Unit groups of group rings over localised integers

Master Thesis

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Contents

Chapter 0

Introduction

0.1 General procedure

We consider the group algebra $\mathbb{Z}_{(p)}G$ of a finite group G over the localisation $\mathbb{Z}_{(p)}$ of the integers at (p), where p is a prime number.

We suppose that we have a Wedderburn embedding

$$\omega_{\mathbb{Z}_{(p)}} : \mathbb{Z}_{(p)}G \xrightarrow{\text{injective}} R_1^{n_1 \times n_1} \times \ldots \times R_t^{n_t \times n_t} =: \Gamma_{(p)} ,$$

where $R_1, \ldots, R_t$ are algebraic number rings over $\mathbb{Z}_{(p)}$. Letting $\Lambda_{(p)} := \omega_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}G)$, we obtain

$$\begin{array}{ccc} \mathbb{Z}_{(p)}G & \xrightarrow[\text{injective}]{\omega_{\mathbb{Z}_{(p)}}} & \Gamma_{(p)} \\ & \searrow^{\sim} & \uparrow \\ & & \Lambda_{(p)} \end{array}$$

We choose $d \geqslant 1$ such that $p^d\Gamma_{(p)} \subseteq \Lambda_{(p)}$. We write $\overline{\Gamma}_{(p)} := \Gamma_{(p)}/p^d\Gamma_{(p)}$ and

$$\varrho : \Gamma_{(p)} \longrightarrow \overline{\Gamma}_{(p)}$$

for the residue class morphism. Let

$$\varphi := \varrho|_{\Lambda_{(2)}} : \Lambda_{(2)} \longrightarrow \overline{\Gamma}_{(2)} .$$

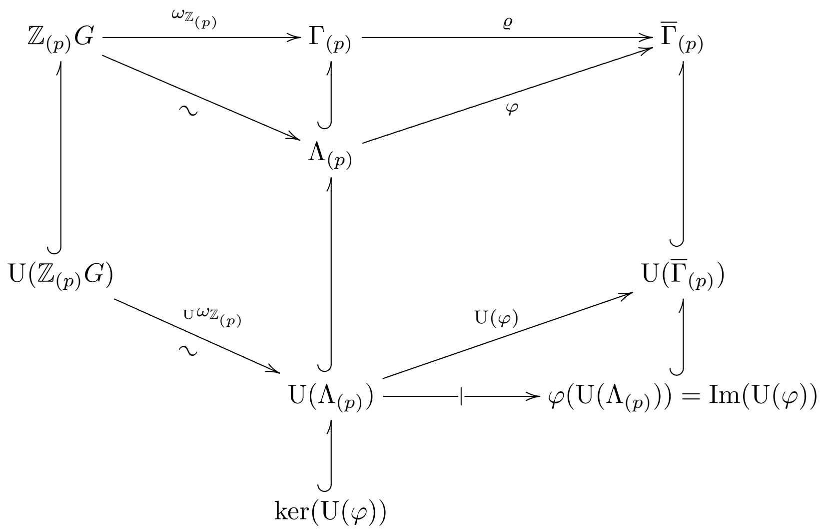
So we have the following commutative diagram of rings and ring morphisms.

$$\begin{array}{ccccc} \mathbb{Z}_{(p)}G & \xrightarrow{\omega_{\mathbb{Z}_{(p)}}} & \Gamma_{(p)} & \xrightarrow{\varrho} & \overline{\Gamma}_{(p)} \\ & \searrow^{\sim} & \uparrow & \nearrow_{\varphi} & \\ & & \Lambda_{(p)} & & \end{array}$$

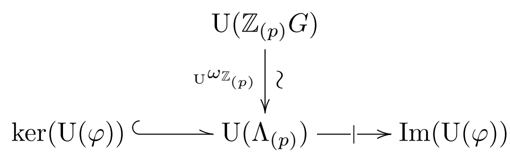
Each ring morphism $\psi : S \longrightarrow T$ restricts to a group morphism on the unit groups

$$\mathrm{U}(\psi) : \mathrm{U}(S) \longrightarrow \mathrm{U}(T) .$$

So we obtain the following diagram.



We end up with the diagram of groups and group morphisms



in which the lower row is a short exact sequence.

So the group $\mathrm{U}(\mathbb{Z}_{(p)}G)$ has a factor group isomorphic to $\mathrm{Im}(\mathrm{U}(\varphi))$ and a corresponding kernel isomorphic to $\ker(\mathrm{U}(\varphi))$.

The infinite group $\ker(\mathrm{U}(\varphi))$ has a simple description as a direct product of congruence subgroups:

$$\begin{aligned} \ker(\mathrm{U}(\varphi)) &= \{1 + p^d\gamma \in \mathrm{U}(\Gamma_{(p)}) : \gamma \in \Gamma_{(p)}\} \\ &= \{1 + p^d\gamma \in \mathrm{U}(\Gamma_{(p)}) : \gamma \in R_1^{n_1\times n_1}\} \times \ldots \times \{1 + p^d\gamma \in \mathrm{U}(\Gamma_{(p)}) : \gamma \in R_t^{n_t\times n_t}\} \\ &\leqslant \mathrm{U}(\Lambda_{(p)}) \leqslant \mathrm{U}(\Gamma_{(p)})\,. \end{aligned}$$

The finite subgroup $\mathrm{Im}(\mathrm{U}(\varphi)) \subseteq \mathrm{U}(\overline{\Gamma}_{(p)})$ is the group we want to investigate, using Magma [**1**].

To do so, we consider the following list of examples.

- $G = \mathrm{S}_3$, $p = 3$
- $G = \mathrm{A}_4$, $p = 2$
- $G = \mathrm{D}_{2p}$, for a prime $p \geqslant 3$
- $G = \mathrm{S}_4$, $p = 2$
- $G = \mathrm{S}_5$, $p = 2$

0.2 The example A_4

We consider the group $G = \mathrm{A}_4$ with $p = 2$. We obtain

$$\Gamma_{(2)} = \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}[\zeta_3] \times \mathbb{Z}_{(2)}^{3\times 3}$$

and thus

$$\mathrm{U}(\Gamma_{(2)}) = \mathrm{U}(\mathbb{Z}_{(2)}) \times \mathrm{U}(\mathbb{Z}_{(2)}[\zeta_3]) \times \mathrm{GL}_3(\mathbb{Z}_{(2)})\,.$$

We have $p^d = 4$ and thus the following description of $\ker(\mathrm{U}(\varphi))$ as direct product of congruence subgroups.

$$\begin{aligned}\ker(\mathrm{U}(\varphi)) =& \ \{1 + 4\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\} \\ =& \ (1 + 4\mathbb{Z}_{(2)}) \times (1 + 4\mathbb{Z}_{(2)}[\zeta_3]) \times \begin{pmatrix} 1 + 4\mathbb{Z}_{(2)} & 4\mathbb{Z}_{(2)} & 4\mathbb{Z}_{(2)} \\ 4\mathbb{Z}_{(2)} & 1 + 4\mathbb{Z}_{(2)} & 4\mathbb{Z}_{(2)} \\ 4\mathbb{Z}_{(2)} & 4\mathbb{Z}_{(2)} & 1 + 4\mathbb{Z}_{(2)} \end{pmatrix}\end{aligned}$$

Moreover, we have

$$|\,\mathrm{Im}(\mathrm{U}(\varphi))| = 1536 = 2^9 \cdot 3\,.$$

The group $\mathrm{Im}(\mathrm{U}(\varphi))$ has a chief series

$$\mathrm{Im}(\mathrm{U}(\varphi)) = H_1 \geqslant H_2 \geqslant H_3 \geqslant \ldots \geqslant H_8 = 1$$

with chief factors

$$\begin{array}{lll} H_1/H_2 \simeq \mathrm{C}_2, & H_2/H_3 \simeq \mathrm{C}_2, & H_3/H_4 \simeq \mathrm{C}_2 \\ H_4/H_5 \simeq \mathrm{C}_3, & H_5/H_6 \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_6/H_7 \simeq \mathrm{C}_2 \times \mathrm{C}_2 \\ H_7/H_8 \simeq \mathrm{C}_2 \times \mathrm{C}_2\,. & & \end{array}$$

The group $\mathrm{Im}(\mathrm{U}(\varphi))$ is solvable and has a derived series

$$\mathrm{Im}(\mathrm{U}(\varphi)) = D_1 \geqslant D_2 \geqslant D_3 \geqslant D_4 = 1$$

with derived factors

$$\begin{aligned} D_1/D_2 &\simeq \mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_6 \\ D_2/D_3 &\simeq \mathrm{C}_2^{\times 4} \\ D_3/D_4 &\simeq \mathrm{C}_2 \times \mathrm{C}_2\,. \end{aligned}$$

In this case, we can still give a presentation of $\mathrm{Im}(\mathrm{U}(\varphi))$, as follows.

$$\begin{aligned}\mathrm{Im}(\mathrm{U}(\varphi)) \simeq \langle m, d, e, k \mid\ & m^2, d^4, e^2, k^6, \\ & [m,d], [d,e], [m,e], [m,k], \\ & [d,k] = {}^k d, [{}^k d, d], [e, k^3], \\ & ({}^k d \cdot e)^2, [e,k]^2 \rangle\,.\end{aligned}$$

For the bigger groups, it was no longer possible to calculate a presentation of $\mathrm{Im}(\mathrm{U}(\varphi))$.

0.3 The example S_5

We consider the group $G = \mathrm{S}_5$ with $p = 2$. We obtain

$$\Gamma_{(2)} = \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{6\times 6}$$

and thus

$$\mathrm{U}(\Gamma_{(2)}) = \mathrm{U}(\mathbb{Z}_{(2)}) \times \mathrm{U}(\mathbb{Z}_{(2)}) \times \mathrm{GL}_4(\mathbb{Z}_{(2)}) \times \mathrm{GL}_4(\mathbb{Z}_{(2)}) \times \mathrm{GL}_5(\mathbb{Z}_{(2)}) \times \mathrm{GL}_5(\mathbb{Z}_{(2)}) \times \mathrm{GL}_6(\mathbb{Z}_{(2)}) \,.$$

We have $p^d = 8$ and thus the following description of $\ker(\mathrm{U}(\varphi))$ as direct product of congruence subgroups.

$$\ker(\mathrm{U}(\varphi)) = \{1 + 8\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\}$$

Moreover, we may decompose as a direct product of subgroups

$$\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \times U_2 \,,$$

where

$$|U_1| = 2^{139} \cdot 3^2 \cdot 5 \cdot 7$$

and

$$|U_2| = 2^{70} \cdot 3^2 \cdot 5 \cdot 7 \,.$$

So we have

$$|\,\mathrm{Im}(\mathrm{U}(\varphi))| = |U_1 \times U_2| = 2^{139} \cdot 3^2 \cdot 5 \cdot 7 \;\cdot\; 2^{70} \cdot 3^2 \cdot 5 \cdot 7 = 2^{209} \cdot 3^4 \cdot 5^2 \cdot 7^2 \,.$$

The group U_1 has a chief series

$$U_1 = H_1 \geqslant H_2 \geqslant H_3 \geqslant \ldots \geqslant H_{41} = 1$$

with chief factors

$$\begin{array}{lll}
H_1/H_2 \simeq \mathrm{A}_8, & H_2/H_3 \simeq \mathrm{C}_2, & H_3/H_4 \simeq \mathrm{C}_2, \\
H_4/H_5 \simeq \mathrm{C}_2, & H_5/H_6 \simeq \mathrm{C}_2, & H_6/H_7 \simeq \mathrm{C}_2, \\
H_7/H_8 \simeq \mathrm{C}_2, & H_8/H_9 \simeq \mathrm{C}_2^{\times 4}, & H_9/H_{10} \simeq \mathrm{C}_2, \\
H_{10}/H_{11} \simeq \mathrm{C}_2, & H_{11}/H_{12} \simeq \mathrm{C}_2^{\times 4}, & H_{12}/H_{13} \simeq C_2, \\
H_{13}/H_{14} \simeq \mathrm{C}_2^{\times 4}, & H_{14}/H_{15} \simeq \mathrm{C}_2, & H_{15}/H_{16} \simeq \mathrm{C}_2^{\times 4}, \\
H_{16}/H_{17} \simeq \mathrm{C}_2, & H_{17}/H_{18} \simeq \mathrm{C}_2^{\times 14}, & H_{18}/H_{19} \simeq \mathrm{C}_2, \\
H_{19}/H_{20} \simeq \mathrm{C}_2^{\times 14}, & H_{20}/H_{21} \simeq \mathrm{C}_2, & H_{21}/H_{22} \simeq \mathrm{C}_2^{\times 4}, \\
H_{22}/H_{23} \simeq \mathrm{C}_2^{\times 4}, & H_{23}/H_{24} \simeq \mathrm{C}_2, & H_{24}/H_{25} \simeq \mathrm{C}_2, \\
H_{25}/H_{26} \simeq \mathrm{C}_2^{\times 4}, & H_{26}/H_{27} \simeq \mathrm{C}_2^{\times 4}, & H_{27}/H_{28} \simeq \mathrm{C}_2^{\times 4}, \\
H_{28}/H_{29} \simeq \mathrm{C}_2^{\times 4}, & H_{29}/H_{30} \simeq \mathrm{C}_2, & H_{30}/H_{31} \simeq \mathrm{C}_2, \\
H_{31}/H_{32} \simeq \mathrm{C}_2, & H_{32}/H_{33} \simeq \mathrm{C}_2^{\times 14}, & H_{33}/H_{34} \simeq \mathrm{C}_2^{\times 14}, \\
H_{34}/H_{35} \simeq \mathrm{C}_2, & H_{35}/H_{36} \simeq \mathrm{C}_2^{\times 4}, & H_{36}/H_{37} \simeq C_2, \\
H_{37}/H_{38} \simeq \mathrm{C}_2^{\times 4}, & H_{38}/H_{39} \simeq \mathrm{C}_2^{\times 4}, & H_{39}/H_{40} \simeq \mathrm{C}_2^{\times 4}, \\
H_{40}/H_{41} \simeq \mathrm{C}_2. & &
\end{array}$$

The group U_2 has a chief series

$$U_2 = H_1 \geqslant H_2 \geqslant H_3 \geqslant \ldots \geqslant H_{14} = 1$$

with chief factors

$$\begin{array}{lll}
H_1/H_2 \simeq \mathrm{A}_8, & H_2/H_3 \simeq \mathrm{C}_2, & H_3/H_4 \simeq \mathrm{C}_2, \\
H_4/H_5 \simeq \mathrm{C}_2, & H_5/H_6 \simeq \mathrm{C}_2, & H_6/H_7 \simeq \mathrm{C}_2^{\times 14}, \\
H_7/H_8 \simeq \mathrm{C}_2^{\times 14}, & H_8/H_9 \simeq \mathrm{C}_2, & H_9/H_{10} \simeq \mathrm{C}_2, \\
H_{10}/H_{11} \simeq \mathrm{C}_2^{\times 14}, & H_{11}/H_{12} \simeq \mathrm{C}_2, & H_{12}/H_{13} \simeq \mathrm{C}_2^{\times 14}, \\
H_{13}/H_{14} \simeq \mathrm{C}_2 \,. & &
\end{array}$$

Neither U_1 nor U_2 is solvable.

In order to treat U_1 with Magma [**1**], we have dissected U_1 into a product of subgroups

$$U_1 = S_1 \cdot S_2 \cdot M_3 \cdot S_3$$

where a Peirce decomposition of $\Lambda_{(2)}$ is used to construct the subgroups $S_1, S_2, S_3, M_3 \leqslant U_1$.

For S_1, S_2, S_3, it was possible to construct a set of generators.

For M_3, a search over all possible candidate elements was performed, deciding in each case whether the candidate element is actually in $\mathrm{Im}(\mathrm{U}(\varphi))$.

For S_1, S_2, S_3, such a search would not have been possible due to time constraints.

Having generators at our disposal in these cases enabled us to let Magma perform the calculation of U_1 within about an hour.

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Thank you!

Chapter 1

Preliminaries

1.1 Conventions

1. Suppose given $a, b \in \mathbb{Z}$. We write $[a, b] := \{z \in \mathbb{Z} : a \leqslant z \leqslant b\}$.
2. We often abbreviate "for all" to "for".
3. Let G be a group. Let $k \in \mathbb{Z}_{\geqslant 0}$ and $X_1, \ldots, X_k \subseteq G$. Then

   $$X_1 \cdot X_2 \cdot \ldots \cdot X_k := \{x_1 \cdot x_2 \cdot \ldots \cdot x_k : x_1 \in X_1, x_2 \in X_2, \ldots, x_k \in X_k\} \subseteq G$$

   We have

   $$X_1 \cdot X_2 \cdot \ldots \cdot X_k \subseteq \langle X_1, X_2, \ldots, X_k \rangle \leqslant G\,.$$

   In particular, if

   $$X_1 \cdot X_2 \cdot \ldots \cdot X_k = G\,,$$

   then

   $$\langle X_1, X_2, \ldots, X_k \rangle = G\,.$$

4. Let R be a ring. Then $\mathrm{U}(R)$ is the group of units of R.
5. Let G be a group. Given $x, y \in G$, we write ${}^{y}x := yxy^{-1}$.
6. For elements a, b in a group, we use the commutator bracket

   $$[a, b] = a^{-1} \cdot b^{-1} \cdot a \cdot b\,.$$

7. Let G be a group. Then $\mathrm{Aut}(G)$ denote the automorphism group of G.
8. Let G be a group. Let $H \leqslant G$ be a subgroup. We write

   $$H \hookrightarrow G$$

   for the inclusion morphism.

9. Let G and H be a groups. Let $\psi : G \longrightarrow H$ be a group morphism. If applicable, we write

   $$\psi :\ H \twoheadrightarrow G$$

   to indicate that ψ is surjective.

1.2 Preliminaries on groups

1.2.1 Semidirect Product

Remark 1. Let N, H be groups with an action of H on N, i.e. a group morphism

$$\begin{aligned} H &\longrightarrow \operatorname{Aut}(N) \\ h &\longmapsto (n \longmapsto {}^h n)\,. \end{aligned}$$

The following applies.

1. We have ${}^h 1 = 1$ for $h \in H$.
2. We have ${}^h(n \cdot \tilde{n}) = {}^h n \cdot {}^h \tilde{n}$ for $h \in H$, $n \in N$.
3. We have ${}^1 n = n$ for $n \in N$.
4. We have ${}^{h \cdot \tilde{h}} n = {}^h n \cdot {}^{\tilde{h}} n$ for $h \in H$, $n \in N$.

We call

$$N \rtimes H := \{(n,h) : n \in N, h \in H\}$$

the *semidirect product* of N and H with the group multiplication

$$(n,h) \cdot (\tilde{n}, \tilde{h}) := (n \cdot {}^h \tilde{n}, h \cdot \tilde{h})\,.$$

Note that in the case of the trivial operation ${}^h n := n$ for $h \in H$ and $n \in N$, we have the direct product $N \rtimes H = N \times H$.

Remark 2. *Suppose given a group G. Suppose given a short exact sequence of groups*

$$N \xrightarrow{\iota} G \xrightarrow{\kappa} H\,.$$

Suppose given a group morphism

$$G \xleftarrow{\eta} H$$

such that $\kappa \circ \eta = \mathrm{id}_H$. Then we have with respect to

$$\begin{aligned} H &\longrightarrow \operatorname{Aut}(N) \\ h &\longmapsto (n \longmapsto \iota^{-1}({}^{\eta(h)}\iota(n)) =: {}^h n) \end{aligned}$$

the group isomorphism

$$\begin{aligned} \chi : N \rtimes H &\xrightarrow{\sim} G \\ (n,h) &\longmapsto \iota(n) \cdot \eta(h) \end{aligned}$$

Proof. We want to show that χ is a group morphism. Let $n, \tilde{n} \in N$ and $h, \tilde{h} \in H$. We obtain

$$\begin{aligned} &\chi\Big((n,h) \cdot (\tilde{n}, \tilde{h})\Big) \\ &= \chi\Big((n \cdot {}^h\tilde{n}, h \cdot \tilde{h})\Big) \\ &= \iota(n \cdot {}^h\tilde{n}) \cdot \eta(h \cdot \tilde{h}) \\ &= \iota(n) \cdot \iota({}^h\tilde{n}) \cdot \eta(h) \cdot \eta(\tilde{h}) \\ &= \iota(n) \cdot {}^{\eta(h)}\iota(\tilde{n}) \cdot \eta(h) \cdot \eta(\tilde{h}) \\ &= \iota(n) \cdot \eta(h) \cdot \iota(\tilde{n}) \cdot \eta(h)^{-1} \cdot \eta(h) \cdot \eta(\tilde{h}) \\ &= \chi\Big((n,h)\Big) \cdot \chi\Big((\tilde{n}, \tilde{h})\Big)\,. \end{aligned}$$

Now we want to show that χ is injective. We have to show that $\ker(\chi) \stackrel{!}{=} 1$. Suppose given $n \in N$, $h \in H$ such that $\chi\Big((n,h)\Big) = 1$. Then

$$\begin{aligned} 1 &= \chi\Big((n,h)\Big) \\ &= \iota(n) \cdot \eta(h) \,. \end{aligned}$$

With κ applied we get

$$\begin{aligned} 1 = \kappa(1) &= \kappa(\iota(n) \cdot \eta(h)) \\ &= \kappa(\iota(n)) \cdot \kappa(\eta(h)) \\ &= 1 \cdot h \,. \end{aligned}$$

So $h = 1$. With that we get

$$1 = \iota(n) \cdot \eta(h) = \iota(n) \cdot \eta(1) = \iota(n) \cdot 1 \,.$$

Since ι is injective, we conclude that $n = 1$. Altogether, $(n,h) = (1,1)$.

Now we want to show that χ is surjective. Let $g \in G$. We *claim* that $g \cdot \eta(\kappa(g))^{-1} \stackrel{!}{\in} \ker(\kappa)$.

We have

$$\kappa(g \cdot \eta(\kappa(g))^{-1}) = \kappa(g) \cdot \kappa(\eta(\kappa(g)))^{-1} = \kappa(g) \cdot \kappa(g)^{-1} = 1_H$$

since κ is a group morphism and $\kappa \circ \eta = \mathrm{id}_H$. This shows the *claim*.

Since we have a short exact sequence there exists a unique $n \in N$ with $\iota(n) = g \cdot \eta(\kappa(g))^{-1}$.

We *claim* that $\chi\Big((n, \kappa(g))\Big) \stackrel{!}{=} g$, for $n \in N$ and $\kappa(g) \in H$.

We have

$$\begin{aligned} \chi\Big((n, \kappa(g))\Big) &= \iota(n) \cdot \eta(\kappa(g)) \\ &= g \cdot \eta(\kappa(g))^{-1} \cdot \eta(\kappa(g)) \\ &= g \,. \end{aligned}$$

This proves the *claim*.

Altogether, χ is a bijective group morphism, hence a group isomorphism. □

1.2.2 Derived series

Let G be a finite group.

Definition 3 (cf. [**5,** §8, p. 100, Definition 8.1])**.** The *derived series* of G is the series of normal subgroups $D_i \trianglelefteq G$, for $i \in [1,k]$,

$$D_k < D_{k-1} < \ldots < D_2 < D_1 = G \,,$$

such that

$$D_{i+1} = [D_i, D_i]$$

for $i \in [1, k-1]$, and such that

$$D_k = [D_k, D_k] \,.$$

So it gets stationary at level k.

Remark 4. *We have*

$$D_k = 1$$

if and only if G is solvable.

Proof. Cf. [**5,** §8, p. 100, Definition 8.1] □

1.2.3 Chief series

Let G be a finite group.

Definition 5 (cf. [**4,** §11, p. 64, Definition 11.6])**.** A *chief series* of G is a finite series of normal subgroups $H_i \trianglelefteqslant G$, for $i \in [1, k]$,

$$1 = H_k \leqslant H_{k-1} \leqslant \ldots \leqslant H_2 \leqslant H_1 = G \,,$$

such that the factor group H_i/H_{i+1} is a minimal normal subgroup in G/H_{i+1} for $i \in [1, k-1]$.

Remark 6. *Minimal normal subgroups are direct products of isomorphic simple groups.*

Proof. Cf. [**4,** §9, p. 51, Theorem 9.12] □

Remark 7. *The set of isoclasses of sub factors of a chief series is independent from the choice of a chief series.*

Proof. Cf. [**4,** §11, p. 63-64, Theorem 11.5 and Definition 11.6] □

1.3 Smith Form

For $A \in \mathbb{Z}^{m\times n}$ there is $S \in \mathbb{Z}^{m\times m}$ invertible and $T \in \mathbb{Z}^{n\times n}$ invertible such that

$$SAT =: D = \begin{pmatrix} d_1 & 0 & \ldots & & & & 0 \\ 0 & d_2 & 0 & \ldots & & & 0 \\ \vdots & 0 & \ddots & & & & \vdots \\ & \vdots & 0 & d_k & 0 & \ldots & 0 \\ & & \vdots & & 0 & & \vdots \\ & & & & & \ddots & \\ 0 & \ldots & & & & & 0 \end{pmatrix} \in \mathbb{Z}^{m\times n}$$

with $d_1, d_2, \ldots, d_k \in \mathbb{Z}_{\geqslant 1}$ and $d_1|d_2|\ldots|d_k$.

Chapter 2

Introductory example

In this introductory chapter we explore the example S_3. Later in the thesis the S_3 will be considered again in the form of D_6; cf. §6.

We want to consider the symmetric group S_3.

We *claim* that we have the following $\mathbb{Q}$-algebra isomorphism of Wedderburn.

$$\begin{aligned} \omega_{\mathbb{Q}} :\ \mathbb{Q}S_3 &\longrightarrow \mathbb{Q} \times \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix} \times \mathbb{Q} \\ (1,2,3) &\longmapsto \left(1,\ \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix},\ 1\right) =: A \\ (1,2) &\longmapsto \left(1,\ \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix},\ -1\right) =: B \end{aligned}$$

Since we have a group isomorphism

$$\begin{aligned} \langle a, b : a^3, b^2, (ba)^2 \rangle &\xrightarrow{\sim} S_3 \\ a &\longmapsto (1,2,3) \\ b &\longmapsto (1,2)\ , \end{aligned}$$

we have to verify that $A^3 = 1$, $B^2 = 1$ and $(BA)^2 = 1$ in order to have the $\mathbb{Q}$-algebra morphism $\omega_{\mathbb{Q}}$.

So

$$
\begin{aligned}
(BA)^2 &= \left(\left(1,\ \begin{pmatrix}-2 & 3\\ -1 & 2\end{pmatrix},\ -1\right)\cdot\left(1,\ \begin{pmatrix}1 & -3\\ 1 & -2\end{pmatrix},\ 1\right)\right)^2\\
&= \left(1\cdot 1,\ \begin{pmatrix}-2 & 3\\ -1 & 2\end{pmatrix}\cdot\begin{pmatrix}1 & -3\\ 1 & -2\end{pmatrix},\ -1\cdot 1\right)^2\\
&= \left(1,\ \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix},\ -1\right)^2\\
&= \left(1^2,\ \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}^2,\ (-1)^2\right)\\
&= \left(1,\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ 1\right) = 1
\end{aligned}
$$

and analogously

$$A^3 = \left(1,\ \begin{pmatrix}1 & -3\\ 1 & -2\end{pmatrix},\ 1\right)^3 = \left(1,\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ 1\right) = 1$$

and

$$B^2 = \left(1,\ \begin{pmatrix}-2 & 3\\ -1 & 2\end{pmatrix},\ -1\right)^2 = \left(1,\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ 1\right) = 1\,.$$

Mapping all the elements of S_3, we get

$$
\begin{aligned}
\mathrm{id} &\longmapsto \left(1,\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ 1\right)\\
(1,2,3) &\longmapsto \left(1,\ \begin{pmatrix}1 & -3\\ 1 & -2\end{pmatrix},\ 1\right)\\
(1,2,3)^2 = (1,3,2) &\longmapsto \left(1,\ \begin{pmatrix}-2 & 3\\ -1 & 1\end{pmatrix},\ 1\right)\\
(1,2) &\longmapsto \left(1,\ \begin{pmatrix}-2 & 3\\ -1 & 2\end{pmatrix},\ -1\right)\\
(1,2,3)\circ(1,2) = (1,3) &\longmapsto \left(1,\ \begin{pmatrix}1 & -3\\ 0 & -1\end{pmatrix},\ -1\right)\\
(1,3,2)\circ(1,2) = (2,3) &\longmapsto \left(1,\ \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix},\ -1\right)\ .
\end{aligned}
$$

Note that the tuple $\big(\mathrm{id}, (1,2,3), (1,3,2), (1,2), (1,3), (2,3)\big)$ of elements of S_3 in $\mathbb{Q}S_3$ is a $\mathbb{Q}$-linear basis of $\mathbb{Q}S_3$, whilst

$$\Big((1,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\ 0)\,,(0,\ \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\ 0)\,,(0,\ \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\ 0)\,,(0,\ \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\ 0)\,,(0,\ \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right),\ 0)\,,(0,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\ 1)\Big)$$

is a $\mathbb{Q}$-linear basis of $\mathbb{Q}\times\begin{pmatrix}\mathbb{Q} & \mathbb{Q}\\ \mathbb{Q} & \mathbb{Q}\end{pmatrix}\times\mathbb{Q}$.

With respect to these two $\mathbb{Q}$-linear bases, the map $\omega_{\mathbb{Q}}$ is described by the following matrix.

$$W := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ 0 & -3 & 3 & 3 & -3 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 1 & -2 & 1 & 2 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}$$

So the columns of W contain the entries of the images of the group elements read out row-wise from left to right.

The determinant of W is

$$\det(W) = 54 > 0\,,$$

hence W is invertible in $\mathbb{Q}^{6\times 6}$, and so $\omega_{\mathbb{Q}}$ is an isomorphism. This proves the *claim*.

Let $\omega_{\mathbb{Z}}$ be the restriction of $\omega_{\mathbb{Q}}$ to $\mathbb{Z}\mathrm{S}_3$ and to $\mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \times \mathbb{Z} =: \Gamma$.

$$\begin{aligned} \omega_{\mathbb{Z}} :\ \mathbb{Z}\mathrm{S}_3 &\longrightarrow \mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \times \mathbb{Z} \\ (1,2,3) &\longmapsto \left(1,\ \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix},\ 1\right) \\ (1,2) &\longmapsto \left(1,\ \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix},\ -1\right) \end{aligned}$$

Then $\omega_{\mathbb{Z}}$ is an injective $\mathbb{Z}$-algebra morphism.

The tuple $\big(\mathrm{id}, (1,2,3), \ldots, (2,3)\big)$ is a $\mathbb{Z}$-linear basis of $\mathbb{Z}\mathrm{S}_3$, whilst

$$\Big(\big(1,\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\ 0\big)\,, \big(0,\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),\ 0\big)\,, \ldots, \big(0,\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\ 1\big)\Big)$$

is a $\mathbb{Z}$-linear basis of $\Gamma = \mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \times \mathbb{Z}$.

With respect to these bases, the map $\omega_{\mathbb{Z}}$ is again described by the matrix W, for which we again have $\det(W) = 54$. But 54 is not invertible in $\mathbb{Z}$, so W is not invertible in $\mathbb{Z}^{6\times 6}$, and thus $\omega_{\mathbb{Z}}$ is not surjective. We want to describe its image,

$$\Lambda \ := \ \omega_{\mathbb{Z}}(\mathbb{Z}\mathrm{S}_3) \ \subseteq \ \mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \times \mathbb{Z} \ = \ \Gamma\,.$$

Note that $\left(a_1,\ \begin{pmatrix} a_2 & a_3 \\ a_4 & a_5 \end{pmatrix},\ a_6\right) \in \mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \times \mathbb{Z}$ is in the image of $\omega_{\mathbb{Z}}$ if and only if there exists

$\begin{pmatrix} b_1 \\ \vdots \\ b_6 \end{pmatrix} \in \mathbb{Z}^{6\times 1}$ such that

$$W \begin{pmatrix} b_1 \\ \vdots \\ b_6 \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} ,$$

i.e. such that

$$\begin{pmatrix} b_1 \\ \vdots \\ b_6 \end{pmatrix} = W^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix}$$

where W^{-1} is formed in $\mathbb{Q}^{6\times 6}$.

Now to remove the denominators in the entries of W^{-1}, we multiply by 6 and obtain

$$6 \cdot (W^{-1}) = 6W^{-1} = \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & -4 & -2 & 6 & 2 & 1 \\ 1 & 2 & 2 & -6 & -4 & 1 \\ 1 & -4 & -2 & 6 & 4 & -1 \\ 1 & 2 & 0 & -6 & -2 & -1 \\ 1 & 2 & 2 & 0 & -2 & -1 \end{pmatrix} \in \mathbb{Z}^{6\times 6}$$

Our equation becomes

$$\begin{pmatrix} 6b_1 \\ \vdots \\ 6b_6 \end{pmatrix} = 6W^{-1} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} .$$

Since we asked for the existence of $\begin{pmatrix} b_1 \\ \vdots \\ b_6 \end{pmatrix} \in \mathbb{Z}^{6\times 1}$ such that this equation holds, our condition is equivalent to

$$6W^{-1} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} \in 6\mathbb{Z}^{6\times 1}$$

This condition amounts to the following list of congruences.

$$\begin{aligned} a_1 + 2a_2 + 2a_5 + a_6 &\equiv_6 0 \\ a_1 - 4a_2 - 2a_3 + 6a_4 + 2a_5 + a_6 &\equiv_6 0 \\ a_1 + 2a_2 + 2a_3 - 6a_4 - 4a_5 + a_6 &\equiv_6 0 \\ a_1 - 4a_2 - 2a_3 + 6a_4 + 4a_5 - a_6 &\equiv_6 0 \\ a_1 + 2a_2 - 6a_4 - 2a_5 - a_6 &\equiv_6 0 \\ a_1 + 2a_2 + 2a_3 - 2a_5 - a_6 &\equiv_6 0 \end{aligned}$$

Note that for $z \in \mathbb{Z}$, the statement $z \equiv_6 0$ is equivalent to $z \equiv_2 0$ and $z \equiv_3 0$.

So reducing the coefficients modulo 2 respectively modulo 3, we obtain the following equivalent list of congruences.

$$a_1 + a_6 \equiv_2 0 \tag{2.1}$$

$$a_1 - a_2 - a_5 + a_6 \equiv_3 0 \tag{2.2}$$
$$a_1 - a_2 + a_3 - a_5 + a_6 \equiv_3 0 \tag{2.3}$$
$$a_1 - a_2 - a_3 - a_5 + a_6 \equiv_3 0 \tag{2.4}$$
$$a_1 - a_2 + a_3 + a_5 - a_6 \equiv_3 0 \tag{2.5}$$
$$a_1 - a_2 + a_5 - a_6 \equiv_3 0 \tag{2.6}$$
$$a_1 - a_2 - a_3 + a_5 - a_6 \equiv_3 0 \tag{2.7}$$

The difference of (2.3) and (2.2) yields

$$a_3 \equiv_3 0\,.$$

Then the sum of (2.4) and (2.5) yields

$$a_1 \equiv_3 a_2\,.$$

The difference of (2.5) and (2.4) yields

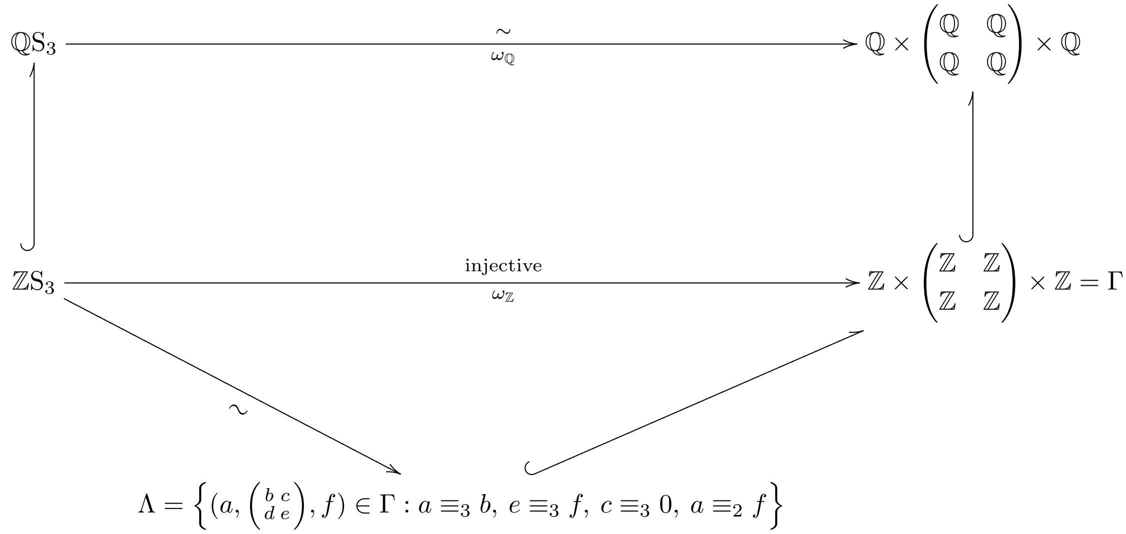
$$a_5 \equiv_3 a_6\,.$$

Altogether, our list of congruences is equivalent to

$$\begin{aligned} a_1 &\equiv_2 a_6 \\ a_3 &\equiv_3 0 \\ a_1 &\equiv_3 a_2 \\ a_5 &\equiv_3 a_6 \end{aligned}$$

With that we see that

$$\omega_{\mathbb{Z}}(\mathbb{Z}\mathrm{S}_3) = \Lambda = \left\{(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f) \in \Gamma : a \equiv_3 b,\ e \equiv_3 f,\ c \equiv_3 0,\ a \equiv_2 f\right\}$$



In order to consider the behaviour at the prime 3 separately, we localise at (3), i.e. we pass from the ground ring $\mathbb{Z}$ to the ground ring

$$\mathbb{Z}_{(3)} := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \not\equiv_3 0\} \subseteq \mathbb{Q}\,,$$

with $\mathrm{U}(\mathbb{Z}_{(3)}) := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, a \not\equiv_3 0, b \not\equiv_3 0\}$.

Analogously to $\omega_{\mathbb{Z}}$ as a restriction of $\omega_{\mathbb{Q}}$ to $\mathbb{Z}$ and to $\Gamma = \mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \times \mathbb{Z}$, we get $\omega_{\mathbb{Z}_{(3)}}$ as a restriction of $\omega_{\mathbb{Q}}$ to $\mathbb{Z}_{(3)}\mathrm{S}_3$ and to $\Gamma_{(3)} := \mathbb{Z}_{(3)} \times \begin{pmatrix} \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)} \\ \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)} \end{pmatrix} \times \mathbb{Z}_{(3)}$:

$$\begin{aligned} \omega_{\mathbb{Z}_{(3)}} : \mathbb{Z}_{(3)}\mathrm{S}_3 &\longrightarrow \mathbb{Z}_{(3)} \;\times\; \begin{pmatrix} \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)} \\ \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)} \end{pmatrix} \;\times\; \mathbb{Z}_{(3)} \\ (1,2,3) &\longmapsto \left(1,\; \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix},\; 1\right) \\ (1,2) &\longmapsto \left(1,\; \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix},\; -1\right) \end{aligned}$$

And so

$$\Lambda_{(3)} \;:=\; \omega_{\mathbb{Z}_{(3)}}(\mathbb{Z}_{(3)}\mathrm{S}_3) \;\subseteq\; \mathbb{Z}_{(3)} \times \begin{pmatrix} \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)} \\ \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)} \end{pmatrix} \times \mathbb{Z}_{(3)} \;=\; \Gamma_{(3)}$$

Note that for all $a, f \in \mathbb{Z}_{(3)}$, we have $a \equiv_2 f$, i.e. $a - f \in 2\mathbb{Z}_{(3)} = \mathbb{Z}_{(3)}$. So $a \equiv_2 f$ is an empty condition. So analogously to the description of Λ, the matrix W also gives

$$\begin{aligned} \Lambda_{(3)} &= \left\{ (a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f) \in \Gamma_{(3)} : a \equiv_3 b,\; e \equiv_3 f,\; c \equiv_3 0,\; a \equiv_2 f \right\} \\ &= \left\{ (a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f) \in \Gamma_{(3)} : a \equiv_3 b,\; e \equiv_3 f,\; c \equiv_3 0 \right\} \end{aligned}$$

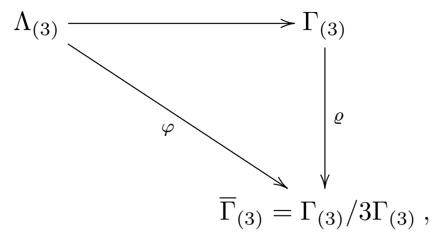
and so

$$\omega_{\mathbb{Z}_{(3)}}|^{\Lambda_{(3)}} : \mathbb{Z}_{(3)}\mathrm{S}_3 \xrightarrow{\sim} \{(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f) \in \Gamma_{(3)} : a \equiv_3 b, e \equiv_3 f, c \equiv_3 0\} = \Lambda_{(3)}\,. \tag{2.8}$$

We consider

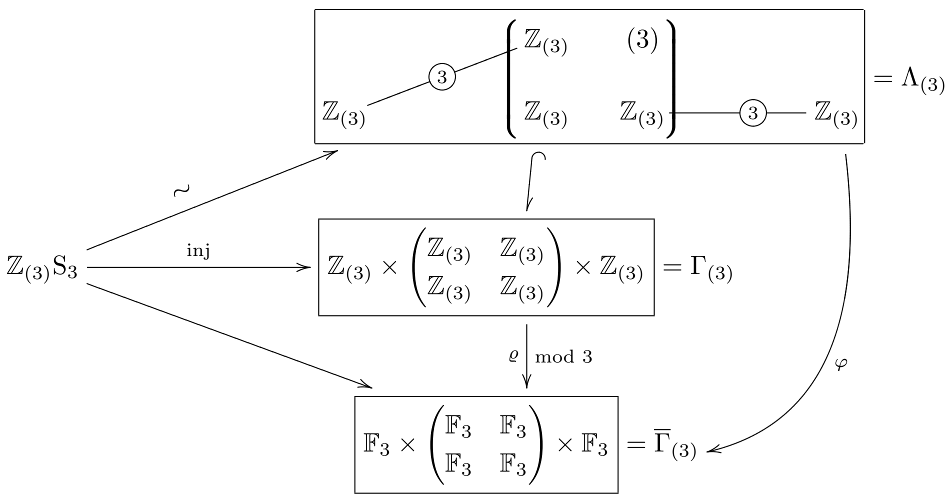
$$\overline{\Gamma}_{(3)} := \Gamma_{(3)}/3\Gamma_{(3)} = \mathbb{F}_3 \times \begin{pmatrix} \mathbb{F}_3 & \mathbb{F}_3 \\ \mathbb{F}_3 & \mathbb{F}_3 \end{pmatrix} \times \mathbb{F}_3$$

and the following commutative diagram of $\mathbb{Z}_{(3)}$-algebras.



where ϱ is the residue class morphism.

So schematically, we have



We aim to give a short exact sequence of groups with middle term isomorphic to $\mathrm{U}(\mathbb{Z}_{(3)}\mathrm{S}_3)$, with known kernel and with cokernel describable via Magma.

Note that $\mathrm{U}(\mathbb{Z}_{(3)}\mathrm{S}_3) \simeq \mathrm{U}(\Lambda_{(3)})$. Restricting the $\mathbb{Z}_{(3)}$-algebra morphism to the respective unit groups, we obtain

$$\ker(\mathrm{U}(\varphi)) \hookrightarrow \mathrm{U}(\Lambda_{(3)}) \xrightarrow{\mathrm{U}(\varphi)} \mathrm{U}(\overline{\Gamma}_{(3)})$$

For the kernel we use the description

$$\ker(\mathrm{U}(\varphi)) = \{1 + 3x : x \in \Gamma_{(3)}\} = (1 + 3\mathbb{Z}_{(3)}) \times \begin{pmatrix} 1 + 3\mathbb{Z}_{(3)} & 1 + 3\mathbb{Z}_{(3)} \\ 1 + 3\mathbb{Z}_{(3)} & 1 + 3\mathbb{Z}_{(3)} \end{pmatrix} \times (1 + 3\mathbb{Z}_{(3)})\,.$$

For the cokernel $\mathrm{Im}(\mathrm{U}(\varphi))$ we need to decide whether or not an element $\xi \in \mathrm{U}(\overline{\Gamma}_{(3)})$ is in $\mathrm{Im}(\mathrm{U}(\varphi))$.

So suppose given $\xi \in \mathrm{U}(\overline{\Gamma}_{(3)})$. We have to decide whether there exists a $x \in \mathrm{U}(\Lambda_{(3)})$ such that $\varphi = \varrho(x) = \xi$.

The problem is that there are infinitely many elements $x \in \Gamma_{(3)}$ with $\varrho(x) = \xi$, and we have to decide whether amongst these inverse images, there is a unit in $\Lambda_{(3)}$, i.e. an element of $\mathrm{U}(\Lambda_{(3)})$. Recall that $\Lambda_{(3)} \subseteq \Gamma_{(3)}$.

It will be shown that locally either all of these inverse images x of ξ are in $\mathrm{U}(\Lambda_{(3)})$, or none; cf. Remark 15(I) below. This argument, as later shown, works over $\mathbb{Z}_{(3)}$, it would not work over $\mathbb{Z}$; cf. Remark 12.

So we can overcome this problem by examining a single chosen element $x \in \Lambda_{(3)}$ with $\varrho(x) = \xi$ and decide whether x is a unit or not.

So Magma is given $\xi \in \mathrm{U}(\overline{\Gamma}_{(3)})$, picks a single element $x \in \Gamma_{(3)}$ with $\varrho(x) = \xi$ and decides whether $x \in \mathrm{U}(\Lambda_{(3)})$. If the answer is yes, then $\xi \in \mathrm{Im}(\mathrm{U}(\varphi))$, if the answer is no, then $\xi \notin \mathrm{Im}(\mathrm{U}(\varphi))$.

In this way, using that $\overline{\Gamma}_{(3)}$ is finite as a set, Magma gives

$$\begin{aligned}\mathrm{Im}(\mathrm{U}(\varphi)) = \Bigg\{ & \left(1, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, 1\right), \left(1, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, -1\right), \left(1, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, 1\right), \\ & \left(1, \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}, -1\right), \left(1, \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}, 1\right), \left(1, \begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}, -1\right), \\ & \left(-1, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, 1\right), \left(-1, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, -1\right), \left(-1, \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}, 1\right), \\ & \left(-1, \begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}, -1\right), \left(-1, \begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}, 1\right), \left(-1, \begin{pmatrix}-1 & 0\\ -1 & -1\end{pmatrix}, -1\right) \Bigg\}.\end{aligned}$$

Via Magma we obtain the set of generators

$$\left\{ \left(1, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, -1\right), \left(-1, \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}, 1\right) \right\}$$

of $\mathrm{Im}(\mathrm{U}(\varphi))$ and then an isomorphism $\mathrm{Im}(\mathrm{U}(\varphi)) \xrightarrow{\sim} \mathrm{S}_3 \times \mathrm{C}_2$.

Altogether, we get the result

$$\begin{array}{ccccccc} \mathrm{U}(\mathbb{Z}_{(3)}\mathrm{S}_3) & \xrightarrow{\sim} & \mathrm{U}(\Lambda_{(3)}) & \longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\; | \;\;\;\;\; & \mathrm{Im}(\mathrm{U}(\varphi)) & \xrightarrow{\sim} & \mathrm{S}_3 \times \mathrm{C}_2 \\ & & \uparrow & & & & \\ & & \mathrm{ker}(\mathrm{U}(\varphi)) = \{1 + 3x : x \in \Gamma_{(3)}\} & & & & \\ & & \uparrow & & & & \\ & & 1 & & & & \end{array}$$

Note that $\mathrm{S}_3 \times \mathrm{C}_2$ is a finite group, while $\{1 + 3x : x \in \Gamma_{(3)}\}$ is infinite.

In there, we have the short exact sequence

$$\mathrm{ker}(\mathrm{U}(\varphi)) \hookrightarrow \mathrm{U}(\Lambda_{(3)}) \twoheadrightarrow \mathrm{Im}\,\mathrm{U}(\varphi)$$

with middle term isomorphic to $\mathrm{U}(\mathbb{Z}_{(3)}\mathrm{S}_{(3)})$, with known kernel and with cokernel described via Magma.

Chapter 3

Theory

3.1 Unit groups of subrings

Lemma 8. *Let B be a ring and $A \subseteq B$ a subring. Suppose that the abelian factor group B/A is finite. Then we have*

$$\mathrm{U}(A) = A \cap \mathrm{U}(B)\,.$$

Proof. Ad $\subseteq$: This follows from $\mathrm{U}(A) \subseteq \mathrm{U}(B)$.

Ad $\supseteq$: Let $u \in \mathrm{U}(B) \cap A$. The map

$$\begin{aligned} \mu_u : B &\longrightarrow B \\ x &\longmapsto x \cdot u \end{aligned}$$

is an abelian group morphism.

We know that $A \cdot u \subseteq A$, because $u \in A$. So the map μ_u can be restricted to a map from A to A.

$$\begin{array}{ccc} B & \xrightarrow[\sim]{\ \mu_u\ } & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{\ \mu_u|_A^A\ } & A \end{array}$$

The map $\mu_u|_A^A$ is injective, as a restriction of an injective map.

