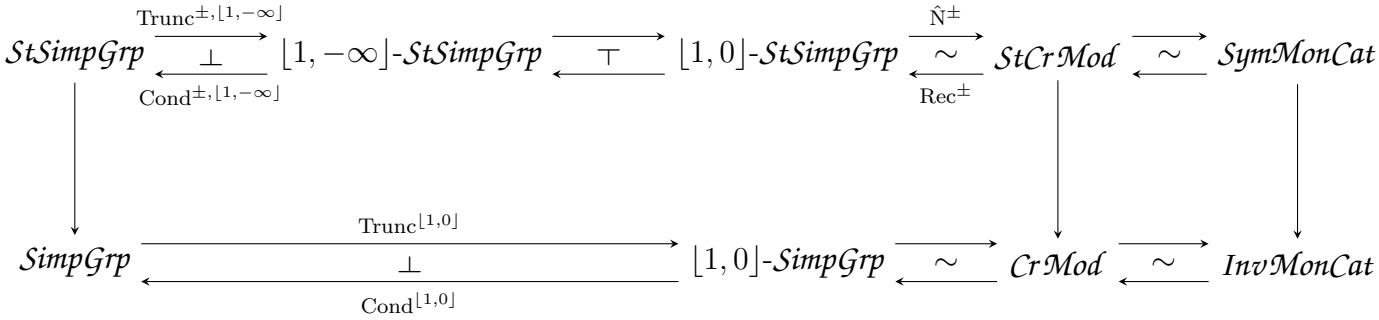
Functors for ordinary and stable simplicial groups

A connection to Conduché's stable crossed modules



Monika Truong

University of Stuttgart
Germany

Functors for ordinary and stable simplicial groups

A connection to Conduché's stable crossed modules

Von der Fakultät Mathematik und Physik
der Universität Stuttgart
zur Erlangung der Würde eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung

Vorgelegt von

Monika Truong
aus Müllheim im Markgräflerland

Hauptberichter:	Priv.-Doz. Dr. Matthias Künzer
Mitberichter:	Prof. Dr. Michael Eisermann
	Prof. Dr. Stefan Schwede
Tag der mündlichen Prüfung:	22.05.2025

Fachbereich Mathematik der Universität Stuttgart
2025

Acknowledgement

First and foremost, I would like to express my sincere gratitude to my supervisor PD Dr. Matthias Künzer for his invaluable guidance, patience and continuous support throughout my thesis and studies.

I am also grateful to PD Dr. Frederik Marks for his interest in my work, as well as for his insightful comments and constructive suggestions.

My thanks extend to Prof. Dr. Michael Eisermann and Prof. Dr. Stefan Schwede for serving as co-examiners and sharing their expertise. I would also like to thank Prof. Dr. Jens Wirth for participating in my doctoral examination committee.

Finally, I wish to express my heartfelt appreciation to my family for their unwavering support and encouragement, which made this achievement possible.

Contents

Abstract **vii**

Zusammenfassung **ix**

Introduction **xi**

0.1 Overview xi
0.2 The ordinary case xii
0.3 The stable case xxiii

1 Prologue **1**

1.1 Conventions 1
1.2 Preliminaries 12

I The ordinary case **25**

2 Simplicial groups **27**

2.1 The simplex category 27
2.2 Composites of morphisms in Δ 42
2.3 The category of simplicial groups 46
2.4 Simplicial subgroups 53
2.5 The décalage functor Dec for simplicial groups 56
2.6 The Moore complex of a simplicial group 59
2.7 Preparations for the Conduché condition 63

3 $\lfloor n,0\rfloor$-simplicial groups **67**
3.1 Pre-$\lfloor n,0\rfloor$-simplicial groups 67
3.2 A description using generators and relations 70
3.3 A construction of pre-$\lfloor n,0\rfloor$-simplicial groups 87
3.4 The category of $\lfloor n,0\rfloor$-simplicial groups 99
3.5 The Moore complex of an $\lfloor n,0\rfloor$-simplicial group 100

4 Simplicial groups and pre-$\lfloor n,0\rfloor$-simplicial groups: the coskeleton functor **105**
4.1 The cut functor $\mathrm{Cut}^{\lfloor n,0\rfloor} : \mathcal{SimpGrp} \to \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$ 105
4.2 The coskeleton functor $\mathrm{Cosk}^{\lfloor n,0\rfloor} : \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp}$ 106
4.3 The adjunction $\mathrm{Cut}^{\lfloor n,0\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,0\rfloor}$ 116

5 $\lfloor n,0\rfloor$-coskeletal simplicial groups **127**
5.1 $\lfloor n,0\rfloor$-coskeletal simplicial groups 127
5.2 The linear order on $\mathcal{P}([0,n])$ 129
5.3 The decomposition of G_n in subgroups $(G\mathrm{N}_k)s_I$ 134
5.4 The Conduché map c_I 149
5.5 The map c_I and the group morphism s_I 156
5.6 The degenerate subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$ 159

6 Simplicial groups and $\lfloor n,0\rfloor$-simplicial groups: the Conduché functor **171**
6.1 The truncation functor $\mathrm{Trunc}^{\lfloor n,0\rfloor} : \mathcal{SimpGrp} \to \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$ 171
6.2 The Conduché functor $\mathrm{Cond}^{\lfloor n,0\rfloor} : \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp}$ 184
6.3 The adjunction $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$ 192
6.4 The Moore complex $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}$ 209

7 The equivalent categories of $\lfloor 1,0\rfloor$-simplicial groups and crossed modules **211**
7.1 The category of $\lfloor 1,0\rfloor$-simplicial groups 211
7.2 The functor $\hat{\mathrm{N}} : \lfloor 1,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{CrMod}$ 216
7.3 The functor $\mathrm{Rec} : \mathcal{CrMod} \to \lfloor 1,0\rfloor\text{-}\mathcal{SimpGrp}$ 218
7.4 The equivalences $\hat{\mathrm{N}}$ and Rec 222
7.5 A remark on simplicial groups and crossed modules 229

7.6 An example for a crossed module 232

8 Crossed modules and invertible monoidal categories 237
8.1 Invertible monoidal categories 237
8.2 The equivalent categories *InvMonCat* and *CrMod* 241

II The stable case 249

9 Stable simplicial groups 251
9.1 The prestable simplex category 251
9.2 Composites of morphisms in $\Delta^{\pm}$ 268
9.3 Prestable simplicial groups 269
9.4 Prestable simplicial subgroups 274
9.5 A functor from Δ to $\Delta^{\pm}$ 276
9.6 The décalage functor $\mathrm{Dec}^{\pm,n}$ for prestable simplicial groups 281
9.7 The Moore complex of a prestable simplicial group 286
9.8 The category of stable simplicial groups 293

10 Simplicial groups and stable simplicial groups 303
10.1 The functor $\hat{\mathrm{Ord}}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$ 303
10.2 The ordinarification functor $\mathrm{Ord}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$, preserving quasi-isomorphisms 308

11 $\lfloor n, -\infty \rfloor$-stable simplicial groups 319
11.1 Pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups 319
11.2 $\lfloor n, -\infty \rfloor$-stable simplicial groups 322
11.3 A description using generators and relations 328
11.4 A construction of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups 351
11.5 The Moore complex of an $\lfloor n, -\infty \rfloor$-stable simplicial group 362

12 Stable simplicial groups and pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups: the coskeleton functor 367
12.1 The cut functor $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}\colon \mathcal{StSimpGrp} \to \mathcal{Pre}\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{PreStSimpGrp}$ 367

12.2 The coskeleton functor
$\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}\colon \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}impGrp \to \mathcal{S}t\mathcal{S}impGrp$ 368
12.3 The adjunction $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$ 375

13 $\lfloor n,-\infty\rfloor$-coskeletal stable simplicial groups **393**
13.1 $\lfloor n,-\infty\rfloor$-coskeletal stable simplicial groups 393
13.2 The decomposition of G_n in subgroups 398
13.3 The degenerate subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$ 406

14 Stable simplicial groups and $\lfloor n,-\infty\rfloor$-stable simplicial groups: the Conduché functor **415**
14.1 The truncation functor $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} : \mathcal{S}t\mathcal{S}impGrp \to \lfloor n,-\infty\rfloor\text{-}\mathcal{S}t\mathcal{S}impGrp$ . . 415
14.2 The Conduché functor $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor} : \lfloor n,-\infty\rfloor\text{-}\mathcal{S}t\mathcal{S}impGrp \to \mathcal{S}t\mathcal{S}impGrp$ . . . 430
14.3 The adjunction $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ 438
14.4 The Moore complex $(H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})\mathrm{N}$ 457

15 $\lfloor n,0\rfloor$-stable simplicial groups **459**
15.1 The category of $\lfloor n,0\rfloor$-stable simplicial groups 459
15.2 A construction of $\lfloor n,0\rfloor$-stable simplicial groups 461
15.3 The inclusion functor
$\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathcal{S}t\mathcal{S}impGrp \to \lfloor n,-\infty\rfloor\text{-}\mathcal{S}t\mathcal{S}impGrp$ 465
15.4 The shrinking functor
$\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,-\infty\rfloor\text{-}\mathcal{S}t\mathcal{S}impGrp \to \lfloor n,0\rfloor\text{-}\mathcal{S}t\mathcal{S}impGrp$ 466
15.5 The adjunction $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor} \dashv \mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$ 473

16 Stable crossed modules **477**
16.1 The category of stable crossed modules 477
16.2 The ordinarification functor $\mathrm{Ord}^{\mathrm{mod}}\colon \mathcal{S}t\mathcal{C}r\mathcal{M}od \to \mathcal{C}r\mathcal{M}od$ 485
16.3 Examples for stable crossed modules 490

17 Construction of the group H_∞ from a stable crossed module **499**
17.1 The group H_n 499
17.2 The group H_∞ 544
17.3 An example for the group H_n via Magma 611

18 The equivalent categories of $\lfloor 1,0\rfloor$-stable simplicial groups and stable crossed modules **629**
18.1 The category of $\lfloor 1,0\rfloor$-stable simplicial groups 629
18.2 The Conduché bracket in a $\lfloor 1,0\rfloor$-stable simplicial group 633
18.3 The functor $\hat{\mathrm{N}}^{\pm}\colon \lfloor 1,0\rfloor$-$\mathcal{StSimpGrp} \to \mathcal{StCrMod}$ 643
18.4 The functor $\mathrm{Rec}^{\pm}\colon \mathcal{StCrMod} \to \lfloor 1,0\rfloor$-$\mathcal{StSimpGrp}$ 645
18.5 The equivalences $\hat{\mathrm{N}}^{\pm}$ and $\mathrm{Rec}^{\pm}$ 653

19 Symmetric monoidal categories **669**
19.1 The category of symmetric monoidal categories 669
19.2 The functor $\mathrm{SymCat}\colon \mathcal{StCrMod} \to \mathcal{SymMonCat}$ 673
19.3 The functor $\mathrm{StCM}\colon \mathcal{SymMonCat} \to \mathcal{StCrMod}$ 677
19.4 The equivalent categories $\mathcal{SymMonCat}$ and $\mathcal{StCrMod}$ 685
19.5 The ordinarification functor $\mathrm{Ord}^{\mathrm{cat}} : \mathcal{SymMonCat} \to \mathcal{InvMonCat}$ 692
19.6 Inner automorphisms 694
19.7 Remarks on dualisation 697

20 Stable simplicial groups, stable crossed modules, simplicial groups and crossed modules **703**

Abstract

I. A simplicial group models a pointed connected topological space up to homotopy.

We may truncate a simplicial group in such a way that the homotopy groups between positions n and 0 are preserved. In this way, we obtain an $\lfloor n, 0 \rfloor$-simplicial group. This process gives a truncation functor from the category of simplicial groups to the category of $\lfloor n, 0 \rfloor$-simplicial groups. We construct the right-adjoint to this truncation functor that preserves homotopy groups, using methods from Conduché.

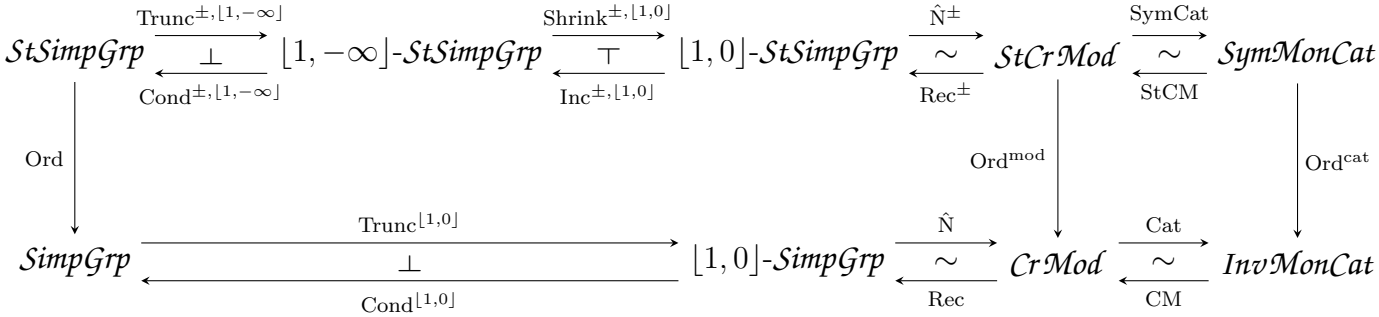
II. A stable simplicial group, also called group spectrum, models a topological spectrum up to homotopy.

We construct adjoint functors between the category of stable simplicial groups and the category of $\lfloor n, -\infty \rfloor$-stable simplicial groups that respect homotopy groups.

The category of $\lfloor 1, 0 \rfloor$-stable simplicial groups is defined as a full subcategory of $\lfloor 1, -\infty \rfloor$-stable simplicial groups. We show that the category of $\lfloor 1, 0 \rfloor$-stable simplicial groups is equivalent to the category of stable crossed modules in the sense of Conduché, using a construction of countably iterated semidirect products.

Zusammenfassung

Wir wollen das folgende Diagramm aus Kategorien und Funktoren konstruieren.



Es bezeichne *SimpGrp* die Kategorie der simplizialen Gruppen. Eine simpliziale Gruppe modelliert einen punktierten zusammenhängenden topologischen Raum bis auf Homotopie.

Wir können eine simpliziale Gruppe geeignet abschneiden, sodass die Homotopiegruppen an den Stellen 1 und 0 erhalten bleiben. Wir erhalten auf diese Weise eine $\lfloor 1, 0 \rfloor$-simpliziale Gruppe. Dieser Abschneideprozess liefert einen Funktor $\mathrm{Trunc}^{\lfloor 1,0 \rfloor}$ von der Kategorie der simplizialen Gruppen in die Kategorie der $\lfloor 1, 0 \rfloor$-simplizialen Gruppen $\lfloor 1, 0 \rfloor$-*SimpGrp*.

Dieser Abschneidefunktor Funktor $\mathrm{Trunc}^{\lfloor 1,0 \rfloor}$ hat einen rechtsadjungierten Funktor $\mathrm{Cond}^{\lfloor 1,0 \rfloor}$, der Homotopiegruppen erhält. Wir konstruieren ihn mit Hilfe von Methoden von Conduché.

Es bezeichne *CrMod* die Kategorie der verschränkten Moduln, welche von J.H.C. Whitehead eingeführt wurden. Die Kategorien *CrMod* und $\lfloor 1, 0 \rfloor$-*SimpGrp* sind äquivalent (Conduché).

Die Kategorie *CrMod* und die Kategorie der invertierbar monoidalen Katgorien *InvMonCat* sind äquivalent (Verdier, Duskin, Brown, Spencer).

Diese Zusammenhänge für gewöhnliche simpliziale Gruppen und verschränkte Moduln sollen in dieser Arbeit für stabile simpliziale Gruppen und stabile verschränkte Moduln nachgebildet werden.

Wir schreiben $\mathcal{StSimpGrp}$ für die Kategorie der stabilen simplizialen Gruppen. Diese wurden von Kan unter den Namen „group spectra“ eingeführt. Er zeigte, dass die Homotopiekategorie der stabilen simplizialen Gruppen äquivalent zur Homotopiekategorie der topologischen Spektren ist.

Wir können eine stabile simpliziale Gruppe geeignet abschneiden, sodass die Homotopiegruppen an den Stellen $\leqslant 1$ erhalten bleiben. Wir erhalten auf diese Weise eine $\lfloor 1, -\infty \rfloor$-stabile simpliziale Gruppe. Wir schreiben $\lfloor 1, -\infty \rfloor$-$\mathcal{StSimpGrp}$ für die Kategorie der $\lfloor 1, -\infty \rfloor$-stabilen simplizialen Gruppen und $\mathrm{Trunc}^{\pm, \lfloor 1, -\infty \rfloor}$ für den Abschneidefunktor.

Zu diesem Abschneidefunktor $\mathrm{Trunc}^{\pm, \lfloor 1, -\infty \rfloor}$ konstruieren wir einen einen rechtsadjungierten Funktor $\mathrm{Cond}^{\pm, \lfloor 1, -\infty \rfloor}$, der Homotopiegruppen erhält.

Aus einer $\lfloor 1, -\infty \rfloor$-stabilen simplizialen Gruppe erhalten wir eine $\lfloor 1, 0 \rfloor$-stabile simpliziale Gruppe durch eine Kernkonstruktion und durch eine Ersetzung aller Gruppen an den Positionen $\leqslant -1$ durch eine triviale Gruppe.

Wir schreiben $\lfloor 1, 0 \rfloor$-$\mathcal{StSimpGrp}$ für die Kategorie der $\lfloor 1, 0 \rfloor$-stabilen simplizialen Gruppen. Wir erhalten so eine Unterkategorie $\lfloor 1, 0 \rfloor$-$\mathcal{StSimpGrp} \subseteq \lfloor 1, -\infty \rfloor$-$\mathcal{StSimpGrp}$. Die genannte Kernkonstruktion liefert einen Funktor $\mathrm{Shrink}^{\pm, \lfloor 1, 0 \rfloor}$, der linksadjungiert zum Inklusionsfunktor $\mathrm{Inc}^{\pm, \lfloor 1, 0 \rfloor}$ ist.

Es bezeichne $\mathcal{StCrMod}$ die Kategorie der stabilen verschränkten Moduln, welche von Conduché eingeführt wurden.

Die Kategorie der stabilen verschränkten Moduln $\mathcal{StCrMod}$ ist äquivalent zur Kategorie $\mathcal{SymMonCat}$ der symmetrisch monoidalen Kategorien.

Wir zeigen, dass die Kategorien $\lfloor 1, 0 \rfloor$-$\mathcal{StSimpGrp}$ und $\mathcal{StCrMod}$ über die Funktoren $\hat{\mathrm{N}}^{\pm}$ und $\mathrm{Rec}^{\pm}$ äquivalent sind. Hierbei liefert $\hat{\mathrm{N}}^{\pm}$ den Moore-Komplex mit zusätzlichen Daten. Für $\mathrm{Rec}^{\pm}$ wird ein abzählbar iteriertes semidirektes Produkt konstruiert.

Die Funktoren $\mathrm{Ord}, \mathrm{Ord}^{\mathrm{mod}}$ und $\mathrm{Ord}^{\mathrm{cat}}$ bilden aus den Kategorien $\mathcal{StSimpGrp}$, $\mathcal{StCrMod}$ und $\mathcal{SymMonCat}$ in ihre gewöhnlichen Entsprechungen $\mathcal{SimpGrp}$, $\mathcal{CrMod}$ und $\mathcal{InvMonCat}$ ab.

Eine stabile simpliziale Gruppe wird vermittels Ord zu einer simplizialen Gruppe, deren Moore-Komplex an den Stellen $\geqslant 0$ erhalten bleibt. Die Konstruktion von Ord wird dabei unter Verwendung von Schnitten gewisser Kerne durchgeführt.

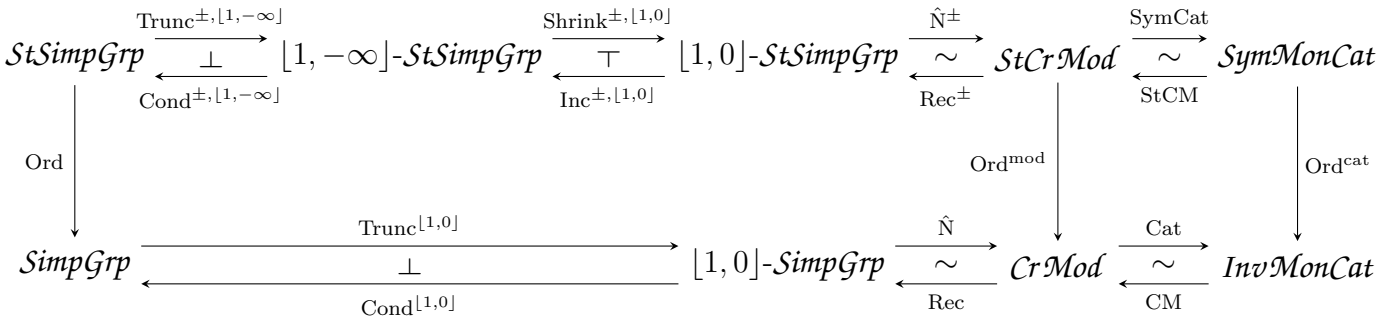
Ein stabiler verschränkter Modul wird vermittels $\mathrm{Ord}^{\mathrm{mod}}$ zu einem verschränkten Modul, indem wir aus der Conduché-Klammer eine Operation konstruieren.

Eine symmetrisch monoidale Kategorie wird vermittels $\mathrm{Ord}^{\mathrm{cat}}$ durch das Vergessen der symmetrischen Verzopfung zu einer invertierbar monoidalen Kategorie.

Introduction

0.1 Overview

Our goal is to establish the following diagram of categories and functors.



We regard the lower row as the "ordinary case" and the upper row as the "stable case". The relations depicted in the ordinary case are largely due to Conduché [**7**] and Brown-Spencer [**4**]. We give an account of the ordinary case and shall use it as a blueprint for the stable case.

In the ordinary case in the lower row, we consider the category of (ordinary) *simplicial groups* $\mathcal{SimpGrp}$ whose homotopy category is equivalent to the homotopy category of pointed connected topological spaces [**17**], [**11**], [**10**]. So in that sense, simplicial groups are a means to study topological spaces.

We may truncate simplicial groups in such a way that the homotopy groups in the positions 1 and 0 are preserved. In this way, we obtain $\lfloor 1,0\rfloor$-*simplicial groups*, which form the category $\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$. This truncation process yields a functor $\mathrm{Trunc}^{\lfloor 1,0\rfloor}$. We show that it is left-adjoint to the functor $\mathrm{Cond}^{\lfloor 1,0\rfloor}$, constructed by Conduché in 1984 [**7**]. In the same work, Conduché has also shown that the category $\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$ is equivalent to the category of

crossed modules $\mathcal{CrMod}$ as defined by J.H.C. Whitehead [**31**].

Furthermore, in 1976, it has been shown by Brown and Spencer, who attribute it to Duskin and Verdier, that $\mathcal{CrMod}$ and the category of *invertible monoidal categories* $\mathcal{InvMonCat}$ are equivalent [**4**].

In the stable case in the upper row, we consider the category of *stable simplicial groups* $\mathcal{StSimpGrp}$. In 1963, Kan has essentially shown that the homotopy category of stable simplicial groups is equivalent to the category of topological spectra [**18**].

To complete the upper row of the diagram, we proceed analogously to the ordinary case:

By truncation of a stable simplicial group, we obtain a $\lfloor 1, -\infty \rfloor$-*stable simplicial group* in the category $\lfloor 1, -\infty \rfloor$-$\mathcal{StSimpGrp}$, where the homotopy groups in positions $\leqslant 1$ are preserved. Using similar methods to the ordinary case, we construct the adjoint functors $\mathrm{Trunc}^{\pm,\lfloor 1,-\infty \rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor 1,-\infty \rfloor}$ between $\mathcal{StSimpGrp}$ and $\lfloor 1, -\infty \rfloor$-$\mathcal{StSimpGrp}$.

By shrinking a $\lfloor 1, -\infty \rfloor$-stable simplicial group via the functor $\mathrm{Shrink}^{\lfloor 1,0 \rfloor}$, we also obtain a $\lfloor 1, 0 \rfloor$-*stable simplicial group* in the category $\lfloor 1, 0 \rfloor$-$\mathcal{StSimpGrp}$, where the homotopy groups in positions 1 and 0 are preserved. We show that the category $\lfloor 1, 0 \rfloor$-$\mathcal{StSimpGrp}$ is equivalent to the category of *stable crossed modules* $\mathcal{StCrMod}$ as defined by Conduché [**7**].

Moreover, the category of stable crossed modules is equivalent to the category of *symmetric monoidal categories* $\mathcal{SymMonCat}$.

Finally, the stable case can be connected to the ordinary case via ordinarification functors $\mathrm{Ord}, \mathrm{Ord}^{\mathrm{mod}}$ and $\mathrm{Ord}^{\mathrm{cat}}$ that map from the categories in upper row to their respective ordinary counterparts in the lower row.

0.2 The ordinary case

0.2.1 Simplicial groups

We consider the simplex category Δ, whose objects are the intervals of integers $[k] = \{z \in \mathbb{Z} \colon 0 \leqslant z \leqslant k\}$ for $k \in \mathbb{Z}_{\geqslant 0}$, and whose morphisms are monotone maps between these linearly ordered intervals.

Then, a *simplicial group* is a functor G from its opposite category Δ^{op} to the category of groups $\mathcal{Grp}$. The category of simplicial groups is denoted by $\mathcal{SimpGrp}$.

We may also regard a simplicial group G as a list of groups G_k for $k \in \mathbb{Z}_{\geqslant 0}$ together with group morphisms $\mathrm{d}_j := \mathrm{d}_j^{G,k} \colon G_k \to G_{k-1}$ and $\mathrm{s}_i := \mathrm{s}_i^{G,k} \colon G_k \to G_{k+1}$, where $i,\ j \in [0, k]$,

between those groups that satisfy the usual simplicial relations.

$$\cdots \underset{\mathrm{s}_k\,,\ldots,\,\mathrm{s}_0}{\overset{\mathrm{d}_{k+1}\,,\ldots,\,\mathrm{d}_0}{\rightleftarrows}} G_k \underset{\mathrm{s}_{k-1}\,,\ldots,\,\mathrm{s}_0}{\overset{\mathrm{d}_k\,,\ldots,\,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_2\,,\mathrm{s}_1\,,\mathrm{s}_0}{\overset{\mathrm{d}_3\,,\mathrm{d}_2\,,\mathrm{d}_1\,,\mathrm{d}_0}{\rightleftarrows}} G_2 \underset{\mathrm{s}_1\,,\mathrm{s}_0}{\overset{\mathrm{d}_2\,,\mathrm{d}_1\,,\mathrm{d}_0}{\rightleftarrows}} G_1 \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1\,,\mathrm{d}_0}{\rightleftarrows}} G_0$$

Its Moore complex $G\mathrm{N}$ is given by groups

$$G\mathrm{N}_k = \bigcap_{i\in[1,k]} \ker \mathrm{d}_i \trianglelefteqslant G_k$$

for $k \in \mathbb{Z}_{\geqslant 0}$, trivial groups for $k \in \mathbb{Z}_{\leqslant -1}$ and differentials $\partial_k = \mathrm{d}_0\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}}$ for $k \in \mathbb{Z}_{\geqslant 1}$.

$$G\mathrm{N} = \left(\ldots \xrightarrow{\partial_4} G\mathrm{N}_3 \xrightarrow{\partial_3} G\mathrm{N}_2 \xrightarrow{\partial_2} G\mathrm{N}_1 \xrightarrow{\partial_1} G\mathrm{N}_0 \xrightarrow{\partial_0} 1 \longrightarrow \ldots\right)$$

Then, for $k \in \mathbb{Z}_{\geqslant 0}$, J. Moore defined the k-th homotopy group of G as the k-th homology group of $G\mathrm{N}$ [**20,** §4]:

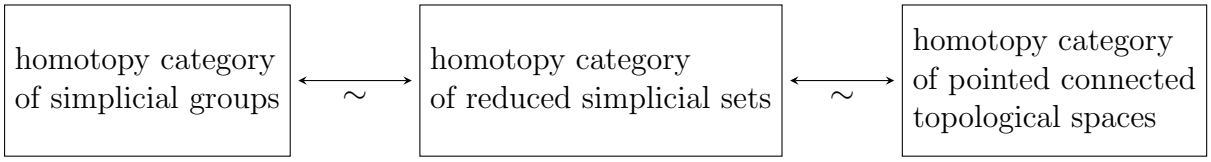
$$G\pi_k := G\mathrm{NH}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}\,.$$

D. Quillen has shown that the category of simplicial groups is a closed simplicial model category [**23,** Ch. II, p. 3.7, Thm. 2]. In particular, we have a homotopy category of simplicial groups.

D. Kan introduced the adjoint pair of functors $G \dashv \overline{W}$ between simplicial groups and reduced simplicial sets, where G plays the role of a 'loop group functor' [**17,** §10, Rem. 10.6], [**11,** Ch. V, §4–5 and Lem. 5.3 (b)]. These functors induce an equivalence between the homotopy category of simplicial groups and the homotopy category of reduced simplicial sets [**11,** Ch. V, Cor. 6.4].

Further, we have that the homotopy category of reduced simplicial groups is equivalent to the homotopy category of pointed connected topological spaces [**10,** Thms. 3.2, 3.4.(ii)].

Altogether, we have an equivalence between the homotopy category of simplicial groups and the homotopy category of pointed connected topological spaces.



So simplicial groups are suitable to perform algebraic topology with.

For instance, Kan has shown a Hurewicz theorem for a given simplicial free group by applying abelianization to the entire simplicial group [**16**].

0.2.2 $\lfloor n,0\rfloor$-simplicial groups

Let $n \in \mathbb{Z}_{\geqslant 1}$. Let $\Delta_{\lfloor n,0\rfloor} \subseteq \Delta$ be the full subcategory whose objects are the intervals $[k]$ for $k \in [0,n]$.

A functor $H\colon (\Delta_{\lfloor n,0\rfloor})^{\mathrm{op}} \to \mathcal{G}rp$ is called a *pre-*$\lfloor n,0\rfloor$*-simplicial group.*

A functor $H\colon (\Delta_{\lfloor n,0\rfloor})^{\mathrm{op}} \to \mathcal{G}rp$ that satisfies the following condition (1) is called an $\lfloor n,0\rfloor$*-simplicial group.*

(1) For I, $J \subseteq [0,n]$ with I, $J \neq \emptyset$ and $I \cup J = [0,n]$, we have the trivial commutator

$$\big[\bigcap_{i\in I} \ker \mathrm{d}_i \,, \bigcap_{j\in J} \ker \mathrm{d}_j\big] = 1 \leqslant H_n\,.$$

This condition (1) is called the *Conduché condition* [**7,** Th. 1.5, Cor. 1.8].

The $\lfloor n,0\rfloor$-simplicial groups form a full subcategory of the pre-$\lfloor n,0\rfloor$-simplicial groups:

$$\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp \;\subseteq\; \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp\,.$$

So, an $\lfloor n,0\rfloor$-simplicial group H is given by a list of groups H_k for $k \in [0,n]$ and group morphisms $\mathrm{d}_j = \mathrm{d}_j^{H,\ell}\colon H_\ell \to H_{\ell-1}$ and $\mathrm{s}_i = \mathrm{s}_i^{H,\ell}\colon H_\ell \to H_{\ell+1}$, where i, $j \in [0,\ell]$, that satisfy the simplicial relations and the Conduché condition.

$$H_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_{n-2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n-1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_1,\mathrm{s}_0}{\overset{\mathrm{d}_2,\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_1 \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_0$$

0.2.3 The functors $\mathrm{Cut}^{\lfloor n,0\rfloor}$ and $\mathrm{Cosk}^{\lfloor n,0\rfloor}$

We aim to construct a pair of adjoint functors between the categories of simplicial groups $\mathcal{S}impGrp$ and the category of $\lfloor n,0\rfloor$-simplicial groups $\lfloor n,0\rfloor$-$\mathcal{S}impGrp$ that preserve homotopy groups; cf. §0.2.5 below. To this end, we first need to consider a pair of adjoint functors between $\mathcal{S}impGrp$ and $\mathcal{P}re$-$\lfloor n,0\rfloor$-$\mathcal{S}impGrp$.

Let $n \in \mathbb{Z}_{\geqslant 1}$. From a simplicial group G we may obtain a pre-$\lfloor n,0\rfloor$-simplicial group $G\,\mathrm{Cut}^{\lfloor n,0\rfloor}$ by simply restricting the functor G from Δ^{op} to $(\Delta_{\lfloor n,0\rfloor})^{\mathrm{op}}$. That is, for $k \in [0,n]$, we let

$$(G\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k := G_k\,.$$

We obtain a functor $\mathrm{Cut}^{\lfloor n,0\rfloor}\colon \mathcal{SimpGrp} \to \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$; cf. Remark 79.

$$G: \quad \cdots \underset{\mathrm{s}_{n+1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+2},\ldots,\mathrm{d}_0}{\rightleftarrows}} G_{n+1} \underset{\mathrm{s}_{n},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+1},\ldots,\mathrm{d}_0}{\rightleftarrows}} G_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n},\ldots,\mathrm{d}_0}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_{n-2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n-1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_{1},\mathrm{s}_0}{\overset{\mathrm{d}_{2},\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} G_1 \underset{\mathrm{s}_{0}}{\overset{\mathrm{d}_{1},\mathrm{d}_0}{\rightleftarrows}} G_0$$

$$G\,\mathrm{Cut}^{\lfloor n,0\rfloor}: \quad G_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n},\ldots,\mathrm{d}_0}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_{n-2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n-1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_{1},\mathrm{s}_0}{\overset{\mathrm{d}_{2},\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} G_1 \underset{\mathrm{s}_{0}}{\overset{\mathrm{d}_{1},\mathrm{d}_0}{\rightleftarrows}} G_0$$

It has a right-adjoint functor $\mathrm{Cosk}^{\lfloor n,0\rfloor}\colon \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp}$, which is called the coskeleton functor; cf. Lemmas 82, 88.

$$\mathcal{SimpGrp} \underset{\mathrm{Cosk}^{\lfloor n,0\rfloor}}{\overset{\mathrm{Cut}^{\lfloor n,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$$

This coskeleton functor goes back to A. Grothendieck and J.-L. Verdier, who construct it using finite projective limits [**1,** Exp. V, 7.1 and Exp. I, 5.1]. There, the coskeleton functor is written as i_{n*}.

Then, J. Duskin [**9,** Chs. 0.7–0.9] constructs $\mathrm{Cosk}^{\lfloor n,0\rfloor}$ by iteration of successive simplicial kernels.

P. Goerss and J. F. Jardine [**11,** Ch. VII, Ex. 1.14] give an account of the construction of Grothendieck and Verdier, using the limit over morphisms $[m] \longrightarrow [k]$ for $m \in [0,n]$ and $k \in \mathbb{Z}_{\geqslant 0}$. So, for a pre-$\lfloor n,0\rfloor$-simplicial group H and for $k \in \mathbb{Z}_{\geqslant 0}$, they describe the group $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$ as consisting of tuples of elements in H_m, where $m \in [0,n]$, satisfying certain conditions.

We use a formula for the group $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$ that works with tuples of elements of H_n only, indexed by morphisms $[n] \longrightarrow [k]$ for $k \in \mathbb{Z}_{\geqslant 0}$. More explicitly, for a pre-$\lfloor n,0\rfloor$-simplicial

group H, we have

$$(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k = \Big\{(h_a)_{a\in\Delta([n],[k])} \in \prod_{a\in\Delta([n],[k])} H_n \colon (h_a)_{a\in\Delta([n],[k])} \text{ is coherent}\Big\} \leqslant \prod_{a\in\Delta([n],[k])} H_n\,,$$

for $k \in [0,n]$. Here, we say that a tuple $(h_a)_{a\in\Delta([n],[k])}$ is *coherent* if we have $h_a H_b = h_{a'} H_{b'}$ for all commutative diagrams

$$\begin{array}{ccc} [n] & \xrightarrow{\;a\;} & [k] \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle a'} \\ [m] & \xrightarrow[\;b'\;]{} & [n] \end{array}$$

in Δ, where $m \in [0,n]$; cf. Definition 80, Lemma 81.

So in comparison to the formula of Goerss and Jardine, we have somewhat shortened the tuples under consideration, which helps us in handling the coskeleton functor.

For $k \in [0,n]$, we note that we have a group isomorphism $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \xrightarrow{\sim} H_k$; cf. Lemma 86.(1).

$$H: \qquad H_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_{n-2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n-1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_1,\mathrm{s}_0}{\overset{\mathrm{d}_2,\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_1 \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_0$$

$$H\mathrm{Cosk}^{\lfloor n,0\rfloor}: \quad \cdots \underset{\mathrm{s}_{n+1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+2},\ldots,\mathrm{d}_0}{\rightleftarrows}} (H\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1} \underset{\mathrm{s}_n,\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+1},\ldots,\mathrm{d}_0}{\rightleftarrows}} (H\mathrm{Cosk}^{\lfloor n,0\rfloor})_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} (H\mathrm{Cosk}^{\lfloor n,0\rfloor})_0$$

$$\simeq \quad \cdots \underset{\mathrm{s}_{n+1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+2},\ldots,\mathrm{d}_0}{\rightleftarrows}} (H\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1} \underset{\mathrm{s}_n,\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+1},\ldots,\mathrm{d}_0}{\rightleftarrows}} H_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_0$$

0.2.4 The degenerate subgroup $G\mathrm{D}_{n+1}$ of an $\lfloor n,0\rfloor$-coskeletal simplicial group G

Let $n \in \mathbb{Z}_{\geqslant 1}$. From an $\lfloor n,0\rfloor$-simplicial group H, we may obtain the simplicial group $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ via the coskeleton functor $\mathrm{Cosk}^{\lfloor n,0\rfloor}$. Then $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ has certain properties

that are inherited from H which are needed to obtain the degenerate subgroup

$$(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1} \;\leqslant\; (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1}$$

which is, in turn, essential to the construction of the functor $\mathrm{Cond}^{\lfloor n,0\rfloor}$; cf. §0.2.5 below. We impose said needed properties on a simplicial group as follows.

We say that a simplicial group G is $\lfloor n,0\rfloor$-*coskeletal* if it satisfies the following conditions $(1,2)$; cf. Definition 90.

(1) We have $\bigcap_{k\in[0,n+1]} \ker \mathrm{d}_k = 1 \leqslant G_{n+1}$.

(2) For $I,\ J \subseteq [0,n]$ with $I,\ J \neq \emptyset$ and $I \cup J = [0,n]$, we have the trivial commutator

$$\big[\bigcap_{i\in I} \ker \mathrm{d}_i\,,\bigcap_{j\in J} \ker \mathrm{d}_j\big] = 1 \leqslant G_n\,.$$

Then, as intended, for an $\lfloor n,0\rfloor$-simplicial group H, the simplicial group $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ is $\lfloor n,0\rfloor$-coskeletal; cf. Remark 91.

Let G be an $\lfloor n,0\rfloor$-coskeletal simplicial group. We shall sketch the construction of the degenerate subgroup $G\mathrm{D}_{n+1}$. We may decompose the group G_{n+1} as a product of subgroups in the form

$$G_{n+1} = \prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{n+1-|I_i|})\mathrm{s}_{I_i}$$

where $I_i \subseteq [0,n]$, where $r = 2^{n+1}$, where $G\mathrm{N}_{n+1-|I_i|}$ are Moore complex terms of G and where s_{I_i} is a composite of the degeneracy maps s_j with $j \in I_i$; cf. §5.3.

For example, if $n = 1$, we have the decomposition

$$G_2 \;=\; (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_0 \cdot G\mathrm{N}_2\,,$$

and if $n = 2$, we have

$$G_3 \;=\; (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_2 \cdot (G\mathrm{N}_2)\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_0 \cdot G\mathrm{N}_3\,;$$

cf. Example 102.

We consider the subset

$$G\mathrm{D}_{n+1} := \prod_{i\in\lfloor r,2\rfloor} (G\mathrm{N}_{n+1-|I_i|})\mathrm{s}_{I_i} \;\subseteq\; G_{n+1}$$

which we obtain by omitting the last Moore complex term $G\mathrm{N}_{n+1}$ from the decomposition of G_{n+1} given above.

Then, using the fact that G is $\lfloor n,0\rfloor$-coskeletal, we show that $G\mathrm{D}_{n+1}$ is a subgroup of G_{n+1}; cf. Lemma 113. To achieve this, we use a technique D. Conduché employed to show that the subgroup of G_{n+1} generated by degenerate elements has trivial intersection with $G\mathrm{N}_{n+1}$ [**7,** Lem. 1.7]. We call $G\mathrm{D}_{n+1}$ the *degenerate subgroup* of G_{n+1}.

For example, if $n=1$, we have the degenerate subgroup

$$G\mathrm{D}_2 \;=\; (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_0 \;\leqslant\; G_2\,,$$

and if $n=2$, we have the degenerate subgroup

$$G\mathrm{D}_3 = (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_2 \cdot (G\mathrm{N}_2)\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_0 \;\leqslant\; G_3\,;$$

cf. Example 114.

In particular, for an $\lfloor n,0\rfloor$-simplicial group H, we have the degenerate subgroup

$$(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1} \;\leqslant\; (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1}\,.$$

As a related result, for a general simplicial group G, A. Mutlu and T. Porter achieved in certain cases a description of the intersection of $G\mathrm{N}_{n+1}$ with the subgroup of G_{n+1} generated by degenerate elements [**21, 22**].

0.2.5 The functors $\mathrm{Trunc}^{\lfloor n,0\rfloor}$ and $\mathrm{Cond}^{\lfloor n,0\rfloor}$

Let $n \in \mathbb{Z}_{\geqslant 1}$. We have the pair of adjoint functors $\mathrm{Cut}^{\lfloor n,0\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,0\rfloor}$ between $\mathcal{SimpGrp}$ and $\mathcal{P}re$-$\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$; cf. §0.2.3.

Based on $\mathrm{Cut}^{\lfloor n,0\rfloor}$ and $\mathrm{Cosk}^{\lfloor n,0\rfloor}$, we shall construct a pair of adjoint functors $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$ between $\mathcal{SimpGrp}$ and $\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$; cf. Lemma 129.

For a simplicial group G, we let $G\mathrm{NB}_n := (G\mathrm{N}_{n+1})\mathrm{d}_0 \trianglelefteqslant G_n$. Then the $\lfloor n,0\rfloor$-simplicial group $G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ has

$$(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k = \begin{cases} G_n/G\mathrm{NB}_n & \text{for } k=n \\ G_n & \text{for } k \in [0,n-1]\,. \end{cases}$$

This yields a functor $\mathrm{Trunc}^{\lfloor n,0\rfloor} \colon \mathcal{SimpGrp} \to \lfloor n,0\rfloor$-$\mathcal{SimpGrp}$.

$$G: \quad \cdots \underset{\mathrm{s}_{n+1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+2},\ldots,\mathrm{d}_0}{\rightleftarrows}} G_{n+1} \underset{\mathrm{s}_n,\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+1},\ldots,\mathrm{d}_0}{\rightleftarrows}} G_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_{n-2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n-1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_1,\mathrm{s}_0}{\overset{\mathrm{d}_2,\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} G_1 \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} G_0$$

$$G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}: \quad G_n/G\mathrm{NB}_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_{n-2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n-1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_1,\mathrm{s}_0}{\overset{\mathrm{d}_2,\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} G_1 \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} G_0$$

This functor $\mathrm{Trunc}^{\lfloor n,0\rfloor}$ preserves the homotopy groups of G in the positions $k \in [0,n]$: we have group isomorphisms $(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\pi_k \xrightarrow{\sim} G\pi_k$ for $k \in [0,n]$; cf. Lemma 121.

In the converse direction from $\lfloor n,0\rfloor$-*SimpGrp* to *SimpGrp*, Conduché constructs a functor as follows [**7**]. Let $\mathit{SimpGrp}^{\lfloor n,0\rfloor}$ be the full subcategory in *SimpGrp* with

$$\mathrm{Ob}(\mathit{SimpGrp}^{\lfloor n,0\rfloor}) = \{G \in \mathrm{Ob}(\mathit{SimpGrp})\colon G\mathrm{N}_i = 1 \text{ for } i \in \mathbb{Z}_{\geqslant n+1}\}\,.$$

In [**7,** Cor. 1.8], Conduché shows that the categories $\mathit{SimpGrp}^{\lfloor n,0\rfloor}$ and $\lfloor n,0\rfloor$-*SimpGrp* are equivalent via the restriction of the truncation functor

$$\mathrm{Trunc}^{\lfloor n,0\rfloor}\big|_{\mathit{SimpGrp}^{\lfloor n,0\rfloor}}\colon \mathit{SimpGrp}^{\lfloor n,0\rfloor} \xrightarrow{\sim} \lfloor n,0\rfloor\text{-}\mathit{SimpGrp}\,.$$

To show the inverse equivalence, he attaches to $H \in \mathrm{Ob}(\lfloor n,0\rfloor\text{-}\mathit{SimpGrp})$ the simplicial subgroup $H\,\mathrm{Cond}^{\lfloor n,0\rfloor} \leqslant H\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \in \mathrm{Ob}(\mathit{SimpGrp}^{\lfloor n,0\rfloor})$ generated by the degenerate elements in $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1}$ [**7,** Th. 1.5].

We shall give an alternative construction to the one given by Conduché in order to be able to prove that

$$\mathrm{Cond}^{\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathit{SimpGrp} \to \mathit{SimpGrp}$$

is right-adjoint to the unrestricted truncation functor $\mathrm{Trunc}^{\lfloor n,0\rfloor}\colon \mathit{SimpGrp} \to \lfloor n,0\rfloor\text{-}\mathit{SimpGrp}$.

It goes as follows. Let H be an $\lfloor n,0\rfloor$-simplicial group. We consider the simplicial group $\tilde{H} := H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ and its degenerate subgroup

$$\tilde{H}\mathrm{D}_{n+1} = (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1} = \tilde{H}_{n+1}\,;$$

cf. Lemma 113. Then the simplicial subgroup $H\,\mathrm{Cond}^{\lfloor n,0\rfloor} \leqslant \tilde{H} = H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ has

$$(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k := \begin{cases} \tilde{H}_k & \text{for } k \in [0,n] \\ \tilde{H}\mathrm{D}_{n+1} & \text{for } k = n+1 \\ \bigcap\limits_{\substack{a\in\,\Delta([n+1],[k]) \\ a \text{ is injective}}} \tilde{H}_a^{-1}(\tilde{H}\mathrm{D}_{n+1}) & \text{for } k \in \mathbb{Z}_{\geqslant n+2}\,. \end{cases}$$

Note that $\tilde{H}_k \simeq H_k$ for $k \in [0,n]$.

This yields the functor $\mathrm{Cond}^{\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp}$.

$$H: \qquad H_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_{n-2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n-1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_1,\mathrm{s}_0}{\overset{\mathrm{d}_2,\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_1 \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_0$$

$$H\,\mathrm{Cond}^{\lfloor n,0\rfloor}: \quad \cdots \underset{\mathrm{s}_{n+2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+3},\ldots,\mathrm{d}_0}{\rightleftarrows}} \bigcap_a \tilde{H}_a^{-1}(\tilde{H}\mathrm{D}_{n+1}) \underset{\mathrm{s}_{n+1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+2},\ldots,\mathrm{d}_0}{\rightleftarrows}} \tilde{H}\mathrm{D}_{n+1} \underset{\mathrm{s}_n,\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+1},\ldots,\mathrm{d}_0}{\rightleftarrows}} \tilde{H}_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} \tilde{H}_0$$

$$\simeq \quad \cdots \underset{\mathrm{s}_{n+2},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+3},\ldots,\mathrm{d}_0}{\rightleftarrows}} \bigcap_a \tilde{H}_a^{-1}(\tilde{H}\mathrm{D}_{n+1}) \underset{\mathrm{s}_{n+1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+2},\ldots,\mathrm{d}_0}{\rightleftarrows}} \tilde{H}\mathrm{D}_{n+1} \underset{\mathrm{s}_n,\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_{n+1},\ldots,\mathrm{d}_0}{\rightleftarrows}} H_n \underset{\mathrm{s}_{n-1},\ldots,\mathrm{s}_0}{\overset{\mathrm{d}_n,\ldots,\mathrm{d}_0}{\rightleftarrows}} \cdots \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1,\mathrm{d}_0}{\rightleftarrows}} H_0$$

This functor $\mathrm{Cond}^{\lfloor n,0\rfloor}$ preserves the homotopy groups of H in the positions $k \in \mathbb{Z}_{\geqslant 0}$: we have group isomorphisms $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\pi_k \xrightarrow{\sim} H\pi_k$ for $k \in \mathbb{Z}_{\geqslant 0}$; cf. Lemma 134. These groups are trivial if $k \geqslant n+1$.

Altogether, building on Conduché [**7**], we have the following result.

Proposition ([**7,** Th. 1.5], Theorem 130, Lemma 134). *The functors* $\mathrm{Trunc}^{\lfloor n,0\rfloor}$ *and* $\mathrm{Cond}^{\lfloor n,0\rfloor}$ *are adjoint functors, where* $\mathrm{Trunc}^{\lfloor n,0\rfloor}$ *is the left-adjoint and* $\mathrm{Cond}^{\lfloor n,0\rfloor}$ *is the right-adjoint. Both functors respect homotopy groups in the sense mentioned above.*

$$\mathcal{SimpGrp} \underset{\mathrm{Cond}^{\lfloor n,0\rfloor}}{\overset{\mathrm{Trunc}^{\lfloor n,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$$

In the particular case of $n = 1$, we obtain the adjoint functors $\mathrm{Trunc}^{\lfloor 1,0\rfloor} \dashv \mathrm{Cond}^{\lfloor 1,0\rfloor}$. In this case, the category $\lfloor 1,0\rfloor$-*SimpGrp* is equivalent to the category of crossed modules *CrMod* and to the category of invertible monoidal category *InvMonCat*; cf. §0.2.6 and §0.2.7 below.

0.2.6 Crossed modules and invertible monoidal categories

A *crossed module* $V = (\mathrm{M}\,,\,\mathrm{B}\,,\,\gamma\,,\,\mathrm{f})$ consists of two groups M, B, an action

$$\gamma\colon \mathrm{B} \to \mathrm{Aut}(\mathrm{M})\,,\ b \mapsto \big(m \mapsto m^b := m(b\gamma)\big)$$

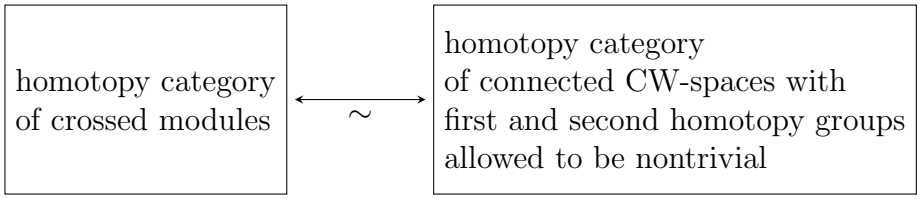
and a group morphism $\mathrm{f}\colon \mathrm{M} \to \mathrm{B}$ that satisfy the following conditions (CM1–2).

(CM1) For $m \in \mathrm{M}$ and $b \in \mathrm{B}$, we have $(m^b)\mathrm{f} = (m\mathrm{f})^b$.

(CM2) For m, $n \in \mathrm{M}$, we have $m^{n\mathrm{f}} = m^n$.

For example, given a group B and a normal subgroup $\mathrm{M} \trianglelefteq \mathrm{B}$, then, with the inclusion map $\mathrm{f}\colon \mathrm{M} \hookrightarrow \mathrm{B}$ and the conjugation action γ of B on M, we get the crossed module $V = (\mathrm{M}\,,\,\mathrm{B}\,,\,\gamma\,,\,\mathrm{f})$.

The notion of a crossed module goes back to J.H.C. Whitehead [**31,** 2.1]. The homotopy category of crossed modules is equivalent to the homotopy category of connected CW-spaces X for which only the first and the second homotopy group are allowed to be nontrivial, i.e. where $X\pi_k \simeq 1$ for $k \geqslant 3$. So crossed modules model homotopy 2-types [**3,** §2.2, p. 41], [**3,** Thm. 2.4.8].



An *invertible monoidal category* is a monoidal category $\mathcal{C}$ together with a unit object I and a tensor product $(\otimes)$ on the objects $\mathrm{Ob}(\mathcal{C})$ and morphisms $\mathrm{Mor}(\mathcal{C})$, where the objects and morphism are invertible with respect to $(\otimes)$. The monoidal structure is understood to be in the strict sense, and so is the inversion.

R. Brown and C.B. Spencer have shown in [**4,** Thm. 1] that the category of crossed modules and the category of invertible monoidal categories are equivalent. They state that this equivalence has been discovered, but not published, by J.-L. Verdier and J. Duskin.

We give an account of this result; cf. Propostion 171.

$$\mathcal{C}r\mathcal{M}od \underset{\text{CM}}{\overset{\text{Cat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{I}nv\mathcal{M}on\mathcal{C}at$$

0.2.7 The equivalent categories of $\lfloor 1, 0 \rfloor$-simplicial groups and crossed modules

Let $n = 1$. We consider $\lfloor 1, 0 \rfloor$-simplicial groups. D. Conduché has shown that the category of $\lfloor 1, 0 \rfloor$-simplicial groups and the category of crossed modules are equivalent [**7,** Cor. 1.8, Cor. 2.7]. We give an account of this equivalence.

Let G be a $\lfloor 1, 0 \rfloor$-simplicial group.

$$G_1 \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1\,,\,\mathrm{d}_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{}}}} G_0$$

We obtain the crossed module $G\hat{\mathrm{N}} = \left(\ker \mathrm{d}_1\,, G_0\,, \boldsymbol{\gamma}\,, \mathrm{d}_0\big|_{\ker \mathrm{d}_1}\right)$ with the action

$$\boldsymbol{\gamma} \colon G_0 \to \mathrm{Aut}(\ker \mathrm{d}_1)\,,\ g \mapsto (m \mapsto m^g = g^- \mathrm{s}_0 \cdot m \cdot g\mathrm{s}_0);$$

cf. Lemma 141. In turn, for a crossed module $V = (\mathrm{M}\,, \mathrm{B}\,, \boldsymbol{\gamma}\,, \mathrm{f})$, we obtain the $\lfloor 1, 0 \rfloor$-simplicial group

$$V\,\mathrm{Rec}\colon \qquad \mathrm{B} \ltimes_{\boldsymbol{\gamma}} \mathrm{M} \underset{\mathrm{s}_0}{\overset{\mathrm{d}_1\,,\,\mathrm{d}_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{}}}} \mathrm{B}\,,$$

where

$$\begin{aligned}
&\mathrm{d}_1\colon \ \mathrm{B} \ltimes \mathrm{M} \to \mathrm{B}\,, \quad (b, m) \mapsto b \\
&\mathrm{d}_0\colon \ \mathrm{B} \ltimes \mathrm{M} \to \mathrm{B}\,, \quad (b, m) \mapsto b \cdot m\mathrm{f} \\
&\mathrm{s}_0\colon \ \mathrm{B} \to \mathrm{B} \ltimes \mathrm{M}\,, \quad b \mapsto (b, 1)\,;
\end{aligned}$$

cf. Lemma 144. These constructions yield mutually inverse equivalences $\hat{\mathrm{N}}$ and Rec between the category of $\lfloor 1, 0 \rfloor$-simplicial groups and the category of crossed modules; cf. Proposition 152.

$$\lfloor 1, 0 \rfloor\text{-}\mathcal{S}imp\mathcal{G}rp \underset{\text{Rec}}{\overset{\hat{\mathrm{N}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{C}r\mathcal{M}od$$

Altogether, the situation in the ordinary case for $n = 1$ can be displayed as follows.

$$\mathcal{SimpGrp} \underset{\mathrm{Cond}^{\lfloor 1,0\rfloor}}{\overset{\mathrm{Trunc}^{\lfloor 1,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor 1,0\rfloor\text{-}\mathcal{SimpGrp} \underset{\mathrm{Rec}}{\overset{\hat{\mathrm{N}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{CrMod} \underset{\mathrm{CM}}{\overset{\mathrm{Cat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{InvMonCat}$$

An alternative construction has been provided by M. Bullejos and A.M. Cegarra [**6,** p. 268] as follows. We consider the full subcategory $\mathcal{SimpGrp}^{\lfloor 1,0\rfloor} \subseteq \mathcal{SimpGrp}$, where

$$\mathrm{Ob}(\mathcal{SimpGrp}^{\lfloor 1,0\rfloor}) = \{G \in \mathrm{Ob}(\mathcal{SimpGrp})\colon G\mathrm{N}_i = 1 \text{ for } i \in \mathbb{Z}_{\geqslant 2}\}\,.$$

Then, they give an equivalence

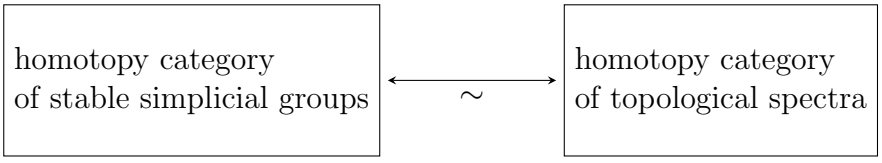
$$\mathcal{SimpGrp}^{\lfloor 1,0\rfloor} \underset{\mathcal{J}}{\overset{\mathcal{N}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{CrMod}\,,$$

such that, for $G \in \mathrm{Ob}(\mathcal{SimpGrp}^{\lfloor 1,0\rfloor})$, we have $G\mathcal{N} = (G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor})\hat{\mathrm{N}}$. The construction of $\mathcal{J}$ uses a subgroup of $(\mathrm{B} \ltimes_\gamma \mathrm{M})^{\times 2}$ and the coskeleton functor $\mathrm{Cosk}^{\lfloor 2,0\rfloor}$.

0.3 The stable case

0.3.1 Stable simplicial groups

In 1963, D. Kan introduced stable simplicial groups under the name "group spectra" [**18,** Def. 5.1]. He shows that the homotopy category of topological spectra and the homotopy category of group spectra are equivalent, expressed as equivalences of homotopy theories [**18,** Prop. 3.7, Prop. 4.7, Cor. 5.6].



In Kan's approach, a group spectrum consists of groups, indexed by $\mathbb{Z}$, and face maps and degeneracy maps between these groups, indexed by $\mathbb{Z}_{\geqslant 0}$, satisfying the usual simplicial relations, such that each element of each group satisfies a finiteness condition [**18,** Def. 4.1].

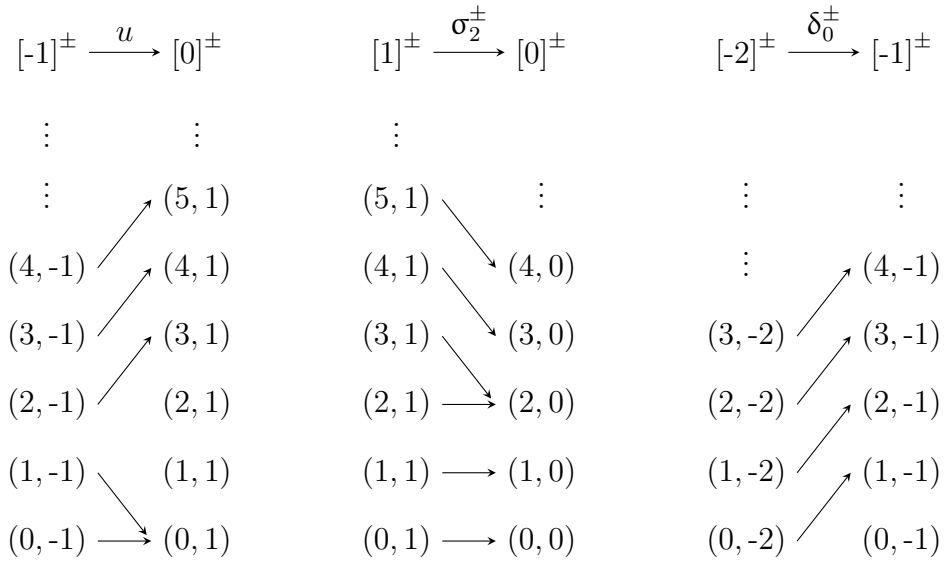
In 2015, M. Stephan establishes a model structure on the category of group spectra **[26,** Thm. 3.12].

Previously, in 2009, F. Weilandt has established a model structure on the category of abelian group spectra **[30]**. In this process, he considers the category $\tilde{\mathbb{Z}}$ whose objects are named $[k]$ for $k \in \mathbb{Z}$, and whose morphisms from $[n]$ to $[k]$, where $n,\ k \in \mathbb{Z}$, are monotone maps $\alpha\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ that satisfy $(i)\alpha = k - n + i$ for almost all $i \in \mathbb{Z}_{\geqslant 0}$ **[30,** Def. 1.1.4]. He shows that Kan's group spectra coincide with contravariant functors from $\tilde{\mathbb{Z}}$ to the category of groups that satisfy a finiteness condition **[30,** Lem. 1.1.7].

We use Weilandt's approach, in slightly different terms, to define stable simplicial groups. We consider the prestable simplex category $\Delta^{\pm}$ whose objects are $[k]^{\pm} = \mathbb{Z}_{\geqslant 0} \times \{k\}$ for $k \in \mathbb{Z}$ and whose morphisms are monotone maps between those objects that satisfy the following condition.

Let $n,\ k \in \mathbb{Z}$. We say that a map $u\colon [n]^{\pm} \to [k]^{\pm}$ is *eventually parallel*, if there exists some $x_0 \in \mathbb{Z}_{\geqslant 0}$ such that $(x, n)u = (x - n + k, k)$ for $x \geqslant x_0$; cf. Definition 172.

We give some examples for morphisms in $\Delta^{\pm}$, i.e. for monotone and eventually parallel maps:



We may consider $\Delta^{\pm}$ as an extension of the simplex category Δ; cf. Lemma 191.

A functor $G\colon (\Delta^{\pm})^{\mathrm{op}} \to \mathcal{G}rp$ is called a *prestable simplicial group*; cf. Definition 182. We write $\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp$ for the category of prestable simplicial groups.

A functor $G\colon (\Delta^{\pm})^{\mathrm{op}} \to \mathcal{G}rp$ that satisfies the following finiteness condition (1) is called a

stable simplicial group; cf. Definition 205.

(1) Let $k \in \mathbb{Z}$. For $g \in G_k$, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_j = 1$ for $j \geqslant \ell$.

We write $\mathcal{StSimpGrp} \subseteq \mathcal{PreStSimpGrp}$ for the full subcategory of stable simplicial groups.

A stable simplicial group G has *face maps* $\mathrm{d}_j^{G,k} := (\delta_j^{\pm})G\colon G_k \to G_{k-1}$ and *degeneracy maps* $\mathrm{s}_i^{G,k} := (\sigma_i^{\pm})G\colon G_k \to G_{k+1}$ for $i,\, j \geqslant 0$.

We may also regard a stable simplicial group G as a list of groups G_k for $k \in \mathbb{Z}$ together with group morphisms $\mathrm{d}_j := \mathrm{d}_j^{G,k}\colon G_k \to G_{k-1}$ (face maps) and $\mathrm{s}_i := \mathrm{s}_i^{G,k}\colon G_k \to G_{k+1}$ (degeneracy maps) between those groups, where $i,\, j \geqslant 0$, such that the usual simplicial relations and the finiteness condition (1) are satisfied. This is the approach that Kan used.

$$\cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_k \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_1 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_{-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

So, compared to an ordinary simplicial group, a stable simplicial group has groups G_k with negative indices, it has infinitely many face maps d_j and degeneracy maps s_i between two groups, and it has to satisfy the additional finiteness condition (1).

For a stable simplicial group G, its Moore complex $G\mathrm{N}$ is given by the groups

$$G\mathrm{N}_k = \bigcap_{i \geqslant 1} \ker \mathrm{d}_i \trianglelefteqslant G_k$$

and the differentials $\partial_k = \mathrm{d}_0\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}}$ for $k \in \mathbb{Z}$.

$$G\mathrm{N} = \left(\ldots \xrightarrow{\partial_4} G\mathrm{N}_3 \xrightarrow{\partial_3} G\mathrm{N}_2 \xrightarrow{\partial_2} G\mathrm{N}_1 \xrightarrow{\partial_1} G\mathrm{N}_0 \xrightarrow{\partial_0} G\mathrm{N}_{-1} \xrightarrow{\partial_{-1}} \ldots\right)$$

Then, for $k \in \mathbb{Z}$, the k-th homotopy group of G is defined as the k-th homology group of $G\mathrm{N}$:

$$G\pi_k := G\mathrm{NH}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}\,.$$

For a prestable simplicial group G, we have the prestable simplicial subgroup $G^{\mathrm{fin}} \leqslant G$, where $G_k^{\mathrm{fin}} = \{g \in G_k\colon g \text{ is finite}\}$, for $k \in \mathbb{Z}$. In fact, G^{fin} is a stable simplicial group.

Then the embedding functor $\mathrm{I}\colon \mathcal{StSimpGrp} \to \mathcal{PreStSimpGrp}$ has a right-adjoint

$$\Phi\colon \mathcal{PreStSimpGrp} \to \mathcal{StSimpGrp}\,,\ \left(G \xrightarrow{\varphi} \tilde{G}\right) \mapsto \left(G^{\mathrm{fin}} \xrightarrow{\varphi^{\mathrm{fin}}} \tilde{G}^{\mathrm{fin}}\right),$$

where φ^{fin} is the restriction of φ to G^{fin} and $\tilde{G}^{\mathrm{fin}}$.

$$\mathit{StSimpGrp} \underset{\Phi}{\overset{\mathrm{I}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathit{PreStSimpGrp}$$

0.3.2 $\lfloor n, -\infty \rfloor$-stable simplicial groups and $\lfloor n, 0 \rfloor$-stable simplicial groups

Let $n \in \mathbb{Z}$. We consider the full subcategory $\Delta^{\pm}_{\lfloor n,-\infty\rfloor} \subseteq \Delta^{\pm}$, where the objects in $\Delta^{\pm}_{\lfloor n,-\infty\rfloor}$ are given by $[k]^{\pm}$ for $k \in \mathbb{Z}_{\leqslant n}$.

A functor $H\colon (\Delta_{\lfloor n,-\infty\rfloor})^{\mathrm{op}} \to \mathit{Grp}$ is called a *pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group*; cf. Definition 226. We write Pre-$\lfloor n, -\infty \rfloor$-$\mathit{PreStSimpGrp}$ for the category of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups.

A functor $H\colon (\Delta_{\lfloor n,-\infty\rfloor})^{\mathrm{op}} \to \mathit{Grp}$ that satisfies the following finiteness condition (1) and the Conduché condition (2) is called an *$\lfloor n, -\infty \rfloor$-stable simplicial group*; cf. Definition 232.

(1) Let $k \in \mathbb{Z}_{\leqslant n}$. For $h \in H_k$, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $h\mathrm{d}_j = 1$ for $j \geqslant \ell$.

(2) For I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$, we have the trivial commutator

$$\Big[\bigcap_{i\in I} \ker \mathrm{d}_i\,, \bigcap_{j\in J} \ker \mathrm{d}_j\Big] = 1 \;\leqslant\; H_n\,.$$

We write $\lfloor n, -\infty \rfloor$-$\mathit{StSimpGrp}$ for the category of $\lfloor n, -\infty \rfloor$-stable simplicial groups. Then $\lfloor n, -\infty \rfloor$-$\mathit{StSimpGrp} \subseteq \mathit{Pre}$-$\lfloor n, -\infty \rfloor$-$\mathit{PreStSimpGrp}$ is a full subcategory.

We may obtain an $\lfloor n, -\infty \rfloor$-stable simplicial group H by listing groups H_k for $k \in \mathbb{Z}_{\leqslant n}$ and group morphisms $\mathrm{d}_j\colon H_k \to H_{k-1}$ and $\mathrm{s}_i\colon H_k \to H_{k+1}$ such that the simplicial relations, the finiteness condition (1) and the Conduché condition (2) are satisfied.

$$H_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_1 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_{-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

Moreover, we say that $H \in \mathrm{Ob}(\lfloor n, -\infty \rfloor$-$\mathit{StSimpGrp})$ is an *$\lfloor n, 0 \rfloor$-stable simplicial group*, if we have $H_k = 1$ for $k \in \mathbb{Z}_{\leqslant -1}$.

$$H_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_1 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_0 \rightleftarrows 1 \rightleftarrows \cdots$$

Then, the category $\lfloor n,0\rfloor$-$\mathcal{StSimpGrp}$ of $\lfloor n,0\rfloor$-stable simplicial groups is a full subcategory of $\lfloor n,-\infty\rfloor$-$\mathcal{StSimpGrp}$. In particular, we have the inclusion functor

$$\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} \to \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\,;$$

cf. Remark 312. For the homotopy groups of an $\lfloor n,0\rfloor$-stable simplicial group H, we have, of course, $H\pi_k = (H\,\mathrm{Inc}^{\pm,\lfloor n,0\rfloor})\pi_k$ for $k \in \mathbb{Z}$.

This inclusion functor has a right-adjoint functor

$$\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \to \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp},$$

where, for $H \in \mathrm{Ob}(\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp})$, we have $H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor} \in \mathrm{Ob}(\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp})$ defined as an $\lfloor n,-\infty\rfloor$-stable simplicial subgroup of H by

$$(H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor})_k := \begin{cases} \bigcap\limits_{\substack{a\in\Delta^{\pm}([-1]^{\pm},[k]^{\pm})\\ a \text{ is injective}}} \ker H_a & \text{for } k \in [0,n] \\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1}\,; \end{cases}$$

cf. Remark 314. This functor also preserves the homotopy groups of H in the positions $k \in [0,n]$: we have $(H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor})\pi_k = H\pi_k$ for $k \in [0,n]$; cf. Lemma 317.

Proposition (Remark 313, Lemmas 317, 320). *The functors* $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}$ *and* $\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$ *are adjoint functors, where* $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}$ *is the left-adjoint and* $\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$ *is the right-adjoint. Both functors preserve homotopy groups in the sense mentioned above.*

$$\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}}{\overset{\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \;\subseteq\; \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}$$

In the following, we shall use both $\lfloor n,-\infty\rfloor$-$\mathcal{StSimpGrp}$ and $\lfloor n,0\rfloor$-$\mathcal{StSimpGrp}$.

We have the pair of adjoint functors $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ between $\mathcal{StSimpGrp}$ and $\lfloor n,-\infty\rfloor$-$\mathcal{StSimpGrp}$, similar to the pair of adjoint functors $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$ between $\mathcal{SimpGrp}$ and $\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$ in the ordinary case; cf. §0.2.5 and §0.3.5 below.

There is no analogue to the pair of adjoint functors $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor} \dashv \mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$ between $\lfloor n,-\infty\rfloor$-$\mathcal{StSimpGrp}$ and $\lfloor n,0\rfloor$-$\mathcal{StSimpGrp}$ in the ordinary case, however.

0.3.3 The functors $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$ and $\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$

Let $n \in \mathbb{Z}$. We aim to construct a pair of adjoint functors between the category of stable simplicial groups *StSimpGrp* and the category of $\lfloor n,-\infty\rfloor$-stable simplicial groups $\lfloor n,-\infty\rfloor$-*StSimpGrp* that preserve homotopy groups; cf. §0.3.5 below. To that end, we proceed in a similar fashion to the ordinary case.

To begin with, we shall construct a pair of adjoint functors $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$ between the category of stable simplicial groups *StSimpGrp* and the category of pre-$\lfloor n,-\infty\rfloor$-prestable simplicial groups *Pre*-$\lfloor n,-\infty\rfloor$-*PreStSimpGrp*.

As an intermediate step, we consider the pair of adjoint functors $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}$ between *PreStSimpGrp* and *Pre*-$\lfloor n,-\infty\rfloor$-*PreStSimpGrp* that are similar in their construction to $\mathrm{Cut}^{\lfloor n,0\rfloor}$ and $\mathrm{Cosk}^{\lfloor n,0\rfloor}$ in the ordinary case; cf. §0.2.3.

$$\mathit{PreStSimpGrp} \underset{\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}}{\overset{\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathit{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathit{PreStSimpGrp}$$

From a prestable simplicial group G we may obtain a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group $G\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}$ by restricting the functor G from $(\Delta^{\pm})^{\mathrm{op}}$ to $(\Delta^{\pm}_{\lfloor n,-\infty\rfloor})^{\mathrm{op}}$. That is, for $k \in \mathbb{Z}_{\leqslant n}$, we let

$$(G\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor})_k := G_k\,.$$

We obtain a functor $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} \colon \mathit{PreStSimpGrp} \to \mathit{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathit{PreStSimpGrp}$; cf. Remark 260.

$$G: \quad \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_{n+1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

$$G\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}: \quad G_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

We construct the functor $\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor} \colon \mathit{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathit{PreStSimpGrp} \to \mathit{PreStSimpGrp}$ that is right-adjoint to $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}$ using coherent tuples, defined similarly to the ordinary case; cf. §0.2.3 and Lemmas 264, 270.

For a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group H and for $k \in \mathbb{Z}$ we let

$$(H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k := \Big\{ (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in \prod_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} H_n \colon (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \text{ is coherent } \Big\}$$
$$\leqslant \prod_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} H_n\,.$$

For $k \in \mathbb{Z}_{\leqslant n}$, we note that we have a group isomorphism $(H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k \xrightarrow{\sim} H_k$; cf. Lemma 268.(1).

$$H : \qquad H_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

$H\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}$:

$$\cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} (H\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_{n+1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} (H\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} (H\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

$$\simeq \qquad \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} (H\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_{n+1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

Then $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}$. Moreover, we consider the pair of adjoint functors $\mathrm{I} \dashv \Phi$; cf. §0.3.1.

$$\mathcal{StSimpGrp} \underset{\Phi}{\overset{\mathrm{I}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{PreStSimpGrp} \underset{\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}}{\overset{\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}$$

We let

$$\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} := \mathrm{I} * \mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} \colon \mathcal{StSimpGrp} \to \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}$$
$$\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} := \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor} * \Phi \colon \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp} \to \mathcal{StSimpGrp}\,.$$

Then we have the pair of adjoint functors $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$.

$$\mathcal{StSimpGrp} \underset{\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}}{\overset{\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}$$

0.3.4 The degenerate subgroup $G\mathrm{D}_{n+1}$ of an $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group G

Let $n \in \mathbb{Z}$. We shall introduce the notion of an $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group in order to define the degenerate subgroup

$$(H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1} \leqslant (H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_{n+1}$$

that is needed for the construction of the functor $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$; cf. §0.3.5 below.

We say that a stable simplicial group G is $\lfloor n, -\infty \rfloor$-*coskeletal* if it satisfies the following properties (1) and (2); cf. Definition 275.

(1) We have $\bigcap\limits_{k\geqslant 0} \ker \mathrm{d}_k = 1 \leqslant G_{n+1}$.
(2) For $I,\, J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$, we have $\big[\bigcap\limits_{i\in I} \ker \mathrm{d}_i\,,\, \bigcap\limits_{j\in J} \ker \mathrm{d}_j\big] = 1 \leqslant G_n$.

Then, as intended, for a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group H, the stable simplicial group
$H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ is $\lfloor n, -\infty \rfloor$-coskeletal; cf. Remark 277.

Let G be an $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group. We give an account of the construction of the degenerate subgroup $G\mathrm{D}_{n+1}$. We may decompose the group G_{n+1} as a product of subgroups in the form

$$\begin{aligned} G_{n+1} &= \prod_{i\in\rfloor\infty,0\rfloor} \big(\bigcap_{j\geqslant i+1} \ker \mathrm{d}_j\big)\mathrm{s}_i \cdot G\mathrm{N}_{n+1} \\ &= \quad \ldots \cdot \big(\bigcap_{j\geqslant 3} \ker \mathrm{d}_j\big)\mathrm{s}_2 \cdot \big(\bigcap_{j\geqslant 2} \ker \mathrm{d}_j\big)\mathrm{s}_1 \cdot \big(\bigcap_{j\geqslant 1} \ker \mathrm{d}_j\big)\mathrm{s}_0 \cdot G\mathrm{N}_{n+1}\,; \end{aligned}$$

cf. §13.2. We consider the subset

$$G\mathrm{D}_{n+1} := \prod_{i\in\rfloor\infty,0\rfloor} \big(\bigcap_{j\geqslant i+1} \ker \mathrm{d}_j\big)\mathrm{s}_i = \quad \ldots \cdot \big(\bigcap_{j\geqslant 3} \ker \mathrm{d}_j\big)\mathrm{s}_2 \cdot \big(\bigcap_{j\geqslant 2} \ker \mathrm{d}_j\big)\mathrm{s}_1 \cdot \big(\bigcap_{j\geqslant 1} \ker \mathrm{d}_j\big)\mathrm{s}_0 \;\subseteq\; G_{n+1}$$

that we obtain by omitting the Moore complex term $G\mathrm{N}_{n+1}$ in the decomposition of G_{n+1} given above. In order to show that $G\mathrm{D}_{n+1}$ is a subgroup of G_{n+1}, we use the décalage functor $\mathrm{Dec}^{\pm}$ and the ordinarification functor Ord to relate it back to the ordinary case; cf. Lemma 287.

xxx

We call $G\mathrm{D}_{n+1}$ the *degenerate subgroup* of G_{n+1}.

For example, for $n = 1$, we have

$$\begin{aligned} G_2 &= \ldots \cdot \big(\bigcap_{j\geqslant 3} \ker \mathrm{d}_j\big)\mathrm{s}_2 \cdot \big(\bigcap_{j\geqslant 2} \ker \mathrm{d}_j\big)\mathrm{s}_1 \cdot \big(\bigcap_{j\geqslant 1} \ker \mathrm{d}_j\big)\mathrm{s}_0 \cdot G\mathrm{N}_2 \\ G\mathrm{D}_2 &= \ldots \cdot \big(\bigcap_{j\geqslant 3} \ker \mathrm{d}_j\big)\mathrm{s}_2 \cdot \big(\bigcap_{j\geqslant 2} \ker \mathrm{d}_j\big)\mathrm{s}_1 \cdot \big(\bigcap_{j\geqslant 1} \ker \mathrm{d}_j\big)\mathrm{s}_0 \qquad \leqslant G_2 \,. \end{aligned}$$

For $n = 2$, we have

$$\begin{aligned} G_3 &= \ldots \cdot \big(\bigcap_{j\geqslant 3} \ker \mathrm{d}_j\big)\mathrm{s}_2 \cdot \big(\bigcap_{j\geqslant 2} \ker \mathrm{d}_j\big)\mathrm{s}_1 \cdot \big(\bigcap_{j\geqslant 1} \ker \mathrm{d}_j\big)\mathrm{s}_0 \cdot G\mathrm{N}_3 \\ G\mathrm{D}_3 &= \ldots \cdot \big(\bigcap_{j\geqslant 3} \ker \mathrm{d}_j\big)\mathrm{s}_2 \cdot \big(\bigcap_{j\geqslant 2} \ker \mathrm{d}_j\big)\mathrm{s}_1 \cdot \big(\bigcap_{j\geqslant 1} \ker \mathrm{d}_j\big)\mathrm{s}_0 \qquad \leqslant G_3 \,. \end{aligned}$$

In particular, for an $\lfloor n, -\infty\rfloor$-stable simplicial group H, we have the degenerate subgroup

$$(H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1} \;\leqslant\; (H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_{n+1}\,.$$

0.3.5 The functors $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ and $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$

Let $n \in \mathbb{Z}$. We have the pair of adjoint functors $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ between $\mathcal{StSimpGrp}$ and $\mathcal{P}re$-$\lfloor n, -\infty\rfloor$-$\mathcal{P}re\mathcal{StSimpGrp}$; cf. §0.3.3.

Based on $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ and $\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$, and using the functors $\mathrm{Trunc}^{\lfloor n,0\rfloor}$ and $\mathrm{Cond}^{\lfloor n,0\rfloor}$ from the ordinary case as a blueprint, we shall construct the pair of adjoint functors $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ between $\mathcal{StSimpGrp}$ and $\lfloor n, -\infty\rfloor$-$\mathcal{StSimpGrp}$.

For a stable simplicial group G, let $G\mathrm{NB}_n := (G\mathrm{N}_{n+1})\mathrm{d}_0 \trianglelefteqslant G_n$. We have the $\lfloor n, -\infty\rfloor$-simplicial group $G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$, where

$$(G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor})_k = \begin{cases} G_n/G\mathrm{NB}_n & \text{for } k = n \\ G_k & \text{for } k \in \mathbb{Z}_{\leqslant n-1}\,. \end{cases}$$

This yields the functor $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}\colon \mathcal{StSimpGrp} \to \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$.

$$G: \qquad \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_{n+1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

$$G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}: \qquad G_n/G\mathrm{NB}_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

This functor $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ has also similar properties to the truncation functor from the ordinary case. Namely, it preserves the homotopy groups of G in the positions $k \in \mathbb{Z}_{\leqslant n}$, i.e. we have group isomorphisms $(G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor})\pi_k \xrightarrow{\sim} G\pi_k$ for $k \in \mathbb{Z}_{\leqslant n}$; cf. Lemma 293.
Moreover, $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ has a right-adjoint functor

$$\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}\colon \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \to \mathcal{StSimpGrp}$$

whose construction proceeds as follows.
Let H be a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group. We consider the stable simplicial group $\tilde{H} := H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$ and the degenerate subgroup

$$\tilde{H}\mathrm{D}_{n+1} \;=\; (H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}})\mathrm{D}_{n+1} \;\leqslant\; (H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}})_{n+1} \;=\; \tilde{H}_{n+1}\,;$$

cf. Lemma 287. Then we have the stable simplicial subgroup

$$H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor} \;\leqslant\; \tilde{H} \;=\; H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,,$$

where

$$(H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k := \begin{cases} \tilde{H}_k & \text{for } k \in \mathbb{Z}_{\leqslant n} \\ \tilde{H}\mathrm{D}_{n+1} & \text{for } k = n+1 \\ \bigcap\limits_{\substack{a \in \Delta^{\pm}([n+1]^{\pm},[k]^{\pm}) \\ a \text{ is injective}}} \tilde{H}_a^{-1}(\tilde{H}\mathrm{D}_{n+1}) & \text{for } k \in \mathbb{Z}_{\geqslant n+2}\,. \end{cases}$$

Note that $\tilde{H}_k \simeq H_k$ for $k \leqslant n$.
This yields the functor $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}\colon \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \to \mathcal{StSimpGrp}$.

$$H: \qquad H_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

$$H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}: \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \bigcap_a \tilde{H}_a^{-1}(\tilde{H}\mathrm{D}_{n+1}) \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \tilde{H}\mathrm{D}_{n+1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \tilde{H}_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \tilde{H}_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

$$\simeq \qquad \cdots \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \bigcap_a \tilde{H}_a^{-1}(\tilde{H}\mathrm{D}_{n+1}) \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \tilde{H}\mathrm{D}_{n+1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_n \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} H_{n-1} \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\rightleftarrows}} \cdots$$

Also, this functor $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ preserves the homotopy groups in the positions $k \in \mathbb{Z}$, i.e. we have group isomorphisms $(H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})\pi_k \xrightarrow{\sim} H\pi_k$ for $k \in \mathbb{Z}$. These groups are trivial if $k \geqslant n+1$.

So we obtain the following result.

Proposition (Theorem 302, Lemma 306). *The functors* $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ *and* $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ *are adjoint functors, where* $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ *is the left-adjoint and* $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ *is the right-adjoint. Both functors respect homotopy groups in the sense mentioned above.*

$$\mathcal{StSimpGrp} \underset{\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}}{\overset{\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$$

Altogether, for $n \in \mathbb{Z}$, we have the following situation.

$$\mathcal{StSimpGrp} \underset{\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}}{\overset{\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}}{\overset{\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}$$

0.3.6 Stable crossed modules and symmetric monoidal categories

We want to consider the relations between stable crossed modules, 2-crossed modules and symmetric monoidal categories.

D. Conduché has introduced the notion of a stable crossed module [**7,** Déf. 3.1]. A *stable crossed module* $S = (\mathrm{M}, \mathrm{B}, \kappa, \mathrm{f})$ consists of two groups M and B, a group morphism $\mathrm{f}\colon \mathrm{M} \to \mathrm{B}$ and a map $\kappa\colon \mathrm{B} \times \mathrm{B} \to \mathrm{M}$, $(b, c) \mapsto (b, c)\kappa =: \llbracket b, c \rrbracket$ that satisfy the following conditions (SCM1–4).

(SCM1) For $b, c \in \mathrm{B}$, we have $\llbracket b, c \rrbracket = \llbracket c, b \rrbracket^{-}$.

(SCM2) For $b, c \in \mathrm{B}$, we have $\llbracket b, c \rrbracket \mathrm{f} = [b, c]$.

(SCM3) For $m, n \in \mathrm{M}$, we have $\llbracket m\mathrm{f}, n\mathrm{f} \rrbracket = [m, n]$.

(SCM4) For $a, b, c \in \mathrm{B}$, we have $\llbracket a, b \cdot c \rrbracket = \llbracket a, b \rrbracket \cdot \llbracket a^b, c \rrbracket$.

The *Conduché bracket* $\llbracket -, = \rrbracket$ plays the role of a commutator of elements in B with values in M, so to speak.

A stable crossed module S is also a crossed module, using the action $m^b := m \cdot \llbracket m\mathrm{f}, b \rrbracket \in \mathrm{M}$, where $m \in \mathrm{M}$, $b \in \mathrm{B}$; cf. Lemma 327 and §0.3.8 below.

There is a connection to 2-*crossed modules*, as introduced by Conduché. A 2-crossed module consists of three groups L, M, N, group morphisms $L \to M \to N$, actions of N on M and L, and a bracket $\{-, =\}\colon M \times M \to L$ that satisfy a list of conditions [**7,** Déf. 2.2, (2.9)].

If we additionally impose the symmetry condition

$$\{b, c\}\{c, b\} = 1 \quad \text{for } b, c \in M$$

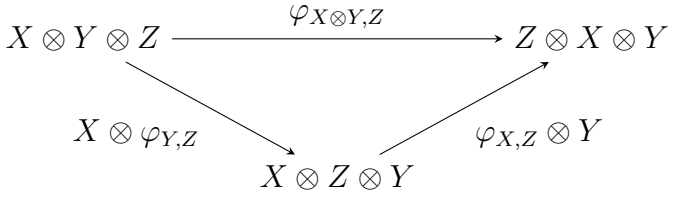
on the bracket, then a 2-crossed module $L \to M \to 1$ is a stable crossed module [**7,** (2.9),(3.9)].

By a *symmetric monoidal category* we understand an invertible monoidal category $\mathcal{C}$ together with a symmetric braiding $\varphi = (\varphi_{X,Y})_{X,Y \in \mathrm{Ob}(\mathcal{C})}$, where $\varphi_{X,Y}\colon X \otimes Y \xrightarrow{\sim} Y \otimes X$ is an isomorphism, natural in X and Y, that satisfies the following conditions (SB1–3).

(SB1) For $X, Y \in \mathrm{Ob}(\mathcal{C})$, we have $\varphi_{X,Y} \blacktriangle \varphi_{Y,X} = \mathrm{id}_{X \otimes Y}$.

(SB2) We have $\varphi_{I,I} = \mathrm{id}_I$.

(SB3) For $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, we have $\varphi_{X \otimes Y, Z} = (X \otimes \varphi_{Y,Z}) \blacktriangle (\varphi_{X,Z} \otimes Y)$.

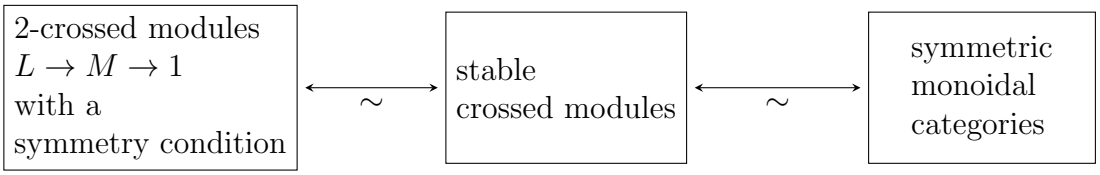


We have a correspondence between stable crossed modules and symmetric monoidal categories. This fact was already known to M. Bullejos, P. Carrasco, A.M. Cegarra [**5,** p. 164, l. 24–28].

We carry out the details in §19.4: We show that the category of stable crossed modules and the category of symmetric monoidal categories are equivalent; cf. Proposition 424.

$$\mathcal{StCrMod} \underset{\text{StCM}}{\overset{\text{SymCat}}{\underset{\sim}{\rightleftarrows}}} \mathcal{SymMonCat}$$

Altogether, we have the following correspondence.



A. Joyal and R. Street [**15,** Rem. 3.1] show that a crossed module $V = (\mathrm{M}, \mathrm{B}, \boldsymbol{\gamma}, \mathrm{f})$ together with a bracket $\{-, =\}\colon \mathrm{B} \times \mathrm{B} \to \mathrm{M}$ that satisfies certain conditions corresponds to an invertible monoidal category $\mathcal{C}$ with a braiding $(\varphi_{X,Y})_{X,Y\in \mathrm{Ob}(\mathcal{C})}$ that satisfies (SB2–3).

An earlier variant of this construction can be found in [**14,** Section 2, Ex. 4].

A version of such a crossed module with a bracket can also be obtained from a 2-crossed module $L \to M \to N$ by letting $N = 1$ [**7,** Déf. 2.2, (2.9)]. It also appears under the name *reduced* 2*-crossed module* in [**2,** §6].

0.3.7 The equivalent categories of $\lfloor 1, 0 \rfloor$-stable simplicial groups and stable crossed modules

Let $n = 1$. Then the category of $\lfloor 1, 0 \rfloor$-stable simplicial groups is equivalent to the category of stable crossed modules. We shall sketch a construction of this equivalence.

Let G be a $\lfloor 1,0 \rfloor$-stable simplicial group.

$$G_1 \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\longrightarrow\!\!\!\!\longleftarrow}} G_0 \underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow\!\!\!\!\longleftarrow}} 1 \underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow\!\!\!\!\longleftarrow}} 1 \underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow\!\!\!\!\longleftarrow}} \cdots$$

Recall that $G\mathrm{N}_1 = \bigcap_{j \geqslant 1} \ker \mathrm{d}_j \trianglelefteqslant G_1$. We obtain the stable crossed module

$$G\hat{\mathrm{N}}^{\pm} = \big(G\mathrm{N}_1\,, G_0\,, \kappa,\, \mathrm{d}_0\big|_{G\mathrm{N}_1}\big)$$

together with the Conduché bracket

$$\kappa\colon G_0 \times G_0 \to G\mathrm{N}_1\,,\ (b,c) \mapsto [\![b,\, c]\!] = (b^{-}\mathrm{s}_0)^{c\mathrm{s}_1} \cdot (b\mathrm{s}_0)^{c\mathrm{s}_0}\,;$$

cf. Remark 395. This yields the functor $\hat{\mathrm{N}}^{\pm}\colon \lfloor 1,0 \rfloor\text{-}\mathcal{StSimpGrp} \to \mathcal{StCrMod}$; cf. Lemma 397.

In turn, for a stable crossed module $S = (\mathrm{M}\,, \mathrm{B}\,, \kappa\,, \mathrm{f})$, we obtain the $\lfloor 1,0 \rfloor$-stable simplicial group $S\,\mathrm{Rec}^{\pm}$ as follows.

$$S\,\mathrm{Rec}^{\pm}: \quad H_\infty \underset{\mathrm{s}_i}{\overset{\mathrm{d}_j}{\longrightarrow\!\!\!\!\longleftarrow}} \mathrm{B} \underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow\!\!\!\!\longleftarrow}} 1 \underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow\!\!\!\!\longleftarrow}} 1 \underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow\!\!\!\!\longleftarrow}} \cdots$$

Here, the group H_∞ may be regarded as an infinitely iterated semidirect product

$$H_\infty = \quad \ldots \ltimes \mathrm{B} \ltimes \mathrm{B} \ltimes \mathrm{B} \ltimes \mathrm{M}\,.$$

More precisely, the group H_∞ is constructed as follows. First, we construct the $(n+1)$-fold iterated semidirect product

$$H_n = \underbrace{\mathrm{B} \ltimes \mathrm{B} \ltimes \ldots \ltimes \mathrm{B}}_{(n+1)\text{-times}} \ltimes \mathrm{M}\,,$$

using the Conduché bracket of S; cf. §17.1.1.

We consider the colimit $\varinjlim_{n} H_n$ of the systems of groups $\big((H_n)_{n \in \mathbb{Z}_{\geqslant -1}}\,, (\iota_{i,j})_{-1 \leqslant i \leqslant j}\big)$, where $\iota_{i,j}\colon H_i \to H_j$ is the embedding morphism.

We consider the set

$$\begin{aligned} H_\infty \;&:=\; \Big\{(\ldots, b_3\,, b_2\,, b_1\,, b_0\,, m) \in \big(\prod_{i \in \mathbb{Z}_{\geqslant 0}} \mathrm{B}\big) \times \mathrm{M}\colon \exists k \in \mathbb{Z}_{\geqslant 0} \text{ with } b_\ell = 1 \text{ for } \ell \geqslant k\Big\} \\ &\subseteq\; \big(\prod_{i \in \mathbb{Z}_{\geqslant 0}} \mathrm{B}\big) \times \mathrm{M} = \quad \ldots \times \mathrm{B} \times \mathrm{B} \times \mathrm{B} \times \mathrm{M}\,. \end{aligned}$$

We have a bijection $\varinjlim_n H_n \xrightarrow{\sim} H_\infty$. Then, by transport of structure along this bijection, H_∞ becomes a group.

The formula for the multiplication in H_∞ is given in Proposition 361.

Moreover, we have the group morphisms

$$\begin{aligned}
\mathrm{d}_0\colon &\quad H_\infty \to \mathrm{B}\,, \quad (\ldots, b_2\,, b_1\,, b_0\,, m) \mapsto b_0 \cdot m\mathrm{f} \\
\mathrm{d}_j\colon &\quad H_\infty \to \mathrm{B}\,, \quad (\ldots, \underbrace{b_j}_{\text{pos. } j}\,, \underbrace{b_{j-1}}_{\text{pos. } j\text{-}1}\,, \ldots, b_2\,, b_1\,, \underbrace{b_0}_{\text{pos. } 0}\,, \underbrace{m}_{\text{pos. -}1}) \mapsto b_j \cdot b_{j-1} \\
\mathrm{s}_i\colon &\quad \mathrm{B} \to H_\infty\,, \quad b \mapsto (\ldots, 1\,, \underbrace{b}_{\text{pos. } i}\,, 1\,, \ldots, \underbrace{1}_{\text{pos. } 0}\,, \underbrace{1}_{\text{pos. -}1})\,,
\end{aligned}$$

for $j \in \mathbb{Z}_{\geqslant 1}$ and $i \in \mathbb{Z}_{\geqslant 0}$; cf. Remark 398. This yields the functor

$$\mathrm{Rec}^{\pm}\colon \mathcal{StCrMod} \to \lfloor 1, 0\rfloor\text{-}\mathcal{StSimpGrp};$$

cf. Lemma 400. We obtain the following proposition.

Proposition (Proposition 405). *We have an equivalence between the category of $\lfloor 1, 0\rfloor$-stable simplicial groups and the category of stable crossed modules via the functors $\hat{\mathrm{N}}^{\pm}$ and $\mathrm{Rec}^{\pm}$.*

$$\lfloor 1, 0\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Rec}^{\pm}}{\overset{\hat{\mathrm{N}}^{\pm}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{StCrMod}$$

Altogether, for $n = 1$, the situation in the stable case can be displayed as follows.

$$\mathcal{StSimpGrp} \underset{\mathrm{Cond}^{\pm,\lfloor 1,-\infty\rfloor}}{\overset{\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor 1, -\infty\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Inc}^{\pm,\lfloor 1,0\rfloor}}{\overset{\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \lfloor 1, 0\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Rec}^{\pm}}{\overset{\hat{\mathrm{N}}^{\pm}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{StCrMod} \underset{\mathrm{StCM}}{\overset{\mathrm{SymCat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{SymMonCat}$$

0.3.8 The ordinarification functors

We consider the categories $\mathcal{StSimpGrp}$, $\mathcal{StCrMod}$ and $\mathcal{SymMonCat}$. Recall that we have the following functors between these categories; cf. §0.3.6, §0.3.7.

$$\mathcal{StSimpGrp} \xrightarrow{\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor} * \mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor} * \hat{\mathrm{N}}^{\pm}} \mathcal{StCrMod} \underset{\mathrm{StCM}}{\overset{\mathrm{SymCat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{SymMonCat}$$

Recall that we have the following functors between their ordinary counterparts; cf. §0.2.7, §0.2.5.

$$\mathcal{SimpGrp} \xrightarrow{\mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}}} \mathcal{CrMod} \underset{\mathrm{CM}}{\overset{\mathrm{Cat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{InvMonCat}$$

We aim to construct ordinarification functors from the former categories to the latter that are compatible with the above-mentioned functors.

We have the functor $\mathrm{Ord}^{\mathrm{cat}} \colon \mathcal{SymMonCat} \to \mathcal{InvMonCat}$ given as follows; cf. Remark 425.

For a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, I, \otimes, \varphi) \in \mathrm{Ob}(\mathcal{SymMonCat})$, we have the invertible monoidal category $\mathcal{C}\mathrm{Ord}^{\mathrm{cat}} = (\mathcal{C}, I, \otimes) \in \mathrm{Ob}(\mathcal{InvMonCat})$ that we obtain by omitting the symmetric braiding φ.

We have the functor $\mathrm{Ord}^{\mathrm{mod}} \colon \mathcal{StCrMod} \to \mathcal{CrMod}$ given as follows; cf. Lemma 329.

For a stable crossed module $S = (\mathrm{M}\,,\, \mathrm{B}\,,\, \kappa\,,\, \mathrm{f}) \in \mathrm{Ob}(\mathcal{StCrMod})$, we have the crossed module $S\mathrm{Ord}^{\mathrm{mod}} = \big(\mathrm{M}\,, \mathrm{B}\,, \gamma\,, \mathrm{f}\big) \in \mathrm{Ob}(\mathcal{CrMod})$, with the action

$$\gamma \colon \mathrm{B} \to \mathrm{Aut}(\mathrm{M})\,,\ b \mapsto \big(m \mapsto m^b := m \cdot [\![m\mathrm{f},\, b]\!]\big)\,.$$

We have the functor $\mathrm{Ord} \colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$ given as follows; cf. Lemma 221.

For a stable simplicial group $G \in \mathrm{Ob}(\mathcal{StSimpGrp})$, we have the simplicial group $G\mathrm{Ord} \in \mathrm{Ob}(\mathcal{SimpGrp})$, where

$$(G\mathrm{Ord})_k := \Big(\bigcap_{i \geqslant k+1} \ker \mathrm{d}_i\Big) \cap \ker\Big(\prod_{i \in \lfloor k,0\rfloor}^{G_k} \mathrm{d}_0\Big) = \Big(\bigcap_{i \geqslant k+1} \ker \mathrm{d}_i\Big) \cap \ker(\underbrace{\mathrm{d}_0 \blacktriangle \mathrm{d}_0 \blacktriangle \ldots \blacktriangle \mathrm{d}_0}_{:\, G_k \to G_{-1}})$$

for $k \in \mathbb{Z}_{\geqslant 0}$. Moreover, this functor has the property that $(G\mathrm{Ord})\pi_k \simeq G\pi_k$ for $i \in \mathbb{Z}_{\geqslant 0}$, where Ord preserves quasi-isomorphisms; cf. Lemma 224.

Proposition (Lemmas 426, 434). *The following diagram is commutative.*

$$\begin{array}{ccccc}
\mathcal{StSimpGrp} & \xrightarrow{\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor} * \mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor} * \hat{\mathrm{N}}^{\pm}} & \mathcal{StCrMod} & \underset{\mathrm{StCM}}{\overset{\mathrm{SymCat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} & \mathcal{SymMonCat} \\
\Big\downarrow{\scriptstyle \mathrm{Ord}} & & \Big\downarrow{\scriptstyle \mathrm{Ord}^{\mathrm{mod}}} & & \Big\downarrow{\scriptstyle \mathrm{Ord}^{\mathrm{cat}}} \\
\mathcal{SimpGrp} & \xrightarrow[\mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}}]{} & \mathcal{CrMod} & \underset{\mathrm{CM}}{\overset{\mathrm{Cat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} & \mathcal{InvMonCat}
\end{array}$$

0.3.9 Questions

(1) One may ask whether there exists a left adjoint to the functor Ord. If so, can this left adjoint be considered as a stabilisation functor from $\mathcal{SimpGrp}$ to $\mathcal{StSimpGrp}$?

(2) Let n, $m \in \mathbb{Z}_{\geqslant 0}$ with $m \leqslant n$.

We consider the category of simplicial groups $\mathcal{SimpGrp}$ and the category of stable simplicial groups $\mathcal{StSimpGrp}$.

Let $\mathcal{SimpGrp}^{\lfloor n,m\rfloor}$ be the full subcategory in $\mathcal{SimpGrp}$ of simplicial groups G having $G\mathrm{N}_i = 1$ for $i \in \mathbb{Z}_{\geqslant 0}\backslash[m,n]$. Similarly, let $\mathcal{StSimpGrp}^{\lfloor n,m\rfloor}$ be the full subcategory in $\mathcal{StSimpGrp}$ of stable simplicial groups G having $G\mathrm{N}_i = 1$ for $i \in \mathbb{Z}\backslash[m,n]$.

We have the restriction $\mathrm{Ord}^{\lfloor n,m\rfloor} := \mathrm{Ord}\,\big|^{\mathcal{SimpGrp}^{\lfloor n,m\rfloor}}_{\mathcal{StSimpGrp}^{\lfloor n,m\rfloor}} : \mathcal{StSimpGrp}^{\lfloor n,m\rfloor} \to \mathcal{SimpGrp}^{\lfloor n,m\rfloor}$; cf. Lemma 222.

Given $m \geqslant 0$, the question arises for which values of $n \geqslant m$ the functor $\mathrm{Ord}^{\lfloor n,m\rfloor}$ is an equivalence.

$$\begin{array}{ccc} \mathcal{StSimpGrp}^{\lfloor n,m\rfloor} & \hookrightarrow & \mathcal{StSimpGrp} \\ \downarrow{\scriptstyle \mathrm{Ord}^{\lfloor n,m\rfloor}} & & \downarrow{\scriptstyle \mathrm{Ord}} \\ \mathcal{SimpGrp}^{\lfloor n,m\rfloor} & \hookrightarrow & \mathcal{SimpGrp} \end{array}$$

(2a) Let $m = 2$ and $n = 3$. In [**7,** Th. 3.4] Conduché shows that the categories $\mathcal{SimpGrp}^{\lfloor 3,2\rfloor}$ and $\mathcal{StCrMod}$ are equivalent. In Proposition 405, we show that the categories $\mathcal{StCrMod}$ and $\lfloor 1,0\rfloor$-$\mathcal{StSimpGrp}$ are equivalent.

For an $\lfloor 1,0\rfloor$-stable simplicial group H, we may use the décalage functor $\mathrm{Dec}^{\pm,2}$ to obtain $(H\,\mathrm{Cond}^{\lfloor 1,0\rfloor})\mathrm{Dec}^{\pm,2} \in \mathrm{Ob}(\mathcal{StSimpGrp}^{\lfloor 3,2\rfloor})$; cf. Remark 197.

In turn, for a stable simplicial group $G \in \mathrm{Ob}(\mathcal{StSimpGrp}^{\lfloor 3,2\rfloor})$, we obtain the $\lfloor 1,0\rfloor$-stable simplicial group $(G\mathrm{Dec}^{\pm,-2})\,\mathrm{Trunc}^{\pm,\lfloor 1,0\rfloor} \in \mathrm{Ob}(\lfloor 1,0\rfloor$-$\mathcal{StSimpGrp})$.

In this way, we may obtain an equivalence between $\lfloor 1,0\rfloor$-$\mathcal{StSimpGrp}$ and $\mathcal{StSimpGrp}^{\lfloor 3,2\rfloor}$.

Altogether, $\mathcal{SimpGrp}^{\lfloor 3,2\rfloor}$ is equivalent to $\lfloor 1,0\rfloor$-$\mathcal{StSimpGrp}$, which in turn is equivalent to $\mathcal{StSimpGrp}^{\lfloor 3,2\rfloor}$.

Therefore, the question arises: Is $\mathrm{Ord}^{\lfloor 3,2\rfloor}$ an equivalence between $\mathcal{StSimpGrp}^{\lfloor 3,2\rfloor}$ and $\mathcal{SimpGrp}^{\lfloor 3,2\rfloor}$?

(2b) Let $m \geqslant 1$ and $n = 2m-1$. Let X, Y be CW-complexes of dimension $\leqslant 2m$. Suppose that X and Y are m-connected. We write $\mathrm{Ho}(\mathcal{CW})$ for the homotopy category of CW-complexes. We consider ${}_{\mathrm{Ho}(\mathcal{CW})}(X,Y)$, the set of morphisms from X to Y in $\mathrm{Ho}(\mathcal{CW})$.

We write $\mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})$ for the Spanier-Whitehead category [**25,** §4]. We consider ${}_{\mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})}(X,Y)$, the set of morphisms from X to Y in $\mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})$.

Then the generalized Freudenthal suspension theorem implies that the canonical map

$$ {}_{\mathrm{Ho}(\mathcal{CW})}(X,Y) \to {}_{\mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})}(X,Y) $$

is bijective [**24,** §8.5, Thm. 11].

So the functor from the homotopy category $\mathrm{Ho}(\mathcal{CW})$ to the Spanier-Whitehead category $\mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})$ becomes fully faithful after restriction to the full subcategory $\mathrm{Ho}(\mathcal{CW})^{\lfloor 2m,m+1\rfloor}$ of m-connected CW-complexes of dimension $\leqslant 2m$. Hence, it yields an equivalence to its full image,

$$ \mathrm{Ho}(\mathcal{CW})^{\lfloor 2m,m+1\rfloor} \xrightarrow{\sim} \mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})^{\lfloor 2m,m+1\rfloor} \subseteq \mathrm{StHo}(\mathcal{CW})\,. $$

In the case $m = 2$, we obtain

$$ \mathrm{Ho}(\mathcal{CW})^{\lfloor 4,3\rfloor} \xrightarrow{\sim} \mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})^{\lfloor 4,3\rfloor}\,. $$

This is to be compared with the equivalence between $\mathcal{SimpGrp}^{\lfloor 3,2\rfloor}$ and $\mathcal{StSimpGrp}^{\lfloor 3,2\rfloor}$. Note that passing from CW-complexes to simplicial groups entails a shift by one in the indexing of the homotopy groups.

In the case $m = 3$, we obtain

$$ \mathrm{Ho}(\mathcal{CW})^{\lfloor 6,4\rfloor} \xrightarrow{\sim} \mathrm{StHo}^{\mathrm{SW}}(\mathcal{CW})^{\lfloor 6,4\rfloor}\,. $$

Gurski, Johnson and Osorno use this in the case $m = 3$ in [**13,** §1.1] to state an equivalence of homotopy theories between stable 2-types and 3-connected unstable 6-types.

Taking the shift in the indexing of the homotopy groups into account one might ask: Is $\mathrm{Ord}^{\lfloor 5,3\rfloor}$ an equivalence between $\mathcal{StSimpGrp}^{\lfloor 5,3\rfloor}$ and $\mathcal{SimpGrp}^{\lfloor 5,3\rfloor}$?

(2c) Or more generally, given $m \geqslant 1$, is $\mathrm{Ord}^{\lfloor 2m-1,m\rfloor}$ an equivalence between $\mathcal{StSimpGrp}^{\lfloor 2m-1,m\rfloor}$ and $\mathcal{SimpGrp}^{\lfloor 2m-1,m\rfloor}$.

Chapter 1

Prologue

1.1 Conventions

1.1.1 Sets

1. Let M be a set. Let $I \subseteq M$ be a subset. We say that I is a *cofinite* subset in M if $M \backslash I$ is finite.

2. Let $\mathbb{Z}$ be the set of integers.

   Let $\mathbb{N} = \{n \in \mathbb{Z} \colon 0 \leqslant n\}$ be the set of natural numbers.

3. For a, $b \in \mathbb{Z}$ with $a \leqslant b$, let $[a, b] := \{z \in \mathbb{Z} \colon a \leqslant z \leqslant b\}$.

4. Let $a \in \mathbb{Z}$. We let $+\infty + a = +\infty$ and $-\infty + a = -\infty$, if it occurs as an interval bound.

5. Let $a \in \mathbb{Z} \cup \{-\infty\}$. We write $\mathbb{Z}_{\geqslant a} := \{z \in \mathbb{Z} \colon z \geqslant a\}$. In particular, $\mathbb{Z}_{\geqslant -\infty} = \mathbb{Z}$. Sometimes, we also write $[a, +\infty] := \mathbb{Z}_{\geqslant a}$.

   Let $a \in \mathbb{Z} \cup \{+\infty\}$. We write $\mathbb{Z}_{\leqslant a} := \{z \in \mathbb{Z} \colon z \leqslant a\}$. In particular, $\mathbb{Z}_{\leqslant +\infty} = \mathbb{Z}$. Sometimes, we also write $[-\infty, a] := \mathbb{Z}_{\leqslant a}$.

6. Let M be a set. We write $\mathcal{P}(M) := \{I \colon I \subseteq M\}$ for the power set of M.

7. For a statement A, we have
$$\chi_A := \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases} .$$

8. Let X, Y be sets. For a map $u\colon X \to Y$, we write $\operatorname{coim} u = Y \setminus \operatorname{im} u$ for the coimage of u.

9. Let X be a set. Suppose given subsets A, $B \subseteq X$.
We write $A \,\dot\cup\, B := A \cup B$ to indicate that $A \cap B = \emptyset$, i.e. to indicate that we have disjoint union of the subsets A, $B \subseteq X$.

1.1.2 Groups

Suppose given groups G, H and M.

1. For $x \in G$, we write x^- for the inverse of x.

2. For x, $y \in G$, we write $x^y := y^- \cdot x \cdot y$ for the conjugate of x by y.

3. For x, $y \in G$, we write $[x, y] := x^- \cdot y^- \cdot x \cdot y$ for the commutator of x and y.

4. We write $G^{(1)} := \big\langle\, [x, y]\colon x,\, y \in G \,\big\rangle$ for the commutator subgroup of G.

5. We have the group morphism
$$!\colon G \to H\,,\ g \mapsto 1\,.$$

6. Let $H \leqslant G$ be a subgroup. We have the embedding morphism
$$\iota_{H,G} := \mathrm{id}_G\big|_H\colon H \to G\,,\ h \mapsto h\,.$$

7. An action of G on M is a group morphism $G \to \mathrm{Aut}(M)$.

8. Suppose given an action $\gamma\colon G \to \mathrm{Aut}(M)$. We have the semidirect product
$$G \ltimes_\gamma M = \big\{(b, m)\colon b \in B\,, m \in M\big\}\,.$$
For $b \in B$ and $m \in M$, we write $m^b := m(b\gamma) \in M$.
The multiplication in $B \ltimes_\gamma M$ is given as follows.
For b, $c \in B$ and m, $n \in M$, we have
$$(b, m) \cdot (c, n) = (b \cdot c, m^b \cdot n)\,.$$

9. In a group, we need a formalism for products that takes the order of the factors under consideration.

   Suppose given $g_i \in G$ for $i \in \mathbb{Z}$. Suppose given a, b, $c \in \mathbb{Z}$.

