An introduction to the elementary theory of Heller triangulated categories

Matthias Künzer

January 24, 2013

Contents

1 Why derived categories?

1.1 Derived functors

Around 1960, Grothendieck struggled with increasingly complicated spectral sequence comparisons ([1]). These spectral sequences arose as follows.

Given left exact functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ between abelian categories ([2]), under normal circumstances ([3]), the derived functor $\mathrm{R}^n(FG)$ of the composite FG can be approximated by the composites of the derived functors $(\mathrm{R}^i F)(\mathrm{R}^j G)$, where $i + j = n$. In other words, we have the Grothendieck spectral sequence, which has E_2-terms $(\mathrm{R}^i F)(\mathrm{R}^j G)$ and converges to $\mathrm{R}^n(FG)$.

For instance, roughly put, the first derivative

$$\mathrm{R}^1(FG)$$

consists of

$$\text{a part of } (\mathrm{R}^1 F)(\mathrm{R}^0 G) \qquad \text{plus} \qquad \text{a part of } (\mathrm{R}^0 F)(\mathrm{R}^1 G)\ ;$$

which can be read as an approximative "product rule for the first derivative" ([4]).

In practice, this is troublesome since it only yields an approximative relationship between

$$\text{the derived functors of the composite} \qquad \text{and} \qquad \text{the composites of the derived functors}\ ,$$

and since, moreover, this approximation is laborious.

Finally, if we want to compose three or more functors and relate their various derivatives, we are stuck.

1.2 Derived functors, renovated

The construction of such a derived functor $\mathrm{R}^i F$ proceeds in three steps.

(1) Resolve injectively.

(2) Apply the functor F.

(3) Take cohomology H^i.

Grothendieck saw that the troubles were caused by the third step and that dropping the third step, one should get a smooth formalism, in which the spectral sequence approximation mentioned above is turned into the simple and precise rule

$$(*) \qquad \mathrm{R}(FG) \ \simeq\ (\mathrm{R}F)(\mathrm{R}G)\ .$$

[1] An example of an assertion of this kind may be seen in [**7,** 6.6.2]. According to Illusie, Grothendieck said: "The second part of EGA III is a mess, so, please, clean this up by introducing derived categories, write the Künneth formula in the general framework of derived categories." [**11,** p. 1108].

[2] An *abelian category* has direct sums, kernels, cokernels, and the homomorphism theorem holds.

[3] The categories $\mathcal{A}$ and $\mathcal{B}$ are supposed to have enough injectives, and F to map injectives to G-acyclics.

[4] More precisely, $\mathrm{R}^1(FG)$ has a two-step filtration, one subfactor of which being a subfactor of $(\mathrm{R}^1 F)(\mathrm{R}^0 G)$, the other being a subfactor of $(\mathrm{R}^0 F)(\mathrm{R}^1 G)$.

The price to pay was the development of this formalism, undertaken by Verdier around 1963 in [**23**].

Since we have dropped taking cohomology H^i, the renovated derived functor $\mathrm{R}F$ now takes values in complexes (over $\mathcal{B}$). So in order to be able to compose, $\mathrm{R}F$ should also take as arguments complexes (over $\mathcal{A}$).

Moreover, in order to ensure the validity of the composition rule $(*)$, one has to formally invert morphisms of complexes that induce isomorphisms in cohomology, called *quasi-isomorphisms.* This process yields the *derived category*

$$\mathrm{D}^+(\mathcal{A}) \ ,$$

having as objects complexes ([5]) over $\mathcal{A}$, and as morphisms fractions

$$f/s \ ,$$

where the numerator f is a morphism of complexes and where the denominator s is a quasi-isomorphism of complexes.

So in full, the formula $(*)$ reads

$$\Big(\mathrm{D}^+(\mathcal{A}) \xrightarrow{\ \mathrm{R}(FG)\ } \mathrm{D}^+(\mathcal{C})\Big) \quad \simeq \quad \Big(\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} \mathrm{D}^+(\mathcal{B}) \xrightarrow{\mathrm{R}G} \mathrm{D}^+(\mathcal{C})\Big) \ . \tag{$*'$}$$

2 Why triangulated categories?

2.1 Verdier triangulated categories

The category $\mathcal{A}$ is abelian.

The derived category $\mathrm{D}^+(\mathcal{A})$ is not abelian ([6]).

There exist hardly any short exact sequences in $\mathrm{D}^+(\mathcal{A})$, only split ones.

As substitute, the image in $\mathrm{D}^+(\mathcal{A})$ of a short exact sequence of complexes $X' \rightarrowtail X \twoheadrightarrow X''$ fits into a diagram

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'^{+1} \ ,$$

called a *distinguished triangle*, where X'^{+1} denotes the complex X', shifted one step to the left ([7]).

$$\begin{tikzcd} & X'' \arrow[dl, dotted] & \\ X' \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

[5] Bounded to the left.

[6] Except if $\mathcal{A}$ is semisimple.

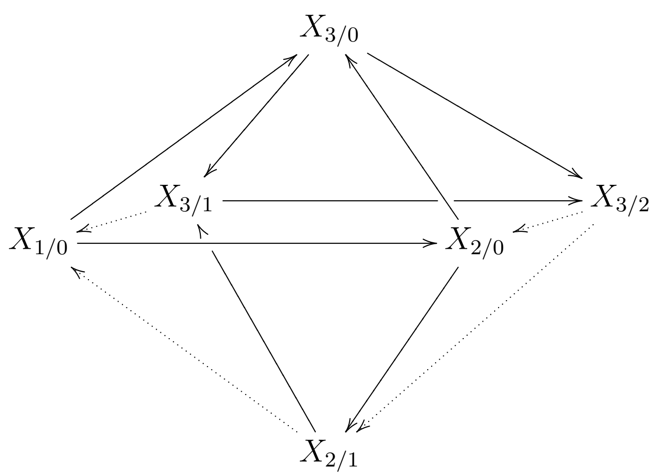
[7] And all differentials negated.