Now we want to show that $\mu_u|_A^A$ is also surjective. So we need to show that

$$A \cdot u = A\,.$$

Since μ_u is bijective, we have

$$\begin{aligned} |B/A| &= |\mu_u(B)/\mu_u(A)| \\ &= |B/\mu_u(A)|\,. \end{aligned}$$

We get the following sequence.

$$\mu_u(A) = Au \hookrightarrow A \hookrightarrow B$$

We obtain the short exact sequence

$$A/\mu_u(A) \longrightarrow B/\mu_u(A) \longrightarrow B/A\,.$$

So

$$|B/\mu_u(A)| = |A/\mu_u(A)| \cdot |B/A|\,.$$

With that we get

$$|A/\mu_u(A)| = 1$$

and thus

$$A = \mu_u(A)\,.$$

Now we know that $\mu_u|_A^A$ is indeed bijective. So there is a $v \in A$ with

$$\left(\mu_u|_A^A\right)(v) = v \cdot u = 1\,.$$

Analogously we obtain a $w \in A$ with $u \cdot w = 1$.

So we get

$$w = v \cdot u \cdot w = v\,.$$

Altogether we have the bilateral inverse $u^{-1} = v = w$. Hence $u \in \mathrm{U}(A)$ □

Remark 9. In Lemma 8, we may not omit the assumption that B/A is a finite abelian group. For example, let $A = \mathbb{Z}$ and $B = \mathbb{Q}$. Then $2 \notin \mathrm{U}(\mathbb{Z}) \subset \mathbb{Z} \cap \mathrm{U}(\mathbb{Q}) \ni 2$.

3.2 Change of representatives of units

Let R be a discrete valuation ring with maximal ideal (π), so $\pi R \overset{\text{max.}}{\subseteq} R$.

Let B be an R-order, i.e. an R-algebra, finitely generated free as an R-module.

Let $A \subseteq B$ be an R-subalgebra.

Remark 10. *We consider the Jacobson radical* $\mathrm{Jac}(B)$ *of* B*.*

(i) *We have* $\pi B \subseteq \mathrm{Jac}(B)$*.*

(ii) *Suppose given* $z \in \mathrm{Jac}(B)$*. Then* $1 + z \in \mathrm{U}(B)$*.*

Proof. Assertion (i) follows e.g. from [**8,** Appendix C.4, p. 146, Lemma 209].

Assertion (ii) follows e.g. from [**8,** Appendix C.1, p. 137, Lemma 182]. □

Lemma 11. *Suppose* B/A *to be a finite abelian group.*

Suppose given $d \geqslant 1$ *with* $\pi^d B \subseteq A \subseteq B$*. We write* $\overline{B} := B/\pi^d B$ *and* $\varrho : B \longrightarrow \overline{B}$, $b \longmapsto b + \pi^d B$*.*

Suppose given $\xi \in \overline{B}$ *and* $x, \tilde{x} \in B$ *such that* $\varrho(x) = \varrho(\tilde{x}) = \xi$*. Then*

$$x \in \mathrm{U}(A) \Longleftrightarrow \tilde{x} \in \mathrm{U}(A)\,.$$

So either all of the inverse images of ξ *in* B *are in* $\mathrm{U}(A)$*, or none.*

Proof. We get the following diagram.

$$\begin{array}{ccc} A & \hookrightarrow & B \\ & \overset{\varphi}{\searrow} & \downarrow \varrho \\ & & \overline{B} = B/\pi^d B \end{array}$$

It suffices to show that

$$x \in \mathrm{U}(A) \overset{!}{\Longrightarrow} \tilde{x} \in \mathrm{U}(A)\,.$$

Now by Lemma 8 we know that

$$\mathrm{U}(A) = \mathrm{U}(B) \cap A\,.$$

So it suffices to show the following two statements:

(1) $x \in A \overset{!}{\Longrightarrow} \tilde{x} \in A$

(2) $x \in \mathrm{U}(B) \overset{!}{\Longrightarrow} \tilde{x} \in \mathrm{U}(B)$

Ad (1): Suppose given $x \in A$. Since $\varrho(x) = \varrho(\tilde{x})$ it follows that $\tilde{x}-x \in \pi^d B \subseteq A$. Then $\tilde{x} = x+(\tilde{x}-x) \in A$.

Ad (2): Suppose given $x \in \mathrm{U}(B)$.

We have $\tilde{x} - x \in \pi^d B \subseteq \pi B$. So we may choose $y \in B$ with $\pi y = \tilde{x} - x$.

Hence

$$\begin{aligned} \tilde{x} &= x + \pi y \\ &= x(1 + \pi x^{-1} y) \end{aligned}$$

Now $x \in \mathrm{U}(B)$ and $1 + \pi x^{-1} y \in \mathrm{U}(B)$, cf. Remark 10(i, ii).

So $\tilde{x} = x(1 + \pi x^{-1} y) \in \mathrm{U}(B)$. □

Remark 12. The proof of Lemma 11 would not work over $\mathbb{Z}$ instead of R, where π is a prime number in $\mathbb{Z}$.

in fact we would not be able to use Remark 10, since e.g. for $B = \mathbb{Z}$,

$$\pi\mathbb{Z} \nsubseteq \mathrm{Jac}(\mathbb{Z}) = 0$$

and thus

$$1 + \pi \nsubseteq \mathrm{U}(\mathbb{Z})$$

3.3 Artin-Wedderburn

Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 2}$ be a prime. We shall work over $\mathbb{Z}_{(p)}$.

Let $S_1, S_2, \ldots, S_t$ be representatives of the isoclasses of the simple $\mathbb{Q}G$-modules.

By the Lemma of Schur [**2,** §5, p. 137, Theorem 5.2.1], $K_j := \mathrm{End}_{\mathbb{Q}G}(S_j)$ is a skew field for $j \in [1, t]$.

Suppose that K_j is commutative for $j \in [1, t]$.

Altogether, $K_j|\mathbb{Q}$ is a finite field extension for $j \in [1, t]$, i.e. K_j is a number field.

Let $R_j \subseteq K_j$ be the integral closure of $\mathbb{Z}_{(p)}$ in K_j for $j \in [1, t]$. Note that R_j is a principal ideal domain.

Let $L_j \subseteq S_j$ be an R_jG-lattice, i.e. a R_jG-submodule with $\operatorname{rk}_{R_j} L_j = \dim_{K_j} S_j =: n_j$ for $j \in [1, t]$. Choose an R_j-linear basis of L_j.

Given $g \in G$ and $j \in [1, t]$, let $\omega_{\mathbb{Z}_{(p)},j}(g) \in R_j^{n_j \times n_j}$ be the describing matrix of the R_j-linear map

$$\begin{aligned} S_j &\longrightarrow S_j, \\ s &\longmapsto g \cdot s \end{aligned}$$

with respect to the chosen R_j-linear basis of L_j.

Let $\omega_{\mathbb{Z}_{(p)}}(g) := (\omega_{\mathbb{Z}_{(p)},1}(g), \ldots, \omega_{\mathbb{Z}_{(p)},t}(g)) \in R_1^{n_1 \times n_1} \times \ldots \times R_t^{n_t \times n_t}$, for $g \in G$.

This yields an injective $\mathbb{Z}_{(p)}$-algebra morphism

$$\omega_{\mathbb{Z}_{(p)}} : \mathbb{Z}_{(p)}G \longrightarrow R_1^{n_1 \times n_1} \times \ldots \times R_t^{n_t \times n_t} .$$

By $\mathbb{Q}$-linear extension, we obtain the $\mathbb{Q}$-algebra isomorphism

$$\omega_{\mathbb{Q}} : \mathbb{Q}G \xrightarrow{\sim} K_1^{n_1 \times n_1} \times \ldots \times K_t^{n_t \times n_t}$$

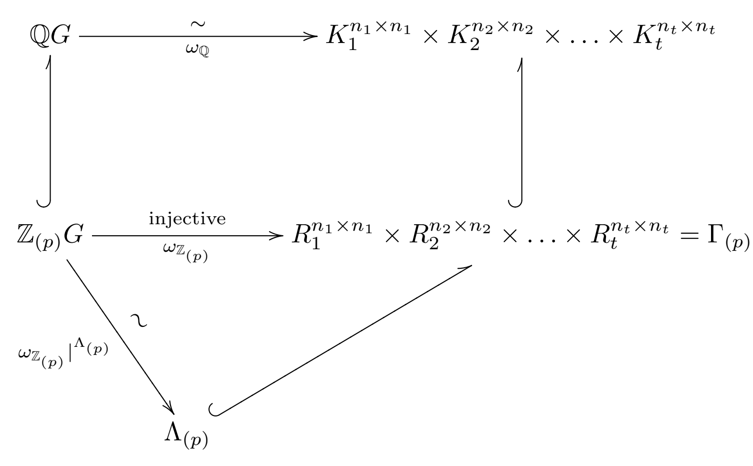
due to Wedderburn and Artin, cf. [**3,** §3, p. 42, Maschkes Theorem (3.14)] and [**2,** §5, p. 138, Theorem 5.2.4].

In the following we will refer to the $\mathbb{Q}$-algebra isomorphism of Wedderburn and Artin as the $\mathbb{Q}$-algebra isomorphism of Wedderburn.

Let

$$\Lambda_{(p)} := \omega_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}G) \subseteq R_1^{n_1 \times n_1} \times \ldots \times R_t^{n_t \times n_t} =: \Gamma_{(p)}$$

Altogether, we have the following commutative diagram.



We have the isomorphism

$$_{\mathrm{U}}\omega_{\mathbb{Z}_{(p)}} := \mathrm{U}(\omega_{\mathbb{Z}_{(p)}}|^{\Lambda_{(p)}}) : \mathrm{U}(\mathbb{Z}_{(p)}G) \xrightarrow{\sim} \mathrm{U}(\Lambda_{(p)})$$

as the restriction of the isomorphism $\omega_{\mathbb{Z}_{(p)}}|^{\Lambda_{(p)}} : \mathbb{Z}_{(p)}G \xrightarrow{\sim} \Lambda_{(p)}$ to their unit groups.

In order to use Lemma 11 we choose $d \geqslant 1$ such that $p^d \Gamma_{(p)} \subseteq \Lambda_{(p)}$. We write

$$\overline{\Gamma}_{(p)} := \Gamma_{(p)} / p^d \Gamma_{(p)} \,. \tag{3.1}$$

For $x \in \Gamma_{(p)}$, we write

$$[x] := x + p^d \Gamma_{(p)} \in \overline{\Gamma}_{(p)} \,.$$

We have the $\mathbb{Z}_{(p)}$-algebra-morphism

$$\begin{aligned} \varrho : \Gamma_{(p)} &\longrightarrow \overline{\Gamma}_{(p)} \\ x &\longmapsto [x] = x + p^d \Gamma_{(p)} \,. \end{aligned}$$

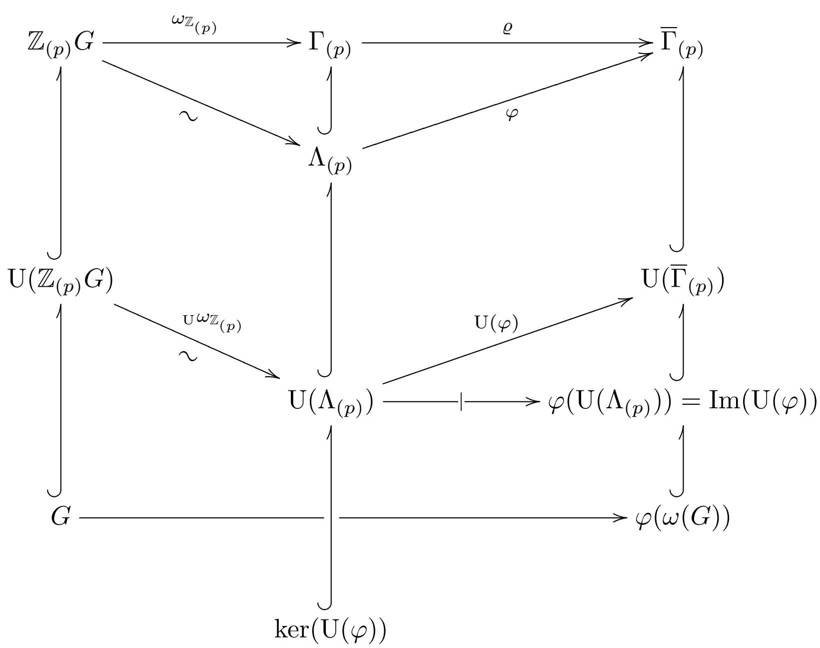
Let

$$\varphi := \varrho|_{\Lambda_{(p)}} : \Lambda_{(p)} \longrightarrow \overline{\Gamma}_{(p)} \,.$$

We have the group morphism

$$\mathrm{U}(\varphi) : \mathrm{U}(\Lambda_{(p)}) \longrightarrow \mathrm{U}(\overline{\Gamma}_{(p)})$$

as a restriction of φ to the unit groups.



From this diagram we have the following diagram of groups.

$$\begin{array}{ccccc} & & \mathrm{U}(\mathbb{Z}_{(p)}G) & & \\ & & {\scriptstyle \mathrm{U}\omega_{\mathbb{Z}_{(p)}}} \downarrow \wr & & \\ \ker(\mathrm{U}(\varphi)) & \hookrightarrow & \mathrm{U}(\Lambda_{(p)}) & \twoheadrightarrow & \mathrm{Im}(\mathrm{U}(\varphi)) \end{array} \tag{3.2}$$

The lower row is a short exact sequence of groups. In particular, $\mathrm{Im}(\mathrm{U}(\varphi))$ is isomorphic to a factor group of $\mathrm{U}(\mathbb{Z}_{(p)}G)$.

Note that

$$\ker(\mathrm{U}(\varphi)) = \{1 + p^d x : x \in \Gamma_{(p)}\} \,.$$

Remark 13. *Suppose given* $x = (x_1, \ldots, x_t) \in \Gamma_{(p)}$ *and* $j \in [1, t]$.

Then $x_j \in R_j^{n_j \times n_j}$, *and so* $\det(x_j) \in R_j$ *for* $j \in [1, t]$.

The following assertions (1–4) *are equivalent.*

(1) *We have* $x \in \mathrm{U}(\Gamma_{(p)})$.

(2) *We have* $\det(x_j) \in \mathrm{U}(R_j)$ *for* $j \in [1, t]$.

(3) *We have* $\mathrm{N}_{K_j|\mathbb{Q}}(\det(x_j)) \in \mathrm{U}(\mathbb{Z}_{(p)})$ *for* $j \in [1, t]$

(4) *We have* $\mathrm{N}_{K_j|\mathbb{Q}}(\det(x_j)) + p\mathbb{Z}_{(p)} \neq 0$ *in* $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ *for* $j \in [1, t]$.

Proof. We have (1) $\Longleftrightarrow$ (2) since a matrix is invertible if and only if it has an invertible determinant.

We have (3) $\Longleftrightarrow$ (4) since an element y of $\mathbb{Z}_{(p)}$ is a unit if and only if it is not contained in $p\mathbb{Z}_{(p)}$.

It remains to show (2) $\overset{!}{\Longleftrightarrow}$ (3).

Suppose given $j \in [1, t]$.

We consider the $\mathbb{Z}_{(p)}$- linear map

$$\begin{aligned} \psi : R_j &\longrightarrow R_j \\ y &\longmapsto \det(x_j) \cdot y\,. \end{aligned}$$

Then $\det(x_j) \in \mathrm{U}(R_j)$ is equivalent to ψ being invertible, which is equivalent to $\det(\psi)$ being invertible in $\mathbb{Z}_{(p)}$.

We have $\mathrm{N}_{K_j|\mathbb{Q}}(\det(x_j)) = \det(\psi)$.

In conclusion we get that $\det(x_j) \in \mathrm{U}(R_j)$ if and only if $\mathrm{N}_{K_j|\mathbb{Q}}(\det(x_j)) \in \mathrm{U}(\mathbb{Z}_{(p)})$. □

Remark 14. We give an alternative argument for assertion (2) in the proof of Lemma 11, in the particular situation $A = \Lambda_{(p)}$ and $B = \Gamma_{(p)}$. Let $x, \tilde{x} \in \Gamma_{(p)}$ such that $x \equiv_{p^d} \tilde{x}$.

We want to show that

$$x \in \mathrm{U}(\Gamma_{(p)}) \overset{!}{\Longrightarrow} \tilde{x} \in \mathrm{U}(\Gamma_{(p)})\,.$$

Suppose that x is a unit in $\Gamma_{(p)}$.

Note that $x_j \equiv_{p^d} \tilde{x}_j$, hence $\det(x_j) \equiv_{p^d} \det(\tilde{x}_j)$, hence $\mathrm{N}_{K_j|\mathbb{Q}}(\det(x_j)) \equiv_{p^d} \mathrm{N}_{K_j|\mathbb{Q}}(\det(\tilde{x}_j))$, for $j \in [1, t]$.

Then we get with Lemma 13 that

$$0 \neq \mathrm{N}_{K_j|\mathbb{Q}}(\det(x_j)) + p\mathbb{Z}_{(p)} = \mathrm{N}_{K_j|\mathbb{Q}}(\det(\tilde{x}_j)) + p\mathbb{Z}_{(p)}$$

for $j \in [1, t]$. Which is equivalent by Lemma 13 to $\tilde{x}$ being a unit in $\Gamma_{(p)}$.

Remark 15. Suppose given $\xi \in \overline{\Gamma}_{(p)}$.

(I) In order to decide whether ξ is contained in $\mathrm{Im}\,\mathrm{U}(\varphi)$, we choose an inverse image $x \in \Gamma_{(p)}$, so $\varrho(x) = \xi$, and test whether $x \in \mathrm{U}(\Lambda_{(p)})$. This is possible by Lemma 11.

(II) To test whether $x \in \mathrm{U}(\Lambda_{(p)})$, we test whether $x \in \Lambda_{(p)}$ and $x \in \mathrm{U}(\Gamma_{(p)})$. This is possible by Lemma 8.

(III) To test whether $x = (x_1, \ldots, x_t) \in \mathrm{U}(\Gamma_{(p)})$, we test whether $\mathrm{N}_{K_j|\mathbb{Q}}(\det(x_j)) + p\mathbb{Z}_{(p)} \neq 0$ in $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ for $j \in [1, t]$. This is possible by Remark 13.

An example in which it happens that $\mathbb{Z}_{(p)} \subset R_j$ is $G = \mathrm{A}_4$, for which we refer to §4.

3.4 Index formula

Let G be a finite group. Let $j \in [1, t]$.

Suppose that we have a Wedderburn isomorphism

$$\omega_{\mathbb{Q}} : \mathbb{Q}G \xrightarrow{\sim} K_1^{n_1 \times n_1} \times \ldots \times K_t^{n_t \times n_t} ,$$

where $K_j|\mathbb{Q}$ is a suitable finite field extension for $j \in [1, t]$.

Let $P_j \subseteq K_j$ be the integral closure of $\mathbb{Z}$ in K_j for $j \in [1, t]$.

Suppose that $\omega_{\mathbb{Q}}$ restricts to

$$\omega_{\mathbb{Z}} : \mathbb{Z}G \xrightarrow{\sim} P_1^{n_1 \times n_1} \times \ldots \times P_t^{n_t \times n_t} =: \Gamma .$$

Let $\Lambda := \omega_{\mathbb{Z}}(\mathbb{Z}G) \subseteq \Gamma$.

We choose $\mathbb{Z}$-linear bases of $\mathbb{Z}G$ and of Γ, respectively.

Let W be he matrix representing $\omega_{\mathbb{Z}}$ with respect to these bases.

Then the index of Λ in Γ is given by

$$|\det(W)| = |\Gamma/\Lambda| ,$$

where Γ/Λ is the abelian factor group.

Write $d_j := [K_j : \mathbb{Q}]$ for $j \in [1, t]$.

Write $\Delta_j := \Delta_{K_j}$ for the discriminant of K_j for $j \in [1, t]$.

Lemma 16. *We have*

$$\begin{aligned} |\det(W)| &= |\Gamma/\Lambda| \\ &= \sqrt{\left| \frac{|G|^{|G|}}{\prod_{j=1}^{t} \Delta_j^{(n_j^2)} n_j^{(n_j^2 d_j)}} \right|} . \end{aligned}$$

Proof. This follows by [**7,** §1.1.2, p. 4, Proposition 1.1.5] □

Remark 17. If $K_j = \mathbb{Q}$ for $j \in [1, t]$, then $\Delta_j = 1$ and $d_j = 1$ for $j \in [1, t]$. So the formula from Lemma 16 simplifies to

$$\begin{aligned} |\det(W)| &= |\Gamma/\Lambda| \\ &= \sqrt{\frac{|G|^{|G|}}{\prod_{j=1}^{t} n_j^{(n_j^2)}}} . \end{aligned}$$

Chapter 4

The group A_4

We intend to illustrate the theory from §3.3 with the example $G = \mathrm{A}_4$. For practical purposes, we start with the Wedderburn isomorphism.

4.1 Theory applied to A_4

We consider the alternating group $G = \mathrm{A}_4$. Let $\zeta := \zeta_3$.

We claim that we have the following $\mathbb{Q}$-algebra isomorphism of Wedderburn.

$$
\begin{aligned}
\omega_{\mathbb{Q}} :\ \mathbb{Q}\mathrm{A}_4 &\xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q}(\zeta) \times \mathbb{Q}^{3\times 3} \\
(1,2)(3,4) &\longmapsto \left(1,\ 1,\ \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 1 \\ -4 & 0 & 1 \end{pmatrix}\right) \\
(1,2,3) &\longmapsto \left(1,\ \zeta,\ \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right)
\end{aligned}
$$

To verify this isomorphism we refer to §A.1.

Let $p = 2$. We shall work over

$$\mathbb{Z}_{(2)} := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \not\equiv_2 0\} \subseteq \mathbb{Q}\,,$$

with $\mathrm{U}(\mathbb{Z}_{(2)}) := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, a \not\equiv_2 0, b \not\equiv_2 0\}$. I.e. we consider the behaviour at the prime 2.

By restriction of $\omega_{\mathbb{Q}}$ we get the injective $\mathbb{Z}_{(2)}$-algebra morphism

$$\omega_{\mathbb{Z}_{(2)}} := \omega_{\mathbb{Q}}\big|_{\mathbb{Z}_{(2)}\mathrm{A}_4}^{\Gamma_{(2)}} : \mathbb{Z}_{(2)}\mathrm{A}_4 \longrightarrow \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}[\zeta] \times \mathbb{Z}_{(2)}^{3\times 3} =: \Gamma_{(2)}\,.$$

Using notation as in §3.3 we get the following left ideals in $\mathbb{Q} \times \mathbb{Q}(\zeta) \times \mathbb{Q}^{3\times 3}$ as representatives of the isoclasses of the simple $\mathbb{Q}\mathrm{A}_4$-modules.

$$S_1 := \mathbb{Q} \times 0 \times 0$$
$$S_2 := 0 \times \mathbb{Q}(\zeta) \times 0$$
$$S_3 := 0 \times 0 \times \begin{pmatrix} \mathbb{Q} & 0 & 0 \\ \mathbb{Q} & 0 & 0 \\ \mathbb{Q} & 0 & 0 \end{pmatrix}$$

An element $g \in \mathrm{A}_4$ acts on an element $s \in S_j$ via $g \cdot s := \omega_{\mathbb{Q}}(g) \cdot s$, where the latter multiplication is to be read in $\mathbb{Q} \times \mathbb{Q}(\zeta) \times \mathbb{Q}^{3\times 3}$.

With that we have the commutative endomorphism algebra $K_j = \mathrm{End}_{\mathbb{Q}\mathrm{A}_4}(S_j)$ for $j \in \{1, 2, 3\}$. In particular, we have the endomorphism algebras

$$K_1 = \mathbb{Q}$$
$$K_2 = \mathbb{Q}(\zeta)$$
$$K_3 = \mathbb{Q}$$

and so $n_1 = \dim_{K_1} S_1 = 1$, $n_2 = \dim_{K_2} S_2 = 1$ and $n_3 = \dim_{K_3} S_3 = 3$.

Since $R_j \subseteq K_j$ is the integral closure of $\mathbb{Z}_{(2)}$ in K_j for $j \in \{1, 2, 3\}$, we obtain

$$R_1 = \mathbb{Z}_{(2)}$$
$$R_2 = \mathbb{Z}_{(2)}[\zeta]$$
$$R_3 = \mathbb{Z}_{(2)} .$$

This confirms that $\Gamma_{(2)} = R_1^{n_1 \times n_1} \times R_2^{n_2 \times n_2} \times R_3^{n_3 \times n_3} = \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}[\zeta] \times \mathbb{Z}_{(2)}^{3\times 3}$

We may choose

$$L_1 := \mathbb{Z}_{(2)} \times 0 \times 0 \qquad \subseteq S_1$$
$$L_2 := 0 \times \mathbb{Z}_{(2)}[\zeta] \times 0 \qquad \subseteq S_2$$
$$L_3 := 0 \times 0 \times \begin{pmatrix} \mathbb{Z}_{(2)} & 0 & 0 \\ \mathbb{Z}_{(2)} & 0 & 0 \\ \mathbb{Z}_{(2)} & 0 & 0 \end{pmatrix} \qquad \subseteq S_3$$

We have $L_j \subseteq S_j$ be an $R_j\mathrm{A}_4$-lattice, i.e. a $R_j\mathrm{A}_4$-submodule with $\mathrm{rk}_{R_j} L_j = \dim_{K_j} S_j =: n_j$ for $j \in [1, t]$.

We have indeed

$$\mathrm{rk}_{R_1} L_1 = \mathrm{rk}_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)} \times 0 \times 0) = 1 = \dim_{\mathbb{Q}}(\mathbb{Q} \times 0 \times 0) = \dim_{K_1} S_1$$
$$\mathrm{rk}_{R_2} L_2 = \mathrm{rk}_{\mathbb{Z}_{(2)}[\zeta]}(0 \times \mathbb{Z}_{(2)}[\zeta] \times 0) = 1 = \dim_{\mathbb{Q}(\zeta)}(0 \times \mathbb{Q}(\zeta) \times 0) = \dim_{K_2} S_2$$
$$\mathrm{rk}_{R_3} L_3 = \mathrm{rk}_{\mathbb{Z}_{(2)}}(0 \times 0 \times \begin{pmatrix} \mathbb{Z}_{(2)} & 0 & 0 \\ \mathbb{Z}_{(2)} & 0 & 0 \\ \mathbb{Z}_{(2)} & 0 & 0 \end{pmatrix}) = 3 = \dim_{\mathbb{Q}}(0 \times 0 \times \begin{pmatrix} \mathbb{Q} & 0 & 0 \\ \mathbb{Q} & 0 & 0 \\ \mathbb{Q} & 0 & 0 \end{pmatrix}) = \dim_{K_3} S_3$$

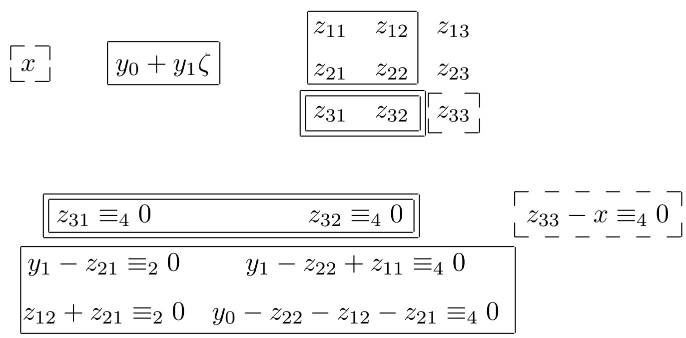
We get the following image of $\omega_{\mathbb{Z}_{(2)}}$.

$$\Lambda_{(2)} := \omega_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)}\mathrm{A}_4) = \left\{ \left(x,\ y_0 + y_1\zeta,\ \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \right) \in \Gamma_{(2)} : \begin{array}{rrr} z_{31} \equiv_4 0 & z_{32} \equiv_4 0 & z_{33} - x \equiv_4 0 \\ y_1 - z_{21} \equiv_2 0 & y_1 - z_{22} + z_{11} \equiv_4 0 & z_{12} + z_{21} \equiv_2 0 \\ y_0 - z_{22} - z_{12} - z_{21} \equiv_4 0 & & \end{array} \right\} \tag{4.1}$$

Here, z_{ij} is short for $z_{i,j}$ etc.

To verify these congruences we refer to §A.2. Here an element in $\mathbb{Z}_{(2)}[\zeta]$ is written in standard form $y_0 + y_1\zeta$ with $y_0, y_1 \in \mathbb{Z}_{(2)}$.

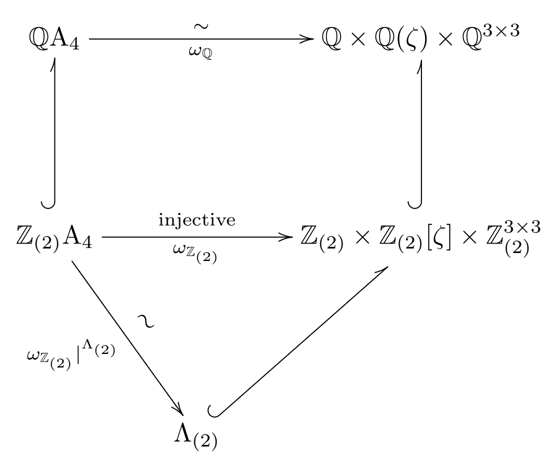
For illustration of the congruences note the following picture.



(4.2)

We have the $\mathbb{Z}_{(2)}$-subalgebra $\Lambda_{(2)} = \omega_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)}\mathrm{A}_4) \subseteq \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}[\zeta] \times \mathbb{Z}_{(2)}^{3\times 3} = \Gamma_{(2)}$.

Altogether, we have the following commutative diagram.



From the congruences in $\Lambda_{(2)}$ one can see that all congruences are satisfied if all matrix entries are divisible by 4. So $2^2 \cdot \Gamma_{(2)} \subseteq \Lambda_{(2)}$. With respect to the notation used in equation (3.1) we may take $d := 2$.

We get $\overline{\Gamma}_{(2)} := \Gamma/2^2\Gamma$.

We have the $\mathbb{Z}_{(2)}$-algebra-morphism

$$\begin{aligned} \varrho : \Gamma_{(2)} &\longrightarrow \overline{\Gamma}_{(2)} \\ x &\longmapsto [x] = x + 2^2\Gamma_{(2)}\ . \end{aligned}$$

Let

$$\varphi := \varrho|_{\Lambda_{(2)}} : \Lambda_{(2)} \longrightarrow \overline{\Gamma}_{(2)} .$$

We have the group morphism

$$\mathrm{U}(\varphi) : \mathrm{U}(\Lambda_{(2)}) \longrightarrow \mathrm{U}(\overline{\Gamma}_{(2)})$$

as a restriction of φ to the unit groups.

We have the group isomorphism

$$_{\mathrm{U}}\omega_{\mathbb{Z}_{(2)}} := \left(\omega_{\mathbb{Z}_{(2)}}|^{\Lambda_{(2)}}\right)|^{\mathrm{U}(\Lambda_{(2)})}_{\mathrm{U}(\mathbb{Z}_{(2)}\mathrm{A}_4)}$$

and thus we have the following diagram of groups.

$$\begin{array}{ccccc} & & \mathrm{U}(\mathbb{Z}_{(2)}\mathrm{A}_4) & & \\ & & {\scriptstyle _{\mathrm{U}}\omega_{\mathbb{Z}_{(2)}}}\downarrow\wr & & \\ \ker(\mathrm{U}(\varphi)) & \hookrightarrow & \mathrm{U}(\Lambda_{(2)}) & \xrightarrow{\mathrm{U}(\varphi)|^{\mathrm{Im}(\mathrm{U}(\varphi))}} & \mathrm{Im}(\mathrm{U}(\varphi)) \end{array} \tag{4.3}$$

The lower row is a short exact sequence of groups. In particular, $\mathrm{Im}(\mathrm{U}(\varphi))$ is a factor group of $\mathrm{U}(\mathbb{Z}_{(2)}\mathrm{A}_4)$.

4.2 Magma application

In the following chapter, we apply Remark 15 to compute $\mathrm{Im}(\mathrm{U}(\varphi))$ by Magma.

In Magma we need some preparations. First need to fix some data for later.

```
Z := Integers();
Z4 := Integers(4); // Z/4Z
Q := Rationals();
Qz<ze> := CyclotomicField(3); // Qz = Q(\zeta_3), ze = zeta_3
Zz := MaximalOrder(Qz); // Zz = Z[zeta_3]

Zz4 := quo<Zz | 4>; // Z[ze]/4Z[ze]
```

We convert an element of $\Lambda_{(2)}$ into a vector with the following convention.

$$\left(x,\ y_0 + y_1\zeta,\ \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}\right) \rightarrowtail (x, y_0, y_1, z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{23}, z_{31}, z_{32}, z_{33})^{\mathrm{t}} \tag{4.4}$$

```
RM := RMatrixSpace(Integers(),12,1);
```

Since we are interested in representatives modulo 4, we choose our coefficients in `M := {-1, 0, 1, 2}`. The data for the vectors with coefficients in `M` are stored in `Coeffs`.

```
// Representatives of Z/(4)
M := {-1,0,1,2};
Coeffs := CartesianPower(M,12);
```

In the next step, we create a list of vectors to be checked.

```
list_of_vectors := [];
for x in Coeffs do
 x_list := [i : i in x];
 y := RM!x_list;
 list_of_vectors cat:= [y];
end for;

//#list_of_vectors; // = 16777216
```

Now we have a list of 16.777.216 vectors with entries in $\mathbb{Z}/4\mathbb{Z}$.

Then we have to decide whether a vector satisfies the list of congruences. Thanks to Remark 15(I), we only need to consider vectors with entries in `M`.

By Remark 15(II), we can check separately that, first, the element is in $\Lambda_{(2)}$ and that, second, the element is invertible in $\mathrm{U}(\Gamma_{(2)})$. Since we store an element as a vector, we have to formulate the congruences accordingly; cf. (4.1), (4.4).

```
// check if vector in \Lambda_(2)
list_improvement1 := [];
for y in list_of_vectors do
 if (y[10,1]) mod 4 eq 0
  and (y[11,1]) mod 4 eq 0
  and (y[12,1] - y[1,1]) mod 4 eq 0
  and (y[3,1] - y[8,1] + y[4,1]) mod 4 eq 0
  and (y[3,1] - y[7,1]) mod 2 eq 0
  and (y[2,1] - y[8,1] - y[5,1] - y[7,1]) mod 4 eq 0
  and (y[5,1] + y[7,1]) mod 2 eq 0
 then
  list_improvement1 := list_improvement1 cat [y];
 end if;
end for;

// #list_improvement1; // = 4096
```

Now we have a list of only 4096 elements left. What is left to check is whether these vectors are in the unit group of $\Gamma_{(2)}$. To do so we use Remark 15(III) and check whether the determinant or norm of our element is not congruent to zero modulo 2.

Note that for $y_0 + y_1\zeta \in \mathbb{Z}_{(2)}[\zeta]$, $y_0, y_1 \in \mathbb{Z}_{(2)}$, we have $\mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}(y_0 + y_1\zeta) = y_0^2 - y_0y_1 + y_1^2$.

```
// check if vectors in U(\Gamma_(2)) via determinant und norm
list_final := [];
for y in list_improvement1 do
 if   not (y[1,1]                                ) mod 2 eq 0  // det of 1x1 block
  and not (y[2,1]^2 - y[2,1] * y[3,1] + y[3,1]^2) mod 2 eq 0  // Norm of
                                                              // 2x2 - zeta - block
  and not (Z!Determinant(Matrix([[y[4,1],y[5,1],y[6,1]],[y[7,1],y[8,1],y[9,1]],
    [y[10,1],y[11,1],y[12,1]]]))                 ) mod 2 eq 0  // det of 3x3 block
 then
  list_final := list_final cat [y];
 end if;
end for;
```

We have narrowed the list of vectors down to `list_final`, such that `#list_final` $= 1536 = 2^9 \cdot 3^1$.

In order to be able to calculate in the group we convert the vectors back into matrices.

We need some preparations.

```
UGq := GL(6,Z4);
```

We can now form $\mathrm{Im}(\mathrm{U}(\varphi))$ by converting the elements of `list_final` back into matrices and then into elements of $\mathrm{GL}_1(\mathbb{Z}/(4)) \times \mathrm{GL}_2(\mathbb{Z}/(4)) \times \mathrm{GL}_3(\mathbb{Z}/(4)) \leqslant \mathrm{GL}_6(\mathbb{Z}/(4)) =$ `UGq` where they form a subgroup.

Note that the 2×2-matrix in the middle represents the multiplication by $u + v\zeta$. I.e. we make use of the injective ring morphism

$$\begin{aligned} \mathbb{Z}_{(2)}[\zeta] &\longrightarrow \mathbb{Z}_{(2)}^{2\times 2} \\ \zeta &\longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} . \end{aligned}$$

```
list_group_elements := [];
for x in list_final do
 y := UGq!DiagonalJoin(<Matrix([[x[1,1]]]),
 Matrix([[x[2,1],-x[3,1]],[x[3,1],x[2,1]-x[3,1]]]), // represents multipl. u + v zeta
 Matrix([[x[4,1],x[5,1],x[6,1]],[x[7,1],x[8,1],x[9,1]],[x[10,1],x[11,1],x[12,1]]])>);
 list_group_elements cat:= [y];
end for;

U_group := sub<UGq | list_group_elements>;

// Order(U_group); // = 1536

ImUphi := U_group;
```

In this case we were able to search the `list_of_vectors` containing all candidates. Since the number of such elements is sufficiently small, we can afford to proceed this way. In later chapters, we will make use of a refined procedure in several steps instead.

4.3 Analysing $\mathrm{Im}(\mathrm{U}(\varphi))$ via Magma

Now we can examine this group $\mathrm{Im}(\mathrm{U}(\varphi))$ more closely. Using Magma, we calculate the derived series, a chief series and a list of generators and a corresponding list of relations.

4.3.1 The derived series of $\mathrm{Im}(\mathrm{U}(\varphi))$

We want to calculate the derived series of $\mathrm{Im}(\mathrm{U}(\varphi))$; cf. §1.2.2.

We can first have Magma perform a few simple tests.

```
Order(ImUphi);       //1536
IsAbelian(ImUphi);  // false
IsSolvable(ImUphi); // true
GroupName(ImUphi);  //C4^2:A4.C2^3
```

Note that the group name alone does not determine the group up to isomorphism.

Since Im(U(φ)) is solvable, the derived series of Im(U(φ)), as calculated by Magma, ends in the trivial group:

```
DS := DerivedSeries(ImUphi);
#DS; // 4

GroupName(quo< DS[1] | DS[2] >); // C2^2*C6
GroupName(quo< DS[2] | DS[3] >); // C2^4
GroupName(quo< DS[3] | DS[4] >); // C2^2
```

We write $D_1 :=$ `DS[1]`, $D_2 :=$ `DS[2]`, $D_3 :=$ `DS[3]`, $D_4 :=$ `DS[4]`.

This means that we have the derived series

$$\mathrm{Im}(\mathrm{U}(\varphi)) = D_1 \geqslant D_2 \geqslant D_3 \geqslant D_4 = 1$$

with

$$\begin{aligned} D_1/D_2 &\simeq \mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_6 \\ D_2/D_3 &\simeq \mathrm{C}_2^{\times 4} \\ D_3/D_4 &\simeq \mathrm{C}_2 \times \mathrm{C}_2 \end{aligned}$$

Magma also gives group names:

```
GroupName(DS[4]); // C1
GroupName(DS[3]); // C2^2
GroupName(DS[2]); // C2^2wrC2:C2
GroupName(DS[1]); // C4^2:A4.C2^3
```

4.3.2 A chief series of Im(U(φ))

We want to calculate a chief series of Im(U(φ)); cf. §1.2.3.

```
CS := ChiefSeries(ImUphi);
#CS;  // 8

GroupName(quo< CS[1] | CS[2] >);  // C2
GroupName(quo< CS[2] | CS[3] >);  // C2
GroupName(quo< CS[3] | CS[4] >);  // C2
GroupName(quo< CS[4] | CS[5] >);  // C3
GroupName(quo< CS[5] | CS[6] >);  // C2^2
GroupName(quo< CS[6] | CS[7] >);  // C2^2
GroupName(quo< CS[7] | CS[8] >);  // C2^2
```

We write $H_1 :=$ `CS[1]`, $H_2 :=$ `CS[2]`, ..., $H_8 :=$ `CS[8]`.

This means that we have the chief series

$$\mathrm{Im}(\mathrm{U}(\varphi)) = H_1 \geqslant H_2 \geqslant H_3 \geqslant \ldots \geqslant H_8 = 1$$

with

$$\begin{array}{lll} H_1/H_2 \simeq \mathrm{C}_2, & H_2/H_3 \simeq \mathrm{C}_2, & H_3/H_4 \simeq \mathrm{C}_2 \\ H_4/H_5 \simeq \mathrm{C}_3, & H_5/H_6 \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_6/H_7 \simeq \mathrm{C}_2 \times \mathrm{C}_2 \\ H_7/H_8 \simeq \mathrm{C}_2 \times \mathrm{C}_2 & & \end{array}$$

4.3.3 Chosen generators for $\mathrm{Im}(\mathrm{U}(\varphi))$

We now want to find a short list of generators of $\mathrm{Im}(\mathrm{U}(\varphi))$ containing group elements having many zero entries in their matrices. In order to bring Magma to suggest such elements, we first sort the list of elements of $\mathrm{Im}(\mathrm{U}(\varphi))$ according to the numbers of zero they contain.

First we give each vector a number to be able to identify them later.

```
index_set := [1 .. 1536];
run_set := [4 .. 12];
yay_Liste := [];
for o in index_set do
yay := 0;
for u in run_set do
if list_final[o,u] eq 0 then;
yay +:= 1;
end if;
end for;
yay_Liste cat:= [<o,yay>];
end for;

#yay_Liste;  // 1536
```

Now we use the sort function of Magma to sort by the number of zeros.

```
C := func<x,y | - x[2] + y[2] >;
Y := yay_Liste;
S := Sort(Y, C);
list_group_elements_new_sorting := [list_group_elements[x[1]] : x in S];
```

We now manually produce a list of generators and verify that it actually generate the group.

```
m_e := list_group_elements_new_sorting[8]; // minus one
list_gen := [list_group_elements_new_sorting[8],
             list_group_elements_new_sorting[11]^-1,
             list_group_elements_new_sorting[1],
             m_e * list_group_elements_new_sorting[2],
             m_e * list_group_elements_new_sorting[10]];
list_gen_num := [<list_group_elements_new_sorting[8],8>,
                 <list_group_elements_new_sorting[11]^-1,11>,
                 <list_group_elements_new_sorting[1],1>,
                 <m_e * list_group_elements_new_sorting[2],2>,
                 <m_e * list_group_elements_new_sorting[10],10>];


U_sub_test := sub<U_group|list_gen>; // with -1
for g in list_group_elements_new_sorting do
 if not g in U_sub_test then
  list_gen cat:= [g];
  list_gen_num cat:= [ <g, Index(list_group_elements_new_sorting, g)> ];
  U_sub_test := sub<U_group | list_gen >;
  print "spy:", #list_gen;
 end if;
end for;
```

```
print Order(sub<U_group | list_gen >); // 1536
# list_gen; // 6
print list_gen; // list of generators
print list_gen_num;

list_gen := Remove(list_gen,5);
list_gen := Remove(list_gen,3);

// Redundancy test:
for i in [1..#list_gen] do
 print Order(sub<U_group | Remove(list_gen,i) >);
end for;
// Results in all being important

#list_gen; // 4
/* list of generators
print list_gen;
[
    [3 0 0 0 0 0]
    [0 3 0 0 0 0]
    [0 0 3 0 0 0]
    [0 0 0 3 0 0]
    [0 0 0 0 3 0]
    [0 0 0 0 0 3],

    [1 0 0 0 0 0]
    [0 1 0 0 0 0]
    [0 0 1 0 0 0]
    [0 0 0 1 0 0]
    [0 0 0 0 1 1]
    [0 0 0 0 0 1],

    [1 0 0 0 0 0]
    [0 1 2 0 0 0]
    [0 2 3 0 0 0]
    [0 0 0 3 0 0]
    [0 0 0 0 1 0]
    [0 0 0 0 0 1],

    [3 0 0 0 0 0]
    [0 2 1 0 0 0]
    [0 3 3 0 0 0]
    [0 0 0 1 1 0]
    [0 0 0 1 0 0]
    [0 0 0 0 0 3]
]
*/
```

So we get the following tuple list of generators of $\mathrm{Im}(\mathrm{U}(\varphi))$.

$$\left(\left[-1,\ -1,\ \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right],\ \left[1,\ 1,\ \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\right],\ \left[1,\ 1+2\zeta,\ \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right],\ \left[-1,\ 2-\zeta,\ \begin{pmatrix}1&1&0\\1&0&0\\0&0&-1\end{pmatrix}\right]\right)$$

4.3.4 Relations for $\mathrm{Im}(\mathrm{U}(\varphi))$

We let Magma calculate a list of relations.

```
U_group_gen_fp := FPGroup(sub<U_group | list_gen >);
U_group_gen_fp;


/* gives:
U_group_gen_fp.1^2 = Id(U_group_gen_fp)
U_group_gen_fp.2^4 = Id(U_group_gen_fp)
U_group_gen_fp.3^2 = Id(U_group_gen_fp)
U_group_gen_fp.4^6 = Id(U_group_gen_fp)
U_group_gen_fp.2^-1 * U_group_gen_fp.1 * U_group_gen_fp.2 * U_group_gen_fp.1
                    = Id(U_group_gen_fp)
U_group_gen_fp.2^-1 * U_group_gen_fp.3 * U_group_gen_fp.2 * U_group_gen_fp.3
                    = Id(U_group_gen_fp)
(U_group_gen_fp.1 * U_group_gen_fp.3)^2 = Id(U_group_gen_fp)
U_group_gen_fp.1 * U_group_gen_fp.4^-1 * U_group_gen_fp.1 * U_group_gen_fp.4
                    = Id(U_group_gen_fp)
U_group_gen_fp.4 * U_group_gen_fp.2^-1 * U_group_gen_fp.4^-1 * U_group_gen_fp.2^-1
                     * U_group_gen_fp.4^-1 * U_group_gen_fp.2 * U_group_gen_fp.4
                    = Id(U_group_gen_fp)
U_group_gen_fp.4^-1 * U_group_gen_fp.2 * U_group_gen_fp.4^-1 * U_group_gen_fp.2^-1
                     * U_group_gen_fp.4^2 * U_group_gen_fp.2 = Id(U_group_gen_fp)
U_group_gen_fp.4^-3 * U_group_gen_fp.3 * U_group_gen_fp.4^3 * U_group_gen_fp.3
                    = Id(U_group_gen_fp)
(U_group_gen_fp.3 * U_group_gen_fp.4 * U_group_gen_fp.2^-1 * U_group_gen_fp.4^-1)^2
                    = Id(U_group_gen_fp)
(U_group_gen_fp.3 * U_group_gen_fp.4 * U_group_gen_fp.3 * U_group_gen_fp.4^-1)^2
                    = Id(U_group_gen_fp)
*/
```

For better clarity we rewrite some of the relations manually.