   9.1. We write

$$\prod_{j \in \lceil a,a \rceil} g_j := g_a \,.$$

   For $b \in \mathbb{Z}_{\geqslant a+1}$, we write

$$\prod_{j \in \lceil a,b \rceil} g_j := \big(\prod_{j \in \lceil a,b-1 \rceil} g_j \big) \cdot g_b \,.$$

   So, for a, $b \in \mathbb{Z}$ with $b \geqslant a+1$, we have

$$\prod_{j \in \lceil a,b \rceil} g_j = g_a \cdot g_{a+1} \cdot \ldots \cdot g_{b-1} \cdot g_b \,.$$

   Moreover, we write

$$\prod_{j \in \lfloor a,a \rfloor} g_j := g_a \,.$$

   For $c \in \mathbb{Z}_{\leqslant a-1}$, we write

$$\prod_{j \in \lfloor a,c \rfloor} g_j := \big(\prod_{j \in \lfloor a,c+1 \rfloor} g_j \big) \cdot g_c \,.$$

   So, for a, $c \in \mathbb{Z}$ with $c \leqslant a-1$, we have

$$\prod_{j \in \lfloor a,c \rfloor} g_j = g_a \cdot g_{a-1} \cdot \ldots \cdot g_{c+1} \cdot g_c \,.$$

   9.2. For $c \in \mathbb{Z}_{\leqslant a}$, we write

$$\prod_{i \in \rfloor a,c \rfloor} g_i := \prod_{i \in \lfloor a-1,c \rfloor} g_i \quad \text{and} \quad \prod_{i \in \lfloor a,c \lfloor} g_i := \prod_{i \in \lfloor a,c+1 \rfloor} g_i \,.$$

   For $b \in \mathbb{Z}_{\geqslant a}$, we write

$$\prod_{i \in \rfloor a,b \rceil} g_i := \prod_{i \in \lceil a+1,b \rceil} g_i \quad \text{and} \quad \prod_{i \in \lceil a,b \lceil} g_i := \prod_{i \in \lceil a,b-1 \rceil} g_i \,.$$

9.3. We write

$$\prod_{j\in\lceil a,b\rfloor} g_j \;:=\; \begin{cases} \prod\limits_{j\in\lceil a,b\rfloor} g_j & \text{if } a\leqslant b \\ 1_G & \text{if } a=b+1 \\ \prod\limits_{j\in\lfloor a-1,b+1\rfloor} g_j^- & \text{if } a\geqslant b+2 \end{cases}$$

$$= \begin{cases} g_a\cdot g_{a+1}\cdot\ldots\cdot g_b & \text{if } a\leqslant b \\ 1_G & \text{if } a=b+1 \\ g_{a-1}^-\cdot g_{a-2}^-\cdot\ldots\cdot g_{b+1}^- & \text{if } a\geqslant b+2\,. \end{cases}$$

Moreover, we write

$$\prod_{j\in\lfloor b,a\rfloor} g_j \;:=\; \begin{cases} \prod\limits_{j\in\lfloor b,a\rfloor} g_j & \text{if } b\geqslant a \\ 1_G & \text{if } b=a-1 \\ \prod\limits_{j\in\lceil b+1,a-1\rceil} g_j^- & \text{if } b\leqslant a-2\,. \end{cases}$$

$$= \begin{cases} g_b\cdot g_{b-1}\cdot\ldots\cdot g_a & \text{if } b\geqslant a \\ 1_G & \text{if } b=a-1 \\ g_{b+1}^-\cdot g_{b+1}^-\cdot\ldots\cdot g_{a-1}^- & \text{if } b\leqslant a-2\,. \end{cases}$$

Then

$$\Big(\prod_{j\in\lfloor b,a\rfloor} g_j\Big)\cdot g_{a-1} = \prod_{j\in\lfloor b,a-1\rfloor} g_j \qquad \text{and} \qquad \Big(\prod_{j\in\lceil a,b\rceil} g_j\Big)\cdot g_{b+1} = \prod_{j\in\lceil a,b+1\rceil} g_j\,.$$

9.4. Suppose that we may choose some $\ell\in\mathbb{Z}_{\geqslant a}$ such that $g_j=1$ for $j\in\mathbb{Z}_{\geqslant\ell+1}$. We write

$$\prod_{i\in\lceil a,\infty\lceil} g_i := \prod_{i\in\lceil a,\ell\rceil} g_i = g_a\cdot g_{a+1}\cdot\ldots\cdot g_{\ell-1}\cdot g_\ell\cdot 1\cdot 1\cdot\ldots.$$

We write

$$\prod_{i\in\rfloor\infty,a\rfloor} g_i := \prod_{i\in\lfloor\ell,a\rfloor} g_i = \ldots\cdot 1\cdot 1\cdot g_\ell\cdot g_{\ell-1}\cdot\ldots\cdot g_{a+1}\cdot g_a\,.$$

9.5. Let $c\leqslant a\leqslant b$. Suppose given $I\subseteq\mathbb{Z}$. The set I can be given implicitly by a condition defining I.

We write

$$\prod_{\substack{i\in\lceil a,b\rceil\\ i\in I}} g_i := g_{i_1}\cdot g_{i_2}\cdot\ldots\cdot g_{i_s}\,,$$

where $\{i_1\,,i_2\,,\ldots\,,i_s\}=[a,b]\cap I$ such that $i_1<i_2<\cdots<i_s\,.$
We write

$$\prod_{\substack{i\in\lfloor a,c\rfloor\\ i\in I}} g_i := g_{i_t}\cdot g_{i_{t-1}}\cdot\ldots\cdot g_{i_1}\,,$$

where $\{i_1\,,i_2\,,\ldots\,,i_t\}=[c,a]\cap I$ such that $i_1<i_2<\cdots<i_t\,.$

10. Suppose given subgroups $U_i\leqslant G$ for $i\in\mathbb{Z}$. Suppose given a, b, $c\in\mathbb{Z}$.

10.1. We write

$$\prod_{j\in\lceil a,a\rceil} U_j := U_a\leqslant G\,.$$

For $b\in\mathbb{Z}_{\geqslant a+1}$, we write

$$\prod_{j\in\lceil a,b\rceil} G_j := \big(\prod_{j\in\lceil a,b-1\rceil} U_j\big)\cdot U_b\ \subseteq G\,.$$

So, for a, $b\in\mathbb{Z}$ with $b\geqslant a+1$, we have

$$\prod_{j\in\lceil a,b\rceil} U_j = U_a\cdot U_{a+1}\cdot\ldots\cdot U_{b-1}\cdot U_b\ \subseteq G\,.$$

Moreover, we write

$$\prod_{j\in\lceil a,a-1\rceil} U_j := 1\leqslant G$$

for the trivial subgroup.
We write

$$\prod_{j\in\lfloor a,a\rfloor} U_j := U_a\leqslant G\,.$$

For $c\in\mathbb{Z}_{\leqslant a-1}$, we write

$$\prod_{j\in\lfloor a,c\rfloor} U_j := \big(\prod_{j\in\lfloor a,c+1\rfloor} U_j\big)\cdot U_c\ \subseteq G\,.$$

So, for a, $c \in \mathbb{Z}$ with $c \leqslant a-1$, we have

$$\prod_{j \in \lfloor a,c \rfloor} U_j = U_a \cdot U_{a-1} \cdot \ldots \cdot U_{c+1} \cdot U_c \ \subseteq G\,.$$

Moreover, we write

$$\prod_{j \in \lfloor a-1,a \rfloor} U_j := 1 \leqslant G$$

for the trivial subgroup.

10.2. We write

$$\begin{aligned} \prod_{i \in \lceil a,\infty \lceil} U_i \ &:= \ \Big\{ \prod_{i \in \lceil a,\infty \lceil} u_i \ : u_i \in U_i\,, \text{there exists } \ell \in \mathbb{Z}_{\leqslant a} \text{ with } u_i = 1 \text{ for } i \in \mathbb{Z}_{\geqslant \ell+1} \Big\} \\ &= \ \Big\{ \prod_{i \in \lceil a,\ell \rceil} u_i \ : u_i \in U_i\,, \ell \in \mathbb{Z}_{\leqslant a} \Big\}\,. \end{aligned}$$

and we write

$$\begin{aligned} \prod_{i \in \rfloor \infty,a \rfloor} U_i \ &:= \ \Big\{ \prod_{i \in \rfloor \infty,a \rfloor} u_i \ : u_i \in U_i\,, \text{there exists } \ell \in \mathbb{Z}_{\geqslant a} \text{ with } u_i = 1 \text{ for } i \in \mathbb{Z}_{\geqslant \ell+1} \Big\} \\ &= \ \Big\{ \prod_{i \in \lfloor \ell,a \rfloor} u_i \ : u_i \in U_i\,, \ell \in \mathbb{Z}_{\geqslant a} \Big\}\,. \end{aligned}$$

11. Suppose given posets $(A, \leqslant)$ and $(B, \leqslant)$. Then the *product order* on the cartesian product $A \times B$ is given as follows.

    For (a,b), $(\tilde{a},\tilde{b}) \in A \times B$, we have

$$(a,b) \leqslant (\tilde{a},\tilde{b}) \quad :\Leftrightarrow \quad (a \leqslant \tilde{a}) \ \wedge \ (b \leqslant \tilde{b})\,.$$

1.1.3 Categories

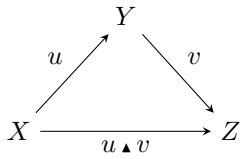
Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$ be a categories.

1. By a *category* $\mathcal{C}$, we understand a small category (with respect to a given universe). I.e. we stipulate that $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets.

2. By writing $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$, we implicitly suppose given objects X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, a morphism $u \in \mathrm{Mor}(\mathcal{C})$ with start object X and target object Y, and a morphism $v \in \mathrm{Mor}(\mathcal{C})$ with start object Y and target object Z.

   For their composite, we write $X \xrightarrow{u \star v} Z$. We pronounce $u \star v$ as 'u comp v'.

To use the symbol ($\blacktriangle$) was introduced to be able to distinguish composition and multiplication. The symbol ($\blacktriangle$) should remind of a commutative triangle.



3. For $X \in \mathrm{Ob}(\mathcal{C})$, sometimes we write $\mathrm{id} := \mathrm{id}_X$ for the identity morphism.

4. Let $n \in \mathbb{N}$. Suppose given $X_i \xrightarrow{u_i} X_{i+1}$ in $\mathcal{C}$ for $i \in [0, n]$.

$$X_0 \xrightarrow{u_0} X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} \ldots \xrightarrow{u_n} X_{n+1}$$

Sometimes, we write

$$\prod_{i \in \lceil 0,n \rceil}^{X_0 \; X_{n+1}} u_i := u_0 \blacktriangle u_1 \blacktriangle \ldots \blacktriangle u_n$$

for their composite.

Let $n \in \mathbb{N}$. Suppose given $Y_{j+1} \xrightarrow{v_j} Y_j$ in $\mathcal{C}$ for $j \in [0, n]$.

$$Y_{n+1} \xrightarrow{v_n} Y_n \xrightarrow{v_{n-1}} Y_{n-1} \xrightarrow{v_{n-2}} \ldots \xrightarrow{v_0} Y_0$$

Sometimes, we write

$$\prod_{j \in \lfloor n,0 \rfloor}^{Y_{n+1} \; Y_0} v_j := v_n \blacktriangle v_{n-1} \blacktriangle \ldots \blacktriangle v_0$$

for their composite.

For $X \in \mathrm{Ob}(\mathcal{C})$, we have $\prod_{i \in \emptyset}^{X \; X} = \mathrm{id}_X$.

5. A *functor* from $\mathcal{C}$ to $\mathcal{D}$ is given by $F := (\mathrm{Mor}(F), \mathrm{Ob}(F))$ where

$$\mathrm{Ob}(F) : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}) \quad \text{and} \quad \mathrm{Mor}(F) : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D}) .$$

A functor is required to satisfy

$$\begin{aligned} u \boldsymbol{s} \, \mathrm{Ob}(F) &= u \, \mathrm{Mor}(F) \, \boldsymbol{s} \\ u \boldsymbol{t} \, \mathrm{Ob}(F) &= u \, \mathrm{Mor}(F) \, \boldsymbol{t} \\ X \boldsymbol{i} \, \mathrm{Mor}(F) &= X \, \mathrm{Ob}(F) \, \boldsymbol{i} \end{aligned}$$

for $u \in \mathrm{Mor}(\mathcal{C})$, $X \in \mathrm{Ob}(\mathcal{C})$, and

$$(u \mathbin{\blacktriangle} v)\,\mathrm{Mor}(F) = u\,\mathrm{Mor}(F) \mathbin{\blacktriangle} v\,\mathrm{Mor}(F)\ ,$$

for $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$.

For $X \in \mathrm{Ob}(\mathcal{C})$, we write $XF := X\,\mathrm{Ob}(F) \in \mathrm{Ob}(\mathcal{D})$. For $u \in \mathrm{Mor}(\mathcal{C})$, we write $uF := u\,\mathrm{Mor}(F) \in \mathrm{Mor}(\mathcal{D})$.

6. For functors $F\colon\ \mathcal{C} \to \mathcal{D}$, $G\colon\ \mathcal{D} \to \mathcal{E}$, we often write $F * G := FG\colon\ \mathcal{C} \to \mathcal{E}$ for their composite.

7. A functor $F\colon\ \mathcal{C} \to \mathcal{D}$ is called *isofunctor* from $\mathcal{C}$ to $\mathcal{D}$ if there exist a functor $G\colon\ \mathcal{D} \to \mathcal{C}$ such that $F * G = \mathrm{id}_{\mathcal{C}}$ and $G * F = \mathrm{id}_{\mathcal{D}}$ hold. Then we write $F^{-} := G$.

   A functor F is an isofunctor if and only if $\mathrm{Ob}(F)$ and $\mathrm{Mor}(F)$ are bijective.

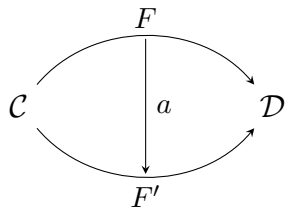
   If $\mathcal{C} = \mathcal{D}$ then an isofunctor $F\colon\ \mathcal{C} \to \mathcal{C}$ is called an *autofunctor*.

8. Suppose given functors F, $F'\colon\ \mathcal{C} \to \mathcal{D}$.

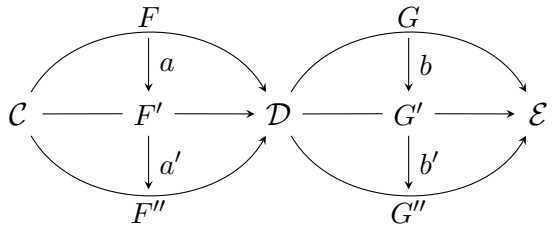
   A *transformation* $F \xrightarrow{a} F'$ is a tuple $a = \big(F, (Xa)_{X \in \mathrm{Ob}(\mathcal{C})}, F'\big)$, where $XF \xrightarrow{Xa} XF'$ in $\mathcal{D}$, such that the diagram

$$\begin{array}{ccc} XF & \xrightarrow{Xa} & XF' \\ {\scriptstyle uF}\downarrow & & \downarrow{\scriptstyle uF'} \\ YF & \xrightarrow[Ya]{} & YF' \end{array}$$

   commutes for $X \xrightarrow{u} Y$ in $\mathcal{C}$. We often write just $a = (Xa)_{X \in \mathrm{Ob}(\mathcal{C})}$.



9. Suppose given functors F, F', $F''\colon \mathcal{C} \to \mathcal{D}$ and G, G', $G''\colon \mathcal{D} \to \mathcal{E}$. Suppose given transformations $F \xrightarrow{a} F' \xrightarrow{a'} F''$ and $G \xrightarrow{b} G' \xrightarrow{b'} G''$.



We write

$$a \blacktriangle a' := \left(XF \xrightarrow{Xa \blacktriangle Xa'} XF''\right)_{X \in \mathrm{Ob}(\mathcal{C})} : F \to F''$$

for the *vertical composite* of the transformations a and a'.

We write

$$a * b = \left(XFG \xrightarrow{X(a*b)} XF'G'\right)_{X \in \mathrm{Ob}(\mathcal{C})} := aG \blacktriangle F'b = Fb \blacktriangle aG' \ : F * G \to F' * G'$$

for the *horizontal composite* of the transformations a and b.

We often write

$$F * b := \mathrm{id}_F * b = Fb = \left(XFG \xrightarrow{(XF)b} XFG'\right)_{X \in \mathrm{Ob}(\mathcal{C})}$$
$$a * G := a * \mathrm{id}_G = aG = \left(XFG \xrightarrow{(Xa)F} XF'G\right)_{X \in \mathrm{Ob}(\mathcal{C})}.$$

Moreover, we have

$$\mathrm{id}_F * \mathrm{id}_G = \mathrm{id}_{F*G}$$

and

$$(a \blacktriangle a') * (b \blacktriangle b') = (a * b) \blacktriangle (a' * b').$$

10. Suppose given functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$. Suppose given transformations $\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to F * G$ and $\eta\colon G * F \to \mathrm{id}_{\mathcal{D}}$ such that we have

$$(G * \varepsilon) \blacktriangle (\eta * G) = \mathrm{id}_G$$
$$(\varepsilon * F) \blacktriangle (F * \eta) = \mathrm{id}_F \ .$$

Then we say that $(F, G, \varepsilon, \eta)$ is an adjunction.

We call F a left adjoint to G and we call G a right adjoint to F. We call ε a unit of the adjunction. We call η a counit of the adjunction.

Sometimes, we write $F \dashv G$.

11. Suppose given functors $F\colon\ \mathcal{C} \to \mathcal{D}$ and $G\colon G \to \mathcal{C}$. Suppose that we have isotransformations
$$\varphi\colon F * G \to \mathrm{id}_{\mathcal{C}} \quad \text{and} \quad \varphi'\colon G * F \to \mathrm{id}_{\mathcal{D}}\ .$$
Then we say that F and G are *equivalences*.

12. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is given as follows. We have $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$. For $X,\ Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, we have ${}_{\mathcal{C}^{\mathrm{op}}}(X, Y) := {}_{\mathcal{C}}(Y, X)$.

    We often write $u^{\mathrm{op}} := u \in {}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ to emphasize that this morphism is in $\mathcal{C}^{\mathrm{op}}$. So if $\big(Y \xrightarrow{u} X\big)$ is a morphism in $\mathcal{C}$, then $\big(X \xrightarrow{u^{\mathrm{op}}} Y\big)$ is a morphism in $\mathcal{C}^{\mathrm{op}}$.

    For $X \xrightarrow{u^{\mathrm{op}}} Y \xrightarrow{v^{\mathrm{op}}} Z$ in $\mathcal{C}^{\mathrm{op}}$, their composite is given by
$$u^{\mathrm{op}} \mathbin{\blacktriangle} v^{\mathrm{op}} := (v \mathbin{\blacktriangle} u)^{\mathrm{op}}\ .$$

13. For a functor $F\colon\ \mathcal{C} \to \mathcal{D}$, its opposite functor F^{op} is given by
$$F^{\mathrm{op}}\colon\ \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\,\mathrm{op}}\,,\ \big(X \xrightarrow{u^{\mathrm{op}}} Y\big) \mapsto \big(XF \xrightarrow{(uF)^{\mathrm{op}}} YF\big)\ .$$

14. We write $[\mathcal{C}, \mathcal{D}]$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$. The set of objects $\mathrm{Ob}\big([\mathcal{C}, \mathcal{D}]\big)$ consists of the functors from $\mathcal{C}$ to $\mathcal{D}$. The set of morphisms $\mathrm{Mor}\big([\mathcal{C}, \mathcal{D}]\big)$ consists of the transformations between such functors.

15. Suppose given a functor $F\colon\ \mathcal{C} \to \mathcal{D}$. Consider the functor categories $[\mathcal{D}, \mathcal{A}]$ and $[\mathcal{C}, \mathcal{A}]$. We have the functor
$$[F, \mathcal{A}]\colon [\mathcal{D}, \mathcal{A}] \to [\mathcal{C}, \mathcal{A}]\,,\ \big(X \xrightarrow{u} Y\big) \mapsto \big(F * X \xrightarrow{Fu} F * Y\big)\ .$$
    Moreover, for $G\colon\ \mathcal{B} \to \mathcal{C}$, we have
$$[F, \mathcal{A}] * [G, \mathcal{A}] = [G * F, \mathcal{A}]\colon [\mathcal{D}, \mathcal{A}] \to [\mathcal{B}, \mathcal{A}]\ .$$

16. Suppose given a functor $F\colon\ \mathcal{C} \to \mathcal{D}$. Consider the functor categories $[\mathcal{A}, \mathcal{C}]$ and $[\mathcal{A}, \mathcal{D}]$. We have the functor
$$[\mathcal{A}, F]\colon [\mathcal{A}, \mathcal{C}] \to [\mathcal{A}, \mathcal{D}]\,,\ \big(X \xrightarrow{u} Y\big) \mapsto \big(X * F \xrightarrow{uF} Y * F\big)\ .$$
    Moreover, for $G\colon\ \mathcal{D} \to \mathcal{E}$, we have
$$[F, \mathcal{A}] * [G, \mathcal{A}] = [\mathcal{A}, F * G]\colon [\mathcal{A}, \mathcal{C}] \to [\mathcal{A}, \mathcal{E}]\ .$$

17. We have the category $\mathcal{S}\mathit{et}$ with sets as objects and with maps between sets as morphisms. Composition is given by composition of maps.

1.1.4 Crossed Modules

1. Suppose given groups M_V and B_V. Suppose given an action

$$\gamma_V \colon \mathrm{B}_V \to \mathrm{Aut}(\mathrm{M}_V)\,,\ b \mapsto \big(m \mapsto m^b := m(b\gamma_V)\big)$$

and a group morphism $\mathrm{f}_V \colon \mathrm{M}_V \to \mathrm{B}_V$ that satisfy (CM1) and (CM2).

(CM1) For $m \in \mathrm{M}_V$ and $b \in \mathrm{B}_V$, we have

$$(m^b)\mathrm{f}_V = (m\mathrm{f}_V)^b\,.$$

(CM2) For n, $m \in \mathrm{M}_V$, we have

$$m^{n\mathrm{f}_V} = m^n\,.$$

Then $V = (\mathrm{M}_V\,,\ \mathrm{B}_V\,,\ \gamma_V\,,\ \mathrm{f}_V)$ is a *crossed module*.

The group M_V is called the *module* of V, the group B_V is called the *base group* of V, the action γ_V is called *action* of V and the group morphism f_V is called the *differential* of V.

2. For a crossed module V, we write $V\pi_0 := \mathrm{B}_V/(\mathrm{M}_V\,\mathrm{f}_V)$ and $V\pi_1 := \ker \mathrm{f}_V$.

3. Suppose given crossed modules V, W. Let $\mu \colon \mathrm{M}_V \to \mathrm{M}_W$ and $\beta \colon \mathrm{B}_V \to \mathrm{B}_W$ be group morphisms that satisfy $(1, 2)$.

(1) For $m \in \mathrm{M}_V$, we have

$$m\mu\,\mathrm{f}_W = m\mathrm{f}_V\beta\,.$$

(2) For $m \in \mathrm{M}_V$ and $b \in \mathrm{B}_V$, we have

$$(m^b)\mu = (m\mu)^{b\beta}\,.$$

Then $(\mu, \beta) \colon V \to W$ is a *crossed module morphism*.

4. Suppose given crossed modules V, W. Let $(\mu, \beta) \colon V \to W$ be a crossed module morphism.

We say that (μ, β) is *injective* if the group morphisms μ and β are injective.

We say that (μ, β) is *surjective* if the group morphisms μ and β are surjective.

We say that (μ, β) is *bijective* if the group morphisms μ and β are bijective.

5. Suppose given a crossed module $V = (\mathrm{M}_V\,,\,\mathrm{B}_V\,,\,\gamma_V\,,\,\mathrm{f}_V)\,.$

   Consider the category $V\mathrm{Cat}$ with

$$\begin{aligned}\mathrm{Ob}(V\mathrm{Cat}) &= \mathrm{B}_V\\ \mathrm{Mor}(V\mathrm{Cat}) &= \mathrm{B}_V \ltimes \mathrm{M}_V\,.\end{aligned}$$

   The source, target and identity is given as follows.

$$\begin{array}{lrcl} s\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b\\ t\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b\cdot m\mathrm{f}_V\\ i\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \leftarrow \mathrm{B}_V\,, & (b,1) & \leftarrow\!\shortmid & b\,. \end{array}$$

   Note that s, i and t are group morphisms; cf. e.g. [**29,** Rem. 24].

   For $b \xrightarrow{(b,m)} b\cdot m\mathrm{f}_V \xrightarrow{(b\cdot m\mathrm{f}_V\,,\,n)} b\cdot(m\cdot n)\mathrm{f}_V$ in $V\mathrm{Cat}$, their composite is given by

$$(b,m)\mathbin{\blacktriangle}(b\cdot m\mathrm{f}_V\,,\,n) = (b,mn)\,;$$

   cf. e.g. [**29,** Rem. 4].

1.2 Preliminaries

1.2.1 Groups

Remark 1 *Suppose given a group H. Suppose given a, b, $c \in H$.*

(1) *We have* $[a\cdot b,c] = [a,c]^b\cdot[b,c]$.

(2) *We have* $[a,b\cdot c] = [a,c]\cdot[a,b]^c$.

Proof. Ad (1). We have

$$[a\cdot b,c] = b^-\cdot a^-\cdot c^-\cdot a\cdot b\cdot c = b^-\cdot(a^-\cdot c^-\cdot a\cdot c)\cdot b\cdot(b^-\cdot c^-\cdot b\cdot c) = [a,c]^b\cdot[b,c]\,.$$

Ad (2). We have

$$[a,b\cdot c] = a^-\cdot c^-\cdot b^-\cdot a\cdot b\cdot c = (a^-\cdot c^-\cdot a\cdot c)\cdot c^-\cdot(a^-\cdot b^-\cdot a\cdot b)\cdot c = [a,c]\cdot[a,b]^c\,.$$

□

Lemma 2 *Suppose given* $G \underset{s}{\overset{p}{\rightleftarrows}} K$ *in* $\mathcal{Grp}$. *Suppose that we have* $s \triangleleft p = \mathrm{id}_K$.

Let $N := \ker p \trianglelefteq G$.

We have the action morphism

$$\gamma\colon K \to \mathrm{Aut}(N)\,,\ k \mapsto (n \mapsto k^- s \cdot n \cdot ks)\,.$$

Consider the semidirect product $K \ltimes_\gamma N =: K \ltimes N$.

We have the group isomorphism

$$\varphi\colon K \ltimes N \to G\,,\ (k,n) \mapsto ks \cdot n\,.$$

Its inverse is given by

$$\varphi^-\colon G \to K \ltimes N\,,\ g \mapsto (gp, g^- ps \cdot g)\,.$$

Proof. We show that φ *is a group morphism.*

For k, $\ell \in K$ and n, $m \in \ker p$, we have

$$\begin{aligned}\big((k,n)\cdot(\ell,m)\big)\varphi = (k\cdot\ell, n^\ell \cdot m)\varphi &= (k\cdot\ell)s \cdot n^\ell \cdot m = ks \cdot \ell s \cdot (\ell s)^- \cdot n \cdot \ell s \cdot m \\ &= ks \cdot n \cdot \ell s \cdot m = (k,n)\varphi \cdot (\ell,m)\varphi\,.\end{aligned}$$

This shows that φ is a group morphism.

We consider the map $\psi\colon G \to K \ltimes \ker p\,,\ g \mapsto (gp, g^- ps \cdot g)$.

We show that ψ *is well-defined.*

For $g \in G$w we have

$$(g^- ps \cdot g)p = g^- psp \cdot gp = (gp)^- \cdot gp = 1\,.$$

This shows $g^- ps \cdot g \in \ker p$. So the map ψ exists.

We show that ψ *is the inverse of* φ.

For $k \in K$ and $m \in \ker p$, we have

$$\begin{aligned}(k,m)\varphi\psi = (ks\cdot m)\psi &= \big((ks\cdot m)p, (ks\cdot m)^- ps \cdot ks \cdot m\big) = (ksp \cdot mp, m^- ps \cdot k^- sps \cdot ks \cdot m) \\ &= (k, k^- s \cdot ks \cdot m) = (k,m)\,.\end{aligned}$$

This shows $\varphi \triangleleft \psi = \mathrm{id}_{K\ltimes\ker p}$.

For $g \in G$, we have

$$g\psi\varphi = (gp, g^- ps \cdot g)\varphi = gps \cdot g^- ps \cdot g = g\,.$$

This shows $\psi \triangleleft \varphi = \mathrm{id}_G$. So we have $\varphi^- = \psi$. □

Definition 3 Let I be a linearly ordered set such that $I \neq \emptyset$.

For $z \in I$, suppose given a group G_z.

For n, $m \in I$ with $n \leqslant m$, suppose given a group morphism $j_{n,m} \colon G_n \to G_m$.

Moreover, for n, m, $\ell \in I$ with $n \leqslant m \leqslant \ell$, suppose that we have $j_{n,m} \vartriangleleft \iota_{m,\ell} = j_{n,\ell} \colon G_n \to G_\ell$ and $j_{n,n} = \mathrm{id}_{G_n} \colon G_n \to G_n$.

We consider the disjoint union $\bigsqcup_{z \in I} G_z = \{(n,g) \colon n \in I\,, g \in G_n\}$, which is a set.

(1) On $\bigsqcup_{z \in I} G_z$, we have the following equivalence relation $(\sim)$.

For (n,g), $(m,h) \in \bigsqcup_{z \in I} G_z$, we have

$$(n,g) \sim (m,h) \;:\Leftrightarrow\; \exists \ell \in I \text{ with } n \leqslant \ell\,, m \leqslant \ell \text{ and } \; g j_{n,\ell} = h \iota_{m,\ell}\,.$$

(2) The group $\varinjlim_z G_z := \bigsqcup_{z \in I} G_z/(\sim)$ is called the *colimit* of $\big((G_z)_{z \in I}\,, (j_{n,m})_{n,m \in I\,, n \leqslant m}\big)$.

For $(n,g) \in \bigsqcup_{z \in I} G_z$, we write $[n,g] := [(n,g)]_\sim$ for its equivalence class.

(3) The multiplication in $\varinjlim_z G_z$ is given as follows.

For $[n,g]$, $[m,h] \in \varinjlim_z G_z$, we have

$$[n,g] \cdot [m,h] := [\ell, g j_{n,\ell} \cdot h j_{m,\ell}]\,,$$

where $\ell \in I$ is chosen such that $n \leqslant \ell$ and $m \leqslant \ell$.

Moreover, we have $1_{\varinjlim_z G_z} = [n,1]$ for each $n \in I$.

Remark 4 We consider the situation of Definition 3. We consider the colimit $\varinjlim_z G_z$; cf. Definition 3.(2).

(1) For $n \in I$, we have the group morphism

$$j_{n,\infty} \colon G_n \to \varinjlim_z G_z\,,\; g \mapsto [n,g]\,.$$

In particular, $[n,g] \cdot [n,\tilde{g}] = [n, g \cdot \tilde{g}]$ for g, $\tilde{g} \in G_n$.

(2) For n, $m \in I$ with $n \leqslant m$, we have

$$j_{n,m} \blacktriangle j_{m,\infty} = j_{n,\infty} \,,$$

since, for $g \in G_n$, we have

$$gj_{n,\infty} = [n, g] = [m, gj_{n,m}] = g(j_{n,m} \blacktriangle j_{m,\infty}) \,.$$

(3) Suppose given a group T.

For $z \in I$, suppose given a group morphism $d_z \colon G_z \to T$.

For n, $m \in I$ with $n \leqslant m$, suppose that we have

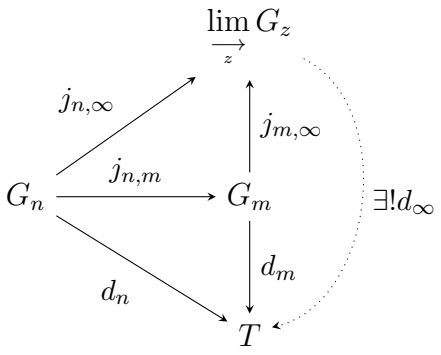
$$j_{n,m} \blacktriangle d_m = d_n \,.$$

Then there exists a unique group morphism $d_\infty \colon \varinjlim_z G_z \to T$ such that

$$j_{n,\infty} \blacktriangle d_\infty = d_n \,,$$

for $n \in I$.

We have $[n, g]d_\infty = gd_n$ for $n \in I$ and $g \in G_n$.



Remark 5 We consider the situation of Definition 3.

For n, $m \in I$ with $n \leqslant m$, suppose that the group morphism $j_{n,m} \colon G_n \to G_m$ is injective.

(1) Suppose given n, $m \in I$ with $n \leqslant m$.

Suppose given $g \in G_n$ and $h \in G_m$ with $[n, g] = [m, h]$.

Then we have

$$gj_{n,m} = h \,.$$

(2) For $n \in I$, the group morphism $j_{n,\infty} \colon G_n \to \varinjlim_z G_z$, $g \mapsto [n, g]$ from Remark 4 is injective.

Lemma 6 *Suppose given a group H.*
Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. For $i \in [0, k]$, suppose given a subgroup $U_i \leqslant H$.
Let $U := \prod_{i \in \lceil 1,k \rceil} U_i = U_1 \cdot U_2 \cdot \ldots \cdot U_k \subseteq H$.
Suppose that we have $u \cdot u' \in U$ for u, $u' \in U$. Then we have $U \leqslant H$.

Proof. Suppose given $u \in U$. It suffices to show that $u^- \overset{!}{\in} U$.
We have $u = \prod_{i \in \lceil 1,k \rceil} u_i$ for some chosen $u_i \in U_i$. So we have

$$u^- = \Big(\prod_{i \in \lceil 1,k \rceil} u_i \Big)^- = \prod_{i \in \lfloor k,1 \rfloor} u_i^- \ \in U \,,$$

since $u_i^- \in U_i \subseteq U$ for $i \in [0, k]$. □

1.2.2 Complexes in groups

Definition 7 (Complex in groups)
Suppose given groups X_k for $k \in \mathbb{Z}$. Suppose given group morphisms $d_k \colon X_k \to X_{k-1}$ for $k \in \mathbb{Z}$.
We call

$$X := \big(\ldots \xrightarrow{d_3^X} X_2 \xrightarrow{d_2^X} X_1 \xrightarrow{d_1^X} X_0 \xrightarrow{d_0^X} X_{-1} \xrightarrow{d_{-1}^X} \ldots \big)$$

a *complex in groups* if $(1, 2)$ hold.

(1) For $k \in \mathbb{Z}$, we have

$$d_k^X \centerdot d_{k-1}^X = ! \,.$$

(2) For $k \in \mathbb{Z}$, we have

$$\operatorname{im} d_k^X \trianglelefteqslant X_{k-1} \,.$$

Definition 8 Suppose given complexes in groups

$$X = \big(\dots \xrightarrow{d_3^X} X_2 \xrightarrow{d_2^X} X_1 \xrightarrow{d_1^X} X_0 \xrightarrow{d_0^X} X_{-1} \xrightarrow{d_{-1}^X} \dots \big)$$
$$Y = \big(\dots \xrightarrow{d_3^Y} Y_2 \xrightarrow{d_2^Y} Y_1 \xrightarrow{d_1^Y} Y_0 \xrightarrow{d_0^Y} Y_{-1} \xrightarrow{d_{-1}^Y} \dots \big) .$$

Suppose given group morphisms

$$\varphi_k \colon X_k \to Y_k$$

for $k \in \mathbb{Z}$.

If we have

$$d_k^X \vartriangle \varphi_{k-1} = \varphi_k \vartriangle d_k^Y$$

for $k \in \mathbb{Z}$ then we call $\varphi := (\varphi_k)_{k\in\mathbb{Z}} \colon X \to Y$ a *morphism of complexes in groups.*

Remark 9 (Category of complexes in group)

We have the category $\mathcal{ComplexGrp}$ *with complexes in groups as objects and morphisms of complexes in groups as morphisms.*

For $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ *in* $\mathcal{ComplexGrp}$*, their composite is given by*

$$\varphi \vartriangle \psi := (\varphi_k \vartriangle \psi_k)_{k\in\mathbb{Z}} \colon X \to Z .$$

For $X \in \mathrm{Ob}\big(\mathcal{ComplexGrp}\big)$*, its identity is given by*

$$\mathrm{id}_X := (\mathrm{id}_{X_k})_{k\in\mathbb{Z}} \colon X \to X .$$

Proof. Suppose given $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ in $\mathcal{ComplexGrp}$.

Let $k \in \mathbb{Z}$. For $x \in X_k$, we have

$$x d_k^X (\varphi_{k-1} \vartriangle \psi_{k-1}) = x\, \varphi_k\, d_k^Y\, \psi_{k-1} = x(\varphi_k \vartriangle \psi_k) d_k^Z .$$

This shows $d_k^X \vartriangle (\varphi_{k-1} \vartriangle \psi_{k-1}) = (\varphi_k \vartriangle \psi_k) \vartriangle d_k^Z$.

So $\varphi \vartriangle \psi = (\varphi_k \vartriangle \psi_k)_{k\in\mathbb{Z}} \colon X \to Z$ is a morphism in complexes in groups.

For $y \in Y_k$, we have

$$y d_k^Y \,\mathrm{id}_{Y_{k-1}} = y\, \mathrm{id}_Y\, d_k^Y .$$

This shows $d_k^Y \vartriangle \mathrm{id}_{Y_{k-1}} = \mathrm{id}_Y \vartriangle d_k^Y$.

So $(\mathrm{id}_{Y_k})_{k\in\mathbb{Z}} \colon Y \to Y$ is a morphism in complexes in groups.

Moreover, we have

$$\varphi \blacktriangle (\mathrm{id}_{Y_k})_{k\in\mathbb{Z}} = (\varphi_k \blacktriangle \mathrm{id}_{Y_k})_{k\in\mathbb{Z}} = (\varphi_k)_{k\in\mathbb{Z}} = \varphi\,,$$

and

$$(\mathrm{id}_{Y_k})_{k\in\mathbb{Z}} \blacktriangle \psi = (\mathrm{id}_{Y_k} \blacktriangle \psi_k)_{k\in\mathbb{Z}} = (\psi_k)_{k\in\mathbb{Z}} = \psi\,.$$

This shows $\mathrm{id}_Y = (\mathrm{id}_{Y_k})_{k\in\mathbb{Z}}$. □

Remark 10 *Suppose given complexes in groups X, Y. Suppose given a morphisms of complexes in groups $\varphi\colon X \to Y$.*

Suppose that φ_k is bijective for $k \in \mathbb{Z}$.

Then we say that φ is bijective. We also say that φ is an isomorphism of complexes in groups*.*

Its inverse is given by $\varphi^- := (\varphi_k^-)_{k\in\mathbb{Z}}\colon Y \to X$.

Proof. Let $k \in \mathbb{Z}$. For $y \in Y_k$, we have

$$y\, d_k^Y\, \varphi_{k-1}^- = y\, \varphi_k^-\, \varphi_k\, d_k^Y\, \varphi_{k-1}^- = y\, \varphi_k^-\, d_k^X\, \varphi_{k-1}\, \varphi_{k-1}^- = y\, \varphi_k^-\, d_k^X\,.$$

This shows $d_k^Y \blacktriangle \varphi_{k-1}^- = \varphi_k^- \blacktriangle d_k^X$ for $k \in \mathbb{Z}$.

So $(\varphi_k^-)_{k\in\mathbb{Z}}\colon Y \to X$ is a morphisms of complexes in groups.

Moreoever, we have

$$\varphi \blacktriangle (\varphi_k^-)_{k\in\mathbb{Z}} = (\varphi_k \blacktriangle \varphi_k^-)_{k\in\mathbb{Z}} = (\mathrm{id}_{X_k})_{k\in\mathbb{Z}} = \mathrm{id}_X\,,$$

and we have

$$(\varphi_k^-)_{k\in\mathbb{Z}} \blacktriangle \varphi = (\varphi_k^- \blacktriangle \varphi_k)_{k\in\mathbb{Z}} = (\mathrm{id}_{Y_k})_{k\in\mathbb{Z}} = \mathrm{id}_Y\ .$$

This shows $\varphi^- = (\varphi_k^-)_{k\in\mathbb{Z}}$. □

Lemma 11 *Suppose given complexes in groups*

$$X = \big(\dots \xrightarrow{d_3^X} X_2 \xrightarrow{d_2^X} X_1 \xrightarrow{d_1^X} X_0 \xrightarrow{d_0^X} X_{-1} \xrightarrow{d_{-1}^X} \dots\big)$$
$$Y = \big(\dots \xrightarrow{d_3^Y} Y_2 \xrightarrow{d_2^Y} Y_1 \xrightarrow{d_1^Y} Y_0 \xrightarrow{d_0^Y} Y_{-1} \xrightarrow{d_{-1}^Y} \dots\big)\,.$$

(1) *For $k \in \mathbb{Z}$, we have the group*

$$X\mathrm{H}_k := \ker d_k^X \,/\, \mathrm{im}\, d_{k+1}^X\,.$$

We call $X\mathrm{H}_k$ the k-th homology group *of X.*

(2) *Suppose given a morphism of complexes in groups* $\varphi = (\varphi_i)_{i\in\mathbb{Z}} \colon X \to Y$.
For $k \in \mathbb{Z}$, *we have the group morphism*

$$\varphi\mathrm{H}_k \colon X\mathrm{H}_k \to Y\mathrm{H}_k\,,\ x \operatorname{im} d^X_{k+1} \mapsto (x\varphi_k) \operatorname{im} d^Y_{k+1}\,.$$

(3) *For* $k \in \mathbb{Z}$, *we have the functor*

$$\mathrm{H}_k \colon \mathcal{ComplexGrp} \to \mathcal{Grp}\,,\ \big(Z \xrightarrow{\varphi} Z'\big) \mapsto \big(Z\mathrm{H}_k \xrightarrow{\varphi\mathrm{H}_k} Z'\mathrm{H}_k\big)\,.$$

Proof. Ad (1). Let $k \in \mathbb{Z}$. Then we have $\operatorname{im} d_{k+1} \trianglelefteq G_k$. Since $\mathrm{d}_{k+1} \vartriangleleft \mathrm{d}_k = !$, we conclude that $\operatorname{im} d_{k+1} \trianglelefteq \ker d_k$.

Ad (2). Let $k \in \mathbb{Z}$. We have to show that $(\operatorname{im} d^X_{k+1})\varphi_k \overset{!}{\subseteq} \operatorname{im} d^Y_{k+1}$.
Suppose given $x \in \operatorname{im} d^X_{k+1}$. Then $x = \tilde{x} d^X_{k+1}$ for some $\tilde{x} \in X_{k+1}$.
We have

$$x\varphi_k = \tilde{x}\, d^X_{k+1}\, \varphi_k = \tilde{x}\varphi_{k+1}\, d^Y_{k+1}\,.$$

This shows $x\varphi_k \in \operatorname{im} d^Y_{k+1}$.
So we have the group morphism $\varphi\mathrm{H}_k \colon X\mathrm{H}_k \to Y\mathrm{H}_k\,,\ x \operatorname{im} \mathrm{d}^X_{k+1} \to (x\varphi_k) \operatorname{im} \mathrm{d}^Y_{k+1}$.

Ad (3). Let $k \in \mathbb{Z}$. Suppose given $Z \xrightarrow{\varphi} Z' \xrightarrow{\varphi'} Z''$ in $\mathcal{ComplexGrp}$.
Note that we have

$$\mathrm{id}_Z\ \mathrm{H}_k \colon Z\mathrm{H}_k \to Z\mathrm{H}_k\,,\ z \operatorname{im} d^Z_{k+1} \mapsto (z\,\mathrm{id}_{Z_k}) \operatorname{im} d^Z_{k+1} = z \operatorname{im} d^Z_{k+1}\,.$$

So $\mathrm{id}_Z\, \mathrm{H}_k = \mathrm{id}_{Z\mathrm{H}_k}$.
We write $\psi := \varphi \vartriangleleft \varphi' \colon Z \to Z'$. For $z \in \ker d^Z_k$, we have

$$\begin{aligned}(z \operatorname{im} d^Z_{k+1})\psi\mathrm{H}_k &= (z\psi_k) \operatorname{im} d^{Z''}_{k+1} = (z\varphi_k\, \varphi'_k) \operatorname{im} d^{Z''}_{k+1} = \big((z\varphi_k) \operatorname{im} d^{Z'}_{k+1}\big)\varphi'\mathrm{H}_k \\ &= (z \operatorname{im} d^Z_{k+1})\varphi\mathrm{H}_k\, \varphi\mathrm{H}'_k\,.\end{aligned}$$

This shows $(\varphi \vartriangleleft \varphi')\mathrm{H}_k = \psi\mathrm{H}_k = \varphi\mathrm{H}_k \vartriangleleft \varphi'\mathrm{H}_k$.
Therefore H_k is a functor. □

Definition 12 (Quasi-isomorphism)
Suppose given complexes in groups X, Y; cf. Definition 7.
Suppose given a morphism of complexes in groups $\varphi \colon X \to Y$; cf. Definition 8.

For $k \in \mathbb{Z}$, we consider the group morphism $\varphi \mathrm{H}_k \colon X\mathrm{H}_k \to Y\mathrm{H}_k$; cf. Lemma 11.(2).

For $k \in \mathbb{Z}$ suppose that $\varphi \mathrm{H}_k$ is a group isomorphism.

Then we call φ a *quasi isomorphism*.

1.2.3 Functors

Remark 13 Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$.

We consider the functors ${}_{\mathcal{D}}\big((-)F, (=)\big) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}et$ and ${}_{\mathcal{C}}\big((-), (=)G\big) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}et$.

For $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$, suppose given a map $\varphi_{X,Y} \colon {}_{\mathcal{D}}(XF, Y) \to {}_{\mathcal{C}}(X, YG)$.

Then (1) and (2) are equivalent.

(1) We have the isotransformation

$$\varphi = (\varphi_{X,Y})_{(X,Y) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D})} \colon {}_{\mathcal{D}}\big((-)F, (=)\big) \xrightarrow{\sim} {}_{\mathcal{C}}\big((-), (=)G\big)$$

between the functors ${}_{\mathcal{D}}\big((-)F, (=)\big)$ and ${}_{\mathcal{C}}\big((-), (=)G\big)$.

(2) (i) The map $\varphi_{X,Y}$ is bijective for $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.

(ii) Suppose given $\big(X \xleftarrow{u} \tilde{X}\big)$ in $\mathcal{C}$ and $\big(Y \xrightarrow{v} \tilde{Y}\big)$ in $\mathcal{D}$.
Then, for $\big(XF \xrightarrow{a} Y\big) \in {}_{\mathcal{D}}(XF, Y)$, we have

$$(uF \blacktriangle a \blacktriangle v)\varphi_{\tilde{X},\tilde{Y}} = u \blacktriangle a\varphi_{X,Y} \blacktriangle vG\,.$$

So we have the following commutative diagram.

$$\begin{array}{ccc} {}_{\mathcal{D}}(XF, Y) & \xrightarrow[\sim]{\varphi_{X,Y}} & {}_{\mathcal{C}}(X, YG) \\ {\scriptstyle uF \blacktriangle (-) \blacktriangle v}\downarrow & & \downarrow{\scriptstyle u \blacktriangle (-) \blacktriangle vG} \\ {}_{\mathcal{D}}(\tilde{X}F, \tilde{Y}) & \xrightarrow[\varphi_{\tilde{X},\tilde{Y}}]{\sim} & {}_{\mathcal{C}}(\tilde{X}, \tilde{Y}G) \end{array}$$

Note that if (1) holds, we have $(X, Y)\varphi = \varphi_{X,Y}$ for $(X, Y) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D})$.

Remark 14 Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given functors $F\colon\ \mathcal{C} \to \mathcal{D}$ and $G\colon\ \mathcal{D} \to \mathcal{C}$.

Suppose that $(F, G, \varepsilon, \eta)$ is an adjunction; cf. §1.1.3 item 10.

Then we have an isotransformation

$$\varphi = \big((X,Y)\varphi\big)_{(X,Y)\in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}\times\mathcal{D})} = (\varphi_{X,Y})_{(X,Y)\in \mathrm{Ob}(\mathcal{C})\times\mathrm{Ob}(\mathcal{D})}$$

between the functors ${}_{\mathcal{D}}\big((-)F,(=)\big)\colon\ \mathcal{C}^{\mathrm{op}}\times\mathcal{D} \to \mathcal{S}et$ and ${}_{\mathcal{C}}\big((-),(=)G\big)\colon\ \mathcal{C}^{\mathrm{op}}\times\mathcal{D} \to \mathcal{S}et$, where for $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$, we have

$$\varphi_{X,Y}\colon {}_{\mathcal{D}}(XF,Y) \to {}_{\mathcal{C}}(X,YG)\,,\ \big(XF \xrightarrow{u} Y\big) \mapsto u\varphi_{X,Y} := \big(X \xrightarrow{X\varepsilon \centerdot uG} YG\big)\,,$$

and where

$$\varphi^{-}_{X,Y}\colon {}_{\mathcal{C}}(X,YG) \to {}_{\mathcal{D}}(XF,Y)\,,\ \big(X \xrightarrow{v} YG\big) \mapsto v\varphi^{-}_{X,Y} = \big(XF \xrightarrow{vF \centerdot Y\eta} Y\big)\,.$$

This bijective map $\varphi_{X,Y}$ is also called the *adjunction bijection* of the adjunction $(F, G, \varepsilon, \eta)$ of $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.

Remark 15 (Adjoint functors)

Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given functors $F\colon\ \mathcal{C} \to \mathcal{D}$ and $G\colon\ \mathcal{D} \to \mathcal{C}$.

Suppose given an isotransformation

$$\varphi = \big((X,Y)\varphi\big)_{(X,Y)\in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}\times\mathcal{D})} = (\varphi_{X,Y})_{(X,Y)\in\, \mathrm{Ob}(\mathcal{C})\times\mathrm{Ob}(\mathcal{D})}\colon {}_{\mathcal{D}}\big((-)F,(=)\big) \xrightarrow{\sim} {}_{\mathcal{C}}\big((-),(=)G\big)$$

cf. Remark 13.

Then we have the transformation $\varepsilon\colon\ \mathrm{id}_{\mathcal{C}} \to F * G$, where

$$X\varepsilon := (\mathrm{id}_{XF})\varphi_{X,XF} : X \to XFG\,,$$

for $X \in \mathrm{Ob}(\mathcal{C})$.

We have the transformation $\eta\colon G * F \to \mathrm{id}_{\mathcal{D}}$, where

$$Y\eta := (\mathrm{id}_{YG})\varphi^{-}_{YG,Y} : YGF \to Y\,,$$

for $Y \in \mathrm{Ob}(\mathcal{D})$.

Moreover, we have

$$(G * \varepsilon) \centerdot (\eta * G) = \mathrm{id}_G$$
$$(\varepsilon * F) \centerdot (F * \eta) = \mathrm{id}_F\ .$$

So $(F, G, \varepsilon, \eta)$ is an adjunction; cf. §1.1.3 item 10.
In particular, we have $F \dashv G$.

Remark 16 Suppose given categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$.
Suppose given functors $F\colon\ \mathcal{C} \to \mathcal{D}$, $F'\colon\ \mathcal{D} \to \mathcal{E}$, $G\colon\ \mathcal{E} \to \mathcal{D}$ and $G\colon\ \mathcal{D} \to \mathcal{C}$.

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

Suppose that we have isotransformations

$$\varphi\colon {}_{\mathcal{D}}\big((-)F, =\big) \xrightarrow{\sim} {}_{\mathcal{C}}\big(-, (=)G\big) \quad \text{and} \quad \varphi'\colon {}_{\mathcal{E}}\big((-)F', =\big) \xrightarrow{\sim} {}_{\mathcal{D}}\big(-, (=)G'\big)\,.$$

So $F \dashv G$ and $F' \dashv G'$; cf. Remark 15.
Writing $H_{\mathcal{C}} := {}_{\mathcal{C}}(-,=)\colon\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}et$, $H_{\mathcal{D}} := {}_{\mathcal{D}}(-,=)$ and $H_{\mathcal{E}} := {}_{\mathcal{E}}(-,=)$, we have

$$\begin{aligned} \varphi\colon &\quad (F^{\mathrm{op}} \times \mathrm{id}_{\mathcal{D}}) * H_{\mathcal{C}} &\xrightarrow{\sim}& \quad (\mathrm{id}_{\mathcal{C}} \times G) * H_{\mathcal{C}} \\ \varphi'\colon &\quad \big((F')^{\mathrm{op}} \times \mathrm{id}_{\mathcal{E}}\big) * H_{\mathcal{E}} &\xrightarrow{\sim}& \quad (\mathrm{id}_{\mathcal{D}} \times G') * H_{\mathcal{D}}\,. \end{aligned}$$

Note that

$$(F^{\mathrm{op}} \times \mathrm{id}_{\mathcal{E}})\varphi'\colon \big((F * F')^{\mathrm{op}} \times \mathrm{id}_{\mathcal{E}}\big) * H_{\mathcal{E}} \xrightarrow{\sim} (\mathrm{id}_{\mathcal{D}} \times G') * H_{\mathcal{D}}\,,$$

and that

$$(\mathrm{id}_{\mathcal{C}} \times G')\varphi\colon (F^{\mathrm{op}} \times G') * H_{\mathcal{D}} \xrightarrow{\sim} \big(\mathrm{id}_{\mathcal{C}} \times (G' * G)\big) * H_{\mathcal{C}}\,.$$

Then we have the isotransformation

$$\psi := (F^{\mathrm{op}} \times \mathrm{id}_{\mathcal{E}})\varphi' \mathbin{\blacktriangle} (\mathrm{id}_{\mathcal{C}} \times G')\varphi\colon {}_{\mathcal{E}}\big((-)FF', =\big) \xrightarrow{\sim} {}_{\mathcal{C}}\big(-, (=)G'G\big)\,.$$

So, for $X \in \mathrm{Ob}(\mathcal{C})$ and $Z \in \mathrm{Ob}(\mathcal{E})$, we have the bijection

$$\psi_{X,Z} = \varphi'_{XF,Z} \mathbin{\blacktriangle} \varphi_{X,ZG'}\colon \varepsilon(XFF',\ Z) \xrightarrow{\sim} \mathcal{C}(X,\ ZG'G)\,.$$

Therefore, we have $F * F' \dashv G' * G$.

Remark 17 *Suppose given categories $\mathcal{C}$, $\mathcal{D}$.*
Suppose given functors $F\colon\ \mathcal{C} \to \mathcal{D}$ and $G\colon\ \mathcal{D} \to \mathcal{C}$.
Suppose that we have isotransformations $\varphi\colon F * G \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ and $\varphi'\colon G * F \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$.

*So F and G are equivalences; cf. §*1.1.3 *item* 11.

We have the isotransformation

$$\psi = \big((X,Y)\psi\big)_{(X,Y)\in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}\times\mathcal{D})} = (\psi_{X,Y})_{(X,Y)\in \mathrm{Ob}(\mathcal{C})\times\mathrm{Ob}(\mathcal{D})} \colon {}_{\mathcal{D}}\big((-)F,=\big) \xrightarrow{\sim} {}_{\mathcal{C}}\big(-,(=)G\big)\,,$$

where

$$\psi_{X,Y} \colon {}_{\mathcal{D}}(XF,Y) \to {}_{\mathcal{C}}(X,YG)\,,\ \big(XF \xrightarrow{a} Y\big) \mapsto \big(X \xrightarrow{(X\varphi)^{-}\,\blacktriangle\, aG} YG\big)\,.$$

So we have $F \dashv G$.

Proof. We show that ψ is an isotransformation.

Suppose given $(X,Y) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$.

We show that $\psi_{X,Y}$ is bijective.

We have that $(X\varphi)^{-}$ is an isomorphism. Moreover, the map

$${}_{\mathcal{D}}(XF,Y) \to {}_{\mathcal{C}}(XFG,YG)\,,\ a \mapsto aG$$

is bijective since G is full and faithful.

So therefore, $\psi_{X,Y}$ is bijective.

Suppose given $X \xleftarrow{u} X'$ in $\mathcal{C}$ and $Y \xrightarrow{v} Y'$ in $\mathcal{D}$. Suppose given $a \in {}_{\mathcal{D}}(XF,Y)$.

We have

$$\begin{aligned} a\big({}_{\mathcal{D}}(uF,v)\,\blacktriangle\,\psi_{X',Y'}\big) &= (uF\,\blacktriangle\,a\,\blacktriangle\,v)\psi_{X',Y'} = (X'\varphi)^{-}\,\blacktriangle\,uFG\,\blacktriangle\,aG\,\blacktriangle\,vG = u\,\blacktriangle\,(X\varphi)^{-}\,\blacktriangle\,aG\,\blacktriangle\,vG \\ &= (X\varphi\,\blacktriangle\,aG)\,{}_{\mathcal{C}}(u,vG) = a\big(\psi_{X,Y}\,\blacktriangle\,{}_{\mathcal{C}}(u,vG)\big)\,. \end{aligned}$$

□

Part I

The ordinary case

Chapter 2

Simplicial groups

2.1 The simplex category

Definition 18 (Simplex category) Let $[n] := \{i \in \mathbb{Z} \colon 0 \leqslant i \leqslant n\}$ for $n \in \mathbb{Z}_{\geqslant 0}$. The subcategory Δ of $\mathcal{S}et$ with

$$\begin{aligned} \mathrm{Ob}(\Delta) &:= \{[n] \colon n \in \mathbb{Z}_{\geqslant 0}\} \\ \mathrm{Mor}(\Delta) &:= \{[n] \xrightarrow{u} [k] \colon k, n \in \mathbb{Z}_{\geqslant 0}, u \text{ is a monotone map.}\} \end{aligned}$$

is called *simplex category*.

Definition 19 We consider the simplex category Δ.

(1) Let $n \in \mathbb{Z}_{\geqslant 1}$. For $k \in [0, n]$, we have the morphism

$$\delta_k^n \colon [n-1] \to [n],\ i \mapsto i\delta_k^n = \begin{cases} i & \text{if } i < k \\ i+1 & \text{if } i \geqslant k\,. \end{cases}$$

Note that δ_k^n is injective. Sometimes, we write $\delta_k := \delta_k^n$.

(2) Let $n \in \mathbb{Z}_{\geqslant 0}$. For $k \in [0, n]$, we have the morphism

$$\sigma_k^n \colon [n+1] \to [n],\ i \mapsto i\sigma_k^n = \begin{cases} i & \text{if } i \leqslant k \\ i-1 & \text{if } i > k\,. \end{cases}$$

Note that σ_k^n is surjective. Sometimes, we write $\sigma_k := \sigma_k^n$.

The following argument in Lemma 20.(2) is due to [**8,** Lem. 10].

Lemma 20 *Suppose given a morphism* $[n] \xrightarrow{u} [k] \in \mathrm{Mor}(\Delta)$.

(1) *Suppose that* u *is an injective monotone map.*
Let $C_u := \{j \in [0,k] \colon j \notin \mathrm{im}\, u\} =: \{j_1, \ldots, j_{k-n}\}$, *where* $0 \leqslant j_1 < \ldots < j_{k-n} \leqslant k$.
Note that $C_u = \mathrm{coim}\, u$.

(1.1) *We have*
$$u = \prod_{\ell \in [1,k-n]}^{[n]\,[k]} \delta_{j_\ell}^{n+\ell} = \delta_{j_1}^{n+1} \blacktriangle \delta_{j_2}^{n+2} \blacktriangle \ldots \blacktriangle \delta_{j_{k-n}}^{k} .$$

(1.2) *Suppose given* $0 \leqslant t_1 < t_2 < \ldots < t_{k-n} \leqslant k$ *such that*
$$u = \prod_{\ell \in [1,k-n]}^{[n]\,[k]} \delta_{t_\ell}^{n+\ell} = \delta_{t_1}^{n+1} \blacktriangle \delta_{t_2}^{n+2} \blacktriangle \ldots \blacktriangle \delta_{t_{k-n}}^{k} .$$
Then we have $C_u = \{t_1, \ldots, t_{k-n}\}$, *i.e. we have* $t_\ell = j_\ell$ *for* $\ell \in [1, k-n]$.

(2) *Suppose that* u *is a surjective monotone map.*
Let $D_u := \{i \in [0, n-1] \colon (i)u = (i+1)u\} =: \{i_1, \ldots, i_{n-k}\}$, *where* $0 \leqslant i_1 < \ldots < i_{n-k} \leqslant n-1$.

(2.1) *We have*
$$u = \prod_{\ell \in \lfloor n-k,1 \rfloor}^{[n]\,[k]} \sigma_{i_\ell}^{k+\ell-1} = \sigma_{i_{n-k}}^{n-1} \blacktriangle \sigma_{i_{n-k-1}}^{n-2} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{k} .$$

(2.2) *Suppose given* $0 \leqslant t_1 < t_2 < \ldots < t_{n-k} \leqslant n-1$ *such that*
$$u = \prod_{\ell \in \lfloor n-k,1 \rfloor}^{[n]\,[k]} \sigma_{t_\ell}^{k+\ell-1} = \sigma_{t_{n-k}}^{n-1} \blacktriangle \sigma_{t_{n-k-1}}^{n-2} \blacktriangle \ldots \blacktriangle \sigma_{t_1}^{k} .$$
Then we have $D_u = \{t_1, \ldots, t_{n-k}\}$, *i.e. we have* $t_\ell = i_\ell$ *for* $\ell \in [1, n-k]$.

Proof. Ad (1.1). Note that $k \geqslant n$, since u is injective.
We show this assertion by induction on $k \in \mathbb{Z}_{\geqslant n}$.
Suppose that $k = n$.

Then we have $u = \mathrm{id}_{[n]}$ and we have $C_u = \emptyset$.

Suppose that $k \geqslant n+1$.

Now we assume that (1.1) holds for all monotone injective maps $[n] \to [k-1]$.

Since $0 \leqslant n < k$, the map u is not surjective. So $C_u \neq \emptyset$.

Let $d := \max C_u = j_{k-n} \in [0,k]$.

We show that $u \blacktriangle \sigma_d^{k-1} \blacktriangle \delta_d^k \overset{!}{=} u$.

Let $x \in [n]$. Note that $xu \neq d$.

Case $xu \leqslant d-1$. We have

$$x(u \blacktriangle \sigma_d^{k-1} \blacktriangle \delta_d^k) = (xu)(\sigma_d^{k-1} \blacktriangle \delta_d^k) = (xu)\delta_d^k = xu\,.$$

Case $xu \geqslant d+1$. We have

$$x(u \blacktriangle \sigma_d^{k-1} \blacktriangle \delta_d^k) = (xu)(\sigma_d^{k-1} \blacktriangle \delta_d^k) = (xu-1)\delta_d^k = xu\,.$$

This shows $u \blacktriangle \sigma_d^{k-1} \blacktriangle \delta_d^k = u$.

We consider the morphism $u \blacktriangle \sigma_d^{k-1} \colon [n] \to [k-1]$ in Δ.

The map $u \blacktriangle \sigma_d^{k-1} = u\big|^{[k]\setminus\{d\}} \blacktriangle \sigma_d^{k-1}\big|_{[k]\setminus\{d\}}$ is injective since u is injective and since $d \in C_u$.

$$\begin{array}{ccccc}
[n] & \xrightarrow{\quad u \quad} & [k] & \xrightarrow{\ \sigma_d^{k-1}\ } & [k-1] \\
a+k-d-1 & \longrightarrow & k & \searrow & \\
\vdots & & \vdots & & k-1 \\
a & \longrightarrow & d+1 & & \vdots \\
\vdots & & d & \longrightarrow & d \\
\vdots & & \vdots & & \vdots
\end{array}$$

We show that $C_u \setminus \{d\} \overset{!}{=} C_{u \blacktriangle \sigma_d^{k-1}}$.

Ad $\subseteq$. Let $x \in C_u$ such that $x < d$. In particular, $x \leqslant k-1$.

Now we *assume* that there exists some $y \in [n]$ such that $x = y(u \blacktriangle \sigma_d^{k-1})$.

Case $yu \leqslant d$. We have

$$x = y(u \blacktriangle \sigma_d^{k-1}) = (yu)\sigma_d^{k-1} = yu\,,$$

which *contradicts* $x \in C_u$.

Case $yu \geqslant d+1$. We have

$$x = y(u \blacktriangle \sigma_d^{k-1}) = (yu)\sigma_d^{k-1} = yu - 1 \geqslant d+1-1 = d\,,$$

which *contradicts* $x < d$.

So $x \in C_{u \blacktriangle \sigma_d^{k-1}}$.

This shows $C_u \backslash \{d\} \subseteq C_{u \blacktriangle \sigma_d^{k-1}}$.

Ad $\supseteq$. Let $x \in C_{u \blacktriangle \sigma_d^{k-1}} \subseteq [k-1]$. We have to show that $x \overset{!}{\in} C_u$ and that $x \overset{!}{\neq} d$.

We *assume* that $x \geqslant d$. Then we have $zu = x+1$ for some $z \in [n]$ since $k \geqslant x+1 \geqslant d+1 > d$.

So we have

$$x = (x+1)\sigma_d^{k-1} = (zu)\sigma_d^{k-1} = z(u \blacktriangle \sigma_d^{k-1})\,,$$

which *contradicts* $x \in C_{u \blacktriangle \sigma_d^{k-1}}$.

This show $x \leqslant d-1$, in particular $x \neq d$.

We *assume* that there exists some $y \in [n]$ such that $yu = x$.

Using $x \leqslant d-1$, we obtain

$$x = x\sigma_d^{k-1} = (yu)\sigma_d^{k-1} = y(u \blacktriangle \sigma_d^{k-1})\,,$$

which *contradicts* $x \in C_{u \blacktriangle \sigma_d^{k-1}}$.

This shows $x \in C_u$.

This shows $C_u \backslash \{d\} \supseteq C_{u \blacktriangle \sigma_d^{k-1}}$.

So we have $C_u \backslash \{d\} = C_{u \blacktriangle \sigma_d^{k-1}}$.

By the induction hypothesis we have

$$u \blacktriangle \sigma_d^{k-1} = \delta_{j_1}^{n+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-n-1}}^{k-1}\,,$$

where $C_{u \blacktriangle \sigma_d^{k-1}} = C_u \backslash \{d\} = \{j_1, \ldots, j_{k-n-1}\}$ and $j_1 < \ldots < j_{k-n-1}$.

So we have

$$u = u \blacktriangle \sigma_d^{k-1} \blacktriangle \delta_d^k = (\delta_{j_1}^{n+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-n-1}}^{k-1}) \blacktriangle \delta_{j_{k-n}}^k\,,$$

where $C_u = C_{u \blacktriangle \sigma_d^{k-1}} \cup \{d\} = \{j_1, \ldots, j_{k-n-1}, j_{k-n}\}$ and $j_1 < \ldots < j_{k-n}$.

Ad (1.2). Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. We show this assertion by induction on $k \in \mathbb{Z}_{\geqslant n}$.

Suppose that $k = n$.

Then we have $u = \prod_{\ell \in \lceil 1,0 \rceil}^{[n]} \delta_{t_\ell}^{n+\ell} = \mathrm{id}_{[n]}$. Moreover, since $u = \mathrm{id}_{[n]}$ is surjective, we have $C_u = \emptyset$.

Suppose that $k \geqslant n+1$.

Now we assume that (1.2) holds for all monotone injective maps $[n] \to [k-1]$.

We write $\tilde{u} := \prod_{\ell \in \lceil 1,k-n-1 \rceil}^{[n]} \delta_{t_\ell}^{n+\ell}$. By induction hypothesis, we have $C_{\tilde{u}} = \{t_1, t_2, \ldots, t_{k-n-1}\}$.

Note that $\tilde{u} \blacktriangle \delta_{t_{k-n}}^k = u$. We *claim* that $C_u \overset{!}{=} C_{\tilde{u}} \cup \{t_{k-n}\}$.

Ad $\subseteq$. Suppose given $x \in C_u$. We may assume that $x \neq t_{k-n}$.

Case $x > t_{k-n}$. Then $x - 1 \geqslant t_{k-n} > t_{k-n-1}$. So $x - 1 \notin C_{\tilde{u}}$.

Therefore, there exists some $y \in [n]$ with $x - 1 = y\tilde{u}$.

We have

$$yu = y(\tilde{u} \blacktriangle \delta_{t_{k-n}}^k) = (y\tilde{u})\delta_{t_{k-n}}^k = (x-1)\delta_{t_{k-n}}^k = x\,.$$

This is impossible, since $x \in C_u$. So this case does not occur.

Case $x < t_{k-n}$. We *assume* that $x \notin C_{\tilde{u}}$. Then there exists some $y \in [n]$ such that $x = y\tilde{u}$.

We have

$$yu = y(\tilde{u} \blacktriangle \delta_{t_{k-n}}^k) = x\delta_{t_{k-n}}^k = x\,,$$

which *contradicts* $x \in C_u$.

This shows that $x \in C_{\tilde{u}} \cup \{t_{k-n}\}$.

Ad $\supseteq$. Suppose given $x \in C_{\tilde{u}} \cup \{t_{k-n}\}$.

Since $t_{k-n} \notin \operatorname{im} \delta_{t_{k-n}}^k$ and $\operatorname{im} u \subseteq \operatorname{im} \delta_{t_{k-n}}^k$, we have $t_{k-n} \in C_u$.

So we have $x \in C_{\tilde{u}}$. In particular, we have $x < t_{k-n}$.

We *assume* that $x \notin C_u$. Then there exists some $y \in [n]$ such that $x = yu$.

We consider $y\tilde{u} \in [0, k-1]$.

If $y\tilde{u} < t_{k-n}$, then we have

$$x = yu = y(\tilde{u} \blacktriangle \delta_{t_{k-n}}^k) = (y\tilde{u})\delta_{t_{k-n}}^k = y\tilde{u}\,,$$

which *contradicts* $x \in C_{\tilde{u}}$.

If $y\tilde{u} \geqslant t_{k-n}$, then we have

$$x = yu = y(\tilde{u} \blacktriangle \delta^k_{t_{k-n}}) = (y\tilde{u})\delta^k_{t_{k-n}} = y\tilde{u} + 1 \geqslant t_{k-n} + 1\,,$$

which *contradicts* $x < t_{k-n}$.

This shows $x \in C_u$.

Ad (2.1). Note that $k \leqslant n$, since u is surjective. We show this assertion by induction on $n - k \in \mathbb{Z}_{\geqslant 0}$.

Suppose that $n - k = 0$.

Then we have $u = \mathrm{id}_{[n]}$ and we have $D_u = \emptyset$.

Suppose that $n - k \geqslant 1$.

Now we assume that (2.1) holds for all monotone surjective maps $[n-1] \to [k]$.

Since $0 \leqslant k < n$, the map u is not injective. So $D_u \neq \emptyset$.

Let $e := \max D_u = i_{n-k}$. Note that $e \in [0, n-1]$ and that $eu = (e+1)u$.

We show that $\sigma^{n-1}_e \blacktriangle \delta^n_{e+1} \blacktriangle u \overset{!}{=} u$.

Let $x \in [n]$.

Case $x \leqslant e$. We have

$$x(\sigma^{n-1}_e \blacktriangle \delta^n_{e+1} \blacktriangle u) = x(\delta^n_{e+1} \blacktriangle u) = xu\,.$$

Case $x = e + 1$. We have

$$x(\sigma^{n-1}_e \blacktriangle \delta^n_{e+1} \blacktriangle u) = e\delta^n_{e+1}u = eu = (e+1)u = xu\,.$$

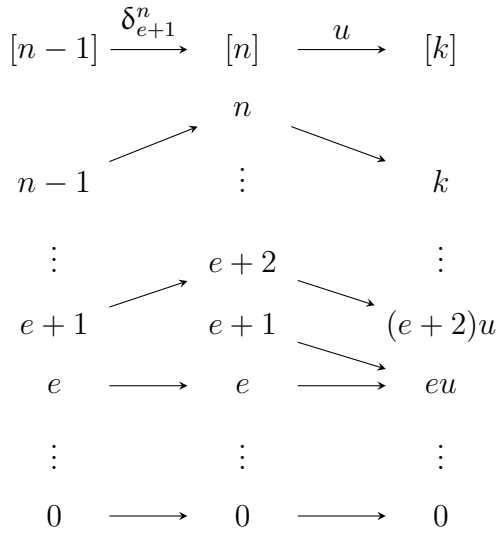
Case $x \geqslant e + 2$. We have

$$x(\sigma^{n-1}_e \blacktriangle \delta^n_{e+1} \blacktriangle u) = (x-1)\delta^n_{e+1}u = xu\,.$$

This shows $\sigma^{n-1}_e \blacktriangle \delta^n_{e+1} \blacktriangle u = u$.

Consider the morphism $\delta^n_{e+1} \blacktriangle u \colon [n-1] \to [k]$ in Δ.

The map $\delta^n_{e+1} \blacktriangle u$ is surjective since u is surjective and since $e \in D_u$.



We show that $D_u\backslash\{e\} \overset{!}{=} D_{\delta^n_{e+1} \blacktriangle u}$.

Ad $\subseteq$. Let $x \in D_u\backslash\{e\}$. Note that $x < e$ since e is maximal in D_u.

We have

$$x(\delta^n_{e+1} \blacktriangle u) = xu = (x+1)u = (x+1)(\delta^n_{e+1} \blacktriangle u)\,.$$

This shows $x \in D_{\delta^n_{e+1} \blacktriangle u}$.

Ad $\supseteq$. Let $x \in D_{\delta^n_{e+1} \blacktriangle u}$. In particular, $x \in [0, n-2]$.

We *assume* that $x \geqslant e+1$. Then

$$(x+1)u = x(\delta^n_{e+1} \blacktriangle u) = (x+1)(\delta^n_{e+1} \blacktriangle u) = (x+2)u\,.$$

So $e < x+1 \in D_u$ which *contradicts* the maximality of e.

We *assume* that $x = e$. Then

$$(x+1)u = xu = x(\delta^n_{e+1} \blacktriangle u) = (x+1)(\delta^n_{e+1} \blacktriangle u) = (x+2)u\,.$$

So $x+1 = e+1 \in D_u$ which *contradicts* the maximality of e.

So we have $x \leqslant e-1$.

We have

$$xu = x(\delta^n_{e+1} \blacktriangle u) = (x+1)(\delta^n_{e+1} \blacktriangle u) = (x+1)u\,.$$

Therefore, we have $x \in D_u$.

This shows $D_u \backslash \{e\} = D_{\delta^n_{e+1} \blacktriangle u}$.

By induction hypothesis we have

$$\delta^n_{e+1} \blacktriangle u = \sigma^{n-2}_{i_{n-k-1}} \blacktriangle \sigma^{n-3}_{i_{n-k-2}} \blacktriangle \ldots \blacktriangle \sigma^k_{i_1},$$

where $D_{\delta^n_{e+1} \blacktriangle u} = \{i_1, \ldots, i_{n-k-1}\}$ and $i_1 < \cdots < i_{n-k-1}$.

So we have

$$u = \sigma^{n-1}_e \blacktriangle \delta^n_{e+1} \blacktriangle u = \sigma^{n-1}_{i_{n-k}} \blacktriangle (\sigma^{n-2}_{i_{n-k-1}} \blacktriangle \sigma^{n-3}_{i_{n-k-2}} \blacktriangle \ldots \blacktriangle \sigma^k_{i_1}),$$

where $D_u = D_{\delta^n_{e+1} \blacktriangle u} \cup \{e\} = \{i_1, \ldots, i_{n-k-1}, i_{n-k}\}$ and $i_1 < \cdots < i_{n-k}$.

Ad (2.2). We show this assertion by induction on $n - k \in \mathbb{Z}_{\geqslant 0}$.

Suppose that $n - k = 0$. Then we have $u = \prod^{[n]}_{\ell \in \lfloor 0,1 \rfloor} \sigma^{n+\ell-1}_{t_\ell} = \mathrm{id}_{[n]}$. Moreover, since $u = \mathrm{id}_{[n]}$ is injective, we have $D_u = \emptyset$.

Suppose that $n - k \geqslant 1$.

Now we assume that (2.2) holds for all monotone surjective maps $[n-1] \to [k]$.

We write $\tilde{u} := \prod^{[k]}_{\ell \in \lfloor n-1-k, 1 \rfloor} \sigma^{k+\ell-1}_{t_\ell}$. By induction hypothesis, we have $D_{\tilde{u}} = \{t_1, t_2, \ldots, t_{k-n-1}\}$.

Note that $u = \sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}$. We *claim* that $D_u \overset{!}{=} D_{\tilde{u}} \cup \{t_{n-k}\}$.

Ad $\subseteq$. Suppose given $x \in D_u$. We may assume that $x \neq t_{n-k}$.

Case $x > t_{n-k}$. We have

$$\begin{aligned} x\tilde{u} &= \big((x+1)\sigma^{n-1}_{t_{n-k}}\big)\tilde{u} = (x+1)(\sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}) = (x+1)u = xu = x(\sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}) = (x\sigma^{n-1}_{t_{n-k}})\tilde{u} \\ &= (x-1)\tilde{u}. \end{aligned}$$

This shows that $x - 1 \in D_{\tilde{u}} = \{t_1, \ldots, t_{n-k-1}\}$.

So we have

$$x \leqslant t_{n-k-1} + 1 \leqslant t_{n-k},$$

which *contradicts* $x > t_{n-k}$. So this case does not occur.

Case $x < t_{n-k}$. We have

$$\begin{aligned} x\tilde{u} &= (x\sigma^{n-1}_{t_{n-k}})\tilde{u} = x(\sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}) = xu = (x+1)u = (x+1)(\sigma^{n-1}_{t_{k-n}} \blacktriangle \tilde{u}) = \big((x+1)\sigma^{n-1}_{t_{n-k}}\big)\tilde{u} \\ &= (x+1)\tilde{u}. \end{aligned}$$

This shows $x \in D_{\tilde{u}} \cup \{t_{n-k}\}$.

Ad $\supseteq$. Suppose given $x \in D_{\tilde{u}} \cup \{t_{n-k}\}$.

Case $x = t_{n-k}$. We have

$$xu = x(\sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}) = x\tilde{u} = \big((x+1)\sigma^{n-1}_{t_{n-k}}\big)\tilde{u} = (x+1)(\sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}) = (x+1)u\,.$$

This shows $x \in D_u$.

Case $x \in D_{\tilde{u}}$. Note that we have $x \leqslant t_{n-k} - 1$.

We have

$$xu = x(\sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}) = (x\sigma^{n-1}_{t_{n-k}})\tilde{u} = x\tilde{u} = (x+1)\tilde{u} = \big((x+1)\sigma^{n-1}_{t_{n-k}}\big)\tilde{u} = (x+1)(\sigma^{n-1}_{t_{n-k}} \blacktriangle \tilde{u}) = (x+1)u\,.$$

This shows $x \in D_u$. □

Remark 21 *Suppose given* $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$.

Suppose that we have

$$u = v \blacktriangle w \quad \text{and} \quad u = \tilde{v} \blacktriangle \tilde{w}$$

with $[n] \xrightarrow{v} [\ell] \xrightarrow{w} [k]$ *and* $[n] \xrightarrow{\tilde{v}} [\tilde{\ell}] \xrightarrow{\tilde{w}} [k]$ *in* $\mathrm{Mor}(\Delta)$, *where* v, $\tilde{v}$ *are surjective and where* w, $\tilde{w}$ *are injective.*

Then we have $\ell = \tilde{\ell}$, $v = \tilde{v}$ *and* $w = \tilde{w}$.

Proof. We show that $\ell \stackrel{!}{=} \tilde{\ell}$.

We consider $\mathrm{coim}\, u = [k] \setminus \mathrm{im}\, u$; cf. §1.1.1 item 8.

We show that $|\,\mathrm{coim}\, u| \stackrel{!}{=} k - \ell$.

Since the morphism $v\colon [n] \to [\ell]$ is surjective, we have

$$\mathrm{im}\, w = \mathrm{im}(v \blacktriangle w) = \mathrm{im}\, u\,.$$

In particular, we have $|\,\mathrm{coim}\, u| = |\,\mathrm{coim}\, w|$.

Since the morphism $w\colon [\ell] \to [k]$ is injective, we have $|\,\mathrm{im}\, w| = \ell + 1$. So $|\,\mathrm{coim}\, w| = (k+1) - (\ell+1) = k - \ell$.

So we have

$$|\,\mathrm{coim}\, u| = |\,\mathrm{coim}\, w| = k - \ell\,.$$

With the same argument, we get $|\,\mathrm{coim}\, u| = k - \tilde{\ell}$.

This shows $\ell = |k - \operatorname{coim} u| = \tilde{\ell}$.

We show that $w \overset{!}{=} \tilde{w}$. Since v and $\tilde{v}$ are surjective, we have

$$\operatorname{coim} w = \operatorname{coim}(v \blacktriangle w) = \operatorname{coim} u = \operatorname{coim}(\tilde{v} \blacktriangle \tilde{w}) = \operatorname{coim} \tilde{w}\,.$$

Hence $\operatorname{im} w = \operatorname{im} \tilde{w}$.

So, since w, $\tilde{w}\colon [\ell] \to [k]$ are injective and monotone, we have $w = \tilde{w}$.

We show that $v \overset{!}{=} \tilde{v}$. We have

$$v \blacktriangle w = u = \tilde{v} \blacktriangle \tilde{w} = \tilde{v} \blacktriangle w\,.$$

Since w is injective, we have $v = \tilde{v}$. □

Corollary 22 *Suppose given a morphism* $\big([n] \xrightarrow{u} [k]\big) \in \operatorname{Mor}(\Delta)$.
Let $\ell := |\operatorname{im} u| - 1$.
Let $D_u := \{i \in [0,n]\colon (i)u = (i+1)u\} =: \{i_1, \ldots, i_{n-\ell}\}$, *where*
$0 \leqslant i_1 < i_2 < \ldots < i_{n-\ell} \leqslant n-1$.
Let $C_u := \{j \in [0,k]\colon j \notin \operatorname{im} u\} =: \{j_1, \ldots, j_{k-\ell}\}$, *where* $0 \leqslant j_1 < j_2 < \ldots < j_{k-\ell} \leqslant k$.
Note that $C_u = \operatorname{coim} u$.

(1) *We have*

$$u = \left(\prod_{t \in \lfloor n-\ell, 1 \rfloor}^{[n]} {}^{[\ell]} \sigma_{i_t}^{\ell+t-1} \right) \blacktriangle \left(\prod_{t \in \lceil 1, k-\ell \rceil}^{[\ell]} {}^{[k]} \delta_{j_t}^{\ell+t} \right) = \sigma_{i_{n-\ell}}^{n-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\ell} \blacktriangle \delta_{j_1}^{\ell+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-\ell}}^{k}\,.$$

(2) *Suppose given* $\ell' \in \big[0, \min\{n,k\}\big]$ *and* $0 \leqslant i'_1 < \ldots < i'_{n-\ell'} \leqslant n-1$ *and* $0 \leqslant j'_1 < \ldots < j'_{k-\ell'} \leqslant k$ *such that*

$$u = \left(\prod_{t \in \lfloor n-\ell', 1 \rfloor}^{[n]} {}^{[\ell']} \sigma_{i'_t}^{\ell'+t-1} \right) \blacktriangle \left(\prod_{t \in \lceil 1, k-\ell' \rceil}^{[\ell']} {}^{[k]} \delta_{j'_t}^{\ell'+t} \right) = \sigma_{i'_{n-\ell'}}^{n-1} \blacktriangle \ldots \blacktriangle \sigma_{i'_1}^{\ell'} \blacktriangle \delta_{j'_1}^{\ell'+1} \blacktriangle \ldots \blacktriangle \delta_{j'_{k-\ell'}}^{k}\,.$$

Then we have $\ell = \ell'$, $D_u = \{i'_1, \ldots, i'_{n-\ell}\}$ *and* $C_u = \{j'_1, \ldots, j'_{k-\ell}\}$.
In particular, we have $i_t = i'_t$ *for* $t \in [1, n-\ell]$ *and* $j_t = j'_t$ *for* $t \in [1, k-\ell]$.

(3) *Suppose given* $\ell' \in \big[0, \min\{n,k\}\big]$ *and* $0 \leqslant i'_1 < \ldots < i'_{n-\ell'} \leqslant n-1$ *and* $0 \leqslant j'_1 < \ldots < j'_{k-\ell'} \leqslant k$ *such that*

$$u = \left(\prod_{t\in\lfloor n-\ell', 1\rfloor}^{[n]\ [\ell']} \sigma_{i'_t}^{\ell'+t-1} \right) \blacktriangle \left(\prod_{t\in\lceil 1, k-\ell'\rceil}^{[\ell']\ [k]} \delta_{j'_t}^{\ell'+t} \right) = \sigma_{i'_{n-\ell'}}^{n-1} \blacktriangle \ldots \blacktriangle \sigma_{i'_1}^{\ell'} \blacktriangle \delta_{j'_1}^{\ell'+1} \blacktriangle \ldots \blacktriangle \delta_{j'_{k-\ell'}}^{k} .$$

Suppose given $\ell'' \in \big[0, \min\{n,k\}\big]$ *and* $0 \leqslant i''_1 < \ldots < i''_{n-\ell''} \leqslant n-1$ *and* $0 \leqslant j''_1 < \ldots < j''_{k-\ell''} \leqslant k$ *such that*

$$u = \left(\prod_{t\in\lfloor n-\ell'', 1\rfloor}^{[n]\ [\ell'']} \sigma_{i''_t}^{\ell''+t-1} \right) \blacktriangle \left(\prod_{t\in\lceil 1, k-\ell''\rceil}^{[\ell'']\ [k]} \delta_{j''_t}^{\ell''+t} \right) = \sigma_{i''_{n-\ell''}}^{n-1} \blacktriangle \ldots \blacktriangle \sigma_{i''_1}^{\ell''} \blacktriangle \delta_{j''_1}^{\ell''+1} \blacktriangle \ldots \blacktriangle \delta_{j''_{k-\ell''}}^{k} .$$

Then we have $\ell' = \ell''$ *and* $i'_t = i''_t$ *for* $t \in [1, n-\ell']$, *and* $j'_t = j''_t$ *for* $t \in [1, k-\ell']$.

Proof. Ad (1). We write $\operatorname{im} u =: \{s_0, \ldots, s_\ell\}$ with $0 \leqslant s_0 < \ldots < s_\ell \leqslant k$.

We consider the map

$$\bar{u} \colon [n] \to [\ell],\ x \mapsto x\bar{u}$$

with $s_{x\bar{u}} := xu$.

Then $\bar{u}$ is surjective, for given $i \in [\ell]$, we find $x \in [u]$ with $xu = s_i$, hence $x\bar{u} = i$. Moreover, $\bar{u}$ is monotone, since given $x, y \in [n]$ with $x \leqslant y$, we have $s_{x\bar{u}} = xu \leqslant yu = s_{y\bar{u}}$, hence $x\bar{u} \leqslant y\bar{u}$.

Let $D_{\bar{u}} = \{i \in [0,n] \colon (i)\bar{u} = (i+1)\bar{u}\}$. *We show that* $D_u \stackrel{!}{=} D_{\bar{u}}$.

Ad $\subseteq$. Suppose given $x \in D_u$. We have

$$s_{x\bar{u}} = xu = (x+1)u = s_{(x+1)\bar{u}} .$$

This shows $x\bar{u} = (x+1)\bar{u}$. So $x \in D_{\bar{u}}$.

Ad $\supseteq$. Suppose given $x \in D_{\bar{u}}$. We have

$$xu = s_{x\bar{u}} = s_{(x+1)\bar{u}} = (x+1)u .$$

So $x \in D_u$.

Therefore, we have $D_u = D_{\bar{u}}$.

Since $\bar{u}$ is surjective, we have, by Lemma 20.(2.1),

$$\bar{u} = \sigma_{i_{n-\ell}}^{n-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\ell} ,$$

where we recall that $i_r \in D_{\bar{u}} = D_u$ for $r \in [1, n-\ell]$.

We consider the map

$$\dot{u}\colon [\ell] \to [k],\ x \mapsto s_x\,.$$

Then $\dot{u}$ is injective and monotone.

We show that $C_u \overset{!}{=} C_{\dot{u}}$.

Ad $\subseteq$. Suppose given $x \in C_u$. We *assume* that $x \notin C_{\dot{u}}$. Then we may choose some $y \in [\ell]$ with $x = y\dot{u}$. We have

$$x = y\dot{u} = s_y \in \operatorname{im} u\,,$$

which *contradicts* $x \in C_u$.

This shows $x \in C_{\dot{u}}$.

Ad $\supseteq$. Suppose given $x \in C_{\dot{u}}$. We *assume* that $x \notin C_u$. Then there exists some $y \in [n]$ such that $x = yu$. We have

$$x = yu = s_{y\bar{u}} \in \operatorname{im} \dot{u}\,,$$

which *contradicts* $x \in C_{\dot{u}}$.

This shows $x \in C_u$.

Therefore, we have $C_u = C_{\dot{u}}$.

Since $\dot{u}$ is injective, we have, by Lemma 20.(1.1),

$$\dot{u} = \delta^{\ell+1}_{j_1} \mathbin{\blacktriangle} \ldots \mathbin{\blacktriangle} \delta^{k}_{j_{k-\ell}}\,,$$

where we recall that $j_r \in C_{\dot{u}} = C_u$ for $r \in [1, k-\ell]$.

Further, for $x \in [n]$, we have

$$x(\bar{u} \mathbin{\blacktriangle} \dot{u}) = (x\bar{u})\dot{u} = s_{x\bar{u}} = xu\,.$$

This shows $\bar{u} \mathbin{\blacktriangle} \dot{u} = u$.

So altogether, we have

$$u = \bar{u} \mathbin{\blacktriangle} \dot{u} = \sigma^{n-1}_{i_{n-\ell}} \mathbin{\blacktriangle} \ldots \mathbin{\blacktriangle} \sigma^{\ell}_{i_1} \mathbin{\blacktriangle} \delta^{\ell+1}_{j_1} \mathbin{\blacktriangle} \ldots \mathbin{\blacktriangle} \delta^{k}_{j_{k-\ell}}\,.$$

Ad (2). We write $\operatorname{im} u =: \{s_0, \ldots, s_\ell\}$ with $0 \leqslant s_0 < \ldots < s_\ell \leqslant k$.

Let

$$\sigma' := \prod_{t \in \lfloor n-\ell', 1 \rfloor}^{[n]} \prod^{[\ell']} \sigma^{\ell'+t-1}_{i'_t} = \sigma^{n-1}_{i'_{n-\ell'}} \mathbin{\blacktriangle} \ldots \mathbin{\blacktriangle} \sigma^{\ell'}_{i'_1}$$

and let

$$\delta' := \prod_{t\in\lceil 1,k-\ell'\rceil}^{[k]}{}^{[\ell']} \delta^{\ell'+t}_{j'_t} = \delta^{\ell'+1}_{j'_1} \blacktriangle \ldots \blacktriangle \delta^{k}_{j'_{k-\ell'}} .$$

We have

$$u = \sigma' \blacktriangle \delta' ,$$

where σ' is surjective and where δ' is injective.

By (1), we have

$$u = \bar{u} \blacktriangle \dot{u}$$

where

$$\bar{u} = \prod_{t\in\lfloor n-\ell,1\rfloor}^{[n]}{}^{[\ell]} \sigma^{\ell+t-1}_{i_t} = \sigma^{n-1}_{i_{n-\ell}} \blacktriangle \ldots \blacktriangle \sigma^{\ell}_{i_1}$$

is surjective, and where

$$\dot{u} = \prod_{t\in\lceil 1,k-\ell\rceil}^{[k]}{}^{[\ell]} \delta^{\ell+t}_{j_t} = \delta^{\ell+1}_{j_1} \blacktriangle \ldots \blacktriangle \delta^{k}_{j_{k-\ell}}$$

is injective.

Then, by Remark 21, we have $\ell = \ell'$, $\sigma' = \bar{u}$ and $\delta' = \dot{u}$. In particular, we have $D_{\bar{u}} = D_{\sigma'}$ and $C_{\dot{u}} = C_{\delta'}$.

Since $\sigma' = \bar{u}$ is surjective, we have, by the proof of (1) and by Lemma 20.(2.2),

$$D_u = D_{\sigma'} = D_{\bar{u}} = \{i \in [0, n-1] \colon (i)\bar{u} = (i+1)\bar{u}\} = \{i'_1 , i'_2 , \ldots , i'_{n-\ell}\} .$$

Since $\delta' = \dot{u}$ is injective, we have, by the proof of (1) and by Lemma 20.(1.2),

$$C_u = C_{\delta'} = C_{\dot{u}} = \{j \in [0, k] \colon j \notin \operatorname{im} \dot{u}\} = \{j'_1 , j'_2 , \ldots , j'_{k-\ell}\} .$$

Ad (3). By (2), we have $\ell = \ell'$, $D_u = \{i'_1 , \ldots , i'_{n-\ell}\}$ and $C_u = \{j'_1, \ldots , j'_{k-\ell}\}$.

Moreover, we have $\ell = \ell''$, $D_u = \{i''_1 , \ldots , i''_{n-\ell}\}$ and $C_u = \{j''_1, \ldots , j''_{k-\ell}\}$.

This shows $\ell' = \ell''$ and $\{i'_1 , \ldots , i'_{n-\ell}\} = \{i''_1 , \ldots , i''_{n-\ell}\}$ and $\{j'_1, \ldots , j'_{k-\ell}\} = \{j''_1, \ldots , j''_{k-\ell}\}$. □

Remark 23 We consider the simplex category $\boldsymbol{\Delta}$.

Suppose given $n \in \mathbb{Z}_{\geqslant 0}$ and $k \in [0, n]$.

For $m \in [0, n-1]$, we have

$$m\delta^n_k = m + \chi_{m\geqslant k} .$$

For $m \in [0, n+1]$, we have

$$m\sigma_k^n = m - \chi_{m>k}\,.$$

Cf. Definition 19, §1.1.1 item 7.

Remark 24 *We consider the simplex category* Δ.

(1) *Let* $n \in \mathbb{Z}_{\geqslant 1}$. *We have*

$$\delta_i^n \blacktriangle \delta_j^{n+1} = \delta_{j-1}^n \blacktriangle \delta_i^{n+1}$$

for $i \in [0, n]$, $j \in [0, n+1]$ *with* $i < j$.

(2) *Let* $n \in \mathbb{Z}_{\geqslant 1}$. *We have*

$$\sigma_j^n \blacktriangle \sigma_i^{n-1} = \sigma_i^n \blacktriangle \sigma_{j-1}^{n-1}$$

for $i \in [0, n-1]$, $j \in [0, n]$ *with* $i < j$.

(3) *Let* $n \in \mathbb{Z}_{\geqslant 0}$. *We have*

$$\delta_j^{n+1} \blacktriangle \sigma_i^n = \begin{cases} \sigma_{i-1}^{n-1} \blacktriangle \delta_j^n & \text{if } n \geqslant 1 \text{ and } j < i \\ \mathrm{id}_{[n]} & \text{if } j = i \text{ or } j = i+1 \\ \sigma_i^{n-1} \blacktriangle \delta_{j-1}^n & \text{if } n \geqslant 1 \text{ and } j > i+1\,, \end{cases}$$

for $i \in [0, n]$, $j \in [0, n+1]$.

Proof. We are using Remark 23.

Ad (1). We have $0 \leqslant i < j \leqslant n+1$. Suppose given $m \in [0, n-1]$.

We have

$$\begin{array}{rcl} m(\delta_i^n \blacktriangle \delta_j^{n+1}) & = & (m + \chi_{m \geqslant i})\delta_j^{n+1} \\ & = & m + \chi_{m \geqslant i} + \chi_{(m+\chi_{m \geqslant i}) \geqslant j} \\ & \overset{j \geqslant i+1}{=} & m + \chi_{m \geqslant i} + \chi_{m+1 \geqslant j} \\ & = & m + \chi_{m \geqslant j-1} + \chi_{m \geqslant i} \\ & \overset{j-1 \geqslant i}{=} & m + \chi_{m \geqslant j-1} + \chi_{(m+\chi_{m \geqslant j-1}) \geqslant i} \\ & = & (m + \chi_{m \geqslant j-1})\delta_i^{n+1} \\ & = & m(\delta_{j-1}^n \blacktriangle \delta_i^{n+1})\,. \end{array}$$

This shows $\delta_i^n \blacktriangle \delta_j^{n+1} = \delta_{j-1}^n \blacktriangle \delta_i^{n+1}$.

Ad (2). We have $0 \leqslant i > j \leqslant n$. Suppose given $m \in [0, n+1]$.

We have

$$\begin{array}{rcl} m(\sigma^n_j \blacktriangle \sigma^{n-1}_i) & = & (m - \chi_{m>j})\sigma^{n-1}_i \\ & = & m - \chi_{m>j} - \chi_{(m-\chi_{m>j})>i} \\ & \overset{i\leqslant j}{=} & m - \chi_{m>j} - \chi_{m>i} \\ & = & m - \chi_{m>i} - \chi_{m>j} \\ & \overset{i\leqslant j}{=} & m - \chi_{m>i} - \chi_{(m-\chi_{m>i})>j-1} \\ & = & (m - \chi_{m>i})\sigma^{n-1}_{j-1} \\ & = & m(\sigma^n_i \blacktriangle \sigma^{n-1}_{j-1})\,. \end{array}$$

This shows $\sigma^n_j \blacktriangle \sigma^{n-1}_i = \sigma^n_i \blacktriangle \sigma^{n-1}_{j-1}$.

Ad (3). We have $n \geqslant 0$, $i \in [0, n]$ and $j \in [0, n+1]$. Suppose given $m \in [0, n]$.

Case $n \geqslant 1$ and $j < i$.

We have

$$\begin{array}{rcl} m(\delta^{n+1}_j \blacktriangle \sigma^n_i) & = & (m + \chi_{m\geqslant j})\sigma^n_i \\ & = & m + \chi_{m\geqslant j} - \chi_{(m+\chi_{m\geqslant j})>i} \\ & \overset{j\leqslant i}{=} & m + \chi_{m\geqslant j} - \chi_{m+1>i} \\ & = & m - \chi_{m>i-1} + \chi_{m\geqslant j} \\ & \overset{j\leqslant i}{=} & m - \chi_{m>i-1} + \chi_{(m-\chi_{m>i-1})\geqslant j} \\ & = & (m - \chi_{m>i-1})\delta^n_j \\ & = & m(\sigma^{n-1}_{i-1} \blacktriangle \delta^n_j)\,. \end{array}$$

This shows $\delta^{n+1}_j \blacktriangle \sigma^n_i = \sigma^{n-1}_{i-1} \blacktriangle \delta^n_j$.

Case $j = i$ or $j = i + 1$.

We have

$$\begin{array}{rcl} m(\delta^{n+1}_j \blacktriangle \sigma^n_i) & = & (m + \chi_{m\geqslant j})\sigma^n_i \\ & = & m + \chi_{m\geqslant j} - \chi_{(m+\chi_{m\geqslant j})>i} \\ & \overset{j=i \text{ or } j=i+1}{=} & m + \chi_{m\geqslant j} - \chi_{m\geqslant j} \\ & = & m\,. \end{array}$$

This shows $\delta^{n+1}_j \blacktriangle \sigma^n_i = \mathrm{id}_{[n]}$.

Case $n \geqslant 1$ and $j > i + 1$.

We have

$$\begin{array}{rcl} m(\delta_j^{n+1} \blacktriangle \sigma_i^n) & = & (m + \chi_{m \geqslant j})\sigma_i^n \\ & = & (m + \chi_{m \geqslant j}) - \chi_{(m+\chi_{m \geqslant j})>i} \\ & \underset{i+1<j}{=} & m + \chi_{m \geqslant j} - \chi_{m>i} \\ & = & m - \chi_{m>i} + \chi_{m \geqslant j} \\ & \underset{i+1<j}{=} & m - \chi_{m>i} + \chi_{(m-\chi_{m>i}+1) \geqslant j} \\ & = & m - \chi_{m>i} + \chi_{(m-\chi_{m>i}) \geqslant j-1} \\ & = & (m - \chi_{m>i})\delta_{j-1}^n \\ & = & m(\sigma_i^{n-1} \blacktriangle \delta_{j-1}^n)\,. \end{array}$$

This shows $\delta_j^{n+1} \blacktriangle \sigma_i^n = \sigma_i^{n-1} \blacktriangle \delta_{j-1}^n$. □

2.2 Composites of morphisms in Δ

We consider the category Δ; cf. Definition 18.

Definition 25

(1) Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given $J \subset [0, n+1]$.
Let $k := |J| \in [0, n]$ and let $J =: \{j_1, j_2, \ldots, j_k\}$, where $0 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$.
We write

$$\delta_J^n := \prod_{r \in \lceil 1,k \rceil}^{[n-k]\ [n]} \delta_{j_r}^{n-k+r} = \delta_{j_1}^{n+1-k} \blacktriangle \delta_{j_2}^{n+2-k} \blacktriangle \ldots \blacktriangle \delta_{j_k}^n \colon [n-k] \to [n]\,.$$

We often write $\delta_J := \delta_J^n$.
Note that δ_J is injective; cf. Definition 19.(1).
Note that $\delta_\emptyset^n = \mathrm{id}_{[n]}$.

(2) Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given $I \subseteq [0, n]$.
Let $k := |I| \in [0, n+1]$ and let $I =: \{i_1, i_2, \ldots, i_k\}$, where $0 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$.
We write

$$\sigma_I^{n+1-k} := \prod_{r \in \lfloor k,1 \rfloor}^{[n+1]\ [n+1-k]} \sigma_{i_r}^{n-k+r} = \sigma_{i_k}^n \blacktriangle \sigma_{i_{k-1}}^{n-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{n+1-k} \colon [n+1] \to [n+1-k]\,.$$

We often write $\sigma_I := \sigma_I^{n+1-k}$.

Note that σ_I is surjective; cf. Definition 19.(2).

Note that $\sigma_{\emptyset}^{n+1} = \mathrm{id}_{[n+1]}$.

Cf. also Definition 37 below.

Remark 26

(1) Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given $J \subset [0,n]$.

We consider the morphism $\delta_J^n \colon [n - |J|] \to [n]$; cf. Definition 25.(1).

We consider the set $C_{\delta_J} := \{j \in [0,n] \colon j \notin \operatorname{im} \delta_J\}$; cf. Lemma 20.(1).

Then we have $J = C_{\delta_J}$; cf. Lemma 20.(1.2).

(2) Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given $I \subseteq [0,n]$.

We consider the morphism $\sigma_I^{n+1-|I|} \colon [n+1] \to [n+1-|I|]$; cf. Definition 25.(2).

We consider the set $D_{\sigma_I} := \{i \in [0,n] \colon (i)\sigma_I = (i+1)\sigma_I\}$; cf. Lemma 20.(2).

Then we have $I = D_{\sigma_I}$; cf. Lemma 20.(2.2).

Remark 27 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given $K \subseteq [n-1]$.*
Suppose given $\ell \in [0,n]$.
We write

$$\begin{aligned} K_1 &:= \{k \in K \colon k \geqslant \ell\} \subseteq [\ell, n-1] \\ K_2 &:= \{k \in K \colon k < \ell\} \subseteq [0, \ell - 1] \\ K_1' &:= \{k+1 \colon k \in K_1\} \subseteq [\ell+1, n]. \end{aligned}$$

Then we have

$$\sigma_\ell^n \blacktriangle \sigma_K^{n-|K|} = \sigma_{K_1'}^{n+1-|K_1|} \blacktriangle \sigma_\ell^{n-|K_1|} \blacktriangle \sigma_{K_2}^{n-|K|} = \sigma_{K_1' \dot\cup \{\ell\} \dot\cup K_2}^{n-|K|} : [n+1] \to [n-|K|].$$

Proof. We write $t := |K_1|$ and $K_1 =: \{k_1, k_2, \ldots, k_t\}$, where $\ell \leqslant k_1 < k_2 < \ldots < k_t \leqslant n-1$.
Note that $K_1 \dot\cup K_2 = K$ and that $K_1' = \{k_1 + 1, k_2 + 1, \ldots, k_t + 1\}$.
Moreover, we have $K_1' \cap K_2 = \emptyset$ and $\ell \notin K_1' \dot\cup K_2$.
So we have the subset $K_1' \dot\cup \{\ell\} \dot\cup K_2 \subset [0,n]$.

We have

$$\sigma_\ell \blacktriangle \sigma_K = \sigma_\ell \blacktriangle \sigma_{K_1 \dot\cup K_2} = \sigma_\ell \blacktriangle \sigma_{K_1} \blacktriangle \sigma_{K_2} = \sigma_\ell \blacktriangle \Big(\prod_{r \in \lfloor t,1 \rfloor} \sigma_{k_r} \Big) \blacktriangle \sigma_{K_2} \overset{24.(2)}{=} \Big(\prod_{r \in \lfloor t,1 \rfloor} \sigma_{k_r+1} \Big) \blacktriangle \sigma_\ell \blacktriangle \sigma_{K_2}$$
$$= \sigma_{K_1'} \blacktriangle \sigma_\ell \blacktriangle \sigma_{K_2} = \sigma_{K_1' \dot\cup \{\ell\} \dot\cup K_2} .$$

□

Remark 28 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given I, $J \subseteq [n]$ where $I \cap J = \emptyset$.*
Suppose given an injective monotone map $u\colon [n] \to [k]$, where $k \in \mathbb{Z}_{\geqslant n}$.
Then we have $(I \dot\cup J)u = Iu \dot\cup Ju$.

Proof. We *assume* that $Iu \cap Ju \neq \emptyset$.
Choose $x \in Iu \cap Ju$. Then we have some $a \in I$ with $au = x$ and we have some $b \in J$ with $bu = x$. Since u is injective, we have $a = b$ and so $a \in I \cap J$. This *contradicts* $I \cap J = \emptyset$.
So $Iu \cap Ju = \emptyset$. This shows the claim. □

Lemma 29 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.*
Suppose given $L \subset [n]$. Then $|L| \in [0,n]$. Suppose given $K \subseteq [n - |L|]$.
We consider the morphisms

$$\begin{array}{rcll} \sigma_L & = & \sigma_L^{n+1-|L|} & : [n+1] \to [n+1-|L|] \\ \sigma_K & = & \sigma_K^{n+1-|L|-|K|} & : [n+1-|L|] \to [n+1-|L|-|K|] \\ \delta_L & = & \delta_L^n & : [n-|L|] \to [n]; \end{array}$$

cf. Definition 25.(1, 2).

(1) *Suppose given $t \in [0, |L|]$. Suppose given $L' \subset [n-t]$ with $|L'| = |L| - t$.*
We consider the map $\delta_{L'} = \delta_{L'}^{n-t}\colon [n-|L|] \to [n-t]$.
We have $K\delta_{L'} \cap L' = \emptyset$.
In particular, we have $K\delta_L \cap L = \emptyset$.

(2) *We consider the set $K\delta_L \dot\cup L \subseteq [n]$; cf.* (1).
We have
$$\sigma_L \blacktriangle \sigma_K = \sigma_{K\delta_L \dot\cup L} : [n+1] \to [n+1-|L|-|K|] .$$

Proof. Ad (1). We *assume* that $K\delta_{L'} \cap L' \neq \emptyset$. We choose $x \in K\delta_{L'} \cap L'$.
In particular, we have $x \in K\delta_{L'}$. So we have $x = y\delta_{L'}$ for a unique $y \in K$.
This shows that $x = y\delta_{L'} \in \operatorname{im} \delta_{L'}$.
Moreover, we have $x \in L' \overset{26.(1)}{=} \{i \in [n-t] \colon i \notin \operatorname{im} \delta_{L'}\}$.
This shows that $x \notin \operatorname{im} \delta_{L'}$ which is a *contradiction*.
So we have $K\delta_{L'} \cap L' = \emptyset$.
Ad (2). We show this claim by induction on $|L| \geqslant 0$.
Let $|L| = 0$. Then $L = \emptyset$. We have $\sigma_L = \mathrm{id}_{[n+1]}$ and $\delta_L = \mathrm{id}_{[n]}$.
So we have $K\delta_L \mathbin{\dot\cup} L = K \mathbin{\dot\cup} \emptyset = K$.
This shows that

$$\sigma_L \mathbin{\blacktriangle} \sigma_K = \mathrm{id}_{[n+1]} \mathbin{\blacktriangle} \sigma_K = \sigma_K = \sigma_{K\delta_L \dot\cup L}\,.$$

Let $|L| \geqslant 1$. Then $n \geqslant 1$. We write $\ell := \max L \in [n]$. Let $\tilde{L} := L \backslash \{\ell\} \subset [n-1]$.
We have $\tilde{L} \subset [n-1]$ with $|\tilde{L}| = |L| - 1 < |L|$ and $K \subseteq [n - |L|] = [(n-1) - |\tilde{L}|]$.
So, by induction hypothesis, we have

$$\sigma_{\tilde{L}} \mathbin{\blacktriangle} \sigma_K = \sigma_{K\delta_{\tilde{L}} \dot\cup \tilde{L}} : [(n-1)+1] \to [(n-1)+1-|\tilde{L}|-|K|]\,.$$

Note that by (1), we have $K\delta_{\tilde{L}} \cap \tilde{L} = \emptyset$.
Moreover, we have

$$\sigma_L = \sigma_{\{\ell\} \dot\cup \tilde{L}} = \sigma_\ell \mathbin{\blacktriangle} \sigma_{\tilde{L}}\,;$$

cf. Definition 25.(2).
We write

$$\begin{aligned} K_1 &:= \Big\{k \in (K\delta_{\tilde{L}} \mathbin{\dot\cup} \tilde{L}) \colon k \geqslant \ell\Big\} \subseteq [\ell, n] \\ K_2 &:= \Big\{k \in (K\delta_{\tilde{L}} \mathbin{\dot\cup} \tilde{L}) \colon k < \ell\Big\} \subseteq [0, \ell-1] \\ K_1' &:= \{k+1 \colon k \in K_1\} \subseteq [\ell+1, n]\,. \end{aligned}$$

By Remark 27, since $K\delta_{\tilde{L}} \mathbin{\dot\cup} \tilde{L} \subseteq [n-1]$, we have

$$\sigma_\ell \mathbin{\blacktriangle} \sigma_{K\delta_{\tilde{L}} \dot\cup \tilde{L}} = \sigma_{K_1' \dot\cup \{\ell\} \dot\cup K_2} \colon [n+1] \to [n - |K\delta_{\tilde{L}} \mathbin{\dot\cup} \tilde{L}|] = [n - |\tilde{L}| - |K|]\,.$$

We claim that $K_1' \dot\cup \{\ell\} \dot\cup K_2 \overset{!}{=} K\delta_L \dot\cup L$.

Note that $K_1 \dot\cup K_2 = K\delta_{\tilde L} \dot\cup \tilde L$ and that $K_1' \dot\cup K_2 = (K_1 \dot\cup K_2)\delta_\ell = (K\delta_{\tilde L} \dot\cup \tilde L)\delta_\ell$.

We have

$$K_1' \dot\cup \{\ell\} \dot\cup K_2 = \big(K_1' \dot\cup K_2\big) \dot\cup \{\ell\} = (K\delta_{\tilde L} \dot\cup \tilde L)\delta_\ell \dot\cup \{\ell\} \overset{28}{=} K\delta_{\tilde L}\delta_\ell \dot\cup \tilde L\delta_\ell \dot\cup \{\ell\}$$
$$\overset{25.(1)}{=} K\delta_L \dot\cup \tilde L \dot\cup \{\ell\} = K\delta_L \dot\cup L\,.$$

This proves the *claim*.

Altogether, we have

$$\sigma_L \blacktriangle \sigma_K = \sigma_\ell \blacktriangle \sigma_{\tilde L} \blacktriangle \sigma_K = \sigma_\ell \blacktriangle \sigma_{K\delta_{\tilde L}\dot\cup\tilde L} = \sigma_{K_1'\dot\cup\{\ell\}\dot\cup K_2} = \sigma_{K\delta_L\dot\cup L}\colon [n+1] \to [n+1-|L|-|K|]\,.$$

□

2.3 The category of simplicial groups

Definition 30 (Simplicial groups)

We consider the category of groups $\mathcal{Grp}$ and the opposite category Δ^{op} of Δ; cf. §1.1.3 item 12.

We write $\mathcal{SimpGrp} := [\Delta^{\mathrm{op}}, \mathcal{Grp}]$ for the category of functors from Δ^{op} to $\mathcal{Grp}$ and transformations between those functors.

