & \equiv_8 & 0 \end{array}$$

The first tie can be skipped since it is always fulfilled and the third one can be written as a tie modulo 4. We obtain the following description of $\mu_2(L_2)$.

$$\mu_2(L_2) = \left\{ (a,b,c) \in R^{\times 3} \,\middle|\, 2a \equiv_8 b+c \text{ and } b \equiv_4 c \right\} \tag{23}$$

Since μ_2 is injective, this results in the following illustration.

$L_2 \qquad \simeq$



$$\begin{pmatrix} a-4b+2c & 0 & 0 \\ 2b+c & a+4c & 0 \\ b & c & a \end{pmatrix} \mapsto (a-4b+2c, a+4c, a)$$

We have shown that $\mu_2(L_2)$ is described in $R^{\times 3}$ by ties. Note that from $2a \equiv_8 b+c$ and $b \equiv_4 c$ we can deduce that $2a \equiv_4 b+c$ and $0 \equiv_4 b-c$. Adding these two ties yields that $2a \equiv_4 2b$, so $a \equiv_2 b$. Since $b \equiv_4 c$, we have in particular that $b \equiv_2 c$. So together, we have $a \equiv_2 b \equiv_2 c$.

So now we can apply Lemma 33 on the $\mathbb{Z}_{(2)}$-algebra $\mu_2(L_2)$ which shows that $\mu_2(L_2)$ is local. We conclude that L_2 is local.

5.3 The $\mathbb{Z}_{(2)}$-algebra L_3

We define the R-subalgebra L_3 of $R^{\times 3}$ as follows.

$$L_3 := \left\{(a,b,c) \in R^{\times 3} \,\middle|\, a+b \equiv_8 2c \text{ and } b \equiv_2 c\right\} \tag{24}$$

Remark 159. *We have $L_2 \simeq L_3$.*

Proof. Suppose given $(a,b,c) \in L_3$. Then we have $a+b \equiv_8 2c$ and $b \equiv_2 c$. We conclude that $a+b \equiv_4 2c$ and $2b \equiv_4 2c$. The difference of these two ties is $a-b \equiv_4 0$.

This shows that for $(a,b,c) \in L_3$, we have $a \equiv_4 b$.

Suppose given $(a,b,c) \in R^{\times 3}$ such that $a+b \equiv_8 2c$ and $b \equiv_4 a$. Then we have $a+b \equiv_4 2c$ and $b-a \equiv_4 0$. The sum of these two ties is $2b \equiv_4 2c$, so $b \equiv_2 c$.

Together we have shown that

$$L_3 = \left\{(a,b,c) \in R^{\times 3} \,\middle|\, a+b \equiv_8 2c \text{ and } a \equiv_4 b\right\}.$$

After permutation of the tuple entries, the right hand side equals the image $\mu_2(L_2)$ of L_2 under the R-algebra monomorphism μ_2; cf. equation (23). So as R-algebras, L_2 and L_3 are isomorphic. $\square$

Corollary 160. *L_3 is a local ring.*

Proof. This follows from Remark 158 and Remark 159. $\square$

5.4 The $\mathbb{Z}_{(2)}$-algebra L_4

We define the R-subalgebra L_4 of $R^{\times 4}$ as follows.

$$L_4 := \left\{(a,b,c,d) \in R^{\times 4} \,\middle|\, a \equiv_2 b \text{ and } b-d \equiv_8 a-c \equiv_4 0\right\} \tag{25}$$

We choose the following R-linear basis of L_4.

$$\Big(\underbrace{(1,1,1,1)}_{S_{4,1}}, \underbrace{(0,2,0,2)}_{S_{4,2}}, \underbrace{(0,0,4,4)}_{S_{4,3}}, \underbrace{(0,0,0,8)}_{S_{4,4}}\Big)$$

This is a commutative R-algebra. We obtain the following multiplication table of L_4.

$S_{4,i} \cdot S_{4,j}$	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$S_{4,1}$	$S_{4,2}$	$S_{4,3}$	$S_{4,4}$
$i=2$	$S_{4,2}$	$2S_{4,2}$	$S_{4,4}$	$2S_{4,4}$
$i=3$	$S_{4,3}$	$S_{4,4}$	$4S_{4,3}$	$4S_{4,4}$
$i=4$	$S_{4,4}$	$2S_{4,4}$	$4S_{4,4}$	$8S_{4,4}$

(26)

We have

$$S_{4,4} = S_{4,2} \cdot S_{4,3}.$$

So as an R-algebra, L_4 is generated by $S_{4,2}$ and $S_{4,3}$.

Moreover, we want to set L_4 in relation to a factor algebra of the polynomial algebra $R[X,Y]$ in two indeterminates X, Y. Using (26), we obtain the following.

$$S_{4,2}^2 - 2S_{4,2} = 0$$
$$S_{4,3}^2 - 4S_{4,3} = 0$$

Denoting by $\mathcal{I}_4 := (X^2-2X, Y^2-4Y)$ the ideal generated by X^2-2X and Y^2-4Y in $R[X,Y]$, we get

$$\begin{array}{rccc} \psi_4\colon & R[X]/\mathcal{I}_4 & \xrightarrow{\sim} & L_4 \\ & X+\mathcal{I}_4 & \longmapsto & (0,2,0,2) \\ & Y+\mathcal{I}_4 & \longmapsto & (0,0,4,4) \end{array}$$

The map ψ_4 is surjective since L_4 as a module over R is generated by $S_{4,2}^0$, $S_{4,2}^1$, $S_{4,3}^1$ and $S_{4,2}^1 S_{4,3}^1$. Since L_4 is free of rank 4 over R like $R[X,Y]/\mathcal{I}_4$ is, we see that ψ_4 is an isomorphism.

5.4.1 L_4 is local: ad hoc method

Remark 161. *The units in L_4 are given as follows.*

$$\mathrm{U}(L_4) = \left\{ (a, a+2b, a+4c, a+2b+4c+8d) \in R^{\times 4} \,\middle|\, a \in \mathrm{U}(R),\ b, c, d \in R \right\}$$

Proof. Ad $\subseteq$. Suppose given $M \in \mathrm{U}(L_4)$. Then there exist $a, b, c, d \in R$ such that

$$M = (a, a+2b, a+4c, a+2b+4c+8d).$$

Since $M \in \mathrm{U}(L_4)$, every entry of M is invertible in R. We conclude that $a \in \mathrm{U}(R)$.

Ad $\supseteq$. Suppose given $M = (a, a+2b, a+4c, a+2b+4c+8d) \in R^{\times 4}$ such that $a \in \mathrm{U}(R)$. Then every entry of M takes the form $a + 2r$ for a certain $r \in R$. In particular, every entry of M is congruent to a modulo 2 and thus M is invertible in $R^{\times 4}$.

We apply Remark 11 to the multiplication by M on $R^{\times 4}$, which is a bijective $\mathbb{Z}_{(2)}$-linear map mapping L_4 to L_4. Note that $\left|R^{\times 4}/L_4\right| = 2 \cdot 4 \cdot 8$. So this map restricts to a bijective linear maps from L_4 to L_4. This shows that $M \in \mathrm{U}(L_4)$. $\square$

Remark 162. *L_4 is a local ring.*

Proof. Suppose given $M_1, M_2 \in L_4$ such that $M_1 \notin \mathrm{U}(L_4)$ and $M_2 \notin \mathrm{U}(L_4)$. It suffices to show that $M_1 + M_2 \overset{!}{\notin} \mathrm{U}(L_4)$; cf. Remark 30.

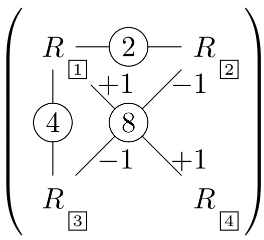
There exist $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in R$ such that $M_1 = (a_1, a_1 + 2b_1, a_1 + 4c_1, a_1 + 2b_1 + 4c_1 + 8d_1)$ and $M_2 = (a_2, a_2 + 2b_2, a_2 + 4c_2, a_2 + 2b_2 + 4c_2 + 8d_2)$. Moreover, we have $a_1 \notin \mathrm{U}(R)$ and $a_2 \notin \mathrm{U}(R)$; cf. Remark 161. The first entry of the sum $M_1 + M_2$ equals $a_1 + a_2$. This is a sum of two non-units in R. Since R is a local ring, we conclude that $a_1 + a_2$ also is a non-unit in R; cf. Remark 30. This shows that $M_1 + M_2 \notin \mathrm{U}(L_4)$; cf. Remark 161. $\square$

5.4.2 L_4 is local: using the radical

The R-algebra L_4 is described in $R^{\times 4}$ by ties. From the ties $a \equiv_2 b$ and $a - c \equiv_4 0$ we conclude that $a \equiv_2 b \equiv_2 c$. Together with the tie $b - d \equiv_4 0$, we have $a \equiv_2 b \equiv_2 c \equiv_2 d$.

So now we can apply Lemma 33 on the $\mathbb{Z}_{(2)}$-algebra L_4 which shows that L_4 is local. We obtain the following illustration of L_4.

$L_4 \quad =$



5.5 The $\mathbb{Z}_{(2)}$-algebra L_5

We define the R-subalgebra A_5 of $R^{4\times 4}$ as follows.

$$A_5 := \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 2c & a+4c & 0 & 0 \\ b+c & 0 & a+2b+2c & 0 \\ -c & -2c+2d & 0 & a+4d \end{pmatrix} \,\middle|\, a,b,c,d \in R \right\} \tag{27}$$
$$= {}_R\left\langle \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \underbrace{\begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&2&0\\0&0&0&0 \end{pmatrix}}_{=:M_1}, \underbrace{\begin{pmatrix} 0&0&0&0\\2&4&0&0\\1&0&2&0\\-1&-2&0&0 \end{pmatrix}}_{=:M_2}, \underbrace{\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&2&0&4 \end{pmatrix}}_{=:M_3} \right\rangle$$

To determine $\mathrm{C}_{R^{4\times 4}}(A_5)$ it suffices to consider the R-linear generators of A_5; cf. Lemma 154. Note that $1_{R^{4\times 4}}$ is a central element in $R^{4\times 4}$. It suffices to consider the R-linear generators of A_5 that are not central in $R^{4\times 4}$.

Suppose given a matrix $M \in \mathrm{C}_{R^{4\times 4}}(A_5)$. We write $M = (a_{i,j})_{i,j\in[1,4]}$ with $a_{i,j} \in R$.

We calculate.

$$M \cdot M_1 = \begin{pmatrix} a_{1,3} & 0 & 2a_{1,3} & 0 \\ a_{2,3} & 0 & 2a_{2,3} & 0 \\ a_{3,3} & 0 & 2a_{3,3} & 0 \\ a_{4,3} & 0 & 2a_{4,3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{1,1}+2a_{3,1} & a_{1,2}+2a_{3,2} & a_{1,3}+2a_{3,3} & a_{1,4}+2a_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_1 \cdot M$$

We conclude that $a_{1,3} = 0$, $a_{2,3} = 0$ and $a_{4,3} = 0$. Furthermore, we conclude that $a_{1,2} + 2a_{3,2} = 0$, $a_{1,4} + 2a_{3,4} = 0$ and $a_{3,3} = a_{1,1} + 2a_{3,1}$.

Using that $a_{1,3} = a_{2,3} = a_{4,3} = 0$, we calculate.

$$M \cdot M_3 = \begin{pmatrix} 0 & 2a_{1,4} & 0 & 4a_{1,4} \\ 0 & 2a_{2,4} & 0 & 4a_{2,4} \\ 0 & 2a_{3,4} & 0 & 4a_{3,4} \\ 0 & 2a_{4,4} & 0 & 4a_{4,4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a_{2,1}+4a_{4,1} & 2a_{2,2}+4a_{4,2} & 0 & 2a_{2,4}+4a_{4,4} \end{pmatrix} = M_3 \cdot M$$

We conclude that $a_{1,4} = 0$, $a_{2,4} = 0$ and $a_{3,4} = 0$. Furthermore, we conclude that $2a_{4,4} = 2a_{2,2} + 4a_{4,2}$ and $2a_{2,1} + 4a_{4,1} = 0$.

Using that $a_{1,3} = a_{2,3} = a_{4,3} = a_{1,4} = a_{2,4} = a_{3,4} = 0$, we calculate.

$$M \cdot M_2 = \begin{pmatrix} 2a_{1,2} & 4a_{1,2} & 0 & 0 \\ 2a_{2,2} & 4a_{2,2} & 0 & 0 \\ 2a_{3,2}+a_{3,3} & 4a_{3,2} & 2a_{3,3} & 0 \\ 2a_{4,2}-a_{4,4} & 4a_{4,2}-2a_{4,4} & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2a_{1,1}+4a_{2,1} & 2a_{1,2}+4a_{2,2} & 0 & 0 \\ a_{1,1}+2a_{3,1} & a_{1,2}+2a_{3,2} & 2a_{3,3} & 0 \\ -a_{1,1}-2a_{2,1} & -a_{1,2}-2a_{2,2} & 0 & 0 \end{pmatrix} = M_2 \cdot M$$

We conclude that $a_{1,2} = 0$ and $a_{3,2} = 0$. Furthermore, we conclude that $2a_{2,2} = 2a_{1,1} + 4a_{2,1}$, $2a_{3,2} + a_{3,3} = a_{1,1} + 2a_{3,1}$, $2a_{4,2} - a_{4,4} = -a_{1,1} - 2a_{2,1}$ and $4a_{4,2} - 2a_{4,4} = -a_{1,2} - 2a_{2,2}$.

Taking into account the vanishing conditions, we see that M takes the following form.

$$M = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & 0 & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & 0 & a_{4,4} \end{pmatrix}$$

Moreover, the vector of the remaining coefficients of M satisfies the following equation.

$$\begin{pmatrix} -1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -4 & 2 \\ 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 \\ -2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{2,2} \\ a_{3,1} \\ a_{3,3} \\ a_{4,1} \\ a_{4,2} \\ a_{4,4} \end{pmatrix} = 0$$

A $\mathbb{Z}$-linear basis of the kernel of this matrix can be obtained for example using Magma. This basis also is a $\mathbb{Z}_{(2)}$-linear basis of the kernel of this matrix. Note that the elements are row vectors in Magma, so we need to transpose the result.

Magma Code 8: L5basis

```
M := Matrix([
[-1, 0, 0,-2,1,0, 0, 0],
[ 0, 0,-2, 0,0,0,-4, 2],
[ 0, 2, 0, 0,0,4, 0, 0],
[-2,-4, 2, 0,0,0, 0, 0],
[ 1, 2, 0, 0,0,0, 2,-1],
[ 0, 0, 2, 0,0,0, 4,-2]
]);
Kernel(Transpose(M));

RSpace of degree 8, dimension 4 over Integer Ring
Echelonized basis:
( 1  0  1  0  1  0  0  1)
( 0  2  4  0  0 -1  0  4)
( 0  0  0  1  2  0  0  0)
( 0  0  0  0  0  0  1  2)
```

Hence we obtain the following description of $\mathrm{C}_{R^{4\times 4}}(A_5)$.

$$L_5 := \mathrm{C}_{R^{4\times 4}}(A_5) = {}_R\left\langle \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=:S_{5,1}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix}}_{=:S_{5,2}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{=:S_{5,3}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}}_{=:S_{5,4}} \right\rangle \tag{28}$$

We obtain the following multiplication table of L_5.

$S_{5,i} \cdot S_{5,j}$	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$S_{5,1}$	$S_{5,2}$	$S_{5,3}$	$S_{5,4}$
$i=2$	$S_{5,2}$	$4S_{5,2}$	0	$4S_{5,4}$
$i=3$	$S_{5,3}$	0	$2S_{5,3}$	0
$i=4$	$S_{5,4}$	$4S_{5,4}$	0	$2S_{5,4}$

(29)

Moreover, we want to set L_5 in relation to a factor ring of the polynomial ring $R[X,Y,Z]$ in three indeterminates X,Y,Z. Using (29), we obtain the following.

$$\begin{aligned} S_{5,2}^2 - 4S_{5,2} &= 0 \\ S_{5,2}S_{5,3} &= 0 \\ S_{5,2}S_{5,4} - 4S_{5,4} &= 0 \\ S_{5,3}^2 - 2S_{5,3} &= 0 \\ S_{5,3}S_{5,4} &= 0 \\ S_{5,4}^2 - 2S_{5,4} &= 0 \end{aligned}$$

Denoting by $\mathcal{I}_5 := (X^2-4X, Y^2-2Y, Z^2-2Z, XY, YZ, XZ-4Z)$ the ideal generated by X^2-4X, Y^2-2Y, Z^2-2Z, XY, YZ and $XZ-4Z$ in $R[X,Y,Z]$, we get

$$\begin{array}{rccc} \psi_5\colon & R[X]/\mathcal{I}_5 & \xrightarrow{\sim} & L_5 \\ & X+\mathcal{I}_5 & \longmapsto & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix} \\ & Y+\mathcal{I}_5 & \longmapsto & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ & Z+\mathcal{I}_5 & \longmapsto & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}$$

The map ψ_5 is surjective since L_5 as a module over R is generated by $S_{5,2}^0$, $S_{5,2}^1$, $S_{5,3}^1$ and $S_{5,4}^1$. Since L_5 is free of rank 4 over R like $R[X,Y,Z]/\mathcal{I}_5$ is, we see that ψ_5 is an isomorphism.

5.5.1 L_5 is local: ad hoc method

Remark 163. *The units in L_5 are given as follows.*

$$\mathrm{U}(L_5) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 2b & a+4b & 0 & 0 \\ c & 0 & a+2c & 0 \\ -b & d & 0 & a+4b+2d \end{pmatrix} \in R^{4\times 4} \;\middle|\; a \in \mathrm{U}(R),\ b,c,d \in R \right\}$$

Proof. Ad $\subseteq$. Suppose given $M \in \mathrm{U}(L_5)$. Then there exist $a,b,c,d \in R$ such that

$$M = \begin{pmatrix} a & 0 & 0 & 0 \\ 2b & a+4b & 0 & 0 \\ c & 0 & a+2c & 0 \\ -b & d & 0 & a+4b+2d \end{pmatrix}.$$

Moreover, we have

$$\begin{aligned} \det(M) &= a \cdot (a+4b) \cdot (a+2c) \cdot (a+4b+2d) \\ &= a^4 + 8a^3b + 2a^3c + 2a^3d + 16a^2b^2 + 16a^2bc + 8a^2bd + 4a^2cd + 32ab^2c + 16abcd \end{aligned}$$

So $\det(M) \equiv_2 a^4$. Since $\det(M) \in \mathrm{U}(R)$, this entails that a^4 is a unit in R. But if a^4 is a unit in R, then a is also a unit in R.

This shows that M is an element of the right hand side.

Ad $\supseteq$. Suppose given

$$M = \begin{pmatrix} a & 0 & 0 & 0 \\ 2b & a+4b & 0 & 0 \\ c & 0 & a+2c & 0 \\ -b & d & 0 & a+4b+2d \end{pmatrix} \in R^{4\times 4} \text{ such that } a \in \mathrm{U}(R).$$

Then $\det(M) \equiv_2 a^4$, but since $a \not\equiv_2 0$, we have $\det(M) \not\equiv_2 0$. So $\det(M) \in \mathrm{U}(R)$. Now, by definition, L_5 is the centralizer of a matrix algebra, viz. of A_5. So we can apply Lemma 28 and we obtain that $M \in \mathrm{U}(L_5)$. $\square$

Remark 164. *L_5 is a local ring.*

Proof. Suppose given $M_1, M_2 \in L_5$ such that $M_1 \notin \mathrm{U}(L_5)$ and $M_2 \notin \mathrm{U}(L_5)$. It suffices to show that $M_1 + M_2 \overset{!}{\notin} \mathrm{U}(L_5)$; cf. Remark 30.

There exist $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in R$ such that

$$M_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 2b_1 & a_1+4b_1 & 0 & 0 \\ c_1 & 0 & a_1+2c_1 & 0 \\ -b_1 & d_1 & 0 & a_1+4b_1+2d_1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 2b_2 & a_2+4b_2 & 0 & 0 \\ c_2 & 0 & a_2+2c_2 & 0 \\ -b_2 & d_2 & 0 & a_2+4b_2+2d_2 \end{pmatrix}.$$

Moreover, we have $a_1 \notin \mathrm{U}(R)$ and $a_2 \notin \mathrm{U}(R)$; cf. Remark 163. Consider the sum

$$M_1 + M_2 = \begin{pmatrix} a_1+a_2 & 0 & 0 & 0 \\ 2b_1+2b_2 & a_1+4b_1+a_2+4b_2 & 0 & 0 \\ c_1+c_2 & 0 & a_1+2c_1+a_2+2c_2 & 0 \\ -b_1-b_2 & d_1+d_2 & 0 & a_1+4b_1+2d_1+a_2+4b_2+2d_2 \end{pmatrix} \in L_5.$$

Since R is a local ring and both a_1 and a_2 are non-units in R, we conclude that $a_1 + a_2$ also is a non-unit in R; cf. Remark 30.

This shows that $M_1 + M_2 \notin \mathrm{U}(L_5)$; cf. Remark 163.

This completes the proof that L_5 is a local ring. $\square$

5.5.2 L_5 is local: using the radical

Define the R-algebra morphism

$$\begin{array}{rccc} \mu_5 \colon & L_5 & \longrightarrow & R^{\times 4} \\ & \begin{pmatrix} a & 0 & 0 & 0 \\ 2b & a+4b & 0 & 0 \\ c & 0 & a+2c & 0 \\ -b & d & 0 & a+4b+2d \end{pmatrix} & \longmapsto & (a+2c,\ a,\ a+4b,\ a+4b+2d). \end{array}$$

This is the R-algebra morphism that maps a matrix in L_5 to the tuple of its diagonal entries in a specific order. Note that μ_5 is an injective R-algebra morphism.

Its image in $R^{\times 4}$ can be described by ties. We get the ties needed to describe the image by inverting the matrix that contains in its rows the entries of the images of the R-linear generators of L_5; cf. equation (28). So we define the matrix $U_5 \in \mathbb{Q}^{4\times 4}$ that has the entries of the elements $(1,1,1,1)$, $(0,0,4,4)$, $(2,0,0,0)$ and $(0,0,0,2)$ as rows and we invert this matrix.

$$U_5 := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad 4 \cdot U_5^{-1} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 4 & -1 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The ties are given by the columns of U_5^{-1}. The factor 4 indicates that the ties are to be read modulo 4. An element $(r_1, r_2, r_3, r_4) \in R^{\times 4}$ fulfills the ties if the following identities hold.

$$\begin{array}{rrrrl} & 4r_2 & & & \equiv_4 \; 0 \\ & -r_2 & +r_3 & & \equiv_4 \; 0 \\ 2r_1 & -2r_2 & & & \equiv_4 \; 0 \\ & & -2r_3 & +2r_4 & \equiv_4 \; 0 \end{array}$$

The first tie can be skipped since it is always fulfilled. The third and the fourth tie can be written as ties modulo 2. We obtain the following description of $\mu_5(L_5)$.

$$\mu_5(L_5) = \left\{(a, b, c, d) \in R^{\times 4} \,\middle|\, a \equiv_2 b,\ c \equiv_2 d \text{ and } b \equiv_4 c\right\}$$

Since μ_5 is injective, this results in the following illustration.

$$\begin{array}{ccc} L_5 & \simeq & \left(R_{\boxed{1}} \text{ — } 2 \text{ — } R_{\boxed{2}} \text{ — } 4 \text{ — } R_{\boxed{3}} \text{ — } 2 \text{ — } R_{\boxed{4}} \right) \\ \begin{pmatrix} a & 0 & 0 & 0 \\ 2b & a+4b & 0 & 0 \\ c & 0 & a+2c & 0 \\ -b & d & 0 & a+4b+2d \end{pmatrix} & \mapsto & (a+2c,\ a,\ a+4b,\ a+4b+2d) \end{array}$$

We have shown that $\mu_5(L_5)$ is described in $R^{\times 4}$ by ties. From the ties $a \equiv_2 b$, $c \equiv_2 d$ and $b \equiv_4 c$ we conclude that $a \equiv_2 b \equiv_2 c \equiv_2 d$.

So now we can apply Lemma 33 on the $\mathbb{Z}_{(2)}$-algebra $\mu_5(L_5)$ which shows that $\mu_5(L_5)$ is local. We conclude that L_5 is local.

5.6 The $\mathbb{Z}_{(2)}$-algebra L_6

We define the R-subalgebra A_6 of $R^{4\times 4}$ as follows.

$$\begin{aligned} A_6 &:= \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ -2c & a+4c & 0 & 0 \\ c & b-c & a+2b+2c & 0 \\ -2d & 0 & 0 & a+4d \end{pmatrix} \,\middle|\, a, b, c, d \in R \right\} \qquad (30) \\ &=_R \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{=:M_4}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{=:M_5}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 4 \end{pmatrix}}_{=:M_6} \right\rangle \end{aligned}$$

To determine $\mathrm{C}_{R^{4\times 4}}(A_6)$ it suffices to consider the R-linear generators of A_6; cf. Lemma 154. Note that $1_{R^{4\times 4}}$ is a central element in $R^{4\times 4}$. It suffices to consider the R-linear generators of A_6 that are not central in $R^{4\times 4}$.

Suppose given a matrix $M \in \mathrm{C}_{R^{4\times 4}}(A_6)$. We write $M = (a_{i,j})_{i,j\in[1,4]}$ with $a_{i,j} \in R$.

We calculate.

$$M \cdot M_4 = \begin{pmatrix} 0 & a_{1,3} & 2a_{1,3} & 0 \\ 0 & a_{2,3} & 2a_{2,3} & 0 \\ 0 & a_{3,3} & 2a_{3,3} & 0 \\ 0 & a_{4,3} & 2a_{4,3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{2,1}+2a_{3,1} & a_{2,2}+2a_{3,2} & a_{2,3}+2a_{3,3} & a_{2,4}+2a_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_4 \cdot M$$

We conclude that $a_{1,3} = 0$, $a_{2,3} = 0$ and $a_{4,3} = 0$. Furthermore, we conclude that $a_{2,1} + 2a_{3,1} = 0$, $a_{3,3} = a_{2,2} + 2a_{3,2}$ and $a_{2,4} + 2a_{3,4} = 0$.

Using that $a_{1,3} = a_{2,3} = a_{4,3} = 0$, we calculate.

$$M \cdot M_6 = \begin{pmatrix} -2a_{1,4} & 0 & 0 & 4a_{1,4} \\ -2a_{2,4} & 0 & 0 & 4a_{2,4} \\ -2a_{3,4} & 0 & 0 & 4a_{3,4} \\ -2a_{4,4} & 0 & 0 & 4a_{4,4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2a_{1,1}+4a_{4,1} & -2a_{1,2}+4a_{4,2} & 0 & -2a_{1,4}+4a_{4,4} \end{pmatrix} = M_6 \cdot M$$

We conclude that $a_{1,4} = 0$, $a_{2,4} = 0$ and $a_{3,4} = 0$. Furthermore, we conclude that $-2a_{4,4} = -2a_{1,1} + 4a_{4,1}$ and $-2a_{1,2} + 4a_{4,2} = 0$.

Using that $a_{1,3} = a_{2,3} = a_{4,3} = a_{1,4} = a_{2,4} = a_{3,4} = 0$, we calculate.

$$\begin{aligned} M \cdot M_5 &= \begin{pmatrix} -2a_{1,2} & 4a_{1,2} & 0 & 0 \\ -2a_{2,2} & 4a_{2,2} & 0 & 0 \\ -2a_{3,2}+a_{3,3} & 4a_{3,2}-a_{3,3} & 2a_{3,3} & 0 \\ -2a_{4,2} & 4a_{4,2} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2a_{1,1}+4a_{2,1} & -2a_{1,2}+4a_{2,2} & 0 & 0 \\ a_{1,1}-a_{2,1}+2a_{3,1} & a_{1,2}-a_{2,2}+2a_{3,2} & 2a_{3,3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_5 \cdot M \end{aligned}$$

We conclude that $a_{1,2} = 0$ and $a_{4,2} = 0$. Furthermore, we conclude that $-2a_{2,2} = -2a_{1,1} + 4a_{2,1}$, $-2a_{3,2} + a_{3,3} = a_{1,1} - a_{2,1} + 2a_{3,1}$ and $4a_{3,2} - a_{3,3} = a_{1,2} - a_{2,2} + 2a_{3,2}$.

Taking into account the vanishing conditions, we see that M takes the following form.

$$M = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & 0 & 0 & a_{4,4} \end{pmatrix}$$

Moreover, the vector of the remaining coefficients of M satisfies the following equation.

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 4 & 2 \\ -2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{2,2} \\ a_{3,1} \\ a_{3,2} \\ a_{3,3} \\ a_{4,1} \\ a_{4,4} \end{pmatrix} = 0$$

A $\mathbb{Z}$-linear basis of the kernel of this matrix can be obtained for example using Magma. This basis also is a $\mathbb{Z}_{(2)}$-linear basis of the kernel of this matrix. Note that the elements are row vectors in Magma, so we need to transpose the result.

Magma Code 9: L6basis

```
M := Matrix([
[ 0, 1, 0,2, 0, 0,0,0],
[ 0, 0,-1,0,-2, 1,0,0],
[-2, 0, 0,0, 0, 0,4,2],
[-2, 4, 2,0, 0, 0,0,0],
[ 1,-1, 0,2, 2,-1,0,0],
[ 0, 0, 1,0, 2,-1,0,0]
]);
Kernel(Transpose(M));
```

```
RSpace of degree 8, dimension 4 over Integer Ring
Echelonized basis:
( 1  0  1  0  0  1  0  1)
( 0  2 -4 -1  0 -4  0  0)
( 0  0  0  0  1  2  0  0)
( 0  0  0  0  0  0  1 -2)
```

We turn the sign of the second and of the last basis element. Moreover, we swap the last two basis elements.

Hence we obtain the following description of $\mathrm{C}_{R^{4\times 4}}(A_6)$.

$$L_6 := \mathrm{C}_{R^{4\times 4}}(A_6) = {}_R\left\langle \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=:S_{6,1}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{=:S_{6,2}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}}_{=:S_{6,3}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{=:S_{6,4}} \right\rangle \tag{31}$$

We obtain the following multiplication table of L_6.

$S_{6,i} \cdot S_{6,j}$	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$S_{6,1}$	$S_{6,2}$	$S_{6,3}$	$S_{6,4}$
$i=2$	$S_{6,2}$	$4S_{6,2}$	0	$4S_{6,4}$
$i=3$	$S_{6,3}$	0	$2S_{6,3}$	0
$i=4$	$S_{6,4}$	$4S_{6,4}$	0	$2S_{6,4}$

(32)

But this is the same multiplication table as the one of L_5; cf. equation (29).

So we conclude the following.

Remark 165. We have $L_5 \simeq L_6$.

Corollary 166. *L_6 is a local ring.*

Note that a matrix

$$\begin{pmatrix} a & 0 & 0 & 0 \\ -2b & a+4b & 0 & 0 \\ b & d & a+4b+2d & 0 \\ -c & 0 & 0 & a+2c \end{pmatrix} \in L_6$$

has the entries of the tuple (a, $a+4b$, $a+4b+2d$ and $a+2c$) on its diagonal. These are exactly the same as those of a general matrix of L_5; cf. the image of μ_5 in §5.5.2. So we can proceed exactly in the same way as in §5.5.2 when projecting a matrix to the tuple of its diagonal entries just by permuting the entries.