First we define

$$\begin{aligned} m &:= \texttt{U\_group\_gen\_fp.1} \\ d &:= \texttt{U\_group\_gen\_fp.2} \\ e &:= \texttt{U\_group\_gen\_fp.3} \\ k &:= \texttt{U\_group\_gen\_fp.4}\,. \end{aligned}$$

Note that for a relation with right hand side 1 we we omit the "$= 1$". So we get, by rewriting the first four relations,

$$m^2, d^4, e^2, k^6\,.$$

We want to rewrite the relations

$$\begin{aligned} k \cdot d^{-1} \cdot k^{-1} \cdot d^{-1} \cdot k^{-1} \cdot d \cdot k \\ k^{-1} \cdot d \cdot k^{-1} \cdot d^{-1} \cdot k^2 \cdot d\,. \end{aligned}$$

Of the first relation we get

$$k \cdot d^{-1} \cdot k^{-1} \cdot d^{-1} \cdot k^{-1} \cdot d \cdot k = k \cdot d^{-1} \cdot k^{-1} \cdot [d, k]$$

So we get

$$[d,k] = k \cdot d \cdot k^{-1} = {}^{k}d \,.$$

We can write

$$k^{-1} \cdot d \cdot k^{-1} \cdot d^{-1} \cdot k^2 \cdot d$$

as

$$d \cdot k^{-1} \cdot d^{-1} \cdot k^2 \cdot d \cdot k^{-1} \,.$$

By multiplication of

$$k \cdot d^{-1} \cdot k^{-1} \cdot d^{-1} \cdot k^{-1} \cdot d \cdot k$$

and

$$d \cdot k^{-1} \cdot d^{-1} \cdot k^2 \cdot d \cdot k^{-1}$$

we get

$$\begin{aligned} &d \cdot k^{-1} \cdot d^{-1} \cdot k^2 \cdot d \cdot k^{-1} \cdot k \cdot d^{-1} \cdot k^{-1} \cdot d^{-1} \cdot k^{-1} \cdot d \cdot k \\ = &\, d \cdot k^{-1} \cdot d^{-1} \cdot k \cdot d^{-1} \cdot k^{-1} \cdot d \cdot k \,. \end{aligned}$$

By rewriting we get

$$1 = [d^{-1}, k] \cdot [d,k]$$

and so we get

$$[d,k] = [k, d^{-1}] = k^{-1} \cdot d \cdot k \cdot d^{-1} = {}^{d}(d^{-1} \cdot k^{-1} \cdot d \cdot k) = {}^{d}[d,k] \,.$$

So d commutes with $[d,k]$.
We get

$$[[d,k],d] = 1$$

so we have, using $[d,k] = {}^{k}d$,

$$[{}^{k}d, d] = 1 \,.$$

By rewriting the remaining all relations as well we get

$$\begin{aligned} \mathrm{U_{fp}} := \langle m,d,e,k \mid\ & m^2, d^4, e^2, k^6, \\ & [m,d], [d,e], [m,e], [m,k], \\ & [d,k] = {}^{k}d, [{}^{k}d, d], [e,k^3], \\ & ({}^{k}d \cdot e)^2, [e,k]^2 \rangle \,. \end{aligned}$$

Then we have the group isomorphism

$$\begin{aligned}
\psi:\ \mathrm{U_{fp}} &\xrightarrow{\sim} \mathrm{Im}(\mathrm{U}(\varphi)) \\
m &\longmapsto \left[-1,\ -1,\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right] \\
d &\longmapsto \left[1,\ 1,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right] \\
e &\longmapsto \left[1,\ 1+2\zeta,\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \\
k &\longmapsto \left[-1,\ 2-\zeta,\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right]
\end{aligned}$$

We verify this assertion using Magma:

```
FG<m,d,e,k> := FreeGroup(4);

list_rel_transformed :=
[m^2 = Id(FG),
d^4 = Id(FG),
e^2 = Id(FG),
k^6 = Id(FG),
(m, d) = Id(FG),
(d, e) = Id(FG),
(m, e) = Id(FG),
(m, k) = Id(FG),
(d, k) = k * d * k^-1,
(k * d * k^-1, d) = Id(FG),
(e, k^3) = Id(FG),
(k * d * k^-1 * e)^2 = Id(FG),
(e, k)^2 = Id(FG)];

FG_factor_u, rho := quo<FG | list_rel_transformed>;
Order(FG_factor_u); //1536

IsIsomorphic(PermutationGroup(FG_factor_u),PermutationGroup(FPGroup(ImUphi))); // true

psi := hom<FG_factor_u -> ImUphi | [<FG_factor_u!m,list_gen[1]>,
                                    <FG_factor_u!d,list_gen[2]>,
                                    <FG_factor_u!e,list_gen[3]>,
                                    <FG_factor_u!k,list_gen[4]>]>;
Order(Kernel(psi)); // 1
Order(Image(psi)); // 1536  So psi is a isomorphism

// We independently test the list of relations holds in ImUphi:
psi_hat := hom<FG -> ImUphi | [<m,list_gen[1]>,<d,list_gen[2]>,
                               <e,list_gen[3]>,<k,list_gen[4]>]>;
```

```
for x in list_rel_transformed do
 if x[1]@psi_hat eq x[2]@psi_hat then
  print "ok";
 else
  print "not ok", x;
 end if;
end for;
```

4.4 Description of $\ker(\mathrm{U}(\varphi))$

Consider the commutative triangle of groups.

$$\begin{array}{ccc} \mathrm{U}(\Gamma_{(2)}) & \xrightarrow{\mathrm{U}(\varrho)} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ \Big\uparrow & \nearrow_{\mathrm{U}(\varphi)} & \\ \mathrm{U}(\Lambda_{(2)}) & & \end{array}$$

Since

$$\ker(\mathrm{U}(\varrho)) \leqslant \mathrm{U}(\Lambda_{(2)}) \ ,$$

we have

$$\ker(\mathrm{U}(\varrho)) = \ker(\mathrm{U}(\varrho)) \cap \mathrm{U}(\Lambda_{(2)}) = \ker(\mathrm{U}(\varphi)) \ .$$

So we have the following commutative diagram.

$$\begin{array}{ccccc} \ker(\mathrm{U}(\varrho)) & \longrightarrow & \mathrm{U}(\Gamma_{(2)}) & \xrightarrow{\mathrm{U}(\varrho)} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ \Big\| & & \Big\uparrow & \nearrow_{\mathrm{U}(\varphi)} & \\ \ker(\mathrm{U}(\varphi)) & \longrightarrow & \mathrm{U}(\Lambda_{(2)}) & & \end{array}$$

With the diagram 4.3 we get a description of the kernel.

$$\ker(\mathrm{U}(\varphi)) = \{1 + 4\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\}$$

4.5 Summary

We have the following diagram of groups, in which the lower row is a short exact sequence.

$$\begin{array}{ccccc} & & \mathrm{U}(\mathbb{Z}_{(2)}\mathrm{A}_4) & & \\ & & {\scriptstyle \mathrm{U}\,\omega_{\mathbb{Z}_{(2)}}} \Big\downarrow \wr & & \\ \ker(\mathrm{U}(\varphi)) & \hookrightarrow & \mathrm{U}(\Lambda_{(2)}) & \twoheadrightarrow & \mathrm{Im}(\mathrm{U}(\varphi)) \end{array}$$

The finite group $\mathrm{Im}(\mathrm{U}(\varphi))$ has order 1536. It is build in §4.2 via Magma. Its derived series, a chief series, chosen generators and a list of relations are given in §4.3.

The infinite group

$$\begin{aligned}\ker(\mathrm{U}(\varphi)) =& \ \{1+4\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\}\\ &= (1+4\mathbb{Z}_{(2)}) \times (1+4\mathbb{Z}_{(2)}[\zeta_3]) \times \begin{pmatrix} (1+4\mathbb{Z}_{(2)}) & 4\mathbb{Z}_{(2)} & 4\mathbb{Z}_{(2)} \\ 4\mathbb{Z}_{(2)} & (1+4\mathbb{Z}_{(2)}) & 4\mathbb{Z}_{(2)} \\ 4\mathbb{Z}_{(2)} & 4\mathbb{Z}_{(2)} & (1+4\mathbb{Z}_{(2)}) \end{pmatrix}\end{aligned}$$

is described in §4.4.

Chapter 5

The group S_4

We consider the symmetric group $G = \mathrm{S}_4$.

5.1 A factor group of $\mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_4)$

5.1.1 Construction of the group morphism $\mathrm{U}(\varphi)$

We claim that we have the following $\mathbb{Q}$-algebra isomorphism of Wedderburn.

$$\begin{aligned}
\omega_{\mathbb{Q}} :\ \mathbb{Q}\mathrm{S}_4 &\longrightarrow \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{2\times 2} \\
(1,2) &\longmapsto \left(-1,\ 1,\ \begin{pmatrix} -11 & -24 & 2 \\ 5 & 11 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -5 & 24 \\ -1 & 5 \end{pmatrix} \right) \\
(1,2,3,4) &\longmapsto \left(-1,\ 1,\ \begin{pmatrix} 26 & 57 & 2 \\ -11 & -24 & -1 \\ -4 & -8 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & -15 \\ 1 & -4 \end{pmatrix} \right)
\end{aligned}$$

To verify this isomorphism we refer to §B.1.

In order to consider the behaviour at the prime 2 separately, we localise at (2), i.e. now we pass from the ground ring $\mathbb{Q}$ to the ground ring

$$\mathbb{Z}_{(2)} := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \not\equiv_2 0\} \subseteq \mathbb{Q}\,,$$

with $\mathrm{U}(\mathbb{Z}_{(2)}) := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, a \not\equiv_2 0, b \not\equiv_2 0\}$.

By restriction, we get the injective map

$$\omega_{\mathbb{Z}_{(2)}} : \mathbb{Z}_{(2)}\mathrm{S}_4 \longrightarrow \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)}^{2\times 2} =: \Gamma_{(2)}$$

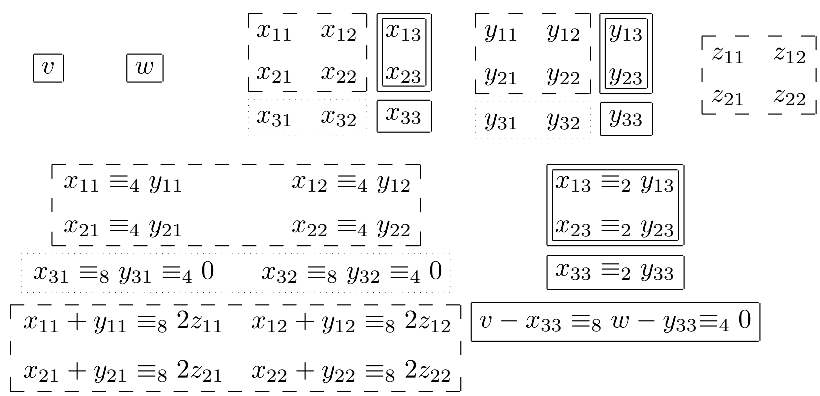
So we get the following.

$$\Lambda_{(2)} := \omega_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)}\mathrm{S}_4) \;=\; \left\{ \left(v,\ w,\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right) \in \Gamma_{(2)} : \begin{array}{rlrlr} x_{11} & \equiv_4 y_{11}, & x_{12} & \equiv_4 y_{12}, & x_{13} \equiv_2 y_{13}, \\ x_{21} & \equiv_4 y_{21}, & x_{22} & \equiv_4 y_{22}, & x_{23} \equiv_2 y_{23}, \\ x_{31} & \equiv_8 y_{31} \equiv_4 0, & x_{32} & \equiv_8 y_{32} \equiv_4 0, & x_{33} \equiv_2 y_{33}, \\ x_{11}+y_{11} & \equiv_8 2z_{11}, & x_{12}+y_{12} & \equiv_8 2z_{12}, & \\ x_{21}+y_{21} & \equiv_8 2z_{21}, & x_{22}+y_{22} & \equiv_8 2z_{22}, & \\ v - x_{33} & \equiv_8 w - y_{33} \equiv_4 0 & & & \end{array} \right\} \tag{5.1}$$

Here, x_{ij} is short for $x_{i,j}$ etc.

To verify these congruences we refer to §B.2.

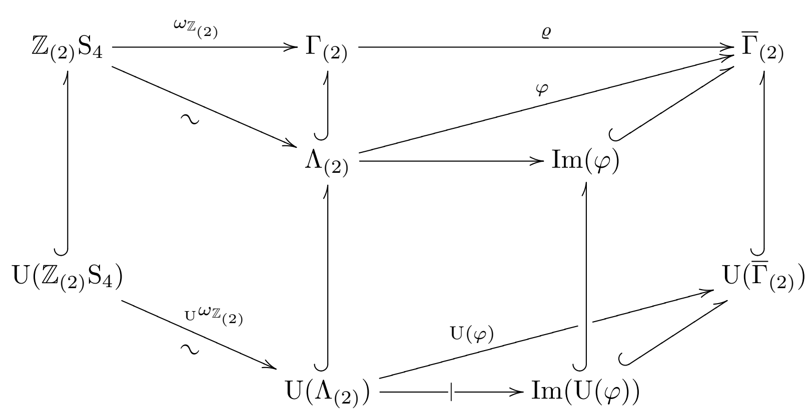
For illustration of the congruences note the following picture.



(5.2)

Now $2^3\Gamma_{(2)} \subseteq \Lambda_{(2)}$ and the exponent 3 is minimal with respect to this property. So we consider $\overline{\Gamma}_{(2)} := \Gamma_{(2)}/2^3\Gamma_{(2)} = \Gamma_{(2)}/8\Gamma_{(2)}$. Let $\varrho : \Gamma_{(2)} \longrightarrow \overline{\Gamma}_{(2)}$ be the residue class map.

We obtain the following commutative diagram.



Note that we have the following diagram of groups.

$$\begin{array}{ccccc} & & \mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_4) & & \\ & & \downarrow {\scriptstyle \mathrm{U}\omega_{\mathbb{Z}_{(2)}}} \wr & & \\ \ker(\mathrm{U}(\varphi)) & \hookrightarrow & \mathrm{U}(\Lambda_{(2)}) & \twoheadrightarrow & \mathrm{Im}(\mathrm{U}(\varphi)) \end{array} \tag{5.3}$$

The lower row is a short exact sequence of groups. In particular, $\mathrm{Im}(\mathrm{U}(\varphi))$ is a factor group of $\mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_4)$.

To calculate $\mathrm{Im}(\mathrm{U}(\varphi))$ with Magma we do not have the possibility to verify for each of the 8^{24} elements of $\overline{\Gamma}_{(2)}$ whether it is is contained in $\mathrm{Im}(\mathrm{U}(\varphi))$ in a reasonable amount of time.

Reminder 18. We recall the following notation. For

$$\left(v,\ w,\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},\ \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix},\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right) = \gamma \in \Gamma_{(2)}$$

we write

$$\gamma + p^d\Gamma_{(p)} = [\gamma] := \left[v,\ w,\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},\ \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix},\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right] \in \overline{\Gamma}_{(2)}\ .$$

Remark 19. *Let $\xi \in \mathrm{Im}(\mathrm{U}(\varphi))$. Let $\gamma \in \Gamma_{(2)}$ such that $[\gamma] = \xi$. Then $\gamma \in \mathrm{U}(\Lambda_{(2)})$.*

Proof. We may choose $\lambda \in \mathrm{U}(\Lambda_{(2)})$ with $\xi = [\lambda]$. Then $[\gamma] = \xi = [\lambda]$. By Lemma 11, we conclude from $\lambda \in \mathrm{U}(\Lambda_{(2)})$ that $\gamma \in \mathrm{U}(\Lambda_{(2)})$. □

5.1.2 Dissecting $\mathrm{Im}(\mathrm{U}(\varphi))$

We will dissect the problem in two steps to verify which elements are in $\mathrm{Im}(\mathrm{U}(\varphi))$.

Step 1. To begin we define the subset

$$\begin{aligned} U_1 := \Bigg\{ & \left[1,\ 1,\ \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} y_{11} & y_{12} & 0 \\ y_{21} & y_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \\ & \det \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \not\equiv_2 0 \text{ and } x_{11} \equiv_4 y_{11},\ x_{12} \equiv_4 y_{12},\ x_{21} \equiv_4 y_{21},\ x_{22} \equiv_4 y_{22}, \\ & x_{11} + y_{11} \equiv_8 2z_{11},\ x_{12} + y_{12} \equiv_8 2z_{12},\ x_{21} + y_{21} \equiv_8 2z_{21},\ x_{22} + y_{22} \equiv_8 2z_{22} \Bigg\}\ . \end{aligned} \tag{5.4}$$

Recalling the picture of (5.2), the remaining congruences are shown in the following figure.

$$
\begin{array}{ccccc}
1 & 1 & \begin{array}{ccc} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{array} & \begin{array}{ccc} y_{11} & y_{12} & 0 \\ y_{21} & y_{22} & 0 \\ 0 & 0 & 1 \end{array} & \begin{array}{cc} z_{11} & z_{12} \\ z_{21} & z_{22} \end{array}
\end{array}
$$

$$
\begin{array}{cc} x_{11} \equiv_4 y_{11} & x_{12} \equiv_4 y_{12} \\ x_{21} \equiv_4 y_{21} & x_{22} \equiv_4 y_{22} \end{array}
$$

$$
\begin{array}{cc} x_{11} + y_{11} \equiv_8 2z_{11} & x_{12} + y_{12} \equiv_8 2z_{12} \\ x_{21} + y_{21} \equiv_8 2z_{21} & x_{22} + y_{22} \equiv_8 2z_{22} \end{array}
$$

We *claim* that $U_1 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$.

By construction we have $U_1 \subseteq \mathrm{Im}(\varphi)$: Every element of U_1 is represented by an element satisfying the congruences describing $\Lambda_{(2)}$, cf. (5.1).

To show that $U_1 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$, we have to verify that each element of U_1 is represented by an element of $\mathrm{U}(\Lambda_{(2)})$.

We have $\mathrm{U}(\Lambda_{(2)}) = \Lambda_{(2)} \cap \mathrm{U}(\Gamma_{(2)})$, cf. Lemma 8.

So we have to show that each element of U_1 is represented by an element in $\Lambda_{(2)}$ that is an invertible element of $\Gamma_{(2)}$.

Suppose given $\left[1,\ 1,\ \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} y_{11} & y_{12} & 0 \\ y_{21} & y_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\right] \in U_1$, represented by the element in brackets; cf. (5.4). We show that

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \equiv_2 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \equiv_2 \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

Suppose given $i, j \in [1, 2]$. Due to the congruences of U_1, we have

$$-2z_{ij} + x_{ij} + y_{ij} \equiv_8 0,\ x_{ij} - y_{ij} \equiv_4 0\,.$$

It follows

$$-2z_{ij} + x_{ij} + y_{ij} \equiv_4 0,\ x_{ij} - y_{ij} \equiv_4 0\,. \tag{5.5}$$

Addition of this congruences yields

$$-2z_{ij} + 2x_{ij} \equiv_4 0$$

and so we have

$$z_{ij} \equiv_2 x_{ij}\,. \tag{5.6}$$

By (5.5), we also have

$$x_{ij} \equiv_2 y_{ij}\,. \tag{5.7}$$

With (5.6) and (5.7) for $i, j \in [1, 2]$, we have

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \equiv_2 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \equiv_2 \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\,.$$

In particular

$$0 \not\equiv_2 \det \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \equiv_2 \det \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv_2 \det \begin{pmatrix} y_{11} & y_{12} & 0 \\ y_{21} & y_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So all these matrices have determinant in $\mathrm{U}(\mathbb{Z}_{(2)})$, hence they are invertible over $\mathbb{Z}_{(2)}$.

This proves the *claim*.

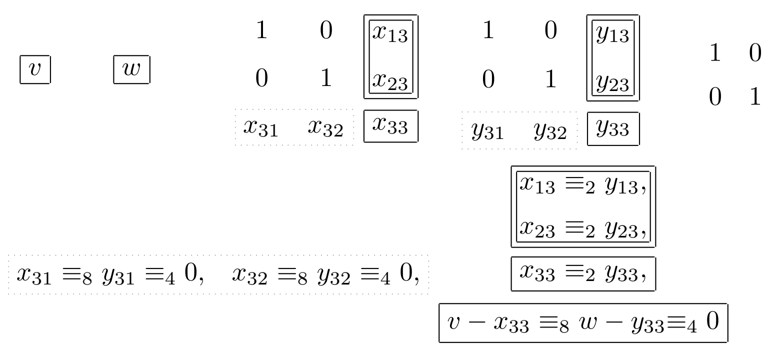
Note that $U_1 \leqslant \mathrm{Im}(\mathrm{U}(\varphi))$. In fact, given elements of U_1 represented by $u_1, u_1' \in \mathrm{U}(\Lambda_{(2)})$, we have $u_1 \cdot u_1' \in \mathrm{U}(\Lambda_{(2)})$, whence $u_1 \cdot u_1'$ satisfies the required congruences. In addition, $u_1 \cdot u_1'$ has the block structure required to represent an element of U_1. So

$$\overline{u_1} \cdot \overline{u_1'} = (u_1 \cdot u_1')\varphi \in U_1$$

Now we want to use U_1 as a factor of $\mathrm{Im}(\mathrm{U}(\varphi))$. We define the subset

$$M_1 := \left\{ \left[v,\ w,\ \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \begin{array}{l} v \not\equiv_2 0 \text{ and } x_{13} \equiv_2 y_{13},\ x_{23} \equiv_2 y_{23},\ x_{31} \equiv_8 y_{31} \equiv_4 0, x_{32} \equiv_8 y_{32} \equiv_4 0, \\ x_{33} \equiv_2 y_{33},\ v - x_{33} \equiv_8 w - y_{33} \equiv_4 0 \end{array} \right\}.$$

Recalling the picture of (5.2), the remaining congruences are shown in the following figure.



We *claim* that $M_1 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$.

By construction we have $M_1 \subseteq \mathrm{Im}(\varphi)$: Every element of M_1 is represented by an element satisfying the congruences describing $\Lambda_{(2)}$, cf. (5.1).

To show that $M_1 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$. We have to verify that each element of M_1 is represented by an element of $\mathrm{U}(\Lambda_{(2)})$.

We have $\mathrm{U}(\Lambda_{(2)}) = \Lambda_{(2)} \cap \mathrm{U}(\Gamma_{(2)})$, cf. Lemma 8.

So we have to show that each element of M_1 is represented by an invertible element of $\Gamma_{(2)}$.

Suppose given $\left[v,\ w,\ \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in M_1$, represented by the element in brackets; cf. (5.4).

Due to the congruences of M_1, we have

$$0 \not\equiv_2 v \equiv_2 x_{33} \equiv_2 y_{33} \equiv_2 w \, .$$

Moreover

$$x_{31} \equiv_2 y_{31} \equiv_2 0 \, , \; x_{32} \equiv_2 y_{32} \equiv_2 0 \, .$$

Hence

$$\begin{aligned} 0 \not\equiv_2 v \equiv_2 w \equiv_2 \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} &\equiv_2 \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \\ \equiv_2 \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & y_{33} \end{pmatrix} &\equiv_2 \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \end{aligned}$$

So all these matrices have determinant in $\mathrm{U}(\mathbb{Z}_{(2)})$, hence they are invertible over $\mathbb{Z}_{(2)}$.

This proves the *claim*.

Further we *claim* that for $u \in \mathrm{Im}(\mathrm{U}(\varphi))$ there exists $u_1 \in U_1$ such that $u_1^{-1} \cdot u \in M_1$.

Suppose given

$$u = \left[v, \; w, \; \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right] \in \mathrm{Im}(\mathrm{U}(\varphi)) \, .$$

Note that $\det \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \not\equiv_2 0$. So we may set

$$u_1 := \left[1, \; 1, \; \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & 0 \\ y_{21} & y_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right] \in U_1 \leqslant \mathrm{Im}(\mathrm{U}(\varphi)) \, .$$

Then

$$u_1^{-1} \cdot u = \left[v, \; w, \; \begin{pmatrix} 1 & 0 & \hat{x}_{13} \\ 0 & 1 & \hat{x}_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & \hat{y}_{13} \\ 0 & 1 & \hat{y}_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in \mathrm{Im}(\mathrm{U}(\varphi)) \, ,$$

for certain $\hat{x}_{13}, \hat{x}_{23}, \hat{y}_{13}, \hat{y}_{23} \in \mathbb{Z}_{(2)}$.

By Remark 19, each element of $\Gamma_{(2)}$ representing $u_1^{-1} \cdot u$ is contained in $\mathrm{U}(\Lambda_{(2)})$. Hence $u_1^{-1} \cdot u \in M_1$.

This proves the *claim*.

So we also know that for $u \in \mathrm{Im}(\mathrm{U}(\varphi))$ there exists an $u_1 \in U_1$ and an $m_1 \in M_1$ with $u = u_1 \cdot m_1$. Hence

$$\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \cdot M_1 \, .$$

Step 2. Now we want to dissect M_1 even further.

We define the subset

$$U_2 := \left\{ \left[v,\ w,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_{33} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y_{33} \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \right. \tag{5.8}$$
$$\left. v \not\equiv_2 0,\ x_{33} \equiv_2 y_{33},\ v - x_{33} \equiv_8 w - y_{33} \equiv_4 0 \right\} .$$

Recalling the picture of (5.2), the remaining congruences are shown in the following figure.

$$\boxed{v} \qquad \boxed{w} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{x_{33}} \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{y_{33}} \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

$$\boxed{x_{33} \equiv_2 y_{33}}$$
$$\boxed{v - x_{33} \equiv_8 w - y_{33} \equiv_4 0}$$

Note that $U_2 \subseteq M_1 \subseteq \mathrm{Im}(\mathrm{U}(\varphi))$.

Note that $U_2 \leqslant \mathrm{Im}(\mathrm{U}(\varphi))$. In fact given elements of U_2 represented by $u_2, u_2' \in \mathrm{U}(\Lambda_{(2)})$, we have $u_2 \cdot u_2' \in \mathrm{U}(\Lambda_{(2)})$ whence $u_2 \cdot u_2'$ satisfies the required congruences. In addition, $u_2 \cdot u_2'$ has the block structure required to represent an element of U_2. So

$$\overline{u_2} \cdot \overline{u_2'} = (u_2 \cdot u_2')\varphi \in U_2 \,.$$

Now we want to use U_2 to factor M_1. We define the set

$$M_2 := \left\{ \left[1,\ 1,\ \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ x_{31} & x_{32} & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ y_{31} & y_{32} & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \right. \tag{5.9}$$
$$\left. x_{13} \equiv_2 y_{13},\ x_{23} \equiv_2 y_{23},\ x_{31} \equiv_8 y_{31} \equiv_4 0,\ x_{32} \equiv_8 y_{32} \equiv_4 0 \right\} .$$

Recalling the picture of (5.2), the remaining congruences are shown in the following figure.

$$1 \qquad 1 \qquad \begin{matrix} 1 & 0 & \boxed{x_{13}} \\ 0 & 1 & \boxed{x_{23}} \\ \boxed{x_{31}} & \boxed{x_{32}} & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & \boxed{y_{13}} \\ 0 & 1 & \boxed{y_{23}} \\ \boxed{y_{31}} & \boxed{y_{32}} & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

$$\boxed{\begin{matrix} x_{13} \equiv_2 y_{13} \\ x_{23} \equiv_2 y_{23} \end{matrix}}$$

$$\boxed{x_{31} \equiv_8 y_{31} \equiv_4 0, \quad x_{32} \equiv_8 y_{32} \equiv_4 0}$$

Note that $M_2 \subseteq M_1 \subseteq \mathrm{Im}(\mathrm{U}(\varphi))$.

Note that M_2 is only a subset of $\mathrm{Im}(\mathrm{U}(\varphi))$.

We *claim* that for $m_1 \in M_1$ there exists $u_2 \in U_2$ such that $u_2^{-1} \cdot m_1 \in M_2$.

For

$$m_1 = \left[v,\ w,\ \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in M_1 \subseteq \mathrm{Im}(\mathrm{U}(\varphi))$$

we set

$$u_2 := \left[v,\ w,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in U_2 \leqslant \mathrm{Im}(\mathrm{U}(\varphi))\ .$$

Then

$$u_2^{-1} \cdot m_1 = \left[1,\ 1,\ \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ \hat{x}_{31} & \hat{x}_{32} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ \hat{y}_{31} & \hat{y}_{32} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

for certain $\hat{x}_{31}, \hat{x}_{32}, \hat{y}_{31}, \hat{y}_{32} \in \mathbb{Z}_{(2)}$.

By Remark 19, each element of $\Gamma_{(2)}$ representing $u_2^{-1} \cdot m_1$ is contained in $\mathrm{U}(\Lambda_{(2)})$. Hence $u_2^{-1} \cdot m_1 \in M_2$.

This proves the *claim*.

So we also know that for $m_1 \in M_1$ there exists an $u_2 \in U_2$ and an $m_2 \in M_2$ with $m_1 = u_2 \cdot m_2$. Hence

$$M_1 = U_2 \cdot M_2\ .$$

We have obtained the following Lemma 20.

Lemma 20. *We have*

$$\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \cdot U_2 \cdot M_2\ .$$

5.2 Calculation of $\mathrm{Im}(\mathrm{U}(\varphi))$ via Magma

5.2.1 Preparations

We have the following $\mathbb{Z}_{(2)}$-linear basis of $\Lambda_{(2)}$:

$$
\begin{aligned}
\mathfrak{B}_{\Lambda_{(2)}} := \Bigg(\;
&\left(1,\ 1,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(4,\ 4,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 2,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&2\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 8,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\4&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\8&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&4&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&8&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&2\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&1\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&1\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&0&2\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}-2&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}8&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&2&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&-2&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&8&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\2&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\-2&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\8&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\right),\\
&\left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&2&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&-2&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \left(0,\ 0,\ \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0&0\\0&8&0\\0&0&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) \Bigg)\\
&=: (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16}, b_{17}, b_{18}, b_{19}, b_{20}, b_{21}, b_{22}, b_{23}, b_{24})
\end{aligned}
\tag{5.10}
$$

We write $b_j\varphi = \overline{b}_j \in \overline{\Gamma}_{(2)}$ for $j \in [1, 24]$.
We obtain the following $\mathbb{Z}_{(2)}$-linear generating set of $\mathrm{Im}(\varphi)$.

$$
\mathfrak{G}_{\mathrm{Im}(\varphi)} = \{\overline{b}_1, \overline{b}_2, \overline{b}_3, \overline{b}_5, \overline{b}_7, \overline{b}_9, \overline{b}_{10}, \overline{b}_{11}, \overline{b}_{12}, \overline{b}_{13}, \overline{b}_{14}, \overline{b}_{16}, \overline{b}_{17}, \overline{b}_{19}, \overline{b}_{20}, \overline{b}_{22}, \overline{b}_{23}\}\,. \tag{5.11}
$$

To deal with Magma, we introduce the following convention to convert the tuple

$$
\lambda = \left(v,\ w,\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},\ \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix},\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right) \in \Lambda_{(2)}
$$

into a vector. We convert the parts as follows.

$$
\begin{aligned}
v &\rightarrowtail v,\\
w &\rightarrowtail w,\\
\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} &\rightarrowtail (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})^{\mathrm{t}},\\
\begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} &\rightarrowtail (y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33})^{\mathrm{t}},\\
\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} &\rightarrowtail (z_{11}, z_{12}, z_{21}, z_{22})^{\mathrm{t}}
\end{aligned}
$$

If we now combine the individual parts in a vector, we obtain the following.

$$
\lambda \rightarrowtail (v, w, x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33}, z_{11}, z_{12}, z_{21}, z_{22})^{\mathrm{t}}
$$

5.2.2 Magma application

To start the Magma procedure, we first need to fix some data for later.

```
Z   := Integers();
Q   := Rationals();
G1  := GL(1,Integers(8));
G2  := GL(1,Integers(8));
G3  := GL(3,Integers(8));
G4  := GL(3,Integers(8));
G5  := GL(2,Integers(8));
UGq := DirectProduct([G1,G2,G3,G4,G5]); // U(\bar Gamma_(2))
U   := sub<UGq|>; // initialise U // at the end this U is Im(U(phi))

RM  := RMatrixSpace(Integers(),24,1);

M8    := {-3,-2,-1,0,1,2,3,4};
M8odd := {-3,-1,1,3};
M8o0  := {-3,-2,-1,1,2,3,4};
M4    := {-1,0,1,2};
M4o0  := {-1,1,2};
M2    := {0,1};
M2o0  := {1};
Coeffs:= CartesianProduct([M8,M2,M4,M2,M2,M8,M4,M8,M4,M8,M4,M8,M4,M8,M4,M8,M4]);
```

Expressing the basis $\mathfrak{B}_{\Lambda_{(2)}}$, cf. (5.10), by the vector convention, we get

```
basis :=
[
RM!Transpose(Matrix([[1,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0]])),     // 1
RM!Transpose(Matrix([[4,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]])),     // 2
RM!Transpose(Matrix([[0,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0]])),     // 3
RM!Transpose(Matrix([[0,8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]])),     // n.u.
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,4,0,0,0,0,0,0,0,0,4,0,0,0,0,0,0]])),     // 5
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,8,0,0,0,0,0,0]])),     // n.u.
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,4,0,0,0,0,0,0,0,0,4,0,0,0,0,0]])),     // 7
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,8,0,0,0,0,0]])),     // n.u.
RM!Transpose(Matrix([[0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0]])),     // 9
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0,0,0]])),     // 10
RM!Transpose(Matrix([[0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0]])),     // 11
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0]])),     // 12
RM!Transpose(Matrix([[0,0,1,0,0,0,0,0,0,0,0, 1, 0,0,0,0,0,0,0,0,1,0,0,0]])),   // 13
RM!Transpose(Matrix([[0,0,2,0,0,0,0,0,0,0,0,-2, 0,0,0,0,0,0,0,0,0,0,0,0]])),   // 14
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0, 8, 0,0,0,0,0,0,0,0,0,0,0,0]])),   // n.u.
RM!Transpose(Matrix([[0,0,0,1,0,0,0,0,0,0,0, 0, 1,0,0,0,0,0,0,0,0,1,0,0]])),   // 16
RM!Transpose(Matrix([[0,0,0,2,0,0,0,0,0,0,0, 0,-2,0,0,0,0,0,0,0,0,0,0,0]])),   // 17
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0, 0, 8,0,0,0,0,0,0,0,0,0,0,0]])),   // n.u.
RM!Transpose(Matrix([[0,0,0,0,0,1,0,0,0,0,0, 0, 0,0, 1,0,0,0,0,0,0,0,1,0]])),  // 19
RM!Transpose(Matrix([[0,0,0,0,0,2,0,0,0,0,0, 0, 0,0,-2,0,0,0,0,0,0,0,0,0]])),  // 20
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0, 0, 0,0, 8,0,0,0,0,0,0,0,0,0]])),  // n.u.
RM!Transpose(Matrix([[0,0,0,0,0,0,1,0,0,0,0, 0, 0,0, 0,1,0,0,0,0,0,0,0,1]])),  // 22
RM!Transpose(Matrix([[0,0,0,0,0,0,2,0,0,0,0, 0, 0,0, 0,-2,0,0,0,0,0,0,0,0]])), // 23
RM!Transpose(Matrix([[0,0,0,0,0,0,0,0,0,0,0, 0, 0,0, 0, 8,0,0,0,0,0,0,0,0]]))  // n.u.
];
```

So we translate the elements of $\mathfrak{B}_{\Lambda_{(2)}}$ as $b_j =$`basis[j]` for $j \in [1, 24]$.

Certain of these elements are mapped to zero in $\overline{\Gamma}_{(2)}$, labelled as n.u., so we have no usage for these for our generating set $\mathfrak{G}_{\mathrm{Im}(\varphi)}$.

Then

$$\begin{aligned}\mathrm{Im}(\varphi) = \Big\{ & \alpha_1\bar{b}_1 + \alpha_2\bar{b}_2 + \alpha_3\bar{b}_3 + \alpha_5\bar{b}_5 + \alpha_7\bar{b}_7 + \alpha_9\bar{b}_9 + \alpha_{10}\bar{b}_{10} + \alpha_{11}\bar{b}_{11} + \alpha_{12}\bar{b}_{12} + \alpha_{13}\bar{b}_{13} \\ & + \alpha_{14}\bar{b}_{14} + \alpha_{16}\bar{b}_{16} + \alpha_{17}\bar{b}_{17} + \alpha_{19}\bar{b}_{19} + \alpha_{20}\bar{b}_{20} + \alpha_{22}\bar{b}_{22} + \alpha_{23}\bar{b}_{23} : \\ & \alpha_1, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{16}, \alpha_{19}, \alpha_{22} \in \{-3,-2,-1,0,1,2,3,4\}, \\ & \alpha_3, \alpha_{10}, \alpha_{12}, \alpha_{14}, \alpha_{17}, \alpha_{20}, \alpha_{23} \in \{-1,0,1,2\}, \\ & \alpha_2, \alpha_5, \alpha_7 \in \{0,1\} \Big\}\end{aligned}$$

The set of appearing coefficient tuples $(\alpha_1, \alpha_2, \alpha_3, \alpha_5, \dots, \alpha_{23})$ is translated to `Coeffs`. We will actually use only certain parts of `Coeffs` in each step of the procedure.

If the procedure was carried out analogously to A_4, cf. §4, then one would have to decide for each of #`Coeffs`= 2^{38} = 274.877.906.944 vectors individually whether it is contained in $\mathrm{Im}(\mathrm{U}(\varphi))$. This is still way too large for Magma.

Now we want to build $\mathrm{Im}(\mathrm{U}(\varphi))$ inside $\mathrm{Im}(\varphi)$. We have $\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \cdot U_2 \cdot M_2$; cf. Lemma 20.

We will start with `U = 1`, then build `U = U_1`, then build `U = <U_1, U_2>` and finally build `U = <U_1, U_2, M_2>`. Since $\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \cdot U_2 \cdot M_2$, we also have built

$$\mathrm{Im}(\mathrm{U}(\varphi)) = \langle U_1, U_2, M_2 \rangle = \mathtt{U}$$

at the end of this procedure.

Step 1. As in (5.4), we have

$$\begin{aligned} U_1 = \Bigg\{ & \left[1,\ 1,\ \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} y_{11} & y_{12} & 0 \\ y_{21} & y_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\right] \in \overline{\Gamma}_{(2)} : \\ & \det \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \not\equiv_2 0 \text{ and } x_{11} \equiv_4 y_{11},\ x_{12} \equiv_4 y_{12},\ x_{21} \equiv_4 y_{21},\ x_{22} \equiv_4 y_{22}, \\ & x_{11} + y_{11} \equiv_8 2z_{11},\ x_{12} + y_{12} \equiv_8 2z_{12},\ x_{21} + y_{21} \equiv_8 2z_{21},\ x_{22} + y_{22} \equiv_8 2z_{22} \Bigg\} \\ = \Big\{ & \bar{b}_1 + \alpha_{13}\bar{b}_{13} + \alpha_{14}\bar{b}_{14} + \alpha_{16}\bar{b}_{16} + \alpha_{17}\bar{b}_{17} + \alpha_{19}\bar{b}_{19} + \alpha_{20}\bar{b}_{20} + \alpha_{22}\bar{b}_{22} + \alpha_{23}\bar{b}_{23} : \\ & \alpha_{13},\ \alpha_{16},\ \alpha_{19},\ \alpha_{22} \in \{-3,-2,-1,0,1,2,3,4\}, \\ & \alpha_{14},\ \alpha_{17},\ \alpha_{20},\ \alpha_{23} \in \{-1,0,1,2\}, \\ & \alpha_{13}\alpha_{22} - \alpha_{16}\alpha_{19} \not\equiv_2 0 \Big\}. \end{aligned}$$

Since we use only a subset of $\mathfrak{G}_{\mathrm{Im}(\varphi)}$ to build U_1, we also consider only the following set `Coeffs5` of corresponding tuples of necessary coefficients.

```
Coeffs5 := CartesianProduct([M8,M8,M8,M8,M4,M4,M4,M4]);

U := sub<UGq | >; // U = 1

for x in Coeffs5 do
 y := basis[1] + x[1] * basis[13] + x[2] * basis[16] + x[3] * basis[19]
 + x[4] * basis[22] + x[5] * basis[14] + x[6] * basis[17] + x[7] * basis[20]
 + x[8] * basis[23];
 if
  not (y[21,1] * y[24,1] - y[23,1] * y[22,1])  mod 2 eq 0 // test for invertibility
 then
  y_mat := UGq!DiagonalJoin(<Matrix([[y[1,1]]]),Matrix([[y[2,1]]]),
  Matrix([[y[3,1],y[4,1],y[5,1]],[y[6,1],y[7,1],y[8,1]],[y[9,1],y[10,1],y[11,1]]]),
  Matrix([[y[12,1],y[13,1],y[14,1]],[y[15,1],y[16,1],y[17,1]],
          [y[18,1],y[19,1],y[20,1]]]),
  Matrix([[y[21,1],y[22,1]],[y[23,1],y[24,1]]])>); // transformation vector to matrix
 // Redundancy test:
  if not y_mat in U then
   U := sub<UGq | GeneratorsSequence(U) cat [y_mat]>;
   print "Order(U) =", Order(U); // current state
  end if;
 end if;
end for; // here: U = U_1
```

Step 2. As in (5.8), we have

$$U_2 = \left\{ \left[v,\ w,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_{33} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y_{33} \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \right.$$
$$\left. v \not\equiv_2 0,\ x_{33} \equiv_2 y_{33},\ v - x_{33} \equiv_8 w - y_{33} \equiv_4 0 \right\}$$
$$= \left\{ \bar{b}_{13} + \bar{b}_{22} + \alpha_1 \bar{b}_1 + \alpha_2 \bar{b}_2 + \alpha_3 \bar{b}_3 : \alpha_1 \in \{-3,-1,1,3\},\ \alpha_2 \in \{0,1\},\ \alpha_3 \in \{-1,0,1,2\} \right\}.$$

Note that $\alpha_1 \not\equiv_2 0$ due to the subset it is taken from. This already yields $v \not\equiv_2 0$.

Since we, again, use only a subset of $\mathfrak{G}_{\mathrm{Im}(\varphi)}$ to build U_2, we also consider only the following set `Coeffs1` of corresponding tuples of necessary coefficients.

```
Coeffs1 := CartesianProduct([M8odd,M2,M4]);

for x in Coeffs1 do
 y := basis[13] + basis[22] + x[1] * basis[1] + x[2] * basis[2] + x[3] * basis[3];
 y_mat := UGq!DiagonalJoin(<Matrix([[y[1,1]]]),Matrix([[y[2,1]]]),
 Matrix([[y[3,1],y[4,1],y[5,1]],[y[6,1],y[7,1],y[8,1]],[y[9,1],y[10,1],y[11,1]]]),
 Matrix([[y[12,1],y[13,1],y[14,1]],[y[15,1],y[16,1],y[17,1]],
                 [y[18,1],y[19,1],y[20,1]]]),
 Matrix([[y[21,1],y[22,1]],[y[23,1],y[24,1]]])>); // transformation vector to matrix
 // Redundancy test:
 if not y_mat in U then
  U := sub<UGq | GeneratorsSequence(U) cat [y_mat]>;
  print "Order(U) =", Order(U); // current state
 end if;
end for; // here: U = <U_1, U_2>
```

Step 3. As in (5.9), we have

$$M_2 = \left\{ \left[1,\ 1,\ \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ x_{31} & x_{32} & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & y_{13} \\ 0 & 1 & y_{23} \\ y_{31} & y_{32} & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \right.$$
$$\left. x_{13} \equiv_2 y_{13},\ x_{23} \equiv_2 y_{23},\ x_{31} \equiv_8 y_{31} \equiv_4 0,\ x_{32} \equiv_8 y_{32} \equiv_4 0 \right\}$$
$$= \left\{ \bar{b}_1 + \bar{b}_{13} + \bar{b}_{22} + \alpha_5 \bar{b}_5 + \alpha_7 \bar{b}_7 + \alpha_9 \bar{b}_9 + \alpha_{10} \bar{b}_{10} + \alpha_{11} \bar{b}_{11} + \alpha_{12} \bar{b}_{12} : \right.$$
$$\left. \alpha_5, \alpha_7 \in \{0,1\},\ \alpha_9, \alpha_{11} \in \{-3,-2,-1,0,1,2,3,4\},\ \alpha_{10}, \alpha_{12} \in \{-1,0,1,2\} \right\}.$$

Since we, again, use only a subset of $\mathfrak{G}_{\mathrm{Im}(\varphi)}$ to build M_2, we also consider only the following set `Coeffs_rest` of corresponding tuples of necessary coefficients.

```
Coeffs_rest := CartesianProduct([M2,M2,M8,M4,M8,M4]);

for x in Coeffs_rest do
 y := basis[1] + basis[13] + basis[22] + x[1] * basis[5] + x[2] * basis[7]
 + x[3] * basis[9] + x[4] * basis[10] + x[5] * basis[11] + x[6] * basis[12];
 y_mat := UGq!DiagonalJoin(<Matrix([[y[1,1]]]),Matrix([[y[2,1]]]),
 Matrix([[y[3,1],y[4,1],y[5,1]],[y[6,1],y[7,1],y[8,1]],[y[9,1],y[10,1],y[11,1]]]),
 Matrix([[y[12,1],y[13,1],y[14,1]],[y[15,1],y[16,1],y[17,1]],
         [y[18,1],y[19,1],y[20,1]]]),
 Matrix([[y[21,1],y[22,1]],[y[23,1],y[24,1]]])>); // transformation vector to matrix
 // Redundancy test:
 if not y_mat in U then
  U := sub<UGq | GeneratorsSequence(U) cat [y_mat]>;
  print "Order(U) =", Order(U); // current state
 end if;
end for; // here: U = <U_1, U_2, M_2> = U_1 * U_2 * M_2 = ImUphi

ImUphi := U;
```

5.3 Analysing $\mathrm{Im}(\mathrm{U}(\varphi))$ via Magma

Now we can examine this group $\mathrm{Im}(\mathrm{U}(\varphi))$ more closely. Using Magma, we calculate the derived series, a chief series and a list of generators.

5.3.1 The derived series of $\mathrm{Im}(\mathrm{U}(\varphi))$

We want to calculate the derived series of $\mathrm{Im}(\mathrm{U}(\varphi))$; cf. §1.2.2.

We can first have Magma perform a few simple tests.

```
Order(ImUphi); // 51539607552 = 2^34 * 3
IsAbelian(ImUphi); // false
IsSolvable(ImUphi); // true
GroupName(ImUphi); // (C2^9*C4^2).C2^6.C2^4.C6.C2^5.C2^5
```

Note that the group name alone does not determine the group map to isomorphism.

Since $\mathrm{Im}(\mathrm{U}(\varphi))$ is solvable, Magma can dissect $\mathrm{Im}(\mathrm{U}(\varphi))$ in its derived series, ending in the trivial group:

```
DS := DerivedSeries(ImUphi);
#DS; // 5

GroupName(quo< DS[1] | DS[2] >);
// C2^9
GroupName(quo< DS[2] | DS[3] >);
// C2*C6
GroupName(quo< DS[3] | DS[4] >);
// C2^8*C4^2
GroupName(quo< DS[4] | DS[5] >);
// C2^7*C4^2
```

We write $D_1 :=$ `DS[1]`, $D_2 :=$ `DS[2]`, ..., $D_5 :=$ `DS[5]`.

This means that we have the derived series

$$\mathrm{Im}(\mathrm{U}(\varphi)) = D_1 \geqslant D_2 \geqslant D_3 \geqslant D_4 \geqslant D_5 = 1$$

with

$$\begin{aligned} D_1/D_2 &\simeq \mathrm{C}_2^{\times 9} \\ D_2/D_3 &\simeq \mathrm{C}_2 \times \mathrm{C}_6 \\ D_3/D_4 &\simeq \mathrm{C}_2^{\times 8} \times \mathrm{C}_4^{\times 2} \\ D_4/D_5 &\simeq \mathrm{C}_2^{\times 7} \times \mathrm{C}_4^{\times 2} \,. \end{aligned}$$

Magma also gives the group names:

```
GroupName(DS[5]); // C1
GroupName(DS[4]); // C2^7*C4^2
GroupName(DS[3]); // (C2^7*C4^4).C2^6.C2^2
GroupName(DS[2]); // (C2^7*C4^4).C2^6.A4.C2^2
GroupName(DS[1]); // (C2^7*C4^4).C2^6.A4.C2^6.C2^5
```

5.3.2 A chief series of $\mathrm{Im}(\mathrm{U}(\varphi))$

We want to calculate the chief series if $\mathrm{Im}(\mathrm{U}(\varphi))$; cf. §1.2.3.

```
CS := ChiefSeries(ImUphi);
#CS;   \\ 26

GroupName(quo< CS[1] | CS[2] >); // C2
GroupName(quo< CS[2] | CS[3] >);    // C2
GroupName(quo< CS[3] | CS[4] >);    // C2
GroupName(quo< CS[4] | CS[5] >);    // C2
GroupName(quo< CS[5] | CS[6] >);    // C2
GroupName(quo< CS[6] | CS[7] >);    // C2
GroupName(quo< CS[7] | CS[8] >);    // C2
GroupName(quo< CS[8] | CS[9] >);    // C2
GroupName(quo< CS[9] | CS[10] >);   // C2
GroupName(quo< CS[10] | CS[11] >);  // C2
GroupName(quo< CS[11] | CS[12] >);  // C2
GroupName(quo< CS[12] | CS[13] >);  // C3
GroupName(quo< CS[13] | CS[14] >);  // C2^2
GroupName(quo< CS[14] | CS[15] >);  // C2^2
GroupName(quo< CS[15] | CS[16] >);  // C2^2
GroupName(quo< CS[16] | CS[17] >);  // C2^2
GroupName(quo< CS[17] | CS[18] >);  // C2^2
GroupName(quo< CS[18] | CS[19] >);  // C2^2
GroupName(quo< CS[19] | CS[20] >);  // C2
GroupName(quo< CS[20] | CS[21] >);  // C2
GroupName(quo< CS[21] | CS[22] >);  // C2
GroupName(quo< CS[22] | CS[23] >);  // C2^2
GroupName(quo< CS[23] | CS[24] >);  // C2^2
GroupName(quo< CS[24] | CS[25] >);  // C2^2
GroupName(quo< CS[25] | CS[26] >);  // C2^2
```

We write $H_1 :=$ `CS[1]`, $H_2 :=$ `CS[2]`, ... $H_{26} :=$ `CS[26]`.