We call $\mathcal{SimpGrp}$ the *(ordinary) category of simplicial groups*.

A functor $(\Delta^{\mathrm{op}} \xrightarrow{G} \mathcal{Grp}) \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ is called an *(ordinary) simplicial group*.

For $n \in \mathbb{N}$, we write $G_n := [n]G \in \mathrm{Ob}(\mathcal{Grp})$.

A transformation $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}\big(\mathcal{SimpGrp}\big)$ between simplicial groups G, $H \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ is called a *simplicial group morphism*.

For $n \in \mathbb{N}$, we write $\varphi_n := [n]\varphi\colon G_n \to H_n$.

For $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$, we write $G_u := (u^{\mathrm{op}})G\colon G_k \to G_n$.

For $G \xrightarrow{\varphi} G' \xrightarrow{\varphi'} G''$ in $\mathcal{SimpGrp}$, their composite is given by their vertical composite of transformations

$$\varphi \blacktriangle \varphi' = \big(G_n \xrightarrow{[n](\varphi\blacktriangle\varphi')} G_n''\big)_{n\in\mathbb{N}} := \big(G_n \xrightarrow{\varphi_n\blacktriangle\varphi_n'} G_n''\big)_{n\in\mathbb{N}}\,.$$

For $G \in \mathrm{Ob}(\mathcal{SimpGrp})$, its identity is given by

$$\mathrm{id}_G = \left(G_n \xrightarrow{\mathrm{id}_{G_n}} G_n\right)_{n\in\mathbb{N}}.$$

Definition 31 (Face maps and degeneracy maps)
We consider the simplex category Δ.
Suppose given a simplicial group $G\colon \Delta^{\mathrm{op}} \to \mathcal{Grp}$.

(1) Suppose given $n \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,n]$. We consider the morphisms

$$\delta^n_j\colon [n-1] \to [n] \quad \text{and} \quad (\delta^n_j)^{\mathrm{op}}\colon [n] \to [n-1]\,,$$

cf. Definition 19.
We call

$$\mathrm{d}^{G,n}_j := G_{\delta^n_j} = \left((\delta^n_j)^{\mathrm{op}}\right)G\colon G_n \to G_{n-1}$$

a *face map*. Sometimes, we abbreviate $\mathrm{d}_j := \mathrm{d}^{G,n}_j$. It is a group morphism.

(2) Suppose given $n \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0,n]$. We consider the morphisms

$$\sigma^n_i\colon [n+1] \to [n] \quad \text{and} \quad (\sigma^n_i)^{\mathrm{op}}\colon [n] \to [n+1]\,;$$

cf. Definition 19.
We call

$$\mathrm{s}^{G,n}_i := G_{\sigma^n_i} = \left((\sigma^n_i)^{\mathrm{op}}\right)G\colon G_n \to G_{n+1}$$

a *degeneracy map*. Sometimes, we abbreviate $\mathrm{s}_i := \mathrm{s}^{G,n}_i$. It is a group morphism.

Remark 32 Suppose given simplicial groups G, $H\colon \Delta^{\mathrm{op}} \to \mathcal{Grp}$.
Suppose given a simplicial group morphism $\varphi\colon G \to H$.
Then we have $(1,2)$.

(1) Let $n \in \mathbb{Z}_{\geqslant 1}$. For $i \in [0,n]$, we have

$$\mathrm{d}^{G,n}_i \vartriangle \varphi_{n-1} = \varphi_n \vartriangle \mathrm{d}^{H,n}_i \ : G_n \to H_{n-1}\,.$$

(2) Let $n \in \mathbb{Z}_{\geqslant 0}$. For $i \in [0,n]$, we have

$$\varphi_n \vartriangle \mathrm{s}^{H,n}_i = \mathrm{s}^{G,n}_i \vartriangle \varphi_{n+1} \ : G_n \to H_{n+1}\,.$$

Remark 33 Suppose given a simplicial group $G\colon \Delta^{\mathrm{op}} \to \mathcal{Grp}$.
Using Remark 24, we have the following equations $(1, 2, 3)$.

(1) Let $n \in \mathbb{Z}_{\geqslant 1}$. We have

$$\mathrm{d}_j^{G,n+1} \vartriangle \mathrm{d}_i^{G,n} = \mathrm{d}_i^{G,n+1} \vartriangle \mathrm{d}_{j-1}^{G,n}$$

for $i \in [0, n]$, $j \in [0, n+1]$ with $i < j$.

(2) Let $n \in \mathbb{Z}_{\geqslant 1}$. We have

$$\mathrm{s}_i^{G,n-1} \vartriangle \mathrm{s}_j^{G,n} = \mathrm{s}_{j-1}^{G,n-1} \vartriangle \mathrm{s}_i^{G,n}$$

for $i \in [0, n-1]$, $j \in [0, n]$ with $i < j$.

(3) Let $n \in \mathbb{Z}_{\geqslant 0}$. We have

$$\mathrm{s}_i^{G,n} \vartriangle \mathrm{d}_j^{G,n+1} = \begin{cases} \mathrm{d}_j^{G,n} \vartriangle \mathrm{s}_{i-1}^{G,n-1} & \text{if } n \geqslant 1 \text{ and } j < i \\ \mathrm{id}_{G_n} & \text{if } j = i \text{ or } j = i+1 \\ \mathrm{d}_{j-1}^{G,n} \vartriangle \mathrm{s}_i^{G,n-1} & \text{if } n \geqslant 1 \text{ and } j > i+1\,, \end{cases}$$

for $i \in [0, n]$, $j \in [0, n+1]$.

Remark 34 *Suppose given a morphism* $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$.
We consider the morphism $\big([k] \xrightarrow{u^{\mathrm{op}}} [n]\big) \in \mathrm{Mor}(\Delta^{\mathrm{op}})$.
We have

$$u^{\mathrm{op}} = \left(\prod_{t \in \lfloor k-\ell, 1 \rfloor}^{[k]\,[\ell]} (\delta_{j_t}^{\ell+t})^{\mathrm{op}} \right) \vartriangle \left(\prod_{t \in \lceil 1, n-\ell \rceil}^{[\ell]\,[n]} (\sigma_{i_t}^{\ell+t-1})^{\mathrm{op}} \right)$$

for some $\ell \geqslant 0$ *and for some* $0 \leqslant i_1 < \ldots < i_{n-\ell} \leqslant n-1$ *and* $0 \leqslant j_1 < \ldots < j_{k-\ell} \leqslant k$.
In particular, for a simplicial group $G\colon \Delta^{\mathrm{op}} \to \mathcal{Grp}$, *we have*

$$G_u = (u^{\mathrm{op}})G = \left(\prod_{t \in \lfloor k-\ell, 1 \rfloor}^{G_k\,G_\ell} \mathrm{d}_{j_t}^{G,\,\ell+t} \right) \vartriangle \left(\prod_{t \in \lceil 1, n-\ell \rceil}^{G_\ell\,G_n} \mathrm{s}_{i_t}^{G,\,\ell+t-1} \right).$$

Proof. By Corollary 22, we have

$$u = \left(\prod_{t \in \lfloor n-\ell, 1 \rfloor}^{[n]\,[\ell]} \sigma_{i_t}^{\ell+t-1} \right) \vartriangle \left(\prod_{t \in \lceil 1, k-\ell \rceil}^{[\ell]\,[k]} \delta_{j_t}^{\ell+t} \right)$$

for some $\ell \geqslant 0$ and for some $0 \leqslant i_1 < \ldots < i_{n-\ell} \leqslant n-1$ and $0 \leqslant j_1 < \ldots < j_{k-\ell} \leqslant k$.
So we have

$$u^{\mathrm{op}} = \left(\left(\prod_{t \in \lfloor n-\ell, 1 \rfloor}^{[n]} {}^{[\ell]} \sigma_{i_t}^{\ell+t-1} \right) \blacktriangle \left(\prod_{t \in \lceil 1, k-\ell \rceil}^{[\ell]} {}^{[k]} \delta_{j_t}^{\ell+t} \right) \right)^{\mathrm{op}} = \left(\prod_{t \in \lfloor k-\ell, 1 \rfloor}^{[k]} {}^{[\ell]} (\delta_{j_t}^{\ell+t})^{\mathrm{op}} \right) \blacktriangle \left(\prod_{t \in \lceil 1, n-\ell \rceil}^{[\ell]} {}^{[n]} (\sigma_{i_t}^{\ell+t-1})^{\mathrm{op}} \right)$$

Moreover, for a simplicial group G, we have

$$\begin{aligned} G_u = (u^{\mathrm{op}})G &= \left(\prod_{t \in \lfloor k-\ell, 1 \rfloor}^{G_\ell} {}^{G_k} \left((\delta_{j_t}^{\ell+t})^{\mathrm{op}} \right) G \right) \blacktriangle \left(\prod_{t \in \lceil 1, n-\ell \rceil}^{G_n} {}^{G_\ell} \left((\sigma_{i_t}^{\ell+t-1})^{\mathrm{op}} \right) G \right) \\ &= \left(\prod_{t \in \lfloor k-\ell, 1 \rfloor}^{G_\ell} {}^{G_k} \mathrm{d}_{j_t}^{\ell+t} \right) \blacktriangle \left(\prod_{t \in \lceil 1, n-\ell \rceil}^{G_n} {}^{G_\ell} \mathrm{s}_{i_t}^{\ell+t-1} \right). \end{aligned}$$

□

Remark 35 *Suppose given simplicial groups G, H.*
For $k \in \mathbb{Z}_{\geqslant 0}$, suppose given simplicial group morphisms $\varphi_k \colon G_k \to H_k$.
Suppose that we have $(1, 2)$.

(1) *For $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0, k]$, we have*

$$\mathrm{d}_j^{G,k} \blacktriangle \varphi_{k-1} = \varphi_k \blacktriangle \mathrm{d}_j^{H,k} \colon G_k \to H_{k-1}.$$

(2) *For $k \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0, k]$, we have*

$$\varphi_k \blacktriangle \mathrm{s}_i^{H,k} = \mathrm{s}_i^{G,k} \blacktriangle \varphi_{k+1} \colon G_k \to H_{k+1}.$$

Then $\varphi := (\varphi_k)_{k \in \mathbb{Z}_{\geqslant 0}} \colon G \to H$ is a simplicial group morphism.

Proof. Let $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $\left([n] \xrightarrow{u} [k] \right)$ in Δ. We consider the group morphisms $G_u \colon G_k \to G_n$ and $H_u \colon H_k \to H_n$.
By Remark 34, we have

$$u^{\mathrm{op}} = \left(\prod_{t \in \lfloor k-\ell, 1 \rfloor}^{[k]} {}^{[\ell]} (\delta_{j_t}^{\ell+t})^{\mathrm{op}} \right) \blacktriangle \left(\prod_{t \in \lceil 1, n-\ell \rceil}^{[\ell]} {}^{[n]} (\sigma_{i_t}^{\ell+t-1})^{\mathrm{op}} \right),$$

hence

$$G_u = \left(\prod_{t\in\lfloor k-\ell,1\rfloor}^{G_k} \mathrm{d}_{j_t}^{G,\,\ell+t} \right) \blacktriangle \left(\prod_{t\in\lceil 1,n-\ell\rceil}^{G_\ell} \mathrm{s}_{i_t}^{G,\,\ell+t-1} \right)$$

and

$$H_u = \left(\prod_{t\in\lfloor k-\ell,1\rfloor}^{H_k} \mathrm{d}_{j_t}^{H,\,\ell+t} \right) \blacktriangle \left(\prod_{t\in\lceil 1,n-\ell\rceil}^{H_\ell} \mathrm{s}_{i_t}^{H,\,\ell+t-1} \right),$$

where $\ell \geqslant 0$ and where $0 \leqslant i_1 < \ldots < i_{n-\ell} \leqslant n-1$ and $0 \leqslant j_1 < \ldots < j_{k-\ell} \leqslant k$.
We have

$$\begin{aligned}
\varphi_k \blacktriangle H_u &= \varphi_k \blacktriangle \left(\prod_{t\in\lfloor k-\ell,1\rfloor}^{H_k} \mathrm{d}_{j_t}^{H,\,\ell+t} \right) \blacktriangle \left(\prod_{t\in\lceil 1,n-\ell\rceil}^{H_\ell} \mathrm{s}_{i_t}^{H,\,\ell+t-1} \right) \\
&\overset{(1)}{=} \left(\prod_{t\in\lfloor k-\ell,1\rfloor}^{G_k} \mathrm{d}_{j_t}^{G,\,\ell+t} \right) \blacktriangle \varphi_{k-(k-\ell)} \blacktriangle \left(\prod_{t\in\lceil 1,n-\ell\rceil}^{H_\ell} \mathrm{s}_{i_t}^{H,\,\ell+t-1} \right) \\
&\overset{(2)}{=} \left(\prod_{t\in\lfloor k-\ell,1\rfloor}^{G_k} \mathrm{d}_{j_t}^{G,\,\ell+t} \right) \blacktriangle \left(\prod_{t\in\lceil 1,n-\ell\rceil}^{G_\ell} \mathrm{s}_{i_t}^{G,\,\ell+t-1} \right) \blacktriangle \varphi_{k-(k-\ell)+(n-\ell)} \\
&= G_u \blacktriangle \varphi_n \,.
\end{aligned}$$

This shows that $\varphi = (\varphi_k)_{k\in\mathbb{Z}_{\geqslant 0}}$ is a simplicial group morphism. □

Remark 36 Suppose given a simplicial group $G\colon \Delta^{\mathrm{op}} \to \mathcal{G}rp$.

$$\cdots \quad G_2 \;\underset{\mathrm{s}_0^{G,1}}{\overset{\mathrm{d}_2^{G,2}}{\underset{\mathrm{d}_0^{G,2}}{\overset{\mathrm{s}_1^{G,1}}{\rightleftarrows}}}}\; G_1 \;\underset{\mathrm{d}_0^{G,1}}{\overset{\mathrm{d}_1^{G,1}}{\rightleftarrows}}\; G_0$$

Using Remark 33, we have in particular the following equations.

$$\begin{array}{llllllllll}
\mathrm{s}_0^{G,0}\blacktriangle\mathrm{s}_0^{G,1} &=& \mathrm{s}_0^{G,0}\blacktriangle\mathrm{s}_1^{G,1}\,, & \mathrm{s}_0^{G,0}\blacktriangle\mathrm{d}_0^{G,1} &=& \mathrm{id}_{G_0}\,, & \mathrm{s}_0^{G,0}\blacktriangle\mathrm{d}_1^{G,1} &=& \mathrm{id}_{G_0} \\
\mathrm{s}_0^{G,1}\blacktriangle\mathrm{d}_0^{G,2} &=& \mathrm{id}_{G_1}\,, & \mathrm{s}_0^{G,1}\blacktriangle\mathrm{d}_1^{G,2} &=& \mathrm{id}_{G_1}\,, & \mathrm{s}_0^{G,1}\blacktriangle\mathrm{d}_2^{G,2} &=& \mathrm{d}_1^{G,1}\blacktriangle\mathrm{s}_0^{G,0} \\
\mathrm{s}_1^{G,1}\blacktriangle\mathrm{d}_0^{G,2} &=& \mathrm{d}_0^{G,1}\blacktriangle\mathrm{s}_0^{G,0}\,, & \mathrm{s}_1^{G,1}\blacktriangle\mathrm{d}_1^{G,2} &=& \mathrm{id}_{G_1}\,, & \mathrm{s}_1^{G,1}\blacktriangle\mathrm{d}_2^{G,2} &=& \mathrm{id}_{G_1} \\
\mathrm{d}_0^{G,2}\blacktriangle\mathrm{d}_0^{G,1} &=& \mathrm{d}_1^{G,2}\blacktriangle\mathrm{d}_0^{G,1}\,, & \mathrm{d}_0^{G,2}\blacktriangle\mathrm{d}_1^{G,1} &=& \mathrm{d}_2^{G,2}\blacktriangle\mathrm{d}_0^{G,1}\,, & \mathrm{d}_1^{G,2}\blacktriangle\mathrm{d}_1^{G,1} &=& \mathrm{d}_2^{G,2}\blacktriangle\mathrm{d}_1^{G,1}\,.
\end{array}$$

Definition 37 Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

Suppose given a simplicial group G; cf. Definition 30.

(1) Suppose given $J \subset [n]$.

Let $k := |J| \in [0,n]$ and let $J =: \{j_1, j_2, \ldots, j_k\}$, where $0 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$.

We consider the map $\delta_J = \delta^n_J = \prod\limits_{r\in\lceil 1,k\rceil}^{[n-k]\quad [n]} \delta^{n-k+r}_{j_r} : [n-k] \to [n]$; cf. Definition 25.(1).

We write

$$\mathrm{d}^{G,n}_J := G_{\delta^n_J} = \big((\delta^n_J)^{\mathrm{op}}\big)G : G_n \to G_{n-k}\,.$$

So we have

$$\begin{aligned}\mathrm{d}^{G,n}_J &= \Big(\prod_{r\in\lceil 1,k\rceil}^{[n-k]\quad [n]} \delta^{n-k+r}_{j_r}\Big)^{\mathrm{op}} G = \Big(\prod_{r\in\lfloor k,1\rfloor}^{[n]\quad [n-k]} (\delta^{n-k+r}_{j_r})^{\mathrm{op}}\Big)G\\ &= \prod_{r\in\lfloor k,1\rfloor}^{G_{n+1}\quad G_{n-k}} \big((\delta^{n-k+r}_{j_r})^{\mathrm{op}}\big)G = \prod_{r\in\lfloor k,1\rfloor}^{G_{n+1}\quad G_{n-k}} \mathrm{d}^{G,n-k+r}_{j_r}\\ &= \mathrm{d}^{G,n}_{j_k} \blacktriangle \mathrm{d}^{G,n-1}_{j_{k-1}} \blacktriangle \ldots \blacktriangle \mathrm{d}^{G,n+1-k}_{j_1}\,.\end{aligned}$$

We often write $\mathrm{d}_J := \mathrm{d}^{G,n}_J$. Note that $\mathrm{d}^{G,n}_{\emptyset} = \mathrm{id}_{G_n}$.

(2) Suppose given $I \subseteq [n]$.

Let $k := |I| \in [0,n+1]$ and let $I =: \{i_1, i_2, \ldots, i_k\}$, where $0 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$.

We consider the map $\sigma_I = \sigma^{n+1-k}_I = \prod\limits_{r\in\lfloor k,1\rfloor}^{[n+1]\quad [n+1-k]} \sigma^{n-k+r}_{i_r} : [n+1] \to [n+1-k]$; cf. Definition 25.(2).

We write

$$\mathrm{s}^{G,n+1-k}_I := G_{\sigma^{n+1-k}_I} = \big((\sigma^{n+1-k}_I)^{\mathrm{op}}\big)G : G_{n+1-k} \to G_{n+1}\,.$$

So we have

$$\begin{aligned}\mathrm{s}^{G,n+1-k}_I &= \Big(\prod_{r\in\lfloor k,1\rfloor}^{[n+1]\quad [n+1-k]} \sigma^{n-k+r}_{i_r}\Big)^{\mathrm{op}} G = \Big(\prod_{r\in\lceil 1,k\rceil}^{[n+1-k]\quad [n+1]} (\sigma^{n-k+r}_{i_r})^{\mathrm{op}}\Big)G\\ &= \prod_{r\in\lceil 1,k\rceil}^{G_{n+1-k}\quad G_{n+1}} \big((\sigma^{n-k+r}_{i_r})^{\mathrm{op}}\big)G = \prod_{r\in\lceil 1,k\rceil}^{G_{n+1-k}\quad G_{n+1}} \mathrm{s}^{G,n-k+r}_{i_r}\\ &= \mathrm{s}^{G,n+1-k}_{i_1} \blacktriangle \mathrm{s}^{G,n+2-k}_{i_2} \blacktriangle \ldots \blacktriangle \mathrm{s}^{G,n}_{i_r}\,.\end{aligned}$$

We often write $\mathrm{s}_I := \mathrm{s}_I^{G,n+1-k}$. Note that $\mathrm{s}_\emptyset^{G,n+1} = \mathrm{id}_{G_{n+1}}$.

Remark 38 *Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given a simplicial group G; cf. Definition* 30.
Suppose given $I \subseteq [n]$. Suppose given $\ell \in [(\max I) + 2, n+1]$. Note that $\ell - |I| \geqslant 1$.
We have

$$\mathrm{s}_I^{G,n+1-|I|} \blacktriangle \mathrm{d}_\ell^{G,n+1} = \mathrm{d}_{\ell-|I|}^{G,n+1-|I|} \blacktriangle \mathrm{s}_I^{G,n-|I|} \colon G_{n+1-|I|} \to G_n \,.$$

Proof. We show this claim by induction on $|I| \in \mathbb{Z}_{\geqslant 0}$.
Let $|I| = 0$. Then we have $I = \emptyset$ and $\mathrm{s}_I = \mathrm{id}$.
So we have

$$\mathrm{s}_I \blacktriangle \mathrm{d}_\ell = \mathrm{d}_\ell = \mathrm{d}_{\ell-|I|} \blacktriangle \mathrm{s}_I \,.$$

Now suppose that $|I| \geqslant 1$. By induction hypothesis, we have

$$\mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{d}_{\tilde{\ell}} = \mathrm{d}_{\tilde{\ell}-1} \blacktriangle \mathrm{s}_{\tilde{I}}$$

for $\tilde{I} \subseteq [0,n]$ with $|\tilde{I}| < |I|$ and for $\tilde{\ell} \in [(\max \tilde{I}) + 2, n+1]$.
Since $|I| \geqslant 1$, we have that $I \neq \emptyset$.
We write $m := \max I \in I$ and we write $\tilde{I} := I \backslash \{m\} \subset I$. Note that $I = \tilde{I} \,\dot\cup\, \{m\}$. So we have

$$\mathrm{s}_I = \mathrm{s}_{\tilde{I} \dot\cup \{m\}} = \mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_m \,.$$

Moreover, note that $|\tilde{I}| = |I| - 1$ and that $\max \tilde{I} \leqslant (\max I) - 1 \leqslant \ell - 3$. So $(\max \tilde{I}) + 2 \leqslant \ell - 1$.
Therefore, we have

$$\mathrm{s}_I \blacktriangle \mathrm{d}_\ell = \mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_m \blacktriangle \mathrm{d}_\ell = \mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{d}_{\ell-1} \blacktriangle \mathrm{s}_m \overset{\text{IH}}{=} \mathrm{d}_{\ell-1-|\tilde{I}|} \blacktriangle \mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_m = \mathrm{d}_{\ell-|I|} \blacktriangle \mathrm{s}_I \,.$$

□

Corollary 39 *We consider the assertions from Lemma* 29.
Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.
Suppose given $L \subset [n]$. Then $|L| \in [0,n]$. Suppose given $K \subseteq [n - |L|]$.
Suppose given a simplicial group G; cf. Definition 30.
We consider the group morphisms

$$\mathrm{s}_L^{G,n+1-|L|} \colon G_{n+1-|L|} \to G_{n+1}$$
$$\mathrm{s}_K^{G,n+1-|L|-|K|} \colon G_{n+1-|L|-|K|} \to G_{n+1-|L|} \,;$$

cf. Definition 37.

We consider the map $\delta_L = \delta_L^{n-|L|} \colon [n-|L|] \to [n]$ *; cf. Definition* 25.(1).

We consider the set $K\delta_L \mathbin{\dot\cup} L \subseteq [n]$.

We have

$$\mathrm{s}_K^{G,n+1-|L|-|K|} \mathbin{\blacktriangle} \mathrm{s}_L^{G,n+1-|L|} = \mathrm{s}_{K\delta_L \dot\cup L}^{G,n+1-|L|-|K|} \colon G_{n+1-|L|-|K|} \to G_{n+1} \,.$$

Proof. We consider the morphisms

$$\begin{aligned} \sigma_L &\colon [n+1] \to [n+1-|L|] \\ \sigma_K &\colon [n+1-|L|] \to [n+1-|L|-|K|] \,; \end{aligned}$$

cf. Definition 25.(1).

Then, by Lemma 29.(2), we have

$$\sigma_L \mathbin{\blacktriangle} \sigma_K = \sigma_{K\delta_L \dot\cup L} \,.$$

So we have

$$\mathrm{s}_K \mathbin{\blacktriangle} \mathrm{s}_L \overset{37.(2)}{=} G_{\sigma_K} \mathbin{\blacktriangle} G_{\sigma_L} = G_{\sigma_L \mathbin{\blacktriangle} \sigma_K} = G_{\sigma_{K\delta_L \dot\cup L}} \overset{37.(2)}{=} \mathrm{s}_{K\delta_L \dot\cup L} \,.$$

□

2.4 Simplicial subgroups

Definition 40 (Simplicial subgroups)

Suppose given simplicial groups G, $H \colon \Delta^{\mathrm{op}} \to \mathcal{G}rp$.

We say that H is a *simplicial subgroup* of G if the conditions $(1, 2)$ hold.

(1) For $n \in \mathbb{Z}_{\geqslant 0}$, we have $H_n \leqslant G_n$.

(2) Suppose given a morphism $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$.
We consider the group morphisms $G_u \colon G_k \to G_n$ and $H_u \colon H_k \to H_n$; cf. Definition 30.
We have
$$H_u = G_u \big|_{H_k}^{H_n} \,.$$

We write $H \leqslant G$ to indicate that H is a simplicial subgroup of G.

Lemma 41 *Suppose given a simplicial group $G\colon \Delta^{\mathrm{op}} \to \mathcal{G}rp$.*
For $n \in \mathbb{Z}_{\geqslant 0}$, suppose given subgroups $H_n \leqslant G_n$.

(1) *For $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$, i.e. $\big([k] \xrightarrow{u^{\mathrm{op}}} [n]\big) \in \mathrm{Mor}(\Delta^{\mathrm{op}})$, suppose that we have $(H_k)G_u \leqslant G_n$.*

Then we have the simplicial group

$$H\colon \Delta^{\mathrm{op}} \to \mathcal{G}rp\,,\ \big([k] \xrightarrow{u^{\mathrm{op}}} [n]\big) \mapsto \big([k]H \xrightarrow{(u^{\mathrm{op}})H} [n]H\big) := \big(H_k \xrightarrow{G_u\big|^{H_n}_{H_k}} H_n\big)\,.$$

Moreover, we have $H \leqslant G$, i.e. H is a simplicial subgroup of G.

(2) *For $n \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,n]$, suppose that we have*

$$(H_n)\mathrm{d}_j^{G,n} \leqslant H_{n-1}\,.$$

For $n \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0,n]$, suppose that we have

$$(H_n)\mathrm{s}_i^{G,n} \leqslant H_{n+1}\,.$$

Then we have the simplicial group

$$H\colon \Delta^{\mathrm{op}} \to \mathcal{G}rp\,,\ \big([k] \xrightarrow{u^{\mathrm{op}}} [n]\big) \mapsto \big([k]H \xrightarrow{(u^{\mathrm{op}})H} [n]H\big) := \big(H_k \xrightarrow{G_u\big|^{H_n}_{H_k}} H_n\big)\,.$$

Moreover, we have $H \leqslant G$, i.e. H is a simplicial subgroup of G.

Proof. Ad (1). Suppose given $[m] \xrightarrow{v^{\mathrm{op}}} [k] \xrightarrow{u^{\mathrm{op}}} [n]$ in Δ^{op}.
We have

$$(\mathrm{id}_{[n]})^{\mathrm{op}}H = \big((\mathrm{id}_{[n]})^{\mathrm{op}}G\big)\big|^{H_n}_{H_n} = \mathrm{id}_{G_n}\big|^{H_n}_{H_n} = \mathrm{id}_{H_n} = \mathrm{id}_{[n]H}\,.$$

We have

$$(v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}})H = \big((v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}})G\big)\big|^{H_n}_{H_m} = (G_v \blacktriangle G_u)\big|^{H_n}_{H_m} = G_v\big|^{H_k}_{H_m} \blacktriangle G_u\big|^{H_n}_{H_k} = (v^{\mathrm{op}})H \blacktriangle (u^{\mathrm{op}})H\,.$$

So H is a functor from Δ^{op} to $\mathcal{G}rp$, i.e. H is a simplicial group.
For $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$, we have, by construction,

$$H_u = (u^{\mathrm{op}})H = G_u\big|^{H_n}_{H_k}\,.$$

So H is a simplicial subgroup of G.

Ad (2). Suppose given a morphism $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$. By (1), it suffices to show that $(H_k)G_u \overset{!}{\leqslant} H_n$.

By Remark 34, we have

$$G_u = \left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k} \prod^{G_\ell} \mathrm{d}_{i_r}^{G,\,\ell+r} \right) \blacktriangle \left(\prod_{q \in \lceil 1, n-\ell \rceil}^{G_\ell} \prod^{G_n} \mathrm{s}_{j_q}^{G,\,\ell+q-1} \right)$$

for some $\ell \geqslant 0$ and $0 \leqslant j_1 < \ldots < j_{n-\ell} \leqslant n-1$ and $0 \leqslant i_1 < \ldots < i_{k-\ell} \leqslant k$.

Using the assumptions given in (2) iteratively, we obtain

$$(H_k)\left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k} \prod^{G_\ell} \mathrm{d}_{i_r}^{G,\,\ell+r} \right) \leqslant H_\ell$$

and

$$(H_\ell)\left(\prod_{q \in \lceil 1, n-\ell \rceil}^{G_\ell} \prod^{G_n} \mathrm{s}_{j_q}^{G,\,\ell+q-1} \right) \leqslant H_n \,.$$

So we have

$$(H_k)G_u = (H_k)\left(\left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k} \prod^{G_\ell} \mathrm{d}_{i_r}^{G,\,\ell+r} \right) \blacktriangle \left(\prod_{q \in \lceil 1, n-\ell \rceil}^{G_\ell} \prod^{G_n} \mathrm{s}_{j_q}^{G,\,\ell+q-1} \right) \right) \leqslant H_n \,.$$

Therefore, we have the group morphism $G_u\big|_{H_k}^{H_n} \colon H_k \to H_n$. Thus, the definition of the group morphism

$$\big([k]H \xrightarrow{(u^{\mathrm{op}})H} [n]H\big) := \big(H_k \xrightarrow{G_u\big|_{H_k}^{H_n}} H_n\big)$$

is possible.

Moreover, by (1), H is a simplicial subgroup of G. □

Remark 42 *Suppose given simplicial groups G, H; cf. Definition* 30.

Suppose given simplicial subgroups $\tilde{G} \leqslant G$ and $\tilde{H} \leqslant H$; cf. Definition 40.

Suppose given a simplicial group morphism $\varphi \colon G \to H$.

Suppose that we have $\tilde{G}_k \, \varphi_k \leqslant \tilde{H}_k$ for $k \in \mathbb{Z}_{\geqslant 0}$.

Then we have the simplicial group morphism

$$\tilde{\varphi} := \varphi\big|_{\tilde{G}}^{\tilde{H}} : \tilde{G} \to \tilde{H}\,,$$

where

$$\tilde{\varphi}_k = \big(\varphi\big|_{\tilde{G}}^{\tilde{H}}\big)_k := \varphi_k\big|_{\tilde{G}_k}^{\tilde{H}_k} : \tilde{G}_k \to \tilde{H}_k\,,\ g \mapsto g\tilde{\varphi}_k = g\varphi_k\,,$$

for $k \in \mathbb{Z}_{\geqslant 0}$.

If $\tilde{G} = G$, *we also write* $\varphi\big|^{\tilde{H}} := \varphi\big|_G^{\tilde{H}} \colon G \to \tilde{H}$.

If $\tilde{H} = H$, *we also write* $\varphi\big|_{\tilde{G}} := \varphi\big|_{\tilde{G}}^{H} \colon \tilde{G} \to H$.

Proof. Suppose given $\big([\ell] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$.

We consider the group morphisms $\tilde{G}_u \colon \tilde{G}_k \to \tilde{G}_\ell$ and $\tilde{H}_u \colon \tilde{H}_k \to \tilde{H}_\ell$.

For $g \in \tilde{G}_k$, we have

$$g(\tilde{\varphi}_k \blacktriangle \tilde{H}_u) = g(\varphi_k \blacktriangle H_u) = g(G_u \blacktriangle \varphi_\ell) = g(\tilde{G}_u \blacktriangle \tilde{\varphi}_\ell)\,.$$

This shows that $\tilde{\varphi}_k \blacktriangle \tilde{H}_u = \tilde{G}_u \blacktriangle \tilde{\varphi}_\ell$.

$$\begin{array}{ccc} \tilde{G}_k & \xrightarrow{\tilde{\varphi}_k} & \tilde{H}_k \\ {\scriptstyle \tilde{G}_u}\downarrow & & \downarrow{\scriptstyle \tilde{H}_u} \\ \tilde{G}_\ell & \xrightarrow[\tilde{\varphi}_\ell]{} & \tilde{H}_\ell \end{array}$$

□

2.5 The décalage functor Dec for simplicial groups

Lemma 43 (The functor dec)

We consider the simplex category Δ.

(1) *Suppose given a morphism* $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$.

We have the morphism $u_+ \in \mathrm{Mor}(\Delta)$*, where*

$$u_+ \colon [n+1] \to [k+1]\,,\ i \mapsto \begin{cases} iu & \text{if } i \in [0,n] \\ k+1 & \text{if } i = n+1\,. \end{cases}$$

(2) *We have the functor*

$$\mathrm{dec} \colon \Delta \to \Delta\,,\ \big([n] \xrightarrow{u} [k]\big) \mapsto \big([n+1] \xrightarrow{u_+} [k+1]\big)\,.$$

Proof. Ad (1). Suppose given $\big([n] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$.

Suppose given x, $y \in [n+1]$ with $x \leqslant y$. We have to show that $xu_+ \overset{!}{\leqslant} yu_+$.

We may assume that $x < y$.

If $y < n+1$, we have

$$xu_+ = xu \leqslant yu = yu_+\,.$$

If $y = n+1$, we have

$$xu_+ = xu < k+1 = yu_+\,.$$

This shows $xu_+ \leqslant yu_+$.

Therefore u_+ is monotone. So $u_+ \in \mathrm{Mor}(\Delta)$.

Ad (2). Suppose given $[n] \xrightarrow{u} [k] \xrightarrow{v} [m]$ in Δ.

For $x \in [n+1]$, we have

$$\begin{aligned} x(\mathrm{id}_{[n]}\,\mathrm{dec}) &= \begin{cases} x & \text{if } x \leqslant n \\ n+1 & \text{if } x = n+1 \end{cases} \\ &= x = x\,\mathrm{id}_{[n+1]} = x\,\mathrm{id}_{[n]\mathrm{dec}}\ . \end{aligned}$$

This shows $\mathrm{id}_{[n]}\,\mathrm{dec} = \mathrm{id}_{[n]\mathrm{dec}}$.

Suppose given $x \in [n+1]$. If $x \leqslant n$, we have

$$x\big((u \blacktriangle v)\mathrm{dec}\big) = xuv = (xu)(v\mathrm{dec}) = x(u\mathrm{dec} \blacktriangle v\mathrm{dec})\,.$$

If $x = n+1$, we have

$$x\big((u \blacktriangle v)\mathrm{dec}\big) = m+1 = (k+1)(v\mathrm{dec}) = x(u\mathrm{dec} \blacktriangle v\mathrm{dec})\,.$$

This shows that $(u \blacktriangle v)\mathrm{dec} = u\mathrm{dec} \blacktriangle v\mathrm{dec}$.

So dec is a functor. □

Lemma 44 (The functor Dec)

We consider the functor $\mathrm{dec}\colon \Delta \to \Delta$ *from Lemma* 43.

We consider the functor $\mathrm{dec}^{\mathrm{op}}\colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$, $\big([k] \xrightarrow{u^{\mathrm{op}}} [n]\big) \mapsto \big([k] \xrightarrow{(u\mathrm{dec})^{\mathrm{op}}} [n]\big)$*; cf.* §1.1.3 item 13.

We have the *ordinary décalage* functor

$$\begin{aligned} \mathrm{Dec} := [\mathrm{dec}^{\mathrm{op}}, \mathit{Grp}]\colon \quad \mathit{SimpGrp} &\to \mathit{SimpGrp} \\ \big(G \xrightarrow{\varphi} H\big) &\mapsto \big(G\mathrm{Dec} \xrightarrow{\varphi\mathrm{Dec}} H\mathrm{Dec}\big) \\ &= \big(\mathrm{dec}^{\mathrm{op}} * G \xrightarrow{\mathrm{dec}^{\mathrm{op}} * \varphi} \mathrm{dec}^{\mathrm{op}} * H\big)\,; \end{aligned}$$

cf. §1.1.3 item 15.

Let $n \in \mathbb{Z}_{\geqslant 0}$. For $G \in \mathrm{Ob}\big(\mathit{SimpGrp}\big)$, we have

$$(G\mathrm{Dec})_n = G_{n+1}\,.$$

For $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}\big(\mathit{SimpGrp}\big)$, we have

$$(\varphi\mathrm{Dec})_n = \varphi_{n+1}\colon G_{n+1} \to H_{n+1}\,.$$

Proof. Suppose given $\big(G \xrightarrow{\varphi} H\big)$ in $\mathit{SimpGrp}$. Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

We have

$$(G\mathrm{Dec})_n = ([n])(\mathrm{dec}^{\mathrm{op}} * G) = ([n+1])G = G_{n+1}\,.$$

We have

$$(\varphi\mathrm{Dec})_n = ([n])(\mathrm{dec}^{\mathrm{op}} * \varphi) = ([n+1])\varphi = \varphi_{n+1}\,.$$

□

Remark 45 *We consider the functor* $\mathrm{Dec}\colon \mathit{SimpGrp} \to \mathit{SimpGrp}$ *from Lemma* 44.

Suppose given a simplicial group $G \in \mathrm{Ob}\big(\mathit{SimpGrp}\big)$.

Let $n \in \mathbb{Z}_{\geqslant 0}$ *and let* $i \in [0, n]$.

We have

$$\begin{aligned} \mathrm{d}_i^{G\mathrm{Dec},n} &= \mathrm{d}_i^{G,n+1} \\ \mathrm{s}_i^{G\mathrm{Dec},n} &= \mathrm{s}_i^{G,n+1}\,. \end{aligned}$$

Proof. We consider the functor $\mathrm{dec}\colon \Delta \to \Delta$ from Lemma 43.

We consider the morphisms $\delta_i^n\colon [n-1] \to [n]$ and $\sigma_i^n\colon [n+1] \to [n]$ in Δ; cf. Definition 19.

Suppose given $x \in [0,n]$.

If $x < i$, then we have

$$x(\delta_i^n \,\mathrm{dec}) = x\delta_i^n = x = x\delta_i^{n+1}\,.$$

If $i < x \leqslant n-1$, then we have

$$x(\delta_i^n \,\mathrm{dec}) = x\delta_i^n = x+1 = x\delta_i^{n+1}\,.$$

If $x = n$, then we have

$$x(\delta_i^n \,\mathrm{dec}) = n(\delta_i^n \,\mathrm{dec}) = n+1 = x+1 = x\delta_i^{n+1}\,.$$

This shows $\delta_i^n \,\mathrm{dec} = \delta_i^{n+1}$.

Suppose given $x \in [0,n+2]$.

If $x \leqslant i$, then we have

$$x(\sigma_i^n \,\mathrm{dec}) = x\sigma_i^n = x = x\sigma_i^{n+1}\,.$$

If $i < x \leqslant n+1$, then we have

$$x(\sigma_i^n \,\mathrm{dec}) = x\sigma_i^n = x-1 = x\sigma_i^{n+1}\,.$$

If $x = n+2$, then we have

$$x(\sigma_i^n \,\mathrm{dec}) = (n+2)(\sigma_i^n \,\mathrm{dec}) = n+1 = x-1 = x\sigma_i^{n+1}$$

This shows $\sigma_i^n \,\mathrm{dec} = \sigma_i^{n+1}$.

So we have

$$\mathrm{d}_i^{G\mathrm{Dec},n} = (\delta_i^n)^{\mathrm{op}}\,(G\mathrm{Dec}) = (\delta_i^n)^{\mathrm{op}}\,(\mathrm{dec}^{\mathrm{op}} * G) = (\delta_i^n \,\mathrm{dec})^{\mathrm{op}}\,G = (\delta_i^{n+1})^{\mathrm{op}}\,G = \mathrm{d}_i^{G,n+1}$$

and

$$\mathrm{s}_i^{G\mathrm{Dec},n} = (\sigma_i^n)^{\mathrm{op}}\,(G\mathrm{Dec}) = (\sigma_i^n)^{\mathrm{op}}\,(\mathrm{dec}^{\mathrm{op}} * G) = (\sigma_i^n \,\mathrm{dec})^{\mathrm{op}}\,G = (\sigma_i^{n+1})^{\mathrm{op}}\,G = \mathrm{s}_i^{G,n+1}\,.$$

□

2.6 The Moore complex of a simplicial group

Definition 46 Let G be a simplicial group; cf. Definition 30.

Let $n \in \mathbb{Z}_{\geqslant 0}$ and let $i \in [0,n]$.

(1) Suppose that $n \geqslant 1$. We write

$$G_{n,i} := \ker \mathrm{d}_i^{G,n} \trianglelefteqslant G_n \,.$$

(2) We write

$$G\mathrm{N}_n := \bigcap_{i\in[1,n]} G_{n,i} \trianglelefteqslant G_n \,.$$

In particular, we have $G\mathrm{N}_0 = G_0$.

(3) Suppose that $n \geqslant 1$. We write

$$G\mathrm{NZ}_n := G\mathrm{N}_n \cap G_{n,0} = \bigcap_{i\in[0,n]} G_{n,i} \trianglelefteqslant G_n \,.$$

Moreover, we write $G\mathrm{NZ}_0 = G_0$.

(4) We write

$$G\mathrm{NB}_n := (G\mathrm{N}_{n+1})\mathrm{d}_0^{G,n+1} = \big(\bigcap_{i\in[1,n+1]} G_{n+1,i}\big)\mathrm{d}_0^{G,n+1} \leqslant G_n \,.$$

Lemma 47 (The Moore complex of a simplicial group)
Let G be a simplicial group; cf. Definition 30.

(1) *For $k \in \mathbb{Z}_{\geqslant 1}$, we have the group morphism*

$$\partial_k^G := \mathrm{d}_0^{G,k}\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}} \colon G\mathrm{N}_k \to G\mathrm{N}_{k-1} \,,\ x \mapsto x\mathrm{d}_0^{G,k} \,.$$

Moreover, we let

$$\partial_0^G := ! \colon G\mathrm{N}_0 \to 1 \,,\ x \mapsto 1 \,.$$

Often, we write $\partial_k := \partial_k^G$ for $k \in \mathbb{Z}_{\geqslant 0}$.

(2) *We have the complex in groups*

$$G\mathrm{N} := \big(\ldots \xrightarrow{\partial_3} G\mathrm{N}_2 \xrightarrow{\partial_2} G\mathrm{N}_1 \xrightarrow{\partial_1} G\mathrm{N}_0 \xrightarrow{!} 1 \xrightarrow{!} \ldots\big) \,;$$

cf. Definition 7.

For $n \in \mathbb{Z}_{\geqslant 0}$, we have $G\mathrm{NB}_n \trianglelefteqslant G_n$; cf. Definition 46.(4).

We call $G\mathrm{N}$ the Moore complex *of G.*

We write

$$G\pi_n := G\mathrm{NH}_n = \ker \partial_n / \operatorname{im} \partial_{n+1} = G\mathrm{NZ}_n / G\mathrm{NB}_n \,.$$

for the n-th homotopy group of G; cf. Lemma 11.(1).

Proof. Ad (1). Let $k \in \mathbb{Z}_{\geqslant 1}$. We have to show that $(G\mathrm{N}_k)\mathrm{d}_0 \overset{!}{\subseteq} G\mathrm{N}_{k-1}$.
Suppose given $x \in G\mathrm{N}_k$. Then $x\mathrm{d}_i = 1$ for $i \in [1,k]$. Suppose given $j \in [1,k-1]$.
We have

$$x\mathrm{d}_0\,\mathrm{d}_j = x\mathrm{d}_{j+1}\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows that $x\mathrm{d}_0 \in G_{k-1,j}$.
So $x\mathrm{d}_0 \in \bigcap\limits_{j\in[1,k-1]} G_{k-1,\,j} = G\mathrm{N}_{k-1}$.
Ad (2). Suppose given $n \in \mathbb{Z}_{\geqslant 1}$.
We show that $\partial_n \mathbin{\blacktriangle} \partial_{n-1} \overset{!}{=} !$.
For $x \in G\mathrm{N}_n$, we have

$$x\partial_n\partial_{n-1} = x\mathrm{d}_0\,\mathrm{d}_0 = x\mathrm{d}_1\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows $\partial_n \mathbin{\blacktriangle} \partial_{n-1} = !$.
We show that $\operatorname{im}\partial_n \overset{!}{\trianglelefteqslant} G_{n-1}$.
Suppose given $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given $y \in G_{n-1}$ and suppose given $x \in \operatorname{im}\partial_n$.
We have to show that $x^y \overset{!}{\in} \operatorname{im}\partial_n$.
We choose $z \in G\mathrm{N}_n$ with $x = z\partial_n = z\mathrm{d}_0$.
Let $\tilde{z} := z^{y\mathrm{s}_0} = y^{-}\mathrm{s}_0 \cdot z \cdot y\mathrm{s}_0 \in G_n$.
Since $z \in G\mathrm{N}_n \trianglelefteqslant G_n$, we also have $\tilde{z} \in G\mathrm{N}_n$.
So we have

$$x^y = (z\mathrm{d}_0)^{y\mathrm{s}_0\mathrm{d}_0} = (z^{y\mathrm{s}_0})\mathrm{d}_0 = \tilde{z}\partial_n\,.$$

This shows $x^y \in \operatorname{im}\partial_n$.
Therefore $\operatorname{im}\partial_n \trianglelefteqslant G_{n-1}$. □

Lemma 48 *Suppose given simplicial groups G, H; cf. Definition 30.*
Suppose given a simplicial group morphism $\varphi\colon G \to H$; cf. Remark 32.

(1) *For $k \in \mathbb{Z}_{\geqslant 0}$, we have the group morphism*

$$\varphi\mathrm{N}_k := \varphi_k\big|^{H\mathrm{N}_k}_{G\mathrm{N}_k}\colon G\mathrm{N}_k \to H\mathrm{N}_k\,,\ x \mapsto x\varphi_k\,.$$

(2) *We consider the complexes in groups* $G\mathrm{N}$ *and* $H\mathrm{N}$*; cf. Lemma* 47.(2).
We have the morphism of complexes in groups

$$\varphi\mathrm{N} := (\varphi\mathrm{N}_k)_{k\in\mathbb{Z}} \colon G\mathrm{N} \to H\mathrm{N}\,,$$

where $\varphi\mathrm{N}_k := !$ *for* $k \leqslant -1$. *Cf. Definition* 8.

(3) *We have the functor*

$$\mathrm{N} \colon \mathcal{SimpGrp} \to \mathcal{ComplexGrp}\,,\ \left(F \xrightarrow{\varphi} F'\right) \mapsto \left(F\mathrm{N} \xrightarrow{\varphi\mathrm{N}} F'\mathrm{N}\right),$$

called Moore complex functor.

Proof. Ad (1). Let $k \in \mathbb{Z}_{\geqslant 0}$. We have to show that $(G\mathrm{N}_k)\varphi_k \overset{!}{\leqslant} H\mathrm{N}_k$.
Suppose given $x \in G\mathrm{N}_k$. For $i \in [1,k]$, we have

$$x\varphi_k\,\mathrm{d}_i^{H,k} = x\mathrm{d}_i^{G,k}\,\varphi_{k-1} = 1\varphi_{k-1} = 1\,.$$

This shows $x\varphi_k \in H_{k,i}$ for $i \in [1,k]$.
So $x\varphi_k \in H\mathrm{N}_k$.
Ad (2). Let $k \in \mathbb{Z}$. We have to show that $\partial_k^G \vartriangle \varphi\mathrm{N}_{k-1} \overset{!}{=} \varphi\mathrm{N}_k \vartriangle \partial_k^H$.
We may assume that $k \geqslant 1$.
For $x \in G\mathrm{N}_k$, we have

$$x\partial_k^G\,(\varphi\mathrm{N}_{k-1}) = x\mathrm{d}_0^{G,k}\,\varphi_{k-1} = x\varphi_k\,\mathrm{d}_0^{H,k} = x(\varphi\mathrm{N}_k)\partial_k^H\,.$$

This shows $\partial_k^G \vartriangle \varphi\mathrm{N}_{k-1} = \varphi\mathrm{N}_k \vartriangle \partial_k^H$.
So $(\varphi\mathrm{N}_k)_{k\in\mathbb{Z}}$ is a morphism of complexes in groups.
Ad (3). Suppose given $F \xrightarrow{\varphi} F' \xrightarrow{\varphi'} F''$ in $\mathcal{SimpGrp}$.
For $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{id}_F\,\mathrm{N}_k = \mathrm{id}_{F_k}\big|_{F\mathrm{N}_k}^{F\mathrm{N}_k} = \mathrm{id}_{F\mathrm{N}_k}\,.$$

For $k \in \mathbb{Z}_{\leqslant -1}$, we have

$$\mathrm{id}_F\,\mathrm{N}_k = ! = \mathrm{id}_{F\mathrm{N}_k}\,.$$

Thus, we have

$$\mathrm{id}_F\,\mathrm{N} = (\mathrm{id}_F\,\mathrm{N}_k)_{k\in\mathbb{Z}} = (\mathrm{id}_{F\mathrm{N}_k})_{k\in\mathbb{Z}} = \mathrm{id}_{F\mathrm{N}}\,.$$

Moreover, for $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N}_k = (\varphi_k \blacktriangle \varphi'_k)\big|_{F\mathrm{N}_k}^{F''\mathrm{N}_k} = \varphi_k\big|_{F\mathrm{N}_k}^{F'\mathrm{N}_k} \blacktriangle \varphi'_k\big|_{F'\mathrm{N}_k}^{F''\mathrm{N}_k} = \varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k\,.$$

For $k \in \mathbb{Z}_{\leqslant -1}$, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N}_k = ! = \varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k\,.$$

Thus, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N} = \big((\varphi \blacktriangle \varphi')\mathrm{N}_k\big)_{k\in\mathbb{Z}} = (\varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k)_{k\in\mathbb{Z}} = (\varphi\mathrm{N}_k)_{k\in\mathbb{Z}} \blacktriangle (\varphi'\mathrm{N}_k)_{k\in\mathbb{Z}} = \varphi\mathrm{N} \blacktriangle \varphi'\mathrm{N}\,.$$

So N is a functor. □

2.7 Preparations for the Conduché condition

We shall give the assertion needed for the verfication of the Conduché condition for the $\lfloor n, 0\rfloor$-simplicial groups obtained from a simplicial group by truncation; cf. §3.4 and Lemma 118 below.

Lemma 49 *Suppose given a simplicial group $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$; cf. Definition* 30.
Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.
For $\ell \in [0, n+1]$, we have

$$G\mathrm{NB}_n = \big(\bigcap_{i\in[0,n+1]\setminus\{\ell\}} G_{n+1,i}\big)\mathrm{d}_\ell^{G,\,n+1}\,;$$

cf. Definition 46.(4).

Proof. For $\ell \in [0, n+1]$, we write

$$U_\ell := \big(\bigcap_{i\in[0,n+1]\setminus\{\ell\}} G_{n+1,i}\big)\mathrm{d}_\ell\,.$$

We show that $G\mathrm{NB}_n \overset{!}{=} U_\ell$ by induction on $\ell \in [0, n+1]$.
Suppose that $\ell = 0$.
Then we have

$$G\mathrm{NB}_n = \big(\bigcap_{i\in[1,n+1]} G_{n+1,i}\big)\mathrm{d}_0 = \big(\bigcap_{i\in[0,n+1]\setminus\{0\}} G_{n+1,i}\big)\mathrm{d}_0 = U_0\,.$$

Suppose that $\ell \in [1, n+1]$.

By induction hypothesis, this assertion holds for $\ell - 1$.

We have to show that $U_{\ell-1} \overset{!}{=} U_\ell$.

Ad $\subseteq$. Suppose given $y \in U_{\ell-1}$. Then we may choose some $x \in \bigcap_{i\in[0,n+1]\setminus\{\ell-1\}} G_{n+1,i}$ such that $y = x\mathrm{d}_{\ell-1}$. Let $\tilde{x} := x^- \cdot x\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-1} \in G_{n+1}$.

We show that $\tilde{x} \overset{!}{\in} \bigcap_{i\in[0,n+1]\setminus\{\ell\}} G_{n+1,i}$.

Suppose given $i \in [0, n+1]\setminus\{\ell\}$. We use Remark 33 for the following calculations.

If $i \leqslant \ell - 2$, then we have

$$\tilde{x}\mathrm{d}_i = x^-\mathrm{d}_i \cdot x\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-1}\,\mathrm{d}_i = 1 \cdot x\mathrm{d}_{\ell-1}\,\mathrm{d}_i\,\mathrm{s}_{\ell-2} = x\mathrm{d}_i\,\mathrm{d}_{\ell-2}\,\mathrm{s}_{\ell-2} = 1\,.$$

If $i = \ell - 1$, then we have

$$\tilde{x}\mathrm{d}_i = x^-\mathrm{d}_{\ell-1} \cdot x\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-1}\,\mathrm{d}_{\ell-1} = (x\mathrm{d}_{\ell-1})^- \cdot x\mathrm{d}_{\ell-1} = 1\,.$$

If $i \geqslant \ell + 1$, then we have

$$\tilde{x}\mathrm{d}_i = x^-\mathrm{d}_i \cdot x\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-1}\,\mathrm{d}_i = 1 \cdot x\mathrm{d}_{\ell-1}\,\mathrm{d}_{i-1}\,\mathrm{s}_{\ell-1} = x\mathrm{d}_i\,\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-1} = 1\,.$$

This shows $\tilde{x} \in G_{n+1,i}$ for $i \in [0, n+1]\setminus\{\ell\}$. So $\tilde{x} \in \bigcap_{i\in[0,n+1]\setminus\{\ell\}} G_{n+1,i}$.

We show that $\tilde{x}\mathrm{d}_\ell \overset{!}{=} y$.

We have

$$\tilde{x}\mathrm{d}_\ell = x^-\mathrm{d}_\ell \cdot x\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-1}\,\mathrm{d}_\ell = 1 \cdot x\mathrm{d}_{\ell-1} = y\,.$$

Altogether, we have

$$y = \tilde{x}\mathrm{d}_\ell \in \big(\bigcap_{i\in[0,n+1]\setminus\{\ell\}} G_{n+1,i}\big)\mathrm{d}_\ell = U_\ell\,.$$

So $U_{\ell-1} \subseteq U_\ell$.

Ad $\supseteq$. Suppose given $y \in U_\ell$. Then we choose some $x \in \bigcap_{i\in[0,n+1]\setminus\{\ell\}} G_{n+1,i}$ such that $y = x\mathrm{d}_\ell$. Let $\tilde{x} := x^- \cdot x\mathrm{d}_\ell\,\mathrm{s}_{\ell-1} \in G_{n+1,i}$.

We show that $\tilde{x} \overset{!}{\in} \bigcap_{i\in[0,n+1]\setminus\{\ell-1\}} G_{n+1,i}$.

Suppose given $i \in [0, n+1]\setminus\{\ell - 1\}$. We use Remark 33 for the following calculations.

If $i \leqslant \ell - 2$, then we have

$$\tilde{x}\mathrm{d}_i = x^{-}\mathrm{d}_i \cdot x\mathrm{d}_\ell\, \mathrm{s}_{\ell-1}\, \mathrm{d}_i = 1 \cdot x\mathrm{d}_\ell\, \mathrm{d}_i\, \mathrm{s}_{\ell-2} = x\mathrm{d}_i\, \mathrm{d}_{\ell-1}\, \mathrm{s}_{\ell-2} = 1\,.$$

If $i = \ell$, then we have

$$\tilde{x}\mathrm{d}_i = x^{-}\mathrm{d}_\ell \cdot x\mathrm{d}_\ell\, \mathrm{s}_{\ell-1}\, \mathrm{d}_\ell = (x\mathrm{d}_\ell)^{-} \cdot x\mathrm{d}_\ell = 1\,.$$

If $i \geqslant \ell + 1$, then we have

$$\tilde{x}\mathrm{d}_i = x^{-}\mathrm{d}_i \cdot x\mathrm{d}_\ell\, \mathrm{s}_{\ell-1}\, \mathrm{d}_i = 1 \cdot x\mathrm{d}_\ell\, \mathrm{d}_{i-1}\, \mathrm{s}_{\ell-1} = x\mathrm{d}_i\, \mathrm{d}_\ell\, \mathrm{s}_{\ell-1} = 1\,.$$

This shows $\tilde{x} \in G_{n+1,i}$ for $i \in [0, n+1]\backslash\{\ell - 1\}$. So $\tilde{x} \in \bigcap_{i\in[0,n+1]\backslash\{\ell-1\}} G_{n+1,i}$.

We show that $\tilde{x}\mathrm{d}_{\ell-1} \overset{!}{=} y$.

We have

$$\tilde{x}\mathrm{d}_{\ell-1} = x^{-}\mathrm{d}_{\ell-1} \cdot x\mathrm{d}_\ell\, \mathrm{s}_{\ell-1}\, \mathrm{d}_{\ell-1} = 1 \cdot x\mathrm{d}_\ell = y\,.$$

Altogether, we have

$$y = \tilde{x}\mathrm{d}_{\ell-1} \in \Big(\bigcap_{i\in[0,n+1]\backslash\{\ell-1\}} G_{n+1,i}\Big)\mathrm{d}_{\ell-1} = U_{\ell-1}\,.$$

So $U_\ell \subseteq U_{\ell-1}$.

This shows $U_{\ell-1} = U_\ell$. □

Lemma 50 *Suppose given a simplicial group G; cf. Definition* 30.

Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given x, $y \in G_n$.

For $i \in [0, n]$, suppose that we have

$$x\mathrm{d}_i^{G,n} = 1 \quad \textit{or} \quad y\mathrm{d}_i^{G,n} = 1\,.$$

Moreover, suppose that there exists an $\ell \in [0, n-1]$ such that

$$x\mathrm{d}_{\ell+1}^{G,n} = 1 \quad \textit{and} \quad y\mathrm{d}_\ell^{G,n} = 1\,.$$

Then we have $[x, y] \in G\mathrm{NB}_n$.

Proof. We choose some $\ell \in [0, n-1]$ such that $x\mathrm{d}_{\ell+1} = 1$ and $y\mathrm{d}_\ell = 1$.

Let $c := [x\mathrm{s}_\ell,\, y\mathrm{s}_{\ell+1}] \in G_{n+1}$.

We show that $c \overset{!}{\in} \bigcap_{i\in[0,n+1]\setminus\{\ell+1\}} G_{n+1,\,i}$.

Let $i \in [0, n+1]\setminus\{\ell+1\}$.

We use Remark 33 for the following calculations.

If $i \leqslant \ell - 1$ then we have

$$c\,\mathrm{d}_i = [x\mathrm{s}_\ell\,\mathrm{d}_i,\ y\mathrm{s}_{\ell+1}\,\mathrm{d}_i] = [x\mathrm{d}_i\,\mathrm{s}_{\ell-1},\ y\mathrm{d}_i\,\mathrm{s}_\ell] = 1$$

since $x\mathrm{d}_i = 1$ or $y\mathrm{d}_i = 1$.

If $i = \ell$ then we have

$$c\,\mathrm{d}_\ell = [x\mathrm{s}_\ell\,\mathrm{d}_\ell,\ y\mathrm{s}_{\ell+1}\,\mathrm{d}_\ell] = [x,\ y\mathrm{d}_\ell\,\mathrm{s}_\ell] = [x, 1] = 1\,.$$

If $i = \ell + 2$ then we have

$$c\,\mathrm{d}_{\ell+2} = [x\mathrm{s}_\ell\,\mathrm{d}_{\ell+2},\ y\mathrm{s}_{\ell+1}\,\mathrm{d}_{\ell+2}] = [x\mathrm{d}_{\ell+1}\,\mathrm{s}_\ell,\ y] = [1, y] = 1\,.$$

If $i \geqslant \ell + 3$ then we have

$$c\,\mathrm{d}_i = [x\mathrm{s}_\ell\,\mathrm{d}_i,\ y\mathrm{s}_{\ell+1}\,\mathrm{d}_i] = [x\mathrm{d}_{i-1}\,\mathrm{s}_\ell,\ y\mathrm{d}_{i-1}\,\mathrm{s}_{\ell+1}] = 1$$

since $x\mathrm{d}_{i-1} = 1$ or $y\mathrm{d}_{i-1} = 1$.

This shows that $c \in \bigcap_{i\in[0,n+1]\setminus\{\ell+1\}} G_{n+1,\,i}$.

We show that $c\,\mathrm{d}_{\ell+1} \overset{!}{=} [x, y]$.

We have

$$c\,\mathrm{d}_{\ell+1} = [x\mathrm{s}_\ell\,\mathrm{d}_{\ell+1},\ y\mathrm{s}_{\ell+1}\,\mathrm{d}_{\ell+1}] = [x, y]\,.$$

Altogether, this shows

$$[x, y] = c\,\mathrm{d}_{\ell+1} \in \big(\bigcap_{i\in[0,n+1]\setminus\{\ell+1\}} G_{n+1,\,i}\big)\mathrm{d}_{\ell+1} \overset{49}{=} G\mathrm{NB}_n\,.$$

□

Chapter 3

$\lfloor n, 0 \rfloor$-simplicial groups

3.1 Pre-$\lfloor n, 0 \rfloor$-simplicial groups

Let $n \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$.

Definition 51 We consider the simplex category Δ; cf. Definition 18.

We have the full subcategory $\Delta_{\lfloor n,0 \rfloor} \subseteq \Delta$, where

$$\mathrm{Ob}\big(\Delta_{\lfloor n,0 \rfloor}\big) := \big\{ [k] \colon k \in [0, n] \big\} .$$

Then note that

$$\mathrm{Mor}\big(\Delta_{\lfloor n,0 \rfloor}\big) := \big\{ [\ell] \xrightarrow{u} [k] \colon \ell, k \in [0, n] \, , u \text{ is a monotone map} \big\} .$$

In particular, we have $\Delta_{\lfloor +\infty,0 \rfloor} = \Delta$.

Definition 52 (Pre-$\lfloor n, 0 \rfloor$-simplicial groups)

We consider the category of groups $\mathcal{G}rp$ and the opposite category $\Delta^{\mathrm{op}}_{\lfloor n,0 \rfloor}$ of $\Delta_{\lfloor n,0 \rfloor}$; cf. §1.1.3 item 12.

We write $\mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{S}impGrp := [\Delta^{\mathrm{op}}_{\lfloor n,0 \rfloor}, \mathcal{G}rp]$ for the category of functors from $\Delta^{\mathrm{op}}_{\lfloor n,0 \rfloor}$ to $\mathcal{G}rp$ and transformations between those functors.

We call $\mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{S}impGrp$ the *category of pre-*$\lfloor n, 0 \rfloor$*-simplicial groups.*

A functor $(\Delta^{\mathrm{op}}_{\lfloor n,0 \rfloor} \xrightarrow{G} \mathcal{G}rp) \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{S}impGrp\big)$ is called a *pre-*$\lfloor n, 0 \rfloor$*-simplicial group.*

For $k \in [0, n]$, we write $G_k := [k]G \in \mathrm{Ob}(\mathcal{G}rp)$.

For $k \in [0,n]$, we write $\varphi_k := [k]\varphi \colon G_k \to G'_k$.

A transformation $\big(G \xrightarrow{\varphi} G'\big) \in \mathrm{Mor}\big(\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp\big)$ between $\lfloor n,0\rfloor$-simplicial groups G, $G' \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp\big)$ is called a *pre-$\lfloor n,0\rfloor$-simplicial group morphism.*

For $\big([\ell] \xrightarrow{u} [k]\big) \in \mathrm{Mor}\big(\Delta_{\lfloor n,0\rfloor}\big)$, we write $G_u := (u^{\mathrm{op}})G \colon G_k \to G_\ell$.

For morphisms $G \xrightarrow{\varphi} G' \xrightarrow{\varphi'} G''$ in $\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp$, their composite is given by their vertical composite of transformations

$$\varphi \mathbin{\blacktriangle} \varphi' = \big(G_k \xrightarrow{[k](\varphi \mathbin{\blacktriangle} \varphi')} G''_k\big)_{k\in[0,n]} := \big(G_k \xrightarrow{\varphi_k \mathbin{\blacktriangle} \varphi'_k} G''_k\big)_{k\in[0,n]} .$$

So $(\varphi \mathbin{\blacktriangle} \varphi')_k = \varphi_k \mathbin{\blacktriangle} \varphi'_k$ for $k \in [0,n]$.

For $G \in \mathrm{Ob}(\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp)$, its identity is given by

$$\mathrm{id}_G = \big(G_k \xrightarrow{\mathrm{id}_{G_k}} G_k\big)_{k\in[0,n]} .$$

So $(\mathrm{id}_G)_k = \mathrm{id}_{G_k}$.

Note that in particular, we have $\mathcal{P}re\text{-}\lfloor +\infty,0\rfloor\text{-}\mathcal{S}impGrp = \mathcal{S}impGrp$.

Definition 53 (Face maps and degeneracy maps)

Suppose given a pre-$\lfloor n,0\rfloor$-simplicial group $G \colon \Delta^{\mathrm{op}}_{\lfloor n,0\rfloor} \to \mathcal{G}rp$.

Recall that $\Delta_{\lfloor n,0\rfloor} \subseteq \Delta$.

For $k \in [1,n]$ and $i \in [0,k]$, we consider the morphism $\delta^k_i \colon [k-1] \to [k]$ in $\Delta_{\lfloor n,0\rfloor}$.

For $k \in [0,n-1]$ and $i \in [0,k]$, we consider the morphism $\sigma^k_i \colon [k+1] \to [k]$ in $\Delta_{\lfloor n,0\rfloor}$; cf. Definition 19.

For $k \in [1,n]$ and $i \in [0,k]$, we have the *face map*

$$\mathrm{d}^{G,k}_i := G_{\delta^k_i} = \big((\delta^k_i)^{\mathrm{op}}\big)G \colon G_k \to G_{k-1} .$$

It is a group morphism.

For $k \in [0,n-1]$ and $i \in [0,k]$, we have the *degeneracy map*

$$\mathrm{s}^{G,k}_i := G_{\sigma^k_i} = \big((\sigma^k_i)^{\mathrm{op}}\big)G \colon G_k \to G_{k+1} .$$

It is a group morphism. Cf. also Definition 31.

Sometimes, we abbreviate $\mathrm{d}_i := \mathrm{d}^{G,k}_i$ and $\mathrm{s}_i := \mathrm{s}^{G,k}_i$.

Remark 54 Suppose given a pre-$\lfloor n, 0 \rfloor$-simplicial group G.
We consider the face maps and degeneracy maps from Definition 53.
Then we have the conditions (1, 2, 3); cf. Remark 24, cf. also Remark 33.

(1) We have

$$\mathrm{d}_j^{G,k+1} \blacktriangle \mathrm{d}_i^{G,k} = \mathrm{d}_i^{G,k+1} \blacktriangle \mathrm{d}_{j-1}^{G,k}$$

for $k \in [0, n-1]$, $i \in [0, k]$, $j \in [0, k+1]$ with $i < j$.

(2) We have

$$\mathrm{s}_i^{G,k-1} \blacktriangle \mathrm{s}_j^{G,k} = \mathrm{s}_{j-1}^{G,k-1} \blacktriangle \mathrm{s}_i^{G,k}$$

for $k \in [1, n-1]$, $i \in [0, k-1]$, $j \in [0, k]$ with $i < j$.

(3) We have

$$\mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} = \begin{cases} \mathrm{d}_j^{G,k} \blacktriangle \mathrm{s}_{i-1}^{G,k-1} & \text{if } k \geqslant 1 \text{ and } j < i \\ \mathrm{id}_{G_k} & \text{if } j = i \text{ or } j = i+1 \\ \mathrm{d}_{j-1}^{G,k} \blacktriangle \mathrm{s}_i^{G,k-1} & \text{if } k \geqslant 1 \text{ and } j > i+1 \end{cases}$$

for $k \in [0, n-1]$, $i \in [0, k]$, $j \in [0, k+1]$.

Remark 55
Suppose given a pre-$\lfloor n, 0 \rfloor$-simplicial group morphism $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}\big(\mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{S}impGrp\big)$.
For $k \in [1, n]$ and $j \in [0, k]$, we have

$$\varphi_k \blacktriangle \mathrm{d}_j^{H,k} = \mathrm{d}_j^{G,k} \blacktriangle \varphi_{k-1} \,.$$

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & H_k \\ {\scriptstyle \mathrm{d}_j^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_j^{H,k}} \\ G_{k-1} & \xrightarrow[\varphi_{k-1}]{} & H_{k-1} \end{array}$$

For $k \in [0, n-1]$ and $i \in [0, k]$, we have

$$\varphi_k \blacktriangle \mathrm{s}_i^{H,k} = \mathrm{s}_i^{G,k} \blacktriangle \varphi_{k+1} \,.$$

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & H_k \\ {\scriptstyle \mathrm{s}_i^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{s}_i^{H,k}} \\ G_{k+1} & \xrightarrow[\varphi_{k+1}]{} & H_{k+1} \end{array}$$

3.2 A description using generators and relations

Let $n \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$. Suppose given m, $\tilde{m} \in [0, n]$.

Definition 56

Let

$$R_{m,\tilde{m}} := \Big\{ \big((\varepsilon_1\,, i_1), (\varepsilon_2\,, i_2), \ldots, (\varepsilon_s\,, i_s)\big) \colon s \in \mathbb{Z}_{\geqslant 0}\,,\ \varepsilon_k \in \{-1, +1\}\,, \text{ where } \sum_{j\in[1,s]} \varepsilon_j = \tilde{m} - m\,,$$
$$i_k \in \big[0, m + \sum_{j\in[1,k]} \varepsilon_j\big] \text{ for } k \in [1, s] \text{ and } 0 \leqslant m + \sum_{j\in[1,k]} \varepsilon_j \leqslant n \text{ for } k \in [0, s] \Big\}\,.$$

An element of $R_{m,\tilde{m}}$ is called an $(m, \tilde{m})$-*index tuple*.

We consider the following relations on $R_{m,\tilde{m}}$, given as subsets of $R_{m,\tilde{m}} \times R_{m,\tilde{m}}$ consisting of pairs of $(m, \tilde{m})$-index tuples of the respective forms.

$$(\overset{++}{\rightsquigarrow}) := \Big\{ \Big(\big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{k-1}\,, i_{k-1}), (\varepsilon_k\,, i_k), (\varepsilon_{k+1}\,, i_{k+1}), (\varepsilon_{k+2}\,, i_{k+2}), \ldots, (\varepsilon_s\,, i_s)\big),$$
$$\big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{k-1}\,, i_{k-1}), (\varepsilon_{k+1}\,, i_{k+1}), (\varepsilon_k\,, i_k + 1), (\varepsilon_{k+2}\,, i_{k+2}), \ldots, (\varepsilon_s\,, i_s)\big) \Big) \colon$$
$$\text{for some } k \in [1, s-1] \text{ with } i_k \geqslant i_{k+1} \text{ and } \varepsilon_k = \varepsilon_{k+1} = +1 \Big\}\,,$$

$$(\overset{--}{\rightsquigarrow}) := \Big\{ \Big(\big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{k-1}\,, i_{k-1}), (\varepsilon_k\,, i_k), (\varepsilon_{k+1}\,, i_{k+1}), (\varepsilon_{k+2}\,, i_{k+2}), \ldots, (\varepsilon_s\,, i_s)\big),$$
$$\big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{k-1}\,, i_{k-1}), (\varepsilon_{k+1}\,, i_{k+1} + 1), (\varepsilon_k\,, i_k), (\varepsilon_{k+2}\,, i_{k+2}), \ldots, (\varepsilon_s\,, i_s)\big) \Big) \colon$$
$$\text{for some } k \in [1, s-1] \text{ with } i_k \leqslant i_{k+1} \text{ and } \varepsilon_k = \varepsilon_{k+1} = -1 \Big\}\,,$$

$$
\begin{aligned}
(\overset{+-;1}{\rightsquigarrow}) := \; & \Big\{ \Big(\big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_k\,,i_k),(\varepsilon_{k+1}\,,i_{k+1}),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big), \\
& \quad \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_{k+1}\,,i_{k+1}-1),(\varepsilon_k\,,i_k),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big)\Big) : \\
& \quad \text{for some } k\in[1,s-1] \text{ with } i_k \leqslant i_{k+1}-1 \text{ and } \varepsilon_k=+1 \text{ and } \varepsilon_{k+1}=-1 \Big\}\,, \\
(\overset{+-;2}{\rightsquigarrow}) := \; & \Big\{ \Big(\big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_k\,,i_k),(\varepsilon_{k+1}\,,i_{k+1}),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big), \\
& \quad \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big)\Big) : \\
& \quad \text{for some } k\in[1,s-1] \text{ with } i_k \in \{i_{k+1}\,,i_{k+1}+1\} \text{ and } \varepsilon_k=+1 \text{ and } \varepsilon_{k+1}=-1 \Big\}\,, \\
(\overset{+-;3}{\rightsquigarrow}) := \; & \Big\{ \Big(\big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_k\,,i_k),(\varepsilon_{k+1}\,,i_{k+1}),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big), \\
& \quad \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_{k+1}\,,i_{k+1}),(\varepsilon_k\,,i_k-1),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big)\Big) : \\
& \quad \text{for some } k\in[1,s-1] \text{ with } i_k \geqslant i_{k+1}+2 \text{ and } \varepsilon_k=+1 \text{ and } \varepsilon_{k+1}=-1 \Big\} \\
\subseteq \; & R_{m,\tilde{m}} \times R_{m,\tilde{m}}\,.
\end{aligned}
$$

Note that all occurring $(m,\tilde{m})$-index tuples are required to be in $R_{m,\tilde{m}}$.

We consider the relation on $R_{m,\tilde{m}}$ given by

$$(\rightsquigarrow) := (\overset{++}{\rightsquigarrow}) \;\cup\; (\overset{--}{\rightsquigarrow}) \;\cup\; (\overset{+-;1}{\rightsquigarrow}) \;\cup\; (\overset{+-;2}{\rightsquigarrow}) \;\cup\; (\overset{+-;3}{\rightsquigarrow}) \subseteq R_{m,\tilde{m}} \times R_{m,\tilde{m}}\,.$$

On $R_{m,\tilde{m}}$, we define the equivalence relation $(\sim)$ to be generated by $(\rightsquigarrow)$.

We write $R_{m,\tilde{m}}/(\sim)$ for the set of equivalence classes.

For $\underline{i} \in R_{m,\tilde{m}}$, we write $[\underline{i}] := \{\underline{j} \in R_{m,\tilde{m}} \colon \underline{j} \sim \underline{i}\} \in R_{m,\tilde{m}}/(\sim)$.

Definition 57 Suppose given $\underline{i} = \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_s\,,i_s)\big) \in R_{m,\tilde{m}}$.

We write

$$
\begin{aligned}
w_0(\underline{i}) &:= -\big|\big\{(j,k)\in[1,s]^{\times 2} \colon j<k \text{ and } \varepsilon_j>\varepsilon_k\big\}\big| \\
w_-(\underline{i}) &:= \sum_{\substack{k\in[1,s]\\ \varepsilon_k=-1}} i_k\,, \quad w_+(\underline{i}) := \sum_{\substack{k\in[1,s]\\ \varepsilon_k=+1}} i_k \\
w_\pm(\underline{i}) &:= w_-(\underline{i}) + w_+(\underline{i})\,.
\end{aligned}
$$

Definition 58

We write $R^0_{m,\tilde{m}} := \big\{ \underline{i} \in R_{m,\tilde{m}} \colon w_0(\underline{i}) = 0 \big\} \subseteq R_{m,\tilde{m}}$.

Note that for $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_s\,, i_s)\big) \in R^0_{m,\tilde{m}}$ we have

$$\underline{i} = \big((-1, i_1), \ldots, (-1, i_\ell), (+1, i_{\ell+1}), \ldots, (+1, i_s)\big)$$

for some $\ell \in [0, s]$.

Remark 59 *Suppose given* $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_s\,, i_s)\big) \in R_{m,\tilde{m}}$.

For $k \in [1, s]$*, we write*

$$\tau^{(\varepsilon_k)}_{i_k} := \begin{cases} \sigma^{r_k}_{i_k} & \text{if } \varepsilon_k = -1 \\ \delta^{r_k}_{i_k} & \text{if } \varepsilon_k = +1\,, \end{cases}$$

where $r_k := m + \sum\limits_{j \in [1,k]} \varepsilon_j$. *So* $\tau^{\varepsilon_k}_{i_k} \colon [r_{k-1}] \to [r_k]$ *in* $\Delta_{\lfloor n,0\rfloor}$.

Moreover, we write

$$\tau_{\underline{i}} := \prod_{k \in \lceil 1,s \rceil}^{[m]\;[\tilde{m}]} \tau^{(\varepsilon_k)}_{i_k} = \tau^{(\varepsilon_1)}_{i_1} \blacktriangle \tau^{(\varepsilon_2)}_{i_2} \blacktriangle \ldots \blacktriangle \tau^{(\varepsilon_s)}_{i_s}\,.$$

We have $\tau_{\underline{i}} \in \mathrm{Mor}\big(\Delta_{\lfloor n,0\rfloor}\big)$.

Proof. Suppose given $k \in [1, s]$. We have to show that $\tau^{(\varepsilon_k)}_{i_k} \overset{!}{\in} \mathrm{Mor}\big(\Delta_{\lfloor n,0\rfloor}\big)$.

Recall that

$$r_k = m + \sum_{j \in [1,k]} \varepsilon_j \overset{56}{\in} [0, n]\,.$$

Case $\varepsilon_k = +1$. We show that $r_k \overset{!}{\in} [1, n]$.

We *assume* that $r_k = 0$. If $k = 1$, then we have

$$0 = r_k = m + \sum_{j \in [1,k]} \varepsilon_j = m + \varepsilon_k = m + 1\,.$$

So $m = -1$, which *contradicts* $m \in \mathbb{Z}_{\geqslant 0}$.

If $k \geqslant 2$, then we have

$$0 = r_k = m + \sum_{j \in [1,k]} \varepsilon_j = m + \Big(\sum_{j \in [1,k-1]} \varepsilon_j \Big) + \varepsilon_k = r_{k-1} + 1\,.$$

So $r_{k-1} = -1$, which *contradicts* $r_{k-1} \in [0, n]$.

This shows that $r_k \in [1, n]$.

Altogether, we have

$$\tau_{i_k}^{(\varepsilon_k)} = \delta_{i_k}^{r_k} \colon [r_k - 1] \to [r_k] \in \mathrm{Mor}\big(\Delta_{\lfloor n,0 \rfloor}\big) \ ,$$

since $r_k \in [1, n]$ and $i_k \in [0, r_k]$.

Case $\varepsilon_k = -1$. We show that $r_k \overset{!}{\in} [0, n-1]$.

We *assume* that $r_k = n$. If $k = 1$, then we have

$$n = r_k = m + \sum_{j \in [1,k]} \varepsilon_j = m + \varepsilon_k = m - 1 \, .$$

So $m = n + 1$, which *contradicts* $m \leqslant n$.

If $k \geqslant 2$, then we have

$$n = r_k = m + \sum_{j \in [1,k]} \varepsilon_j = m + \Big(\sum_{j \in [1,k-1]} \varepsilon_j \Big) + \varepsilon_k = r_{k-1} - 1 \, .$$

So $r_{k-1} = n + 1$, which *contradicts* $r_{k-1} \in [0, n]$.

This shows that $r_k \in [0, n-1]$.

Altogether, we have

$$\tau_{i_k}^{(\varepsilon_k)} = \sigma_{i_k}^{r_k} \colon [r_k + 1] \to [r_k] \in \mathrm{Mor}\big(\Delta_{\lfloor n,0 \rfloor}\big) \ ,$$

since $r_k \in [0, n-1]$ and $i_k \in [0, r_k]$. □

Remark 60 *Suppose given* $\underline{i}$, $\underline{j} \in R_{m,\tilde{m}}$.

(1) *We have*

$$(\underline{i} \overset{++}{\rightsquigarrow} \underline{j}) \vee (\underline{i} \overset{--}{\rightsquigarrow} \underline{j}) \Rightarrow \ \big(w_0(\underline{i}) = w_0(\underline{j})\big) \wedge \big(w_\pm(\underline{i}) < w_\pm(\underline{j})\big) \, ,$$
$$(\underline{i} \overset{+-;1}{\rightsquigarrow} \underline{j}) \vee (\underline{i} \overset{+-;2}{\rightsquigarrow} \underline{j}) \vee (\underline{i} \overset{+-;3}{\rightsquigarrow} \underline{j}) \Rightarrow \ w_0(\underline{i}) < w_0(\underline{j}) \, .$$

(2) *We have*

$$\underline{i} \sim \underline{j} \ \Rightarrow \ \tau_{\underline{i}} = \tau_{\underline{j}} \, .$$

Proof. Ad (1). If $\underline{i} \overset{++}{\rightsquigarrow} \underline{j}$, then we have

$$w_0(\underline{i}) = w_0(\underline{j}) \, , \ w_-(\underline{i}) = w_-(\underline{j}) \text{ and } w_+(\underline{i}) = w_+(\underline{j}) - 1 \, .$$

So $w_\pm(\underline{i}) < w_\pm(\underline{j})$.

If $\underline{i} \overset{--}{\rightsquigarrow} \underline{j}$, then we have

$$w_0(\underline{i}) = w_0(\underline{j}),\ w_-(\underline{i}) = w_-(\underline{j}) - 1 \text{ and } w_+(\underline{i}) = w_+(\underline{j}).$$

So $w_\pm(\underline{i}) < w_\pm(\underline{j})$.

If $\underline{i} \overset{+-,1}{\rightsquigarrow} \underline{j}$ or $\underline{i} \overset{+-,3}{\rightsquigarrow} \underline{j}$, then we have

$$w_0(\underline{i}) = w_0(\underline{j}) - 1.$$

So $w_0(\underline{i}) < w_0(\underline{j})$.

If $\underline{i} \overset{+-,2}{\rightsquigarrow} \underline{j}$, then we write $\underline{i} = \big((\varepsilon_1, i_1), \ldots, (\varepsilon_s, i_s)\big)$ and obtain

$$w_0(\underline{i}) = w_0(\underline{j}) - 1 - \Big|\big\{(a,b) \in [1,s]^{\times 2} \colon a < b \text{ and } \varepsilon_a > \varepsilon_b \text{ and } \big|\{a,b\} \cap \{k, k+1\}\big| = 1\big\}\Big|$$

for some $k \in [1, s-1]$. So $w_0(\underline{i}) < w_0(\underline{j})$.

Altogether, this shows the assertion.

Ad (2). We may assume that $\underline{i} \rightsquigarrow \underline{j}$.

We write $\underline{i} = \big((\varepsilon_1, i_1), \ldots, (\varepsilon_s, i_s)\big)$ and $\underline{j} = \big((\tilde{\varepsilon}_1, j_1), \ldots, (\tilde{\varepsilon}_t, j_t)\big)$.

Case $\underline{i} \overset{++}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$(\tilde{\varepsilon}_k, j_k) = \begin{cases} (\varepsilon_k, i_k) & \text{for } k \in [1,s]\backslash\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}, i_{\ell+1}) & \text{for } k = \ell \\ (\varepsilon_\ell, i_\ell + 1) & \text{for } k = \ell + 1 \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \geqslant i_{\ell+1}$, and $\varepsilon_\ell = \varepsilon_{\ell+1} = +1$.

So $\underline{j} = \big((\varepsilon_1, i_1), \ldots, (\varepsilon_{\ell-1}, i_{\ell-1}), (\varepsilon_{\ell+1}, i_{\ell+1}), (\varepsilon_\ell, i_\ell + 1), (\varepsilon_{\ell+2}, i_{\ell+2}), \ldots, (\varepsilon_s, i_s)\big)$.

Let $r := m + \sum\limits_{k \in [1,\ell-1]} \varepsilon_k$.

Since $i_{\ell+1} < i_\ell + 1$, we have

$$\begin{aligned} \tau_{\underline{j}} &= \prod_{k \in \lceil 1,s \rceil}^{[m]} {}^{[\tilde{m}]}\tau_{j_k}^{(\tilde{\varepsilon}_k)} \\ &= \Big(\prod_{k \in \lceil 1,\ell-1 \rceil}^{[m]} {}^{[r]}\tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \tau_{i_{\ell+1}}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell+1}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k \in \lceil \ell+2,s \rceil}^{[r+2]} {}^{[\tilde{m}]}\tau_{i_k}^{(\varepsilon_k)}\Big) \end{aligned}$$

$$
\begin{aligned}
&\overset{59}{=} \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\delta_{i_{\ell+1}} \blacktriangle \delta_{i_\ell+1}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r+2]\ [\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{24.(1)}{=} \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\delta_{i_\ell} \blacktriangle \delta_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r+2]\ [\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{59}{=} \quad \prod_{k\in\lceil 1,s\rceil}^{[m]\ [\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \\
&= \quad \tau_{\underline{i}} \,.
\end{aligned}
$$

Case $\underline{i} \overset{--}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$
(\tilde{\varepsilon}_k \,, j_k) = \begin{cases} (\varepsilon_k \,, i_k) & \text{for } k \in [1,s]\backslash\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1} \,, i_{\ell+1}+1) & \text{for } k = \ell \\ (\varepsilon_\ell \,, i_\ell) & \text{for } k = \ell+1 \end{cases}
$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \leqslant i_{\ell+1}$, and $\varepsilon_\ell = \varepsilon_{\ell+1} = -1$.

So $\underline{j} = \big((\varepsilon_1 \,, i_1), \ldots, (\varepsilon_{\ell-1} \,, i_{\ell-1}), (\varepsilon_{\ell+1} \,, i_{\ell+1}+1), (\varepsilon_\ell \,, i_\ell), (\varepsilon_{\ell+2} \,, i_{\ell+2}), \ldots, (\varepsilon_s \,, i_s)\big)$.

Let $r := m + \sum\limits_{k\in[1,\ell-1]} \varepsilon_k$.

Since $i_\ell < i_{\ell+1} + 1$, we have

$$
\begin{aligned}
\tau_{\underline{j}} \quad &= \quad \prod_{k\in\lceil 1,s\rceil}^{[m]\ [\tilde{m}]} \tau_{j_k}^{(\tilde{\varepsilon}_k)} \\
&= \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle \tau_{i_{\ell+1}+1}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r-2]\ [\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{59}{=} \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\sigma_{i_{\ell+1}+1} \blacktriangle \sigma_{i_\ell}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r-2]\ [\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{24.(2)}{=} \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\sigma_{i_\ell} \blacktriangle \sigma_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r-2]\ [\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{59}{=} \quad \prod_{k\in\lceil 1,s\rceil}^{[m]\ [\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \\
&= \quad \tau_{\underline{i}} \,.
\end{aligned}
$$

Case $\underline{i} \overset{+-,1}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$(\tilde{\varepsilon}_k \,, j_k) = \begin{cases} (\varepsilon_k \,, i_k) & \text{for } k \in [1, s] \backslash \{\ell, \ell+1\} \\ (\varepsilon_{\ell+1} \,, i_{\ell+1} - 1) & \text{for } k = \ell \\ (\varepsilon_\ell \,, i_\ell) & \text{for } k = \ell + 1 \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \leqslant i_{\ell+1} - 1$, and $\varepsilon_\ell = +1$, $\varepsilon_{\ell+1} = -1$.

So $\underline{j} = \big((\varepsilon_1 \,, i_1), \ldots, (\varepsilon_{\ell-1} \,, i_{\ell-1}), (\varepsilon_{\ell+1} \,, i_{\ell+1} - 1), (\varepsilon_\ell \,, i_\ell), (\varepsilon_{\ell+2} \,, i_{\ell+2}), \ldots, (\varepsilon_s \,, i_s)\big)$.

Let $r := m + \sum\limits_{k \in [1, \ell-1]} \varepsilon_k$.

Since $i_\ell < i_{\ell+1}$, we have

$$\begin{aligned}
\tau_{\underline{j}} &= \prod_{k \in \lceil 1, s \rceil}^{[m][\tilde{m}]} \tau_{j_k}^{(\tilde{\varepsilon}_k)} \\
&= \Big(\prod_{k \in \lceil 1, \ell-1 \rceil}^{[m][r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle \tau_{i_{\ell+1}-1}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k \in \lceil \ell+2, s \rceil}^{[r][\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{59}{=} \Big(\prod_{k \in \lceil 1, \ell-1 \rceil}^{[m][r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\sigma_{i_{\ell+1}-1} \blacktriangle \delta_{i_\ell}) \blacktriangle \Big(\prod_{k \in \lceil \ell+2, s \rceil}^{[r][\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{24.(3)}{=} \Big(\prod_{k \in \lceil 1, \ell-1 \rceil}^{[m][r]} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\delta_{i_\ell} \blacktriangle \sigma_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k \in \lceil \ell+2, s \rceil}^{[r][\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{59}{=} \prod_{k \in \lceil 1, s \rceil}^{[m][\tilde{m}]} \tau_{i_k}^{(\varepsilon_k)} \\
&= \tau_{\underline{i}} \,.
\end{aligned}$$

Case $\underline{i} \overset{+-,2}{\rightsquigarrow} \underline{j}$. Then $s - 2 = t$ and we have

$$(\tilde{\varepsilon}_k \,, j_k) = \begin{cases} (\varepsilon_k \,, i_k) & \text{for } k \in [1, \ell-1] \\ (\varepsilon_{k+2} \,, i_{k+2}) & \text{for } k \in [\ell, s-2] \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \in \{i_{\ell+1} \,, i_{\ell+1} + 1\}$, and $\varepsilon_\ell = +1$, $\varepsilon_{\ell+1} = -1$.

So $\underline{j} = \big((\varepsilon_1 \,, i_1), \ldots, (\varepsilon_{\ell-1} \,, i_{\ell-1}), (\varepsilon_{\ell+2} \,, i_{\ell+2}), \ldots, (\varepsilon_s \,, i_s)\big)$.

Let $r := m + \sum\limits_{k \in [1, \ell-1]} \varepsilon_k$.

Since $i_\ell \in \{i_{\ell+1}\,, i_{\ell+1}+1\}$, we have

$$\begin{aligned}
\tau_{\underline{j}} &= \prod_{k\in\lceil 1,s-2\rfloor}^{[m]\ [\tilde m]} \tau_{j_k}^{(\tilde\varepsilon_k)} \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil \ell,s-2\rfloor}^{[r]\ [\tilde m]} \tau_{i_{k+2}}^{(\varepsilon_{k+2})}\Big) \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]\ [\tilde m]} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \mathrm{id}_{[r]} \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]\ [\tilde m]} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{24.(3)}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle (\delta_{i_\ell} \blacktriangle \sigma_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]\ [\tilde m]} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{59}{=} \prod_{k\in\lceil 1,s\rfloor}^{[m]\ [\tilde m]} \tau_{i_k}^{(\varepsilon_k)} \\
&= \tau_{\underline{i}}\,.
\end{aligned}$$

Case $\underline{i} \overset{+-,3}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$(\tilde\varepsilon_k\,, j_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s]\backslash\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1}) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell - 1) & \text{for } k = \ell+1 \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \geqslant i_{\ell+1}+2$, and $\varepsilon_\ell = +1$, $\varepsilon_{\ell+1} = -1$.
So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}), (\varepsilon_\ell\,, i_\ell - 1), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.
Let $r := m + \sum\limits_{k\in[1,\ell-1]} \varepsilon_k$.
Since $i_\ell > i_{\ell+1}+1$, we have

$$\begin{aligned}
\tau_{\underline{j}} &= \prod_{k\in\lceil 1,s\rfloor}^{[m]\ [\tilde m]} \tau_{j_k}^{(\tilde\varepsilon_k)} \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]\ [r]} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \tau_{i_{\ell+1}}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell - 1}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]\ [\tilde m]} \tau_{i_k}^{(\varepsilon_k)}\Big)
\end{aligned}$$

$$\begin{aligned}
&\overset{59}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]}\prod^{[r]}\tau_{i_k}^{(\varepsilon_k)}\Big)\blacktriangle(\sigma_{i_{\ell+1}}\blacktriangle\delta_{i_\ell-1})\blacktriangle\Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r]}\prod^{[\tilde m]}\tau_{i_k}^{(\varepsilon_k)}\Big)\\
&\overset{24.(3)}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]}\prod^{[r]}\tau_{i_k}^{(\varepsilon_k)}\Big)\blacktriangle(\delta_{i_\ell}\blacktriangle\sigma_{i_{\ell+1}})\blacktriangle\Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r]}\prod^{[\tilde m]}\tau_{i_k}^{(\varepsilon_k)}\Big)\\
&\overset{59}{=} \prod_{k\in\lceil 1,s\rceil}^{[m]}\prod^{[\tilde m]}\tau_{i_k}^{(\varepsilon_k)}\\
&= \tau_{\underline{i}}\,.
\end{aligned}$$

Altogether, this shows $\tau_{\underline{i}}=\tau_{\underline{j}}$. □

Remark 61 *For every nonempty subset $\emptyset\neq S\subseteq R^0_{m,\tilde m}$, there exists some $\underline{t}\in S$ with $w_\pm(\underline{t})=\max\big\{w_\pm(\underline{\ell})\colon \underline{\ell}\in S\big\}$.*

Proof. Suppose given $\underline{i}=\big((\varepsilon_1,i_1),\ldots,(\varepsilon_s,i_s)\big)\in R^0_{m,\tilde m}$.

Since $w_0(\underline{i})=0$, we get

$$\varepsilon_k=\begin{cases}-1 & \text{for } k\in[1,\ell]\\ +1 & \text{for } k\in[\ell+1,s]\,,\end{cases}$$

for $\ell:=\big|\big\{j\in[1,s]\colon\varepsilon_j=-1\big\}\big|\in[0,s]$.

We show that $w_-(\underline{i})\overset{!}{\leqslant} m^2$.

Note that

$$0\overset{56}{\leqslant} m+\sum_{k\in[1,\ell]}\varepsilon_k=m-\ell\,.$$

So we have $\ell\leqslant m$.

Moreover, for $k\in[1,\ell]$, note that we have

$$i_k\overset{56}{\leqslant} m+\sum_{j\in[1,k]}\varepsilon_j=m-k\leqslant m\,.$$

So we have

$$w_-(\underline{i})\overset{57}{=}\sum_{\substack{k\in[1,s]\\ \varepsilon_k=-1}}i_k=\sum_{k\in[1,\ell]}i_k\leqslant\sum_{k\in[1,\ell]}m=\ell\cdot m\leqslant m^2\,.$$

We show that $w_+(\underline{i})\overset{!}{\leqslant}\tilde m^2$.

Note that

$$0 \overset{56}{\leqslant} m + \sum_{k\in[1,\ell]} \varepsilon_k = m + \big(\sum_{k\in[1,s]} \varepsilon_k\big) - \big(\sum_{k\in[\ell+1,s]} \varepsilon_k\big) \overset{56}{=} m + (\tilde{m} - m) - (s-\ell) = \tilde{m} - (s-\ell)\,.$$

So we have $s - \ell \leqslant \tilde{m}$.

Moreover, for $k \in [\ell+1, s]$, note that we have

$$i_k \overset{56}{\leqslant} m + \sum_{j\in[1,k]} \varepsilon_j \leqslant m + \sum_{j\in[1,s]} \varepsilon_j \overset{56}{=} m + (\tilde{m} - m) = \tilde{m}\,.$$

So we have

$$w_+(\underline{i}) \overset{57}{=} \sum_{\substack{k\in[1,s]\\ \varepsilon_k=+1}} i_k = \sum_{k\in[\ell+1,s]} i_k \leqslant \sum_{k\in[\ell+1,s]} \tilde{m} = (s-\ell)\cdot\tilde{m} \leqslant \tilde{m}^2\,.$$

Altogether, this shows $w_\pm(\underline{i}) \overset{57}{=} w_-(\underline{i}) + w_+(\underline{i}) \leqslant m^2 + \tilde{m}^2$.

So, since the set $w_\pm\big(R^0_{m,\tilde{m}}\big) \subseteq \mathbb{Z}$ is bounded above by $m^2+\tilde{m}^2$, there exists some $\underline{t} \in S$ with $w_\pm(\underline{t}) = \max\big\{w_\pm(\underline{\ell})\colon \underline{\ell} \in S\big\}$. □

Definition 62 Suppose given $\underline{i} = \big((\varepsilon_1, i_1), \ldots, (\varepsilon_s, i_s)\big) \in R_{m,\tilde{m}}$.

Let $\ell := \big|\{k \in [1,s]\colon \varepsilon_k = -1\}\big| \in [0,s]$.

We say that $\underline{i}$ is a *standard* $(m,\tilde{m})$*-index tuple*, if the conditions $(1,2,3)$ hold.

(1) We have
$$\varepsilon_k = \begin{cases} -1 & \text{for } k \in [1,\ell] \\ +1 & \text{for } k \in [\ell+1, s]\,. \end{cases}$$

(2) We have
$$m - 1 \geqslant i_1 > i_2 > \ldots > i_\ell \geqslant 0\,.$$

(3) We have
$$0 \leqslant i_{\ell+1} < i_{\ell+2} < \ldots < i_s \leqslant \tilde{m}\,.$$

Remark 63 Suppose given $\underline{i} \in R_{m,\tilde{m}}$ with $w_0(\underline{i}) < 0$.

Then $\underline{i}$ is not a standard $(m,\tilde{m})$-index tuple since the property (1) of Definition 62 is not satisfied.

Definition 64 Suppose given $\underline{i}, \underline{j} \in R_{m,\tilde{m}}$.

We write

$$\underline{i} \approx\!\!\!\!\rightarrow \underline{j} \;:\Leftrightarrow \text{There exists some } k \in \mathbb{Z}_{\geqslant 1} \text{ and } \underline{i}_{(a)} \in R_{m,\tilde{m}} \text{ for } a \in [1,k] \text{ such that the conditions } (1,2) \text{ hold.}$$

(1) We have $\underline{i}_{(1)} = \underline{i}$ and $\underline{i}_{(k)} = \underline{j}$.

(2) We have $\underline{i}_{(a)} \rightsquigarrow \underline{i}_{(a+1)}$ for $a \in [1, k-1]$.

Remark 65 *Suppose given $\underline{i} \in R_{m,\tilde{m}}$.*

(1) *Suppose that $w_0(\underline{i}) < 0$.*
Then there exists some $\underline{j} \in [\underline{i}]$ with $\underline{i} \rightsquigarrow \underline{j}$ and $w_0(\underline{i}) < w_0(\underline{j})$.

(2) *There exists some $\underline{j} \in [\underline{i}]$ with $\underline{i} \approx\!\!\!\!\rightarrow \underline{j}$ and $w_0(\underline{j}) = 0$.*

(3) *Suppose that $\underline{i}$ is not standard and that $w_0(\underline{i}) = 0$.*
Then there exists some $\underline{j} \in [\underline{i}]$ with $\underline{i} \rightsquigarrow \underline{j}$ and $w_0(\underline{j}) = 0$ and $w_{\pm}(\underline{i}) < w_{\pm}(\underline{j})$.

(4) *Suppose that $w(\underline{i}) = 0$, i.e, we have $\underline{i} \in R^0_{m,\tilde{m}}$; cf. Definition 58.*
Then there exists some standard $\underline{j} \in [\underline{i}]$ with $\underline{i} \approx\!\!\!\!\rightarrow \underline{j}$; cf. Definitions 62, 64.

(5) *There exists some standard $\underline{j} \in [\underline{i}]$ with $\underline{i} \approx\!\!\!\!\rightarrow \underline{j}$.*

Proof. We write $\underline{i} = \big((\varepsilon_1, i_1), \ldots, (\varepsilon_s, i_s)\big)$.

Ad (1). Since $w_0(\underline{i}) < 0$, there exists some $\ell \in [1, s-1]$ with $\varepsilon_\ell = +1$ and $\varepsilon_{\ell+1} = -1$.

So

$$\underline{i} = \big((\varepsilon_1, i_1), \ldots, (\varepsilon_{\ell-1}, i_{\ell-1}), (+1, i_\ell), (-1, i_{\ell+1}), (\varepsilon_{\ell+2}, i_{\ell+2}), \ldots, (\varepsilon_s, i_s)\big).$$

Case $i_\ell < i_{\ell+1}$.

Then we consider

$$\underline{j} := \big((\varepsilon_1, i_1), \ldots, (\varepsilon_{\ell-1}, i_{\ell-1}), (\varepsilon_{\ell+1}, i_{\ell+1} - 1), (\varepsilon_\ell, i_\ell), (\varepsilon_{\ell+2}, i_{\ell+2}), \ldots, (\varepsilon_s, i_s)\big).$$

Note that

$$0 \leqslant i_{\ell+1} - 1 \leqslant \big(m + \sum_{k \in [1, \ell+1]} \varepsilon_k\big) - 1 \leqslant m + \big(\sum_{k \in [1, \ell-1]} \varepsilon_k\big) + \varepsilon_{\ell+1} \leqslant n,$$

and that

$$0 \leqslant i_\ell \leqslant i_{\ell+1} - 1 \leqslant m + \big(\sum_{k\in[1,\ell-1]} \varepsilon_k \big) + \varepsilon_{\ell+1} \leqslant m + \big(\sum_{k\in[1,\ell-1]} \varepsilon_k \big) + \varepsilon_{\ell+1} + \varepsilon_k \leqslant n\,.$$

This shows that $\underline{j} \in R_{m,\tilde{m}}$.

We have $\underline{i} \overset{+-;1}{\rightsquigarrow} \underline{j}$; cf. Definition 56.

Moreover, by Remark 60.(1), we have $w_0(\underline{i}) < w_0(\underline{j})$.

Case $i_\ell \in \{i_{\ell+1}\,, i_{\ell+1}+1\}$.

Then we consider

$$\underline{j} := \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)\,.$$

For $\ell' \in [\ell+2, s]$, note that

$$0 \leqslant i_{\ell'} \leqslant m + \sum_{k\in[1,\ell']} \varepsilon_k = \big(m + \sum_{k\in[1,\ell-1]} \varepsilon_k\big) + \big(m + \sum_{k\in[\ell+2,\ell']} \varepsilon_k\big) \leqslant n\,.$$

This shows that $\underline{j} \in R_{m,\tilde{m}}$.

We have $\underline{i} \overset{+-;2}{\rightsquigarrow} \underline{j}$; cf. Definition 56.

Moreover, by Remark 60.(1), we have $w_0(\underline{i}) < w_0(\underline{j})$.

Case $i_\ell > i_{\ell+1} + 1$.

Then we consider

$$\underline{j} := \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}), (\varepsilon_\ell\,, i_\ell - 1), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)\,.$$

Note that

$$0 \leqslant i_{\ell+1} \leqslant i_\ell - 2 \leqslant \big(m + \sum_{k\in[1,\ell]} \varepsilon_k\big) - 2 = \big(m + \sum_{k\in[1,\ell-1]} \varepsilon_k\big) - 1 = \big(m + \sum_{k\in[1,\ell-1]} \varepsilon_k\big) + \varepsilon_{\ell+1} \leqslant n\,,$$

and that

$$0 \leqslant i_\ell - 1 \leqslant m + \big(\sum_{k\in[1,\ell-1]} \varepsilon_k\big) = m + \big(\sum_{k\in[1,\ell-1]} \varepsilon_k\big) + \varepsilon_{\ell+1} + \varepsilon_\ell \leqslant n\,.$$

This shows that $\underline{j} \in R_{m,\tilde{m}}$.

We have $\underline{i} \overset{+-,3}{\rightsquigarrow} \underline{j}$; cf. Definition 56.

Moreover, by Remark 60.(1), we have $w_0(\underline{i}) < w_0(\underline{j})$.

Ad (2). If $w_0(\underline{i}) = 0$ then we have $\underline{i} \approx\!\!\!\!\rightarrow \underline{i}$ with $w_0(\underline{i}) = 0$.

So we may assume that $w_0(\underline{i}) < 0$.

We *assume* that such an element $\underline{j} \in [\underline{i}]$ does not exist.

We write $\underline{i}_{(0)} := \underline{i}$. Since $\underline{i} \approx\!\!\!\!\rightarrow \underline{i}_{(0)}$, we have $w_0(\underline{i}_{(0)}) < 0$.

By (1), there exists some $\underline{i}_{(1)} \in [\underline{i}_{(0)}]$ such that $\underline{i}_{(0)} \rightsquigarrow \underline{i}_{(1)}$ where $w_0(\underline{i}_{(0)}) < w_0(\underline{i}_{(1)})$.

Then $\underline{i} \approx\!\!\!\!\rightarrow \underline{i}_{(1)}$. By the assumption, we have $w_0(\underline{i}_{(1)}) < 0$.

By (1), there exists some $\underline{i}_{(2)} \in [\underline{i}_{(1)}]$ such that $\underline{i}_{(1)} \rightsquigarrow \underline{i}_{(2)}$ where $w_0(\underline{i}_{(1)}) < w_0(\underline{i}_{(2)})$.

Then $\underline{i} \approx\!\!\!\!\rightarrow \underline{i}_{(2)}$. By the assumption, we have $w_0(\underline{i}_{(2)}) < 0$.

By iterated use of (1), we obtain $(m, \tilde{m})$-index tuples $\underline{i}_{(k)} \in [\underline{i}]$ for $k \in \mathbb{Z}_{\geqslant 0}$ with the following property.

For $k \in \mathbb{Z}_{\geqslant 0}$, we have $\underline{i}_{(k)} \rightsquigarrow \underline{i}_{(k+1)}$ and $w_0(\underline{i}_{(k)}) < w_0(\underline{i}_{(k+1)}) < 0$.

We obtain a strictly ascending chain of infinite length

$$w_0(\underline{i}_{(0)}) < w_0(\underline{i}_{(1)}) < w_0(\underline{i}_{(2)}) < \ldots < 0\,,$$

which is a *contradiction*.

Ad (3). Let $\ell = \big|\{k \in [1, s] \colon \varepsilon_k = -1\}\big|$.

Since $\underline{i}$ is a non-standard $(m, \tilde{m})$-index tuple with $w_0(\underline{i}) = 0$, we recall that $\underline{i}$ satisfies

$$\big((\mathrm{i}) \wedge \neg(\mathrm{ii})\big) \ \vee \ \big((\mathrm{i}) \wedge \neg(\mathrm{iii})\big)\,,$$

where the properties $(\mathrm{i}, \mathrm{ii}, \mathrm{iii})$ are given as follows; cf. Definition 62.

(i) We have

$$\varepsilon_k = \begin{cases} -1 & \text{for } k \in [1, \ell] \\ +1 & \text{for } k \in [\ell + 1, s]\,. \end{cases}$$

(ii) We have

$$m - 1 \geqslant i_1 > i_2 > \ldots > i_\ell \geqslant 0\,.$$

(iii) We have

$$0 \leqslant i_{\ell+1} < i_{\ell+2} < \ldots < i_s \leqslant \tilde{m}\,.$$

Case (i) $\wedge\, \neg$(ii). Then there exists some $t \in [1, \ell-1]$ such that $\varepsilon_t = \varepsilon_{t+1} = -1$ and $i_t \leqslant i_{t+1}$. We consider

$$\underline{j} := \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{t-1}\,, i_{t-1}), (\varepsilon_{t+1}\,, i_{t+1}+1), (\varepsilon_t\,, i_t), (\varepsilon_{t+2}\,, i_{t+2}), \ldots, (\varepsilon_s\,, i_s)\big)\,.$$

Note that

$$0 \leqslant i_{t+1}+1 \leqslant \big(m + \sum_{k\in[1,t+1]} \varepsilon_k\big) + 1 = m + \big(\sum_{k\in[1,t-1]} \varepsilon_k\big) + \varepsilon_{t+1} \leqslant n\,,$$

and that

$$0 \leqslant i_t \leqslant i_{t+1} \leqslant m + \sum_{k\in[1,t+1]} \varepsilon_k \leqslant m + \big(\sum_{k\in[1,t-1]} \varepsilon_k\big) + \varepsilon_t + \varepsilon_{t+1} \leqslant n\,.$$

This shows that $\underline{j} \in R_{m,\tilde{m}}$.

We have $\underline{i} \overset{--}{\rightsquigarrow} \underline{j}$; cf. Definition 56.

So, by Remark 60.(1), we have $0 = w_0(\underline{i}) = w_0(\underline{j})$ and $w_\pm(\underline{i}) < w_\pm(\underline{j})$.

Case (i) $\wedge\, \neg$(iii). Then there exists some $t \in [\ell+1, s-1]$ such that $\varepsilon_t = \varepsilon_{t+1} = +1$ and $i_t \geqslant i_{t+1}$.

We consider

$$\underline{j} := \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{t-1}\,, i_{t-1}), (\varepsilon_{t+1}\,, i_{t+1}), (\varepsilon_t\,, i_t+1), (\varepsilon_{t+2}\,, i_{t+2}), \ldots, (\varepsilon_s\,, i_s)\big)\,.$$

Note that

$$0 \leqslant i_{t+1} \leqslant i_t \leqslant m + \big(\sum_{k\in[1,t]} \varepsilon_k\big) = m + \big(\sum_{k\in[1,t-1]} \varepsilon_k\big) + \varepsilon_{t+1} \leqslant n\,,$$

and that

$$0 \leqslant i_t + 1 \leqslant \big(m + \sum_{k\in[1,t]} \varepsilon_k\big) + 1 = m + \big(\sum_{k\in[1,t-1]} \varepsilon_k\big) + \varepsilon_{t+1} + \varepsilon_t \leqslant n\,.$$

This shows that $\underline{j} \in R_{m,\tilde{m}}$.

We have $\underline{i} \overset{++}{\rightsquigarrow} \underline{j}$; cf. Definition 56.

So, by Remark 60.(1), we have $0 = w_0(\underline{i}) = w_0(\underline{j})$ and $w_\pm(\underline{i}) < w_\pm(\underline{j})$.

Ad (4). We consider the set

$$T := \big\{\underline{\ell} \in R^0_{m,\tilde{m}} \colon\ \exists \underline{j} \in [\underline{\ell}] \text{ with } \underline{\ell} \rightsquigarrows \underline{j} \text{ and } \underline{j} \text{ is standard } \big\} \subseteq R^0_{m,\tilde{m}}\,.$$

Let $S := R^0_{m,\tilde{m}} \backslash T \subseteq R^0_{m,\tilde{m}}$. We have to show that $S \overset{!}{=} \emptyset$.

We *assume* that $S \neq \emptyset$.

Then, by Remark 61, we may choose some $\underline{t} \in S$ with $w_\pm(\underline{t}) = \max\big\{w_\pm(\underline{\ell}) \colon \underline{\ell} \in S\big\}$.

Case: $\underline{t}$ is standard.

Then we have $\underline{t} \rightsquigarrow \underline{t}$ where $\underline{t}$ is standard. This shows $\underline{t} \in T$, which *contradicts* $\underline{t} \in S$.

Case: $\underline{t}$ is not standard.

Then, by (3), since $w_0(\underline{t}) = 0$, there exists some $\underline{k} \in [\underline{t}]$ with $\underline{t} \rightsquigarrow \underline{k}$ and $w_0(\underline{k}) = 0$ and $w_\pm(\underline{t}) < w_\pm(\underline{k})$.

Then, since $w_\pm(\underline{t}) = \max\big\{w_\pm(\underline{\ell}) \colon \underline{\ell} \in S\big\}$, we have $\underline{k} \notin S$, i.e. $\underline{k} \in T$.

So there exists some standard $\underline{r} \in [\underline{k}]$ with $\underline{k} \rightsquigarrow \underline{r}$.

This shows $\underline{t} \rightsquigarrow \underline{k} \rightsquigarrow \underline{r}$, hence $\underline{t} \in T$, which *contradicts* $\underline{t} \in S$.

Altogether, this shows $S = \emptyset$.