Now any morphism $X_{1/0} \longrightarrow X_{2/0}$ in $\mathrm{D}^+(\mathcal{A})$ fits into such a distinguished triangle $X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{2/1} \longrightarrow X_{1/0}^{+1}$, and this completion is unique up to isomorphism ([8]). We call $X_{2/1}$ the *cone* of the morphism $X_{1/0} \longrightarrow X_{2/0}$ ([9]).

The compatibility of taking cones with composition is expressed by the following *Verdier octahedron* ([10]), in which $X_{0^{+1}/i} = X_{i/0}^{+1}$ for $1 \leqslant i \leqslant 3$.

$$
\begin{tikzcd}
 & & & & 0 \\
 & & & 0 \arrow[r] & X_{0^{+1}/3} \arrow[u] \\
 & & 0 \arrow[r] & X_{3/2} \arrow[r] \arrow[u] & X_{0^{+1}/2} \arrow[u] \\
 & 0 \arrow[r] & X_{2/1} \arrow[r] \arrow[u] & X_{3/1} \arrow[r] \arrow[u] & X_{0^{+1}/1} \arrow[u] \\
0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] & X_{2/0} \arrow[r] \arrow[u] & X_{3/0} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
$$

Here $X_{j/i}$ is the cone on $X_{i/0} \longrightarrow X_{j/0}$ for $1 \leqslant i < j \leqslant 3$. Moreover, $X_{3/2}$ is the cone on $X_{2/1} \longrightarrow X_{3/1}$.



A theory of Verdier triangulated categories was developed by Verdier **[23]**, which plays the same role for $\mathrm{D}^+(\mathcal{A})$ as the theory of abelian categories plays for $\mathcal{A}$.

Here, a *Verdier triangulated category* is a triple $(\mathcal{D}, \mathsf{T}, \Xi)$, consisting of an additive category $\mathcal{D}$, an automorphism $\mathsf{T} = (-)^{+1}$ of $\mathcal{D}$, called *shift,* and a set Ξ of distinguished triangles, satisfying a list of axioms, including the existence of a Verdier octahedron on each pair of composable morphisms. Then, Ξ is called a *Verdier triangulation* on $(\mathcal{D}, \mathsf{T})$.

[8] In contrast to what we are used to from kernels and cokernels in abelian categories, this isomorphism is not uniquely determined in general.

[9] This notion is motivated by the homotopy category of CW-complexes, which becomes a Verdier triangulated category after Spanier-Whitehead stabilisation, where this cone is an actual geometrically constructed cone.

[10] This alternative, non-octahedral form of this diagram was observed in [**3,** 1.1.14].

For example, the derived category $\mathrm{D}^+(\mathcal{A})$ is Verdier triangulated. Also already the homotopy category $\mathrm{K}^+(\mathcal{A})$, obtained as the category of complexes ([11]) modulo split acyclic complexes, is Verdier triangulated. More generally, the stable category of a Frobenius category ([12]) is Verdier triangulated; cf. [**9,** Th. 2.6].

One of the axioms records a curious phenomenon, without parallel in the context of abelian categories. For every distinguished triangle

$$X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{2/1} \longrightarrow X_{1/0}^{+1}\,,$$

we get the rotated distinguished triangle

$$X_{2/1}^{-1} \longrightarrow X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{2/1} \qquad ({}^{13})\,.$$

In a Verdier triangulated category, the cone of a morphism is at the same time a substitute for its kernel and its cokernel, but only in a weak form ([14]). Therefore, a Verdier triangulated category is weakly abelian, i.e. it is an additive category in which each morphism is and has a weak kernel and a weak cokernel.

A set Ξ of distinguished triangles that satisfies the list of axioms except possibly for the existence of a Verdier octahedron on each diagram $X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{3/0}$, is called a *Puppe triangulation* on $(\mathcal{D}, \mathsf{T})$ [**21**]. Cf. §3.1 below.

2.2 Exact functors between Verdier triangulated categories

A *strictly exact functor* between Verdier triangulated categories is a shiftcompatible additive functor that maps distinguished triangles to distinguished triangles; i.e. that preserves cones.

Derived functors, such as the functor $\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} \mathrm{D}^+(\mathcal{B})$ from §1.2, are strictly exact.

An *exact functor* between Verdier triangulated categories $(\mathcal{D}, \mathsf{T}, \Xi)$ and $(\mathcal{D}', \mathsf{T}', \Xi')$ is a pair (V, a) consisting of an additive functor $V : \mathcal{D} \longrightarrow \mathcal{D}'$ and an isotransformation $a : \mathsf{T}\,V \longrightarrow V\,\mathsf{T}'$ such that each distinghuished triangle in $\mathcal{D}$, mapped via V and isomorphically replaced via a, yields a distinguished triangle of $\mathcal{D}'$.

So $V : \mathcal{D} \longrightarrow \mathcal{D}'$ is strictly exact if and only if $(V, 1)$ is exact.