In particular, this means, we have the chief series

$$\mathrm{Im}(\mathrm{U}(\varphi)) = H_1 \geqslant H_2 \geqslant H_3 \geqslant \ldots \geqslant H_{26} = 1$$

with

$$\begin{array}{lll}
H_1/H_2 \simeq \mathrm{C}_2, & H_2/H_3 \simeq \mathrm{C}_2, & H_3/H_4 \simeq \mathrm{C}_2 \\
H_4/H_5 \simeq \mathrm{C}_2, & H_5/H_6 \simeq \mathrm{C}_2, & H_6/H_7 \simeq \mathrm{C}_2 \\
H_7/H_8 \simeq \mathrm{C}_2, & H_8/H_9 \simeq \mathrm{C}_2, & H_9/H_{10} \simeq \mathrm{C}_2 \\
H_{10}/H_{11} \simeq \mathrm{C}_2, & H_{11}/H_{12} \simeq \mathrm{C}_2, & H_{12}/H_{13} \simeq C_3 \\
H_{13}/H_{14} \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_{14}/H_{15} \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_{15}/H_{16} \simeq \mathrm{C}_2 \times \mathrm{C}_2 \\
H_{16}/H_{17} \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_{17}/H_{18} \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_{18}/H_{19} \simeq \mathrm{C}_2 \times \mathrm{C}_2 \\
H_{19}/H_{20} \simeq \mathrm{C}_2, & H_{20}/H_{21} \simeq \mathrm{C}_2, & H_{21}/H_{22} \simeq \mathrm{C}_2 \\
H_{22}/H_{23} \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_{23}/H_{24} \simeq \mathrm{C}_2 \times \mathrm{C}_2, & H_{24}/H_{25} \simeq \mathrm{C}_2 \times \mathrm{C}_2 \\
H_{25}/H_{26} \simeq \mathrm{C}_2 \times \mathrm{C}_2\,. & &
\end{array}$$

Magma also gives the group names:

```
GroupName(CS[26]); // C1
GroupName(CS[25]); // C2^2
GroupName(CS[24]); // C2^4
GroupName(CS[23]); // C2^2*C4^2
GroupName(CS[22]); // C2^4*C4^2
GroupName(CS[21]); // C2^5*C4^2
GroupName(CS[20]); // C2^6*C4^2
GroupName(CS[19]); // C2^7*C4^2
GroupName(CS[18]); // C2^9*C4^2
GroupName(CS[17]); // C2^8.C2^6.C2
GroupName(CS[16]); // C2^10.C2^6.C2
GroupName(CS[15]); // C2^9.C2^6.C2^4
GroupName(CS[14]); // (C2^5*C4^5).C2^6
GroupName(CS[13]); // (C2^6*C4^5).C2^6.C2
GroupName(CS[12]); // (C2^7*C4^4).C2^6.A4
GroupName(CS[11]); // (C2^7*C4^4).C2^6.A4.C2
GroupName(CS[10]); // (C2^7*C4^4).C2^6.A4.C2^2
GroupName(CS[9]);  // (C2^9*C4^2).C2^6.C2^4.C6.C2^2
GroupName(CS[8]);  // (C2^7*C4^4).C2^6.A4.C2^4
GroupName(CS[7]);  // (C2^9*C4^2).C2^6.C2^4.C6.C2^4
GroupName(CS[6]);  // (C2^9*C4^2).C2^6.C2^4.C6.C2^5
GroupName(CS[5]);  // (C2^9*C4^2).C2^6.C2^4.C6.C2^6
GroupName(CS[4]);  // (C2^7*C4^4).C2^6.A4.C2^6.C2^2
GroupName(CS[3]);  // (C2^9*C4^2).C2^6.C2^4.C6.C2^6.C2^2
GroupName(CS[2]);  // (C2^9*C4^2).C2^6.C2^4.C6.C2^6.C2^3
GroupName(CS[1]);  // (C2^7*C4^4).C2^6.A4.C2^6.C2^5
```

5.3.3 Chosen generators for $\mathrm{Im}(\mathrm{U}(\varphi))$

We now want to try to force Magma to choose generators selected by us, in the hope that Magma can then give us some more information about the group. To do this, we first choose this list of generators ourselves. The attempt will be to find as many producers as possible that contain a lot of zeros.

```
one := RM!Transpose(Matrix([[1, 1, 1,0,0,0,1,0,0,0,1, 1,0,0,0,1,0,0,0,1, 1,0,0,1]]));

blockify := function(y)
 return DiagonalJoin(<Matrix([[y[1,1]]]),Matrix([[y[2,1]]]),
 Matrix([[y[3,1],y[4,1],y[5,1]],[y[6,1],y[7,1],y[8,1]],[y[9,1],y[10,1],y[11,1]]]),
 Matrix([[y[12,1],y[13,1],y[14,1]],[y[15,1],y[16,1],y[17,1]],
         [y[18,1],y[19,1],y[20,1]]]),
 Matrix([[y[21,1],y[22,1]],[y[23,1],y[24,1]]])>);
end function;

Vtest := sub<ImUphi | [UGq!blockify(one + basis[i]):i in [3,5,7,9,11,14,16,19,23]]>;
Factorisation(Order(Vtest)); // Results in 2^28 * 3^1
                             // so Order(ImUphi) / Order(Vtest) equals 64

Vtest2 := sub<ImUphi | Vtest, [UGq!blockify(3*one), UGq!blockify(5*one)] >;
Factorisation(Order(Vtest2)); // Results in 2^30 * 3^1
                              // so Order(ImUphi) / Order(Vtest2) equals 16

new := RM!Transpose(Matrix([[1,5, 1,0,0,0,1,0,0,0,1, 1,0,0,0,1,0,0,0,5, 1,0,0,1]]));
Vtest3 := sub< ImUphi | Vtest2, [blockify(new)]>;
Factorisation(Order(Vtest3)); // Results in 2^31 * 3^1
                              // so Order(ImUphi) / Order(Vtest3) equals 8

list_new_elements := [];
for i1 in [0..7] do
 for i2 in [0..1] do
  for i3 in [0..3] do
   for i13 in [0..7] do
    for i14 in [0..3] do
     for i22 in [0..7] do
      for i23 in [0..3] do
       mat := i1 * blockify(basis[1]) + i2 * blockify(basis[2])
               + i3 * blockify(basis[3]) + i13 * blockify(basis[13])
               + i14 * blockify(basis[14]) + i22 * blockify(basis[22])
               + i23 * blockify(basis[23]);
        if Determinant(mat) mod 2 eq 1 then //  test if mat invertible in Lambda_(2)
 mat_gr := UGq!mat;
 if not mat_gr in Vtest3 then
  list_new_elements cat:= [mat_gr];
 end if;
end if;
      end for;
     end for;
    end for;
   end for;
  end for;
 end for;
end for;
```

```
Vtest_inter := Vtest3;
count := 0;
for g in list_new_elements do
 count +:= 1;
 if not g in Vtest_inter then
  Vtest_inter := sub< ImUphi | Vtest_inter, g>;
  print Order(Vtest_inter);
 end if;
end for;

print Vtest_inter eq ImUphi; // true

#(list_new_elements); // 7168
```

We have now `Vtest_inter = ImUphi`, and the list of generators Magma uses for `Vtest_inter` will now be put in `list_gen`.

To find the generators and to make the magma code as simple as possible and thus minimize the program runtime, we will first sort the set of vectors.

First we give each vector a number to be able to identify them later.

```
index_Set := [1 .. 7168];
index_SetRow := [1 .. 10];
index_SetEntries := [1 .. 10];
yay_List := [];

    for o in index_Set do
yay := 0;
  for u in index_SetRow do
     for v in index_SetEntries do
        if list_new_elements[o,u] eq 0 then;
yay +:= 1;
end if;
     end for;
  end for;
yay_List cat:= [<o,yay>];
end for;

#yay_List; // 7168
```

Not we use the sort function of Magma, to sort by the number of zeros.

```
C := func<x,y | - x[2] + y[2] >;
Y := yay_List;
S := Sort(Y, C);
list_group_elements_new_sorting := [list_new_elements[x[1]] : x in S];
```

```
Vtest_inter := Vtest3;
for g in list_group_elements_new_sorting do
 if not g in Vtest_inter then
  Vtest_inter := sub< ImUphi | Vtest_inter, g>;
  print Order(Vtest_inter);
 end if;
end for;

// Vtest_inter is our current group of generators

list_gen := [];
for i in [1..#Generators(Vtest_inter)] do
 list_gen cat:= [Vtest_inter.i];
end for;

// Redundancy test:
for i in [1..#list_gen] do
 print Order(sub<Vtest_inter | Remove(list_gen,i)>);
end for;
// Results in 2,3,4,5,6 and 9 being redundant
list_gen[2];
list_gen[3];
list_gen[4];
list_gen[5];
list_gen[6];
list_gen[9];

list_gen_red := Remove(list_gen,9);

// Test for further redundancy:
for i in [1..#list_gen_red] do
 print Order(sub<Vtest_inter | Remove(list_gen_red,i)>);
end for;
// Results in 2,3,4 and 5 being redundant

list_gen_reda := Remove(list_gen_red,5);

// Test for further redundancy:
for i in [1..#list_gen_reda] do
 print Order(sub<Vtest_inter | Remove(list_gen_reda,i)>);
end for;
// Results in 2 and 3 being redundant

list_gen_redaa := Remove(list_gen_reda,3);

// Test for further redundancy:
for i in [1..#list_gen_redaa] do
 print Order(sub<Vtest_inter | Remove(list_gen_redaa,i)>);
end for;
// Results in all other being important


final_list_gen := list_gen_redaa;
```

So we have the following list of generators.

$$g_1 := \left[1,\ 3,\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&3\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\right] \quad g_2 := \left[1,\ 1,\ \begin{pmatrix}1&0&0\\0&1&0\\4&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&1&0\\4&0&1\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\right]$$

$$g_3 := \left[1,\ 1,\ \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\right] \quad g_4 := \left[1,\ 1,\ \begin{pmatrix}3&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\right]$$

$$g_5 := \left[1,\ 1,\ \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&1\\0&1\end{pmatrix}\right] \quad g_6 := \left[1,\ 1,\ \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0\\1&1\end{pmatrix}\right]$$

$$g_7 := \left[3,\ 3,\ \begin{pmatrix}3&0&0\\0&3&0\\0&0&3\end{pmatrix},\ \begin{pmatrix}3&0&0\\0&3&0\\0&0&3\end{pmatrix},\ \begin{pmatrix}3&0\\0&3\end{pmatrix}\right] \quad g_8 := \left[5,\ 5,\ \begin{pmatrix}5&0&0\\0&5&0\\0&0&5\end{pmatrix},\ \begin{pmatrix}5&0&0\\0&5&0\\0&0&5\end{pmatrix},\ \begin{pmatrix}5&0\\0&5\end{pmatrix}\right]$$

$$g_9 := \left[1,\ 5,\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&5\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\right] \quad g_{10} := \left[1,\ 1,\ \begin{pmatrix}1&0&0\\0&5&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&5&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\right]$$

$$g_{11} := \left[1,\ 1,\ \begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0\\0&3\end{pmatrix}\right] \quad g_{12} := \left[1,\ 1,\ \begin{pmatrix}1&0&0\\0&5&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&5&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0\\0&5\end{pmatrix}\right]$$

Using these elements as generators, we obtain

$$\mathrm{Im}(\mathrm{U}(\varphi)) = \langle g_1, g_2, \ldots, g_{12} \rangle \,.$$

5.4 Description of $\ker(\mathrm{U}(\varphi))$

Consider the commutative triangle of groups.

$$\begin{array}{ccc} \mathrm{U}(\Gamma_{(2)}) & \xrightarrow{\mathrm{U}(\varrho)} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ \uparrow & \nearrow_{\mathrm{U}(\varphi)} & \\ \mathrm{U}(\Lambda_{(2)}) & & \end{array}$$

Since

$$\ker(\mathrm{U}(\varrho)) \leqslant \mathrm{U}(\Lambda_{(2)}) \,,$$

we have

$$\ker(\mathrm{U}(\varrho)) = \ker(\mathrm{U}(\varrho)) \cap \mathrm{U}(\Lambda_{(2)}) = \ker(\mathrm{U}(\varphi)) \,.$$

So we have the following commutative diagram.

$$\begin{array}{ccccc} \ker(\mathrm{U}(\varrho)) & \longrightarrow & \mathrm{U}(\Gamma_{(2)}) & \xrightarrow{\mathrm{U}(\varrho)} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ \| & & \uparrow & \nearrow_{\mathrm{U}(\varphi)} & \\ \ker(\mathrm{U}(\varphi)) & \longrightarrow & \mathrm{U}(\Lambda_{(2)}) & & \end{array}$$

With the diagram 5.3 we get a description of the kernel.

$$
\begin{aligned}
&\ker(\mathrm{U}(\varphi)) \\
&\quad = \{1 + 8\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\} \\
&\quad = \left\{ \left(1+8v,\ 1+8w,\ \left(\begin{smallmatrix} 1+8x_{11} & 8x_{12} & 8x_{13} \\ 8x_{21} & 1+8x_{22} & 8x_{23} \\ 8x_{31} & 8x_{32} & 1+8x_{33} \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1+8y_{11} & 8y_{12} & 8y_{13} \\ 8y_{21} & 1+8y_{22} & 8y_{23} \\ 8y_{31} & 8y_{32} & 1+8y_{33} \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1+8z_{11} & 8z_{12} \\ 8z_{21} & 1+8z_{22} \end{smallmatrix}\right)\right) \in \mathrm{U}(\Gamma_{(2)}) : \right. \\
&\qquad \left. \left(v,\ w,\ \left(\begin{smallmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{smallmatrix}\right),\ \left(\begin{smallmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{smallmatrix}\right),\ \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right)\right) \in \Gamma_{(2)} \right\}
\end{aligned}
$$

5.5 Summary

We have the following diagram of groups, in which the lower row is a short exact sequence.

$$
\begin{array}{ccccc}
 & & \mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_4) & & \\
 & & \mathrm{U}\omega_{\mathbb{Z}_{(2)}} \big\downarrow \wr & & \\
\ker(\mathrm{U}(\varphi)) & \hookrightarrow & \mathrm{U}(\Lambda_{(2)}) & \longmapsto & \mathrm{Im}(\mathrm{U}(\varphi))
\end{array}
$$

The finite group $\mathrm{Im}(\mathrm{U}(\varphi))$ has order $2^{34} \cdot 3$. It is build in §5.2 via Magma. Its derived series, a chief series and chosen generators are given in §5.3.

The infinite group $\ker(\mathrm{U}(\varphi)) = \{1 + 8\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\}$ is described in §5.4.

Chapter 6

The group D_{2p}

Let $p \in \mathbb{Z}_{\geqslant 3}$ be a prime. We consider the dihedral group $G = \mathrm{D}_{2p}$.

Note that in the case $p = 3$, we obtain $G = \mathrm{D}_{2\cdot 3} \simeq \mathrm{S}_3$. Thus for $p = 3$ the following calculation is essentially a short version of §2.

6.1 Preparations

We will make use of the description of Simon Klenk of the group ring $\mathbb{Z}_{(p)}\mathrm{D}_{2p}$; cf. **[6]**.

Let $\vartheta = \vartheta_p := \zeta_p + \zeta_p^{-1} - 2$; cf. [**6,** §1, p. 1, Definition 17]. Let $\Gamma_{(p)} := \mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}[\vartheta]^{2\times 2} \times \mathbb{Z}_{(p)}$.

Remark 21. Suppose given $a \in \mathbb{Z}_{(p)} \subseteq \mathbb{Z}_{(p)}[\vartheta]$ and $b \in \mathbb{Z}_{(p)}[\vartheta]$. Then

$$a \equiv_\vartheta b$$

means

$$a - b \in \vartheta\mathbb{Z}_{(p)}[\vartheta]\ .$$

Lemma 22.

(1) *We have the following isomorphism of rings.*

$$\begin{aligned} \mathbb{F}_p &\xrightarrow{\sim} \mathbb{Z}[\vartheta]/\vartheta\mathbb{Z}[\vartheta] \\ 1 &\longmapsto 1 + \vartheta\mathbb{Z}[\vartheta] \end{aligned}$$

(2) *We have the following equality.*

$$\vartheta\mathbb{Z}_{(p)}[\vartheta] \cap \mathbb{Z}_{(p)} = p\mathbb{Z}_{(p)}$$

Proof. Ad (1). This follows by cf. [**6,** §1.4.1, p. 10, Lemma 27.(i)] localised at (p).

Ad (2). This follows by cf. [**6,** §1.4.1, p. 10, Lemma 27.(ii)] localised at (p). □

Lemma 23. *The ring* $\mathbb{Z}_{(p)}[\vartheta]$ *is a discrete valuation ring with maximal ideal generated by* ϑ*. We have*

$$(p) = (\vartheta)^{\frac{p-1}{2}} \tag{6.1}$$

as ideals in $\mathbb{Z}_{(p)}[\vartheta]$*.*

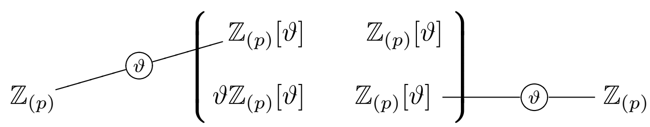
Proof. Cf. [**6,** §1.4.1, p. 12, Lemma 29]. □

We consider

$$\Lambda_{(p)} := \left\{ \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f \right) \in \mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}[\vartheta]^{2\times 2} \times \mathbb{Z}_{(p)} \,:\, a \equiv_\vartheta b, d \equiv_\vartheta 0, e \equiv_\vartheta f \right\} \subseteq \Gamma_{(p)} \,, \tag{6.2}$$

cf. [**6,** §4.1, p. 44, Notation 68].

Schematically, we have the following picture of $\Lambda_{(p)}$.



(6.3)

We consider the dihedral group of order $2p$,

$$\mathrm{D}_{2p} := \langle x, y : x^p, y^2, (yx)^2 \rangle \,;$$

cf. [**6,** §3.1, p. 36, Definition 54]. Note that in D_{2p}, we have

$$yxyx = 1 \text{ and } y = y^{-1} \,,$$

so

$${}^{y}x = x^{-1} \,.$$

Proposition 24 (Klenk)**.** We have the following Wedderburn isomorphism, restricted to $\mathbb{Z}_{(p)}\mathrm{D}_{2p}$ and to $\Lambda_{(p)}$.

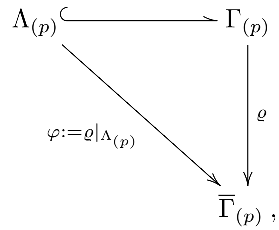
$$\begin{aligned} \omega_{\mathbb{Z}_{(p)}}|^{\Lambda_{(p)}} \,:\, \mathbb{Z}_{(p)}\mathrm{D}_{2p} &\xrightarrow{\sim} \Lambda_{(p)} \\ x &\longmapsto \left(1, \begin{pmatrix} 1 & 1 \\ \vartheta & \vartheta+1 \end{pmatrix}, 1\right) \\ y &\longmapsto \left(1, \begin{pmatrix} 1 & 0 \\ \vartheta & -1 \end{pmatrix}, -1\right) \,. \end{aligned}$$

Proof. Cf. [**6,** §4.1, p. 46, Theorem 70]. □

Let $\overline{\Gamma}_{(p)} := \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \times \left(\mathbb{Z}_{(p)}[\vartheta]/\vartheta\mathbb{Z}_{(p)}[\vartheta]\right)^{2\times 2} \times \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$.

Note that we calculate modulo p in the first and third factors, whereas we calculate modulo ϑ in the second factor. This is a slight generalisation of the procedure explained in §3. Cf. also §6.2 below.

We consider the following commutative diagram of $\mathbb{Z}_{(p)}$-algebras.



where ϱ is the residue class morphism, mapping

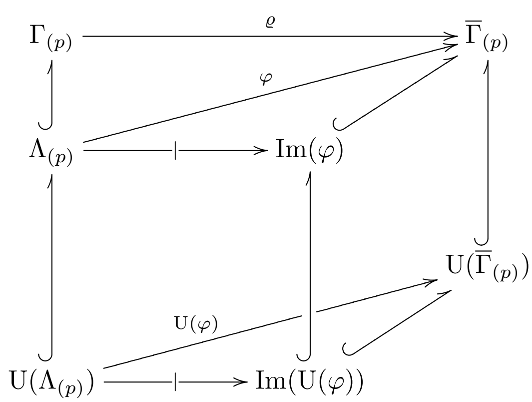
$$\left(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f\right) \longmapsto \left(a + p\mathbb{Z}_{(p)}, \left(\begin{smallmatrix} b+\vartheta\mathbb{Z}_{(p)}[\vartheta] & c+\vartheta\mathbb{Z}_{(p)}[\vartheta] \\ d+\vartheta\mathbb{Z}_{(p)}[\vartheta] & e+\vartheta\mathbb{Z}_{(p)}[\vartheta] \end{smallmatrix}\right), f + p\mathbb{Z}_{(p)}\right) =: \left[a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f\right]$$

Remark 25. *We have the following isomorphism of $\mathbb{F}_p$-algebras.*

$$\mathbb{F}_p \times \mathbb{F}_p^{2\times 2} \times \mathbb{F}_p \xrightarrow{\sim} \overline{\Gamma}_{(p)}$$
$$\left(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f\right) = \left(a + p\mathbb{Z}, \left(\begin{smallmatrix} b+p\mathbb{Z} & c+p\mathbb{Z} \\ d+p\mathbb{Z} & e+p\mathbb{Z} \end{smallmatrix}\right), f + p\mathbb{Z}\right) \longmapsto \left(a + p\mathbb{Z}_{(p)}, \left(\begin{smallmatrix} b+\vartheta\mathbb{Z}_{(p)}[\vartheta] & c+\vartheta\mathbb{Z}_{(p)}[\vartheta] \\ d+\vartheta\mathbb{Z}_{(p)}[\vartheta] & e+\vartheta\mathbb{Z}_{(p)}[\vartheta] \end{smallmatrix}\right), f + p\mathbb{Z}_{(p)}\right)$$

Proof. This follows using Lemma 22(1). □

We obtain the following commutative diagram.



6.2 Description of $\ker(\mathrm{U}(\varphi))$

Remark 26. *We get the following description of the kernel of φ.*

$$\ker(\mathrm{U}(\varphi)) \overset{!}{=} \left\{ \left(1 + pa,\ \begin{pmatrix} 1+\vartheta b & \vartheta c \\ \vartheta d & 1+\vartheta e \end{pmatrix},\ 1 + pf\right) \in \mathrm{U}(\Gamma_{(p)}) : a, f \in \mathbb{Z}_{(p)},\ b, c, d, e \in \mathbb{Z}_{(p)}[\vartheta] \right\} = \ker(\mathrm{U}(\varrho))$$

Here, the element $\left(1 + pa,\ \begin{pmatrix} 1+\vartheta b & \vartheta c \\ \vartheta d & 1+\vartheta e \end{pmatrix},\ 1 + pf\right)$ *is invertible in* $\Gamma_{(p)}$ *for* $a, f \in \mathbb{Z}_{(p)}, b, c, d, e \in \mathbb{Z}_{(p)}[\vartheta]$.

Proof. To show that $\left(1 + pa,\ \begin{pmatrix} 1+\vartheta b & \vartheta c \\ \vartheta d & 1+\vartheta e \end{pmatrix},\ 1 + pf\right)$ is invertible in $\Gamma_{(p)}$ for $a, f \in \mathbb{Z}_{(p)},\ b, c, d, e \in \mathbb{Z}_{(p)}[\vartheta]$, we remark the following.

1. We have $\det \begin{pmatrix} 1+\vartheta b & \vartheta c \\ \vartheta d & 1+\vartheta e \end{pmatrix} \equiv_\vartheta 1$.

2. We have $\mathrm{U}(\mathbb{Z}_{(p)}[\vartheta]) = \mathbb{Z}_{(p)}[\vartheta]\backslash(\vartheta)$. This is true due to $\mathbb{Z}_{(p)}[\vartheta]$ being a discrete valuation ring with maximal ideal (ϑ); cf. Lemma 23.

So

$$\det\begin{pmatrix} 1+\vartheta b & \vartheta c \\ \vartheta d & 1+\vartheta e \end{pmatrix}$$

is invertible in $\mathbb{Z}_{(p)}[\vartheta]$. Hence

$$\left(1+pa,\ \begin{pmatrix} 1+\vartheta b & \vartheta c \\ \vartheta d & 1+\vartheta e \end{pmatrix},\ 1+pf\right)$$

is invertible in $\Gamma_{(p)}$.

To prove that this description represents the kernel of $\mathrm{U}(\varphi)$, the following property must be shown.

By construction we have

$$\ker(\mathrm{U}(\varphi)) = \ker(\mathrm{U}(\varrho|_{\Lambda_{(p)}})) = \ker(\mathrm{U}(\varrho)|_{\mathrm{U}(\Lambda_{(p)})}) = \ker(\mathrm{U}(\varrho)) \cap \mathrm{U}(\Lambda_{(p)})\,.$$

So it suffices to show that $\ker(\mathrm{U}(\varrho)) \leqslant \mathrm{U}(\Lambda_{(p)})$. So we need to show that the congruences of $\Lambda_{(p)}$ are fulfilled for the elements of $\ker(\mathrm{U}(\varrho))$. Recall the description (6.2) of $\Lambda_{(p)}$ via congruences. We have

$$\begin{aligned} (1+pa)-(1+\vartheta b) = pa - \vartheta b &\equiv_\vartheta 0 \\ (1+pf)-(1+\vartheta e) = pf - \vartheta e &\equiv_\vartheta 0 \\ \vartheta d &\equiv_\vartheta 0 \end{aligned}$$

due to $p \equiv_\vartheta 0$; cf. Lemma 22(2). □

6.3 A description of $\mathrm{Im}(\mathrm{U}(\varphi))$

We have

$$\begin{aligned}
\mathrm{U}(\Lambda_{(p)}) &\overset{\text{Lemma 8}}{=} \Lambda_{(p)} \cap \mathrm{U}(\Gamma_{(p)}) \\
&\overset{\text{Lemma 23}}{=} \Lambda_{(p)} \cap \left\{ \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\right) \in \Gamma_{(p)} :\ a, f \in \mathbb{Z}_{(p)}\backslash(p),\ \det\begin{pmatrix} b & c \\ d & e \end{pmatrix} \in \mathbb{Z}_{(p)}[\vartheta]\backslash(\vartheta) \right\} \\
&= \left\{ \left(a, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, f\right) \in \Gamma_{(p)} :\ a \equiv_\vartheta b, e \equiv_\vartheta f, d \equiv_\vartheta 0,\ a, f \in \mathbb{Z}_{(p)}\backslash(p), \right. \\
&\qquad\qquad \left. \det\begin{pmatrix} b & c \\ d & e \end{pmatrix} \in \mathbb{Z}_{(p)}[\vartheta]\backslash(\vartheta) \right\} \\
&= \left\{ \left(a, \begin{pmatrix} a+\vartheta\tilde{b} & c \\ \vartheta\tilde{d} & f+\vartheta\tilde{e} \end{pmatrix}, f\right) \in \Gamma_{(p)} :\ a, f \in \mathbb{Z}_{(p)}\backslash(p),\ \tilde{b}, c, \tilde{d}, \tilde{e} \in \mathbb{Z}_{(p)}[\vartheta], \right. \\
&\qquad\qquad \left. \det\begin{pmatrix} a+\vartheta\tilde{b} & c \\ \vartheta\tilde{d} & f+\vartheta\tilde{e} \end{pmatrix} \in \mathbb{Z}_{(p)}[\vartheta]\backslash(\vartheta) \right\} \\
&= \left\{ \left(a, \begin{pmatrix} a+\vartheta\tilde{b} & c \\ \vartheta\tilde{d} & f+\vartheta\tilde{e} \end{pmatrix}, f\right) \in \Gamma_{(p)} :\ a, f \in \mathbb{Z}_{(p)}\backslash(p),\ \tilde{b}, c, \tilde{d}, \tilde{e} \in \mathbb{Z}_{(p)}[\vartheta] \right\}.
\end{aligned}$$

We obtain

$$\begin{aligned}\mathrm{Im}(\mathrm{U}(\varphi)) &= \left\{ \left[a, \begin{pmatrix} a+\vartheta\tilde{b} & c \\ \vartheta\tilde{d} & f+\vartheta\tilde{e} \end{pmatrix}, f \right] \in \overline{\Gamma}_{(p)} \,:\, a, f \in \mathbb{Z}_{(p)} \backslash (p),\ \tilde{b}, c, \tilde{d}, \tilde{e} \in \mathbb{Z}_{(p)}[\vartheta] \right\} \\ &= \left\{ \left[a, \begin{pmatrix} a & c \\ 0 & f \end{pmatrix}, f \right] \in \overline{\Gamma}_{(p)} \,:\, a, f \in \mathbb{Z}_{(p)} \backslash (p),\ c \in \mathbb{Z}_{(p)}[\vartheta] \right\} .\end{aligned}$$

Applying Remark 25, we can rewrite $\mathrm{Im}(\mathrm{U}(\varphi))$ as

$$\mathrm{Im}(\mathrm{U}(\varphi)) = \left\{ \left[a, \begin{pmatrix} a & c \\ 0 & f \end{pmatrix}, f \right] \in \overline{\Gamma}_{(p)} \,:\, a, f \in \{1, \ldots, p-1\},\ c \in \{0, \ldots, p-1\} \right\} .$$

Note that $\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathbb{F}_p^{2\times 2} : \alpha, \delta \in \mathbb{F}_p \backslash \{0\} \right\} = \mathrm{U}\left(\begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p \\ 0 & \mathbb{F}_p \end{pmatrix} \right)$.

We have the group isomorphism

$$\begin{aligned}\mathrm{U}\left(\begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p \\ 0 & \mathbb{F}_p \end{pmatrix} \right) &\xrightarrow{\sim} \mathrm{Im}(\mathrm{U}(\varphi)) \\ \begin{pmatrix} a+p\mathbb{Z} & c+p\mathbb{Z} \\ 0 & f+p\mathbb{Z} \end{pmatrix} &\longmapsto \left[a, \begin{pmatrix} a & c \\ 0 & f \end{pmatrix}, f \right]\end{aligned}$$

where surjectivity follows from the description of $\mathrm{Im}(\mathrm{U}(\varphi))$ given above and injectivity follows using Lemma 22(1).

We have the following short exact sequence of groups.

$$\begin{array}{ccccc} \{\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \beta \in \mathbb{F}_p\} & \longrightarrow & \mathrm{U}\left(\begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p \\ 0 & \mathbb{F}_p \end{pmatrix} \right) & \longrightarrow & \mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p) \\ & & \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} & \longmapsto & (\alpha, \delta)\ . \end{array}$$

Note that we have the group morphism

$$\begin{aligned}\mathrm{U}\left(\begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p \\ 0 & \mathbb{F}_p \end{pmatrix} \right) &\xleftarrow{\eta} \mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p) \\ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} &\longleftarrow\!\shortmid (\alpha, \delta)\ ,\end{aligned}$$

so that we can use Remark 2.

We have

$$\left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \beta \in \mathbb{F}_p \right\} \simeq (\mathbb{F}_p, +) \simeq \mathrm{C}_p$$

and

$$\mathrm{U}(\mathbb{F}_p) \simeq \mathrm{C}_{p-1}\ .$$

With the notation of the semi-direct product, Remark 2 gives

$$\mathrm{U}\left(\begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p \\ 0 & \mathbb{F}_p \end{pmatrix}\right) \simeq (\mathbb{F}_p, +) \rtimes (\mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p)) \simeq \mathrm{C}_p \rtimes (\mathrm{C}_{p-1} \times \mathrm{C}_{p-1}) \,,$$

with the action defining on $(\mathbb{F}_p, +) \rtimes (\mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p))$ derived from

$$\begin{aligned} {}^{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & \alpha\beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

to be

$${}^{(\alpha,\delta)}\beta = \alpha\beta\delta^{-1}$$

with $(\alpha, \delta) \in \mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p)$ and $\beta \in \mathbb{F}_p$.

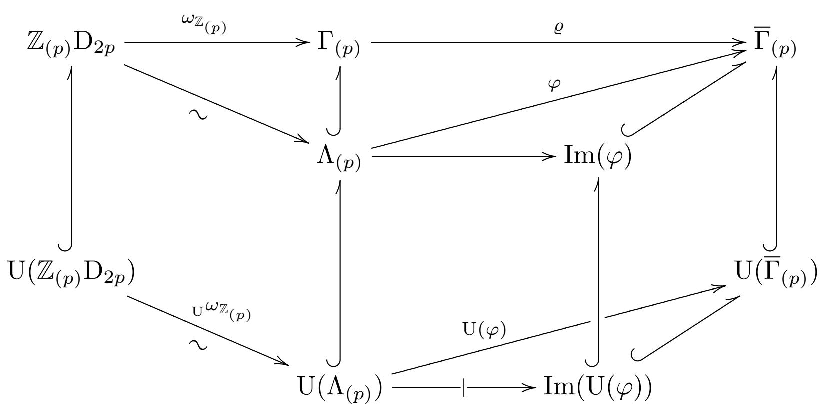
Note that the rules of the action become

1. ${}^{(\alpha,\delta)}0 = \alpha 0 \delta^{-1} = 0$ for $(\alpha, \delta) \in \mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p)$.
2. ${}^{(\alpha,\delta)}(\beta + \tilde{\beta}) = {}^{(\alpha,\delta)}\beta + {}^{(\alpha,\delta)}\tilde{\beta}$ for $(\alpha, \delta) \in \mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p)$ and $\beta \in \mathbb{F}_p$.
3. ${}^{(1,1)}\beta = 1 \cdot \beta \cdot 1^{-1} = \beta$ for $\beta \in \mathbb{F}_p$.
4. ${}^{(\alpha,\delta)\cdot(\tilde{\alpha},\tilde{\delta})}\beta = {}^{(\alpha\tilde{\alpha},\delta\tilde{\delta})}\beta = (\alpha\tilde{\alpha})\beta(\delta\tilde{\delta})^{-1} = \alpha\tilde{\alpha}\beta\tilde{\delta}^{-1}\delta^{-1} = \alpha({}^{(\tilde{\alpha},\tilde{\delta})}\beta)\delta^{-1} = {}^{(\alpha,\delta)}\left({}^{(\tilde{\alpha},\tilde{\delta})}\beta\right)$ for $(\alpha, \delta), (\tilde{\alpha}, \tilde{\delta}) \in \mathrm{U}(\mathbb{F}_p) \times \mathrm{U}(\mathbb{F}_p)$ and $\beta \in \mathbb{F}_p$.

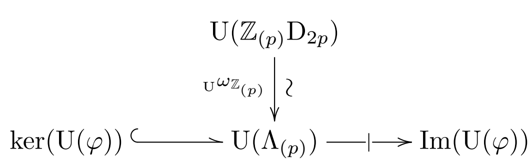
6.4 Summary

In summary we have obtained the following Lemma 27.

Lemma 27. *We obtain the following commutative diagram.*



We have the following diagram of groups, in which the lower row is a short exact sequence.



We have

$$\ker(\mathrm{U}(\varphi)) = \left\{ \left(1+pa,\ \begin{pmatrix} 1+\vartheta b & \vartheta c \\ \vartheta d & 1+\vartheta e \end{pmatrix},\ 1+pf \right) \in \mathrm{U}(\Gamma_{(p)}) : a,\dots,f \in \mathbb{Z}_{(p)} \right\}$$

and

$$\mathrm{Im}(\mathrm{U}(\varphi)) \simeq \mathrm{U}\left(\begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p \\ 0 & \mathbb{F}_p \end{pmatrix} \right) \simeq \mathrm{C}_p \rtimes (\mathrm{C}_{p-1} \times \mathrm{C}_{p-1}) \,.$$

Chapter 7

The group S_5

We consider the symmetric group $G = \mathrm{S}_5$.

7.1 A factor group of $\mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_5)$

7.1.1 Construction of the group morphism $\mathrm{U}(\varphi)$

We claim that we have the following $\mathbb{Q}$-algebra isomorphism of Wedderburn.

$$\omega_{\mathbb{Q}} : \mathbb{Q}\mathrm{S}_5 \longrightarrow \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^{4\times 4} \times \mathbb{Q}^{4\times 4} \times \mathbb{Q}^{5\times 5} \times \mathbb{Q}^{5\times 5} \times \mathbb{Q}^{6\times 6}$$

$$(1,2) \longmapsto \left(-1,\ 1,\ \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 2 & 4 & -4 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -2 & 1 \\ 1 & 1 & 0 & 1 & -2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -3 & -64 & 42 & -12 & -28 \\ 0 & 11 & -5 & 0 & 0 \\ 0 & 24 & -11 & 0 & 0 \\ 3 & 67 & -41 & 10 & 21 \\ -1 & -11 & 8 & -3 & -6 \end{pmatrix}, \begin{pmatrix} -5 & -1850 & -294 & -860 & -600 & -110 \\ 2 & 1025 & 161 & 476 & 328 & 64 \\ -4 & -1680 & -265 & -780 & -540 & -100 \\ -5 & -2627 & -413 & -1220 & -841 & -164 \\ 3 & 1419 & 224 & 659 & 456 & 86 \\ 0 & 134 & 21 & 62 & 42 & 9 \end{pmatrix} \right)$$

$$(1,2,3,4) \longmapsto \left(1,\ 1,\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 6 & 6 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -2 & 1 \\ 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 2 & 2 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 3 & 60 & -38 & 10 & 22 \\ 2 & 40 & -28 & 9 & 20 \\ 5 & 99 & -69 & 22 & 49 \\ 4 & 104 & -73 & 23 & 55 \\ 1 & 9 & -6 & 2 & 3 \end{pmatrix}, \begin{pmatrix} -7 & -3540 & -560 & -1644 & -1138 & -212 \\ 8 & 4408 & 698 & 2049 & 1422 & 270 \\ -13 & -6987 & -1103 & -3246 & -2243 & -426 \\ -18 & -9984 & -1581 & -4641 & -3221 & -612 \\ 7 & 3861 & 610 & 1794 & 1241 & 236 \\ 3 & 1668 & 263 & 775 & 535 & 103 \end{pmatrix} \right)$$

To verify this isomorphism we refer to §C.1.

In order to consider the behaviour at the prime 2 separately, we localise at (2), i.e. now we pass from the ground ring $\mathbb{Q}$ to the ground ring

$$\mathbb{Z}_{(2)} := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \not\equiv_2 0\} \subseteq \mathbb{Q}\,,$$

with $\mathrm{U}(\mathbb{Z}_{(2)}) := \{\frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \in \mathbb{Z}, a \not\equiv_2 0, b \not\equiv_2 0\}$.
By restriction we get the injective map

$$\omega_{\mathbb{Z}_{(2)}} : \mathbb{Z}_{(2)}\mathrm{S}_5 \longrightarrow \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{4\times 4} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{5\times 5} \times \mathbb{Z}_{(2)}^{6\times 6} =: \Gamma_{(2)}\,.$$

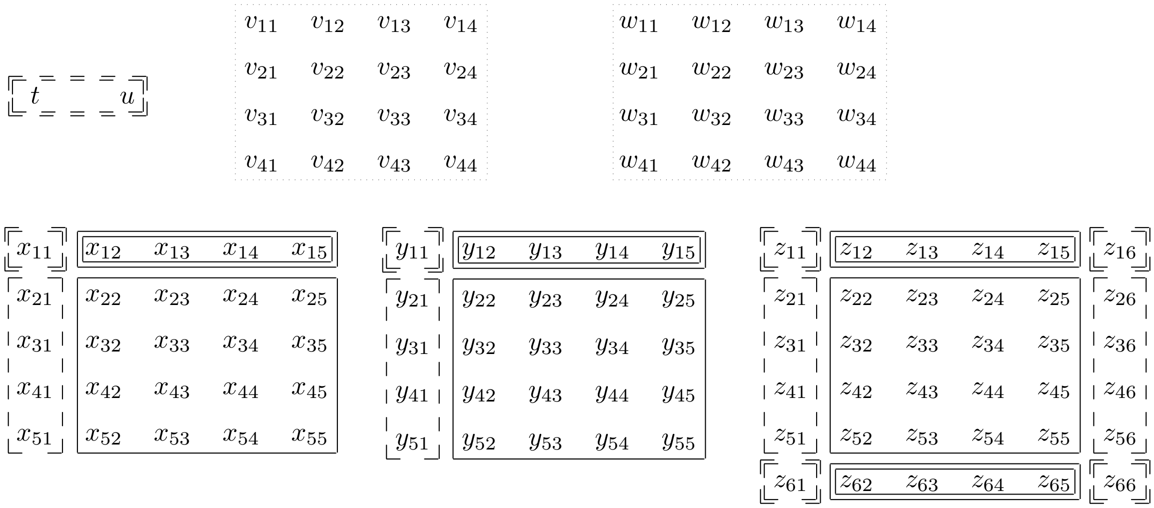
So we get the following.

$$\begin{aligned}
\Lambda_{(2)} &:= \omega_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)}\mathrm{S}_5) \\
&= \left\{ \left(t,\ u,\ \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}, \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix}, \right.\right. \\
&\qquad \left.\begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ y_{21} & y_{22} & y_{23} & y_{24} & y_{25} \\ y_{31} & y_{32} & y_{33} & y_{34} & y_{35} \\ y_{41} & y_{42} & y_{43} & y_{44} & y_{45} \\ y_{51} & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix}, \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} \end{pmatrix} \right) \in \Gamma_{(2)} : \\
&\qquad \left.\begin{array}{llll}
v_{11} \equiv_2 w_{11}, & v_{12} \equiv_2 w_{12}, & v_{13} \equiv_2 w_{13}, & v_{14} \equiv_2 w_{14}, \\
v_{21} \equiv_2 w_{21}, & v_{22} \equiv_2 w_{22}, & v_{23} \equiv_2 w_{23}, & v_{24} \equiv_2 w_{24}, \\
v_{31} \equiv_2 w_{31}, & v_{32} \equiv_2 w_{32}, & v_{33} \equiv_2 w_{33}, & v_{34} \equiv_2 w_{34}, \\
v_{41} \equiv_2 w_{41}, & v_{42} \equiv_2 w_{42}, & v_{43} \equiv_2 w_{43}, & v_{44} \equiv_2 w_{44}, \\
x_{22} + y_{22} \equiv_8 2z_{22}, & x_{23} + y_{23} \equiv_8 2z_{23}, & x_{24} + y_{24} \equiv_8 2z_{24}, & x_{25} + y_{25} \equiv_8 2z_{25}, \\
x_{32} + y_{32} \equiv_8 2z_{32}, & x_{33} + y_{33} \equiv_8 2z_{33}, & x_{34} + y_{34} \equiv_8 2z_{34}, & x_{35} + y_{35} \equiv_8 2z_{35}, \\
x_{42} + y_{42} \equiv_8 2z_{42}, & x_{43} + y_{43} \equiv_8 2z_{43}, & x_{44} + y_{44} \equiv_8 2z_{44}, & x_{45} + y_{45} \equiv_8 2z_{45}, \\
x_{52} + y_{52} \equiv_8 2z_{52}, & x_{53} + y_{53} \equiv_8 2z_{53}, & x_{54} + y_{54} \equiv_8 2z_{54}, & x_{55} + y_{55} \equiv_8 2z_{55}, \\
y_{22} \equiv_2 z_{22}, & y_{23} \equiv_2 z_{23}, & y_{24} \equiv_2 z_{24}, & y_{25} \equiv_2 z_{25}, \\
y_{32} \equiv_2 z_{32}, & y_{33} \equiv_2 z_{33}, & y_{34} \equiv_2 z_{34}, & y_{35} \equiv_2 z_{35}, \\
y_{42} \equiv_2 z_{42}, & y_{43} \equiv_2 z_{43}, & y_{44} \equiv_2 z_{44}, & y_{45} \equiv_2 z_{45}, \\
y_{52} \equiv_2 z_{52}, & y_{53} \equiv_2 z_{53}, & y_{54} \equiv_2 z_{54}, & y_{55} \equiv_2 z_{55}, \\
x_{12} \equiv_2 0, & x_{13} \equiv_2 0, & x_{14} \equiv_2 0, & x_{15} \equiv_2 0, \\
y_{12} \equiv_2 0, & y_{13} \equiv_2 0, & y_{14} \equiv_2 0, & y_{15} \equiv_2 0, \\
z_{12} \equiv_2 0, & z_{13} \equiv_2 0, & z_{14} \equiv_2 0, & z_{15} \equiv_2 0, \\
x_{12} + y_{12} \equiv_8 2z_{12}, & x_{13} + y_{13} \equiv_8 2z_{13}, & x_{14} + y_{14} \equiv_8 2z_{14}, & x_{15} + y_{15} \equiv_8 2z_{15}, \\
y_{12} \equiv_4 2z_{62}, & y_{13} \equiv_4 2z_{63}, & y_{14} \equiv_4 2z_{63}, & y_{15} \equiv_4 2z_{65}, \\
z_{26} \equiv_2 0, & z_{36} \equiv_2 0, & z_{46} \equiv_2 0, & z_{56} \equiv_2 0, \\
x_{21} - y_{21} \equiv_4 z_{26}, & x_{31} - y_{31} \equiv_4 z_{36}, & x_{41} - y_{41} \equiv_4 z_{46}, & x_{51} - y_{51} \equiv_4 z_{56}, \\
y_{21} \equiv_2 z_{21}, & y_{31} \equiv_2 z_{31}, & y_{41} \equiv_2 z_{41}, & y_{51} \equiv_2 z_{51}, \\
t \equiv_2 u, & z_{16} \equiv_2 0, & x_{11} \equiv_2 z_{66}, & x_{11} - y_{11} \equiv_4 z_{16}, \\
u - y_{11} \equiv_4 2z_{61}, & \multicolumn{3}{l}{t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0}
\end{array} \right\}
\end{aligned} \tag{7.1}$$

Here, v_{ij} is short for $v_{i,j}$ etc.

To verify these congruences we refer to §C.2.

For illustration of the congruences note the following picture.



$$v_{ij} \equiv_2 w_{ij},\ i,j \in [1,4]$$

$$x_{ij} + y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5] \qquad y_{ij} \equiv_2 z_{ij},\ i,j \in [2,5],$$

$$x_{1i} \equiv_2 0,\ i \in [2,5] \qquad y_{1i} \equiv_2 0,\ i \in [2,5] \qquad z_{1i} \equiv_2 0,\ i \in [2,5]$$

$$x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5] \qquad y_{1i} \equiv_4 2z_{6i},\ i \in [2,5]$$

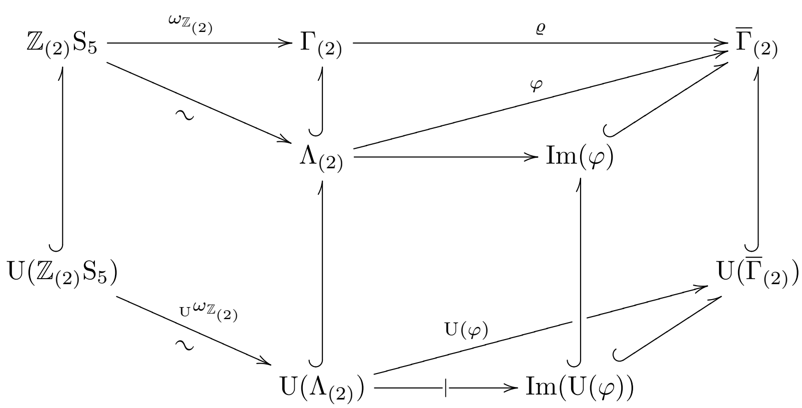
$$z_{i6} \equiv_2 0,\ i \in [2,5] \qquad x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5] \qquad y_{i1} \equiv_2 z_{i1},\ i \in [2,5]$$

$$t \equiv_2 u \qquad z_{16} \equiv_2 0 \qquad x_{11} \equiv_2 z_{66} \qquad x_{11} - y_{11} \equiv_4 z_{16}$$

$$u - y_{11} \equiv_4 2z_{61} \qquad t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 \tag{7.2}$$

Now $2^3\Gamma_{(2)} \subseteq \Lambda_{(2)}$ and the exponent of 3 is minimal with respect to this property. So we consider $\overline{\Gamma}_{(2)} := \Gamma_{(2)}/2^3\Gamma_{(2)} = \Gamma_{(2)}/8\Gamma_{(2)}$. Let $\varrho : \Gamma_{(2)} \longrightarrow \overline{\Gamma}_{(2)}$ be the residue class map.

We obtain the following commutative diagram.



Note that we have the following diagram of groups.

$$
\begin{array}{ccccc}
 & & \mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_5) & & \\
 & & {\scriptstyle \mathrm{U}\,\omega_{\mathbb{Z}_{(2)}}}\big\downarrow \wr & & \\
\ker(\mathrm{U}(\varphi)) & \hookrightarrow & \mathrm{U}(\Lambda_{(2)}) & \twoheadrightarrow & \mathrm{Im}(\mathrm{U}(\varphi))
\end{array}
\tag{7.3}
$$

The lower row is a short exact sequence of groups. In particular, $\mathrm{Im}(\mathrm{U}(\varphi))$ is a factor group of $\mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_5)$.

To calculate $\mathrm{Im}(\mathrm{U}(\varphi))$ with Magma we do not have the possibility to verify for each of the 8^{120} elements of $\overline{\Gamma}_{(2)}$ whether it is contained in $\mathrm{Im}(\mathrm{U}(\varphi))$.

Reminder 28. We recall the following notation. For

$$\big(t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) = \gamma \in \Gamma_{(2)}$$

we write

$$\gamma + 8\Gamma_{(2)} =: [\gamma] = \big[t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \overline{\Gamma}_{(2)}\ .$$

Remark 29. *Let $\xi \in \mathrm{Im}(\mathrm{U}(\varphi))$. Let $\gamma \in \Gamma_{(2)}$ such that $[\gamma] = \xi$. Then $\gamma \in \mathrm{U}(\Lambda_{(2)})$.*

Proof. We may choose $\lambda \in \mathrm{U}(\Lambda_{(2)})$ with $\xi = [\lambda]$. Then $[\gamma] = \xi = [\lambda]$. By Lemma 11, we conclude from $\lambda \in \mathrm{U}(\Lambda_{(2)})$ that $\gamma \in \mathrm{U}(\Lambda_{(2)})$. □

7.1.2 Dissecting $\mathrm{Im}(\mathrm{U}(\varphi))$

Remark 30. *Note that* $\ker(\mathrm{U}(\rho)) = 8\Gamma_{(2)} + 1$. *Suppose given*

$$\mathrm{U}(\Gamma_{(2)}) \geqslant B \geqslant 8\Gamma_{(2)} + 1$$

and

$$\mathrm{U}(\Gamma_{(2)}) \supseteq A\ .$$

Then

$$\overline{A} \cap \overline{B} = \overline{A \cap B}\ .$$

Proof. First we show $\overline{A} \cap \overline{B} \supseteq \overline{A \cap B}$.

Suppose given $x \in A \cap B$. So $[x] \in \overline{A \cap B}$.

We need to show that $[x] \in \overline{A} \cap \overline{B}$.

We have $x \in A \cap B$ and thus $x \in A$ and $x \in B$. So $[x] \in \overline{A}$ and $[x] \in \overline{B}$. And so

$$[x] \in \overline{A} \cap \overline{B}\ .$$

Second we show $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$.