Ad (5). By (2), we may choose some $\underline{j} \in [\underline{i}]$ with $\underline{i} \rightsquigarrow \underline{j}$ and $w_0(\underline{j}) = 0$. By (4), we may choose some standard $\underline{k} \in [\underline{j}]$ with $\underline{j} \rightsquigarrow \underline{k}$.

This shows $\underline{i} \rightsquigarrow \underline{j} \rightsquigarrow \underline{k}$, hence $\underline{i} \rightsquigarrow \underline{k}$. □

Remark 66 *Suppose given $\underline{i}$, $\underline{j} \in R_{m,\tilde{m}}$ where $\underline{i}$ and $\underline{j}$ are standard.*

We have

$$\tau_{\underline{j}} = \tau_{\underline{i}} \;\Rightarrow\; \underline{j} = \underline{i}\,.$$

Proof. We write $\underline{i} = \big((\varepsilon'_1, i_1), \ldots, (\varepsilon'_s, i_s)\big)$ and $\underline{j} = \big((\varepsilon''_1, j_1), \ldots, (\varepsilon''_t, j_t)\big) \in R_{m,\tilde{m}}$.

Let $a := \big|\{i_k \in [1,s] \colon \varepsilon'_k = -1\}\big|$ and let $b := \big|\{j_k \in [1,t] \colon \varepsilon''_k = -1\}\big|$.

By Definition 62, we have

$$\varepsilon'_k = \begin{cases} -1 & \text{for } k \in [1,a] \\ +1 & \text{for } k \in [a+1,s]\,, \end{cases}$$

where

$$m - 1 \geqslant i_1 > i_2 > \ldots > i_a \geqslant 0$$

and where

$$0 \leqslant i_{a+1} < i_{a+2} < \ldots < i_s \leqslant \tilde{m}\,.$$

Note that we have

$$\tilde{m} - m \overset{56}{=} \sum_{j\in[1,s]} \varepsilon'_j = \big(\sum_{j\in[1,a]} \varepsilon'_j\big) + \big(\sum_{j\in[a+1,s]} \varepsilon'_j\big) = -a + (s-a) = s - 2a\,.$$

Let $\ell' := m - a \geqslant 0$. Then we have $a = m - \ell'$.

Note that $\tilde{m} - \ell' = \tilde{m} - m + a = s - a \geqslant 0$.

Moreover, we have

$$\tilde{m} - \ell' + a = \tilde{m} - (m-a) + a = (\tilde{m} - m) + 2a = s - 2a + 2a = s\,.$$

We have

$$\begin{aligned}
\tau_{\underline{i}} &\overset{59}{=} \prod_{k\in\lceil 1,s\rfloor}^{[m]\;[\tilde{m}]} \tau_{i_k}^{(\varepsilon'_k)} \\
&= \Big(\prod_{k\in\lceil 1,a\rfloor}^{[m]\;[m-a]} \tau_{i_k}^{(-1)}\Big) \blacktriangle \Big(\prod_{k\in\lceil a+1,s\rfloor}^{[m-a]\;[\tilde{m}]} \tau_{i_k}^{(+1)}\Big) \\
&\overset{59}{=} \Big(\prod_{k\in\lceil 1,a\rfloor}^{[m]\;[m-a]} \sigma_{i_k}\Big) \blacktriangle \Big(\prod_{k\in\lceil a+1,s\rfloor}^{[m-a]\;[\tilde{m}]} \delta_{i_k}\Big) \\
&= \Big(\prod_{k\in\lceil 1,m-\ell'\rceil}^{[m]\;[\ell']} \sigma_{i_k}\Big) \blacktriangle \Big(\prod_{k\in\lceil a+1,\tilde{m}-\ell'+a\rceil}^{[\ell']\;[\tilde{m}]} \delta_{i_k}\Big) \\
&= \Big(\prod_{k\in\lfloor m-\ell',1\rfloor}^{[m]\;[\ell']} \sigma_{i_{a-k+1}}\Big) \blacktriangle \Big(\prod_{k\in\lceil 1,\tilde{m}-\ell'\rceil}^{[\ell']\;[\tilde{m}]} \delta_{i_{k+a}}\Big)\,.
\end{aligned}$$

Moreover, by Definition 62, we have

$$\varepsilon''_k = \begin{cases} -1 & \text{for } k \in [1,b] \\ +1 & \text{for } k \in [b+1,t]\,, \end{cases}$$

where

$$m - 1 \geqslant j_1 > j_2 > \ldots > j_b \geqslant 0$$

and where

$$0 \leqslant j_{b+1} < j_{b+2} < \ldots < j_t \leqslant \tilde{m}\,.$$

Note that we have

$$\tilde{m} - m \overset{56}{=} \sum_{j\in[1,s]} \varepsilon''_j = \big(\sum_{j\in[1,b]} \varepsilon''_j\big) + \big(\sum_{j\in[b+1,t]} \varepsilon''_j\big) = -b + (t-b) = t - 2b\,.$$

Let $\ell'' := m - b \geqslant 0$. Then we have $b = m - \ell''$.

Note that $\tilde{m} - \ell'' = \tilde{m} - m + b = t - b \geqslant 0$.

Moreover, we have

$$\tilde{m} - \ell'' + b = \tilde{m} - (m - b) + b = (\tilde{m} - m) + 2b = t - 2b + 2b = t\,.$$

We have

$$\begin{aligned}
\tau_{\underline{j}} &\overset{59}{=} \prod_{k \in \lceil 1, t \rceil}^{[m][\tilde{m}]} \tau_{j_k}^{(\varepsilon''_k)} \\
&= \Big(\prod_{k \in \lceil 1, b \rceil}^{[m][m-b]} \tau_{j_k}^{(-1)} \Big) \blacktriangle \Big(\prod_{k \in \lceil b+1, t \rceil}^{[m-b][\tilde{m}]} \tau_{j_k}^{(+1)} \Big) \\
&\overset{59}{=} \Big(\prod_{k \in \lceil 1, b \rceil}^{[m][m-b]} \sigma_{j_k} \Big) \blacktriangle \Big(\prod_{k \in \lceil b+1, t \rceil}^{[m-b][\tilde{m}]} \delta_{j_k} \Big) \\
&= \Big(\prod_{k \in \lceil 1, m-\ell'' \rceil}^{[m][\ell'']} \sigma_{j_k} \Big) \blacktriangle \Big(\prod_{k \in \lceil b+1, \tilde{m}-\ell''+b \rceil}^{[\ell''][\tilde{m}]} \delta_{j_k} \Big) \\
&= \Big(\prod_{k \in \lfloor m-\ell'', 1 \rfloor}^{[m][\ell'']} \sigma_{j_{b-k+1}} \Big) \blacktriangle \Big(\prod_{k \in \lceil 1, \tilde{m}-\ell'' \rceil}^{[\ell''][\tilde{m}]} \delta_{j_{k+b}} \Big)\,.
\end{aligned}$$

By Corollary 22.(3), since $\tau_{\underline{i}} = \tau_{\underline{j}}$, we have $\ell = \ell'$ and thus $a = m - \ell' = m - \ell'' = b$.

So $s = \tilde{m} - \ell' + a = \tilde{m} - \ell'' + b = t$. Moreover, we have $i_k = j_k$ for $k \in [1, m - \ell'] = [1, a]$ and $i_k = j_k$ for $k \in [1 + a, \tilde{m} - \ell' + a] = [a + 1, s]$.

So we have $(\varepsilon'_k\,, i_k) = (\varepsilon''_k\,, j_k)$ for $k \in [1, s] = [1, t]$. This shows $\underline{i} = \underline{j}$. □

Lemma 67 *Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given m, $\tilde{m} \in [0, n]$.*

Suppose given $\underline{i}$, $\underline{j} \in R_{m,\tilde{m}}$.

We have

$$\underline{i} \sim \underline{j} \;\Leftrightarrow\; \tau_{\underline{i}} = \tau_{\underline{j}}\,.$$

Proof. Ad $\Rightarrow$. By Remark 60.(2), we have $\tau_{\underline{i}} = \tau_{\underline{j}}$.

Ad $\Leftarrow$. By Remark 65.(5), we may choose some standard $\underline{k} \in [\underline{i}]$ and we may choose some standard $\underline{\ell} \in [\underline{j}]$.

By Remark 60.(2), we have $\tau_{\underline{k}} = \tau_{\underline{i}}$ and $\tau_{\underline{j}} = \tau_{\underline{\ell}}$. So $\tau_{\underline{k}} = \tau_{\underline{\ell}}$.

Then, by Remark 66, we have $\underline{k} = \underline{\ell}$.
This shows that $\underline{i} \sim \underline{k} = \underline{\ell} \sim \underline{j}$. □

3.3 A construction of pre-$\lfloor n, 0 \rfloor$-simplicial groups

Let $n \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$.

Remark 68 *Suppose given m, m', $m'' \in [0, n]$.*
Suppose given $\underline{i} \in R_{m,m'}$ and $\underline{j} \in R_{m',m''}$; cf. Definition 56.
Let $\underline{i} = \big((\varepsilon_1 , i_1), \ldots, (\varepsilon_r , i_r)\big)$ and $\underline{j} = \big((\varepsilon'_1 , j_1), \ldots, (\varepsilon'_{r'} , j_{r'})\big)$.
Then we have the (m, m'')-index tuple $\underline{i} \sqcup \underline{j} = \big((\varepsilon''_1 , \ell_1) , \ldots, (\varepsilon''_{r+r'} , \ell_{r+r'})\big) \in R_{m,m''}$ defined by

$$(\varepsilon''_k , \ell_k) := \begin{cases} (\varepsilon_k , i_k) & \text{for } k \in [1, r] \\ (\varepsilon'_{k-r} , j_{k-r}) & \text{for } k \in [r+1, r+r'] . \end{cases}$$

So

$$\underline{i} \sqcup \underline{j} = \big((\varepsilon_1 , i_1), \ldots, (\varepsilon_r , i_r), (\varepsilon'_1 , j_1), \ldots, (\varepsilon'_{r'} , j_{r'})\big) .$$

Proof. We have to show that $\sum\limits_{j \in [1, r+r']} \varepsilon''_j \overset{!}{=} m'' - m$; cf. Definition 56.

We have

$$\sum_{j \in [1, r+r']} \varepsilon''_j = \big(\sum_{j \in [1,r]} \varepsilon_j \big) + \big(\sum_{j \in [1,r']} \varepsilon_j \big) = (m' - m) + (m'' - m') = m'' - m .$$

Suppose given $k \in [1, r + r']$. We have to show that $0 \overset{!}{\leqslant} \ell_k \overset{!}{\leqslant} m + \sum\limits_{q \in [1,k]} \varepsilon''_q$; cf. Definition 56.

Case $k \in [1, r]$. We have

$$0 \leqslant \ell_k = i_k \overset{56}{\leqslant} m + \sum_{q \in [1,k]} \varepsilon_q = m + \sum_{q \in [1,k]} \varepsilon''_q .$$

Case $k \in [r + 1, r + r']$. Note that we have $m' - m = \sum\limits_{q \in [1,r]} \varepsilon_q$; cf. Definition 56.

We have

$$\begin{aligned} 0 \leqslant \ell_k = j_{k-r} \overset{56}{\leqslant} m' + \sum_{q\in[1,k-r]} \varepsilon'_q &= m + \big(\sum_{q\in[1,r]} \varepsilon_q\big) + \big(\sum_{q\in[1,k-r]} \varepsilon'_q\big) \\ &= m + \big(\sum_{q\in[1,r]} \varepsilon_q\big) + \big(\sum_{q\in[r+1,k]} \varepsilon'_{q-r}\big) \\ &= m + \sum_{q\in[1,k]} \varepsilon''_q \,. \end{aligned}$$

This shows that $0 \leqslant \ell_k \leqslant m + \sum_{q\in[1,k]} \varepsilon''_q$.

Suppose given $k \in [0, r+r']$. We have to show that $m + \sum_{q\in[1,k]} \varepsilon''_q \overset{!}{\in} [0,n]$; cf. Definition 56.

Case $k \in [0,r]$. We have

$$m + \sum_{q\in[1,k]} \varepsilon''_q = m + \sum_{q\in[1,k]} \varepsilon_q \overset{56}{\in} [0,n]\,.$$

Case $k \in [r+1, r+r']$. Note that we have $m' - m = \sum_{q\in[1,r]} \varepsilon_q$; cf. Definition 56.

We have

$$\begin{aligned} m + \sum_{q\in[1,k]} \varepsilon''_q &= m' - \big(\sum_{q\in[1,r]} \varepsilon_q\big) + \big(\sum_{q\in[1,k]} \varepsilon''_q\big) = m' - \big(\sum_{q\in[1,r]} \varepsilon_q\big) + \big(\sum_{q\in[1,r]} \varepsilon_q\big) + \big(\sum_{q\in[1,k-r]} \varepsilon'_q\big) \\ &= m' + \sum_{q\in[1,k-r]} \varepsilon'_q \overset{56}{\in} [0,n]\,. \end{aligned}$$

This shows that $m + \sum_{q\in[1,k]} \varepsilon''_q \in [0,n]$.

Altogether, we have $\underline{i} \sqcup \underline{j} \in R_{m,m''}$. □

Remark 69 *Suppose given* m, m', $m'' \in [0,n]$.

Supose given $\underline{i} \in R_{m,m'}$ *and* $\underline{j} \in R_{m',m''}$.

We consider the morphisms $\tau_{\underline{i}}$, $\tau_{\underline{j}} \in \mathrm{Mor}\big(\Delta_{\lfloor n,0\rfloor}\big)$*; cf. Remark* 59.

Then $\tau_{\underline{i}}$ *and* $\tau_{\underline{j}}$ *are composable and we have*

$$\tau_{\underline{i}} \blacktriangle \tau_{\underline{j}} = \tau_{\underline{i}\sqcup\underline{j}}\,.$$

Proof. We write $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (i_r\,, i_r)\big)$ and $\underline{j} = \big((\varepsilon'_1\,, j_1), \ldots, (\varepsilon'_{r'}\,, j_{r'})\big)$.
Then we have

$$\tau_{\underline{i}} = \prod_{k\in\lceil 1,r\rceil}^{[m]\,[m']} \tau_{i_k}^{(\varepsilon_k)} \quad \text{and} \quad \tau_{\underline{j}} = \prod_{k\in\lceil 1,r'\rceil}^{[m']\,[m'']} \tau_{j_k}^{(\varepsilon'_k)}\,.$$

Moreover, we have $\underline{i} \sqcup \underline{j} = \big((\varepsilon''_1\,, \ell_1), \ldots, (\varepsilon''_{r+r'}\,, \ell_{r+r'})\big)$ where

$$(\varepsilon''_k\,, \ell_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1, r] \\ (\varepsilon'_{k-r}\,, j_{k-r}) & \text{for } k \in [r+1, r+r']\,; \end{cases}$$

cf. Remark 68.

We obtain

$$\begin{aligned} \tau_{\underline{i}\sqcup\underline{j}} &= \prod_{k\in\lceil 1,r+r'\rceil}^{[m]\,[m'']} \tau_{\ell_k}^{(\varepsilon''_k)} \\ &= \Big(\prod_{k\in\lceil 1,r\rceil}^{[m]\,[m']} \tau_{\ell_k}^{(\varepsilon''_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil r+1,r+r'\rceil}^{[m']\,[m'']} \tau_{\ell_k}^{(\varepsilon''_k)}\Big) \\ &= \Big(\prod_{k\in\lceil 1,r\rceil}^{[m]\,[m']} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil 1,r'\rceil}^{[m']\,[m'']} \tau_{j_k}^{(\varepsilon'_k)}\Big) \\ &= \tau_{\underline{i}} \blacktriangle \tau_{\underline{j}}\,. \end{aligned}$$

□

Remark 70 *Suppose given* $\big([m] \xrightarrow{u} [m']\big) \in \mathrm{Mor}\big(\Delta_{\lfloor n,0\rfloor}\big)$.
Then there exists a unique standard $\underline{a} \in R_{m,m'}$ *with* $u = \tau_{\underline{a}}$*; cf. Definitions* 56, 62 *and Remark* 59.

Proof. By Corollary 22.$(1, 3)$, we have

$$u = \Big(\prod_{k\in\lfloor m-\ell,1\rfloor}^{[m]\,[\ell]} \sigma_{i_k}^{\ell+k-1}\Big) \blacktriangle \Big(\prod_{k\in\lceil 1,m'-\ell\rceil}^{[\ell]\,[m']} \delta_{j_k}^{\ell+k}\Big) = \sigma_{i_{m-\ell}}^{m-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\ell} \blacktriangle \delta_{j_1}^{\ell+1} \blacktriangle \ldots \blacktriangle \delta_{j_{m'-\ell}}^{m'}$$

for some uniquely determined $\ell \in \big[0, \min\{m, m'\}\big]$, $0 \leqslant i_1 < i_2 < \ldots < i_{m-\ell} \leqslant m-1$ and $0 \leqslant j_1 < j_2 < \ldots < j_{m'-\ell} \leqslant m'$.
Let the tuple $\underline{a} = \big((\varepsilon_1\,, a_1), \ldots, (\varepsilon_{m+m'-2\ell}\,, a_{m+m'-2\ell})\big)$ be given as follows.

Let

$$(\varepsilon_k\,,a_k) := \begin{cases} (-1, i_{m-\ell-k+1}) & \text{for } k \in [1, m-\ell] \\ (+1, j_{k-m+\ell}) & \text{for } k \in [m-\ell+1, m+m'-2\ell]\,. \end{cases}$$

Then $\underline{a} = \big((-1, i_{m-\ell}), \ldots, (-1, i_1), (+1, j_1), \ldots, (+1, j_{m'-\ell})\big)$.

We show that $\underline{a} \overset{!}{\in} R_{m,m'}$.

We have

$$\sum_{t\in[1,m+m'-2\ell]} \varepsilon_t = (m-\ell)\cdot(-1) + (m'-\ell)\cdot(+1) = m'-m\,.$$

Suppose given $k \in [0, m+m'-2\ell]$. We show that $m + \sum\limits_{t\in[1,k]} \varepsilon_t \overset{!}{\in} [0,n]$.

If $k \in [0, m-\ell]$, then we have

$$m + \sum_{t\in[1,k]} \varepsilon_t = m-k \in \big[m-(m-\ell), m-0\big] = [\ell, m] \subseteq [0,n]\,.$$

If $k \in [m-\ell+1, m+m'-2\ell]$, then we have

$$\begin{aligned} m + \sum_{t\in[1,k]} \varepsilon_t &= m + \big(\sum_{t\in[1,m-\ell]} \varepsilon_t\big) + \big(\sum_{t\in[m-\ell+1,k]} \varepsilon_t\big) = m-(m-\ell) + \big(k-(m-\ell)\big) \\ &= 2\ell - m + k \in \big[2\ell - m + (m-\ell+1), 2\ell - m + (m+m'-2\ell)\big] = [\ell+1, m'] \subseteq [0,n]\,. \end{aligned}$$

This shows that $m + \sum\limits_{t\in[1,k]} \varepsilon_t \in [0,n]$.

Suppose given $k \in [1, m+m'-2\ell]$. We show that $0 \overset{!}{\leqslant} a_k \overset{!}{\leqslant} m + \sum\limits_{t\in[1,k]} \varepsilon_t$.

For $s \in [1, m-\ell]$, note that

$$0 \leqslant i_s \leqslant (m-1) - \big((m-\ell)-s\big) = s+\ell-1\,.$$

So, if $k \in [1, m-\ell]$, then we have

$$0 \leqslant a_k = i_{m-\ell-k+1} \leqslant (m-\ell-k+1) + \ell - 1 = m-k = m + \sum_{t\in[1,k]} \varepsilon_t\,.$$

If $k \in k \in [m-\ell+1, m+m'-2\ell]$, then we have

$$0 \leqslant a_k = j_{k-m+\ell} \leqslant m' - \big((m'-\ell) - (k-m+\ell)\big) = 2\ell + k - m = m - (m-\ell) + \big(k - (m-\ell)\big)$$
$$= m + \sum_{t\in[1,k]} \varepsilon_t \,.$$

So $\underline{a} \in R_{m,m'}$; cf. Definition 56.

Moreover, note that $\underline{a}$ is standard; cf. Definition 62.

We have

$$\tau_{\underline{a}} = \prod_{k\in\lceil 1,m+m'-2\ell\rceil}^{[m]\,[m']} \tau_{a_k}^{(\varepsilon_k)} = \left(\prod_{k\in\lfloor m-\ell,1\rfloor}^{[m]\,[\ell]} \sigma_{i_k}^{\ell+k-1} \right) \blacktriangle \left(\prod_{k\in\lceil 1,m'-\ell\rceil}^{[\ell]\,[m']} \delta_{j_k}^{\ell+k} \right) = u \,;$$

cf. Remark 59. □

Lemma 71 (Construction of a pre-$\lfloor n,0\rfloor$-simplicial group)

For $k \in [0,n]$, suppose given a group G_k .

For $k \in [1,n]$ and $j \in [0,k]$, suppose given a group morphism $d_j^k \colon G_k \to G_{k-1}$.

For $k \in [0,n-1]$ and $i \in [0,k]$, suppose given a group morphism $s_i^k \colon G_k \to G_{k+1}$.

Suppose that the following properties (i, ii, iii) *are satisfied.*

(i) *We have*

$$d_j^{k+1} \blacktriangle d_i^k = d_i^{k+1} \blacktriangle d_{j-1}^k$$

for $k \in [1,n-1]$, $i \in [0,k]$, $j \in [0,k+1]$ with $i<j$.

(ii) *We have*

$$s_i^{k-1} \blacktriangle s_j^k = s_{j-1}^{k-1} \blacktriangle s_i^k$$

for $k \in [1,n-1]$, $i \in [0,k-1]$, $j \in [0,k]$ with $i<j$.

(iii) *We have*

$$s_i^k \blacktriangle d_j^{k+1} = \begin{cases} d_j^k \blacktriangle s_{i-1}^{k-1} & \text{if } k \geqslant 1 \text{ and } j<i \\ \mathrm{id}_{G_k} & \text{if } j=i \text{ or } j=i+1 \\ d_{j-1}^k \blacktriangle s_i^{k-1} & \text{if } k \geqslant 1 \text{ and } j>i+1 \end{cases}$$

for $k \in [0,n-1]$, $i \in [0,k]$, $j \in [0,k+1]$.

Cf. also Remark 54.

Let $k \in [1,n]$. *For* $j \in [0,k]$, *we write*

$$t_j^{(+1),k} := d_j^k \colon G_k \to G_{k-1}\,.$$

We often write $t_j^{(+1)} := t_j^{(+1),k}$.

Let $k \in [0,n-1]$. *For* $i \in [0,k]$, *we write*

$$t_i^{(-1),k} := s_i^k \colon G_k \to G_{k+1}\,.$$

We often write $t_i^{(-1)} := t_i^{(-1),k}$.

Moreover, for $\underline{b} = \big((\varepsilon_1\,,b_1),\ldots,(\varepsilon_{\tilde{s}}\,,b_{\tilde{s}})\big) \in R_{m,m'}$, *where* m, $m' \in [0,n]$, *we write*

$$t_{\underline{b}} := \prod_{p\in\lfloor \tilde{s},1\rfloor}^{G_{m'}\quad G_m} t_{b_p}^{(\varepsilon_p)} = t_{b_{\tilde{s}}}^{(\varepsilon_{\tilde{s}})} \blacktriangle \ldots \blacktriangle t_{b_1}^{(\varepsilon_1)}\,.$$

(1) *Suppose given* $\underline{b}$, $\underline{c} \in R_{m,m'}$, *where* m, $m' \in [0,n]$. *Suppose that* $\underline{b} \sim \underline{c}$; *cf. Definition* 56.
We have
$$t_{\underline{b}} = t_{\underline{c}}\,.$$

(2) *Suppose given* $\underline{b} \in R_{m,m'}$ *and* $\underline{c} \in R_{m',m''}$, *where* m, m', $m'' \in [0,n]$.
We consider the (m,m'')*-index tuple* $\underline{b} \sqcup \underline{c}$; *cf. Remark* 68.
We have
$$t_{\underline{b}\sqcup\underline{c}} = t_{\underline{c}} \blacktriangle t_{\underline{b}} \colon G_{m''} \to G_m\,.$$

(3) *There exists a unique pre-*$\lfloor n,0\rfloor$*-simplicial group* G *such that* G *actually maps* $[k]$ *to* G_k *for* $k \in \mathbb{Z}_{\geqslant 0}$ *and such that* $\mathrm{d}_j^{G,k} = d_j^k$ *for* $k \in [1,n]$, $j \in [0,k]$ *and such that* $\mathrm{s}_i^{G,k} = s_i^k$ *for* $k \in [0,n-1]$, $i \in [0,k]$.
The pre-$\lfloor n,0\rfloor$*-simplicial group* G *is given as follows.*
For $[k] \in \mathrm{Ob}\big(\Delta_{\lfloor n,0\rfloor}\big)$, *we have*
$$[k]G = G_k\,.$$
For $\big([m] \xrightarrow{u} [m']\big) \in \mathrm{Mor}\big(\Delta_{\lfloor n,0\rfloor}\big)$, *there exists by Remark* 70 *a unique standard* $\underline{i} \in R_{m,m'}$ *with* $u = \tau_{\underline{i}}$.
We have
$$(u^{\mathrm{op}})G = (\tau_{\underline{i}}^{\mathrm{op}})G = t_{\underline{i}}\,.$$

Proof. Ad (1). We may assume that $\underline{b} \rightsquigarrow \underline{c}$; cf. Definition 56.

We write $\underline{b} = \big((\varepsilon_1\,,b_1),\ldots,(\varepsilon_q\,,b_q)\big)\,.$

Case $\underline{b} \overset{++}{\rightsquigarrow} \underline{c}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,,b_1),\ldots,(\varepsilon_{\ell-1}\,,b_{\ell-1}),(\varepsilon_{\ell+1}\,,b_{\ell+1}),(\varepsilon_\ell\,,b_\ell+1),(\varepsilon_{\ell+2}\,,b_{\ell+2}),\ldots,(\varepsilon_q\,,b_q)\big)$$

for some $\ell \in [1,q-1]$ with $b_\ell \geqslant b_{\ell+1}$ and $\varepsilon_\ell = \varepsilon_{\ell+1} = +1$.

Let $r := m + \sum\limits_{p\in[1,\ell-1]} \varepsilon_p\,.$

Since $b_{\ell+1} < b_\ell + 1\,,$ we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_{r+2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle t_{b_\ell+1}^{(\varepsilon_\ell)} \blacktriangle t_{b_{\ell+1}}^{(\varepsilon_{\ell+1})} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_{r+2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle (d_{b_\ell+1}^{r+2} \blacktriangle d_{b_{\ell+1}}^{r+1}) \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&\overset{\text{(i)}}{=} \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_{r+2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle (d_{b_{\ell+1}}^{r+2} \blacktriangle d_{b_\ell}^{r+1}) \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\;\;G_m} t_{b_p}^{(\varepsilon_p)} \\
&= t_{\underline{b}}\,.
\end{aligned}$$

Case $\underline{b} \overset{--}{\rightsquigarrow} \underline{c}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,,b_1),\ldots,(\varepsilon_{\ell-1}\,,b_{\ell-1}),(\varepsilon_{\ell+1}\,,b_{\ell+1}+1),(\varepsilon_\ell\,,b_\ell),(\varepsilon_{\ell+2}\,,b_{\ell+2}),\ldots,(\varepsilon_q\,,b_q)\big)$$

for some $\ell \in [1,q-1]$ with $b_\ell \leqslant b_{\ell+1}$ and $\varepsilon_\ell = \varepsilon_{\ell+1} = -1$.

Let $r := m + \sum\limits_{p\in[1,\ell-1]} \varepsilon_p\,.$

Since $b_\ell < b_{\ell+1} + 1\,,$ we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_{r-2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle t_{b_\ell}^{(\varepsilon_\ell)} \blacktriangle t_{b_{\ell+1}+1}^{(\varepsilon_{\ell+1})} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_{r-2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle s_{b_\ell}^{r-2} \blacktriangle s_{b_{\ell+1}+1}^{r-1} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big)
\end{aligned}$$

$$\begin{aligned} &\overset{\text{(ii)}}{=} \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \, G_{r-2}} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle s_{b_{\ell+1}}^{r-2} \blacktriangle s_{b_\ell}^{r-1} \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \, G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &= \prod_{p \in \lfloor q, 1 \rfloor}^{G_{m'} \, G_m} t_{b_p}^{(\varepsilon_p)} \\ &= t_{\underline{b}} \, . \end{aligned}$$

Case $\underline{i} \overset{+-;1}{\rightsquigarrow} \underline{j}$. Then we have

$$\underline{c} = \big((\varepsilon_1 \, , b_1), \ldots, (\varepsilon_{\ell-1} \, , b_{\ell-1}), (\varepsilon_{\ell+1} \, , b_{\ell+1} - 1), (\varepsilon_\ell \, , b_\ell), (\varepsilon_{\ell+2} \, , b_{\ell+2}), \ldots, (\varepsilon_q \, , b_q) \big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \leqslant b_{\ell+1} - 1$ and $\varepsilon_\ell = +1 \, , \, \varepsilon_{\ell+1} = -1$.

Let $r := m + \sum\limits_{k \in [1, \ell-1]} \varepsilon_k \, .$

Since $b_\ell < b_{\ell+1} \, ,$ we have

$$\begin{aligned} t_{\underline{c}} &= \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \, G_r} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle t_{b_\ell}^{(\varepsilon_\ell)} \blacktriangle t_{b_{\ell+1}-1}^{(\varepsilon_{\ell+1})} \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \, G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &= \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \, G_r} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle d_{b_\ell}^{r} \blacktriangle s_{b_{\ell+1}-1}^{r-1} \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \, G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &\overset{\text{(iii)}}{=} \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \, G_r} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle s_{b_{\ell+1}}^{r} \blacktriangle d_{b_\ell}^{r+1} \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \, G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &= \prod_{p \in \lfloor q, 1 \rfloor}^{G_{m'} \, G_m} t_{b_p}^{(\varepsilon_p)} \\ &= t_{\underline{b}} \, . \end{aligned}$$

Case $\underline{i} \overset{+-;2}{\rightsquigarrow} \underline{j}$. Then we have

$$\underline{c} = \big((\varepsilon_1 \, , b_1), \ldots, (\varepsilon_{\ell-1} \, , b_{\ell-1}), (\varepsilon_{\ell+2} \, , b_{\ell+2}), \ldots, (\varepsilon_q \, , b_q) \big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \in \{ b_{\ell+1} \, , b_{\ell+1} + 1 \}$ and $\varepsilon_\ell = +1 \, , \, \varepsilon_{\ell+1} = -1$.

Let $r := m + \sum\limits_{p \in [1, \ell-1]} \varepsilon_p \, .$

Since $b_\ell \in \{b_{\ell+1}\,, b_{\ell+1}+1\}$, we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\quad G_r} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\quad G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\quad G_r} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle \mathrm{id}_{G_r} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\quad G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&\overset{\text{(iii)}}{=} \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\quad G_r} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle s_{b_{\ell+1}}^r \blacktriangle d_{b_\ell}^{r+1} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\quad G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\quad G_m} t_{b_p}^{(\varepsilon_p)} \\
&= t_{\underline{b}}\,.
\end{aligned}$$

Case $\underline{i} \overset{+-,3}{\rightsquigarrow} \underline{j}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_{\ell-1}\,, b_{\ell-1}), (\varepsilon_{\ell+1}\,, b_{\ell+1}), (\varepsilon_\ell\,, b_\ell - 1), (\varepsilon_{\ell+2}\,, b_{\ell+2}), \ldots, (\varepsilon_q\,, b_q)\big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \geqslant b_{\ell+1}+2$ and $\varepsilon_\ell = +1\,,\ \varepsilon_{\ell+1} = -1$.

Let $r := m + \sum\limits_{q\in[1,\ell-1]} \varepsilon_q\,.$

Since $b_\ell > b_{\ell+1}+1\,,$ we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\quad G_r} t_{b_q}^{(\varepsilon_q)}\Big) \blacktriangle t_{b_\ell-1}^{(\varepsilon_\ell)} \blacktriangle t_{b_{\ell+1}}^{(\varepsilon_{\ell+1})} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\quad G_m} t_{b_q}^{(\varepsilon_q)}\Big) \\
&= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\quad G_r} t_{b_q}^{(\varepsilon_q)}\Big) \blacktriangle d_{b_\ell-1}^r \blacktriangle s_{b_{\ell+1}}^{r-1} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\quad G_m} t_{b_q}^{(\varepsilon_q)}\Big) \\
&\overset{\text{(iii)}}{=} \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\quad G_r} t_{b_q}^{(\varepsilon_q)}\Big) \blacktriangle s_{b_{\ell+1}}^r \blacktriangle d_{b_\ell}^{r+1} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\quad G_m} t_{b_q}^{(\varepsilon_q)}\Big) \\
&= \prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\quad G_m} t_{b_p}^{(\varepsilon_p)} \\
&= t_{\underline{b}}\,.
\end{aligned}$$

Altogether, this shows $t_{\underline{b}} = t_{\underline{c}}\,.$

Ad (2). We write $\underline{b} = \big((\varepsilon_1\,,b_1),\ldots,(\varepsilon_q\,,b_q)\big)$ and $\underline{c} = \big((\varepsilon'_1\,,c_1),\ldots,(\varepsilon'_{q'}\,,c_{q'})\big)$.

Then we have $\underline{b}\sqcup\underline{c} = \big((\varepsilon_1\,,b_1),\ldots,(\varepsilon_q\,,b_q),(\varepsilon'_1\,,c_1),\ldots,(\varepsilon'_{q'}\,,c_{q'})\big)$; cf. Remark 68.

We have

$$t_{\underline{b}\sqcup\underline{c}} = \Big(\prod_{p\in\lfloor q',1\rfloor}^{G_{m''}\quad G_{m'}} t_{c_p}^{(\varepsilon'_p)}\Big)\blacktriangle\Big(\prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\quad G_m} t_{b_p}^{(\varepsilon_p)}\Big) = t_{\underline{c}}\blacktriangle t_{\underline{b}}\,.$$

Ad (3). For $\big([m]\xrightarrow{u}[m']\big)\in\mathrm{Mor}\big(\Delta_{\lfloor n,0\rfloor}\big)$, there exists by Remark 70 a unique standard $\underline{i}\in R_{m,m'}$ with $u=\tau_{\underline{i}}$.

We construct a pre-$\lfloor n,0\rfloor$-simplicial group G as follows.

For $k\in[0,n]$, let

$$[k]G := G_k\,.$$

For $\big([m]\xrightarrow{u}[m']\big)\in\mathrm{Mor}(\Delta_{\lfloor n,0\rfloor})$, there exists by Remark 70 a unique standard $\underline{i}\in R_{m,m'}$ with $u=\tau_{\underline{i}}$.

Let

$$(u^{\mathrm{op}})G := (\tau_{\underline{i}}^{\mathrm{op}})G = t_{\underline{i}}\,.$$

Suppose given $[m]\xrightarrow{u}[m']\xrightarrow{v}[m'']$ in $\Delta_{\lfloor n,0\rfloor}$.

We have to show that $\big((\mathrm{id}_{[m]})^{\mathrm{op}}\big)G \overset{!}{=} \mathrm{id}_{G_m}$.

By Remark 70, we have $\mathrm{id}_{[m]}=\tau_{\underline{a}}$, for a unique standard $\underline{a}\in R_{m,m}$. Note that $\underline{a}=(\ )$ is the empty tuple.

We have

$$\big((\mathrm{id}_{[m]})^{\mathrm{op}}\big)G = (\tau_{\underline{a}}^{\mathrm{op}})G = t_{\underline{a}} = \prod_{p\in\lfloor 0,1\rfloor}^{G_m\quad G_m} t_{a_p}^{(\varepsilon_p)} = \mathrm{id}_{G_m}\,.$$

We have to show that $(u\blacktriangle v)^{\mathrm{op}}\,G \overset{!}{=} (v^{\mathrm{op}})G\blacktriangle(u^{\mathrm{op}})G$.

By Remark 70, there exists a unique standard $\underline{i}\in R_{m,m'}$ with $u=\tau_{\underline{i}}$ and there exists a unique standard $\underline{j}\in R_{m',m''}$ with $v=\tau_{\underline{j}}$.

We write $\underline{i} = \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_r\,,i_r)\big)$ and $\underline{j} = \big((\varepsilon'_1\,,j_1),\ldots,(\varepsilon'_{r'}\,,j_{r'})\big)$.

We consider $\underline{i}\sqcup\underline{j} = \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_r\,,i_r),(\varepsilon'_1\,,j_1),\ldots,(\varepsilon'_{r'}\,,j_{r'})\big)\in R_{m,m''}$; cf. Remark 68.

By Remark 65.(5), there exists some standard $\underline{a}\in[\underline{i}\sqcup\underline{j}]$. By Remark 60.(2), we have $\tau_{\underline{i}\sqcup\underline{j}}=\tau_{\underline{a}}$.

We have

$$(u \blacktriangle v)^{\mathrm{op}} G = (\tau_{\underline{i}} \blacktriangle \tau_{\underline{j}})^{\mathrm{op}} G \overset{69}{=} (\tau_{\underline{i} \sqcup \underline{j}})^{\mathrm{op}} G = \tau_{\underline{a}}^{\mathrm{op}} G = t_{\underline{a}} \overset{(1)}{=} t_{\underline{i} \sqcup \underline{j}} \overset{(2)}{=} t_{\underline{j}} \blacktriangle t_{\underline{i}}$$
$$= (\tau_{\underline{j}}^{\mathrm{op}}) G \blacktriangle (\tau_{\underline{i}}^{\mathrm{op}}) G = (v^{\mathrm{op}}) G \blacktriangle (u^{\mathrm{op}}) G \,.$$

□

Remark 72 (Construction of a simplicial group)

Suppose that $n = +\infty$. For $k \in \mathbb{Z}_{\geqslant 0}$, suppose given a group G_k.

For $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0, k]$, suppose given a group morphism $d_j^k \colon G_k \to G_{k-1}$.

For $k \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0, k]$, suppose given a group morphism $s_i^k \colon G_k \to G_{k+1}$.

Suppose that the following properties (i′, ii′, iii′) are satisfied.

(i′) We have

$$d_j^{k+1} \blacktriangle d_i^k = d_i^{k+1} \blacktriangle d_{j-1}^k$$

for $k \in \mathbb{Z}_{\geqslant 1}$, $i \in [0, k]$, $j \in [0, k+1]$ with $i < j$.

(ii′) We have

$$s_i^{k-1} \blacktriangle s_j^k = s_{j-1}^{k-1} \blacktriangle s_i^k$$

for $k \in \mathbb{Z}_{\geqslant 1}$, $i \in [0, k-1]$, $j \in [0, k]$ with $i < j$.

(iii′) We have

$$s_i^k \blacktriangle d_j^{k+1} = \begin{cases} d_j^k \blacktriangle s_{i-1}^{k-1} & \text{if } k \geqslant 1 \text{ and } j < i \\ \mathrm{id}_{G_k} & \text{if } j = i \text{ or } j = i+1 \\ d_{j-1}^k \blacktriangle s_i^{k-1} & \text{if } k \geqslant 1 \text{ and } j > i+1 \end{cases}$$

for $k \in \mathbb{Z}_{\geqslant 0}$, $i \in [0, k]$, $j \in [0, k+1]$.

Cf. also Remark 33.

In fact, in this case $n = +\infty$, the pre-$\lfloor n, 0 \rfloor$-simplicial group G given in Lemma 71.(3) is a simplicial group; cf. Definition 30.

Lemma 73 (Construction of a pre-$\lfloor n, 0 \rfloor$-simplicial group morphism)

Suppose given pre-$\lfloor n, 0 \rfloor$-simplicial groups G and $\tilde{G}$.

For $k \in [0, n]$, suppose given group morphisms $\varphi_k \colon G_k \to \tilde{G}_k$.

Suppose that the following properties (1, 2) *hold.*

(1) *For $k\in[1,n]$ and $j\in[0,k]$, we have* $\varphi_k \blacktriangle \mathrm{d}_j^{\tilde{G},k} = \mathrm{d}_j^{G,k} \blacktriangle \varphi_{k-1}$.

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & \tilde{G}_k \\ {\scriptstyle \mathrm{d}_j^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_j^{\tilde{G},k}} \\ G_{k-1} & \xrightarrow[\varphi_{k-1}]{} & \tilde{G}_{k-1} \end{array}$$

(2) *For $k\in[0,n-1]$ and $i\in[0,k]$, we have* $\varphi_k \blacktriangle \mathrm{s}_i^{\tilde{G},k} = \mathrm{s}_i^{G,k} \blacktriangle \varphi_{k+1}$.

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & \tilde{G}_k \\ {\scriptstyle \mathrm{s}_i^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{s}_i^{\tilde{G},k}} \\ G_{k+1} & \xrightarrow[\varphi_{k+1}]{} & \tilde{G}_{k+1} \end{array}$$

Then $\varphi := \left(G_k \xrightarrow{\varphi_k} \tilde{G}_k\right)_{k\in[0,n]}$ is a pre-$\lfloor n,0\rfloor$-simplicial group morphism from G to $\tilde{G}$.

If $n=+\infty$, then G and $\tilde{G}$ are simplicial groups and φ is a simplicial group morphism; cf. Definition 30 *and Remark* 72*.*

Proof. Suppose given $\left([m]\xrightarrow{u}[m']\right)$ in $\Delta_{\lfloor n,0\rfloor}$.

Recall that by Corollary 22.(1), we have

$$u = \left(\prod_{t\in\lfloor m-\ell,1\rfloor}^{[m]}{}^{[\ell]}\ \sigma_{i_t}^{\ell+t-1}\right) \blacktriangle \left(\prod_{t\in\lceil 1,m'-\ell\rceil}^{[\ell]}{}^{[m']}\ \delta_{j_t}^{\ell+t}\right).$$

for some $\ell\in[0,n]$ and $0\leqslant i_1<i_2<\ldots<i_{n-\ell}\leqslant m$ and $0\leqslant j_1<j_2<\ldots<j_{m'-\ell}\leqslant m'$.

We have

$$\begin{aligned} \varphi_{m'} \blacktriangle (u^{\mathrm{op}})\tilde{G} &= \varphi_{m'} \blacktriangle \left(\left(\prod_{t\in\lfloor m'-\ell,1\rfloor}^{[m']}{}^{[\ell]}\ (\delta_{j_t}^{\ell+t})^{\mathrm{op}}\right) \blacktriangle \left(\prod_{t\in\lceil 1,m-\ell\rceil}^{[\ell]}{}^{[m]}(\sigma_{i_t}^{\ell+t-1})^{\mathrm{op}}\right)\right)\tilde{G} \\ &\overset{53}{=} \varphi_{m'} \blacktriangle \left(\prod_{t\in\lfloor m'-\ell,1\rfloor}^{\tilde{G}_{m'}}{}^{\tilde{G}_\ell}\ \mathrm{d}_{j_t}^{\tilde{G},\ell+t}\right) \blacktriangle \left(\prod_{t\in\lceil 1,m-\ell\rceil}^{[\tilde{G}_\ell}{}^{\tilde{G}_m}\ \mathrm{s}_{i_t}^{\tilde{G},\ell+t-1}\right) \\ &\overset{(1)}{=} \left(\prod_{t\in\lfloor m'-\ell,1\rfloor}^{G_{m'}}{}^{G_\ell}\ \mathrm{d}_{j_t}^{G,\ell+t}\right) \blacktriangle \varphi_{m'-(m'-\ell)} \blacktriangle \left(\prod_{t\in\lceil 1,m-\ell\rceil}^{\tilde{G}_\ell}{}^{\tilde{G}_m}\ \mathrm{s}_{i_t}^{\tilde{G},\ell+t-1}\right) \end{aligned}$$

$$\begin{aligned}
&\overset{(2)}{=} \Big(\prod_{t\in\lfloor m'-\ell,1\rfloor}^{G_{m'}\ G_\ell} \mathrm{d}_{j_t}^{G,\ell+t} \Big) \blacktriangle \Big(\prod_{t\in\lceil 1,m-\ell\rceil}^{G_\ell\ G_m} \mathrm{s}_{i_t}^{G,\ell+t-1} \Big) \blacktriangle \varphi_{\ell+(m-\ell)} \\
&\overset{53}{=} \Big(\Big(\prod_{t\in\lfloor m'-\ell,1\rfloor}^{[m']\ [\ell]} (\delta_{j_t}^{\ell+t})^{\mathrm{op}} \Big) \blacktriangle \Big(\prod_{t\in\lceil 1,m-\ell\rceil}^{[\ell]\ [m]} (\sigma_{i_t}^{\ell+t-1})^{\mathrm{op}} \Big) \Big) G \blacktriangle \varphi_m \\
&= (u^{\mathrm{op}})G \blacktriangle \varphi_m \,.
\end{aligned}$$

$$\begin{array}{ccc}
G_{m'} & \xrightarrow{\varphi_{m'}} & \tilde{G}_{m'} \\
{\scriptstyle (u^{\mathrm{op}})G}\downarrow & & \downarrow{\scriptstyle (u^{\mathrm{op}})\tilde{G}} \\
G_m & \xrightarrow[\varphi_m]{} & \tilde{G}_m
\end{array}$$

□

3.4 The category of $\lfloor n,0\rfloor$-simplicial groups

Let $n \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$.

Definition 74 ($\lfloor n,0\rfloor$-simplicial group)

Suppose that $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given a pre-$\lfloor n,0\rfloor$-simplicial group $G\colon \Delta_{\lfloor n,0\rfloor}^{\mathrm{op}} \to \mathcal{G}rp$; cf. Definition 52.

We call G an $\lfloor n,0\rfloor$-*simplicial group* if the following *Conduché condition* (on $\lfloor n,0\rfloor$-simplicial groups) (1) is satisfied.

(1) For subsets $I,\ J \subseteq [0,n]$ with $I,\ J \neq \emptyset$ and $I \cup J = [0,n]$, we have

$$\big[\bigcap_{i\in I} \ker \mathrm{d}_i^{G,n}\,, \bigcap_{j\in J} \ker \mathrm{d}_j^{G,n}\big] = 1\,.$$

Cf. Definition 53 and [**7,** Thm. 1.5, Cor. 1.8].

We have the full subcategory $\lfloor n,0\rfloor$-$\mathcal{S}impGrp \subseteq \mathcal{P}re$-$\lfloor n,0\rfloor$-$\mathcal{S}impGrp$ defined by

$$\mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp\big) := \big\{ \big(\Delta_{\lfloor n,0\rfloor}^{\mathrm{op}} \xrightarrow{G} \mathcal{G}rp\big) \colon \text{ G is an } \lfloor n,0\rfloor\text{-simplicial group} \big\}\,.$$

We call $\lfloor n,0\rfloor$-$\mathcal{S}impGrp$ the *category of* $\lfloor n,0\rfloor$-*simplicial groups*.

A transformation between $\lfloor n,0\rfloor$-simplicial groups is called an $\lfloor n,0\rfloor$-*simplicial group morphism*.

For the example $n = 1$, cf. § 7.4 below.

Remark 75 Suppose given $\lfloor n, 0 \rfloor$-simplicial groups G, $\tilde{G}$.
For $k \in [0, n]$, suppose given group morphisms $\varphi_k \colon G_k \to \tilde{G}_k$.
Suppose that the following properties $(1, 2)$ hold.

(1) For $k \in [1, n]$ and $j \in [0, k]$, we have $\varphi_k \vartriangle \mathrm{d}_j^{\tilde{G},k} = \mathrm{d}_j^{G,k} \vartriangle \varphi_{k-1}$.

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & \tilde{G}_k \\ {\scriptstyle \mathrm{d}_j^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_j^{\tilde{G},k}} \\ G_{k-1} & \xrightarrow[\varphi_{k-1}]{} & \tilde{G}_{k-1} \end{array}$$

(2) For $k \in [0, n-1]$ and $i \in [0, k]$, we have $\varphi_k \vartriangle \mathrm{s}_i^{\tilde{G},k} = \mathrm{s}_i^{G,k} \vartriangle \varphi_{k+1}$.

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & \tilde{G}_k \\ {\scriptstyle \mathrm{s}_i^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{s}_i^{\tilde{G},k}} \\ G_{k+1} & \xrightarrow[\varphi_{k+1}]{} & \tilde{G}_{k+1} \end{array}$$

Then $\varphi := \left(G_k \xrightarrow{\varphi_k} \tilde{G}_k\right)_{k \in [0,n]}$ is an $\lfloor n, 0 \rfloor$-simplicial group morphism from G to $\tilde{G}$; cf. Definition 74, Lemma 73.

3.5 The Moore complex of an $\lfloor n, 0 \rfloor$-simplicial group

Let $n \in \mathbb{Z}_{\geqslant 1}$.

Definition 76 Let G be a pre-$\lfloor n, 0 \rfloor$-simplicial group. Let $k \in [0, n]$ and let $j \in [0, k]$.

(1) We write

$$G_{k,j} := \ker \mathrm{d}_j^{G,k} \trianglelefteqslant G_k .$$

(2) We write

$$G\mathrm{N}_k := \bigcap_{i \in [1,k]} G_{k,i} \trianglelefteqslant G_k \, .$$

(3) We write

$$G\mathrm{NZ}_k := G\mathrm{N}_k \cap G_{k,0} = \bigcap_{i \in [0,k]} G_{k,i} \trianglelefteqslant G_k \, .$$

(4) Suppose that $k \in [0, n-1]$. We write

$$G\mathrm{NB}_k := (G\mathrm{N}_{k+1})\mathrm{d}_0^{G,k+1} = \big(\bigcap_{i \in [1,k+1]} G_{k+1,i} \big)\mathrm{d}_0^{G,k+1} \leqslant G_k \, .$$

Here, in § 3.5, we will apply this Definition 76 to an $\lfloor n, 0 \rfloor$-simplicial group.

Lemma 77 (The Moore complex of an $\lfloor n, 0 \rfloor$-simplicial group)
Let G be an $\lfloor n, 0 \rfloor$-simplicial group.

(1) *For $k \in [1, n]$, we have the group morphism*

$$\partial_k^G := \mathrm{d}_0^{G,k}\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}} \colon G\mathrm{N}_k \to G\mathrm{N}_{k-1} \, , \ x \mapsto x\mathrm{d}_0^{G,k} \, .$$

Sometimes, we write $\partial_k := \partial_k^G$.

(2) *We have the complex in groups*

$$G\mathrm{N} := \big(\ldots \xrightarrow{!} 1 \xrightarrow{!} G\mathrm{N}_n \xrightarrow{\partial_n} G\mathrm{N}_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} G\mathrm{N}_0 \xrightarrow{!} 1 \xrightarrow{!} \ldots \big);$$

cf. Definition 7.
In particular, for $k \in [0, n-1]$, we have $G\mathrm{NB}_k \trianglelefteqslant G_k$; cf. Definition 76.(4).
We call $G\mathrm{N}$ the Moore complex *of G.*
For $k \in [0, n]$, we write

$$G\pi_k := G\mathrm{NH}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}$$

for the k-th homotopy group of G; cf. Lemma 11.(1).
For $k \in [0, n-1]$, note that we have

$$G\pi_k = G\mathrm{NZ}_k / G\mathrm{NB}_k \, .$$

Moreover, we have $G\pi_n = G\mathrm{NZ}_n$.

Proof. Ad (1). Let $k \in [1, n]$. We have to show that $(G\mathrm{N}_k)\mathrm{d}_0 \overset{!}{\subseteq} G\mathrm{N}_{k-1}$.
Suppose given $x \in G\mathrm{N}_k$. Then $x\mathrm{d}_i = 1$ for $i \in [1, k]$. Suppose given $j \in [1, k-1]$.
We have

$$x\mathrm{d}_0\,\mathrm{d}_j = x\mathrm{d}_{j+1}\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows $x\mathrm{d}_0^{G,k} \in G_{k-1,j}$.
So $x\mathrm{d}_0 \in \bigcap_{j\in[1,k-1]} G_{k-1,\,j} = G\mathrm{N}_{k-1}$.

Ad (2). Let $k \in [1, n]$. *We show that* $\partial_k \mathbin{\blacktriangle} \partial_{k-1} \overset{!}{=} !$.
For $x \in G\mathrm{N}_k$, we have

$$x\partial_k\partial_{k-1} = x\mathrm{d}_0\,\mathrm{d}_0 = x\mathrm{d}_1\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows $\partial_k \mathbin{\blacktriangle} \partial_{k-1} = !$.
We show that $\operatorname{im}\partial_k \overset{!}{\trianglelefteq} G_{k-1}$.
Suppose given $x \in \operatorname{im}\partial_k$ and suppose given $y \in G_{k-1}$. We have to show that $x^y \overset{!}{\in} \operatorname{im}\partial_k$.
We may write $x = z\partial_k = z\mathrm{d}_0^{G,k}$ for some $z \in G\mathrm{N}_k$.
Let $\tilde{z} := z^{y\mathrm{s}_0} = y^-\mathrm{s}_0 \cdot z \cdot y\mathrm{s}_0 \in G_k$.
Since $z \in G\mathrm{N}_k \trianglelefteq G_k$, we also have $\tilde{z} \in G\mathrm{N}_k$.
So we have

$$x^y = (z\mathrm{d}_0)^{y\mathrm{s}_0\mathrm{d}_0} = (z^{y\mathrm{s}_0})\mathrm{d}_0 = \tilde{z}\partial_k\,.$$

This shows that $x^y \in \operatorname{im}\partial_k$.
Therefore, we have $\operatorname{im}\partial_k \trianglelefteq G_{k-1}$. □

Lemma 78 *Suppose given $\lfloor n, 0 \rfloor$-simplicial groups G, H. Suppose given an $\lfloor n, 0 \rfloor$-simplicial group morphism $\varphi\colon G \to H$.*

(1) *For $k \in [0, n]$, we have the group morphism*

$$\varphi\mathrm{N}_k := \varphi_k\big|^{H\mathrm{N}_k}_{G\mathrm{N}_k}\colon G\mathrm{N}_k \to H\mathrm{N}_k\,,\ x \mapsto x\varphi_k\,.$$

(2) *We consider the complexes in groups $G\mathrm{N}$ and $H\mathrm{N}$; cf. Lemma 77.(2).*

We have the morphism of complexes in groups

$$\varphi \mathrm{N} := (\varphi \mathrm{N}_k)_{k \in \mathbb{Z}} \colon G\mathrm{N} \to H\mathrm{N}\,,$$

where $\varphi \mathrm{N}_k := !$ *for* $k \geqslant n+1$ *and for* $k \leqslant -1$.

(3) *We have the functor*

$$\mathrm{N} \colon \lfloor n, 0 \rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{ComplexGrp}\,,\ \left(F \xrightarrow{\varphi} F'\right) \mapsto \left(F\mathrm{N} \xrightarrow{\varphi \mathrm{N}} F'\mathrm{N}\right).$$

Proof. Ad (1). Let $k \in [0, n]$. We have to show that $(G\mathrm{N}_k)\varphi_k \overset{!}{\leqslant} H\mathrm{N}_k$.
Suppose given $x \in G\mathrm{N}_k$. For $i \in [1, k]$, we have

$$x\varphi_k \, \mathrm{d}_i^{H,k} = x \mathrm{d}_i^{G,k} \, \varphi_{k-1} = 1\varphi_{k-1} = 1\,.$$

This shows $x\varphi_k \in H_{k,i}$ for $i \in [1, k]$.
So $x\varphi_k \in H\mathrm{N}_k$.
Ad (2). Let $k \in \mathbb{Z}$. We have to show that $\partial_k^G \mathbin{\blacktriangle} \varphi \mathrm{N}_{k-1} \overset{!}{=} \varphi \mathrm{N}_k \mathbin{\blacktriangle} \partial_k^H$.
We may assume that $k \in [1, n]$.
For $x \in G\mathrm{N}_k$, we have

$$x\partial_k^G \, (\varphi \mathrm{N}_{k-1}) = x \mathrm{d}_0^{G,k} \, \varphi_{k-1} = x\varphi_k \, \mathrm{d}_0^{H,k} = x(\varphi \mathrm{N}_k)\partial_k^H\,.$$

This shows $\partial_k^G \mathbin{\blacktriangle} \varphi \mathrm{N}_{k-1} = \varphi \mathrm{N}_k \mathbin{\blacktriangle} \partial_k^H$.
So $(\varphi \mathrm{N}_k)_{k \in \mathbb{Z}}$ is a morphism of complexes in groups.
Ad (3). Suppose given $F \xrightarrow{\varphi} F' \xrightarrow{\varphi'} F''$ in $\lfloor n, 0 \rfloor$-$\mathcal{SimpGrp}$.
For $k \in [0, n]$, we have

$$\mathrm{id}_F \, \mathrm{N}_k = \mathrm{id}_{F_k} \big|_{F\mathrm{N}_k}^{F\mathrm{N}_k} = \mathrm{id}_{F\mathrm{N}_k}\,.$$

For $\in \mathbb{Z} \backslash [0, n]$, we have

$$\mathrm{id}_F \, \mathrm{N}_k = ! = \mathrm{id}_{F\mathrm{N}_k}\,.$$

Thus, we have

$$\mathrm{id}_F \, \mathrm{N} = (\mathrm{id}_F \, \mathrm{N}_k)_{k \in \mathbb{Z}} = (\mathrm{id}_{F\mathrm{N}_k})_{k \in \mathbb{Z}} = \mathrm{id}_{F\mathrm{N}}\,.$$

Moreover, for $k \in [0, n]$, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N}_k = (\varphi_k \blacktriangle \varphi'_k)\big|_{F\mathrm{N}_k}^{F''\mathrm{N}_k} = \big(\varphi_k\big|_{F\mathrm{N}_k}^{F'\mathrm{N}_k}\big) \blacktriangle \big(\varphi'_k\big|_{F'\mathrm{N}_k}^{F''\mathrm{N}_k}\big) = \varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k\,.$$

For $k \in \mathbb{Z}\backslash[0, n]$, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N}_k = ! = \varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k\,.$$

Thus, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N} = \big((\varphi \blacktriangle \varphi')\mathrm{N}_k\big)_{k\in\mathbb{Z}} = (\varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k)_{k\in\mathbb{Z}} = (\varphi\mathrm{N}_k)_{k\in\mathbb{Z}} \blacktriangle (\varphi'\mathrm{N}_k)_{k\in\mathbb{Z}} = \varphi\mathrm{N} \blacktriangle \varphi'\mathrm{N}\,.$$

So N is a functor. □

Chapter 4

Simplicial groups and pre-$\lfloor n, 0 \rfloor$-simplicial groups: the coskeleton functor

Let $n \in \mathbb{Z}_{\geqslant 0}$.

4.1 The cut functor $\mathrm{Cut}^{\lfloor n,0 \rfloor} : \mathcal{SimpGrp} \to \mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{SimpGrp}$

Remark 79 (The cut functor $\mathrm{Cut}^{\lfloor n,0 \rfloor}$)
Let $\mathrm{I}_n \colon \Delta^{\mathrm{op}}_{\lfloor n,0 \rfloor} \to \Delta^{\mathrm{op}}$, $\big([k] \xrightarrow{u} [\ell]\big) \mapsto \big([k] \xrightarrow{u} [\ell]\big)$ be the inclusion functor.
We have the *cut functor*

$$\begin{array}{rccl} \mathrm{Cut}^{\lfloor n,0 \rfloor} \colon & \mathcal{SimpGrp} & \to & \mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{SimpGrp} \\ & \big(G \xrightarrow{\varphi} \tilde{G}\big) & \mapsto & \big(G\,\mathrm{Cut}^{\lfloor n,0 \rfloor} \xrightarrow{\varphi\,\mathrm{Cut}^{\lfloor n,0 \rfloor}} \tilde{G}\,\mathrm{Cut}^{\lfloor n,0 \rfloor}\big) := \big(\mathrm{I}_n * G \xrightarrow{\mathrm{I}_n * \varphi} \mathrm{I}_n * \tilde{G}\big)\,; \end{array}$$

cf. Definitions 30, 52.

So we have

$$\begin{aligned} G\,\mathrm{Cut}^{\lfloor n,0 \rfloor} &= G\big|_{\Delta^{\mathrm{op}}_{\lfloor n,0 \rfloor}} \\ \varphi\,\mathrm{Cut}^{\lfloor n,0 \rfloor} &= \big(\big(G_k \xrightarrow{\varphi_k} \tilde{G}_k\big)_{k \in \mathbb{Z}_{\geqslant 0}}\big)\,\mathrm{Cut}^{\lfloor n,0 \rfloor} = \big(G_k \xrightarrow{\varphi_k} \tilde{G}_k\big)_{k \in [0,n]}\,. \end{aligned}$$

4.2 The coskeleton functor $\mathrm{Cosk}^{\lfloor n,0 \rfloor} : \mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{S}impGrp \to \mathcal{S}impGrp$

Definition 80 Suppose given a pre-$\lfloor n, 0 \rfloor$-simplicial group G; cf. Definition 52.

For $k \in \mathbb{Z}_{\geqslant 0}$, we consider the set

$$ {}_{\Delta}([n],[k]) = \big\{ a \in \mathrm{Mor}(\Delta) : a\boldsymbol{s} = [n]\,, a\boldsymbol{t} = [k] \big\} = \big\{ [n] \xrightarrow{a} [k] : a \text{ is monotone} \big\}\,. $$

Let $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given a tuple $(g_a)_{a \in {}_{\Delta}([n],[k])} \in \prod\limits_{a \in {}_{\Delta}([n],[k])} G_n$.

We call $(g_a)_{a \in {}_{\Delta}([n],[k])}$ *coherent* if we have

$$ g_a G_b = g_{a'} G_{b'} $$

for $m \in [0, n]$ and for $a,\, a' \in {}_{\Delta}([n],[k])$, $b,\, b' \in {}_{\Delta}\big([m],[n]\big)$ with $b \blacktriangle a = b' \blacktriangle a'$.

$$ \begin{array}{ccc} [n] & \xrightarrow{\ a\ } & [k] \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle a'} \\ [m] & \xrightarrow[\ b'\ ]{} & [n] \end{array} $$

Sometimes, we abbreviate $(g_a)_a := (g_a)_{a \in {}_{\Delta}([n],[k])}$. Note that $g_a \in G_n$ for each $[n] \xrightarrow{a} [k]$.

Lemma 81 (The simplicial group $G\,\mathrm{Cosk}^{\lfloor n,0 \rfloor}$)

Suppose given a pre-$\lfloor n, 0 \rfloor$-simplicial group G; cf. Definition 52.

(1) *For $k \in \mathbb{Z}_{\geqslant 0}$, we have the subgroup*

$$ \begin{aligned} (G\,\mathrm{Cosk}^{\lfloor n,0 \rfloor})_k &:= \Big\{ (g_a)_{a \in {}_{\Delta}([n],[k])} \in \prod_{a \in {}_{\Delta}([n],[k])} G_n \ :\ (g_a)_{a \in {}_{\Delta}([n],[k])} \text{ is coherent } \Big\} \\ &\leqslant \prod_{a \in {}_{\Delta}([n],[k])} G_n\,; \end{aligned} $$

cf. Definition 80.

(2) *Suppose given $\big([k] \xrightarrow{u} [\ell]\big) \in \mathrm{Mor}(\Delta)$; cf. Definition* 18.

We consider the groups $(G\,\mathrm{Cosk}^{\lfloor n,0 \rfloor})_k$ and $(G\,\mathrm{Cosk}^{\lfloor n,0 \rfloor})_\ell$; cf. (1).

We have the group morphism

$$(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u \colon (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell \to (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,,\ (g_a)_{a\in\Delta([n],[\ell])} \mapsto (g_{b\blacktriangle u})_{b\in\Delta([n],[k])}\,.$$

(3) *We have the simplicial group*

$$G\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \colon \Delta^{\mathrm{op}} \to \mathcal{G}rp\,,\ \big([\ell] \xrightarrow{u^{\mathrm{op}}} [k]\big) \mapsto \big((G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell \xrightarrow{(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u} (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\big)\,;$$

cf. Definition 30.

Proof. Ad (1). Suppose given $(g_a)_{a\in\Delta([n],[k])}\,,\ (h_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,.$

Note that $\big((h_a)_{a\in\Delta([n],[k])}\big)^- = (h_a^-)_{a\in\Delta([n],[k])}\,.$

We have

$$(g_a)_{a\in\Delta([n],[k])} \cdot \big((h_a)_{a\in\Delta([n],[k])}\big)^- = (g_a)_{a\in\Delta([n],[k])} \cdot (h_a^-)_{a\in\Delta([n],[k])} = (g_a \cdot h_a^-)_{a\in\Delta([n],[k])}\,.$$

We have to show that $(g_a \cdot h_a^-)_{a\in\Delta([n],[k])}$ is coherent.

Suppose given $m \in [0,n]$ and suppose given $a,\ a' \in {}_\Delta([n],[k]),\ b,\ b' \in {}_\Delta([m],[n])$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have

$$(g_a \cdot h_a^-)G_b = g_a G_b \cdot (h_a G_b)^- = g_{a'} G_{b'} \cdot (h_{a'} G_{b'})^- = (g_{a'} \cdot h_{a'}^-)G_{b'}\,.$$

This shows $(g_a \cdot h_a^-)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,.$

Ad (2). *We show that* $(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u$ *is a well-defined map.*

Suppose given a coherent tuple $(g_a)_{a\in\Delta([n],[\ell])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell\,.$ We have to show that $(g_{b\blacktriangle u})_{b\in\Delta([n],[k])}$ is coherent.

Suppose given $b,\ b' \in {}_\Delta([n],[k])$ and suppose given $m \in [0,n],\ c,\ c' \in {}_\Delta([m],[n])$, with $c \blacktriangle b = c' \blacktriangle b'\,.$

Then we have

$$c \blacktriangle (b \blacktriangle u) = c' \blacktriangle (b' \blacktriangle u)\,.$$

Since $(b \blacktriangle u),\ (b' \blacktriangle u) \in {}_\Delta([n],[\ell])$ and since $(g_a)_{\in\Delta([n],[\ell])}$ is coherent, we have

$$(g_{b\blacktriangle u})G_c = (g_{b'\blacktriangle u})G_{c'}\,.$$

So $(g_{b\blacktriangle u})_{b\in\Delta([n],[k])}$ is coherent.

We show that $(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u$ *is a group morphism.*

Suppose given $(g_a)_{\in\Delta([n],[\ell])}$, $(h_a)_{\in\Delta([n],[\ell])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell$.

We have

$$\begin{aligned}
&= \big((g_a)_{a\in\Delta([n],[\ell])} \cdot (h_a)_{a\in\Delta([n],[\ell])}\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u \\
&= \big((g_a \cdot h_a)_{a\in\Delta([n],[\ell])}\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u \\
&= (g_{b\blacktriangle u} \cdot h_{b\blacktriangle u})_{b\in\Delta([n],[k])} \\
&= (g_{b\blacktriangle u})_{b\in\Delta([n],[k])} \cdot (h_{b\blacktriangle u})_{b\in\Delta([n],[k])} \\
&= \big((g_a)_{a\in\Delta([n],[\ell])}\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u \cdot \big((h_a)_{a\in\Delta([n],[\ell])}\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u\,.
\end{aligned}$$

Ad (3). Suppose given $[k] \xrightarrow{u} [\ell] \xrightarrow{v} [m]$ in Δ. We consider $[m] \xrightarrow{v^{\mathrm{op}}} [\ell] \xrightarrow{u^{\mathrm{op}}} [k]$ in Δ^{op}.

For $(g_a)_{a\in\Delta([n],[m])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_m$, we have

$$\begin{aligned}
\big((g_a)_a\big)\big((\mathrm{id}^{\mathrm{op}}_{[m]})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big) &= \big((g_a)_a\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{\mathrm{id}_{[m]}} = (g_{a\blacktriangle\,\mathrm{id}_{[m]}})_a = (g_a)_a \\
&= \big((g_a)_a\big)\,\mathrm{id}_{(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_m}\,.
\end{aligned}$$

This shows $(\mathrm{id}_{[m]})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor} = \mathrm{id}_{(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_m}$.

For $(g_a)_{a\in\Delta([n],[m])}$, we have

$$\begin{aligned}
&\big((g_a)_a\big)\big((v^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \blacktriangle (u^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big) \\
&= \big((g_a)_a\big)\big((G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_v \blacktriangle (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u\big) \\
&= \big((g_{b\blacktriangle v})_{b\in\Delta([n],[\ell])}\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u \\
&= (g_{(c\blacktriangle u)\blacktriangle v})_{c\in\Delta([n],[k])} \\
&= (g_{c\blacktriangle(u\blacktriangle v)})_{c\in\Delta([n],[k])} \\
&= \big((g_a)_a\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{(u\blacktriangle v)} \\
&= \big((g_a)_a\big)\Big(\big((u\blacktriangle v)^{\mathrm{op}}\big)G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\Big) \\
&= \big((g_a)_a\big)\big((v^{\mathrm{op}}\blacktriangle u^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)\,.
\end{aligned}$$

This shows $(v^{\mathrm{op}}\blacktriangle u^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor} = (v^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \blacktriangle (u^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$.

So $G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ is a simplicial group. □

Lemma 82 (The coskeleton functor $\mathrm{Cosk}^{\lfloor n,0\rfloor}$)

For a pre-$\lfloor n,0\rfloor$*-simplicial group* $G \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp\big)$*, we consider the simplicial group* $G\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \in \mathrm{Ob}\big(\mathcal{S}impGrp\big)$*; cf. Lemma* 81.(3).

4.2. THE COSKELETON FUNCTOR $\mathrm{Cosk}^{\lfloor n,0\rfloor} : \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp \to \mathcal{S}impGrp$

We have the coskeleton functor

$$\mathrm{Cosk}^{\lfloor n,0\rfloor} \colon \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp \to \mathcal{S}impGrp\,,\ \big(G \xrightarrow{\varphi} \tilde{G}\big) \mapsto \big(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \xrightarrow{\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}} \tilde{G}\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)\,,$$

where, for $k \in [0,n]$*, we have*

$$(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \colon (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \to (\tilde{G}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,,\ (g_a)_{a\in\Delta([n],[k])} \mapsto (g_a\varphi_n)_{a\in\Delta([n],[k])}\,.$$

Proof. Suppose given $\big(G \xrightarrow{\varphi} \tilde{G}\big)$ in $\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp$. Let $k \in [0,n]$.

We show that $\varphi(\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$ *is a well-defined map.*

Suppose given $(g_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$.

We have to show that $(g_a\varphi_n)_{a\in\Delta([n],[k])} \overset{!}{\in} (\tilde{G}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$, i.e. that $(g_a\varphi_n)_{a\in\Delta([n],[k])}$ is coherent.

For $a,\, a' \in {}_\Delta([n],[k])$, suppose given $m \in [0,n]$ and $b,\, b' \in {}_\Delta([m],[n])$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have to show that $(g_a\varphi_n)\tilde{G}_b \overset{!}{=} (g_{a'}\varphi_n)\tilde{G}_{b'}$.

We have

$$(g_a\varphi_n)\tilde{G}_b = (g_aG_b)\varphi_m = (g_{a'}G_{b'})\varphi_m = (g_{a'}\varphi_n)\tilde{G}_{b'}\,.$$

$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & \tilde{G}_n \\ {\scriptstyle G_b}\downarrow & & \downarrow{\scriptstyle \tilde{G}_b} \\ G_m & \xrightarrow[\varphi_m]{} & \tilde{G}_m \end{array} \qquad\qquad \begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & \tilde{G}_n \\ {\scriptstyle G_{b'}}\downarrow & & \downarrow{\scriptstyle \tilde{G}_{b'}} \\ G_n & \xrightarrow[\varphi_m]{} & \tilde{G}_n \end{array}$$

We show that $\varphi(\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$ *is a group morphism.*

Suppose given $(g_a)_{a\in\Delta([n],[k])}\,,\ (h_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$.

We have

$$\begin{aligned}\big((g_a)_a \cdot (h_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k &= \big((g_a \cdot h_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k = \big((g_a \cdot h_a)\varphi_n\big)_a = (g_a\varphi_n \cdot h_a\varphi_n)_a \\ &= (g_a\varphi_n)_a \cdot (h_a\varphi_n)_a = \big((g_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \cdot \big((h_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,.\end{aligned}$$

We show that $\mathrm{Cosk}^{\lfloor n,0\rfloor}$ *is a functor.*

Suppose given $G \xrightarrow{\varphi} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{\tilde{G}}$ in $\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp$.

For $k \in [0,n]$ and for $(g_a)_{a\in{}_\Delta([n],[k])} \in (G\operatorname{Cosk}^{\lfloor n,0\rfloor})_k$, we have

$$(g_a)_a\big((\mathrm{id}_G)\operatorname{Cosk}^{\lfloor n,0\rfloor}\big)_k = \big(g_a(\mathrm{id}_G)_n\big)_a = (g_a)_a = (g_a)_a\,\mathrm{id}_{(G\operatorname{Cosk}^{\lfloor n,0\rfloor})_k}\,.$$

This shows $(\mathrm{id}_G)\operatorname{Cosk}^{\lfloor n,0\rfloor} = \mathrm{id}_{G\operatorname{Cosk}^{\lfloor n,0\rfloor}}$.

For $k \in [0,n]$ and for $(g_a)_{a\in{}_\Delta([n],[k])} \in (G\operatorname{Cosk}^{\lfloor n,0\rfloor})_k$, we have

$$\begin{aligned}(g_a)_a\big((\varphi \mathbin{\blacktriangle} \tilde\varphi)\operatorname{Cosk}^{\lfloor n,0\rfloor}\big)_k &= \big(g_a(\varphi \mathbin{\blacktriangle} \tilde\varphi)_n\big)_a = \big(g_a(\varphi_n \mathbin{\blacktriangle} \tilde\varphi_n)\big)_a = \big((g_a\varphi_n)_a\big)(\tilde\varphi\operatorname{Cosk}^{\lfloor n,0\rfloor})_k\\ &= (g_a)_a(\varphi\operatorname{Cosk}^{\lfloor n,0\rfloor})_k(\tilde\varphi\operatorname{Cosk}^{\lfloor n,0\rfloor})_k = (g_a)_a(\varphi\operatorname{Cosk}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \tilde\varphi\operatorname{Cosk}^{\lfloor n,0\rfloor})_k\,.\end{aligned}$$

This shows $(\varphi \mathbin{\blacktriangle} \tilde\varphi)\operatorname{Cosk}^{\lfloor n,0\rfloor} = \varphi\operatorname{Cosk}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \tilde\varphi\operatorname{Cosk}^{\lfloor n,0\rfloor}$. □

Example 83 Let $n = 1$. Suppose given a pre-$\lfloor 1,0\rfloor$-simplicial group H; cf. Definition 52. We consider the simplicial group $H\operatorname{Cosk}^{\lfloor 1,0\rfloor}$; cf. Lemma 81.

(1) For $k \in \{1,2\}$, we consider the set

$${}_\Delta([0],[k]) = \big\{a \in \mathrm{Mor}(\Delta) : a\boldsymbol{s} = [0]\,, a\boldsymbol{t} = [k]\big\} = \big\{[0] \xrightarrow{a} [k] : a \text{ is monotone}\big\}\,;$$

cf. Definition 80.

We have

$$\begin{aligned}{}_\Delta([0],[1]) &= \{\delta^1_0\,, \delta^1_1\,\}\\ {}_\Delta([0],[2]) &= \{\delta^1_1 \mathbin{\blacktriangle} \delta^2_2\,, \delta^1_0 \mathbin{\blacktriangle} \delta^2_2\,, \delta^1_0 \mathbin{\blacktriangle} \delta^2_1\,\}\,;\end{aligned}$$

cf. Corollary 22.

We write $\beta^2_0 := \delta^1_1 \mathbin{\blacktriangle} \delta^2_2$, $\beta^2_1 := \delta^1_0 \mathbin{\blacktriangle} \delta^2_2$ and $\beta^2_2 := \delta^1_0 \mathbin{\blacktriangle} \delta^2_1$. Then ${}_\Delta([0],[2]) = \{\beta^2_0\,, \beta^2_1\,, \beta^2_2\}$.

(2) For $k \in \{0,1,2\}$, we consider the set

$${}_\Delta([1],[k]) = \big\{a \in \mathrm{Mor}(\Delta) : a\boldsymbol{s} = [1]\,, a\boldsymbol{t} = [k]\big\} = \big\{[1] \xrightarrow{a} [k] : a \text{ is monotone}\big\}\,;$$

cf. Definition 80.

We have

$$\begin{aligned}{}_\Delta([1],[0]) &= \{\sigma^0_0\,\}\\ {}_\Delta([1],[1]) &= \{\mathrm{id}_{[1]}\,, \sigma^0_0 \mathbin{\blacktriangle} \delta^1_1\,, \sigma^0_0 \mathbin{\blacktriangle} \delta^1_0\,\}\\ {}_\Delta([1],[2]) &= \{\delta^2_0\,, \delta^2_1\,, \delta^2_2\,, \sigma^0_0 \mathbin{\blacktriangle} \delta^1_1 \mathbin{\blacktriangle} \delta^2_2\,, \sigma^0_0 \mathbin{\blacktriangle} \delta^1_0 \mathbin{\blacktriangle} \delta^2_2\,, \sigma^0_0 \mathbin{\blacktriangle} \delta^1_0 \mathbin{\blacktriangle} \delta^2_1\,\}\,;\end{aligned}$$

cf. Corollary 22.

We write $\alpha_0^1 := \sigma_0^0 \blacktriangle \delta_1^1$ and $\alpha_1^1 := \sigma_0^0 \blacktriangle \delta_0^1$. Then ${}_\Delta([1],[1]) = \{\mathrm{id}_{[1]}, \alpha_0^1, \alpha_1^1\}$.

So, for $h \in (H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_1$, we have

$$h = (h_{\mathrm{id}_{[1]}}, h_{\alpha_0^1}, h_{\alpha_1^1}) \in \prod_{a \in {}_\Delta([1],[1])} H_1 .$$

Moreover, we write $\alpha_0^2 := \sigma_0^0 \blacktriangle \delta_1^1 \blacktriangle \delta_2^2$, $\alpha_1^2 := \sigma_0^0 \blacktriangle \delta_0^1 \blacktriangle \delta_2^2$ and $\alpha_2^2 := \sigma_0^0 \blacktriangle \delta_0^1 \blacktriangle \delta_1^2$.

Then ${}_\Delta([1],[2]) = \{\delta_0^2, \delta_1^2, \delta_2^2, \alpha_0^2, \alpha_1^2, \alpha_2^2\}$.

So, for $h \in (H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2$, we have

$$h = (h_{\delta_0^2}, h_{\delta_1^2}, h_{\delta_2^2}, h_{\alpha_0^2}, h_{\alpha_1^2}, h_{\alpha_2^2}) \in \prod_{a \in {}_\Delta([1],[2])} H_1 .$$

From the coherence condition, we obtain e.g. the following assertions.

Since $\delta_0^1 \blacktriangle \delta_0^2 = \delta_0^1 \blacktriangle \delta_1^2$, we have $h_{\delta_0^2}\, \mathrm{d}_0^{H,1} = h_{\delta_1^1}\, \mathrm{d}_0^{H,1}$.

Since $\mathrm{id}_{[1]} \blacktriangle \alpha_0^2 = (\sigma_0^0 \blacktriangle \delta_1^1) \blacktriangle \delta_2^2$, we have $h_{\alpha_0^2} = h_{\alpha_0^2}\, \mathrm{id}_{H_1} = h_{\delta_2^2}\, H_{\sigma_0^0 \blacktriangle \delta_1^1} = h_{\delta_2^2}\, \mathrm{d}_1^{H,1}\, \mathrm{s}_0^{H,0}$.

(3) We consider the group

$$\begin{aligned}
(H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2 &\overset{81.(1)}{=} \Big\{ (h_a)_{a \in {}_\Delta([1],[2])} \in \prod_{a \in {}_\Delta([1],[2])} H_1 : (h_a)_{a \in {}_\Delta([1],[2])} \text{ is coherent } \Big\} \\
&\overset{(2)}{=} \Big\{ h = (h_{\delta_0^2}, h_{\delta_1^2}, h_{\delta_2^2}, h_{\alpha_0^2}, h_{\alpha_1^2}, h_{\alpha_2^2}) \in \prod_{a \in {}_\Delta([1],[2])} H_1 : h \text{ is coherent } \Big\} .
\end{aligned}$$

We consider its subgroup $(H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2 \leqslant (H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2$; cf. Definition 46.(2).

We show that

$$(H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2 \overset{!}{=} \big\{ (h_{\delta_0}, 1, 1, 1, 1, 1) \colon h_{\delta_0} \in H\mathrm{NZ}_1 \big\} \simeq H\mathrm{NZ}_1 .$$

Ad $\leqslant$. Suppose given $h = (h_{\delta_0^2}, h_{\delta_1^2}, h_{\delta_2^2}, h_{\alpha_0^2}, h_{\alpha_1^2}, h_{\alpha_2^2}) \in (H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2$.

We have

$$\begin{aligned}
1 &= h\mathrm{d}_1^{H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor},2} \\
&= (h_{\delta_0^2}, h_{\delta_1^2}, h_{\delta_2^2}, h_{\alpha_0^2}, h_{\alpha_1^2}, h_{\alpha_2^2})(H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_{\delta_1^2} \\
&\overset{81.(2)}{=} (h_{\mathrm{id}_{[1]} \blacktriangle \delta_1^2}, h_{\alpha_0^1 \blacktriangle \delta_1^2}, h_{\alpha_1^1 \blacktriangle \delta_1^2}) \\
&= (h_{\delta_1^2}, h_{\alpha_0^2}, h_{\alpha_2^2}) .
\end{aligned}$$

This shows that $h_{\delta_1^2} = h_{\alpha_0^2} = h_{\alpha_2^2} = 1\,.$

We have

$$\begin{array}{rcl} 1 & = & h\mathrm{d}_2^{H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor},2} \\ & = & (h_{\delta_0^2}\,, h_{\delta_1^2}\,, h_{\delta_2^2}\,, h_{\alpha_0^2}\,, h_{\alpha_1^2}\,, h_{\alpha_2^2})(H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_{\delta_2^2} \\ & \overset{81.(2)}{=} & (h_{\mathrm{id}_{[1]}\,\blacktriangle\,\delta_2^2}\,, h_{\alpha_0^1\,\blacktriangle\,\delta_2^2}\,, h_{\alpha_1^1\,\blacktriangle\,\delta_2^2}) \\ & = & (h_{\delta_2^2}\,, h_{\alpha_0^2}\,, h_{\alpha_1^2})\,. \end{array}$$

This shows that $h_{\delta_2^2} = h_{\alpha_0^2} = h_{\alpha_1^2} = 1\,.$

It remains to show that $h_{\delta_0^2} \overset{!}{\in} H\mathrm{NZ}_1\,.$

We have $\delta_0^1 \blacktriangle \delta_0^2 = \delta_0^1 \blacktriangle \delta_1^2\,.$ Since h is coherent, we have

$$h_{\delta_0^2}\,\mathrm{d}_0^{H,2} = h_{\delta_0^2}\,H_{\delta_0^1} \overset{80}{=} h_{\delta_1^2}\,H_{\delta_0^1} = 1\,H_{\delta_0^1} = 1\,.$$

We have $\delta_1^1 \blacktriangle \delta_0^2 = \delta_0^1 \blacktriangle \delta_2^2\,.$ Since h is coherent, we have

$$h_{\delta_0^2}\,\mathrm{d}_1^{H,2} = h_{\delta_0^2}\,H_{\delta_1^1} \overset{80}{=} h_{\delta_2^2}\,H_{\delta_0^1} = 1\,H_{\delta_0^1} = 1\,.$$

This shows that $h_{\delta_0^2} \in H_{1,0} \cap H_{1,1} \overset{76.(3)}{=} H\mathrm{NZ}_1\,.$

So we have $h = (h_{\delta_0^2}\,, 1, 1, 1, 1, 1)\,,$ where $h_{\delta_0^2} \in H\mathrm{NZ}_1\,.$

Ad $\geqslant$. Suppose given $x \in H\mathrm{NZ}_1\,.$

We have to show that the tuple

$$(h_{\delta_0^2}\,, h_{\delta_1^2}\,, h_{\delta_2^2}\,, h_{\alpha_0^2}\,, h_{\alpha_1^2}\,, h_{\alpha_2^2}) := (x, 1, 1, 1, 1, 1) \in \prod_{a \in {}_\Delta([1],[2])} H_1$$

is coherent.

Let $m = 0$. For $b \in {}_\Delta([0],[1]) = \{\delta_0^1\,, \delta_1^1\,\}$ and $a \in {}_\Delta([1],[2]) = \{\delta_0^2\,, \delta_1^2\,, \delta_2^2\,, \alpha_0^2\,, \alpha_1^2\,, \alpha_2^2\}$, we consider $b \blacktriangle a$.

We obtain the following composition table.

$\blacktriangle$	δ_0^2	δ_1^2	δ_2^2	α_0^2	α_1^2	α_2^2
δ_0^1	β_2^2	β_2^2	β_1^2	β_0^2	β_1^2	β_2^2
δ_1^1	β_1^2	β_0^2	β_0^2	β_0^2	β_1^2	β_2^2

We have

$$\beta_0^2 = \quad \delta_0^1 \blacktriangle \alpha_0^2 = \delta_1^1 \blacktriangle \delta_1^2 = \delta_1^1 \blacktriangle \delta_2^2 = \delta_1^1 \blacktriangle \alpha_0^2 \,.$$

So we have to show that

$$h_{\alpha_0^2} H_{\delta_0^1} \overset{!}{=} h_{\delta_1^2} H_{\delta_1^1} \overset{!}{=} h_{\delta_2^2} H_{\delta_1^1} \overset{!}{=} h_{\alpha_0^2} H_{\delta_1^1} \,.$$

We have $h_{\alpha_0^2} = h_{\delta_1^2} = h_{\delta_2^2} = 1\,.$

So

$$1H_{\delta_0^1} = 1H_{\delta_1^1} = 1H_{\delta_1^1} = 1H_{\delta_1^1} = 1\,.$$

We have

$$\beta_1^2 = \quad \delta_0^1 \blacktriangle \delta_2^2 = \delta_0^1 \blacktriangle \alpha_1^2 = \delta_1^1 \blacktriangle \delta_0^2 = \delta_1^1 \blacktriangle \alpha_1^2 \,.$$

So we have to show that

$$h_{\delta_2^2} H_{\delta_0^1} \overset{!}{=} h_{\alpha_1^2} H_{\delta_0^1} \overset{!}{=} h_{\delta_0^2} H_{\delta_1^1} \overset{!}{=} h_{\alpha_1^2} H_{\delta_1^1} \,.$$

Since we have $h_{\delta_2^2} = h_{\alpha_1^2} = 1\,,$ it remains to show that $h_{\delta_0^2} H_{\delta_1^1} \overset{!}{=} 1\,.$

Indeed, we have

$$h_{\delta_0^2} H_{\delta_1^1} = x\mathrm{d}_1^{H,1} = 1\,,$$

since $x \in H\mathrm{NZ}_1\,.$

We have

$$\beta_2^2 = \quad \delta_0^1 \blacktriangle \delta_0^2 = \delta_0^1 \blacktriangle \delta_1^2 = \delta_0^1 \blacktriangle \alpha_2^2 = \delta_1^1 \blacktriangle \alpha_2^2 \,.$$

So we have to show that

$$h_{\delta_0^2} H_{\delta_0^1} \overset{!}{=} h_{\delta_1^2} H_{\delta_0^1} \overset{!}{=} h_{\alpha_2^2} H_{\delta_0^1} \overset{!}{=} h_{\alpha_2^2} H_{\delta_1^1} \,.$$

Since we have $h_{\delta_1^2} = h_{\alpha_2^2} = 1\,,$ it remains to show that $h_{\delta_0^2} H_{\delta_0^1} \overset{!}{=} 1\,.$

Indeed, we have

$$h_{\delta_0^2} H_{\delta_0^1} = x\mathrm{d}_0^{H,1} = 1\,,$$

since $x \in H\mathrm{NZ}_1\,.$

Let $m = 1$. For $b \in {}_{\Delta}([1],[1]) = \{\mathrm{id}_{[1]}\,, \alpha_0^1\,, \alpha_1^1\}$ and $a \in {}_{\Delta}([1],[2]) = \{\delta_0^2\,, \delta_1^2\,, \delta_2^2\,, \alpha_0^2\,, \alpha_1^2\,, \alpha_2^2\}$, we consider $b \blacktriangle a$.

We obtain the following composition table.

$\blacktriangle$	δ_0^2	δ_1^2	δ_2^2	α_0^2	α_1^2	α_2^2
$\mathrm{id}_{[1]}$	δ_0^2	δ_1^2	δ_2^2	α_0^2	α_1^2	α_2^2
α_0^1	α_1^2	α_0^2	α_0^2	α_0^2	α_1^2	α_2^2
α_1^1	α_2^2	α_2^2	α_1^2	α_0^2	α_1^2	α_2^2

We have

$$\alpha_0^2 = \quad \alpha_0^1 \blacktriangle \delta_1^2 = \alpha_0^1 \blacktriangle \delta_2^2 = \alpha_0^1 \blacktriangle \alpha_0^2 = \alpha_1^1 \blacktriangle \alpha_0^2 = \mathrm{id}_{[1]} \blacktriangle \alpha_0^2 \,.$$

So we have to show that

$$h_{\delta_1^2} H_{\alpha_0^1} \overset{!}{=} h_{\delta_2^2} H_{\alpha_0^1} \overset{!}{=} h_{\alpha_0^2} H_{\alpha_0^1} \overset{!}{=} h_{\alpha_0^2} H_{\alpha_1^1} \overset{!}{=} h_{\alpha_0^2} H_{\mathrm{id}_{[1]}} \,.$$

We have $h_{\delta_1^2} = h_{\delta_2^2} = h_{\alpha_0^2} = 1$.

So

$$1 H_{\alpha_0^1} = 1 H_{\alpha_0^1} = 1 H_{\alpha_0^1} = 1 H_{\alpha_1^1} = 1 H_{\mathrm{id}_{[1]}} = 1 \,.$$

We have

$$\alpha_1^2 = \quad \alpha_0^1 \blacktriangle \delta_0^2 = \mathrm{id}_{[1]} \blacktriangle \alpha_1^2 = \alpha_0^1 \blacktriangle \alpha_1^2 = \alpha_1^1 \blacktriangle \alpha_1^2 = \alpha_1^1 \blacktriangle \delta_2^2 \,.$$

So we have to show that

$$h_{\delta_0^2} H_{\alpha_0^1} \overset{!}{=} h_{\alpha_1^2} H_{\mathrm{id}_{[1]}} \overset{!}{=} h_{\alpha_1^2} H_{\alpha_0^1} \overset{!}{=} h_{\alpha_1^2} H_{\alpha_1^1} \overset{!}{=} h_{\delta_2^2} H_{\alpha_1^1} \,.$$

Since $h_{\alpha_1^2} = h_{\delta_2^2} = 1$, it remains to show that $h_{\delta_0^2} H_{\alpha_0^1} \overset{!}{=} 1$.

We have

$$h_{\delta_0^2} H_{\alpha_0^1} = x H_{\sigma_0^0 \blacktriangle \delta_1^1} = x H_{\delta_1^1} H_{\sigma_0^0} = x \mathrm{d}_1^{H,1} \mathrm{s}_0^{H,0} = 1 \mathrm{s}_0^{H,0} = 1 \,,$$

since $x \in H\mathrm{NZ}_1$.

We have

$$\alpha_2^2 = \quad \alpha_1^1 \blacktriangle \delta_0^2 = \alpha_1^1 \blacktriangle \delta_1^2 = \mathrm{id}_{[1]} \blacktriangle \alpha_2^2 = \alpha_1^1 \blacktriangle \alpha_2^2 = \alpha_0^1 \blacktriangle \alpha_2^2 \,.$$

So we have to show that

$$h_{\delta_0^2} H_{\alpha_1^1} \overset{!}{=} h_{\delta_1^2} H_{\alpha_1^1} \overset{!}{=} h_{\alpha_2^2} H_{\mathrm{id}_{[1]}} \overset{!}{=} h_{\alpha_2^2} H_{\alpha_1^1} \overset{!}{=} h_{\alpha_2^2} H_{\alpha_0^1} \,.$$

Since $h_{\delta_1^2} = h_{\alpha_2^2} = 1$, it remains to show that $h_{\delta_0^2} H_{\alpha_1^1} \overset{!}{=} 1$.

We have

$$h_{\delta_0^2} H_{\alpha_1^1} = xH_{\sigma_0^0 \blacktriangle \delta_0^1} = xH_{\delta_0^1} H_{\sigma_0^0} = x\mathrm{d}_0^{H,1}\,\mathrm{s}_0^{H,0} = 1\mathrm{s}_0^{H,0} = 1\,,$$

since $x \in H\mathrm{NZ}_1$.

Altogether, this shows that the tuple $(x,1,1,1,1,1)$ is coherent.

So we have $(H\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2 = \big\{(h_{\delta_0^2}, 1,1,1,1,1) \colon h_{\delta_0^2} \in H\mathrm{NZ}_1\big\} \simeq H\mathrm{NZ}_1$.

Remark 84 *Let $k \in \mathbb{Z}_{\geqslant n+1}$. Suppose given $a \in {}_{\Delta}([n],[k])$.*
There exists some $b \in {}_{\Delta}([n],[k-1])$ and some $j \in [k-n-1,k]$ with $a = b \blacktriangle \delta_j^k$.

Proof. By Corollary 22.(1), we have

$$a = \left(\prod_{t\in\lfloor n-\ell,1\rfloor}^{[n]\;\;[\ell]} \sigma_{i_t}^{\ell+t-1}\right) \blacktriangle \left(\prod_{t\in\lceil 1,k-\ell\rceil}^{[\ell]\;\;[k]} \delta_{j_t}^{\ell+t}\right) \colon [n] \to [k]\,,$$

where $\ell = |\operatorname{im} a| - 1$, where $0 \leqslant i_1 < i_2 < \ldots < i_{n-\ell} \leqslant n$ and where

$$\{j_1,\ldots,j_{k-\ell}\} = \{j \in [0,k] \colon j \notin \operatorname{im} u\} \text{ and } 0 \leqslant j_1 < j_2 < \ldots < j_{k-\ell} \leqslant k\,.$$

Note that $\ell \leqslant n \leqslant k-1$. Note that $0 \leqslant k-n-1 \leqslant k-\ell-1 \leqslant j_{k-\ell} \leqslant k$.

Let

$$b := \left(\prod_{t\in\lfloor n-\ell,1\rfloor}^{[n]\;\;[\ell]} \sigma_{i_t}^{\ell+t-1}\right) \blacktriangle \left(\prod_{t\in\lceil 1,k-\ell-1\rceil}^{[\ell]\;\;[k-1]} \delta_{j_t}^{\ell+t}\right) \colon [n] \to [k-1]\,.$$

Then we have $a = b \blacktriangle \delta_{k-\ell}^k$ and $j_{k-\ell} \in [k-n-1,k]$. □

Lemma 85 *Let H be a pre-$\lfloor n,0\rfloor$-simplicial group.*
We consider the simplicial group $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$; cf. Lemma 81.(3).
For $k \in \mathbb{Z}_{\geqslant n+2}$, we have $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_k = 1$; cf. Definition 46.(2).

Proof. Suppose given $k \in \mathbb{Z}_{\geqslant n+2}$. Suppose given $(h_a)_{a\in{}_{\Delta}([n],[k])} \in (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_k$.

We have to show that $(h_a)_{a\in{}_{\Delta}([n],[k])} \overset{!}{=} 1$.

Suppose given $\tilde{a} \in {}_{\Delta}([n],[k])$. We have to show that $h_{\tilde{a}} \overset{!}{=} 1 \in H_n$.

By Remark 84, we have $\tilde{a} = b \blacktriangle \delta_j^k$ for some $b \in {}_{\Delta}([n],[k-1])$ and for some $j \in [k-n-1,k]$.

Note that $j \in [1,k]$ since $1 \leqslant k-n-1$.

We have

$$1 = \big((h_a)_{a\in\Delta([n],[k])}\big)\mathrm{d}_j^{H\mathrm{Cosk}^{\lfloor n,0\rfloor},k} \overset{81.(2)}{=} (h_{c\blacktriangle\delta_j^k})_{c\in\Delta([n],[k-1])}\,.$$

In particular, we have $1 = h_{b\blacktriangle\delta_j^k} = h_{\tilde{a}}$.

So we have $(h_a)_{a\in\Delta([n],[k])} = 1$.

This shows that $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_k = 1$. □

4.3 The adjunction $\mathrm{Cut}^{\lfloor n,0\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,0\rfloor}$

We consider the functors

$$\begin{array}{lrcll}
\mathrm{Cut}^{\lfloor n,0\rfloor}\colon & \mathcal{SimpGrp} & \to & \mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}; & \text{cf. Remark 79}\\
\mathrm{Cosk}^{\lfloor n,0\rfloor}\colon & \mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} & \to & \mathcal{SimpGrp}; & \text{cf. Lemma 82.}
\end{array}$$

Lemma 86 (The counit η)

Suppose given a pre-$\lfloor n,0\rfloor$-simplicial group $G \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$.

We consider the pre-$\lfloor n,0\rfloor$-simplicial group $G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}$.

For $k \in [0,n]$, note that we have

$$(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k = (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,;$$

cf. Lemma 81.(1).

For $[k] \xrightarrow{u} [\ell]$ in $\Delta_{\lfloor n,0\rfloor}$, note that we have

$$\begin{array}{rcl}
(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor})_u = (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u\colon \ (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell & \to & (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\\
(g_a)_{a\in\Delta([n],[\ell])} & \mapsto & (g_{b\blacktriangle u})_{b\in\Delta([n],[k])}\,;
\end{array}$$

cf. Lemma 81.(2).

(1) *For $k \in [0,n]$, we have the group isomorphism*

$$(G\vartheta)_k\colon G_k \to (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,,\ x \mapsto (xG_a)_{a\in\Delta([n],[k])}\,.$$

(2) *We have the pre-$\lfloor n,0\rfloor$-simplicial group isomorphism*

$$G\vartheta := \big(G_k \xrightarrow{(G\vartheta)_k} (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\big)_{k\in[0,n]}\colon G \to G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,;$$

cf. (1).

(3) *We have the isotransformation*

$$\vartheta = \big(\tilde{G} \xrightarrow{\tilde{G}\vartheta} \tilde{G}\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}\big)_{\tilde{G}\in \mathrm{Ob}(\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp)} \colon\ \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp} \to \mathrm{Cosk}^{\lfloor n,0\rfloor} * \mathrm{Cut}^{\lfloor n,0\rfloor}\,;$$

cf. (2).

Moreover, we have the isotransformation

$$\eta := \vartheta^{-} \colon\ \mathrm{Cosk}^{\lfloor n,0\rfloor} * \mathrm{Cut}^{\lfloor n,0\rfloor} \to \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp}\ .$$

In particular, for $k \in [0,n]$*, we have the group isomorphism*

$$(G\eta)_k = \big((G\vartheta)_k\big)^{-} \colon (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \to G_k\,,\ (g_a)_{a\in\Delta([n],[k])} \mapsto \big((g_a)_a\big)(G\eta)_k =: \tilde{g}\,,$$

where $(\tilde{g}G_a)_{a\in\Delta([n],[k])} = (g_a)_{a\in\Delta([n],[k])}$ *; cf.* (1).

Proof. Ad (1). Suppose given $k \in [0,n]$.

We show that $(G\vartheta)_k$ *is a well-defined map.*

Suppose given $x \in G$. We have to show that $(xG_a)_{a\in\Delta([n],[k])}$ is coherent.

Suppose given $a,\ a' \in {}_\Delta([n],[k])$. Suppose given $m \in [0,n]$ and $b,\ b' \in {}_\Delta([m],[n])$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have

$$(xG_a)G_b = xG_{b\blacktriangle a} = xG_{b'\blacktriangle a'} = (xG_{a'})G_{b'}\,.$$

We show that $(G\vartheta)_k$ *is a group morphism.*

For $x,\ y \in G_k$, we have

$$\begin{aligned}(x\cdot y)(G\vartheta)_k &= \big((x\cdot y)G_a\big)_{a\in\Delta([n],[k])} = (xG_a \cdot yG_a)_{a\in\Delta([n],[k])} = (xG_a)_{a\in\Delta([n],[k])}\cdot(yG_a)_{a\in\Delta([n],[k])}\\ &= x(G\vartheta)_k \cdot y(G\vartheta)_k\,.\end{aligned}$$

We show that $(G\vartheta)_k$ *is injective.*

Suppose given $x \in \ker\big((G\vartheta)_k\big)$. Then we have $xG_a = 1$ for $a \in {}_\Delta([n],[k])$.

We choose $\hat{a} \in {}_\Delta([n],[k])$ to be surjective. We choose $\hat{b} \in {}_\Delta([k],[n])$ such that $\hat{b}\blacktriangle\hat{a} = \mathrm{id}_{[k]}$.

Then we have

$$1 = x(G_{\hat{a}}\blacktriangle G_{\hat{b}}) = x(G_{\hat{b}\blacktriangle\hat{a}}) = xG_{\mathrm{id}_{[k]}} = x\,.$$

So $(G\vartheta)_k$ is injective.

We show that $(G\vartheta)_k$ is surjective.

Suppose given $(g_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$.

We choose $\hat{a} \in {}_\Delta([n],[k])$ to be surjective. We choose $\hat{b} \in {}_\Delta([k],[n])$ such that $\hat{b} \blacktriangle \hat{a} = \mathrm{id}_{[k]}$.

For $a \in {}_\Delta([n],[k])$, note that we have

$$(a \blacktriangle \hat{b}) \blacktriangle \hat{a} = a = \mathrm{id}_{[n]} \blacktriangle a\,.$$

$$\begin{array}{ccc} [n] & \xrightarrow{\hat{a}} & [k] \\ {\scriptstyle a\blacktriangle\hat{b}}\uparrow & & \uparrow{\scriptstyle a} \\ [n] & \xrightarrow[\mathrm{id}_{[n]}]{} & [n] \end{array}$$

Since $(g_a)_a$ is coherent, we have

$$(g_{\hat{a}})G_{a\blacktriangle\hat{b}} = (g_a)G_{\mathrm{id}_{[n]}} = g_a$$

for $a \in {}_\Delta([n],[k])$.

Let $x := g_{\hat{a}}G_{\hat{b}} \in G_k$. Then we have

$$x(G\vartheta)_k = (xG_a)_a = \big((g_{\hat{a}}G_{\hat{b}})G_a\big)_a = \big((g_{\hat{a}})G_{a\blacktriangle\hat{b}}\big)_a = (g_a)_a\,.$$

So $(G\vartheta)_k$ is surjective.

Ad (2). *We show that $G\vartheta = \big((G\vartheta)_k\big)_{k\in[0,n]}$ is a pre-$\lfloor n,0\rfloor$-simplicial group morphism.*

Suppose given $[k] \xrightarrow{u} [\ell]$ in $\mathbf{\Delta}_{\lfloor n,0\rfloor}$.

We consider the group isomorphisms $(G\vartheta)_k$ and $(G\vartheta)_\ell$ from (1).

For $g \in G_\ell$, we have

$$\begin{aligned} g\big((G\vartheta)_\ell \blacktriangle (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u\big) &\overset{(1)}{=} \big((gG_a)_{a\in{}_\Delta([n],[\ell])}\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u \overset{81.(2)}{=} (gG_{b\blacktriangle u})_{b\in{}_\Delta([n],[k])} \\ &= (gG_uG_b)_{b\in{}_\Delta([n],[k])} = (gG_u)(G\vartheta)_k = g(G_u \blacktriangle (G\vartheta)_k)\,. \end{aligned}$$

This shows $(G\vartheta)_\ell \blacktriangle (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u = G_u \blacktriangle (G\vartheta)_k$.

$$\begin{array}{ccc} G_\ell & \xrightarrow{(G\vartheta)_\ell} & (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell \\ {\scriptstyle G_u}\downarrow & & \downarrow{\scriptstyle (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u} \\ G_k & \xrightarrow[(G\vartheta)_k]{} & (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \end{array}$$

So $G\vartheta$ is a pre-$\lfloor n,0\rfloor$-simplicial group morphism.

Moreover, by (1), $G\vartheta$ is a pre-$\lfloor n,0\rfloor$-simplicial group isomorphism.

Ad (3). Suppose given $\tilde{G} \xrightarrow{\varphi} \tilde{\tilde{G}}$ in $\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp$.

Suppose given $k \in [0,n]$. We consider the group morphisms $(\tilde{G}\vartheta)_k$ and $(\tilde{\tilde{G}}\vartheta)_k$ from (1). For $g \in \tilde{G}_k$, we have

$$\begin{aligned} g\big((\tilde{G}\vartheta)_k \blacktriangle (\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k\big) &\overset{(1)}{=} \big((g\tilde{G}_a)_{a\in\Delta([n],[k])}\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k \\ &\overset{79}{=} \big((g\tilde{G}_a)_{a\in\Delta([n],[k])}\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \\ &\overset{82}{=} \big((g\tilde{G}_a)\varphi_n\big)_{a\in\Delta([n],[k])} \\ &= \big((g\varphi_k)\tilde{\tilde{G}}_a\big)_{a\in\Delta([n],[k])} \\ &\overset{(1)}{=} (g\varphi_k)(\tilde{\tilde{G}}\vartheta)_k \\ &= g\big(\varphi_k \blacktriangle (\tilde{\tilde{G}}\vartheta)_k\big)\,. \end{aligned}$$

This shows $(\tilde{G}\vartheta)_k \blacktriangle (\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k = \varphi_k \blacktriangle (\tilde{\tilde{G}}\vartheta)_k$.

Therefore, we have $\tilde{G}\vartheta \blacktriangle (\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}) = \varphi \blacktriangle \tilde{\tilde{G}}\vartheta$.

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{G}\vartheta} & \tilde{G}\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}} \\ \tilde{\tilde{G}} & \xrightarrow[\tilde{\tilde{G}}\vartheta]{} & \tilde{\tilde{G}}\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor} \end{array}$$

So ϑ is a transformation.

Moreover, by (2), ϑ is an isotransformation. Thus we have the isotransformation $\eta := \vartheta^{-}$. □

Lemma 87 (The unit ε)

Suppose given a simplicial group $G \in \mathrm{Ob}(\mathcal{SimpGrp})$.

We consider the simplicial group $G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$.

For $k \in \mathbb{Z}_{\geqslant 0}$, *note that we have*

$$(G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k = \Big\{(g_a)_{a\in\Delta([n],[k])} \in \prod_{a\in\,\Delta([n],[k])} G_n \colon (g_a)_{a\in\Delta([n],[k])} \text{ is coherent } \Big\};$$

cf. Lemma 81.(1).

For $[k] \xrightarrow{u} [\ell]$ *in* Δ^{op}, *note that we have*

$$\begin{aligned}(G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u \colon\ (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell &\to (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\\ (g_a)_{a\in\,\Delta([n],[\ell])} &\mapsto (g_{b\blacktriangle u})_{b\in\,\Delta([n],[k])};\end{aligned}$$

cf. Lemma 81.(2).

(1) *For* $k \in \mathbb{Z}_{\geqslant 0}$, *we have the group morphism*

$$(G\varepsilon)_k \colon G_k \to (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k,\ g \mapsto (gG_a)_{a\in\,\Delta([n],[k])};$$

cf. also Lemma 86.(1).

(2) *We have the simplicial group morphism*

$$G\varepsilon := \big(G_k \xrightarrow{(G\varepsilon)_k} (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\big)_{k\in\mathbb{Z}_{\geqslant 0}} \colon G \to G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor};$$

cf. (1).

(3) *We have the transformation*

$$\varepsilon = \big(\tilde{G} \xrightarrow{\tilde{G}\varepsilon} \tilde{G}\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_{\tilde{G}\in\mathrm{Ob}(\mathcal{SimpGrp})} \colon \mathrm{id}_{\mathcal{SimpGrp}} \to \mathrm{Cut}^{\lfloor n,0\rfloor} * \mathrm{Cosk}^{\lfloor n,0\rfloor};$$

cf. (2).

Proof. Ad (1). Suppose given $k \in \mathbb{Z}_{\geqslant 0}$.

We show that $(G\varepsilon)_k$ *is a well-defined map.*

4.3. THE ADJUNCTION $\mathrm{Cut}^{\lfloor n,0\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,0\rfloor}$

Suppose given $x \in G$. We have to show that $(xG_a)_{a\in\Delta([n],[k])}$ coherent; cf. Definition 80.

Suppose given $a,\ a' \in {}_\Delta([n],[k])$. Suppose given $m \in [0,n]$ and $b,\ b' \in {}_\Delta([m],[n])$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have

$$(xG_a)G_b = xG_{b\blacktriangle a} = xG_{b'\blacktriangle a'} = (xG_{a'})G_{b'}\,.$$

We show that $(G\varepsilon)_k$ *is a group morphism.*

For $x,\ y \in G_k$, we have

$$\begin{aligned}(x\cdot y)(G\varepsilon)_k &= \big((x\cdot y)G_a\big)_{a\in\Delta([n],[k])} = (xG_a\cdot yG_a)_{a\in\Delta([n],[k])} = (xG_a)_{a\in\Delta([n],[k])}\cdot(yG_a)_{a\in\Delta([n],[k])}\\ &= x(G\varepsilon)_k\cdot y(G\varepsilon)_k\,.\end{aligned}$$

Ad (2). *We show that* $G\varepsilon$ *is a simplicial group morphism.*

Suppose given $[k] \xrightarrow{u} [\ell]$ in $\mathbf{\Delta}$.

We consider the group morphisms $(G\varepsilon)_k$ and $(G\varepsilon)_\ell$ from (1).

For $g \in G_\ell$, we have

$$\begin{aligned}g\big((G\varepsilon)_\ell \blacktriangle (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u\big) &\overset{(1)}{=} \big((gG_a)_{a\in{}_\Delta([n],[\ell])}\big)(G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u = (gG_{b\blacktriangle u})_{b\in{}_\Delta([n],[k])}\\ &= (gG_uG_b)_{b\in{}_\Delta([n],[k])} \overset{(1)}{=} (gG_u)(G\varepsilon)_k = g(G_u\blacktriangle(G\varepsilon)_k)\,.\end{aligned}$$

This shows $(G\varepsilon)_\ell \blacktriangle (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u = G_u \blacktriangle (G\varepsilon)_k$.

$$\begin{array}{ccc} G_\ell & \xrightarrow{(G\varepsilon)_\ell} & (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_\ell \\ {\scriptstyle G_u}\downarrow & & \downarrow{\scriptstyle (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_u} \\ G_k & \xrightarrow[(G\varepsilon)_k]{} & (G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \end{array}$$

So $G\varepsilon$ is a simplicial group morphism.

Ad (3). Suppose given $\tilde{G} \xrightarrow{\varphi} \tilde{\tilde{G}}$ in $\mathcal{SimpGrp}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. For $g \in \tilde{G}_k$, we have

$$\begin{aligned} g\big((\tilde{G}\varepsilon)_k \blacktriangle (\varphi \operatorname{Cut}^{\lfloor n,0\rfloor} \operatorname{Cosk}^{\lfloor n,0\rfloor})_k\big) &\overset{(1)}{=} \big((g\tilde{G}_a)_{a\in\Delta([n],[k])}\big)(\varphi \operatorname{Cut}^{\lfloor n,0\rfloor} \operatorname{Cosk}^{\lfloor n,0\rfloor})_k \\ &\overset{82}{=} \big((g\tilde{G}_a)(\varphi \operatorname{Cut}^{\lfloor n,0\rfloor})_n\big)_{a\in\Delta([n],[k])} \\ &\overset{79}{=} \big((g\tilde{G}_a)\varphi_n\big)_{a\in\Delta([n],[k])} \\ &= \big((g\varphi_k)\tilde{\tilde{G}}_a\big)_{a\in\Delta([n],[k])} \\ &\overset{(1)}{=} (g\varphi_k)(\tilde{\tilde{G}}\varepsilon)_k \\ &= g\big(\varphi_k \blacktriangle (\tilde{\tilde{G}}\varepsilon)_k\big)\,. \end{aligned}$$

This shows $(\tilde{G}\varepsilon)_k \blacktriangle (\varphi \operatorname{Cut}^{\lfloor n,0\rfloor} \operatorname{Cosk}^{\lfloor n,0\rfloor})_k = \varphi_k \blacktriangle (\tilde{\tilde{G}}\varepsilon)_k$.