5.7 The $\mathbb{Z}_{(2)}$-algebra L_7

We define the R-subalgebra A_7 of $R^{8\times 8}$ as follows.

$$
A_7 := {}_R\Big\langle
\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\1&4&0&0&0&0&2&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&-2&2&2&2&0&-2&2\\0&1&-1&-1&-1&0&1&-1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&2\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\1&4&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&-2&2&2&2&0&-2&2\\0&1&-1&-1&-1&0&1&-1\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&0\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&4&0&0&0&0&1\\0&0&0&0&0&0&0&-1\\0&2&-2&2&2&0&0&0\\0&-1&1&-1&-1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&2\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-1&0\\1&0&4&0&0&0&0&2\\0&0&0&0&0&0&1&0\\0&2&-2&2&2&0&0&0\\0&-1&1&-1&-1&0&0&0\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&0\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&-1\\1&0&0&4&0&0&-2&1\\0&2&2&-2&2&0&2&0\\0&-1&-1&1&-1&0&-1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&2\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&0\\1&0&0&4&0&0&-1&2\\0&2&2&-2&2&0&2&0\\0&-1&-1&1&-1&0&-1&0\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&0\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&1&-1\\1&2&2&2&2&0&0&2\\0&-1&-1&-1&0&2&0&-1\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&2\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&1&-1\\1&2&2&2&2&0&0&2\\0&-1&-1&-1&0&2&0&-1\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&2\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-2&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&0\\1&0&0&0&2&4&0&0\\0&0&0&0&0&0&4&0\\0&0&0&0&0&0&0&0\end{pmatrix},
\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&-2\\0&0&0&0&0&0&0&-2\\0&0&0&0&0&0&0&0\\1&0&0&0&2&4&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&4\end{pmatrix}
\Big\rangle
$$

To determine $\mathrm{C}_{R^{8\times 8}}(A_7)$ it suffices to consider the R-linear generators of A_7; cf. Lemma 154. Note that $1_{R^{8\times 8}}$ is a central element in $R^{8\times 8}$. It suffices to consider the R-linear generators of A_7 that are not central in $R^{8\times 8}$.

We define

$$L_7 := \mathrm{C}_{R^{8\times 8}}(A_7). \tag{33}$$

5.7.1 L_7 is local: a Magma calculation

We use Magma to calculate generators of L_7. To see these eight generators, print the variable `gen` of the Magma code "L7" in §5.7.2 below.

Next we consider the associative $\mathbb{F}_2$-algebra (denoted by `A` in the Magma code "L7" in §5.7.2 below) generated by these eight elements. This is the same as the factor ring $L_7/(2\cdot L_7)$ treated as an algebra.

We let Magma determine the structure constants, thus we can define the $\mathbb{F}_2$-algebra in Magma. Its Jacobson radical is an algebra of dimension seven over $\mathbb{F}_2$ which is one less than the algebra itself. Thus the factor algebra `A` modulo its Jacobson radical has dimension one, hence it is a field.

This shows that the $\mathbb{F}_2$-algebra under consideration is local; cf. [Mül13, Remark 192]. We conclude the following.

Remark 167. The R-algebra L_7 is local.

5.7.2 The Magma code

Magma Code 10: L7blocks

```
ConvertMatToVec := function(A);
// row-wise
    vec := &cat[[A[j][i] : i in [1..NumberOfColumns(A)]] : j in [1..
   NumberOfRows(A)]];
    return RMatrixSpace(Z,#vec,1)!vec;
end function;
```

```
blowup := function(A)
// A integer square matrix
    m := NumberOfRows(A);
    B := RMatrixSpace(Z,m*m,m*m)!0;
    I := DiagonalMatrix([1 : i in [1..m]]);
    for i in [0..m-1] do
        for j in [0..m-1] do
            InsertBlock(~B,I*A[i+1,j+1],1+i*m,1+j*m);
        end for;
    end for;
    return B;
end function;

blowupId := function(A)
// A integer square matrix
    At := Transpose(A);
    m := NumberOfRows(A);
    B := RMatrixSpace(Z,m*m,m*m)!0;
    for i in [0..m-1] do
        InsertBlock(~B,At,1+i*m,1+i*m);
    end for;
    return B;
end function;

M1 := Matrix([
[1,0,0,0,0,0,0,0],
[0,1,0,0,0,0,0,0],
[0,0,1,0,0,0,0,0],
[0,0,0,1,0,0,0,0],
[0,0,0,0,1,0,0,0],
[0,0,0,0,0,1,0,0],
[0,0,0,0,0,0,1,0],
[0,0,0,0,0,0,0,1]
]);

M2 := Matrix([
[0, 0, 0, 0, 0,0, 0, 0],
[1, 4, 0, 0, 0,0, 2, 0],
[0, 0, 0, 0, 0,0, 0,-1],
[0, 0, 0, 0, 0,0, 0,-1],
[0,-2, 2, 2, 2,0,-2, 2],
[0, 1,-1,-1,-1,0, 1,-1],
[0, 0, 0, 0, 0,0, 0, 0],
[0, 0, 0, 0, 0,0, 0, 2]
]);

M3 := Matrix([
[0, 0, 0, 0, 0,0,0, 0],
[0, 0, 0, 0, 0,0,0, 0],
[1, 0, 4, 0, 0,0,0, 1],
[0, 0, 0, 0, 0,0,0,-1],
[0, 2,-2, 2, 2,0,0, 0],
```

```
[0,-1, 1,-1,-1,0,0, 0],
[0, 0, 0, 0, 0,0,0, 0],
[0, 0, 0, 0, 0,0,0, 2]
]);

M4 := Matrix([
[0, 0, 0, 0, 0,0, 0, 0],
[0, 0, 0, 0, 0,0, 0, 0],
[0, 0, 0, 0, 0,0, 0,-1],
[1, 0, 0, 4, 0,0,-2, 1],
[0, 2, 2,-2, 2,0, 2, 0],
[0,-1,-1, 1,-1,0,-1, 0],
[0, 0, 0, 0, 0,0, 0, 0],
[0, 0, 0, 0, 0,0, 0, 2]]);

M5 := Matrix([
[0, 0, 0, 0,0,0, 0, 0],
[0, 0, 0, 0,0,0,-1, 0],
[0, 0, 0, 0,0,0, 0,-1],
[0, 0, 0, 0,0,0, 1,-1],
[1, 2, 2, 2,2,0, 0, 2],
[0,-1,-1,-1,0,2, 0,-1],
[0, 0, 0, 0,0,0, 2, 0],
[0, 0, 0, 0,0,0, 0, 2]
]);

M6 := Matrix([
[0,0,0,0,0,0, 0,0],
[0,0,0,0,0,0,-2,0],
[0,0,0,0,0,0, 0,0],
[0,0,0,0,0,0, 2,0],
[0,0,0,0,0,0, 0,0],
[1,0,0,0,2,4, 0,0],
[0,0,0,0,0,0, 4,0],
[0,0,0,0,0,0, 0,0]
]);

M7 := Matrix([
[1,0,0,0,0,0,0,0],
[0,1,0,0,0,0,0,0],
[0,0,1,0,0,0,0,0],
[0,0,0,1,0,0,0,0],
[0,0,0,0,1,0,0,0],
[0,0,0,0,0,1,0,0],
[0,0,0,0,0,0,1,0],
[0,0,0,0,0,0,0,1]
]);

M8 := Matrix([
[0, 0, 0, 0, 0,0, 0, 0],
[1, 4, 0, 0, 0,0, 1, 0],
[0, 0, 0, 0, 0,0, 0, 0],
[0, 0, 0, 0, 0,0, 1, 0],
```

```
[0,-2, 2, 2, 2,0,-2, 2],
[0, 1,-1,-1,-1,0, 1,-1],
[0, 0, 0, 0, 0,0, 2, 0],
[0, 0, 0, 0, 0,0, 0, 0]
]);

M9 := Matrix([
[0, 0, 0, 0, 0,0, 0,0],
[0, 0, 0, 0, 0,0,-1,0],
[1, 0, 4, 0, 0,0, 0,2],
[0, 0, 0, 0, 0,0, 1,0],
[0, 2,-2, 2, 2,0, 0,0],
[0,-1, 1,-1,-1,0, 0,0],
[0, 0, 0, 0, 0,0, 2,0],
[0, 0, 0, 0, 0,0, 0,0]
]);

M10 := Matrix([
[0, 0, 0, 0, 0,0, 0,0],
[0, 0, 0, 0, 0,0,-1,0],
[0, 0, 0, 0, 0,0, 0,0],
[1, 0, 0, 4, 0,0,-1,2],
[0, 2, 2,-2, 2,0, 2,0],
[0,-1,-1, 1,-1,0,-1,0],
[0, 0, 0, 0, 0,0, 2,0],
[0, 0, 0, 0, 0,0, 0,0]]);

M11 := Matrix([
[0, 0, 0, 0,0,0, 0, 0],
[0, 0, 0, 0,0,0,-1, 0],
[0, 0, 0, 0,0,0, 0,-1],
[0, 0, 0, 0,0,0, 1,-1],
[1, 2, 2, 2,2,0, 0, 2],
[0,-1,-1,-1,0,2, 0,-1],
[0, 0, 0, 0,0,0, 2, 0],
[0, 0, 0, 0,0,0, 0, 2]
]);

M12 := Matrix([
[0,0,0,0,0,0,0, 0],
[0,0,0,0,0,0,0, 0],
[0,0,0,0,0,0,0,-2],
[0,0,0,0,0,0,0,-2],
[0,0,0,0,0,0,0, 0],
[1,0,0,0,2,4,0, 0],
[0,0,0,0,0,0,0, 0],
[0,0,0,0,0,0,0, 4]
]);

Mtup := [M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,M12];
Mvec := [ConvertMatToVec(Mtup[i]): i in [1..#Mtup]];
Mvecmatrix := HorizontalJoin([Mvec[i] : i in [1..#Mvec]]);
```

```
M := VerticalJoin([blowupId(Mtup[i])-blowup(Mtup[i]) : i in [1..#Mtup]]);

D,S,T := SmithForm(M);
k := #[i : i in [1..#Diagonal(D)] | Diagonal(D)[i] ne 0];
// The following matrix has in its columns a Z-basis of the kernel of M
Ker := SubmatrixRange(T,1,k+1,64,64);
// We extract the columns of Ker
colKer := [RMatrixSpace(Z,64,1)!ElementToSequence(SubmatrixRange(Ker,1,i,
   NumberOfRows(Ker),i)) : i in [1..NumberOfColumns(Ker)]];

D1,S1,T1 := SmithForm(Mvecmatrix);
D2,S2,T2 := SmithForm(Ker);

// We are looking for U such that Mvecmatrix =! Ker . U
//           < = >             S2.S1^-1.D1 =! D2 . (T2^-1 . U . T1)

U := T2*Transpose(Solution(Transpose(D2),Transpose(S2*S1^-1*D1)))*T1^-1;
D3,S3,T3 := SmithForm(U);

// Then Ker.S3^-1 has in its columns a basis of the module generated
// by the columns of Ker. Manually we create a matrix B3 that uses
// as many columns of Mvecmatrix as possible.
B := Ker*S3^-1;
// columns 4 and 6 of B add up to Mvec[2]
B1 := HorizontalJoin(RemoveColumn(B,6),Mvec[2]);
// columns 6 and 8 of B1 add up to Mvec[8]
B2 := HorizontalJoin(RemoveColumn(B1,6),Mvec[8]);
// (column 5) + 2*(column 3) of B2 add up to Mvec[11]
B3 := HorizontalJoin(RemoveColumn(B2,5),Mvec[11]);

U1 := Transpose(Solution(Transpose(B),Transpose(B3)));
U2 := Transpose(Solution(Transpose(Ker),Transpose(B3)));
// both are invertible over R, thus we found a change-of-basis-matrix

// only column of Ker which is not generated by columns of Mvecmatrix
v := ElementToSequence(SubmatrixRange(B3,1,6,64,6));
vMat := RMatrixSpace(Z,8,8)!v;
```

Magma Code 11: L7

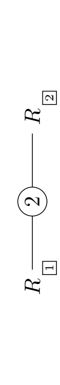
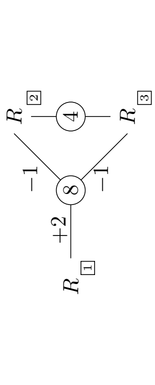
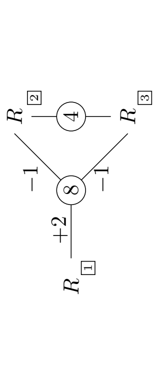
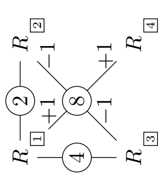
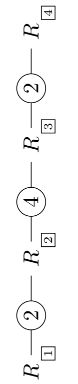
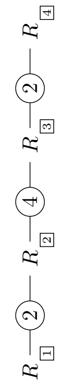
```
load pre;
load L7blocks;

// B3 is basis matrix

gen := [RMatrixSpace(Z,8,8)!ElementToSequence(SubmatrixRange(B3,1,i,
   NumberOfRows(B3),i)) : i in [1..NumberOfColumns(B3)]];

// structure constants modulo 2
Coeff := [[ElementToSequence(Transpose(Solution(Transpose(B3),Transpose(
   ConvertMatToVec(gen[i]*gen[j]))))) : j in [1..#gen]] : i in [1..#gen
   ]];
A := AssociativeAlgebra<GF(2), #gen | Coeff>;
```

```
print A;
Associative Algebra of dimension 8 with base ring GF(2)

print JacobsonRadical(A);
Associative Algebra of dimension 7 with base ring GF(2)

print PrimitiveIdempotents(MatrixAlgebra(A));
[
    [1 0 0 0 0 0 0 0]
    [0 1 0 0 0 0 0 0]
    [0 0 1 0 0 0 0 0]
    [0 0 0 1 0 0 0 0]
    [0 0 0 0 1 0 0 0]
    [0 0 0 0 0 1 0 0]
    [0 0 0 0 0 0 1 0]
    [0 0 0 0 0 0 0 1]
]
```

In the last line of the latter code we ask Magma for the idempotents in A. The only output is the identity matrix (Magma omits the zero matrix which is the second idempotent in A). This also shows that A does not contain non-trivial idempotents.

5.8 Overview

We summarize the results from §5.1 to §5.7 in a table. We provide the description of the $\mathbb{Z}_{(2)}$-algebras by ties and an isomorphic factor algebra as far as we have determined them.

name	matrix algebra	illustration	factor algebra
L_1	${}_R\left\langle \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&2\end{pmatrix} \right\rangle$		$R[X]/(X^2-2X)$
L_2	${}_R\left\langle \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-4&0&0\\2&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}2&0&0\\1&4&0\\0&1&0\end{pmatrix} \right\rangle$		$R[X]/(X^3-6X^2+8X)$
L_3			$R[X]/(X^3-6X^2+8X)$
L_4			$R[X]/(X^2-2X, Y^2-4Y)$
L_5	${}_R\left\langle \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&0\\2&4&0&0\\0&0&0&0\\-1&0&0&4\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&2&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&2\end{pmatrix} \right\rangle$		$R[X,Y,Z]/\ (X^2-4X, Y^2-2Y, Z^2-2Z, XY, YZ, XZ-4Z)$
L_6	${}_R\left\langle \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&0\\-2&4&0&0\\1&0&4&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-1&0&0&2\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&2&0\\0&0&0&0\end{pmatrix} \right\rangle$		$R[X,Y,Z]/\ (X^2-4X, Y^2-2Y, Z^2-2Z, XY, YZ, XZ-4Z)$
L_7	R-subalgebra of $R^{8\times 8}$ of rank 8 over R; cf. §5.7		

Chapter 6: Example $\mathbb{Z}_{(2)}\,\mathrm{S}_4$

Let $R := \mathbb{Z}_{(2)}$. We have $\mathrm{frac}(R) = \mathbb{Q}$. We define $\Gamma := R^{2\times 2} \times R^{3\times 3} \times R^{3\times 3} \times R \times R$. We denote the full diagonal in Γ by Δ.

6.1 Wedderburn: $\mathbb{Z}_{(2)}\,\mathrm{S}_4 \overset{\sim}{\to} \Omega$

We now want to look at the group algebra $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ of the symmetric group S_4 over the ground ring $\mathbb{Z}_{(2)}$ which is a discrete valuation ring, in particular a principal ideal domain as requested for the construction of tori. By Maschke's theorem, the group algebra $\mathbb{Q}\,\mathrm{S}_4$ is semisimple. The isomorphism given by the Artin-Wedderburn theorem may take the following form; cf. [Kün01, p. 22f].

$$
\begin{array}{rrcl}
\omega\colon & \mathbb{Q}\,\mathrm{S}_4 & \overset{\sim}{\to} & \mathbb{Q}^{2\times 2} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q} \times \mathbb{Q} \\
 & (1,2) & \mapsto & \left(\left(\begin{smallmatrix} -5 & 24 \\ -1 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} -11 & -24 & 2 \\ 5 & 11 & -1 \\ 0 & 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right), -1, \ 1 \right) \\
 & (1,2,3,4) & \mapsto & \left(\left(\begin{smallmatrix} 4 & -15 \\ 1 & -4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 26 & 57 & 2 \\ -11 & -24 & -1 \\ -4 & -8 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 1 \end{smallmatrix}\right), -1, \ 1 \right)
\end{array}
$$

Since ω is a $\mathbb{Q}$-algebra isomorphism and $\langle (1,2), (1,2,3,4) \rangle = \mathrm{S}_4$, this characterizes ω uniquely. The last factor corresponds to the trivial representation. The last but one factor corresponds to the sign representation.

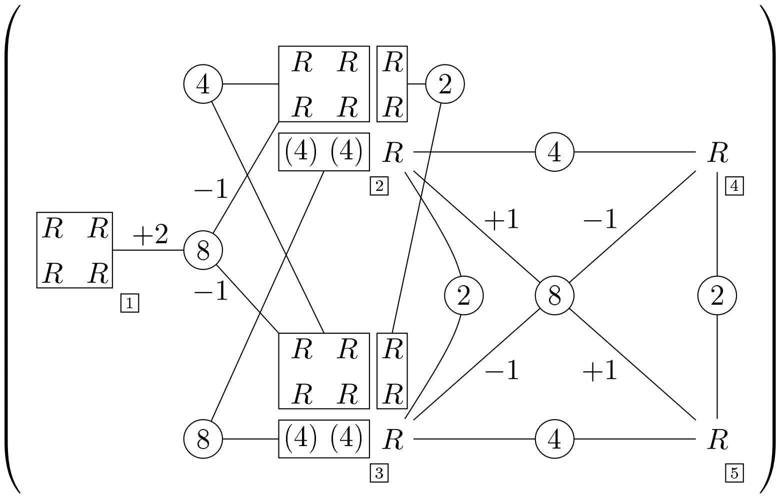
We define ω^{r} as the map ω restricted in the domain to $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ and in the co-domain to Γ. We obtain the following diagram.

$$
\begin{tikzcd}
\mathbb{Q}\,\mathrm{S}_4 \arrow[rr, "\omega", "\sim"'] & & \mathbb{Q}^{2\times 2} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q} \times \mathbb{Q} \\
R\,\mathrm{S}_4 \arrow[u, hook] \arrow[rr, "\omega^{\mathrm{r}}"] \arrow[dr, "\sim"'] & & \Gamma \arrow[u, hook] \\
 & \omega^{\mathrm{r}}(R\,\mathrm{S}_4) =: \Omega \arrow[ur, hook] &
\end{tikzcd}
$$

Note that Ω is an isomorphic copy of $R\,\mathrm{S}_4$. Choosing ω as above, the image $\Omega = \omega^{\mathrm{r}}(R\,\mathrm{S}_4)$ has the following description in Γ via ties, i.e. via congruences of matrix entries; cf. [Kün01, p. 22].

$$
\Omega = \left\{ \left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}, \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}, d, e \right) \in \Gamma \;\middle|\;
\begin{array}{rcll}
2a_{i,j} - b_{i,j} - c_{i,j} & \equiv_8 & 0 & \text{for } i,j \in \{1,2\} \\
b_{i,j} - c_{i,j} & \equiv_4 & 0 & \text{for } i,j \in \{1,2\} \\
b_{i,3} - c_{i,3} & \equiv_2 & 0 & \text{for } i \in \{1,2,3\} \\
b_{3,j} - c_{3,j} & \equiv_8 & 0 & \text{for } j \in \{1,2\} \\
b_{3,j} & \equiv_4 & 0 & \text{for } j \in \{1,2\} \\
c_{3,j} & \equiv_4 & 0 & \text{for } j \in \{1,2\} \\
b_{3,3} - d & \equiv_4 & 0 & \\
c_{3,3} - e & \equiv_4 & 0 & \\
d - e & \equiv_2 & 0 & \\
b_{3,3} - c_{3,3} - d + e & \equiv_8 & 0 &
\end{array}
\right\} \tag{34}
$$

All these conditions we can visualize as in the example $\mathbb{Z}_{(3)}\,\mathrm{S}_3$ in §1. Then we obtain the following illustration of Ω.

$\Omega =$ 

The boxed numbers on the bottom right of each block indicate the order of the blocks when writing them in the description as above (i.e. as a tuple of matrices).

The intersection $\Omega \cap \Delta$ is a maximal commutative R-subalgebra of Ω; cf. Lemma 121.

In $\Omega \cap \Delta$, we have the idempotents

$$e := \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0 \right),$$

$$e' := \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0 \right),$$

$$f := \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 1, 1 \right).$$

Note that we have

$$ee' = e'e = ef = fe = e'f = fe' = 0.$$

We will see later that e, e' and f are primitive in Ω; cf. Remark 168.(2) below. So the sum $1_\Omega = e+e'+f$ is an orthogonal decomposition of 1_Ω into primitive idempotents in Ω which is contained in $\Omega \cap \Delta$. We obtain the following Peirce decomposition of Ω.

$$\Omega = e\Omega e \oplus e'\Omega e' \oplus f\Omega f \oplus e\Omega e' \oplus e\Omega f \oplus e'\Omega e \oplus e'\Omega f \oplus f\Omega e \oplus f\Omega e' \tag{35}$$

We give an R-linear basis of the R-algebra Ω sorted by the Peirce components with respect to the idempotents e, e' and f.

Peirce component	R-linear basis of Peirce component
$e\Omega e$	$b_1 := \left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_2 := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}-2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $b_3 := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}8&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$
$e'\Omega e'$	$b_4 := \left(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_5 := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&2&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&-2&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $b_6 := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&8&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$
$f\Omega f$	$b_7 := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right), 1, 1\right)$ $\quad b_8 := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&2\end{smallmatrix}\right), 0, 2\right)$ $b_9 := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 4, 4\right)$ $\quad b_{10} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 8\right)$
$e\Omega e'$	$b_{11} := \left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_{12} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&2&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&-2&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $b_{13} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&8&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$
$e\Omega f$	$b_{14} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_{15} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&2\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$
$e'\Omega e$	$b_{16} := \left(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_{17} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\2&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\-2&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $b_{18} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\8&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$
$e'\Omega f$	$b_{19} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&1\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&1\\0&0&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_{20} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&2\\0&0&0\end{smallmatrix}\right), 0, 0\right)$
$f\Omega e$	$b_{21} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_{22} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\8&0&0\end{smallmatrix}\right), 0, 0\right)$
$f\Omega e'$	$b_{23} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right), 0, 0\right)$ $\quad b_{24} := \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&8&0\end{smallmatrix}\right), 0, 0\right)$

We define the R-linear basis $\mathcal{B} := (b_i : i \in [1, 24])$ of Ω.

6.2 Primitivity of certain idempotents in Ω

Keep the notation of §6.1.

Remark 168.

(1) *The $\mathbb{Z}_{(2)}$-algebras $e\Omega e$, $e'\Omega e'$ and $f\Omega f$ are local.*

(2) *The idempotents e, e' and f are primitive in Ω.*

Proof. Ad (1). Consider the Peirce component $e\Omega e$. We have the following isomorphism of R-algebras.

$$\begin{array}{ccc} e\Omega e & \xrightarrow{\sim} & \left\{(a,b,c) \in R^{\times 3} \,\middle|\, 2a \equiv_8 b + c \text{ and } b \equiv_4 c\right\} \\ \left(\left(\begin{smallmatrix}a&\\&0\end{smallmatrix}\right), \left(\begin{smallmatrix}b&&\\&0&\\&&0\end{smallmatrix}\right), \left(\begin{smallmatrix}c&&\\&0&\\&&0\end{smallmatrix}\right), 0, 0\right) & \longmapsto & (a,b,c) \end{array}$$

Recall that (b_1, b_2, b_3) is an R-linear basis of $e\Omega e$. The images of these three elements form the R-linear basis $((1,1,1), (0,2,-2), (0,0,8))$ of the right hand side. So this is in fact an isomorphism of R-algebras.

The right hand side equals $\mu_2(L_2)$ of §5.2.2 where we have shown that this $\mathbb{Z}_{(2)}$-algebra is local. So $e\Omega e$ also is local.

Consider the Peirce component $e'\Omega e'$. As an R-algebra, this component is isomorphic to $e\Omega e$, in particular it is local.

Consider the Peirce component $f\Omega f$. We have the following isomorphism of R-algebras.

$$\begin{array}{ccc} f\Omega f & \stackrel{\sim}{\longrightarrow} & \left\{(a,b,c,d) \in R^{\times 4} \,\middle|\, a \equiv_2 b,\ a \equiv_4 c \text{ and } a+d \equiv_8 b+c\right\} \\ \left(\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & a \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & b \end{pmatrix}, c, d\right) & \longmapsto & (a,b,c,d) \end{array}$$

Recall that (b_7, b_8, b_9, b_{10}) is an R-linear basis of $f\Omega f$. The images of these four elements form the R-linear basis $((1,1,1,1),(0,2,0,2),(0,0,4,4),(0,0,0,8))$ of the right hand side.

To verify that this is an isomorphism using the description of Ω in (34), note that if (a,b,c,d) is contained in the right hand side, than (a,b,c,d) also satisfies the ties $b \equiv_4 d$ and $c \equiv_2 d$.

So the right hand side equals L_4 of §5.4; cf. (25). We have shown that L_4 is local; cf. Remark 162. So $f\Omega f$ also is local.

Ad (2). By (1), the R-algebras $e\Omega e$, $e'\Omega e'$ and $f\Omega f$ are local. Applying Remark 139.(2), we obtain that e, e' and f are primitive in Ω. $\square$

6.3 Tori in $\mathfrak{l}(\Omega)$

Keep the notation of §6.1.

An R-linear basis of $\mathrm{Z}(\Omega)$, the center of Ω, can be chosen as follows.

$$\begin{array}{lcl} \text{basis element} & & \text{expressed in the basis } \mathcal{B} \text{ of } \Omega \\ \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, 1, 1\right) & = & b_1 + b_4 + b_7 \\ \left(\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}, \begin{pmatrix} -2 & & \\ & -2 & \\ & & -2 \end{pmatrix}, 2, -2\right) & = & b_2 + b_5 + 2b_7 - 2b_8 \\ \left(\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 8 & & \\ & 8 & \\ & & 8 \end{pmatrix}, 0, 8\right) & = & b_3 + b_6 + 4b_8 \\ \left(\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, 4, 4\right) & = & b_9 \\ \left(\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, 0, 8\right) & = & b_{10} \end{array} \tag{36}$$

Recall the Peirce decomposition $1_\Omega = e + e' + f$ from §6.1. Note that $e = b_1$, $e' = b_4$ and $f = b_7$.

As an R-submodule of $\Omega \cap \Delta$, define

$$\mathfrak{t}_0 := {}_R\langle b_1 + b_4 + b_7,\ b_2 + b_5 + 2b_7 - 2b_8,\ b_3 + b_6 + 4b_8,\ b_9, b_{10},\ e,\ e',\ f\rangle.$$

Then $\mathfrak{t}_0$ is a Lie subalgebra of $\mathfrak{l}(\Omega \cap \Delta)$.

We obtain that $\mathfrak{t}_0$ is an integral torus in $\mathfrak{l}(\Omega)$; cf. Lemma 129.

We can shorten this generating set to an R-linear basis of $\mathfrak{t}_0$ as follows.

$$\mathfrak{t}_0 = {}_R\langle b_1,\ b_4,\ b_7,\ b_2 + b_5 - 2b_8,\ b_3 + b_6 + 4b_8,\ b_9,\ b_{10}\rangle \tag{37}$$

By intersecting Ω with Δ, we get the full diagonal of Ω in Γ; cf. Definition 117. Then $\mathfrak{l}(\Omega \cap \Delta)$ is a maximal rational torus in $\mathfrak{l}(\Omega)$; cf. Lemma 120. So we define

$$T := \Omega \cap \Delta.$$

This is a maximal commutative subalgebra of Ω; cf. Lemma 121.

An R-linear basis of the R-algebra T is given as follows.

$$\mathcal{B}_T := (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$$

Note that $e, e', f \in T$ and the Peirce components $e\Omega e, e'\Omega e'$ and $f\Omega f$ are commutative. So T is a direct sum of Peirce components of Ω; cf. also Lemma 140.(1).

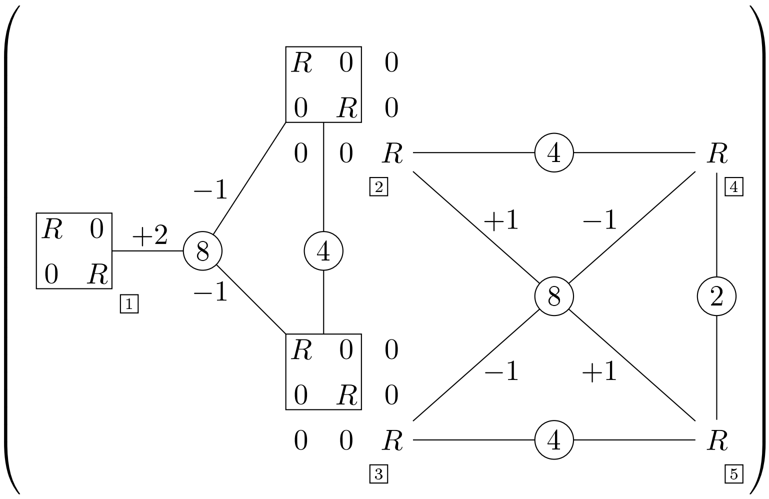
$$T = e\Omega e \oplus e'\Omega e' \oplus f\Omega f = Te \oplus Te' \oplus Tf \tag{38}$$

We can describe elements of T as tuples of matrices with ties; cf. equation (34).

$$T = \left\{ \left(\begin{pmatrix} a_{1,1} & 0 \\ 0 & a_{2,2} \end{pmatrix}, \begin{pmatrix} b_{1,1} & 0 & 0 \\ 0 & b_{2,2} & 0 \\ 0 & 0 & b_{3,3} \end{pmatrix}, \begin{pmatrix} c_{1,1} & 0 & 0 \\ 0 & c_{2,2} & 0 \\ 0 & 0 & c_{3,3} \end{pmatrix}, d, e \right) \in \Gamma \;\middle|\; \begin{array}{rcll} 2a_{i,i} - b_{i,i} - c_{i,i} & \equiv_8 & 0 & \text{for } i \in \{1,2\} \\ b_{i,i} - c_{i,i} & \equiv_4 & 0 & \text{for } i \in \{1,2\} \\ b_{3,3} - c_{3,3} & \equiv_2 & 0 & \\ b_{3,3} - d & \equiv_4 & 0 & \\ c_{3,3} - e & \equiv_4 & 0 & \\ d - e & \equiv_2 & 0 & \\ b_{3,3} - c_{3,3} - d + e & \equiv_8 & 0 & \end{array} \right\}$$

This leads to the following illustration of T where we omit the tie $b_{3,3} - c_{3,3} \equiv_2 0$ since this tie is already implied by the other ties for T.