Suppose given $x, y \in \mathrm{U}(\Gamma_{(2)})$ such that $x \in A$, $y \in B$ and $[x] = [y] \in \overline{A} \cap \overline{B}$. We have to show that $[x] \overset{!}{\in} \overline{A \cap B}$. So $x - y =: 8z$, with $z \in \Gamma_{(2)}$.

Then

$$\begin{aligned} x \cdot y^{-1} &= (x - y + y) \cdot y^{-1} \\ &= 8zy^{-1} + 1 \in 8\Gamma_{(2)} + 1\ . \end{aligned}$$

So we get, with $x \cdot y^{-1} \in 8\Gamma_{(2)} + 1 \leqslant B$ and $y \in B$, that

$$x = x \cdot y^{-1} \cdot y \in B\,.$$

With $x \in A$ we have

$$x \in A \cap B\,.$$

So

$$[x] \in \overline{A \cap B}\,.$$

□

Remark 31. Suppose given $\xi \in \overline{\mathrm{U}(\Gamma_{(2)})} = \mathrm{U}(\overline{\Gamma}_{(2)})$. Suppose given $x \in \Gamma_{(2)}$ with $\xi = [x]$.
Note that by Lemma 11, we have

$$\xi \in \overline{\mathrm{U}(\Lambda_{(2)})} \Longleftrightarrow x \in \mathrm{U}(\Lambda_{(2)})\,,$$

with

$$\overline{\mathrm{U}(\Lambda_{(2)})} = \mathrm{Im}(\mathrm{U}(\varphi))\,.$$

In fact, to show "$\Longrightarrow$" we may choose $y \in \mathrm{U}(\Lambda_{(2)}) \subseteq \Gamma_{(2)}$ with $\xi = [y]$. We conclude by Lemma 11 that $x \in \mathrm{U}(\Lambda_{(2)})$.
Cf. also Remark 15.

To verify which elements are in $\mathrm{Im}(\mathrm{U}(\varphi))$ we will dissect the problem in two disjunctive partial problems.
Step 1. We define the subset

$$U_{2,0} := \left\{ \left(1,\ 1,\ \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}, \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \right.\right.$$
$$\left.\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \Gamma_{(2)} : \tag{7.4}$$
$$\left. \det \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix} \not\equiv_2 0\,, v_{ij} \equiv_2 w_{ij},\ i,j \in [1,4] \right\}\,.$$

$$U_2 := \overline{U_{2,0}} = \left\{ \left[1,\ 1,\ \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix},\ \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \right.\right.$$

$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \tag{7.5}$$

$$\left. \det \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix} \not\equiv_2 0\,, v_{ij} \equiv_2 w_{ij},\ i,j \in [1,4] \right\}.$$

We *claim* that $U_2 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$.

By construction we have $U_2 \subseteq \mathrm{Im}(\varphi)$: Every element of U_2 is represented by an element satisfying the congruences describing $\Lambda_{(2)}$, cf. (7.1).

To show that $U_2 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$, we have to verify that each element of U_2 is represented by an element of $\mathrm{U}(\Lambda_{(2)})$.

We have

$$\mathrm{U}(\Lambda_{(2)}) = \Lambda_{(2)} \cap \mathrm{U}(\Gamma_{(2)})\,, \tag{7.6}$$

cf. Lemma 8.

So we have to show that each element of U_2 is represented by an invertible element of $\Gamma_{(2)}$.

Suppose given $\big[1,\ 1,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\big] \in U_2$, represented by the element in brackets; cf. (7.5)

We have $v_{ij} \equiv_2 w_{ij}$ for $i,j \in [1,4]$, hence

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} \equiv_2 \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix}.$$

as a congruence in $\mathbb{Z}_{(2)}^{4\times 4}$. In particular we get

$$\det \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} \equiv_2 \det \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix} \not\equiv_2 0\,.$$

So both these matrices have determinant in $\mathrm{U}(\mathbb{Z}_{(2)})$, hence they are invertible over $\mathbb{Z}_{(2)}$.

This proves the *claim*.

Step 2. We define the subset

$$U_{1,0} = \left\{ \left(t,\ u,\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ \begin{pmatrix} x_{11}&x_{12}&x_{13}&x_{14}&x_{15}\\x_{21}&x_{22}&x_{23}&x_{24}&x_{25}\\x_{31}&x_{32}&x_{33}&x_{34}&x_{35}\\x_{41}&x_{42}&x_{43}&x_{44}&x_{45}\\x_{51}&x_{52}&x_{53}&x_{54}&x_{55} \end{pmatrix}, \right.\right.$$

$$\left.\begin{pmatrix} y_{11}&y_{12}&y_{13}&y_{14}&y_{15}\\y_{21}&y_{22}&y_{23}&y_{24}&y_{25}\\y_{31}&y_{32}&y_{33}&y_{34}&y_{35}\\y_{41}&y_{42}&y_{43}&y_{44}&y_{45}\\y_{51}&y_{52}&y_{53}&y_{54}&y_{55} \end{pmatrix},\ \begin{pmatrix} z_{11}&z_{12}&z_{13}&z_{14}&z_{15}&z_{16}\\z_{21}&z_{22}&z_{23}&z_{24}&z_{25}&z_{26}\\z_{31}&z_{32}&z_{33}&z_{34}&z_{35}&z_{36}\\z_{41}&z_{42}&z_{43}&z_{44}&z_{45}&z_{46}\\z_{51}&z_{52}&z_{53}&z_{54}&z_{55}&z_{56}\\z_{61}&z_{62}&z_{63}&z_{64}&z_{65}&z_{66} \end{pmatrix} \right) \in \Gamma_{(2)} :$$

$$\left. \begin{array}{llll} t \not\equiv_2 0,\ \det \begin{pmatrix} z_{22}&z_{23}&z_{24}&z_{25}\\z_{32}&z_{33}&z_{34}&z_{35}\\z_{42}&z_{43}&z_{44}&z_{45}\\z_{52}&z_{53}&z_{54}&z_{55} \end{pmatrix} \not\equiv_2 0\,, & & & \\ x_{ij}+y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5], & y_{ij} \equiv_2 z_{ij},\ i,j\in[2,5], & & \\ x_{1i} \equiv_2 0,\ i\in[2,5], & y_{1i} \equiv_2 0,\ i\in[2,5], & z_{1i}\equiv_2 0,\ i\in[2,5], & \\ x_{1i}+y_{1i} \equiv_8 2z_{1i}, i\in[2,5], & y_{1i}\equiv_4 2z_{6i},\ i\in[2,5], & & \\ z_{i6}\equiv_2 0,\ i\in[2,5], & x_{i1}-y_{i1}\equiv_4 z_{i6},\ i\in[2,5], & y_{i1}\equiv_2 z_{i1},\ i\in[2,5], & \\ t\equiv_2 u, & z_{16}\equiv_2 0, & x_{11}\equiv_2 z_{66}, & x_{11}-y_{11}\equiv_4 z_{16}, \\ u-y_{11}\equiv_4 2z_{61}, & t+u+x_{11}+y_{11}\ \equiv_8 2z_{11}+2z_{66}\equiv_4 0 & & \end{array} \right\} . \tag{7.7}$$

Then we have

$$U_1 := \overline{U_{1,0}} = \left\{ \left[t,\ u,\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ \begin{pmatrix} x_{11}&x_{12}&x_{13}&x_{14}&x_{15}\\x_{21}&x_{22}&x_{23}&x_{24}&x_{25}\\x_{31}&x_{32}&x_{33}&x_{34}&x_{35}\\x_{41}&x_{42}&x_{43}&x_{44}&x_{45}\\x_{51}&x_{52}&x_{53}&x_{54}&x_{55} \end{pmatrix}, \right.\right.$$

$$\left.\begin{pmatrix} y_{11}&y_{12}&y_{13}&y_{14}&y_{15}\\y_{21}&y_{22}&y_{23}&y_{24}&y_{25}\\y_{31}&y_{32}&y_{33}&y_{34}&y_{35}\\y_{41}&y_{42}&y_{43}&y_{44}&y_{45}\\y_{51}&y_{52}&y_{53}&y_{54}&y_{55} \end{pmatrix},\ \begin{pmatrix} z_{11}&z_{12}&z_{13}&z_{14}&z_{15}&z_{16}\\z_{21}&z_{22}&z_{23}&z_{24}&z_{25}&z_{26}\\z_{31}&z_{32}&z_{33}&z_{34}&z_{35}&z_{36}\\z_{41}&z_{42}&z_{43}&z_{44}&z_{45}&z_{46}\\z_{51}&z_{52}&z_{53}&z_{54}&z_{55}&z_{56}\\z_{61}&z_{62}&z_{63}&z_{64}&z_{65}&z_{66} \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} :$$

$$\left. \begin{array}{llll} t \not\equiv_2 0,\ \det \begin{pmatrix} z_{22}&z_{23}&z_{24}&z_{25}\\z_{32}&z_{33}&z_{34}&z_{35}\\z_{42}&z_{43}&z_{44}&z_{45}\\z_{52}&z_{53}&z_{54}&z_{55} \end{pmatrix} \not\equiv_2 0\,, & & & \\ x_{ij}+y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5], & y_{ij} \equiv_2 z_{ij},\ i,j\in[2,5], & & \\ x_{1i} \equiv_2 0,\ i\in[2,5], & y_{1i} \equiv_2 0,\ i\in[2,5], & z_{1i}\equiv_2 0,\ i\in[2,5], & \\ x_{1i}+y_{1i} \equiv_8 2z_{1i}, i\in[2,5], & y_{1i}\equiv_4 2z_{6i},\ i\in[2,5], & & \\ z_{i6}\equiv_2 0,\ i\in[2,5], & x_{i1}-y_{i1}\equiv_4 z_{i6},\ i\in[2,5], & y_{i1}\equiv_2 z_{i1},\ i\in[2,5], & \\ t\equiv_2 u, & z_{16}\equiv_2 0, & x_{11}\equiv_2 z_{66}, & x_{11}-y_{11}\equiv_4 z_{16}, \\ u-y_{11}\equiv_4 2z_{61}, & t+u+x_{11}+y_{11}\ \equiv_8 2z_{11}+2z_{66}\equiv_4 0 & & \end{array} \right\} . \tag{7.8}$$

We *claim* that $U_1 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$.

By construction we have $U_1 \subseteq \mathrm{Im}(\varphi)$: Every element of U_1 is represented by an element satisfying the congruences describing $\Lambda_{(2)}$, cf. (7.1).

To show that $U_1 \overset{!}{\subseteq} \mathrm{Im}(\mathrm{U}(\varphi))$, we have to verify that each element of U_1 is represented by an element of $\mathrm{U}(\Lambda_{(2)})$.

Remark 32. *Suppose given* $\lambda = \big(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \Lambda_{(2)}$.

(1) *We have* $t \equiv_2 u \equiv_2 x_{11} \equiv_2 y_{11} \equiv_2 z_{11} \equiv_2 z_{66}$.

(2) *We have*

$$\begin{pmatrix} x_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & x_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & x_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix} \equiv_2 \begin{pmatrix} y_{22} & y_{23} & y_{24} & y_{25} \\ y_{32} & y_{33} & y_{34} & y_{35} \\ y_{42} & y_{43} & y_{44} & y_{45} \\ y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix} \equiv_2 \begin{pmatrix} z_{22} & z_{23} & z_{24} & z_{25} \\ z_{32} & z_{33} & z_{34} & z_{35} \\ z_{42} & z_{43} & z_{44} & z_{45} \\ z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} .$$

(3) *We have*

$$\lambda \in \mathrm{U}(\Lambda_{(2)}) \Longleftrightarrow t \not\equiv_2 0 \ \text{ and } \ \det \begin{pmatrix} z_{22} & z_{23} & z_{24} & z_{25} \\ z_{32} & z_{33} & z_{34} & z_{35} \\ z_{42} & z_{43} & z_{44} & z_{45} \\ z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \not\equiv_2 0$$

Proof. Ad (1). We need to show that

$$t \equiv_2 u \equiv_2 x_{11} \equiv_2 y_{11} \equiv_2 z_{11} \equiv_2 z_{66} .$$

We have due to the congruences of U_1

$$t \equiv_2 u . \tag{7.9}$$

With

$$z_{16} \equiv_2 0 \ \text{ and } \ x_{11} - y_{11} \equiv_2 z_{16}$$

we get

$$x_{11} \equiv_2 y_{11} . \tag{7.10}$$

With

$$2z_{11} + 2z_{66} \equiv_4 0$$

we get

$$z_{11} \equiv_2 z_{66} . \tag{7.11}$$

With

$$u - y_{11} \equiv_4 2z_{61} \equiv_2 0$$

we get

$$u \equiv_2 y_{11} . \tag{7.12}$$

We have

$$x_{11} \equiv_2 z_{66} . \tag{7.13}$$

So we have

$$t \overset{(7.9)}{\equiv_2} u \overset{(7.12)}{\equiv_2} y_{11} \overset{(7.10)}{\equiv_2} x_{11} \overset{(7.13)}{\equiv_2} z_{66} \overset{(7.11)}{\equiv_2} z_{11} .$$

Ad (2). We show that in $\mathbb{Z}_{(2)}^{4\times 4}$ we have

$$\begin{pmatrix} x_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & x_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & x_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix} \equiv_2 \begin{pmatrix} y_{22} & y_{23} & y_{24} & y_{25} \\ y_{32} & y_{33} & y_{34} & y_{35} \\ y_{42} & y_{43} & y_{44} & y_{45} \\ y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix} \equiv_2 \begin{pmatrix} z_{22} & z_{23} & z_{24} & z_{25} \\ z_{32} & z_{33} & z_{34} & z_{35} \\ z_{42} & z_{43} & z_{44} & z_{45} \\ z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} .$$

Suppose given $i, j \in [2,5]$. Due to the congruences of U_1 we have

$$x_{ij} + y_{ij} \equiv_8 2z_{ij}$$

it follows

$$x_{ij} \equiv_2 y_{ij} .$$

And we have

$$y_{ij} \equiv_2 z_{ij} .$$

So we have that

$$\begin{pmatrix} x_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & x_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & x_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix} \equiv_2 \begin{pmatrix} y_{22} & y_{23} & y_{24} & y_{25} \\ y_{32} & y_{33} & y_{34} & y_{35} \\ y_{42} & y_{43} & y_{44} & y_{45} \\ y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix} \equiv_2 \begin{pmatrix} z_{22} & z_{23} & z_{24} & z_{25} \\ z_{32} & z_{33} & z_{34} & z_{35} \\ z_{42} & z_{43} & z_{44} & z_{45} \\ z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} .$$

Ad (3).

Ad $\Rightarrow$. Since $z_{12} \equiv_2 z_{13} \equiv_2 z_{14} \equiv_2 z_{15} \equiv_2 0$ and $z_{16} \equiv_2 0$ and $z_{26} \equiv_2 z_{36} \equiv_2 z_{46} \equiv_2 z_{56} \equiv_2 0$, invertibility of λ implies not only $t \not\equiv_2 0$, but also $\det \begin{pmatrix} z_{22} & z_{23} & z_{24} & z_{25} \\ z_{32} & z_{33} & z_{34} & z_{35} \\ z_{42} & z_{43} & z_{44} & z_{45} \\ z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \not\equiv_2 0$.

Ad $\Leftarrow$. We have $\mathrm{U}(\Lambda_{(2)}) = \Lambda_{(2)} \cap \mathrm{U}(\Gamma_{(2)})$, cf. Lemma 8.

We want to show that this tuple λ is invertible in $\Gamma_{(2)}$.

We have

$$x_{12} \equiv_2 x_{13} \equiv_2 x_{14} \equiv_2 x_{15} \equiv_2 0,$$
$$y_{12} \equiv_2 y_{13} \equiv_2 y_{14} \equiv_2 y_{15} \equiv_2 0$$

and

$$z_{12} \equiv_2 z_{13} \equiv_2 z_{14} \equiv_2 z_{15} \equiv_2 z_{16} \equiv_2 0,$$
$$z_{16} \equiv_2 z_{26} \equiv_2 z_{36} \equiv_2 z_{46} \equiv_2 z_{56} \equiv_2 0$$

directly due to the congruences of U_1. So we have that $(x_{ij})_{i,j\in[1,5]}$, $(y_{ij})_{i,j\in[1,5]}$, $(z_{ij})_{i,j\in[1,6]}$ are all lower block triangular matrices modulo 2.

In particular, by (2) we get

$$\det\begin{pmatrix} x_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & x_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & x_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix} \equiv_2 \det\begin{pmatrix} y_{22} & y_{23} & y_{24} & y_{25} \\ y_{32} & y_{33} & y_{34} & y_{35} \\ y_{42} & y_{43} & y_{44} & y_{45} \\ y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix} \equiv_2 \det\begin{pmatrix} z_{22} & z_{23} & z_{24} & z_{25} \\ z_{32} & z_{33} & z_{34} & z_{35} \\ z_{42} & z_{43} & z_{44} & z_{45} \\ z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \not\equiv_2 0\,.$$

So λ is invertible in $\Gamma_{(2)}$. □

Suppose given $\big[t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in U_1$, represented by the element in brackets; cf. (7.8). By Remark 32, it is represented by an element in $\mathrm{U}(\Lambda_{(2)})$.

This proves the *claim*.

Step 3: We *claim* that $U_1, U_2 \trianglelefteqslant \mathrm{Im}(\mathrm{U}(\varphi))$.

We define the following normal subgroups of $\mathrm{U}(\Gamma_{(2)})$,

$$\tilde{U}_{1,0} := \left\{ \begin{array}{l} \big(t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \mathrm{U}(\Gamma_{(2)}) : \\ (v_{ij})_{i,j\in[1,4]} \equiv_8 (w_{ij})_{i,j\in[1,4]} \equiv_8 \mathrm{E}_4, \\ t \not\equiv_2 0, u \not\equiv_2 0,\ (x_{ij})_{i,j\in[1,5]}, (y_{ij})_{i,j\in[1,5]} \in \mathrm{GL}_5(\mathbb{Z}_{(2)}),\ (z_{ij})_{i,j\in[1,6]} \in \mathrm{GL}_6(\mathbb{Z}_{(2)}) \end{array} \right\} \geqslant 1 + 8\Gamma_{(2)}$$

and

$$\tilde{U}_{2,0} := \left\{ \begin{array}{l} \big(t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \mathrm{U}(\Gamma_{(2)}) : \\ t \equiv_8 u \equiv_8 1,\ (x_{ij})_{i,j\in[1,5]} \equiv_8 (y_{ij})_{i,j\in[1,5]} \equiv_8 \mathrm{E}_5,\ (z_{ij})_{i,j\in[1,6]} \equiv_8 \mathrm{E}_6, \\ (v_{ij})_{i,j\in[1,4]}, (w_{ij})_{i,j\in[1,4]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}) \end{array} \right\} \geqslant 1 + 8\Gamma_{(2)}\,.$$

We have

$$\tilde{U}_{1,0} \cap \tilde{U}_{2,0} = 1 + 8\Gamma_{(2)}$$

by construction. So

$$\overline{\tilde{U}_{1,0}} \cap \overline{\tilde{U}_{2,0}} \overset{\text{Remark 30}}{=} \overline{\tilde{U}_{1,0} \cap \tilde{U}_{2,0}} = \overline{1 + 8\Gamma_{(2)}} = 1\,.$$

We define the corresponding images of $\tilde{U}_{1,0}$ and $\tilde{U}_{2,0}$ under the residue class map as

$$\begin{aligned}
\tilde{U}_1 := \overline{\tilde{U}_{1,0}} = \ & \Big\{ \big[t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \mathrm{U}(\overline{\Gamma}_{(2)}) : \\
& (v_{ij})_{i,j\in[1,4]} \equiv_8 \mathrm{E}_4, (w_{ij})_{i,j\in[1,4]} \equiv_8 \mathrm{E}_4, \\
& t \not\equiv_2 0, u \not\equiv_2 0,\ (x_{ij})_{i,j\in[1,5]}, (y_{ij})_{i,j\in[1,5]} \in \mathrm{GL}_5(\mathbb{Z}_{(2)}),\ (z_{ij})_{i,j\in[1,6]} \in \mathrm{GL}_6(\mathbb{Z}_{(2)}) \Big\} \\
= \ & \Big\{ \big[t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \mathrm{U}(\overline{\Gamma}_{(2)}) : \\
& (v_{ij})_{i,j\in[1,4]} = (w_{ij})_{i,j\in[1,4]} = \mathrm{E}_4, \\
& t \not\equiv_2 0, u \not\equiv_2 0,\ (x_{ij})_{i,j\in[1,5]}, (y_{ij})_{i,j\in[1,5]} \in \mathrm{GL}_5(\mathbb{Z}_{(2)}),\ (z_{ij})_{i,j\in[1,6]} \in \mathrm{GL}_6(\mathbb{Z}_{(2)}) \Big\} \\
\leqslant \ & \mathrm{U}(\overline{\Gamma}_{(2)})
\end{aligned}$$

and

$$\begin{aligned}
\tilde{U}_2 := \overline{\tilde{U}_{2,0}} = \ & \Big\{ \big[t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \mathrm{U}(\overline{\Gamma}_{(2)}) : \\
& t \equiv_8 1, u \equiv_8 1,\ (x_{ij})_{i,j\in[1,5]} \equiv_8 \mathrm{E}_5, (y_{ij})_{i,j\in[1,5]} \equiv_8 \mathrm{E}_5,\ (z_{ij})_{i,j\in[1,6]} \equiv_8 \mathrm{E}_6, \\
& (v_{ij})_{i,j\in[1,4]}, (w_{ij})_{i,j\in[1,4]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}) \Big\} \\
= \ & \Big\{ \big[t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \mathrm{U}(\overline{\Gamma}_{(2)}) : \\
& t = u = 1,\ (x_{ij})_{i,j\in[1,5]} = (y_{ij})_{i,j\in[1,5]} = \mathrm{E}_5,\ (z_{ij})_{i,j\in[1,6]} = \mathrm{E}_6, \\
& (v_{ij})_{i,j\in[1,4]}, (w_{ij})_{i,j\in[1,4]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}) \Big\} \leqslant \mathrm{U}(\overline{\Gamma}_{(2)})\,.
\end{aligned}$$

So

$$\begin{aligned}
1 &= \overline{\tilde{U}_{1,0} \cap \tilde{U}_{2,0}} \\
&= \overline{\tilde{U}_{1,0}} \cap \overline{\tilde{U}_{2,0}} \\
&= \tilde{U}_1 \cap \tilde{U}_2\,.
\end{aligned} \tag{7.14}$$

Since $1 + 8\Gamma_{(2)} \leqslant \tilde{U}_{1,0}, \tilde{U}_{2,0} \trianglelefteqslant \mathrm{U}(\Gamma_{(2)})$ and since $\mathrm{U}(\rho) : \mathrm{U}(\Gamma_{(2)}) \longrightarrow \mathrm{U}(\overline{\Gamma}_{(2)})$ is surjective with kernel $1 + 8\Gamma_{(2)}$, we have

$$\tilde{U}_1, \tilde{U}_2 \trianglelefteqslant \mathrm{U}(\overline{\Gamma}_{(2)})\,.$$

We claim that

$$U_1 = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 \tag{7.15}$$

and

$$U_2 = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_2 \tag{7.16}$$

First we show (7.16).

Regarding Remark 30 let

$$B = \tilde{U}_{2,0}$$

so $\overline{B} = \overline{\tilde{U}_{2,0}} = \tilde{U}_2$ with $1 + 8\Gamma_{(2)} \leqslant B \leqslant \mathrm{U}(\Gamma_{(2)})$ and let

$$A = \mathrm{U}(\Lambda_{(2)}) \overset{(7.6)}{=} \mathrm{U}(\Gamma_{(2)}) \cap \Lambda_{(2)}$$

so $\overline{A} = \overline{\mathrm{U}(\Lambda_{(2)})} = \mathrm{Im}(\mathrm{U}(\varphi))$ with $A \subseteq \mathrm{U}(\Gamma_{(2)})$.

We have

$$1 + 8\Gamma_{(2)} \subseteq \Lambda_{(2)}$$

as we take from (7.1). So we have

$$1 + 8\Gamma_{(2)} \leqslant \mathrm{U}(\Gamma_{(2)}) \cap \Lambda_{(2)} \leqslant \mathrm{U}(\Gamma_{(2)})\,.$$

Note that

$$\begin{aligned} \mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{2,0} = \Bigg\{ & \big(t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \Lambda_{(2)} : \\ & t \equiv_8 u \equiv_8 1,\ (x_{ij})_{i,j\in[1,5]} \equiv_8 (y_{ij})_{i,j\in[1,5]} \equiv_8 \mathrm{E}_5,\ (z_{ij})_{i,j\in[1,6]} \equiv_8 \mathrm{E}_6, \\ & (v_{ij})_{i,j\in[1,4]}, (w_{ij})_{i,j\in[1,4]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}) \Bigg\} \\ \overset{\text{Step 1}}{=} \Bigg\{ & \big(t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \Gamma_{(2)} : \\ & t \equiv_8 u \equiv_8 1,\ (x_{ij})_{i,j\in[1,5]} \equiv_8 (y_{ij})_{i,j\in[1,5]} \equiv_8 \mathrm{E}_5,\ (z_{ij})_{i,j\in[1,6]} \equiv_8 \mathrm{E}_6, \\ & (v_{ij})_{i,j\in[1,4]} \equiv_2 (w_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}) \Bigg\} \end{aligned}$$

We have

$$\overline{A} \cap \overline{B} = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_2$$

and

$$\overline{A \cap B} = \overline{\mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{2,0}}\,.$$

So with Remark 30 we have

$$\mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_2 = \overline{A} \cap \overline{B} = \overline{A \cap B} = \overline{\mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{2,0}}\,.$$

So we get

$$\begin{aligned}
\overline{\mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{2,0}} = \Bigg\{ & \big[t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \mathrm{U}(\Gamma_{(2)}) : \\
& t \equiv_8 u \equiv_8 1,\ (x_{ij})_{i,j\in[1,5]} \equiv_8 (y_{ij})_{i,j\in[1,5]} \equiv_8 \mathrm{E}_5,\ (z_{ij})_{i,j\in[1,6]} \equiv_8 \mathrm{E}_6, \\
& (v_{ij})_{i,j\in[1,4]} \equiv_2 (w_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}) \Bigg\} \\
= \Bigg\{ & \big[1,\ 1,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\big] \in \mathrm{U}(\Gamma_{(2)}) : \\
& (v_{ij})_{i,j\in[1,4]} \equiv_2 (w_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}) \Bigg\} \\
= & U_2
\end{aligned}$$

This shows (7.16).

Second we show (7.15), i.e.

$$U_1 \overset{!}{=} \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 \ .$$

Regarding Remark 30, let

$$B = \tilde{U}_{1,0} \ ,$$

so $\overline{B} = \overline{\tilde{U}_{1,0}} = \tilde{U}_1$ with $1 + 8\Gamma_{(2)} \leqslant B \leqslant \mathrm{U}(\Gamma_{(2)})$, and let

$$A = \mathrm{U}(\Lambda_{(2)}) \overset{(7.6)}{=} \mathrm{U}(\Gamma_{(2)}) \cap \Lambda_{(2)} \ ,$$

so $\overline{A} = \overline{\mathrm{U}(\Lambda_{(2)})} = \mathrm{Im}(\mathrm{U}(\varphi))$ with $A \subseteq \mathrm{U}(\Gamma_{(2)})$.

We have

$$1 + 8\Gamma_{(2)} \subseteq \Lambda_{(2)}$$

as we take from (7.1). So we have

$$1 + 8\Gamma_{(2)} \leqslant \mathrm{U}(\Gamma_{(2)}) \cap \Lambda_{(2)} \leqslant \mathrm{U}(\Gamma_{(2)}) \ .$$

We have

$$\overline{A} \cap \overline{B} = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1$$

and

$$\overline{A \cap B} = \overline{\mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{1,0}} \ .$$

So with Remark 30 we have

$$\mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 = \overline{A} \cap \overline{B} = \overline{A \cap B} = \overline{\mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{1,0}} \ .$$

Note that

$$
\begin{aligned}
\mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{1,0} = \Bigg\{ & \big(t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \Lambda_{(2)} : \\
& (v_{ij})_{i,j\in[1,4]} \equiv_8 (w_{ij})_{i,j\in[1,4]} \equiv_8 \mathrm{E}_4, \\
& t \not\equiv_2 0, u \not\equiv_2 0,\ (x_{ij})_{i,j\in[1,5]}, (y_{ij})_{i,j\in[1,5]} \in \mathrm{GL}_5(\mathbb{Z}_{(2)}),\ (z_{ij})_{i,j\in[1,6]} \in \mathrm{GL}_6(\mathbb{Z}_{(2)}) \Bigg\} \\
\overset{\text{Remark 32}}{=} \Bigg\{ & \big(t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \Gamma_{(2)} : \\
& (v_{ij})_{i,j\in[1,4]} \equiv_8 (w_{ij})_{i,j\in[1,4]} \equiv_8 \mathrm{E}_4,\ t \not\equiv_2 0,\ (z_{ij})_{i,j\in[2,5]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}), \\
& x_{ij} + y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5];\ y_{ij} \equiv_2 z_{ij},\ i,j \in [2,5]; \\
& x_{1i} \equiv_2 0,\ i \in [2,5];\ y_{1i} \equiv_2 0,\ i \in [2,5];\ z_{1i} \equiv_2 0,\ i \in [2,5]; \\
& x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5];\ y_{1i} \equiv_4 2z_{6i},\ i \in [2,5]; \\
& z_{i6} \equiv_2 0,\ i \in [2,5];\ x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5];\ y_{i1} \equiv_2 z_{i1},\ i \in [2,5]; \\
& t \equiv_2 u,\ z_{16} \equiv_2 0,\ x_{11} \equiv_2 z_{66},\ x_{11} - y_{11} \equiv_4 z_{16}, \\
& u - y_{11} \equiv_4 2z_{61},\ t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 \Bigg\}
\end{aligned}
$$

So we get

$$
\begin{aligned}
\overline{\mathrm{U}(\Lambda_{(2)}) \cap \tilde{U}_{1,0}} = \Bigg\{ & \big[t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \overline{\Gamma}_{(2)} : \\
& (v_{ij})_{i,j\in[1,4]} \equiv_8 (w_{ij})_{i,j\in[1,4]} \equiv_8 \mathrm{E}_4,\ t \not\equiv_2 0,\ (z_{ij})_{i,j\in[2,5]} \in \mathrm{GL}_4(\mathbb{Z}_{(2)}), \\
& x_{ij} + y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5];\ y_{ij} \equiv_2 z_{ij},\ i,j \in [2,5]; \\
& x_{1i} \equiv_2 0,\ i \in [2,5];\ y_{1i} \equiv_2 0,\ i \in [2,5];\ z_{1i} \equiv_2 0,\ i \in [2,5]; \\
& x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5];\ y_{1i} \equiv_4 2z_{6i},\ i \in [2,5]; \\
& z_{i6} \equiv_2 0,\ i \in [2,5];\ x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5];\ y_{i1} \equiv_2 z_{i1},\ i \in [2,5]; \\
& t \equiv_2 u,\ z_{16} \equiv_2 0,\ x_{11} \equiv_2 z_{66},\ x_{11} - y_{11} \equiv_4 z_{16}, \\
& u - y_{11} \equiv_4 2z_{61},\ t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 \Bigg\} \\
= \Bigg\{ & \big[t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \in \overline{\Gamma}_{(2)} : \\
& t \not\equiv_2 0,\ \det(z_{ij})_{i,j\in[2,5]} \not\equiv_2 0, \\
& x_{ij} + y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5];\ y_{ij} \equiv_2 z_{ij},\ i,j \in [2,5]; \\
& x_{1i} \equiv_2 0,\ i \in [2,5];\ y_{1i} \equiv_2 0,\ i \in [2,5];\ z_{1i} \equiv_2 0,\ i \in [2,5]; \\
& x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5];\ y_{1i} \equiv_4 2z_{6i},\ i \in [2,5]; \\
& z_{i6} \equiv_2 0,\ i \in [2,5];\ x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5];\ y_{i1} \equiv_2 z_{i1},\ i \in [2,5]; \\
& t \equiv_2 u,\ z_{16} \equiv_2 0,\ x_{11} \equiv_2 z_{66},\ x_{11} - y_{11} \equiv_4 z_{16}, \\
& u - y_{11} \equiv_4 2z_{61},\ t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 \Bigg\} \\
= U_1 &
\end{aligned}
$$

This shows (7.15).

We have $\mathrm{Im}(\mathrm{U}(\varphi)) \leqslant \mathrm{U}(\overline{\Gamma}_{(2)})$ and $\tilde{U}_1 \trianglelefteq \mathrm{U}(\overline{\Gamma}_{(2)})$.

We have

$$U_1 = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 \leqslant \tilde{U}_1$$

and

$$U_1 = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 \trianglelefteq \mathrm{Im}(\mathrm{U}(\varphi))$$

and

$$U_2 = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_2 \leqslant \tilde{U}_2$$

and

$$U_2 = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_2 \trianglelefteq \mathrm{Im}(\mathrm{U}(\varphi)) \, .$$

We obtain the following commutative diagrams.

$$\begin{array}{ccc} \mathrm{Im}(\mathrm{U}(\varphi)) & \xrightarrow{\leqslant} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ {\scriptstyle\trianglelefteq}\uparrow & & \uparrow{\scriptstyle\trianglelefteq} \\ U_1 & \xrightarrow{\leqslant} & \tilde{U}_1 \end{array}$$

$$\begin{array}{ccc} \mathrm{Im}(\mathrm{U}(\varphi)) & \xrightarrow{\leqslant} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ {\scriptstyle\trianglelefteq}\uparrow & & \uparrow{\scriptstyle\trianglelefteq} \\ U_2 & \xrightarrow{\leqslant} & \tilde{U}_2 \end{array}$$

Step 4: We want to show that

$$U_1 \cap U_2 = 1 \, .$$

In fact,

$$U_1 \cap U_2 \overset{\substack{(7.15)\\(7.16)}}{=} \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 \cap \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_2 = \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 \cap \tilde{U}_2 \overset{(7.14)}{=} 1 \, .$$

Now we want to show that

$$U_1 \cdot U_2 = \mathrm{Im}(\mathrm{U}(\varphi)) \, .$$

Recall the following picture of congruences sorted in two areas; cf. (7.2).

$$\boxed{t \qquad u} \qquad \begin{array}{|cccc|} \hline v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \\ \hline \end{array} \qquad \begin{array}{|cccc|} \hline w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \\ \hline \end{array}$$

$$\begin{array}{|ccccc|} \hline x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \\ \hline \end{array} \qquad \begin{array}{|ccccc|} \hline y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ y_{21} & y_{22} & y_{23} & y_{24} & y_{25} \\ y_{31} & y_{32} & y_{33} & y_{34} & y_{35} \\ y_{41} & y_{42} & y_{43} & y_{44} & y_{45} \\ y_{51} & y_{52} & y_{53} & y_{54} & y_{55} \\ \hline \end{array} \qquad \begin{array}{|cccccc|} \hline z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} \\ \hline \end{array}$$

$$\boxed{v_{ij} \equiv_2 w_{ij},\ i,j \in [1,4]}$$

$$\begin{array}{|llll|} \hline x_{ij} + y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5] & y_{ij} \equiv_2 z_{ij},\ i,j \in [2,5], & & \\ x_{1i} \equiv_2 0,\ i \in [2,5] & y_{1i} \equiv_2 0,\ i \in [2,5] & z_{1i} \equiv_2 0,\ i \in [2,5] & \\ x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5] & y_{1i} \equiv_4 2z_{6i},\ i \in [2,5] & & \\ z_{i6} \equiv_2 0,\ i \in [2,5] & x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5] & y_{i1} \equiv_2 z_{i1},\ i \in [2,5] & \\ t \equiv_2 u & z_{16} \equiv_2 0 & x_{11} \equiv_2 z_{66} & x_{11} - y_{11} \equiv_4 z_{16} \\ u - y_{11} \equiv_4 2z_{61} & t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 & & \\ \hline \end{array}$$

Suppose given $[\lambda] \in \mathrm{Im}(\mathrm{U}(\varphi))$. We write

$$\lambda =: (t,\ u,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}) \in \mathrm{U}(\Lambda_{(2)})$$

Let

$$\begin{aligned} \lambda_1 &:= (t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}) \in \mathrm{U}(\Gamma_{(2)}) \\ \lambda_2 &:= (1,\ 1,\ (v_{ij})_{i,j\in[1,4]},\ (w_{ij})_{i,j\in[1,4]},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6) \in \mathrm{U}(\Gamma_{(2)}) \end{aligned}$$

We have $\lambda_1, \lambda_2 \in \Lambda_{(2)} \cap \mathrm{U}(\Gamma_{(2)}) = \mathrm{U}(\Lambda_{(2)})$ as we take from the congruences. We have $\lambda_1 \cdot \lambda_2 = \lambda$. So we have

$$[\lambda_1] \cdot [\lambda_2] = [\lambda]$$

with $[\lambda_1] \in \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_1 \overset{(7.15)}{=} U_1$ and $[\lambda_2] \in \mathrm{Im}(\mathrm{U}(\varphi)) \cap \tilde{U}_2 \overset{(7.16)}{=} U_2$.
Altogether we get $\mathrm{Im}(\mathrm{U}(\varphi))$ as the internal direct product

$$\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \times U_2\,.$$

To calculate U_1 via Magma we will dissect the problem in two steps to verify which elements are in U_1.
Step 1. To begin we define the subset

$$\tilde{S}_1 := \left\{ \left[1,\ 1,\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix}, \right.\right.$$
$$\left.\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & y_{42} & y_{43} & y_{44} & y_{45} \\ 0 & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} & 0 \\ 0 & z_{32} & z_{33} & z_{34} & z_{35} & 0 \\ 0 & z_{42} & z_{43} & z_{44} & z_{45} & 0 \\ 0 & z_{52} & z_{53} & z_{54} & z_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} :$$
$$\left. \det \begin{pmatrix} x_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & x_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & x_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix} \not\equiv_2 0,\ \det \begin{pmatrix} y_{22} & y_{23} & y_{24} & y_{25} \\ y_{32} & y_{33} & y_{34} & y_{35} \\ y_{42} & y_{43} & y_{44} & y_{45} \\ y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix} \not\equiv_2 0,\ \det \begin{pmatrix} z_{22} & z_{23} & z_{24} & z_{25} \\ z_{32} & z_{33} & z_{34} & z_{35} \\ z_{42} & z_{43} & z_{44} & z_{45} \\ z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \not\equiv_2 0 \right\}$$
$$\leqslant \tilde{U}_1 \leqslant \mathrm{U}(\overline{\Gamma}_{(2)})$$

Let

$$\tilde{S}_{1,0} := \mathrm{U}(\rho)^{-1}(\tilde{S}_1) \geqslant \ker(\mathrm{U}(\rho)) = 8\Gamma_{(2)} + 1\,.$$

Note that $\tilde{S}_{1,0} \leqslant \mathrm{U}(\Gamma_{(2)})$.
Then

$$\overline{\tilde{S}_{1,0}} = \mathrm{U}(\rho)(\tilde{S}_{1,0}) = \tilde{S}_1\,.$$

Let

$$S_{1,0} := \tilde{S}_{1,0} \cap \mathrm{U}(\Lambda_{(2)}) \leqslant \mathrm{U}(\Lambda_{(2)})\,.$$

Note that

$$\begin{aligned} S_{1,0} &= \tilde{S}_{1,0} \cap \mathrm{U}(\Lambda_{(2)}) \\ &\overset{(7.6)}{=} \tilde{S}_{1,0} \cap \mathrm{U}(\Gamma_{(2)}) \cap \Lambda_{(2)} \\ &= \tilde{S}_{1,0} \cap \Lambda_{(2)}\,. \end{aligned}$$

We have

$$\begin{array}{ccl} S_1 & := & \overline{S_{1,0}} \\ & = & \overline{\tilde{S}_{1,0} \cap \mathrm{U}(\Lambda_{(2)})} \\ & \overset{\text{Remark 30}}{=} & \overline{\tilde{S}_{1,0}} \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & = & \tilde{S}_1 \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & \leqslant & \tilde{U}_1 \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & \overset{(7.15)}{=} & U_1\,. \end{array}$$

Note that

$$S_1 = \left\{ \left[1,\ 1,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0&0&0\\0&x_{22}&x_{23}&x_{24}&x_{25}\\0&x_{32}&x_{33}&x_{34}&x_{35}\\0&x_{42}&x_{43}&x_{44}&x_{45}\\0&x_{52}&x_{53}&x_{54}&x_{55}\end{pmatrix}, \right.\right.$$

$$\left.\begin{pmatrix}1&0&0&0&0\\0&y_{22}&y_{23}&y_{24}&y_{25}\\0&y_{32}&y_{33}&y_{34}&y_{35}\\0&y_{42}&y_{43}&y_{44}&y_{45}\\0&y_{52}&y_{53}&y_{54}&y_{55}\end{pmatrix},\ \begin{pmatrix}1&0&0&0&0&0\\0&z_{22}&z_{23}&z_{24}&z_{25}&0\\0&z_{32}&z_{33}&z_{34}&z_{35}&0\\0&z_{42}&z_{43}&z_{44}&z_{45}&0\\0&z_{52}&z_{53}&z_{54}&z_{55}&0\\0&0&0&0&0&1\end{pmatrix}\right] \in \overline{\Gamma}_{(2)} : \tag{7.17}$$

$$\left. \det\begin{pmatrix}z_{22}&z_{23}&z_{24}&z_{25}\\z_{32}&z_{33}&z_{34}&z_{35}\\z_{42}&z_{43}&z_{44}&z_{45}\\z_{52}&z_{53}&z_{54}&z_{55}\end{pmatrix} \not\equiv_2 0,\ x_{ij}+y_{ij} \equiv_8 2z_{ij} \text{ for } i,j\in[2,5],\ y_{ij}\equiv_2 z_{ij} \text{ for } i,j\in[2,5] \right\}.$$

We define the subset

$$M_1 := \left\{ \left[t,\ u,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}x_{11}&x_{12}&x_{13}&x_{14}&x_{15}\\x_{21}&1&0&0&0\\x_{31}&0&1&0&0\\x_{41}&0&0&1&0\\x_{51}&0&0&0&1\end{pmatrix}, \right.\right.$$

$$\left.\begin{pmatrix}y_{11}&y_{12}&y_{13}&y_{14}&y_{15}\\y_{21}&1&0&0&0\\y_{31}&0&1&0&0\\y_{41}&0&0&1&0\\y_{51}&0&0&0&1\end{pmatrix},\ \begin{pmatrix}z_{11}&z_{12}&z_{13}&z_{14}&z_{15}&z_{16}\\z_{21}&1&0&0&0&z_{26}\\z_{31}&0&1&0&0&z_{36}\\z_{41}&0&0&1&0&z_{46}\\z_{51}&0&0&0&1&z_{56}\\z_{61}&z_{62}&z_{63}&z_{64}&z_{65}&z_{66}\end{pmatrix}\right] \in \overline{\Gamma}_{(2)} :$$

$$\left.\begin{array}{llll} t\not\equiv_2 0, & & & \\ x_{1i}\equiv_2 0,\ i\in[2,5], & y_{1i}\equiv_2 0,\ i\in[2,5], & z_{1i}\equiv_2 0,\ i\in[2,5], & \\ x_{1i}+y_{1i}\equiv_8 2z_{1i}, i\in[2,5], & y_{1i}\equiv_4 2z_{6i},\ i\in[2,5], & & \\ z_{i6}\equiv_2 0,\ i\in[2,5], & x_{i1}-y_{i1}\equiv_4 z_{i6},\ i\in[2,5], & y_{i1}\equiv_2 z_{i1},\ i\in[2,5], & \\ t\equiv_2 u, & z_{16}\equiv_2 0, & x_{11}\equiv_2 z_{66}, & x_{11}-y_{11}\equiv_4 z_{16}, \\ u-y_{11}\equiv_4 2z_{61}, & t+u+x_{11}+y_{11}\ \equiv_8 2z_{11}+2z_{66}\equiv_4 0 & & \end{array}\right\}.$$

Recalling the picture of (7.2), the remaining congruences are shown in the following picture.

$$\begin{array}{ccc} \boxed{t \quad u} & \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} & \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \end{array}$$

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & 0 & 0 & 0 \\ x_{31} & 0 & 1 & 0 & 0 \\ x_{41} & 0 & 0 & 1 & 0 \\ x_{51} & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{ccccc} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ y_{21} & 1 & 0 & 0 & 0 \\ y_{31} & 0 & 1 & 0 & 0 \\ y_{41} & 0 & 0 & 1 & 0 \\ y_{51} & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccccc} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & 1 & 0 & 0 & 0 & z_{26} \\ z_{31} & 0 & 1 & 0 & 0 & z_{36} \\ z_{41} & 0 & 0 & 1 & 0 & z_{46} \\ z_{51} & 0 & 0 & 0 & 1 & z_{56} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} \end{array}$$

$$\boxed{\begin{array}{lll} x_{1i} \equiv_2 0,\ i \in [2,5] & y_{1i} \equiv_2 0,\ i \in [2,5] & z_{1i} \equiv_2 0,\ i \in [2,5] \\ x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5] & y_{1i} \equiv_4 2z_{6i},\ i \in [2,5] & \end{array}}$$

$$\boxed{z_{i6} \equiv_2 0,\ i \in [2,5] \qquad x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5] \qquad y_{i1} \equiv_2 z_{i1},\ i \in [2,5]}$$

$$\boxed{\begin{array}{llll} t \equiv_2 u & z_{16} \equiv_2 0 & x_{11} \equiv_2 z_{66} & x_{11} - y_{11} \equiv_4 z_{16} \\ u - y_{11} \equiv_4 2z_{61} & t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 & & \end{array}}$$

We want to show that

$$U_1 \overset{!}{=} S_1 \cdot M_1 \, .$$

We have

$$U_1 \supseteq M_1$$

and

$$U_1 \supseteq S_1 \, .$$

So we need to show that

$$U_1 \overset{!}{\subseteq} S_1 \cdot M_1 \, .$$

Let $u_1 \in U_1$ with

$$u_1 = \big[t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big] \, .$$

Let

$$\hat{u}_1 = \big(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ (x_{ij})_{i,j\in[1,5]},\ (y_{ij})_{i,j\in[1,5]},\ (z_{ij})_{i,j\in[1,6]}\big) \in \mathrm{U}(\Lambda_{(2)}) \, .$$

We write

$$\hat{s}_1 := \left(1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 \\ 0 & (x_{ij})_{i,j\in[2,5]} \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & (y_{ij})_{i,j\in[2,5]} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & (z_{ij})_{i,j\in[2,5]} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \in S_{1,0}\ .$$

So

$$\hat{s}_1^{-1}\cdot\hat{u}_1 = \left(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & (x_{1j})_{j\in[2,5]} \\ (\tilde{x}_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & (y_{1j})_{j\in[2,5]} \\ (\tilde{y}_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & (z_{1j})_{j\in[2,5]} & z_{16} \\ (\tilde{z}_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (\tilde{z}_{i6})_{i\in[2,5]} \\ z_{61} & (z_{6j})_{j\in[2,5]} & z_{66} \end{pmatrix}\right),$$

for certain $(\tilde{x}_{i1})_{i\in[2,5]}, (\tilde{y}_{i1})_{i\in[2,5]}, (\tilde{z}_{i1})_{i\in[2,5]}, (\tilde{z}_{i6})_{i\in[2,5]} \in \mathbb{Z}_{(2)}^{4\times 1}$.
We consider

$$s_1 := [\hat{s}_1],\ u_1 = [\hat{u}_1]\ .$$

It remains to show that

$$s_1^{-1}\cdot u_1 \overset{!}{\in} M_1\ .$$

We have

$$s_1^{-1} = [\hat{s}_1]^{-1} = [\hat{s}_1^{-1}]\ .$$

Since

$$\hat{s}_1 \in S_{1,0} = \tilde{S}_{1,0}\cap \mathrm{U}(\Lambda_{(2)})$$

we have

$$\hat{s}_1^{-1} \in S_{1,0} \leqslant \mathrm{U}(\Lambda_{(2)})\ .$$

And we have

$$u_1 = [\hat{u}_1]$$

with

$$\hat{u}_1 \in \mathrm{U}(\Lambda_{(2)})\ .$$

Now we have

$$s_1^{-1}\cdot u_1 = [\hat{s}_1^{-1}]\cdot[\hat{u}_1] = [\hat{s}_1^{-1}\cdot\hat{u}_1] \in M_1\ ,$$

since $\hat{s}_1^{-1}\cdot\hat{u}_1 \in \mathrm{U}(\Lambda_{(2)})$ and since $\hat{s}_1^{-1}\cdot\hat{u}_1$ has the form required by the definition of M_1. Hence

$$u_1 = s_1\cdot(s_1^{-1}\cdot u_1) \in S_1\cdot M_1\ .$$

So we know that for $u_1 \in U_1$ there exists an $s_1 \in S_1$ and an $m_1 \in M_1$ with $u_1 = s_1\cdot m_1$. Hence

$$U_1 = S_1\cdot M_1\ .$$

Step 2. Now we want to dissect M_1 even further.