Therefore, we have $\tilde{G}\varepsilon \blacktriangle \varphi \operatorname{Cut}^{\lfloor n,0\rfloor} \operatorname{Cosk}^{\lfloor n,0\rfloor} = \varphi \blacktriangle \tilde{\tilde{G}}\varepsilon$.

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{G}\varepsilon} & \tilde{G} \operatorname{Cut}^{\lfloor n,0\rfloor} \operatorname{Cosk}^{\lfloor n,0\rfloor} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi \operatorname{Cut}^{\lfloor n,0\rfloor} \operatorname{Cosk}^{\lfloor n,0\rfloor}} \\ \tilde{\tilde{G}} & \xrightarrow[\tilde{\tilde{G}}\varepsilon]{} & \tilde{\tilde{G}} \operatorname{Cut}^{\lfloor n,0\rfloor} \operatorname{Cosk}^{\lfloor n,0\rfloor} \end{array}$$

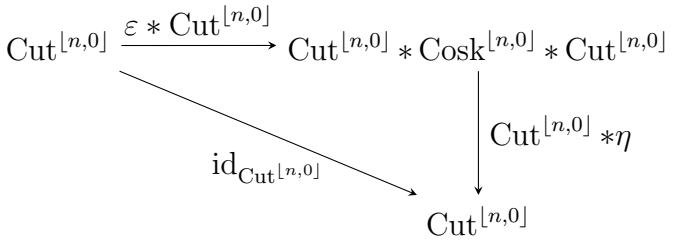
So ε is a transformation. □

Lemma 88 (The adjoint functors $\operatorname{Cut}^{\lfloor n,0\rfloor}$ and $\operatorname{Cosk}^{\lfloor n,0\rfloor}$)
We consider the transformations

$$\begin{array}{lrcll} \eta\colon & \operatorname{Cosk}^{\lfloor n,0\rfloor} * \operatorname{Cut}^{\lfloor n,0\rfloor} & \to & \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp}; & \text{cf. Lemma 86.(3)} \\ \varepsilon\colon & \mathrm{id}_{\mathcal{S}impGrp} & \to & \operatorname{Cut}^{\lfloor n,0\rfloor} * \operatorname{Cosk}^{\lfloor n,0\rfloor}; & \text{cf. Lemma 87.(3).} \end{array}$$

Then $(\operatorname{Cut}^{\lfloor n,0\rfloor}, \operatorname{Cosk}^{\lfloor n,0\rfloor}, \varepsilon, \eta)$ *is an adjunction; cf.* §1.1.3 *item* 10.
In particular, we have $\operatorname{Cut}^{\lfloor n,0\rfloor} \dashv \operatorname{Cosk}^{\lfloor n,0\rfloor}$.

Proof. We show that $(\varepsilon * \operatorname{Cut}^{\lfloor n,0\rfloor}) \blacktriangle (\operatorname{Cut}^{\lfloor n,0\rfloor} * \eta) \overset{!}{=} \mathrm{id}_{\operatorname{Cut}^{\lfloor n,0\rfloor}}$.



Suppose given $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$.

We show that $G\big((\varepsilon * \mathrm{Cut}^{\lfloor n,0\rfloor}) \vartriangle (\mathrm{Cut}^{\lfloor n,0\rfloor} * \eta)\big) \overset{!}{=} \mathrm{id}_{G\,\mathrm{Cut}^{\lfloor n,0\rfloor}}$.

Suppose given $k \in [0,n]$. Suppose given $x \in G_k$.

We have

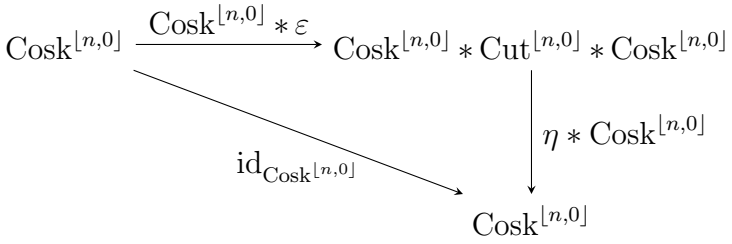
$$\begin{aligned}
& x\Big(G\big((\varepsilon * \mathrm{Cut}^{\lfloor n,0\rfloor}) \vartriangle (\mathrm{Cut}^{\lfloor n,0\rfloor} * \eta)\big)\Big)_k \\
= \quad & x\big(G(\varepsilon * \mathrm{Cut}^{\lfloor n,0\rfloor}) \vartriangle G(\mathrm{Cut}^{\lfloor n,0\rfloor} * \eta)\big)_k \\
= \quad & x\Big(\big((G\varepsilon)\,\mathrm{Cut}^{\lfloor n,0\rfloor}\big)_k \vartriangle \big((G\,\mathrm{Cut}^{\lfloor n,0\rfloor})\eta\big)_k\Big) \\
\overset{87.(2,1)}{=} \quad & \big((xG_a)_{a\in\Delta([n],[k])}\big)\big((G\,\mathrm{Cut}^{\lfloor n,0\rfloor})\eta\big)_k \\
\overset{86.(3,1)}{=} \quad & x \\
= \quad & x\,\mathrm{id}_{G_k} \\
= \quad & x(\mathrm{id}_{G\,\mathrm{Cut}^{\lfloor n,0\rfloor}})_k\,.
\end{aligned}$$

This shows $\Big(G\big((\varepsilon * \mathrm{Cut}^{\lfloor n,0\rfloor}) \vartriangle (\mathrm{Cut}^{\lfloor n,0\rfloor} * \eta)\big)\Big)_k = (\mathrm{id}_{G\,\mathrm{Cut}^{\lfloor n,0\rfloor}})_k$.

So we have $G\big((\varepsilon * \mathrm{Cut}^{\lfloor n,0\rfloor}) \vartriangle (\mathrm{Cut}^{\lfloor n,0\rfloor} * \eta)\big) = \mathrm{id}_{G\,\mathrm{Cut}^{\lfloor n,0\rfloor}}$.

Therefore, we have $(\varepsilon * \mathrm{Cut}^{\lfloor n,0\rfloor}) \vartriangle (\mathrm{Cut}^{\lfloor n,0\rfloor} * \eta) = \mathrm{id}_{\mathrm{Cut}^{\lfloor n,0\rfloor}}$.

We show that $(\mathrm{Cosk}^{\lfloor n,0\rfloor} * \varepsilon) \vartriangle (\eta * \mathrm{Cosk}^{\lfloor n,0\rfloor}) \overset{!}{=} \mathrm{id}_{\mathrm{Cosk}^{\lfloor n,0\rfloor}}$.



Suppose given $G \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{S}impGrp\big)$.

We show that $G\big((\mathrm{Cosk}^{\lfloor n,0\rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,0\rfloor})\big) \overset{!}{=} \mathrm{id}_{G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $(x_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$; cf. Lemma 81.(1).

For $c \in {}_\Delta([n],[n])$ and $b \in {}_\Delta([n],[k])$, note that we have

$$c \blacktriangle b = \mathrm{id}_{[n]} \blacktriangle (c \blacktriangle b)\,.$$

$$\begin{array}{ccc} [n] & \xrightarrow{\;b\;} & [k] \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle c\blacktriangle b} \\ [n] & \xrightarrow[\mathrm{id}_{[n]}]{} & [n] \end{array}$$

So, since $(x_a)_{a\in\Delta([n],[k])}$ is coherent, we have

$$x_b\, G_c = x_{c\blacktriangle b}\, G_{\mathrm{id}_{[n]}} = x_{c\blacktriangle b}$$

for $c \in {}_\Delta([n],[n])$ and $b \in {}_\Delta([n],[k])$; cf. Definition 80.

We have

$$\begin{aligned}
& \big((x_a)_{a\in\Delta([n],[k])}\big)\Big(G\big((\mathrm{Cosk}^{\lfloor n,0\rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,0\rfloor})\big)\Big)_k \\
= \quad & \big((x_a)_{a\in\Delta([n],[k])}\big)\big(G(\mathrm{Cosk}^{\lfloor n,0\rfloor} * \varepsilon) \blacktriangle G(\eta * \mathrm{Cosk}^{\lfloor n,0\rfloor})\big)_k \\
= \quad & \big((x_a)_{a\in\Delta([n],[k])}\big)\Big(\big((G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\varepsilon\big)_k \blacktriangle \big((G\eta)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k\Big) \\
\overset{87.(1)}{=} \quad & \Big(\Big(\big((x_a)_{a\in\Delta([n],[k])}\big)(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_b\Big)_{b\in\Delta([n],[k])}\Big)\big((G\eta)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \\
\overset{81.(2)}{=} \quad & \Big(\big((x_{c\blacktriangle b})_{c\in\Delta([n],[n])}\big)_{b\in\Delta([n],[k])}\Big)\big((G\eta)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \\
= \quad & \Big(\big((x_b\, G_c)_{c\in\Delta([n],[n])}\big)_{b\in\Delta([n],[k])}\Big)\big((G\eta)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \\
\overset{82}{=} \quad & \Big(\big((x_b\, G_c)_{c\in\Delta([n],[n])}\big)(G\eta)_n\Big)_{b\in\Delta([n],[k])} \\
\overset{86.(3,1)}{=} \quad & (x_b)_{b\in\Delta([n],[k])} \\
= \quad & \big((x_a)_{a\in\Delta([n],[k])}\big)\,\mathrm{id}_{(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k}\,.
\end{aligned}$$

This shows $\Big(G\big((\mathrm{Cosk}^{\lfloor n,0\rfloor} * \varepsilon) \mathbin{\blacktriangle} (\eta * \mathrm{Cosk}^{\lfloor n,0\rfloor})\big)\Big)_k = \mathrm{id}_{(G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k}$.

So we have $G\big((\mathrm{Cosk}^{\lfloor n,0\rfloor} * \varepsilon) \mathbin{\blacktriangle} (\eta * \mathrm{Cosk}^{\lfloor n,0\rfloor})\big) = \mathrm{id}_{G\,\mathrm{Cosk}^{\lfloor n,0\rfloor}}$.

Therefore, we have $(\mathrm{Cosk}^{\lfloor n,0\rfloor} * \varepsilon) \mathbin{\blacktriangle} (\eta * \mathrm{Cosk}^{\lfloor n,0\rfloor}) = \mathrm{id}_{\mathrm{Cosk}^{\lfloor n,0\rfloor}}$.

Altogether, this proves the lemma. □

Remark 89 We consider the adjunction $(\mathrm{Cut}^{\lfloor n,0\rfloor}, \mathrm{Cosk}^{\lfloor n,0\rfloor}, \varepsilon, \eta)$; cf. Lemma 88.

For $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ and $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$, we have the adjunction bijection

$$\begin{array}{rcl} \alpha_{G,H}\colon\ {}_{\mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}(G\,\mathrm{Cut}^{\lfloor n,0\rfloor}, H) & \tilde{\leftrightarrow} & {}_{\mathcal{SimpGrp}}(G, H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}) \\ \big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{\varphi} H\big) & \mapsto & \big(G \xrightarrow{G\varepsilon \mathbin{\blacktriangle} \varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}} H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big) \\ \big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{\psi\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} H\eta} H\big) & \leftarrow\!\shortmid & \big(G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)\,; \end{array}$$

cf. Remark 14.

Suppose given $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ and suppose given $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$.

(1) Suppose given a pre-$\lfloor n,0\rfloor$-simplicial group morphism $G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{\varphi} H$.
Then we have the simplicial group morphism

$$\varphi\alpha_{G,H} = G\varepsilon \mathbin{\blacktriangle} \varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\colon G \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,,$$

where, for $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$(\varphi\alpha_{G,H})_k = (G\varepsilon)_k \mathbin{\blacktriangle} (\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\colon G_k \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,,\ x \mapsto \big((xG_a)\varphi_n\big)_{a\in\Delta([n],[k])}\,.$$

Cf. Lemmas 87.(1), 82.

(2) Suppose given a simplicial group morphism $G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$.
Then we have the pre-$\lfloor n,0\rfloor$-simplicial group morphism

$$\psi\alpha^{-}_{G,H} = \psi\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} H\eta\colon G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to H\,,$$

where, for $k \in [0,n]$, we have

$$(\psi\alpha^{-}_{G,H})_k = (\psi\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k \mathbin{\blacktriangle} (H\eta)_k\colon (G\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k \to H_k\,,\ x \mapsto (x\psi_k)(H\eta)_k =: \tilde{x}\,,$$

where $x\psi_k = (\tilde{x}H_a)_{a\in\Delta([n],[k])}$. Cf. Lemma 86.(3, 1).

Chapter 5

$\lfloor n, 0 \rfloor$-coskeletal simplicial groups

Suppose given a simplicial group G that satisfies the technical condition of being $\lfloor n, 0 \rfloor$-coskeletal; cf. Definition 90 below.

In this chapter, we shall construct the degenerate subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$ that is needed for the functor $\mathrm{Cond}^{\lfloor n,0 \rfloor}$ from $\lfloor n, 0 \rfloor$-simplicial groups to simplicial groups; cf. Lemma 126 below.

5.1 $\lfloor n, 0 \rfloor$-coskeletal simplicial groups

Definition 90 ($\lfloor n, 0 \rfloor$-coskeletal simplicial groups)

Let $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given simplicial group G.

We say that G is $\lfloor n, 0 \rfloor$-*coskeletal* if the properties $(1, 2)$ hold.

(1) We have $G\mathrm{NZ}_{n+1} = 1$; cf. Definition 46.(3).

(2) For I, $J \subseteq [0, n]$ with I, $J \neq \emptyset$ and $I \cup J = [0, n]$, we have

$$[\bigcap_{i \in I} G_{n,i} \, , \bigcap_{j \in J} G_{n,j}] = 1 \, ;$$

cf. also property (1) in Definition 74.

Property (2) is also called the *Conduché condition* on $\lfloor n, 0 \rfloor$-coskeletal simplicial groups; cf. [**7,** Thm. 1.5].

Remark 91 *Let $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given an $\lfloor n,0\rfloor$-simplicial group H; cf. Definition 74. We consider the simplicial group $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$; cf. Lemma 81.(3). Then $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ is $\lfloor n,0\rfloor$-coskeletal.*

Proof. We show the properties $(1,2)$ from Definition 90.
Ad (1). Suppose given $x := (x_a)_{a\in{}_{\Delta}([n],[n+1])} \in (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{NZ}_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1}$.
Then, for $i \in [0,n+1]$, we have

$$(1)_{b\in{}_{\Delta}([n],[n])} = 1 = x\mathrm{d}_i^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor},n+1} = \big((x_a)_{a\in{}_{\Delta}([n],[n+1])}\big)(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{\delta_i} \overset{81.(2)}{=} (x_{b\blacktriangle\delta_i})_{b\in{}_{\Delta}([n],[n])}\,.$$

So we have

$$1 = x_{b\blacktriangle\delta_i}$$

for $b \in {}_{\Delta}([n],[n])$ and $i \in [0,n+1]$.
Suppose given $a \in {}_{\Delta}([n],[n+1])$.
By Remark 84, there exists some $b \in {}_{\Delta}([n],[n])$ and some $k \in [0,n+1]$ with $a = b \blacktriangle \delta_k^{n+1}$.
Therefore, we have

$$x_a = x_{b\blacktriangle\delta_k} = 1\,.$$

This shows

$$x = (x_a)_{a\in{}_{\Delta}([n],[n+1])} = (1)_{a\in{}_{\Delta}([n],[n+1])} = 1\,.$$

So $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{NZ}_{n+1} = 1$.
This shows (1).
Ad (2). Suppose given $I,\ J \subseteq [0,n]$ with $I,\ J = \emptyset$ and $I \cup J = [0,n]$.
We have to show that

$$\big[\bigcap_{i\in I}(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n,i}\,,\bigcap_{j\in J}(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n,j}\big] \overset{!}{=} 1\,.$$

We consider pre-$\lfloor n,0\rfloor$-simplicial group $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}$; cf. Remark 79.
Recall that we have

$$(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor})_n = (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_n\,;$$

cf. Lemma 86.

Moreover, we consider the pre-$\lfloor n,0\rfloor$-simplicial group isomorphism

$$H\eta = (H\vartheta)^- \colon H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{\sim} H\,;$$

cf. Lemma 86.(3, 1, 2).

In particular, we have the group isomorphism

$$(H\eta)_n \colon (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_n \to H_n\,.$$

Suppose given $x \in \bigcap\limits_{i\in I}(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n,i}$ and $y \in \bigcap\limits_{j\in J}(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n,j}$.

We show that $[x,y] \overset{!}{=} 1$. Suppose given $i \in I$. We have

$$\big(x(H\eta_n)\big)\mathrm{d}_i^{H,n} \overset{55}{=} (x\mathrm{d}_i^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor},n})(H\eta)_{n-1} = 1(H\eta)_{n-1} = 1\,.$$

This shows $x(H\eta)_n \in H_{n,i}$ for $i \in I$.

Similarly, we have $y(H\eta)_n \in H_{n,j}$ for $j \in J$.

So

$$[x,y](H\eta)_n = \big[x(H\eta)_n\,, y(H\eta)_n\big] \overset{74.(1)}{=} 1\,.$$

Therefore, we have

$$[x,y] = 1\big((H\eta)_n\big)^- = 1\,.$$

This shows (2). □

5.2 The linear order on $\mathcal{P}\big([0,n]\big)$

Lemma 92 *Suppose given $n \in \mathbb{Z}_{\geqslant -1}$.*

For I, $J \in \mathcal{P}\big([0,n]\big)$, we let

$$I \leqslant J \;:\Leftrightarrow\; \max(I\backslash J) \leqslant \max(J\backslash I)\,,$$

where $\max(\emptyset) := -1$.

Then we have a linear order $(\leqslant)$ on $\mathcal{P}\big([0,n]\big)$.

Moreover, we have

$$I < J \;\Leftrightarrow\; \max(I\backslash J) < \max(J\backslash I)\,.$$

This order ($\leqslant$) on $\mathcal{P}\big([0,n]\big)$ can be explained by attaching a binary number to each subset I of $[0,n]$ by picking the digit 1 in position i if i is contained in I and by ordering the resulting binary numbers the usual way. Cf. Example 93.(3) below.

Proof. Reflexivity. Suppose given $I \in \mathcal{P}\big([0,n]\big)$. We have

$$\max(I \backslash I) = \max(\emptyset) = \max(I \backslash I)\,.$$

This shows $I \leqslant I$.

Identitivity. Suppose given $I,\ J \in \mathcal{P}\big([0,n]\big)$ with $I \leqslant J$ and $J \leqslant I$.

We *assume* that $I \neq J$. Then $I \backslash J \neq \emptyset$ or $J \backslash I \neq \emptyset$.

We may assume that $I \backslash J \neq \emptyset$.

We write $x := \max(I \backslash J)$ and $y := \max(J \backslash I)$. Then we have

$$x \leqslant y \quad \text{and} \quad y \leqslant x\,.$$

So $x \leqslant y \leqslant x$. This shows $x = y$.

Since $I \backslash J \neq \emptyset$, we have $x \in I \backslash J$. So $x \geqslant 0$.

Moreover, we have $y = x \geqslant 0$. So $y \neq -1$. Hence $J \backslash I \neq \emptyset$ and $y \in J \backslash I$.

But we have $x \in I \backslash J$ and $x = y \in J$, which is a *contradiction.*

This shows $I = J$.

Transitivity. First, we show the following assertion (i).

(i) Suppose given $I,\ J \in \mathcal{P}\big([0,n]\big)$.

Then we have

$$I \leqslant J \ \Rightarrow\ \max I \leqslant \max J\,.$$

Ad (i). Suppose given $I,\ J \in \mathcal{P}\big([0,n]\big)$ with $I \leqslant J$.

We *assume* that $x := \max I > \max J$. Then we have $x \neq -1$ and thus $x \in I \backslash J$.

So

$$x = \max(I \backslash J) \leqslant \max(J \backslash I) \leqslant \max J\,,$$

which *contradicts* $x > \max J$.

So we have $\max I \leqslant \max J$. This shows (i).

Suppose given I, J, $K \in \mathcal{P}\big([0,n]\big)$ with $I \leqslant J$ and $J \leqslant K$.

Then we have

$$\max(I\backslash J) \leqslant \max(J\backslash I) \quad \text{and} \quad \max(J\backslash K) \leqslant \max(K\backslash J)\,.$$

Moreover, by (i), we have $\max I \leqslant \max J \leqslant \max K$.

We have to show that $I \overset{!}{\leqslant} K$, i.e. we have to show that $\max(I\backslash K) \overset{!}{\leqslant} \max(K\backslash I)$.

We show this claim by induction on $|I|$.

Let $|I| = 0$. Then we have $I = \emptyset$, $I\backslash K = \emptyset$ and $K\backslash I = K$.

We have

$$\max(I\backslash K) = \max(\emptyset) = -1 \leqslant \max K = \max(K\backslash I)\,.$$

Let $|I| \geqslant 1$. By induction, we have the following assertion (IH).

(IH) Suppose given $\tilde{I} \in \mathcal{P}\big([0,n]\big)$ with $|\tilde{I}| < |I|$. For $\tilde{J}$, $\tilde{K} \in \mathcal{P}\big([0,n]\big)$ with $\tilde{I} \leqslant \tilde{J}$ and $\tilde{J} \leqslant \tilde{K}$, we have $\tilde{I} \leqslant \tilde{K}$.

We have to show that $I \overset{!}{\leqslant} K$.

Case $\max I < \max K$. Then we have $\max(K) \in K\backslash I$ and thus $\max(K\backslash I) = \max K$.

So

$$\max(I\backslash K) \leqslant \max I < \max K = \max(K\backslash I)\,.$$

This shows $I \leqslant K$.

Case $\max I = \max K$. Then, by (i), we have $m := \max I = \max J = \max K$ with $m \geqslant 0$ and $m \in I \cap J \cap K$.

Let $\tilde{I} := I\backslash\{m\}$, $\tilde{J} := J\backslash\{m\}$ and let $\tilde{K} := K\backslash\{m\}$. Then $|\tilde{I}| < |I|$.

Note that $\tilde{I}\backslash\tilde{J} = I\backslash J$ and that $\tilde{J}\backslash\tilde{I} = J\backslash I$.

We have

$$\max(\tilde{I}\backslash\tilde{J}) = \max(I\backslash J) \leqslant \max(J\backslash I) = \max(\tilde{J}\backslash\tilde{I})\,.$$

This shows $\tilde{I} \leqslant \tilde{J}$.

Note that $\tilde{J}\backslash\tilde{K} = J\backslash K$ and that $\tilde{K}\backslash\tilde{J} = K\backslash J$.

We have

$$\max(\tilde{J}\backslash\tilde{K}) = \max(J\backslash K) \leqslant \max(K\backslash J) = \max(\tilde{K}\backslash\tilde{J})\,.$$

This shows $\tilde{J} \leqslant \tilde{K}$.

Then, by (IH), we have $\tilde{I} \leqslant \tilde{K}$. So $\max(\tilde{I} \backslash \tilde{K}) \leqslant \max(\tilde{K} \backslash \tilde{I})$.

Moreover, note that $\tilde{I} \backslash \tilde{K} = I \backslash K$ and that $\tilde{K} \backslash \tilde{I} = K \backslash I$.

Therefore, we have

$$\max(I \backslash K) = \max(\tilde{I} \backslash \tilde{K}) \leqslant \max(\tilde{K} \backslash \tilde{I}) = \max(K \backslash I)\,.$$

This shows $I \leqslant K$.

So $(\leqslant)$ is a linear order.

We show that $I < J \overset{!}{\Leftrightarrow} \max(I \backslash J) < \max(J \backslash I)\,.$

It suffices to show that $I = J \overset{!}{\Leftrightarrow} \max(I \backslash J) = \max(J \backslash I)\,.$

Ad $\Rightarrow$. We have $I \backslash J = \emptyset = J \backslash I$. So $\max(I \backslash J) = -1 = \max(J \backslash I)$.

Ad $\Leftarrow$. Since $\max(I \backslash J) \leqslant \max(J \backslash I)$ and $\max(J \backslash I) \leqslant \max(I \backslash J)$, we have $I \leqslant J$ and $J \leqslant I$.

By the identitivity of $(\leqslant)$, it follows that $I = J$. □

Example 93

(1) Suppose given $n \in \mathbb{Z}_{\geqslant -1}$. For $I \in \mathcal{P}\big([0, n]\big)$, we have

$$\emptyset \leqslant I \leqslant [0, n]\,.$$

(2) Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. For $I \in \mathcal{P}\left([0, n]\right) \backslash \{\emptyset\}$, we have

$$\emptyset < \{0\} \leqslant I \leqslant [0, n]\,.$$

(3) In $\mathcal{P}\big([0, 2]\big)$, we have

$$\emptyset < \{0\} < \{1\} < \{0, 1\} < \{2\} < \{0, 2\} < \{1, 2\} < \{0, 1, 2\}\,.$$

Remark 94 *Suppose given* $n \in \mathbb{Z}_{\geqslant 0}$. *We consider the set* $\mathcal{P}\left([0, n]\right)$.
Suppose given I, $J \in \mathcal{P}\left([0, n]\right)$.

(1) *We have*

$$\max I < \max J \;\Rightarrow\; I < J\,.$$

(2) *We have*

$$I \leqslant J \;\Rightarrow\; \max I \leqslant \max J\,.$$

(3) *Suppose given* $K \subseteq [0,n]\backslash(I \cup J)$. *We have*

$$I < J \;\Leftrightarrow\; I \mathbin{\dot\cup} K < J \mathbin{\dot\cup} K\,.$$

(4) *Suppose given* $K \subseteq I \cap J$. *We have*

$$I < J \;\Leftrightarrow\; I\backslash K < J\backslash K\,.$$

(5) *Let* $k \in \mathbb{Z}_{\geqslant n}$. *Suppose given an injective monotone map* $u\colon [0,n] \to [0,k]$. *We have*

$$I < J \;\Leftrightarrow\; Iu < Ju\,.$$

Proof. Ad (1). Suppose that we have $\max I < \max J$. Then we have $\max J \notin I$. So $\max(J\backslash I) = \max J$. Therefore, we have

$$\max(I\backslash J) \leqslant \max I < \max J = \max(J\backslash I)\,.$$

This shows that $I < J$.

Ad (2). We *assume* that $\max I > \max J$. Then, by (1), we have $I > J$, which *contradicts* $I \leqslant J$. So $\max I \leqslant \max J$.

Cf. also assertion (i) in the proof of Lemma 92.

Ad (3). We have

$$I\backslash J = (I \mathbin{\dot\cup} K)\backslash(J \mathbin{\dot\cup} K) \quad \text{and} \quad J\backslash I = (J \mathbin{\dot\cup} K)\backslash(I \mathbin{\dot\cup} K)\,.$$

So we have

$$\max(I\backslash J) < \max(J\backslash I) \quad\Leftrightarrow\quad \max\big((I \mathbin{\dot\cup} K)\backslash(J \mathbin{\dot\cup} K)\big) < \max\big((J \mathbin{\dot\cup} K)\backslash(I \mathbin{\dot\cup} K)\big)\,.$$

This shows that $I < J \;\Leftrightarrow\; I \mathbin{\dot\cup} K < J \mathbin{\dot\cup} K$.

Ad (4). We have

$$I\backslash J = (I\backslash K)\backslash(J\backslash K) \quad \text{and} \quad J\backslash I = (J\backslash K)\backslash(I\backslash K)\,.$$

So we have

$$\max(I\backslash J) < \max(J\backslash I) \quad\Leftrightarrow\quad \max\big((I\backslash K)\backslash(J\backslash K)\big) < \max\big((J\backslash K)\backslash(I\backslash K)\big)\,.$$

This shows that $I < J \Leftrightarrow I\backslash K < J\backslash K$.

Ad (5). Since u is injective and monotone, we have

$$\max(I\backslash J) < \max(J\backslash I) \quad \Leftrightarrow \quad \big(\max(I\backslash J)\big)u < \big(\max(J\backslash I)\big)u$$

and we have

$$\big(\max(I\backslash J)\big)u = \max\big((I\backslash J)u\big) = \max(Iu\backslash Ju)$$

and

$$\big(\max(J\backslash I)\big)u = \max\big((J\backslash I)u\big) = \max(Ju\backslash Iu)\,.$$

Therefore, we have

$$\max(I\backslash J) < \max(J\backslash I) \quad \Leftrightarrow \quad \max(Iu\backslash Ju) < \max(Ju\backslash Iu)\,.$$

This shows that $I < J \Leftrightarrow Iu < Ju$. □

5.3 The decomposition of G_n in subgroups $(G\mathrm{N}_k)s_I$

Definition 95 Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given a simplicial group G; cf. Definition 30.

Suppose given $I \subseteq [0, n]$.

We write

$$G_{n;I} := \bigcap_{i \in I} G_{n,i} \trianglelefteq G_n\,;$$

cf. Definition 46.(1).

Remark 96 Suppose given a simplicial group G; cf. Definition 30.

Let $n \in \mathbb{Z}_{\geqslant 0}$. For $k \in [0, n]$, we have

$$G_{n;\{k\}} = G_{n,k}\,;$$

cf. Definition 46.(1).

We have

$$G_{n;\emptyset} = G_n\,;$$

cf. § 1.1.1 item 2.

Moreover, we have

$$G_{n;[1,n]} = G\mathrm{N}_n \quad \text{and} \quad G_{n;[0,n]} = G\mathrm{NZ}_n\,;$$

cf. Definition 46.(2, 3).

Remark 97 *Suppose given a simplicial group G; cf. Definition* 30.
Recall that $G_0 = G\mathrm{N}_0$; cf. Definition 46.(2).
Let $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given $k \in [2, n+1]$.

(1) *We have the surjective map*

$$p^G_{n,k} \colon G_{n;[k,n]} \to G_{n;[k-1,n]}\,,\ g \mapsto gp^G_{n,k} := g^- \mathrm{d}^{G,n}_{k-1}\,\mathrm{s}^{G,n-1}_{k-2} \cdot g\,.$$

(2) *For $g \in G_{n;[k,n]}$, we have*

$$g = (g\mathrm{d}^{G,n}_{k-1})\mathrm{s}^{G,n-1}_{k-2} \cdot (g^- \mathrm{d}^{G,n}_{k-1}\,\mathrm{s}^{G,n-1}_{k-2} \cdot g)\,,$$

where $g\mathrm{d}^{G,n}_{k-1} \in G_{n-1;[k-1,n-1]}$ and where $g^- \mathrm{d}^{G,n}_{k-1}\,\mathrm{s}^{G,n-1}_{k-2} \cdot g = gp^G_{n,k} \in G_{n;[k-1,n]}$.
In particular, we have

$$G_{n;[k,n]} = \big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}^{G,n-1}_{k-2} \cdot G_{n;[k-1,n]}\,.$$

(3) *We have*

$$\big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}^{G,n-1}_{k-2} \ \cap\ G_{n;[k-1,n]} = 1\,.$$

(4) *For $g \in G_{n;[k,n]}$ there exists a unique $y \in G_{n-1;[k-1,n-1]}$ and a unique $z \in G_{n;[k-1,n]}$ such that*

$$g = y\mathrm{s}^{G,n-1}_{k-2} \cdot z\,.$$

Moreover, we have $z = gp^G_{n,k}$.

Proof. Ad (1). *We show that $p^G_{n,k}$ is a well-defined map.*
Suppose given $g \in G_{n;[k,n]}$. Suppose given $j \in [k-1, n]$.
If $j = k-1$, then we have

$$(g^- \mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g)\mathrm{d}_{k-1} = g^- \mathrm{d}_{k-1} \cdot g\mathrm{d}_{k-1} = 1\,.$$

If $j \in [k, n]$, then we have

$$(g^-\,\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g)\mathrm{d}_j = g^- \mathrm{d}_{k-1}\mathrm{d}_{j-1}\mathrm{s}_{k-2} \cdot g\mathrm{d}_j = g^-\,\mathrm{d}_j\,\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g\mathrm{d}_j = 1\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot 1 = 1\,.$$

This shows that $gp^G_{n,k} = g^-\,\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g \in G_{n;[k-1,n]}$.
So $p^G_{n,k}$ is a well-defined map.

We show that $p^G_{n,k}$ is surjective.

Suppose given $x \in G_{n;[k-1,n]} \leqslant G_{n;[k,n]}$.

We have

$$xp^G_{n,k} = x\mathrm{d}_{k-1}\mathrm{s}_{k-2} \cdot x = 1\mathrm{s}_{k-2} \cdot x = x\,.$$

So $p^G_{n,k}$ is surjective.

Ad (2). Ad $\subseteq$. Suppose given $g \in G_{n;[k,n]}$.

We have

$$g = (g\mathrm{d}_{k-1}\,\mathrm{s}_{k-2}) \cdot (g^-\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g) \overset{!}{\in} \big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2} \cdot G_{n;[k-1,n]}\,.$$

By (1), we have $g^-\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g \in G_{n;[k-1,n]}$.

It remains to show that $g\mathrm{d}_{k-1} \overset{!}{\in} G_{n-1;[k-1,n-1]}$.

Suppose given $j \in [k-1, n-1]$. Note that $j+1 \in [k,n]$.

We have

$$(g\,\mathrm{d}_{k-1})\mathrm{d}_j = g\,\mathrm{d}_{j+1}\,\mathrm{d}_{k-1} = 1\mathrm{d}_{k-1} = 1\,.$$

This shows that $g\,\mathrm{d}_{k-1} \in G_{n-1;[k-1,n]}$.

Ad $\supseteq$. It suffices to show that $G_{n;[k,n]} \overset{!}{\supseteq} \big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2}$.

Suppose given $x \in G_{n-1;[k-1,n-1]}$. For $j \in [k,n]$, we have

$$x\mathrm{s}_{k-1}\,\mathrm{d}_j = x\mathrm{d}_{j-1}\,\mathrm{s}_{k-1} = 1\mathrm{s}_{k-1} = 1\,.$$

This shows that $x\mathrm{s}_{k-1} \in G_{n;[k,n]}$. So we have $G_{n;[k,n]} \supseteq \big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2}$.

Ad (3). Suppose given $x \in \big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2} \cap G_{n;[k-1,n]}$.

Then we have $x = y\mathrm{s}_{k-2}$ for some chosen $y \in G_{n-1;[k-1,n-1]}$.

We have

$$1 = x\mathrm{d}_{k-1} = y\mathrm{s}_{k-2}\,\mathrm{d}_{k-1} = y\,.$$

So

$$x = y\mathrm{s}_{k-2} = 1\mathrm{s}_{k-2} = 1\,.$$

This shows $\big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2} \cap G_{n;[k-1,n]} = 1$.

Ad (4). Suppose given $g \in G_{n;[k,n]}$.

We write $Y := G_{n-1;[k-1,n-1]}$ and we write $Z := G_{n;[k-1,n]}$.
By (2), we have $g = y\mathrm{s}_{k-2} \cdot z$, where $y := g\mathrm{d}_{k-1} \in Y$ and $z := g^{-}\mathrm{d}_{k-1}\mathrm{s}_{k-2} \cdot g = gp^{G}_{n,k} \in Z$.
It remains to show uniqueness.
Suppose given $\tilde{y} \in Y$ and $\tilde{z} \in Z$ such that $g = \tilde{y}\mathrm{s}_{k-2} \cdot \tilde{z}$.
We claim that $y \overset{!}{=} \tilde{y}$ and that $z \overset{!}{=} \tilde{z}$.
We have

$$y\mathrm{s}_{k-2} \cdot z = g = \tilde{y}\mathrm{s}_{k-2} \cdot \tilde{z}\,.$$

So we have

$$\underbrace{(\tilde{y}^{-} \cdot y)}_{\in Y}\mathrm{s}_{k-2} = \underbrace{\tilde{z} \cdot z^{-}}_{\in Z} \in (Y\mathrm{s}_{k-2}) \cap Z\,.$$

By (3), we have $(Y\mathrm{s}_{k-2}) \cap Z = 1$. Therefore, we have $y\mathrm{s}_{k-2} = \tilde{y}\mathrm{s}_{k-2}$ and $z = \tilde{z}$.
Hence also $y = y\mathrm{s}_{k-2}\,\mathrm{d}_{k-2} = \tilde{y}\mathrm{s}_{k-2}\,\mathrm{d}_{k-2} = \tilde{y}$.
This shows the claim. □

Example 98 Suppose given a simplicial group G; cf. Definition 30.
Let $n \in \mathbb{Z}_{\geqslant 1}$. We give some examples for Remark 97.

(1) Let $k = n + 1$. We have

$$G_{n;[n+1,n]} = G_{n;\emptyset} = G_n = G_{n-1}\,\mathrm{s}_{n-1} \cdot G_{n,n}\,.$$

(2) Suppose that $n \geqslant 2$. Let $k = n$. We have

$$G_{n;[n,n]} = G_{n,n} = \big(G_{n-1,n-1}\big)\mathrm{s}_{n-2} \cdot G_{n;[n-1,n]}\,.$$

(3) Let $k = 2$. We have

$$G_{n;[2,n]} = \big(G_{n-1;[1,n-1]}\big)\mathrm{s}_0 \cdot G_{n;[1,n]} = (G\mathrm{N}_{n-1})\,\mathrm{s}_0 \cdot G\mathrm{N}_n\,;$$

cf. Definition 46.(2).

Remark 99 *Suppose given a simplicial group G; cf. Definition* 30.

(1) *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. We consider the set $\mathcal{P}\big([0,n]\big)$. Let $r := \big|\mathcal{P}([0,n])\big| = 2^{n+1}$. We write $\mathcal{P}([0,n]) =: \{I_1, I_2, \ldots, I_r\}$, where $I_1 < I_2 < \ldots < I_r$.*

We have

$$\begin{aligned}\mathcal{P}\big([0,n]\big) &= \big\{I\in\mathcal{P}\big([0,n]\big)\colon n\in I\big\}\ \dot\cup\ \big\{I\in\mathcal{P}\big([0,n]\big)\colon n\notin I\big\}\\ &= \Big(\mathcal{P}\big([0,n]\big)\backslash\mathcal{P}\big([0,n-1]\big)\Big)\ \dot\cup\ \mathcal{P}\big([0,n-1]\big)\,.\end{aligned}$$

Moreover, for $J\in\mathcal{P}\big([0,n]\big)\backslash\mathcal{P}\big([0,n-1]\big)$ *and* $I\in\mathcal{P}\big([0,n-1]\big)$*, we have* $J>I$*; cf. Lemma* 92.

In particular, we have

$$\mathcal{P}\big([0,n-1]\big)=\{I_1\,,I_2\,,\ldots,I_t\}\quad\text{and}\quad\mathcal{P}\big([0,n]\big)\backslash\mathcal{P}\big([0,n-1]\big)=\{I_{t+1}\,,I_{t+2}\,,\ldots,I_r\}\,,$$

where $t:=\big|\mathcal{P}([0,n-1])\big|=2^n$*.*

(2) *Suppose given* $n\in\mathbb{Z}_{\geqslant 1}$*. Suppose given* $m\in[0,n-1]$*. We consider the set* $\mathcal{P}\big([0,m]\big)$*.*
Let $s:=\big|\mathcal{P}\big([0,m]\big)\big|=2^{m+1}$*.*
We write $\mathcal{P}\big([0,m]\big)=:\{I_1\,,I_2\,,\ldots\,,I_s\}$ *where* $I_1<I_2<\ldots<I_s$*; cf. Lemma* 92*.*
We consider the set $\mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big)$*.*
Then $\big|\mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big)\big|=s$*.*
We write $\mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big)=:\{J_1\,,J_2\,,\ldots\,,J_s\}$ *where* $J_1<J_2<\ldots<J_s$*.*
Then $J_t=I_t\,\dot\cup\,\{m+1\}$ *and* $|J_t|=|I_t|+1$ *for* $t\in[1,s]$*.*
We have

$$\Big(\prod_{i\in\lfloor s,1\rfloor}(G\mathrm{N}_{n-|I_i|-1})\mathrm{s}_{I_i}\Big)\,\mathrm{s}_{m+1}=\prod_{i\in\lfloor s,1\rfloor}(G\mathrm{N}_{n-|J_i|})\,\mathrm{s}_{J_i}\,;$$

cf. Definitions 46.(2)*,* 37.(2)*.*

Proof. Ad (1). Suppose given $I\in\mathcal{P}\big([0,n]\big)\backslash\mathcal{P}\big([0,n-1]\big)$ and $J\in\mathcal{P}\big([0,n-1]\big)$.
Then we have $n\in I$ and $n\notin J$.
So

$$\max(I\backslash J)=n>n-1\geqslant\max(J\backslash I)\,.$$

This shows $I>J$.
Ad (2). We have

$$\big|\mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big)\big|=2^{m+2}-2^{m+1}=2^{m+1}=s\,.$$

We show the following properties (i, ii) first.

(i) We have $\mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big) = \big\{I \,\dot\cup\, \{m+1\} \colon I \in \mathcal{P}\big([0,m]\big)\big\}$.

(ii) For $I\,,\, \tilde I \in \mathcal{P}\big([0,m]\big)$ with $I < \tilde I$, we have $I \,\dot\cup\, \{m+1\} < \tilde I \,\dot\cup\, \{m+1\}$.

Ad (i). Ad $\subseteq$. Suppose given $J \in \mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big)$.
Let $I := J\backslash\{m+1\} \in \mathcal{P}\big([0,m]\big)$. Then we have $J = I \,\dot\cup\, \{m+1\}$.
Ad $\supseteq$. Suppose given $I \in \mathcal{P}\big([0,m]\big)$.
Then we have $I \,\dot\cup\, \{m+1\} \in \big\{J \in \mathcal{P}\big([0,m+1]\big) \colon m+1 \in J\big\} = \mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big)$.
Ad (ii). By Remark 94.(3), we have

$$I_1 \,\dot\cup\, \{m+1\} < I_2 \,\dot\cup\, \{m+1\} < \ldots < I_s \,\dot\cup\, \{m+1\}\,,$$

and by (i), each element of $\mathcal{P}\big([0,m+1]\big)\backslash\mathcal{P}\big([0,m]\big)$ appears in this chain.
So we have $J_t = I_t \,\dot\cup\, \{m+1\}$ for $t \in [1,s]$.
In particular, this also shows $|J_t| = |I_t| + 1$ for $t \in [1,s]$.
We have

$$\Big(\prod_{i\in\lfloor s,1\rfloor} (G\mathrm{N}_{n-|I_i|-1})\mathrm{s}_{I_i}\Big)\,\mathrm{s}_{m+1} = \prod_{i\in\lfloor s,1\rfloor} (G\mathrm{N}_{n-(|I_i|+1)})(\mathrm{s}_{I_i} \blacktriangle \mathrm{s}_{m+1}) \overset{37.(2)}{=} \prod_{i\in\lfloor s,1\rfloor} (G\mathrm{N}_{n-(|I_i|+1)})\,\mathrm{s}_{I_i \,\dot\cup\, \{m+1\}}$$
$$= \prod_{i\in\lfloor s,1\rfloor} (G\mathrm{N}_{n-|J_i|})\,\mathrm{s}_{J_i}\,.$$

□

Lemma 100 *Suppose given a simplicial group G; cf. Definition* 30.
Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given $k \in [1,n+1]$.
Let $r := \big|\mathcal{P}\big([0,k-2]\big)\big| = 2^{k-1}$.
We write $\mathcal{P}\big([0,k-2]\big) =: \{I_1\,, I_2\,, \ldots\,, I_r\}$ where $I_1 < I_2 < \ldots < I_r$; cf. Lemma 92.

(1) *We have*

$$G_{n;[k,n]} = \prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{n-|I_i|})\,\mathrm{s}_{I_i}\,;$$

cf. Definition 37.(2).

(2) *Suppose given* $g \in G_{n;[k,n]}$. *Then there exist unique elements* $x_i \in G\mathrm{N}_{n-|I_i|}$ *for* $i \in [1,r]$ *such that*

$$g = \prod_{i\in\lfloor r,1\rfloor} x_i \,\mathrm{s}_{I_i} = x_r \,\mathrm{s}_{I_r} \cdot x_{r-1}\,\mathrm{s}_{I_{r-1}} \cdot \ldots \cdot x_1\,\mathrm{s}_{I_1}\,.$$

If $k \geqslant 2$, *then we also have*

$$g = \Big(\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i}\Big)\mathrm{s}_{k-2} \cdot \Big(\prod_{\lfloor t,1\rfloor} x_i \mathrm{s}_{I_i}\Big)\,,$$

where $t := \big|\mathcal{P}([0,k-3])\big| = 2^{k-2}$.

(3) *For* $j \in [2,k]$, *we consider the surjective map from Remark* 97.(1)

$$p^G_{n,j} \colon G_{n;[j,n]} \to G_{n;[j-1,n]}\,,\ g \mapsto g p^G_{n,j} = g^{-}\mathrm{d}_{j-1}\mathrm{s}_{j-2} \cdot g\,.$$

We have the surjective map

$$\pi^G_{n,k} := \prod_{j\in\lfloor k,2\rfloor}^{G_{n;[k,n]}\ {}^{G\mathrm{N}_n}} p^G_{n,j} = p^G_{n,k} \blacktriangle p^G_{n,k-1} \blacktriangle \ldots \blacktriangle p^G_{n,2} \,:\, G_{n;[k,n]} \to G\mathrm{N}_n\,.$$

Suppose given $g \in G_{n;[k,n]}$. *We consider its unique decomposition from* (2)*:*

$$g = \prod_{i\in\lfloor r,1\rfloor} x_i \,\mathrm{s}_{I_i} = x_r \,\mathrm{s}_{I_r} \cdot x_{r-1}\,\mathrm{s}_{I_{r-1}} \cdot \ldots \cdot x_1\,\mathrm{s}_{I_1}\,,$$

where $x_i \in G\mathrm{N}_{n-|I_i|}$ *for* $i \in [1,r]$.
We have $g\pi^G_{n,k} = x_1$.

Proof. Ad (1). We consider the product order $(\leqslant)$ on $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$; cf. §1.1.1 item 11.
We consider the subposet

$$P := \big\{(n,k) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \colon k \in [1,n+1]\big\} \subseteq \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}\,.$$

We show this claim by induction on P.
Suppose given $(n,k) \in P$. By the induction hypothesis, this claim holds for $(\tilde{n},\tilde{k})$ with $(\tilde{n},\tilde{k}) < (n,k)$.
Case $k = 1$. We have

$$\mathcal{P}\big([0,-1]\big) = \mathcal{P}\big(\emptyset\big) = \big\{\emptyset\big\} =: \{I_1\} \quad \text{and} \quad r = 1\,.$$

So we have

$$\prod_{i\in\lfloor 1,1\rfloor} (G\mathrm{N}_{n-|I_i|})\,\mathrm{s}_{I_i} = (G\mathrm{N}_{n-0})\mathrm{s}_\emptyset \overset{37.(2)}{=} (G\mathrm{N}_n)\,\mathrm{id}_{G_n} = G\mathrm{N}_n \overset{97.(2)}{=} G_{n,[1,n]}\,.$$

This shows the claim for $k = 1$.

Case $k \geqslant 2$. Since $k \in [1, n+1]$, we have $n \geqslant k-1 \geqslant 1$.

Recall that we write $\mathcal{P}\big([0,k-2]\big) = \{I_1\,, I_2\,, \ldots\,, I_r\}$ where $I_1 < I_2 < \ldots < I_r\,.$

Let $t := \big|\mathcal{P}\big([0,k-2]\big)\backslash\mathcal{P}\big([0,k-3]\big)\big| = 2^{k-2}$.

We write $\mathcal{P}\big([0,k-2]\big)\backslash\mathcal{P}\big([0,k-3]\big) =: \{J_1\,, J_2\,, \ldots\,, J_t\}$, where $J_1 < \ldots < J_t\,.$

We write $\mathcal{P}\big([0,k-3]\big) =: \{\tilde{J}_1\,, \tilde{J}_2\,, \ldots\,, \tilde{J}_t\}$, where $\tilde{J}_1 < \ldots < \tilde{J}_t\,.$

By Remark 99.(1), we have

$$\mathcal{P}\big([0,k-2]\big) = \Big(\mathcal{P}\big([0,k-2]\big)\backslash\mathcal{P}\big([0,k-3]\big)\Big) \;\cup\; \mathcal{P}\big([0,k-3]\big)$$

and $J > \tilde{J}$ for $J \in \mathcal{P}\big([0,k-2]\big)\backslash\mathcal{P}\big([0,k-3]\big)$ and $\tilde{J} \in \mathcal{P}\big([0,k-3]\big)\,.$

So we have

$$\tilde{J}_1 < \ldots < \tilde{J}_t < J_1 < \ldots < J_t\,,$$

and therefore $I_s = \tilde{J}_s$ and $I_{s+t} = J_s \overset{99.(2)}{=} \tilde{J}_s \,\dot\cup\, \{k-2\}$ for $s \in [1,t]$.

We show the following claims (i, ii).

(i) We have

$$\big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2} = \prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|J_i|})\,\mathrm{s}_{J_i}\,.$$

(ii) We have

$$G_{n;[k-1,n]} = \prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|\tilde{J}_i|})\,\mathrm{s}_{\tilde{J}_i}\,.$$

Ad (i). We consider the group $G_{n-1;[k-1,n-1]} \leqslant G_{n-1}\,.$

Since $(n-1, k-1) < (n,k)$, we have, by induction hypothesis,

$$G_{n-1;[k-1,n-1]} = \prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-1-|\tilde{J}_i|})\,\mathrm{s}_{\tilde{I}_i}\,,$$

remarking that $t = \big|\mathcal{P}\big([0,(k-1)-2]\big)\big| = \big|\mathcal{P}\big([0,k-3]\big)\big| = 2^{k-2}$ and that $\mathcal{P}\big([0,k-3]\big) = \{\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_t\}$, where $\tilde{J}_1 < \ldots < \tilde{J}_t$.

Moreover, by Remark 99.(2), we have

$$\Big(\prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-1-|\tilde{J}_i|})\,\mathrm{s}_{\tilde{J}_i}\Big)\,\mathrm{s}_{k-2} = \prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|J_i|})\,\mathrm{s}_{J_i}\,.$$

This shows

$$\big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2} = \prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|J_i|})\,\mathrm{s}_{J_i}\,.$$

Ad (ii). We consider the group $G_{n;[k-1,n]} \leqslant G_n$.

Since $(n,k-1) < (n,k)$, we have, by induction hypothesis,

$$G_{n;[k-1,n]} = \prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|\tilde{J}_i|})\,\mathrm{s}_{\tilde{J}_i}\,,$$

remarking that $t = \big|\,\mathcal{P}\,\big([0,(k-1)-2]\big)\big| = \big|\mathcal{P}\big([0,k-3]\big)\big| = 2^{k-2}$ and that $\mathcal{P}\big([0,k-3]\big) = \{\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_t\}$, where $\tilde{J}_1 < \ldots < \tilde{J}_t$.

Altogether, this shows

$$\begin{aligned}
G_{n;[k,n]} &\overset{97.(2)}{=} \big(G_{n-1;[k-1,n-1]}\big)\mathrm{s}_{k-2}\cdot G_{n;[k-1,n]}\\
&\overset{(\mathrm{i},\,\mathrm{ii})}{=} \big(\prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|J_i|})\mathrm{s}_{J_i}\big)\cdot\big(\prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|\tilde{J}_i|})\mathrm{s}_{\tilde{J}_i}\big)\\
&= \big(\prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|I_{i+s}|})\,\mathrm{s}_{I_{i+t}}\big)\cdot\big(\prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|I_i|})\,\mathrm{s}_{I_i}\big)\\
&= \big(\prod_{i\in\lfloor r,t+1\rfloor} (G\mathrm{N}_{n-|I_i|})\,\mathrm{s}_{I_i}\big)\cdot\big(\prod_{i\in\lfloor t,1\rfloor} (G\mathrm{N}_{n-|I_i|})\,\mathrm{s}_{I_i}\big)\\
&= \prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{n-|I_i|})\,\mathrm{s}_{I_i}\,.
\end{aligned}$$

Ad (2). We consider the product decomposition of g in r factors. By (1), it remains to show uniqueness.

Suppose that we have $g = \prod_{i\in\lfloor r,1\rfloor} x_i\,\mathrm{s}_{I_i}$ for some elements $x_i \in G\mathrm{N}_{n-|I_i|}$ for $i \in [1,r]$.

Suppose that we have $g = \prod_{i\in\lfloor r,1\rfloor} y_i\,\mathrm{s}_{I_i}$ for some elements $y_i \in G\mathrm{N}_{n-|I_i|}$ for $i \in [1,r]$.

We have to show that $x_i \overset{!}{=} y_i$ for $i \in [1,r]$.

We consider the product order $(\leqslant)$ on $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$; cf. §1.1.1 item 11.

We consider the subposet

$$P := \big\{(n,k) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \colon k \in [1, n+1]\big\} \subseteq \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}\,.$$

We show this claim by induction on P.

Suppose given $(n,k) \in P$. By the induction hypothesis, this claim holds for $(\tilde{n}, \tilde{k})$ with $(\tilde{n}, \tilde{k}) < (n,k)$.

Case $k = 1$. Then

$$\mathcal{P}\big([0,-1]\big) = \{\emptyset\} =: \{I_1\} \quad \text{and} \quad r = 2^0 = 1\,.$$

So we have

$$g = \prod_{i \in \lfloor r,1 \rfloor} x_i\, \mathrm{s}_{I_i} = x_1 \mathrm{s}_\emptyset = x_1\,,$$

and

$$g = \prod_{i \in \lfloor r,1 \rfloor} y_i\, \mathrm{s}_{I_i} = y_1 \mathrm{s}_\emptyset = y_1\,.$$

This shows $x_1 = y_1$.

Case $k \geqslant 2$. Since $k \in [1, n+1]$, we have $n \geqslant k+1 \geqslant 1$.

Note that $t = 2^{k-2} = \frac{r}{2}$. We consider the set $\mathcal{P}\big([0,k-2]\big) = \{I_1\,, \ldots, I_t\,, I_{t+1}\,, \ldots, I_r\}$.

By Remark 97.(2), we have

$$\begin{aligned}\mathcal{P}\big([0,k-2]\big) &= \big\{I \in \mathcal{P}\big([0,k-2]\big) \colon k-2 \in I\big\} \;\dot\cup\; \big\{I \in \mathcal{P}\big([0,k-2]\big) \colon k-2 \notin I\big\} \\ &= \Big(\mathcal{P}\big([0,k-2]\big) \backslash \mathcal{P}\big([0,k-3]\big)\Big) \;\dot\cup\; \mathcal{P}\big([0,k-3]\big)\,,\end{aligned}$$

where for $I \in \mathcal{P}\big([0,k-2]\big) \backslash \mathcal{P}\big([0,k-3]\big)$ and $J \in \mathcal{P}\big([0,k-3]\big)$, we have $I > J$; cf. Lemma 92.

Then note that we have $I_i \in \mathcal{P}\big([0,k-2]\big) \backslash \mathcal{P}\big([0,k-3]\big)$ for $i \in [t+1, r]$ and $I_i \in \mathcal{P}\big([0,k-3]\big)$ for $i \in [1,t]$.

Moreover, for $i \in [1,t]$, note that we have $I_{i+t} = I_i \,\dot\cup\, \{k-2\}$ and that

$$\mathrm{s}_{I_{i+t}} = \mathrm{s}_{I_i \dot\cup \{k-2\}} = \mathrm{s}_{I_i} \blacktriangleleft \mathrm{s}_{k-2}\,;$$

cf. Definition 37.(2).

We have

$$g = \prod_{i\in\lfloor r,1\rfloor} x_i\,\mathrm{s}_{I_i} = \Big(\prod_{i\in\lfloor r,t+1\rfloor} x_i\,\mathrm{s}_{I_i}\Big)\cdot\Big(\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i}\Big) = \Big(\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_{i+t}}\Big)\cdot\Big(\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i}\Big)$$
$$= \Big(\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,(\mathrm{s}_{I_i}\blacktriangle\mathrm{s}_{k-2})\Big)\cdot\Big(\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i}\Big) = \Big(\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i}\Big)\mathrm{s}_{k-2}\cdot\Big(\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i}\Big)\,.$$

We write $x := \prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i} \in G_{n-1}$ and we write $\tilde{x} := \prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i} \in G_n$.

We show that $x \overset{!}{\in} G_{n-1,[k-1,n-1]}$.

Suppose given $\ell \in [k-1,n-1]$. For $i\in[1,t]$, note that we have $\ell \geqslant k-1 \geqslant (\max I_i)+2$ and that $\ell - |I_i| \geqslant \ell-(k-2) \geqslant 1$.

Therefore, we have

$$x\mathrm{d}_\ell = \big(\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i}\big)\mathrm{d}_\ell = \prod_{i\in\lfloor t,1\rfloor} x_{i+t}(\mathrm{s}_{I_i}\blacktriangle\mathrm{d}_\ell) \overset{38}{=} \prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,(\mathrm{d}_{\ell-|I_i|}\blacktriangle\mathrm{s}_{I_i}) = \prod_{i\in\lfloor t,1\rfloor} 1\,\mathrm{s}_{I_i} = 1\,.$$

This shows that $x \in G_{n-1,[k-1,n-1]}$.

We show that $\tilde{x} \overset{!}{\in} G_{n,[k-1,n]}$.

Suppose given $\ell \in [k-1,n]$. For $i\in[1,t]$, note that we have $\ell \geqslant k-1 \geqslant (\max I_i)+2$ and that $\ell - |I_i| \geqslant \ell-(k-2) \geqslant 1$.

We have

$$\tilde{x}\mathrm{d}_\ell = \big(\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i}\big)\mathrm{d}_\ell = \prod_{i\in\lfloor t,1\rfloor} x_i(\mathrm{s}_{I_i}\blacktriangle\mathrm{d}_\ell) \overset{38}{=} \prod_{i\in\lfloor t,1\rfloor} x_i(\mathrm{d}_{\ell-|I_t|}\blacktriangle\mathrm{s}_{I_i}) = \prod_{i\in\lfloor t,1\rfloor} 1\,\mathrm{s}_{I_i} = 1\,.$$

This shows that $\tilde{x} \in G_{n,[k-1,n]}$.

Therefore, we have

$$g = x\,\mathrm{s}_{k-2}\cdot\tilde{x}\,,$$

where $x \in G\mathrm{N}_{n-1,[k-1,n-1]}$ and where $\tilde{x} \in G_{n,[k-1,n]}$.

Similarly, we have

$$g = y\,\mathrm{s}_{k-2}\cdot\tilde{y}$$

for $y := \prod_{i\in\lfloor t,1\rfloor} y_{i+t}\,\mathrm{s}_{I_i} \in G\mathrm{N}_{n-1,[k-1,n-1]}$ and $\tilde{y} := \prod_{i\in\lfloor t,1\rfloor} y_i \in G_{n,[k-1,n]}$.

Moreover, by Remark 97.(4), there exists a unique $z \in G\mathrm{N}_{n-1,[k-1,n-1]}$ and a unique $\tilde{z} \in G_{n,[k-1,n]}$ such that $g = z\,\mathrm{s}_{k-2}\cdot\tilde{z}$.

This shows $x = z = y$ and $\tilde{x} = \tilde{z} = \tilde{y}$.

In particular, we have

$$\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i} = x = y = \prod_{i\in\lfloor t,1\rfloor} y_{i+t}\,\mathrm{s}_{I_i}\,.$$

Now recall that $t = 2^{k-2}$ and that $r = 2^{k-1}$.

Since $(n-1,k-1) < (n,k)$, we have, by induction hypothesis, $x_{i+t} = y_{i+t}$ for $i \in [1,t]$.

Moreover, we have

$$\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i} = \tilde{x} = \tilde{y} = \prod_{i\in\lfloor t,1\rfloor} y_i\,\mathrm{s}_{I_i}\,.$$

Since $(n,k-1) < (n,k)$, we have, by induction hypothesis, $x_i = y_i$ for $i \in [1,t]$.

Altogether, this shows $x_i = y_i$ for $i \in [1,r]$.

Finally, by Remark 97.(4), we have that $x = \prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i} \in G_{n-1;[k-1,n-1]}$ and that $\tilde{x} = \prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i} \in G_{n;[k-1,n]}$ are unique.

Ad (3). We show that $g\pi^G_{n,k} \overset{!}{=} x_1$ by induction on $k \in [1,n+1]$.

Let $k = 1$. Then we have $r = 2^{1-1} = 1$.

By (2), we have

$$g\pi^G_{n,1} = g\,\mathrm{id}_{G_{n;[1,n]}} = g = \prod_{i\in\lfloor 1,1\rfloor} x_i\,\mathrm{s}_{I_i} = x_1\,\mathrm{s}_{I_1} \overset{93.(1)}{=} x_1\,,$$

where $x_1 \in G\mathrm{N}_n$.

Now suppose that $k \in [2,n+1]$.

Recall that we have $\mathcal{P}([0,k-2]) = \{I_1, I_2, \ldots, I_r\}$, where $I_1 < I_2 < \ldots < I_r$.

We consider the set $\mathcal{P}([0,k-3])$. Let $t := \big|\mathcal{P}([0,k-3])\big| = 2^{k-2}$.

By Remark 99.(1), we have $\mathcal{P}([0,k-3]) = \{I_1, I_2, \ldots, I_t\}$.

By induction hypothesis, we have the following assertion (IH).

(IH) Suppose given $\tilde{g} \in G_{n;[k-1,n]}$. We consider its decomposition $\tilde{g} = \prod_{i\in\lfloor t,1\rfloor} \tilde{x}_i\mathrm{s}_{I_i}$, where $\tilde{x}_i \in G\mathrm{N}_{n-|I_i|}$ for $i \in [1,t]$; cf. (2). Then we have $\tilde{g}\pi^G_{n,k-1} = \tilde{x}_1$.

Let $\tilde{g} := gp^G_{n,k} = g^-\mathrm{d}_{k-1}\mathrm{s}_{k-2}\cdot g \in G_{n;[k-1,n]}$.

We consider its decomposition $\tilde{g} = \prod_{i\in\lfloor t,1\rfloor} \tilde{x}_i \mathrm{s}_{I_i}$, where $\tilde{x}_i \in G\mathrm{N}_{n-|I_i|}$ for $i \in [1,t]$; cf. (1, 2).

By (IH), we have $\tilde{g}\pi^G_{n,k-1} = \tilde{x}_1$.

We have

$$g\pi^G_{n,k} = g\big(p^G_{n,k} \blacktriangleleft \pi^G_{n,k-1}\big) = \tilde{g}\pi^G_{n,k-1} \overset{\text{(IH)}}{=} \tilde{x}_1 .$$

We have

$$g = g\mathrm{d}_{k-1}\mathrm{s}_{k-2} \cdot (g^{-}\mathrm{d}_{k-1}\mathrm{s}_{k-2} \cdot g) = g\mathrm{d}_{k-1}\mathrm{s}_{k-2} \cdot \tilde{g} ,$$

where $g\mathrm{d}_{k-1} \in G_{n-1,[k-1,n-1]}$ and where $\tilde{g} \in G_{n;[k-1,n]}$.

Moreover, by (2), we have

$$g = \big(\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i}\big)\mathrm{s}_{k-2} \cdot \big(\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i}\big) ,$$

where $\prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i} \in G_{n-1;[k-1,n-1]}$ and where $\prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i} \in G_{n;[k-1,n]}$.

So, by Remark 97.(4), we have

$$g\mathrm{d}_{k-1} = \prod_{i\in\lfloor t,1\rfloor} x_{i+t}\,\mathrm{s}_{I_i} \quad \text{and} \quad \tilde{g} = \prod_{i\in\lfloor t,1\rfloor} x_i\,\mathrm{s}_{I_i} .$$

So we have

$$\prod_{i\in\lfloor t,1\rfloor} \tilde{x}_i\mathrm{s}_{I_i} = \tilde{g} = \prod_{i\in\lfloor t,1\rfloor} x_i\mathrm{s}_{I_i} .$$

In particular, by (2), it follows that

$$\tilde{x}_1 = \tilde{x}_1\,\mathrm{s}_\emptyset = \tilde{x}_1\,\mathrm{s}_{I_1} = x_1\mathrm{s}_{I_1} = x_1\,\mathrm{s}_\emptyset = x_1 .$$

Therefore, we have $g\pi^G_{n,k} = \tilde{x}_1 = x_1$. □

Corollary 101 *Suppose given a simplicial group G; cf. Definition* 30.
Let $n \in \mathbb{Z}_{\geqslant 0}$. Let $r := \big|\mathcal{P}\big([0,n-1]\big)\big| = 2^n$.
We write $\mathcal{P}\big([0,n-1]\big) =: \{I_1, I_2, \ldots, I_r\}$, where $I_1 < I_2 < \ldots < I_r$; cf. Lemma 92.

(1) *We have*

$$G_n = \prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{n-|I_i|})\,\mathrm{s}_{I_i} ;$$

cf. Definition 37.(2).

(2) *Suppose given $g \in G_n$. Then there exist unique elements $x_i \in G\mathrm{N}_{n-|I_i|}$ for $i \in [1,r]$ such that*

$$g = \prod_{i\in\lfloor r,1\rfloor} x_i \,\mathrm{s}_{I_i} = x_r \,\mathrm{s}_{I_r} \cdot x_{r-1} \,\mathrm{s}_{I_{r-1}} \cdot \ldots \cdot x_1 \,\mathrm{s}_{I_1} .$$

(3) *We have the surjective map*

$$\pi_n^G := \pi_{n,n+1}^G = \prod_{j\in\lfloor n+1,2\rfloor}^{G_n} {}^{G\mathrm{N}_n} p_{n,j}^G : G_n \to G\mathrm{N}_n ;$$

cf. Remark 97.(1).

Suppose given $g \in G_n$. We consider its unique decomposition from (2)*:*

$$g = \prod_{i\in\lfloor r,1\rfloor} x_i \,\mathrm{s}_{I_i} = x_r \,\mathrm{s}_{I_r} \cdot x_{r-1} \,\mathrm{s}_{I_{r-1}} \cdot \ldots \cdot x_1 \,\mathrm{s}_{I_1} .$$

Then we have $g\pi_n^G = x_1$.

Proof. This is Lemma 100.(1, 2, 3) for $k = n + 1$. □

Example 102 We give some examples for Corollary 101.(1) and Lemma 100.(1).
Suppose given a simplicial group G. We use Remark 97.(2) for the following calculations.

(0) Let $n = 0$. We have

$$G_0 = G\mathrm{N}_0 ;$$

cf. Definition 46.(2).

According to Corollary 101.(1), we write $\mathcal{P}\big([0,-1]\big) = \{\emptyset\} = \{I_1\}$. We obtain again

$$G_0 = \prod_{i\in\lfloor 1,1\rfloor} (G\mathrm{N}_{0-|I_i|})\mathrm{s}_{I_i} = (G\mathrm{N}_{0-|\emptyset|})\mathrm{s}_\emptyset = G\mathrm{N}_0 .$$

(1) Let $n = 1$. Applying Remark 97.(2), we obtain

$$G_1 = (G\mathrm{N}_0)\mathrm{s}_0 \cdot G\mathrm{N}_1 .$$

According to Corollary 101.(1), we write $\mathcal{P}\big([0,0]\big) = \big\{\emptyset, \{0\}\big\}$ with

$$\underbrace{\emptyset}_{=:I_1} < \underbrace{\{0\}}_{=:I_2} .$$

We obtain again

$$G_1 = \prod_{i\in\lfloor 2,1\rfloor} (G\mathrm{N}_{1-|I_i|})\mathrm{s}_{I_i} = (G\mathrm{N}_{1-|\{0\}|})\mathrm{s}_{\{0\}} \cdot (G\mathrm{N}_{1-|\emptyset|})\mathrm{s}_{\emptyset} = (G\mathrm{N}_0)\mathrm{s}_0 \cdot G\mathrm{N}_1\,.$$

(2) Let $n=2$. Applying Remark 97.(2), we obtain

$$\begin{array}{rcccc} G_2 & = & G_1\,\mathrm{s}_1 & \cdot & G_{2,2} \\ & \overset{(1)}{=} & \big((G\mathrm{N}_0)\mathrm{s}_0 \cdot G\mathrm{N}_1\big)\mathrm{s}_1 & \cdot & (G\mathrm{N}_1)\mathrm{s}_0 \cdot G\mathrm{N}_2 \\ & = & (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_1 & \cdot & (G\mathrm{N}_1)\mathrm{s}_0 \cdot G\mathrm{N}_2\,. \end{array}$$

According to Corollary 101.(1), we write $\mathcal{P}\big([0,1]\big) = \big\{\emptyset,\{0\},\{1\},\{0,1\}\big\}$ with

$$\underbrace{\emptyset}_{=:I_1} < \underbrace{\{0\}}_{=:I_2} < \underbrace{\{1\}}_{=:I_3} < \underbrace{\{0,1\}}_{=:I_4}\,.$$

We obtain again

$$\begin{aligned} G_2 &= \prod_{i\in\lfloor 4,1\rfloor} (G\mathrm{N}_{2-|I_i|})\mathrm{s}_{I_i} \\ &= (G\mathrm{N}_{2-|\{0,1\}|})\mathrm{s}_{\{0,1\}} \cdot (G\mathrm{N}_{2-|\{1\}|})\mathrm{s}_{\{1\}} \cdot (G\mathrm{N}_{2-|\{0\}|})\mathrm{s}_{\{0\}} \cdot (G\mathrm{N}_{2-|\emptyset|})\mathrm{s}_{\emptyset} \\ &= (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_0 \cdot G\mathrm{N}_2\,. \end{aligned}$$

(3) Let $n=3$. Applying Remark 97.(2), we obtain

$$\begin{array}{rcccc} G_3 & = & G_2\,\mathrm{s}_2 & \cdot & G_{3,3} \\ & \overset{(2)}{=} & \big((G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_0 \cdot G\mathrm{N}_2\big)\mathrm{s}_2 & \cdot & G_{2,2}\,\mathrm{s}_1 \cdot G_{3;[2,3]} \\ & = & (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_2 \cdot (G\mathrm{N}_2)\mathrm{s}_2 & \cdot & \big((G\mathrm{N}_1)\mathrm{s}_0 \cdot G\mathrm{N}_2\big)\mathrm{s}_1 \cdot \big((G\mathrm{N}_2)\mathrm{s}_0 \cdot G\mathrm{N}_3\big) \\ & = & (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_2 \cdot (G\mathrm{N}_2)\mathrm{s}_2 & \cdot & (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_0 \cdot G\mathrm{N}_3\,. \end{array}$$

According to Corollary 101.(1), we write

$$\mathcal{P}\big([0,2]\big) = \big\{\emptyset,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\big\}$$

with

$$\underbrace{\emptyset}_{=:I_1} < \underbrace{\{0\}}_{=:I_2} < \underbrace{\{1\}}_{=:I_3} < \underbrace{\{0,1\}}_{=:I_4} < \underbrace{\{2\}}_{=:I_5} < \underbrace{\{0,2\}}_{=:I_6} < \underbrace{\{1,2\}}_{=:I_7} < \underbrace{\{0,1,2\}}_{=:I_8}\,.$$

We obtain again

$$\begin{aligned} G_3 &= \prod_{i\in\lfloor 8,1\rfloor} (G\mathrm{N}_{3-|I_i|})\mathrm{s}_{I_i} \\ &= (G\mathrm{N}_{3-|\{0,1,2\}|})\mathrm{s}_{\{0,1,2\}} \cdot (G\mathrm{N}_{3-|\{1,2\}|})\mathrm{s}_{\{1,2\}} \cdot (G\mathrm{N}_{3-|\{0,2\}|})\mathrm{s}_{\{0,2\}} \cdot (G\mathrm{N}_{3-|\{2\}|})\mathrm{s}_{\{2\}} \\ &\quad \cdot (G\mathrm{N}_{3-|\{0,1\}|})\mathrm{s}_{\{0,1\}} \cdot (G\mathrm{N}_{3-|\{1\}|})\mathrm{s}_{\{1\}} \cdot (G\mathrm{N}_{3-|\{0\}|})\mathrm{s}_{\{0\}} \cdot (G\mathrm{N}_{3-|\emptyset|})\mathrm{s}_{\emptyset} \\ &= (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_2 \cdot (G\mathrm{N}_2)\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_0 \cdot G\mathrm{N}_3\,. \end{aligned}$$

Remark 103 Suppose given a simplicial group G; cf. Definition 30.

Instead of writing G_n as a product of subgroups as in Corollary 101.(1), one can also use iterated semidirect products.

We consider e.g. the group G_2. We have the following short exact sequences with respective coretractions.

$$G_{2,2} \hookrightarrow G_2 \underset{\mathrm{d}_2}{\overset{\mathrm{s}_1}{\rightleftarrows}} G_1$$

$$G_{2,[1,2]} \hookrightarrow G_{2,2} \underset{\mathrm{d}_1}{\overset{\mathrm{s}_0}{\rightleftarrows}} G_{1,1}$$

$$G_{1,1} \hookrightarrow G_1 \underset{\mathrm{d}_1}{\overset{\mathrm{s}_0}{\rightleftarrows}} G_0$$

These yield the isomorphies

$$\begin{aligned} G_2 &\simeq G_1 \ltimes G_{2,2} \\ G_{2,2} &\simeq G_{1,1} \ltimes G_{2,[1,2]} \\ G_1 &\simeq G_0 \ltimes G_{1,1} \end{aligned}$$

wirh semidirect products resulting from the short exact sequences above.

So we have

$$G_2 \simeq (G_0 \ltimes G_{1,1}) \ltimes (G_{1,1} \ltimes G_{2,[1,2]}) = (G\mathrm{N}_0 \ltimes G\mathrm{N}_1) \ltimes (G\mathrm{N}_1 \ltimes G\mathrm{N}_2)\,.$$

5.4 The Conduché map c_I

The following construction is due to Conduché [**7,** Lemme 1.7].

Definition 104 (Conduché map)

Suppose given a simplicial group G; cf. Definition 30. Let $n \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $I \subseteq [0, n-1]$. We shall define the *Conduché map* $\mathrm{c}_I^{n-|I|} \colon G_{n-|I|} \to G_n$ by recursion on $|I|$.

(1) We consider the empty subset $\emptyset \subseteq [0, n-1]$. Then we have the map

$$\mathrm{c}_\emptyset^n := \mathrm{id}_{G_n} \colon G_n \to G_n\,,\ g \mapsto g\mathrm{c}_\emptyset^n = g\,.$$

(2) Suppose that $I \neq \emptyset$.

We write $r := |I|$ and $I := \{i_1\,, i_2\,, \ldots, i_r\}$, where $i_1 < i_2 < \ldots < i_r$.

Moreover, we write $\tilde{I} := I \backslash \{i_r\} = I \backslash \{\max I\}$.

Let $m := \max \tilde{I}$. We let $\max \emptyset := -1$.

Then we let

$$\mathrm{c}_I^{n-r} \colon G_{n-r} \to G_n\,,\ g \mapsto g\mathrm{c}_I^{n-r} := \prod_{k \in \lfloor i_r\,, m+1 \rfloor} \left((g\mathrm{c}_{\tilde{I}}^{n-r})\,\mathrm{s}_k \right)^{(-1)^{ir-k}}\,.$$

We often write $\mathrm{c}_I := \mathrm{c}_I^{n-r}$.

Example 105 We give some examples for the Conduché map from Definition 104.

Suppose given a simplicial group G.

(1) Let $n = \mathbb{Z}_{\geqslant 0}$. For $g_n \in G_n$, we have

$$g_n\,\mathrm{c}_\emptyset \overset{104.(1)}{=} g_n\,.$$

(2) Let $n \in \mathbb{Z}_{\geqslant 1}$. We consider the set $I = \{0\} =: \{i_1\}$. We have $|I| = 1$.

Then we have $\tilde{I} = \{0\} \backslash \{0\} = \emptyset$ and $m = \max \emptyset = -1$; cf. Lemma 92.

For $g_{n-1} \in G_{n-1}$, we have

$$g_{n-1}\,\mathrm{c}_{\{0\}} \overset{104.(2)}{=} \prod_{k \in \lfloor 0,0 \rfloor} ((g_{n-1}\,\mathrm{c}_\emptyset)\mathrm{s}_k)^{(-1)^{0-k}} \overset{(1)}{=} (g_{n-1}\,\mathrm{s}_0)^{(-1)^0} = g_{n-1}\,\mathrm{s}_0\,.$$

(3) Let $n = 4$. We consider the set $I = \{0, 2\}$, where we write

$$\underbrace{0}_{=:i_1} < \underbrace{2}_{=:i_2} .$$

Then we have $\tilde{I} = \{0, 2\}\backslash\{2\} = \{0\}$ and $m = \max\{0\} = 0$.

For $g_2 \in G_2$, we have

$$\begin{aligned} g_2\,\mathrm{c}_{\{0,2\}} &\overset{104.(2)}{=} \prod_{k\in\lfloor 2,1\rfloor} \big((g_2\,\mathrm{c}_{\{0\}})\mathrm{s}_k\big)^{(-1)^{2-k}} \overset{(2)}{=} \big((g_2\mathrm{s}_0)\mathrm{s}_2\big)^{(-1)^0} \cdot \big((g_2\mathrm{s}_0)\mathrm{s}_1\big)^{(-1)^1} \\ &\overset{(2)}{=} g_2\,\mathrm{s}_0\mathrm{s}_2 \cdot (g_2\,\mathrm{s}_0\mathrm{s}_1)^- . \end{aligned}$$

(4) Let $n = 5$. We consider the set $I = \{0, 2, 4\}$, where we write

$$\underbrace{0}_{=:i_1} < \underbrace{2}_{=:i_2} < \underbrace{4}_{=:i_3} .$$

Then we have $\tilde{I} = \{0, 2, 4\}\backslash\{4\} = \{0, 2\}$ and $m = \max\{0, 2\} = 2$.

For $g_2 \in G_2$, we have

$$\begin{aligned} g_2\,\mathrm{c}_{\{0,2,4\}} &\overset{104.(2)}{=} \prod_{k\in\lfloor 4,3\rfloor} \big((g_2\,\mathrm{c}_{\{0,2\}})\mathrm{s}_k\big)^{(-1)^{2-k}} \\ &= \big((g_2\,\mathrm{c}_{\{0,2\}})\mathrm{s}_4\big)^{(-1)^0} \cdot \big((g_2\,\mathrm{c}_{\{0,2\}})\mathrm{s}_3\big)^{(-1)^1} \\ &\overset{(3)}{=} (g_2\mathrm{s}_0\mathrm{s}_2 \cdot (g_2\mathrm{s}_0\mathrm{s}_1)^-)\,\mathrm{s}_4 \cdot \big((g_2\mathrm{s}_0\mathrm{s}_2 \cdot (g_2\mathrm{s}_0\mathrm{s}_1)^-)\,\mathrm{s}_3\big)^- \\ &= g_2\mathrm{s}_0\mathrm{s}_2\mathrm{s}_4 \cdot (g_2\mathrm{s}_0\mathrm{s}_1\mathrm{s}_4)^- \cdot g_2\mathrm{s}_0\mathrm{s}_1\mathrm{s}_3 \cdot (g_2\mathrm{s}_0\mathrm{s}_2\mathrm{s}_3)^- . \end{aligned}$$

(5) Let $n = 6$. We consider the set $I = \{1\} =: \{i_1\}$.

Then we have $\tilde{I} = \{1\}\backslash\{1\} = \emptyset$ and $m = \max\emptyset = -1$.

For $g_4 \in G_4$, we have

$$g_4\,\mathrm{c}_{\{1\}} \overset{104.(2)}{=} \prod_{k\in\lfloor 1,0\rfloor} ((g_4\,\mathrm{c}_\emptyset)\mathrm{s}_k)^{(-1)^{1-k}} \overset{(1)}{=} (g_4\mathrm{s}_1)^{(-1)^0} \cdot (g_4\mathrm{s}_0)^{(-1)^1} = g_4\mathrm{s}_1 \cdot (g_4\mathrm{s}_0)^- .$$

(6) Let $n = 6$. We consider the set $I = \{1, 4\}$, where we write

$$\underbrace{1}_{=:i_1} < \underbrace{4}_{=:i_2} .$$

Then we have $\tilde{I} = \{1,4\}\backslash\{4\} = \{1\}$ and $m = \max\{1\} = 1$.

For $g_4 \in G_4$, we have

$$\begin{aligned} g_4\,\mathrm{c}_{\{1,4\}} &\overset{104.(2)}{=} \prod_{k\in\lfloor 4,2\rfloor} \big((g_4\,\mathrm{c}_{\{1\}})\mathrm{s}_k\big)^{(-1)^{4-k}} \\ &= \big((g_4\,\mathrm{c}_{\{1\}})\mathrm{s}_4\big)^{(-1)^0} \cdot \big((g_4\,\mathrm{c}_{\{1\}})\mathrm{s}_3\big)^{(-1)^1} \cdot \big((g_4\,\mathrm{c}_{\{1\}})\mathrm{s}_2\big)^{(-1)^2} \\ &\overset{(5)}{=} (g_4\mathrm{s}_1 \cdot (g_4\mathrm{s}_0)^-)\,\mathrm{s}_4 \cdot ((g_4\mathrm{s}_1 \cdot (g_4\mathrm{s}_0)^-)\mathrm{s}_3)^- \cdot (g_4\mathrm{s}_1 \cdot (g_4\mathrm{s}_0)^-)\,\mathrm{s}_2 \\ &= g_4\mathrm{s}_1\mathrm{s}_4 \cdot (g_4\mathrm{s}_0\mathrm{s}_4)^- \cdot g_4\mathrm{s}_0\mathrm{s}_3 \cdot (g_4\mathrm{s}_1\mathrm{s}_3)^- \cdot g_4\mathrm{s}_1\mathrm{s}_2 \cdot (g_4\mathrm{s}_0\mathrm{s}_2)^-\,. \end{aligned}$$

Remark 106 *Suppose given a simplicial group G; cf. Definition* 30.

Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given $I \subseteq [0,n]$. Let $J := [1,n+1]\backslash\{i+1 \colon i \in I\} \subseteq [1,n+1]$.

Suppose given $x \in G\mathrm{N}_{n+1-|I|}$. We consider the map $\mathrm{c}_I^{n+1-|I|} \colon G_{n+1-|I|} \to G_{n+1}$; cf. Definition 104.

We have

$$x\mathrm{c}_I^{n+1-|I|} \in G_{n+1;J}\,;$$

cf. Definition 95.

Proof. We show this claim by induction on $n \in \mathbb{Z}_{\geqslant -1}$.

Suppose that $n = -1$. Then we have $I \subseteq [0,-1]$ and so $I = \emptyset$ and $J = [1,0]\backslash\emptyset = \emptyset$.

Suppose given $x \in G\mathrm{N}_{n+1-0} = G\mathrm{N}_0$.

Then we have

$$x\mathrm{c}_I = x\mathrm{c}_\emptyset \overset{104.(1)}{=} x \quad \text{and} \quad G\mathrm{N}_0 = G_0 \overset{95}{=} G_{0;\emptyset}\,.$$

This shows that $x\mathrm{c}_I \in G_{0;\emptyset}$.

Now suppose that $n \geqslant 1$. By induction, we have the following assertion (IH).

(IH) Suppose given $\tilde{I} \subseteq [0,n-1]$. Let $\tilde{J} := [1,n]\backslash\{i+1 \colon i \in \tilde{I}\}$.

Suppose given $y \in G\mathrm{N}_{n-|\tilde{I}|}$. Then we have $y\mathrm{c}_{\tilde{I}}^{n-|\tilde{I}|} \in G_{n;\tilde{J}}$.

Suppose given $x \in G\mathrm{N}_{n+1-|I|}$.

We have to show that $(x\mathrm{c}_I)\mathrm{d}_j \overset{!}{=} 1$ for $j \in J = [1,n+1]\backslash\{i+1 \colon i \in I\}$.

Note that $I \neq \emptyset$. We write $\ell := \max I \in I \subseteq [0,n]$.

Let $\tilde{I} = I\backslash\{\ell\}$ and let $\tilde{J} := [1, n+1]\backslash\{i+1 \colon i \in \tilde{I}\} = J \dot\cup \{\ell+1\}$. Then we have $|\tilde{I}| = |I| - 1$ and $\tilde{I} \subseteq [0, n-1]$.

We write $m := \max \tilde{I} \geqslant -1$. Note that $m \leqslant \ell - 1$.

Suppose given $j \in J$.

Case $j \in [\ell+1, n+1]$. Then we have $2 \leqslant \ell + 2 \leqslant j$, since $j \neq \ell + 1$. So for $k \in [m+1, \ell]$, we have $j \geqslant k+2$ and therefore

$$\mathrm{s}_k \vartriangleleft \mathrm{d}_j = \mathrm{d}_{j-1} \vartriangleleft \mathrm{s}_k \,.$$

We show that $j - 1 \overset{!}{\in} \tilde{J}$.

We have $j - 1 \in [1, n]$ since $j \geqslant 2$. Moreover, we have $j - 2 \notin \tilde{I}$ since $j - 2 \geqslant \ell > \max \tilde{I}$. So $j - 1 \notin \{i+1 \colon i \in \tilde{I}\}$.

This shows that $j - 1 \in [1, n]\backslash\{i+1 \colon i \in \tilde{I}\} = \tilde{J}$.

Then, by induction hypothesis (IH), we have $(x\mathrm{c}_{\tilde{I}})\mathrm{d}_{j-1} = 1$.

So we have

$$\begin{aligned}
(x\mathrm{c}_I)\mathrm{d}_j &\overset{104.(2)}{=} \Big(\prod_{k \in \lfloor \ell, m+1 \rfloor} ((x\mathrm{c}_{\tilde{I}})\mathrm{s}_k)^{(-1)^{\ell-k}}\Big)\mathrm{d}_j = \prod_{k \in \lfloor \ell, m+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{s}_k \vartriangleleft \mathrm{d}_j) \\
&= \prod_{k \in \lfloor \ell, m+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{d}_{j-1} \vartriangleleft \mathrm{s}_k) = \prod_{k \in \lfloor \ell, m+1 \rfloor} \Big(\big((x\mathrm{c}_{\tilde{I}})\mathrm{d}_{j-1}\big)^{(-1)^{\ell-k}}\Big)\, \mathrm{s}_k \\
&\overset{(\mathrm{IH})}{=} \prod_{k \in \lfloor \ell, m+1 \rfloor} \Big(1^{(-1)^{\ell-k}}\Big)\, \mathrm{s}_k = 1 \,.
\end{aligned}$$

Case $j \in [m+1, \ell]$. We show that $j \overset{!}{\geqslant} m + 2$.

We *assume* that $j = m + 1$.

If $\tilde{I} = \emptyset$, then we have $m = -1$ and so $j = 0$ which *contradicts* $j \in [1, n+1]$.

If $\tilde{I} \neq \emptyset$, then we have $m \in \tilde{I} \subseteq I$ and so $j = m+1 \in \{i+1 \colon i \in I\}$ which *contradicts* $j \in J$.

This shows $j \geqslant m + 2$.

Note that $j \in \tilde{J}$. By induction hypothesis (IH), we have $(x\mathrm{c}_{\tilde{I}})\mathrm{d}_j = 1$.

If $j - 2 \geqslant m + 1$, then we have $j - 2 \notin \tilde{I}$ and thus $j - 1 \in \tilde{J}$. Therefore, by the induction hypothesis (IH), we have $(x\mathrm{c}_{\tilde{I}})\mathrm{d}_{j-1} = 1$.

For $k \in [j+1, \ell]$, we have $j < k$, and so $\mathrm{s}_k \vartriangleleft \mathrm{d}_j = \mathrm{d}_j \vartriangleleft \mathrm{s}_{k-1}$.

For $k \in [j-1, j]$, we have $j \in \{k, k+1\}$, and so $\mathrm{s}_k \vartriangleleft \mathrm{d}_j = \mathrm{id}_{G_n}$.

For $k \in [m+1, j-2]$, we have $j \geqslant k+2$, and so $\mathrm{s}_k \blacktriangle \mathrm{d}_j = \mathrm{d}_{j-1} \blacktriangle \mathrm{s}_k$.

So we have

$$
\begin{aligned}
& (x\mathrm{c}_I)\mathrm{d}_j \\
= \ & \Big(\prod_{k \in \lfloor \ell, m+1 \rfloor} ((x\mathrm{c}_{\tilde{I}})\mathrm{s}_k)^{(-1)^{\ell-k}} \Big) \mathrm{d}_j \\
= \ & \prod_{k \in \lfloor \ell, m+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{s}_k \blacktriangle \mathrm{d}_j) \\
= \ & \Big(\prod_{k \in \lfloor \ell, j+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{s}_k \blacktriangle \mathrm{d}_j) \Big) \cdot \Big(\prod_{k \in \lfloor j, j-1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{s}_k \blacktriangle \mathrm{d}_j) \Big) \cdot \Big(\prod_{k \in \lfloor j-2, m+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{s}_k \blacktriangle \mathrm{d}_j) \Big) \\
= \ & \Big(\prod_{k \in \lfloor \ell, j+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{d}_j \blacktriangle \mathrm{s}_{k-1}) \Big) \cdot \Big(\prod_{k \in \lfloor j, j-1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} \, \mathrm{id}_{G_n} \Big) \cdot \Big(\prod_{k \in \lfloor j-2, m+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{d}_{j-1} \blacktriangle \mathrm{s}_k) \Big) \\
= \ & \Big(\prod_{k \in \lfloor \ell, j+1 \rfloor} \Big(\big((x\mathrm{c}_{\tilde{I}})\mathrm{d}_j \big)^{(-1)^{\ell-k}} \Big) \mathrm{s}_{k-1} \Big) \cdot \big((x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-j}} \cdot (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-j+1}} \big) \\
& \qquad \cdot \Big(\prod_{k \in \lfloor j-2, m+1 \rfloor} \Big(\big((x\mathrm{c}_{\tilde{I}})\mathrm{d}_{j-1} \big)^{(-1)^{\ell-k}} \Big) \mathrm{s}_k \Big) \\
\overset{\text{(IH)}}{=} \ & \Big(\prod_{k \in \lfloor \ell, j+1 \rfloor} \Big(1^{(-1)^{\ell-k}} \Big) \mathrm{s}_{k-1} \Big) \cdot 1 \cdot \Big(\prod_{k \in \lfloor j-2, m+1 \rfloor} \Big(1^{(-1)^{\ell-k}} \Big) \mathrm{s}_k \Big) \\
= \ & 1 \cdot 1 \cdot 1 \\
= \ & 1 \, .
\end{aligned}
$$

Case $j \in [1, m]$. For $k \in [m+1, \ell]$, we have $j \leqslant m < k$, and so

$$\mathrm{s}_k \blacktriangle \mathrm{d}_j = \mathrm{d}_j \blacktriangle \mathrm{s}_{k-1} \, .$$

Moreover, note that $j \in \tilde{J}$ since $J \subseteq \tilde{J}$.

Then, by induction hypothesis (IH), we have $(x\mathrm{c}_{\tilde{I}})\mathrm{d}_j = 1$.

So we have

$$
\begin{aligned}
(x\mathrm{c}_I)\mathrm{d}_j \ &= \ \Big(\prod_{k \in \lfloor \ell, m+1 \rfloor} ((x\mathrm{c}_{\tilde{I}})\mathrm{s}_k)^{(-1)^{\ell-k}} \Big) \mathrm{d}_j = \prod_{k \in \lfloor \ell, m+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{s}_k \blacktriangle \mathrm{d}_j) \\
&= \ \prod_{k \in \lfloor \ell, m+1 \rfloor} (x\mathrm{c}_{\tilde{I}})^{(-1)^{\ell-k}} (\mathrm{d}_j \blacktriangle \mathrm{s}_{k-1}) = \prod_{k \in \lfloor \ell, m+1 \rfloor} \Big(\big((x\mathrm{c}_{\tilde{I}})\mathrm{d}_j \big)^{(-1)^{\ell-k}} \Big) \mathrm{s}_{k-1} \\
&\overset{\text{(IH)}}{=} \ \prod_{k \in \lfloor \ell, m+1 \rfloor} \Big(1^{(-1)^{\ell-k}} \Big) \mathrm{s}_k = 1 \, .
\end{aligned}
$$

Altogether, this shows that $(x\mathrm{c}_I)\mathrm{d}_j = 1$ in each case, as claimed. □

Remark 107 *Let $n \in \mathbb{Z}_{\geqslant 0}$.*
Suppose given a simplicial group G; cf. Definition 30.
Suppose given $I \subseteq [0,n]$ with $I \neq \emptyset$. Suppose given $x \in G\mathrm{N}_{n+1-|I|}$. Let $I' := [0,n]\backslash I$.
We consider the map $\mathrm{c}_I^{n+1-|I|} \colon G_{n+1-|I|} \to G_{n+1}$; cf. Definition 104.(1).
We have

$$(x\mathrm{c}_I)\mathrm{d}_0 \in G_{n,I'}\,.$$

Proof. By Remark 106, we have $x\mathrm{c}_I \in G_{n+1,J}$, where $J = [1,n+1]\backslash\{i+1 \colon i \in I\}$.
Suppose given $k \in I'$. We show that $k+1 \overset{!}{\in} J$.
We have $k \in [0,n]$, hence $k+1 \in [1,n+1]$.
Moreover, we have $k+1 \notin \{i+1 \colon i \in I\}$ since $k \notin I$.
This shows $k+1 \in J$.
So we have

$$(x\mathrm{c}_I)\mathrm{d}_0\,\mathrm{d}_k = (x\mathrm{c}_I)\mathrm{d}_{k+1}\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows $(x\mathrm{c}_I)\mathrm{d}_0 \in G_{n,I'}$. □

Lemma 108 *Let $n \in \mathbb{Z}_{\geqslant 0}$.*
Suppose given an $\lfloor n,0 \rfloor$-coskeletal simplicial group G; cf. Definition 90.
Suppose given I, $J \subseteq [0,n]$ such that I, $J \neq \emptyset$ and such that $I \cap J = \emptyset$.
Suppose given $x \in G\mathrm{N}_{n+1-|I|}$ and suppose given $y \in G\mathrm{N}_{n+1-|J|}$.
Then we have

$$[x\mathrm{c}_I\,, y\mathrm{c}_J] = 1 \in G_{n+1}\,;$$

cf. Definition 104.

Proof. By Remark 106 we have $x\mathrm{c}_I \in G_{n;\tilde{I}}$, where $\tilde{I} := [1,n+1]\backslash\{i+1 \colon i \in I\}$, and we have $y\mathrm{c}_J \in G_{n;\tilde{J}}$, where $\tilde{J} := [1,n+1]\backslash\{j+1 \colon j \in J\}$.
Moreover, note that we have $\{i+1 \colon i \in I\} \cap \{j+1 \colon j \in J\} = \emptyset$ since $I \cap J = \emptyset$.

So

$$\begin{aligned}\tilde{I} \cup \tilde{J} &= \big([1, n+1] \backslash \{i+1 \colon i \in I\}\big) \cup \big([1, n+1] \backslash \{j+1 \colon j \in J\}\big) \\ &= [1, n+1] \backslash \big(\{i+1 \colon i \in I\} \cap \{j+1 \colon j \in J\}\big) \\ &= [1, n+1] \backslash \emptyset \\ &= [1, n+1] \,.\end{aligned}$$

Therefore, for $k \in [1, n+1]$, we have

$$[x\mathrm{c}_I \,, y\mathrm{c}_J] \,\mathrm{d}_k = [(x\mathrm{c}_I)\mathrm{d}_k \,, (y\mathrm{c}_J)\mathrm{d}_k] = 1$$

since $k \in \tilde{I}$ and so $(x\mathrm{c}_I)\mathrm{d}_k = 1$ or $k \in \tilde{J}$ and so $(y\mathrm{c}_J)\mathrm{d}_k = 1$.

By Remark 107, we have

$$(x\mathrm{c}_I)\mathrm{d}_0 \in G_{n;I'} \,,$$

where $I' = [0, n] \backslash I$, and we have

$$(y\mathrm{c}_J)\mathrm{d}_0 \in G_{n;J'} \,,$$

where $J' = [0, n] \backslash J$.

Note that

$$I' \cup J' = \big([0, n] \backslash I\big) \cup \big([0, n] \backslash J\big) = [0, n] \backslash \big(I \cap J\big) = [0, n] \backslash \emptyset = [0, n] \,.$$

Note that $I' \neq \emptyset$ since $J \neq \emptyset$ and $J \subseteq I'$.

Moreover, note that $J' \neq \emptyset$ since $I \neq \emptyset$ and $I \subseteq J'$.

Then we have

$$[x\mathrm{c}_I \,, y\mathrm{c}_J] \,\mathrm{d}_0 = [(x\mathrm{c}_I)\mathrm{d}_0 \,, (y\mathrm{c}_J)\mathrm{d}_0] \in [G_{n;I'} \,, G_{n;J'}] \overset{90.(2)}{=} 1 \,.$$

Altogether, we have $[x\mathrm{c}_I \,, y\mathrm{c}_J] \,\mathrm{d}_k = 1$ for $k \in [0, n+1]$.

This shows $[x\mathrm{c}_I \,, y\mathrm{c}_J] \in G\mathrm{NZ}_{n+1} \overset{90.(1)}{=} 1$.

Therefore, we have $[x\mathrm{c}_I \,, y\mathrm{c}_J] = 1$. □

5.5 The map c_I and the group morphism s_I

Definition 109 Suppose given a simplicial group G; cf. Definition 30.

Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given $I \in \mathcal{P}\,([0, n-1])$.

We consider sets $J \in \mathcal{P}([0, n-1])$ with $J < I$; cf. Lemma 92.

We write

$$G\mathrm{D}_{n,I} := \left\langle (G\mathrm{N}_{n-|J|})\mathrm{s}_J \colon \emptyset < J < I \right\rangle \leqslant G_n .$$

Lemma 110 *Suppose given a simplicial group G; cf. Definition 30.*

Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given $I \in \mathcal{P}([0,n])$. Let $r := |I|$. We write $I =: \{i_1, i_2, \ldots, i_r\}$, where $i_1 < i_2 < \ldots < i_r$.

Suppose given $g \in G\mathrm{N}_{n+1-r}$. Then we have

$$g\mathrm{c}_I = g\mathrm{s}_I \cdot x \ \in G_{n+1}$$

for some $x \in G\mathrm{D}_{n+1,I}$; cf. Definitions 104, 109.

Proof. We show this claim by induction on $n \in \mathbb{Z}_{\geqslant -1}$.

Let $n = -1$. Then we have $I = \emptyset$ and $G\mathrm{D}_{0,\emptyset} = 1$.

We have $g \in G_0$ and

$$g\mathrm{s}_I \overset{37.(2)}{=} g\,\mathrm{id}_{G_0} \overset{104.(1)}{=} g\mathrm{c}_I = g\mathrm{c}_I \cdot 1 .$$

Now suppose that $n \geqslant 0$. By induction, we have the following assertion (IH).

(IH) Suppose given $K \in \mathcal{P}([0,\ell])$, where $\ell < n$. Suppose given $\tilde{g} \in G_{\ell+1-|K|}$.
Then we have

$$\tilde{g}\mathrm{c}_K = \tilde{g}\mathrm{s}_K \cdot \tilde{x} \ \in G_{\ell+1}$$

for some $\tilde{x} \in G\mathrm{D}_{\ell+1,K}$.

Case $I = \emptyset$. We have $r = 0$ and $g \in G\mathrm{N}_{n+1}$.

So

$$g\mathrm{s}_I \overset{37.(2)}{=} g\,\mathrm{id}_{G_{n+1}} \overset{104.(1)}{=} g\mathrm{c}_I = g\mathrm{c}_I \cdot 1 .$$

Case $I \neq \emptyset$. We have $r \geqslant 1$.

We write

$$\tilde{I} := I \backslash \{i_r\} = I \backslash \{\max I\} \in \mathcal{P}([0, n-1]) \quad \text{and} \quad m := \max \tilde{I} ,$$

where we recall that $\max \emptyset = -1$; cf. Lemma 92. Note that $i_r \geqslant m+1$.

By (IH), since $n - 1 < n$, we have

$$g\mathrm{c}_{\tilde{I}} = g\mathrm{s}_{\tilde{I}} \cdot \tilde{x} \ \in G_n$$

for some $\tilde{x} \in G\mathrm{D}_{n,\tilde{I}}$.

For $k \in [m+1, i_r]$, we write $J_k := \tilde{I} \dot\cup \{k\}$. Note that $J_{i_r} = I$.

Moreover, note that

$$\mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_k \overset{37.(2)}{=} \mathrm{s}_{\tilde{I}\dot\cup\{k\}} = \mathrm{s}_{J_k}\,.$$

We have

$$\begin{aligned}
g\mathrm{c}_I &\overset{104.(2)}{=} \prod_{k\in\lfloor i_r, m+1\rfloor} ((g\mathrm{c}_{\tilde{I}})\mathrm{s}_k)^{(-1)^{i_r-k}} \\
&= (g\mathrm{c}_{\tilde{I}})\mathrm{s}_{i_r} \cdot \left(\prod_{k\in\lfloor i_r-1, m+1\rfloor} ((g\mathrm{c}_{\tilde{I}})\mathrm{s}_k)^{(-1)^{i_r-k}}\right) \\
&\overset{(\mathrm{IH})}{=} (g\mathrm{s}_{\tilde{I}}\cdot\tilde{x})\mathrm{s}_{i_r} \cdot \left(\prod_{k\in\lfloor i_r-1, m+1\rfloor} ((g\mathrm{s}_{\tilde{I}}\cdot\tilde{x})\mathrm{s}_k)^{(-1)^{i_r-k}}\right) \\
&= g(\mathrm{s}_{\tilde{I}}\blacktriangle\mathrm{s}_{i_r})\cdot\tilde{x}\mathrm{s}_{i_r} \cdot \left(\prod_{k\in\lfloor i_r-1, m+1\rfloor} (g(\mathrm{s}_{\tilde{I}}\blacktriangle\mathrm{s}_k)\cdot\tilde{x}\mathrm{s}_k)^{(-1)^{i_r-k}}\right) \\
&= g\mathrm{s}_I\cdot\tilde{x}\mathrm{s}_{i_r} \cdot \left(\prod_{k\in\lfloor i_r-1, m+1\rfloor} (g\mathrm{s}_{J_k}\cdot\tilde{x}\mathrm{s}_k)^{(-1)^{i_r-k}}\right)
\end{aligned}\,.$$

It remains to show that $g\mathrm{s}_{J_k} \overset{!}{\in} G\mathrm{D}_{n+1,I}$ for $k \in [m+1, i_r-1]$ and that $\tilde{x}\mathrm{s}_k \overset{!}{\in} G\mathrm{D}_{n+1,I}$ for $k \in [m+1, i_r]$.

We show that $g\mathrm{s}_{J_k} \overset{!}{\in} G\mathrm{D}_{n+1,I}$ *for* $k \in [m+1, i_r-1]$.

Let $k \in [m+1, i_r-1]$. We have

$$J_k = \tilde{I} \dot\cup \{k\} \overset{94.(1)}{<} \tilde{I} \dot\cup \{i_r\} = I\,.$$

This shows that

$$g\mathrm{s}_{J_k} \in (G\mathrm{N}_{n-r})\mathrm{s}_{J_k} \subseteq \big\langle (G\mathrm{N}_{n+1-|J|})\mathrm{s}_J \colon \emptyset < J < I\big\rangle = G\mathrm{D}_{n+1,I}\,.$$

We show that $\tilde{x}\mathrm{s}_k \overset{!}{\in} G\mathrm{D}_{n+1,I}$ *for* $k \in [m+1, i_r]$.

Suppose given $k \in [m+1, i_r]$. We have

$$\begin{aligned}
\tilde{x}\mathrm{s}_k \in \big(G\mathrm{D}_{n,\tilde{I}}\big)\,\mathrm{s}_k &= \Big(\big\langle (G\mathrm{N}_{n-|\tilde{J}|})\mathrm{s}_{\tilde{J}} \colon \emptyset < \tilde{J} < \tilde{I}\big\rangle\Big)\mathrm{s}_k = \big\langle (G\mathrm{N}_{n-|\tilde{J}|})\mathrm{s}_{\tilde{J}}\,\mathrm{s}_k \colon \emptyset < \tilde{J} < \tilde{I}\big\rangle \\
&= \big\langle (G\mathrm{N}_{n-|\tilde{J}|})\mathrm{s}_{\tilde{J}\dot\cup\{k\}} \colon \emptyset < \tilde{J} < \tilde{I}\big\rangle\,.
\end{aligned}$$

Moreover, for $\tilde{J} \in \mathcal{P}([0,n-1])$ with $\tilde{J} < \tilde{I}$, we have

$$\tilde{J} \mathbin{\dot\cup} \{k\} \overset{94.(3)}{<} \tilde{I} \mathbin{\dot\cup} \{k\} \overset{94.(2)}{\leqslant} \tilde{I} \mathbin{\dot\cup} \{i_r\} = I \subseteq [0,n]\,.$$

This shows that $\tilde{x}\mathrm{s}_k \in \big\langle (G\mathrm{N}_{n+1-|J|})\mathrm{s}_J \colon \emptyset < J < I \big\rangle = G\mathrm{D}_{n+1,I}$.
Altogether this shows that

$$x := \tilde{x}\mathrm{s}_{i_r} \cdot \left(\prod_{k \in \lfloor i_r - 1\,, m+1 \rfloor} (g\mathrm{s}_{J_k} \cdot \tilde{x}\mathrm{s}_k)^{(-1)^{i_r - k}} \right) \in G\mathrm{D}_{n+1,I}\,,$$

and we have $g\mathrm{c}_I = g\mathrm{s}_I \cdot x$. □

5.6 The degenerate subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$

Lemma 111 *Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.*
Suppose given I, $J \subseteq [0,n]$ such that $I \cap J \neq \emptyset$. Suppose that $I < J$; cf. Lemma 92.
We write $\tilde{n} := n - |I \cap J|$. Note that $I \cap J \subset [0,n]$. In particular, we have $\tilde{n} \geqslant 0$.
We consider the injective monotone map

$$\delta^n_{I\cap J} \colon [0,\tilde{n}] \to [0,n]\,;$$

cf. Definition 25.(1).
Note that $I \cap J = \{k \in [0,n] \colon k \notin \operatorname{im} \delta^n_{I\cap J}\}$; cf. Remark 26.(1).
We consider the monotone bijective map

$$\tilde{\delta} := \delta^n_{I\cap J}\big|^{\operatorname{im}\delta^n_{I\cap J}} : [0,\tilde{n}] \to [0,n]\backslash(I\cap J)\,.$$

We consider its inverse

$$\tilde{\delta}^- = \big(\delta^n_{I\cap J}\big|^{\operatorname{im}\delta^n_{I\cap J}}\big)^- : [0,n]\backslash(I\cap J) \to [0,\tilde{n}]\,.$$

We consider the sets

$$\tilde{I} := \big(I\backslash(I\cap J)\big)\tilde{\delta}^- \subseteq [0,\tilde{n}] \quad \text{and} \quad \tilde{J} := \big(J\backslash(I\cap J)\big)\tilde{\delta}^- \subseteq [0,\tilde{n}].$$

Note that $n-|I|=\tilde{n}-|\tilde{I}|$ *and that* $n-|J|=\tilde{n}-|\tilde{J}|$.
Moreover, let $\tilde{m}:=\max\tilde{I}$. *Recall that* $\max\emptyset=-1$*; cf. Lemma* 92.
Suppose given a simplicial group G*; cf. Definition* 30.
Suppose given $x\in G\mathrm{N}_{n+1-|I|}$ *and suppose given* $y\in G\mathrm{N}_{n+1-|J|}$.
We consider the group morphisms

$$\begin{array}{rcrcl}
\mathrm{s}_I & = & \mathrm{s}_I^{G,n+1-|I|}\colon & G_{n+1-|I|} & \to\ G_{n+1}\\
\mathrm{s}_{\tilde{I}} & = & \mathrm{s}_{\tilde{I}}^{G,n+1-|I|}\colon & G_{n+1-|I|} & \to\ G_{\tilde{n}+1}\\
\mathrm{s}_J & = & \mathrm{s}_J^{G,n+1-|J|}\colon & G_{n+1-|J|} & \to\ G_{n+1}\\
\mathrm{s}_{\tilde{J}} & = & \mathrm{s}_{\tilde{J}}^{G,n+1-|J|}\colon & G_{n+1-|J|} & \to\ G_{\tilde{n}+1}\\
\mathrm{s}_{I\cap J} & = & \mathrm{s}_{I\cap J}^{G,\tilde{n}+1}\colon & G_{\tilde{n}+1} & \to\ G_{n+1}\,;
\end{array}$$

cf. Definition 37.(1). *Note that* $x\mathrm{s}_I$, $y\mathrm{s}_J\in G_{n+1}$.

(1) *We have* $\max\tilde{I}<\max\tilde{J}$.

(2) *Suppose given* $K\subseteq[0,\tilde{m}]$. *We have* $K\tilde{\delta}\,\dot\cup\,(I\cap J)<J$.

(3) *We have* $\mathrm{s}_{\tilde{I}}\mathbin{\blacktriangle}\mathrm{s}_{I\cap J}=\mathrm{s}_I$ *and* $\mathrm{s}_{\tilde{J}}\mathbin{\blacktriangle}\mathrm{s}_{I\cap J}=\mathrm{s}_J$.
So
$$(x\mathrm{s}_I)^{y\mathrm{s}_J}=\big((x\mathrm{s}_{\tilde{I}})^{y\mathrm{s}_{\tilde{J}}}\big)\mathrm{s}_{I\cap J}\in G_{n+1}\,.$$

(4) *We have*
$$(G\mathrm{N}_{n+1-|I|})\mathrm{s}_{\tilde{I}}\ \leqslant\ G_{\tilde{n}+1;[\tilde{m}+2,\tilde{n}+1]}\,;$$
cf. Definition 95.

(5) *We have*
$$(x\mathrm{s}_I)^{y\mathrm{s}_J}\in G\mathrm{D}_{n+1,J}\,;$$
cf. Definition 109.

Proof. Ad (1). Since $\tilde{\delta}$ is monotone and bijective, it suffices to show that

$$\max\big(I\backslash(I\cap J)\big)\overset{!}{<}\max\big(J\backslash(I\cap J)\big)\,.$$

We *assume* that $\max\big(I\backslash(I\cap J)\big)\geqslant\max\big(J\backslash(I\cap J)\big)$.

We have $\max\big(I\backslash(I\cap J)\big) \neq \max\big(J\backslash(I\cap J)\big)$ since $\big(I\backslash(I\cap J)\big)\cap\big(J\backslash(I\cap J)\big) = \emptyset$ and since not both $I\backslash(I\cap J)$ and $J\backslash(I\cap J)$ are empty, because $I \neq J$.

So we have $\max\big(I\backslash(I\cap J)\big) > \max\big(J\backslash(I\cap J)\big)$.

By Remark 94.(1), we have $I\backslash(I\cap J) > J\backslash(I\cap J)$.

Then, by Remark 94.(4), it follows that $I > J$, which *contradicts* $I < J$.

This shows that $\max\big(I\backslash(I\cap J)\big) < \max\big(J\backslash(I\cap J)\big)$.

So we have $\max\tilde{I} < \max\tilde{J}$.

Ad (2). Note that $\tilde{m} \leqslant \tilde{n}+1$ since $\tilde{I} \subseteq [0,\tilde{n}+1]$.

So we have

$$K \subseteq [0,\tilde{m}] \subseteq [0,\tilde{n}+1]\,.$$

We have

$$\max K \leqslant \tilde{m} = \max\tilde{I} \overset{(1)}{<} \max\tilde{J}\,.$$

By Remark 94.(1), we have $K < \tilde{J}$.

Since $\tilde{\delta}$ is injective and monotone, we have, by Remark 94.(5), $K\tilde{\delta} < \tilde{J}\tilde{\delta}$.