2.3 Stability properties of the Verdier formalism

Adjoints of exact functors are exact [**20,** App. 2, Prop. 11] [**12,** 1.6]. Already Grothendieck and Deligne observed in Algebraic Geometry the appearance of exact functors that are not

[11] Bounded to the left.

[12] A *Frobenius category* is an exact category with a sufficient supply of relatively bijective objects.

[13] With a sign inserted.

[14] In the notation above, $X_{2/0} \longrightarrow X_{2/1}$ is a weak cokernel of $X_{1/0} \longrightarrow X_{2/0}$, i.e. it satisfies the universal property of a cokernel, except for uniqueness of the induced morphism. Moreover, $X_{2/1}^{-1} \longrightarrow X_{1/0}$ is a weak kernel of $X_{1/0} \longrightarrow X_{2/0}$, i.e. satisfies the universal property of a kernel, except for uniqueness of the induced morphism.

derived functors, but adjoints to derived functors ([15]).

The *Karoubi hull* of an additive category is the universal additive category whose idempotents split [**13,** III.II]. We may form Karoubi hulls within the context of Verdier triangulated categories and exact functors, as shown by BALMER and SCHLICHTING [**1**].

The *localisation* of a category at a subset of its morphisms is the universal category such that morphisms of this subset become invertible. For instance, the derived category $\mathrm{D}^+(\mathcal{A})$ is the localisation of the homotopy category $\mathrm{K}^+(\mathcal{A})$ at the subset of quasiisomorphisms, i.e. at the subset of morphisms with acyclic cone. We may form the localisation of a Verdier triangulated category at the subset of morphisms with cone in a given thick subcategory ([16]) within the realm of Verdier triangulated categories and exact functors [**23**][**22,** Prop. 1.3].

3 Heller triangulated categories

3.1 Heller's original theorem

Let $\mathcal{D}$ be a weakly abelian category; cf. §2.1. The *Freyd category* $\hat{\mathcal{D}}$ is the universal abelian category containing $\mathcal{D}$ [**6**]. Reducing modulo the full additive subcategory of projective objects, we obtain the stable category $\underline{\hat{\mathcal{D}}}$, which is Verdier triangulated ([17]).

Now suppose $\mathcal{D}$ to carry a shift functor T. Then $\underline{\hat{\mathcal{D}}}$ carries two shift functors, a first one induced by T, a second one given by the Verdier triangulated structure on $\underline{\hat{\mathcal{D}}}$.

HELLER discovered a bijection between the set of Puppe triangulations Ξ on $(\mathcal{D}, \mathsf{T})$ and the set of isomorphisms from the first shift functor to the third power of the second shift functor satisfying an extra condition ([18]) [**10,** Th. 16.4].

So such an isomorphism between these shift functors can be made responsable for a Puppe triangulation, as the extra datum needed to upgrade a weakly abelian category with shift $(\mathcal{D}, \mathsf{T})$ to a Puppe triangulated category $(\mathcal{D}, \mathsf{T}, \Xi)$.

So we could just as well include this isomorphism instead of Ξ in our data.

3.2 Extending Heller's theorem

Let $\mathcal{D}$ be a weakly abelian category. Let T be an automorphism of $\mathcal{D}$.

In order to extend HELLER's result from Puppe triangulations to Verdier triangulations and beyond, all we need is a suitable replacement for $\underline{\hat{\mathcal{D}}}$.

[15] The functor $\mathrm{R}f^!$, constructed for certain morphisms f of schemes, is only abusively written with a "R"; cf. [**8,** Exp. XVIII, Th. 3.1.4].

[16] A *thick* subcategory is a full subcategory closed under shift, forming cones and taking summands.

[17] The reason being that $\mathcal{D}$ is a big enough full subcategory in $\hat{\mathcal{D}}$ consisting of bijective objects, so that $\hat{\mathcal{D}}$ is a Frobenius abelian category. Cf. [**9,** §2.1].

[18] Such an isomorphism can be pre- and postcomposed with the Verdier shift on $\underline{\hat{\mathcal{D}}}$; the condition is that the result of precomposition is the negative of the result of postcomposition.

A *weak square* in $\mathcal{D}$ is a commutative quadrangle that is at the same time a weak pullback and a weak pushout ([19]). A weak square is marked as

$$\begin{tikzcd} {} \arrow[r] & {} \\ {} \arrow[u] \arrow[r] \arrow[ur, phantom, "+"] & {} \arrow[u] \end{tikzcd}.$$

Alternatively, a commutative quadrangle is a weak square if and only if its diagonal sequence is exact in the middle when viewed in $\hat{\mathcal{D}}$.

Let $\mathcal{D}^+(\bar{\Delta}_2^\#)$ be the category of diagrams in $\mathcal{D}$ of the form

$$\begin{tikzcd}
 & & & & 0 \arrow[r] & \cdots \\
 & & & 0 \arrow[r] & X_{0^{+1}/2} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & \cdots \\
 & & 0 \arrow[r] & X_{2/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{0^{+1}/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & \cdots \\
 & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{2/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u] & \\
0 \arrow[r] & X_{0/2^{-1}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{1/2^{-1}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u] & & \\
\vdots \arrow[u] \arrow[ur, phantom, "+"] & \vdots \arrow[u] \arrow[ur, phantom, "+"] & \vdots \arrow[u] & & &
\end{tikzcd},$$

where we do not require any relation between $X_{0^{+1}/2}$ and $X_{2/0}^{+1}$ etc. Viewed in the abelian category $\hat{\mathcal{D}}$, this is just the category of acyclic complexes consisting of objects in $\mathcal{D}$.