$T = \Omega \cap \Delta =$



By identifying $\mathfrak{l}(\Omega) := \Omega$ as R-modules and equipping $\mathfrak{l}(\Omega)$ with the commutator Lie bracket

$$\begin{array}{rrcl} [-,=]\colon & \mathfrak{l}(\Omega) \times \mathfrak{l}(\Omega) & \longrightarrow & \mathfrak{l}(\Omega) \\ & (x, y) & \longmapsto & [x,y] := xy - yx, \end{array}$$

$\mathfrak{l}(\Omega)$ becomes a Lie algebra over R. Similarly we obtain the Lie algebra $\mathfrak{l}(T)$ over R.

We want to verify the maximality of the rational torus $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$, using Lemma 112. It suffices to show that

$$\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \stackrel{!}{=} \mathfrak{l}(T).$$

Ad $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \stackrel{!}{\subseteq} \mathfrak{l}(T)$. Suppose given

$$x := \left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}, \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}, d, e \right) \in \mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)).$$

We obtain

$$0 = [x, b_1] = xb_1 - b_1x = \left(\begin{pmatrix} 0 & -a_{1,2} \\ a_{2,1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -b_{1,2} & -b_{1,3} \\ b_{2,1} & 0 & 0 \\ b_{3,1} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -c_{1,2} & -c_{1,3} \\ c_{2,1} & 0 & 0 \\ c_{3,1} & 0 & 0 \end{pmatrix}, 0, 0 \right),$$

$$0 = [x, b_4] = xb_4 - b_4x = \left(\begin{pmatrix} 0 & a_{1,2} \\ -a_{2,1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_{1,2} & 0 \\ -b_{2,1} & 0 & -b_{2,3} \\ 0 & b_{3,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & c_{1,2} & 0 \\ -c_{2,1} & 0 & -c_{2,3} \\ 0 & c_{3,2} & 0 \end{pmatrix}, 0, 0 \right).$$

This shows $x \in \mathfrak{l}(T)$.

Ad $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{\supseteq} \mathfrak{l}(T)$. Since elements of T are tuples of diagonal matrices, $\mathfrak{l}(T)$ is an abelian Lie algebra over R, showing $\mathfrak{l}(T) \subseteq \mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$.

So we have shown that $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) = \mathfrak{l}(T)$ and thus we have verified the maximality of the rational torus $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$ by direct calculation.

Now we want to show that $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$ is not an integral torus.

We have the element

$$b_8 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, 0, 2 \right) \in \mathfrak{l}(T).$$

We want to determine the describing matrix $(\operatorname{ad}_{\mathfrak{l}(\Omega)} b_8)_{\mathcal{B},\mathcal{B}}$ of the adjoint endomorphism $\operatorname{ad}_{\mathfrak{l}(\Omega)} b_8$ with respect to the R-linear basis $\mathcal{B}$ of $\mathfrak{l}(\Omega)$. This requires the Lie brackets $[b_8, b_j]$ for $j \in [1, 24]$. The matrix $(\operatorname{ad}_{\mathfrak{l}(\Omega)} b_8)_{\mathcal{B},\mathcal{B}}$ contains on position (i, j) the coefficient $\alpha_{i,j} \in R$ for $i, j \in [1, 24]$ such that $[b_8, b_j] = \sum_{i \in [1,24]} \alpha_{i,j} b_i$ for $j \in [1, 24]$.

We have $b_8 e = e b_8 = 0$ and $e' b_8 = b_8 e' = 0$ and $f b_8 = b_8 f$. This means that there are only zeros in the columns corresponding to the basis elements of $e\Omega e$, $e\Omega e'$, $e'\Omega e$, $e'\Omega e'$ and $f\Omega f$.

For the remaining basis elements of $e\Omega f$, $e'\Omega f$, $f\Omega e$ and $f\Omega e'$, we obtain the following.

$$\begin{array}{rcr}
[b_8, b_{14}] & = & -b_{15} \\
[b_8, b_{15}] & = & -2b_{15} \\
[b_8, b_{19}] & = & -b_{20} \\
[b_8, b_{20}] & = & -2b_{20} \\
[b_8, b_{21}] & = & b_{22} \\
[b_8, b_{22}] & = & 2b_{22} \\
[b_8, b_{23}] & = & b_{24} \\
[b_8, b_{24}] & = & 2b_{24}
\end{array}$$

Thus we obtain

$$(\operatorname{ad}_{\mathfrak{l}(\Omega)} b_8)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix}.$$

This matrix is diagonalizable over $\mathbb{Q}$, e.g. we have the diagonalizing matrix $Y \in \mathrm{GL}_{24}(\mathbb{Q})$ as follows.

$$Y := \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1
\end{pmatrix} \tag{39}$$

We have $\det(Y) = -16$. This implies that $\mathrm{v}_2(\det(Y)) = 4 \neq 0$. In particular, $Y \notin \mathrm{GL}_{24}(R)$ since -16 is not a unit in $\mathbb{Z}_{(2)}$. We write $A := (\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_8)_{\mathcal{B},\mathcal{B}}$. The first twenty columns of Y form an R-linear basis of $(\mathrm{E}_A(0)) \cap R^{24\times 1}$, the twenty-first and twenty-second column form an R-linear basis of $(\mathrm{E}_A(-2)) \cap R^{24\times 1}$ and the two rightmost columns form an R-linear basis of $(\mathrm{E}_A(2)) \cap R^{24\times 1}$ which we will confirm with the following Magma code.

We will construct matrices W, D, S and T using the notation of Lemma 49. For $\lambda \in \{0, -2, 2\}$ a $\mathbb{Z}$-linear basis and thus also a $\mathbb{Z}_{(2)}$-linear basis of the eigenmodule $\mathrm{E}_A(\lambda) = \mathrm{E}_{(\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_8)_{\mathcal{B},\mathcal{B}}}(\lambda)$ can then be obtained from the first twenty (for $\lambda = 0$) resp. two (for $\lambda = -2$ or $\lambda = 2$) columns of the corresponding matrix S^{-1}. Change the considered eigenvalue in the first line of the Magma code.

Magma Code 12: z2s4EigenmoduleBasis

```
lambda := 0; // eigenvalues are 0, -2 and 2.

A := RMatrixSpace(Rationals(),24,24)!0;
A[15,14] := -1; A[15,15] := -2;
A[20,19] := -1; A[20,20] := -2;
A[22,21] := 1; A[22,22] := 2;
A[24,23] := 1; A[24,24] := 2;

W := Transpose(BasisMatrix(Eigenspace(Transpose(A),lambda)));
for i in [1..NumberOfColumns(W)] do
   MultiplyColumn(~W,2^(-1*(Minimum([Valuation(W[j][i],2):j in [1..
      NumberOfRows(W)]]))), i);
end for;
D,S,T := SmithForm(W);
print S^-1;
```

By Corollary 48.(1) we conclude that $A = (\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_8)_{\mathcal{B},\mathcal{B}}$ is not diagonalizable over $\mathbb{Z}_{(2)}$.

We conclude that $\mathfrak{l}(T)$ is not an integral torus in $\mathfrak{l}(\Omega)$.

For the sake of completeness, we give Y^{-1} and the matrix product $Y^{-1} \cdot (\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_8)_{\mathcal{B},\mathcal{B}} \cdot Y$. We have

$$Y^{-1} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&1
\end{pmatrix} \in \mathrm{GL}_{24}(\mathbb{Q}).$$

We have

$$Y^{-1} \cdot (\mathrm{ad}_{\mathfrak{l}(\Omega)}\, b_8)_{\mathcal{B},\mathcal{B}} \cdot Y = \begin{pmatrix}
0\\
&0\\
&&0\\
&&&0\\
&&&&0\\
&&&&&0\\
&&&&&&0\\
&&&&&&&0\\
&&&&&&&&0\\
&&&&&&&&&0\\
&&&&&&&&&&0\\
&&&&&&&&&&&0\\
&&&&&&&&&&&&0\\
&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&&&&-2\\
&&&&&&&&&&&&&&&&&&&&&-2\\
&&&&&&&&&&&&&&&&&&&&&&2\\
&&&&&&&&&&&&&&&&&&&&&&&2
\end{pmatrix}.$$

Alternatively we can use the function “rdiag” from Magma Code 4.

6.4 The integral core of the standard torus $\mathfrak{l}(T)$ in $\mathfrak{l}(\Omega)$

Keep the notation of §6.1 and §6.3.

Since we found out that $\mathfrak{l}(T)$ is not an integral torus in $\mathfrak{l}(\Omega)$, the question for a maximal integral torus in $\mathfrak{l}(\Omega)$ arises. We recall the definition of the integral core of a rational torus $\mathfrak{l}(T)$ in $\mathfrak{l}(\Omega)$; cf. Definition 130.

We have an orthogonal decomposition of the identity element of Ω into primitive idempotents in Ω, viz. $1 = e + e' + f$. By Corollary 126, we know that $\mathrm{ad}_{\mathfrak{l}(\Omega)}(e), \mathrm{ad}_{\mathfrak{l}(\Omega)}(e')$ and $\mathrm{ad}_{\mathfrak{l}(\Omega)}(f)$ are R-diagonalizable. In fact, the describing matrices of these three maps with respect to the R-linear basis $\mathcal{B}$ of Ω are already diagonal.

Alternatively, to verify the R-diagonalizability in Magma, type the following instructions.

Magma Code 13: z2s4RDiagIdempotents

```
load pre;
load z2s4Init1;
load definitions;
load z2s4Init2;

rdiag(admatrix(b[1]),2);
rdiag(admatrix(b[4]),2);
rdiag(admatrix(b[7]),2);
```

To determine the integral core of $\mathfrak{l}(T)$ in $\mathfrak{l}(\Omega)$, we use Magma.

Calculating the integral core starting with an arbitrarily chosen basis

Recall that $\mathcal{B}_T = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$ is an R-linear basis of T. For $i \in [1, 10]$, we denote by $A_i := \mathrm{ad}_{\mathfrak{l}(\Omega)}(b_i)_{\mathcal{B},\mathcal{B}}$ the describing matrix of the adjoint endomorphism $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_i)$ with respect to the basis $\mathcal{B}$. Let $A = (A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10})$ be the tuple of these ten matrices. We will start the Partitions Algorithm 94 using this cd-tuple on $R^{24\times 24}$.

Since the calculations lasted several hours, we give the output here. However, the duration strongly depends on the choice of the basis.

Magma Code 14: z2s4IntegralCore

```
load pre;
load z2s4Init1;
load definitions;
load z2s4Init2;
load partalgo;

time Partalgo(A);

l =  13 , there are 27644437 partitions to check.
List of finest partitions contains 1 element(s).
Partitions in L_eff:
[
    [ 1, 2, 3, 2, 4, 5, 2, 4, 5, 2, 1, 2, 3 ]
]
A Z_(p)-linear basis of the diagonalizability locus is given by the
   columns of the following matrix.
[ 1  0  0  0  0  0  0]
[ 0  0  0  0  0  0  1]
[ 0  0  0  0  0  1  0]
[ 0  1  0  0  0  0  0]
[ 0  0  0  0  0  0  1]
[ 0  0  0  0  0  1  0]
[ 0  0  1  0  0  0  0]
[ 0  0  0  0  0  4 -2]
[ 0  0  0  1  0  0  0]
[ 0  0  0  0  1  0  0]
Time: 41690.534
```

This is to be interpreted as follows.

$$\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) = {}_R\langle b_1, b_4, b_7, b_9, b_{10}, b_3 + b_6 + 4b_8, b_2 + b_5 - 2b_8\rangle \tag{40}$$

We want to show that $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{=} \mathfrak{t}_0$.

We have

$$\begin{aligned}
b_1 &= \left(\begin{pmatrix}1&\\&0\end{pmatrix}, \begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}, \begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}, 0, 0\right) = e \\
b_4 &= \left(\begin{pmatrix}0&\\&1\end{pmatrix}, \begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix}, \begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix}, 0, 0\right) = e' \\
b_7 &= \left(\begin{pmatrix}0&\\&0\end{pmatrix}, \begin{pmatrix}0&&\\&0&\\&&1\end{pmatrix}, \begin{pmatrix}0&&\\&0&\\&&1\end{pmatrix}, 1, 1\right) = f \\
b_9 &= \left(\begin{pmatrix}0&\\&0\end{pmatrix}, \begin{pmatrix}0&&\\&0&\\&&0\end{pmatrix}, \begin{pmatrix}0&&\\&0&\\&&0\end{pmatrix}, 4, 4\right) \in \mathrm{Z}(\Omega) \\
b_{10} &= \left(\begin{pmatrix}0&\\&0\end{pmatrix}, \begin{pmatrix}0&&\\&0&\\&&0\end{pmatrix}, \begin{pmatrix}0&&\\&0&\\&&0\end{pmatrix}, 0, 8\right) \in \mathrm{Z}(\Omega) \\
b_3 + b_6 + 4b_8 &= \left(\begin{pmatrix}0&\\&0\end{pmatrix}, \begin{pmatrix}0&&\\&0&\\&&0\end{pmatrix}, \begin{pmatrix}8&&\\&8&\\&&8\end{pmatrix}, 0, 8\right) \in \mathrm{Z}(\Omega) \\
b_2 + b_5 - 2b_8 &= \left(\begin{pmatrix}0&\\&0\end{pmatrix}, \begin{pmatrix}2&&\\&2&\\&&0\end{pmatrix}, \begin{pmatrix}-2&&\\&-2&\\&&-4\end{pmatrix}, 0, -4\right)
\end{aligned}$$

Note that $b_2 + b_5 - 2b_8 + 2f$ is a central element in Ω, so $b_2 + b_5 - 2b_8 \in \mathfrak{t}_0$.

This shows that $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \subseteq \mathfrak{t}_0$.

To see that $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{\supseteq} \mathfrak{t}_0$, note that e, e' and f and the elements of the R-linear basis of $\mathrm{Z}(\Omega)$ of (36) are contained in $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$; cf. (36) and (40).

Hence we obtain $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) = \mathfrak{t}_0$. This means that in this example, we have equality in Question 135.

Moreover, note that the list of finest partitions found in the algorithm consisted of one element. So here we have an affirmative example for Question 93.

Calculating the integral core using Remark 59

We will use Remark 59 to achieve a better runtime of the algorithm and to show that the duration strongly depends on the choice of the basis. This time we start with the R-linear basis of $\mathfrak{t}_0$ we found in equation (37). We extend it to an R-linear basis of T.

$$\mathcal{C} := (b_1,\, b_4,\, b_7,\, b_2 + b_5 - 2b_8,\, b_3 + b_6 + 4b_8,\, b_9,\, b_{10},\, b_5,\, b_6,\, b_8)$$

In our implementation of the Partitions Algorithm, the matrices of the cd-tuple under consideration that are R-diagonalizable are considered separately; cf. Algorithm 94. So using the cd-tuple that contains the describing matrices of the elements of $\mathcal{C}$ with respect to the basis $\mathcal{B}$ of $\mathfrak{l}(\Omega)$, the main part of the algorithm is executed only for the describing matrices of the adjoint endomorphisms $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_5)$, $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_6)$ and $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_8)$. The following Magma code shows the results.

Magma Code 15: z2s4IntegralCore2

```
load pre;
load z2s4Init1;
load definitions;
load z2s4Init2;
load partalgo;

C := [b[1],b[4],b[7],SubTup(b[2]+b[5],b[8]+b[8]),
   AddTup([b[3],b[6],ScalMultTup(b[8],4)]),b[9],b[10],b[5],b[6],b[8]];
List := [RMBQ!admatrix(X) : X in C];
time Partalgo(List);

l = 9 , there are 21147 partitions to check.
List of finest partitions contains 0 element(s).
```

```
There is no non-trivial linear combination of the given matrices that is
   Z_(p)-diagonalizable.
A Z_(p)-linear basis of the diagonalizability locus is given by the
   columns of the following matrix.
[1 0 0 0 0 0 0]
[0 1 0 0 0 0 0]
[0 0 1 0 0 0 0]
[0 0 0 1 0 0 0]
[0 0 0 0 1 0 0]
[0 0 0 0 0 1 0]
[0 0 0 0 0 0 1]
[0 0 0 0 0 0 0]
[0 0 0 0 0 0 0]
[0 0 0 0 0 0 0]
Time: 5.257
```

Here the choice of the basis reduced to runtime by a factor of about 7900.

Remark 169. We have the element $x := \left(\left(\begin{smallmatrix} 0 & \\ & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & & \\ & 0 & \\ & & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 & & \\ & 8 & \\ & & 8 \end{smallmatrix}\right), 0, 0 \right) = b_3 + b_6 + 4b_8 - b_{10} \in \mathrm{Z}(\Omega)$. So we obtain $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x) = 0$.

We project x on the Peirce component $e\Omega e$ of Ω. Then we have

$$exe = \left(\left(\begin{smallmatrix} 0 & \\ & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & & \\ & 0 & \\ & & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 & & \\ & 0 & \\ & & 0 \end{smallmatrix}\right), 0, 0 \right) = b_3.$$

But we see that $\mathrm{ad}_{\mathfrak{l}(\Omega)}(exe)$ is not diagonalizable over R since $b_3 \in \mathfrak{l}(T)$ but $b_3 \notin \mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ as we see e.g. using the R-linear basis of $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ we found in equation (40).

This shows that in general, for a discrete valuation ring and an R-order Ω, the R-diagonalizability of $\mathrm{ad}_{\mathfrak{l}(\Omega)}(x)$ does not imply the R-diagonalizability of $\mathrm{ad}_{\mathfrak{l}(\Omega)}(exe)$ for $x \in \mathrm{Z}(\Omega)$ and an idempotent e in a orthogonal decomposition of 1_Ω into primitive idempotents where every idempotent is in Δ.

Note that e and x are elements of $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$, but ex is not. So $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ is not an R-subalgebra of Ω.

6.5 Decompositions of Ω

Let Ω be defined as in §6.1. Let T be defined as in §6.3. Let the R-linear basis $\mathcal{B}$ of Ω be defined as in §6.1. Let the R-linear basis $\mathcal{B}_T$ of T be defined as in §6.3. Let the primitive idempotents e, e' and f be defined as in §6.1.

We are interested in a decomposition of Ω into indecomposable submodules. On the one hand, we will decompose Ω as a T-T-bimodule. On the other hand, we will decompose $\mathfrak{l}(\Omega)$ as an $\mathfrak{l}(T)$-Lie module.

6.5.1 A decomposition of Ω into T-T-sub-bimodules

Recall the Peirce decomposition of Ω; cf. (35). In the following we will show that this decomposition of Ω is already a decomposition into indecomposable T-T-sub-bimodules of Ω.

The $\mathbb{Z}_{(2)}$-algebras $e\Omega e$, $e'\Omega e'$ and $f\Omega f$ are commutative, so the T-T-bimodules $Te = e\Omega e$, $Te' = e'\Omega e'$ and $Tf = f\Omega f$ are indecomposable; cf. Lemma 140.(2), using Lemma 121 and Remark 168.(1).

So it remains to show the indecomposability of the T-T-bimodules where two different idempotents are involved.

Ad $e'\Omega f$.

We want to show that $e'\Omega f$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_1 := (b_{19}, b_{20})$ which is an R-linear basis of $e'\Omega f$.

For a better distinction between the basis elements of Ω and the basis elements of $e'\Omega f$, we write

$$x_1 := b_{19} = \left(\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&0\\0&0&1\\0&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&0\\0&0&1\\0&0&0 \end{smallmatrix}\right), 0, 0 \right) \quad x_2 := b_{20} = \left(\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&0\\0&0&0\\0&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&0\\0&0&2\\0&0&0 \end{smallmatrix}\right), 0, 0 \right).$$

Thus we obtain $\mathcal{B}_1 = (x_1, x_2)$.

It suffices to show that the endomorphism ring $\mathrm{End}_{T\text{-}T}(e'\Omega f)$ is a local ring; cf. Lemma 35.

First we will give a description of this ring and then we will determine all elements in this ring. Afterwards we will see that $\mathrm{End}_{T\text{-}T}(e'\Omega f)$ is local.

We have

$$\begin{aligned} \mathrm{End}_{T\text{-}T}(e'\Omega f) = \big\{ h \in \mathrm{End}_R(e'\Omega f) \mid\ & h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,10],\ j \in [1,2] \text{ and} \\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,10],\ j \in [1,2] \big\}. \end{aligned}$$

For $i \in [1,10]$ we define $M_{\mathcal{B}_1,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $e'\Omega f$ from the left with respect to the basis $\mathcal{B}_1$. For $j \in [1,10]$ we define $M_{\mathcal{B}_1,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $e'\Omega f$ from the right with respect to the basis $\mathcal{B}_1$.

Furthermore, we have the following diagram.

$$\begin{tikzcd} h \arrow[r, mapsto] & h_{\mathcal{B}_1,\mathcal{B}_1} \\ \mathrm{End}_R(e'\Omega f) \arrow[r, "\varphi_1", "\sim"'] & R^{2\times 2} \\ \mathrm{End}_{T\text{-}T}(e'\Omega f) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_1(\mathrm{End}_{T\text{-}T}(e'\Omega f)) =: E_1 \arrow[u, hook] \end{tikzcd}$$

Here the map $\varphi_1\colon \mathrm{End}_R(e'\Omega f) \to R^{2\times 2}$ is the R-algebra isomorphism sending a map $h \in \mathrm{End}_R(e'\Omega f)$ to its describing matrix in the algebra of 2×2-matrices over R with respect to the basis $\mathcal{B}_1$. Since φ_1 is an R-algebra morphism, E_1 is a subalgebra of $R^{2\times 2}$.

Then we have

$$\begin{aligned} \mathrm{End}_{T\text{-}T}(e'\Omega f) \simeq E_1 = \big\{ M \in R^{2\times 2} \mid & M \cdot M_{\mathcal{B}_1,i,\mathrm{l}} = M_{\mathcal{B}_1,i,\mathrm{l}} \cdot M \text{ for } i \in [1,10] \text{ and} \\ & M \cdot M_{\mathcal{B}_1,j,\mathrm{r}} = M_{\mathcal{B}_1,j,\mathrm{r}} \cdot M \text{ for } j \in [1,10] \big\}. \end{aligned} \tag{41}$$

First we will give the products $b_i \cdot x_j$ and $x_j \cdot b_i$ where $i \in [1,10]$ and $j \in [1,2]$.

We use the results to calculate the matrices $M_{\mathcal{B}_1,i,\mathrm{l}}$ and $M_{\mathcal{B}_1,i,\mathrm{r}}$ for $i \in [1,10]$.

i	$b_i \cdot x_1$	$b_i \cdot x_2$	$M_{\mathcal{B}_1,i,\mathrm{l}}$	$x_1 \cdot b_i$	$x_2 \cdot b_i$	$M_{\mathcal{B}_1,i,\mathrm{r}}$
$1,2,3$	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$
4	x_1	x_2	$\left(\begin{smallmatrix} 1&0\\0&1 \end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$
5	$2x_1 - 2x_2$	$-2x_2$	$\left(\begin{smallmatrix} 2&0\\-2&-2 \end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$
6	$4x_2$	$8x_2$	$\left(\begin{smallmatrix} 0&0\\4&8 \end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$
7	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$	x_1	x_2	$\left(\begin{smallmatrix} 1&0\\0&1 \end{smallmatrix}\right)$
8	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$	x_2	$2x_2$	$\left(\begin{smallmatrix} 0&0\\1&2 \end{smallmatrix}\right)$
$9,10$	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$

(42)

Note that $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ are central elements in $R^{2\times 2}$, so we can omit these matrices in the description of E_1. Also note that $M_{\mathcal{B}_1,6,\mathrm{l}} = 4\cdot M_{\mathcal{B}_1,8,\mathrm{r}}$, so we have just to consider one of these two matrices. Thus we have

$$E_1 = \left\{ M \in R^{2\times 2} \,\middle|\, M\cdot\begin{pmatrix} 2 & 0\\ -2 & -2\end{pmatrix} = \begin{pmatrix} 2 & 0\\ -2 & -2\end{pmatrix}\cdot M \text{ and } M\cdot\begin{pmatrix} 0 & 0\\ 1 & 2\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 1 & 2\end{pmatrix}\cdot M \right\}.$$

But $\left(\begin{smallmatrix}2&0\\-2&-2\end{smallmatrix}\right) = \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right) + (-2)\cdot\left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right)$ where $\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$ is a central element in $R^{2\times 2}$. Thus for $M\in R^{2\times 2}$ we have $M\cdot\left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right)\cdot M$ if and only if $M\cdot\left(\begin{smallmatrix}2&0\\-2&-2\end{smallmatrix}\right) = \left(\begin{smallmatrix}2&0\\-2&-2\end{smallmatrix}\right)\cdot M$ and so

$$E_1 = \left\{ M \in R^{2\times 2} \,\middle|\, M\cdot\begin{pmatrix} 0 & 0\\ 1 & 2\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 1 & 2\end{pmatrix}\cdot M \right\}. \tag{43}$$

But this is $\mathrm{C}_{R^{2\times 2}}(A_1)$ of §5.1. So $E_1 = L_1$; cf. (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we conclude that E_1 is local and thus $e'\Omega f$ is indecomposable as a T-T-bimodule.

Ad $e\Omega f$.

We want to show that $e\Omega f$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_2 := (b_{14}, b_{15})$ which is an R-linear basis of $e\Omega f$.

We write

$$x_3 := b_{14} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right) \quad x_4 := b_{15} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&2\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right).$$

Thus we obtain $\mathcal{B}_2 = (x_3, x_4)$.

It suffices to show that the endomorphism ring $\mathrm{End}_{T\text{-}T}(e\Omega f)$ is a local ring; cf. Lemma 35.

We have

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(e\Omega f) = \big\{h \in \mathrm{End}_R(e\Omega f) \mid\ & h(b_i x_j) = b_i h(x_j) \text{ for } i\in[1,10],\ j\in[3,4] \text{ and}\\ & h(x_j b_i) = h(x_j) b_i \text{ for } i\in[1,10],\ j\in[3,4]\big\}.\end{aligned}$$

For $i\in[1,10]$ we define $M_{\mathcal{B}_2,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $e\Omega f$ from the left with respect to the basis $\mathcal{B}_2$. For $j\in[1,10]$ we define $M_{\mathcal{B}_2,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $e\Omega f$ from the right with respect to the basis $\mathcal{B}_2$.

Furthermore, we have the following diagram.

$$\begin{tikzcd} h \arrow[r, mapsto] & h_{\mathcal{B}_2,\mathcal{B}_2} \\ \mathrm{End}_R(e\Omega f) \arrow[r, "\varphi_2", "\sim"'] & R^{2\times 2} \\ \mathrm{End}_{T\text{-}T}(e\Omega f) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_2(\mathrm{End}_{T\text{-}T}(e\Omega f)) =: E_2 \arrow[u, hook] \end{tikzcd}$$

Here the map $\varphi_2\colon \mathrm{End}_R(e\Omega f)\to R^{2\times 2}$ is the R-algebra isomorphism sending a map $h\in\mathrm{End}_R(e\Omega f)$ to its describing matrix in the algebra of 2×2-matrices over R with respect to the basis $\mathcal{B}_2$. Since φ_2 is an R-algebra morphism, E_2 is a subalgebra of $R^{2\times 2}$.

Then we have

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(e\Omega f) \simeq E_2 = \big\{M\in R^{2\times 2}\ |\ & M\cdot M_{\mathcal{B}_2,i,\mathrm{l}} = M_{\mathcal{B}_2,i,\mathrm{l}}\cdot M \text{ for } i\in[1,10] \text{ and}\\ & M\cdot M_{\mathcal{B}_2,j,\mathrm{r}} = M_{\mathcal{B}_2,j,\mathrm{r}}\cdot M \text{ for } j\in[1,10]\big\}.\end{aligned} \tag{44}$$

We determine the matrices $M_{\mathcal{B}_2,i,\mathrm{l}}$ and $M_{\mathcal{B}_2,i,\mathrm{r}}$ for $i \in [1,10]$.

i	$b_i \cdot x_3$	$b_i \cdot x_4$	$M_{\mathcal{B}_2,i,\mathrm{l}}$	$x_3 \cdot b_i$	$x_4 \cdot b_i$	$M_{\mathcal{B}_2,i,\mathrm{r}}$
1	x_3	x_4	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
2	$2x_3 - 2x_4$	$-2x_4$	$\left(\begin{smallmatrix}2&0\\-2&-2\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
3	$4x_4$	$8x_4$	$\left(\begin{smallmatrix}0&0\\4&8\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
$4,5,6$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
7	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	x_3	x_4	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$
8	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	x_4	$2x_4$	$\left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right)$
$9,10$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$

(45)

But these matrices are (up to permutation) exactly the same as the ones for E_1; cf. (42). So we obtain

$$E_2 = \left\{ M \in R^{2\times 2} \;\middle|\; M \cdot \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \cdot M \right\}, \tag{46}$$

cf. equation (43). But this is $\mathrm{C}_{R^{2\times 2}}(A_1)$ of §5.1. So $E_2 = L_1$; cf. (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we conclude that E_2 is local and thus $e\Omega f$ is indecomposable as a T-T-bimodule.

Ad $f\Omega e$.

We want to show that $f\Omega e$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_3 := (b_{21}, b_{22})$ which is an R-linear basis of $f\Omega e$.

We write

$$x_5 := b_{21} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right), 0, 0 \right) \quad x_6 := b_{22} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\8&0&0\end{smallmatrix}\right), 0, 0 \right).$$

Thus we obtain $\mathcal{B}_3 = (x_5, x_6)$.

It suffices to show that the endomorphism ring $\mathrm{End}_{T\text{-}T}(f\Omega e)$ is a local ring; cf. Lemma 35.

We have

$$\begin{aligned} \mathrm{End}_{T\text{-}T}(f\Omega e) = \big\{ h \in \mathrm{End}_R(f\Omega e) \mid\; & h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,10],\ j \in [5,6] \text{ and} \\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,10],\ j \in [5,6] \big\}. \end{aligned}$$

For $i \in [1,10]$ we define $M_{\mathcal{B}_3,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $f\Omega e$ from the left with respect to the basis $\mathcal{B}_3$. For $j \in [1,10]$ we define $M_{\mathcal{B}_3,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $f\Omega e$ from the right with respect to the basis $\mathcal{B}_3$.

Furthermore, we have the following diagram.

$$\begin{tikzcd} h \arrow[r, mapsto] & h_{\mathcal{B}_3,\mathcal{B}_3} \\ \mathrm{End}_R(f\Omega e) \arrow[r, "\varphi_3", "\sim"'] & R^{2\times 2} \\ \mathrm{End}_{T\text{-}T}(f\Omega e) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_3(\mathrm{End}_{T\text{-}T}(f\Omega e)) =: E_3 \arrow[u, hook] \end{tikzcd}$$

Here the map $\varphi_3\colon \mathrm{End}_R(f\Omega e) \to R^{2\times 2}$ is the R-algebra isomorphism sending a map $h \in \mathrm{End}_R(f\Omega e)$ to its describing matrix in the algebra of 2×2-matrices over R with respect to the basis $\mathcal{B}_3$. Since φ_3 is an R-algebra morphism, E_3 is a subalgebra of $R^{2\times 2}$.