To begin we define the subset

$$\tilde{S}_2 := \left\{ \left[1,\ 1,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&x_{12}&x_{13}&x_{14}&x_{15}\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \right.\right.$$

$$\left.\left. \begin{pmatrix}1&y_{12}&y_{13}&y_{14}&y_{15}\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&z_{12}&z_{13}&z_{14}&z_{15}&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&z_{62}&z_{63}&z_{64}&z_{65}&1\end{pmatrix} \right] \in \mathrm{U}(\overline{\Gamma}_{(2)}) \right\}$$

$$\leqslant \tilde{U}_1$$

$$\leqslant \mathrm{U}(\overline{\Gamma}_{(2)})\ .$$

Let

$$\tilde{S}_{2,0} := \mathrm{U}(\rho)^{-1}(\tilde{S}_2) \leqslant \tilde{U}_{1,0} \leqslant \mathrm{U}(\overline{\Gamma}_{(2)})\ .$$

Then

$$\overline{\tilde{S}_{2,0}} = \mathrm{U}(\rho)(\tilde{S}_{2,0}) = \tilde{S}_2\ .$$

Let

$$S_{2,0} := \tilde{S}_{2,0} \cap \mathrm{U}(\Lambda_{(2)}) \leqslant \mathrm{U}(\Lambda_{(2)})\ .$$

We have

$$\begin{array}{rcl} S_2 & = & \overline{S_{2,0}} \\ & = & \overline{\tilde{S}_{2,0} \cap \mathrm{U}(\Lambda_{(2)})} \\ & \overset{\text{Remark 30}}{=} & \overline{\tilde{S}_{2,0}} \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & = & \tilde{S}_2 \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & \leqslant & \tilde{U}_1 \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & \overset{(7.15)}{=} & U_1\ . \end{array}$$

Note that

$$S_2 = \left\{ \left[1,\ 1,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&x_{12}&x_{13}&x_{14}&x_{15}\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \right.\right.$$

$$\left.\begin{pmatrix}1&y_{12}&y_{13}&y_{14}&y_{15}\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&z_{12}&z_{13}&z_{14}&z_{15}&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&z_{62}&z_{63}&z_{64}&z_{65}&1\end{pmatrix} \right] \in \mathrm{U}(\overline{\Gamma}_{(2)}) : \tag{7.18}$$

$$x_{1i} \equiv_2 0,\ i \in [2,5];\ y_{1i} \equiv_2 0,\ i \in [2,5];\ z_{1i} \equiv_2 0,\ i \in [2,5];$$

$$\left. x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5];\ y_{1i} \equiv_4 2z_{6i},\ i \in [2,5] \right\} .$$

We have

$$S_2 \subseteq M_1$$

since $S_{2,0} \leqslant \mathrm{U}(\Lambda_{(2)})$ and since elements of S_2 have the form required by the definition of M_1.
We define the subset

$$M_2 := \left\{ \left[t,\ u,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}x_{11}&0&0&0&0\\x_{21}&1&0&0&0\\x_{31}&0&1&0&0\\x_{41}&0&0&1&0\\x_{51}&0&0&0&1\end{pmatrix}, \right.\right.$$

$$\left.\begin{pmatrix}y_{11}&0&0&0&0\\y_{21}&1&0&0&0\\y_{31}&0&1&0&0\\y_{41}&0&0&1&0\\y_{51}&0&0&0&1\end{pmatrix}, \begin{pmatrix}z_{11}&0&0&0&0&z_{16}\\z_{21}&1&0&0&0&z_{26}\\z_{31}&0&1&0&0&z_{36}\\z_{41}&0&0&1&0&z_{46}\\z_{51}&0&0&0&1&z_{56}\\z_{61}&0&0&0&0&z_{66}\end{pmatrix} \right] \in \overline{\Gamma}_{(2)} :$$

$$\left.\begin{array}{llll} t \not\equiv_2 0, & & & \\ z_{i6} \equiv_2 0,\ i \in [2,5], & x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5], & y_{i1} \equiv_2 z_{i1},\ i \in [2,5], & \\ t \equiv_2 u, & z_{16} \equiv_2 0, & x_{11} \equiv_2 z_{66}, & x_{11} - y_{11} \equiv_4 z_{16}, \\ u - y_{11} \equiv_4 2z_{61}, & t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 & & \end{array} \right\} .$$

Recalling the picture of (7.2), the remaining congruences are shown in the following picture.

$$
\begin{array}{ccccc}
 & & & & \\
t & = & = & = & u
\end{array}
\qquad
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
$$

$$
\begin{array}{ccccc}
x_{11} & 0 & 0 & 0 & 0 \\
x_{21} & 1 & 0 & 0 & 0 \\
x_{31} & 0 & 1 & 0 & 0 \\
x_{41} & 0 & 0 & 1 & 0 \\
x_{51} & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{ccccc}
y_{11} & 0 & 0 & 0 & 0 \\
y_{21} & 1 & 0 & 0 & 0 \\
y_{31} & 0 & 1 & 0 & 0 \\
y_{41} & 0 & 0 & 1 & 0 \\
y_{51} & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{cccccc}
z_{11} & 0 & 0 & 0 & 0 & z_{16} \\
z_{21} & 1 & 0 & 0 & 0 & z_{26} \\
z_{31} & 0 & 1 & 0 & 0 & z_{36} \\
z_{41} & 0 & 0 & 1 & 0 & z_{46} \\
z_{51} & 0 & 0 & 0 & 1 & z_{56} \\
z_{61} & 0 & 0 & 0 & 0 & z_{66}
\end{array}
$$

$$
z_{i6} \equiv_2 0,\ i \in [2,5] \qquad x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5] \qquad y_{i1} \equiv_2 z_{i1},\ i \in [2,5]
$$

$$
\begin{array}{llll}
t \equiv_2 u & z_{16} \equiv_2 0 & x_{11} \equiv_2 z_{66} & x_{11} - y_{11} \equiv_4 z_{16} \\
u - y_{11} \equiv_4 2z_{61} & t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 & &
\end{array}
$$

We *claim* that

$$M_1 \overset{!}{=} S_2 \cdot M_2 \,.$$

We have

$$M_1 \supseteq S_2$$

and

$$M_1 \supseteq M_2 \,.$$

We show that

$$M_1 \overset{!}{\supseteq} S_2 \cdot M_2 \,.$$

Let

$$[\hat{s}_2] \in S_2$$

with

$$\hat{s}_2 = \left(1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & (x_{1j})_{j\in[2,5]} \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & (y_{1j})_{j\in[2,5]} \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & (z_{1j})_{j\in[2,5]} & 0 \\ 0 & \mathrm{E}_4 & 0 \\ 0 & (z_{6j})_{j\in[2,5]} & 1 \end{pmatrix} \right) \in S_{2,0} \,.$$

Let

$$[\hat{m}_2] \in M_2$$

with

$$\hat{m}_2 = \left(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & 0 \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & 0 \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & 0 & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & 0 & z_{66} \end{pmatrix}\right) .$$

We have

$$\hat{s}_2 \cdot \hat{m}_2 = \left(t,\, u,\, \mathrm{E}_4,\, \mathrm{E}_4,\, \begin{pmatrix} \tilde{x}_{11} & (x_{1j})_{j\in[2,5]} \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} \tilde{y}_{11} & (y_{1j})_{j\in[2,5]} \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} \tilde{z}_{11} & (z_{1j})_{j\in[2,5]} & \tilde{z}_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ \tilde{z}_{61} & (z_{6j})_{j\in[2,5]} & \tilde{z}_{66} \end{pmatrix}\right),$$

for certain $\tilde{x}_{11}, \tilde{y}_{11}, \tilde{z}_{11}, \tilde{z}_{16}, \tilde{z}_{61}, \tilde{z}_{66} \in \mathbb{Z}_{(2)}$.
We have $\hat{s}_2 \in S_{2,0} \leqslant \mathrm{U}(\Lambda_{(2)})$ and we have $\hat{m}_2 \in \mathrm{U}(\Lambda_{(2)})$. So

$$\hat{s}_2 \cdot \hat{m}_2 \in \mathrm{U}(\Lambda_{(2)}) .$$

Since $\hat{s}_2 \cdot \hat{m}_2$ has the form required by the definition of M_1, we obtain

$$[\hat{s}_2] \cdot [\hat{m}_2] = [\hat{s}_2 \cdot \hat{m}_2] \in M_1 .$$

So we have

$$M_1 \supseteq S_2 \cdot M_2 .$$

So we need to show that

$$M_1 \overset{!}{\subseteq} S_2 \cdot M_2 .$$

Let

$$m_1 = \left[t,\, u,\, \mathrm{E}_4,\, \mathrm{E}_4,\, \begin{pmatrix} x_{11} & (x_{1j})_{j\in[2,5]} \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} y_{11} & (y_{1j})_{j\in[2,5]} \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} z_{11} & (z_{1j})_{j\in[2,5]} & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & (z_{6j})_{j\in[2,5]} & z_{66} \end{pmatrix}\right] \in M_1$$

and let

$$\hat{m}_1 = \left(t,\, u,\, \mathrm{E}_4,\, \mathrm{E}_4,\, \begin{pmatrix} x_{11} & (x_{1j})_{j\in[2,5]} \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} y_{11} & (y_{1j})_{j\in[2,5]} \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} z_{11} & (z_{1j})_{j\in[2,5]} & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & (z_{6j})_{j\in[2,5]} & z_{66} \end{pmatrix}\right).$$

Note that

$$m_1 = [\hat{m}_1] .$$

Let

$$\hat{s}_2 = \left(1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & (x_{1j})_{j\in[2,5]} \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & (y_{1j})_{j\in[2,5]} \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & (z_{1j})_{j\in[2,5]} & 0 \\ 0 & \mathrm{E}_4 & 0 \\ 0 & (z_{6j})_{j\in[2,5]} & 1 \end{pmatrix}\right) \in S_{2,0} .$$

So

$$\begin{aligned} \hat{s}_2^{-1} \cdot \hat{m}_1 = &\left(1,\, 1,\, \mathrm{E}_4,\, \mathrm{E}_4,\, \begin{pmatrix} 1 & (-x_{1j})_{j\in[2,5]} \\ 0 & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} 1 & (-y_{1j})_{j\in[2,5]} \\ 0 & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} 1 & (-z_{1j})_{j\in[2,5]} & 0 \\ 0 & \mathrm{E}_4 & 0 \\ 0 & (-z_{6j})_{j\in[2,5]} & 1 \end{pmatrix}\right) \\ &\cdot \left(t,\, u,\, \mathrm{E}_4,\, \mathrm{E}_4,\, \begin{pmatrix} x_{11} & (x_{1j})_{j\in[2,5]} \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} y_{11} & (y_{1j})_{j\in[2,5]} \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} z_{11} & (z_{1j})_{j\in[2,5]} & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & (z_{6j})_{j\in[2,5]} & z_{66} \end{pmatrix}\right) \\ = &\left(t,\, u,\, \mathrm{E}_4,\, \mathrm{E}_4,\, \begin{pmatrix} \tilde{x}_{11} & 0 \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} \tilde{y}_{11} & 0 \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix}, \begin{pmatrix} \tilde{z}_{11} & 0 & \tilde{z}_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ \tilde{z}_{61} & 0 & \tilde{z}_{66} \end{pmatrix}\right), \end{aligned}$$

for certain $\tilde{x}_{11}, \tilde{y}_{11}, \tilde{z}_{11}, \tilde{z}_{16}, \tilde{z}_{61}, \tilde{z}_{66} \in \mathbb{Z}_{(2)}$.

We have $\hat{s}_2 \in S_{2,0} \leqslant \mathrm{U}(\Lambda_{(2)})$, so $\hat{s}_2^{-1} \in \mathrm{U}(\Lambda_{(2)})$, and we have $\hat{m}_1 \in \mathrm{U}(\Lambda_{(2)})$. So

$$\hat{s}_2^{-1} \cdot m_1 \in \mathrm{U}(\Lambda_{(2)}) \, .$$

Since $\hat{s}_2^{-1} \cdot \hat{m}_1$ has the form required by the definition of M_2, we obtain $[\hat{s}_2^{-1} \cdot \hat{m}_1] \in M_2$. Hence

$$m_1 = [\hat{m}_1] = [\hat{s}_2] \cdot [\hat{s}_2^{-1} \cdot \hat{m}_1] \in S_2 \cdot M_2 \, .$$

This proves the *claim*.

Now we have achieved the factorisation

$$U_1 = S_1 \cdot M_1 = S_1 \cdot S_2 \cdot M_2 \, .$$

Step 3. Now we want to dissect M_2 even further.

To begin we define the subset

$$\tilde{S}_3 := \left\{ \left[1,\ 1,\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 & 0 \\ x_{31} & 0 & 1 & 0 & 0 \\ x_{41} & 0 & 0 & 1 & 0 \\ x_{51} & 0 & 0 & 0 & 1 \end{pmatrix}, \right.\right.$$
$$\left.\left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ y_{21} & 1 & 0 & 0 & 0 \\ y_{31} & 0 & 1 & 0 & 0 \\ y_{41} & 0 & 0 & 1 & 0 \\ y_{51} & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ z_{21} & 1 & 0 & 0 & 0 & z_{26} \\ z_{31} & 0 & 1 & 0 & 0 & z_{36} \\ z_{41} & 0 & 0 & 1 & 0 & z_{46} \\ z_{51} & 0 & 0 & 0 & 1 & z_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \in \mathrm{U}(\overline{\Gamma}_{(2)}) \right\} \leqslant \tilde{U}_1 \leqslant \mathrm{U}(\overline{\Gamma}_{(2)}) \, .$$

Let

$$\tilde{S}_{3,0} := \mathrm{U}(\rho)^{-1}(\tilde{S}_3) \leqslant \tilde{U}_{1,0} \leqslant \mathrm{U}(\Gamma_{(2)}) \, .$$

Then

$$\overline{\tilde{S}_{3,0}} = \mathrm{U}(\rho)(\tilde{S}_{3,0}) = \tilde{S}_3 \, .$$

Let

$$S_{3,0} := \tilde{S}_{3,0} \cap \mathrm{U}(\Lambda_{(2)}) \leqslant \mathrm{U}(\Lambda_{(2)}) \, .$$

We have

$$\begin{array}{rcl} S_3 & := & \overline{S_{3,0}} \\ & = & \overline{\tilde{S}_{3,0} \cap \mathrm{U}(\Lambda_{(2)})} \\ & \overset{\text{Remark 30}}{=} & \overline{\tilde{S}_{3,0}} \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & = & \tilde{S}_3 \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & \leqslant & \tilde{U}_1 \cap \overline{\mathrm{U}(\Lambda_{(2)})} \\ & \overset{(7.15)}{=} & U_1 \, . \end{array}$$

Note that

$$S_3 = \left\{ \left[1,\ 1,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&0\\x_{21}&1&0&0&0\\x_{31}&0&1&0&0\\x_{41}&0&0&1&0\\x_{51}&0&0&0&1\end{pmatrix}, \right.\right.$$

$$\left. \begin{pmatrix}1&0&0&0&0\\y_{21}&1&0&0&0\\y_{31}&0&1&0&0\\y_{41}&0&0&1&0\\y_{51}&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&0&0\\z_{21}&1&0&0&0&z_{26}\\z_{31}&0&1&0&0&z_{36}\\z_{41}&0&0&1&0&z_{46}\\z_{51}&0&0&0&1&z_{56}\\0&0&0&0&0&1\end{pmatrix} \right] \in \mathrm{U}(\overline{\Gamma}_{(2)}) : \tag{7.19}$$

$$\left. z_{i6} \equiv_2 0, i \in [2,5];\quad x_{i1} - y_{i1} \equiv_4 z_{i6}, i \in [2,5];\quad y_{i1} \equiv_2 z_{i1}, i \in [2,5] \right\}.$$

We have

$$S_3 \subseteq M_2$$

since $S_{3,0} \leqslant \mathrm{U}(\Lambda_{(2)})$ and since elements of S_3 have the form required by the definition of M_2.
We define the subset

$$M_3 := \left\{ \left[t,\ u,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}x_{11}&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \right.\right.$$

$$\left. \begin{pmatrix}y_{11}&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}z_{11}&0&0&0&0&z_{16}\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\z_{61}&0&0&0&0&z_{66}\end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \tag{7.20}$$

$$\left. \begin{array}{llll} t \not\equiv_2 0, & & & \\ t \equiv_2 u, & z_{16} \equiv_2 0, & x_{11} \equiv_2 z_{66}, & x_{11} - y_{11} \equiv_4 z_{16}, \\ u - y_{11} \equiv_4 2z_{61}, & t + u + x_{11} + y_{11} & \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 & \end{array} \right\}.$$

Recalling the picture of (7.2), the remaining congruences are shown in the following picture.

$$
\boxed{\begin{array}{cc} t & u \end{array}}
\qquad
\begin{array}{cccc}
1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{cccc}
1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{array}
$$

$$
\begin{array}{ccccc}
\boxed{x_{11}} & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{ccccc}
\boxed{y_{11}} & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{cccccc}
\boxed{z_{11}} & 0 & 0 & 0 & 0 & \boxed{z_{16}}\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
\boxed{z_{61}} & 0 & 0 & 0 & 0 & \boxed{z_{66}}
\end{array}
$$

$$
\boxed{\begin{array}{llll}
t \equiv_2 u & z_{16} \equiv_2 0 & x_{11} \equiv_2 z_{66} & x_{11} - y_{11} \equiv_4 z_{16}\\
u - y_{11} \equiv_4 2z_{61} & \multicolumn{3}{l}{t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0}
\end{array}}
$$

We *claim* that

$$M_2 \overset{!}{=} M_3 \cdot S_3 \, .$$

We have

$$M_2 \supseteq S_3$$

and

$$M_2 \supseteq M_3 \, .$$

We show that

$$M_2 \overset{!}{\supseteq} M_3 \cdot S_3 \, .$$

Let

$$[\hat{s}_3] \in S_3$$

with

$$\hat{s}_3 = \left(1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 \\ (x_{i1})_{i \in [2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ (y_{i1})_{i \in [2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ (z_{i1})_{i \in [2,5]} & \mathrm{E}_4 & (z_{i6})_{i \in [2,5]} \\ 0 & 0 & 1 \end{pmatrix} \right) .$$

Let

$$[\hat{m}_3] \in M_3$$

with

$$\hat{m}_3 = \left(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & 0 \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & 0 \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & 0 & z_{16} \\ 0 & \mathrm{E}_4 & 0 \\ z_{61} & 0 & z_{66} \end{pmatrix}\right) .$$

We have

$$\hat{m}_3 \cdot \hat{s}_3 = \left(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & 0 \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & 0 \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & 0 & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & 0 & z_{66} \end{pmatrix}\right) .$$

We have $\hat{s}_3 \in S_{3,0} \leqslant \mathrm{U}(\Lambda_{(2)})$ and we have $\hat{m}_3 \in \mathrm{U}(\Lambda_{(2)})$. So

$$\hat{m}_3 \cdot \hat{s}_3 \in \mathrm{U}(\Lambda_{(2)}) .$$

Since $\hat{m}_3 \cdot \hat{s}_3$ has the form required by the definition of M_2, we obtain

$$[\hat{m}_3] \cdot [\hat{s}_3] = [\hat{m}_3 \cdot \hat{s}_3] \in M_2 .$$

So we have

$$M_2 \supseteq M_3 \cdot S_3 .$$

So we need to show that

$$M_2 \overset{!}{\subseteq} M_3 \cdot S_3 .$$

Suppose given

$$m_2 = \left[t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & 0 \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & 0 \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & 0 & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & 0 & z_{66} \end{pmatrix}\right] \in M_2 .$$

Write

$$\hat{m}_2 = \left(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & 0 \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & 0 \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & 0 & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & 0 & z_{66} \end{pmatrix}\right) .$$

Note that

$$m_2 = [\hat{m}_2] .$$

Let

$$\hat{s}_3 = \left(1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ 0 & 0 & 1 \end{pmatrix}\right) .$$

So

$$\begin{aligned} \hat{m}_2 \cdot \hat{s}_3^{-1} &= \left(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & 0 \\ (x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & 0 \\ (y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & 0 & z_{16} \\ (z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (z_{i6})_{i\in[2,5]} \\ z_{61} & 0 & z_{66} \end{pmatrix}\right) \\ &\quad \cdot \left(1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 \\ (-x_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ (-y_{i1})_{i\in[2,5]} & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ (-z_{i1})_{i\in[2,5]} & \mathrm{E}_4 & (-z_{i6})_{i\in[2,5]} \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \left(t,\ u,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} x_{11} & 0 \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} y_{11} & 0 \\ 0 & \mathrm{E}_4 \end{pmatrix},\ \begin{pmatrix} z_{11} & 0 & z_{16} \\ 0 & \mathrm{E}_4 & 0 \\ z_{61} & 0 & z_{66} \end{pmatrix}\right) . \end{aligned}$$

We have $\hat{s}_3 \in S_{3,0} \leqslant \mathrm{U}(\Lambda_{(2)})$, so $\hat{s}_3^{-1} \in \mathrm{U}(\Lambda_{(2)})$, and we have $\hat{m}_2 \in \mathrm{U}(\Lambda_{(2)})$. So

$$\hat{m}_2 \cdot \hat{s}_3^{-1} \in \mathrm{U}(\Lambda_{(2)}) \, .$$

Since $\hat{m}_2 \cdot \hat{s}_3^{-1}$ has the form required by the definition of M_3, we obtain $[\hat{m}_2 \cdot \hat{s}_3^{-1}] \in M_3$. Hence

$$m_2 = [\hat{m}_2] = [\hat{m}_2 \cdot \hat{s}_3^{-1}] \cdot [\hat{s}_3] \in M_3 \cdot S_3 \, .$$

This proves the *claim*.

Remark 33. Let

$$\varepsilon := \left(1,\ 1,\ \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1 \end{pmatrix} \right) .$$

Note that $\overline{\varepsilon} \in \mathrm{Im}(\varphi)$ is an idempotent and thus

$$M_3 = \mathrm{U}(\overline{\varepsilon}\,\mathrm{Im}(\varphi)\overline{\varepsilon}) + (1_{\overline{\Gamma}_{(2)}} - \overline{\varepsilon})$$

is a subgroup of $\mathrm{Im}(\mathrm{U}(\varphi))$.

Now we have achieved the factorisation of subgroups

$$U_1 = S_1 \cdot M_1 = S_1 \cdot S_2 \cdot M_2 = S_1 \cdot S_2 \cdot M_3 \cdot S_3 \, . \tag{7.21}$$

Remark 34. *Since $U_1 \leqslant \mathrm{Im}(\mathrm{U}(\varphi))$ is a subgroup, we have*

$$U_1 = S_1 \cdot S_2 \cdot M_3 \cdot S_3 \leqslant \langle S_1, S_2, S_3, M_3 \rangle \leqslant U_1 \, ,$$

so

$$U_1 = \langle S_1, S_2, S_3, M_3 \rangle \, .$$

We will use this fact when calculating U_1 via Magma.

We have obtained the following Lemma 35.

Lemma 35. *We have*

$$\begin{aligned} \mathrm{Im}(\mathrm{U}(\varphi)) &= U_1 \times U_2 = (S_1 \cdot S_2 \cdot M_3 \cdot S_3) \times U_2 \\ &= \langle S_1, S_2, S_3, M_3 \rangle \times U_2 \, . \end{aligned}$$

7.2 Calculation of $\mathrm{Im}(\mathrm{U}(\varphi))$ via Magma

7.2.1 Preparations

Suppose given $k \in [1,7]$ and $i,j \in [1,n_k]$. Let $e_{k;i,j} \in \Gamma_{(2)}$ be the matrix tuple having entry 1 in the kth matrix in position (i,j), and 0 elsewhere.

For example, we have

$$e_{4;2,3} = \left(0,\ 0,\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right)$$

and

$$e_{7;5,6} = \left(0,\ 0,\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0 \end{pmatrix} \right)$$

and

$$e_{3;4,1} = \left(0,\ 0,\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right).$$

This notation can now be used to form linear combinations to represent matrix tuples with several entries not equal to 1. For example, we have

$$5e_{3;4,1} + (-2)e_{3;4,2} + 2e_{4;2,3} + e_{7;5,6} = \left(0,\ 0,\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\5&-2&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&2&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0 \end{pmatrix} \right)$$

and

$$e_{3;1,1} + e_{4;1,1} = \left(0,\ 0,\ \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right).$$

We have the following $\mathbb{Z}_{(2)}$-linear basis of $\Lambda_{(2)}$, using the notation $e_{k;i,j}$ defined above:

$$
\mathfrak{B}_{\Lambda_{(2)}} := \left(
\begin{array}{r}
e_{3;1,1}+e_{4;1,1}, 2e_{4;1,1},\ e_{3;1,2}+e_{4;1,2}, 2e_{4;1,2},\ e_{3;1,3}+e_{4;1,3}, 2e_{4;1,3},\ e_{3;1,4}+e_{4;1,4}, 2e_{4;1,4}, \\
e_{3;2,1}+e_{4;2,1}, 2e_{4;2,1},\ e_{3;2,2}+e_{4;2,2}, 2e_{4;2,2},\ e_{3;2,3}+e_{4;2,3}, 2e_{4;2,3},\ e_{3;2,4}+e_{4;2,4}, 2e_{4;2,4}, \\
e_{3;3,1}+e_{4;3,1}, 2e_{4;3,1},\ e_{3;3,2}+e_{4;3,2}, 2e_{4;3,2},\ e_{3;3,3}+e_{4;3,3}, 2e_{4;3,3},\ e_{3;3,4}+e_{4;3,4}, 2e_{4;3,4}, \\
e_{3;4,1}+e_{4;4,1}, 2e_{4;4,1},\ e_{3;4,2}+e_{4;4,2}, 2e_{4;4,2},\ e_{3;4,3}+e_{4;4,3}, 2e_{4;4,3},\ e_{3;4,4}+e_{4;4,4}, 2e_{4;4,4}, \\
e_{5;2,2}+e_{6;2,2}+e_{7;2,2}, -2e_{5;2,2}+2e_{6;2,2}, 8e_{5;2,2}, \\
e_{5;2,3}+e_{6;2,3}+e_{7;2,3}, -2e_{5;2,3}+2e_{6;2,3}, 8e_{5;2,3}, \\
e_{5;2,4}+e_{6;2,4}+e_{7;2,4}, -2e_{5;2,4}+2e_{6;2,4}, 8e_{5;2,4}, \\
e_{5;2,5}+e_{6;2,5}+e_{7;2,5}, -2e_{5;2,5}+2e_{6;2,5}, 8e_{5;2,5}, \\
e_{5;3,2}+e_{6;3,2}+e_{7;3,2}, -2e_{5;3,2}+2e_{6;3,2}, 8e_{5;3,2}, \\
e_{5;3,3}+e_{6;3,3}+e_{7;3,3}, -2e_{5;3,3}+2e_{6;3,3}, 8e_{5;3,3}, \\
e_{5;3,4}+e_{6;3,4}+e_{7;3,4}, -2e_{5;3,4}+2e_{6;3,4}, 8e_{5;3,4}, \\
e_{5;3,5}+e_{6;3,5}+e_{7;3,5}, -2e_{5;3,5}+2e_{6;3,5}, 8e_{5;3,5}, \\
e_{5;4,2}+e_{6;4,2}+e_{7;4,2}, -2e_{5;4,2}+2e_{6;4,2}, 8e_{5;4,2}, \\
e_{5;4,3}+e_{6;4,3}+e_{7;4,3}, -2e_{5;4,3}+2e_{6;4,3}, 8e_{5;4,3}, \\
e_{5;4,4}+e_{6;4,4}+e_{7;4,4}, -2e_{5;4,4}+2e_{6;4,4}, 8e_{5;4,4}, \\
e_{5;4,5}+e_{6;4,5}+e_{7;4,5}, -2e_{5;4,5}+2e_{6;4,5}, 8e_{5;4,5}, \\
e_{5;5,2}+e_{6;5,2}+e_{7;5,2}, -2e_{5;5,2}+2e_{6;5,2}, 8e_{5;5,2}, \\
e_{5;5,3}+e_{6;5,3}+e_{7;5,3}, -2e_{5;5,3}+2e_{6;5,3}, 8e_{5;5,3}, \\
e_{5;5,4}+e_{6;5,4}+e_{7;5,4}, -2e_{5;5,4}+2e_{6;5,4}, 8e_{5;5,4}, \\
e_{5;5,5}+e_{6;5,5}+e_{7;5,5}, -2e_{5;5,5}+2e_{6;5,5}, 8e_{5;5,5}, \\
-2e_{5;1,2}+2e_{6;1,2}+e_{7;6,2}, 4e_{6;1,2}+2e_{7;1,2}, 4e_{5;1,2}+4e_{6;1,2}, 8e_{5;1,2}, \\
-2e_{5;1,3}+2e_{6;1,3}+e_{7;6,3}, 4e_{6;1,3}+2e_{7;1,3}, 4e_{5;1,3}+4e_{6;1,3}, 8e_{5;1,3}, \\
-2e_{5;1,4}+2e_{6;1,4}+e_{7;6,4}, 4e_{6;1,4}+2e_{7;1,4}, 4e_{5;1,4}+4e_{6;1,4}, 8e_{5;1,4}, \\
-2e_{5;1,5}+2e_{6;1,5}+e_{7;6,5}, 4e_{6;1,5}+2e_{7;1,5}, 4e_{5;1,5}+4e_{6;1,5}, 8e_{5;1,5}, \\
e_{5;2,1}+e_{6;2,1}+e_{7;2,1}, 2e_{5;2,1}+2e_{6;2,1}, 2e_{5;2,1}+2e_{7;2,6}, 4e_{5;2,1}, \\
e_{5;3,1}+e_{6;3,1}+e_{7;3,1}, 2e_{5;3,1}+2e_{6;3,1}, 2e_{5;3,1}+2e_{7;3,6}, 4e_{5;3,1}, \\
e_{5;4,1}+e_{6;4,1}+e_{7;4,1}, 2e_{5;4,1}+2e_{6;4,1}, 2e_{5;4,1}+2e_{7;4,6}, 4e_{5;4,1}, \\
e_{5;5,1}+e_{6;5,1}+e_{7;5,1}, 2e_{5;5,1}+2e_{6;5,1}, 2e_{5;5,1}+2e_{7;5,6}, 4e_{5;5,1}, \\
e_{1;1,1}+e_{2;1,1}+e_{5;1,1}+e_{6;1,1}+e_{7;1,1}+e_{7;6,6}, \\
2e_{5;1,1}+2e_{6;1,1}+2e_{7;1,1}+e_{7;6,1}, \\
-2e_{5;1,1}+2e_{6;1,1}+e_{7;6,1}, \\
-2e_{1;1,1}+2e_{5;1,1}+2e_{7;1,6}, \\
2e_{1;1,1}+2e_{2;1,1}+2e_{5;1,1}+2e_{6;1,1}, \\
4e_{2;1,1}+4e_{6;1,1}, \\
4e_{1;1,1}+4e_{2;1,1}, \\
8e_{1;1,1}
\end{array}
\right)
\tag{7.22}
$$

The vertical markings indicate to which part of the picture (7.2) the respective part of $\mathfrak{B}_{\Lambda_{(2)}}$ belongs.

We enumerate the elements of $\mathfrak{B}_{\Lambda_{(2)}}$ as follows.

$$\mathfrak{B}_{\Lambda_{(2)}} =: (b_1, b_2, \ldots, b_{119}, b_{120})$$

So

$$\begin{aligned}
b_1 = e_{3;1,1} + e_{4;1,1}, b_2 = 2e_{4;1,1},\ b_3 = e_{3;1,2} + e_{4;1,2}, b_4 = 2e_{4;1,2},\ b_5 = e_{3;1,3} + e_{4;1,3}, b_6 = 2e_{4;1,3},\\
b_7 = e_{3;1,4} + e_{4;1,4}, b_8 = 2e_{4;1,4}, b_9 = e_{3;2,1} + e_{4;2,1}, b_{10} = 2e_{4;2,1},\ b_{11} = e_{3;2,2} + e_{4;2,2}, b_{12} = 2e_{4;2,2},\\
b_{13} = e_{3;2,3} + e_{4;2,3}, b_{14} = 2e_{4;2,3},\ b_{15} = e_{3;2,4} + e_{4;2,4}, b_{16} = 2e_{4;2,4}, b_{17} = e_{3;3,1} + e_{4;3,1}, b_{18} = 2e_{4;3,1},\\
b_{19} = e_{3;3,2} + e_{4;3,2}, b_{20} = 2e_{4;3,2},\ b_{21} = e_{3;3,3} + e_{4;3,3}, b_{22} = 2e_{4;3,3},\ b_{23} = e_{3;3,4} + e_{4;3,4}, b_{24} = 2e_{4;3,4},\\
b_{25} = e_{3;4,1} + e_{4;4,1}, b_{26} = 2e_{4;4,1},\ b_{27} = e_{3;4,2} + e_{4;4,2}, b_{28} = 2e_{4;4,2},\ b_{29} = e_{3;4,3} + e_{4;4,3}, b_{30} = 2e_{4;4,3},\\
b_{31} = e_{3;4,4} + e_{4;4,4}, b_{32} = 2e_{4;4,4},\\
b_{33} = e_{5;2,2} + e_{6;2,2} + e_{7;2,2}, b_{34} = -2e_{5;2,2} + 2e_{6;2,2}, b_{35} = 8e_{5;2,2},\\
b_{36} = e_{5;2,3} + e_{6;2,3} + e_{7;2,3}, b_{37} = -2e_{5;2,3} + 2e_{6;2,3}, b_{38} = 8e_{5;2,3},\\
b_{39} = e_{5;2,4} + e_{6;2,4} + e_{7;2,4}, b_{40} = -2e_{5;2,4} + 2e_{6;2,4}, b_{41} = 8e_{5;2,4},\\
b_{42} = e_{5;2,5} + e_{6;2,5} + e_{7;2,5}, b_{43} = -2e_{5;2,5} + 2e_{6;2,5}, b_{44} = 8e_{5;2,5},\\
b_{45} = e_{5;3,2} + e_{6;3,2} + e_{7;3,2}, b_{46} = -2e_{5;3,2} + 2e_{6;3,2}, b_{47} = 8e_{5;3,2},\\
b_{48} = e_{5;3,3} + e_{6;3,3} + e_{7;3,3}, b_{49} = -2e_{5;3,3} + 2e_{6;3,3}, b_{50} = 8e_{5;3,3},\\
b_{51} = e_{5;3,4} + e_{6;3,4} + e_{7;3,4}, b_{52} = -2e_{5;3,4} + 2e_{6;3,4}, b_{53} = 8e_{5;3,4},\\
b_{54} = e_{5;3,5} + e_{6;3,5} + e_{7;3,5}, b_{55} = -2e_{5;3,5} + 2e_{6;3,5}, b_{56} = 8e_{5;3,5},\\
b_{57} = e_{5;4,2} + e_{6;4,2} + e_{7;4,2}, b_{58} = -2e_{5;4,2} + 2e_{6;4,2}, b_{59} = 8e_{5;4,2},\\
b_{60} = e_{5;4,3} + e_{6;4,3} + e_{7;4,3}, b_{61} = -2e_{5;4,3} + 2e_{6;4,3}, b_{62} = 8e_{5;4,3},\\
b_{63} = e_{5;4,4} + e_{6;4,4} + e_{7;4,4}, b_{64} = -2e_{5;4,4} + 2e_{6;4,4}, b_{65} = 8e_{5;4,4},\\
b_{66} = e_{5;4,5} + e_{6;4,5} + e_{7;4,5}, b_{67} = -2e_{5;4,5} + 2e_{6;4,5}, b_{68} = 8e_{5;4,5},\\
b_{69} = e_{5;5,2} + e_{6;5,2} + e_{7;5,2}, b_{70} = -2e_{5;5,2} + 2e_{6;5,2}, b_{71} = 8e_{5;5,2},\\
b_{72} = e_{5;5,3} + e_{6;5,3} + e_{7;5,3}, b_{73} = -2e_{5;5,3} + 2e_{6;5,3}, b_{74} = 8e_{5;5,3},\\
b_{75} = e_{5;5,4} + e_{6;5,4} + e_{7;5,4}, b_{76} = -2e_{5;5,4} + 2e_{6;5,4}, b_{77} = 8e_{5;5,4},\\
b_{78} = e_{5;5,5} + e_{6;5,5} + e_{7;5,5}, b_{79} = -2e_{5;5,5} + 2e_{6;5,5}, b_{80} = 8e_{5;5,5},\\
b_{81} = -2e_{5;1,2} + 2e_{6;1,2} + e_{7;6,2}, b_{82} = 4e_{6;1,2} + 2e_{7;1,2}, b_{83} = 4e_{5;1,2} + 4e_{6;1,2}, b_{84} = 8e_{5;1,2},\\
b_{85} = -2e_{5;1,3} + 2e_{6;1,3} + e_{7;6,3}, b_{86} = 4e_{6;1,3} + 2e_{7;1,3}, b_{87} = 4e_{5;1,3} + 4e_{6;1,3}, b_{88} = 8e_{5;1,3},\\
b_{89} = -2e_{5;1,4} + 2e_{6;1,4} + e_{7;6,4}, b_{90} = 4e_{6;1,4} + 2e_{7;1,4}, b_{91} = 4e_{5;1,4} + 4e_{6;1,4}, b_{92} = 8e_{5;1,4},\\
b_{93} = -2e_{5;1,5} + 2e_{6;1,5} + e_{7;6,5}, b_{94} = 4e_{6;1,5} + 2e_{7;1,5}, b_{95} = 4e_{5;1,5} + 4e_{6;1,5}, b_{96} = 8e_{5;1,5},\\
b_{97} = e_{5;2,1} + e_{6;2,1} + e_{7;2,1}, b_{98} = 2e_{5;2,1} + 2e_{6;2,1}, b_{99} = 2e_{5;2,1} + 2e_{7;2,6}, b_{100} = 4e_{5;2,1},\\
b_{101} = e_{5;3,1} + e_{6;3,1} + e_{7;3,1}, b_{102} = 2e_{5;3,1} + 2e_{6;3,1}, b_{103} = 2e_{5;3,1} + 2e_{7;3,6}, b_{104} = 4e_{5;3,1},\\
b_{105} = e_{5;4,1} + e_{6;4,1} + e_{7;4,1}, b_{106} = 2e_{5;4,1} + 2e_{6;4,1}, b_{107} = 2e_{5;4,1} + 2e_{7;4,6}, b_{108} = 4e_{5;4,1},\\
b_{109} = e_{5;5,1} + e_{6;5,1} + e_{7;5,1}, b_{110} = 2e_{5;5,1} + 2e_{6;5,1}, b_{111} = 2e_{5;5,1} + 2e_{7;5,6}, b_{112} = 4e_{5;5,1},\\
b_{113} = e_{1;1,1} + e_{2;1,1} + e_{5;1,1} + e_{6;1,1} + e_{7;1,1} + e_{7;6,6},\\
b_{114} = 2e_{5;1,1} + 2e_{6;1,1} + 2e_{7;1,1} + e_{7;6,1},\\
b_{115} = -2e_{5;1,1} + 2e_{6;1,1} + e_{7;6,1},\\
b_{116} = -2e_{1;1,1} + 2e_{5;1,1} + 2e_{7;1,6},\\
b_{117} = 2e_{1;1,1} + 2e_{2;1,1} + 2e_{5;1,1} + 2e_{6;1,1},\\
b_{118} = 4e_{2;1,1} + 4e_{6;1,1},\\
b_{119} = 4e_{1;1,1} + 4e_{2;1,1},\\
b_{120} = 8e_{1;1,1}
\end{aligned}$$

We write $b_j\varphi =: \overline{b}_j \in \mathrm{Im}(\varphi) \subseteq \overline{\Gamma}_{(2)}$ for $j \in [1, 120]$.

We obtain the following $\mathbb{Z}_{(2)}$-linear generating set of $\mathrm{Im}(\varphi)$.

$$
\begin{aligned}
\mathfrak{G}_{\mathrm{Im}(\varphi)} = \{&\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_{32}, \\
&\overline{b}_{33}, \overline{b}_{34}, \overline{b}_{36}, \overline{b}_{37}, \overline{b}_{39}, \overline{b}_{40}, \overline{b}_{42}, \overline{b}_{43}, \overline{b}_{45}, \overline{b}_{46}, \overline{b}_{48}, \overline{b}_{49}, \overline{b}_{51}, \overline{b}_{52}, \overline{b}_{54}, \overline{b}_{55}, \\
&\overline{b}_{57}, \overline{b}_{58}, \overline{b}_{60}, \overline{b}_{61}, \overline{b}_{63}, \overline{b}_{64}, \overline{b}_{66}, \overline{b}_{67}, \overline{b}_{69}, \overline{b}_{70}, \overline{b}_{72}, \overline{b}_{73}, \overline{b}_{75}, \overline{b}_{76}, \overline{b}_{78}, \overline{b}_{79}, \\
&\overline{b}_{81}, \overline{b}_{82}, \overline{b}_{83}, \overline{b}_{85}, \overline{b}_{86}, \overline{b}_{87}, \overline{b}_{89}, \overline{b}_{90}, \overline{b}_{91}, \overline{b}_{93}, \overline{b}_{94}, \overline{b}_{95}, \\
&\overline{b}_{97}, \ldots, \overline{b}_{112} \\
&\overline{b}_{113}, \ldots, \overline{b}_{119}\} \,.
\end{aligned}
\tag{7.23}
$$

7.2.2 Generators for U_2

Remark 36. *U_2 has the following set of generators.*

$$
\begin{aligned}
&\{1_{\overline{\Gamma}_{(2)}} + \overline{b}_i : i \in \{3,4,5,6,7,8,9,10,\ 13,14,15,16,17,18,19,20,\ 23,24,25,26,27,28,29,30\}\} \\
&\ \cup \{1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_1, 1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_{11}, 1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_{21}, 1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_{31} : \sigma \in \{-2, 2, 4\}\} \\
&\ \cup \{1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_2, 1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_{12}, 1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_{22}, 1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_{32} : \tau \in \{-1, 1, 2\}\}
\end{aligned}
\tag{7.24}
$$

Proof. Note that the elements in the claimed set of generators are invertible and in $\mathrm{Im}(\varphi)$, hence in $\mathrm{Im}(\mathrm{U}(\varphi))$; cf. Remark 15(II). By their block structure, they are contained in U_2.

Suppose given

$$
\xi := \left[\ 1,\ 1,\ \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix},\ \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in U_2 \,.
$$

Note that replacing representatives modulo 8, we may suppose that the matrix entries of ξ are in $\mathbb{Z}$.

We want to show that ξ is a product of the elements in the claimed list of generators above.

Note that

$$
\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .
$$

Using this calculation blockwise, we get

$$
\begin{aligned}
(1_{\overline{\Gamma}_{(2)}} + \overline{b}_9)^{-1} \cdot (1_{\overline{\Gamma}_{(2)}} + \overline{b}_3) \cdot (1_{\overline{\Gamma}_{(2)}} + \overline{b}_9)^{-1} &= \left[\ 1,\ 1,\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \\
(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{19})^{-1} \cdot (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{13}) \cdot (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{19})^{-1} &= \left[\ 1,\ 1,\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \\
(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{29})^{-1} \cdot (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{23}) \cdot (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{29})^{-1} &= \left[\ 1,\ 1,\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right]
\end{aligned}
$$

After reordering the rows we may assume that $v_{11} \not\equiv_2 0$.

Choose $v'_{11} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $v'_{11} \cdot v_{11} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (v'_{11} - 1)\bar{b}_1 = \left[\ 1,\ 1,\ \begin{pmatrix} v_{11}^{-1} & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{pmatrix},\ *,\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right],$$

we may suppose that $v_{11} = 1$. The asterisk stands for a matrix that does not need to be calculated.

Using

$$\begin{gathered} (1_{\overline{\Gamma}_{(2)}} + \bar{b}_9)^{-v_{21}} \\ (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{17})^{-v_{31}} \\ (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{25})^{-v_{41}} \end{gathered}$$

to eliminate the entries in the first column in block 3, we may suppose

$$\xi = \left[\ 1,\ 1,\ \begin{pmatrix} 1 & v_{12} & v_{13} & v_{14} \\ 0 & v_{22} & v_{23} & v_{24} \\ 0 & v_{32} & v_{33} & v_{34} \\ 0 & v_{42} & v_{43} & v_{44} \end{pmatrix},\ \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in U_2\,.$$

After reordering the rows we may assume that $v_{22} \not\equiv_2 0$.

Choose $v'_{22} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $v'_{22} \cdot v_{22} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (v'_{22} - 1)\bar{b}_{11} = \left[\ 1,\ 1,\ \begin{pmatrix} 1 & & 0 & 0 & 0 \\ 0 & v_{22}^{-1} & 0 & 0 \\ 0 & & 0 & 1 & 0 \\ 0 & & 0 & 0 & 1 \end{pmatrix},\ *,\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right],$$

we may suppose that $v_{22} = 1$.

Using

$$\begin{gathered} (1_{\overline{\Gamma}_{(2)}} + \bar{b}_3)^{-v_{12}} \\ (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{19})^{-v_{32}} \\ (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{27})^{-v_{42}} \end{gathered}$$

to eliminate the entries in the second column in block 3, we may suppose

$$\xi = \left[\ 1,\ 1,\ \begin{pmatrix} 1 & 0 & v_{13} & v_{14} \\ 0 & 1 & v_{23} & v_{24} \\ 0 & 0 & v_{33} & v_{34} \\ 0 & 0 & v_{43} & v_{44} \end{pmatrix},\ \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in U_2\,.$$

After reordering the rows we may assume that $v_{33} \not\equiv_2 0$.

Choose $v'_{33} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $v'_{33} \cdot v_{33} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (v'_{33} - 1)\overline{b}_{21} = \left[\ 1,\ 1,\ \begin{pmatrix} 1\,0 & & 0\,0 \\ 0\,1 & & 0\,0 \\ 0\,0 & v_{33}^{-1} & 0 \\ 0\,0 & & 0\,1 \end{pmatrix},\ *,\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right],$$

we may suppose that $v_{33} = 1$.

Using

$$\begin{array}{c} (1_{\overline{\Gamma}_{(2)}} + \overline{b}_5)^{-v_{13}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{13})^{-v_{23}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{29})^{-v_{43}} \end{array}$$

to eliminate the entries in the third column in block 3, we may suppose

$$\xi = \left[\ 1,\ 1,\ \begin{pmatrix} 1\ 0\ 0\ v_{14} \\ 0\ 1\ 0\ v_{24} \\ 0\ 0\ 1\ v_{34} \\ 0\ 0\ 0\ v_{44} \end{pmatrix},\ \begin{pmatrix} w_{11}\ w_{12}\ w_{13}\ w_{14} \\ w_{21}\ w_{22}\ w_{23}\ w_{24} \\ w_{31}\ w_{32}\ w_{33}\ w_{34} \\ w_{41}\ w_{42}\ w_{43}\ w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in U_2\,.$$

Note that we have $v_{44} \not\equiv_2 0$.

Choose $v'_{44} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $v'_{44} \cdot v_{44} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (v'_{44} - 1)\overline{b}_{31} = \left[\ 1,\ 1,\ \begin{pmatrix} 1\,0\,0 & 0 \\ 0\,1\,0 & 0 \\ 0\,0\,1 & 0 \\ 0\,0\,0 & v_{44}^{-1} \end{pmatrix},\ *,\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right],$$

we may suppose that $v_{44} = 1$.

Using

$$\begin{array}{c} (1_{\overline{\Gamma}_{(2)}} + \overline{b}_7)^{-v_{14}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{15})^{-v_{24}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{23})^{-v_{34}} \end{array}$$

to eliminate the entries in the fourth column in block 3, we may suppose

$$\begin{aligned} \xi &= \left[\ 1,\ 1,\ \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \end{pmatrix},\ \begin{pmatrix} w_{11}\ w_{12}\ w_{13}\ w_{14} \\ w_{21}\ w_{22}\ w_{23}\ w_{24} \\ w_{31}\ w_{32}\ w_{33}\ w_{34} \\ w_{41}\ w_{42}\ w_{43}\ w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \\ &= \left[\ 1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} w_{11}\ w_{12}\ w_{13}\ w_{14} \\ w_{21}\ w_{22}\ w_{23}\ w_{24} \\ w_{31}\ w_{32}\ w_{33}\ w_{34} \\ w_{41}\ w_{42}\ w_{43}\ w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in U_2\,. \end{aligned}$$

For $i, j \in [1, 4]$, we have $v_{ij} \equiv_2 w_{ij}$ by (7.2). And so we have for $i = j$ that $v_{ii} \equiv_2 w_{ii} = 1$ and for $i \neq j$ that $v_{ij} \equiv_2 w_{ij} = 0$.