Let $\mathcal{D}^+(\bar{\Delta}_3^\#)$ be the category of diagrams in $\mathcal{D}$ of the form

$$\begin{tikzcd}
 & & & & & 0 \arrow[r] & \cdots \\
 & & & & 0 \arrow[r] & X_{0^{+1}/3} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & \cdots \\
 & & & 0 \arrow[r] & X_{3/2} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{0^{+1}/2} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & \cdots \\
 & & 0 \arrow[r] & X_{2/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{3/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{0^{+1}/1} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & \cdots \\
 & 0 \arrow[r] & X_{1/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{2/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{3/0} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u] & \\
0 \arrow[r] & X_{0/3^{-1}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{1/3^{-1}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & X_{2/3^{-1}} \arrow[r] \arrow[u] \arrow[ur, phantom, "+"] & 0 \arrow[u] & & \\
\vdots \arrow[u] \arrow[ur, phantom, "+"] & \vdots \arrow[u] \arrow[ur, phantom, "+"] & \vdots \arrow[u] \arrow[ur, phantom, "+"] & \vdots \arrow[u] & & &
\end{tikzcd},$$

[19] The respective universal property is supposed to hold, except for the uniqueness of the induced morphism.

where we do not require any relation between $X_{0^{+1}/3}$ and $X_{3/0}^{+1}$ etc.

Etc.

For $n \geqslant 0$, we let

$$\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)}$$

be the reduction of $\mathcal{D}^+(\bar{\Delta}_n^\#)$ modulo the full additive subcategory of diagrams all of whose morphisms split. This category carries two shift functors, the outer shift $[-]^{+1}$ and the inner shift $[-^{+1}]$, characterised by, respectively,

$$\begin{array}{rcl} ([X]^{+1})_{\beta/\alpha} & = & X_{\alpha^{+1}/\beta} \\ ([X^{+1}])_{\beta/\alpha} & = & (X_{\beta/\alpha})^{+1} \end{array}$$

for $X \in \operatorname{Ob} \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^\#)$. In other words, the outer shift pulls the whole diagram down left, the inner shift applies T pointwise.

Then

$$\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)} \qquad \simeq \qquad \underline{\hat{\mathcal{D}}}\ ,$$

where

the outer shift	corresponds to	the third power of the Verdier shift ,
the inner shift	corresponds to	the functor induced by T .

So we can transport an isomorphism as in Heller's theorem from §3.1 to an isomorphism

$$[-]^{+1} \xrightarrow[\sim]{\vartheta_2} [-^{+1}]$$

from the outer to the inner shift functor on $\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)}$.

Using $\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)}$ as a replacement for $\underline{\hat{\mathcal{D}}}$ will enable us, in §3.3 below, to extend from $\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)}$ to $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)}$ for $n \geqslant 0$, so as to include octahedra and bigger diagrams [**3,** 1.1.14], and to drop the extra condition on the isomorphism mentioned in §3.1.

3.3 Heller triangulated categories

Let $\mathcal{D}$ be a weakly abelian category. Let T be an automorphism of $\mathcal{D}$.

Let a *Heller triangulation* on $(\mathcal{D}, \mathsf{T})$ be a tuple $\vartheta = (\vartheta_n)_{n \geqslant 0}$ of isomorphisms $\vartheta_n : [-]^{+1} \longrightarrow [-^{+1}]$ from the outer shift $[-]^{+1}$ to the inner shift $[-^{+1}]$ on $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)}$ satisfying compatibilities with quasicyclic operations ([20]) and with folding ([21]).

[20] Deleting and doubling rows and columns in a periodic manner yield functors $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} \xrightarrow{p^\#} \underline{\mathcal{D}^+(\bar{\Delta}_m^\#)}$. We require that $X\vartheta_n \underline{p}^\# = X\underline{p}^\# \vartheta_m$ for $X \in \operatorname{Ob} \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^\#)$.

[21] Suppose given $n \geqslant 0$ and $X \in \operatorname{Ob} \underline{\mathcal{D}^+(\bar{\Delta}_{2n+1}^\#)} = \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_{2n+1}^\#)$. We can canonically ([22]) construct an object $X\underline{\mathfrak{f}}_n \in \operatorname{Ob} \underline{\mathcal{D}^+(\bar{\Delta}_{n+1}^\#)} = \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_{n+1}^\#)$ that has $(X\underline{\mathfrak{f}}_n)_{i/0} = X_{n+i/i-1}$ for $1 \leqslant i \leqslant n+1$; the diagram $X\underline{\mathfrak{f}}_n$ involves direct sums of objects occurring in X. The operation $\underline{\mathfrak{f}}_n$ can be turned into a functor from $\underline{\mathcal{D}^+(\bar{\Delta}_{2n+1}^\#)}$ to $\underline{\mathcal{D}^+(\bar{\Delta}_{n+1}^\#)}$. We require that $X\vartheta_{2n+1}\underline{\mathfrak{f}}_n = X\underline{\mathfrak{f}}_n\vartheta_{n+1}$. Cf. [**3,** 1.1.13].

[22] Up to sign.

A *Heller triangulated category* then is a triple $(\mathcal{D}, \mathsf{T}, \vartheta)$ consisting of a weakly abelian category $\mathcal{D}$, an automorphism T of $\mathcal{D}$ and a Heller triangulation ϑ on $(\mathcal{D}, \mathsf{T})$ [**15,** Def. 1.5.(ii.1)]. Often, we write just $\mathcal{D} := (\mathcal{D}, \mathsf{T}, \vartheta)$.