Then we have

$$\operatorname{End}_{T\text{-}T}(f\Omega e) \simeq E_3 = \big\{M \in R^{2\times 2} \;|M \cdot M_{\mathcal{B}_3,i,\mathrm{l}} = M_{\mathcal{B}_3,i,\mathrm{l}} \cdot M \text{ for } i \in [1,10] \text{ and } M \cdot M_{\mathcal{B}_3,j,\mathrm{r}} = M_{\mathcal{B}_3,j,\mathrm{r}} \cdot M \text{ for } j \in [1,10]\big\}. \tag{47}$$

We determine the matrices $M_{\mathcal{B}_3,i,\mathrm{l}}$ and $M_{\mathcal{B}_3,i,\mathrm{r}}$ for $i \in [1,10]$.

i	$b_i \cdot x_5$	$b_i \cdot x_6$	$M_{\mathcal{B}_3,i,\mathrm{l}}$	$x_5 \cdot b_i$	$x_6 \cdot b_i$	$M_{\mathcal{B}_3,i,\mathrm{r}}$
1	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	x_5	x_6	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$
2	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	$2x_5 - 2x_6$	$-2x_6$	$\left(\begin{smallmatrix}2&0\\-2&-2\end{smallmatrix}\right)$
3	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	$4x_6$	$8x_6$	$\left(\begin{smallmatrix}0&0\\4&8\end{smallmatrix}\right)$
$4,5,6$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
7	x_5	x_6	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
8	x_6	$2x_6$	$\left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
$9,10$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$

(48)

But these matrices are (up to permutation) exactly the same as the ones for E_1; cf. (42). So we obtain

$$E_3 = \left\{ M \in R^{2\times 2} \,\middle|\, M \cdot \begin{pmatrix}0&0\\1&2\end{pmatrix} = \begin{pmatrix}0&0\\1&2\end{pmatrix} \cdot M \right\}, \tag{49}$$

cf. equation (43). But this is $\mathrm{C}_{R^{2\times 2}}(A_1)$ of §5.1. So $E_3 = L_1$; cf. (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we conclude that E_3 is local and thus $f\Omega e$ is indecomposable as a T-T-bimodule.

Ad $f\Omega e'$.

We want to show that $f\Omega e'$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_4 := (b_{23}, b_{24})$ which is an R-linear basis of $f\Omega e'$.

We write

$$x_7 := b_{23} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right), 0, 0 \right) \quad x_8 := b_{24} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&8&0\end{smallmatrix}\right), 0, 0 \right).$$

Thus we obtain $\mathcal{B}_4 = (x_7, x_8)$.

It suffices to show that the endomorphism ring $\operatorname{End}_{T\text{-}T}(f\Omega e')$ is a local ring; cf. Lemma 35.

We have

$$\operatorname{End}_{T\text{-}T}(f\Omega e') = \big\{h \in \operatorname{End}_R(f\Omega e') \mid h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,10],\ j \in [7,8] \text{ and } h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,10],\ j \in [7,8]\big\}.$$

For $i \in [1,10]$ we define $M_{\mathcal{B}_4,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $f\Omega e'$ from the left with respect to the basis $\mathcal{B}_4$. For $j \in [1,10]$ we define $M_{\mathcal{B}_4,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $f\Omega e'$ from the right with respect to the basis $\mathcal{B}_4$.

Furthermore, we have the following diagram.

$$\begin{tikzcd}
h \arrow[r, mapsto] & h_{\mathcal{B}_4,\mathcal{B}_4} \\
\operatorname{End}_R(f\Omega e') \arrow[r, "\varphi_4", "\sim"'] & R^{2\times 2} \\
\operatorname{End}_{T\text{-}T}(f\Omega e') \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_4(\operatorname{End}_{T\text{-}T}(f\Omega e')) =: E_4 \arrow[u, hook]
\end{tikzcd}$$

Here the map $\varphi_4\colon \operatorname{End}_R(f\Omega e') \to R^{2\times 2}$ is the R-algebra isomorphism sending a map $h \in \operatorname{End}_R(f\Omega e')$ to its describing matrix in the algebra of 2×2-matrices over R with respect to the basis $\mathcal{B}_4$. Since φ_4 is an R-algebra morphism, E_4 is a subalgebra of $R^{2\times 2}$.

Then we have

$$\begin{aligned}\operatorname{End}_{T\text{-}T}(f\Omega e') \simeq E_4 = \big\{M \in R^{2\times 2} \mid & M \cdot M_{\mathcal{B}_4,i,\mathrm{l}} = M_{\mathcal{B}_4,i,\mathrm{l}} \cdot M \text{ for } i \in [1,10] \text{ and} \\ & M \cdot M_{\mathcal{B}_4,j,\mathrm{r}} = M_{\mathcal{B}_4,j,\mathrm{r}} \cdot M \text{ for } j \in [1,10]\big\}.\end{aligned} \tag{50}$$

We determine the matrices $M_{\mathcal{B}_4,i,\mathrm{l}}$ and $M_{\mathcal{B}_4,i,\mathrm{r}}$ for $i \in [1,10]$.

i	$b_i \cdot x_7$	$b_i \cdot x_8$	$M_{\mathcal{B}_4,i,\mathrm{l}}$	$x_7 \cdot b_i$	$x_8 \cdot b_i$	$M_{\mathcal{B}_4,i,\mathrm{r}}$
$1,2,3$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
4	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	x_7	x_8	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$
5	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	$2x_7 - 2x_8$	$-2x_8$	$\left(\begin{smallmatrix}2&0\\-2&-2\end{smallmatrix}\right)$
6	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	$4x_8$	$8x_8$	$\left(\begin{smallmatrix}0&0\\4&8\end{smallmatrix}\right)$
7	x_7	x_8	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
8	x_8	$2x_8$	$\left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$
$9,10$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	0	0	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$

(51)

But these matrices are (up to permutation) exactly the same as the ones for E_1; cf. (42). So we obtain

$$E_4 = \left\{M \in R^{2\times 2} \,\middle|\, M \cdot \begin{pmatrix}0&0\\1&2\end{pmatrix} = \begin{pmatrix}0&0\\1&2\end{pmatrix} \cdot M\right\}, \tag{52}$$

cf. equation (43). But this is $\mathrm{C}_{R^{2\times 2}}(A_1)$ of §5.1. So $E_4 = L_1$; cf. (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we conclude that E_4 is local and thus $f\Omega e'$ is indecomposable as a T-T-bimodule.

Ad $e\Omega e'$.

We want to show that $e\Omega e'$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_5 := (b_{11}, b_{12}, b_{13})$ which is an R-linear basis of $e\Omega e'$.

We write

$$x_9 := b_{11} = \left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right) \qquad x_{10} := b_{12} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&2&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&-2&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right)$$
$$x_{11} := b_{13} = \left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&8&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0\right).$$

Thus we obtain $\mathcal{B}_5 = (x_9, x_{10}, x_{11})$.

It suffices to show that the endomorphism ring $\operatorname{End}_{T\text{-}T}(e\Omega e')$ is a local ring; cf. Lemma 35.

We have

$$\begin{aligned}\operatorname{End}_{T\text{-}T}(e\Omega e') = \big\{h \in \operatorname{End}_R(e\Omega e') \mid & h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,10],\ j \in [9,11] \text{ and} \\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,10],\ j \in [9,11]\big\}.\end{aligned}$$

For $i \in [1,10]$ we define $M_{\mathcal{B}_5,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $e\Omega e'$ from the left with respect to the basis $\mathcal{B}_5$. For $j \in [1,10]$ we define $M_{\mathcal{B}_5,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $e\Omega e'$ from the right with respect to the basis $\mathcal{B}_5$.

Furthermore, we have the following diagram.

$$\begin{tikzcd}
h \arrow[r, mapsto] & h_{\mathcal{B}_5,\mathcal{B}_5} \\
\mathrm{End}_R(e\Omega e') \arrow[r, "\varphi_5", "\sim"'] & R^{3\times 3} \\
\mathrm{End}_{T\text{-}T}(e\Omega e') \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_5(\mathrm{End}_{T\text{-}T}(e\Omega e')) =: E_5 \arrow[u, hook]
\end{tikzcd}$$

Here the map $\varphi_5\colon \mathrm{End}_R(e\Omega e') \to R^{3\times 3}$ is the R-algebra isomorphism sending a map $h \in \mathrm{End}_R(e\Omega e')$ to its describing matrix in the algebra of 3×3-matrices over R with respect to the basis $\mathcal{B}_5$. Since φ_5 is an R-algebra morphism, E_5 is a subalgebra of $R^{3\times 3}$.

Then we have

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(e\Omega e') \simeq E_5 = \big\{M \in R^{3\times 3}\ | M\cdot M_{\mathcal{B}_5,i,\mathrm{l}} = M_{\mathcal{B}_5,i,\mathrm{l}} \cdot M \text{ for } i \in [1,10] \text{ and}\\ M\cdot M_{\mathcal{B}_5,j,\mathrm{r}} = M_{\mathcal{B}_5,j,\mathrm{r}} \cdot M \text{ for } j \in [1,10]\big\}.\end{aligned} \tag{53}$$

We determine the matrices $M_{\mathcal{B}_5,i,\mathrm{l}}$ and $M_{\mathcal{B}_5,i,\mathrm{r}}$ for $i \in [1,10]$.

i	$b_i \cdot x_9$	$b_i \cdot x_{10}$	$b_i \cdot x_{11}$	$M_{\mathcal{B}_5,i,\mathrm{l}}$	$x_9 \cdot b_i$	$x_{10} \cdot b_i$	$x_{11} \cdot b_i$	$M_{\mathcal{B}_5,i,\mathrm{r}}$
1	x_9	x_{10}	x_{11}	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
2	x_{10}	$2x_{10}+x_{11}$	$-2x_{11}$	$\begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
3	x_{11}	$-2x_{11}$	$8x_{11}$	$\begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
4	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	x_9	x_{10}	x_{11}	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
5	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	x_{10}	$2x_{10}+x_{11}$	$-2x_{11}$	$\begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix}$
6	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	x_{11}	$-2x_{11}$	$8x_{11}$	$\begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix}$
$7,8,9,10$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

(54)

We omit the matrices that are central in $R^{3\times 3}$. It remains

$$E_5 = \left\{M \in R^{3\times 3} \,\middle|\, M\cdot\begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix} = \begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix}\cdot M \text{ and } M\cdot\begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix}\cdot M\right\}. \tag{55}$$

But this is $\mathrm{C}_{R^{2\times 2}}(A_2)$ of §5.2. So $E_5 = L_2$; cf. (21) in §5.2. We have shown that L_2 is local; cf. Remark 158. So we conclude that E_5 is local and thus $e\Omega e'$ is indecomposable as a T-T-bimodule.

Ad $e'\Omega e$.

We want to show that $e'\Omega e$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_6 := (b_{16}, b_{17}, b_{18})$ which is an R-linear basis of $e'\Omega e$.

We write

$$x_{12} := b_{16} = \left(\begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, 0, 0\right) \qquad x_{13} := b_{17} = \left(\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\2&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\-2&0&0\\0&0&0\end{pmatrix}, 0, 0\right)$$
$$x_{14} := b_{18} = \left(\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\8&0&0\\0&0&0\end{pmatrix}, 0, 0\right).$$

Thus we obtain $\mathcal{B}_6 = (x_{12}, x_{13}, x_{14})$.

It suffices to show that the endomorphism ring $\mathrm{End}_{T\text{-}T}(e'\Omega e)$ is a local ring; cf. Lemma 35.

We have

$$\mathrm{End}_{T\text{-}T}(e'\Omega e) = \big\{h \in \mathrm{End}_R(e'\Omega e) \mid h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,10],\ j \in [12,14] \text{ and } h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,10],\ j \in [12,14]\big\}.$$

For $i \in [1,10]$ we define $M_{\mathcal{B}_6,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $e'\Omega e$ from the left with respect to the basis $\mathcal{B}_6$. For $j \in [1,10]$ we define $M_{\mathcal{B}_6,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $e'\Omega e$ from the right with respect to the basis $\mathcal{B}_6$.

Furthermore, we have the following diagram.

$$\begin{tikzcd}
h \arrow[r, mapsto] & h_{\mathcal{B}_6,\mathcal{B}_6} \\
\mathrm{End}_R(e'\Omega e) \arrow[r, "\varphi_6", "\sim"'] & R^{3\times 3} \\
\mathrm{End}_{T\text{-}T}(e'\Omega e) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_6(\mathrm{End}_{T\text{-}T}(e'\Omega e)) =: E_6 \arrow[u, hook]
\end{tikzcd}$$

Here the map $\varphi_6\colon \mathrm{End}_R(e'\Omega e) \to R^{3\times 3}$ is the R-algebra isomorphism sending a map $h \in \mathrm{End}_R(e'\Omega e)$ to its describing matrix in the algebra of 3×3-matrices over R with respect to the basis $\mathcal{B}_6$. Since φ_6 is an R-algebra morphism, E_6 is a subalgebra of $R^{3\times 3}$.

Then we have

$$\mathrm{End}_{T\text{-}T}(e'\Omega e) \simeq E_6 = \big\{M \in R^{3\times 3} \mid M \cdot M_{\mathcal{B}_6,i,\mathrm{l}} = M_{\mathcal{B}_6,i,\mathrm{l}} \cdot M \text{ for } i \in [1,10] \text{ and } M \cdot M_{\mathcal{B}_6,j,\mathrm{r}} = M_{\mathcal{B}_6,j,\mathrm{r}} \cdot M \text{ for } j \in [1,10]\big\}. \tag{56}$$

We determine the matrices $M_{\mathcal{B}_6,i,\mathrm{l}}$ and $M_{\mathcal{B}_6,i,\mathrm{r}}$ for $i \in [1,10]$.

(57)

i	$b_i \cdot x_{12}$	$b_i \cdot x_{13}$	$b_i \cdot x_{14}$	$M_{\mathcal{B}_6,i,\mathrm{l}}$	$x_{12} \cdot b_i$	$x_{13} \cdot b_i$	$x_{14} \cdot b_i$	$M_{\mathcal{B}_6,i,\mathrm{r}}$
1	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	x_{12}	x_{13}	x_{14}	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
2	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	x_{13}	$2x_{13}+x_{14}$	$-2x_{14}$	$\begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix}$
3	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	x_{14}	$-2x_{14}$	$8x_{14}$	$\begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix}$
4	x_{12}	x_{13}	x_{14}	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
5	x_{13}	$2x_{13}+x_{14}$	$-2x_{14}$	$\begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
6	x_{14}	$-2x_{14}$	$8x_{14}$	$\begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
7, 8, 9, 10	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	0	0	0	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

But these matrices are (up to permutation) exactly the same as the ones for E_5; cf. (54). So we obtain

$$E_6 = \left\{M \in R^{3\times 3} \,\middle|\, M \cdot \begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix} = \begin{pmatrix}0&0&0\\1&2&0\\0&1&-2\end{pmatrix} \cdot M \text{ and } M \cdot \begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\1&-2&8\end{pmatrix} \cdot M\right\}, \tag{58}$$

cf. equation (55). But this is $\mathrm{C}_{R^{2\times 2}}(A_2)$ of §5.2. So $E_6 = L_2$; cf. (21) in §5.2. We have shown that L_2 is local; cf. Remark 158. So we conclude that E_6 is local and thus $e'\Omega e$ is indecomposable as a T-T-bimodule.

We summarize.

We obtain the following decomposition of Ω into a direct sum of T-T-sub-bimodules of Ω.

$$\begin{aligned}
\Omega = \; & {}_R\Big\langle \Big(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}-2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \\
& \quad \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}8&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&2&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&-2&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \\
& \quad \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&8&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right), 1, 1 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&2\end{smallmatrix}\right), 0, 2 \Big), \\
& \quad \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 4, 4 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 8 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&1\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&1\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&2\\0&0&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&2\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\8&0&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&8&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&2&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&-2&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \\
& \quad \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&8&0\\0&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle \\
& \oplus {}_R\Big\langle \Big(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\2&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\-2&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big), \\
& \quad \Big(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\8&0&0\\0&0&0\end{smallmatrix}\right), 0, 0 \Big) \Big\rangle
\end{aligned}$$

In this decomposition, all summands but T are indecomposable as T-T-sub-bimodules of Ω.

This decomposition is the same as the following decomposition where we write the indecomposable T-T-bimodules as Peirce components.

$$\Omega = e\Omega e \oplus e'\Omega e' \oplus f\Omega f \oplus e'\Omega f \oplus e\Omega f \oplus f\Omega e \oplus f\Omega e' \oplus e\Omega e' \oplus e'\Omega e \tag{59}$$

By applying Lemma 144 we conclude that all summands in this decomposition are non-isomorphic to each other.

So we have found out that the Peirce decomposition of §6.1 is already a decomposition of Ω into indecomposable T-T-sub-bimodules of Ω; cf. equation (35) on page 125.

Recall that $Te = e\Omega e$, $Te' = e'\Omega e'$ and $Tf = f\Omega f$ are local; cf. Remark 168.(1). By Lemma 25, the T-T-endomorphism rings $\mathrm{End}_{T\text{-}T}(e\Omega e)$, $\mathrm{End}_{T\text{-}T}(e'\Omega e')$ and $\mathrm{End}_{T\text{-}T}(f\Omega f)$ also are local.

So we were able to show that all indecomposable summands have local T-T-endomorphism rings.

6.5.2 A decomposition of $\mathfrak{l}(\Omega)$ into $\mathfrak{l}(T)$-Lie submodules

Keep the notation of §6.5.1.

We consider the Lie algebra $\mathfrak{l}(\Omega)$ as an $\mathfrak{l}(T)$-Lie module over its Lie subalgebra $\mathfrak{l}(T)$. We are interested in a decomposition of the $\mathfrak{l}(T)$-Lie module $\mathfrak{l}(\Omega)$ into indecomposable $\mathfrak{l}(T)$-Lie submodules.

Recall that T is commutative. We conclude that $\mathfrak{l}(T)$ is an abelian Lie algebra over R, hence T is a trivial $\mathfrak{l}(T)$-Lie module. This entails that T decomposes into $\mathfrak{l}(T)$-Lie submodules of rank 1 over R.

Let $X \subseteq T$ be such a summand of rank 1 over R. Then X is a trivial $\mathfrak{l}(T)$-Lie module and so $\mathrm{End}_{\mathfrak{l}(T)}(X) = \mathrm{End}_R(X) \simeq R$. This shows in particular that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(X)$ is a local ring.

It remains to decompose the Peirce components where two different idempotents are involved. The calculations and arguments are similar to those in §6.5.1.

Ad $e'\Omega f$.

We want to show that $e'\Omega f$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_1$ is an R-linear basis of $e'\Omega f$.

It suffices to show that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega f)$ is a local ring; cf. Lemma 35.

We have

$$\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega f) = \left\{ h \in \mathrm{End}_R(e'\Omega f) \,\middle|\, h([b_i, x_j]) = [b_i, h(x_j)] \text{ for } i \in [1,10],\ j \in [1,2] \right\}.$$

For $i \in [1,10]$ we define $M_{\mathcal{B}_1,i}$ to be the describing matrix of the Lie bracket with b_i on $e'\Omega f$ with respect to the basis $\mathcal{B}_1$, i.e.

$$M_{\mathcal{B}_1,i} := \left(\mathrm{ad}_{e'\Omega f}\, b_i\right)_{\mathcal{B}_1,\mathcal{B}_1} \in R^{2\times 2} \text{ for } i \in [1,10].$$

Furthermore, we have the following diagram.

$$\begin{tikzcd}
h \arrow[r, mapsto] & h_{\mathcal{B}_1,\mathcal{B}_1} \\
\mathrm{End}_R(e'\Omega f) \arrow[r, "\widetilde{\varphi}_1", "\sim"'] & R^{2\times 2} \\
\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega f) \arrow[u, hook] \arrow[r, "\sim"'] & \widetilde{\varphi}_1(\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega f)) =: \widetilde{E}_1 \arrow[u, hook]
\end{tikzcd}$$

Here the map $\widetilde{\varphi}_1\colon \mathrm{End}_R(e'\Omega f) \to R^{2\times 2}$ is the isomorphism of R-algebras sending a map $h \in \mathrm{End}_R(e'\Omega f)$ to its describing matrix in the R-algebra of 2×2-matrices over R with respect to the basis $\mathcal{B}_1$. Since $\widetilde{\varphi}_1$ is a morphism of R-algebras, $\widetilde{E}_1$ is an R-subalgebra of $R^{2\times 2}$.

Then we have

$$\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega f) \simeq \widetilde{E}_1 = \left\{ M \in R^{2\times 2} \,\middle|\, M \cdot M_{\mathcal{B}_1,i} = M_{\mathcal{B}_1,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{60}$$

Now note that $(\mathrm{ad}_{e'\Omega f}\, b_i)(x) = [b_i, x] = b_i x - x b_i$ for $x \in e'\Omega f$. So we obtain the following identity.

$$M_{\mathcal{B}_1,i} = \left(\mathrm{ad}_{e'\Omega f}\, b_i\right)_{\mathcal{B}_1,\mathcal{B}_1} = M_{\mathcal{B}_1,i,\mathrm{l}} - M_{\mathcal{B}_1,i,\mathrm{r}} \text{ for } i \in [1,10] \tag{61}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_1,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_1,i,\mathrm{r}} = 0$; cf. (42). Thus we conclude that

$$M_{\mathcal{B}_1,i} \in \{M_{\mathcal{B}_1,i,\mathrm{l}}\,, -M_{\mathcal{B}_1,i,\mathrm{l}}\,, M_{\mathcal{B}_1,i,\mathrm{r}}\,, -M_{\mathcal{B}_1,i,\mathrm{r}}\}. \tag{62}$$

Using (41), (60), (61) and (62) we obtain that $E_1 = \widetilde{E}_1$. But $E_1 = \mathrm{C}_{R^{2\times 2}}(A_1) = L_1$; cf. (18) and (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we have shown that $\widetilde{E}_1$ is local. In particular, $\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega f)$ is local; cf. (60).

Ad $e\Omega f$.

We want to show that $e\Omega f$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_2$ is an R-linear basis of $e\Omega f$.

It suffices to show that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(e\Omega f)$ is a local ring; cf. Lemma 35.

Define the isomorphism of R-algebras $\widetilde{\varphi}_2\colon \operatorname{End}_R(e\Omega f) \to R^{2\times 2}$ by $\widetilde{\varphi}_2(h) := h_{\mathcal{B}_2,\mathcal{B}_2} \in R^{2\times 2}$ for $h \in \operatorname{End}_R(e\Omega f)$. Define the R-subalgebra $\widetilde{E}_2 := \widetilde{\varphi}_2(\operatorname{End}_{\mathfrak{l}(T)}(e\Omega f))$ of $R^{2\times 2}$. Define

$$M_{\mathcal{B}_2,i} := (\operatorname{ad}_{e\Omega f} b_i)_{\mathcal{B}_2,\mathcal{B}_2} \in R^{2\times 2} \text{ for } i \in [1,10].$$

Then $M_{\mathcal{B}_2,i} = M_{\mathcal{B}_2,i,\mathrm{l}} - M_{\mathcal{B}_2,i,\mathrm{r}}$ for $i \in [1,10]$. We have

$$\operatorname{End}_{\mathfrak{l}(T)}(e\Omega f) \simeq \widetilde{E}_2 = \left\{ M \in R^{2\times 2} \,\middle|\, M \cdot M_{\mathcal{B}_2,i} = M_{\mathcal{B}_2,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{63}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_2,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_2,i,\mathrm{r}} = 0$; cf. (45). We conclude that $\widetilde{E}_2 = E_2 = E_1$; cf. (43), (44), (46) and (63). But $E_1 = \mathrm{C}_{R^{2\times 2}}(A_1) = L_1$; cf. (18) and (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we have shown that $\widetilde{E}_2$ is local. In particular, $\operatorname{End}_{\mathfrak{l}(T)}(e\Omega f)$ is local; cf. (63).

Ad $f\Omega e$.

We want to show that $f\Omega e$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_3$ is an R-linear basis of $f\Omega e$.

It suffices to show that the endomorphism ring $\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e)$ is a local ring; cf. Lemma 35.

Define the isomorphism of R-algebras $\widetilde{\varphi}_3\colon \operatorname{End}_R(f\Omega e) \to R^{2\times 2}$ by $\widetilde{\varphi}_3(h) := h_{\mathcal{B}_3,\mathcal{B}_3} \in R^{2\times 2}$ for $h \in \operatorname{End}_R(f\Omega e)$. Define the R-subalgebra $\widetilde{E}_3 := \widetilde{\varphi}_3(\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e))$ of $R^{2\times 2}$. Define

$$M_{\mathcal{B}_3,i} := (\operatorname{ad}_{f\Omega e} b_i)_{\mathcal{B}_3,\mathcal{B}_3} \in R^{2\times 2} \text{ for } i \in [1,10].$$

Then $M_{\mathcal{B}_3,i} = M_{\mathcal{B}_3,i,\mathrm{l}} - M_{\mathcal{B}_3,i,\mathrm{r}}$ for $i \in [1,10]$. We have

$$\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e) \simeq \widetilde{E}_3 = \left\{ M \in R^{2\times 2} \,\middle|\, M \cdot M_{\mathcal{B}_3,i} = M_{\mathcal{B}_3,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{64}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_3,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_3,i,\mathrm{r}} = 0$; cf. (48). We conclude that $\widetilde{E}_3 = E_3 = E_1$; cf. (43), (47), (49) and (64). But $E_1 = \mathrm{C}_{R^{2\times 2}}(A_1) = L_1$; cf. (18) and (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we have shown that $\widetilde{E}_3$ is local. In particular, $\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e)$ is local; cf. (64).

Ad $f\Omega e'$.

We want to show that $f\Omega e'$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_4$ is an R-linear basis of $f\Omega e'$.

It suffices to show that the endomorphism ring $\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e')$ is a local ring; cf. Lemma 35.

Define the isomorphism of R-algebras $\widetilde{\varphi}_4\colon \operatorname{End}_R(f\Omega e') \to R^{2\times 2}$ by $\widetilde{\varphi}_4(h) := h_{\mathcal{B}_4,\mathcal{B}_4} \in R^{2\times 2}$ for $h \in \operatorname{End}_R(f\Omega e')$. Define the R-subalgebra $\widetilde{E}_4 := \widetilde{\varphi}_4(\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e'))$ of $R^{2\times 2}$. Define

$$M_{\mathcal{B}_4,i} := \left(\operatorname{ad}_{f\Omega e'} b_i\right)_{\mathcal{B}_4,\mathcal{B}_4} \in R^{2\times 2} \text{ for } i \in [1,10].$$

Then $M_{\mathcal{B}_4,i} = M_{\mathcal{B}_4,i,\mathrm{l}} - M_{\mathcal{B}_4,i,\mathrm{r}}$ for $i \in [1,10]$. We have

$$\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e') \simeq \widetilde{E}_4 = \left\{ M \in R^{2\times 2} \,\middle|\, M \cdot M_{\mathcal{B}_4,i} = M_{\mathcal{B}_4,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{65}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_4,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_4,i,\mathrm{r}} = 0$; cf. (51). We conclude that $\widetilde{E}_4 = E_4 = E_1$; cf. (43), (50), (52) and (65). But $E_1 = \mathrm{C}_{R^{2\times 2}}(A_1) = L_1$; cf. (18) and (19) in §5.1. We have shown that L_1 is local; cf. Remark 156. So we have shown that $\widetilde{E}_4$ is local. In particular, $\operatorname{End}_{\mathfrak{l}(T)}(f\Omega e')$ is local; cf. (65).

Ad $e\Omega e'$.

We want to show that $e\Omega e'$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_5$ is an R-linear basis of $e\Omega e'$.

It suffices to show that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(e\Omega e')$ is a local ring; cf. Lemma 35.

Define the isomorphism of R-algebras $\widetilde{\varphi}_5\colon \mathrm{End}_R(e\Omega e') \to R^{3\times 3}$ by $\widetilde{\varphi}_5(h) := h_{\mathcal{B}_5,\mathcal{B}_5} \in R^{3\times 3}$ for $h \in \mathrm{End}_R(e\Omega e')$. Define the R-subalgebra $\widetilde{E}_5 := \widetilde{\varphi}_5(\mathrm{End}_{\mathfrak{l}(T)}(e\Omega e'))$ of $R^{3\times 3}$. Define

$$M_{\mathcal{B}_5,i} := (\mathrm{ad}_{e\Omega e'}\, b_i)_{\mathcal{B}_5,\mathcal{B}_5} \in R^{3\times 3} \text{ for } i \in [1,10].$$

Then $M_{\mathcal{B}_5,i} = M_{\mathcal{B}_5,i,\mathrm{l}} - M_{\mathcal{B}_5,i,\mathrm{r}}$ for $i \in [1,10]$. We have

$$\mathrm{End}_{\mathfrak{l}(T)}(e\Omega e') \simeq \widetilde{E}_5 = \left\{ M \in R^{3\times 3} \,\middle|\, M \cdot M_{\mathcal{B}_5,i} = M_{\mathcal{B}_5,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{66}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_5,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_5,i,\mathrm{r}} = 0$; cf. (54). We conclude that $\widetilde{E}_5 = E_5$; cf. (53) and (66). But $E_5 = \mathrm{C}_{R^{3\times 3}}(A_2) = L_2$; cf. (20) and (21) in §5.2. We have shown that L_2 is local; cf. Remark 158. So we have shown that $\widetilde{E}_5$ is local. In particular, $\mathrm{End}_{\mathfrak{l}(T)}(e\Omega e')$ is local; cf. (66).

Ad $e'\Omega e$.

We want to show that $e'\Omega e$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_6$ is an R-linear basis of $e'\Omega e$.

It suffices to show that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega e)$ is a local ring; cf. Lemma 35.

Define the isomorphism of R-algebras $\widetilde{\varphi}_6\colon \mathrm{End}_R(e'\Omega e) \to R^{3\times 3}$ by $\widetilde{\varphi}_6(h) := h_{\mathcal{B}_6,\mathcal{B}_6} \in R^{3\times 3}$ for $h \in \mathrm{End}_R(e'\Omega e)$. Define the R-subalgebra $\widetilde{E}_6 := \widetilde{\varphi}_6(\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega e))$ of $R^{3\times 3}$. Define

$$M_{\mathcal{B}_6,i} := (\mathrm{ad}_{e'\Omega e}\, b_i)_{\mathcal{B}_6,\mathcal{B}_6} \in R^{3\times 3} \text{ for } i \in [1,10].$$

Then $M_{\mathcal{B}_6,i} = M_{\mathcal{B}_6,i,\mathrm{l}} - M_{\mathcal{B}_6,i,\mathrm{r}}$ for $i \in [1,10]$. We have

$$\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega e) \simeq \widetilde{E}_6 = \left\{ M \in R^{3\times 3} \,\middle|\, M \cdot M_{\mathcal{B}_6,i} = M_{\mathcal{B}_6,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{67}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_6,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_6,i,\mathrm{r}} = 0$; cf. (57). We conclude that $\widetilde{E}_6 = E_6 = E_5$; cf. (53), (55), (58) and (67). But $E_5 = \mathrm{C}_{R^{3\times 3}}(A_2) = L_2$; cf. (20) and (21) in §5.2. We have shown that L_2 is local; cf. Remark 158. So we have shown that $\widetilde{E}_6$ is local. In particular, $\mathrm{End}_{\mathfrak{l}(T)}(e'\Omega e)$ is local; cf. (67).