Choose $w'_{11} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $w'_{11} \cdot w_{11} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(w'_{11} - 1)\bar{b}_2 = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} w_{11}^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right],$$

we may suppose that $w_{11} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{10})^{-\frac{w_{21}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{18})^{-\frac{w_{31}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{26})^{-\frac{w_{41}}{2}}$$

to eliminate the entries in the first column in block 4, we may suppose

$$\xi = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & w_{12} & w_{13} & w_{14} \\ 0 & w_{22} & w_{23} & w_{24} \\ 0 & w_{32} & w_{33} & w_{34} \\ 0 & w_{42} & w_{43} & w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in U_2\ .$$

Choose $w'_{22} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $w'_{22} \cdot w_{22} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(w'_{22} - 1)\bar{b}_{12} = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & w_{22}^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right],$$

we may suppose that $w_{22} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{4})^{-\frac{w_{12}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{20})^{-\frac{w_{32}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{28})^{-\frac{w_{42}}{2}}$$

to eliminate the entries in the second column in block 4, we may suppose

$$\xi = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & w_{13} & w_{14} \\ 0 & 1 & w_{23} & w_{24} \\ 0 & 0 & w_{33} & w_{34} \\ 0 & 0 & w_{43} & w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in U_2\ .$$

Choose $w'_{33} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $w'_{33} \cdot w_{33} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(w'_{33} - 1)\bar{b}_{22} = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w_{33}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6\ \right],$$

we may suppose that $w_{33} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_6)^{-\frac{w_{13}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{14})^{-\frac{w_{23}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{30})^{-\frac{w_{43}}{2}}$$

to eliminate the entries in the third column in block 4, we may suppose

$$\xi = \left[1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} 1\,0\,0\,w_{14} \\ 0\,1\,0\,w_{24} \\ 0\,0\,1\,w_{34} \\ 0\,0\,0\,w_{44} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6 \right] \in U_2\,.$$

Choose $w'_{44} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $w'_{44} \cdot w_{44} \equiv_8 1$.
Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(w'_{44} - 1)\overline{b}_{32} = \left[1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} 1\,0\,0 & 0 \\ 0\,1\,0 & 0 \\ 0\,0\,1 & 0 \\ 0\,0\,0 & w_{44}^{-1} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6 \right]\,,$$

we may suppose that $w_{44} = 1$.
Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_8)^{-\frac{w_{14}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{16})^{-\frac{w_{24}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{24})^{-\frac{w_{34}}{2}}$$

to eliminate the entries in the fourth column in block 4, we may suppose

$$\begin{aligned} \xi &= \left[1,\ 1,\ \mathrm{E}_4,\ \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6 \right] \\ &= \left[1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6 \right] \in U_2\,. \end{aligned}$$

□

7.2.3 Generators for $U_1 = \langle S_1, S_2, S_3, M_3 \rangle$

7.2.3.1 Generators for S_1

Remark 37. *S_1 has the following set of generators.*

$$\begin{aligned} &\{1_{\overline{\Gamma}_{(2)}} + \overline{b}_i : i \in \{36, 37,\ 39, 40,\ 42, 43,\ 45, 46,\ \ 51, 52,\ 54, 55,\ 57, 58,\ 60, 61, \\ &\qquad 66, 67,\ 69, 70,\ 72, 73,\ 75, 76\}\} \\ &\cup \{1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_{33}, 1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_{48}, 1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_{63}, 1_{\overline{\Gamma}_{(2)}} + \sigma\overline{b}_{78} : \sigma \in \{-2, 2, 4\}\} \\ &\cup \{1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_{34}, 1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_{49}, 1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_{64}, 1_{\overline{\Gamma}_{(2)}} + \tau\overline{b}_{79} : \tau \in \{-1, 1, 2\}\} \end{aligned} \tag{7.25}$$

Proof. Note that the elements in the claimed set of generators are invertible and in $\mathrm{Im}(\varphi)$, hence in $\mathrm{Im}(\mathrm{U}(\varphi))$; cf. Remark 15(II). By their block structure, they are contained in S_1.

Suppose given

$$\xi = \left[1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & y_{42} & y_{43} & y_{44} & y_{45} \\ 0 & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} & 0 \\ 0 & z_{32} & z_{33} & z_{34} & z_{35} & 0 \\ 0 & z_{42} & z_{43} & z_{44} & z_{45} & 0 \\ 0 & z_{52} & z_{53} & z_{54} & z_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \in S_1 .$$

Note that replacing representatives modulo 8, we may suppose that the matrix entries of ξ are in $\mathbb{Z}$.

We want to show that ξ is a product of the elements in the claimed list of generators above.

Note that

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Using this calculation blockwise, we get

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{45})^{-1} \cdot (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{36}) \cdot (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{45})^{-1} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}]$$

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{60})^{-1} \cdot (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{51}) \cdot (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{60})^{-1} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}]$$

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{75})^{-1} \cdot (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{66}) \cdot (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{75})^{-1} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}]$$

The asterisk stands for matrices that do not need to be calculated.

After reordering the rows we may assume that $z_{22} \not\equiv_2 0$.

Choose $z'_{22} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $z'_{22} \cdot z_{22} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (z'_{22} - 1)\bar{b}_{33} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}] ,$$

we may suppose that $z_{22} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{45})^{-z_{32}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{57})^{-z_{42}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{69})^{-z_{52}}$$

to eliminate the entries in the second column in block 7, we may suppose

$$\xi = \left[1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & y_{42} & y_{43} & y_{44} & y_{45} \\ 0 & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & z_{23} & z_{24} & z_{25} & 0 \\ 0 & 0 & z_{33} & z_{34} & z_{35} & 0 \\ 0 & 0 & z_{43} & z_{44} & z_{45} & 0 \\ 0 & 0 & z_{53} & z_{54} & z_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \in S_1 .$$

After reordering the rows we may assume that $z_{33} \not\equiv_2 0$.

Choose $z'_{33} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $z'_{33} \cdot z_{33} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (z'_{33} - 1)\overline{b}_{48} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{33}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}],$$

we may suppose that $z_{33} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{36})^{-z_{23}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{60})^{-z_{43}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{72})^{-z_{53}}$$

to eliminate the entries in the third column in block 7, we may suppose

$$\xi = \left[1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & y_{42} & y_{43} & y_{44} & y_{45} \\ 0 & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & z_{24} & z_{25} & 0 \\ 0 & 0 & 1 & z_{34} & z_{35} & 0 \\ 0 & 0 & 0 & z_{44} & z_{45} & 0 \\ 0 & 0 & 0 & z_{54} & z_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \in S_1 .$$

After reordering the rows we may assume that $z_{44} \not\equiv_2 0$.

Choose $z'_{44} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $z'_{44} \cdot z_{44} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (z'_{44} - 1)\overline{b}_{63} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{44}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}],$$

we may suppose that $z_{44} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{39})^{-z_{24}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{51})^{-z_{34}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{75})^{-z_{54}}$$

to eliminate the entries in the fourth column in block 7, we may suppose

$$\xi = \left[1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & y_{42} & y_{43} & y_{44} & y_{45} \\ 0 & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&z_{25}&0 \\ 0&0&1&0&z_{35}&0 \\ 0&0&0&1&z_{45}&0 \\ 0&0&0&0&z_{55}&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right] \in S_1 .$$

We have $z_{55} \not\equiv_2 0$.

Choose $z'_{55} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $z'_{55} \cdot z_{55} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + (z'_{55} - 1)\overline{b}_{78} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ *,\ \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&z_{55}^{-1}&0 \\ 0&0&0&0&0&1 \end{pmatrix}],$$

we may suppose that $z_{55} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{42})^{-z_{25}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{54})^{-z_{35}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{66})^{-z_{45}}$$

to eliminate the entries in the fifth column in block 7, we may suppose

$$\begin{aligned} \xi &= \left[1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & y_{42} & y_{43} & y_{44} & y_{45} \\ 0 & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right] \\ &= \left[1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & y_{42} & y_{43} & y_{44} & y_{45} \\ 0 & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \mathrm{E}_6 \right] \in S_1 . \end{aligned}$$

For $i, j \in [2, 5]$, we have $y_{ij} \equiv_2 z_{ij}$ by (7.2). And so we have for $i = j$ that $y_{ii} \equiv_2 z_{ii} = 1$ and for $i \neq j$ that $y_{ij} \equiv_2 z_{ij} = 0$.

Choose $y'_{22} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $y'_{22} \cdot y_{22} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(y_{22}' - 1)\overline{b}_{34} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{22}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_6]\,,$$

we may suppose that $y_{22} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{46})^{-\frac{y_{32}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{58})^{-\frac{y_{42}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{70})^{-\frac{y_{52}}{2}}$$

to eliminate the entries in the second column in block 6, we may suppose

$$\xi = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & y_{23} & y_{24} & y_{25} \\ 0 & 0 & y_{33} & y_{34} & y_{35} \\ 0 & 0 & y_{43} & y_{44} & y_{45} \\ 0 & 0 & y_{53} & y_{54} & y_{55} \end{pmatrix},\ \mathrm{E}_6\ \right] \in S_1\,.$$

Choose $y_{33}' \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $y_{33}' \cdot y_{33} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(y_{33}' - 1)\overline{b}_{49} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & y_{33}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_6]\,,$$

we may suppose that $y_{33} = 1$.

Using

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{37})^{-\frac{y_{23}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{61})^{-\frac{y_{43}}{2}}$$
$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{73})^{-\frac{y_{53}}{2}}$$

to eliminate the entries in the third column in block 6, we may suppose

$$\xi = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_{24} & y_{25} \\ 0 & 0 & 1 & y_{34} & y_{35} \\ 0 & 0 & 0 & y_{44} & y_{45} \\ 0 & 0 & 0 & y_{54} & y_{55} \end{pmatrix},\ \mathrm{E}_6\ \right] \in S_1\,.$$

Choose $y_{44}' \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $y_{44}' \cdot y_{44} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(y'_{44} - 1)\overline{b}_{64} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & y_{44}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_6]\,,$$

we may suppose that $y_{44} = 1$.

Using

$$\begin{gathered} (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{40})^{-\frac{y_{24}}{2}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{52})^{-\frac{y_{34}}{2}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{76})^{-\frac{y_{54}}{2}} \end{gathered}$$

to eliminate the entries in the fourth column in block 6, we may suppose

$$\xi = \left[\ 1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & y_{25} \\ 0 & 0 & 1 & 0 & y_{35} \\ 0 & 0 & 0 & 1 & y_{45} \\ 0 & 0 & 0 & 0 & y_{55} \end{pmatrix},\ \mathrm{E}_6\ \right] \in S_1\,.$$

Choose $y'_{55} \in \{-1, 1, 3, 5\} \subseteq \mathbb{Z}$ such that $y'_{55} \cdot y_{55} \equiv_8 1$.

Using

$$1_{\overline{\Gamma}_{(2)}} + \frac{1}{2}(y'_{55} - 1)\overline{b}_{79} = [1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ *,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & y_{55}^{-1} \end{pmatrix},\ \mathrm{E}_6]\,,$$

we may suppose that $y_{55} = 1$.

Using

$$\begin{gathered} (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{43})^{-\frac{y_{25}}{2}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{55})^{-\frac{y_{35}}{2}} \\ (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{67})^{-\frac{y_{45}}{2}} \end{gathered}$$

to eliminate the entries in the fifth column in block 6, we may suppose

$$\begin{aligned} \xi &= \left[\ 1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \mathrm{E}_6\ \right] \\ &= \left[\ 1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & x_{42} & x_{43} & x_{44} & x_{45} \\ 0 & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_6\ \right] \in S_1\,. \end{aligned}$$

For $i, j \in [2,5]$, we have $x_{ij} + y_{ij} \equiv_8 2z_{ij}$ by (7.2). And we have for $i = j$ that $x_{ij} \equiv_8 2z_{ii} - y_{ii} = 2 - 1 = 1$ and for $i \neq j$ that $x_{ij} \equiv_8 2z_{ij} - y_{ij} = 2 \cdot 0 - 0 = 0$.

So

$$\xi = \left[\; 1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix},\ \mathrm{E}_5,\ \mathrm{E}_6 \;\right] = \left[\; 1,\ 1,\ \mathrm{E}_4,\ \mathrm{E}_4,\ \mathrm{E}_5,\ \mathrm{E}_5,\ \mathrm{E}_6 \;\right] \in S_1 .$$

□

7.2.3.2 Generators for S_2

Remark 38. *S_2 has the following set of generators.*

$$\begin{aligned} \{&1_{\overline{\Gamma}_{(2)}} + \bar{b}_{81}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{82}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{83}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{85}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{86}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{87}, \\ &1_{\overline{\Gamma}_{(2)}} + \bar{b}_{89}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{90}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{91}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{93}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{94}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{95}\} \end{aligned} \tag{7.26}$$

Proof. Note that the elements in the claimed set of generators are invertible and in $\mathrm{Im}(\varphi)$, hence in $\mathrm{Im}(\mathrm{U}(\varphi))$; cf. Remark 15(II). By their block structure, they are contained in S_2.

Suppose given

$$\xi := \left[\; 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} & z_{15} & 0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0 & z_{62} & z_{63} & z_{64} & z_{65} & 1 \end{pmatrix} \;\right] \in S_2 .$$

Note that replacing representatives modulo 8, we may suppose that the matrix entries of ξ are in $\mathbb{Z}$.

We want to show that ξ is a product of the elements in the claimed list of generators above.

Note that

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{82} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&4&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&2&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{86} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&4&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&2&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{90} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&4&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&2&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{94} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&4 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&2&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right].$$

Since $z_{12}, z_{13}, z_{14}, z_{15}$ are divisible by 2, we can, after left multiplication with

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{82})^{-\frac{z_{12}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{86})^{-\frac{z_{13}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{90})^{-\frac{z_{14}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{94})^{-\frac{z_{15}}{2}}$$

suppose that $z_{12} = z_{13} = z_{14} = z_{15} = 0$ and

$$\xi = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&x_{12}&x_{13}&x_{14}&x_{15} \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&y_{12}&y_{13}&y_{14}&y_{15} \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&z_{62}&z_{63}&z_{64}&z_{65}&1 \end{pmatrix} \right].$$

Note that

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{81} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{85} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{89} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{93} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}\ \right].$$

So we can, after left multiplication with

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{81})^{-z_{62}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{85})^{-z_{63}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{89})^{-z_{64}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{93})^{-z_{65}}$$

suppose that $z_{62} = z_{63} = z_{64} = z_{65} = 0$ and

$$\begin{aligned} \xi &= \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\ \right] \\ &= \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathrm{E}_6\ \right]. \end{aligned}$$

For $i \in [2, 5]$, we have $y_{1i} \equiv_4 2z_{6i}$ by (7.2) and $z_{6i} = 0$, thus $y_{1i} \equiv_4 0$; cf. (7.2).

Note that

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{83} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&4&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&4&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{87} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&4&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&4&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{91} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&4&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&4&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{95} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&4\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&4\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6\ \right].$$

Since $y_{12}, y_{13}, y_{14}, y_{15}$ are divisible by 4, we can, after left multiplication with

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{83})^{-\frac{y_{12}}{4}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{87})^{-\frac{y_{13}}{4}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{91})^{-\frac{y_{14}}{4}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{95})^{-\frac{y_{15}}{4}}$$

suppose that $y_{12} = y_{13} = y_{14} = y_{15} = 0$ and

$$\begin{aligned} \xi &= \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&x_{12}&x_{13}&x_{14}&x_{15}\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6\ \right] \\ &= \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&x_{12}&x_{13}&x_{14}&x_{15}\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6\ \right]. \end{aligned}$$

For $i \in [2, 5]$, we have $x_{1i} + y_{1i} \equiv_8 z_{1i}$ by (7.2) and $y_{1i} = 0$ and $z_{1i} = 0$, thus we get $x_{1i} \equiv_8 0$. So

$$\begin{aligned} \xi &= \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6\ \right] \\ &= \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \mathrm{E}_5, \mathrm{E}_5, \mathrm{E}_6\ \right]. \end{aligned}$$

□

7.2.3.3 Generators for S_3

Remark 39. *S_3 has the following set of generators.*

$$\{1_{\overline{\Gamma}_{(2)}} + \overline{b}_i \mid i \in [97, 112]\} \tag{7.27}$$

Proof. Note that the elements in the claimed set of generators are invertible and in $\mathrm{Im}(\varphi)$, hence in $\mathrm{Im}(\mathrm{U}(\varphi))$; cf. Remark 15(II). By their block structure, they are contained in S_3.

Suppose given

$$\xi := \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & 0&0&0&0 \\ x_{21} & 1&0&0&0 \\ x_{31} & 0&1&0&0 \\ x_{41} & 0&0&1&0 \\ x_{51} & 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1 & 0&0&0&0 \\ y_{21} & 1&0&0&0 \\ y_{31} & 0&1&0&0 \\ y_{41} & 0&0&1&0 \\ y_{51} & 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1 & 0&0&0&0 & 0 \\ z_{21} & 1&0&0&0 & z_{26} \\ z_{31} & 0&1&0&0 & z_{36} \\ z_{41} & 0&0&1&0 & z_{46} \\ z_{51} & 0&0&0&1 & z_{56} \\ 0 & 0&0&0&0 & 1 \end{pmatrix} \right] \in S_3 \,.$$

Note that replacing representatives modulo 8, we may suppose that the matrix entries of ξ are in $\mathbb{Z}$.

We want to show that ξ is a product of the elements in the claimed list of generators above.

Note that

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{97} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{101} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 1&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 1&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{105} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \overline{b}_{109} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 1&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 1&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \right] .$$

So we can, after right multiplication with

$$(1_{\overline{\Gamma}_{(2)}} + \overline{b}_{97})^{-z_{21}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{101})^{-z_{31}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{105})^{-z_{41}}, (1_{\overline{\Gamma}_{(2)}} + \overline{b}_{109})^{-z_{51}}$$

suppose that $z_{21} = z_{31} = z_{41} = z_{51} = 0$ and

$$\xi = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1 & 0&0&0&0 \\ x_{21} & 1&0&0&0 \\ x_{31} & 0&1&0&0 \\ x_{41} & 0&0&1&0 \\ x_{51} & 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1 & 0&0&0&0 \\ y_{21} & 1&0&0&0 \\ y_{31} & 0&1&0&0 \\ y_{41} & 0&0&1&0 \\ y_{51} & 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 & 0 \\ 0&1&0&0&0 & z_{26} \\ 0&0&1&0&0 & z_{36} \\ 0&0&0&1&0 & z_{46} \\ 0&0&0&0&1 & z_{56} \\ 0&0&0&0&0 & 1 \end{pmatrix} \right] \in S_3 \,.$$

For $i \in [2,5]$, we have $z_{i6} \equiv_2 0$ by (7.2).

Note that

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{99} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 2&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&2\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{103} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 2&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&2\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{107} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 2&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&2\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}\ \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{111} = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 2&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&2\\ 0&0&0&0&0&1 \end{pmatrix}\ \right].$$

Since $z_{26}, z_{36}, z_{46}, z_{56}$ are divisible by 2, we can, after right multiplication with

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{99})^{-\frac{z_{26}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{103})^{-\frac{z_{36}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{107})^{-\frac{z_{46}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{111})^{-\frac{z_{56}}{2}}$$

suppose that $z_{26} = z_{36} = z_{46} = z_{56} = 0$ and

$$\xi = \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ x_{21}&1&0&0&0\\ x_{31}&0&1&0&0\\ x_{41}&0&0&1&0\\ x_{51}&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\ y_{21}&1&0&0&0\\ y_{31}&0&1&0&0\\ y_{41}&0&0&1&0\\ y_{51}&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}\ \right]$$

$$= \left[\ 1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ x_{21}&1&0&0&0\\ x_{31}&0&1&0&0\\ x_{41}&0&0&1&0\\ x_{51}&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\ y_{21}&1&0&0&0\\ y_{31}&0&1&0&0\\ y_{41}&0&0&1&0\\ y_{51}&0&0&0&1 \end{pmatrix}, \mathrm{E}_6\ \right] \in S_3\,.$$

For $i \in [2,5]$, we have $y_{i1} \equiv_2 z_{i1} = 0$; cf. (7.2).

Note that

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{98} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 2&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 2&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6 \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{102} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 2&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 2&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6 \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{106} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 2&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 2&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6 \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{110} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 2&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 2&0&0&0&1 \end{pmatrix}, \mathrm{E}_6 \right] .$$

Since $y_{21}, y_{31}, y_{41}, y_{51}$ are divisible by 2, we can, after right multiplication with

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{98})^{-\frac{y_{21}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{102})^{-\frac{y_{31}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{106})^{-\frac{y_{41}}{2}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{110})^{-\frac{y_{51}}{2}}$$

suppose that $y_{21} = y_{31} = y_{41} = y_{51} = 0$ and

$$\begin{aligned} \xi &= \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ x_{21}&1&0&0&0 \\ x_{31}&0&1&0&0 \\ x_{41}&0&0&1&0 \\ x_{51}&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_6 \right] \\ &= \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0 \\ x_{21}&1&0&0&0 \\ x_{31}&0&1&0&0 \\ x_{41}&0&0&1&0 \\ x_{51}&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6 \right] \in S_3 . \end{aligned}$$

For $i \in [2,5]$, we have $x_{i1} \equiv_4 y_{i1} + z_{i6} = 0 + 0 = 0$; cf. (7.2).

Note that

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{100} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 4&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6 \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{104} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 4&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6 \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{108} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 4&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6 \right]$$

$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{112} = \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 4&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6 \right] .$$

Since $x_{21}, x_{31}, x_{41}, x_{51}$ are divisible by 4, we can, after right multiplication with

$$(1_{\overline{\Gamma}_{(2)}} + \bar{b}_{100})^{-\frac{x_{21}}{4}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{104})^{-\frac{x_{31}}{4}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{108})^{-\frac{x_{41}}{4}}, (1_{\overline{\Gamma}_{(2)}} + \bar{b}_{112})^{-\frac{x_{51}}{4}}$$

suppose that $x_{21} = x_{31} = x_{41} = x_{51} = 0$ and

$$\begin{aligned} \xi &= \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \mathrm{E}_5, \mathrm{E}_6 \right] \\ &= \left[1, 1, \mathrm{E}_4, \mathrm{E}_4, \mathrm{E}_5, \mathrm{E}_5, \mathrm{E}_6 \right] \in S_3 . \end{aligned}$$

□

7.2.4 Magma application

To start the Magma procedure, we first need to fix some data for later.

```
Z := Integers();
Q := Rationals();
UGq := GL(26,Integers(8));
U := sub<UGq|>; // initial U // after the procedure, we have U = ImUphi

RM := RMatrixSpace(Integers(),120,1);

M8 := {-3,-2,-1,0,1,2,3,4};
M8odd := {-3,-1,1,3};
M4 := {-1,0,1,2};
M4o0 := {-1,1,2};
M2 := {0,1};

// function to convert the vector back into a matrix
MakeMatrix := function(y);
  return UGq!DiagonalJoin(<Matrix([[y[1,1]]]), Matrix([[y[2,1]]]),
            Matrix([[y[3,1],y[4,1],y[5,1],y[6,1]],[y[7,1],y[8,1],y[9,1],
              y[10,1]],[y[11,1],y[12,1],y[13,1],y[14,1]],[y[15,1],y[16,1],
              y[17,1],y[18,1]]]),
            Matrix([[y[19,1],y[20,1],y[21,1],y[22,1]],[y[23,1],y[24,1],
              y[25,1],y[26,1]],[y[27,1],y[28,1],y[29,1],y[30,1]],[y[31,1],
              y[32,1],y[33,1],y[34,1]]]),
            Matrix([[y[35,1],y[36,1],y[37,1],y[38,1],y[39,1]],[y[40,1],
              y[41,1],y[42,1],y[43,1],y[44,1]],[y[45,1],y[46,1],y[47,1],
              y[48,1],y[49,1]],[y[50,1],y[51,1],
              y[52,1],y[53,1],y[54,1]],[y[55,1],y[56,1],y[57,1],y[58,1],y[59,1]]]),
            Matrix([[y[60,1],y[61,1],y[62,1],y[63,1],y[64,1]],[y[65,1],
              y[66,1],y[67,1],y[68,1],y[69,1]],[y[70,1],y[71,1],y[72,1],
              y[73,1],y[74,1]],[y[75,1],y[76,1],y[77,1],y[78,1],
              y[79,1]],[y[80,1],y[81,1],y[82,1],y[83,1],y[84,1]]]),
            Matrix([[y[85,1],y[86,1],y[87,1],y[88,1],y[89,1],y[90,1]],
              [y[91,1],y[92,1],y[93,1],y[94,1],y[95,1],y[96,1]],[y[97,1],
              y[98,1],y[99,1],y[100,1],y[101,1],y[102,1]],[y[103,1],
              y[104,1],y[105,1],y[106,1],y[107,1],y[108,1]],[y[109,1],
              y[110,1],y[111,1],y[112,1],y[113,1],y[114,1]],[y[115,1],
              y[116,1],y[117,1],y[118,1],y[119,1],y[120,1]]])>);
end function;

// function to transpose matricies
T := function(x)
 return RM!Transpose(Matrix([x]));
end function;
```

Expressing the basis $\mathfrak{B}_{\Lambda_{(2)}}$, cf. (7.22), by the vector convention, we get

```
basis :=
[
// Block number:
//  1  2  3 3 3 3  3 3 3 3  3 3 3 3  3 3 3 3  4 4 4 4  4 4 4 4  4 4 4 4  4 4 4 4  5 5 5 5 5  5 5 5 5 5  5 5 5 5 5  5 5 5 5 5  5 5 5 5 5  6 6 6 6 6  6 6 6 6 6  6 6 6 6 6  6 6 6 6 6  6 6 6 6 6  7 7 7 7 7 7  7 7 7 7 7 7  7 7 7 7 7 7  7 7 7 7 7 7  7 7 7 7 7 7  7 7 7 7 7 7
T([0, 0, 1,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 1,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 1
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 2,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 2

T([0, 0, 0,1,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,1,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 3
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,2,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 4

T([0, 0, 0,0,1,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,1,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 5
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,2,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 6

T([0, 0, 0,0,0,1, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,1, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 7
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,2, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 8


T([0, 0, 0,0,0,0, 1,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 1,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 9
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 2,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 10

T([0, 0, 0,0,0,0, 0,1,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,1,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 11
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,2,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 12

T([0, 0, 0,0,0,0, 0,0,1,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,1,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 13
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,2,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 14

T([0, 0, 0,0,0,0, 0,0,0,1, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,1, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 15
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,2, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 16


T([0, 0, 0,0,0,0, 0,0,0,0, 1,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 1,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 17
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 2,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 18

T([0, 0, 0,0,0,0, 0,0,0,0, 0,1,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,1,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 19
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,2,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 20

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,1,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,1,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 27
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,2,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 28

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0, 1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,1,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 33
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,-2,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,2,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 34
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0, 8,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,-2,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,2,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 37
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0, 8,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,-2,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,2,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 40
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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,-2, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,2, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 43
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0, 8, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage


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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,-2,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,2,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 46
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0, 8,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,-2,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,2,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 49
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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,-2,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,2,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 52
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0, 8,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,-2, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,2, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 55
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0, 8, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage


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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,-2,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,2,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 58
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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,-2,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,2,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 64
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0, 8,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,-2, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,2, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 67
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0, 8, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage


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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,-2,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,2,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 70
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0, 8,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,-2,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,2,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 73
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0, 8,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,-2,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,2,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 76
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0, 8,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,-2, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,2, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 79
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0, 8, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage


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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,4,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,2,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 82
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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0, 8,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0, 0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,4,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,2,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 90
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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0, 8,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage

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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,4, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,2,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 94
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 4, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,4, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 95
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 8, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // no usage


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T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 98
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,2, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 99
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 4,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 100

T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 1,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 101
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 102
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,2, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 103
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 4,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 104

T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 1,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 105
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 106
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,2, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 107
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 4,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 108

T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 1,0,0,0,0,0, 0,0,0,0,0,0]), // 109
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 110
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,2, 0,0,0,0,0,0]), // 111
T([0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 4,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 112


T([ 1, 1, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,  1,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 1,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,1]), // 113
T([ 0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,  2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 1,0,0,0,0,0]), // 114
T([ 0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, -2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 1,0,0,0,0,0]), // 115
T([-2, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,  2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,2, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 116
T([ 2, 2, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,  2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 2,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 117
T([ 0, 4, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,  0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 4,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 118
T([ 4, 4, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,  0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0]), // 119
T([ 8, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,  0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0])  // no usage
];
```

So we translate the elements of $\mathfrak{B}_{\Lambda_{(2)}}$ as $b_j =$`basis[j]` for $j \in [1, 120]$.

Certain of these elements are mapped to zero in $\overline{\Gamma}_{(2)}$, so we have no usage for these for our generating tuple $\mathfrak{G}_{\mathrm{Im}(\varphi)}$; cf. (7.23).

Then $\mathrm{Im}(\varphi)$ is the $\mathbb{Z}/8$-linear span of $\mathfrak{G}_{\mathrm{Im}(\varphi)}$.

We let α_i be the coefficient of $\bar{b}_i$. We need not use the whole list $\alpha_i \in \{-3, -2, -1, 0, 1, 2, 3, 4\}$ for each of them, because certain elements are mapped to the same element in $\overline{\Gamma}_{(2)}$.

For example, we have $\bar{b}_2 = 2\bar{e}_{4;1,1}$, so with $\alpha_2 \in \{-1, 0, 1, 2\}$ we get

$$\alpha_2 \bar{b}_2 \in \{-2e_{4;1,1}, 0, 2e_{4;1,1}, 4e_{4;1,1}\} = \{z \cdot 2\bar{e}_{4;1,1} : z \in \mathbb{Z}/8\}\,.$$

So we get

$$\begin{aligned}
\mathrm{Im}(\varphi) = \Big\{ & \alpha_1\bar{b}_1 + \alpha_2\bar{b}_2 + \alpha_3\bar{b}_3 + \alpha_4\bar{b}_4 + \alpha_5\bar{b}_5 + \alpha_6\bar{b}_6 + \alpha_7\bar{b}_7 + \alpha_8\bar{b}_8 + \alpha_9\bar{b}_9 + \alpha_{10}\bar{b}_{10} + \alpha_{11}\bar{b}_{11} + \alpha_{12}\bar{b}_{12} \\
& + \alpha_{13}\bar{b}_{13} + \alpha_{14}\bar{b}_{14} + \alpha_{15}\bar{b}_{15} + \alpha_{16}\bar{b}_{16} + \alpha_{17}\bar{b}_{17} + \alpha_{18}\bar{b}_{18} + \alpha_{19}\bar{b}_{19} + \alpha_{20}\bar{b}_{20} + \alpha_{21}\bar{b}_{21} + \alpha_{22}\bar{b}_{22} \\
& + \alpha_{23}\bar{b}_{23} + \alpha_{24}\bar{b}_{24} + \alpha_{25}\bar{b}_{25} + \alpha_{26}\bar{b}_{26} + \alpha_{27}\bar{b}_{27} + \alpha_{28}\bar{b}_{28} + \alpha_{29}\bar{b}_{29} + \alpha_{30}\bar{b}_{30} + \alpha_{31}\bar{b}_{31} + \alpha_{32}\bar{b}_{32} : \\
& \alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{15}, \alpha_{17}, \alpha_{19}, \alpha_{21}, \alpha_{23}, \alpha_{25}, \alpha_{27}, \alpha_{29}, \alpha_{31} \in \{-3, -2, -1, 0, 1, 2, 3, 4\}, \\
& \alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}, \alpha_{14}, \alpha_{16}, \alpha_{18}, \alpha_{20}, \alpha_{22}, \alpha_{24}, \alpha_{26}, \alpha_{28}, \alpha_{30}, \alpha_{32} \in \{-1, 0, 1, 2\} \Big\} \\
\oplus \Big\{ & \alpha_{33}\bar{b}_{33} + \alpha_{34}\bar{b}_{34} + \alpha_{36}\bar{b}_{36} + \alpha_{37}\bar{b}_{37} + \alpha_{39}\bar{b}_{39} + \alpha_{40}\bar{b}_{40} + \alpha_{42}\bar{b}_{42} + \alpha_{43}\bar{b}_{43} + \alpha_{45}\bar{b}_{45} \\
& + \alpha_{46}\bar{b}_{46} + \alpha_{48}\bar{b}_{48} + \alpha_{49}\bar{b}_{49} + \alpha_{51}\bar{b}_{51} + \alpha_{52}\bar{b}_{52} + \alpha_{54}\bar{b}_{54} + \alpha_{55}\bar{b}_{55} + \alpha_{57}\bar{b}_{57} + \alpha_{58}\bar{b}_{58} \\
& + \alpha_{60}\bar{b}_{60} + \alpha_{61}\bar{b}_{61} + \alpha_{63}\bar{b}_{63} + \alpha_{64}\bar{b}_{64} + \alpha_{66}\bar{b}_{66} + \alpha_{67}\bar{b}_{67} + \alpha_{69}\bar{b}_{69} + \alpha_{70}\bar{b}_{70} + \alpha_{72}\bar{b}_{72} \\
& + \alpha_{73}\bar{b}_{73} + \alpha_{75}\bar{b}_{75} + \alpha_{76}\bar{b}_{76} + \alpha_{78}\bar{b}_{78} + \alpha_{79}\bar{b}_{79} : \\
& \alpha_{33}, \alpha_{36}, \alpha_{39}, \alpha_{42}, \alpha_{45}, \alpha_{48}, \alpha_{51}, \alpha_{54}, \alpha_{57}, \alpha_{60}, \alpha_{63}, \\
& \qquad \alpha_{66}, \alpha_{69}, \alpha_{72}, \alpha_{75}, \alpha_{78} \in \{-3, -2, -1, 0, 1, 2, 3, 4\}, \\
& \alpha_{34}, \alpha_{37}, \alpha_{40}, \alpha_{43}, \alpha_{46}, \alpha_{49}, \alpha_{52}, \alpha_{55}, \alpha_{58}, \alpha_{61}, \alpha_{64}, \alpha_{67}, \alpha_{70}, \alpha_{73}, \alpha_{76}, \alpha_{79} \in \{-1, 0, 1, 2\} \Big\} \\
\oplus \Big\{ & \alpha_{81}\bar{b}_{81} + \alpha_{82}\bar{b}_{82} + \alpha_{83}\bar{b}_{83} + \alpha_{85}\bar{b}_{85} + \alpha_{86}\bar{b}_{86} + \alpha_{87}\bar{b}_{87} + \alpha_{89}\bar{b}_{89} + \alpha_{90}\bar{b}_{90} + \alpha_{91}\bar{b}_{91} \\
& + \alpha_{93}\bar{b}_{93} + \alpha_{94}\bar{b}_{94} + \alpha_{95}\bar{b}_{95} : \\
& \alpha_{81}, \alpha_{85}, \alpha_{89}, \alpha_{93} \in \{-3, -2, -1, 0, 1, 2, 3, 4\}, \\
& \alpha_{82}, \alpha_{86}, \alpha_{90}, \alpha_{94} \in \{-1, 0, 1, 2\}, \\
& \alpha_{83}, \alpha_{87}, \alpha_{91}, \alpha_{95} \in \{0, 1\} \Big\} \\
\oplus \Big\{ & \alpha_{97}\bar{b}_{97} + \alpha_{98}\bar{b}_{98} + \alpha_{99}\bar{b}_{99} + \alpha_{100}\bar{b}_{100} + \alpha_{101}\bar{b}_{101} + \alpha_{102}\bar{b}_{102} + \alpha_{103}\bar{b}_{103} + \alpha_{104}\bar{b}_{104} + \alpha_{105}\bar{b}_{105} \\
& + \alpha_{106}\bar{b}_{106} + \alpha_{107}\bar{b}_{107} + \alpha_{108}\bar{b}_{108} + \alpha_{109}\bar{b}_{109} + \alpha_{110}\bar{b}_{110} + \alpha_{111}\bar{b}_{111} + \alpha_{112}\bar{b}_{112} : \\
& \alpha_{97}, \alpha_{101}, \alpha_{105}, \alpha_{109} \in \{-3, -2, -1, 0, 1, 2, 3, 4\}, \\
& \alpha_{98}, \alpha_{99}, \alpha_{102}, \alpha_{103}, \alpha_{106}, \alpha_{107}, \alpha_{110}, \alpha_{111} \in \{-1, 0, 1, 2\}, \\
& \alpha_{100}, \alpha_{104}, \alpha_{108}, \alpha_{112} \in \{0, 1\} \Big\} \\
\oplus \Big\{ & \alpha_{113}\bar{b}_{113} + \alpha_{114}\bar{b}_{114} + \alpha_{115}\bar{b}_{115} + \alpha_{116}\bar{b}_{116} + \alpha_{117}\bar{b}_{117} + \alpha_{118}\bar{b}_{118} + \alpha_{119}\bar{b}_{119} : \\
& \alpha_{113}, \alpha_{114}, \alpha_{115} \in \{-3, -2, -1, 0, 1, 2, 3, 4\}, \\
& \alpha_{116}, \alpha_{117} \in \{-1, 0, 1, 2\}, \\
& \alpha_{118}, \alpha_{119} \in \{0, 1\} \Big\}\,.
\end{aligned} \tag{7.28}$$

The vertical markings indicate to which part of the picture (7.2) the respective part of $\mathrm{Im}(\varphi)$ belongs, analogously to (7.22).

Now we want to build $\mathrm{Im}(\mathrm{U}(\varphi))$ inside $\mathrm{Im}(\varphi)$. We have

$$\begin{aligned}\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \times U_2 = (S_1 \cdot S_2 \cdot M_3 \cdot S_3) \times U_2 &= \langle S_1, S_2, M_3, S_3 \rangle \times U_2 \\ &= \langle S_1, S_2, S_3, M_3 \rangle \times U_2\,;\end{aligned}$$

cf. Lemma 35.

We will start building `U_2` separately.

Part 1. As in (7.5), we have

$$U_2 = \left\{ \left[1,\ 1,\ \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}, \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \right.\right.$$

$$\left.\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} :$$

$$\left. \det \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix} \not\equiv_2 0\,, v_{ij} \equiv_2 w_{ij},\ i,j \in [1,4] \right\} .$$

We will use the generators calculated in §7.2.2.

To begin with we define U_2 as the trivial group.

```
U_2 := sub<UGq | >;
```

Then we add the generators defined in Remark 36.

These generators of U_2 are

$$\begin{aligned}&\{1_{\overline{\Gamma}_{(2)}} + \bar{b}_i : i \in \{3,4,5,6,7,8,9,10,\ 13,14,15,16,17,18,19,20,\ 23,24,25,26,27,28,29,30\}\} \\ &\cup \{1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_1, 1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_{11}, 1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_{21}, 1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_{31} : \sigma \in \{-2,2,4\}\} \\ &\cup \{1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_2, 1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_{12}, 1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_{22}, 1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_{32} : \tau \in \{-1,1,2\}\}\end{aligned}$$

by (7.24).

```
// one vector written as basis vectors
one := basis[113] + (basis[33] + basis[48] + basis[63] + basis[78])
       + (basis[1] + basis[11] + basis[21] + basis[31]);

y_list := [one + basis[i] : i in [3,4,5,6,7,8,9,10,
                                  13,14,15,16,17,18,19,20,
                                  23,24,25,26,27,28,29,30]];
y_list cat:= [one + 2 * x * basis[i] : x in M4o0, i in [1,11,21,31]];
y_list cat:= [one + x * basis[i] : x in M4o0, i in [2,12,22,32]];
U_2 := sub<UGq | GeneratorsSequence(U_2) cat [MakeMatrix(y) : y in y_list]>;
// Factorisation(Order(U_2)); // 2^70 * 3^2 * 5 * 7 = 371886360525984560578560
```

To calculate `U_1` we will start with `U_1` = 1, then build `U_1 = S_1`, then build `U_1 = <S_1, S_2>`, then build `U_1 = <S_1, S_2, S_3>` and finally build `U_1 = <S_1, S_2, S_3, M_3>`.

Part 2. As in (7.8), we have

$$U_1 = \left\{ \left[t,\ u,\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} x_{11}&x_{12}&x_{13}&x_{14}&x_{15}\\x_{21}&x_{22}&x_{23}&x_{24}&x_{25}\\x_{31}&x_{32}&x_{33}&x_{34}&x_{35}\\x_{41}&x_{42}&x_{43}&x_{44}&x_{45}\\x_{51}&x_{52}&x_{53}&x_{54}&x_{55} \end{pmatrix}, \begin{pmatrix} y_{11}&y_{12}&y_{13}&y_{14}&y_{15}\\y_{21}&y_{22}&y_{23}&y_{24}&y_{25}\\y_{31}&y_{32}&y_{33}&y_{34}&y_{35}\\y_{41}&y_{42}&y_{43}&y_{44}&y_{45}\\y_{51}&y_{52}&y_{53}&y_{54}&y_{55} \end{pmatrix}, \begin{pmatrix} z_{11}&z_{12}&z_{13}&z_{14}&z_{15}&z_{16}\\z_{21}&z_{22}&z_{23}&z_{24}&z_{25}&z_{26}\\z_{31}&z_{32}&z_{33}&z_{34}&z_{35}&z_{36}\\z_{41}&z_{42}&z_{43}&z_{44}&z_{45}&z_{46}\\z_{51}&z_{52}&z_{53}&z_{54}&z_{55}&z_{56}\\z_{61}&z_{62}&z_{63}&z_{64}&z_{65}&z_{66} \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \right.$$

$$\left. \begin{array}{llll} t \not\equiv_2 0,\ \det \begin{pmatrix} z_{22}&z_{23}&z_{24}&z_{25}\\z_{32}&z_{33}&z_{34}&z_{35}\\z_{42}&z_{43}&z_{44}&z_{45}\\z_{52}&z_{53}&z_{54}&z_{55} \end{pmatrix} \not\equiv_2 0\,, &&& \\ x_{ij}+y_{ij} \equiv_8 2z_{ij},\ i,j\in[2,5], & y_{ij}\equiv_2 z_{ij},\ i,j\in[2,5], && \\ x_{1i}\equiv_2 0,\ i\in[2,5], & y_{1i}\equiv_2 0,\ i\in[2,5], & z_{1i}\equiv_2 0,\ i\in[2,5], & \\ x_{1i}+y_{1i}\equiv_8 2z_{1i}, i\in[2,5], & y_{1i}\equiv_4 2z_{6i},\ i\in[2,5], && \\ z_{i6}\equiv_2 0,\ i\in[2,5], & x_{i1}-y_{i1}\equiv_4 z_{i6},\ i\in[2,5], & y_{i1}\equiv_2 z_{i1},\ i\in[2,5], & \\ t\equiv_2 u, & z_{16}\equiv_2 0, & x_{11}\equiv_2 z_{66}, & x_{11}-y_{11}\equiv_4 z_{16}, \\ u-y_{11}\equiv_4 2z_{61}, & t+u+x_{11}+y_{11} \ \equiv_8 2z_{11}+2z_{66}\equiv_4 0 && \end{array} \right\}.$$

Now we will start to build $U_1 = \langle S_1, S_2, S_3, M_3\rangle$; cf. Remark 34

Step 1. We add S_1 to the generators of U_1.

As in (7.17), we have

$$S_1 = \left\{ \left[1,\ 1,\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&x_{22}&x_{23}&x_{24}&x_{25}\\0&x_{32}&x_{33}&x_{34}&x_{35}\\0&x_{42}&x_{43}&x_{44}&x_{45}\\0&x_{52}&x_{53}&x_{54}&x_{55} \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&y_{22}&y_{23}&y_{24}&y_{25}\\0&y_{32}&y_{33}&y_{34}&y_{35}\\0&y_{42}&y_{43}&y_{44}&y_{45}\\0&y_{52}&y_{53}&y_{54}&y_{55} \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&z_{22}&z_{23}&z_{24}&z_{25}&0\\0&z_{32}&z_{33}&z_{34}&z_{35}&0\\0&z_{42}&z_{43}&z_{44}&z_{45}&0\\0&z_{52}&z_{53}&z_{54}&z_{55}&0\\0&0&0&0&0&1 \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \right.$$

$$\left. \det \begin{pmatrix} z_{22}&z_{23}&z_{24}&z_{25}\\z_{32}&z_{33}&z_{34}&z_{35}\\z_{42}&z_{43}&z_{44}&z_{45}\\z_{52}&z_{53}&z_{54}&z_{55} \end{pmatrix} \not\equiv_2 0,\ x_{ij}+y_{ij}\equiv_8 2z_{ij} \text{ for } i,j\in[2,5],\ y_{ij}\equiv_2 z_{ij} \text{ for } i,j\in[2,5] \right\}$$

We will use the generators calculated in §7.2.3.1.

To begin with we define U_1 as the trivial group.

```
U_1 := sub<UGq | >;
```

Then we add the generators defined in Remark 37.

These generators of S_1 are

$$\{1_{\overline{\Gamma}_{(2)}} + \bar{b}_i : i \in \{36,37,\ 39,40,\ 42,43,\ 45,46,\ \ 51,52,\ 54,55,\ 57,58,\ 60,61,\ 66,67,\ 69,70,\ 72,73,\ 75,76\}\}$$
$$\cup\, \{1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_{33}, 1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_{48}, 1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_{63}, 1_{\overline{\Gamma}_{(2)}} + \sigma\bar{b}_{78} : \sigma \in \{-2,2,4\}\}$$
$$\cup\, \{1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_{34}, 1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_{49}, 1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_{64}, 1_{\overline{\Gamma}_{(2)}} + \tau\bar{b}_{79} : \tau \in \{-1,1,2\}\}$$

by (7.25).

So we are adding those to `U_1`.

```
// one vector written as basis vectors
one := basis[113] + (basis[33] + basis[48] + basis[63] + basis[78])
       + (basis[1] + basis[11] + basis[21] + basis[31]);

// S_1 is added to U_1
y_list := [one + basis[i] : i in [36,37, 39,40, 42,43, 45,46,  51,52, 54,55,
                                  57,58, 60,61, 66,67, 69,70, 72,73, 75,76]];
y_list cat:= [one + 2 * x * basis[i] : x in M4o0, i in [33,48,63,78]];
y_list cat:= [one + x * basis[i] : x in M4o0, i in [34,49,64,79]];
U_1 := sub<UGq | GeneratorsSequence(U_1) cat [MakeMatrix(y) : y in y_list]>;
// here: U_1 = <S_1>
// Factorisation(Order(U_1)); // 2^70 * 3^2 * 5 * 7
```

Step 2. We add S_2 to the generators of U_1.

As in (7.18), we have

$$S_2 = \left\{ \left[1,\ 1,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&x_{12}&x_{13}&x_{14}&x_{15}\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&y_{12}&y_{13}&y_{14}&y_{15}\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&z_{12}&z_{13}&z_{14}&z_{15}&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&z_{62}&z_{63}&z_{64}&z_{65}&1\end{pmatrix} \right] \in \mathrm{U}(\overline{\Gamma}_{(2)}) : \begin{array}{l} x_{1i} \equiv_2 0, i \in [2,5];\ \ y_{1i} \equiv_2 0, i \in [2,5];\ \ z_{1i} \equiv_2 0, i \in [2,5]; \\ x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5];\ \ y_{1i} \equiv_4 2z_{6i}, i \in [2,5] \end{array} \right\}.$$

We will use the generators calculated in §7.2.3.2.

We want to add the generators defined in Remark 38 to U_1.

These generators of S_2 are

$$\{1_{\overline{\Gamma}_{(2)}} + \bar{b}_{81}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{82}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{83}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{85}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{86}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{87},$$
$$1_{\overline{\Gamma}_{(2)}} + \bar{b}_{89}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{90}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{91}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{93}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{94}, 1_{\overline{\Gamma}_{(2)}} + \bar{b}_{95}\}$$

by (7.26).

So we are adding those to `U_1`.

```
// S_2 is added to U_1
y_list cat:= [one + basis[i] : i in [81,82,83,  85,86,87,  89,90,91,  93,94,95]];
U_1 := sub<UGq | GeneratorsSequence(U_1) cat [MakeMatrix(y) : y in y_list]>;
// here: U_1 = <S_1, S_2>
// Factorisation(Order(U_1)); // 2^94 * 3^2 * 5 * 7
```

Step 3. We build `U_1 = <S_1, S_2, S_3>`

As in (7.19), we have

$$S_3 = \left\{ \left[1,\ 1,\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0&0&0\\x_{21}&1&0&0&0\\x_{31}&0&1&0&0\\x_{41}&0&0&1&0\\x_{51}&0&0&0&1\end{pmatrix},\right.\right.$$

$$\left.\begin{pmatrix}1&0&0&0&0\\y_{21}&1&0&0&0\\y_{31}&0&1&0&0\\y_{41}&0&0&1&0\\y_{51}&0&0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0&0&0&0\\z_{21}&1&0&0&0&z_{26}\\z_{31}&0&1&0&0&z_{36}\\z_{41}&0&0&1&0&z_{46}\\z_{51}&0&0&0&1&z_{56}\\0&0&0&0&0&1\end{pmatrix}\right] \in \mathrm{U}(\overline{\Gamma}_{(2)}) :$$

$$\left. z_{i6} \equiv_2 0,\ i \in [2,5];\ x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5];\ y_{i1} \equiv_2 z_{i1},\ i \in [2,5] \right\}.$$

We will use the generators calculated in §7.2.3.3.

We want to add the generators defined in Remark 39 to U_1.

These generators of S_2 are

$$\{1_{\overline{\Gamma}_{(2)}} + \bar{b}_i \mid i \in [97, 112]\}$$

by (7.27).

So we are adding those to `U_1`.

```
// S_3 is added to U_1
y_list cat:= [one + basis[i] : i in [97,98,99,100,101,102,103,104,105,106,107,
                                     108,109,110,111,112]];
U_1 := sub<UGq | GeneratorsSequence(U_1) cat [MakeMatrix(y) : y in y_list]>;
// here: U_1 = <S_1, S_2, S_3>
// Factorisation(Order(U_1)); // 2^135 * 3^2 * 5 * 7
```

Step 4. We build `U_1 = <S_1, S_2, S_3, M_3>`

As in (7.20), we have

$$M_3 = \left\{ \left[t,\ u,\ \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} x_{11}&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} y_{11}&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} z_{11}&0&0&0&0&z_{16}\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ z_{61}&0&0&0&0&z_{66} \end{pmatrix} \right] \in \overline{\Gamma}_{(2)} : \begin{array}{llll} t \not\equiv_2 0, &&& \\ t \equiv_2 u, & z_{16} \equiv_2 0, & x_{11} \equiv_2 z_{66}, & x_{11} - y_{11} \equiv_4 z_{16}, \\ u - y_{11} \equiv_4 2z_{61}, & t + u + x_{11} + y_{11} \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 && \end{array} \right\}.$$

Since we use only $\{\bar{b}_i : i \in [113, 119]\} \subseteq \mathfrak{G}_{\mathrm{Im}(\varphi)}$ to build M_3, we also consider only the following set `Coeffs` of corresponding tuples of necessary coefficients; cf. (7.28). Cf. also Remark 34.

```
Coeffs  := CartesianProduct([M8odd, M8,M8, M4,M4, M2,M2]);

// M_3 is added to U_1
for x in Coeffs  do
y := (one - basis[113])
      + x[1] * basis[113] + x[2] * basis[114] + x[3] * basis[115] + x[4] * basis[116]
      + x[5] * basis[117] + x[6] * basis[118] + x[7] * basis[119] ;
 y_mat := MakeMatrix(y);
 U_1 := sub<UGq | GeneratorsSequence(U_1) cat [y_mat]>;
end for;
// here: U_1 = <S_1, S_2, S_3, M_3> = S_1 * S_2 * M_3 * S_3

// Factorisation(Order(U_1));
// 2^139 * 3^2 * 5 * 7 = 219522960548035821549492226746382308574494720
```

The resulting group $\mathrm{Im}(\mathrm{U}(\varphi))$ is given in Magma as

```
ImUphi := DirectProduct(U_2, U_1);
```

7.3 Analysing $\mathrm{Im}(\mathrm{U}(\varphi))$ via Magma

Now we can examine this group $\mathrm{Im}(\mathrm{U}(\varphi))$ more closely. We will look at U_1 and U_2 separately. Using Magma, we calculate the derived series and a chief series.