For example, the derived category $\mathrm{D}^+(\mathcal{A})$ is Heller triangulated. Also the homotopy category $\mathrm{K}^+(\mathcal{A})$ is Heller triangulated. More generally, the stable category of a Frobenius category is Heller triangulated. Cf. [**15,** Cor. 4.7][**18,** Prop. 36] ([23]).

3.4 n-triangles

Suppose given a Heller triangulated category $(\mathcal{D}, \mathsf{T}, \vartheta)$. Suppose given $n \geqslant 0$.

The *base* of a diagram $X \in \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^{\#})$ is its subdiagram

$$(X_{1/0} \longrightarrow X_{2/0} \longrightarrow \cdots \longrightarrow X_{n-1/0} \longrightarrow X_{n/0}) \qquad \in \operatorname{Ob} \mathcal{D}(\dot{\Delta}_n)$$

on the linearly ordered set $\dot{\Delta}_n := \{1, 2, \ldots, n\}$.

A diagram $X \in \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^{\#})$ is called an *n-triangle* if $X\vartheta_n = 1$. A morphism $X \xrightarrow{f} Y$ between n-triangles X and Y is called *periodic* if $[f]^{+1} = [f^{+1}]$.

The restriction functor to the base, mapping from the category of n-triangles and periodic morphisms to $\mathcal{D}(\dot{\Delta}_n)$, is full [**15,** Lem. 3.2]. If all idempotents split in $\mathcal{D}$, then it is also surjective on objects [**15,** Lem. 3.1] ([24]).

Such triangles are stable under quasicyclic operations and under folding [**15,** Lem. 3.4.(1, 2)].

3.5 Retrieving the Verdier context in the Heller context

Suppose given a Heller triangulated category $(\mathcal{D}, \mathsf{T}, \vartheta)$ in which all idempotents split.

Let Ξ be the set of 2-triangles in $\mathcal{D}$. Then the triple $(\mathcal{D}, \mathsf{T}, \Xi)$ is a Verdier triangulated category [**15,** Prop. 3.6] ([25]).

Each 3-triangle is a Verdier octahedron; cf. §2.1. However, not every Verdier octahedron is a 3-triangle [**17,** Lem. 6] ([26]).

$$\begin{array}{rcl} \{\text{distinguished triangles}\} & = & \{\text{2-triangles}\} \\ \{\text{Verdier octahedra}\} & \supseteq & \{\text{3-triangles}\} \end{array}$$

3.6 Exact functors between Heller triangulated categories

Suppose given Heller triangulated categories $(\mathcal{D}, \mathsf{T}, \vartheta)$ and $(\mathcal{D}', \mathsf{T}', \vartheta')$.

[23] Cf. also [**17,** Prop. 22.(1).]

[24] More generally, this holds if $\mathcal{D}$ is a closed Heller triangulated category [**18,** Lem. 20]. Cf. also [**15,** Rem. 3.3].

[25] More generally, this holds if $\mathcal{D}$ is a closed Heller triangulated category [**18,** Rem. 18].

[26] Not even when requiring that it contains the triangles described in [**3,** 1.1.13]; cf. [**17,** Rem. 7].

A *strictly exact functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a shiftcompatible additive functor $V : \mathcal{D} \longrightarrow \mathcal{D}'$ that respects weak squares, and that satisfies

$$X\vartheta_n \underline{V^+(\bar{\Delta}_n^\#)} \;=\; X\underline{V^+(\bar{\Delta}_n^\#)}\vartheta'_n$$

for all $n \geqslant 0$ and all $X \in \operatorname{Ob}\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob}\mathcal{D}^+(\bar{\Delta}_n^\#)$, where $\underline{V^+(\bar{\Delta}_n^\#)}$ acts by pointwise application of V.

An *exact functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a pair (V, a) consisting of an additive functor $V : \mathcal{D} \longrightarrow \mathcal{D}'$ respecting weak squares, and an isotransformation $a : \mathsf{T}\,V \longrightarrow V\,\mathsf{T}'$ such that

$$X\vartheta_n \underline{V^+(\bar{\Delta}_n^\#)} \cdot X\underline{a^+(\bar{\Delta}_n^\#)} \;=\; X\underline{V^+(\bar{\Delta}_n^\#)}\vartheta'_n$$

for all $n \geqslant 0$ and all $X \in \operatorname{Ob}\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob}\mathcal{D}^+(\bar{\Delta}_n^\#)$.

So $V : \mathcal{D} \longrightarrow \mathcal{D}'$ is strictly exact if and only if $(V, 1)$ is exact.

3.7 Stability properties of the Heller formalism

Adjoints of exact functors are exact [**18,** Prop. 28].

We may form the Karoubi hull within the context of Heller triangulated categories and exact functors [**18,** Prop. 12].

We may form the localisation at the subset of morphisms with cone in a given thick subcategory within the context of Heller triangulated categories and exact functors [**18,** Prop. 38].

The derived functor $\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} \mathrm{D}^+(\mathcal{B})$ from §1.2 is exact, using that $\mathcal{A}$ is supposed to have enough injectives ([27]).

It is also possible to characterise exactness of a functor, in a manner similar to §2.2, by preservation of n-triangles [**18,** Prop. 25]. The reason behind that possibility is that closed ([28]) Heller triangulated categories can, alternatively, be defined via sets of n-triangles for $n \geqslant 0$ with suitable preservation properties with respect to quasicyclic operations and folding, as Thomas informed me.