So, by Remark 94.(3), it follows that

$$K\tilde{\delta}\,\dot\cup\,(I\cap J) < \tilde{J}\tilde{\delta}\,\dot\cup\,(I\cap J) = \Big(\big(J\backslash(I\cap J)\big)\tilde{\delta}^{-}\,\tilde{\delta}\Big)\,\dot\cup\,(I\cap J) = \big(J\backslash(I\cap J)\big)\,\dot\cup\,(I\cap J) = J\,,$$

as claimed in (2).

Ad (3). *We show that* $I\cap J \subset [n]$.

We *assume* that $I\cap J = [n]$. This implies that $I = J = [n]$ which *contradicts* $I < J$; cf. Lemma 92.

We have

$$\tilde{I}\tilde{\delta}\,\dot\cup\,(I\cap J) = \big(I\backslash(I\cap J)\big)\tilde{\delta}^{-}\tilde{\delta}\,\dot\cup\,(I\cap J) = \big(I\backslash(I\cap J)\big)\,\dot\cup\,(I\cap J) = I\,.$$

Moreover, since $I\cap J \subset [n]$, and $\tilde{I} \subseteq [\tilde{n}] = \big[n - |I\cap J|\big]$, we have

$$\mathrm{s}_{\tilde{I}} \mathbin{\blacktriangle} \mathrm{s}_{I\cap J} \overset{39}{=} \mathrm{s}_{\tilde{I}\tilde{\delta}\,\dot\cup\,(I\cap J)} = \mathrm{s}_I \colon G_{n+1-|I|} \to G_{n+1}\,.$$

Similarly, we have $\mathrm{s}_{\tilde{J}} \mathbin{\blacktriangle} \mathrm{s}_{I\cap J} = \mathrm{s}_J \colon G_{n+1-|J|} \to G_{n+1}$.

Ad (4). We show that $(G\mathrm{N}_{n+1-|I|})\mathrm{s}_{\tilde{I}} \overset{!}{\leqslant} G_{\tilde{n}+1;[\tilde{m}+2,\tilde{n}+1]}$.

Note that $\tilde{I} \subseteq [0,\tilde{m}]$, hence $|\tilde{I}| \leqslant \tilde{m}+1$.

Suppose given $g \in G\mathrm{N}_{n+1-|I|}$. Suppose given $\ell \in [\tilde{m}+2, \tilde{n}+1]$. So $\ell - |\tilde{I}| \geqslant \tilde{m}+2-|\tilde{I}| \geqslant 1$.

Then, by Remark 38, we have

$$\mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{d}_{\ell} = \mathrm{d}_{\ell-|\tilde{I}|} \blacktriangle \mathrm{s}_{\tilde{I}} .$$

So we have

$$(g\mathrm{s}_{\tilde{I}})\mathrm{d}_{\ell} = g(\mathrm{d}_{\ell-|\tilde{I}|})\mathrm{s}_{\tilde{I}} = 1\mathrm{s}_{\tilde{I}} = 1 .$$

This shows that $(G\mathrm{N}_{n+1-|I|})\mathrm{s}_{\tilde{I}} \leqslant G_{\tilde{n}+1;[\tilde{m}+2,\tilde{n}+1]}$.

Ad (5). We have

$$(x\mathrm{s}_I)^{y\mathrm{s}_J} \overset{(3)}{=} \big((x\mathrm{s}_{\tilde{I}})^{y\mathrm{s}_{\tilde{J}}}\big)\mathrm{s}_{I\cap J} .$$

Moreover, we have

$$x\mathrm{s}_{\tilde{I}} \ \in (G\mathrm{N}_{n+1-|I|})\mathrm{s}_{\tilde{I}} \overset{(4)}{\in} G_{\tilde{n}+1;[\tilde{m}+2,\tilde{n}+1]} .$$

Since $G_{\tilde{n}+1;[\tilde{m}+2,\tilde{n}+1]} \trianglelefteqslant G_{\tilde{n}+1}$, we have

$$(x\mathrm{s}_{\tilde{I}})^{y\mathrm{s}_{\tilde{J}}} \ \in G_{\tilde{n}+1;[\tilde{m}+2,\tilde{n}+1]} .$$

We write $r := |\mathcal{P}([0,\tilde{m}])| = 2^{\tilde{m}+1}$ and $\mathcal{P}([0,\tilde{m}]) =: \{K_1, K_2, \ldots, K_r\}$, where $K_1 < K_2 < \ldots < K_r$. Note that $K_r = [0,\tilde{m}]$; cf. Example 93.(1).

By Lemma 100.(1), we have

$$\Big(\tilde{G}_{n+1,[\tilde{m}+2,\tilde{n}+1]}\Big)\,\mathrm{s}_{I\cap J} \overset{100.(1)}{=} \Big(\prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{\tilde{n}+1-|K_i|})\mathrm{s}_{K_i}\Big)\mathrm{s}_{I\cap J} = \prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{\tilde{n}+1-|K_i|})(\mathrm{s}_{K_i} \blacktriangle \mathrm{s}_{I\cap J}) .$$

Let $i \in [1,r]$. Note that $I\cap J \subset [0,n]$ and that $K_i \subseteq [0,\tilde{m}] \subseteq [0,\tilde{n}]$.

By Corollary 39, we have $\mathrm{s}_{K_i} \blacktriangle \mathrm{s}_{I\cap J} = \mathrm{s}_{K_i\tilde{\delta}\,\dot\cup\,(I\cap J)} \colon G_{\tilde{n}+1-|K_i|} \to G_{n+1}$.

By (2), we have $K_i\tilde{\delta}\,\dot\cup\,(I\cap J) < J$. Since $I\cap J \neq \emptyset$, we also have $\emptyset < K_i\tilde{\delta}\,\dot\cup\,(I\cap J)$.

So altogether, we have

$$\begin{aligned}(x\mathrm{s}_I)^{y\mathrm{s}_J} = \big((x\mathrm{s}_{\tilde{I}})^{y\mathrm{s}_{\tilde{J}}}\big)\mathrm{s}_{I\cap J} \in \big(G_{n+1,[\tilde{m}+2,n+1]}\big)\,\mathrm{s}_{I\cap J} &= \prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{n+1-|K_i|})(\mathrm{s}_{K_i} \blacktriangle \mathrm{s}_{I\cap J}) \\ &\overset{39}{=} \prod_{i\in\lfloor r,1\rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i\tilde{\delta}\,\dot\cup\,(I\cap J)} \\ &\overset{(2),109}{\subseteq} G\mathrm{D}_{n+1,J} .\end{aligned}$$

□

Remark 112 *Let $n \in \mathbb{Z}_{\geqslant 1}$.*
Suppose given an $\lfloor n, 0 \rfloor$-coskeletal simplicial group G; cf. Definition 90.
Suppose given I, $J \in \mathcal{P}\big([0,n]\big)$ such that $\emptyset < I < J$; cf. Lemma 92.
Suppose given $x \in G\mathrm{N}_{n+1-|I|}$ and suppose given $y \in G\mathrm{N}_{n+1-|J|}$.
We consider the group morphisms

$$\mathrm{s}_I = \mathrm{s}_I^{G,n+1-|I|} \colon G_{n+1-|I|} \to G_{n+1}$$
$$\mathrm{s}_J = \mathrm{s}_J^{G,n+1-|J|} \colon G_{n+1-|J|} \to G_{n+1}\,;$$

cf. Definition 37.(2).
We have

$$x\mathrm{s}_I \cdot y\mathrm{s}_J = y\mathrm{s}_J \cdot \tilde{x}$$

for some $\tilde{x} \in G\mathrm{D}_{n+1,J}$; cf. Definition 109.

Proof. We show this claim by induction on $I \in \big\{K \in \mathcal{P}\big([0,n]\big) \colon \emptyset < K < J\big\}$.
Case $I \cap J \neq \emptyset$. Then, by Lemma 111.(5), we have

$$\tilde{x} := (x\mathrm{s}_I)^{y\mathrm{s}_J} \in G\mathrm{D}_{n+1,J}\,.$$

We have

$$x\mathrm{s}_I \cdot y\mathrm{s}_J = y\mathrm{s}_J \cdot (x\mathrm{s}_I)^{y\mathrm{s}_J} = y\mathrm{s}_J \cdot \tilde{x}\,.$$

Case $I \cap J = \emptyset$. By Lemma 110, we have

$$y\mathrm{s}_J = y\mathrm{c}_J \cdot y'$$

for some $y' \in G\mathrm{D}_{n+1,J}$ and we have

$$x\mathrm{s}_I = x\mathrm{c}_I \cdot x'$$

for some $x' \in G\mathrm{D}_{n+1,I}$.
Note that $x' \in G\mathrm{D}_{n+1,I} \leqslant G\mathrm{D}_{n+1,J}$ and that $x\mathrm{s}_I \in G\mathrm{D}_{n+1,J}$ since $I < J$; cf. Definition 109.
We write

$$x' = \prod_{b \in \lfloor a,1 \rfloor} x'_b\, \mathrm{s}_{A_b} \in G\mathrm{D}_{n+1,I}\,,$$

where $a \in \mathbb{Z}_{\geqslant 0}$ and $A_b \subseteq [0,n]$ with $\emptyset < A_b < I$ for $b \in [1,a]$, and where $x'_b \in G\mathrm{N}_{n+1-|A_b|}$.
Let $b \in [1,a]$. By induction hypothesis, we have

$$x'_b \, \mathrm{s}_{A_b} \cdot y\mathrm{s}_J = y\mathrm{s}_J \cdot \tilde{x}'_b$$

for some $\tilde{x}'_b \in G\mathrm{D}_{n+1,J}$.
So we have

$$x' \cdot y\mathrm{s}_J = \Big(\prod_{b \in \lfloor a,1 \rfloor} x'_b \, \mathrm{s}_{A_b} \Big) \cdot y\mathrm{s}_J = y\mathrm{s}_J \cdot \Big(\prod_{b \in \lfloor a,1 \rfloor} \tilde{x}'_b \Big) ,$$

where $\tilde{x}' := \prod_{b \in \lfloor a,1 \rfloor} x'_b \in G\mathrm{D}_{n+1,J}$.
Therefore, we have

$$\begin{aligned} x\mathrm{s}_I \cdot y\mathrm{s}_J &= (x\mathrm{c}_I \cdot x') \cdot y\mathrm{s}_J = x\mathrm{c}_I \cdot y\mathrm{s}_J \cdot \tilde{x}' = x\mathrm{c}_I \cdot (y\mathrm{c}_J \cdot y') \cdot \tilde{x}' \overset{108}{=} y\mathrm{c}_J \cdot x\mathrm{c}_I \cdot y' \cdot \tilde{x}' \\ &= \big(y\mathrm{s}_J \cdot (y')^-\big) \cdot x\mathrm{c}_I \cdot y' \cdot \tilde{x}' = y\mathrm{s}_J \cdot \big((y')^- \cdot x\mathrm{s}_I \cdot (x')^- \cdot y' \cdot \tilde{x}'\big) , \end{aligned}$$

where $\tilde{x} := (y')^- \cdot x\mathrm{s}_I \cdot (x')^- \cdot y' \cdot \tilde{x}' \in G\mathrm{D}_{n+1,J}$. □

Lemma 113 (The degenerate subgroup $G\mathrm{D}_{n+1}$)
Let $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given an $\lfloor n,0\rfloor$-coskeletal simplicial group G; cf. Definition 90.
We consider the decomposition of the group G_{n+1} from Corollary 101*, where*

$$G_{n+1} = \prod_{i \in \lfloor r,1 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} ,$$

where $r = 2^{n+1}$, $\mathcal{P}\big([0,n]\big) = \{K_1, K_2, \ldots, K_r\}$ and $K_1 < K_2 < \ldots < K_r$.
Recall that $K_1 = \emptyset$, $K_2 = \{0\}$ and that $K_r = [0,n]$; cf. Example 93.(2).
Then, for $\ell \in [2,r]$, we have the subgroup

$$\prod_{i \in \lfloor \ell,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} \leqslant G_{n+1} .$$

In particular, we have the subgroup

$$G\mathrm{D}_{n+1} := \prod_{i \in \lfloor r,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} \leqslant G_{n+1} .$$

We call $G\mathrm{D}_{n+1}$ the degenerate subgroup *of G_{n+1}.*

Proof. We show this claim by induction on $\ell \in [2, r]$.

Let $\ell = 2$. We have

$$\prod_{i \in \lfloor 2,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} = (G\mathrm{N}_{n+1-|K_2|})\mathrm{s}_{K_2} = (G\mathrm{N}_n)\mathrm{s}_0 \leqslant G_{n+1}\,.$$

Now suppose that $\ell \geqslant 3$. By induction hypothesis, we have the subgroup

$$\prod_{i \in \lfloor \ell-1,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} \leqslant G_{n+1}\,.$$

Hence $\prod_{i \in \lfloor \ell-1,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} = G\mathrm{D}_{n+1,K_\ell}$; cf. Definition 109.

Suppose given $g,\, h \in \prod_{i \in \lfloor \ell,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i}$.

We write $g = \prod_{i \in \lfloor \ell,2 \rfloor} x_i\,\mathrm{s}_{K_i}$, where $x_i \in G\mathrm{N}_{n+1-|K_i|}$ for $i \in [2, \ell]$.

We write $h = \prod_{i \in \lfloor \ell,2 \rfloor} y_i\,\mathrm{s}_{K_i}$, where $y_i \in G\mathrm{N}_{n+1-|K_i|}$ for $i \in [2, \ell]$.

We write $J := K_\ell$.

We show that $g \cdot h \overset{!}{\in} \prod_{i \in \lfloor \ell,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i}$.

By Remark 112, since G is $\lfloor n, 0 \rfloor$-coskeletal, we have, for $i \in [2, \ell-1]$,

$$x_i\,\mathrm{s}_{K_i} \cdot y_\ell\,\mathrm{s}_J = y_\ell\,\mathrm{s}_J \cdot \tilde{x}_i$$

for some $\tilde{x}_i \in G\mathrm{D}_{n+1,J}$.

So we have

$$\begin{aligned} g \cdot h &= \Big(\prod_{i \in \lfloor \ell,2 \rfloor} x_i\,\mathrm{s}_{K_i}\Big) \cdot \Big(\prod_{i \in \lfloor \ell,2 \rfloor} y_i\,\mathrm{s}_{K_i}\Big) = x_\ell\,\mathrm{s}_J \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} x_i\,\mathrm{s}_{K_i}\Big) \cdot y_\ell\,\mathrm{s}_J \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} y_i\,\mathrm{s}_{K_i}\Big) \\ &= x_\ell\,\mathrm{s}_J \cdot y_\ell\,\mathrm{s}_J \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} \tilde{x}_i\Big) \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} y_i\,\mathrm{s}_{K_i}\Big) = (x_\ell\,y_\ell)\mathrm{s}_J \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} \tilde{x}_i\Big) \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} y_i\,\mathrm{s}_{K_i}\Big)\,. \end{aligned}$$

Note that $\emptyset < K_i < J$ for $i \in [2, \ell-2]$. So, by Definition 109, we have

$$z := \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} \tilde{x}_i\Big) \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} y_i\,\mathrm{s}_{K_i}\Big) \in G\mathrm{D}_{n+1,J}\,.$$

Since $\prod_{\lfloor \ell-1,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} = G\mathrm{D}_{n+1,J} \leqslant G_{n+1}$, we have

$$z = \prod_{i \in \lfloor \ell-1,2 \rfloor} z_i \, \mathrm{s}_{K_i} ,$$

where $z_i \in G\mathrm{N}_{n+1-|K_i|}$ for $i \in [2, \ell-1]$.

Moreover, we have $z_\ell := x_\ell \, y_\ell \in G\mathrm{N}_{n+1-|K_\ell|}$. Therefore, we have

$$\begin{aligned} g \cdot h &= (x_\ell \, y_\ell)\mathrm{s}_J \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} \tilde{x}_i \Big) \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} y_i \, \mathrm{s}_{K_i} \Big) = z_\ell \, \mathrm{s}_{K_\ell} \cdot z = z_\ell \, \mathrm{s}_{K_\ell} \cdot \Big(\prod_{i \in \lfloor \ell-1,2 \rfloor} z_i \, \mathrm{s}_{K_i} \Big) \\ &= \prod_{i \in \lfloor \ell,2 \rfloor} z_i \, \mathrm{s}_{K_i} \in \prod_{i \in \lfloor \ell,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} . \end{aligned}$$

Altogether, this shows that we have the subgroup $\prod_{i \in \lfloor \ell,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} \leqslant G_{n+1}$; cf. Lemma 6.

□

Example 114

(1) Let $n = 1$. Suppose given a $\lfloor 1, 0 \rfloor$-coskeletal simplicial group G.

We consider the decomposition of the group G_2 from Corollary 101, where

$$G_2 = (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_0 \cdot G\mathrm{N}_2 ;$$

cf. also Example 102.(2).

Then, by Lemma 113, we have the subgroup

$$G\mathrm{D}_2 = (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_1 \cdot (G\mathrm{N}_1)\mathrm{s}_0 \leqslant G_2 .$$

(2) Let $n = 2$. Suppose given a $\lfloor 2, 0 \rfloor$-coskeletal simplicial group G.

We consider the decomposition of the group G_3 from Corollary 101, where

$$G_3 = (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_2 \cdot (G\mathrm{N}_2)\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_0 \cdot G\mathrm{N}_3 ;$$

cf. also Example 102.(3).

Then, by Lemma 113, we have the subgroup

$$G\mathrm{D}_3 = (G\mathrm{N}_0)\mathrm{s}_0\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_1\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_2 \cdot (G\mathrm{N}_2)\mathrm{s}_2 \cdot (G\mathrm{N}_1)\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_1 \cdot (G\mathrm{N}_2)\mathrm{s}_0 \leqslant G_3 .$$

Remark 115 Let $n \in \mathbb{Z}_{\geqslant 1}$.

We write $r := \big|\mathcal{P}\big([0,n]\big)\big| = 2^{n+1}$ and we write $\mathcal{P}\big([0,n]\big) =: \{K_1, K_2, K_3, \ldots, K_r\}$, where $\emptyset = K_1 < K_2 < \ldots < K_r = [0,n]$.

Let $\ell \in [3,r]$. We write $I := K_\ell$.

Suppose given a simplicial group G; cf. Definition 30.

We consider the subgroup

$$G\mathrm{D}_{n+1,I} = \big\langle (G\mathrm{N}_{n+1-|J|})\mathrm{s}_J \colon \emptyset < J < I \big\rangle \leqslant G_{n+1}$$

from Definition 109.

Suppose that G is $\lfloor n,0 \rfloor$-coskeletal; cf. Definition 90.

Then, by Lemma 113, we have

$$G\mathrm{D}_{n+1,I} = \prod_{i \in \lfloor \ell-1,2 \rfloor} (G\mathrm{N}_{n+1-|K_i|})\mathrm{s}_{K_i} \leqslant G_{n+1}\,.$$

Lemma 116 *Let $n \in \mathbb{Z}_{\geqslant 1}$.*

Suppose given an $\lfloor n,0 \rfloor$-coskeletal simplicial group G; cf. Definition 90.

We consider the subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$; cf. Lemma 113.

We consider the surjective map $\pi^G_{n+1} \colon G_{n+1} \to G\mathrm{N}_{n+1}$ from Corollary 101.(3).

Suppose given $g \in G_{n+1}$.

(1) *We have* $g \in G\mathrm{D}_{n+1} \quad \Leftrightarrow \quad g\pi^G_{n+1} = 1\,.$

(2) *We have* $g \in G\mathrm{N}_{n+1} \quad \Leftrightarrow \quad g\pi^G_{n+1} = g\,.$

(3) *We have* $g \in G\mathrm{D}_{n+1} \quad \Leftrightarrow \quad g\pi^G_{n+1}\mathrm{d}^{G,n+1}_0 = 1\,.$

Proof. Ad (1). Let $r := 2^{n+1}$. Let $\mathcal{P}([0,n]) =: \{I_1, I_2, \ldots, I_r\}$, where

$$\emptyset = I_1 < I_2 < \ldots < I_r = [0,n]\,.$$

By Corollary 101.(2), since $g \in G_{n+1}$, we have

$$g = \prod_{i \in \lfloor r,1 \rfloor} x_i\, \mathrm{s}_{I_i}\,,$$

where $x_i \in G\mathrm{N}_{n+1-|I_i|}$ for $i \in [1,r]$.

Ad $\Rightarrow$. Suppose that $g \in G\mathrm{D}_{n+1} \leqslant G_{n+1}$.

Since $g \in G\mathrm{D}_{n+1}$, we have

$$g \overset{113}{=} \prod_{i\in\lfloor r,2\rfloor} \tilde{x}_i \,\mathrm{s}_{I_i} = \Big(\prod_{i\in\lfloor r,2\rfloor} \tilde{x}_i \,\mathrm{s}_{I_i}\Big)\cdot 1 = \Big(\prod_{i\in\lfloor r,2\rfloor} \tilde{x}_i \,\mathrm{s}_{I_i}\Big)\cdot 1\mathrm{s}_{I_1}\,,$$

where $\tilde{x}_i \in G\mathrm{N}_{n+1-|I_i|}$ for $i \in [2,r]$.

Then, by Corollary 101.(2), we have $x_i = \tilde{x}_i$ for $i \in [2,r]$ and $x_1 = 1$.

This shows that $g\pi^G_{n+1} \overset{101.(3)}{=} x_1 = 1$.

Ad $\Leftarrow$. Suppose that $g\pi^G_{n+1} = 1$. Then we have $x_1 = g\pi^G_{n+1} \overset{101.(3)}{=} 1$.

So we have

$$g = \prod_{i\in\lfloor r,1\rfloor} x_i \,\mathrm{s}_{I_i} = \Big(\prod_{i\in\lfloor r,2\rfloor} x_i \,\mathrm{s}_{I_i}\Big)\cdot 1\mathrm{s}_{I_1} = \prod_{i\in\lfloor r,2\rfloor} x_i \,\mathrm{s}_{I_i} \overset{113}{\in} G\mathrm{D}_{n+1}\,.$$

Ad (2). Ad $\Rightarrow$. Suppose that $g \in G\mathrm{N}_{n+1} \leqslant G_{n+1}$.

Let $r := 2^{n+1}$. Let $\mathcal{P}([0,n]) =: \{I_1, I_2, \ldots, I_r\}$, where $\emptyset = I_1 < I_2 < \ldots < I_r = [0,n]$.

By Corollary 101.(2), since $g \in G_{n+1}$, we have

$$g = \prod_{i\in\lfloor r,1\rfloor} x_i \,\mathrm{s}_{I_i}\,,$$

where $x_i \in G\mathrm{N}_{n+1-|I_i|}$ for $i \in [1,r]$.

Since $g \in G\mathrm{N}_{n+1}$, we have

$$g = 1\cdot g = \Big(\prod_{i\in\lfloor r,2\rfloor} 1\mathrm{s}_{I_i}\Big)\cdot g\,.$$

Then, by Corollary 101.(2), we have $x_i = 1$ for $i \in [2,r]$ and $x_1 = g$.

This shows that $g\pi^G_{n+1} \overset{101.(3)}{=} x_1 = g$.

Ad $\Leftarrow$. Suppose that $g = g\pi^G_{n+1}$. We have $g = g\pi^G_{n+1} \in G\mathrm{N}_{n+1}$.

Ad (3). Ad $\Rightarrow$. Suppose that $g \in G\mathrm{D}_{n+1}$. By (1), we have $g\pi^G_{n+1} = 1$.

So we have $g\pi^G_{n+1}\mathrm{d}^{G,n+1}_0 = 1$.

Ad $\Leftarrow$. Suppose that $g\pi^G_{n+1}\mathrm{d}^{G,n+1}_0 = 1$. Recall that $g\pi^G_{n+1} \in G\mathrm{N}_{n+1}$.

So we have $g\pi^G_{n+1} \in G\mathrm{N}_{n+1} \cap \ker \mathrm{d}^{G,n+1}_0 \overset{46.(3)}{=} G\mathrm{NZ}_{n+1}$.

Since G is $\lfloor n, 0\rfloor$-coskeletal, we have that $G\mathrm{NZ}_{n+1} = 1$; cf. Definition 90.(1).

This shows that $g\pi^G_{n+1} = 1$. So, by (1), we have $g \in G\mathrm{D}_{n+1}$. □

Remark 117 *Suppose given simplicial groups G, H; cf. Definition* 30. *Suppose given a simplicial group morphism $\varphi\colon G \to H$.*

We consider the functor $\mathrm{N}\colon \mathcal{SimpGrp} \to \mathcal{ComplexGrp}$*; cf. Lemma* 48.(3).

We have

$$\pi^G_{n+1} \vartriangleleft \varphi\mathrm{N}_{n+1} = \varphi_{n+1} \vartriangleleft \pi^H_{n+1} : G_{n+1} \to H\mathrm{N}_{n+1};$$

cf. Corollary 101.(3).

$$\begin{array}{ccc} G_{n+1} & \xrightarrow{\varphi_{n+1}} & H_{n+1} \\ {\scriptstyle \pi^G_{n+1}}\downarrow & & \downarrow{\scriptstyle \pi^H_{n+1}} \\ G\mathrm{N}_{n+1} & \xrightarrow[\varphi\mathrm{N}_{n+1}]{} & H\mathrm{N}_{n+1} \end{array}$$

Proof. Suppose given $j \in [2, n+2]$.

We consider the maps

$$p^G_{n+1,j}\colon G_{n+1;[j,n+1]} \to G_{n+1,[j-1,n+1]} \quad \text{and} \quad p^H_{n+1,j}\colon H_{n+1;[j,n+1]} \to H_{n+1,[j-1,n+1]};$$

cf. Remark 97.(1).

For $g \in G_{n+1;[j,n+1]}$, we have

$$(gp^G_{n+1,j})\varphi_{n+1} \overset{97.(1)}{=} (g^- \mathrm{d}^{G,n+1}_{j-1}\mathrm{s}^{G,n}_{j-2} \cdot g)\varphi_{n+1} = (g\varphi_{n+1})^- \mathrm{d}^{H,n+1}_{j-1}\mathrm{s}^{H,n}_{j-2} \cdot g\varphi_{n+1} \overset{97.(1)}{=} (g\varphi_{n+1})p^H_{n+1,j}.$$

So, for $g \in G_{n+1}$, we have

$$\begin{aligned} g(\pi^G_{n+1} \vartriangleleft \varphi\mathrm{N}_{n+1}) &\overset{101.(3)}{=} g\Big(\Big(\prod_{j\in\lfloor n+2,2\rfloor}^{G\mathrm{N}_{n+1}} p^G_{n+1,j}\Big) \vartriangleleft \varphi\mathrm{N}_{n+1}\Big) = \Big(g\Big(\prod_{j\in\lfloor n+2,2\rfloor}^{G\mathrm{N}_{n+1}} p^G_{n+1,j}\Big)\Big)\varphi_{n+1} \\ &= (g\varphi_{n+1})\Big(\prod_{j\in\lfloor n+2,2\rfloor}^{H\mathrm{N}_{n+1}} p^H_{n+1,j}\Big) \overset{101.(3)}{=} g(\varphi_{n+1} \vartriangleleft \pi^H_{n+1}). \end{aligned}$$

This shows that $\pi^G_{n+1} \vartriangleleft \varphi\mathrm{N}_{n+1} = \varphi_{n+1} \vartriangleleft \pi^H_{n+1}$. □

Chapter 6

Simplicial groups and $\lfloor n, 0 \rfloor$-simplicial groups: the Conduché functor

6.1 The truncation functor $\mathrm{Trunc}^{\lfloor n,0 \rfloor} : \mathit{SimpGrp} \to \lfloor n, 0 \rfloor\text{-}\mathit{SimpGrp}$

Let $n \in \mathbb{Z}_{\geqslant 1}$.

Lemma 118 (The $\lfloor n, 0 \rfloor$-simplicial group $G\,\mathrm{Trunc}^{\lfloor n,0 \rfloor}$)

Suppose given a simplicial group G; cf. Definition 30.

We consider the group $G_n/G\mathrm{NB}_n$; cf. Definition 46.(4)*, Lemma* 47.(2).

For $k \in [0, n-1]$, we consider the group G_k.

(1) *For $j \in [0, n]$, we have the group morphism*

$$\bar{\mathrm{d}}_j \colon G_n/G\mathrm{NB}_n \to G_{n-1}\,,\ gG\mathrm{NB}_n \mapsto (gG\mathrm{NB}_n)\bar{\mathrm{d}}_j = g\mathrm{d}_j^{G,n}\,.$$

For $i \in [0, n-1]$, we have the group morphism

$$\bar{\mathrm{s}}_i \colon G_{n-1} \to G_n/G\mathrm{NB}_n\,,\ g \mapsto g\bar{\mathrm{s}}_i = (g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\,.$$

(2) *Using Lemma* 71.(3)*, we have the pre-$\lfloor n, 0 \rfloor$-simplicial group $G\,\mathrm{Trunc}^{\lfloor n,0 \rfloor}$ given as follows.*

For $k \in [0,n-1]$, we have

$$(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k = [k](G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}) := G_k$$
$$(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n = [n](G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}) := G_n/G\mathrm{NB}_n\,.$$

For $k \in [1,n-1]$ and $j \in [0,k]$, we have

$$\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} := \mathrm{d}_j^{G,k} \colon G_k \to G_{k-1}\,.$$

For $j \in [0,n]$, we have

$$\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} := \bar{\mathrm{d}}_j \colon G_n/G\mathrm{NB}_n \to G_{n-1}\,;$$

cf. (1).

For $k \in [0,n-2]$ and $i \in [0,k]$, we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} := \mathrm{s}_i^{G,k} \colon G_k \to G_{k+1}\,.$$

For $i \in [0,n-1]$, we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1} := \bar{\mathrm{s}}_i \colon G_{n-1} \to G_n/G\mathrm{NB}_n\,;$$

cf. (1).

$$G_n/G\mathrm{NB}_n \underset{\bar{\mathrm{s}}_i}{\overset{\bar{\mathrm{d}}_j}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i^{G,n-2}}{\overset{\mathrm{d}_j^{G,n-1}}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i^{G,1}}{\overset{\mathrm{d}_j^{G,2}}{\rightleftarrows}} G_1 \underset{\mathrm{s}_i^{G,0}}{\overset{\mathrm{d}_j^{G,1}}{\rightleftarrows}} G_0$$

(3) *The pre-$\lfloor n,0\rfloor$-simplicial group $G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ given in* (2) *is an $\lfloor n,0\rfloor$-simplicial group. We call $G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ the* truncation of G at $\lfloor n,0\rfloor$.

Proof. Ad (1). Suppose given $j \in [0,n]$.

We show that $\bar{\mathrm{d}}_j$ is well-defined. We have to show that $(G\mathrm{NB}_n)\mathrm{d}_j^{G,n} \overset{!}{=} 1$.

Suppose given $x \in G\mathrm{NB}_n$. We choose some $y \in G\mathrm{N}_{n+1}$ such that $x = y\mathrm{d}_0^{G,n+1}$; cf. Definition 46.(4).

We have

$$x\mathrm{d}_j^{G,n} = y\mathrm{d}_0^{G,n+1}\mathrm{d}_j^{G,n} = y\mathrm{d}_{j+1}^{G,n+1}\mathrm{d}_0^{G,n} = 1\,.$$

This shows $(G\mathrm{NB}_k)\mathrm{d}_j^{G,n} = 1$.

Therefore, we have the group morphism

$$\bar{\mathrm{d}}_j\colon G_n/G\mathrm{NB}_n \to G_{n-1}\,,\ gG\mathrm{NB}_n \mapsto g\mathrm{d}_j^{G,n}\,.$$

Ad (2). We use Lemma 71.(3) to show that $G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ is a pre-$\lfloor n,0\rfloor$-simplicial group.

To that end, we have to verify the properties (i, ii, iii) from Lemma 71.

Ad (i). Suppose given $k \in [1, n-1]$. Suppose given $i \in [0,k]$ and $j \in [0,k+1]$ with $i < j$.

Case $k \in [1, n-2]$. We have

$$\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \mathrm{d}_j^{G,k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G,k} \overset{33.(1)}{=} \mathrm{d}_i^{G,k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G,k} = \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}\,.$$

Case $k = n-1$. For $g \in G_n$, we have

$$\begin{aligned}(gG\mathrm{NB}_n)(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}) &= (gG\mathrm{NB}_n)(\bar{\mathrm{d}}_j \mathbin{\blacktriangle} \mathrm{d}_i^{G,n-1}) = (g\mathrm{d}_j^{G,n})\mathrm{d}_i^{G,n-1}\\ &\overset{33.(1)}{=} (g\mathrm{d}_i^{G,n})\mathrm{d}_{j-1}^{G,n-1} = (gG\mathrm{NB}_n)(\bar{\mathrm{d}}_i \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G,n-1})\\ &= (gG\mathrm{NB}_n)(\mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k})\,.\end{aligned}$$

This shows $\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}$.

Ad (ii). Suppose given $k \in [1, n-1]$. Suppose given $i \in [0, k-1]$ and $j \in [0,k]$ with $i < j$.

Case $k \in [1, n-2]$. We have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \mathrm{s}_i^{G,k-1} \mathbin{\blacktriangle} \mathrm{s}_j^{G,k} \overset{33.(2)}{=} \mathrm{s}_{j-1}^{G,k-1} \mathbin{\blacktriangle} \mathrm{s}_i^{G,k} = \mathrm{s}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}\,.$$

Case $k = n-1$. For $g \in G_{n-1}$, we have

$$\begin{aligned}g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}) &= g(\mathrm{s}_i^{G,n-2} \mathbin{\blacktriangle} \bar{\mathrm{s}}_j) = (g\mathrm{s}_i^{G,n-2}\mathrm{s}_j^{G,n-1})G\mathrm{NB}_n\\ &\overset{33.(2)}{=} (g\mathrm{s}_{j-1}^{G,n-2}\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n = g(\mathrm{s}_{j-1}^{G,n-2} \mathbin{\blacktriangle} \bar{\mathrm{s}}_i)\\ &= g(\mathrm{s}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k})\,.\end{aligned}$$

This shows $\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \mathrm{s}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}$.

Ad (iii). Suppose given $k \in [0, n-1]$. Suppose given $i \in [0,k]$ and $j \in [0,k+1]$.

Case $k \in [0, n-2]$. If $k \geqslant 1$ and $j < i$, then we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \mathbin{\blacktriangle} \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} = \mathrm{s}_i^{G,k} \mathbin{\blacktriangle} \mathrm{d}_j^{G,k+1} \overset{33.(3)}{=} \mathrm{d}_j^{G,k} \mathbin{\blacktriangle} \mathrm{s}_{i-1}^{G,k-1} = \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \mathbin{\blacktriangle} \mathrm{s}_{i-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1}\,.$$

If $j \in \{i, i+1\}$, then we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} = \mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} \overset{33.(3)}{=} \mathrm{id}_{G_k} = \mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k}\,.$$

If $k \geqslant 1$ and $j > i+1$, then we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} = \mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} \overset{33}{=} \mathrm{d}_{j-1}^{G,k} \blacktriangle \mathrm{s}_i^{G,k-1} = \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1}\,.$$

Case $k = n-1$. Suppose given $g \in G_{n-1}$. We have

$$g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1}) = g(\bar{\mathrm{s}}_i \blacktriangle \bar{\mathrm{d}}_j) = \big((g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\big)\bar{\mathrm{d}}_j = g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n}\,.$$

If $k \geqslant 1$ and $j < i$, then we have

$$\begin{aligned} g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1}) &= g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n} \overset{33.(3)}{=} g\mathrm{d}_j^{G,n-1}\mathrm{s}_{i-1}^{G,n-2} \\ &= g(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{s}_{i-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1})\,. \end{aligned}$$

If $j \in \{i, i+1\}$, then we have

$$g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1}) = g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n} \overset{33.(3)}{=} g\,\mathrm{id}_{G_k} = g\,\mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k}\,.$$

If $k \geqslant 1$ and $j > i+1$, then we have

$$\begin{aligned} g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1}) &= g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n} \overset{33.(3)}{=} g\mathrm{d}_{j-1}^{G,n-1}\mathrm{s}_i^{G,n-2} \\ &= g(\mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1})\,. \end{aligned}$$

Altogether, we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k+1} = \begin{cases} \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{s}_{i-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} & \text{if } k \geqslant 1 \text{ and } j < i \\ \mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k} & \text{if } j = i \text{ or } j = i+1 \\ \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k-1} & \text{if } k \geqslant 1 \text{ and } j > i+1\,. \end{cases}$$

Ad (3). Suppose given I, $J \subseteq [0,n]$ with I, $J \neq \emptyset$ and $I \cup J = [0,n]$.

We have to show that $\big[\bigcap_{i\in I} \ker \bar{\mathrm{d}}_i\,,\, \bigcap_{j\in J} \ker \bar{\mathrm{d}}_j\big] \overset{!}{=} 1$; cf. Definition 74.

Suppose given $xG\mathrm{NB}_n \in \bigcap_{i\in I} \ker \bar{\mathrm{d}}_i$ with $x \in G_n$. Then we have $x\mathrm{d}_i^{G,n} = 1$ for $i \in I$; cf. (1).

Suppose given $yG\mathrm{NB}_n \in \bigcap_{j\in J} \ker \bar{\mathrm{d}}_j$ with $y \in G_n$. Then we have $y\mathrm{d}_j^{G,n} = 1$ for $j \in J$; cf. (1).

We use Lemma 50 to show that $[x,y] \overset{!}{\in} G\mathrm{NB}_n$.

Suppose given $k \in [0,n]$. Since $I \cup J = [0,n]$, we have that $x\mathrm{d}_k^{G,n} = 1$ or $y\mathrm{d}_k^{G,n} = 1$.

Case $n \in I \cap J$. Since $I \cup J = [0,n]$, we may assume that $n-1 \in J$.

Let $\ell := n-1$. Then we have $x\mathrm{d}_{\ell+1}^{G,n} = x\mathrm{d}_n^{G,n} = 1$ and $y\mathrm{d}_\ell^{G,n} = y\mathrm{d}_{n-1}^{G,n} = 1$.

Case $n \notin I \cap J$. But since $I \cup J = [0,n]$, we may assume that $n \in I$.

Using $J \neq \emptyset$, we let $\ell := \max J \leqslant n-1$. Then $\ell + 1 \in I$.

So we have $x\mathrm{d}_{\ell+1}^{G,n} = 1$ and $y\mathrm{d}_\ell^{G,n} = 1$.

Therefore, by Lemma 50, we have $[x,y] \in G\mathrm{NB}_n$.

This shows $\big[x G\mathrm{NB}_n\,, yG\mathrm{NB}_n\big] = 1$.

Therefore, we have $\big[\bigcap_{i\in I} \ker \bar{\mathrm{d}}_i\,,\, \bigcap_{j\in J} \ker \bar{\mathrm{d}}_j\big] = 1$. □

Lemma 119 (The $\lfloor n,0\rfloor$-simplicial group morphism $\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$)
Suppose given simplicial groups G and H; cf. Definition 30.
We consider the $\lfloor n,0\rfloor$-simplicial groups $G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ and $H\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$; cf. Lemma 118.(3).
Suppose given a simplicial group morphism $\varphi\colon G \to H$.
For $k \in [0,n]$, we consider the group morphism $\varphi_k\colon G_k \to H_k$; cf. Remark 32.

(1) *We have the group morphism*

$$(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n\colon G_n/G\mathrm{NB}_n \to H_n/H\mathrm{NB}_n\,,\ xG\mathrm{NB}_n \mapsto (x\varphi_n)H\mathrm{NB}_n\,.$$

Moreover, for $k \in [0,n-1]$, we let

$$(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k := \varphi_k\colon G_k \to H_k\,.$$

(2) *We have the $\lfloor n,0\rfloor$-simplicial group morphism*

$$\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} = \big((G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \xrightarrow{(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k} (H\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k\big)_{k\in[0,n]}\colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,;$$

cf. (1).

Proof. Ad (1). *We show that* $\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ *is well-defined.*

It suffices to show that $(G\mathrm{NB}_n)\varphi_n \overset{!}{\subseteq} H\mathrm{NB}_n$.

Suppose given $x \in G\mathrm{NB}_n$. We choose some $y \in G\mathrm{N}_{n+1}$ such that $x = y\mathrm{d}_0^{G,n+1}$.

Let $z := y\varphi_{n+1} \in H_{n+1}$.

For $k \in [1, n+1]$, we have

$$z\mathrm{d}_k^{H,n+1} = y\varphi_{n+1}\mathrm{d}_k^{H,n+1} \overset{32}{=} y\mathrm{d}_k^{G,n+1}\varphi_n = 1\,.$$

This shows $z \in H\mathrm{N}_{n+1}$.

Moreover, we have

$$z\mathrm{d}_0^{H,n+1} = y\varphi_{n+1}\mathrm{d}_0^{H,n+1} \overset{32}{=} y\mathrm{d}_0^{G,n+1}\varphi_n = x\varphi_n\,.$$

This shows $x\varphi_n = z\mathrm{d}_0^{H,n+1} \in (H\mathrm{N}_{n+1})\mathrm{d}_0^{H,n+1} = H\mathrm{NB}_n$.

So we have $(G\mathrm{NB}_n)\varphi_n \subseteq H\mathrm{NB}_n$.

Ad (2). We use Lemma 73 to show that $\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ is an $\lfloor n,0\rfloor$-simplicial group morphism.

Suppose given $k \in [1,n]$ and $j \in [0,k]$.

Case $k \in [1, n-1]$. We have

$$(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \blacktriangle \mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \varphi_k \blacktriangle \mathrm{d}_j^{H,k} \overset{32}{=} \mathrm{d}_j^{G,k} \blacktriangle \varphi_{k-1} = \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{k-1}\,.$$

Case $k = n$. For $g \in G_n$, we have

$$\begin{aligned}
(gG\mathrm{NB}_n)\big((\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \blacktriangle \mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}\big) &= \big((g\varphi_n)H\mathrm{NB}_n\big)\mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \\
&\overset{118.(2,1)}{=} g\varphi_n\mathrm{d}_j^{H,n} \\
&\overset{32}{=} g\mathrm{d}_j^{G,n}\varphi_{n-1} \\
&= (g\mathrm{d}_j^{G,n})(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{n-1} \\
&\overset{118.(2,1)}{=} (gG\mathrm{NB}_n)\big(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{k-1}\big)\,.
\end{aligned}$$

This shows $(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \blacktriangle \mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{k-1}$.

Suppose given $k \in [0, n-1]$ and $i \in [0,k]$.

Case $k \in [0, n-2]$. We have

$$(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \blacktriangle \mathrm{s}_i^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \varphi_k \blacktriangle \mathrm{s}_i^{H,k} \overset{32}{=} \mathrm{s}_i^{G,k} \blacktriangle \varphi_{k+1} = \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{k+1}\,.$$

Case $k = n-1$. For $g \in G_{n-1}$, we have

$$\begin{array}{rcl} g\big((\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \blacktriangle \mathrm{s}_i^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k}\big) & = & (g\varphi_{n-1})\mathrm{s}_i^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1} \\ & \overset{118.(2,1)}{=} & (g\varphi_{n-1}\mathrm{s}_i^{H,n-1})H\mathrm{NB}_n \\ & \overset{32}{=} & (g\mathrm{s}_i^{G,n-1}\varphi_n)H\mathrm{NB}_n \\ & \overset{(1)}{=} & \big((g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\big)(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n \\ & \overset{118.(2,1)}{=} & g\big(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{k+1}\big)\,. \end{array}$$

This shows $(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \blacktriangle \mathrm{s}_i^{H\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} = \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},k} \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{k+1}$.

So, by Lemma 73, $\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ is an $\lfloor n,0\rfloor$-simplicial group morphism. □

Lemma 120 (The truncation functor $\mathrm{Trunc}^{\lfloor n,0\rfloor}$)

We have the truncation functor

$$\begin{array}{rccl} \mathrm{Trunc}^{\lfloor n,0\rfloor}\colon & \mathcal{SimpGrp} & \to & \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \\ & \big(G \xrightarrow{\varphi} H\big) & \mapsto & \big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \xrightarrow{\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor}} H\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)\,; \end{array}$$

cf. Lemma 118.(3), 119.(2).

Proof. Suppose given $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$.

We show that $(\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \overset{!}{=} \mathrm{id}_{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}}$.

Suppose given $k \in [0,n]$.

Case $k \in [0,n-1]$. We have

$$\big((\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_k \overset{119.(1)}{=} \mathrm{id}_{G_k} \overset{118.(2)}{=} \mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k}\,.$$

Case $k = n$. For $g \in G_n$, we have

$$\begin{array}{rcl} (gG\mathrm{NB}_n)\big((\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_k & \overset{119.(1)}{=} & (g\,\mathrm{id}_{G_n})G\mathrm{NB}_n = gG\mathrm{NB}_n = (gG\mathrm{NB}_n)\,\mathrm{id}_{G/G\mathrm{NB}_n} \\ & \overset{118.(2)}{=} & (gG\mathrm{NB}_n)\,\mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k}\,. \end{array}$$

This shows $\big((\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_k = \mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k}$.

So we have $(\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,0\rfloor} = \mathrm{id}_{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}}$.

Suppose given $F \xrightarrow{\varphi} G \xrightarrow{\tilde{\varphi}} H$ in $\mathcal{SimpGrp}$.

We show that $(\varphi \mathbin{\blacktriangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \overset{!}{=} \varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$.

Suppose given $k \in [0,n]$.

Let $\hat{\varphi} := \varphi \mathbin{\blacktriangle} \tilde{\varphi}\colon F \to H$. Then note that we have $\hat{\varphi}_k = \varphi_k \mathbin{\blacktriangle} \tilde{\varphi}_k\colon F_k \to H_k$.

Case $k \in [0,n-1]$. We have

$$\begin{aligned}\big((\varphi \mathbin{\blacktriangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_k &= (\hat{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \overset{119.(1)}{=} \hat{\varphi}_k = \varphi_k \mathbin{\blacktriangle} \tilde{\varphi}_k \overset{119.(1)}{=} (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \mathbin{\blacktriangle} (\tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \\ &= (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k\,.\end{aligned}$$

Case $k = n$. For $x \in F_n$, we have

$$\begin{aligned}(xF\mathrm{NB}_n)\big((\varphi \mathbin{\blacktriangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_k &= (xF\mathrm{NB}_n)(\hat{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n \\ &\overset{119.(1)}{=} (x\hat{\varphi}_n)H\mathrm{NB}_n \\ &= (x\varphi_n\tilde{\varphi}_n)H\mathrm{NB}_n \\ &\overset{119.(1)}{=} \big((x\varphi_n)G\mathrm{NB}_n\big)(\tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n \\ &\overset{119.(1)}{=} (xF\mathrm{NB}_n)\big((\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n \mathbin{\blacktriangle} (\tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n\big) \\ &= (xF\mathrm{NB}_n)(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k\,.\end{aligned}$$

This shows $\big((\varphi \mathbin{\blacktriangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_k = (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k$.

So we have $(\varphi \mathbin{\blacktriangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,0\rfloor} = \varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$. □

Lemma 121 (The Moore complex of $G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$)

Let $n \in \mathbb{Z}_{\geqslant 1}$. *For a simplicial group* G, *we consider its Moore complex*

$$G\mathrm{N} = \big(\ldots \xrightarrow{\partial_3^G} G\mathrm{N}_2 \xrightarrow{\partial_2^G} G\mathrm{N}_1 \xrightarrow{\partial_1^G} G\mathrm{N}_0 \xrightarrow{!} 1 \xrightarrow{!} \ldots\big);$$

cf. Lemma 47.(2).

(1) *Suppose given a simplicial group* G; *cf. Definition* 30.

We consider the $\lfloor n,0\rfloor$*-simplicial group* $G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$; *cf. Lemmas* 120, 118.

Recall that

$$(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k = \begin{cases} G_n/G\mathrm{NB}_n & \text{for } k = n \\ G_k & \text{for } k \in [0,n-1]\,. \end{cases}$$

We consider its Moore complex $(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}$*; cf. Lemma* 77.(2).

We have

$$(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_k = \begin{cases} G\mathrm{N}_n/G\mathrm{NB}_n & \text{for } k = n \\ G\mathrm{N}_k & \text{for } k \in [0, n-1] \\ 1 & \text{for } k \in \mathbb{Z}\backslash[0,n]\,. \end{cases}$$

For $k \in \mathbb{Z}$*, we write* $\bar{\partial}_k := \partial_k^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}}$.

Then we have

$$\bar{\partial}_n\colon G\mathrm{N}_n/G\mathrm{NB}_n \to G\mathrm{N}_{n-1}\,,\ gG\mathrm{NB}_n \mapsto (gG\mathrm{NB}_n)\bar{\partial}_n = g\mathrm{d}_0^{G,n}\,.$$

Note that $\operatorname{im}\bar{\partial}_n = (G\mathrm{N}_n)\mathrm{d}_0^G = \operatorname{im}\partial_n^G$.

For $k \in [1, n-1]$*, we have*

$$\bar{\partial}_k = \mathrm{d}_0^{G,k}\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}}\colon G\mathrm{N}_k \to G\mathrm{N}_{k-1}\,,\ g \mapsto g\bar{\partial}_k = g\mathrm{d}_0^{G,k}\,.$$

Note that $\bar{\partial}_k = \partial_k^G$ *for* $k \in [1, n-1]$.

So, we have $\operatorname{im}\bar{\partial}_k = \operatorname{im}\partial_k^G$ *for* $k \in [1, n]$.

Moreover, for $k \in \mathbb{Z}\backslash[1,n]$*, we have*

$$\bar{\partial}_k = !\ : G\mathrm{N}_k \mapsto G\mathrm{N}_{k-1}\,,\ g \mapsto 1\,.$$

So we have

$$(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N} = \ldots \xrightarrow{!} 1 \xrightarrow{!} G\mathrm{N}_n/G\mathrm{NB}_n \xrightarrow{\bar{\partial}_n} G\mathrm{N}_{n-1} \xrightarrow{\bar{\partial}_{n-1}} \ldots \xrightarrow{\bar{\partial}_1} G\mathrm{N}_0 \xrightarrow{!} 1 \xrightarrow{!} \ldots$$

(2) *Suppose given a simplicial group* G*; cf. Definition* 30.

We have the morphism of complexes in groups

$$G\psi\colon G\mathrm{N} \to (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}$$

that is given as follows.

For $k = n$*, we have*

$$(G\psi)_n\colon G\mathrm{N}_n \to (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n\,,\ g \mapsto gG\mathrm{NB}_n\,.$$

For $k \in [0, n-1]$, we have

$$(G\psi)_k = \mathrm{id}_{G\mathrm{N}_k} \colon G\mathrm{N}_k \to G\mathrm{N}_k\,,\ g \mapsto g\,.$$

Moreover, for $k \in \mathbb{Z}\backslash[0,n]$, we have

$$(G\psi)_k = ! : G\mathrm{N}_k \to 1\,,\ g \mapsto 1\,.$$

$$\begin{array}{ccccccccccccccccc}
\ldots & \xrightarrow{} & G\mathrm{N}_{n+1} & \xrightarrow{\partial^G_{n+1}} & G\mathrm{N}_n & \xrightarrow{\partial^G_n} & G\mathrm{N}_{n-1} & \xrightarrow{\partial^G_{n-1}} & \ldots & \xrightarrow{\partial^G_1} & G\mathrm{N}_0 & \xrightarrow{!} & 1 & \xrightarrow{!} & \ldots \\
 & & \downarrow{\scriptstyle !} & & \downarrow{\scriptstyle (G\psi)_n} & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_{n-1}}} & & & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_0}} & & \downarrow{\scriptstyle !} & & \\
\ldots & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & G\mathrm{N}_n/G\mathrm{NB}_n & \xrightarrow[\bar\partial_n]{} & G\mathrm{N}_{n-1} & \xrightarrow[\bar\partial_{n-1}]{} & \ldots & \xrightarrow[\bar\partial_1]{} & G\mathrm{N}_0 & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & \ldots
\end{array}$$

(3) *We consider the functors*

$$\begin{array}{rrcll}
\mathrm{N}\colon & \textit{SimpGrp} & \to & \textit{ComplexGrp}\,; & \text{cf. Lemma 48.(3)} \\
\mathrm{Trunc}^{\lfloor n,0\rfloor}\colon & \textit{SimpGrp} & \to & \lfloor n,0\rfloor\text{-}\textit{SimpGrp}\,; & \text{cf. Lemma 120} \\
\mathrm{N}\colon & \lfloor n,0\rfloor\text{-}\textit{SimpGrp} & \to & \textit{ComplexGrp}\,; & \text{cf. Lemma 78.(3)}\ .
\end{array}$$

We have the transformation

$$\psi = \big(G\mathrm{N} \xrightarrow{G\psi} (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}\big)_{G\in\mathrm{Ob}(\textit{SimpGrp})} : \mathrm{N} \to \mathrm{Trunc}^{\lfloor n,0\rfloor} * \mathrm{N};$$

cf. (2).

(4) *Suppose given a simplicial group G; cf. Definition* 30.
Suppose that we have $G\mathrm{N}_k = 1$ for $k \in \mathbb{Z}_{\geqslant n+1}$; cf. Definition 46.(2).
In particular, we have $G\mathrm{NB}_n = (G\mathrm{N}_{n+1})\mathrm{d}_0^{G,n+1} = 1$; cf. Definition 46.(4).
Then the morphism of complexes in groups $G\psi$ given in (2) *is an isomorphism.*

$$\begin{array}{ccccccccccccccccc}
\ldots & \xrightarrow{!} & 1 & \xrightarrow{!} & G\mathrm{N}_n & \xrightarrow{\partial^G_n} & G\mathrm{N}_{n-1} & \xrightarrow{\partial^G_{n-1}} & \ldots & \xrightarrow{\partial^G_1} & G\mathrm{N}_0 & \xrightarrow{!} & 1 & \xrightarrow{!} & \ldots \\
 & & \downarrow{\scriptstyle !} & & \wr\downarrow{\scriptstyle (G\psi)_n} & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_{n-1}}} & & & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_0}} & & \downarrow{\scriptstyle !} & & \\
\ldots & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & G\mathrm{N}_n/1 & \xrightarrow[\bar\partial_n]{} & G\mathrm{N}_{n-1} & \xrightarrow[\bar\partial_{n-1}]{} & \ldots & \xrightarrow[\bar\partial_1]{} & G\mathrm{N}_0 & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & \ldots
\end{array}$$

(5) *Suppose given a simplicial group G; cf. Definition* 30.

We have $\ker \bar\partial_n = G\mathrm{NZ}_n/G\mathrm{NB}_n$ *as subgroups of* $G\mathrm{N}_n/G\mathrm{NB}_n$.

(6) *Suppose given a simplicial group G; cf. Definition* 30.

Let $k \in [0,n]$. *We have the group isomorphism* $(G\psi)\mathrm{H}_k\colon G\mathrm{NH}_k \xrightarrow{\sim} (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{NH}_k$; *cf. Lemma* 11. *So in particular, we have*

$$(G\psi)\mathrm{H}_k\colon G\pi_k \xrightarrow{\sim} (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\pi_k\,;$$

cf. Lemma 77.(2).

Proof. Ad (1). It suffices to show that $(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n \overset{!}{=} G\mathrm{N}_n/G\mathrm{NB}_n$ as subgroups of $G_n/G\mathrm{NB}_n$.

Ad $\leqslant$. Suppose given $g \in G_n$ such that

$$gG\mathrm{NB}_n \in (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n = \bigcap_{i\in[1,n]} (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_{n,i} \leqslant (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n = G_n/G\mathrm{NB}_n\,.$$

Then, for $j \in [1,n]$, we have

$$1 = (gG\mathrm{NB}_n)\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \overset{118.(2)}{=} g\mathrm{d}_j^{G,n}\,.$$

So $g \in G\mathrm{N}_n$. This shows $(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n \leqslant G\mathrm{N}_n/G\mathrm{NB}_n$.

Ad $\geqslant$. Suppose given $g \in G\mathrm{N}_n$. Then, for $j \in [1,n]$, we have

$$1 = g\mathrm{d}_j^{G,n} \overset{118.(2)}{=} (gG\mathrm{NB}_n)\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n}\,.$$

So $gG\mathrm{NB}_n \in (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n$. This shows $G\mathrm{N}_n/G\mathrm{NB}_n \leqslant (G\,\mathrm{Trunc})\mathrm{N}_n$.

Altogether, we have $(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n = G\mathrm{N}_n/G\mathrm{NB}_n$.

Ad (2). Suppose given $k \in \mathbb{Z}$. We have to show that $\partial_k^G \mathbin{\blacktriangle} (G\psi)_{k-1} \overset{!}{=} (G\psi)_k \mathbin{\blacktriangle} \bar\partial_k$.

It suffices to consider $k \in [1, n+1]$.

Case $k \in [1, n-1]$. For $g \in G\mathrm{N}_k$, we have

$$g\big(\partial_k^G \mathbin{\blacktriangle} (G\psi)_{k-1}\big) = (g\mathrm{d}_0^{G,k})\,\mathrm{id}_{G\mathrm{N}_{k-1}} = (g\,\mathrm{id}_{G\mathrm{N}_k})\mathrm{d}_0^{G,k} = g\big((G\psi)_k \mathbin{\blacktriangle} \bar\partial_k\big)\,.$$

Case $k = n$. For $g \in G\mathrm{N}_n$, we have

$$g\big(\partial_n^G \blacktriangle (G\psi)_{n-1}\big) = (g\mathrm{d}_0^{G,n})\,\mathrm{id}_{G\mathrm{N}_{n-1}} = g\mathrm{d}_0^{G,n} \overset{(1)}{=} (gG\mathrm{NB}_n)\bar{\partial}_n = g\big((G\psi)_n \blacktriangle \bar{\partial}_n\big)\,.$$

Case $k = n+1$. Suppose given $g \in G\mathrm{N}_{n+1}$. Note that $g\mathrm{d}_0^{G,n+1} \in G\mathrm{NB}_n$. We have

$$g\big(\partial_{n+1}^G \blacktriangle (G\psi)_n\big) = (g\mathrm{d}_0^{G,n+1})G\mathrm{NB}_n = 1G\mathrm{NB}_n = \big(g(G\psi)_{n+1}\big)\,! = g\big((G\psi)_{n+1} \blacktriangle \bar{\partial}_{n+1}\big)\,.$$

Ad (3). Suppose given a simplicial group morphism $\varphi\colon G \to H$.

We have to show that $G\psi \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N} \overset{!}{=} \varphi\mathrm{N} \blacktriangle H\psi$.

$$\begin{array}{ccc} G\mathrm{N} & \xrightarrow{G\psi} & (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N} \\ {\scriptstyle \varphi\mathrm{N}}\downarrow & & \downarrow{\scriptstyle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}} \\ H\mathrm{N} & \xrightarrow[H\psi]{} & (H\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N} \end{array}$$

Suppose given $k \in \mathbb{Z}$. We have to show that $(G\psi)_k \blacktriangle (\rho\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_k \overset{!}{=} \varphi\mathrm{N}_k \blacktriangle (H\psi)_k$.

Case $k \in [0, n-1]$. For $g \in G\mathrm{N}_k$, we have

$$\begin{aligned} g\big((G\psi)_k \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_k\big) &= (g\,\mathrm{id}_{G\mathrm{N}_k})\big((\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_k\big) = g\varphi_k \\ &= \big(g(\varphi\mathrm{N}_k)\big)\,\mathrm{id}_{H\mathrm{N}_k} = g\big(\varphi\mathrm{N}_k \blacktriangle (H\psi)_k\big)\,. \end{aligned}$$

Case $k = n$. For $g \in G\mathrm{N}_n$, we have

$$\begin{aligned} g\big((G\psi)_n \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n\big) &= (gG\mathrm{NB}_n)\big((\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n\big) = (g\varphi_n)H\mathrm{NB}_n = (g\varphi_n)(H\psi)_n \\ &= g\big(\varphi\mathrm{N}_n \blacktriangle (H\psi)_n\big)\,. \end{aligned}$$

Case $k \in \mathbb{Z}\backslash[0,n]$. We have

$$(G\psi)_k \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_k = !\ \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_k = !\ = \varphi\mathrm{N}_k \blacktriangle\ !\ = \varphi\mathrm{N}_k \blacktriangle (H\psi)_k\,.$$

Ad (4). For $k \in [0, n-1]$, note that we have

$$(G\psi)_k = \mathrm{id}_{G\mathrm{N}_k}\colon G\mathrm{N}_k \to G\mathrm{N}_k\,.$$

For $k \in \mathbb{Z}\backslash[0,n]$, note that we have

$$(G\psi)_k = ! : 1 \to 1\,.$$

Moreover, note that

$$(G\psi)_n\colon G\mathrm{N}_n \to G\mathrm{N}_n/G\mathrm{NB}_n\,,\ g \mapsto gG\mathrm{NB}_n$$

is a group isomorphism since we have $G\mathrm{NB}_n = (G\mathrm{N}_{n+1})\mathrm{d}_0^{G,n+1} = 1\mathrm{d}_0^{G\mathrm{N}_{n+1}} = 1$.

This shows that $G\psi$ is an isomorphism of complexes in groups.

Ad (5). Ad $\leqslant$. Suppose given $x \in G\mathrm{N}_n$, such that $xG\mathrm{NB}_n \in \ker\bar{\partial}_n$.

We have

$$1 = (xG\mathrm{NB}_n)\bar{\partial}_n \overset{(1)}{=} x\mathrm{d}_0^{G,n} = x\partial_n^G\,.$$

This shows that $x \in G\mathrm{NZ}_n$.

So we have $\ker\bar{\partial}_n \leqslant G\mathrm{NZ}_n/G\mathrm{NB}_n$.

Ad $\geqslant$. Suppose given $y \in G\mathrm{NZ}_n$. In particular, we have $y \in G\mathrm{N}_n$.

We consider $yG\mathrm{NB}_n \in G\mathrm{NZ}_n/G\mathrm{NB}_n$.

For $i \in [1,n]$, we have

$$(yG\mathrm{NB}_n)\mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \overset{118.(2)}{=} y\mathrm{d}_i^{G,n} = 1\,.$$

This shows that $yG\mathrm{NB}_n \in (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}_n$.

Moreover, we have

$$(yG\mathrm{NB}_n)\bar{\partial}_n \overset{(1)}{=} y\mathrm{d}_0^{G,n} = y\partial_n^G = 1\,.$$

This shows that $yG\mathrm{NB}_n \in \ker\bar{\partial}_n$.

So we have $G\mathrm{NZ}_n/G\mathrm{NB}_n \leqslant \ker\bar{\partial}_n$.

Altogether, this shows that $\ker\bar{\partial}_n = G\mathrm{NZ}_n/G\mathrm{NB}_n$.

Ad (6). *Case* $k \in [0,n-1]$. We have

$$\begin{array}{rccl}(G\psi)\mathrm{H}_k\colon & \ker\partial_k^G/\operatorname{im}\partial_{k+1}^G & \to & \ker\bar{\partial}_k/\operatorname{im}\bar{\partial}_{k+1} \\ & g\operatorname{im}\partial_{k+1}^G & \mapsto & \big(g(G\psi)_k\big)\operatorname{im}\bar{\partial}_{k+1} \overset{(2)}{=} g\operatorname{im}\bar{\partial}_{k+1} \overset{(1)}{=} g\operatorname{im}\partial_{k+1}^G\,.\end{array}$$

So $(G\psi)\mathrm{H}_k$ is an isomorphism for $k \in [0,n-1]$.

Case $k = n$. We have

$$(G\psi)\mathrm{H}_n\colon \ker\partial_n^G/\operatorname{im}\partial_{n+1}^G \to \ker\bar{\partial}_n/\operatorname{im}\bar{\partial}_{n+1}\,,\ g\operatorname{im}\partial_{n+1}^G \mapsto \big(g(G\psi)_n\big)\operatorname{im}\bar{\partial}_{n+1} \overset{(2)}{=} (gG\mathrm{NB}_n)\operatorname{im}\bar{\partial}_{n+1}\,.$$

We show that $(G\psi)\mathrm{H}_n$ *is injective.*

Suppose given $g \in \ker \partial_n^G$ such that $(g \operatorname{im} \partial_{n+1}^G)\big((G\psi)\mathrm{H}_n\big) = 1$.

I.e. we have that $g(G\mathrm{NB}_n) \operatorname{im} \bar{\partial}_{n+1} = 1$.

Note that we have $\bar{\partial}_{n+1} = ! : 1 \to G\mathrm{N}_n/G\mathrm{NB}_n$. So we have $gG\mathrm{NB}_n = 1$, i.e. we have

$$g \in G\mathrm{NB}_n = (G\mathrm{N}_{n+1})\mathrm{d}_0^{G,n+1} = \operatorname{im} \partial_n^G .$$

This shows that $g \operatorname{im} \partial_{n+1}^G = 1 \operatorname{im} \partial_{n+1}^G$.

So $(G\psi)\mathrm{H}_n$ is injective.

We show that $(G\psi)\mathrm{H}_n$ *is surjective.*

Suppose given $g \in G\mathrm{N}_n$ such that $gG\mathrm{NB}_n \in \ker \bar{\partial}_n \overset{(5)}{=} G\mathrm{NZ}_n/G\mathrm{NB}_n$.

Then we have $g \in G\mathrm{NZ}_n = \ker \partial_n^G$.

We consider the element $(gG\mathrm{NB}_n) \operatorname{im} \bar{\partial}_{n+1} \in (G \operatorname{Trunc}^{\lfloor n,0\rfloor})\mathrm{H}_n$.

We have

$$g \operatorname{im} \partial_{n+1}^G \in G\mathrm{NZ}_n/G\mathrm{NB}_n = G\mathrm{NH}_n$$

and

$$(g \operatorname{im} \partial_{n+1}^G)\big((G\psi)\mathrm{H}_n\big) = \big(g(G\psi)_n\big) \operatorname{im} \bar{\partial}_{n+1} = \big(gG\mathrm{NB}_n\big) \operatorname{im} \bar{\partial}_{n+1} .$$

So $(G\psi)\mathrm{H}_n$ is surjective.

This shows that $(G\psi)\mathrm{H}_n$ is a group isomorphism. □

6.2 The Conduché functor $\mathrm{Cond}^{\lfloor n,0\rfloor} : \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp}$

Remark 122 *Let* $n \in \mathbb{Z}_{\geqslant 1}$.

Suppose given an $\lfloor n,0\rfloor$*-coskeletal simplicial group* G*; cf. Definition* 90.

Suppose given $g_n \in G_n$.

For $\ell \in [0,n]$*, we have* $g_n \, \mathrm{s}_\ell^{G,n} \in G\mathrm{D}_{n+1}$*; cf. Lemma* 113.

Proof. By Corollary 101, we have

$$G_n = \prod_{k \in \lfloor r,1\rfloor} (G\mathrm{N}_{n-|I_k|})\mathrm{s}_{I_k} ,$$

where $r = 2^n$, where $\mathcal{P}\big([0,n-1]\big) = \{I_1\,, I_2\,, \ldots, I_r\}$ and where $I_1 < I_2 < \ldots < I_r\,.$
So we may write

$$g_n = \prod_{k\in\lfloor r,1\rfloor} x_k\,\mathrm{s}_{I_k}$$

where $x_k \in G\mathrm{N}_{n-|I_k|}\,.$
We have

$$g_n\,\mathrm{s}_\ell = \Big(\prod_{k\in\lfloor r,1\rfloor} x_k\,\mathrm{s}_{I_k}\Big)\mathrm{s}_\ell = \prod_{k\in\lfloor r,1\rfloor} x_k(\mathrm{s}_{I_k}\mathbin{\blacktriangle}\mathrm{s}_\ell)\,.$$

Let $k\in[1,r]$. Note that $\{\ell\}\subset[n]$ since $n\geqslant 1$. Note that $I_k\subseteq[n-1] = \big[n-|\{\ell\}|\big]$.
Suppose given $\ell\in[0,n]$. Then, by Corollary 39, we have

$$\mathrm{s}_{I_k}\mathbin{\blacktriangle}\mathrm{s}_\ell = \mathrm{s}_{I_k}\mathbin{\blacktriangle}\mathrm{s}_{\{\ell\}} = \mathrm{s}_{I_k\delta_{\{\ell\}}\,\dot\cup\,\{\ell\}} : G_{n-|I_k|}\to G_{n+1}\,.$$

Let $J_k := I_k\delta_{\{\ell\}}\,\dot\cup\,\{\ell\}\in\mathcal{P}\big([0,n]\big)$. Note that we have $J_k\neq\emptyset$ since $\ell\in J_k\,.$
This shows that $J_k\in\mathcal{P}\big([0,n]\big)\backslash\{\emptyset\}\,.$
So we have

$$g_n\,\mathrm{s}_\ell = \prod_{k\in\lfloor r,1\rfloor} x_k(\mathrm{s}_{I_k}\mathbin{\blacktriangle}\mathrm{s}_\ell) = \prod_{k\in\lfloor r,1\rfloor} x_k\mathrm{s}_{J_k}\in G\mathrm{D}_{n+1}\,.$$

□

Lemma 123 (The simplicial subgroup $G^\vee$)
Let $n\in\mathbb{Z}_{\geqslant 1}\,.$ Suppose given an $\lfloor n,0\rfloor$-coskeletal simplicial group G; cf. Definition 90.
We consider the degenerate subgroup $G\mathrm{D}_{n+1}\leqslant G_{n+1}$ from Lemma 113.
Suppose given $k\in\mathbb{Z}_{\geqslant 0}\,.$ Let $A_k := \Big\{[n+1]\xrightarrow{a}[k]\colon\ a \text{ is injective and monotone}\Big\}.$
We write

$$G^\vee_k := \bigcap_{a\in A_k} G_a^{-1}(G\mathrm{D}_{n+1})\leqslant G_k\,.$$

So, if $k\in[0,n]$, then we have $A_k = \emptyset$. Moreover, we have $A_{n+1} = \{\mathrm{id}_{[n+1]}\}\,.$
So we have

$$G^\vee_k = \begin{cases} G_k & \text{if } \ k\in[0,n] \\ G\mathrm{D}_{n+1} & \text{if } \ k = n+1 \\ \{g\in G_k\colon gG_a\in G\mathrm{D}_{n+1} \text{ for } a\in A_k\} & \text{if } \ k\in\mathbb{Z}_{\geqslant n+2}\,. \end{cases}$$

We have that $G^\vee \leqslant G$ is a simplicial subgroup; cf. Definition 40, Lemma 41.(1).

Proof. Suppose given $\big([\ell] \xrightarrow{u} [k]\big) \in \mathrm{Mor}(\Delta)$. We have to show that $(G_k^\vee)G_u \overset{!}{\leqslant} G_\ell^\vee$.

Suppose given $g \in G_k^\vee$. Suppose given $b \in A_\ell$. We have to show that $(gG_u)G_b \overset{!}{\in} G\mathrm{D}_{n+1}$.

We consider the monotone map $\big([n+1] \xrightarrow{b \blacktriangle u} [k]\big) \in \mathrm{Mor}(\Delta)$.

Let $m := |\operatorname{im}(b \blacktriangle u)| - 1$. By Corollary 22.(1), we may write

$$b \blacktriangle u = \bar{v} \blacktriangle \dot{v} \colon [n+1] \to [k],$$

where $\bar{v} \colon [n+1] \to [m]$ is monotone and surjective, and where $\dot{v} \colon [m] \to [k]$ is monotone and injective. Note that $m \in [0, n+1]$ since $\bar{v}$ is surjective.

Case $m = n+1$. Then we have $\bar{v} = \mathrm{id}_{[n+1]}$.

So $b \blacktriangle u = \mathrm{id}_{[n+1]} \blacktriangle \dot{v} = \dot{v} : [n+1] \to [k]$ is injective.

This shows that $b \blacktriangle u \in A_k$. Therefore, we have

$$(gG_u)G_b = gG_{b \blacktriangle u} \in (G_k^\vee)G_{b \blacktriangle u} = \Big(\bigcap_{a \in A_k} G_a^{-1}(G\mathrm{D}_{n+1}) \Big) G_{b \blacktriangle u} \leqslant G\mathrm{D}_{n+1}.$$

Case $m \in [0, n]$. By Lemma 20.(2.1), we have

$$\bar{v} = \prod_{r \in \lfloor n+1-m, 1 \rfloor}^{[n+1]} {}^{[m]}\, \sigma_{i_r}^{m+r-1},$$

where $I := \{i_1, i_2, \ldots, i_{n+1-m}\} = \{i \in [0,n] \colon i\bar{v} = (i+1)\bar{v}\}$, and where $0 \leqslant i_1 < i_2 < \ldots < i_{n+1-m} \leqslant n$.

Note that $\bar{v} = \sigma_I^m \colon [n+1] \to [m]$; cf. Definition 25.(2). So we have

$$G_{\bar{v}} = G_{\sigma_I} \overset{37.(2)}{=} \mathrm{s}_I : G_m \to G_{n+1}.$$

Note that $I \neq \emptyset$ since $m \leqslant n < n+1$ and that $\max I \in [0,n]$. Let $\tilde{I} := I \backslash \{\max I\} \subset I$. Then $I = \tilde{I} \,\dot\cup\, \{\max I\}$.

We have

$$G_{\bar{v}} = \mathrm{s}_I = \mathrm{s}_{\tilde{I} \dot\cup \{\max I\}} = \mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_{\max I} \colon G_m \to G_{n+1}.$$

Moreover, note that $(gG_{\dot{v}})\mathrm{s}_{\tilde{I}} \in G_n$.

Then we have

$$(gG_u)G_b = gG_{b\blacktriangle u} = gG_{\bar{v}\blacktriangle \dot{v}} = (gG_{\dot{v}})G_{\bar{v}} = (gG_{\dot{v}})(\mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_{\max I}) = \big((gG_{\dot{v}})\mathrm{s}_{\tilde{I}}\big)\mathrm{s}_{\max I} \overset{122}{\in} G\mathrm{D}_{n+1}\,.$$

Altogether, this shows that $(gG_u)G_b \in G\mathrm{D}_{n+1}$.

So we have $(G_k^\vee)G_u \leqslant G_\ell^\vee$.

This shows that $G^\vee \leqslant G$ is a simplicial subgroup; cf. Lemma 41.(1). □

Lemma 124 (The simplicial group morphism $\varphi^\vee$)

Let $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given $\lfloor n,0\rfloor$-coskeletal simplicial groups G, H; cf. Definition 90.

Suppose given a simplicial group morphism $\varphi\colon G \to H$.

We consider the simplicial subgroups $G^\vee \leqslant G$ and $H^\vee \leqslant H$ from Lemma 123.

Then we have the simplicial group morphism

$$\varphi^\vee := \varphi\big|_{G^\vee}^{H^\vee} : G^\vee \to H^\vee\,,$$

where

$$\varphi_k^\vee = \big(\varphi\big|_{G^\vee}^{H^\vee}\big)_k = \varphi_k\big|_{G_k^\vee}^{H_k^\vee} : G_k^\vee \to H_k^\vee\,,\ g \mapsto g\varphi_k^\vee = g\varphi_k\,,$$

for $k \in \mathbb{Z}_{\geqslant 0}$; cf. Remark 42.

Proof. By Remark 42, it suffices to show that $(G_k^\vee)\varphi_k \overset{!}{\leqslant} H_k^\vee$ for $k \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$.

We show that $(G\mathrm{N}_k)\varphi_k \overset{!}{\leqslant} H\mathrm{N}_k$ for $k \in \mathbb{Z}_{\geqslant 0}$.

Note that we have $(G\mathrm{N}_0)\varphi_0 = G_0\varphi_0 \leqslant H_0 = H\mathrm{N}_0$.

Suppose that $k \geqslant 1$. Suppose given $x \in G\mathrm{N}_k$. For $j \in [1,k]$, we have

$$(x\varphi_k)\mathrm{d}_j^{H,k} = (x\mathrm{d}_j^{G,k})\varphi_{k-1} = 1\varphi_{k-1} = 1\,.$$

This shows that $x\varphi_k \in H\mathrm{N}_k$. Cf. also Lemma 48.(1).

We show that $(G\mathrm{D}_{n+1})\varphi_{n+1} \overset{!}{\leqslant} H\mathrm{D}_{n+1}$.

Suppose given $x \in G\mathrm{D}_{n+1}$. Then we may write

$$x = \prod_{i\in\lfloor r,2\rfloor} x_i\,\mathrm{s}_{I_i}\,,$$

where $r = 2^{n+1}$, where $\mathcal{P}\big([0,n]\big) = \{I_1, I_2, \ldots, I_r\}$, where $I_1 < I_2 < \ldots < I_r$, and where $x_i \in G\mathrm{N}_{n+1-|I_i|}$ for $i \in [1,r]$; cf. Lemma 113.

For $i \in [1,r]$, recall that we have $x_i\,\varphi_{n+1-|I_i|} \in H\mathrm{N}_{n+1-|I_i|}$.

We have

$$x\varphi_{n+1} = \Big(\prod_{i\in\lfloor r,2\rfloor} x_i\,\mathrm{s}_{I_i}\Big)\varphi_{n+1} = \prod_{i\in\lfloor r,2\rfloor} x_i(\mathrm{s}_{I_i} \blacktriangle \varphi_{n+1}) = \prod_{i\in\lfloor r,2\rfloor} x_i(\varphi_{n+1-|I_i|} \blacktriangle \mathrm{s}_{I_i})$$
$$= \prod_{i\in\lfloor r,2\rfloor} (x_i\,\varphi_{n+1-|I_i|})\mathrm{s}_{I_i} \overset{113}{\in} H\mathrm{D}_{n+1}\,.$$

Suppose given $g_k \in G_k^\vee$.

We show that $(G_k^\vee)\varphi_k \overset{!}{\leqslant} H_k^\vee$.

Suppose given $a \in A_k = \Big\{[n+1] \xrightarrow{u} [k]\colon\ u \text{ is injective and monotone}\Big\}$; cf. Lemma 123.

We have $g_k\,G_a \in G\mathrm{D}_{n+1}$ and so

$$(g_k\,\varphi_k)H_a = (g_k\,G_a)\varphi_{n+1} \in (G\mathrm{D}_{n+1})\varphi_{n+1} \leqslant H\mathrm{D}_{n+1}\,.$$

This shows that $g_k\,\varphi_k \in H_a^{-1}(H\mathrm{D}_{n+1})$.

So $g_k\,\varphi_k \in H_k^\vee$. This shows that $G_k^\vee\varphi_k \leqslant H_k^\vee$.

Therefore, $\varphi^\vee\colon G^\vee \to H^\vee$ is a simplicial group morphism. □

Remark 125 (The simplicial group $H\,\mathrm{Cond}^{\lfloor n,0\rfloor}$)

Let $n \in \mathbb{Z}_{\geqslant 1}$.

(1) Suppose given an $\lfloor n,0\rfloor$-simplicial group H; cf. Definition 74.

We consider the $\lfloor n,0\rfloor$-coskeletal simplicial group $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$; cf. Lemma 81.(1) and Remark 91.

We consider the degenerate subgroup $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1}$; cf. Lemma 113.

We consider the simplicial subgroup $(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^\vee \leqslant H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$; cf. Lemma 123.

We write

$$H\,\mathrm{Cond}^{\lfloor n,0\rfloor} := (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^\vee\,.$$

So, for $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned}(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k &= \bigcap_{\substack{a\in\Delta([n+1],[k])\\ a \text{ is injective}}} (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_a^{-1}\big((H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}\big)\\ &= \begin{cases}(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k & \text{if } k\in[0,n]\\ (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1} & \text{if } k = n+1\\ \bigcap\limits_{\substack{a\in\Delta([n+1],[k])\\ a \text{ is injective}}} (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_a^{-1}\big((H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}\big) & \text{if } k\in\mathbb{Z}_{\geqslant n+2}\,.\end{cases}\end{aligned}$$

So, given $[k] \xrightarrow{u} [\ell]$ in $\mathbf{\Delta}$, we have

$$\begin{aligned}(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_u\colon\ (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_\ell &\to (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k\\ (h_a)_{a\in\Delta([n],[\ell])} &\mapsto (h_{b\vartriangleleft u})_{b\in\Delta([n],[k])}\,.\end{aligned}$$

In particular, for $k \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0,k]$, we have

$$\begin{aligned}\mathrm{s}_i^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},k}\colon\ (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k &\to (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_{k+1}\\ (h_a)_{a\in\Delta([n],[k])} &\mapsto \big((h_a)_{a\in\Delta([n],[k])}\mathrm{s}_i^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},k} = (h_{b\vartriangleleft\sigma_i^k})_{b\in\Delta([n],[k+1])}\,.\end{aligned}$$

For $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,k]$, we have

$$\begin{aligned}\mathrm{d}_j^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},k}\colon\ (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k &\to (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_{k-1}\\ (h_a)_{a\in\Delta([n],[k])} &\mapsto \big((h_a)_{a\in\Delta([n],[k])}\mathrm{d}_j^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},k} = (h_{b\vartriangleleft\delta_j^k})_{b\in\Delta([n],[k-1])}\,.\end{aligned}$$

Cf. Lemmas 124, 81.(1).

(2) Suppose given $\lfloor n,0\rfloor$-simplicial groups G, H; cf. Definition 74.
Suppose given a simplicial group morphism $\varphi\colon G \to H$.
By (1), we have the simplicial groups

$$G\,\mathrm{Cond}^{\lfloor n,0\rfloor} = (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^\vee \quad\text{and}\quad H\,\mathrm{Cond}^{\lfloor n,0\rfloor} = (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^\vee.$$

By Lemma 124, we have the simplicial group morphism

$$(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^\vee\colon (G\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^\vee \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^\vee\,.$$

We write

$$\varphi\,\mathrm{Cond}^{\lfloor n,0\rfloor} := (\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^{\vee} \colon G\,\mathrm{Cond}^{\lfloor n,0\rfloor} \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\ .$$

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$ and suppose given $(g_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$.

Then we have

$$\big((g_a)_a\big)(\varphi\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k = \big((g_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})^{\vee}_k = \big((g_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \overset{82}{\equiv} (g_a\varphi_n)_a\,.$$

So, for $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$(\varphi\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \colon (G\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \to (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k\,,\ (g_a)_{a\in\Delta([n],[k])} \mapsto (g_a\varphi_n)_{a\in\Delta([n],[k])}\,.$$

Lemma 126 (The Conduché functor $\mathrm{Cond}^{\lfloor n,0\rfloor}$)
Let $n \in \mathbb{Z}_{\geqslant 1}$. We have the Conduché functor

$$\begin{array}{rccl} \mathrm{Cond}^{\lfloor n,0\rfloor}\colon & \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} & \to & \mathcal{SimpGrp}\,, \\ & \big(G \xrightarrow{\varphi} H\big) & \mapsto & \big(G\,\mathrm{Cond}^{\lfloor n,0\rfloor} \xrightarrow{\varphi\,\mathrm{Cond}^{\lfloor n,0\rfloor}} H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)\,; \end{array}$$

cf. Remark 125.(1, 2).

Proof. Suppose given $G \in \mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$.

Let $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $(g_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$.

We have

$$(g_a)_{a\in\Delta([n],[k])}\big((\mathrm{id}_G)\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)_k = (g_a\,\mathrm{id}_{G_n})_{a\in\Delta([n],[k])} = (g_a)_{a\in\Delta([n],[k])}\,.$$

This shows that $\big((\mathrm{id}_G)\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)_k = \mathrm{id}_{(G\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k}$.

So we have $(\mathrm{id}_G)\,\mathrm{Cond}^{\lfloor n,0\rfloor} = \mathrm{id}_{G\,\mathrm{Cond}^{\lfloor n,0\rfloor}}$.

Suppose given $G \xrightarrow{\varphi} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{\tilde{G}}$ in $\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$.

Let $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $(g_a)_{a\in\Delta([n],[k])} \in (G\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$.

We have

$$\begin{aligned} (g_a)_{a\in\Delta([n],[k])}\big((\varphi \blacktriangle \tilde{\varphi})\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)_k &= \big(g_a(\varphi_n \blacktriangle \tilde{\varphi}_n)\big)_{a\in\Delta([n],[k])} \\ &= \big((g_a\varphi_n)\tilde{\varphi}_n\big)_{a\in\Delta([n],[k])} \\ &= (g_a\varphi_n)_{a\in\Delta([n],[k])}(\tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \\ &= (g_a)_{a\in\Delta([n],[k])}\big((\varphi\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \blacktriangle (\tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k\big)\,. \end{aligned}$$

This shows that $\big((\varphi \mathbin{\triangle} \tilde{\varphi})\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)_k = (\varphi\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \mathbin{\triangle} (\tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$.
So we have $(\varphi \mathbin{\triangle} \tilde{\varphi})\,\mathrm{Cond}^{\lfloor n,0\rfloor} = \varphi\,\mathrm{Cond}^{\lfloor n,0\rfloor} \mathbin{\triangle} \tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,0\rfloor}$. □

Remark 127 *Let $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given an $\lfloor n,0\rfloor$-simplicial group H; cf. Definition* 74.
We consider the functor $\mathrm{Cond}^{\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp}$; cf. Lemma 126.
For $k \in \mathbb{Z}_{\geqslant n+1}$, we have

$$(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_k = 1\,;$$

cf. Definition 46.(2).

Proof. Suppose given $k \in \mathbb{Z}_{\geqslant n+1}$.
Case $k = n+1$. Suppose given

$$x \in (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1} \leqslant (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_{n+1} \overset{125.(1)}{=} (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}\,.$$

Then we have

$$x = \prod_{i\in\lfloor r,2\rfloor} x_i\,\mathrm{s}_{I_i} = \Big(\prod_{i\in\lfloor r,2\rfloor} x_i\,\mathrm{s}_{I_i}\Big)\cdot 1\,,$$

where $r = \big|\mathcal{P}([0,n])\big| = 2^{n+1}$, where $\mathcal{P}([0,n]) =: \{I_1, I_2, \ldots, I_r\}$, where $\emptyset = I_1 < \ldots < I_r = [0,n]$ and where $x_i \in (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1-|I_i|}$ for $i \in [2,r]$ and $1 \in (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1}$; cf. Lemma 113.
Moreover, since $x \in (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1}$, we have

$$x = 1\cdot x = \Big(\prod_{i\in\lfloor r,2\rfloor} 1\mathrm{s}_{I_i}\Big)\cdot x\,.$$

Then, by Corollary 101, we have $x_i = 1$ for $i \in [2,r]$ and we have $1 = x$.
This shows that $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1} = 1$.
Case $k \geqslant n+2$. We have $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \overset{125.(1)}{\leqslant} (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k$.
So $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_k \leqslant (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_k \overset{85}{=} 1$.
This shows that $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_k = 1$ if $k \geqslant 2$. □

6.3 The adjunction $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$

Lemma 128 *Let $n \in \mathbb{Z}_{\geqslant 1}$.*

Recall that we have $\lfloor n,0\rfloor$-$\mathcal{SimpGrp} \subseteq \mathcal{P}re$-$\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$; cf. Definition 74.

Let $\mathrm{I}\colon \lfloor n,0\rfloor$-$\mathcal{SimpGrp} \to \mathcal{P}re$-$\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$ be the inclusion functor.

(1) *We consider the functor $\mathrm{Cut}^{\lfloor n,0\rfloor}\colon \mathcal{SimpGrp} \to \mathcal{P}re$-$\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$; cf. Remark 79.*

For a simplicial group G, recall that $(G\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k = G_k$ for $k \in [0,n]$.

We consider the functor $\mathrm{Trunc}^{\lfloor n,0\rfloor}\colon \mathcal{SimpGrp} \to \lfloor n,0\rfloor$-$\mathcal{SimpGrp}$; cf. Lemma 120.

For a simplicial group G, recall that

$$(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k = \begin{cases} G_k & \text{for } k \in [0,n-1] \\ G_n/G\mathrm{NB}_n & \text{for } k = n\,; \end{cases}$$

cf. Lemma 118.

We have the transformation

$$\rho = \big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{G\rho} G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_{G\in\mathrm{Ob}(\mathcal{SimpGrp})}\colon\ \mathrm{Cut}^{\lfloor n,0\rfloor} \to \mathrm{Trunc}^{\lfloor n,0\rfloor} * \mathrm{I}\,,$$

where the pre-$\lfloor n,0\rfloor$-simplicial group morphism $G\rho\colon G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ is given as follows.

For $k \in [0,n-1]$, we have

$$(G\rho)_k := \mathrm{id}_{G_k}\colon G_k \to G_k\,.$$

Moreover, we have

$$(G\rho)_n\colon G_n \to G_n/G\mathrm{NB}_n\,,\ \ g \mapsto gG\mathrm{NB}_n\,.$$

(2) *We consider the functors*

$$\mathrm{Cond}^{\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp} \quad \text{and} \quad \mathrm{Cosk}^{\lfloor n,0\rfloor}\colon \mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp};$$

cf. Lemmas 126, 82.

For an $\lfloor n,0\rfloor$-simplicial group H, recall that $H\,\mathrm{Cond}^{\lfloor n,0\rfloor} \leqslant H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$. Cf. Remark 125.(1).

We have the transformation

$$\iota = \big(H\,\mathrm{Cond}^{\lfloor n,0\rfloor} \xrightarrow{H\iota} H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_{H\in\mathrm{Ob}(\lfloor n,0\rfloor\text{-}\mathit{SimpGrp})} \colon\ \mathrm{Cond}^{\lfloor n,0\rfloor} \to \mathrm{I} * \mathrm{Cosk}^{\lfloor n,0\rfloor}\,,$$

where the simplicial group morphism $H\iota \colon H\,\mathrm{Cond}^{\lfloor n,0\rfloor} \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ *is given as follows.*

For $k \in \mathbb{Z}_{\geqslant 0}$*, we have the embedding group morphism*

$$(H\iota)_k \colon (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\,,\ (h_a)_{a\in\Delta([n],[k])} \mapsto (h_a)_{a\in\Delta([n],[k])}\,;$$

cf. Lemma 82.

Note that we have $(H\iota)_k = \mathrm{id}_{(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k}$ *for* $k \in [0,n]$*; cf. Remark* 125.

(3) *Suppose given a simplicial group* G *and suppose given an* $\lfloor n,0\rfloor$*-simplicial group* H*; cf. Definitions* 30, 74.

Suppose given a simplicial group morphism $c \colon G \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$*.*

Suppose that we have $(G_{n+1})c_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}$*; cf. Lemmma* 113.

Then we have the simplicial group morphism

$$c^\vee := c\Big|^{H\mathrm{Cond}^{\lfloor n,0\rfloor}} \colon G \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,;$$

cf. Remark 42.

Moreover, we have $c^\vee \mathbin{\blacktriangle} H\iota = c \colon G \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$*; cf.* (2).

(4) *Suppose given a simplicial group* G *and suppose given an* $\lfloor n,0\rfloor$*-simplicial group* H*; cf. Definitions* 30, 74.

Suppose given a pre-$\lfloor n,0\rfloor$*-simplicial group morphism* $b \colon G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to H$*.*

Suppose that we have $(G\mathrm{NB}_n)b_n = 1$*.*

Then we have the $\lfloor n,0\rfloor$*-simplicial group morphism*

$$b^\wedge \colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H\,,$$

where

$$b^\wedge_k := b_k \colon G_k \to H_k$$

for $k \in [0,n-1]$*, and where*

$$b^\wedge_n \colon G_n/G\mathrm{NB}_n \to H_n\,,\ gG\mathrm{NB}_n \mapsto (gG\mathrm{NB}_n)\,b^\wedge_n := gb_n\,.$$

Moreover, we have $b = G\rho \mathbin{\blacktriangle} b^\wedge \colon G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to H$*; cf.* (1).

Proof. Ad (1). Suppose given a simplicial group G.

We use Lemma 73 to show that $G\rho = \big((G\rho)_k\big)_{k\in[0,n]}$ *is a pre-*$\lfloor n,0\rfloor$*-simplicial group morphism.*

It suffices to show that

$$(G\rho)_n \bullet \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \overset{!}{=} \mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n} \bullet (G\rho)_{n-1}$$

for $j \in [0,n]$, and that

$$(G\rho)_{n-1} \bullet \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1} \overset{!}{=} \mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n-1} \bullet (G\rho)_n$$

for $i \in [0,n-1]$.

Suppose given $j \in [0,n]$. For $g \in G_n$, we have

$$\begin{aligned} g\big((G\rho)_n \bullet \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n}\big) &= (gG\mathrm{NB}_n)\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \overset{118}{=} g\mathrm{d}_j^{G,n} = (g\mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n})\,\mathrm{id}_{G_{n-1}} \\ &= g\big(\mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n} \bullet (G\rho)_{n-1}\big)\,. \end{aligned}$$

This shows that $(G\rho)_n \bullet \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} = \mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n} \bullet (G\rho)_{n-1}$.

Suppose given $i \in [0,n-1]$. For $g \in G_n$, we have

$$\begin{aligned} g\big((G\rho)_{n-1} \bullet \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1}\big) &= (g(\mathrm{id}_{G_{n-1}})\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1}) \overset{118}{=} (g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n = (g\mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n-1})\rho_n \\ &= g\big(\mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n-1} \bullet (G\rho)_n\big)\,. \end{aligned}$$

This shows that $(G\rho)_{n-1} \bullet \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1} = \mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n-1} \bullet (G\rho)_n$.

So, by Lemma 73, $G\rho$ is a pre-$\lfloor n,0\rfloor$-simplicial group morphism.

We show that ρ *is a transformation.*

Suppose given a simplicial group morphism $\varphi\colon G \to \tilde{G}$.

We have to show that $G\rho \bullet \varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \overset{!}{=} \varphi\,\mathrm{Cut}^{\lfloor n,0\rfloor} \bullet \tilde{G}\rho$.

$$\begin{array}{ccc} G\,\mathrm{Cut}^{\lfloor n,0\rfloor} & \xrightarrow{G\rho} & G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \\ {\scriptstyle \varphi\,\mathrm{Cut}^{\lfloor n,0\rfloor}}\downarrow & & \downarrow{\scriptstyle \varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor}} \\ \tilde{G}\,\mathrm{Cut}^{\lfloor n,0\rfloor} & \xrightarrow[\tilde{G}\rho]{} & \tilde{G}\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \end{array}$$

6.3. THE ADJUNCTION $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$

Suppose given $k \in [0,n]$.

Case $k \in [0,n-1]$. For $g \in (G\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k = G_k$, we have

$$g\big((G\rho)_k \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k\big) = g(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_k \overset{119}{=} g\varphi_k = (g\varphi_k)(\tilde{G}\rho)_k = g\big((\varphi\,\mathrm{Cut}^{\lfloor n,0\rfloor})_k \blacktriangle (\tilde{G}\rho)_k\big)\,.$$

Case $k = n$. For $g \in (G\,\mathrm{Cut}^{\lfloor n,0\rfloor})_n = G_n$, we have

$$\begin{aligned} g\big((G\rho)_n \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n\big) &= (gG\mathrm{NB}_n)(\varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor})_n \overset{119}{=} (g\varphi_n)\tilde{G}\mathrm{NB}_n = (g\varphi_n)(\tilde{G}\rho)_n \\ &= g\big((\varphi\,\mathrm{Cut}^{\lfloor n,0\rfloor})_n \blacktriangle (\tilde{G}\rho)_n\big)\,. \end{aligned}$$

This shows that $G\rho \blacktriangle \varphi\,\mathrm{Trunc}^{\lfloor n,0\rfloor} = \varphi\,\mathrm{Cut}^{\lfloor n,0\rfloor} \blacktriangle \tilde{G}\rho$.

So ρ is a transformation.

Ad (2). *We show that* ι *is a transformation.*

Suppose given an $\lfloor n,0\rfloor$-simplicial group morphism $\psi\colon H \to \tilde{H}$.

We have to show that $H\iota \blacktriangle \psi\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \overset{!}{=} \psi\,\mathrm{Cond}^{\lfloor n,0\rfloor} \blacktriangle \tilde{H}\iota$.

$$\begin{array}{ccc} H\,\mathrm{Cond}^{\lfloor n,0\rfloor} & \xrightarrow{H\iota} & H\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \\ {\scriptstyle \psi\,\mathrm{Cond}^{\lfloor n,0\rfloor}}\downarrow & & \downarrow{\scriptstyle \psi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}} \\ \tilde{H}\,\mathrm{Cond}^{\lfloor n,0\rfloor} & \xrightarrow[\tilde{H}\iota]{} & \tilde{H}\,\mathrm{Cosk}^{\lfloor n,0\rfloor} \end{array}$$

Suppose given $(h_a)_{a\in\Delta([n],[k])} \in (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$. We have

$$\begin{aligned} (h_a)_a\,\big((H\iota)_k \blacktriangle (\psi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k\big) &= (h_a)_a\,(\psi\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_k \overset{125.(2)}{=} (h_a)_a\,(\psi\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \\ &= (h_a)_a\,\big((\psi\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \blacktriangle (\tilde{H}\iota)_k\big)\,. \end{aligned}$$

This shows that $H\iota \blacktriangle \psi\,\mathrm{Cosk}^{\lfloor n,0\rfloor} = \psi\,\mathrm{Cond}^{\lfloor n,0\rfloor} \blacktriangle \tilde{H}\iota$.

So ι is a transformation.

Ad (3). By Remark 42, it suffices to show that $(G_k)c_k \overset{!}{\leqslant} (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$ for $k \in \mathbb{Z}_{\geqslant 0}$.

Let $k \in \mathbb{Z}_{\geqslant 0}$. For a monotone injective map $a\colon [n+1] \to [k]$, we have

$$(G_k\,c_k)(H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_a = (G_k\,G_a)c_{n+1} \leqslant (G_{n+1})c_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}\,.$$

This shows that $(G_k)c_k \leqslant (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$; cf. Remark 125.

Then, by Remark 42, we have the simplicial group morphism

$$c^\vee = c\Big|^{H\mathrm{Cond}^{\lfloor n,0\rfloor}} : G \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\ .$$

Let $k \in \mathbb{Z}_{\geqslant 0}$. For $g \in G_k$, we have

$$g(c^\vee \blacktriangle H\iota)_k = g\big(c_k^\vee \blacktriangle (H\iota)_k\big) = (gc_k)(H\iota)_k = gc_k\,.$$

This shows that $(c^\vee \blacktriangle H\iota)_k = c_k$.

So we have $c^\vee \blacktriangle H\iota = c$.

Ad (4). The group morphism $b_n^\wedge \colon G_n/G\mathrm{NB}_n \to H_n$, $(gb_n)G\mathrm{NB}_n \mapsto gb_n$ is well-defined since we have $(G\mathrm{NB}_n)b_n = 1$.

We use Lemma 73 to show that $b^\wedge = (b_k^\wedge)_{k\in[0,n]}$ is an $\lfloor n,0\rfloor$-simplicial group morphism.

It suffices to shows that

$$b_n^\wedge \blacktriangle \mathrm{d}_j^{H,n} \overset{!}{=} \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \blacktriangle b_{n-1}^\wedge$$

for $j \in [0,n]$, and that

$$b_{n-1}^\wedge \blacktriangle \mathrm{s}_i^{H,n-1} \overset{!}{=} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1} \blacktriangle b_n^\wedge$$

for $i \in [0,n-1]$.

Suppose given $j \in [0,n]$. For $gG\mathrm{NB}_n \in G_n/G\mathrm{NB}_n$, we have

$$\begin{aligned} gG\mathrm{NB}_n(b_n^\wedge \blacktriangle \mathrm{d}_j^{H,n}) &= (gb_n)\mathrm{d}_j^{H,n} = (g\mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n})b_{n-1} = (g\mathrm{d}_j^{G,n})b_{n-1} \\ &\overset{118}{=} \big((gG\mathrm{NB}_n)\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n}\big)b_{n-1} = gG\mathrm{NB}_n(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \blacktriangle b_{n-1}^\wedge)\,. \end{aligned}$$

This shows that $b_n^\wedge \blacktriangle \mathrm{d}_j^{H,n} = \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n} \blacktriangle b_{n-1}^\wedge$.

Suppose given $i \in [0,n-1]$. For $g \in G_{n-1}$, we have

$$\begin{aligned} g(b_{n-1}^\wedge \blacktriangle \mathrm{s}_i^{H,n-1}) = (gb_{n-1})\mathrm{s}_i^{H,n-1} = (g\mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,0\rfloor},n-1})b_n = (g\mathrm{s}_i^{G,n-1})b_n = \big((g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\big)b_n^\wedge \\ \overset{118}{=} g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1} \blacktriangle b_n^\wedge)\,. \end{aligned}$$

This shows that $b_{n-1}^\wedge \blacktriangle \mathrm{s}_i^{H,n-1} = \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},n-1} \blacktriangle b_n^\wedge$.

Then, by Lemma 73, $b^\wedge \colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H$ is an $\lfloor n,0\rfloor$-simplicial group morphism.

Let $k \in [0, n-1]$. For $g \in G_k$, we have

$$g(G\rho \mathbin{\blacktriangle} b^{\wedge})_k = g\big((G\rho)_k \mathbin{\blacktriangle} b^{\wedge}_k\big) = (g\,\mathrm{id}_{G_k})b_k = gb_k\,.$$

Let $k = n$. For $g \in G_n$, we have

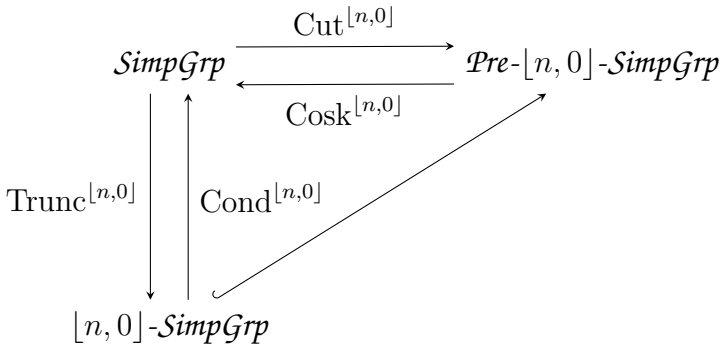
$$g(G\rho \mathbin{\blacktriangle} b^{\wedge})_n = g\big((G\rho)_n \mathbin{\blacktriangle} b^{\wedge}_n\big) = (gG\mathrm{NB}_n)b^{\wedge}_n = gb_n\,.$$

This shows that $(G\rho \mathbin{\blacktriangle} b^{\wedge})_k = b_k$ for $k \in [0, n]$.

So we have $G\rho \mathbin{\blacktriangle} b^{\wedge} = b$. □

Lemma 129 *Suppose given* $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ *and suppose given* $H \in \mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$. *We consider the following functors.*

$\mathrm{Cut}^{\lfloor n,0\rfloor}$:	$\mathcal{SimpGrp}$	$\to$	$\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$;	cf. Remark 79
$\mathrm{Cosk}^{\lfloor n,0\rfloor}$:	$\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$	$\to$	$\mathcal{SimpGrp}$;	cf. Lemma 82
$\mathrm{Trunc}^{\lfloor n,0\rfloor}$:	$\mathcal{SimpGrp}$	$\to$	$\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$;	cf. Lemma 120
$\mathrm{Cond}^{\lfloor n,0\rfloor}$:	$\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$	$\to$	$\mathcal{SimpGrp}$;	cf. Lemma 126



We consider the adjunction bijection concerning $\mathrm{Cut}^{\lfloor n,0\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,0\rfloor}$ *from Remark* 89

$$\begin{array}{rrcl}
\alpha_{G,H}\colon & {}_{\mathcal{P}re\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor},\ H\big) & \widetilde{\leftrightarrow} & {}_{\mathcal{SimpGrp}}\big(G,\ H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big) \\
& \big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{\varphi} H\big) & \mapsto & \big(G \xrightarrow{G\varepsilon \mathbin{\blacktriangle} \varphi\,\mathrm{Cosk}^{\lfloor n,0\rfloor}} H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big) \\
& \big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{\psi\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} H\eta} H\big) & \leftarrow\!\shortmid & \big(G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)\,.
\end{array}$$

We consider the pre-$\lfloor n,0\rfloor$*-simplicial group morphism* $G\rho\colon G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$ *; cf. Lemma* 128.(1).

We consider the simplicial group morphism $H\iota\colon H\,\mathrm{Cond}^{\lfloor n,0\rfloor} \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ *; cf. Lemma* 128.(2). *We have have the bijection*

$$\beta_{G,H}\colon {}_{\lfloor n,0\rfloor}\text{-}\mathcal{SimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, H\big) \xrightarrow{\sim} {}_{\mathcal{SimpGrp}}\big(G,\, H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)$$

given as follows.

For an $\lfloor n,0\rfloor$*-simplicial group morphism* $\varphi\colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H$*, we have the simplicial group morphism*

$$\varphi\beta_{G,H} := \big((G\rho \blacktriangle \varphi)\alpha_{G,H}\big)\big|^{H\mathrm{Cond}^{\lfloor n,0\rfloor}} : G \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,,$$

where, for $k \in \mathbb{Z}_{\geqslant 0}$*, we have*

$$\begin{array}{rccl}(\varphi\beta_{G,H})_k\colon & G_k & \to & (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \\ & g & \mapsto & g(\varphi\beta_{G,H})_k = g\big((G\rho \blacktriangle \varphi)\alpha_{G,H}\big)_k = g\Big((G\varepsilon)_k \blacktriangle \big((G\rho \blacktriangle \varphi)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k\Big)\,.\end{array}$$

For a simplicial group morphism $\psi\colon G \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}$*, we have the* $\lfloor n,0\rfloor$*-simplicial group morphism*

$$\psi\beta^{-}_{G,H}\colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H\,,$$

where, for $k \in [0, n-1]$*, we have*

$$(\psi\beta^{-}_{G,H})_k := \big((\psi \blacktriangle H\iota)\alpha^{-}_{G,H}\big)_k : G_k \to H_k\,,\ g \mapsto g\big((\psi \blacktriangle H\iota)\alpha^{-}_{G,H}\big)_k = g\big((\psi \blacktriangle H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \blacktriangle (H\eta)\big)_k$$

and where

$$\begin{array}{rccll}(\psi\beta^{-}_{G,H})_n\colon & G_n/G\mathrm{NB}_n & \to & H_n & \\ & gG\mathrm{NB}_n & \mapsto & (gG\mathrm{NB}_n)(\psi\beta^{-}_{G,H})_n & := g\big((\psi \blacktriangle H\iota)\alpha^{-}_{G,H}\big)_n \\ & & & & = g\big((\psi \blacktriangle H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \blacktriangle (H\eta)\big)_n\,.\end{array}$$

Moreover, the following diagram is commutative.

$$\begin{array}{ccc} {}_{\mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor},\, H\big) & \xrightarrow[\sim]{\alpha_{G,H}} & {}_{\mathcal{SimpGrp}}\big(G,\, H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big) \\ G\rho \blacktriangle (-)\uparrow & & \uparrow (-) \blacktriangle H\iota \\ {}_{\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, H\big) & \xrightarrow[\beta_{G,H}]{\sim} & {}_{\mathcal{SimpGrp}}\big(G,\, H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)\end{array}$$

So, for an $\lfloor n,0\rfloor$*-simplicial group morphism* $\varphi\colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H$*, we have*

$$\varphi\beta_{G,H} \mathbin{\blacktriangle} H\iota = (G\rho \mathbin{\blacktriangle} \varphi)\alpha_{G,H}\,.$$

For a simplicial group morphism $\psi\colon G \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}$*, we have*

$$G\rho \mathbin{\blacktriangle} \psi\beta^{-}_{G,H} = (\psi \mathbin{\blacktriangle} H\iota)\alpha^{-}_{G,H}\,.$$

Proof. We construct the map $\beta_{G,H}$*.*

We consider the map

$$\begin{array}{rccl}\tilde{\beta}_{G,H}\colon & {}_{\lfloor n,0\rfloor}\text{-}\mathcal{SimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, H\big) & \to & \mathcal{SimpGrp}\big(G,\, H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big) \\ & \big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \xrightarrow{\varphi} H\big) & \mapsto & \varphi\tilde{\beta}_{G,H} := \big(G \xrightarrow{(G\rho\mathbin{\blacktriangle}\varphi)\alpha_{G,H}} H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)\,.\end{array}$$

Suppose given $\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \xrightarrow{\varphi} H\big) \in {}_{\lfloor n,0\rfloor}\text{-}\mathcal{SimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, H\big)$.

We show that $(G_{n+1})(\varphi\tilde{\beta}_{G,H})_{n+1} \overset{!}{\leqslant} (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}$*; cf. Lemma* 113.

We consider the simplicial group morphism $\varphi\tilde{\beta}_{G,H} = (G\rho \mathbin{\blacktriangle} \varphi)\alpha_{G,H}\colon G \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$.

By Remark 117, we have

$$\pi^{G}_{n+1} \mathbin{\blacktriangle} (\varphi\tilde{\beta}_{G,H})\mathrm{N}_{n+1} = (\varphi\tilde{\beta}_{G,H})_{n+1} \mathbin{\blacktriangle} \pi^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}}_{n+1} \colon G_{n+1} \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1}\,.$$

Moreover, we have

$$\partial^{G}_{n+1} \mathbin{\blacktriangle} (\varphi\tilde{\beta}_{G,H})\mathrm{N}_{n} = (\varphi\tilde{\beta}_{G,H})\mathrm{N}_{n+1} \mathbin{\blacktriangle} \partial^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}}_{n+1} \colon G\mathrm{N}_{n+1} \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_{n}\,,$$

and we have

$$\iota_{G\mathrm{N}_n,G_n} \mathbin{\blacktriangle} (\varphi\tilde{\beta}_{G,H})_n = (\varphi\tilde{\beta}_{G,H})\mathrm{N}_n \mathbin{\blacktriangle} \iota_{(H\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_n,(H\mathrm{Cosk}^{\lfloor n,0\rfloor})_n} \colon G\mathrm{N}_n \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_n\,.$$

Suppose given $g \in G_n$. Note that for $a\colon [n] \to [n]$ in Δ, we have

$$G_a \mathbin{\blacktriangle} (G\rho)_n \mathbin{\blacktriangle} \varphi_n = (G\rho)_n \mathbin{\blacktriangle} (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor})a \mathbin{\blacktriangle} \varphi_n = (G\rho)_n \mathbin{\blacktriangle} \varphi_n \mathbin{\blacktriangle} H_a\,.$$

So we have

$$\begin{array}{rcl} g\big((\varphi\tilde{\beta}_{G,H})_n \mathbin{\blacktriangle} (H\eta)_n\big) & = & g\Big((G\varepsilon)_n \mathbin{\blacktriangle} \big((G\rho \mathbin{\blacktriangle} \varphi)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_n \mathbin{\blacktriangle} (H\eta)_n\Big) \\ & \overset{87.(2)}{=} & \big((gG_a)_{a\in\Delta([n],[n])}\big)\Big(\big((G\rho \mathbin{\blacktriangle} \varphi)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_n \mathbin{\blacktriangle} (H\eta)_n\Big) \\ & \overset{82}{=} & \Big(gG_a\big((G\rho)_n \mathbin{\blacktriangle} \varphi_n\big)\big)_{a\in\Delta([n],[n])}\Big)(H\eta)_n \\ & = & \Big(\big(g((G\rho)_n \mathbin{\blacktriangle} \varphi_n)H_a\big)_{a\in\Delta([n],[n])}\Big)(H\eta)_n \\ & \overset{86.(3)}{=} & g\big((G\rho)_n \mathbin{\blacktriangle} \varphi_n\big)\,. \end{array}$$

This shows that

$$(\varphi\tilde{\beta}_{G,H})_n \blacktriangle (H\eta)_n = (G\rho)_n \blacktriangle \varphi_n \,.$$

We write

$$\begin{array}{lll} \pi'_{n+1} & := \pi_{n+1}^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}} : & (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1} \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1} \\ \partial'_{n+1} & := \partial_{n+1}^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}} : & (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1} \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_n \\ \iota' & := \iota_{(H\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_n,(H\mathrm{Cosk}^{\lfloor n,0\rfloor})_n} : & (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_n \to (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_n \,. \end{array}$$

Then we have the following commutative diagram.

$$\begin{array}{ccccccccc} G_{n+1} & \xrightarrow{\pi^G_{n+1}} & G\mathrm{N}_{n+1} & \xrightarrow{\partial^G_{n+1}} & G\mathrm{N}_n & \xrightarrow{\iota_{G\mathrm{N}_n,G_n}} & G_n & \xrightarrow{(G\rho)_n \blacktriangle \varphi_n} & H_n \\ \downarrow{\scriptstyle (\varphi\tilde{\beta}_{G,H})_{n+1}} & & \downarrow{\scriptstyle (\varphi\tilde{\beta}_{G,H})\mathrm{N}_{n+1}} & & \downarrow{\scriptstyle (\varphi\tilde{\beta}_{G,H})\mathrm{N}_n} & & \downarrow{\scriptstyle (\varphi\tilde{\beta}_{G,H})_n} & \nearrow{\scriptstyle (H\eta)_n} & \\ (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_{n+1} & \xrightarrow[\pi'_{n+1}]{} & (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1} & \xrightarrow[\partial'_{n+1}]{} & (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{N}_n & \xrightarrow[\iota']{} & (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})_n & & \end{array}$$

Note that π^G_{n+1} and π'_{n+1} are just maps; cf. Corollary 101.(3).