We summarize.

We obtain the following decomposition of $\mathfrak{l}(\Omega)$ into a direct sum of $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$.

$$\mathfrak{l}(\Omega) = T \oplus e\Omega e' \oplus e\Omega f \oplus e'\Omega e \oplus e'\Omega f \oplus f\Omega e \oplus f\Omega e'$$

In this decomposition, all summands but T are indecomposable as $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$. The summand T is a trivial $\mathfrak{l}(T)$-Lie submodule of $\mathfrak{l}(\Omega)$.

Note that outside of the rational torus $\mathfrak{l}(T)$ resp. T, the decomposition of $\mathfrak{l}(\Omega)$ into indecomposable $\mathfrak{l}(T)$-Lie submodules and the decomposition of Ω into indecomposable T-T-sub-bimodules coincide; cf. equation (59) on page 142.

In addition we have shown that in a decomposition of $\mathfrak{l}(\Omega)$ as an $\mathfrak{l}(T)$-Lie module into indecomposable summands, all these summands have local $\mathfrak{l}(T)$-endomorphism rings.

6.6 Magma

The following two codes are used for calculations with $\Omega \simeq \mathbb{Z}_{(2)}\,\mathrm{S}_4$ in Magma. However, note that initialization files such as “pre” and “definitions” are required; cf. Magma Codes 3 and 4.

Magma Code 16: z2s4Init1

```
// global definitions
Sizes := [2,3,3,1,1]; // sizes of blocks
nb := #Sizes; // number of blocks
nt := 19; // number of ties needed to describe Omega
rt := &+Sizes; // rank of torus
rl := &+[Sizes[i]^2 : i in [1..nb]]; // rank of Omega
prime := 2; // R is Z localized at the prime number 2
e := 8; // ties that describe Omega are given mod e

RM := RMatrixSpace(Z,rl,rt);
RMQ := RMatrixSpace(Q,rl,rt);
RV := RMatrixSpace(Z,rl,1);
RQV := VectorSpace(Q,rl);
RM2 := RMatrixSpace(Z,nt,rl);
RMB := RMatrixSpace(Z,rl,rl);
RMBQ := KMatrixSpace(Q,rl,rl);
RMVQ := KMatrixSpace(Q,rl,1);

Ties_Omega :=   // Ties mod e that describe Omega,
                // given in the rows of this matrix
    RM2!Matrix([
        [ 0, 0, 0, 0,0,0,0,0,0,0,2,0, 0, 0, 0,0, 0, 0,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,0,0,2, 0, 0, 0,0, 0, 0,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,0,0,0, 0, 0, 0,0, 0, 0,0, 2, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,0,0,0, 0, 0, 0,0, 0, 0,0, 0, 2,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,0,1,0, 0, 0, 0,0, 0, 0,0,-1, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,0,0,1, 0, 0, 0,0, 0, 0,0, 0,-1,0,0, 0],
        [-2, 0, 0, 0,1,0,0,0,0,0,0,0, 0, 1, 0,0, 0, 0,0, 0, 0,0,0, 0],
        [ 0,-2, 0, 0,0,1,0,0,0,0,0,0, 0, 0, 1,0, 0, 0,0, 0, 0,0,0, 0],
        [ 0, 0,-2, 0,0,0,0,1,0,0,0,0, 0, 0, 0,0, 1, 0,0, 0, 0,0,0, 0],
        [ 0, 0, 0,-2,0,0,0,0,1,0,0,0, 0, 0, 0,0, 0, 1,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,2,0,0,0,0,0,0,0, 0,-2, 0,0, 0, 0,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,2,0,0,0,0,0,0, 0, 0,-2,0, 0, 0,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,2,0,0,0,0, 0, 0, 0,0,-2, 0,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,2,0,0,0, 0, 0, 0,0, 0,-2,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,4,0,0,0,0,0, 0, 0, 0,4, 0, 0,0, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,4,0,0, 0, 0, 0,0, 0, 0,4, 0, 0,0,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,0,0,0, 4, 0, 0,0, 0, 0,0, 0, 0,4,0, 0],
        [ 0, 0, 0, 0,0,0,0,0,0,0,0,0,-1, 0, 0,0, 0, 0,0, 0, 0,1,1,-1],
        [ 0, 0, 0, 0,0,0,0,0,0,0,0,0, 0, 0, 0,0, 0, 0,0, 0, 0,2,0,-2]
    ]);
```

Magma Code 17: z2s4Init2

```
// R-linear basis of T
b := [];
// e Omega e
b[1] := CoerceGamma([1,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0]);
b[2] := CoerceGamma([0,0,0,0,2,0,0,0,0,0,0,0,0,-2,0,0,0,0,0,0,0,0,0,0]);
```

```
b[3] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,8,0,0,0,0,0,0,0,0,0,0]);
// e' Omega e'
b[4] := CoerceGamma([0,0,0,1,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0]);
b[5] := CoerceGamma([0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0,-2,0,0,0,0,0,0]);
b[6] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,8,0,0,0,0,0,0]);
// f Omega f
b[7] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,1,1]);
b[8] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0,2]);
b[9] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4,4]);
b[10] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,8]);

// describing matrices of the adjoint endomorphisms of the elements
// of b with respect to the basis Basis_Omega which is defined in
// the file "definitions"
A := [RMBQ!admatrix(x) : x in b];
```

Chapter 7: Example $\mathbb{Z}_{(2)}\,\mathrm{S}_5$

Let $R := \mathbb{Z}_{(2)}$. We have $\mathrm{frac}(R) = \mathbb{Q}$. We define $\Gamma := R \times R \times R \times R \times R^{2\times 2} \times R^{2\times 2} \times R^{3\times 3}$. We denote the full diagonal in Γ by Δ.

7.1 The Morita-reduced version Ω of $\mathbb{Z}_{(2)}\,\mathrm{S}_5$

This example is about the group algebra $\mathbb{Z}_{(2)}\,\mathrm{S}_5$ over the ground ring $\mathbb{Z}_{(2)}$ which is a discrete valuation ring. By Maschke's theorem, the group algebra $\mathbb{Q}\,\mathrm{S}_4$ is semisimple. The isomorphism given by the Artin-Wedderburn theorem may take the following form; cf. [Kün01, p. 29f].

$$
\begin{array}{llccccc}
\omega\colon & \mathbb{Q}\,\mathrm{S}_5 \xrightarrow{\sim} & \mathbb{Q} \times \mathbb{Q} \times & \mathbb{Q}^{4\times 4} & \times\ \mathbb{Q}^{4\times 4} & \times & \mathbb{Q}^{5\times 5} \\
 & & \times & \mathbb{Q}^{5\times 5} & \times\ \mathbb{Q}^{6\times 6} & &
\end{array}
$$

$$
(1,2) \mapsto \left(-1, \quad 1, \quad \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 & 2 & 4 & -4 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -2 & 1 \\ 1 & 1 & 0 & 1 & -2 \end{pmatrix}, \right.
$$
$$
\left. \begin{pmatrix} -3 & -64 & 42 & -12 & -28 \\ 0 & 11 & -5 & 0 & 0 \\ 0 & 24 & -11 & 0 & 0 \\ 3 & 67 & -41 & 10 & 21 \\ -1 & -11 & 8 & -3 & -6 \end{pmatrix}, \quad \begin{pmatrix} -5 & -1850 & -294 & -860 & -600 & -110 \\ 2 & 1025 & 161 & 476 & 328 & 64 \\ -4 & -1680 & -265 & -780 & -540 & -100 \\ -5 & -2627 & -413 & -1220 & -841 & -164 \\ 3 & 1419 & 224 & 659 & 456 & 86 \\ 0 & 134 & 21 & 62 & 42 & 9 \end{pmatrix} \right)
$$

$$
(1,2,3,4,5) \mapsto \left(1, \quad 1, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 & 6 & 6 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -2 & 1 \\ 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 2 & 2 & -1 \end{pmatrix}, \right.
$$
$$
\left. \begin{pmatrix} 3 & 60 & -38 & 10 & 22 \\ 2 & 40 & -28 & 9 & 20 \\ 5 & 99 & -69 & 22 & 49 \\ 4 & 104 & -73 & 23 & 55 \\ 1 & 9 & -6 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} -7 & -3540 & -560 & -1644 & -1138 & -212 \\ 8 & 4408 & 698 & 2049 & 1422 & 270 \\ -13 & -6987 & -1103 & -3246 & -2243 & -426 \\ -18 & -9984 & -1581 & -4641 & -3221 & -612 \\ 7 & 3861 & 610 & 1794 & 1241 & 236 \\ 3 & 1668 & 263 & 775 & 535 & 103 \end{pmatrix} \right)
$$

Since ω is a $\mathbb{Q}$-algebra isomorphism and $\langle (1,2), (1,2,3,4,5) \rangle = \mathrm{S}_5$, this characterizes ω uniquely. The first factor corresponds to the sign representation. The second factor corresponds to the trivial representation.

We write

$$\Gamma_1 := R \times R \times R^{4\times 4} \times R^{4\times 4} \times R^{5\times 5} \times R^{5\times 5} \times R^{6\times 6}.$$

Using the Wedderburn embedding ω, we get that $R\,\mathrm{S}_5 \simeq \omega(R\,\mathrm{S}_5) \subseteq \Gamma_1$. As rings, $\omega(R\,\mathrm{S}_5)$ is Morita equivalent to the Morita reduced ring $\omega(R\,\mathrm{S}_5)^{\mathrm{red}} =: \Omega$ which can be described as follows.

$$\Omega = \left\{ \left(a, b, c, d, \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix}, \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix}, \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix} \right) \in \Gamma \;\middle|\; \begin{array}{rcll} a & \equiv_2 & b & \\ c & \equiv_2 & d & \\ e_{i,2} + f_{i,2} & \equiv_8 & 2g_{i,2} & \text{for } i \in \{1,2\} \\ f_{2,j} & \equiv_2 & g_{2,j} & \text{for } j \in \{1,2\} \\ e_{k,1} - f_{k,1} & \equiv_4 & g_{k,3} & \text{for } k \in \{1,2\} \\ f_{1,2} & \equiv_4 & 2g_{3,2} & \\ e_{1,1} & \equiv_2 & g_{3,3} & \\ b - f_{1,1} & \equiv_4 & 2g_{3,1} & \\ a + b + e_{1,1} + f_{1,1} & \equiv_8 & 2g_{1,1} + 2g_{3,3} & \\ g_{1,1} + g_{3,3} & \equiv_2 & 0 & \\ e_{1,2} & \equiv_2 & 0 & \\ f_{1,2} & \equiv_2 & 0 & \\ g_{1,l} & \equiv_2 & 0 & \text{for } l \in \{2,3\} \\ g_{2,3} & \equiv_2 & 0 & \end{array} \right\} \tag{68}$$

The intersection $\Omega \cap \Delta$ is a maximal commutative R-subalgebra of Ω; cf. Lemma 121.

In $\Omega \cap \Delta$, we have the idempotents

$$\begin{aligned} e &:= \left(1, 1, 0, 0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), \\ f &:= \left(0, 0, 0, 0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \\ g &:= \left(0, 0, 1, 1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right). \end{aligned}$$

Note that we have

$$ef = eg = fe = fg = ge = gf = 0.$$

We will see later that e, f and g are primitive in Ω; cf. Remark 170.(2) and Remark 171.(2) below. So the sum $1_\Omega = e + f + g$ is an orthogonal decomposition of 1_Ω into primitive idempotents in Ω which is contained in $\Omega \cap \Delta$. We obtain the following Peirce decomposition of Ω.

$$\Omega = e\Omega e \oplus f\Omega f \oplus g\Omega g \oplus e\Omega f \oplus e\Omega g \oplus f\Omega e \oplus f\Omega g \oplus g\Omega e \oplus g\Omega f \tag{69}$$

We give an R-linear basis of the R-algebra Ω sorted by Peirce components with respect to the idempotents e, f and g.

Peirce component	R-linear basis of Peirce component
$e\Omega e$	$b_1 := \left(1,1,0,0,\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right)\right)$ $b_2 := \left(0,4,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_3 := \left(0,0,0,0,\left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_4 := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_5 := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&2\end{smallmatrix}\right)\right)$ $b_6 := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&4\end{smallmatrix}\right)\right)$ $b_7 := \left(0,2,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}-2&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&2\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_8 := \left(0,0,0,0,\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}2&0&0\\0&0&0\\1&0&0\end{smallmatrix}\right)\right)$
$f\Omega f$	$b_9 := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_{10} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&2&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_{11} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&4&0\\0&0&0\end{smallmatrix}\right)\right)$
$g\Omega g$	$b_{12} := \left(0,0,1,1,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_{13} := \left(0,0,0,2,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$
$f\Omega e$	$b_{14} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_{15} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&2\\0&0&0\end{smallmatrix}\right)\right)$ $b_{16} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\2&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_{17} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&4\\0&0&0\end{smallmatrix}\right)\right)$
$e\Omega f$	$b_{18} := \left(0,0,0,0,\left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-2\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&-1&0\end{smallmatrix}\right)\right)$ $b_{19} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&2&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_{20} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&4&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right)$ $b_{21} := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&2&0\end{smallmatrix}\right)\right)$
$e\Omega g = 0$	–
$f\Omega g = 0$	–
$g\Omega e = 0$	–
$g\Omega f = 0$	–

We define the R-linear basis $\mathcal{B} := (b_i : i \in [1, 21])$ of Ω.

Omitting the zero components we obtain the following Peirce decomposition of Ω; cf. equation (69).

$$\Omega = e\Omega e \oplus f\Omega f \oplus g\Omega g \oplus e\Omega f \oplus f\Omega e \tag{70}$$

Note that the Peirce component $e\Omega e$ is not commutative, e.g. $b_7 b_8 \neq b_8 b_7$. In the previous examples in §1 and in §6, all considered Peirce components of the form $x\Omega x$ for a primitive idempotent x were commutative.

So in Ω we have the primitive idempotent e such that $e\Omega e$ is not commutative. This entails that there cannot exist an orthogonal decomposition $1_\Omega = \sum_{i\in[1,l]} e_i$ such that $e_i\Omega e_i \subseteq \Delta$ for $i \in [1, l]$; cf. Lemma 176 below, using Remark 170.(1) and Remark 171.(1) below.

The reason for this phenomenon is that at the prime 2 there exists a decomposition number for S_5 that is bigger than 1, viz. 2.

7.2 Primitivity of certain idempotents in Ω

Keep the notation of §7.1 and of §7.3.

Remark 170.

(1) *The $\mathbb{Z}_{(2)}$-algebras $f\Omega f$ and $g\Omega g$ are local.*

(2) *The idempotents f and g are primitive in Ω.*

Proof. Ad (1). Consider the Peirce component $f\Omega f$. We have the following isomorphism of R-algebras.

$$\begin{array}{ccc} f\Omega f & \xrightarrow{\sim} & \{(a,b,c) \in R^{\times 3} \mid a+b \equiv_8 2c \text{ and } b \equiv_2 c\} \\ \left(0,0,0,0,\left(\begin{smallmatrix}0 & \\ & a\end{smallmatrix}\right),\left(\begin{smallmatrix}0 & \\ & b\end{smallmatrix}\right),\left(\begin{smallmatrix}0 & & \\ & c & \\ & & 0\end{smallmatrix}\right)\right) & \longmapsto & (a,b,c) \end{array}$$

Recall that (b_9, b_{10}, b_{11}) is an R-linear basis of $f\Omega f$. The images of these three elements form the R-linear basis $((1,1,1),(0,4,2),(0,0,4))$ of the right hand side. So this is in fact an isomorphism of R-algebras.

The right hand side equals L_3 of §5.3; cf. (24). We have shown that this $\mathbb{Z}_{(2)}$-algebra is local; cf. Corollary 160. So $f\Omega f$ also is local.

Consider the Peirce component $g\Omega g$. We have the following isomorphism of R-algebras.

$$\begin{array}{ccc} g\Omega g & \xrightarrow{\sim} & \{(a,b) \in R^{\times 2} \mid a \equiv_2 b\} \\ \left(0,0,a,b,\left(\begin{smallmatrix}0 & \\ & 0\end{smallmatrix}\right),\left(\begin{smallmatrix}0 & \\ & 0\end{smallmatrix}\right),\left(\begin{smallmatrix}0 & & \\ & 0 & \\ & & 0\end{smallmatrix}\right)\right) & \longmapsto & (a,b) \end{array}$$

Recall that (b_{12}, b_{13}) is an R-linear basis of $g\Omega g$. The images of these two elements form the R-linear basis $((1,1),(0,2))$ of the right hand side. So this is in fact an isomorphism of R-algebras.

The right hand side equals $\mu_1(L_1)$ of §5.1.2 where we have shown that this $\mathbb{Z}_{(2)}$-algebra is local. So $g\Omega g$ also is local.

Ad (2). By (1), the R-algebras $f\Omega f$ and $g\Omega g$ are local. Applying Remark 139.(2), we obtain that the idempotents f and g are primitive in Ω. □

Remark 171.

(1) *The $\mathbb{Z}_{(2)}$-algebra $e\Omega e$ is local.*

(2) *The idempotent e is primitive in Ω.*

Proof. Ad (1). We define the following injective morphism of R-algebras.

$$\begin{array}{ccc} \varrho\colon e\Omega e & \longrightarrow & R \times R \times R \times R \times R^{2\times 2} \\ \left(a,b,0,0,\begin{pmatrix}c & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}d & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}f_{1,1} & 0 & f_{1,3}\\ 0 & 0 & 0\\ f_{3,1} & 0 & f_{3,3}\end{pmatrix}\right) & \longmapsto & \left(a,b,c,d,\begin{pmatrix}f_{1,1} & f_{1,3}\\ f_{3,1} & f_{3,3}\end{pmatrix}\right) \end{array}$$

We write $\varrho(e\Omega e) =: \Xi$. To show that $e\Omega e$ is local, it suffices to show that Ξ is local. Ξ has the R-linear basis $(\xi_i : i \in [1,8])$ where $x_i := \varrho(b_i)$ for $i \in [1,8]$. These elements are

$$\begin{array}{rclrcl} \xi_1 &=& \left(1,1,1,1,\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) & \xi_2 &=& \left(0,4,0,0,\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\right) \\ \xi_3 &=& \left(0,0,4,0,\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\right) & \xi_4 &=& \left(0,0,0,4,\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\right) \\ \xi_5 &=& \left(0,0,0,0,\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\right) & \xi_6 &=& \left(0,0,0,0,\begin{pmatrix}0 & 0\\ 0 & 4\end{pmatrix}\right) \\ \xi_7 &=& \left(0,2,0,-2,\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\right) & \xi_8 &=& \left(0,0,2,2,\begin{pmatrix}2 & 0\\ 1 & 0\end{pmatrix}\right) \end{array}$$

We choose an orthogonal decomposition of $1_{K\Xi}$ into centrally primitive idempotents in $K\Xi$. We have $1_{K\Xi} = \sum_{i\in[1,5]} \varepsilon_i$ with ε_i defined as follows for $i \in [1,5]$.

$$\varepsilon_1 := \left(1,0,0,0,\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right) \quad \varepsilon_2 := \left(0,1,0,0,\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right) \quad \varepsilon_3 := \left(0,0,1,0,\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right)$$
$$\varepsilon_4 := \left(0,0,0,1,\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right) \quad \varepsilon_5 := \left(0,0,0,0,\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

We obtain the following R-subalgebras of $R \times R \times R \times R \times R^{2\times 2}$.

$$\varepsilon_1\Xi = \left\{ \left(r, 0, 0, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \,\middle|\, r \in R \right\} = \begin{pmatrix} \underset{\boxed{1}}{R} & \underset{\boxed{2}}{0} & \underset{\boxed{3}}{0} & \underset{\boxed{4}}{0} & \underset{\boxed{5}}{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

$$\varepsilon_2\Xi = \left\{ \left(0, r, 0, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \,\middle|\, r \in R \right\} = \begin{pmatrix} \underset{\boxed{1}}{0} & \underset{\boxed{2}}{R} & \underset{\boxed{3}}{0} & \underset{\boxed{4}}{0} & \underset{\boxed{5}}{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

$$\varepsilon_3\Xi = \left\{ \left(0, 0, r, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \,\middle|\, r \in R \right\} = \begin{pmatrix} \underset{\boxed{1}}{0} & \underset{\boxed{2}}{0} & \underset{\boxed{3}}{R} & \underset{\boxed{4}}{0} & \underset{\boxed{5}}{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

$$\varepsilon_4\Xi = \left\{ \left(0, 0, 0, r, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \,\middle|\, r \in R \right\} = \begin{pmatrix} \underset{\boxed{1}}{0} & \underset{\boxed{2}}{0} & \underset{\boxed{3}}{0} & \underset{\boxed{4}}{R} & \underset{\boxed{5}}{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

$$\begin{aligned} \varepsilon_5\Xi &= \left\{ \left(0, 0, 0, 0, \begin{pmatrix} r & s \\ t & u \end{pmatrix} \right) \,\middle|\, r, s, t, u \in R \text{ and } r \equiv_2 u \text{ and } s \equiv_2 0 \right\} \\ &= \begin{pmatrix} \underset{\boxed{1}}{0} & \underset{\boxed{2}}{0} & \underset{\boxed{3}}{0} & \underset{\boxed{4}}{0} & \underset{\boxed{5}}{\overset{2}{\begin{matrix} R & (2) \\ R & R \end{matrix}}} \end{pmatrix} \end{aligned}$$

Note that for $i \in [1, 4]$ we have $\varepsilon_i\Xi \simeq R$. So we obtain $\mathrm{Jac}(\varepsilon_i\Xi) \simeq (2)$ for $i \in [1, 4]$. We want to determine $\mathrm{Jac}(\varepsilon_5\Xi)$.

We write

$$\varepsilon_5\Xi \xrightarrow[\sim]{\eta} \begin{pmatrix} \overset{2}{\begin{matrix} R & (2) \\ R & R \end{matrix}} \end{pmatrix} = {}_R\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle =: E$$

where η is the projection on the last component. Note that

$$2E = {}_R\left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right\rangle = \begin{pmatrix} \overset{4}{\begin{matrix} (2) & (4) \\ (2) & (2) \end{matrix}} \end{pmatrix}. \tag{71}$$

We define the following R-subalgebra of E.

$$\begin{pmatrix} (2) & (2) \\ R & (2) \end{pmatrix} = {}_R\left\langle \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}}_{=:A_1}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}}_{=:A_2}, \underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}}_{=:A_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{=:A_4} \right\rangle =: I.$$

We have the R-algebra morphism

$$\begin{aligned} E &\to \mathbb{F}_2 \times \mathbb{F}_2 \\ \begin{pmatrix} a & 2b \\ c & d \end{pmatrix} &\mapsto (a + (2), d + (2)) \,. \end{aligned}$$

Note that I is the kernel of this R-algebra morphism and thus I is an ideal of E.

We have the following multiplication table of I.

$A_i \cdot A_j$	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$2A_1$	0	$2A_3$	0
$i=2$	0	$2A_2$	0	$2A_4$
$i=3$	0	$2A_3$	0	A_1
$i=4$	$2A_4$	0	A_2	0

This shows that

$$I^2 = {}_R\left\langle \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}}_{=2A_3=:A_5}, \underbrace{\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}}_{=2A_4=:A_6} \right\rangle = \begin{pmatrix} (2) & (4) \\ (2) & (2) \end{pmatrix}.$$

To determine I^3, we multiply each of the basis elements of I^2 with each of the basis elements of I. It suffices to consider $A_i \cdot A_j$ for $i \in [1,4]$ and $j \in \{1,2,5,6\}$ since $I \cdot I^2 = I^2 \cdot I = I^3$.

$A_i \cdot A_j$	$j=1$	$j=2$	$j=5$	$j=6$
$i=1$	$2A_1$	0	$2A_5$	0
$i=2$	0	$2A_2$	0	$2A_6$
$i=3$	0	A_5	0	$2A_1$
$i=4$	A_6	0	$2A_2$	0

This shows that

$$I^3 = {}_R\left\langle \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right\rangle = \begin{pmatrix} (4) & (4) \\ (2) & (4) \end{pmatrix}.$$

In particular, we obtain that $I^3 \subseteq 2E$; cf. equation (71). Now we can apply [Mül13, Lemma 213.(ii)]. This shows that $I \subseteq \mathrm{Jac}(E)$.

Since $|E/I| = 2$ and $\mathrm{Jac}(E) \subset E$, we conclude that already $I = \mathrm{Jac}(E)$; cf. [Mül13, Lemma 187].

Then the ideal $\eta^{-1}(I) =: \widetilde{I}$ is the Jacobson radical of $\varepsilon_5\Xi$.

Now $\mathrm{Jac}(\Xi) = \Xi \cap \bigoplus_{i\in[1,5]} \mathrm{Jac}(\varepsilon_i\Xi)$; cf. [Mül13, Proposition 222], that is

$$\mathrm{Jac}(\Xi) = \left\{ \left(a, b, c, d, \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix} \right) \in \Xi \,\middle|\, a, b, c, d, f_{1,1}, f_{2,2} \in (2) \right\}.$$

Note that the entries of the elements $\xi_2, \ldots, \xi_8$ are all divisible by 2. So $\mathrm{Jac}(\Xi)$ has the R-linear basis

$$(2 \cdot \xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \xi_5,\ \xi_6,\ \xi_7,\ \xi_8).$$

Thus $\Xi/\mathrm{Jac}(\Xi)$ is isomorphic to $\mathbb{F}_2$, in particular it is a field. So by [Mül13, Lemma 192] we see that Ξ is local.

Since ϱ is an isomorphism of R-algebras, we conclude that $e\Omega e$ is local.

Ad (2). By (1), the R-algebra $e\Omega e$ is local. Applying Remark 139.(2), we obtain that the idempotent e is primitive in Ω. $\square$

7.3 Tori in $\mathfrak{l}(\Omega)$

Keep the notation of §7.1.

An R-linear basis of $\mathrm{Z}(\Omega)$, the center of Ω, can be chosen as follows.

$$\begin{array}{lcl} \text{basis element} & & \text{expressed in the basis } \mathcal{B} \text{ of } \Omega \\ \left(1,1,0,0,\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&&\\&1&\\&&1\end{smallmatrix}\right)\right) & = & b_1+b_9 \\ \left(0,4,0,0,\left(\begin{smallmatrix}4&\\&4\end{smallmatrix}\right),\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}2&&\\&2&\\&&2\end{smallmatrix}\right)\right) & = & b_2+b_3-b_5+b_6+4b_9-b_{10} \\ \left(0,0,0,0,\left(\begin{smallmatrix}4&\\&4\end{smallmatrix}\right),\left(\begin{smallmatrix}4&\\&4\end{smallmatrix}\right),\left(\begin{smallmatrix}4&&\\&4&\\&&4\end{smallmatrix}\right)\right) & = & b_3+b_4+b_6+4b_9 \\ \left(0,0,0,0,\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}8&\\&8\end{smallmatrix}\right),\left(\begin{smallmatrix}0&&\\&0&\\&&0\end{smallmatrix}\right)\right) & = & 2b_4-2b_5+b_6+2b_{10}-b_{11} \\ \left(0,0,0,0,\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}4&&\\&4&\\&&4\end{smallmatrix}\right)\right) & = & 2b_5+b_{11} \\ \left(0,0,1,1,\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&&\\&0&\\&&0\end{smallmatrix}\right)\right) & = & b_{12} \\ \left(0,0,0,2,\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&\\&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&&\\&0&\\&&0\end{smallmatrix}\right)\right) & = & b_{13} \end{array} \tag{72}$$

Recall the Peirce decomposition $1_\Omega = e+f+g$ from §7.1. Note that $e=b_1$, $f=b_9$ and $g=b_{12}$.

As an R-submodule of $\Omega\cap\Delta$, define

$$\begin{aligned} \mathfrak{t}_0 := {}_R\langle & b_1+b_9,\ b_2+b_3-b_5+b_6+4b_9-b_{10},\ b_3+b_4+b_6+4b_9,\ 2b_4-2b_5+b_6+2b_{10}-b_{11}, \\ & 2b_5+b_{11},\ b_{12},\ b_{13},\ e,\ f,\ g\rangle. \end{aligned}$$

Then $\mathfrak{t}_0$ is a Lie subalgebra of $\mathfrak{l}(\Omega\cap\Delta)$.

We obtain that $\mathfrak{t}_0$ is an integral torus in $\mathfrak{l}(\Omega)$; cf. Lemma 129.

We can shorten this generating set to an R-linear basis of $\mathfrak{t}_0$ as follows.

$$\mathfrak{t}_0 = {}_R\langle b_1,\ b_9,\ b_{12},\ b_{13},\ b_2-b_4-b_5-b_{10},\ b_3+b_4+b_6,\ 2b_4+b_6+2b_{10},\ 2b_5+b_{11}\rangle \tag{73}$$

By intersecting Ω with Δ, we get the full diagonal of Ω in Γ; cf. Definition 117. Then $\mathfrak{l}(\Omega\cap\Delta)$ is a maximal rational torus in $\mathfrak{l}(\Omega)$; cf. Lemma 120. So we define

$$T := \Omega\cap\Delta.$$

This is a maximal commutative subalgebra of Ω; cf. Lemma 121.

An R-linear basis of the R-algebra T is given as follows.

$$\mathcal{B}_T := (b_1,b_2,b_3,b_4,b_5,b_6,b_9,b_{10},b_{11},b_{12},b_{13})$$

Note that $f\Omega f$ and $g\Omega g$ are contained in Δ, in particular, $f\Omega f$ and $g\Omega g$ are commutative R-algebras. So $Tf = f\Omega f$ and $Tg = g\Omega g$; cf. Lemma 140.(1). But note that

$$T \subset e\Omega e \oplus f\Omega f \oplus g\Omega g$$

is in fact a proper inclusion.

For example, we have the element

$$b_7 = \left(0,2,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}-2&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&2\\0&0&0\\0&0&0\end{smallmatrix}\right)\right) \in e\Omega e\backslash T.$$

Furthermore, the matrix $\left(\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_7)\right)_{\mathcal{B},\mathcal{B}}$ is not even diagonalizable over $\mathbb{Q}$. To verify this, type the following instructions in Magma.