7.3.1 The derived series of U_1

We want to calculate the derived series of U_1; cf. §1.2.2.

Since U_1 is not solvable, the derived series calculated by Magma ends in a perfect group $\neq 1$.

```
DS := DerivedSeries(U_1);
#DS; // 2

GroupName(quo< DS[1] | DS[2] >); // C2^8
```

We write $D_1 :=$ `DS[1]`, $D_2 :=$ `DS[2]`.

This means that we have the derived series

$$U_1 = D_1 \geqslant D_2$$

with

$$D_1/D_2 \simeq \mathrm{C}_2^{\times 8}\,.$$

Magma confirms that $D_2 \neq 1$ and that D_2 is perfect in still another way:

```
IsAbelian(U_1); // false
IsPerfect(DS[2]); // true
```

Magma also gives the group names:

```
GroupName(DS[2]);
// (C2^45*C4^5).C2^4.C2.C2^4.C2^5.C2^14.C2^5.C2.C2^4.C2^4.C2^4.C2^4.C2^4.A8
```

7.3.2 The derived series of U_2

We want to calculate the derived series of U_2; cf. §1.2.2.

Since U_2 is not solvable, the derived series calculated by Magma ends in a perfect group $\neq 1$.

```
DS := DerivedSeries(U_2);
#DS; // 2

GroupName(quo<DS[1]|DS[2]>);  // C2^4
```

We write $D_1 :=$ `DS[1]`, $D_2 :=$ `DS[2]`.

This means that we have the derived series

$$U_2 = D_1 \geqslant D_2$$

with

$$D_1/D_2 \simeq \mathrm{C}_2^{\times 4}\,.$$

Magma confirms that $D_2 \neq 1$ and that D_2 is perfect in still another way:

```
IsAbelian(U_2);   // false
IsPerfect(DS[2]); // true
```

Magma also gives the group names:

```
GroupName(DS[2]);  // C2^31.C2^14.C2.C2^14.A8
GroupName(U_2); // C2^31.C2^14.C2.C2^14.C2^4.A8
```

7.3.3 A chief series of U_1

We want to calculate a chief series of U_1; cf. §1.2.3.

```
CS := ChiefSeries(U_1);
#CS; // 41

GroupName(quo< CS[ 1] | CS[ 2] >); // A8
GroupName(quo< CS[ 2] | CS[ 3] >); // C2
GroupName(quo< CS[ 3] | CS[ 4] >); // C2
GroupName(quo< CS[ 4] | CS[ 5] >); // C2
GroupName(quo< CS[ 5] | CS[ 6] >); // C2
GroupName(quo< CS[ 6] | CS[ 7] >); // C2
GroupName(quo< CS[ 7] | CS[ 8] >); // C2
GroupName(quo< CS[ 8] | CS[ 9] >); // C2^4
GroupName(quo< CS[ 9] | CS[10] >); // C2
GroupName(quo< CS[10] | CS[11] >); // C2
GroupName(quo< CS[11] | CS[12] >); // C2^4
GroupName(quo< CS[12] | CS[13] >); // C2
GroupName(quo< CS[13] | CS[14] >); // C2^4
GroupName(quo< CS[14] | CS[15] >); // C2
GroupName(quo< CS[15] | CS[16] >); // C2^4
GroupName(quo< CS[16] | CS[17] >); // C2
GroupName(quo< CS[17] | CS[18] >); // C2^14
GroupName(quo< CS[18] | CS[19] >); // C2
GroupName(quo< CS[19] | CS[20] >); // C2^14
GroupName(quo< CS[20] | CS[21] >); // C2
GroupName(quo< CS[21] | CS[22] >); // C2^4
GroupName(quo< CS[22] | CS[23] >); // C2^4
GroupName(quo< CS[23] | CS[24] >); // C2
GroupName(quo< CS[24] | CS[25] >); // C2
GroupName(quo< CS[25] | CS[26] >); // C2^4
GroupName(quo< CS[26] | CS[27] >); // C2^4
GroupName(quo< CS[27] | CS[28] >); // C2^4
GroupName(quo< CS[28] | CS[29] >); // C2^4
GroupName(quo< CS[29] | CS[30] >); // C2
GroupName(quo< CS[30] | CS[31] >); // C2
GroupName(quo< CS[31] | CS[32] >); // C2
GroupName(quo< CS[32] | CS[33] >); // C2^14
GroupName(quo< CS[33] | CS[34] >); // C2^14
GroupName(quo< CS[34] | CS[35] >); // C2
GroupName(quo< CS[35] | CS[36] >); // C2^4
GroupName(quo< CS[36] | CS[37] >); // C2
GroupName(quo< CS[37] | CS[38] >); // C2^4
GroupName(quo< CS[38] | CS[39] >); // C2^4
GroupName(quo< CS[39] | CS[40] >); // C2^4
GroupName(quo< CS[40] | CS[41] >); // C2
```

We write $H_1 :=$ `CS[1]`, $H_2 :=$ `CS[2]`, ..., $H_{41} :=$ `CS[41]`.

In particular, this means, we have the chief series

$$U_1 = H_1 \geqslant H_2 \geqslant H_3 \geqslant \ldots \geqslant H_{41} = 1$$

with

$$
\begin{array}{lll}
H_1/H_2 \simeq \mathrm{A}_8, & H_2/H_3 \simeq \mathrm{C}_2, & H_3/H_4 \simeq \mathrm{C}_2, \\
H_4/H_5 \simeq \mathrm{C}_2, & H_5/H_6 \simeq \mathrm{C}_2, & H_6/H_7 \simeq \mathrm{C}_2, \\
H_7/H_8 \simeq \mathrm{C}_2, & H_8/H_9 \simeq \mathrm{C}_2^{\times 4}, & H_9/H_{10} \simeq \mathrm{C}_2, \\
H_{10}/H_{11} \simeq \mathrm{C}_2, & H_{11}/H_{12} \simeq \mathrm{C}_2^{\times 4}, & H_{12}/H_{13} \simeq C_2, \\
H_{13}/H_{14} \simeq \mathrm{C}_2^{\times 4}, & H_{14}/H_{15} \simeq \mathrm{C}_2, & H_{15}/H_{16} \simeq \mathrm{C}_2^{\times 4}, \\
H_{16}/H_{17} \simeq \mathrm{C}_2, & H_{17}/H_{18} \simeq \mathrm{C}_2^{\times 14}, & H_{18}/H_{19} \simeq \mathrm{C}_2, \\
H_{19}/H_{20} \simeq \mathrm{C}_2^{\times 14}, & H_{20}/H_{21} \simeq \mathrm{C}_2, & H_{21}/H_{22} \simeq \mathrm{C}_2^{\times 4}, \\
H_{22}/H_{23} \simeq \mathrm{C}_2^{\times 4}, & H_{23}/H_{24} \simeq \mathrm{C}_2, & H_{24}/H_{25} \simeq \mathrm{C}_2, \\
H_{25}/H_{26} \simeq \mathrm{C}_2^{\times 4}, & H_{26}/H_{27} \simeq \mathrm{C}_2^{\times 4}, & H_{27}/H_{28} \simeq \mathrm{C}_2^{\times 4}, \\
H_{28}/H_{29} \simeq \mathrm{C}_2^{\times 4}, & H_{29}/H_{30} \simeq \mathrm{C}_2, & H_{30}/H_{31} \simeq \mathrm{C}_2, \\
H_{31}/H_{32} \simeq \mathrm{C}_2, & H_{32}/H_{33} \simeq \mathrm{C}_2^{\times 14}, & H_{33}/H_{34} \simeq \mathrm{C}_2^{\times 14}, \\
H_{34}/H_{35} \simeq \mathrm{C}_2, & H_{35}/H_{36} \simeq \mathrm{C}_2^{\times 4}, & H_{36}/H_{37} \simeq C_2, \\
H_{37}/H_{38} \simeq \mathrm{C}_2^{\times 4}, & H_{38}/H_{39} \simeq \mathrm{C}_2^{\times 4}, & H_{39}/H_{40} \simeq \mathrm{C}_2^{\times 4}, \\
H_{40}/H_{41} \simeq \mathrm{C}_2, & &
\end{array}
$$

Concerning $H_1/H_2 \simeq \mathrm{A}_8$, note that $\mathrm{A}_8 \simeq \mathrm{GL}_4(\mathbb{F}_2)$:

```
IsIsomorphic(GL(4,2),AlternatingGroup(8)); // true
```

7.3.4 A chief series of U_2

We want to calculate a chief series of U_2; cf. §1.2.3.

```
CS := ChiefSeries(U_2);
#CS; // 14

GroupName(quo<CS[ 1]|CS[ 2]>); // A8
GroupName(quo<CS[ 2]|CS[ 3]>); // C2
GroupName(quo<CS[ 3]|CS[ 4]>); // C2
GroupName(quo<CS[ 4]|CS[ 5]>); // C2
GroupName(quo<CS[ 5]|CS[ 6]>); // C2
GroupName(quo<CS[ 6]|CS[ 7]>); // C2^14
GroupName(quo<CS[ 7]|CS[ 8]>); // C2^14
GroupName(quo<CS[ 8]|CS[ 9]>); // C2
GroupName(quo<CS[ 9]|CS[10]>); // C2
GroupName(quo<CS[10]|CS[11]>); // C2^14
GroupName(quo<CS[11]|CS[12]>); // C2
GroupName(quo<CS[12]|CS[13]>); // C2^14
GroupName(quo<CS[13]|CS[14]>); // C2
```

We write $H_1 :=$ `CS[1]`, $H_2 :=$ `CS[2]`, ..., $H_{14} :=$ `CS[14]`.

In particular, this means, we have the chief series

$$U_2 = H_1 \geqslant H_2 \geqslant H_3 \geqslant \ldots \geqslant H_{14} = 1$$

with

$$\begin{array}{lll} H_1/H_2 \simeq \mathrm{A}_8, & H_2/H_3 \simeq \mathrm{C}_2, & H_3/H_4 \simeq \mathrm{C}_2, \\ H_4/H_5 \simeq \mathrm{C}_2, & H_5/H_6 \simeq \mathrm{C}_2, & H_6/H_7 \simeq \mathrm{C}_2^{\times 14}, \\ H_7/H_8 \simeq \mathrm{C}_2^{\times 14}, & H_8/H_9 \simeq \mathrm{C}_2, & H_9/H_{10} \simeq \mathrm{C}_2, \\ H_{10}/H_{11} \simeq \mathrm{C}_2^{\times 14}, & H_{11}/H_{12} \simeq \mathrm{C}_2, & H_{12}/H_{13} \simeq C_2^{\times 14}, \\ H_{13}/H_{14} \simeq \mathrm{C}_2 & & \end{array}$$

7.4 Description of $\ker(\mathrm{U}(\varphi))$

Consider the commutative triangle of groups.

$$\begin{array}{ccc} \mathrm{U}(\Gamma_{(2)}) & \xrightarrow{\mathrm{U}(\varrho)} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ \big\uparrow & \nearrow_{\mathrm{U}(\varphi)} & \\ \mathrm{U}(\Lambda_{(2)}) & & \end{array}$$

Since

$$\ker(\mathrm{U}(\varrho)) \leqslant \mathrm{U}(\Lambda_{(2)}) \, ,$$

we have

$$\ker(\mathrm{U}(\varrho)) = \ker(\mathrm{U}(\varrho)) \cap \mathrm{U}(\Lambda_{(2)}) = \ker(\mathrm{U}(\varphi)) \, .$$

So we have the following commutative diagram.

$$\begin{array}{ccccc} \ker(\mathrm{U}(\varrho)) & \longrightarrow & \mathrm{U}(\Gamma_{(2)}) & \xrightarrow{\mathrm{U}(\varrho)} & \mathrm{U}(\overline{\Gamma}_{(2)}) \\ \Big\| & & \big\uparrow & \nearrow_{\mathrm{U}(\varphi)} & \\ \ker(\mathrm{U}(\varphi)) & \longrightarrow & \mathrm{U}(\Lambda_{(2)}) & & \end{array}$$

With the diagram 7.3 we get a description of the kernel.

$$\ker(\mathrm{U}(\varphi)) = \{1 + 8\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\}$$

7.5 Summary

We have the following diagram of groups, in which the lower row is a short exact sequence.

$$\begin{array}{ccccc} & & \mathrm{U}(\mathbb{Z}_{(2)}\mathrm{S}_5) & & \\ & & {\scriptstyle \mathrm{U}\omega_{\mathbb{Z}_{(2)}}} \big\downarrow \wr & & \\ \ker(\mathrm{U}(\varphi)) & \hookrightarrow & \mathrm{U}(\Lambda_{(2)}) & \twoheadrightarrow & \mathrm{Im}(\mathrm{U}(\varphi)) \end{array}$$

The finite group $\mathrm{Im}(\mathrm{U}(\varphi))$ decomposes into the direct product of two subgroups $\mathrm{Im}(\mathrm{U}(\varphi)) = U_1 \times U_2$ with

$$|U_1| = 2^{139} \cdot 3^2 \cdot 5 \cdot 7 = 219522960548035821549492226746382308574494720$$

and

$$|U_2| = 2^{70} \cdot 3^2 \cdot 5 \cdot 7 = 371886360525984560578560\,.$$

So we get

$$|\,\mathrm{Im}(\mathrm{U}(\varphi))| = |U_2 \times U_1| = 2^{70} \cdot 3^2 \cdot 5 \cdot 7 \;\cdot\; 2^{139} \cdot 3^2 \cdot 5 \cdot 7 = 2^{209} \cdot 3^4 \cdot 5^2 \cdot 7^2\,.$$

U_1 and U_2 are built in §7.2 via Magma. Their derived series and chief series are given in §7.3.

The infinite group $\ker(\mathrm{U}(\varphi)) = \;\{1 + 8\gamma \in \mathrm{U}(\Gamma_{(2)}) : \gamma \in \Gamma_{(2)}\}$ is described in §7.4.

Appendix A

Code for the group A_4

A.1 Verification of the isomorphism of Wedderburn in A_4

We verify that $\omega_{\mathbb{Q}} : \mathbb{Q}A_4 \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q}(\zeta) \times \mathbb{Q}^{3\times 3}$ as given in §4 is a $\mathbb{Q}$-algebra isomorphism.

```
Q := Rationals();
Qz<ze> := CyclotomicField(3); // Qz = ℚ(ζ3), ze = ζ3
G := AlternatingGroup(4);
n := Order(G);
print G.1, G.2; // G.1 = (1,2)(3,4), G.2 = (1,2,3)

G_fp, phi := FPGroup(G); // phi: Isomorphism from G_fp to G
                         // G_fp has type: finitely presented group
                         // G has type: permutations group
rel := [x[1] * x[2]^-1 : x in Relations(G_fp)];
    // list of relations in G_fp.1, G_fp.2

mat1_1 := GL(1,Q)!Matrix([[1]]); // matrixes for image of G_fp.1
mat2_1 := GL(1,Qz)!Matrix([[1]]);
mat3_1 := GL(3,Q)!Matrix([[-1,0,0],[-2,-1,1],[-4,0,1]]);

mat1_2 := GL(1,Q)!Matrix([[1]]); // matrixes for image of G_fp.2
mat2_2 := GL(1,Qz)!Matrix([[ze]]);
mat3_2 := GL(3,Q)!Matrix([[-1,1,0],[-1,0,1],[0,0,1]]);

// Wedderburn components:
psi1 := hom< G_fp -> GL(1,Q) | [<G_fp.1,mat1_1>,<G_fp.2,mat1_2>] >;
psi2 := hom< G_fp -> GL(1,Qz) | [<G_fp.1,mat2_1>,<G_fp.2,mat2_2>] >;
psi3 := hom< G_fp -> GL(3,Q) | [<G_fp.1,mat3_1>,<G_fp.2,mat3_2>] >;

verify1 := &and[x@psi1 eq GL(1,Q)!1 : x in rel]; // testing relations in image
verify2 := &and[x@psi2 eq GL(1,Qz)!1 : x in rel];
verify3 := &and[x@psi3 eq GL(3,Q)!1 : x in rel];

verify := &and[verify1,verify2,verify3];
print verify; // Wedderburn well-defined, Q-algebra morphism

G_fp_list := [x@@phi : x in G]; // list of inverse images in G_fp
```

```
// We construct the representing matrix W in Q^12x12 to omega_Q
W := MatrixRing(Q,n)!0;
for i in [1..n] do // fill column i of W
 t1 := ElementToSequence(G_fp_list[i]@psi1);
 for j in [1..1] do
  W[j,i] := t1[j];
 end for;
 t2 := &cat[ElementToSequence(x) : x in ElementToSequence(G_fp_list[i]@psi2)];
 // e.g.: ElementToSequence(ze + 2) gives [2,1];
 for j in [2..3] do
  W[j,i] := t2[j-1];
 end for;
 t3 := ElementToSequence(G_fp_list[i]@psi3);
 for j in [4..12] do
  W[j,i] := t3[j-3];
 end for;
end for;

Determinant(W); // Gives 12288
Factorisation(Z!Determinant(W));// Gives [ <2, 12>, <3, 1> ]


verify_invertible := IsInvertible(W);
print verify_invertible; // Wedderburn bijective

W_int := MatrixRing(Z,n)!W;
```

We can now compare the calculated determinant with the theoretical determinant and thus check whether the determinant determined by Magma is correct.

Remark 40. Since $\omega_{\mathbb{Q}}$ restricts to $\omega_{\mathbb{Z}} : \mathbb{Z}\mathrm{A}_4 \longrightarrow \mathbb{Z} \times \mathbb{Z}[\zeta] \times \mathbb{Z}^{3\times 3}$ and since $|\Lambda_{\mathbb{Q}(\zeta)}| = |-3| = 3$ and $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 2$, we obtain

$$\begin{aligned}
|\det(\mathtt{W})| &= |\Gamma/\Lambda| \\
&= \sqrt{\left| \frac{|G|^{|G|}}{\prod_{j=1}^{t} \Delta_j^{(n_j^2)} n_j^{(n_j^2 d_j)}} \right|} \\
&= \sqrt{\left| \frac{12^{12}}{(1^{(1^2)} \cdot 1^{(1^2 \cdot 1)}) \cdot (3^{(1^2)} \cdot 1^{(1^2 \cdot 2)}) \cdot (1^{(3^2)} \cdot 3^{(3^2 \cdot 1)}))} \right|} \\
&= 2^{12} \cdot 3^1
\end{aligned}$$

Cf. Lemma 16.

This confirms our Magma calculation.

A.2 Verification of the congruences describing the image of $\mathbb{Z}_{(2)}\mathrm{A}_4$

We verify the list of congruences of $\Lambda_{(2)}$; cf. Equation (4.1).

$$\Lambda_{(2)} = \omega_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)}\mathrm{A}_4) = \left\{ \left(x,\ y_0 + y_1\zeta,\ \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \right) \in \Gamma_{(2)} : \begin{array}{rrr} z_{31} \equiv_4 0 & z_{32} \equiv_4 0 & z_{33} - x \equiv_4 0 \\ y_1 - z_{21} \equiv_2 0 & y_1 - z_{22} + z_{11} \equiv_4 0 & z_{12} + z_{21} \equiv_2 0 \\ y_0 - z_{22} - z_{12} - z_{21} \equiv_4 0 & & \end{array} \right\}$$

Remark 41. To convert the congruences into a matrix, we used the following algorithm. Recall the conversion of elements in $\Lambda_{(2)}$ into a vector; cf. (4.4).

$$\left(x,\ y_0 + y_1\zeta,\ \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \right) \rightarrowtail (x, y_0, y_1, z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{23}, z_{31}, z_{32}, z_{33})^{\mathrm{t}}$$

We will look at the congruences all modulo 4. So we get the following list of congruences.

$$\begin{array}{rrr} z_{31} \equiv_4 0 & z_{32} \equiv_4 0 & z_{33} - x \equiv_4 0 \\ 2y_1 - 2z_{21} \equiv_4 0 & y_1 - z_{22} + z_{11} \equiv_4 0 & 2z_{12} + 2z_{21} \equiv_4 0 \\ y_0 - z_{22} - z_{12} - z_{21} \equiv_4 0 & & \end{array}$$

Note that we will use the Matrix `W` from A.1.

```
// Matrix of congruences, all modulo 4
ties := Matrix([
[0, 0,0, 0,0,0,0,0,0,1,0,0],
[0, 0,0, 0,0,0,0,0,0,0,1,0],
[-1, 0,0, 0,0,0,0,0,0,0,0,1],
[0, 0,2, 0,0,0,-2,0,0,0,0,0],
[0, 0,1, 1,0,0,0,-1,0,0,0,0],
[0, 0,0, 0,2,0,2,0,0,0,0,0],
[0, 1,0, 0,-1,0,-1,-1,0,0,0,0]
]);

// verification of the congruences
RMatrixSpace(Integers(4),NumberOfRows(ties),12)!(ties * W_int);
// Solution: Zero-matrix

SmithForm(ties);
```

The product of the elementary divisors is in fact equal to

$$2^2 = 4^7 \cdot 2^{-12}\,.$$

Appendix B

Code for the group S_4

B.1 Verification of the isomorphism of Wedderburn in S_4

We verify that $\omega_{\mathbb{Q}} : \mathbb{Q}S_4 \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{2\times 2}$ as given in §5 is a $\mathbb{Q}$-algebra isomorphism.

```
Q := Rationals();
Z := Integers();
G := SymmetricGroup(4);
n := Order(G);
print G.1, G.2; // G.1 = (1, 2, 3, 4), G.2 = (1, 2)


G_fp, phi := FPGroup(G); // phi: Isomorphism from G_fp to G
                         // G_fp has type: finitely presented group
                         // G has type: permutations group
rel := [x[1] * x[2]^-1 : x in Relations(G_fp)];
    // list of relations in G_fp.1, G_fp.2


mat1_2 := GL(1,Q)!Matrix([[-1]]); // matrixes for image of G_fp.1
mat2_2 := GL(1,Q)!Matrix([[1]]);
mat3_2 := GL(3,Q)!Matrix([[-11,-24,2],[5,11,-1],[0,0,-1]]);
mat4_2 := GL(3,Q)!Matrix([[1,0,0],[1,-1,1],[0,0,1]]);
mat5_2 := GL(2,Q)!Matrix([[-5,24],[-1,5]]);

mat1_1 := GL(1,Q)!Matrix([[-1]]); // matrixes for image of G_fp.2
mat2_1 := GL(1,Q)!Matrix([[1]]);
mat3_1 := GL(3,Q)!Matrix([[26,57,2],[-11,-24,-1],[-4,-8,-1]]);
mat4_1 := GL(3,Q)!Matrix([[-2,1,0],[-3,0,1],[-4,0,1]]);
mat5_1 := GL(2,Q)!Matrix([[4,-15],[1,-4]]);


// Wedderburn components:
psi1 := hom< G_fp -> GL(1,Q) | [<G_fp.1,mat1_1>,<G_fp.2,mat1_2>] >;
psi2 := hom< G_fp -> GL(1,Q) | [<G_fp.1,mat2_1>,<G_fp.2,mat2_2>] >;
psi3 := hom< G_fp -> GL(3,Q) | [<G_fp.1,mat3_1>,<G_fp.2,mat3_2>] >;
psi4 := hom< G_fp -> GL(3,Q) | [<G_fp.1,mat4_1>,<G_fp.2,mat4_2>] >;
psi5 := hom< G_fp -> GL(2,Q) | [<G_fp.1,mat5_1>,<G_fp.2,mat5_2>] >;
```

```
// testing relations in image:
verify1 := &and[x@psi1 eq GL(1,Q)!1 : x in rel];
verify2 := &and[x@psi2 eq GL(1,Q)!1 : x in rel];
verify3 := &and[x@psi3 eq GL(3,Q)!1 : x in rel];
verify4 := &and[x@psi4 eq GL(3,Q)!1 : x in rel];
verify5 := &and[x@psi5 eq GL(2,Q)!1 : x in rel];


verify := &and[verify1,verify2,verify3,verify4,verify5];
print verify; // Wedderburn well-defined, Q-algebra morphism


G_fp_list := [x@@phi : x in G]; // list of inverse images in G_fp

// We construct the representing matrix W in Q^24x24 to omega_Q
W := MatrixRing(Q,n)!0;
for i in [1..n] do // fill column i of W
t1 := ElementToSequence(G_fp_list[i]@psi1);
for j in [1..1] do
W[j,i] := t1[j];
end for;
t2 := ElementToSequence(G_fp_list[i]@psi2);
for j in [2..2] do
W[j,i] := t2[j-1];
end for;
t3 := ElementToSequence(G_fp_list[i]@psi3);
for j in [3..11] do
W[j,i] := t3[j-2];
end for;
t4 := ElementToSequence(G_fp_list[i]@psi4);
for j in [12..20] do
W[j,i] := t4[j-11];
end for;
t5 := ElementToSequence(G_fp_list[i]@psi5);
for j in [21..24] do
 W[j,i] := t5[j-20];
end for;
 end for;

Determinant(W); // Gives 463856467968
Factorisation(Z!Determinant(W)); // Gives [ <2, 34>, <3, 3> ]



verify_invertible := IsInvertible(W);
print verify_invertible; // Wedderburn bijective

W_int := MatrixRing(Z,n)!W;
```

We can now compare the calculated determinant with the theoretical determinant and thus check whether the determinant determined by magma is correct.

Remark 42. Let n_j the dimension of the tuple entries from $\Lambda_{(2)}$ for $j \in [1,5]$.

$$|\det(\mathtt{W})| = \sqrt{\left|\frac{|G|^{|G|}}{\prod_{j=1}^{t} n_j^{(n_j^2)}}\right|} = \sqrt{\frac{24^{24}}{1^1 \cdot 1^1 \cdot 3^9 \cdot 3^9 \cdot 2^4}} = 2^{34} \cdot 3^3$$

Cf. Remark 17.

This confirms our Magma calculation.

B.2 Verification of the congruences describing the image of $\mathbb{Z}_{(2)}\mathrm{S}_4$

We verify the list of congruences of $\Lambda_{(2)}$; cf. Equation (5.1).

$$\Lambda_{(2)} = \omega_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)}\mathrm{S}_4) = \left\{ \left(v,\ w,\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right) \in \Gamma_{(2)} : \begin{array}{rlrlrl} x_{11} & \equiv_4 y_{11}, & x_{12} & \equiv_4 y_{12}, & x_{13} \equiv_2 y_{13}, \\ x_{21} & \equiv_4 y_{21}, & x_{22} & \equiv_4 y_{22}, & x_{23} \equiv_2 y_{23}, \\ x_{31} & \equiv_8 y_{31} \equiv_4 0, & x_{32} & \equiv_8 y_{32} \equiv_4 0, & x_{33} \equiv_2 y_{33}, \\ x_{11}+y_{11} & \equiv_8 2z_{11}, & x_{12}+y_{12} & \equiv_8 2z_{12}, & \\ x_{21}+y_{21} & \equiv_8 2z_{21}, & x_{22}+y_{22} & \equiv_8 2z_{22}, & \\ v - x_{33} & \equiv_8 w - y_{33} \equiv_4 0 & & & \end{array} \right\}$$

Note that we will use the vector notation of $\lambda \in \Lambda_{(2)}$; cf. 4.4.

We will look at the congruences all modulo 8. So we get the following list of congruences.

$$\begin{array}{rrrr} 2x_{11} - 2y_{11} \equiv_8 0 & 2x_{12} - 2y_{12} \equiv_8 0 & 4x_{13} - 4y_{13} \equiv_8 0 & 2x_{21} - 2y_{21} \equiv_8 0 \\ 2x_{22} - 2y_{22} \equiv_8 0 & 4x_{23} - 4y_{23} \equiv_8 0 & x_{31} - y_{31} \equiv_8 0 & 2y_{31} \equiv_8 0 \\ x_{32} - y_{32} \equiv_8 0 & 2y_{32} \equiv_8 0 & 4x_{33} - 4y_{33} \equiv_8 0 & x_{11} + y_{11} - 2z_{11} \equiv_8 0 \\ x_{12} + y_{12} - 2z_{12} \equiv_8 0 & x_{21} + y_{21} - 2z_{21} \equiv_8 0 & x_{22} + y_{22} - 2z_{22} \equiv_8 0 & \\ v - x_{33} - w + y_{33} \equiv_8 0 & 2w - 2y_{33} \equiv_8 0 & & \end{array}$$

Note that we will use the Matrix `W_int` from B.1.

```
// Matrix of congruences, all modulo 8
ties := Matrix([
[4, 4, 0,0,0,0,0,0,0,0,0,0,  0,0,0,0,0,0,0,0,0,0,  0,0,0,0],
[1,-1, 0,0,0,0,0,0,0,0,0,-1, 0,0,0,0,0,0,0,0,0,1,  0,0,0,0],
[2, 0, 0,0,0,0,0,0,0,0,0,-2, 0,0,0,0,0,0,0,0,0,0,  0,0,0,0],
[0, 2, 0,0,0,0,0,0,0,0,0,0,  0,0,0,0,0,0,0,0,0,-2, 0,0,0,0],
[0, 0, 0,0,4,0,0,0,0,0,0,0,  0,0,4,0,0,0,0,0,0,0,  0,0,0,0],
[0, 0, 0,0,0,0,0,4,0,0,0,0,  0,0,0,0,0,4,0,0,0,0,  0,0,0,0],
[0, 0, 0,0,0,0,0,0,2,0,0,0,  0,0,0,0,0,0,0,0,0,0,  0,0,0,0],
[0, 0, 0,0,0,0,0,0,0,2,0,0,  0,0,0,0,0,0,0,0,0,0,  0,0,0,0],
[0, 0, 0,0,0,0,0,0,0,0,0,0,  0,0,0,0,0,0,0,2,0,0,  0,0,0,0],
[0, 0, 0,0,0,0,0,0,0,0,0,0,  0,0,0,0,0,0,0,0,2,0,  0,0,0,0],
[0, 0, 0,0,0,0,0,0,1,0,0,0,  0,0,0,0,0,0,0,-1,0,0, 0,0,0,0],
[0, 0, 0,0,0,0,0,0,0,1,0,0,  0,0,0,0,0,0,0,0,-1,0, 0,0,0,0],
[0, 0, 2,0,0,0,0,0,0,0,0,0,  -2,0,0,0,0,0,0,0,0,0, 0,0,0,0],
[0, 0, 0,2,0,0,0,0,0,0,0,0,  0,-2,0,0,0,0,0,0,0,0, 0,0,0,0],
[0, 0, 0,0,0,2,0,0,0,0,0,0,  0,0,0,-2,0,0,0,0,0,0, 0,0,0,0],
[0, 0, 0,0,0,0,2,0,0,0,0,0,  0,0,0,0,-2,0,0,0,0,0, 0,0,0,0],
[0, 0, 1,0,0,0,0,0,0,0,0,0,  1,0,0,0,0,0,0,0,0,0,  -2,0,0,0],
[0, 0, 0,1,0,0,0,0,0,0,0,0,  0,1,0,0,0,0,0,0,0,0,  0,-2,0,0],
[0, 0, 0,0,0,1,0,0,0,0,0,0,  0,0,0,1,0,0,0,0,0,0,  0,0,-2,0],
[0, 0, 0,0,0,0,1,0,0,0,0,0,  0,0,0,0,1,0,0,0,0,0,  0,0,0,-2]]);

// verification of the congruences
RMatrixSpace(Integers(8),NumberOfRows(ties),24)!(ties * W_int);
// Solution: Zero-matrix

SmithForm(ties);
```

The product of the nonzero elementary divisors, multiplied with 8 to the number of zero rows, is in fact

$$2^3 \cdot 4^7 \cdot 8^3 = 8^{20} \cdot 2^{-34} \, .$$

Appendix C

Code for the group S_5

C.1 Verification of the isomorphism of Wedderburn in S_5

We verify that $\omega_{\mathbb{Q}} : \mathbb{Q}S_5 \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^{4\times 4} \times \mathbb{Q}^{4\times 4} \times \mathbb{Q}^{5\times 5} \times \mathbb{Q}^{5\times 5} \times \mathbb{Q}^{6\times 6}$ as given in §7 is a $\mathbb{Q}$-algebra isomorphism.

```
Q := Rationals();
Z := Integers();
G := SymmetricGroup(5);
n := Order(G); // 120

print G.1, G.2; // G.1 = (1,2,3,4,5), G.2 = (1,2)



G_fp, phi := FPGroup(G); // phi: Isomorphism from G_fp to G
                         // G_fp has type: finitely presented group
                         // G has type: permutations group
rel := [x[1] * x[2]^-1 : x in Relations(G_fp)];
    // list of relations in G_fp.1, G_fp.2


// matrixes for image of G_fp.1
mat1_1 := GL(1,Q)!Matrix([[1]]);
mat2_1 := GL(1,Q)!Matrix([[1]]);
mat3_1 := GL(4,Q)!Matrix([[0,0,0,1],[-1,0,0,-1],[0,-1,0,1],[0,0,-1,-1]]);
mat4_1 := GL(4,Q)!Matrix([[0,0,0,1],[-1,0,0,-1],[0,-1,0,1],[0,0,-1,-1]]);
mat5_1 := GL(5,Q)!Matrix([[3,4,6,6,-2],[0,0,0,1,0],[-1,-1,-1,-2,1],
                          [0,0,-1,-1,-1],[1,1,2,2,-1]]);
mat6_1 := GL(5,Q)!Matrix([[3,60,-38,10,22],[2,40,-28,9,20],[5,99,-69,22,49],
                          [4,104,-73,23,55],[1,9,-6,2,3]]);
mat7_1 := GL(6,Q)!Matrix([[-7,-3540,-560,-1644,-1138,-212],[8,4408,698,2049,1422,270],
                [-13,-6987,-1103,-3246,-2243,-426],[-18,-9984,-1581,-4641,-3221,-612],
                [7,3861,610,1794,1241,236],[3,1668,263,775,535,103]]);
```

```
// matrixes for image of G_fp.2
mat1_2 := GL(1,Q)!Matrix([[-1]]);
mat2_2 := GL(1,Q)!Matrix([[1]]);
mat3_2 := GL(4,Q)!Matrix([[-1,0,0,-1],[0,-1,0,1],[0,0,-1,-1],[0,0,0,1]]);
mat4_2 := GL(4,Q)!Matrix([[1,0,0,1],[0,1,0,-1],[0,0,1,1],[0,0,0,-1]]);
mat5_2 := GL(5,Q)!Matrix([[3,4,2,4,-4],[0,-1,-1,0,0],[0,0,1,0,0],
                          [-1,-1,-1,-2,1],[1,1,0,1,-2]]);
mat6_2 := GL(5,Q)!Matrix([[-3,-64,42,-12,-28],[0,11,-5,0,0],[0,24,-11,0,0],
                          [3,67,-41,10,21],[-1,-11,8,-3,-6]]);
mat7_2 := GL(6,Q)!Matrix([[-5,-1850,-294,-860,-600,-110],[2,1025,161,476,328,64],
                  [-4,-1680,-265,-780,-540,-100],[-5,-2627,-413,-1220,-841,-164],
                  [3,1419,224,659,456,86],[0,134,21,62,42,9]]);

// Wedderburn components
psi1 := hom< G_fp -> GL(1,Q) | [<G_fp.1,mat1_1>,<G_fp.2,mat1_2>] >;
psi2 := hom< G_fp -> GL(1,Q) | [<G_fp.1,mat2_1>,<G_fp.2,mat2_2>] >;
psi3 := hom< G_fp -> GL(4,Q) | [<G_fp.1,mat3_1>,<G_fp.2,mat3_2>] >;
psi4 := hom< G_fp -> GL(4,Q) | [<G_fp.1,mat4_1>,<G_fp.2,mat4_2>] >;
psi5 := hom< G_fp -> GL(5,Q) | [<G_fp.1,mat5_1>,<G_fp.2,mat5_2>] >;
psi6 := hom< G_fp -> GL(5,Q) | [<G_fp.1,mat6_1>,<G_fp.2,mat6_2>] >;
psi7 := hom< G_fp -> GL(6,Q) | [<G_fp.1,mat7_1>,<G_fp.2,mat7_2>] >;

// testing relations in image
verify1 := &and[x@psi1 eq GL(1,Q)!1 : x in rel];
verify2 := &and[x@psi2 eq GL(1,Q)!1 : x in rel];
verify3 := &and[x@psi3 eq GL(4,Q)!1 : x in rel];
verify4 := &and[x@psi4 eq GL(4,Q)!1 : x in rel];
verify5 := &and[x@psi5 eq GL(5,Q)!1 : x in rel];
verify6 := &and[x@psi6 eq GL(5,Q)!1 : x in rel];
verify7 := &and[x@psi7 eq GL(6,Q)!1 : x in rel];

verify := &and[verify1,verify2,verify3,verify4,verify5,verify6,verify7];
print verify; // Wedderburn well-defined, Q-algebra morphism

G_fp_list := [x@@phi : x in G]; // list of inverse images in G_fp


// We construct the representing matrix W in Q^120x120 to omega_Q
W := MatrixRing(Q,n)!0;
for i in [1..n] do // fill column i of W
 t1 := ElementToSequence(G_fp_list[i]@psi1);
 for j in [1..1] do
  W[j,i] := t1[j];
 end for;
 t2 := ElementToSequence(G_fp_list[i]@psi2);
 for j in [2..2] do
  W[j,i] := t2[j-1];
 end for;
 t3 := ElementToSequence(G_fp_list[i]@psi3);
 for j in [3..18] do
  W[j,i] := t3[j-2];
 end for;
 t4 := ElementToSequence(G_fp_list[i]@psi4);
```

```
 for j in [19..34] do
  W[j,i] := t4[j-18];
 end for;
 t5 := ElementToSequence(G_fp_list[i]@psi5);
 for j in [35..59] do
  W[j,i] := t5[j-34];
 end for;
 t6 := ElementToSequence(G_fp_list[i]@psi6);
 for j in [60..84] do
  W[j,i] := t6[j-59];
 end for;
 t7 := ElementToSequence(G_fp_list[i]@psi7);
 for j in [85..120] do
  W[j,i] := t7[j-84];
 end for;
end for;
Factorisation(Z!Determinant(W)); // Gives: [ <2, 130>, <3, 42>, <5, 35> ]

verify_invertible := IsInvertible(W);
print verify_invertible; // Wedderburn bijective

W_int := MatrixRing(Z,n)!W;
```

We can now compare the calculated determinant with the theoretical determinant and thus check whether the determinant determined by Magma is correct.

Remark 43. Let n_j the dimension of the tuple entries from $\Lambda_{(2)}$ for $j \in [1,5]$.

$$|\det(\mathtt{W})| = \sqrt{\left|\frac{|G|^{|G|}}{\prod_{j=1}^{t} n_j^{(n_j^2)}}\right|} = \sqrt{\frac{120^{120}}{1^1 \cdot 1^1 \cdot 4^{16} \cdot 4^{16} \cdot 5^{25} \cdot 5^{25} \cdot 6^{36}}} = 2^{130} \cdot 3^{42} \cdot 5^{35}$$

Cf. Remark 17.

This confirms our Magma calculation.

C.2 Verification of the congruences describing the image of $\mathbb{Z}_{(2)}\mathrm{S}_5$

We verify the list of congruences of $\Lambda_{(2)}$; cf. Equation (7.1).

$$\Lambda_{(2)} = \omega_{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)}\mathrm{S}_5)$$
$$= \left\{ \left(t,\ u,\ \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}, \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix}, \right.\right.$$
$$\left.\begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ y_{21} & y_{22} & y_{23} & y_{24} & y_{25} \\ y_{31} & y_{32} & y_{33} & y_{34} & y_{35} \\ y_{41} & y_{42} & y_{43} & y_{44} & y_{45} \\ y_{51} & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix}, \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} \end{pmatrix} \right) \in \Gamma_{(2)} :$$
$$\begin{array}{llll} v_{ij} \equiv_2 w_{ij},\ i,j \in [1,4], & & & \\ x_{ij} + y_{ij} \equiv_8 2z_{ij},\ i,j \in [2,5], & y_{ij} \equiv_2 z_{ij},\ i,j \in [2,5], & & \\ x_{1i} \equiv_2 0,\ i \in [2,5], & y_{1i} \equiv_2 0,\ i \in [2,5], & z_{1i} \equiv_2 0,\ i \in [2,5], & \\ x_{1i} + y_{1i} \equiv_8 2z_{1i}, i \in [2,5], & y_{1i} \equiv_4 2z_{6i},\ i \in [2,5], & & \\ z_{i6} \equiv_2 0,\ i \in [2,5], & x_{i1} - y_{i1} \equiv_4 z_{i6},\ i \in [2,5], & y_{i1} \equiv_2 z_{i1},\ i \in [2,5], & \\ t \equiv_2 u, & z_{16} \equiv_2 0, & x_{11} \equiv_2 z_{66}, & x_{11} - y_{11} \equiv_4 z_{16}, \\ u - y_{11} \equiv_4 2z_{61}, & t + u + x_{11} + y_{11} \ \equiv_8 2z_{11} + 2z_{66} \equiv_4 0 & & \end{array} \right\}$$

Note that we will use the vector notation of $\lambda \in \Lambda_{(2)}$; cf. 4.4. We will look at the congruences all modulo 8.

Note that we will use the Matrix `W_int` from C.1.

```
// Matrix of congruences, all modulo 8
ties := Matrix([
[0, 0, 4,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, -4,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,4,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,-4,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,4,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,-4,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,4, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,-4, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],

[0, 0, 0,0,0,0, 4,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, -4,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,4,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,-4,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,4,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,-4,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,4, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,-4, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],

[0, 0, 0,0,0,0, 0,0,0,0, 4,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, -4,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,4,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,-4,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,4,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,-4,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,4, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,-4, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],

[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 4,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, -4,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,4,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,-4,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,4,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,-4,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,4, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,-4, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],

[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,-2,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,-2,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,1,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,1,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,-2,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,1, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,1, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,-2,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],

[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,-2,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,-2,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,1,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,1,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,-2,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,1, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,1, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,-2,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],

[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,-2,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,-2,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,1,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,1,0, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,-2,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,1, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,1, 0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,-2,0, 0,0,0,0,0,0, 0,0,0,0,0,0],

[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,1,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,-2,0,0,0,0, 0,0,0,0,0,0],
[0, 0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,0,0,0, 0,0,1,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,-2,0,0,0, 0,0,0,0,0,0],
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]);

// verification of the congruences
RMatrixSpace(Integers(8),NumberOfRows(ties),120)!(ties * W_int); // Solution: Zero-matrix
IsZero(RMatrixSpace(Integers(8),NumberOfRows(ties),120)!(ties * W_int)); // Gives: true

D := SmithForm(ties);
[D[i,i] : i in [1..NumberOfRows(ties)]];
```

The product of the nonzero elementary divisors, multiplied with 8 to the number of elementary divisors divisible by 8, is in fact

$$2^{10} \cdot 4^{47} \cdot 8^{9} = 8^{87} \cdot 2^{-130}\,.$$

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German summary

Wir betrachten die Gruppenalgebra $\mathbb{Z}_{(p)}G$ einer endlichen Gruppe G über der Lokalisierung $\mathbb{Z}_{(p)}$ der ganzen Zahlen bei (p), wobei p eine Primzahl ist.

Wir nehmen an, wir haben eine Wedderburn-Einbettung

$$\omega_{\mathbb{Z}_{(p)}} : \mathbb{Z}_{(p)}G \xrightarrow{\text{injektiv}} R_1^{n_1 \times n_1} \times \ldots \times R_t^{n_t \times n_t} =: \Gamma_{(p)} \, ,$$

wobei $R_1, \ldots, R_t$ algebraische Zahlenringe über $\mathbb{Z}_{(p)}$ sind. Sei $\Lambda_{(p)} := \omega_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}G)$, so erhalten wir

$$\begin{array}{ccc} \mathbb{Z}_{(p)}G & \xrightarrow[\text{injektiv}]{\omega_{\mathbb{Z}_{(p)}}} & \Gamma_{(p)} \\ & \searrow^{\sim} & \uparrow \\ & & \Lambda_{(p)} \end{array}$$

Wir wählen $d \geqslant 1$ so, dass $p^d\Gamma_{(p)} \subseteq \Lambda_{(p)}$. Wir schreiben $\overline{\Gamma}_{(p)} := \Gamma_{(p)}/p^d\Gamma_{(p)}$ und

$$\varrho : \Gamma_{(p)} \longrightarrow \overline{\Gamma}_{(p)}$$

für den Restklassenmorphismus. Sei

$$\varphi := \varrho|_{\Lambda_{(2)}} : \Lambda_{(2)} \longrightarrow \overline{\Gamma}_{(2)} \, .$$

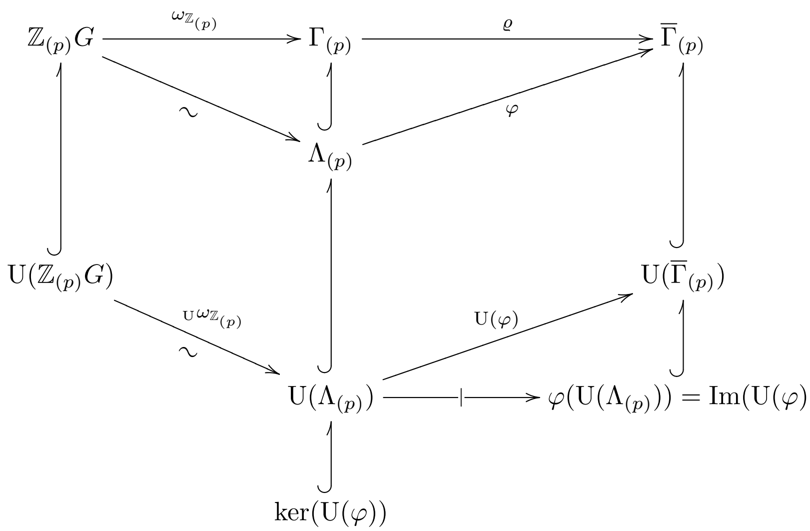
Wir haben also das folgende kommutative Diagramm von Ringen und Ringmorphismen.

$$\begin{array}{ccccc} \mathbb{Z}_{(p)}G & \xrightarrow{\omega_{\mathbb{Z}_{(p)}}} & \Gamma_{(p)} & \xrightarrow{\varrho} & \overline{\Gamma}_{(p)} \\ & \searrow^{\sim} & \uparrow & \nearrow_{\varphi} & \\ & & \Lambda_{(p)} & & \end{array}$$

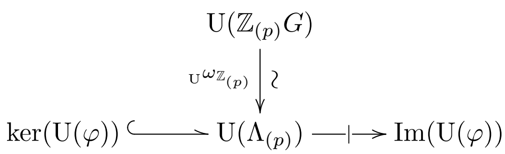
Jeder Ringmorphismus $\psi : S \longrightarrow T$ wird zu einem Gruppenmorphismus auf den Einheitsgruppen eingeschränkt,

$$\mathrm{U}(\psi) : \mathrm{U}(S) \longrightarrow \mathrm{U}(T) \, .$$

Also erhalten wir das folgende Diagramm.



Es ergibt sich das folgende Diagramm von Gruppen und Gruppenmorphismen



wobei die untere Zeile eine kurz exakte Sequenz ist.

Die unendliche Gruppe $\ker(\mathrm{U}(\varphi))$ lässt sich einfach als direktes Produkt von Kongruenzuntergruppen beschreiben:

$$\begin{aligned}\ker(\mathrm{U}(\varphi)) &= \{1 + p^d\gamma \in \mathrm{U}(\Gamma_{(p)}) : \gamma \in \Gamma_{(p)}\} \\ &= \{1 + p^d\gamma \in \mathrm{U}(\Gamma_{(p)}) : \gamma \in R_1^{n_1 \times n_1}\} \times \ldots \times \{1 + p^d\gamma \in \mathrm{U}(\Gamma_{(p)}) : \gamma \in R_t^{n_t \times n_t}\} \\ &\leqslant \mathrm{U}(\Lambda_{(p)}) \leqslant \mathrm{U}(\Gamma_{(p)})\,.\end{aligned}$$

Die endliche Untergruppe $\mathrm{Im}(\mathrm{U}(\varphi)) \subseteq \mathrm{U}(\overline{\Gamma}_{(p)})$ ist die Gruppe, die wir mit Magma **[1]** untersucht haben.

Zu diesem Zweck betrachten wir die folgende Liste von Beispielen.

- $G = \mathrm{S}_3$, $p = 3$
- $G = \mathrm{A}_4$, $p = 2$
- $G = \mathrm{D}_{2p}$, für eine Primzahl $p \geqslant 3$
- $G = \mathrm{S}_4$, $p = 2$
- $G = \mathrm{S}_5$, $p = 2$

Versicherung

Hiermit versichere ich,

1. dass ich meine Arbeit selbstständig verfasst habe,
2. dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,
3. dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und
4. dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, den 06.12.2024

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