Suppose given $x \in G_{n+1}$.

Note that we have $x\pi^G_{n+1} \in G\mathrm{N}_{n+1}$. So we have $x\pi^G_{n+1}\mathrm{d}_0^{G,n+1} \in G\mathrm{NB}_n$.

So we have

$$(x\pi^G_{n+1}\mathrm{d}_0^{G,n+1})(G\rho)_n = (x\pi^G_{n+1}\mathrm{d}_0^{G,n+1})G\mathrm{NB}_n = 1G\mathrm{NB}_n \,.$$

We have

$$\begin{aligned} & \big(x(\varphi\tilde{\beta}_{G,H})_{n+1}\big)\pi_{n+1}^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}}\ \mathrm{d}_0^{H\,\mathrm{Cosk}^{\lfloor n,0\rfloor},n+1} \\ &= \big(x(\varphi\tilde{\beta}_{G,H})_{n+1}\big)(\pi'_{n+1} \blacktriangle \partial'_{n+1} \blacktriangle \iota') \\ &= x\big((\varphi\tilde{\beta}_{G,H})_{n+1} \blacktriangle \pi'_{n+1} \blacktriangle \partial'_{n+1} \blacktriangle \iota'\big) \\ &= x\Big(\pi^G_{n+1} \blacktriangle \partial^G_{n+1} \blacktriangle \iota_{G\mathrm{N}_n,G_n} \blacktriangle \big((G\rho)_n \blacktriangle \varphi_n\big) \blacktriangle (H\eta)^-_n\Big) \\ &= (x\pi^G_{n+1}\mathrm{d}_0^{G,n+1})(G\rho)_n\,\big(\varphi_n \blacktriangle (H\eta)^-_n\big) \\ &= 1G\mathrm{NB}_n\big(\varphi_n \blacktriangle (H\eta)^-_n\big) \\ &= 1 \,. \end{aligned}$$

Recall that $H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}$ is $\lfloor n,0\rfloor$-coskeletal; cf. Remark 91.

Then, by Lemma 116.(3), we have $x(\varphi\tilde{\beta}_{G,H})_{n+1} \in (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}$.
This shows that $(G_{n+1})(\varphi\tilde{\beta}_{G,H})_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,0\rfloor})\mathrm{D}_{n+1}$.
So, by Lemma 128.(3), we have the simplicial group morphism

$$(\varphi\tilde{\beta}_{G,H})^{\vee} = (\varphi\tilde{\beta}_{G,H})\Big|^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}} : G \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor},$$

where

$$(\varphi\tilde{\beta}_{G,H})^{\vee} \blacktriangle H\iota = \varphi\tilde{\beta}_{G,H} : G \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\ .$$

We obtain the well-defined map

$$\beta_{G,H} : {}_{\lfloor n,0\rfloor}\text{-}\mathcal{SimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\ H\big) \to \mathcal{SimpGrp}\big(G,\ H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big),\ \varphi \mapsto \varphi\beta_{G,H} = (\varphi\tilde{\beta}_{G,H})^{\vee}.$$

Then we have

$$\varphi\beta_{G,H} = (\varphi\beta)\Big|^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}} = \big((G\rho \blacktriangle \varphi)\alpha_{G,H}\big)\Big|^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}}$$

and

$$\varphi\beta_{G,H} \blacktriangle H\iota = \varphi\tilde{\beta}_{G,H} = (G\rho \blacktriangle \varphi)\alpha_{G,H}.$$

We construct the map $\beta^{-}_{G,H}$.
We consider the map

$$\begin{array}{rcl} \tilde{\beta}^{\#}_{G,H} : \ \mathcal{SimpGrp}\big(G,\ H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big) & \to & \mathcal{Pre}\text{-}{}_{\lfloor n,0\rfloor}\text{-}\mathcal{SimpGrp}\big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor},\ H\big) \\ \big(G \xrightarrow{\psi} H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big) & \mapsto & \psi\tilde{\beta}^{\#}_{G,H} := \big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{(\psi \blacktriangle H\iota)\alpha^{-}_{G,H}} H\big). \end{array}$$

Suppose given $\big(G \xrightarrow{\psi} H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big) \in \mathcal{SimpGrp}\big(G,\ H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)$.
We show that $(G\mathrm{NB}_n)(\psi\tilde{\beta}^{\#}_{G,H})_n \overset{!}{=} 1$.
Suppose given $x \in G\mathrm{NB}_n$. Then we have $x = y\mathrm{d}_0^{G,n+1}$ for some chosen $y \in G\mathrm{N}_{n+1}$.
Note that $y\psi_{n+1} \in (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1} \overset{127}{=} 1$.
So we have

$$\begin{aligned} x(\psi\tilde{\beta}^{\#}_{G,H})_n &= (y\mathrm{d}_0)\big((\psi \blacktriangle H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \blacktriangle H\eta\big)_n = (y\mathrm{d}_0)\psi_n\,(H\eta)_n = (y\psi_{n+1})\mathrm{d}_0\,(H\eta)_n \\ &= 1\mathrm{d}_0\,(H\eta)_n = 1\,. \end{aligned}$$

This shows that $(G\mathrm{NB}_n)(\psi\tilde{\beta}^{\#}_{G,H})_n = 1$.

So, by Lemma 128.(4), we have the $\lfloor n,0\rfloor$-simplicial group morphism

$$(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}\colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\to H\,,$$

where, for $k\in[0,n-1]$, we have

$$(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_k:=(\psi\tilde{\beta}^{\#}_{G,H})_k : G_k\to H_k\,,\ g\mapsto g(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_k=g(\psi\tilde{\beta}^{\#}_{G,H})_k$$

and where

$$(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_n\colon G_n/G\mathrm{NB}_n\to H_n\,,\ gG\mathrm{NB}_n\mapsto(gG\mathrm{NB}_n)(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_n:=g(\psi\tilde{\beta}^{\#}_{G,H})_n\,.$$

Moreover, we have

$$\psi\tilde{\beta}^{\#}_{G,H}=G\rho_{\blacktriangle}\,(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}:G\,\mathrm{Cut}^{\lfloor n,0\rfloor}\to H\,.$$

Let

$$\beta^{\#}_{G,H}\colon \mathit{SimpGrp}\big(G,\,H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big)\to{}_{\lfloor n,0\rfloor}\text{-}\mathit{SimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\,H\big)\,,\ \psi\mapsto\psi\beta^{\#}_{G,H}:=(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}\,.$$

Then we have

$$(\psi\blacktriangle H\iota)\alpha^{-}_{G,H}=\psi\tilde{\beta}^{\#}_{G,H}=G\rho_{\blacktriangle}\,(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}=G\rho_{\blacktriangle}\,\psi\beta^{\#}_{G,H}\,.$$

We want to show that $\beta^{\#}_{G,H}$ is the inverse bijection of $\beta_{G,H}$.

We show that $\beta_{G,H}\blacktriangle\beta^{\#}_{G,H}\overset{!}{=}\mathrm{id}$.

Suppose given an $\lfloor n,0\rfloor$-simplicial group morphism $\varphi\colon G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\to H$.

We have to show that $\varphi(\beta_{G,H}\blacktriangle\beta^{\#}_{G,H})\overset{!}{=}\varphi$.

If suffices to show that $G\rho_{\blacktriangle}\big(\varphi(\beta_{G,H}\blacktriangle\beta^{\#}_{G,H})\big)\overset{!}{=}G\rho_{\blacktriangle}\,\varphi$.

We have

$$G\rho_{\blacktriangle}\big(\varphi(\beta_{G,H}\blacktriangle\beta^{\#}_{G,H})\big)=G\rho_{\blacktriangle}\,(\varphi\beta_{G,H})\beta^{\#}_{G,H}=(\varphi\beta_{G,H}\blacktriangle H\iota)\alpha^{-}_{G,H}=\big((G\rho_{\blacktriangle}\,\varphi)\alpha_{G,H}\big)\alpha^{-}_{G,H}=G\rho_{\blacktriangle}\,\varphi\,.$$

So we have $\varphi(\beta_{G,H}\blacktriangle\beta^{\#}_{G,H})=\varphi$.

This shows that $\beta_{G,H}\blacktriangle\beta^{\#}_{G,H}=\mathrm{id}$.

We show that $\beta^{\#}_{G,H}\blacktriangle\beta_{G,H}\overset{!}{=}\mathrm{id}$.

Suppose given a simplicial group morphism $\psi\colon G\to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}$.

6.3. THE ADJUNCTION $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$

We have to show that $\psi(\beta^{\#}_{G,H} \blacktriangle \beta_{G,H}) \overset{!}{=} \psi$.

If suffices to show that $\big(\psi(\beta^{\#}_{G,H} \blacktriangle \beta_{G,H})\big) \blacktriangle H\iota \overset{!}{=} \psi \blacktriangle H\iota$.

We have

$$\big(\psi(\beta^{\#}_{G,H} \blacktriangle \beta_{G,H})\big) \blacktriangle H\iota = (\psi\beta^{\#}_{G,H})\beta_{G,H} \blacktriangle H\iota = (G\rho \blacktriangle \psi\beta^{\#}_{G,H})\alpha_{G,H} = \big((\psi \blacktriangle H\iota)\alpha^{-}_{G,H}\big)\alpha_{G,H} = \psi \blacktriangle H\iota\,.$$

So we have $\psi(\beta^{\#}_{G,H} \blacktriangle \beta_{G,H}) = \psi$.

This shows that $\beta^{\#}_{G,H} \blacktriangle \beta_{G,H} = \mathrm{id}$.

Altogether, this shows that $\beta_{G,H}$ is a bijection, where $\beta^{-}_{G,H} = \beta^{\#}_{G,H}$. □

Theorem 130 *We have the isotransformation*

$$\beta = (\beta_{G,H})_{(G,H)\in \mathrm{Ob}(\mathcal{SimpGrp}^{\mathrm{op}} \times \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp})} : {}_{\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\big((-)\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, =\,\big) \xrightarrow{\sim} {}_{\mathcal{SimpGrp}}\big(-, (=)\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\big)\,;$$

cf. Lemma 129.

In particular, we have $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$ *; cf.* §1.1.3 *item* 10 *and Remark* 13.

$$\mathcal{SimpGrp} \underset{\mathrm{Cond}^{\lfloor n,0\rfloor}}{\overset{\mathrm{Trunc}^{\lfloor n,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$$

Proof. Suppose given $\big(G \xleftarrow{u} \tilde{G}\big)$ in $\mathcal{SimpGrp}$ and $\big(H \xrightarrow{v} \tilde{H}\big)$ in $\lfloor n,0\rfloor$-$\mathcal{SimpGrp}$.

We have to show that the following diagram is commutative.

$$\begin{array}{ccc} {}_{\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, H\big) & \xrightarrow[\sim]{\beta_{G,H}} & {}_{\mathcal{SimpGrp}}\big(G,\, H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big) \\ \Big\downarrow u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle (-) \blacktriangle v & & \Big\downarrow u \blacktriangle (-) \blacktriangle v\,\mathrm{Cond}^{\lfloor n,0\rfloor} \\ {}_{\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\big(\tilde{G}\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, \tilde{H}\big) & \xrightarrow[\beta_{\tilde{G},\tilde{H}}]{\sim} & {}_{\mathcal{SimpGrp}}\big(\tilde{G},\, \tilde{H}\,\mathrm{Cond}^{\lfloor n,0\rfloor}\big) \end{array}$$

Suppose given $\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \xrightarrow{\varphi} H\big) \in {}_{\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor},\, H\big)$.

We have to show that $(u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle \varphi \blacktriangle v)\beta_{\tilde{G},\tilde{H}} \overset{!}{=} u \blacktriangle \varphi\beta_{G,H} \blacktriangle v\,\mathrm{Cond}^{\lfloor n,0\rfloor}$.

Let $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $\tilde{g} \in \tilde{G}_k$.

We have

$$\begin{aligned}
\tilde{g}\big((u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle \varphi \blacktriangle v)\beta_{\tilde{G},\tilde{H}}\big)_k &\overset{129}{=} \tilde{g}\big(\tilde{G}\varepsilon \blacktriangle (\tilde{G}\rho \blacktriangle u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle \varphi \blacktriangle v)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \\
&= \big(\tilde{g}(\tilde{G}\varepsilon)_k\big)\big((\tilde{G}\rho \blacktriangle u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle \varphi \blacktriangle v)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \\
&\overset{87.(1)}{=} (\tilde{g}\tilde{G}_a)_{a\in\Delta([n],[k])}\big((\tilde{G}\rho \blacktriangle u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle \varphi \blacktriangle v)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \\
&\overset{82}{=} \Big(\tilde{g}\tilde{G}_a\big((\tilde{G}\rho)_n \blacktriangle (u\,\mathrm{Trunc})_n \blacktriangle \varphi_n \blacktriangle v_n\big)\Big)_{a\in\Delta([n],[k])} \\
&\overset{128.(1)}{=} \Big((\tilde{g}\tilde{G}_a)\tilde{G}\mathrm{NB}_n\big((u\,\mathrm{Trunc})_n \blacktriangle \varphi_n \blacktriangle v_n\big)\Big)_{a\in\Delta([n],[k])} \\
&\overset{119}{=} \big((\tilde{g}\tilde{G}_a u_n)G\mathrm{NB}_n\,(\varphi_n \blacktriangle v_n)\big)_{a\in\Delta([n],[k])} \\
&= \Big(\big((\tilde{g}\tilde{G}_a u_n)G\mathrm{NB}_n\big)\varphi_n v_n\Big)_{a\in\Delta([n],[k])}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\tilde{g}(u \blacktriangle \varphi\beta_{G,H} \blacktriangle v\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k &= \tilde{g}\big(u_k \blacktriangle (\varphi\beta_{G,H})_k \blacktriangle (v\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k\big) \\
&\overset{129}{=} \tilde{g}u_k\Big((G\varepsilon)_k \blacktriangle \big((G\rho \blacktriangle \varphi)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \blacktriangle (v\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k\Big) \\
&\overset{87.(1)}{=} (\tilde{g}u_k G_a)_{a\in\Delta([n],[k])}\Big(\big((G\rho \blacktriangle \varphi)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \blacktriangle (v\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k\Big) \\
&\overset{82}{=} \Big(\tilde{g}u_k G_a\big((G\rho)_n \blacktriangle \varphi_n\big)\Big)_{a\in\Delta([n],[k])}(v\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k \\
&\overset{125.(2)}{=} \Big(\tilde{g}u_k G_a\big((G\rho)_n \blacktriangle \varphi_n\big)v_n\Big)_{a\in\Delta([n],[k])} \\
&\overset{128.(1)}{=} \Big(\big((\tilde{g}u_k G_a)G\mathrm{NB}_n\big)\varphi_n v_n\Big)_{a\in\Delta([n],[k])} \\
&= \Big(\big((\tilde{g}\tilde{G}_a u_n)G\mathrm{NB}_n\big)\varphi_n v_n\Big)_{a\in\Delta([n],[k])}.
\end{aligned}$$

This shows that $\big((u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle \varphi \blacktriangle v)\beta_{\tilde{G},\tilde{H}}\big)_k = (u \blacktriangle \varphi\beta_{G,H} \blacktriangle v\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k$.

So we have $(u\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \blacktriangle \varphi \blacktriangle v)\beta_{\tilde{G},\tilde{H}} = u \blacktriangle \varphi\beta_{G,H} \blacktriangle v\,\mathrm{Cond}^{\lfloor n,0\rfloor}$.

The claim can also be shown using the isotransformation α.

□

Lemma 131 (The counit $\hat{\eta}$)

Let $n \in \mathbb{Z}_{\geqslant 1}$.

(1) *Suppose given an* $\lfloor n,0\rfloor$*-simplicial group* H*; cf. Definition* 74.

We consider the pre-$\lfloor n,0\rfloor$*-simplicial group morphisms*

$$(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho\colon H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$$

and

$$(H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor}\colon H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}\,;$$

cf. Lemma 128.(1, 2).

We consider the isomorphism of pre-$\lfloor n,0\rfloor$*-simplicial groups*

$$H\eta\colon H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor} \to H\,;$$

cf. Lemma 86.

We have that $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho$ *and* $(H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor}$ *are isomorphisms of pre-*$\lfloor n,0\rfloor$*-simplicial groups.*

So we have the isomorphism of $\lfloor n,0\rfloor$*-simplicial groups*

$$\big((H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho\big)^{-} \mathbin{\triangle} (H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\triangle} H\eta \;:\; H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H\,;$$

cf. Definition 74.

(2) *A counit* $\hat{\eta}$ *of the adjunction* $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$ *is given by*

$$\hat{\eta} = \big(H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \xrightarrow{H\hat{\eta}} H\big)_{H\in \mathrm{Ob}(\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp})}\colon \mathrm{Cond}^{\lfloor n,0\rfloor} * \mathrm{Trunc}^{\lfloor n,0\rfloor} \to \mathrm{id}_{\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}\,,$$

where

$$H\hat{\eta} = \big((H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho\big)^{-} \mathbin{\triangle} (H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\triangle} H\eta \;:\; H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \to H$$

for $H \in \mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$*; cf.* (1).

The counit $\hat{\eta}$ *is an isotransformation.*

$$\begin{array}{ccc}
H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor} & \xrightarrow[\sim]{H\hat{\eta}} & H \\
\Big\uparrow{\scriptstyle (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho}\ \wr & & \wr\ \Big\uparrow{\scriptstyle H\eta} \\
H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor} & \xrightarrow[(H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor}]{\sim} & H\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\,\mathrm{Cut}^{\lfloor n,0\rfloor}
\end{array}$$

Proof. Ad (1). *We show that* $(H\operatorname{Cond}^{\lfloor n,0\rfloor})\rho$ *is an isomorphism of pre-*$\lfloor n,0\rfloor$*-simplicial groups. Cf. Lemma* 128.(1).

For $k\in[0,n-1]$, recall that we have $\big((H\operatorname{Cond}^{\lfloor n,0\rfloor})\rho\big)_k=\mathrm{id}_{(H\operatorname{Cond}^{\lfloor n,0\rfloor})_k}$.

So it remains to show that

$$\begin{array}{rccl}\big((H\operatorname{Cond}^{\lfloor n,0\rfloor})\rho\big)_n\colon & (H\operatorname{Cond}^{\lfloor n,0\rfloor})_n & \to & (H\operatorname{Cond}^{\lfloor n,0\rfloor})_n/(H\operatorname{Cond}^{\lfloor n,0\rfloor})\mathrm{NB}_n\\ & (h_a)_{a\in\Delta([n],[n])} & \mapsto & \big((h_a)_{a\in\Delta([n],[n])}\big)(H\operatorname{Cond}^{\lfloor n,0\rfloor})\mathrm{NB}_n\end{array}$$

is a group isomorphism; cf. Remark 125.(1).

To that end, it suffices to show that $(H\operatorname{Cond}^{\lfloor n,0\rfloor})\mathrm{NB}_n\overset{!}{=}1$.

We have

$$(H\operatorname{Cond}^{\lfloor n,0\rfloor})\mathrm{NB}_n\overset{76.(4)}{=}\big((H\operatorname{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1}\big)\mathrm{d}_0^{H\operatorname{Cond}^{\lfloor n,0\rfloor},n+1}\overset{127}{=}1\mathrm{d}_0^{H\operatorname{Cond}^{\lfloor n,0\rfloor},n+1}=1\,.$$

So $\big((H\operatorname{Cond}^{\lfloor n,0\rfloor})\rho\big)_n$ is a group isomorphism.

This shows that $(H\operatorname{Cond}^{\lfloor n,0\rfloor})\rho$ is an isomorphism of pre-$\lfloor n,0\rfloor$-simplicial groups.

We show that

$$(H\iota)\operatorname{Cut}^{\lfloor n,0\rfloor}\colon H\operatorname{Cond}^{\lfloor n,0\rfloor}\operatorname{Cut}^{\lfloor n,0\rfloor}\to H\operatorname{Cosk}^{\lfloor n,0\rfloor}\operatorname{Cut}^{\lfloor n,0\rfloor}$$

is an isomorphism of pre-$\lfloor n,0\rfloor$*-simplicial groups. Cf. Lemma* 128.(2).

Let $k\in[0,n]$. Recall that we have

$$(H\operatorname{Cond}^{\lfloor n,0\rfloor}\operatorname{Cut}^{\lfloor n,0\rfloor})_k\overset{79}{=}(H\operatorname{Cond}^{\lfloor n,0\rfloor})_k\overset{125.(1)}{=}(H\operatorname{Cosk}^{\lfloor n,0\rfloor})_k\overset{79}{=}(H\operatorname{Cosk}^{\lfloor n,0\rfloor}\operatorname{Cut}^{\lfloor n,0\rfloor})_k\,.$$

Moreover, for $(h_a)_{a\in\Delta([n],[k])}\in(H\operatorname{Cond}^{\lfloor n,0\rfloor}\operatorname{Cut}^{\lfloor n,0\rfloor})_k$, note that we have

$$\big((h_a)_a\big)\big((H\iota)\operatorname{Cut}^{\lfloor n,0\rfloor}\big)_k\overset{79}{=}\big((h_a)_a\big)(H\iota)_k\overset{128.(2)}{=}(h_a)_a\,.$$

So

$$\big((H\iota)\operatorname{Cut}^{\lfloor n,0\rfloor}\big)_k=(H\iota)_k\colon(H\operatorname{Cond}^{\lfloor n,0\rfloor})_k\to(H\operatorname{Cosk}^{\lfloor n,0\rfloor})_k\,,\ (h_a)_{a\in\Delta([n],[k])}\mapsto(h_a)_{a\in\Delta([n],[k])}$$

is a group isomorphism.

Therefore $(H\iota)\operatorname{Cut}^{\lfloor n,0\rfloor}$ is an isomorphism of pre-$\lfloor n,0\rfloor$-simplicial groups.

Ad (2). For an $\lfloor n,0\rfloor$-simplicial group H, we consider the bijective map from Lemma 129

$$\beta_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},H} \colon {}_{\lfloor n,0\rfloor}\text{-}\mathcal{SimpGrp}\big(H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor}, H\big) \xrightarrow{\sim} {}_{\mathcal{SimpGrp}}(H\,\mathrm{Cond}^{\lfloor n,0\rfloor}, H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\,.$$

By Remark 15, we have the counit $\hat{\eta}\colon \mathrm{Cond}^{\lfloor n,0\rfloor} * \mathrm{Trunc}^{\lfloor n,0\rfloor} \to \mathrm{id}_{\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}}$ of the adjunction $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$, where

$$H\hat{\eta} := (\mathrm{id}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}})\beta^{-}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},H}$$

for $H \in \mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$.
So, for $H \in \mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$, we have to show that

$$(\mathrm{id}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}})\beta^{-}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},H} \overset{!}{=} \big((H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho\big)^{-} \mathbin{\blacktriangle} (H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} H\eta\,.$$

Suppose given $H \in \mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}\big)$. We have

$$\begin{aligned}
&(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho \mathbin{\blacktriangle} (\mathrm{id}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}})\beta^{-}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},H}\\
&\overset{129}{=} (\mathrm{id}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}} \mathbin{\blacktriangle} H\iota)\alpha^{-}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},H}\\
&= (H\iota)\alpha^{-}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},H}\\
&\overset{129}{=} (H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} H\eta\,.
\end{aligned}$$

This shows that

$$\big((H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\rho\big)^{-} \mathbin{\blacktriangle} (H\iota)\,\mathrm{Cut}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} H\eta = (\mathrm{id}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}})\beta^{-}_{H\,\mathrm{Cond}^{\lfloor n,0\rfloor},H} = H\hat{\eta}\,.$$

By (1), $H\hat{\eta}$ is an isomorphism. □

Corollary 132 *Let $n \in \mathbb{Z}_{\geqslant 1}$. The functor $\mathrm{Trunc}^{\lfloor n,0\rfloor}\colon \mathcal{SimpGrp} \to \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$ is dense.*

Proof. Suppose given an $\lfloor n,0\rfloor$-simplicial group H. We consider the $\lfloor n,0\rfloor$-simplicial group $H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor}$.

Then, by Lemma 131.(2), we have the $\lfloor n,0\rfloor$-simplicial group isomorphism

$$(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \xrightarrow[\sim]{H\hat{\eta}} H\,.$$

Therefore, $\mathrm{Trunc}^{\lfloor n,0\rfloor}$ is dense. □

Remark 133 (The unit $\hat{\varepsilon}$)

We consider the unit

$$\varepsilon\colon \mathrm{id}_{\mathcal{SimpGrp}} \to \mathrm{Cut}^{\lfloor n,0\rfloor} * \mathrm{Cosk}^{\lfloor n,0\rfloor}$$

of the adjunction $(\mathrm{Cut}^{\lfloor n,0\rfloor}, \mathrm{Cosk}^{\lfloor n,0\rfloor}, \varepsilon, \eta)$; cf. Lemma 87.

We consider the transformation

$$\rho = \big(G\,\mathrm{Cut}^{\lfloor n,0\rfloor} \xrightarrow{G\rho} G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\big)_{G\in\mathrm{Ob}(\mathcal{SimpGrp})}\colon\ \mathrm{Cut}^{\lfloor n,0\rfloor} \to \mathrm{Trunc}^{\lfloor n,0\rfloor} * \mathrm{I}\,;$$

cf. Lemma 128.

Then, the unit of the adjunction $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$ with respect to the isotransformation β from Theorem 130 is given by

$$\hat{\varepsilon} = \big(G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,\mathrm{Cond}^{\lfloor n,0\rfloor} \xrightarrow{G\hat{\varepsilon}} G\big)_{G\in\mathrm{Ob}(\mathcal{SimpGrp})}\colon\ \mathrm{id}_{\mathcal{SimpGrp}} \to \mathrm{Trunc}^{\lfloor n,0\rfloor} * \mathrm{Cond}^{\lfloor n,0\rfloor}\,,$$

where, for $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$, we have

$$G\hat{\varepsilon} = \big(G\varepsilon \mathbin{\blacktriangle} (G\rho)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)\big|^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,\mathrm{Cond}^{\lfloor n,0\rfloor}}\ : G \to G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,\mathrm{Cond}^{\lfloor n,0\rfloor}\ .$$

So, for $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ and $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$(G\hat{\varepsilon})_k\colon G_k \to (G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,\mathrm{Cond}^{\lfloor n,0\rfloor})_k\,,\ g \mapsto \big((gG_a)G\mathrm{NB}_n\big)_{a\in\Delta([n],[k])}\,.$$

Proof. By Remark 15, the unit of the adjunction $\mathrm{Trunc}^{\lfloor n,0\rfloor} \dashv \mathrm{Cond}^{\lfloor n,0\rfloor}$ with respect to β is given by $\hat{\varepsilon}$, where

$$\begin{aligned}
G\hat{\varepsilon} &\overset{130}{=} (\mathrm{id}_{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}})\beta_{G,G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}} \\
&\overset{129}{=} \big((G\rho \mathbin{\blacktriangle} \mathrm{id}_{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}})\alpha_{G,G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}}\big)\big|^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,\mathrm{Cond}^{\lfloor n,0\rfloor}} \\
&= \big((G\rho)\alpha_{G,G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}}\big)\big|^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,\mathrm{Cond}^{\lfloor n,0\rfloor}} \\
&\overset{89}{=} \big(G\varepsilon \mathbin{\blacktriangle} (G\rho)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)\big|^{G\,\mathrm{Trunc}^{\lfloor n,0\rfloor}\,\mathrm{Cond}^{\lfloor n,0\rfloor}}\,,
\end{aligned}$$

for $G \in \mathrm{Ob}(\mathcal{SimpGrp})$.

So, for $G \in \mathrm{Ob}(\mathcal{SimpGrp})$, $k \in \mathbb{Z}_{\geqslant 0}$ and $g \in G_k$, we have

$$\begin{aligned}
g(G\hat{\varepsilon})_k &= g\big(G\varepsilon \mathbin{\blacktriangle} (G\rho)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \overset{87.(1)}{=} \big((gG_a)_{a\in\Delta([n],[k])}\big)\big((G\rho)\,\mathrm{Cosk}^{\lfloor n,0\rfloor}\big)_k \\
&\overset{82}{=} \big((gG_a)(G\rho)_n\big)_{a\in\Delta([n],[k])} \overset{128.(1)}{=} \big((gG_a)G\mathrm{NB}_n\big)_{a\in\Delta([n],[k])}\,.
\end{aligned}$$

□

6.4 The Moore complex $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}$

Lemma 134 *Let $n \in \mathbb{Z}_{\geqslant 1}$. Suppose given an $\lfloor n,0\rfloor$-simplicial group H; cf. Definition 74.*

We consider its Moore complex $H\mathrm{N}$; cf. Lemma 77.(2).

We consider the simplicial group $H\,\mathrm{Cond}^{\lfloor n,0\rfloor}$; cf. Remark 125. We consider its Moore complex $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}$; cf. Lemma 47.(2).

We consider the morphism of complexes in groups

$$(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\psi\colon (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N} \to (H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor})\mathrm{N}$$

from Lemma 121.(2). It is an isomorphism.

We consider the isomorphism of $\lfloor n,0\rfloor$-simplicial groups

$$H\hat{\eta}\colon H\,\mathrm{Cond}^{\lfloor n,0\rfloor}\,\mathrm{Trunc}^{\lfloor n,0\rfloor} \xrightarrow{\sim} H$$

from Lemma 131.(2).

We have the isomorphism of complexes in groups

$$\varphi := (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\psi \mathbin{\blacktriangle} (H\hat{\eta})\mathrm{N}\colon (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N} \xrightarrow{\sim} H\mathrm{N}\,.$$

$$\begin{array}{ccccccccccccc}
\ldots & \xrightarrow{!} & 1 & \xrightarrow{!} & (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_n & \xrightarrow{\partial_n^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}}} & \ldots & \xrightarrow{\partial_1^{H\,\mathrm{Cond}^{\lfloor n,0\rfloor}}} & (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_0 & \xrightarrow{!} & 1 & \xrightarrow{!} & \ldots \\
 & & \downarrow{\scriptstyle !} & & \wr\downarrow{\scriptstyle \varphi_n} & & & & \wr\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle !} & & \\
\ldots & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & H\mathrm{N}_n & \xrightarrow[\partial_n^H]{} & \ldots & \xrightarrow[\partial_1^H]{} & H\mathrm{N}_0 & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & \ldots
\end{array}$$

Proof. By Remark 127, we have $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_k = 1$ for $k \in \mathbb{Z}_{\geqslant n+1}$.

So, by Lemma 121.(4), we have that $(H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\psi$ is an isomorphism of complexes in groups.

Moreover, $(H\hat{\eta})\mathrm{N}$ is isomorphism of complexes in groups, since $H\hat{\eta}$ is an isomorphism of $\lfloor n,0\rfloor$-simplicial groups.

Therefore, $\varphi = (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\psi \mathbin{\blacktriangle} (H\hat{\eta})\mathrm{N}$ is an isomorphism of complexes in groups. □

Chapter 7

The equivalent categories of $\lfloor 1,0 \rfloor$-simplicial groups and crossed modules

7.1 The category of $\lfloor 1,0 \rfloor$-simplicial groups

Remark 135 ($\lfloor 1,0 \rfloor$-simplicial group)

We consider the simplex category Δ; cf. Definition 18.

We consider the full subcategory $\Delta_{\lfloor 1,0 \rfloor} \subseteq \Delta$; cf. Definition 51.

We consider the category of $\lfloor 1,0 \rfloor$-simplicial groups $\lfloor 1,0 \rfloor$-$\mathcal{SimpGrp}$; cf. Definition 74.

Suppose given a $\lfloor 1,0 \rfloor$-simplicial group $G \colon \Delta^{\mathrm{op}}_{\lfloor 1,0 \rfloor} \to \mathcal{Grp}$.

Then we have the face maps $\mathrm{d}_0^{G,1} \colon G_1 \to G_0$, $\mathrm{d}_1^{G,1} \colon G_1 \to G_0$ and the degeneracy map $\mathrm{s}_0^{G,0} \colon G_0 \to G_1$; cf. Definition 53.

In this context of $\lfloor 1,0 \rfloor$-simplicial groups, we often write $\mathrm{s}_0^G := \mathrm{s}_0^{G,0}$, $\mathrm{d}_0^G := \mathrm{d}_0^{G,1}$ and $\mathrm{d}_1^G := \mathrm{d}_1^{G,1}$.

These maps satisfy the equations

$$\mathrm{s}_0^G \mathbin{\triangleleft} \mathrm{d}_0^G = \mathrm{id}_{G_0}$$
$$\mathrm{s}_0^G \mathbin{\triangleleft} \mathrm{d}_1^G = \mathrm{id}_{G_0};$$

cf. Remark 54.(3).

Morover, we have

$$[\ker \mathrm{d}_0^G, \ker \mathrm{d}_1^G] = 1\,;$$

cf. Definition 74.

Remark 136 ($\lfloor 1,0 \rfloor$-simplicial group morphism)

Suppose given $\lfloor 1,0 \rfloor$-simplicial groups G, $H\colon \Delta^{\mathrm{op}}_{\lfloor 1,0 \rfloor} \to \mathcal{G}rp$; cf. Remark 135.

Suppose given a $\lfloor 1,0 \rfloor$-simplicial group morphism $\varphi = \left(G_k \xrightarrow{\varphi_k} H_k\right)_{k\in[0,1]}\colon G \to H$; cf. Definition 74.

Then we have the following equations; cf. Remark 55.

We have

$$\begin{aligned}
\varphi_0 \vartriangleleft \mathrm{s}_0^H &= \mathrm{s}_0^G \vartriangleleft \varphi_1 \\
\mathrm{d}_0^G \vartriangleleft \varphi_0 &= \varphi_1 \vartriangleleft \mathrm{d}_0^H \\
\mathrm{d}_1^G \vartriangleleft \varphi_0 &= \varphi_1 \vartriangleleft \mathrm{d}_1^H\,.
\end{aligned}$$

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_0} & H_0 \\ {\scriptstyle \mathrm{s}_0^G}\downarrow & & \downarrow{\scriptstyle \mathrm{s}_0^H} \\ G_1 & \xrightarrow[\varphi_1]{} & H_1 \end{array} \qquad \begin{array}{ccc} G_0 & \xrightarrow{\varphi_0} & H_0 \\ {\scriptstyle \mathrm{d}_0^G}\uparrow & & \uparrow{\scriptstyle \mathrm{d}_0^H} \\ G_1 & \xrightarrow[\varphi_1]{} & H_1 \end{array} \qquad \begin{array}{ccc} G_0 & \xrightarrow{\varphi_0} & H_0 \\ {\scriptstyle \mathrm{d}_1^G}\uparrow & & \uparrow{\scriptstyle \mathrm{d}_1^H} \\ G_1 & \xrightarrow[\varphi_1]{} & H_1 \end{array}$$

Remark 137 (Construction of a $\lfloor 1,0 \rfloor$-simplicial group)

Suppose given groups G_0, G_1. Suppose given group morphisms d_0, $d_1\colon G_1 \to G_0$ and $s_0\colon G_0 \to G_1$.

$$G_1 \underset{d_0}{\overset{d_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}} G_0 \qquad \text{(with } s_0 \text{ between } d_1, d_0\text{)}$$

Suppose that the conditions $(1,2)$ hold.

(1) We have $s_0 \vartriangleleft d_0 = \mathrm{id}_{G_0}$ and $s_0 \vartriangleleft d_1 = \mathrm{id}_{G_0}$.

(2) We have $[\ker d_0, \ker d_1] = 1$.

Then, by Lemma 71.(3), we have the pre-$\lfloor 1,0 \rfloor$-simplicial group G given as follows.

We have

$$\begin{aligned} [0]G = G_0\,,\ [1]G = G_1 \\ \mathrm{s}_0^G = s_0\,,\ \mathrm{d}_0^G = d_0\,,\ \mathrm{d}_1^G = d_1\,. \end{aligned}$$

Since condition (2) holds, we have that G is a $\lfloor 1,0 \rfloor$-simplicial group; cf. Definition 307 and Remark 135.

Remark 138 (Construction of a $\lfloor 1,0 \rfloor$-simplicial group morphism)
Suppose given $\lfloor 1,0 \rfloor$-simplicial groups G and H.

$$G_1 \underset{\mathrm{d}_0^H}{\overset{\mathrm{d}_1^H}{\underset{\longleftarrow}{\overset{\longrightarrow}{\overset{\mathrm{s}_0^G}{}}}}} G_0 \qquad\qquad H_1 \underset{\mathrm{d}_0^H}{\overset{\mathrm{d}_1^H}{\underset{\longleftarrow}{\overset{\longrightarrow}{\overset{\mathrm{s}_0^H}{}}}}} H_0$$

Suppose given group morphisms $\varphi_1 \colon G_1 \to H_1$ and $\varphi_0 \colon G_0 \to H_0$ such that $(1,2,3)$ hold.

(1) We have $\quad \mathrm{d}_0^G \vartriangleleft \varphi_0 = \varphi_1 \vartriangleleft \mathrm{d}_0^H\,.$

(2) We have $\quad \mathrm{d}_1^G \vartriangleleft \varphi_0 = \varphi_1 \vartriangleleft \mathrm{d}_1^H\,.$

(3) We have $\quad \mathrm{s}_0^G \vartriangleleft \varphi_1 = \varphi_0 \vartriangleleft \mathrm{s}_0^H\,.$

$$\begin{array}{ccc} G_1 & \underset{\mathrm{d}_0^G}{\overset{\mathrm{d}_1^G}{\underset{\longleftarrow}{\overset{\longrightarrow}{\overset{\mathrm{s}_0^G}{}}}}} & G_0 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ H_1 & \underset{\mathrm{d}_0^H}{\overset{\mathrm{d}_1^H}{\underset{\longleftarrow}{\overset{\longrightarrow}{\overset{\mathrm{s}_0^H}{}}}}} & H_0 \end{array}$$

Then, by Lemma 73 and Definition 74, $\varphi := \big(G_k \xrightarrow{\varphi_k} H_k\big)_{k\in[0,1]}$ is a $\lfloor 1,0 \rfloor$-simplicial group morphism from G to H.

Remark 139 (The truncation functor $\mathrm{Trunc}^{\lfloor 1,0 \rfloor}$)
Suppose given simplicial groups G, H; cf. Definition 30.

(1) Recall that $G\mathrm{NB}_1 \overset{46.(4)}{=} \big(\ker \mathrm{d}_1^{G,2} \cap \ker \mathrm{d}_2^{G,2}\big)\mathrm{d}_0^{G,2} \overset{47.(2)}{\trianglelefteq} G_1\,.$

By Lemma 118, we have the $\lfloor 1,0 \rfloor$-simplicial group $G\,\mathrm{Trunc}^{\lfloor 1,0 \rfloor}$, where

$$\begin{aligned} (G\,\mathrm{Trunc}^{\lfloor 1,0 \rfloor})_0 &= G_0\,, \\ (G\,\mathrm{Trunc}^{\lfloor 1,0 \rfloor})_1 &= G_1/G\mathrm{NB}_1\,, \end{aligned}$$

and where

$$\begin{aligned}
&\mathrm{s}_0^{G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}} \colon G_1/G\mathrm{NB}_1 \leftarrow G_0\,,\ (x\mathrm{s}_0^{G,0})G\mathrm{NB}_1 \leftarrow\!\shortmid\, x\\
&\mathrm{d}_0^{G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}} \colon G_1/G\mathrm{NB}_1 \to G_0\,,\ gG\mathrm{NB}_1 \mapsto g\mathrm{d}_0^{G,1}\\
&\mathrm{d}_1^{G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}} \colon G_1/G\mathrm{NB}_1 \to G_0\,,\ gG\mathrm{NB}_1 \mapsto g\mathrm{d}_1^{G,1}\,.
\end{aligned}$$

The $\lfloor 1,0 \rfloor$-simplicial group $G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}$ is called the truncation at $\lfloor 1,0 \rfloor$ of G.

(2) Suppose given a simplicial group morphism $\varphi \colon G \to H$.
We consider the $\lfloor 1,0 \rfloor$-simplicial groups $G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}$ and $H\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}$; cf. (1).
By Lemma 119, we have the $\lfloor 1,0 \rfloor$-simplicial group morphism

$$\varphi\,\mathrm{Trunc}^{\lfloor 1,0\rfloor} \colon G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor} \to H\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}\,,$$

where

$$\begin{aligned}
&(\varphi\,\mathrm{Trunc}^{\lfloor 1,0\rfloor})_1 \colon G_1/G\mathrm{NB}_1 \to H_1/H\mathrm{NB}_1\,, xG\mathrm{NB}_1 \mapsto (x\varphi_1)H\mathrm{NB}_1\\
&(\varphi\,\mathrm{Trunc}^{\lfloor 1,0\rfloor})_0 \colon G_0 \to H_0\,, x \mapsto x\varphi_0\,.
\end{aligned}$$

(3) By Lemma 120, we have the functor

$$\begin{array}{rccl}
\mathrm{Trunc}^{\lfloor 1,0\rfloor}\colon & \mathit{SimpGrp} & \to & \lfloor 1,0 \rfloor\text{-}\mathit{SimpGrp}\\
& \big(G \xrightarrow{\varphi} H\big) & \mapsto & \big(G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor} \xrightarrow{\varphi\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}} H\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}\big)\,;
\end{array}$$

cf. $(1,2)$.

Remark 140 (The functor $\mathrm{Cond}^{\lfloor 1,0\rfloor}$)
Suppose given $\lfloor 1,0 \rfloor$-simplicial groups G, H; cf. Remark 135.

(1) By Remark 125.(1), we have the simplicial group $G\,\mathrm{Cond}^{\lfloor 1,0\rfloor}$, where

$$(G\,\mathrm{Cond}^{\lfloor 1,0\rfloor})_k = \begin{cases}
(G\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_k & \text{for } k \in [0,1]\\
(G\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{D}_2 & \text{for } k = 2\\
\bigcap\limits_{\substack{a \in \Delta([2],[k])\\ a \text{ is injective}}} (G\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_a^{-1}\big((G\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{D}_2\big) & \text{for } k \in \mathbb{Z}_{\geqslant 3}\,.
\end{cases}$$

Concerning $G\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor}$ cf. Example 83. Moreover, recall from Example 114.(1) that we have

$$(G\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor})\mathrm{D}_2 = \big((G\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor})\mathrm{N}_0\big)\mathrm{s}_0\mathrm{s}_1 \cdot \big((G\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor})\mathrm{N}_1\big)\mathrm{s}_1 \cdot \big((G\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor})\mathrm{N}_1\big)\mathrm{s}_0 \cdot (G\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor})\mathrm{N}_2\,.$$

For $k \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0,k]$, we have

$$\begin{array}{rccl} \mathrm{s}_i^{G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor},k}\colon & (G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_k & \to & (G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_{k+1} \\ & (g_a)_{a \in \Delta([1],[k])} & \mapsto & \big((g_a)_{a \in \Delta([1],[k])}\big)\mathrm{s}_i^{G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor},k} = (g_{b \blacktriangle \sigma_i^k})_{b \in \Delta([1],[k+1])}\,. \end{array}$$

For $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,k]$, we have

$$\begin{array}{rccl} \mathrm{d}_j^{G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor},k}\colon & (G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_k & \to & (G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_{k-1} \\ & (g_a)_{a \in \Delta([1],[k])} & \mapsto & \big((g_a)_{a \in \Delta([1],[k])}\big)\mathrm{d}_j^{G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor},k} = (g_{b \blacktriangle \delta_j^k})_{b \in \Delta([1],[k-1])}\,. \end{array}$$

(2) Suppose given a $\lfloor 1,0 \rfloor$-simplicial group morphism $\varphi\colon G \to H$.

We consider the simplicial groups $G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}$ and $H\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}$; cf. (1).

By Remark 125.(2), we have the simplicial group morphism

$$\varphi\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}\colon G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor} \to H\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}\,,$$

where, for $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$\varphi\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}\colon (G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_k \to (H\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_k\,,\ (g_a)_{a \in \Delta([1],[k])} \mapsto (g_a \varphi_1)_{a \in \Delta([1],[k])}\,.$$

(3) By Lemma 126, we have the functor

$$\begin{array}{rccl} \mathrm{Cond}^{\lfloor 1,0 \rfloor}\colon & \lfloor 1,0 \rfloor\text{-}\mathit{SimpGrp} & \to & \mathit{SimpGrp} \\ & \big(G \xrightarrow{\varphi} H\big) & \mapsto & \big(G\,\mathrm{Cond}^{\lfloor 1,0 \rfloor} \xrightarrow{\varphi\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}} H\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}\big)\,; \end{array}$$

cf. (1, 2).

7.2 The functor $\hat{\mathrm{N}}\colon \lfloor 1,0 \rfloor$-*SimpGrp* $\to$ *CrMod*

Lemma 141 (The crossed module $G\hat{\mathrm{N}}$)

Suppose given a $\lfloor 1,0 \rfloor$-simplicial group G; cf. Remark 135.

We consider the groups

$$G\mathrm{N}_1 \overset{76.(2)}{=} \ker \mathrm{d}_1^G \trianglelefteq G_1 \text{ and } G_0\,.$$

We consider the group morphisms

$$\gamma_G\colon G_0 \to \mathrm{Aut}(G\mathrm{N}_1)\,,\ g \mapsto \big(m \mapsto m^g := g^-\mathrm{s}_0^G \cdot m \cdot g\mathrm{s}_0^G\big)$$
$$\mathrm{d}_0^G\big|_{G\mathrm{N}_1}\colon G\mathrm{N}_1 \to G_0\,,\ m \mapsto m\mathrm{d}_0^G\,.$$

Then $G\hat{\mathrm{N}} = (\mathrm{M}_{G\hat{\mathrm{N}}}\,, \mathrm{B}_{G\hat{\mathrm{N}}}\,, \gamma_{G\hat{\mathrm{N}}}\,, \mathrm{f}_{G\hat{\mathrm{N}}}) := (G\mathrm{N}_1\,, G_0, \gamma_G\,, \mathrm{d}_0^G\big|_{G\mathrm{N}_1})$ is a crossed module.

Proof. We write $\mathrm{s}_0 := \mathrm{s}_0^G$, $\mathrm{d}_0 := \mathrm{d}_0^G$ and $\mathrm{d}_1 := \mathrm{d}_1^G$.

Ad (CM1). For $m \in G\mathrm{N}_1$ and $g \in G_0$, we have

$$(m^g)(\mathrm{d}_0\big|_{G\mathrm{N}_1}) = \big(g^-\mathrm{s}_0 \cdot m \cdot g\mathrm{s}_0\big)\mathrm{d}_0 = g^-\mathrm{s}_0\mathrm{d}_0 \cdot m\mathrm{d}_0 \cdot g\mathrm{s}_0\mathrm{d}_0 = g^- \cdot m\mathrm{d}_0 \cdot g = \big(m(\mathrm{d}_0\big|_{G\mathrm{N}_1})\big)^g\,.$$

Ad (CM2). Suppose given m, $n \in G\mathrm{N}_1$. We have

$$(n\mathrm{d}_0\mathrm{s}_0 \cdot n^-)\mathrm{d}_0 = n\mathrm{d}_0\mathrm{s}_0\mathrm{d}_0 \cdot n^-\mathrm{d}_0 = n\mathrm{d}_0 \cdot n\mathrm{d}_0 = 1\,.$$

This shows $n\mathrm{d}_0\mathrm{s}_0 \cdot n^- \in \ker \mathrm{d}_0$. So we have $[n\mathrm{d}_0\mathrm{s}_0 \cdot n^-, m] = 1$; cf. Remark 135.

We have

$$\begin{aligned} m^{n(\mathrm{d}_0|_{G\mathrm{N}_1})} &= (n\mathrm{d}_0\mathrm{s}_0)^- \cdot m \cdot n\mathrm{d}_0\mathrm{s}_0 = (n\mathrm{d}_0\mathrm{s}_0)^- \cdot m \cdot n\mathrm{d}_0\mathrm{s}_0 \cdot n^- \cdot n \\ &= (n\mathrm{d}_0\mathrm{s}_0)^- \cdot n\mathrm{d}_0\mathrm{s}_0 \cdot n^- \cdot m \cdot n = m^n\,. \end{aligned}$$

□

Lemma 142 (The crossed module morphism $\varphi\hat{\mathrm{N}}$)

Suppose given a $\lfloor 1,0 \rfloor$-simplicial group morphism $\varphi\colon G \to H$; cf. Remark 136.

$$\begin{array}{ccc} & \xrightarrow{\mathrm{d}_1^G} & \\ G_1 & \xleftarrow{\mathrm{s}_0^G} & G_0 \\ & \xrightarrow{\mathrm{d}_0^G} & \\ \varphi_1 \downarrow & & \downarrow \varphi_0 \\ & \xrightarrow{\mathrm{d}_1^H} & \\ H_1 & \xleftarrow{\mathrm{s}_0^H} & H_0 \\ & \xrightarrow{\mathrm{d}_0^H} & \end{array}$$

We consider the crossed modules

$$G\hat{\mathrm{N}} = \big(G\mathrm{N}_1\,, G_0\,, \gamma_G, \mathrm{d}_0^G\big|_{G\mathrm{N}_1}\big) \quad \text{and} \quad H\hat{\mathrm{N}} = \big(H\mathrm{N}_1\,, H_0\,, \gamma_H, \mathrm{d}_0^H\big|_{H\mathrm{N}_1}\big);$$

cf. Lemma 141.

We consider the group morphisms

$$\begin{aligned} \varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}\colon &\ G\mathrm{N}_1 \to H\mathrm{N}_1\,, \quad && m \mapsto m\varphi_1 \\ \varphi_0\colon &\ G_0 \to H_0\,, && g \mapsto g\varphi_0\,. \end{aligned}$$

Then $\varphi\hat{\mathrm{N}} = \big(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}, \varphi_0\big)\colon G\hat{\mathrm{N}} \to H\hat{\mathrm{N}}$ *is a crossed module morphism; cf.* §1.1.4 *item* 3.

Proof. For $m \in G\mathrm{N}_1$, we have

$$m\varphi_1\mathrm{d}_1^H = m\mathrm{d}_1^G\varphi_0 = 1\varphi_0 = 1\,.$$

This shows $m\varphi_1 \in H\mathrm{N}_1$. So $\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}$ is a well-defined group morphism.

For $m \in G\mathrm{N}_1$, we have

$$m\big(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} \triangleleft \mathrm{d}_0^H\big|_{H\mathrm{N}_1}\big) = m\varphi_1\mathrm{d}_0^H = m\mathrm{d}_0^G\varphi_0 = m\big(\mathrm{d}_0^G\big|_{G\mathrm{N}_1} \triangleleft \varphi_0\big)\,.$$

This shows $\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} \triangleleft \mathrm{d}_0^H\big|_{H\mathrm{N}_1} = \mathrm{d}_0^G\big|_{G\mathrm{N}_1} \triangleleft \varphi_0$.

$$\begin{array}{ccc} G\mathrm{N}_1 & \xrightarrow{\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}} & H\mathrm{N}_1 \\ {\scriptstyle \mathrm{d}_0^G\big|_{G\mathrm{N}_1}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_0^H\big|_{H\mathrm{N}_1}} \\ G_0 & \xrightarrow[\varphi_0]{} & H_0 \end{array}$$

For $m \in G\mathrm{N}_1$ and $g \in G_0$, we have

$$\begin{aligned} (m^g)\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} &= (g^-\mathrm{s}_0^G \cdot m \cdot g\mathrm{s}_0^G)\varphi_1 = g^-\mathrm{s}_0^G\varphi_1 \cdot m\varphi_1 \cdot g\mathrm{s}_0^G\varphi_1 = (g\varphi_0)^-\mathrm{s}_0^H \cdot m\varphi_1 \cdot g\varphi_0\mathrm{s}_0^H \\ &= (m\varphi_1)^{g\varphi_0} = \big(m\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}\big)^{g\varphi_0}\,. \end{aligned}$$

This shows that $\big(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}, \varphi_0\big)$ is a crossed module morphism. □

Lemma 143 (The functor $\hat{\mathrm{N}}$)

We have the functor

$$\hat{\mathrm{N}} \colon \lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{CrMod} \quad \big(G \xrightarrow{\varphi} H\big) \mapsto \big(G\hat{\mathrm{N}} \xrightarrow{\varphi\hat{\mathrm{N}}} H\hat{\mathrm{N}}\big)\,;$$

cf. Lemmas 141, 142.

Proof. Suppose given $G \in \mathrm{Ob}\big(\lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp}\big)$.

Recall that we have $(\mathrm{id}_G)_k = \mathrm{id}_{G_k}$ for $k \in [0,1]$. So we have the crossed module morphism

$$(\mathrm{id}_G)\hat{\mathrm{N}} = \big(\,\mathrm{id}_{G_1}\big|_{G\mathrm{N}_1}^{G\mathrm{N}_1}, \mathrm{id}_{G_0}\,\big) = (\mathrm{id}_{G\mathrm{N}_1}, \mathrm{id}_{G_0}) = \mathrm{id}_{G\hat{\mathrm{N}}}\,;$$

cf. Lemma 142.

Suppose given $G \xrightarrow{\varphi} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{\tilde{G}}$ in $\lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp}$. We have

$$\begin{aligned}(\varphi \mathbin{\blacktriangle} \tilde{\varphi})\hat{\mathrm{N}} &= \big((\varphi_1 \mathbin{\blacktriangle} \tilde{\varphi}_1)\big|_{G\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1}, \varphi_0 \mathbin{\blacktriangle} \tilde{\varphi}_0\big) = \big(\varphi_1\big|_{G\mathrm{N}_1}^{\tilde{G}\mathrm{N}_1} \mathbin{\blacktriangle} \tilde{\varphi}_1\big|_{\tilde{G}\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1}, \varphi_0 \mathbin{\blacktriangle} \tilde{\varphi}_0\big)\\ &= \big(\varphi_1\big|_{G\mathrm{N}_1}^{\tilde{G}\mathrm{N}_1}, \varphi_0\big) \mathbin{\blacktriangle} \big(\tilde{\varphi}_1\big|_{\tilde{G}\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1}, \tilde{\varphi}_0\big) = \varphi\hat{\mathrm{N}} \mathbin{\blacktriangle} \tilde{\varphi}\hat{\mathrm{N}}\,.\end{aligned}$$

Therefore, $\hat{\mathrm{N}}$ is a functor. □

7.3 The functor $\mathrm{Rec} \colon \mathcal{CrMod} \to \lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp}$

Lemma 144 (The $\lfloor 1,0 \rfloor$-simplicial group $V\mathrm{Rec}$)

Suppose given a crossed module $V = (\mathrm{M}_V\,, \mathrm{B}_V\,, \gamma_V\,, \mathrm{f}_V)$.

We consider the semidirect product $\mathrm{B}_V \ltimes \mathrm{M}_V := \mathrm{B}_V \ltimes_{\gamma_V} \mathrm{M}_V$ *; cf.* §1.1.2 *item* 8.

We have the $\lfloor 1,0 \rfloor$*-simplicial group* $V\mathrm{Rec}$ *given as follows.*

We have

$$(V\mathrm{Rec})_0 = \mathrm{B}_V\,,\ (V\mathrm{Rec})_1 = \mathrm{B}_V \ltimes \mathrm{M}_V\,.$$

Moreover, we have

$$\begin{aligned}
\mathrm{s}_0^{V\mathrm{Rec}} \colon &\quad \mathrm{B}_V \ltimes \mathrm{M}_V \leftarrow \mathrm{B}_V\,, &(b,1) &\quad\leftarrow\quad b\\
\mathrm{d}_0^{V\mathrm{Rec}} \colon &\quad \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, &(b,m) &\quad\mapsto\quad b \cdot m\mathrm{f}_V\\
\mathrm{d}_1^{V\mathrm{Rec}} \colon &\quad \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, &(b,m) &\quad\mapsto\quad b\,.
\end{aligned}$$

Proof. We write $\mathrm{s}_0 := \mathrm{s}_0^{V\mathrm{Rec}}$, $\mathrm{d}_0 := \mathrm{d}_0^{V\mathrm{Rec}}$ and $\mathrm{d}_1 := \mathrm{d}_1^{V\mathrm{Rec}}$.

The maps s_0 and d_0 are group morphism.

We show that d_1 *is a group morphism.* For (b,m), $(\tilde{b},\tilde{m}) \in \mathrm{B}_V \ltimes \mathrm{M}_V$, we have

$$\begin{aligned}\big((b,m)\cdot(\tilde{b},\tilde{m})\big)\mathrm{d}_1 &= (b\tilde{b}, m^{\tilde{b}}\,\tilde{m})\mathrm{d}_1 = b\tilde{b}\cdot(m^{\tilde{b}}\,\tilde{m})\mathrm{f}_V \overset{(\mathrm{CM1})}{=} b\tilde{b}\cdot(m\mathrm{f}_V)^{\tilde{b}}\,\tilde{m}\mathrm{f}_V = (b\cdot m\mathrm{f}_V)\cdot(\tilde{b}\cdot\tilde{m}\mathrm{f}_V)\\ &= (b,m)\mathrm{d}_1\cdot(\tilde{b},\tilde{m})\mathrm{d}_1\,.\end{aligned}$$

We use Remark 137 to show that $V\mathrm{Rec}$ is a $\lfloor 1,0\rfloor$-simplicial group.

For $b \in \mathrm{B}_V$, we have

$$b\mathrm{s}_0\,\mathrm{d}_0 = (b,1)\mathrm{d}_0 = b\cdot 1\mathrm{f}_V = b\,.$$

This shows $\mathrm{s}_0 \vartriangleleft \mathrm{d}_0 = \mathrm{id}_{\mathrm{B}_V}$.

For $b \in \mathrm{B}_V$, we have

$$b\mathrm{s}_0\,\mathrm{d}_1 = (b,1)\mathrm{d}_1 = b\,.$$

This shows $\mathrm{s}_0 \vartriangleleft \mathrm{d}_1 = \mathrm{id}_{\mathrm{B}_V}$.

Suppose given $x \in \ker \mathrm{d}_1$. Then we have $x = (1,m)$ for some $m \in \mathrm{M}_V$.

Suppose given $y \in \ker \mathrm{d}_0$. Then we have $y = (n^{-}\mathrm{f}_V, n)$ for some $n \in \mathrm{M}_V$.

We have

$$\begin{aligned}x\cdot y = (1,m)\cdot(n^{-}\mathrm{f}_V\,,\,n) &= (n^{-}\mathrm{f}_V\,,\,m^{n^{-}\mathrm{f}_V}\cdot n) \overset{(\mathrm{CM2})}{=} (n^{-}\mathrm{f}_V\,,\,n\cdot m) = (n^{-}\mathrm{f}_V\,,\,n)\cdot(1,m)\\ &= y\cdot x\,.\end{aligned}$$

This shows $[\ker \mathrm{d}_0, \ker \mathrm{d}_1] = 1$. □

Remark 145 Suppose given a crossed module $V = (\mathrm{M}_V\,,\,\mathrm{B}_V\,,\,\gamma_V\,,\,\mathrm{f}_V)$.

We consider the category $V\mathrm{Cat}$; cf. **[29,** Rem. 4], cf. also Remark 165.(1).

Recall that we have

$$\begin{aligned}\mathrm{Ob}(V\mathrm{Cat}) &= \mathrm{B}_V\\ \mathrm{Mor}(V\mathrm{Cat}) &= \mathrm{B}_V \ltimes \mathrm{M}_V\,.\end{aligned}$$

and that the source, target and identity are given as follows.

$$\begin{array}{lllll} s\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b\\ t\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b\cdot m\mathrm{f}_V\\ i\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \leftarrow \mathrm{B}_V\,, & (b,1) & \leftarrow\!\shortmid & b\,.\end{array}$$

In the context of Lemma 144, we have $s = \mathrm{d}_1$, $t = \mathrm{d}_0$ and $i = \mathrm{s}_0$. So the constructions $V\mathrm{Rec}$ and $V\mathrm{Cat}$ are similar.

Lemma 146 (The $\lfloor 1,0 \rfloor$-simplicial group morphism $\rho\mathrm{Rec}$)

Suppose given crossed modules $V = (\mathrm{M}_V, \mathrm{B}_V, \gamma_V, \mathrm{f}_V)$ *and* $W = (\mathrm{M}_W, \mathrm{B}_W, \gamma_W, \mathrm{f}_W)$. *Suppose given a crossed module morphism* $\rho = (\mu, \beta)\colon V \to W$.

We consider the $\lfloor 1,0 \rfloor$*-simplicial groups* $V\mathrm{Rec}$ *and* $W\mathrm{Rec}$*; cf. Lemma* 144.

We have the $\lfloor 1,0 \rfloor$*-simplicial group morphism* $\rho\mathrm{Rec}\colon V\mathrm{Rec} \to W\mathrm{Rec}$ *given as follows.*

We have

$$\begin{aligned} (\rho\mathrm{Rec})_1\colon \quad & \mathrm{B}_V \ltimes \mathrm{M}_V \;\to\; \mathrm{B}_W \ltimes \mathrm{M}_W\,, \quad (b,m) \mapsto (b\beta, m\mu) \\ (\rho\mathrm{Rec})_0 := \beta\colon \quad & \mathrm{B}_V \;\to\; \mathrm{B}_W\,, \qquad\qquad b \mapsto b\beta\,. \end{aligned}$$

Proof. We show that $(\rho\mathrm{Rec})_1$ *is a group morphism.*

For b, $c \in \mathrm{B}_V$ and m, $n \in \mathrm{M}_V$, we have

$$\begin{aligned} \big((b,m)\cdot(c,n)\big)(\rho\mathrm{Rec})_1 &= (b\cdot c, m^c\cdot n)(\rho\mathrm{Rec})_1 = \big((b\cdot c)\beta, (m^c\cdot n)\mu\big) = \big(b\beta\cdot c\beta, (m\mu)^{c\beta}\cdot n\mu\big) \\ &= (b\beta, m\mu)\cdot(c\beta, n\mu) = (b,m)(\rho\mathrm{Rec})_1 \cdot (c,n)(\rho\mathrm{Rec})_1\,. \end{aligned}$$

We use Remark 138 to show that $\rho\mathrm{Rec}$ is $\lfloor 1,0 \rfloor$-simplicial group morphism.

For $b \in \mathrm{B}_V$ and $m \in \mathrm{M}_V$, we have

$$(b,m)\big(\mathrm{d}_1^{V\mathrm{Rec}} \vartriangleleft (\rho\mathrm{Rec})_0\big) = b(\rho\mathrm{Rec})_0 = b\beta = (b\beta, m\mu)\mathrm{d}_1^{W\mathrm{Rec}} = (b,m)\big((\rho\mathrm{Rec})_1 \vartriangleleft \mathrm{d}_1^{W\mathrm{Rec}}\big)\,.$$

This shows $\mathrm{d}_1^{V\mathrm{Rec}} \vartriangleleft (\rho\mathrm{Rec})_0 = (\rho\mathrm{Rec})_1 \vartriangleleft \mathrm{d}_1^{W\mathrm{Rec}}$.

For $b \in \mathrm{B}_V$ and $m \in \mathrm{M}_V$, we have

$$\begin{aligned} (b,m)\big(\mathrm{d}_0^{V\mathrm{Rec}} \vartriangleleft (\rho\mathrm{Rec})_0\big) &= (b\cdot m\mathrm{f}_V)(\rho\mathrm{Rec})_0 = b\beta\cdot m\mathrm{f}_V\beta = b\beta\cdot m\mu\mathrm{f}_W = (b\beta, m\mu)\mathrm{d}_0^{W\mathrm{Rec}} \\ &= (b,m)\big((\rho\mathrm{Rec})_1 \vartriangleleft \mathrm{d}_0^{W\mathrm{Rec}}\big)\,. \end{aligned}$$

This shows $\mathrm{d}_0^{V\mathrm{Rec}} \vartriangleleft (\rho\mathrm{Rec})_0 = (\rho\mathrm{Rec})_1 \vartriangleleft \mathrm{d}_0^{W\mathrm{Rec}}$.

For $b \in \mathrm{B}_V$, we have

$$b\,\mathrm{s}_0^{V\mathrm{Rec}}\,(\rho\mathrm{Rec})_1 = (b,1)(\rho\mathrm{Rec})_1 = (b\beta, 1) = (b\beta)\mathrm{s}_0^{W\mathrm{Rec}} = b\,(\rho\mathrm{Rec})_0\,\mathrm{s}_0^{W\mathrm{Rec}}\,.$$

This shows $\mathrm{s}_0^{V\mathrm{Rec}} \vartriangleleft (\rho\mathrm{Rec})_1 = (\rho\mathrm{Rec})_0 \vartriangleleft \mathrm{s}_0^{W\mathrm{Rec}}$. □

Lemma 147 (The functor Rec)
We have the reconstruction functor

$$\mathrm{Rec}\colon \mathcal{CrMod} \to \lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp}\,,\ \big(V \xrightarrow{\rho} W\big) \mapsto \big(V\mathrm{Rec} \xrightarrow{\rho\mathrm{Rec}} W\mathrm{Rec}\big)\,;$$

cf. Lemmas 144, 146.

Proof. Suppose given a crossed module $V = (\mathrm{M}_V\,,\,\mathrm{B}_V\,,\,\gamma_V\,,\,\mathrm{f}_V)$.
We consider the identity crossed module morphism $\mathrm{id}_V = (\mathrm{id}_{\mathrm{M}_V}\,,\mathrm{id}_{\mathrm{B}_V})\colon V \to V$.
We have

$$\big((\mathrm{id}_V)\mathrm{Rec}\big)_0 = \mathrm{id}_{\mathrm{B}_V}\ .$$

Moreover, for $(b,m) \in (V\mathrm{Rec})_1 = \mathrm{B}_V \ltimes \mathrm{M}_V$, we have

$$(b,m)\big((\mathrm{id}_V)\mathrm{Rec}\big)_1 = (b\,\mathrm{id}_{\mathrm{B}_V}\,,m\,\mathrm{id}_{\mathrm{M}_V}) = (b,m) = (b,m)\,\mathrm{id}_{\mathrm{B}_V \ltimes \mathrm{M}_V}\ .$$

So $\big((\mathrm{id}_V)\mathrm{Rec}\big)_1 = \mathrm{id}_{\mathrm{B}_V \ltimes \mathrm{M}_V}$.
This shows that $(\mathrm{id}_V)\mathrm{Rec} = \mathrm{id}_{V\mathrm{Rec}}$.
Suppose given $V \xrightarrow{\rho} \tilde{V} \xrightarrow{\tilde{\rho}} \tilde{\tilde{V}}$ in $\mathcal{CrMod}$.
We have $\rho = (\mu,\beta)$, where $\mu\colon \mathrm{M}_V \to \mathrm{M}_{\tilde{V}}$ and $\beta\colon \mathrm{B}_V \to \mathrm{B}_{\tilde{V}}$.
We have $\tilde{\rho} = (\tilde{\mu},\tilde{\beta})$, where $\tilde{\mu}\colon \mathrm{M}_{\tilde{V}} \to \mathrm{M}_{\tilde{\tilde{V}}}$ and $\tilde{\beta}\colon \mathrm{B}_{\tilde{V}} \to \mathrm{B}_{\tilde{\tilde{V}}}$.
Note that $\rho \blacktriangle \tilde{\rho} = (\mu,\beta) \blacktriangle (\tilde{\mu},\tilde{\beta}) = (\mu \blacktriangle \tilde{\mu}, \beta \blacktriangle \tilde{\beta})$.
We have

$$\Big((\rho \blacktriangle \tilde{\rho})\mathrm{Rec}\big)_0 = \beta \blacktriangle \tilde{\beta} = (\rho\mathrm{Rec})_0 \blacktriangle (\tilde{\rho}\mathrm{Rec})_0\,.$$

Moreover, for $(b,m) \in (V\mathrm{Rec})_1 = \mathrm{B}_V \ltimes \mathrm{M}_V$, we have

$$(b,m)\Big((\rho \blacktriangle \tilde{\rho})\mathrm{Rec}\big)_1 = (b\beta\tilde{\beta}, m\mu\tilde{\mu}) = (b\beta, m\mu)(\tilde{\rho}\mathrm{Rec})_1 = (b,m)\big((\rho\mathrm{Rec})_1 \blacktriangle (\tilde{\rho}\mathrm{Rec})_1\big)\,.$$

So $\big((\rho \blacktriangle \tilde{\rho})\mathrm{Rec}\big)_1 = (\rho\mathrm{Rec})_1 \blacktriangle (\tilde{\rho}\mathrm{Rec})_1$.
This shows that $(\rho \blacktriangle \tilde{\rho})\mathrm{Rec} = \rho\mathrm{Rec} \blacktriangle \tilde{\rho}\mathrm{Rec}$. Therefore, Rec is a functor. □

7.4 The equivalences $\hat{\mathrm{N}}$ and Rec

We consider the functors

$$\begin{array}{lrcll} \hat{\mathrm{N}}\colon & \lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp} & \to & \mathcal{CrMod}\,; & \text{cf. Lemma 143} \\ \mathrm{Rec}\colon & \mathcal{CrMod} & \to & \lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp}\,; & \text{cf. Lemma 147}\,. \end{array}$$

Remark 148 (The $\lfloor 1,0 \rfloor$-simplicial group $G\hat{\mathrm{N}}\,\mathrm{Rec}$)

(1) Suppose given a $\lfloor 1,0 \rfloor$-simplicial group G. We consider the $\lfloor 1,0 \rfloor$-simplicial group $G\hat{\mathrm{N}}\,\mathrm{Rec}$.

Recall that we have the crossed module

$$G\hat{\mathrm{N}} = (G\mathrm{N}_1, G_0, \gamma_G, \mathrm{d}_0^G\big|_{G\mathrm{N}_1})$$

with

$$\gamma_G\colon G_0 \to \mathrm{Aut}(G\mathrm{N}_1)\,,\ g \mapsto (m \mapsto m^g = g^-\mathrm{s}_0^G \cdot m \cdot g\mathrm{s}_0^G)\,;$$

cf. Lemma 141.

Then, by Lemma 144, we have the $\lfloor 1,0 \rfloor$-simplicial group $G\hat{\mathrm{N}}\,\mathrm{Rec}$, where

$$\begin{aligned} (G\hat{\mathrm{N}}\,\mathrm{Rec})_1 &= G_0 \ltimes_{\gamma_G} G\mathrm{N}_1 \\ (G\hat{\mathrm{N}}\,\mathrm{Rec})_0 &= G_0\,, \end{aligned}$$

and where

$$\begin{aligned} &\mathrm{s}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}}\colon G_0 \ltimes_{\gamma_G} G\mathrm{N}_1 \leftarrow G_0,\ (g,1) \leftarrow\!\shortmid\, g \\ &\mathrm{d}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}}\colon G_0 \ltimes_{\gamma_G} G\mathrm{N}_1 \to G_0,\ (g,m) \mapsto g \cdot m\mathrm{d}_0^G \\ &\mathrm{d}_1^{G\hat{\mathrm{N}}\,\mathrm{Rec}}\colon G_0 \ltimes_{\gamma_G} G\mathrm{N}_1 \to G_0,\ (g,m) \mapsto g\,. \end{aligned}$$

(2) Suppose given a $\lfloor 1,0 \rfloor$-simplicial group morphism $G \xrightarrow{\varphi} H$. We consider the $\lfloor 1,0 \rfloor$-simplicial groups $G\hat{\mathrm{N}}\,\mathrm{Rec}$ and $H\hat{\mathrm{N}}\,\mathrm{Rec}$; cf. (1).

We consider the $\lfloor 1,0 \rfloor$-simplicial group morphism $\varphi\hat{\mathrm{N}}\,\mathrm{Rec}\colon G\hat{\mathrm{N}}\,\mathrm{Rec} \to H\hat{\mathrm{N}}\,\mathrm{Rec}$.

Recall that we have the crossed module morphism

$$\varphi\hat{\mathrm{N}} = \big(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}, \varphi_0\big)\colon G\hat{\mathrm{N}} \to H\hat{\mathrm{N}}\,;$$

cf. Lemma 142.

Then, by Lemma 146, we have

$$\begin{aligned}
&(\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_1\colon && G_0 \ltimes_{\gamma_G} G\mathrm{N}_1 \to H_0 \ltimes_{\gamma_H} H\mathrm{N}_1\,, && (g,m)\mapsto (g\varphi_0, m\varphi_1)\\
&(\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_0 = \varphi_0\colon && G_0 \to H_0\,, && g\mapsto g\varphi_0\,.
\end{aligned}$$

Lemma 149 (The functor $\hat{\mathrm{N}} * \mathrm{Rec}$)

(1) *Suppose given a* $\lfloor 1,0\rfloor$*-simplicial group* G*.*

We have the $\lfloor 1,0\rfloor$*-simplicial group isomorphism* $G\psi\colon G\hat{\mathrm{N}}\,\mathrm{Rec}\to G$*, where*

$$\begin{aligned}
&(G\psi)_1\colon && G_0 \ltimes G\mathrm{N}_1 \to G_1\,, && (g,m)\mapsto g\mathrm{s}_0^G\cdot m\\
&(G\psi)_0 := \mathrm{id}_{G_0}\colon && G_0\to G_0\,, && g\mapsto g\,.
\end{aligned}$$

Moreover, we have

$$(G\psi)_1^-\colon G_1\to G_0\ltimes G\mathrm{N}_1\,,\ g\mapsto (g\mathrm{d}_1^G, g^-\mathrm{d}_1^G\mathrm{s}_0^G\cdot g)\,.$$

(2) *We have the isotransformation*

$$\psi = \big(G\hat{\mathrm{N}}\,\mathrm{Rec}\xrightarrow{G\psi} G\big)_{G\in\mathrm{Ob}(\lfloor 1,0\rfloor\text{-}\mathcal{SimpGrp})}\colon \hat{\mathrm{N}}*\mathrm{Rec}\to \mathrm{id}_{\lfloor 1,0\rfloor\text{-}\mathcal{SimpGrp}}\,;$$

cf. (1).

Proof. Ad (1). We write $\mathrm{s}_0 := \mathrm{s}_0^G$, $\mathrm{d}_0 := \mathrm{d}_0^G$ and $\mathrm{d}_1 := \mathrm{d}_1^G$.

We show that $(G\psi)_1$ *is a group morphism.* We abbreviate $\psi := (G\psi)_1$.

For g, $h\in G_0$ and m, $n\in G\mathrm{N}_1$, we have

$$\begin{aligned}
\big((g,m)\cdot(h,n)\big)\psi &= (g\cdot h, m^h\cdot n)\psi = (g\cdot h)\mathrm{s}_0\cdot m^h\cdot n = g\mathrm{s}_0\cdot h\mathrm{s}_0\cdot h^-\mathrm{s}_0\cdot m\cdot h\mathrm{s}_0\cdot n\\
&= (g\mathrm{s}_0\cdot m)\cdot(h\mathrm{s}_0\cdot n) = (g,m)\psi\cdot(h,n)\psi\,.
\end{aligned}$$

So ψ is a group morphism.

We consider the map $\psi'\colon G_1\to G_0\ltimes G\mathrm{N}_1\,,\ g\mapsto (g\mathrm{d}_1, g^-\mathrm{d}_1\mathrm{s}_0\cdot g)$.

We show that ψ' *is well-defined.*

For $g\in G_1$, we have

$$(g^-\mathrm{d}_1\mathrm{s}_0\cdot g)\mathrm{d}_1 = g^-\mathrm{d}_1\mathrm{s}_0\mathrm{d}_1\cdot g\mathrm{d}_1 = g^-\mathrm{d}_1\cdot g\mathrm{d}_1 = 1\,.$$

So $g^-\mathrm{d}_1\mathrm{s}_0\cdot g\in G\mathrm{N}_1$. Therefore, the map ψ' exists.

We show that ψ' is the inverse of ψ.

For $g \in G_0$ and $m \in G\mathrm{N}_1$, we have

$$\begin{aligned}(g,m)\psi\psi' &= (g\mathrm{s}_0 \cdot m)\psi' = \big(g\mathrm{s}_0\mathrm{d}_1 \cdot m\mathrm{d}_1\,,\,(g\mathrm{s}_0 \cdot m)^-\mathrm{d}_1\mathrm{s}_0 \cdot g\mathrm{s}_0 \cdot m\big)\\ &= (g, m^-\mathrm{d}_1\mathrm{s}_0 \cdot g^-\mathrm{s}_0\mathrm{d}_1\mathrm{s}_0 \cdot g\mathrm{s}_0 \cdot m) = (g, g^-\mathrm{s}_0 \cdot g\mathrm{s}_0 \cdot m) = (g,m)\,.\end{aligned}$$

This shows $\psi \mathbin{\triangle} \psi' = \mathrm{id}_{G_0 \ltimes G\mathrm{N}_1}$.

For $g \in G_1$, we have

$$g\psi'\psi = (g\mathrm{d}_1\,,\, g^-\mathrm{d}_1\mathrm{s}_0 \cdot g)\psi = g\mathrm{d}_1\mathrm{s}_0 \cdot g^-\mathrm{d}_1\mathrm{s}_0 \cdot g = g\,.$$

This shows $\psi' \mathbin{\triangle} \psi = \mathrm{id}_{G_1}$. So we have $\psi' = \psi^-$.

Altogether, $\psi = (G\psi)_1$ is a group isomorphism with inverse $(G\psi)^- = \psi'$.

We show that $G\psi$ is a $\lfloor 1,0 \rfloor$-simplicial group morphism.

For $g \in G_0$ and $m \in G\mathrm{N}_1$, we have

$$(g,m)\big((G\psi)_1 \mathbin{\triangle} \mathrm{d}_0^G\big) = (g\mathrm{s}_0^G \cdot m)\mathrm{d}_0^G = g \cdot m\mathrm{d}_0^G = (g \cdot m\mathrm{d}_0^G)\,\mathrm{id}_{G_0} = (g,m)(\mathrm{d}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}} \mathbin{\triangle} \mathrm{id}_{G_0})\,.$$

This shows $(G\psi)_1 \mathbin{\triangle} \mathrm{d}_0^G = \mathrm{d}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}} \mathbin{\triangle} (G\psi)_0$.

$$\begin{array}{ccc} G_0 \ltimes G\mathrm{N}_1 & \xrightarrow{\mathrm{d}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}}} & G_0 \\ {\scriptstyle (G\psi)_1}\downarrow & & \downarrow{\scriptstyle (G\psi)_0} \\ G_1 & \xrightarrow[\mathrm{d}_0^G]{} & G_0 \end{array}$$

For $g \in G_0$ and $m \in G\mathrm{N}_1$, we have

$$(g,m)\big((G\psi)_1 \mathbin{\triangle} \mathrm{d}_1^G\big) = (g\mathrm{s}_0^G \cdot m)\mathrm{d}_1^G = g \cdot 1 = g\,\mathrm{id}_{G_0} = (g,m)(\mathrm{d}_1^{G\hat{\mathrm{N}}\,\mathrm{Rec}} \mathbin{\triangle} \mathrm{id}_{G_0})\,.$$

This shows $(G\psi)_1 \mathbin{\triangle} \mathrm{d}_1^G = \mathrm{d}_1^{G\hat{\mathrm{N}}\,\mathrm{Rec}} \mathbin{\triangle} (G\psi)_0$.

$$\begin{array}{ccc} G_0 \ltimes G\mathrm{N}_1 & \xrightarrow{\mathrm{d}_1^{G\hat{\mathrm{N}}\,\mathrm{Rec}}} & G_0 \\ {\scriptstyle (G\psi)_1}\downarrow & & \downarrow{\scriptstyle (G\psi)_0} \\ G_1 & \xrightarrow[\mathrm{d}_1^G]{} & G_0 \end{array}$$

For $g \in G_0$, we have

$$g\big(\mathrm{s}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}} \mathbin{\triangle} (G\psi)_1\big) = (g,1)(G\psi)_1 = g\mathrm{s}_0^G \cdot 1 = g\mathrm{s}_0^G = g(\mathrm{id}_{G_0} \mathbin{\triangle} \mathrm{s}_0^G)\,.$$

This shows $\mathrm{s}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}} \mathbin{\triangle} (G\psi)_1 = (G\psi)_0 \mathbin{\triangle} \mathrm{s}_0^G$.

$$\begin{array}{ccc} G_0 \ltimes G\mathrm{N}_1 & \xleftarrow{\mathrm{s}_0^{G\hat{\mathrm{N}}\,\mathrm{Rec}}} & G_0 \\ {\scriptstyle (G\psi)_1}\downarrow & & \downarrow{\scriptstyle (G\psi)_0} \\ G_1 & \xleftarrow[\mathrm{s}_0^G]{} & G_0 \end{array}$$

So, by Remark 138, $G\psi$ is a $\lfloor 1,0 \rfloor$-simplicial group morphism.

Since id_{G_0} and $G\psi$ are group isomorphisms we have that $G\psi$ is a $\lfloor 1,0 \rfloor$-simplicial group isomorphism.

Ad (2). Suppose given $G \xrightarrow{\varphi} H$ in $\lfloor 1,0 \rfloor$-*SimpGrp*. We have

$$(G\psi)_0 \mathbin{\triangle} \varphi_0 = \mathrm{id}_{G_0} \mathbin{\triangle} \varphi_0 = \varphi_0 \overset{148.(2)}{=} (\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_0 = (\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_0 \mathbin{\triangle} \mathrm{id}_{H_0} = (\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_0 \mathbin{\triangle} (H\psi)_0\,.$$

$$\begin{array}{ccc} G_0 & \xrightarrow{(G\psi)_0} & G_0 \\ {\scriptstyle (\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_0}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ H_0 & \xrightarrow[(H\psi)_0]{} & H_0 \end{array}$$

Moreover, for $g \in G_0$, $m \in G\mathrm{N}_1$, we have

$$\begin{array}{rcl} (g,m)\big((G\psi)_1 \mathbin{\triangle} \varphi_1\big) & = & (g\mathrm{s}_0^G \cdot m)\varphi_1 = g\mathrm{s}_0^G\varphi_1 \cdot m\varphi_1 = g\varphi_0\mathrm{s}_0^H \cdot m\varphi_1 = (g\varphi_0, m\varphi_1)(H\psi)_1 \\ & \overset{148.(2)}{=} & (g,m)\big((\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_1 \mathbin{\triangle} (H\psi)_1\big)\,. \end{array}$$

This shows $(G\psi)_1 \mathbin{\triangle} \varphi_1 = (\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_1 \mathbin{\triangle} (H\psi)_1$.

$$\begin{array}{ccc} G_0 \ltimes G\mathrm{N}_1 & \xrightarrow{(G\psi)_1} & G_1 \\ {\scriptstyle (\varphi\hat{\mathrm{N}}\,\mathrm{Rec})_1}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ H_0 \ltimes H\mathrm{N}_1 & \xrightarrow[(H\psi)_1]{} & H_1 \end{array}$$

So we have $G\psi \mathbin{\triangle} \varphi = \varphi \hat{\mathrm{N}}\,\mathrm{Rec} \mathbin{\triangle} H\psi$.

$$\begin{array}{ccc} G\hat{\mathrm{N}}\,\mathrm{Rec} & \xrightarrow{G\psi} & G \\ {\scriptstyle \varphi\hat{\mathrm{N}}\,\mathrm{Rec}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ H\hat{\mathrm{N}}\,\mathrm{Rec} & \xrightarrow[H\psi]{} & H \end{array}$$

So ψ is a transformation. Using (1), we conclude that ψ is an isotransformation. □

Remark 150 (The crossed module $V\,\mathrm{Rec}\,\hat{\mathrm{N}}$)

(1) Suppose given a crossed module $V = (\mathrm{M}_V\,,\,\mathrm{B}_V\,,\,\boldsymbol{\gamma}_V\,,\,\mathrm{f}_V)$. We consider the crossed module $V\,\mathrm{Rec}\,\hat{\mathrm{N}}$.

Recall that we have the $\lfloor 1,0 \rfloor$-simplicial group $V\,\mathrm{Rec}$, where

$$(V\,\mathrm{Rec})_0 = \mathrm{B}_V\,,\ (V\,\mathrm{Rec})_1 = \mathrm{B}_V \ltimes_{\boldsymbol{\gamma}_V} \mathrm{M}_V\,,$$

and where

$$\begin{array}{lrcl} \mathrm{s}_0^{V\,\mathrm{Rec}}\colon & \mathrm{B}_V \ltimes_{\boldsymbol{\gamma}_V} \mathrm{M}_V \leftarrow \mathrm{B}_V\,, & (b,1) & \leftarrow\!\shortmid\ b \\ \mathrm{d}_0^{V\,\mathrm{Rec}}\colon & \mathrm{B}_V \ltimes_{\boldsymbol{\gamma}_V} \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto\ b \cdot m\mathrm{f}_V \\ \mathrm{d}_1^{V\,\mathrm{Rec}}\colon & \mathrm{B}_V \ltimes_{\boldsymbol{\gamma}_V} \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto\ b\,; \end{array}$$

cf. Lemma 144.

Then, by Lemma 141, we have the crossed module

$$\begin{aligned} V\,\mathrm{Rec}\,\hat{\mathrm{N}} &= (\mathrm{M}_{V\mathrm{Rec}\hat{\mathrm{N}}}\,,\,\mathrm{B}_{V\mathrm{Rec}\hat{\mathrm{N}}}\,,\,\boldsymbol{\gamma}_{V\mathrm{Rec}\hat{\mathrm{N}}}\,,\,\mathrm{f}_{V\mathrm{Rec}\hat{\mathrm{N}}}) \\ &= \left((V\mathrm{Rec})\mathrm{N}_1\,,\mathrm{B}_V\,,\gamma_{V\mathrm{Rec}}\,,\mathrm{d}_0^{V\,\mathrm{Rec}}\big|_{(V\mathrm{Rec})\mathrm{N}_1}\right) \\ &= \left(1 \ltimes \mathrm{M}_V\,,\mathrm{B}_V\,,\gamma_{V\mathrm{Rec}}\,,\mathrm{d}_0^{V\mathrm{Rec}}\big|_{1\ltimes \mathrm{M}_V}\right). \end{aligned}$$

Note that

$$\mathrm{M}_{V\mathrm{Rec}\hat{\mathrm{N}}} = \ker \mathrm{d}_1^{V\mathrm{Rec}} = 1 \ltimes \mathrm{M}_V = \{(1,m) \in \mathrm{B}_V \ltimes_{\boldsymbol{\gamma}_V} \mathrm{M}_V \colon m \in \mathrm{M}_V\} \leqslant \mathrm{B}_V \ltimes_{\boldsymbol{\gamma}_V} \mathrm{M}_V\,,$$

and that

$$\begin{array}{lll} \boldsymbol{\gamma}_{V\mathrm{Rec}\hat{\mathrm{N}}} = \boldsymbol{\gamma}_{V\mathrm{Rec}}\colon & \mathrm{B}_V \to \mathrm{Aut}(1 \ltimes \mathrm{M}_V)\,, & b \mapsto \left((1,m) \mapsto (1,m)^{b\mathrm{s}_0^{V\mathrm{Rec}}} = (1,m)^{(1,b)} = (1,m^b)\right) \\ \mathrm{f}_{V\mathrm{Rec}\hat{\mathrm{N}}} = \mathrm{d}_0^{V\mathrm{Rec}}\big|_{1\ltimes \mathrm{M}_V}\colon & 1 \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (1,m) \mapsto m\mathrm{f}_V\,. \end{array}$$

(2) Suppose given a crossed module morphism $\rho = (\mu, \beta)\colon V \to W$; cf. Definition 324.

We consider the crossed modules $V\,\mathrm{Rec}\,\hat{\mathrm{N}}$ and $W\,\mathrm{Rec}\,\hat{\mathrm{N}}$; cf. (1). We consider the crossed module morphism $\rho\,\mathrm{Rec}\,\hat{\mathrm{N}}\colon V\,\mathrm{Rec}\,\hat{\mathrm{N}} \to W\,\mathrm{Rec}\,\hat{\mathrm{N}}$.

Recall that we have the $\lfloor 1, 0 \rfloor$-simplicial group morphism $\rho\,\mathrm{Rec}\colon V\,\mathrm{Rec} \to W\,\mathrm{Rec}$, where

$$(\rho\,\mathrm{Rec})_1\colon\ \mathrm{B}_V \ltimes_{\gamma_V} \mathrm{M}_V \to \mathrm{B}_W \ltimes_{\gamma_W} \mathrm{M}_W\,,\quad (b, m) \mapsto (b\beta, m\mu)$$

$$(\rho\,\mathrm{Rec})_0\colon\ \mathrm{B}_V \to \mathrm{B}_W\,,\qquad\qquad b \mapsto b\beta\,;$$

cf. Lemma 146.

Then, by Lemma 142, we have the crossed module morphism

$$\rho\,\mathrm{Rec}\,\hat{\mathrm{N}} = \Big((\rho\,\mathrm{Rec})_1\big|_{(V\,\mathrm{Rec})\mathrm{N}_1}^{(W\,\mathrm{Rec})\mathrm{N}_1}, (\rho\,\mathrm{Rec})_0\Big) = \Big((\rho\,\mathrm{Rec})_1\big|_{1\ltimes \mathrm{M}_V}^{1\ltimes \mathrm{M}_W}, \beta\Big)\colon V\,\mathrm{Rec}\,\hat{\mathrm{N}} \to W\,\mathrm{Rec}\,\hat{\mathrm{N}}\,.$$

Note that

$$(\rho\,\mathrm{Rec})_1\big|_{1\ltimes \mathrm{M}_V}^{1\ltimes \mathrm{M}_W}\colon 1 \ltimes \mathrm{M}_V \to 1 \ltimes \mathrm{M}_W\,, (1, m) \mapsto (1, m\mu)\,.$$

Lemma 151 (The functor $\mathrm{Rec} * \hat{\mathrm{N}}$)

(1) *Suppose given a crossed module* $V = (\mathrm{M}_V\,,\, \mathrm{B}_V\,,\, \gamma_V\,,\, \mathrm{f}_V)$.

We have the crossed module isomorphism $V\hat{\psi} := (\pi_{\mathrm{M}_V}, \mathrm{id}_{\mathrm{B}_V})\colon V\,\mathrm{Rec}\,\hat{\mathrm{N}} \to V$*, where*

$$\pi_{\mathrm{M}_V}\colon 1 \ltimes \mathrm{M}_V \xrightarrow[\sim]{} \mathrm{M}_V\,,\ (1, m) \mapsto m\,.$$

(2) *We have the isotransformation*

$$\hat{\psi} = \big(V\,\mathrm{Rec}\,\hat{\mathrm{N}} \xrightarrow{V\hat{\psi}} V\big)_{V \in \mathrm{Ob}(\mathcal{CrMod})}\colon\ \mathrm{Rec} * \hat{\mathrm{N}} \to \mathrm{id}_{\mathcal{CrMod}}\,;$$

cf. (1).

Proof. Ad (1). Note that π_{M_V} and $\mathrm{id}_{\mathrm{B}_V}$ are group isomorphisms.

For $m \in \mathrm{M}_V$, we have

$$(1, m)(\pi_{\mathrm{M}_V} \triangleleft \mathrm{f}_V) = m\mathrm{f}_V \overset{150.(1)}{=} (1, m)\mathrm{f}_{V\mathrm{Rec}\hat{\mathrm{N}}} = (1, m)(\mathrm{f}_{V\mathrm{Rec}\hat{\mathrm{N}}} \triangleleft \mathrm{id}_{\mathrm{B}_V})\,.$$

This shows $\pi_{\mathrm{M}_V} \triangleleft \mathrm{f}_V = \mathrm{f}_{V\mathrm{Rec}\hat{\mathrm{N}}} \triangleleft \mathrm{id}_{\mathrm{B}_V}$.

$$\begin{array}{ccc} \mathrm{M}_{V\,\mathrm{Rec}\,\hat{\mathrm{N}}} & \xrightarrow{\pi_{\mathrm{M}_V}} & \mathrm{M}_V \\ {\scriptstyle \mathrm{f}_{V\mathrm{Rec}\hat{\mathrm{N}}}}\downarrow & & \downarrow{\scriptstyle \mathrm{f}_V} \\ \mathrm{B}_{V\,\mathrm{Rec}\,\hat{\mathrm{N}}} & \xrightarrow[\mathrm{id}_{\mathrm{B}_V}]{} & \mathrm{B}_V \end{array}$$

Moreover, for $m \in \mathrm{M}_V$ and $b \in \mathrm{B}_{V\mathrm{Rec}\hat{\mathrm{N}}} = \mathrm{B}_V$, we have

$$\big((1,m)^b\big)\pi_{\mathrm{M}_V} \overset{150.(1)}{=} (1,m^b)\pi_{\mathrm{M}_V} = m^b = \big((1,m)\pi_{\mathrm{M}_V}\big)^{b\,\mathrm{id}_{\mathrm{B}_V}}.$$

So $(\pi_{\mathrm{M}_V}, \mathrm{id}_{\mathrm{B}_V})$ is a crossed module isomorphism; cf. §1.1.4 items 3, 4.

Ad (2). Suppose given $V \xrightarrow{\rho} W$ in $\mathcal{C}r\mathcal{M}od$. Then $\rho = (\mu, \beta)$ where $\mu\colon \mathrm{M}_V \to \mathrm{M}_W$ and $\beta\colon \mathrm{B}_V \to \mathrm{B}_W$. We write $(\hat{\mu}, \hat{\beta}) := \Big((\rho\,\mathrm{Rec})_1\big|^{1\ltimes \mathrm{M}_W}_{1\ltimes \mathrm{M}_V}, \beta\Big) \overset{150.(2)}{=} \rho\,\mathrm{Rec}\,\hat{\mathrm{N}}\colon V\,\mathrm{Rec}\,\hat{\mathrm{N}} \to W\,\mathrm{Rec}\,\hat{\mathrm{N}}$.

For $m \in \mathrm{M}_V$, we have

$$(1,m)(\pi_{\mathrm{M}_V} \blacktriangle \mu) = m\mu = (1,m\mu)\pi_{\mathrm{M}_W} \overset{150.(1)}{=} (1,m)\Big((\rho\,\mathrm{Rec})_1\big|^{1\ltimes \mathrm{M}_W}_{1\ltimes \mathrm{M}_V} \blacktriangle \pi_{\mathrm{M}_W}\Big) = (1,m)(\hat{\mu} \blacktriangle \pi_{\mathrm{M}_W}).$$

This shows $\pi_{\mathrm{M}_V} \blacktriangle \mu = \hat{\mu} \blacktriangle \pi_{\mathrm{M}_W}$.

$$\begin{array}{ccc} 1 \ltimes \mathrm{M}_V & \xrightarrow{\pi_{\mathrm{M}_V}} & \mathrm{M}_V \\ {\scriptstyle \hat{\mu}}\downarrow & & \downarrow{\scriptstyle \mu} \\ 1 \ltimes \mathrm{M}_W & \xrightarrow[\pi_{\mathrm{M}_W}]{} & \mathrm{M}_W \end{array}$$

For $b \in \mathrm{B}_V$, we have

$$b(\mathrm{id}_{\mathrm{B}_V} \blacktriangle \beta) = b\beta = b(\hat{\beta} \blacktriangle \mathrm{id}_{\mathrm{B}_W}).$$

This shows $\mathrm{id}_{\mathrm{B}_V} \blacktriangle \beta = \hat{\beta} \blacktriangle \mathrm{id}_{\mathrm{B}_W}$.

$$\begin{array}{ccc} \mathrm{B}_V & \xrightarrow{\mathrm{id}_{\mathrm{B}_V}} & \mathrm{B}_V \\ {\scriptstyle \hat{\beta}}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathrm{B}_W & \xrightarrow[\mathrm{id}_{\mathrm{B}_W}]{} & \mathrm{B}_W \end{array}$$

So we have

$$\begin{aligned} V\hat{\psi} \blacktriangle \rho &= (\pi_{\mathrm{M}_V}, \mathrm{id}_{\mathrm{B}_V}) \blacktriangle (\mu, \beta) = (\pi_{\mathrm{M}_V} \blacktriangle \mu, \mathrm{id}_{\mathrm{B}_V} \blacktriangle \beta) = (\hat{\mu} \blacktriangle \pi_{\mathrm{M}_W}, \hat{\beta} \blacktriangle \mathrm{id}_{\mathrm{B}_W}) \\ &= (\hat{\mu}, \hat{\beta}) \blacktriangle (\pi_{\mathrm{M}_W}, \mathrm{id}_{\mathrm{B}_W}) = \rho\,\mathrm{Rec}\,\hat{\mathrm{N}} \blacktriangle W\hat{\psi}. \end{aligned}$$

$$\begin{array}{ccc} V\,\mathrm{Rec}\,\hat{\mathrm{N}} & \xrightarrow{V\hat{\psi}} & V \\ {\scriptstyle \rho\,\mathrm{Rec}\,\hat{\mathrm{N}}}\downarrow & & \downarrow{\scriptstyle \rho} \\ W\,\mathrm{Rec}\,\hat{\mathrm{N}} & \xrightarrow[W\hat{\psi}]{} & W \end{array}$$

So $\hat{\psi}$ is a transformation. Using (1), we conclude that $\hat{\psi}$ is an isotransformation. □

Proposition 152 (The equivalences $\hat{\mathrm{N}}$ and Rec)

By Lemma 149.(2)*, we have the isotransformation* $\psi\colon \hat{\mathrm{N}} * \mathrm{Rec} \xrightarrow[\sim]{} \mathrm{id}_{\lfloor 1,0\rfloor\text{-}\mathcal{SimpGrp}}$.

By Lemma 151.(2)*, we have the isotransformation* $\hat{\psi}\colon \mathrm{Rec} * \hat{\mathrm{N}} \xrightarrow[\sim]{} \mathrm{id}_{\mathcal{CrMod}}$.

So $\hat{\mathrm{N}}$ *and* Rec *are equivalences. In particular, the categories* $\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$ *and* $\mathcal{CrMod}$ *are equivalent.*

> The equivalence between $\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$ and $\mathcal{CrMod}$ has also been shown by Conduché [**7,** Cor. 1.8, Cor. 2.7].

7.5 A remark on simplicial groups and crossed modules

Remark 153 We consider the functors

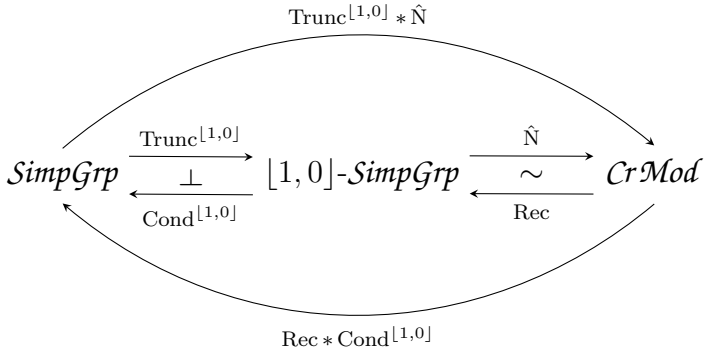
$\mathrm{Trunc}^{\lfloor 1,0\rfloor}\colon$	$\mathcal{SimpGrp}$	$\to$	$\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$;	cf. Remark 139.(3)
$\mathrm{Cond}^{\lfloor 1,0\rfloor}\colon$	$\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$	$\to$	$\mathcal{SimpGrp}$;	cf. Lemma 126
$\hat{\mathrm{N}}\colon$	$\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$	$\to$	$\mathcal{CrMod}$;	cf. Lemma 143
$\mathrm{Rec}\colon$	$\mathcal{CrMod}$	$\to$	$\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$;	cf. Lemma 147.

Recall that we have $\mathrm{Trunc}^{\lfloor 1,0\rfloor} \dashv \mathrm{Cond}^{\lfloor 1,0\rfloor}$; cf. Theorem 130.

Recall that the functors $\hat{\mathrm{N}}$ and Rec are mutually inverse equivalences, up to isotransformation; cf. Proposition 152. So, by Remark 17, we have $\hat{\mathrm{N}} \dashv \mathrm{Rec}$.

We consider the functors

$$\mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}}\colon \mathcal{SimpGrp} \to \mathcal{CrMod} \quad \text{and} \quad \mathrm{Rec} * \mathrm{Cond}^{\lfloor 1,0\rfloor}\colon \mathcal{CrMod} \to \mathcal{SimpGrp}.$$



Then, by Remark 16, we have $\mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}} \dashv \mathrm{Rec} * \mathrm{Cond}^{\lfloor 1,0\rfloor}$.

Remark 154 Suppose given a crossed module $V = (\mathrm{M}_V\,,\,\mathrm{B}_V\,,\,\gamma_V\,,\,\mathrm{f}_V)$; cf. §1.1.4 item 1.

(1) We consider the $\lfloor 1,0 \rfloor$-simplicial group $V\,\mathrm{Rec}$; cf. Lemma 147.

We have

$$(V\,\mathrm{Rec})_0 = \mathrm{B}_V \quad \text{and} \quad (V\,\mathrm{Rec})_1 = \mathrm{B}_V \ltimes \mathrm{M}_V\,,$$

and we have

$$\begin{array}{lllll} \mathrm{s}_0^{V\,\mathrm{Rec}}\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \leftarrow \mathrm{B}_V\,, & (b,1) & \leftarrow\!\shortmid & b \\ \mathrm{d}_0^{V\,\mathrm{Rec}}\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b \cdot m\mathrm{f}_V \\ \mathrm{d}_1^{V\,\mathrm{Rec}}\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b\,; \end{array}$$

cf. Lemma 144.

Moreover, we have

$$\begin{array}{lcll} (V\,\mathrm{Rec})\mathrm{N}_0 & = & (V\,\mathrm{Rec})_0 & = \mathrm{B}_V \\ (V\,\mathrm{Rec})\mathrm{N}_1 & \overset{76.(2)}{=} & \ker \mathrm{d}_1^{V\,\mathrm{Rec}} & \simeq \mathrm{M}_V \\ (V\,\mathrm{Rec})\mathrm{N}_2 & \overset{77.(2)}{=} & 1\,. & \end{array}$$

(2) We consider the simplicial group $V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0 \rfloor}$; cf. Lemma 126.

We consider the group $(V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_2$.

We have

$$\begin{array}{ll} & (V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})_2 \\ \overset{103}{\simeq} & \big((V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})\mathrm{N}_0 \ltimes (V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})\mathrm{N}_1\big) \ltimes \big((V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})\mathrm{N}_1 \ltimes (V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0 \rfloor})\mathrm{N}_2\big) \\ \overset{134}{\simeq} & \big((V\,\mathrm{Rec})\mathrm{N}_0 \ltimes (V\,\mathrm{Rec})\mathrm{N}_1\big) \ltimes \big((V\,\mathrm{Rec})\mathrm{N}_1 \ltimes (V\,\mathrm{Rec})\mathrm{N}_2\big) \\ \overset{(1)}{\simeq} & (\mathrm{B}_V \ltimes \mathrm{M}_V) \ltimes (\mathrm{M}_V \ltimes 1)\,. \end{array}$$

(3) We consider the simplicial group $V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor}$; cf. Lemma 82.

We consider the isotransformation $\eta\colon \mathrm{Cosk}^{\lfloor 1,0 \rfloor} * \mathrm{Cut}^{\lfloor 1,0 \rfloor} \to \mathrm{id}_{\mathcal{P}re\text{-}\lfloor 1,0 \rfloor\text{-}\mathcal{S}impGrp}$ from Lemma 86.(3).

In particular, we have the isomorphism

$$(V\mathrm{Rec})\eta\colon V\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0 \rfloor}\,\mathrm{Cut}^{\lfloor 1,0 \rfloor} \xrightarrow{\sim} V\mathrm{Rec}$$

in pre-$\lfloor 1,0 \rfloor$-simplicial groups, hence in $\lfloor 1,0 \rfloor$-simplicial groups.

So we also have the group isomorphisms

$$\big((V\,\mathrm{Rec})\eta\big)\mathrm{N}_0\colon (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor}\,\mathrm{Cut}^{\lfloor 1,0\rfloor})\mathrm{N}_0 = (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_0 \xrightarrow{\sim} (V\,\mathrm{Rec})\mathrm{N}_0$$
$$\big((V\,\mathrm{Rec})\eta\big)\mathrm{N}_1\colon (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor}\,\mathrm{Cut}^{\lfloor 1,0\rfloor})\mathrm{N}_1 = (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_1 \xrightarrow{\sim} (V\,\mathrm{Rec})\mathrm{N}_1\,;$$

cf. Lemma 78.(3).

We consider the group $(V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2$.

We have

$$\begin{aligned}
&(V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2\\
&\overset{103}{\simeq} \big((V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_0 \ltimes (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_1\big) \ltimes \big((V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_1 \ltimes (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2\big)\\
&\simeq \big((V\,\mathrm{Rec})\mathrm{N}_0 \ltimes (V\,\mathrm{Rec})\mathrm{N}_1\big) \ltimes \big((V\,\mathrm{Rec})\mathrm{N}_1 \ltimes (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2\big)\\
&\overset{(1)}{\simeq} (\mathrm{B}_V \ltimes \mathrm{M}_V) \ltimes \big(\mathrm{M}_V \ltimes (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2\big)\,.
\end{aligned}$$

(4) We consider the group $(V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2$.

We consider the the subgroup $(V\,\mathrm{Rec})\mathrm{NZ}_1 \leqslant (V\,\mathrm{Rec})_1 = \mathrm{B}_V \ltimes \mathrm{M}_V$; cf. Definition 76.(4).

We have

$$\begin{aligned}
(V\,\mathrm{Rec})\mathrm{NZ}_1 &= \ker \mathrm{d}_0^{V\,\mathrm{Rec},1} \cap \ker \mathrm{d}_1^{V\,\mathrm{Rec},1}\\
&\overset{(1)}{=} \big\{(b,m) \in \mathrm{B}_V \ltimes \mathrm{M}_V\colon b = 1 \text{ and } \ m\mathrm{f}_V = 1\big\}\\
&\simeq 1 \ltimes \ker \mathrm{f}_V\\
&\simeq \ker \mathrm{f}_V\,.
\end{aligned}$$

We consider the subgroup $(V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2 \leqslant (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2$; cf. Definition 76.(2).

We have $(V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2 \overset{83.(3)}{\simeq} (V\,\mathrm{Rec})\mathrm{NZ}_1 \simeq \ker \mathrm{f}_V$.

(5) We have

$$(V\,\mathrm{Rec}\,\mathrm{Cond}^{\lfloor 1,0\rfloor})_2 \overset{(2)}{\simeq} (\mathrm{B}_V \ltimes \mathrm{M}_V) \ltimes (\mathrm{M}_V \ltimes 1)\,.$$

and

$$(V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})_2 \overset{(3)}{\simeq} (\mathrm{B}_V \ltimes \mathrm{M}_V) \ltimes \big(\mathrm{M}_V \ltimes (V\,\mathrm{Rec}\,\mathrm{Cosk}^{\lfloor 1,0\rfloor})\mathrm{N}_2\big) \overset{(4)}{\simeq} (\mathrm{B}_V \ltimes \mathrm{M}_V) \ltimes (\mathrm{M}_V \ltimes \ker \mathrm{f}_V)\,.$$

Since we have $\ker \mathrm{f}_V \neq 1$ in general, this shows that

$$(V \operatorname{Rec} \operatorname{Cond}^{\lfloor 1,0 \rfloor})_2 \not\simeq (V \operatorname{Rec} \operatorname{Cosk}^{\lfloor 1,0 \rfloor})_2$$

in general.

Remark 155 In **[27,** (5.23),(6.5),(6.11)], S. Thomas considers the functor Trunc, which, in our notation, is $\operatorname{Trunc}^{\lfloor 1,0 \rfloor} * \hat{\mathrm{N}} \colon \mathcal{SimpGrp} \to \mathcal{CrMod}$.

He constructs a right adjoint to Trunc, which he calls Cosk and which we obtain as $\operatorname{Rec} * \operatorname{Cond}^{\lfloor 1,0 \rfloor} \colon \mathcal{CrMod} \to \mathcal{SimpGrp}$. Cf. Remark 153.

He considers a crossed module $V = (\mathrm{M}_V\,,\, \mathrm{B}_V\,,\, \gamma_V\,,\, \mathrm{f}_V)$, which, in his notation, is written as $\operatorname{Mp} V = \mathrm{M}_V$ and $\operatorname{Gp} V = \mathrm{B}_V$.

In **[27,** (6.15),(6.13),(6.14)], he obtains e.g. $(\operatorname{Cosk} V)_2 \simeq \operatorname{Mp} V_{\,2} \rtimes \operatorname{Gp} V$.

In **[27,** (6.21)], he calculates $\operatorname{Mp} V_{\,2} \rtimes \operatorname{Gp} V \simeq \operatorname{Mp} V \rtimes (\operatorname{Mp} V \rtimes \operatorname{Gp} V)$.

So

$$(\operatorname{Cosk} V)_2 \simeq \operatorname{Mp} V \rtimes (\operatorname{Mp} V \rtimes \operatorname{Gp} V)\,.$$

Cf. Remark 154.(2).

Note that Cosk in **[27,** (6.15)] is *not* $\operatorname{Rec} * \operatorname{Cosk}^{\lfloor 1,0 \rfloor}$.

7.6 An example for a crossed module

Concerning properties of crossed modules and examples for crossed modules, we refer e.g. to **[28]**.

Here, we include an example for the occurrence of crossed modules which we have learned from W. Messing **[19]**.

It turns out that this example actually covers all crossed modules; cf. Remark 157.

Moreover, we give a stable variant of this example in Example 333.

Example 156 (Cf. **[19]**) Suppose given a group G.

Suppose given normal subgroups $N,\ M \trianglelefteq G$ with $[N, M] = 1$.

We consider the group G/N. We consider the group morphism

$$f \colon M \to G/N\,,\ m \mapsto mN\,.$$

Consider the action morphism

$$\alpha\colon G \to \mathrm{Aut}(M)\,,\ g \mapsto (m \mapsto m^g = g^- \cdot m \cdot g)\,.$$

For $m \in M$ and $n \in N$, we have

$$m(n\alpha) = n^- \cdot m \cdot n = m \cdot n^- \cdot n = m\,.$$

So $n\alpha = \mathrm{id}_n$.

This shows $N\alpha = 1$.

Therefore, we have the action morphism

$$\gamma\colon G/N \to \mathrm{Aut}(M)\,,\ gN \mapsto (m \mapsto m^{gN} := m^g = g^- \cdot m \cdot g)\,.$$

We show that $(M, G/N, \gamma, f)$ is a crossed module.

Ad (CM1). For $m \in M$ and $g \in G$, we have

$$(m^{gN})f = m^g N = (mN)^{gN} = (mf)^{gN}\,.$$

Ad (CM2). For m, $m' \in M$, we have

$$m^{m'} = m^{m'N} = m^{m'f}\,.$$

So $(M, G/N, \gamma, f)$ is a crossed module.

Remark 157

Suppose given a crossed module $V = (\mathrm{M}_V\,,\ \mathrm{B}_V\,,\ \boldsymbol{\gamma}_V\,,\ \mathrm{f}_V)$.