Magma Code 18: z2s5NonDiagonalizableElement

```
load pre;
load z2s5Init1;
load definitions;
load z2s5Init2;

rdiag(admatrix(b[7]),0);
```

We can describe elements of T as tuples of matrices with ties; cf. equation (68).

$$T = \left\{ \left(a, b, c, d, \begin{pmatrix} e_{1,1} & 0 \\ 0 & e_{2,2} \end{pmatrix}, \begin{pmatrix} f_{1,1} & 0 \\ 0 & f_{2,2} \end{pmatrix}, \begin{pmatrix} g_{1,1} & 0 & 0 \\ 0 & g_{2,2} & 0 \\ 0 & 0 & g_{3,3} \end{pmatrix} \right) \in \Gamma \;\middle|\; \begin{array}{rcl} a & \equiv_2 & b \\ c & \equiv_2 & d \\ e_{2,2} + f_{2,2} & \equiv_8 & 2g_{2,2} \\ f_{2,2} & \equiv_2 & g_{2,2} \\ e_{1,1} & \equiv_4 & f_{1,1} \\ e_{1,1} & \equiv_2 & g_{3,3} \\ b & \equiv_4 & f_{1,1} \\ a + b + e_{1,1} + f_{1,1} & \equiv_8 & 2g_{1,1} + 2g_{3,3} \\ 2g_{1,1} + 2g_{3,3} & \equiv_4 & 0 \\ g_{1,1} & \equiv_2 & 0 \end{array} \right\}$$

Recall that $\mathfrak{l}(\Omega)$ and $\mathfrak{l}(T)$ denote the commutator Lie algebras over R of Ω and T, respectively.

We want to verify the maximality of the rational torus $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$, using Lemma 112. It suffices to show that

$$\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{=} \mathfrak{l}(T).$$

Ad $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{\subseteq} \mathfrak{l}(T)$. Suppose given

$$x := \left(a, b, c, d, \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix}, \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix}, \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix} \right) \in \mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)).$$

We obtain

$$0 = [x, b_1] = xb_1 - b_1x = \left(0, 0, 0, 0, \begin{pmatrix} 0 & -e_{1,2} \\ e_{2,1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -f_{1,2} \\ f_{2,1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -g_{1,2} & 0 \\ g_{2,1} & 0 & g_{2,3} \\ 0 & -g_{3,2} & 0 \end{pmatrix} \right),$$

$$0 = [x, b_4] = xb_4 - b_4x = \left(0, 0, 0, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -4f_{1,2} \\ 4f_{2,1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2g_{1,2} & -2g_{1,3} \\ 2g_{2,1} & 0 & 0 \\ 2g_{3,1} & 0 & 0 \end{pmatrix} \right).$$

This shows $x \in \mathfrak{l}(T)$.

Ad $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{\supseteq} \mathfrak{l}(T)$. Since elements of T are tuples of diagonal matrices, $\mathfrak{l}(T)$ is an abelian Lie algebra over R, showing $\mathfrak{l}(T) \subseteq \mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$.

So we have shown that $\mathfrak{c}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) = \mathfrak{l}(T)$ and thus we have verified the maximality of the rational torus $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$ by direct calculation.

Now we want to show that $\mathfrak{l}(T) \subseteq \mathfrak{l}(\Omega)$ is not an integral torus.

We have the element

$$b_2 = \left(0, 4, 0, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \in \mathfrak{l}(T).$$

We want to determine the describing matrix $(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}}$ of the adjoint endomorphism $\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2$ with respect to the R-linear basis $\mathcal{B}$ of $\mathfrak{l}(\Omega)$. This requires the Lie brackets $[b_2, b_j]$ for $j \in [1, 21]$. The matrix $(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}}$ contains on position (i, j) the coefficient $\alpha_{i,j} \in R$ for $i, j \in [1, 21]$ such that $[b_2, b_j] = \sum_{i\in[1,21]} \alpha_{i,j} b_i$ for $j \in [1, 21]$.

We have $b_2 f = f b_2 = 0$ and $g b_2 = b_2 g = 0$ and $e b_2 = b_2 e$. This means that there are only zeros in the columns corresponding to the basis elements of $f\Omega f$, $g\Omega g$ and $e\Omega e$.

Recall that $e\Omega g$, $g\Omega e$, $f\Omega g$ and $g\Omega f$ are all zero.

For the remaining basis elements of $e\Omega f$ and $f\Omega e$, we obtain the following.

$$
\begin{array}{rcr}
[b_2, b_{14}] &=& -b_{16} \\
[b_2, b_{15}] &=& 0 \\
[b_2, b_{16}] &=& -2b_{16} \\
[b_2, b_{17}] &=& 0 \\
[b_2, b_{18}] &=& 0 \\
[b_2, b_{19}] &=& b_{20} \\
[b_2, b_{20}] &=& 2b_{20} \\
[b_2, b_{21}] &=& 0
\end{array}
$$

Thus we obtain

$$
(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.
$$

This matrix is diagonalizable over $\mathbb{Q}$, e.g. we have the diagonalizing matrix $Y \in \mathrm{GL}_{21}(\mathbb{Q})$ as follows.

$$
Y := \begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} \qquad (74)
$$

We have $\det(Y) = 4$. This implies that $\mathrm{v}_2(\det(Y)) = 2 \neq 0$. In particular, $Y \notin \mathrm{GL}_{21}(R)$ since 4 is not a unit in $\mathbb{Z}_{(2)}$. We write $A := (\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}}$. The first column of Y forms an R-linear basis of $(\mathrm{E}_A(2)) \cap R^{21\times 1}$, the second column forms an R-linear basis of $(\mathrm{E}_A(-2)) \cap R^{21\times 1}$ and the columns number three to twenty-one form an R-linear basis of $(\mathrm{E}_A(0)) \cap R^{21\times 1}$ which we will confirm with the following Magma code.

We will construct matrices W, D, S and T using the notation of Lemma 49. For $\lambda \in \{2, -2, 0\}$ a $\mathbb{Z}$-linear basis and thus also a $\mathbb{Z}_{(2)}$-linear basis of the eigenmodule $\mathrm{E}_A(\lambda) = \mathrm{E}_{(\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}}}(\lambda)$ can then be obtained from the first column (for $\lambda = 2$ or $\lambda = -2$) resp. from the first nineteen columns (for

$\lambda = 0$) of the corresponding matrix S^{-1}. Change the considered eigenvalue in the first line of the Magma code.

Magma Code 19: z2s5EigenmoduleBasis

```
lambda := 2; // eigenvalues are 2, -2 and 0.

A := RMatrixSpace(Rationals(),21,21)!0;
A[16,14] := -1; A[16,16] := -2;
A[20,19] := 1; A[20,20] := 2;

W := Transpose(BasisMatrix(Eigenspace(Transpose(A),lambda)));
for i in [1..NumberOfColumns(W)] do
   MultiplyColumn(~W,2^(-1*(Minimum([Valuation(W[j][i],2):j in [1..
      NumberOfRows(W)]]))), i);
end for;
D,S,T := SmithForm(W);
print S^-1;
```

By Corollary 48.(1) we conclude that $A = (\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}}$ is not diagonalizable over $\mathbb{Z}_{(2)}$.

We conclude that $\mathfrak{l}(T)$ is not an integral torus in $\mathfrak{l}(\Omega)$.

For the sake of completeness, we give Y^{-1} and the matrix product $Y^{-1} \cdot (\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}} \cdot Y$. We have

$$Y^{-1} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} \in \mathrm{GL}_{21}(\mathbb{Q}).$$

We have

$$Y^{-1} \cdot (\mathrm{ad}_{\mathfrak{l}(\Omega)} b_2)_{\mathcal{B},\mathcal{B}} \cdot Y = \begin{pmatrix}
2\\
&-2\\
&&0\\
&&&0\\
&&&&0\\
&&&&&0\\
&&&&&&0\\
&&&&&&&0\\
&&&&&&&&0\\
&&&&&&&&&0\\
&&&&&&&&&&0\\
&&&&&&&&&&&0\\
&&&&&&&&&&&&0\\
&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&&&0\\
&&&&&&&&&&&&&&&&&&&&0
\end{pmatrix}.$$

Alternatively we can use the function “rdiag” from Magma Code 4.

7.4 The integral core of the standard torus $\mathfrak{l}(T)$ in $\mathfrak{l}(\Omega)$

Keep the notation of §7.1 and §7.3.

Since we found out that $\mathfrak{l}(T)$ is not an integral torus in $\mathfrak{l}(\Omega)$, the question for a maximal integral torus in $\mathfrak{l}(\Omega)$ arises. We recall the definition of the integral core of a rational torus $\mathfrak{l}(T)$ in $\mathfrak{l}(\Omega)$; cf. Definition 130.

We have an orthogonal decomposition of the identity element of Ω into primitive idempotents in Ω, viz. $1 = e + f + g$. By Corollary 126, we know that $\mathrm{ad}_{\mathfrak{l}(\Omega)}(e), \mathrm{ad}_{\mathfrak{l}(\Omega)}(f)$ and $\mathrm{ad}_{\mathfrak{l}(\Omega)}(g)$ are R-diagonalizable. In fact, the describing matrices of these three maps with respect to the R-linear basis $\mathcal{B}$ of Ω are already diagonal.

Alternatively, to verify the R-diagonalizability in Magma, type the following instructions.

Magma Code 20: z2s5RDiagIdempotents

```
load pre;
load z2s5Init1;
load definitions;
load z2s5Init2;

rdiag(admatrix(b[1]),2);
rdiag(admatrix(b[9]),2);
rdiag(admatrix(b[12]),2);
```

To determine the integral core of $\mathfrak{l}(T)$ in $\mathfrak{l}(\Omega)$, we use Magma.

Calculating the integral core starting with an arbitrarily chosen basis

Recall that $\mathcal{B}_T = (b_1, b_2, b_3, b_4, b_5, b_6, b_9, b_{10}, b_{11}, b_{12}, b_{13})$ is an R-linear basis of T. For $i \in [1, 6] \cup [9, 13]$, we denote by $A_i := \mathrm{ad}_{\mathfrak{l}(\Omega)}(b_i)_{\mathcal{B},\mathcal{B}}$ the describing matrix of the adjoint endomorphism $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_i)$ with respect to the basis $\mathcal{B}$. Let $A = (A_1, A_2, A_3, A_4, A_5, A_6, A_9, A_{10}, A_{11}, A_{12}, A_{13})$ be the tuple of these eleven matrices. We will start the Partitions Algorithm 94 using this cd-tuple on $R^{21\times 21}$.

The code together with its output is as follows.

Magma Code 21: z2s5IntegralCore

```
load pre;
load z2s5Init1;
load definitions;
load z2s5Init2;
load partalgo;

time Partalgo([A[1],A[2],A[3],A[4],A[5],A[6],A[9],A[10],A[11],A[12],A
   [13]]);

l =  11 , there are 678570 partitions to check.
List of finest partitions contains 1 element(s).
Partitions in L_eff:
[
    [ 1, 2, 1, 1, 1, 2, 3, 3, 3, 2, 3 ]
]
A Z_(p)-linear basis of the diagonalizability locus is given by the
   columns of the following matrix.
[ 1  0  0  0  0  0  0  0]
[ 0  0  0  0  2  0  0 -1]
[ 0  0  0  0  0 -1  2  1]
```

```
[ 0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  0  1]
[ 0  0  0  0  1  0  1  0]
[ 0  1  0  0  0  0  0  0]
[ 0  0  0  0  0  2 -2 -1]
[ 0  0  0  0  1  0  0  0]
[ 0  0  1  0  0  0  0  0]
[ 0  0  0  1  0  0  0  0]
Time: 327.087
```

This is to be interpreted as follows.

$$\begin{aligned}\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) = {}_R\langle & b_1,\, b_9,\, b_{12},\, b_{13},\, 2b_2+b_6+b_{11},\, -b_3+b_4+2b_{10}, \\ & 2b_3+b_6-2b_{10},\, -b_2+b_3+b_5-b_{10}\rangle\end{aligned} \tag{75}$$

We want to show that $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{=} \mathfrak{t}_0$.

We have

$$\begin{aligned}
b_1 &= \left(1,1,0,0,\begin{pmatrix}1&\\&0\end{pmatrix},\begin{pmatrix}1&\\&0\end{pmatrix},\begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix}\right) = e\\
b_9 &= \left(0,0,0,0,\begin{pmatrix}0&\\&1\end{pmatrix},\begin{pmatrix}0&\\&1\end{pmatrix},\begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix}\right) = f\\
b_{12} &= \left(0,0,1,1,\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}0&&\\&0&\\&&0\end{pmatrix}\right) = g\\
b_{13} &= \left(0,0,0,2,\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}0&&\\&0&\\&&0\end{pmatrix}\right) \in \mathrm{C}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega))\\
2b_2+b_6+b_{11} &= \left(0,8,0,0,\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}4&&\\&4&\\&&4\end{pmatrix}\right) \in \mathrm{C}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega))\\
-b_3+b_4+2b_{10} &= \left(0,0,0,0,\begin{pmatrix}-4&\\&0\end{pmatrix},\begin{pmatrix}4&\\&8\end{pmatrix},\begin{pmatrix}0&&\\&4&\\&&0\end{pmatrix}\right)\\
2b_3+b_6-2b_{10} &= \left(0,0,0,0,\begin{pmatrix}8&\\&0\end{pmatrix},\begin{pmatrix}0&\\&-8\end{pmatrix},\begin{pmatrix}4&&\\&-4&\\&&4\end{pmatrix}\right)\\
-b_2+b_3+b_5-b_{10} &= \left(0,-4,0,0,\begin{pmatrix}4&\\&0\end{pmatrix},\begin{pmatrix}0&\\&-4\end{pmatrix},\begin{pmatrix}2&&\\&-2&\\&&2\end{pmatrix}\right)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
-b_3+b_4+2b_{10} &= \underbrace{\left(4,4,0,0,\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}8&\\&8\end{pmatrix},\begin{pmatrix}4&&\\&4&\\&&4\end{pmatrix}\right)}_{\in \mathrm{C}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega))} - \underbrace{\left(4,4,0,0,\begin{pmatrix}4&\\&0\end{pmatrix},\begin{pmatrix}4&\\&0\end{pmatrix},\begin{pmatrix}4&&\\&0&\\&&4\end{pmatrix}\right)}_{=4e} \in \mathfrak{t}_0,\\
2b_3+b_6-2b_{10} &= \underbrace{\left(0,0,0,0,\begin{pmatrix}8&\\&8\end{pmatrix},\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}4&&\\&4&\\&&4\end{pmatrix}\right)}_{\in \mathrm{C}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega))} - \underbrace{\left(0,0,0,0,\begin{pmatrix}0&\\&8\end{pmatrix},\begin{pmatrix}0&\\&8\end{pmatrix},\begin{pmatrix}0&&\\&8&\\&&0\end{pmatrix}\right)}_{=8f} \in \mathfrak{t}_0,\\
-b_2+b_3+b_5-b_{10} &= \underbrace{\left(0,-4,0,0,\begin{pmatrix}4&\\&4\end{pmatrix},\begin{pmatrix}0&\\&0\end{pmatrix},\begin{pmatrix}2&&\\&2&\\&&2\end{pmatrix}\right)}_{\in \mathrm{C}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(\Omega))} - \underbrace{\left(0,0,0,0,\begin{pmatrix}0&\\&4\end{pmatrix},\begin{pmatrix}0&\\&4\end{pmatrix},\begin{pmatrix}0&&\\&4&\\&&0\end{pmatrix}\right)}_{=4f} \in \mathfrak{t}_0.
\end{aligned}$$

Altogether we have shown that the R-linear basis of $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$ given in (75) is contained in $\mathfrak{t}_0$; cf. equation (73).

This shows that $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \subseteq \mathfrak{t}_0$.

To see that $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) \overset{!}{\supseteq} \mathfrak{t}_0$, note that e, f and g and the elements of the R-linear basis of $\mathrm{Z}(\Omega)$ of (72) are contained in $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T))$; cf. (72) and (75).

Hence we obtain $\mathrm{Cor}_{\mathfrak{l}(\Omega)}(\mathfrak{l}(T)) = \mathfrak{t}_0$. This means that in this example, we have equality in Question 135.

Moreover, note that the list of finest partitions found in the algorithm consisted of one element. So here we have an affirmative example for Question 93.

Calculating the integral core using Remark 59

We will use Remark 59 to achieve a better runtime of the algorithm and to show that the duration strongly depends on the choice of the basis. This time we start with the R-linear basis of $\mathfrak{t}_0$ we found in equation (73). We extend it to an R-linear basis of T.

$$\mathcal{C} := (b_1,\, b_9,\, b_{12},\, b_{13},\, b_2 - b_4 - b_5 - b_{10},\, b_3 + b_4 + b_6,\, 2b_4 + b_6 + 2b_{10},\, 2b_5 + b_{11},\, b_2,\, b_3,\, b_5)$$

In our implementation of the Partitions Algorithm, the matrices of the cd-tuple under consideration that are R-diagonalizable are considered separately; cf. Algorithm 94. So using the cd-tuple that contains the describing matrices of the elements of $\mathcal{C}$ with respect to the basis $\mathcal{B}$ of $\mathfrak{l}(\Omega)$, the main part of the algorithm is executed only for the describing matrices of the adjoint endomorphisms $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_2)$, $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_3)$ and $\mathrm{ad}_{\mathfrak{l}(\Omega)}(b_5)$. The following Magma code shows the results.

Magma Code 22: z2s5IntegralCore2

```
load pre;
load z2s5Init1;
load definitions;
load z2s5Init2;
load partalgo;

C := [b[1],b[9],b[12],b[13],SubTup(b[2],b[4]+b[5]+b[10]),
     b[3]+b[4]+b[6],b[4]+b[4]+b[6]+b[10]+b[10],
     b[5]+b[5]+b[11],
     b[2],b[3],b[5]];
List := [RMBQ!admatrix(X) : X in C];
time Partalgo(List);

l = 9 , there are 21147 partitions to check.
List of finest partitions contains 0 element(s).
There is no non-trivial linear combination of the given matrices that is
   Z_(p)-diagonalizable.
A Z_(p)-linear basis of the diagonalizability locus is given by the
   columns of the following matrix.
[1 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0]
[0 0 0 1 0 0 0 0]
[0 0 0 0 1 0 0 0]
[0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
Time: 5.741
```

Here the choice of the basis reduced to runtime by a factor of about 57.

7.5 Decompositions of Ω

Let Ω be defined as in §7.1. Let T be defined as in §7.3. Let the R-linear basis $\mathcal{B}$ of Ω be defined as in §7.1. Let the R-linear basis $\mathcal{B}_T$ of T be defined as in §7.3. Let the primitive idempotents e, f and g be defined as in §7.1.

We are interested in a decomposition of Ω into indecomposable submodules. On the one hand, we will decompose Ω as a T-T-bimodule. On the other hand, we will decompose $\mathfrak{l}(\Omega)$ as an $\mathfrak{l}(T)$-Lie module.

7.5.1 A decomposition of Ω into T-T-sub-bimodules

Recall the Peirce decomposition of Ω; cf. (70). In the following we will show that this decomposition of Ω is already a decomposition into indecomposable T-T-sub-bimodules of Ω.

The $\mathbb{Z}_{(2)}$-algebras $f\Omega f$ and $g\Omega g$ are commutative, so the T-T-bimodules $Tf = f\Omega f$ and $Tg = g\Omega g$ are indecomposable; cf. Lemma 140.(2), using Lemma 121 and Remark 170.(1). Since $e\Omega e$ is not commutative as a $\mathbb{Z}_{(2)}$-algebra, we cannot show the indecomposability of $e\Omega e$ in the same way.

The methods to show the indecomposability of the T-T-bimodules $f\Omega e$, $e\Omega f$ and $e\Omega e$ will be the same as those we applied in §6.5.1.

Ad $f\Omega e$.

We want to show that $f\Omega e$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_1 := (b_{14}, b_{15}, b_{16}, b_{17})$ which is an R-linear basis of $f\Omega e$.

For a better distinction between the basis elements of Ω and the basis elements of $f\Omega e$, we write

$$x_1 := b_{14} = \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right)\right) \qquad x_2 := b_{15} = \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&2\\0&0&0\end{smallmatrix}\right)\right)$$
$$x_3 := b_{16} = \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\2&0&0\\0&0&0\end{smallmatrix}\right)\right) \qquad x_4 := b_{17} = \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&0\\0&0&4\\0&0&0\end{smallmatrix}\right)\right).$$

Thus we obtain $\mathcal{B}_1 = (x_1, x_2, x_3, x_4)$.

It suffices to show that the endomorphism ring $\mathrm{End}_{T\text{-}T}(f\Omega e)$ is a local ring; cf. Lemma 35.

We have

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(f\Omega e) = \big\{h \in \mathrm{End}_R(f\Omega e) \mid\ & h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,6]\cup[9,13],\ j \in [1,4] \text{ and}\\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,6]\cup[9,13],\ j \in [1,4]\big\}.\end{aligned}$$

For $i \in [1,6]\cup[9,13]$ we define $M_{\mathcal{B}_1,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $f\Omega e$ from the left with respect to the basis $\mathcal{B}_1$. For $j \in [1,6]\cup[9,13]$ we define $M_{\mathcal{B}_1,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $f\Omega e$ from the right with respect to the basis $\mathcal{B}_1$.

Furthermore, we have the following diagram.

$$\begin{tikzcd}
h \arrow[r, mapsto] & h_{\mathcal{B}_1,\mathcal{B}_1} \\
\mathrm{End}_R(f\Omega e) \arrow[r, "\varphi_1", "\sim"'] & R^{4\times 4} \\
\mathrm{End}_{T\text{-}T}(f\Omega e) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_1(\mathrm{End}_{T\text{-}T}(f\Omega e)) =: E_1 \arrow[u, hook]
\end{tikzcd}$$

Here the map $\varphi_1\colon \mathrm{End}_R(f\Omega e) \to R^{4\times 4}$ is the R-algebra isomorphism sending a map $h \in \mathrm{End}_R(f\Omega e)$ to its describing matrix in the algebra of 4×4-matrices over R with respect to the basis $\mathcal{B}_1$. Since φ_1 is an R-algebra morphism, E_1 is a subalgebra of $R^{4\times 4}$.

Then we have

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(f\Omega e) \simeq E_1 = \big\{M \in R^{4\times 4}\ |\ & M \cdot M_{\mathcal{B}_1,i,\mathrm{l}} = M_{\mathcal{B}_1,i,\mathrm{l}} \cdot M \text{ for } i \in [1,6]\cup[9,13] \text{ and}\\ & M \cdot M_{\mathcal{B}_1,j,\mathrm{r}} = M_{\mathcal{B}_1,j,\mathrm{r}} \cdot M \text{ for } j \in [1,6]\cup[9,13]\big\}.\end{aligned} \tag{76}$$

First we will give the products $b_i \cdot x_j$ and $x_j \cdot b_i$ where $i \in [1,6]\cup[9,13]$ and $j \in [1,4]$.

We use the results to calculate the matrices $M_{\mathcal{B}_1,i,\mathrm{l}}$ and $M_{\mathcal{B}_1,i,\mathrm{r}}$ for $i \in [1,6] \cup [9,13]$.

i	$b_i \cdot x_1$	$b_i \cdot x_2$	$b_i \cdot x_3$	$b_i \cdot x_4$	$M_{\mathcal{B}_1,i,\mathrm{l}}$
$1,2,3,4,5,6$	0	0	0	0	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$
9	x_1	x_2	x_3	x_4	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$
10	$2x_2 + x_3 - x_4$	$4x_2 - x_4$	$2x_3$	$2x_4$	$\begin{pmatrix} 0&0&0&0 \\ 2&4&0&0 \\ 1&0&2&0 \\ -1&-1&0&2 \end{pmatrix} =: M_1$
11	$2x_3$	$2x_4$	$4x_3$	$4x_4$	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 2&0&4&0 \\ 0&2&0&4 \end{pmatrix} =: M_2$
$12,13$	0	0	0	0	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$

(77)

i	$x_1 \cdot b_i$	$x_2 \cdot b_i$	$x_3 \cdot b_i$	$x_4 \cdot b_i$	$M_{\mathcal{B}_1,i,\mathrm{r}}$
1	x_1	x_2	x_3	x_4	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$
2	x_3	0	$2x_3$	0	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&2&0 \\ 0&0&0&0 \end{pmatrix} =: M_3$
3	$4x_1 - 2x_2 - x_3 + x_4$	0	$2x_3$	0	$\begin{pmatrix} 4&0&0&0 \\ -2&0&0&0 \\ -1&0&2&0 \\ 1&0&0&0 \end{pmatrix} =: M_4$
4	$2x_2 + x_3 - x_4$	$4x_2 - 2x_4$	$2x_3$	0	$\begin{pmatrix} 0&0&0&0 \\ 2&4&0&0 \\ 1&0&2&0 \\ -1&-2&0&0 \end{pmatrix} =: M_5$
5	x_3	x_4	$2x_3$	$2x_4$	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&2&0 \\ 0&1&0&2 \end{pmatrix} =: M_6$
6	0	$2x_4$	0	$4x_4$	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&2&0&4 \end{pmatrix} =: M_7$
$9,10,11,12,13$	0	0	0	0	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$

(78)

Note that we have the following equalities.

$$\begin{aligned} 2M_1 &= 2M_5 + M_7 \\ M_2 &= 2M_3 + M_7 \\ M_4 &= 4 \cdot 1_{R^{4\times 4}} - M_5 - M_7 \\ 2M_6 &= 2M_3 + M_7 \end{aligned}$$

Thus we can skip M_1, M_2, M_4 and M_6 in the description of E_1. Moreover, we omit the matrices that are central in $R^{4\times 4}$. It remains

$$E_1 = \left\{ M \in R^{4\times 4} \,\middle|\, M \cdot M_j = M_j \cdot M \text{ for } j \in \{3,5,7\} \right\}. \tag{79}$$

But this is $\mathrm{C}_{R^{4\times 4}}(A_5)$ of §5.5. So $E_1 = L_5$; cf. (28) in §5.5. We have shown that L_5 is local; cf. Remark 164. So we conclude that E_1 is local and thus $f\Omega e$ is indecomposable as a T-T-bimodule.

Ad $e\Omega f$.

We want to show that $e\Omega f$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_2 := (b_{18}, b_{19}, b_{20}, b_{21})$ which is an R-linear basis of $e\Omega f$.

We write

$$x_5 := b_{18} = \left(0,0,0,0, \left(\begin{smallmatrix} 0&2\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&-2\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&0\\0&0&0\\0&-1&0 \end{smallmatrix}\right)\right) \qquad x_6 := b_{19} = \left(0,0,0,0, \left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&4\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&2&0\\0&0&0\\0&0&0 \end{smallmatrix}\right)\right)$$

$$x_7 := b_{20} = \left(0,0,0,0, \left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&4&0\\0&0&0\\0&0&0 \end{smallmatrix}\right)\right) \qquad x_8 := b_{21} = \left(0,0,0,0, \left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&0\\0&0&0\\0&2&0 \end{smallmatrix}\right)\right).$$

Thus we obtain $\mathcal{B}_2 = (x_5, x_6, x_7, x_8)$.

It suffices to show that the endomorphism ring $\mathrm{End}_{T\text{-}T}(e\Omega f)$ is a local ring; cf. Lemma 35.

We have

$$\begin{aligned} \mathrm{End}_{T\text{-}T}(e\Omega f) = \big\{ h \in \mathrm{End}_R(e\Omega f) \mid\ & h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,6] \cup [9,13],\ j \in [5,8] \text{ and} \\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,6] \cup [9,13],\ j \in [5,8] \big\}. \end{aligned}$$

For $i \in [1,6] \cup [9,13]$ we define $M_{\mathcal{B}_2,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $e\Omega f$ from the left with respect to the basis $\mathcal{B}_2$. For $j \in [1,6] \cup [9,13]$ we define $M_{\mathcal{B}_2,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $e\Omega f$ from the right with respect to the basis $\mathcal{B}_2$.

Furthermore, we have the following diagram.

$$\begin{tikzcd} h \arrow[r, mapsto] & h_{\mathcal{B}_2,\mathcal{B}_2} \\ \mathrm{End}_R(e\Omega f) \arrow[r, "\varphi_2", "\sim"'] & R^{4\times 4} \\ \mathrm{End}_{T\text{-}T}(e\Omega f) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_2(\mathrm{End}_{T\text{-}T}(e\Omega f)) =: E_2 \arrow[u, hook] \end{tikzcd}$$

Here the map $\varphi_2\colon \mathrm{End}_R(e\Omega f) \to R^{4\times 4}$ is the R-algebra isomorphism sending a map $h \in \mathrm{End}_R(e\Omega f)$ to its describing matrix in the algebra of 4×4-matrices over R with respect to the basis $\mathcal{B}_2$. Since φ_2 is an R-algebra morphism, E_2 is a subalgebra of $R^{4\times 4}$.

Then we have

$$\begin{aligned} \mathrm{End}_{T\text{-}T}(e\Omega f) \simeq E_2 = \big\{ M \in R^{4\times 4} \mid\ & M \cdot M_{\mathcal{B}_2,i,\mathrm{l}} = M_{\mathcal{B}_2,i,\mathrm{l}} \cdot M \text{ for } i \in [1,6] \cup [9,13] \text{ and} \\ & M \cdot M_{\mathcal{B}_2,j,\mathrm{r}} = M_{\mathcal{B}_2,j,\mathrm{r}} \cdot M \text{ for } j \in [1,6] \cup [9,13] \big\}. \end{aligned} \tag{80}$$

We determine the matrices $M_{\mathcal{B}_2,i,\mathrm{l}}$ and $M_{\mathcal{B}_2,i,\mathrm{r}}$ for $i \in [1,6] \cup [9,13]$.

(81)

i	$b_i \cdot x_5$	$b_i \cdot x_6$	$b_i \cdot x_7$	$b_i \cdot x_8$	$M_{\mathcal{B}_2,i,\mathrm{l}}$
1	x_5	x_6	x_7	x_8	$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$
2	0	x_7	$2x_7$	0	$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&2&0\\0&0&0&0 \end{pmatrix} =: M_8$
3	$4x_5 + 2x_6 - x_7 + 2x_8$	x_7	$2x_7$	0	$\begin{pmatrix} 4&0&0&0\\2&0&0&0\\-1&1&2&0\\2&0&0&0 \end{pmatrix} =: M_9$
4	$-2x_6 + x_7$	$4x_6 - x_7$	$2x_7$	0	$\begin{pmatrix} 0&0&0&0\\-2&4&0&0\\1&-1&2&0\\0&0&0&0 \end{pmatrix} =: M_{10}$
5	$-x_8$	x_7	$2x_7$	$2x_8$	$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&2&0\\-1&0&0&2 \end{pmatrix} =: M_{11}$
6	$-2x_8$	0	0	$4x_8$	$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\-2&0&0&4 \end{pmatrix} =: M_{12}$
$9, 10, 11, 12, 13$	0	0	0	0	$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$

i	$x_5 \cdot b_i$	$x_6 \cdot b_i$	$x_7 \cdot b_i$	$x_8 \cdot b_i$	$M_{\mathcal{B}_2,i,\mathrm{r}}$
$1,2,3,4,5,6$	0	0	0	0	$\left(\begin{smallmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{smallmatrix}\right)$
9	x_5	x_6	x_7	x_8	$\left(\begin{smallmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)$
10	$-2x_6+x_7-x_8$	$4x_6-x_7$	$2x_7$	$2x_8$	$\left(\begin{smallmatrix}0&0&0&0\\-2&4&0&0\\1&-1&2&0\\-1&0&0&2\end{smallmatrix}\right) =: M_{13}$
11	$-2x_8$	$2x_7$	$4x_7$	$4x_8$	$\left(\begin{smallmatrix}0&0&0&0\\0&0&0&0\\0&2&4&0\\-2&0&0&4\end{smallmatrix}\right) =: M_{14}$
$12,13$	0	0	0	0	$\left(\begin{smallmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{smallmatrix}\right)$

(82)

Note that we have the following equalities.

$$\begin{aligned} M_9 &= 4 \cdot 1_{R^{4\times 4}} - M_{10} - M_{12} \\ 2M_{11} &= 2M_8 + M_{12} \\ 2M_{13} &= 2M_{10} + M_{12} \\ M_{14} &= 2M_8 + M_{12} \end{aligned}$$

Thus we can skip M_9, M_{11}, M_{13} and M_{14} in the description of E_2. Moreover, we omit the matrices that are central in $R^{4\times 4}$. It remains

$$E_2 = \left\{ M \in R^{4\times 4} \,\middle|\, M \cdot M_i = M_i \cdot M \text{ for } i \in \{8, 10, 12\} \right\}. \tag{83}$$

But this is $\mathrm{C}_{R^{4\times 4}}(A_6)$ of §5.6. So $E_2 = L_6$; cf. (31) in §5.6. We have shown that L_6 is local; cf. Corollary 166. So we conclude that E_2 is local and thus $e\Omega f$ is indecomposable as a T-T-bimodule.