3.8 Advantages of ϑ

Having n-triangles at our disposal allows constructions that have not been possible within the Verdier context. For instance, given two 3-triangles, a morphism between the bases can be prolonged to a morphism between the 3-triangles. This is no longer true, in general, once we replace "3-triangles" by "Verdier octahedra" [**17,** Lem. 6].

But why should we work primarily with ϑ, and only secondarily with n-triangles? A possible answer is that usage of ϑ allows low-effort proofs of the stability properties of the Heller formalism explained in §3.7; cf. [**18,** §2.2, §6, §5.2].

---

[27] Somewhat provisionally still, we may use [**18,** Prop. 28, Prop. 36], [**17,** Cor. 21], [**15,** Cor. 4.9] to arrive there. It would be preferable to use the derived functor construction via ind-categories along the lines of [**8,** Exp. XVII, §1.2].

[28] A Heller triangulated category is called *closed* if it is closed under taking cones in its Karoubi hull.

Of course, the price to pay is to get accustomed to the administration of the n-triangles being done by a tuple of isomorphisms ϑ.

3.9 An amusing observation

Suppose given Heller triangulated category $(\mathcal{D}, \mathsf{T}, \vartheta)$ in which all idempotents split ([29]).

A commutative quadrangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "x"] \arrow[r, "f"'] & Y \arrow[u, "y"'] \end{tikzcd}$$

in $\mathcal{D}$ is called a *dweak square* ([30]) if its diagonal sequence

$$X \xrightarrow{(f\ x)} Y \oplus X' \xrightarrow{\left(\begin{smallmatrix} y \\ -f' \end{smallmatrix}\right)} Y'$$

appears as part of a 2-triangle. So a dweak square is in particular a weak square; cf. §3.2.

Alternatively, a commutative quadrangle is a dweak square if and only if it appears in some n-triangle for some $n \geqslant 0$.

Any corner $\uparrow_{\rightarrow}$ can be completed to a dweak square. This completion is unique up to non-unique isomorphism. Accordingly in the dual situation.

Suppose given $n \geqslant 1$. Consider the set Chain_n of isoclasses of diagrams of the form $X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$ in $\mathcal{D}$, i.e. the set of isoclasses in $\mathcal{D}(\dot{\Delta}_n)$. We obtain two bijections

$$\sigma,\, \tau \;:\; \mathrm{Chain}_n \xrightarrow{\sim} \mathrm{Chain}_n$$

as follows.

Let τ map the isoclass of $X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$ to the isoclass of $X_1^{+1} \longrightarrow X_2^{+1} \longrightarrow \cdots \longrightarrow X_{n-1}^{+1} \longrightarrow X_n^{+1}$.

Let σ be defined as follows. Suppose given $X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$. Prolong this diagram by $X_n \longrightarrow 0$. Complete to dweak squares along $X_1 \longrightarrow 0$, yielding a new row $0 \longrightarrow X'_2 \longrightarrow \cdots \longrightarrow X'_{n-1} \longrightarrow X'_n \longrightarrow W_1$. Complete to dweak squares along $X'_2 \longrightarrow 0$, yielding a new row $0 \longrightarrow X''_3 \longrightarrow \cdots \longrightarrow X''_{n-1} \longrightarrow X''_n \longrightarrow W_2$. Etc. Then let σ map the isoclass of $X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$ to the isoclass of $W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{n-1} \longrightarrow W_n$.

Elementary properties of n-triangles force $\sigma = \tau$.

If we only require $\mathcal{D}$ to be Verdier triangulated, both σ and τ are still definable, but it is unclear to me whether they coincide ([31]).

[29] More generally, the following holds if $\mathcal{D}$ is a closed Heller triangulated category.

[30] An abbreviation for "distinguished weak square". Also known as *homotopy cartesian square*, as *homotopy bicartesian square*, or as *Mayer-Vietoris square.*

[31] Suppose that [**3,** 1.1.13] holds in our Verdier triangulated category. Then σ and τ coincide if $n \in \{1, 2, 3\}$.

4 Remarks on spectral sequences

4.1 Four indices

Suppose given an abelian category $\mathcal{A}$. Suppose given a filtered complex M with values in $\mathcal{A}$, i.e. a chain of monomorphisms

$$M(-\infty) \rightarrowtail \cdots \rightarrowtail M(i) \rightarrowtail M(i+1) \rightarrowtail \cdots \rightarrowtail M(+\infty),$$

indexed by $\{-\infty\} \sqcup \mathbf{Z} \sqcup \{+\infty\}$, that satisfies certain technical conditions ([32]).

We shall use the linearly ordered set

$$\bar{\mathbf{Z}}_\infty \;:=\; \{\, i^{+k} \;:\; i \in \{-\infty\} \sqcup \mathbf{Z} \sqcup \{+\infty\},\; k \in \mathbf{Z} \,\}\;,$$

where formally i^{+k} is defined as the pair (i,k), and where $i^{+k} \leqslant j^{+\ell}$ if $k < \ell$ or ($k = \ell$ and $i \leqslant j$).

Taking M as a base, we can form a diagram that is, morally, an ∞-triangle. It consists of shifted subfactor complexes $M(\beta/\alpha)$ for $\beta^{-1} \leqslant \alpha \leqslant \beta \leqslant \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$ and is called *spectral object* $\operatorname{Sp}(M)$ of M ([33]). For $\gamma/\alpha \leqslant \delta/\beta$, i.e. $\gamma \leqslant \delta$ and $\alpha \leqslant \beta$, the induced morphism

$$M(\gamma/\alpha) \longrightarrow M(\delta/\beta)$$

appears in this diagram $\operatorname{Sp}(M)$.