Ad $e\Omega e$.

We want to show that $e\Omega e$ is indecomposable as a T-T-bimodule.

Define $\mathcal{B}_3 := (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$ which is an R-linear basis of $e\Omega e$.

We write

$$\begin{aligned}
x_9 := b_1 &= \left(1,1,0,0, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right)\right) & x_{10} := b_2 &= \left(0,4,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right) \\
x_{11} := b_3 &= \left(0,0,0,0, \left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right) & x_{12} := b_4 &= \left(0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)\right) \\
x_{13} := b_5 &= \left(0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&2\end{smallmatrix}\right)\right) & x_{14} := b_6 &= \left(0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&4\end{smallmatrix}\right)\right) \\
x_{15} := b_7 &= \left(0,2,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}-2&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&2\\0&0&0\\0&0&0\end{smallmatrix}\right)\right) & x_{16} := b_8 &= \left(0,0,0,0, \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\\1&0&0\end{smallmatrix}\right)\right).
\end{aligned}$$

Thus we obtain $\mathcal{B}_3 = (x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16})$.

It suffices to show that the endomorphism ring $\mathrm{End}_{T\text{-}T}(e\Omega e)$ is a local ring; cf. Lemma 35.

We have

$$\begin{aligned} \mathrm{End}_{T\text{-}T}(e\Omega e) = \big\{ h \in \mathrm{End}_R(e\Omega e) \mid\ & h(b_i x_j) = b_i h(x_j) \text{ for } i \in [1,6] \cup [9,13],\ j \in [9,16] \text{ and} \\ & h(x_j b_i) = h(x_j) b_i \text{ for } i \in [1,6] \cup [9,13],\ j \in [9,16] \big\}. \end{aligned}$$

For $i \in [1,6] \cup [9,13]$ we define $M_{\mathcal{B}_3,i,\mathrm{l}}$ to be the describing matrix of the multiplication by b_i on $e\Omega e$ from the left with respect to the basis $\mathcal{B}_3$. For $j \in [1,6] \cup [9,13]$ we define $M_{\mathcal{B}_3,j,\mathrm{r}}$ to be the describing matrix of the multiplication by b_j on $e\Omega e$ from the right with respect to the basis $\mathcal{B}_3$.

Furthermore, we have the following diagram.

$$\begin{tikzcd}
h \arrow[r, mapsto] & h_{\mathcal{B}_3,\mathcal{B}_3} \\
\mathrm{End}_R(e\Omega e) \arrow[r, "\varphi_3", "\sim"'] & R^{8\times 8} \\
\mathrm{End}_{T\text{-}T}(e\Omega e) \arrow[u, hook] \arrow[r, "\sim"'] & \varphi_3(\mathrm{End}_{T\text{-}T}(e\Omega e)) =: E_3 \arrow[u, hook]
\end{tikzcd}$$

Here the map $\varphi_3\colon \mathrm{End}_R(e\Omega e) \to R^{8\times 8}$ is the R-algebra isomorphism sending a map $h \in \mathrm{End}_R(e\Omega e)$ to its describing matrix in the algebra of 8×8-matrices over R with respect to the basis $\mathcal{B}_3$. Since φ_3 is an R-algebra morphism, E_3 is a subalgebra of $R^{8\times 8}$.

Then we have

$$\begin{aligned}\mathrm{End}_{T\text{-}T}(e\Omega e) \simeq E_3 = \big\{M \in R^{8\times 8} \mid & M \cdot M_{\mathcal{B}_3,i,\mathrm{l}} = M_{\mathcal{B}_3,i,\mathrm{l}} \cdot M \text{ for } i \in [1,6] \cup [9,13] \text{ and} \\ & M \cdot M_{\mathcal{B}_3,j,\mathrm{r}} = M_{\mathcal{B}_3,j,\mathrm{r}} \cdot M \text{ for } j \in [1,6] \cup [9,13]\big\}.\end{aligned} \tag{84}$$

A multiplication table for the products $b_i \cdot b_j$ for $i, j \in [1,13]$ is given in Appendix B.

We use it to calculate the matrices $M_{\mathcal{B}_3,i,\mathrm{l}}$ and $M_{\mathcal{B}_3,i,\mathrm{r}}$ for $i \in [1,6] \cup [9,13]$. Note that for $i \in [1,6]$, the first six columns of $M_{\mathcal{B}_3,i,\mathrm{l}}$ and $M_{\mathcal{B}_3,i,\mathrm{r}}$ coincide since $b_j \in T$ for $j \in [1,6]$ and T is commutative.

i	$M_{\mathcal{B}_3,i,\mathrm{r}}$	$M_{\mathcal{B}_3,i,\mathrm{l}}$
1	$\begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}$
2	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 1&4&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \\ 0&-2&2&2&2&0&-2&2 \\ 0&1&-1&-1&-1&0&1&-1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&2 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 1&4&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&-2&2&2&2&0&-2&2 \\ 0&1&-1&-1&-1&0&1&-1 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$
3	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&4&0&0&0&0&1 \\ 0&0&0&0&0&0&0&-1 \\ 0&2&-2&2&2&0&0&0 \\ 0&-1&1&-1&-1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&2 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 1&0&4&0&0&0&0&2 \\ 0&0&0&0&0&0&1&0 \\ 0&2&-2&2&2&0&0&0 \\ 0&-1&1&-1&-1&0&0&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$
4	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1 \\ 1&0&0&4&0&0&-2&1 \\ 0&2&2&-2&2&0&2&0 \\ 0&-1&-1&1&-1&0&-1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&2 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&4&0&0&-1&2 \\ 0&2&2&-2&2&0&2&0 \\ 0&-1&-1&1&-1&0&-1&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$
5	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&1&-1 \\ 1&2&2&2&2&0&0&2 \\ 0&-1&-1&-1&0&2&0&-1 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&2 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&1&-1 \\ 1&2&2&2&2&0&0&2 \\ 0&-1&-1&-1&0&2&0&-1 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&2 \end{pmatrix}$
6	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-2&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&2&4&0&0 \\ 0&0&0&0&0&0&4&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-2 \\ 0&0&0&0&0&0&0&-2 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&2&4&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&4 \end{pmatrix}$
$9, 10, 11, 12, 13$	0	0

(85)

We omit the matrices that are central in $R^{8\times 8}$. It remains

$$E_3 = \big\{M \in R^{8\times 8} \mid M \cdot M_{\mathcal{B}_3,i,\mathrm{l}} = M_{\mathcal{B}_3,i,\mathrm{l}} \cdot M \text{ for } i \in [2,6] \text{ and } M \cdot M_{\mathcal{B}_3,j,\mathrm{r}} = M_{\mathcal{B}_3,j,\mathrm{r}} \cdot M \text{ for } j \in [2,6]\big\}. \tag{86}$$

But this is $\mathrm{C}_{R^{8\times 8}}(A_7)$ of §5.7. So $E_3 = L_7$; cf. (33) in §5.7. We have shown that L_7 is local; cf. Remark 167. So we conclude that E_3 is local and thus $e\Omega e$ is indecomposable as a T-T-bimodule.

We summarize.

We have shown that the Peirce decomposition

$$\Omega = e\Omega e \oplus f\Omega f \oplus g\Omega g \oplus e\Omega f \oplus f\Omega e \tag{87}$$

as of (70) is a decomposition of Ω into indecomposable T-T-sub-bimodules.

By applying Lemma 144 we conclude that all summands are non-isomorphic to each other as T-T-bimodules.

Moreover, we have seen that T itself is not a sum of indecomposables in this decomposition.

Recall that $Tf = f\Omega f$ and $Tg = g\Omega g$ are local; cf. Lemma 140.(1) and Remark 170.(1). By Lemma 25, the T-T-endomorphism rings $\mathrm{End}_{T\text{-}T}(f\Omega f)$ and $\mathrm{End}_{T\text{-}T}(g\Omega g)$ also are local.

So we were able to show that all indecomposable summands have local T-T-endomorphism rings.

Remark 172. *There exists a discrete valuation ring R, a split R-order Ω' in a completely split R-order Γ' such that, letting Δ' be the full diagonal in Γ' and $T' := \Omega' \cap \Delta'$, the following holds.*

There exists an orthogonal decomposition $1_{\Omega'} = e_1 + e_2$ of $1_{\Omega'}$ into primitive idempotents in Ω' such that $e_1, e_2 \in T'$ and such that $e_1\Omega' e_2$ is a decomposable T'-T'-bimodule.

Proof. Define $\varepsilon := \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)\right) \in K\Omega$. Then, letting $e_1 := \varepsilon f$ and $e_2 := \varepsilon e$, the sum $1_{\varepsilon\Omega} = e_1 + e_2$ is an orthogonal decomposition of 1_Ω into primitive idempotents in $\Omega' := \varepsilon\Omega$. We have $T' := \varepsilon T = {}_R\langle e_1, e_2, \varepsilon b_2\rangle$.

As a T'-T'-bimodule, we can decompose $e_1\Omega' e_2$ as follows.

$$e_1\Omega' e_2 = {}_R\langle \varepsilon b_{14}\rangle \oplus {}_R\langle \varepsilon b_{15}\rangle$$

This shows that $e_1\Omega' e_2$ is decomposable as a T'-T'-bimodule; cf. Question 145. □

Moreover, consider the Peirce diagonal $\widetilde{T} := e_1\Omega' e_1 \oplus e_2\Omega' e_2$. Then $e_1(K\Omega')e_2$ is a simple $K\widetilde{T}$-$K\widetilde{T}$-bimodule, whence $e_1\Omega' e_2$ is an indecomposable $\widetilde{T}$-$\widetilde{T}$-bimodule.

Question 173. Suppose given an R-order Ω' and an orthogonal decomposition $1_{\Omega'} = \sum_{i\in[1,n]} e_i$ of $1_{\Omega'}$ into primitive idempotents in Ω'. Suppose given $i, j \in [1,n]$ such that $e_i\Omega' e_j \neq 0$.

We ask whether $e_i\Omega' e_j$ is indecomposable as a bimodule over the Peirce diagonal $\bigoplus_{i\in[1,n]} e_i\Omega' e_i$.

7.5.2 A decomposition of $\mathfrak{l}(\Omega)$ into $\mathfrak{l}(T)$-Lie submodules

Keep the notation of §7.5.1.

We consider the Lie algebra $\mathfrak{l}(\Omega)$ as an $\mathfrak{l}(T)$-Lie module over its Lie subalgebra $\mathfrak{l}(T)$. We are interested in a decomposition of the $\mathfrak{l}(T)$-Lie module $\mathfrak{l}(\Omega)$ into indecomposable $\mathfrak{l}(T)$-Lie submodules.

In contrast to the examples $\mathbb{Z}_{(3)}\,\mathrm{S}_3$ in §1 and $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ in §6, there exists an indecomposable projective Ω-module with a non-commutative endomorphism ring.

As we will see in the following, one consequence is that the decomposition of $\mathfrak{l}(\Omega)$ into $\mathfrak{l}(T)$-Lie submodules is not as closely connected to a decomposition of Ω into indecomposable T-T-bimodules as it has been in the previous examples.

We know that $f\Omega f$ and $g\Omega g$ are contained in T, so $f\Omega f$ and $g\Omega g$ are trivial $\mathfrak{l}(T)$-Lie modules. Hence $f\Omega f$ and $g\Omega g$ decompose into $\mathfrak{l}(T)$-Lie submodules of rank 1 over R.

Let $X \subseteq f\Omega f$ be such a summand of rank 1 over R. Then X is a trivial $\mathfrak{l}(T)$-Lie module and so $\mathrm{End}_{\mathfrak{l}(T)}(X) = \mathrm{End}_R(X) \simeq R$. This shows in particular that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(X)$ is a local ring. Similarly we see that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(X)$ is a local ring for a summand $X \subseteq g\Omega g$ of rank 1 over R.

We will show first that $f\Omega e$ and $e\Omega f$ are indecomposable as $\mathfrak{l}(T)$-Lie modules.

Then it remains to decompose $e\Omega e$. As an R-module, $e\Omega e$ is of rank 8. We will find a decomposition into six indecomposable $\mathfrak{l}(T)$-Lie submodules.

Ad $f\Omega e$.

We want to show that $f\Omega e$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_1$ is an R-linear basis of $f\Omega e$.

It suffices to show that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(f\Omega e)$ is a local ring; cf. Lemma 35.

Define the isomorphism of R-algebras $\widetilde{\varphi}_1\colon \mathrm{End}_R(f\Omega e) \to R^{4\times 4}$ by $\widetilde{\varphi}_1(h) := h_{\mathcal{B}_1,\mathcal{B}_1} \in R^{4\times 4}$ for $h \in \mathrm{End}_R(f\Omega e)$. Define the R-subalgebra $\widetilde{E}_1 := \widetilde{\varphi}_1(\mathrm{End}_{\mathfrak{l}(T)}(f\Omega e))$ of $R^{4\times 4}$. Define

$$M_{\mathcal{B}_1,i} := (\mathrm{ad}_{f\Omega e}\, b_i)_{\mathcal{B}_1,\mathcal{B}_1} \in R^{4\times 4} \text{ for } i \in [1,10].$$

Then $M_{\mathcal{B}_1,i} = M_{\mathcal{B}_1,i,\mathrm{l}} - M_{\mathcal{B}_1,i,\mathrm{r}}$ for $i \in [1,10]$. We have

$$\mathrm{End}_{\mathfrak{l}(T)}(f\Omega e) \simeq \widetilde{E}_1 = \left\{ M \in R^{4\times 4} \,\middle|\, M \cdot M_{\mathcal{B}_1,i} = M_{\mathcal{B}_1,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{88}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_1,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_1,i,\mathrm{r}} = 0$; cf. (77) and (78). So we obtain $\widetilde{E}_1 = E_1$; cf. (76) and (88). But $E_1 = \mathrm{C}_{R^{4\times 4}}(A_5) = L_5$; cf. (27) and (28) in §5.5. We have shown that L_5 is local; cf. Remark 164. So we have shown that $\widetilde{E}_1$ is local. In particular, $\mathrm{End}_{\mathfrak{l}(T)}(f\Omega e)$ is local; cf. (88).

Ad $e\Omega f$.

We want to show that $e\Omega f$ is indecomposable as an $\mathfrak{l}(T)$-Lie module.

Recall that $\mathcal{B}_2$ is an R-linear basis of $e\Omega f$.

It suffices to show that the endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(e\Omega f)$ is a local ring; cf. Lemma 35.

Define the isomorphism of R-algebras $\widetilde{\varphi}_2\colon \mathrm{End}_R(e\Omega f) \to R^{4\times 4}$ by $\widetilde{\varphi}_2(h) := h_{\mathcal{B}_2,\mathcal{B}_2} \in R^{4\times 4}$ for $h \in \mathrm{End}_R(e\Omega f)$. Define the R-subalgebra $\widetilde{E}_2 := \widetilde{\varphi}_2(\mathrm{End}_{\mathfrak{l}(T)}(e\Omega f))$ of $R^{4\times 4}$. Define

$$M_{\mathcal{B}_2,i} := (\mathrm{ad}_{e\Omega f}\, b_i)_{\mathcal{B}_2,\mathcal{B}_2} \in R^{4\times 4} \text{ for } i \in [1,10].$$

Then $M_{\mathcal{B}_2,i} = M_{\mathcal{B}_2,i,\mathrm{l}} - M_{\mathcal{B}_2,i,\mathrm{r}}$ for $i \in [1,10]$. We have

$$\mathrm{End}_{\mathfrak{l}(T)}(e\Omega f) \simeq \widetilde{E}_2 = \left\{ M \in R^{4\times 4} \,\middle|\, M \cdot M_{\mathcal{B}_2,i} = M_{\mathcal{B}_2,i} \cdot M \text{ for } i \in [1,10] \right\}. \tag{89}$$

Recall that for $i \in [1,10]$ we have $M_{\mathcal{B}_2,i,\mathrm{l}} = 0$ or $M_{\mathcal{B}_2,i,\mathrm{r}} = 0$; cf. (81) and (82). So we obtain $\widetilde{E}_2 = E_2$; cf. (80) and (89). But $E_2 = \mathrm{C}_{R^{4\times 4}}(A_6) = L_6$; cf. (30) and (31) in §5.6. We have shown that L_6 is local; cf. Corollary 166. So we have shown that $\widetilde{E}_2$ is local. In particular, $\mathrm{End}_{\mathfrak{l}(T)}(e\Omega f)$ is local; cf. (89).

Ad $e\Omega e$.

Recall the R-linear basis $\mathcal{B}_T = (b_1, b_2, b_3, b_4, b_5, b_6, b_9, b_{10}, b_{11}, b_{12}, b_{13})$ of T. The $\mathfrak{l}(T)$-Lie submodule $e\Omega e$ of Ω is generated by the elements $b_1, \ldots, b_8$. In the following we will show that a decomposition of $e\Omega e$ into a direct sum of indecomposable $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$ is given as follows.

$$e\Omega e = {}_R\langle b_1\rangle \oplus {}_R\langle b_2\rangle \oplus {}_R\langle b_5\rangle \oplus {}_R\langle b_6\rangle \oplus {}_R\langle b_7, -b_2 + b_4 + 2b_7\rangle \oplus {}_R\langle b_8, -b_3 - b_4 + 2b_8\rangle \tag{90}$$

In fact we will show that all these summands have local $\mathfrak{l}(T)$-endomorphism rings.

First we will show that all these summands are in fact $\mathfrak{l}(T)$-Lie modules.

The summands ${}_R\langle b_1\rangle$, ${}_R\langle b_2\rangle$, ${}_R\langle b_5\rangle$ and ${}_R\langle b_6\rangle$ are contained in the torus T. Since T is commutative, we obtain that $[b_i, t] = 0$ for $i \in \{1, 2, 5, 6\}$ and for $t \in T$. Hence these four summands are in fact $\mathfrak{l}(T)$-Lie submodules.

We consider $T_7 := {}_R\langle b_7, -b_2 + b_4 + 2b_7\rangle$. We write $c := -b_2 + b_4 + 2b_7$. We calculate the Lie brackets $[b_7, b_i]$ and $[c, b_i]$ for $i \in [1, 6] \cup [9, 13]$.

i	$[b_7, b_i]$	$[c, b_i]$
1	0	0
2	$-c$	$-2c$
3	$-c$	$-2c$
4	$-c$	$-2c$
5	0	0
6	$2c$	$4c$
$9, 10, 11, 12, 13$	0	0

(91)

Note that $[c, b_i] = [-b_2 + b_4, b_i] + [2b_7, b_i] = 0 + 2[b_7, b_i]$ for $i \in [1, 6] \cup [9, 13]$ since T is commutative.

This shows that T_7 is an $\mathfrak{l}(T)$-Lie submodule of $\mathfrak{l}(\Omega)$.

Next we consider $T_8 := {}_R\langle b_8, -b_3 - b_4 + 2b_8\rangle$. We write $d := -b_3 - b_4 + 2b_8$. We calculate the Lie brackets $[b_8, b_i]$ and $[d, b_i]$ for $i \in [1, 6] \cup [9, 13]$.

i	$[b_8, b_i]$	$[d, b_i]$
1	0	0
2	d	$2d$
3	d	$2d$
4	d	$2d$
5	0	0
6	$-2d$	$-4d$
$9, 10, 11, 12, 13$	0	0

(92)

Note that $[d, b_i] = [-b_3 - b_4, b_i] + [2b_8, b_i] = 0 + 2[b_8, b_i]$ for $i \in [1, 6] \cup [9, 13]$ since T is commutative.

This shows that T_8 is an $\mathfrak{l}(T)$-Lie submodule of $\mathfrak{l}(\Omega)$.

To see that (90) is in fact a decomposition of $e\Omega e$, we have to show that the right hand side contains $e\Omega e$. But we have

$$\begin{aligned} b_3 &= -(-b_3 - b_4 + 2b_8) + 2b_8 - (-b_2 + b_4 + 2b_7) + 2b_7 - b_2 \\ b_4 &= (-b_2 + b_4 + 2b_7) - 2b_7 + b_2 \end{aligned}$$

and for $i \in \{1, 2, 5, 6, 7, 8\}$, the element b_i is one of the generators of the right hand side of (90).

Note that the $\mathfrak{l}(T)$-endomorphism ring of an $\mathfrak{l}(T)$-Lie module of rank 1 over R is isomorphic to R and thus, in particular, is local. So it remains to show that the $\mathfrak{l}(T)$-Lie modules T_7 and T_8 have local $\mathfrak{l}(T)$-endomorphism rings.

We consider the $\mathfrak{l}(T)$-Lie module $T_7 = {}_R\langle b_7, c\rangle$.

The $\mathfrak{l}(T)$-endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(T_7)$ is isomorphic to the ring of 2×2-matrices that commute with $\left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right)$; cf. (91). But this is $\mathrm{C}_{R^{2\times 2}}(A_1) = L_1$; cf. (18) and (19) in §5.1. We have shown in Remark 156 that L_1 is local. Hence $\mathrm{End}_{\mathfrak{l}(T)}(T_7) \simeq L_1$ is local.

We consider the $\mathfrak{l}(T)$-Lie module $T_8 = {}_R\langle b_8, d\rangle$.

The $\mathfrak{l}(T)$-endomorphism ring $\mathrm{End}_{\mathfrak{l}(T)}(T_8)$ is isomorphic to the ring of 2×2-matrices that commute with $\left(\begin{smallmatrix}0&0\\1&2\end{smallmatrix}\right)$; cf. (92). But this is $\mathrm{C}_{R^{2\times 2}}(A_1) = L_1$; cf. (18) and (19) in §5.1. We have shown in Remark 156 that L_1 is local. Hence $\mathrm{End}_{\mathfrak{l}(T)}(T_8) \simeq L_1$ is local.

We summarize.

We obtain the following decomposition of $\mathfrak{l}(\Omega)$ into a direct sum of $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$.

$$\mathfrak{l}(\Omega) = {}_R\langle b_1\rangle \oplus {}_R\langle b_2\rangle \oplus {}_R\langle b_5\rangle \oplus {}_R\langle b_6\rangle \oplus T_7 \oplus T_8 \oplus e\Omega f \oplus f\Omega e \oplus f\Omega f \oplus g\Omega g$$

In this decomposition, all summands but $f\Omega f$ and $g\Omega g$ are indecomposable as $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$. The summands $f\Omega f$ and $g\Omega g$ are trivial $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$.

Note that the Peirce components of Ω that have trivial intersection with T are indecomposable as $\mathfrak{l}(T)$-Lie submodules of $\mathfrak{l}(\Omega)$ and as T-T-sub-bimodules of Ω; cf. equation (87) on page 166.

In addition we have shown that in a decomposition of $\mathfrak{l}(\Omega)$ as an $\mathfrak{l}(T)$-Lie module into indecomposable summands, all these summands have local $\mathfrak{l}(T)$-endomorphism rings.

7.6 Magma

The following two codes are used for calculations with $\Omega \simeq \mathbb{Z}_{(2)}\,\mathrm{S}_5$ in Magma. However, note that initialization files such as "pre" and "definitions" are required; cf. Magma Codes 3 and 4.

Magma Code 23: z2s5Init1

```
// global definitions
Sizes := [1,1,1,1,2,2,3]; // sizes of blocks
nb := #Sizes; // number of blocks
nt := 18; // number of ties needed to describe Omega
rt := &+Sizes; // rank of torus
rl := &+[Sizes[i]^2 : i in [1..nb]]; // rank of Omega
prime := 2; // R is Z localized at the prime number 2
e := 8; // ties that describe Omega are given mod e

RM := RMatrixSpace(Z,rl,rt);
RMQ := RMatrixSpace(Q,rl,rt);
RV := RMatrixSpace(Z,rl,1);
RQV := VectorSpace(Q,rl);
RM2 := RMatrixSpace(Z,nt,rl);
RMB := RMatrixSpace(Z,rl,rl);
RMBQ := KMatrixSpace(Q,rl,rl);
RMVQ := KMatrixSpace(Q,rl,1);

Ties_Omega := // Ties mod e that describe Omega,
             // given in the rows of this matrix
   RM2!Matrix([
      [0, 0,4,-4,0,0,0,0, 0,0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [0, 0,0, 0,0,1,0,0, 0,1, 0,0, 0,-2, 0, 0, 0, 0, 0, 0, 0],
      [0, 0,0, 0,0,0,0,1, 0,0, 0,1, 0, 0, 0, 0,-2, 0, 0, 0, 0],
      [0, 0,0, 0,0,0,0,0, 0,0, 4,0, 0, 0, 0,-4, 0, 0, 0, 0, 0],
```

```
      [0, 0,0, 0,0,0,0,0, 0,0, 0,4, 0, 0, 0, 0,-4, 0, 0, 0, 0],
      [0, 0,0, 0,2,0,0,0,-2,0, 0,0, 0, 0,-2, 0, 0, 0, 0, 0, 0],
      [0, 0,0, 0,0,0,2,0, 0,0,-2,0, 0, 0, 0, 0, 0,-2, 0, 0, 0],
      [0, 0,0, 0,0,0,0,0, 0,2, 0,0, 0, 0, 0, 0, 0, 0, 0,-4, 0],
      [4,-4,0, 0,0,0,0,0, 0,0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [0, 0,0, 0,4,0,0,0, 0,0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0,-4],
      [0, 2,0, 0,0,0,0,0,-2,0, 0,0, 0, 0, 0, 0, 0, 0,-4, 0, 0],
      [1, 1,0, 0,1,0,0,0, 1,0, 0,0,-2, 0, 0, 0, 0, 0, 0, 0,-2],
      [0, 0,0, 0,0,0,0,0, 0,0, 0,0, 4, 0, 0, 0, 0, 0, 0, 0,-4],
      [0, 0,0, 0,0,4,0,0, 0,0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [0, 0,0, 0,0,0,0,0, 0,4, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [0, 0,0, 0,0,0,0,0, 0,0, 0,0, 0, 4, 4, 0, 0, 0, 0, 0, 0],
      [0, 0,0, 0,0,0,0,0, 0,0, 0,0, 0, 0, 0, 0, 0, 4, 0, 0, 0],
      [0, 0,0, 0,0,0,0,0, 0,0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
   ]);
```

Magma Code 24: z2s5Init2

```
// R-linear basis of Omega
b := [];
// e Omega e, diagonal part
b[1] := CoerceGamma([1,1,0,0,1,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,1]);
b[2] := CoerceGamma([0,4,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0]);
b[3] := CoerceGamma([0,0,0,0,4,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0]);
b[4] := CoerceGamma([0,0,0,0,0,0,0,0,4,0,0,0,2,0,0,0,0,0,0,0,0]);
b[5] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,2]);
b[6] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4]);
// e Omega e, non diagonal part
b[7] := CoerceGamma([0,2,0,0,0,0,0,0,-2,0,0,0,0,0,2,0,0,0,0,0,0]);
b[8] := CoerceGamma([0,0,0,0,2,0,0,0,2,0,0,0,2,0,0,0,0,0,1,0,0]);
// f Omega f
b[9] := CoerceGamma([0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,1,0,0,0,0]);
b[10] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,4,0,0,0,0,2,0,0,0,0]);
b[11] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4,0,0,0,0]);
// g Omega g
b[12] := CoerceGamma([0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
b[13] := CoerceGamma([0,0,0,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
// f Omega e
b[14] := CoerceGamma([0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,1,0,0,0,0,0]);
b[15] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,2,0,0,0]);
b[16] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0]);
b[17] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4,0,0,0]);
// e Omega f
b[18] := CoerceGamma([0,0,0,0,0,2,0,0,0,-2,0,0,0,0,0,0,0,0,0,-1,0]);
b[19] := CoerceGamma([0,0,0,0,0,0,0,0,0,4,0,0,0,2,0,0,0,0,0,0,0]);
b[20] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,4,0,0,0,0,0,0,0]);
b[21] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0]);

// describing matrices of the adjoint endomorphisms of the elements
// of b with respect to the basis Basis_Omega which is defined in
// the file "definitions"
A := [RMBQ!admatrix(x) : x in b];

// the center of Omega is generated by the following
// seven elements
```

```
c := [];
c[1] := CoerceGamma([1,1,0,0,1,0,0,1,1,0,0,1,1,0,0,0,1,0,0,0,1]);
c[2] := CoerceGamma([0,4,0,0,4,0,0,4,0,0,0,0,2,0,0,0,2,0,0,0,2]);
c[3] := CoerceGamma([0,0,0,0,4,0,0,4,4,0,0,4,2,0,0,0,2,0,0,0,2]);
c[4] := CoerceGamma([0,0,0,0,0,0,0,0,8,0,0,8,0,0,0,0,0,0,0,0,0]);
c[5] := CoerceGamma([0,0,0,0,0,0,0,0,0,0,0,0,4,0,0,0,4,0,0,0,4]);
c[6] := CoerceGamma([0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
c[7] := CoerceGamma([0,0,0,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
```

Chapter A: On Krull-Schmidt for projectives

Let A be a ring and let $\mathcal{B}$ be the category of A-modules.

Proposition 174. *Let $X, Y \in \operatorname{Ob}\mathcal{B}$ such that $X \simeq Y$. Suppose given direct summands X' of X and Y' of Y and an isomorphism $\varphi\colon X \xrightarrow{\sim} Y$. Denote by $\pi_{X'}$ the projection map from X onto X' and denote by $\iota_{X'}$ the inclusion map from X' into X. Likewise we define $\pi_{Y'}$ and $\iota_{Y'}$. Suppose that $\psi := \pi_{Y'} \circ \varphi \circ \iota_{X'}$ is an isomorphism.*

Then we have the maps

$$\begin{aligned} \tau\colon \ker \pi_{Y'} &\longrightarrow X/X' \\ y &\longmapsto \varphi^{-1}(y) + X' \end{aligned}$$

and

$$\begin{aligned} \sigma\colon X/X' &\longrightarrow \ker \pi_{Y'} \\ x + X' &\longmapsto \varphi(x) - (\varphi \circ \iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x). \end{aligned}$$

Moreover, $\tau \circ \sigma = \mathrm{id}_{X/X'}$ and $\sigma \circ \tau = \mathrm{id}_{\ker \pi_{Y'}}$.

$$\begin{tikzcd} X' \arrow[rr, "\sim", "\psi"'] \arrow[d, bend left, "\iota_{X'}"] & & Y' \arrow[d, bend left, "\iota_{Y'}"] \\ X \arrow[rr, "\sim", "\varphi"'] \arrow[u, bend left, "\pi_{X'}"] & & Y \arrow[u, bend left, "\pi_{Y'}"] \end{tikzcd}$$

$$\begin{tikzcd} X/X' \arrow[r, shift left, "\sigma"] \arrow[r, phantom, "\sim"] & \ker \pi_{Y'} \arrow[l, shift left, "\tau"] \end{tikzcd}$$

Proof.