Let $M\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha) \in \operatorname{Ob}\mathcal{A}$ be defined as the image of H^0 of this morphism, i.e.

$$M(\gamma/\alpha)\mathrm{H}^0 \twoheadrightarrow M\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha) \rightarrowtail M(\delta/\beta)\mathrm{H}^0 \qquad ({}^{34})\;.$$

These objects $M\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)$ assemble to a big diagram with values in $\mathcal{A}$, the *spectral sequence*

$$M\mathrm{E}$$

of M ([35]).

Suppose given $\varepsilon^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \varepsilon \leqslant \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$. We obtain the short exact sequence

$$M\mathrm{E}(\varepsilon/\beta/\!\!/\gamma/\alpha) \rightarrowtail M\mathrm{E}(\varepsilon/\beta/\!\!/\delta/\alpha) \twoheadrightarrow M\mathrm{E}(\varepsilon/\gamma/\!\!/\delta/\alpha)\;,$$

which can be made responsible for all exact sequences in general spectral sequences known to me. Cf. [**15,** Lem. 3.9], generalising a particular case of [**24,** §II.4.2.6].

Dropping certain "initial terms" ([36]) from the spectral sequence $M\mathrm{E}$, we obtain the *proper spectral sequence*

$$M\dot{\mathrm{E}}$$

of M.

[32] Viz. $M(-\infty) = 0$, $M(i) \longrightarrow M(i+1)$ being pointwise split and the whole filtration being pointwise almost everywhere constant. Cf. [**16,** §3.1].

[33] This term has been coined by Verdier; cf. [**24,** §II.4].

[34] This definition slightly generalises the definition given in [**5,** App.]. The original definition in [**24,** §II.4.2.3] was closer to classical terminology, as found in [**4,** §XV.1].

[35] The classical spectral sequence terms are amongst the terms $M\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)$; cf. [**16,** §3.5].

[36] E_1-terms and similar ones; cf. [**16,** §3.6].

4.2 Comparisons

4.2.1 Grothendieck spectral sequences

Maintain the situation of §1.1. So $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$.

Suppose given $X \in \operatorname{Ob}\mathcal{A}$. Resolve X injectively. This yields a complex with values in $\mathcal{A}$. Apply F pointwise. This yields a complex with values in $\mathcal{B}$. Resolve this complex injectively, via the method of Cartan-Eilenberg [**4,** §XVII.1]. This yields a double complex with values in $\mathcal{B}$. Apply G pointwise. This yields a double complex with values in $\mathcal{C}$.

The *total complex* of this resulting double complex is obtained by forming direct sums over its diagonals. Replacing an increasing number of rows in this double complex by zero rows, and then taking the total complex, we obtain a descending chain of subcomplexes filtering our original total complex.

This filtered complex gives rise to the Grothendieck spectral sequence $X\mathrm{E}^{\mathrm{Gr}}_{F,G}$ via the method of §4.1. This yields a functor $\mathrm{E}^{\mathrm{Gr}}_{F,G}$ on $\mathcal{A}$, mapping to the category of spectral sequences with values in $\mathcal{C}$.

So we had to "resolve X twice", with an intermediate application of F, and a final application of G, to carry out this construction.

4.2.2 First comparison

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, $\mathcal{C}$ ([37]). Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $X' \in \operatorname{Ob}\mathcal{A}'$. Let $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor such that $(X,-)F$ and $(-,X')F$ are left exact. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be a left exact functor. Suppose further conditions to hold; see [**16,** §5.1].

$$
\begin{array}{c}
\phantom{\mathcal{A}}X \quad\;\; X'\\
\mathcal{A} \times \mathcal{A}'\\
\downarrow F\\
\mathcal{B}\\
\downarrow G\\
\mathcal{C}
\end{array}
$$

We have Grothendieck spectral sequence functors,

$$
\begin{array}{lll}
\mathrm{E}^{\mathrm{Gr}}_{(X,-)F,\,G} & \text{for} & \mathcal{A}' \xrightarrow{(X,-)F} \mathcal{B} \xrightarrow{G} \mathcal{C}\ , \\
\mathrm{E}^{\mathrm{Gr}}_{(-,X')F,\,G} & \text{for} & \mathcal{A} \xrightarrow{(-,X')F} \mathcal{B} \xrightarrow{G} \mathcal{C}\ .
\end{array}
$$

We evaluate the former at X' and the latter at X. Then the proper Grothendieck spectral sequences are isomorphic, i.e.

$$
X'\,\dot{\mathrm{E}}^{\mathrm{Gr}}_{(X,-)F,\,G} \;\simeq\; X\,\dot{\mathrm{E}}^{\mathrm{Gr}}_{(-,X')F,\,G}\ ;
$$

cf. [**16,** Th. 31]. So instead of "resolving X' twice", we may just as well "resolve X twice".

[37] Of which $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ are supposed to have enough injectives.

4.2.3 Second comparison

Suppose given abelian categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{C}$ ([38]). Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}'$ be a left exact functor. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor such that $(Y, -)G$ is left exact.

Let $B \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y such that $(B^k, -)G$ is exact for all $k \geqslant 0$. Let $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$ be an injective resolution of X. Suppose further conditions to hold; see [**16,** §6.1].

$$
\begin{array}{ccc}
 & & \scriptstyle X \\
 & & \mathcal{A} \\
 & \scriptstyle Y & \downarrow{\scriptstyle F} \\
 & \mathcal{B} & \times \; \mathcal{B}' \\
 & & \downarrow{\scriptstyle G} \\
 & & \mathcal{C}
\end{array}
$$

We have the Grothendieck spectral sequence functor

$$\mathrm{E}^{\mathrm{Gr}}_{F,(Y,-)G} \quad \text{for} \quad \mathcal{A} \xrightarrow{F} \mathcal{B}' \xrightarrow{(Y,-)G} \mathcal{C}\ ,$$

which we evaluate at X.

On the other hand, we can consider the double complex $(B, AF)G$, where the indices of B count rows and the indices of A count columns. As described in §4.2.1, we can associate a spectral sequence to a double complex, in this case named $\mathrm{E}_{\mathrm{I}}\big((B, AF)G\big)$.

Then the proper spectral sequences are isomorphic,

$$X\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,(Y,-)G} \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}\big((B, AF)G\big)\ .$$

So instead of "resolving X twice", we may just as well "resolve X once and Y once".

4.2.4 Applications

The comparisons in §4.2.2 and §4.2.3 may be used to reprove the following two theorems of Beyl.

The first theorem allows acyclic objects to be alternatively used to calculate Grothendieck spectral sequences [**2,** Th. 3.4]; cf. [**16,** Th. 40].

The second theorem allows the Hochschild-Serre-Hopf spectral sequence to be calculated with injective or, equivalently, with projective resolutions; the former fitting in the context of Grothendieck spectral sequences, the second being apt for manipulating concrete representing cocycles of cohomology classes; cf. [**2,** Th. 3.5], [**16,** Th. 52, 53].

Further applications can be found in [**16,** §8] ([39]).

[38] Of which $\mathcal{A}$ and $\mathcal{B}'$ are supposed to have enough injectives.

[39] If we were to reduce complexity in the assertions of §4.2, then, in the spirit of §1.2, we should directly work with suitably defined derived categories of double complexes; I do not know how to do that. We would probably get an additional shift functor.

References

[1] Balmer, P., Schlichting, M., *Idempotent completion of triangulated categories,* J. Alg. 236 (2), p. 819–834, 2001.

[2] Beyl, R., *The spectral sequence of a group extension,* Bull. Sc. math., 2e série, 105, p. 417–434, 1981.

[3] Bernstein, J.; Beilinson, A.A.; Deligne, P., *Faisceaux pervers*, Astérisque 100, 1982.

[4] Cartan, H., Eilenberg, S., *Homological Algebra,* Princeton, 1956.

[5] Deligne, P., *Décompositions dans la catégorie dérivée,* Proc. Symp. Pure Math. 55 (1), p. 115–127, 1994.

[6] Freyd, P., *Stable Homotopy,* Proc. Conf. Categorical Algebra, La Jolla, p. 121–172, Springer, 1965.

[7] Grothendieck, A. (redigé en coll. avec Dieudonné, J.), *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents,* Pub. Math. IHÉS 17, 1963.

[8] Grothendieck, A. et al., *Théorie des topos et cohomologie étale des schémas*, SGA 4, SLN 269+270+305, 1972.

[9] Happel, D., *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras,* LMS LN 119, 1988.

[10] Heller, A., *Stable homotopy categories,* Bull. Am. Math. Soc. 74, p. 28–63, 1968.

[11] Illusie, L., *Reminiscences of Grothendieck and His School,* Notices AMS, Vol. 57 (9), October 2010.

[12] Keller, B., Vossieck, D., *Sous les catégories dérivées,* C. R. Acad. Sci. Paris, Série I, p. 225–228, 1987.

[13] Karoubi, M., *K-théorie,* Conférences au séminaire d'été de Montréal No. 36, Presses de Montr2éal, 1969.

[14] Künzer, M., *On derived categories,* diploma thesis, Stuttgart, 1996.

[15] Künzer, M., *Heller triangulated categories,* Homol. Homot. Appl. 9 (2), p. 233–320, 2007.

[16] Künzer, M., *Comparison of spectral sequences involving bifunctors,* Documenta Math. 13, p. 677–737, 2008.

[17] Künzer, M., *Nonisomorphic Verdier octahedra on the same base,* J. Homotopy and Related Structures 4 (1), p. 7–38, 2009.

[18] Künzer, M., *On exact functors for Heller triangulated categories,* preprint, arxiv 1301.0031, 2012.

[19] Maltsiniotis, G., *Catégories triangulées supérieures*, Pré-preprint, www.math.jussieu.fr/∼maltsin/textes.html, 2005.

[20] Margolis, H.R., *Spectra and the Steenrod algebra,* North-Holland, 1983.

[21] Puppe, D., *On the formal structure of stable homotopy,* Coll. on Algebraic Topology, Aarhus, p. 65–71, 1962.

[22] Rickard, J., *Derived categories and stable equivalence,* J. Pure Appl. Alg. 61, p. 303–317, 1989.

[23] Verdier, J.L., *Catégories Derivées*, published in SGA 4 1/2, SLN 569, p. 262–311, 1977 (written 1963).

[24] Verdier, J.L., *Des catégories derivées des catégories abéliennes*, éd. G. Maltsiniotis, Astérisque 239, 1996 (written 1967).

Matthias Künzer
University of Stuttgart
Fachbereich Mathematik
Pfaffenwaldring 57
D-70569 Stuttgart
kuenzer@mathematik.uni-stuttgart.de
w5.mathematik.uni-stuttgart.de/fachbereich/Kuenzer/