σ is well-defined.

We have to show that $\varphi(x) - (\varphi \circ \iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x) \overset{!}{\in} \ker \pi_{Y'}$ for $x \in X$.
Suppose given $x \in X$. We calculate.

$$\begin{aligned} \pi_{Y'}\big(\varphi(x) - (\varphi \circ \iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x)\big) &= (\pi_{Y'} \circ \varphi)(x) - (\underbrace{\pi_{Y'} \circ \varphi \circ \iota_{X'}}_{=\psi} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x) \\ &= (\pi_{Y'} \circ \varphi)(x) - (\mathrm{id}_{Y'} \circ \pi_{Y'} \circ \varphi)(x) = 0 \end{aligned}$$

This shows that $\sigma(x) \in \ker \pi_{Y'}$ for $x \in X$.

Moreover, we have to show that $\sigma(x) \overset{!}{=} 0$ for $x \in X'$. Suppose given $x \in X'$. Then

$$(\iota_{X'} \circ \pi_{X'})(x) = x.$$

Since φ is bijective, it suffices to show that

$$(\iota_{X'} \circ \pi_{X'})(x) = x \overset{!}{=} (\iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x).$$

Because of the injectivity of $\iota_{X'}$, it suffices to show that $\pi_{X'}(x) \overset{!}{=} (\psi^{-1} \circ \pi_{Y'} \circ \varphi)(x)$. Since ψ is an isomorphism, it suffices to show that $(\psi \circ \pi_{X'})(x) \overset{!}{=} (\pi_{Y'} \circ \varphi)(x)$.

But now we have

$$(\psi \circ \pi_{X'})(x) = (\pi_{Y'} \circ \varphi \circ \iota_{X'} \circ \pi_{X'})(x) = (\pi_{Y'} \circ \varphi)(\iota_{X'} \circ \pi_{X'}(x)) = (\pi_{Y'} \circ \varphi)(x).$$

Taking both together, we have shown that σ is well-defined.

We have $\tau \circ \sigma = \mathrm{id}_{X/X'}$.

Suppose given $x + X' \in X/X'$. We calculate.

$$\begin{aligned}(\tau \circ \sigma)(x + X') &= \tau(\varphi(x) - (\varphi \circ \iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x)) \\ &= \varphi^{-1}\big(\varphi(x) - (\varphi \circ \iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x)\big) + X' \\ &= x - \underbrace{(\iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(x)}_{\in X'} + X' \\ &= x + X'\end{aligned}$$

We have $\sigma \circ \tau = \mathrm{id}_{\ker \pi_{Y'}}$.

Suppose given $y \in \ker \pi_{Y'}$. We calculate.

$$\begin{aligned}(\sigma \circ \tau)(y) &= \sigma(\varphi^{-1}(y) + X') \\ &= (\varphi \circ \varphi^{-1})(y) - (\varphi \circ \iota_{X'} \circ \psi^{-1} \circ \pi_{Y'} \circ \varphi)(\varphi^{-1}(y)) \\ &= y - \underbrace{(\varphi \circ \iota_{X'} \circ \psi^{-1} \circ \pi_{Y'})(y)}_{=0} \\ &= y\end{aligned}$$

□

Lemma 175. *Suppose given $m \geq 0$ and indecomposables $X_i \in \mathrm{Ob}\,\mathcal{B}$ for $i \in [1,m]$. Suppose given $n \geq 0$ and indecomposables $Y_j \in \mathrm{Ob}\,\mathcal{B}$ for $j \in [1,n]$ such that $\mathrm{End}_{\mathcal{B}}(Y_j)$ is local for $j \in [1,n]$. Suppose that*

$$\bigoplus_{i \in [1,m]} X_i \simeq \bigoplus_{j \in [1,n]} Y_j.$$

Then $m = n$ and there exists a permutation $\rho \in \mathrm{S}_n$ such that $X_i \simeq Y_{\rho(i)}$ for $i \in [1,n]$.

Proof. Choose an isomorphism $\varphi \colon \bigoplus_{i \in [1,m]} X_i \to \bigoplus_{j \in [1,n]} Y_j$. We denote the projection map from $\bigoplus_{i \in [1,m]} X_i$ onto X_i by π_{X_i} for $i \in [1,m]$ and we denote the inclusion map from X_i into $\bigoplus_{i \in [1,m]} X_i$ by ι_{X_i}. Likewise we define π_{Y_j} and ι_{Y_j} for $j \in [1,n]$.

Claim 1. Idempotents split in $\mathcal{B}$; cf. [Mül13, Definition 140]

Suppose given $V \in \mathrm{Ob}\,\mathcal{B}$. Suppose given an idempotent $\varepsilon \in \mathrm{End}_{\mathcal{B}}(V)$. Define $W := \mathrm{im}(\varepsilon)$ and $W' := \ker(\varepsilon)$. We have the inclusion maps $\iota \colon W' \to V$ and $\dot{\varepsilon} \colon W \to V$. We denote by $\overline{\varepsilon} \colon V \to W$ the map ε restricted in the codomain to W. Then the following diagram commutes.

$$\begin{tikzcd} W' \arrow[r, "\iota"] & V \arrow[rr, "\varepsilon"] \arrow[dr, "\overline{\varepsilon}"'] & & V \\ & & W \arrow[ur, "\dot{\varepsilon}"'] & \end{tikzcd}$$

We obtain the identities

$$\dot{\varepsilon} \circ \overline{\varepsilon} = \varepsilon \tag{93}$$

and

$$\varepsilon \circ \iota = 0. \tag{94}$$

Since ε is an idempotent, we have

$$\dot{\varepsilon} \circ \overline{\varepsilon} \circ \dot{\varepsilon} \circ \overline{\varepsilon} \overset{(93)}{=} \varepsilon^2 = \varepsilon \overset{(93)}{=} \dot{\varepsilon} \circ \overline{\varepsilon} = \dot{\varepsilon} \circ \mathrm{id}_W \circ \overline{\varepsilon}.$$

Since $\overline{\varepsilon}$ is surjective and $\dot{\varepsilon}$ is injective, we have

$$\overline{\varepsilon} \circ \dot{\varepsilon} = \mathrm{id}_W\,. \tag{95}$$

We define $u := (1-\varepsilon)\,|^{W'}$, i.e. u is the map $1-\varepsilon$ restricted in the codomain to W'. Then the following diagram commutes.

$$\begin{tikzcd}
V \arrow[dr, "1-\varepsilon"] \arrow[dd, "u"'] & & \\
& V \arrow[r, "\varepsilon"] & V \\
W' \arrow[ur, "\iota"] & &
\end{tikzcd}$$

Thus we have the following equation.

$$\iota \circ u = 1 - \varepsilon. \tag{96}$$

Moreover, we have $\iota \circ u \circ \varepsilon \overset{(96)}{=} (1-\varepsilon) \circ \varepsilon = 0$ and by the injectivity of ι, we obtain

$$u \circ \varepsilon = 0. \tag{97}$$

Now we can construct maps between $W \oplus W'$ and V as follows.

$$\begin{tikzcd}
W \oplus W' \arrow[r, bend left, "\begin{pmatrix} \dot{\varepsilon} & \iota \end{pmatrix}"] & V \arrow[l, bend left, "\begin{pmatrix} \overline{\varepsilon} \\ u \end{pmatrix}"]
\end{tikzcd}$$

We have $\begin{pmatrix} \overline{\varepsilon} \\ u \end{pmatrix} \circ \begin{pmatrix} \dot{\varepsilon} & \iota \end{pmatrix} = \mathrm{id}_{W \oplus W'}$.

We will determine the entries of the matrix $\left(\begin{smallmatrix} \overline{\varepsilon} \\ u \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} \dot{\varepsilon} & \iota \end{smallmatrix}\right)$. We have

$$\begin{pmatrix} \overline{\varepsilon} \\ u \end{pmatrix} \circ \begin{pmatrix} \dot{\varepsilon} & \iota \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} \circ \dot{\varepsilon} & \overline{\varepsilon} \circ \iota \\ u \circ \dot{\varepsilon} & u \circ \iota \end{pmatrix}$$

For the entry on position (1,1), we have $\overline{\varepsilon} \circ \dot{\varepsilon} \overset{(95)}{=} \mathrm{id}_W = 1$.

For the entry on position (1,2), we have $\dot{\varepsilon} \circ \overline{\varepsilon} \circ \iota \overset{(93)}{=} \varepsilon \circ \iota \overset{(94)}{=} 0$. Using the injectivity of $\dot{\varepsilon}$, we obtain $\overline{\varepsilon} \circ \iota = 0$.

For the entry on position (2,1), we have $\iota \circ u \circ \dot{\varepsilon} \circ \overline{\varepsilon} \overset{(93),(96)}{=} (1-\varepsilon) \circ \varepsilon = 0$. Using the surjectivity of $\overline{\varepsilon}$ and the injectivity of ι, we get $u \circ \dot{\varepsilon} = 0$.

For the entry on position (2,2), we have $\iota \circ u \circ \iota \overset{(96)}{=} (1-\varepsilon) \circ \iota \overset{(94)}{=} \iota = \iota \circ \mathrm{id}_{W'}$. Using the injectivity of ι, we get $u \circ \iota = \mathrm{id}_{W'} = 1$.

Putting the matrix together, we have shown that

$$\begin{pmatrix} \overline{\varepsilon} \\ u \end{pmatrix} \circ \begin{pmatrix} \dot{\varepsilon} & \iota \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} \circ \dot{\varepsilon} & \overline{\varepsilon} \circ \iota \\ u \circ \dot{\varepsilon} & u \circ \iota \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_{W \oplus W'}\,.$$

We have $\begin{pmatrix}\dot\varepsilon & \iota\end{pmatrix} \circ \begin{pmatrix}\overline{\varepsilon} \\ u\end{pmatrix} = \mathrm{id}_V$.

We calculate.

$$\begin{pmatrix}\dot\varepsilon\ \iota\end{pmatrix} \circ \begin{pmatrix}\overline{\varepsilon} \\ u\end{pmatrix} = \dot\varepsilon \circ \overline{\varepsilon} + \iota \circ u \overset{(93),(96)}{=} \varepsilon + (1-\varepsilon) = 1 = \mathrm{id}_V\,.$$

This shows that we found an isomorphism between V and $W \oplus W'$. We obtain the following diagram.

$$\begin{tikzcd} V \arrow[r, "f := \begin{pmatrix}\overline{\varepsilon} \\ u\end{pmatrix}", "\sim"'] \arrow[d, "\varepsilon"'] & W \oplus W' \arrow[d, "\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}"] \\ V \arrow[r, "\sim", "\begin{pmatrix}\overline{\varepsilon} \\ u\end{pmatrix}"'] & W \oplus W' \end{tikzcd} \tag{98}$$

Diagram (98) *commutes.*

We have $\begin{pmatrix}\overline{\varepsilon} \\ u\end{pmatrix} \circ \varepsilon = \begin{pmatrix}\overline{\varepsilon}\circ\varepsilon \\ u\circ\varepsilon\end{pmatrix} \overset{(93),(97)}{=} \begin{pmatrix}\overline{\varepsilon}\circ\dot\varepsilon\circ\overline{\varepsilon} \\ 0\end{pmatrix} \overset{(95)}{=} \begin{pmatrix}\overline{\varepsilon} \\ 0\end{pmatrix}$.

We have $\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix} \circ \begin{pmatrix}\overline{\varepsilon} \\ u\end{pmatrix} = \begin{pmatrix}\overline{\varepsilon} \\ 0\end{pmatrix}$.

This shows that Diagram (98) commutes.

Altogether, this *proves Claim 1.*

Claim 2. There exists $k \in [1,n]$ *such that* $\pi_{Y_k} \circ \varphi \circ \iota_{X_m}$ *is an isomorphism.*

Define $e := \varphi \circ \iota_{X_m} \circ \pi_{X_m} \circ \varphi^{-1}$. This is an idempotent on $\bigoplus_{j\in[1,n]} Y_j$ since $\pi_{X_m} \circ \iota_{X_m} = \mathrm{id}_{X_m}$. Furthermore, we have $e \neq 0$ since $X_m \neq 0$ and thus $\iota_{X_m} \circ \pi_{X_m} \neq 0$. We write $e = (e_{i,j})_{i,j\in[1,n]}$ as a matrix such that $e_{i,j} \colon Y_j \to Y_i$ for $i, j \in [1,n]$. Then there exist $k, l \in [1,n]$ such that $e_{k,l}$ is an isomorphism; cf. [Mül13, Lemma 139] and Claim 1. We write $\psi := \pi_{Y_k} \circ \varphi \circ \iota_{X_m}$.

We can write the map $e_{k,l}$ as follows.

$$\begin{aligned} e_{k,l} &= \pi_{Y_k} \circ e \circ \iota_{Y_l} \\ &= \pi_{Y_k} \circ \varphi \circ \iota_{X_m} \circ \pi_{X_m} \circ \varphi^{-1} \circ \iota_{Y_l} \\ &= \psi \circ \left(\pi_{X_m} \circ \varphi^{-1} \circ \iota_{Y_l}\right). \end{aligned}$$

Thus ψ is a retraction of $\pi_{X_m} \circ \varphi^{-1} \circ \iota_{Y_l} \circ e_{k,l}^{-1}$, so

$$\mathrm{id}_{Y_k} = \psi \circ \underbrace{\pi_{X_m} \circ \varphi^{-1} \circ \iota_{Y_l} \circ e_{k,l}^{-1}}_{=:\tilde\psi}\,.$$

It follows that ψ is surjective, $\tilde\psi$ is injective and $\tilde\psi \circ \psi$ is an idempotent on X_m. Thus $\mathrm{im}(\tilde\psi \circ \psi) = \mathrm{im}(\tilde\psi)$ is a summand of X_m. Since X_m is indecomposable and $Y_k \neq 0$, we obtain that $\tilde\psi(Y_k) = X_m$. Thus $\tilde\psi$ has to be surjective.

We conclude that $\tilde\psi$ is bijective. The same holds for $\psi = \tilde\psi^{-1}$.

This *proves Claim 2.*

We proceed by induction on n.

For $n = 0$, we have that $0 \simeq \bigoplus_{i\in[1,m]} X_i$ as modules. Since X_i is indecomposable for $i \in [1,m]$, we conclude that $m = 0$. The trivial permutation on the empty set $\rho = 1_{\mathrm{S}_0} = \mathrm{id}_\emptyset$ satisfies the required property.

Suppose now that $n \geq 1$ and the claim is known for $n-1$. Then, by Claim 2, we find $k \in [1,n]$ and an isomorphism $\psi\colon X_m \xrightarrow{\sim} Y_k$. We are in the following situation.

$$\begin{tikzcd}
X_m \arrow[r, "\sim", "\psi"'] \arrow[d, shift left, "\iota_{X_m}"] & Y_k \arrow[d, shift left, "\iota_{Y_k}"] \\
\bigoplus\limits_{i\in[1,m]} X_i \arrow[u, shift left, "\pi_{X_m}"] \arrow[r, "\sim", "\varphi"'] & \bigoplus\limits_{j\in[1,n]} Y_j \arrow[u, shift left, "\pi_{Y_k}"]
\end{tikzcd}$$

Here we can apply Proposition 174. Thus we find an isomorphism $\sigma\colon \bigoplus_{i\in[1,m]} X_i/X_m \xrightarrow{\sim} \ker(\pi_{Y_k})$. We need to show that

$$\bigoplus_{i\in[1,m-1]} X_i \simeq \bigoplus_{j\in[1,n]\setminus\{k\}} Y_j.$$

But we have the following.

$$\begin{array}{ccccc}
\bigoplus\limits_{i\in[1,m-1]} X_i & \xrightarrow{\ \sim\ } & \bigoplus\limits_{i\in[1,m]} X_i/X_m & \xrightarrow[\sigma]{\ \sim\ } & \ker(\pi_{Y_k}) = \bigoplus\limits_{j\in[1,n]\setminus\{k\}} Y_j \\
(x_i)_{i\in[1,m-1]} & \longmapsto & (x_1,\ldots,x_{m-1},0)+X_m & &
\end{array}$$

Then we let $\rho(m) := k$ and we define ρ on all integers between 1 and $m-1$ as given by induction. $\square$

Lemma 176. *Suppose given an orthogonal decomposition $1_A = \sum_{i\in[1,n]} e_i$ of 1_A into primitive idempotents in A. Suppose given an orthogonal decomposition $1_A = \sum_{i\in[1,m]} \tilde{e}_i$ of 1_A into primitive idempotents in A. Suppose that e_iAe_i is local for $i \in [1,n]$.*

Then $m = n$ and there exist $u \in \mathrm{U}(A)$ and $\sigma \in \mathrm{S}_n$ such that $u^{-1}\tilde{e}_iu = e_{\sigma(i)}$ for $i \in [1,n]$.

Proof. Since e_i is a primitive idempotent for $i \in [1,n]$, the A-module Ae_i is indecomposable for $i \in [1,n]$. Likewise we get that the A-module $A\tilde{e}_i$ is indecomposable for $i \in [1,m]$. Moreover, we know that $\mathrm{End}_A(Ae_i) \simeq e_iAe_i$ is local for $i \in [1,n]$.

Since $\bigoplus_{i\in[1,n]} Ae_i \simeq A \simeq \bigoplus_{i\in[1,m]} A\tilde{e}_i \simeq A$ we can apply Lemma 175. We obtain that $m = n$ and we find a permutation $\sigma \in \mathrm{S}_n$ such that $A\tilde{e}_i \simeq Ae_{\sigma(i)}$ for $i \in [1,n]$. Thus we get isomorphisms φ_i for $i \in [1,n]$ such that

$$\begin{aligned}
\varphi_i\colon A\tilde{e}_i &\xrightarrow{\sim} Ae_{\sigma(i)} \\
\tilde{e}_i &\longmapsto \varphi_i(e_i).
\end{aligned}$$

We define

$$u := \sum_{i\in[1,n]} \varphi_i(\tilde{e}_i) \quad \text{and} \quad v := \sum_{i\in[1,n]} \varphi_i^{-1}(e_{\sigma(i)}).$$

Furthermore, we have $\tilde{e}_i\varphi_i(\tilde{e}_i) = \varphi_i(\tilde{e}_i\tilde{e}_i) = \varphi_i(\tilde{e}_i)$ and thus $u \in \tilde{e}_iAe_{\sigma(i)}$ for $i \in [1,n]$. Similarly we can show that $v \in e_{\sigma(i)}A\tilde{e}_i$ for $i \in [1,n]$.

Step 1: We have $uv = 1$ and $vu = 1$.

We have

$$\begin{aligned} uv &= \sum_{i\in[1,n]} \sum_{j\in[1,n]} \underbrace{\varphi_i(\tilde{e}_i)\varphi_j^{-1}(e_{\sigma(j)})}_{\in \tilde{e}_i A e_{\sigma(i)} e_{\sigma(j)} A \tilde{e}_j} \\ &= \sum_{i\in[1,n]} \varphi_i(\tilde{e}_i)\varphi_i^{-1}(e_{\sigma(i)}) \\ &= \sum_{i\in[1,n]} \varphi_i^{-1}(\underbrace{\varphi_i(\tilde{e}_i)}_{\in \tilde{e}_i A e_{\sigma(i)}} e_{\sigma(i)}) \\ &= \sum_{i\in[1,n]} \varphi_i^{-1}(\varphi_i(\tilde{e}_i)) = 1 \end{aligned}$$

and

$$\begin{aligned} vu &= \sum_{j\in[1,n]} \sum_{i\in[1,n]} \underbrace{\varphi_j^{-1}(e_{\sigma(j)})\varphi_i(\tilde{e}_i)}_{\in e_{\sigma(j)} A \tilde{e}_j \tilde{e}_i A e_{\sigma(i)}} \\ &= \sum_{i\in[1,n]} \varphi_i^{-1}(e_{\sigma(i)})\varphi_i(\tilde{e}_i) \\ &= \sum_{i\in[1,n]} \varphi_i(\underbrace{\varphi_i^{-1}(e_{\sigma(i)})}_{\in e_{\sigma(i)} A \tilde{e}_i} \tilde{e}_i) \\ &= \sum_{i\in[1,n]} \varphi_i(\varphi_i^{-1}(e_{\sigma(i)})) = 1. \end{aligned}$$

This completes Step 1. In particular, this shows that u is a unit in A.

Step 2: We have $u^{-1}\tilde{e}_i u = e_{\sigma(i)}$ for $i \in [1, n]$.

It suffices to show that $\tilde{e}_i u = u e_{\sigma(i)}$ for $i \in [1, n]$. Suppose given $i \in [1, n]$. We calculate.

$$\begin{aligned} \tilde{e}_i u &= \sum_{j\in[1,n]} \tilde{e}_i \underbrace{\varphi_j(\tilde{e}_j)}_{\in \tilde{e}_j A e_{\sigma(j)}} = \varphi_i(\tilde{e}_i) \\ u e_{\sigma(i)} &= \sum_{j\in[1,n]} \underbrace{\varphi_j(\tilde{e}_j)}_{\in \tilde{e}_j A e_{\sigma(j)}} e_{\sigma(i)} = \varphi_i(\tilde{e}_i) \end{aligned}$$

This shows that $\tilde{e}_i u = u e_{\sigma(i)}$ for $i \in [1, n]$. *This completes Step 2.* □

Chapter B: Multiplication table for the Peirce diagonal of the Morita-reduced version of $\mathbb{Z}_2\,\mathrm{S}_5$

Let Ω, the idempotents $e, f, g \in \Omega$ and the R-linear basis $\mathcal{B}$ of Ω be defined as in §7.1. Then the tuple $(b_i : i \in [1, 13])$ is an R-linear basis of the Peirce diagonal $e\Omega e \oplus f\Omega f \oplus g\Omega g$. We give the multiplication table of this Peirce diagonal.

$b_i \cdot b_j$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$	$j=11$	$j=12$	$j=13$
$i=1$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	0	0	0	0	0
$i=2$	b_2	$4b_2-2b_5+b_6$	$2b_5-b_6$	$2b_5-b_6$	$2b_5-b_6$	0	$b_2+b_4-2b_5+b_6+2b_7$	$2b_5-b_6$	0	0	0	0	0
$i=3$	b_3	$2b_5-b_6$	$4b_3-2b_5+b_6$	$2b_5-b_6$	$2b_5-b_6$	0	$-b_2+b_4+2b_7$	$2b_3$	0	0	0	0	0
$i=4$	b_4	$2b_5-b_6$	$2b_5-b_6$	$4b_4-2b_5+b_6$	$2b_5-b_6$	0	$-b_2-b_4+2b_5-b_6+2b_7$	$2b_4$	0	0	0	0	0
$i=5$	b_5	$2b_5-b_6$	$2b_5-b_6$	$2b_5-b_6$	$2b_5$	$2b_6$	$-b_2+b_4+2b_7$	$-b_3-b_4+2b_5-b_6+2b_8$	0	0	0	0	0
$i=6$	b_6	0	0	0	$2b_6$	$4b_6$	0	$-2b_3-2b_4+4b_8$	0	0	0	0	0
$i=7$	b_7	$2b_2-2b_5+b_6$	0	$-2b_4+2b_5-b_6$	$-b_2+b_4+2b_7$	$-2b_2+2b_4+4b_7$	$b_2+b_4-2b_5+b_6$	$-b_4+2b_5-b_6$	0	0	0	0	0
$i=8$	b_8	$-b_3-b_4+2b_5-b_6+2b_8$	$b_3-b_4+2b_8$	$-b_3+b_4+2b_8$	$-b_3-b_4+2b_5-b_6+2b_8$	0	$-b_2+b_5+2b_7$	$2b_8$	0	0	0	0	0
$i=9$	0	0	0	0	0	0	0	0	b_9	b_{10}	b_{11}	0	0
$i=10$	0	0	0	0	0	0	0	0	b_{10}	$4b_{10}-b_{11}$	$2b_{11}$	0	0
$i=11$	0	0	0	0	0	0	0	0	b_{11}	$2b_{11}$	$4b_{11}$	0	0
$i=12$	0	0	0	0	0	0	0	0	0	0	0	b_{12}	b_{13}
$i=13$	0	0	0	0	0	0	0	0	0	0	0	b_{13}	$2b_{13}$

References

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[CR81] Charles W. Curtis and Irving Reiner. Methods of Representation Theory. Vol. I. 1981.

[Gro07] Jonathan L. Gross. *Combinatorial Methods with Computer Applications.* CRC Press, 2007.

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer Graduate Texts in Mathematics 9, 1972.

[Kün01] Matthias Künzer. *Ties for the integral group ring of the symmetric group. 2nd edition with corrections.* Dissertation, University of Bielefeld, 2001.

[Kün15] Matthias Künzer. *Liealgebren.* Lecture notes, University of Stuttgart, 2015.

[Mül13] Franziska Müller. *Some local presentations for tensor products of simple modules of the symmetric group.* Diploma thesis, University of Stuttgart, 2013.

[Ste12] Nico Stein. *Adelman's Abelianisation of an Additive Category.* Bachelor's thesis, University of Stuttgart, Germany, 2012.

[Zor11] Zorn (username). *math.stackexchange.com: Diagonalizable transformation restricted to an invariant subspace is diagonalizable*, 2011. URL `http://math.stackexchange.com/questions/78090`, last checked: Apr 27 2017.

Zusammenfassung

In der Theorie der Lie-Algebren werden Tori dazu verwendet, die halbeinfachen komplexen Lie-Algebren zu klassifizieren. Dabei haben die Elemente eines Torus die Eigenschaft, dass der zugehörige adjungierte Endomorphismus halbeinfach, d.h. diagonalisierbar ist.

Wir verallgemeinern die bekannten Begriffe der Lie-Algebren über (algebraisch abgeschlossenen) Körpern nun auf Lie-Algebren über Hauptidealbereichen.

Im Folgenden sei R ein Hauptidealbereich mit Quotientenkörper K, es sei M ein endlich erzeugt freier R-Modul und $\varphi\colon M \to M$ eine R-lineare Abbildung.

Der Begriff der Diagonalisierbarkeit von φ spaltet sich in zwei unterschiedliche Definitionen auf: Es heiße φ diagonalisierbar über R, falls es eine R-lineare Basis von M gibt, die aus Eigenvektoren von φ besteht. Wir sagen, dass φ diagonalisierbar über K ist, falls es eine K-lineare Basis von $K \otimes_R M$ gibt, die aus Eigenvektoren von $K \otimes_R \varphi$ besteht. Hier impliziert die erste Variante der Diagonalisierbarkeit die letztere.

Sei $\mathfrak{g}$ eine Lie-Algebra über R. Dann heiße $\mathfrak{t} \subseteq \mathfrak{g}$ ganzzahliger Torus, falls der adjungierte Endomorphismus $\mathrm{ad}_{\mathfrak{g}}(t)$ für alle $t \in \mathfrak{t}$ diagonalisierbar über R ist. Es heiße $\mathfrak{t} \subseteq \mathfrak{g}$ rationaler Torus, falls der adjungierte Endomorphismus $\mathrm{ad}_{\mathfrak{g}}(t)$ für alle $t \in \mathfrak{t}$ diagonalisierbar über K ist. Somit ist jeder ganzzahlige Torus auch ein rationaler Torus.

Analog zur Theorie der halbeinfachen komplexen Lie-Algebren zeigen wir, dass jeder rationale Torus eine abelsche Lie-Algebra über R ist. Zudem ist ein rationaler Torus in $\mathfrak{g}$ maximal in $\mathfrak{g}$, falls er seinem Zentralisator in $\mathfrak{g}$ gleicht. In Beispielen zeigen wir, dass maximale rationale Tori im Allgemeinen nicht eindeutig sind.

Wir können einen maximalen rationalen Torus $\mathfrak{t}$ in $\mathfrak{g}$ dazu verwenden, den $\mathfrak{t}$-Modul $\mathfrak{g}$ in unzerlegbare Teilmoduln zu zerlegen.

Sei nun Γ ein direktes Produkt von Matrixringen über R. Sei Ω eine Teilalgebra, für die $K \otimes_R (\Gamma/\Omega) = 0$ gilt. Ein solches Ω bezeichnen wir als Split-R-Ordnung.

Sei Δ die Teilalgebra von Γ, die aus Tupeln von Diagonalmatrizen besteht. Dann ist $\Delta \cap \Omega$ eine kommutative Teilalgebra von Ω. Wir bilden die Kommutator-Lie-Algebren $\mathfrak{l}(\Omega \cap \Delta)$ und $\mathfrak{l}(\Omega)$. Unser Standardbeispiel für einen maximalen rationalen Torus in $\mathfrak{l}(\Omega)$ ist dann die Lie-Algebra $\mathfrak{l}(\Omega \cap \Delta)$.

In dieser Arbeit untersuchen wir beispielhaft die Gruppenringe $\mathbb{Z}_{(3)}\,\mathrm{S}_3$, $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ und $\mathbb{Z}_{(2)}\,\mathrm{S}_5$. Das Bild eines solchen Gruppenrings RG unter einem Wedderburn-Isomorphismus ist dann eine Split-R-Ordnung. So können wir Tori in isomorphen Kopien der betrachteten Gruppenringe untersuchen, wobei wir im Falle $\mathbb{Z}_{(2)}\,\mathrm{S}_5$ auf eine Morita-reduzierte Version zurückgreifen. Im Beispiel $\mathbb{Z}_{(3)}\,\mathrm{S}_3$ ist der Standardtorus bereits ein ganzzahliger Torus. In den Beispielen $\mathbb{Z}_{(2)}\,\mathrm{S}_4$ und $\mathbb{Z}_{(2)}\,\mathrm{S}_5$ ermitteln wir einen ganzzahligen Torus, der im Standardtorus maximal ist.

Dazu wählen wir eine R-lineare Basis des rationalen Torus und bestimmen, welche R-Linearkombinationen von diesen Basiselementen wieder adjungierte Endomorphismen haben, die über R diagonalisierbar sind. Dies ist von vornherein ein unendliches Problem. Wir stellen jedoch fest, dass die Eigenmoduln des adjungierten Endomorphismus einer solchen Linearkombination eng mit den Eigenmoduln der adjungierten Endomorphismen der Basiselemente zusammenhängen. Daher können wir die Untersuchung auf eine endliche Menge beschränken.

Wir konstruieren einen Algorithmus, der für eine endliche Liste von k kommutierenden, über K diagonalisierbaren Matrizen eine Basis des R-Teilmoduls von R^k ausgibt, der die Koeffizientenvektoren enthält, deren zugehörige Linearkombination der Matrizen über R diagonalisierbar ist.

Erklärung

Ich versichere, dass ich die vorliegende Masterarbeit selbstständig und lediglich unter Benutzung der angegebenen Quellen verfasst habe. Alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen wurden als solche gekennzeichnet. Ich erkläre weiterhin, dass diese Arbeit weder vollständig noch in wesentlichen Teilen im Rahmen eines anderen Prüfungsverfahrens eingereicht wurde. Das elektronische Exemplar stimmt mit dieser Arbeit überein.

Ort, Datum

Unterschrift