(1) *We have group morphisms*

$$j\colon \mathrm{M}_V \to \mathrm{B}_V \ltimes \mathrm{M}_V\,,\ m \mapsto (1, m)$$
$$k\colon \mathrm{M}_V \to \mathrm{B}_V \ltimes \mathrm{M}_V\,,\ m \mapsto (m\mathrm{f}_V\,, m^-)\,.$$

(2) *We consider the groups*

$$G := \mathrm{B}_V \ltimes \mathrm{M}_V$$
$$M := (\mathrm{M}_V)j = \{(1, m)\colon m \in \mathrm{M}_V\} \trianglelefteqslant G$$
$$N := (\mathrm{M}_V)k = \{(m\mathrm{f}_V\,, m^-)\colon m \in \mathrm{M}_V\} \trianglelefteqslant G\,.$$

We have $[N, M] = 1$.

Moreover, we have the group isomorphism

$$\ell\colon \mathrm{B}_V \to G/N\,,\ b \mapsto (b, 1)N\,.$$

(3) *By Example* 156*, we have the crossed module* $(M, G/N, \gamma, f)$ *with*

$$\gamma\colon G/N \to \mathrm{Aut}(M)\,,\ (b,m)N \mapsto \Big((1,x) \mapsto (1,x)^{(b,m)N} = (1,x)^{(b,m)}\Big)$$
$$f\colon M \to G/N\,,\ (1,m) \mapsto (1,m)N\,.$$

We have the crossed module isomorphism

$$(j\big|^M, \ell)\colon (\mathrm{M}_V\,,\, \mathrm{B}_V\,,\, \boldsymbol{\gamma}_V\,,\, \mathrm{f}_V) \to (M, G/N, \gamma, f)\,.$$

So every crossed module is, up to isomorphism, of the form as constructed in Example 156*.*

Proof. Ad (1). For m, $m' \in \mathrm{M}_V$, we have

$$(m \cdot m')j = (1, m \cdot m') = (1,m) \cdot (1,m') = mj \cdot m'j\,.$$

So j is a group morphism.

For m, $m' \in \mathrm{M}_V$, we have

$$\begin{aligned} mk \cdot m'k &= (m\mathrm{f}_V\,, m^-) \cdot \big(m'\mathrm{f}_V\,, (m')^-\big) = \big(m\mathrm{f}_V \cdot m'\mathrm{f}_V\,, (m^-)^{m'\mathrm{f}_V} \cdot (m')^-\big) \\ &\overset{(\mathrm{CM2})}{=} \big((m \cdot m')\mathrm{f}_V\,, (m^-)^{m'} \cdot (m')^-\big) = \big((m \cdot m')\mathrm{f}_V\,, (m')^- \cdot m^-\big) \\ &= \big((m \cdot m')\mathrm{f}_V\,, (m \cdot m')^-\big) = (m \cdot m')k\,. \end{aligned}$$

So k is a group morphism.

Ad (2). We have $M \trianglelefteqslant G$ by construction.

For m, $n \in \mathrm{M}_V$ and b, $c \in \mathrm{B}_V$, note that we have

$$\begin{aligned} (c,n)^{(b,m)} &= (b,m)^- \cdot (c,n) \cdot (b,m) = \big(b^-, (m^-)^{b^-}\big) \cdot (c,n) \cdot (b,m) \\ &= \big(b^- \cdot c, (m^-)^{b^- \cdot c} \cdot n\big) \cdot (b,m) = \Big(b^- \cdot c \cdot b, \big((m^-)^{b^- \cdot c} \cdot n\big)^b \cdot m\Big) \\ &= \big(c^b, (m^-)^{(c^b)} \cdot n^b \cdot m\big)\,. \end{aligned}$$

So, for x, $m \in \mathrm{M}_V$ and $b \in \mathrm{B}_V$, we have

$$\begin{aligned} (x\mathrm{f}_V\,, x^-)^{(b,m)} &= \big((x\mathrm{f}_V)^b, (m^-)^{\left((x\mathrm{f}_V)^b\right)} \cdot (x^-)^b \cdot m\big) \\ &\overset{(\mathrm{CM1})}{=} \big((x^b)\mathrm{f}_V\,, (m^-)^{(x^b)\mathrm{f}_V} \cdot (x^-)^b \cdot m\big) \\ &\overset{(\mathrm{CM2})}{=} \big((x^b)\mathrm{f}_V\,, (m^-)^{(x^b)} \cdot (x^-)^b \cdot m\big) \\ &= \big((x^b)\mathrm{f}_V\,, (x^b)^- \cdot m^- \cdot x^b \cdot (x^b)^- \cdot m\big) \\ &= \big((x^b)\mathrm{f}_V\,, (x^b)^-\big) \in N\,. \end{aligned}$$

This shows $N \trianglelefteqslant G$.

In particular, for x, $m \in \mathrm{M}_V$, we have

$$(x\mathrm{f}_V\,, x^-)^{(1,m)} = (x\mathrm{f}_V\,, x^-)\,.$$

This shows $[N, M] = 1$.

One can also show these assertions by noting that we have the invertible monoidal category $V\mathrm{Cat}$ and that we have $N = \ker \boldsymbol{t}_{V\mathrm{Cat}} \trianglelefteqslant G$ and $M = \ker \boldsymbol{s}_{V\mathrm{Cat}} \trianglelefteqslant G$; cf. Remark 165.(1). Then, by **[29,** Lem. 39, Lem. 25], we have $[N, M] = 1$.

For b, $c \in \mathrm{B}_V$, we have

$$(b \cdot c)\ell = (b \cdot c, 1)N = \big((b,1) \cdot (c,1)\big)N = (b,1)N \cdot (c,1)N = b\ell \cdot c\ell\,.$$

This shows that ℓ is a group morphism.

Suppose given $b \in \mathrm{B}_V$ with $(b,1)N = (1,1)N$.

Then there exists some $m \in \mathrm{M}_V$ such that

$$(m\mathrm{f}_V\,, m^-) = (b,1) \cdot (1,1)^- = (b,1)\,.$$

So we have $m = 1$, and therefore $b = 1$.

This shows that ℓ is injective.

Suppose given $b \in \mathrm{B}_V$ and $m \in \mathrm{M}_V$.

We consider $(b,m)N \in G/N$ and we consider $b \cdot m\mathrm{f}_V \in \mathrm{B}_V$.

We have

$$\begin{aligned}(b \cdot m\mathrm{f}_V)\ell &= (b \cdot m\mathrm{f}_V\,, 1)N = (b \cdot m\mathrm{f}_V\,, m^m \cdot m^-)N \overset{(\mathrm{CM2})}{=} (b \cdot m\mathrm{f}_V\,, m^{m\mathrm{f}_V} \cdot m^-)N \\ &= \big((b,m) \cdot (m\mathrm{f}_V\,, m^-)\big)N = (b,m)N\,.\end{aligned}$$

This shows that ℓ is surjective.

So altogether, ℓ is a group isomorphism.

Ad (3). For $m \in \mathrm{M}_V$, we have

$$\begin{aligned}m(\mathrm{f}_V \triangleleft \ell) = (m\mathrm{f}_V)\ell = (m\mathrm{f}_V\,, 1)N = \big((m\mathrm{f}_V\,, 1) \cdot (m^-\mathrm{f}_V\,, m)\big)N = (1,m)N = (1,m)f \\ = m(j\big|^M \triangleleft f)\,.\end{aligned}$$

This shows $j\big|^M \triangleleft f = \mathrm{f}_V \triangleleft \ell$.

$$\begin{array}{ccc} \mathrm{M}_V & \xrightarrow[\sim]{j\big|^M} & M \\ {\scriptstyle \mathrm{f}_V}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{B}_V & \xrightarrow[\ell]{\sim} & G/N \end{array}$$

For $m \in \mathrm{M}_V$ and $b \in \mathrm{B}_V$, we have

$$(mj\big|^M)^{b\ell} = (1,m)^{(b,1)N} = (1,m)^{(b,1)} = (b^-,1)\cdot(1,m)\cdot(b,1) = (1,m^b) = (m^b)j\big|^M .$$

So $(j\big|^M, \ell)$ is a crossed module morphism.

Since $j\big|^M$ and ℓ are group isomorphisms, we have that $(j\big|^M, \ell)$ is a crossed module isomorphism. $\square$

Chapter 8

Crossed modules and invertible monoidal categories

8.1 Invertible monoidal categories

We recall the notion of an invertible monoidal category and the notion of a monoidal functor from [**29,** §2.1, §2.2].

Definition 158 (Invertible monoidal categories)

Let $\mathcal{C}$ be a category. Suppose given a functor

$$(\otimes)\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}\,,\ \big(\big(X\xrightarrow{u}X'\big),\big(Y\xrightarrow{v}Y'\big)\big)\mapsto\big(X\otimes X'\xrightarrow{u\otimes v}Y\otimes Y'\big)$$

and suppose given an object $I\in\mathrm{Ob}(\mathcal{C})$ such that the following conditions $(1,2,3)$ hold.

(1) For $u\in\mathrm{Mor}(\mathcal{C})$, we have

$$u\otimes\mathrm{id}_I=u=\mathrm{id}_I\otimes u\,.$$

(2) For u, v, $w\in\mathrm{Mor}(\mathcal{C})$, we have

$$(u\otimes v)\otimes w=u\otimes(v\otimes w)\,.$$

(3) For $u\in\mathrm{Mor}(\mathcal{C})$, there exists some $v\in\mathrm{Mor}(\mathcal{C})$ such that

$$u\otimes v=\mathrm{id}_I=v\otimes u\,.$$

Then we call $(\mathcal{C}, I, \otimes)$ an *invertible monoidal category*. Cf. [**29,** Def. 12].
We often write $\mathcal{C} := (\mathcal{C}, I, \otimes)$.
The functor $(\otimes)$ is called its *tensor product*.
Further, for $u \in \mathrm{Mor}(\mathcal{C})$ and $X \in \mathrm{Ob}(\mathcal{C})$, we shall often write

$$u \otimes X := u \otimes \mathrm{id}_X \quad \text{and} \quad X \otimes u := \mathrm{id}_X \otimes u\,.$$

Remark 159 Suppose given an invertible monoidal category $(\mathcal{C}, I, \otimes)$.

(1) Suppose that we have an object $\tilde{I} \in \mathrm{Ob}(\mathcal{C})$ such that $u \otimes \mathrm{id}_{\tilde{I}} = \mathrm{id}_{\tilde{I}} \otimes u$ for $u \in \mathrm{Mor}(\mathcal{C})$.
Then $I = \tilde{I}$.
So the object I is uniquely determined.
We call I the *unit object* of $\mathcal{C}$.
See [**29,** Rem. 15].

(2) Suppose given $u \in \mathrm{Mor}(\mathcal{C})$. Suppose given v, $w \in \mathrm{Mor}(\mathcal{C})$ such that

$$u \otimes v = \mathrm{id}_I = v \otimes u \quad \text{and} \quad u \otimes w = \mathrm{id}_I = w \otimes u\,.$$

Then we have $v = w$.
We call $u^{\otimes -} := v$ the *tensor inverse* of u.
Moreover, for $X \in \mathrm{Ob}(\mathcal{C})$, we have the object $X^{\otimes -}$ that has $\mathrm{id}_{X^{\otimes -}} = (\mathrm{id}_X)^{\otimes -}$.
See [**29,** Rem. 20].

(3) For X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)\,.$$

See [**29,** Def. 12, Rem. 14].

(4) For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$X \otimes I = X = I \otimes X\,.$$

See [**29,** Def. 12, Rem. 14].

(5) For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we shall write

$$X^Y := Y^{\otimes -} \otimes X \otimes Y \quad \text{and} \quad [X, Y] := X^{\otimes -} \otimes Y^{\otimes -} \otimes X \otimes Y\,.$$

For u, $v \in \mathrm{Mor}(\mathcal{C})$, we shall write

$$u^v := v^{\otimes -} \otimes u \otimes v \quad \text{and} \quad [u, v] := u^{\otimes -} \otimes v^{\otimes -} \otimes u \otimes v\,.$$

Moreover, for $X \in \mathrm{Ob}(\mathcal{C})$ and $u \in \mathrm{Mor}(\mathcal{C})$, we shall write

$$u^X := X^{\otimes -} \otimes u \otimes X = \mathrm{id}_{X^{\otimes -}} \otimes u \otimes \mathrm{id}_X\ .$$

Remark 160 Suppose given an invertible monoidal category $(\mathcal{C}, I, \otimes)$.
We have that every morphism in $\mathcal{C}$ is an isomorphism; cf. [**29,** Cor. 27].

(1) For $\left(X \overset{u}{\longrightarrow} Y\right) \in \mathrm{Mor}(\mathcal{C})$, we have

$$u^{\otimes -} = Y^{\otimes -} \otimes u^- \otimes X^{\otimes -} \colon X^{\otimes -} \to Y^{\otimes -}\,;$$

cf. [**29,** Lem. 26.(1)].

(2) For $\left(X \overset{u}{\longrightarrow} Y\right) \in \mathrm{Mor}(\mathcal{C})$, we have

$$u^- = Y \otimes u^{\otimes -} \otimes X \colon Y \to X\,;$$

cf. [**29,** Lem. 26.(2)].

(3) For $(X \overset{u}{\longrightarrow} Y \overset{v}{\longrightarrow} Z)$ in $\mathcal{C}$, we have

$$u \mathbin{\blacktriangle} v = u \otimes Y^{\otimes -} \otimes v\,;$$

cf. [**29,** Rem. 28].

(4) For $\left(I \overset{u}{\longrightarrow} X\right)$, $\left(Y \overset{v}{\longrightarrow} I\right)$ in $\mathcal{C}$, we have

$$u \otimes v = v \otimes u\,;$$

cf. [**29,** Lem. 25].

Definition 161 (Monoidal functors)
Suppose given invertible monoidal categories $(\mathcal{C}, I_{\mathcal{C}}, \otimes)$ and $(\mathcal{D}, I_{\mathcal{D}}, \otimes)$.
Suppose given a functor $F \colon \mathcal{C} \to \mathcal{D}$.
We call F a *monoidal functor* if we have

$$(u \otimes v)F = uF \otimes vF$$

for u, $v \in \mathrm{Mor}(\mathcal{C})$. Cf. [**29,** Def. 31, Rem. 32.(2)].

Remark 162 Suppose given invertible monoidal categories $(\mathcal{C}, I_{\mathcal{C}}, \otimes)$ and $(\mathcal{D}, I_{\mathcal{D}}, \otimes)$. Suppose given a monoidal functor $F\colon\ \mathcal{C} \to \mathcal{D}$.

(1) We have

$$I_{\mathcal{C}}F = I_{\mathcal{D}}\,.$$

(2) For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X \otimes Y)F = XF \otimes YF\,.$$

(3) For $X \in \mathrm{Ob}(\mathcal{C})$ and $u \in \mathrm{Mor}(\mathcal{C})$, we have

$$(X^{\otimes -})F = (XF)^{\otimes -} \quad \text{and} \quad (u^{\otimes -})F = (uF)^{\otimes -}\,.$$

See [**29,** Def. 31, Rem. 32.(2), Rem. 33].

Definition 163 (The category of invertible monoidal categories)

We have the category $\mathit{InvMonCat}$, called the *category of invertible monoidal categories*, having invertible monoidal categories as objects, and having monoidal functors as morphisms. Cf. [**29,** Def. 38].

Remark 164 *Suppose given an invertible monoidal category $(\mathcal{C}, I, \otimes)$.*
Suppose given $A \in \mathrm{Ob}(\mathcal{C})$ and $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$.
We have

$$A \otimes (u \mathbin{\blacktriangle} v) = (A \otimes u) \mathbin{\blacktriangle} (A \otimes v) \quad \text{and} \quad (u \mathbin{\blacktriangle} v) \otimes A = (u \otimes A) \mathbin{\blacktriangle} (v \otimes A)\,.$$

Proof. We have

$$A \otimes (u \mathbin{\blacktriangle} v) = (\mathrm{id}_A \mathbin{\blacktriangle} \mathrm{id}_A) \otimes (u \mathbin{\blacktriangle} v) = (\mathrm{id}_A \otimes u) \mathbin{\blacktriangle} (\mathrm{id}_A \otimes v) = (A \otimes u) \mathbin{\blacktriangle} (A \otimes v)\,.$$

Moreover, we have

$$(u \mathbin{\blacktriangle} v) \otimes A = (u \mathbin{\blacktriangle} v) \otimes (\mathrm{id}_A \mathbin{\blacktriangle} \mathrm{id}_A) = (u \otimes \mathrm{id}_A) \mathbin{\blacktriangle} (v \otimes \mathrm{id}_A) = (u \otimes A) \mathbin{\blacktriangle} (v \otimes A)\,.$$

□

8.2 The equivalent categories *InvMonCat* and *CrMod*

In [**29,** Lem. 39], we have constructed the functor $\mathrm{Cat}\colon \mathcal{CrMod} \to \mathcal{InvMonCat}$.

In [**29,** Lem. 42], we have constructed the functor $\mathrm{CM}\colon \mathcal{InvMonCat} \to \mathcal{CrMod}$.

In [**29,** Prop. 43], we have considered the composites

$$\mathrm{Cat} * \mathrm{CM}\colon \mathcal{CrMod} \to \mathcal{CrMod} \text{ and } \mathrm{CM} * \mathrm{Cat}\colon \mathcal{InvMonCat} \to \mathcal{InvMonCat}\,.$$

Moreover, for a crossed module V, we have constructed a crossed module isomorphism $V\,\mathrm{Cat}\,\mathrm{CM} \xrightarrow{\sim} V$.

For an invertible monoidal category $\mathcal{C}$, we have constructed a monoidal isofunctor $\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} \xrightarrow{\sim} \mathcal{C}$.

Here, we shall show that these isomorphisms yield isotransformations $\mathrm{Cat} * \mathrm{CM} \xrightarrow{\sim} \mathrm{id}_{\mathcal{CrMod}}$ and $\mathrm{CM} * \mathrm{Cat} \xrightarrow{\sim} \mathrm{id}_{\mathcal{InvMonCat}}$.

As a consequence, the functors $\mathrm{Cat}\colon \mathcal{CrMod} \to \mathcal{InvMonCat}$ and $\mathrm{CM}\colon \mathcal{InvMonCat} \to \mathcal{CrMod}$ are equivalences of categories; cf. Proposition 171 below.

Remark 165 (The functor Cat)

(1) Let $V = (\mathrm{M}_V\,, \mathrm{B}_V\,, \partial_V\,, \mathrm{f}_V)$ be a crossed module.

We consider the invertible monoidal category $V\mathrm{Cat}$, where

$$\mathrm{Ob}(V\mathrm{Cat}) = \mathrm{B}_V$$
$$\mathrm{Mor}(V\mathrm{Cat}) = \mathrm{B}_V \ltimes_{\partial_V} \mathrm{M}_V = \mathrm{B}_V \ltimes \mathrm{M}_V\,.$$

For $b \xrightarrow{(b,m)} b \cdot m\mathrm{f}_V \xrightarrow{(b\cdot m\mathrm{f}_V\,, m')} b \cdot (mm')\mathrm{f}_V$ in $V\mathrm{Cat}$, their composite is given by

$$(b, m) \vartriangle (b \cdot m\mathrm{f}_V\,, m') = (b, m \cdot m')\,.$$

Source, target and identity are given as follows.

$$\begin{array}{lllll} s\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b \\ t\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) & \mapsto & b \cdot m\mathrm{f}_V \\ i\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \leftarrow \mathrm{B}_V\,, & (b,1) & \leftarrow\!\shortmid & b\,. \end{array}$$

Cf. [**29,** Rem. 4]. Note that s, i and t are group morphisms; cf. e.g. [**29,** Rem. 24].

The tensor product $(\cdot)\colon V\mathrm{Cat}\times V\mathrm{Cat}\to V\mathrm{Cat}$ is given by the group multiplication $(\cdot)$ in the group $\mathrm{Ob}(V\mathrm{Cat}) = \mathrm{B}_V$, respectively given by the group multiplication $(\cdot)$ in the semidirect product $\mathrm{Mor}(V\mathrm{Cat}) = \mathrm{B}_V \ltimes \mathrm{M}_V$:

$$\begin{array}{ccccccc} (\,\cdot\,)\colon & V\mathrm{Cat} & \times & V\mathrm{Cat} & \longrightarrow & V\mathrm{Cat} \\ & \left(\begin{array}{c} b \\ \downarrow (b,m) \\ b\cdot m\mathrm{f}_V \end{array} \right. & , & \left. \begin{array}{c} c \\ \downarrow (c,n) \\ c\cdot n\mathrm{f}_V \end{array} \right) & \longmapsto & \left(\begin{array}{c} b\cdot c \\ \downarrow (b,m)\cdot(c,n) \\ b\cdot m\mathrm{f}_V\cdot c\cdot n\mathrm{f}_V \end{array} \right). \end{array}$$

Cf. [**29,** Rem. 29].

(2) Suppose given crossed modules V, W. Let $\rho := (\mu,\beta)\colon V\to W$ be a crossed module morphism.

Consider the invertible monoidal categories $V\mathrm{Cat}$ and $W\mathrm{Cat}$.

We have the monoidal functor

$$\rho\,\mathrm{Cat}\colon V\mathrm{Cat}\to W\mathrm{Cat}\,,\ \big(b \xrightarrow{(b,m)} b\cdot m\mathrm{f}_V\big)\mapsto \big(b\beta \xrightarrow{(b\beta,m\mu)} (b\cdot m\mathrm{f}_V)\beta\big)\,;$$

cf. [**29,** Lem. 39.(1)].

(3) We have the functor

$$\mathrm{Cat}\colon \mathcal{C}r\mathcal{M}od\to \mathit{InvMonCat}\,,\ \big(V\xrightarrow{\rho} W\big)\mapsto \big(V\mathrm{Cat}\xrightarrow{\rho\,\mathrm{Cat}} W\mathrm{Cat}\big)\,;$$

cf. [**29,** Lem. 39.(2)].

Remark 166 (The functor CM)

(1) Let $\mathcal{C} = (\mathcal{C}, I, \otimes)$ be an invertible monoidal category.

We consider the crossed module

$$\mathcal{C}\,\mathrm{CM} = (\mathrm{M}_{\mathcal{C}\,\mathrm{CM}}\,,\,\mathrm{B}_{\mathcal{C}\,\mathrm{CM}}\,,\,\gamma_{\mathcal{C}\,\mathrm{CM}}\,,\,\mathrm{f}_{\mathcal{C}\,\mathrm{CM}}) = \big(\ker \boldsymbol{s}, \mathrm{Ob}(\mathcal{C})\,,\gamma,\boldsymbol{t}\,\big|_{\ker \boldsymbol{s}}\big)\,,$$

where

$$\gamma\colon\ \mathrm{Ob}(\mathcal{C})\to \mathrm{Aut}(\ker \boldsymbol{s})\,,\ X\mapsto (u\mapsto X^{\otimes -}\otimes u\otimes X)\,;$$

cf. [**29,** Lem. 40.(2)].

Note that we have the group morphisms

$$\begin{aligned} \boldsymbol{s}\colon\ \mathrm{Mor}(\mathcal{C})\to \mathrm{Ob}(\mathcal{C})\ ,\ \big(X\xrightarrow{u} Y\big)\mapsto u\,\boldsymbol{s} = X \\ \boldsymbol{t}\colon\ \mathrm{Mor}(\mathcal{C})\to \mathrm{Ob}(\mathcal{C})\ ,\ \big(X\xrightarrow{u} Y\big)\mapsto u\,\boldsymbol{t} = Y\,. \end{aligned}$$

In particular, we have $\ker \boldsymbol{s} = \big\{\big(I\xrightarrow{u} Y\big)\in \mathrm{Mor}(\mathcal{C})\,\big\}$.

(2) Suppose given invertible monoidal categories $\mathcal{C} = (\mathcal{C}, I_{\mathcal{C}}, \otimes)$, $\mathcal{D} = (\mathcal{D}, I_{\mathcal{D}}, \otimes)$.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor.

We consider the crossed modules $\mathcal{C}\,\mathrm{CM}$ and $\mathcal{D}\,\mathrm{CM}$.

We have the crossed module morphism $F\,\mathrm{CM} =: (\mu_F, \beta_F)\colon \mathcal{C}\,\mathrm{CM} \to \mathcal{D}\,\mathrm{CM}$, where

$$\begin{aligned} \mu_F &\colon \ker s_{\mathcal{C}} \to \ker s_{\mathcal{D}}\,,\ \left(I_{\mathcal{C}} \xrightarrow{u} Y\right) \mapsto \left(I_{\mathcal{D}} \xrightarrow{uF} YF\right) \\ \beta_F &\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})\ , X \mapsto XF\,. \end{aligned}$$

cf. [**29,** Lem. 42.(1)].

(3) We have the functor

$$\mathrm{CM}\colon \mathit{InvMonCat} \to \mathit{CrMod}\,,\ \left(\mathcal{C} \xrightarrow{F} \mathcal{D}\right) \mapsto \left(\mathcal{C}\,\mathrm{CM} \xrightarrow{F\,\mathrm{CM}} \mathcal{D}\,\mathrm{CM}\right);$$

cf. [**29,** Lem. 42.(2)].

Remark 167 (The crossed module $V\mathrm{Cat}\,\mathrm{CM}$)

(1) Let $V = (\mathrm{M}_V, \mathrm{B}_V, \gamma_V, \mathrm{f}_V)$ be a crossed module. We consider the crossed module $V\mathrm{Cat}\,\mathrm{CM}$.

We have the invertible monoidal category $V\mathrm{Cat}$, where

$$\begin{aligned} \mathrm{Ob}(V\mathrm{Cat}) &= \mathrm{B}_V \\ \mathrm{Mor}(V\mathrm{Cat}) &= \mathrm{B}_V \ltimes \mathrm{M}_V \end{aligned}$$

and where we have the group morphisms

$$\begin{array}{lll} s\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) \mapsto b \qquad\qquad \text{(source)} \\ t\colon & \mathrm{B}_V \ltimes \mathrm{M}_V \to \mathrm{B}_V\,, & (b,m) \mapsto b \cdot m\mathrm{f}_V \quad \text{(target)} \\ i\colon & \mathrm{B}_V \to \mathrm{B}_V \ltimes \mathrm{M}_V\,, & b \mapsto (1,b) \qquad\qquad \text{(identity)}\,; \end{array}$$

cf. Remark 165.

Then, by Remark 166.(1), we have the crossed module

$$V\mathrm{Cat}\,\mathrm{CM} = (\mathrm{M}_{V\mathrm{Cat}\,\mathrm{CM}}, \mathrm{B}_{V\mathrm{Cat}\,\mathrm{CM}}, \gamma_{V\mathrm{Cat}\,\mathrm{CM}}, \mathrm{f}_{V\mathrm{Cat}\,\mathrm{CM}})\,,$$

where

$$\begin{aligned} \mathrm{M}_{V\mathrm{Cat}\,\mathrm{CM}} &= \ker s = \{(1,m) \in \mathrm{B}_V \ltimes \mathrm{M}_V \colon m \in \mathrm{M}_V\} = 1 \ltimes \mathrm{M}_V \\ \mathrm{B}_{V\mathrm{Cat}\,\mathrm{CM}} &= \mathrm{Ob}(V\mathrm{Cat}) = \mathrm{B}_V \end{aligned}$$

and where

$$\gamma_{V\mathrm{Cat}\,\mathrm{CM}}\colon \mathrm{B}_V \to \mathrm{Aut}(\ker s)\,,\ b \mapsto \big((1,m) \mapsto (b,1)^- \cdot (1,m) \cdot (b,1) = (1,m^b)\big)$$
$$\mathrm{f}_{V\mathrm{Cat}\,\mathrm{CM}} = t\big|_{\ker s}\colon\ \ker s \to \mathrm{B}_V\,,\ (1,m) \mapsto m\mathrm{f}_V\,.$$

(2) Let $\rho = (\mu,\beta)\colon V \to W$ be a crossed module morphism. Then we have the group morphisms $\mu\colon \mathrm{M}_V \to \mathrm{M}_W$ and $\beta\colon \mathrm{B}_V \to \mathrm{B}_W$.

We consider the crossed modules $V\mathrm{Cat}\,\mathrm{CM}$ and $W\mathrm{Cat}\,\mathrm{CM}$; cf. (1). We consider the crossed module morphism $\rho\,\mathrm{Cat}\,\mathrm{CM}\colon V\mathrm{Cat}\,\mathrm{CM} \to W\mathrm{Cat}\,\mathrm{CM}$.

Recall that we have the monoidal functor

$$\rho\,\mathrm{Cat}\colon V\mathrm{Cat} \to W\mathrm{Cat}\,,\ \big(b \xrightarrow{(b,m)} b\cdot m\mathrm{f}_V\big) \mapsto \big(b\beta \xrightarrow{(b\beta,m\mu)} b\beta\cdot(m\mu)\mathrm{f}_W\big)\,;$$

cf. Remark 165.(2).

Then, by Remark 166.(2), we have the crossed module morphism

$$\rho\,\mathrm{Cat}\,\mathrm{CM} = (\mu_{\rho\,\mathrm{Cat}}\,,\beta_{\rho\,\mathrm{Cat}})\colon V\mathrm{Cat}\,\mathrm{CM} \to W\mathrm{Cat}\,\mathrm{CM}\,,$$

where

$$\mu_{\rho\,\mathrm{Cat}}\colon 1 \ltimes \mathrm{M}_V \to 1 \ltimes \mathrm{M}_W\,,\ (1,m) \mapsto (1,m\mu)$$
$$\beta_{\rho\,\mathrm{Cat}}\colon \mathrm{B}_V \to \mathrm{B}_W\,,\ b \mapsto b\beta\,.$$

Lemma 168 (The functor $\mathrm{Cat} * \mathrm{CM}$)

(1) *Let V be a crossed module. By* **[29,** Prop. 43.(1)*], we have the crossed module isomorphism*

$$(\pi_{\mathrm{M}_V}\,,\mathrm{id}_{\mathrm{B}_V})\colon V\mathrm{Cat}\,\mathrm{CM} \xrightarrow{\sim} V\,,$$

where $\pi_{\mathrm{M}_V}\colon 1 \ltimes \mathrm{M}_V \to \mathrm{M}_V\,,\ (1,m) \mapsto m$.

(2) *We have the isotransformation*

$$\psi := \big(V\mathrm{Cat}\,\mathrm{CM} \xrightarrow{(\pi_{\mathrm{M}_V}\,,\mathrm{id}_{\mathrm{B}_V})} V\big)_{V\in\mathrm{Ob}(\mathcal{C}r\mathcal{M}od)}\colon\ \mathrm{Cat} * \mathrm{CM} \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}r\mathcal{M}od}\ .$$

Proof. Ad (2). Suppose given $\big(V \xrightarrow{\rho} W\big)$ in $\mathcal{C}r\mathcal{M}od$.

We have $\rho = (\mu,\beta)$, where $\mu\colon \mathrm{M}_V \to \mathrm{M}_W$ and $\beta\colon \mathrm{B}_V \to \mathrm{B}_W$. Recall that we write $(\mu_{\rho\,\mathrm{Cat}}\,,\beta_{\rho\,\mathrm{Cat}}) := \rho\,\mathrm{Cat}\,\mathrm{CM}$; cf. Remark 167.(2).

We show that the following diagram is commutative.

$$\begin{array}{ccc} V\,\mathrm{Cat}\,\mathrm{CM} & \xrightarrow{(\pi_{\mathrm{M}_V},\mathrm{id}_{\mathrm{B}_V})} & V \\ \Big\downarrow{\scriptstyle \rho\,\mathrm{Cat}\,\mathrm{CM}} & & \Big\downarrow{\scriptstyle \rho} \\ W\,\mathrm{Cat}\,\mathrm{CM} & \xrightarrow[(\pi_{\mathrm{M}_W},\mathrm{id}_{\mathrm{B}_W})]{} & W \end{array}$$

For $(1,m) \in 1 \ltimes \mathrm{M}_V$, we have

$$(1,m)(\pi_{\mathrm{M}_V} \blacktriangleleft \mu) = m\mu = (1, m\mu)\pi_{\mathrm{M}_W} \overset{167.(2)}{=} (1,m)(\mu_{\rho\,\mathrm{Cat}} \blacktriangleleft \pi_{\mathrm{M}_W})\,.$$

This shows $\pi_{\mathrm{M}_V} \blacktriangleleft \mu = \mu_{\rho\,\mathrm{Cat}} \blacktriangleleft \pi_{\mathrm{M}_W}$.

For $b \in \mathrm{B}_V$, we have

$$b(\mathrm{id}_{\mathrm{B}_V} \blacktriangleleft \beta) = b\beta = (b\beta)\,\mathrm{id}_{\mathrm{B}_W} = b(\beta_{\rho\,\mathrm{Cat}} \blacktriangleleft \mathrm{id}_{\mathrm{B}_W})\,.$$

This shows $\mathrm{id}_{\mathrm{B}_V} \blacktriangleleft \beta = \beta_{\rho\,\mathrm{Cat}} \blacktriangleleft \mathrm{id}_{\mathrm{B}_W}$.

So we have

$$\begin{aligned}(\pi_{\mathrm{M}_V}, \mathrm{id}_{\mathrm{B}_V}) \blacktriangleleft \rho &= (\pi_{\mathrm{M}_V}, \mathrm{id}_{\mathrm{B}_V}) \blacktriangleleft (\mu,\beta) = (\pi_{\mathrm{M}_V} \blacktriangleleft \mu, \mathrm{id}_{\mathrm{B}_V} \blacktriangleleft \beta) = (\mu_{\rho\,\mathrm{Cat}} \blacktriangleleft \pi_{\mathrm{M}_W}, \beta_{\rho\,\mathrm{Cat}} \blacktriangleleft \mathrm{id}_{\mathrm{B}_W}) \\ &= (\mu_{\rho\,\mathrm{Cat}}, \beta_{\rho\,\mathrm{Cat}}) \blacktriangleleft (\pi_{\mathrm{M}_W}, \mathrm{id}_{\mathrm{B}_W}) = \rho\,\mathrm{Cat}\,\mathrm{CM} \blacktriangleleft (\pi_{\mathrm{M}_W}, \mathrm{id}_{\mathrm{B}_W})\,.\end{aligned}$$

So ψ is a transformation. By (1), we conclude that ψ is an isotransformation. □

Remark 169 (The invertible monoidal category $\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}$)

(1) Let $(\mathcal{C}, I, \otimes)$ be an invertible monoidal category. We consider the invertible monoidal category $\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}$.

By Remark 166.(1), we have the crossed module $\mathcal{C}\,\mathrm{CM} = (\mathrm{M}_{\mathcal{C}\,\mathrm{CM}}, \mathrm{B}_{\mathcal{C}\,\mathrm{CM}}, \gamma_{\mathcal{C}\,\mathrm{CM}}, \mathrm{f}_{\mathcal{C}\,\mathrm{CM}})$, where

$$\begin{aligned}\mathrm{M}_{\mathcal{C}\,\mathrm{CM}} &= \ker \mathcal{s} = \left\{ \left(I \xrightarrow{u} Y\right) \in \mathrm{Mor}(\mathcal{C}) \right\} \\ \mathrm{B}_{\mathcal{C}\,\mathrm{CM}} &= \mathrm{Ob}(\mathcal{C})\end{aligned}$$

and where

$$\begin{aligned}\gamma_{\mathcal{C}\,\mathrm{CM}} &\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Aut}(\ker \mathcal{s})\,,\ X \mapsto (u \mapsto X^{\otimes -} \otimes u \otimes X) \\ \mathrm{f}_{\mathcal{C}\,\mathrm{CM}} &= \mathcal{t}\,\big|_{\ker \mathcal{s}} \colon \ker \mathcal{s} \to \mathrm{Ob}(\mathcal{C})\ ,\ \left(I \xrightarrow{u} Y\right) \mapsto Y\,.\end{aligned}$$

Then, by Remark 165.(1), we have the invertible monoidal category $\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}$, where

$$\begin{aligned}\mathrm{Ob}(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}) &= \mathrm{B}_{\mathcal{C}\,\mathrm{CM}} = \mathrm{Ob}(\mathcal{C})\\ \mathrm{Mor}(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}) &= \mathrm{B}_{\mathcal{C}\,\mathrm{CM}} \ltimes \mathrm{M}_{\mathcal{C}\,\mathrm{CM}} = \mathrm{Ob}(\mathcal{C}) \ltimes \ker \mathcal{s}\ .\end{aligned}$$

Moreover, we have the group morphisms

$$\begin{aligned}\mathcal{s}\colon&\ \mathrm{Ob}(\mathcal{C}) \ltimes \ker \mathcal{s} \to \mathrm{Ob}(\mathcal{C})\ ,\ \ \big(X, (I \xrightarrow{u} Y)\big) \mapsto X && \text{(source)}\\ \mathcal{t}\colon&\ \mathrm{Ob}(\mathcal{C}) \ltimes \ker \mathcal{s} \to \mathrm{Ob}(\mathcal{C})\ ,\ \ \big(X, (I \xrightarrow{u} Y)\big) \mapsto X \otimes Y && \text{(target)}\\ \mathcal{i}\colon&\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}) \ltimes \ker \mathcal{s}\,,\ \ X \mapsto \big(X, (I \xrightarrow{\mathrm{id}_I} I)\big) && \text{(identity)}\,.\end{aligned}$$

(2) Let $(\mathcal{C}, I_{\mathcal{C}}\,, \otimes)$ and $(\mathcal{D}, I_{\mathcal{D}}\,, \otimes)$ be invertible monoidal categories. Suppose given a monoidal functor $F\colon\ \mathcal{C} \to \mathcal{D}$.

We consider the invertible monoidal categories $\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}$ and $\mathcal{D}\,\mathrm{CM}\,\mathrm{Cat}$; cf. (1). We consider the monoidal functor $F\,\mathrm{CM}\,\mathrm{Cat}\colon\ \mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} \to \mathcal{D}\,\mathrm{CM}\,\mathrm{Cat}$.

By Remark 166.(2), we have the crossed module morphism

$$F\,\mathrm{CM} = (\mu_F\,, \beta_F)\colon\ \mathcal{C}\,\mathrm{CM} \to \mathcal{D}\,\mathrm{CM}\,,$$

where

$$\begin{aligned}\mu_F\colon&\ \ker \mathcal{s}_{\mathcal{C}} \to \ker \mathcal{s}_{\mathcal{D}}\,,\ \big(I_{\mathcal{C}} \xrightarrow{u} Y\big) \mapsto \big(I_{\mathcal{D}} \xrightarrow{uF} YF\big)\\ \beta_F\colon&\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})\ ,\ X \mapsto XF\,.\end{aligned}$$

Then, by Remark 165.(2), we have the monoidal functor

$$\begin{aligned}F\,\mathrm{CM}\,\mathrm{Cat}\colon\qquad \mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} &\ \to\ \mathcal{D}\,\mathrm{CM}\,\mathrm{Cat}\\ \Big(X \xrightarrow{(X, I_{\mathcal{C}} \xrightarrow{u} Y)} X \otimes Y\Big) &\ \mapsto\ \Big(XF \xrightarrow{(XF, I_{\mathcal{D}} \xrightarrow{uF} YF)} XF \otimes YF\Big)\,.\end{aligned}$$

Lemma 170 (The functor $\mathrm{CM} * \mathrm{Cat}$)

(1) *Let $\mathcal{C}$ be an invertible monoidal category. By* [**29,** Prop. 43.(2)]*, we have the monoidal isofunctor*

$$\mathcal{C}\,\varphi\colon\ \mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} \to \mathcal{C}\,,\ \Big(X \xrightarrow{(X, I_{\mathcal{C}} \xrightarrow{u} Y)} X \otimes Y\Big) \mapsto \big(X \xrightarrow{X \otimes u} X \otimes Y\big)\,.$$

(2) *We have the isotransformation*

$$\varphi := \big(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} \xrightarrow{\mathcal{C}\,\varphi} \mathcal{C}\big)_{\mathcal{C} \in \mathrm{Ob}(\mathit{InvMonCat})}\colon\ \mathrm{CM} * \mathrm{Cat} \xrightarrow{\sim} \mathrm{id}_{\mathit{InvMonCat}}\ .$$

Proof. Ad (2). Suppose given $\left(\mathcal{C} \xrightarrow{F} \mathcal{D}\right)$ in *InvMonCat*.
We show that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} & \xrightarrow{\mathcal{C}\,\varphi} & \mathcal{C} \\ {\scriptstyle F\,\mathrm{CM}\,\mathrm{Cat}}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{D}\,\mathrm{CM}\,\mathrm{Cat} & \xrightarrow[\mathcal{D}\,\varphi]{} & \mathcal{D} \end{array}$$

Suppose given $(X,u) = \left(X, (I_{\mathcal{C}} \xrightarrow{u} Y)\right) \in \mathrm{Mor}(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}) = \mathrm{Ob}(\mathcal{C}) \ltimes \ker \mathcal{s}_{\mathcal{D}}$. We have

$$(X,u)(\mathcal{C}\,\varphi \blacktriangle F) = (X \otimes u)F = XF \otimes uF = (XF, uF)\,\mathcal{D}\,\varphi \overset{169}{=} (X,u)(F\,\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} \blacktriangle \mathcal{D}\,\varphi)\,.$$

This shows $\mathcal{C}\,\varphi \blacktriangle F = F\,\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} \blacktriangle \mathcal{D}\,\varphi$.
So φ is a transformation. By (1), we conclude that φ is an isotransformation. □

Proposition 171 (The equivalences Cat and CM)
By Lemma 168.(2)*, we have the isotransformation* $\psi\colon\ \mathrm{Cat} * \mathrm{CM} \xrightarrow{\sim} \mathrm{id}_{\mathcal{CrMod}}$.
By Lemma 170.(2)*, we have the isotransformation* $\varphi\colon\ \mathrm{CM} * \mathrm{Cat} \xrightarrow{\sim} \mathrm{id}_{\mathcal{InvMonCat}}$.
So Cat *and* CM *are equivalences. In particular, the categories* *CrMod* *and* *InvMonCat* *are equivalent.*

$$\mathcal{CrMod} \underset{\mathrm{CM}}{\overset{\mathrm{Cat}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{InvMonCat}$$

Part II

The stable case

Chapter 9

Stable simplicial groups

9.1 The prestable simplex category

In the following construction of the prestable simplex category $\Delta^{\pm}$, we follow Weilandt [**30,** Def. 1.1.4], with slightly different notations.

Definition 172 (Prestable simplex category)

For $z \in \mathbb{Z}$, we write $[z]^{\pm} := \mathbb{Z}_{\geqslant 0} \times \{z\}$.

Let z, $w \in \mathbb{Z}$. For a map $[z]^{\pm} \xrightarrow{u} [w]^{\pm}$, we consider the following properties $(1, 2)$.

(1) For i, $i' \in \mathbb{Z}_{\geqslant 0}$ with $i \leqslant i'$ and j, $j' \in \mathbb{Z}_{\geqslant 0}$ with $(i, z)u = (j, w)$ and $(i', z)u = (j', w)$, we have $j \leqslant j'$.

(2) There exists some $i_0 \in \mathbb{Z}_{\geqslant 0}$ such that $(i, z)u = (i - z + w, w)$ for $i \geqslant i_0$.

A map $[z]^{\pm} \xrightarrow{u} [w]^{\pm}$ satisfying the property (1) is called *monotone*.

A map $[z]^{\pm} \xrightarrow{u} [w]^{\pm}$ satisfying the property (2) is called *eventually parallel*.

The subcategory $\Delta^{\pm}$ of $\mathcal{S}et$ with

$$\mathrm{Ob}\big(\Delta^{\pm}\big) := \{[z]^{\pm} \colon z \in \mathbb{Z}\}$$
$$\mathrm{Mor}\big(\Delta^{\pm}\big) := \{[z]^{\pm} \xrightarrow{u} [w]^{\pm} \colon z, w \in \mathbb{Z}, u \text{ is monotone and eventually parallel}\}$$

is called the *prestable simplex category*.

Suppose given a morphism $[z]^{\pm} \xrightarrow{u} [w]^{\pm}$ in $\mathrm{Mor}(\Delta^{\pm})$.

We consider the map

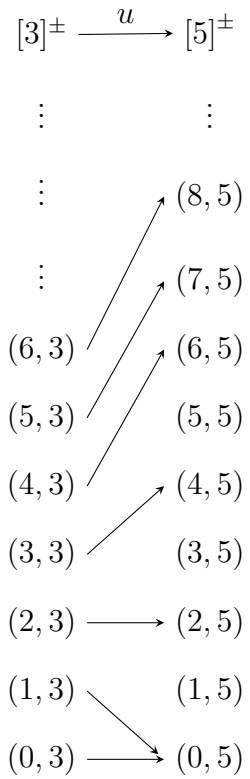
$$u^{\vee}\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}\,,\ x \mapsto s_x\,,\quad \text{where}\ \ (s_x, w) := (x, z)u\,.$$

So we have

$$(x, z)u = (xu^{\vee}, w) \quad \text{for}\ \ x \in \mathbb{Z}_{\geqslant 0}\,.$$

Often, we simply write $(xu, w) := (xu^{\vee}, w)$ by abuse of notation.

Example 173 We consider the prestable simplex category $\Delta^{\pm}$.
We consider the map $[3]^{\pm} \xrightarrow{u} [5]^{\pm}$ given as follows.



The map u is monotone.
Moreover, we have $(i, 3) = (i - 3 + 5, 5)$ for $i \geqslant 4 =: i_0$. So u is eventually parallel.
Therefore, we have $u \in \mathrm{Mor}(\Delta^{\pm})$.

Definition 174 We consider the prestable simplex category $\Delta^{\pm}$.

Let $z \in \mathbb{Z}$. For $k \in \mathbb{Z}_{\geqslant 0}$, we have the morphism

$$\delta_k^{\pm,z} \colon [z-1]^{\pm} \to [z]^{\pm},\ (i, z-1) \mapsto (i, z-1)\delta_k^{\pm,z} := \begin{cases} (i,z) & \text{if } i < k \\ (i+1,z) & \text{if } i \geqslant k\,. \end{cases}$$

Sometimes, we write $\delta_k := \delta_k^{\pm,z}$.

Let $z \in \mathbb{Z}$. For $k \in \mathbb{Z}_{\geqslant 0}$, we have the morphism

$$\sigma_k^{\pm,z} \colon [z+1]^{\pm} \to [z]^{\pm},\ (i, z+1) \mapsto (i, z+1)\sigma_k^{\pm,z} := \begin{cases} (i,z) & \text{if } i \leqslant k \\ (i-1,z) & \text{if } i > k\,. \end{cases}$$

Sometimes, we write $\sigma_k := \sigma_k^{\pm,z}$.

Lemma 175 *Suppose given a morphism $[n]^{\pm} \xrightarrow{u} [k]^{\pm}$ in $\mathrm{Mor}(\Delta^{\pm})$.*

(1) *Suppose that u is injective.*

Let $C_u := \{j \in \mathbb{Z}_{\geqslant 0} \colon (j,k) \notin \mathrm{im}\, u\} = \{j_1, \ldots, j_{k-n}\}$, where $0 \leqslant j_1 < \ldots < j_{k-n}$.

(1.1) *We have*

$$u = \prod_{\ell \in \lceil 1, k-n \rceil}^{[n]^{\pm}} \delta_{j_\ell}^{\pm, n+\ell}{}^{[k]^{\pm}} = \delta_{j_1}^{\pm,n+1} \blacktriangle \delta_{j_2}^{\pm,n+2} \blacktriangle \ldots \blacktriangle \delta_{j_{k-n}}^{\pm,k}\,.$$

(1.2) *Suppose given $0 \leqslant t_1 < t_2 < \ldots < t_{k-n}$ such that*

$$u = \prod_{\ell \in \lceil 1, k-n \rceil}^{[n]^{\pm}} \delta_{t_\ell}^{\pm, n+\ell}{}^{[k]^{\pm}} = \delta_{t_1}^{\pm,n+1} \blacktriangle \delta_{t_2}^{\pm,n+2} \blacktriangle \ldots \blacktriangle \delta_{t_{k-n}}^{\pm,k}\,.$$

Then we have $C_u = \{t_1, \ldots, t_{k-n}\}$, i.e. we have $t_\ell = j_\ell$ for $\ell \in [1, k-n]$.

(2) *Suppose that u is surjective.*

Let $D_u := \{j \in \mathbb{Z}_{\geqslant 0} \colon (j,n)u = (j+1,n)u\} = \{i_1, \ldots, i_{n-k}\}$, where $0 \leqslant i_1 < \ldots < i_{n-k}$.

(2.1) *We have*

$$u = \prod_{\ell \in \lfloor n-k, 1 \rfloor}^{[n]^{\pm}} \sigma_{i_\ell}^{\pm, k+\ell-1}{}^{[k]^{\pm}} = \sigma_{i_{n-k}}^{\pm,n-1} \blacktriangle \sigma_{i_{n-k-1}}^{\pm,n-2} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm,k}\,.$$

(2.2) *Suppose given* $0 \leqslant t_1 < t_2 < \ldots < t_{n-k}$ *such that*

$$u = \prod_{\ell \in \lfloor n-k,1 \rfloor}^{[n]^{\pm}} \sigma_{t_\ell}^{\pm,k+\ell-1} = \sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \sigma_{t_{n-k-1}}^{\pm,n-2} \blacktriangle \ldots \blacktriangle \sigma_{t_1}^{\pm,k} .$$

Then we have $D_u = \{t_1, \ldots, t_{n-k}\}$ *, i.e. we have* $t_\ell = i_\ell$ *for* $\ell \in [1, n-k]$.

Proof. Ad (1.1). Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. We show this assertion by induction on $k \in \mathbb{Z}_{\geqslant n}$.

Let $k = n$.

Then we have $u = \mathrm{id}_{[n]^{\pm}}$ and we have $C_u = \emptyset$.

Let $k \geqslant n+1$.

Now we assume that (1.1) holds for all injective morphisms $[n]^{\pm} \to [k-1]^{\pm}$.

Since $k - n \geqslant 1$, the morphism u is not surjective. So $C_u \neq \emptyset$.

Let $d := \max C_u = j_{k-n} \in \mathbb{Z}_{\geqslant 0}$.

We show that $u \blacktriangle \sigma_d^{\pm,k-1} \blacktriangle \delta_d^{\pm,k} \stackrel{!}{=} u$.

Let $x \in \mathbb{Z}_{\geqslant 0}$. Note that $(xu, k) = (x, n)u \neq (d, k)$. So $xu \neq d$.

Case $xu \leqslant d-1$. We have

$$(x,n)(u \blacktriangle \sigma_d^{\pm,k-1} \blacktriangle \delta_d^{\pm,k}) = (xu,k)(\sigma_d^{\pm,k-1} \blacktriangle \delta_d^{\pm,k}) = (xu,k-1)\delta_d^{\pm,n} = (xu,k) = (x,n)u .$$

Case $xu \geqslant d+1$. We have

$$(x,n)(u \blacktriangle \sigma_d^{\pm,k-1} \blacktriangle \delta_d^{\pm,k}) = (xu,k)(\sigma_d^{\pm,k-1} \blacktriangle \delta_d^{\pm,k}) = (xu-1,k-1)\delta_d^{\pm,k} = (xu,k) = (x,n)u .$$

This shows $u \blacktriangle \sigma_d^{\pm,k-1} \blacktriangle \delta_d^{\pm,k} = u$.

Consider the morphism $u \blacktriangle \sigma_d^{\pm,k-1} \colon [n]^{\pm} \to [k-1]^{\pm}$ in $\Delta^{\pm}$. The map $u \blacktriangle \sigma_d^{\pm,k-1}$ is injective since u is injective and since $d \in C_u$.

We show that $C_u \backslash \{d\} \stackrel{!}{=} C_{u \blacktriangle \sigma_d^{\pm,k-1}}$.

Ad $\subseteq$. Let $x \in C_u \backslash \{d\}$. Note that $x < d$ since d is maximal in C_u .

Now we *assume* that we may choose some $y \in \mathbb{Z}_{\geqslant 0}$ such that $(x, k-1) = (y, n)(u \blacktriangle \sigma_d^{\pm,k-1}) = (yu, k)\sigma_d^{\pm,k-1}$.

Case $yu \leqslant d$. We have

$$(x, k-1) = (yu, k)\sigma_d^{\pm,k-1} = (yu, k-1) .$$

So $x = yu$. Therefore $(x,k) = (yu,k) = (y,n)u$, which *contradicts* $x \in C_u$.

Case $yu \geqslant d+1$. We have

$$(x, k-1) = (yu, k)\sigma_d^{\pm,k-1} = (yu-1, k-1)\,.$$

So $x = yu - 1 \geqslant d$, which *contradicts* $x < d$.

So $x \in C_{u \blacktriangle \sigma_d^{\pm,k-1}}$.

This shows $C_u \backslash \{d\} \subseteq C_{u \blacktriangle \sigma_d^{\pm,k-1}}$.

Ad $\supseteq$. Let $x \in C_{u \blacktriangle \sigma_d^{\pm,k-1}}$. We have to show that $x \overset{!}{\in} C_u$ and that $x \overset{!}{\neq} d$.

Case $x \leqslant d$. We *assume* that we may choose some $y \in \mathbb{Z}_{\geqslant 0}$ such that $(x,k) = (y,n)u$.

We have

$$(x, k-1) = (x,k)\sigma_d^{\pm,k-1} = (y,n)(u \blacktriangle \sigma_d^{\pm,k-1})\,,$$

which *contradicts* $x \in C_{u \blacktriangle \sigma_d^{\pm,k-1}}$. So $x \in C_u$.

Case $x \geqslant d+1$. We may choose some $z \in \mathbb{Z}_{\geqslant 0}$ with $(z,n)u = (x+1,k)$ since $x+1 \geqslant d+2 > d$.

So we have

$$(x, k-1) = (x+1, k)\sigma_d^{\pm,k-1} = (z,n)(u \blacktriangle \sigma_d^{\pm,k-1})\,,$$

This is impossible since $x \in C_{u \blacktriangle \sigma_d^{\pm,k-1}}$. So this case does not occur.

This shows $x \in C_u$.

Now we *assume* that $x = d$.

We have $d+1 \notin C_u$ since $d = \max C_u$.

Therefore, we may choose some $w \in \mathbb{Z}_{\geqslant 0}$ with $(d+1,k) = (w,n)u$. So we have

$$(x, k-1) = (d, k-1) = (d+1, k)\sigma_d^{\pm,k-1} = (w,n)(u \blacktriangle \sigma_d^{\pm,k-1})\,,$$

which *contradicts* $x \in C_{u \blacktriangle \sigma_d^{\pm,k-1}}$.

This shows $C_u \backslash \{d\} \supseteq C_{u \blacktriangle \sigma_d^{\pm,k-1}}$.

So we have $C_u \backslash \{d\} = C_{u \blacktriangle \sigma_d^{\pm,k-1}}$.

By induction hypothesis we have

$$u \blacktriangle \sigma_d^{\pm,k-1} = \prod_{\ell \in \lceil 1, k-n-1 \rceil}^{[n]^\pm \; [k-1]^\pm} \delta_{j_\ell}^{\pm,n+\ell} = \delta_{j_1}^{\pm,n+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-n-1}}^{\pm,k-1}\,,$$

where $C_{u \blacktriangle \sigma_d^{\pm,k-1}} = C_u \backslash \{d\} = \{j_1, \ldots, j_{k-n-1}\}$ and $j_1 < \ldots < j_{k-n-1}$.

So we have

$$u = u \blacktriangle \sigma_d^{\pm,k-1} \blacktriangle \delta_d^{\pm,k} = (\delta_{j_1}^{\pm,n+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-n-1}}^{\pm,k-1}) \blacktriangle \delta_{j_{k-n}}^{\pm,k},$$

where $C_u = C_{u \blacktriangle \sigma_d^{\pm,k-1}} \cup \{d\} = \{j_1, \ldots, j_{k-n-1}, j_{k-n}\}$ and $j_1 < \ldots < j_{k-n}$.

Ad (1.2). Suppose given $n \in \mathbb{Z}_{\geqslant 0}$. We show this assertion by induction on $k \in \mathbb{Z}_{\geqslant n}$.

Suppose that $k = n$.

Then we have $u = \prod_{\ell \in \lceil 1,0 \rceil}^{[n]^\pm} \delta_{t_\ell}^{\pm,n+\ell} = \mathrm{id}_{[n]^\pm}$. Moreover, since $u = \mathrm{id}_{[n]^\pm}$ is surjective, we have $C_u = \emptyset$.

Suppose that $k \geqslant n+1$.

Now we assume that (1.2) holds for all injective morphisms $[n]^\pm \to [k-1]^\pm$.

We write $\tilde{u} := \prod_{\ell \in \lceil 1,k-n-1 \rceil}^{[k-1]^\pm} \delta_{t_\ell}^{\pm,n+\ell}$. Note that $u = \tilde{u} \blacktriangle \delta_{t_{k-n}}^{\pm,k}$.

By induction hypothesis, we have $C_{\tilde{u}} = \{t_1, t_2, \ldots, t_{k-n-1}\}$.

We have to show that $C_u \stackrel{!}{=} C_{\tilde{u}} \cup \{t_{k-n}\}$.

Ad $\subseteq$. Suppose given $x \in C_u$. We may assume that $x \neq t_{k-n}$.

Case $x > t_{k-n}$. Then $x - 1 \geqslant t_{k-n} > t_{k-n-1}$. So $x - 1 \notin C_{\tilde{u}}$.

Therefore, we may choose some $y \in \mathbb{Z}_{\geqslant 0}$ with $(x-1, k-1) = (y, n)\tilde{u}$.

We have

$$(y,n)u = (y,n)(\tilde{u} \blacktriangle \delta_{t_{k-n}}^{\pm,k}) = (x-1, k-1)\delta_{t_{k-n}}^{\pm,k} = (x,k).$$

This is impossible since $x \in C_u$. So this case does not occur.

Case $x < t_{k-n}$. We *assume* that $x \notin C_{\tilde{u}}$. Then we may choose some $y \in \mathbb{Z}_{\geqslant 0}$ with $(x, k-1) = (y, n)\tilde{u}$.

We have

$$(y,n)u = (y,n)(\tilde{u} \blacktriangle \delta_{t_{k-n}}^{\pm,k}) = (x, k-1)\delta_{t_{k-n}}^{\pm,k} = (x,k),$$

which *contradicts* $x \in C_u$.

This shows $xC_{\tilde{u}} \subseteq\in C_{\tilde{u}} \cup \{t_{k-n}\}$.

Ad $\supseteq$. Suppose given $x \in C_{\tilde{u}} \cup \{t_{k-n}\}$.

Case $x = t_{k-n}$. Note that we have $(t_{k-n}, k) \notin \mathrm{im}(\delta_{t_{k-n}}^{\pm,k})$.

In particular, we have $(t_{k-n}, k) \notin \mathrm{im}(\tilde{u} \blacktriangle \delta^{\pm,k}_{t_{k-n}}) = \mathrm{im}\, u$. So $t_{k-n} \in C_u$.

Case $x \in C_{\tilde{u}}$. Then we have $x < t_{k-n}$.

We *assume* that $x \notin C_u$. Then we may choose some $y \in \mathbb{Z}_{\geqslant 0}$ with $(x,k) = (y,n)u$.

We have $(y,n)\tilde{u} = (y\tilde{u}, k-1)$ and we consider $y\tilde{u} \in \mathbb{Z}_{\geqslant 0}$.

If $y\tilde{u} < t_{k-n}$, then we have

$$(x,k) = (y,n)u = (y,n)(\tilde{u} \blacktriangle \delta^{\pm,k}_{t_{k-n}}) = (y\tilde{u}, k-1)\delta^{\pm,k}_{t_{k-n}} = (y,n)\tilde{u}\,,$$

which *contradicts* $x \in C_{\tilde{u}}$.

If $y\tilde{u} \geqslant t_{k-n}$, then we have

$$(x,k) = (y,n)u = (y,n)(\tilde{u} \blacktriangle \delta^{\pm,k}_{t_{k-n}}) = (y\tilde{u}, k-1)\delta^{\pm,k}_{t_{k-n}} = (y\tilde{u}+1, k)\,.$$

This shows $x = y\tilde{u} + 1 > t_{k-n}$, which *contradicts* $x < t_{k-n}$.

Therefore, we have $x \in C_u$.

Ad (2.1). Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. We show this assertion by induction on $n \in \mathbb{Z}_{\geqslant k}$.

Suppose that $n = k$.

Then we have $u = \mathrm{id}_{[n]^\pm}$ and we have $D_u = \emptyset$.

Suppose that $n \geqslant k+1$.

Now we assume that (2.1) holds for all surjective morphisms $[n-1]^\pm \to [k]^\pm$.

Since $k < n$, the map u is not injective. So $D_u \neq \emptyset$.

Let $e := \max D_u = i_{n-k}$. Note that $(e,n)u = (e+1,n)u$.

We show that $\sigma^{\pm,n-1}_e \blacktriangle \delta^{\pm,n}_{e+1} \blacktriangle u \overset{!}{=} u$.

Let $x \in \mathbb{Z}_{\geqslant 0}$.

Case $x \leqslant e$. We have

$$(x,n)(\sigma^{\pm,n-1}_e \blacktriangle \delta^{\pm,n}_{e+1} \blacktriangle u) = (x,n-1)(\delta^{\pm,n}_{e+1} \blacktriangle u) = (x,n)u\,.$$

Case $x = e+1$. We have

$$(x,n)(\sigma^{\pm,n-1}_e \blacktriangle \delta^{\pm,n}_{e+1} \blacktriangle u) = (e,n-1)(\delta^{\pm,n}_{e+1} \blacktriangle u) = (e,n)u = (x,n)u\,.$$

Case $x \geqslant e+2$. We have

$$(x,n)(\sigma^{\pm,n-1}_e \blacktriangle \delta^{\pm,n}_{e+1} \blacktriangle u) = (x-1,n-1)(\delta^{\pm,n}_{e+1} \blacktriangle u) = (x,n)u\,.$$

This shows $\sigma_e^{\pm,n-1} \blacktriangle \delta_{e+1}^{\pm,n} \blacktriangle u = u$.

We consider the morphism $\delta_{e+1}^{\pm,n} \blacktriangle u \colon [n-1]^{\pm} \to [k]^{\pm}$ in $\Delta^{\pm}$. The map $\delta_{e+1}^{\pm,n} \blacktriangle u$ is surjective since u is surjective and since $e \in D_u$.

We show that $D_u \backslash \{e\} \stackrel{!}{=} D_{\delta_{e+1}^{\pm,n} \blacktriangle u}$.

Ad $\subseteq$. Let $x \in D_u \backslash \{e\}$. Note that $x < e$ since e is maximal in D_u.

We have

$$(x, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u) = (x, n)u = (x+1, n)u = (x+1, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u)\,.$$

This shows $x \in D_{\delta_{e+1}^{\pm,n} \blacktriangle u}$.

Ad $\supseteq$. Let $x \in D_{\delta_{e+1}^{\pm,n} \blacktriangle u}$.

We *assume* that $x \geqslant e+1$. Then

$$(x+1, n)u = (x, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u) = (x+1, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u) = (x+2, n)u\,.$$

So $e < x+1 \in D_u$ which *contradicts* the maximality of e.

We *assume* that $x = e$. Then

$$(x+1, n)u = (x, n)u = (x, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u) = (x+1, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u) = (x+2, n)u\,.$$

So $e < x+1 \in D_u$ which *contradicts* the maximality of e.

So we have $x \leqslant e-1$. In particular, $x \neq e$.

We have

$$(x, n)u = (x, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u) = (x+1, n-1)(\delta_{e+1}^{\pm,n} \blacktriangle u) = (x+1, n)u\,.$$

Therefore, we have $x \in D_u$.

This shows $D_u \backslash \{e\} = D_{\delta_{e+1}^{\pm,n} \blacktriangle u}$.

By induction hypothesis we have

$$\delta_{e+1}^{\pm,n} \blacktriangle u = \sigma_{i_{n-k-1}}^{\pm,n-2} \blacktriangle \sigma_{i_{n-k-2}}^{\pm,n-3} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm,k}\,,$$

since $D_{\delta_{e+1}^{\pm,n} \blacktriangle u} = \{i_1, \ldots, i_{n-k-1}\}$ and $i_1 < \cdots < i_{n-k-1}$.

So we have

$$u = \sigma_e^{\pm,n-1} \blacktriangle \delta_{e+1}^{\pm,n} \blacktriangle u = \sigma_{i_{n-k}}^{\pm,n-1} \blacktriangle (\sigma_{i_{n-k-1}}^{\pm,n-2} \blacktriangle \sigma_{i_{n-k-2}}^{\pm,n-3} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm,k})\,,$$

where $D_u = D_{\delta_{e+1}^{\pm,n} \blacktriangle u} \cup \{e\} = \{i_1, \ldots, i_{n-k-1}, i_{n-k}\}$ and $i_1 < \cdots < i_{n-k}$.

Ad (2.2). Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. We show this assertion by induction on $n \in \mathbb{Z}_{\geqslant k}$.

Suppose that $n = k$. Then we have $u = \prod_{\ell \in \lfloor 0,1 \rfloor}^{[n]^{\pm}} \sigma_{t_\ell}^{\pm,n+\ell-1} = \mathrm{id}_{[n]^{\pm}}$. Moreover, since $u = \mathrm{id}_{[n]^{\pm}}$ is injective, we have $D_u = \emptyset$.

Suppose that $n \geqslant k+1$.

Now we assume that (2.2) holds for all surjective morphisms $[n-1]^{\pm} \to [k]^{\pm}$.

We write $\tilde{u} := \prod_{\ell \in \lfloor n-1-k,1 \rfloor}^{[k]^{\pm}} \sigma_{t_\ell}^{\pm,k+\ell-1}$. Note that $u = \sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \tilde{u}$.

By induction hypothesis, we have $D_{\tilde{u}} = \{t_1, t_2, \ldots, t_{k-n-1}\}$.

We have to show that $D_u \overset{!}{=} D_{\tilde{u}} \cup \{t_{n-k}\}$.

Ad $\subseteq$. Suppose given $x \in D_u$. We may assume that $x \neq t_{n-k}$.

Case $x > t_{n-k}$. We have

$$\begin{aligned}(x-1,n-1)\tilde{u} &= (x,n)(\sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \tilde{u}) = (x,n)u = (x+1,n)u = (x+1,n)(\sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \tilde{u}) \\ &= (x,n-1)\tilde{u}.\end{aligned}$$

This shows $x - 1 \in D_{\tilde{u}} = \{t_1, \ldots, t_{n-k-1}\}$.

So we have

$$x \leqslant t_{n-k-1} + 1 \leqslant t_{n-k},$$

which is impossible, since $x > t_{n-k}$. So this case does not occur.

Case $x < t_{n-k}$. We have

$$(x,n-1)\tilde{u} = (x,n)(\sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \tilde{u}) = (x,n)u = (x+1,n)u = (x+1,n)(\sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \tilde{u}) = (x+1,n-1)\tilde{u}.$$

This shows $x \in D_{\tilde{u}} \subseteq D_{\tilde{u}} \cup \{t_{n-k}\}$.

Ad $\supseteq$. Suppose given $x \in D_{\tilde{u}} \cup \{t_{n-k}\}$.

Case $x = t_{n-k}$. Then we have $x \in D_u$.

Case $x \in D_{\tilde{u}}$. Note that we have $x \leqslant t_{n-k-1} \leqslant t_{n-k} - 1$.

We have

$$(x,n)u = (x,n)(\sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \tilde{u}) = (x,n-1)\tilde{u} = (x+1,n-1)\tilde{u} = (x+1,n)(\sigma_{t_{n-k}}^{\pm,n-1} \blacktriangle \tilde{u}) = (x+1,n)u.$$

This shows $x \in D_u$. □

Remark 176 *Suppose given* $\big([n]^{\pm} \xrightarrow{u} [k]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$.

Suppose that we have

$$u = v \blacktriangle w \quad \text{and} \quad u = \tilde{v} \blacktriangle \tilde{w}$$

with $[n]^{\pm} \xrightarrow{v} [\ell]^{\pm} \xrightarrow{w} [k]^{\pm}$ *and* $[n]^{\pm} \xrightarrow{\tilde{v}} [\tilde{\ell}]^{\pm} \xrightarrow{\tilde{w}} [k]^{\pm}$ *in* $\mathrm{Mor}(\Delta^{\pm})$ *, where* v*,* $\tilde{v}$ *are surjective and where* w*,* $\tilde{w}$ *are injective.*

Then we have $\ell = \tilde{\ell}$*,* $v = \tilde{v}$ *and* $w = \tilde{w}$ *.*

Proof. We show that $\ell \overset{!}{=} \tilde{\ell}$*.*

We consider $\mathrm{coim}\, u = \{(j,k) \in [k]^{\pm} \setminus \mathrm{im}\, u \colon j \in \mathbb{Z}_{\geqslant 0}\} \subseteq [k]^{\pm}$; cf. §1.1.1 item 8.

We show that $|\mathrm{coim}\, u| \overset{!}{=} k - \ell$.

Since the morphism $w \colon [\ell]^{\pm} \to [k]^{\pm}$ is eventually parallel, we may choose some $i_0 \in \mathbb{Z}_{\geqslant 0}$ with

$$(i,\ell)w = (i - \ell + k, k) \quad \text{for } i \geqslant i_0 \,.$$

Since w is monotone, we have $iw \leqslant i_0\, w = i_0 - \ell + k$ for $i \in [0, i_0]$.

Therefore, we have the injective map $w' := w\Big|_{[0,i_0]\times\{\ell\}}^{[0,i_0-\ell+k]\times\{k\}}$.

We recall that $\mathrm{coim}\, w' = \big([0, i_0 - \ell + k] \times \{k\}\big) \setminus \big([0, i_0] \times \{k\}\big)$ and that $\mathrm{coim}\, w = [k]^{\pm} \setminus \mathrm{im}\, w$; cf. §1.1.1 item 8.

Note that $\mathrm{coim}\, w' = \mathrm{coim}\, w$.

Moreover, since $v \colon [n]^{\pm} \to [\ell]^{\pm}$ is surjective, note that we have

$$\mathrm{im}\, w = \mathrm{im}(v \blacktriangle w) = \mathrm{im}\, u \,.$$

So

$$\mathrm{coim}\, u = \mathrm{coim}\, w = \mathrm{coim}\, w' \,.$$

In particular, we have

$$\begin{aligned} |\mathrm{coim}\, u| = |\mathrm{coim}\, w'| &= \big|[0, i_0 - \ell + k]\big| - \big|[0, i_0]\big| = (i_0 - \ell + k + 1) - (i_0 + 1) \\ &= k - \ell \,. \end{aligned}$$

With the same argument, we get $|\mathrm{coim}\, u| = k - \tilde{\ell}$.

This shows $\ell = |\mathrm{coim}\, u| - k = \tilde{\ell}$.

We show that $w \overset{!}{=} \tilde{w}$*.*

Since v and $\tilde{v}$ are surjective, we have

$$\operatorname{coim} w = \operatorname{coim}(v \blacktriangle w) = \operatorname{coim} u = \operatorname{coim}(\tilde{v} \blacktriangle \tilde{w}) = \operatorname{coim} \tilde{w} .$$

So, since w, $\tilde{w} \colon [\ell]^{\pm} \to [k]^{\pm}$ are injective and monotone and since $[k]^{\pm} \setminus \operatorname{im} w = [k]^{\pm} \setminus \operatorname{im} \tilde{w}$, we have $w = \tilde{w}$.

We show that $v \overset{!}{=} \tilde{v}$.

We have

$$v \blacktriangle w = u = \tilde{v} \blacktriangle \tilde{w} = \tilde{v} \blacktriangle w .$$

Since w is injective, we have $v = \tilde{v}$.

□

Corollary 177 *Suppose given a morphism* $\left([n]^{\pm} \xrightarrow{u} [k]^{\pm}\right) \in \operatorname{Mor}(\Delta^{\pm})$.
Let $t := \left|[k]^{\pm} \setminus \operatorname{im} u\right|$. *Let* $\ell := k - t$.
Let $D_u := \left\{ i \in \mathbb{Z}_{\geqslant 0} \colon (i,n)u = (i+1,n)u \right\} = \{i_1, \ldots, i_{n-\ell}\}$, *where* $0 \leqslant i_1 < i_2 < \ldots < i_{n-\ell}$.
Let $C_u := \{ j \in \mathbb{Z}_{\geqslant 0} \colon (j,k) \notin \operatorname{im} u \} = \{j_1, \ldots, j_{k-\ell}\}$, *where* $0 \leqslant j_1 < j_2 < \ldots < j_{k-\ell}$.
Note that $|C_u| - |D_u| = k - n$, *that* $t = k - \ell = |C_u|$ *and that* $n - \ell = |D_u|$. *In particular, we have* $\ell \leqslant \min\{n,k\}$.

(1) *We have*

$$u = \left(\prod_{r \in \lfloor n-\ell, 1 \rfloor}^{[n]^{\pm}} {}^{[\ell]^{\pm}} \sigma_{i_r}^{\pm, \ell+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1, k-\ell \rceil}^{[\ell]^{\pm}} {}^{[k]^{\pm}} \delta_{j_r}^{\pm, \ell+r} \right) = \sigma_{i_{n-\ell}}^{\pm,n-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm,\ell} \blacktriangle \delta_{j_1}^{\pm,\ell+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-\ell}}^{\pm,k} .$$

(2) *Suppose given* $\ell' \in \mathbb{Z}_{\leqslant \min\{n,k\}}$ *and* $0 \leqslant i'_1 < \ldots < i'_{n-\ell'}$ *and* $0 \leqslant j'_1 < \ldots < j'_{k-\ell'}$ *such that*

$$\begin{aligned} u &= \left(\prod_{r \in \lfloor n-\ell', 1 \rfloor}^{[n]^{\pm}} {}^{[\ell']^{\pm}} \sigma_{i'_r}^{\pm, \ell'+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1, k-\ell' \rceil}^{[\ell']^{\pm}} {}^{[k]^{\pm}} \delta_{j'_r}^{\pm, \ell'+r} \right) \\ &= \sigma_{i'_{n-\ell'}}^{\pm,n-1} \blacktriangle \ldots \blacktriangle \sigma_{i'_1}^{\pm,\ell'} \blacktriangle \delta_{j'_1}^{\pm,\ell'+1} \blacktriangle \ldots \blacktriangle \delta_{j'_{k-\ell'}}^{\pm,k} . \end{aligned}$$

Then we have $\ell = \ell'$, $D_u = \{i'_1, \ldots, i'_{n-\ell}\}$ *and* $C_u = \{j'_1, \ldots, j'_{k-\ell}\}$.

(3) *Suppose given* $\ell' \in \mathbb{Z}_{\leqslant \min\{n,k\}}$ *and* $0 \leqslant i'_1 < \ldots < i'_{n-\ell'}$ *and* $0 \leqslant j'_1 < \ldots < j'_{k-\ell'}$ *such that*

$$u = \left(\prod_{r \in \lfloor n-\ell', 1 \rfloor}^{[n]^\pm} {}^{[\ell']^\pm} \sigma^{\pm,\ell'+r-1}_{i'_r} \right) \blacktriangle \left(\prod_{r \in \lceil 1, k-\ell' \rceil}^{[\ell']^\pm} {}^{[k]^\pm} \delta^{\pm,\ell'+r}_{j'_r} \right)$$
$$= \sigma^{\pm,n-1}_{i'_{n-\ell'}} \blacktriangle \ldots \blacktriangle \sigma^{\pm,\ell'}_{i'_1} \blacktriangle \delta^{\pm,\ell'+1}_{j'_1} \blacktriangle \ldots \blacktriangle \delta^{\pm,k}_{j'_{k-\ell'}} .$$

Suppose given $\ell'' \in \mathbb{Z}_{\leqslant \min\{n,k\}}$ *and* $0 \leqslant i''_1 < \ldots < i''_{n-\ell''}$ *and* $0 \leqslant j''_1 < \ldots < j''_{k-\ell''}$ *such that*

$$u = \left(\prod_{r \in \lfloor n-\ell'', 1 \rfloor}^{[n]^\pm} {}^{[\ell'']^\pm} \sigma^{\pm,\ell''+r-1}_{i''_r} \right) \blacktriangle \left(\prod_{r \in \lceil 1, k-\ell'' \rceil}^{[\ell'']^\pm} {}^{[k]^\pm} \delta^{\pm,\ell''+r}_{j''_r} \right)$$
$$= \sigma^{\pm,n-1}_{i''_{n-\ell''}} \blacktriangle \ldots \blacktriangle \sigma^{\pm,\ell''}_{i''_1} \blacktriangle \delta^{\pm,\ell''+1}_{j''_1} \blacktriangle \ldots \blacktriangle \delta^{\pm,k}_{j''_{k-\ell''}} .$$

Then we have $\ell' = \ell''$, *and* $i'_r = i''_r$ *for* $r \in [1, n-\ell']$, *and* $j'_r = j''_r$ *for* $r \in [1, k-\ell']$.

Proof. Ad (1). Since u is eventually parallel, we may choose some $i_0 \in \mathbb{Z}_{\geqslant 0}$ such that

$$(i,n)u = (i-n+k, k) \quad \text{for } i \geqslant i_0 .$$

We write $\operatorname{im} u =: \{(s_r\,, k) \colon r \in \mathbb{Z}_{\geqslant 0}\} \subseteq [k]^\pm$ with $s_r < s_{r'}$ for $r < r'$.

We consider the map

$$\bar{u} \colon [n]^\pm \to [\ell]^\pm,\ (x,n) \mapsto (x,n)\bar{u} = (x\bar{u}, \ell)$$

with $(s_{x\bar{u}}\,, k) := (x,n)u = (xu, k)$.

Let $D_{\bar{u}} := \{i \in \mathbb{Z}_{\geqslant 0} \colon (i,n)\bar{u} = (i+1,n)\bar{u}\}$.

We show that $D_u \overset{!}{=} D_{\bar{u}}$.

Ad $\subseteq$. Suppose given $x \in D_u$. We have

$$(s_{x\bar{u}}, k) = (x,n)u = (x+1,n)u = (s_{(x+1)\bar{u}}, k) .$$

This shows $s_{x\bar{u}} = s_{(x+1)\bar{u}}$. So $(x,n)\bar{u} = (x+1,n)\bar{u}$.

Therefore, we have $x \in D_{\bar{u}}$.

Ad $\supseteq$. Suppose given $x \in D_{\bar{u}}$. We have

$$(x,n)u = (s_{x\bar{u}}\,, k) = (s_{(x+1)\bar{u}}\,, k) = (x+1,n)u .$$

So $x \in D_u$.

Therefore, we have $D_u = D_{\bar{u}}$.

We show that $\bar{u}$ is surjective and monotone.

For $i \in \mathbb{Z}_{\geqslant 0}$, we may choose $x \in \mathbb{Z}_{\geqslant 0}$ with $(xu, k) = (x, n)u = (s_i, k)$. Therefore, we have $xu = s_i$, and hence $(x, n)\bar{u} = (x\bar{u}, \ell) = (i, \ell)$. So $\bar{u}$ is surjective.

For $x, y \in \mathbb{Z}_{\geqslant 0}$ with $x \leqslant y$, we have $s_{x\bar{u}} = xu \leqslant yu = s_{y\bar{u}}$, hence $x\bar{u} \leqslant y\bar{u}$. So $\bar{u}$ is monotone.

We show that $\bar{u}$ is eventually parallel.

Suppose given $i \geqslant i_0$.

Note that we have $s_{i\bar{u}} = iu = s_{iu-t}$ and thus $(i, n)\bar{u} = (i\bar{u}, \ell) = (iu - t, \ell)$.

So we have

$$(i, n)\bar{u} = (iu - t, \ell) = (i - n + k - t, \ell) = (i - n + \ell, \ell) \quad \text{for } i \geqslant i_0 .$$

Hence $\bar{u}$ is eventually parallel.

Therefore, by Lemma 175.(2.1), we have

$$\bar{u} = \sigma^{\pm,n-1}_{i_{n-\ell}} \blacktriangle \ldots \blacktriangle \sigma^{\pm,\ell}_{i_1} ,$$

where we recall that $i_r \in D_{\bar{u}} = D_u$ for $r \in [1, n - \ell]$.

We consider the map

$$\dot{u} \colon [\ell]^{\pm} \to [k]^{\pm},\ (x, \ell) \mapsto (x, \ell)\dot{u} := (s_x, k) .$$

Then $\dot{u}$ is injective and monotone.

Let $C_{\dot{u}} := \{j \in \mathbb{Z}_{\geqslant 0} \colon (j, k) \notin \operatorname{im} \dot{u}\}$.

We show that $C_u \overset{!}{=} C_{\dot{u}}$.

Ad $\subseteq$. Suppose given $x \in C_u$. We *assume* that $x \notin C_{\dot{u}}$. Then we may choose some $(y, \ell) \in [\ell]^{\pm}$ with $(x, k) = (y, \ell)\dot{u}$. We have

$$(x, k) = (y, \ell)\dot{u} = (s_y , k) \in \operatorname{im} u ,$$

which *contradicts* $x \in C_u$.

This shows $x \in C_{\dot{u}}$.

Ad $\supseteq$. Suppose given $x \in C_{\dot{u}}$. We *assume* that $x \notin C_u$. Then we may choose some $(y, n) \in [n]^{\pm}$ with $(x, k) = (y, n)u$. We have

$$(x, k) = (y, n)u = (s_{y\bar{u}} , k) = (y\bar{u}, \ell)\dot{u} \in \operatorname{im} \dot{u} ,$$

which *contradicts* $x \in C_{\dot{u}}$.

This shows $x \in C_u$.

Therefore, we have $C_u = C_{\dot{u}}$.

We show that $\dot{u}$ is eventually parallel.

Let $j_0 := i_0 - n + k - t = i_0 - |D_u| \in \mathbb{Z}_{\geqslant 0}$. Suppose given $j \in \mathbb{Z}_{\geqslant 0}$ with $j \geqslant j_0$. Then we have $j = j_0 + a = i_0 - n + k - t + a$ for some $a \in \mathbb{Z}_{\geqslant 0}$.

We consider the morphism $\bar{u}$. We have

$$(i_0 + a, n)\bar{u} = (i_0 + a - n + k - t, \ell) = (j, \ell)\,.$$

So $(i_0 + a)\bar{u} = j$.

Therefore, we have

$$\begin{aligned}(j,\ell)\dot{u} = (j\dot{u}, k) = \big((i_0+a)\bar{u}\dot{u}, k\big) = (s_{(i_0+a)\bar{u}}\,, k) = \big((i_0+a)u, k\big) = (i_0 + a - n + k, k)\\ = \big((i_0 + a - n + k - t) - (k - t) + k, k\big) = (j - \ell + k, k)\,.\end{aligned}$$

So $\dot{u}$ is eventually parallel.

Therefore, by Lemma 175.(1.1), we have

$$\dot{u} = \delta^{\pm,\ell+1}_{j_1} \blacktriangle \ldots \blacktriangle \delta^{\pm,k}_{j_{k-\ell}}\,,$$

where we recall that $j_r \in C_{\dot{u}} = C_u$ for $r \in [1, k - \ell]$.

Further, for $x \in \mathbb{Z}_{\geqslant 0}$, we have

$$(x, n)(\bar{u} \blacktriangle \dot{u}) = (x\bar{u}, \ell)\dot{u} = (s_{x\bar{u}}\,, k) = (xu, k) = (x, n)u\,.$$

This shows $\bar{u} \blacktriangle \dot{u} = u$.

So altogether, we have

$$u = \bar{u} \blacktriangle \dot{u} = \sigma^{\pm,n-1}_{i_{n-\ell}} \blacktriangle \ldots \blacktriangle \sigma^{\pm,\ell}_{i_1} \blacktriangle \delta^{\pm,\ell+1}_{j_1} \blacktriangle \ldots \blacktriangle \delta^{\pm,k}_{j_{k-\ell}}\,.$$

Ad (2). Let

$$\sigma' := \prod_{r \in \lfloor n-\ell', 1 \rfloor}^{[n]^\pm \; [\ell']^\pm} \sigma^{\pm,\ell'+r-1}_{i'_r} = \sigma^{\pm,n-1}_{i'_{n-\ell'}} \blacktriangle \ldots \blacktriangle \sigma^{\pm,\ell'}_{i'_1}$$

and let

$$\delta' := \prod_{r \in \lceil 1, k-\ell' \rceil}^{[\ell']^\pm \; [k]^\pm} \delta^{\pm,\ell'+r}_{j'_r} = \delta^{\pm,\ell'+1}_{j'_1} \blacktriangle \ldots \blacktriangle \delta^{\pm,k}_{j'_{k-\ell'}}\,.$$

We have

$$u = \sigma' \blacktriangle \delta' \,,$$

where σ' is surjective and where δ' is injective.

By (1), we have

$$u = \bar{u} \blacktriangle \dot{u} \,,$$

where

$$\bar{u} = \prod_{r \in \lfloor n-\ell, 1 \rfloor}^{[n]^{\pm}} \sigma_{i_r}^{\pm, \ell+r-1} = \sigma_{i_{n-\ell}}^{\pm, n-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm, \ell}$$

is surjective, and where

$$\dot{u} = \prod_{r \in \lceil 1, k-\ell \rceil}^{[\ell]^{\pm}} \delta_{j_r}^{\pm, \ell+r} = \delta_{j_1}^{\pm, \ell+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-\ell}}^{\pm, k}$$

is injective.

Then, by Remark 176, we have $\ell = \ell'$, $\sigma' = \bar{u}$ and $\delta' = \dot{u}$. In particular, we have $D_{\bar{u}} = D_{\sigma'}$ and $C_{\dot{u}} = C_{\delta'}$.

Since $\sigma' = \bar{u}$ is surjective, we have, by the proof of (1) and by Lemma 175.(2.2),

$$D_u = D_{\sigma'} = D_{\bar{u}} = \{ i \in \mathbb{Z}_{\geqslant 0} \colon (i,n)\bar{u} = (i+1,n)\bar{u} \} = \{ i'_1 \,, i'_2 \,, \ldots, i'_{n-\ell} \} \,.$$

Since $\delta' = \dot{u}$ is injective, we have, by the proof of (1) and by Lemma 175.(1.2),

$$C_u = C_{\delta'} = C_{\dot{u}} = \{ j \in \mathbb{Z}_{\geqslant 0} \colon (j,k) \notin \operatorname{im} \dot{u} \} = \{ j'_1 \,, j'_2 \,, \ldots, j'_{k-\ell} \} \,.$$

Ad (3). By (2), we have $\ell = \ell'$, $D_u = \{ i'_1 \,, \ldots, i'_{n-\ell} \}$ and $C_u = \{ j'_1, \ldots, j'_{k-\ell} \}$.

Moreover, by (2), we have $\ell = \ell''$, $D_u = \{ i''_1 \,, \ldots, i''_{n-\ell} \}$ and $C_u = \{ j''_1, \ldots, j''_{k-\ell} \}$.

This shows $\ell' = \ell''$ and $\{ i'_1 \,, \ldots, i'_{n-\ell} \} = \{ i''_1 \,, \ldots, i''_{n-\ell} \}$ and $\{ j'_1, \ldots, j'_{k-\ell} \} = \{ j''_1, \ldots, j''_{k-\ell} \}$. □

Remark 178 We consider the prestable simplex category $\Delta^{\pm}$.

Suppose given $z \in \mathbb{Z}$ and suppose given $k \in \mathbb{Z}_{\geqslant 0}$.

For $m \in \mathbb{Z}_{\geqslant 0}$, we have

$$(m, z-1)\delta_k^{\pm, z} = (m + \chi_{m \geqslant k} \,, z) \,.$$

For $m \in \mathbb{Z}_{\geqslant 0}$, we have

$$(m, z+1)\sigma_k^{\pm, n} = (m - \chi_{m > k} \,, z) \,.$$

Cf. Definition 174.

Remark 179 *We consider the prestable simplex category $\Delta^{\pm}$. Let $z \in \mathbb{Z}$.*

(1) *We have*

$$\delta_i^{\pm,z} \blacktriangle \delta_j^{\pm,z+1} = \delta_{j-1}^{\pm,z} \blacktriangle \delta_i^{\pm,z+1}$$

for i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

(2) *We have*

$$\sigma_j^{\pm,z} \blacktriangle \sigma_i^{\pm,z-1} = \sigma_i^{\pm,z} \blacktriangle \sigma_{j-1}^{\pm,z-1}$$

for i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

(3) *For i, $j \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z} = \begin{cases} \sigma_{i-1}^{\pm,z-1} \blacktriangle \delta_j^{\pm,z} & \text{if } j < i \\ \mathrm{id}_{[z]^{\pm}} & \text{if } j = i \text{ or } j = i+1 \\ \sigma_i^{\pm,z-1} \blacktriangle \delta_{j-1}^{\pm,z} & \text{if } j > i+1\,. \end{cases}$$

Proof. Cf. also the proof of Remark 24.

We are using Remark 178.

Recall that we write $(m, k-1)\delta_i^{\pm,k} = (m\delta_i^{\pm,k}, k)$ and $(m, k+1)\sigma_j^{\pm,k} = (m\sigma_j^{\pm,k}, k)$ for i, $j \in \mathbb{Z}_{\geqslant 0}$.

Ad (1). Suppose given i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$. Suppose given $m \in \mathbb{Z}_{\geqslant 0}$.

We have

$$\begin{aligned} m(\delta_i^{\pm,z} \blacktriangle \delta_j^{\pm,z+1}) &= (m + \chi_{m \geqslant i})\delta_j^{\pm,z+1} \\ &= m + \chi_{m \geqslant i} + \chi_{(m+\chi_{m\geqslant i}) \geqslant j} \\ &\overset{j \geqslant i+1}{=} m + \chi_{m \geqslant i} + \chi_{m+1 \geqslant j} \\ &= m + \chi_{m \geqslant j-1} + \chi_{m \geqslant i} \\ &\overset{j-1 \geqslant i}{=} m + \chi_{m \geqslant j-1} + \chi_{(m+\chi_{m\geqslant j-1}) \geqslant i} \\ &= (m + \chi_{m \geqslant j-1})\delta_i^{\pm,z+1} \\ &= m(\delta_{j-1}^{\pm,z} \blacktriangle \delta_i^{\pm,z+1})\,. \end{aligned}$$

So

$$\begin{aligned} (m, z-1)(\delta_i^{\pm,z} \blacktriangle \delta_j^{\pm,z+1}) &= \big(m(\delta_i^{\pm,z} \blacktriangle \delta_j^{\pm,z+1}), z+1\big) = \big(m(\delta_{j-1}^{\pm,z} \blacktriangle \delta_i^{\pm,z+1}), z+1\big) \\ &= (m, z-1)(\delta_{j-1}^{\pm,z} \blacktriangle \delta_i^{\pm,z+1})\,. \end{aligned}$$

This shows $\delta_i^{\pm,z} \blacktriangle \delta_j^{\pm,z+1} = \delta_{j-1}^{\pm,z} \blacktriangle \delta_i^{\pm,z+1}$.

Ad (2). Suppose given i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$. Suppose given $m \in \mathbb{Z}_{\geqslant 0}$.

We have

$$\begin{aligned} m(\sigma_j^{\pm,z} \blacktriangle \sigma_i^{\pm,z-1}) &= (m - \chi_{m>j})\sigma_i^{z-1} \\ &= m - \chi_{m>j} - \chi_{(m-\chi_{m>j})>i} \\ &\overset{i<j}{=} m - \chi_{m>j} - \chi_{m>i} \\ &= m - \chi_{m>i} - \chi_{m>j} \\ &\overset{i<j}{=} m - \chi_{m>i} - \chi_{(m-\chi_{m>i})>j-1} \\ &= (m - \chi_{m>i})\sigma_{j-1}^{\pm,z-1} \\ &= m(\sigma_i^{\pm,z} \blacktriangle \sigma_{j-1}^{\pm,z-1}) . \end{aligned}$$

So

$$\begin{aligned} (m, z+1)(\sigma_j^{\pm,z} \blacktriangle \sigma_i^{\pm,z-1}) = \big(m(\sigma_j^{\pm,z} \blacktriangle \sigma_i^{\pm,z-1}), z-1\big) = \big(m(\sigma_i^{\pm,z} \blacktriangle \sigma_{j-1}^{\pm,z-1}), z-1\big) \\ = (m, z+1)(\sigma_i^{\pm,z} \blacktriangle \sigma_{j-1}^{\pm,z-1}) . \end{aligned}$$

This shows $\sigma_j^{\pm,z} \blacktriangle \sigma_i^{\pm,z-1} = \sigma_i^{\pm,z} \blacktriangle \sigma_{j-1}^{\pm,z-1}$.

Ad (3). Suppose given i, $j \in \mathbb{Z}_{\geqslant 0}$. Suppose given $m \in \mathbb{Z}_{\geqslant 0}$.

Case $j < i$.

We have

$$\begin{aligned} m(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}) &= (m + \chi_{m\geqslant j})\sigma_i^{\pm,z} \\ &= m + \chi_{m\geqslant j} - \chi_{(m+\chi_{m\geqslant j})>i} \\ &\overset{j\leq i}{=} m + \chi_{m\geqslant j} - \chi_{m+1>i} \\ &= m - \chi_{m>i-1} + \chi_{m\geqslant j} \\ &\overset{j\leq i}{=} m - \chi_{m>i-1} + \chi_{(m-\chi_{m>i-1})\geqslant j} \\ &= (m - \chi_{m>i-1})\delta_j^{\pm,z} \\ &= m(\sigma_{i-1}^{\pm,z-1} \blacktriangle \delta_j^{\pm,z}) . \end{aligned}$$

So

$$(m, z)(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}) = \big(m(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}), z\big) = \big(m(\sigma_{i-1}^{\pm,z-1} \blacktriangle \sigma_j^{\pm,z}), z\big) = (m, z)(\sigma_{i-1}^{\pm,z-1} \blacktriangle \delta_j^{\pm,z}) .$$

This shows $\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z} = \sigma_{i-1}^{\pm,z-1} \blacktriangle \delta_j^{\pm,z}$.

Case $j = i$ or $j = i + 1$.

We have

$$\begin{array}{rcl} m(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}) & = & (m+\chi_{m\geqslant j})\sigma_i^z \\ & = & m+\chi_{m\geqslant j}-\chi_{(m+\chi_{m\geqslant j})>i} \\ & \underset{j=i \text{ or } j=i+1}{=} & m+\chi_{m\geqslant j}-\chi_{m\geqslant j} \\ & = & m\,. \end{array}$$

So

$$(m,z)(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}) = \big(m(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}), z\big) = (m,z)\,.$$

This shows $\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z} = \mathrm{id}_{[z]^\pm}$.

Case $j > i+1$.

We have

$$\begin{array}{rcl} m(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}) & = & (m+\chi_{m\geqslant j})\sigma_i^{\pm,z} \\ & = & (m+\chi_{m\geqslant j})-\chi_{(m+\chi_{m\geqslant j})>i} \\ & \underset{i+1<j}{=} & m+\chi_{m\geqslant j}-\chi_{m>i} \\ & = & m-\chi_{m>i}+\chi_{m\geqslant j} \\ & \underset{i+1<j}{=} & m-\chi_{m>i}+\chi_{(m-\chi_{m>i}+1)\geqslant j} \\ & = & m-\chi_{m>i}+\chi_{(m-\chi_{m>i})\geqslant j-1} \\ & = & (m-\chi_{m>i})\delta_{j-1}^{\pm,z} \\ & = & m(\sigma_i^{\pm,z-1} \blacktriangle \delta_{j-1}^{\pm,z})\,. \end{array}$$

So

$$\begin{aligned} (m,z)(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}) &= \big(m(\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z}), z\big) = \big(m(\sigma_i^{\pm,z-1} \blacktriangle \delta_{j-1}^{\pm,z}), z\big) \\ &= (m,z)(\sigma_i^{\pm,z-1} \blacktriangle \delta_{j-1}^{\pm,z})\,. \end{aligned}$$

This shows $\delta_j^{\pm,z+1} \blacktriangle \sigma_i^{\pm,z} = \sigma_i^{\pm,z-1} \blacktriangle \delta_{j-1}^{\pm,z}$. □

9.2 Composites of morphisms in $\Delta^\pm$

We consider the category $\Delta^\pm$; cf. Definition 172.

Definition 180 Let $n \in \mathbb{Z}$. Suppose given a finite set $I \subseteq \mathbb{Z}_{\geqslant 0}$.

Let $k := |I| \in \mathbb{Z}_{\geqslant 0}$ and let $I =: \{i_1, i_2, \ldots, i_k\}$, where $0 \leqslant i_1 < i_2 < \ldots < i_k$.

(1) We write

$$\delta_I^{\pm,n} := \prod_{r\in\lceil 1,k\rceil}^{[n-k]^\pm \to [n]^\pm} \delta_{i_r}^{\pm,n-k+r} = \delta_{i_1}^{\pm,n-k+1} \blacktriangle \delta_{i_2}^{\pm,n-k+2} \blacktriangle \ldots \blacktriangle \delta_{i_k}^{\pm,n} \colon [n-k]^\pm \to [n]^\pm\,.$$

Note that $\delta_I^{\pm,n}$ is injective; cf. Definition 174. Note that $\delta_\emptyset^{\pm,n} = \mathrm{id}_{[n]^\pm}$.

(2) We write

$$\sigma_I^{\pm,n} := \prod_{r\in\lfloor k,1\rfloor}^{[n+k]^\pm} \sigma_{i_r}^{\pm,n+r-1} = \sigma_{i_k}^{\pm,n+k-1} \blacktriangle \sigma_{i_{k-1}}^{\pm,n+k-2} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm,n} \colon [n+k]^\pm \to [n]^\pm .$$

Note that $\sigma_I^{\pm,n}$ is surjective; cf. Definition 174. Note that $\sigma_\emptyset^{\pm,n} = \mathrm{id}_{[n]^\pm}$.

Cf. also Definition 187 below.

Remark 181 Suppose given $n \in \mathbb{Z}$. Suppose given a finite set $I \subseteq \mathbb{Z}_{\geqslant 0}$. Let $k := |I|$.

(1) We consider the morphism $\delta_I^{\pm,n} \colon [n-k]^\pm \to [n]^\pm$; cf. Definition 180.(1).
We consider the set $C_{\delta_I^{\pm,n}} := \{j \in \mathbb{Z}_{\geqslant 0} \colon (j,n) \notin \operatorname{im} \delta_I^{\pm,n}\}$; cf. Lemma 175.(1).
Then we have $I = C_{\delta_I^{\pm,n}}$; cf. Lemma 175.(1.2).

(2) We consider the morphism $\sigma_I^{\pm,n} \colon [n+k]^\pm \to [n]^\pm$; cf. Definition 180.(2).
We consider the set $D_{\sigma_I^{\pm,n}} := \{i \in \mathbb{Z}_{\geqslant 0} \colon (i,n+k)\sigma_I^{\pm,n} = (i+1,n+k)\sigma_I^{\pm,n}\}$; cf. Lemma 175.(2).
Then we have $I = D_{\sigma_I^{\pm,n}}$; cf. Lemma 175.(2.2).

9.3 Prestable simplicial groups

Definition 182 (Prestable simplicial group)

We consider the category of groups $\mathcal{Grp}$ and the opposite category $(\Delta^\pm)^{\mathrm{op}}$ of $\Delta^\pm$; cf. §1.1.3 item 12.

We write

$$\mathcal{PreStSimpGrp} := [(\Delta^\pm)^{\mathrm{op}}, \mathcal{Grp}]$$

for the category of functors from $(\Delta^\pm)^{\mathrm{op}}$ to $\mathcal{Grp}$ and transformations between those functors.

We call $\mathcal{PreStSimpGrp}$ the *category of prestable simplicial groups*.

A functor $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$ is called a *prestable simplicial group*.

For $z \in \mathbb{Z}$, we write $G_z := ([z]^\pm)G \in \mathrm{Ob}(\mathcal{Grp})$.

For $\big([z]^{\pm} \xrightarrow{u} [w]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$, we write $G_u := (u^{\mathrm{op}})G\colon G_w \to G_z$, which is a group morphism.

A transformation $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}\big(\mathcal{PreStSimpGrp}\big)$ between prestable simplicial groups G, $H \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$ is called a *prestable simplicial group morphism.*

For $z \in \mathbb{Z}$, we write $\varphi_z := ([z]^{\pm})\varphi\colon G_z \to H_z$, which is a group morphism.

Note that the condition on a tuple $\big(G_z \xrightarrow{\varphi_z} H_z\big)_{z\in\mathbb{Z}}$ to be a prestable simplicial group morphism writes: For every $\big([z]^{\pm} \xrightarrow{u} [w]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$, we have $\varphi_z \blacktriangle H_u = G_u \blacktriangle \varphi_w$.

$$\begin{array}{ccc} G_z & \xrightarrow{\varphi_z} & H_z \\ {\scriptstyle G_u}\downarrow & & \downarrow{\scriptstyle H_u} \\ G_w & \xrightarrow[\varphi_w]{} & H_w \end{array}$$

For $G \xrightarrow{\varphi} G' \xrightarrow{\varphi'} G''$ in $\mathcal{PreStSimpGrp}$, their composite is given by their vertical composite of transformations

$$\varphi \blacktriangle \varphi' = \big(G_z \xrightarrow{([z]^{\pm})(\varphi \blacktriangle \varphi')} G''_z\big)_{z\in\mathbb{Z}} := \big(G_z \xrightarrow{\varphi_z \blacktriangle \varphi'_z} G''_z\big)_{z\in\mathbb{Z}}.$$

In particular, $(\varphi \blacktriangle \varphi')_z = \varphi_z \blacktriangle \varphi'_z$ for $z \in \mathbb{Z}$.

For $G \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$, its identity is given by

$$\mathrm{id}_G = \big(G_z \xrightarrow{\mathrm{id}_{G_z}} G_z\big)_{z\in\mathbb{Z}}.$$

In particular, $(\mathrm{id}_G)_z = \mathrm{id}_{G_z}$ for $z \in \mathbb{Z}$.

Definition 183 (Face maps and degeneracy maps)

Consider the prestable simplex category $\Delta^{\pm}$.

Suppose given $z \in \mathbb{Z}$ and suppose given $k \in \mathbb{Z}_{\geqslant 0}$.

We consider the morphisms $\delta_k^{\pm,z}\colon [z-1]^{\pm} \to [z]^{\pm}$ and $\sigma_k^{\pm,z}\colon [z+1]^{\pm} \to [z]^{\pm}$; cf. Definition 174.

We consider the morphisms $(\delta_k^{\pm,z})^{\mathrm{op}}\colon [z]^{\pm} \to [z-1]^{\pm}$ and $(\sigma_k^{\pm,z})^{\mathrm{op}}\colon [z]^{\pm} \to [z+1]^{\pm}$ in $(\Delta^{\pm})^{\mathrm{op}}$.

Suppose given a simplicial group $G\colon (\Delta^{\pm})^{\mathrm{op}} \to \mathcal{Grp}$.

We call

$$\mathrm{d}_k^{G,z} := G_{\delta_k^{\pm,z}} = \big((\delta_k^{\pm,z})^{\mathrm{op}}\big)G\colon G_z \to G_{z-1}$$

a *face map*, and we call

$$\mathrm{s}_k^{G,\,z} := G_{\sigma_k^{\pm,z}} = \big((\sigma_k^{\pm,z})^{\mathrm{op}}\big)G \colon G_z \to G_{z+1}$$

a *degeneracy map*, which are group morphisms.
Sometimes, we abbreviate $\mathrm{d}_k := \mathrm{d}_k^{G,\,z}$ and $\mathrm{s}_k := \mathrm{s}_k^{G,\,z}$.

Remark 184 Suppose given prestable simplicial groups G, $H \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$. Suppose given a simplicial group morphism $\varphi \colon G \to H$. Let $z \in \mathbb{Z}$.
Then we have $(1, 2)$.

(1) For $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\varphi_z \vartriangle \mathrm{s}_i^{H,\,z} = \mathrm{s}_i^{G,\,z} \vartriangle \varphi_{z+1} \,:\, G_z \to H_{z+1}\,.$$

(2) For $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{d}_j^{G,\,z} \vartriangle \varphi_{z-1} = \varphi_z \vartriangle \mathrm{d}_j^{H,\,z} \,:\, G_z \to H_{z-1}\,.$$

Remark 185 Suppose given a prestable simplicial group $G \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$. Let $z \in \mathbb{Z}$. Using Remark 179, we have the following equations $(1, 2, 3)$.

(1) For i, $j \in \mathbb{Z}_{\geqslant 0}$, with $i < j$, we have

$$\mathrm{d}_j^{G,\,z+1} \vartriangle \mathrm{d}_i^{G,\,z} = \mathrm{d}_i^{G,\,z+1} \vartriangle \mathrm{d}_{j-1}^{G,\,z}\,.$$

(2) For i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$, we have

$$\mathrm{s}_i^{G,\,z-1} \vartriangle \mathrm{s}_j^{G,\,z} = \mathrm{s}_{j-1}^{G,\,z-1} \vartriangle \mathrm{s}_i^{G,\,z}\,.$$

(3) For i, $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{s}_i^{G,\,z} \vartriangle \mathrm{d}_j^{G,\,z+1} = \begin{cases} \mathrm{d}_j^{G,\,z} \vartriangle \mathrm{s}_{i-1}^{G,\,z-1} & \text{if } j < i \\ \mathrm{id}_{G_z} & \text{if } j \in \{i, i+1\} \\ \mathrm{d}_{j-1}^{G,\,z} \vartriangle \mathrm{s}_i^{G,\,z-1} & \text{if } j > i+1\,. \end{cases}$$

Remark 186 *Suppose given a prestable simplicial group $G \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$. Suppose given a morphism $\big([n]^{\pm} \xrightarrow{u} [k]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$.*

Let $t := \big|[k]^{\pm} \setminus \operatorname{im} u\big|$. *Let* $\ell := k - t$.
Let $D_u := \big\{i \in \mathbb{Z}_{\geqslant 0} \colon (i,n)u = (i+1,n)u\big\} = \{i_1, \ldots, i_{n-\ell}\}$, *where* $0 \leqslant i_1 < i_2 < \ldots < i_{n-\ell}$.
Let $C_u := \{j \in \mathbb{Z}_{\geqslant 0} \colon (j,k) \notin \operatorname{im} u\} = \{j_1, \ldots, j_{k-\ell}\}$, *where* $0 \leqslant j_1 < j_2 < \ldots < j_{k-\ell}$.
Note that $\ell \leqslant \min\{n,k\}$. *We have*

$$\begin{aligned} G_u = (u^{\mathrm{op}})G &= \left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k \; G_\ell} \mathrm{d}_{j_r}^{G,\,\ell+r} \right) \blacktriangle \left(\prod_{r \in \lceil 1, n-\ell \rceil}^{G_\ell \; G_n} \mathrm{s}_{i_r}^{G,\,\ell+r-1} \right) \\ &= \mathrm{d}_{j_{k-\ell}}^{G,k} \blacktriangle \ldots \blacktriangle \mathrm{d}_{j_1}^{G,\ell+1} \blacktriangle \mathrm{s}_{i_1}^{G,\ell} \blacktriangle \ldots \blacktriangle \mathrm{s}_{i_{n-\ell}}^{G,n-1} . \end{aligned}$$

Proof. By Corollary 177.(1), we have

$$u = \left(\prod_{r \in \lfloor n-\ell, 1 \rfloor}^{[n]^{\pm} \; [\ell]^{\pm}} \sigma_{i_r}^{\pm,\ell+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1, k-\ell \rceil}^{[\ell]^{\pm} \; [k]^{\pm}} \delta_{j_r}^{\pm,\ell+r} \right).$$

So we have

$$\begin{aligned} (u^{\mathrm{op}})G &= \left(\left(\left(\prod_{r \in \lfloor n-\ell, 1 \rfloor}^{[n]^{\pm} \; [\ell]^{\pm}} \sigma_{i_r}^{\pm,\ell+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1, k-\ell \rceil}^{[\ell]^{\pm} \; [k]^{\pm}} \delta_{j_r}^{\pm,\ell+r} \right) \right)^{\mathrm{op}} \right) G \\ &= \left(\left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{[k]^{\pm} \; [\ell]^{\pm}} (\delta_{j_r}^{\pm,\ell+r})^{\mathrm{op}} \right) \blacktriangle \left(\prod_{r \in \lceil 1, n-\ell \rceil}^{[\ell]^{\pm} \; [n]^{\pm}} (\sigma_{i_r}^{\pm,\ell+r-1})^{\mathrm{op}} \right) \right) G \\ &= \left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k \; G_\ell} \big((\delta_{j_r}^{\pm,\ell+r})^{\mathrm{op}} \big) G \right) \blacktriangle \left(\prod_{r \in \lceil 1, n-\ell \rceil}^{G_\ell \; G_n} \big((\sigma_{i_r}^{\pm,\ell+r-1})^{\mathrm{op}} \big) G \right) \\ &= \left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k \; G_\ell} \mathrm{d}_{j_r}^{G,\ell+r} \right) \blacktriangle \left(\prod_{r \in \lceil 1, n-\ell \rceil}^{G_\ell \; G_n} \mathrm{s}_{i_r}^{G,\ell+r-1} \right). \end{aligned}$$

□

Definition 187 Suppose given a prestable simplicial group G; cf. Definition 182.
Let $n \in \mathbb{Z}$. Suppose given a finite set $I \subseteq \mathbb{Z}_{\geqslant 0}$.
Let $k := |I| \in \mathbb{Z}_{\geqslant 0}$ and let $I =: \{i_1, i_2, \ldots, i_k\}$, where $0 \leqslant i_1 < i_2 < \ldots < i_k$.

(1) We consider the map

$$\delta_I^{\pm,n} = \prod_{r \in \lceil 1,k \rceil}^{[n-k]^{\pm} \; [n]^{\pm}} \delta_{i_r}^{\pm,n-k+r} \; : [n-k]^{\pm} \to [n]^{\pm};$$

cf. Definition 180.(1).

We write

$$\mathrm{d}_I^{G,n} := G_{\delta_I^{\pm,n}} = \big((\delta_I^{\pm,n})^{\mathrm{op}}\big)G : G_n \to G_{n-k}\,.$$

So we have

$$\begin{aligned}\mathrm{d}_I^{G,n} &= \Big(\Big(\prod_{r\in\lceil 1,k\rceil}^{[n-k]^\pm}\prod_{}^{[n]^\pm}\delta_{i_r}^{\pm,n-k+r}\Big)^{\mathrm{op}}\Big)G = \Big(\prod_{r\in\lfloor k,1\rfloor}^{[n]^\pm}\prod_{}^{[n-k]^\pm}(\delta_{i_r}^{\pm,n-k+r})^{\mathrm{op}}\Big)G\\ &= \prod_{r\in\lfloor k,1\rfloor}^{G_n}\prod_{}^{G_{n-k}}\big((\delta_{i_r}^{\pm,n-k+r})^{\mathrm{op}}\big)G = \prod_{r\in\lfloor k,1\rfloor}^{G_n}\prod_{}^{G_{n-k}}\mathrm{d}_{i_r}^{G,n-k+r}\\ &= \mathrm{d}_{i_k}^{G,n}\blacktriangle\mathrm{d}_{i_{k-1}}^{G,n-1}\blacktriangle\ldots\blacktriangle\mathrm{d}_{i_1}^{G,n-k+1}\,.\end{aligned}$$

We often write $\mathrm{d}_J := \mathrm{d}_J^{G,n}$. Note that $\mathrm{d}_\emptyset^{G,n} = \mathrm{id}_{G_n}$.

(2) We consider the map

$$\sigma_I^{\pm,n} = \prod_{r\in\lfloor k,1\rfloor}^{[n+k]^\pm}\prod_{}^{[n]^\pm}\sigma_{i_r}^{\pm,n+r-1} \;: [n+k]^\pm \to [n]^\pm\,;$$

cf. Definition 180.(2).

We write

$$\mathrm{s}_I^{G,n} := G_{\sigma_I^{\pm,n}} = \big((\sigma_I^{\pm,n})^{\mathrm{op}}\big)G : G_{n+k} \to G_n\,.$$

So we have

$$\begin{aligned}\mathrm{s}_I^{G,n} &= \Big(\Big(\prod_{r\in\lfloor k,1\rfloor}^{[n+k]^\pm}\prod_{}^{[n]^\pm}\sigma_{i_r}^{\pm,n+r-1}\Big)^{\mathrm{op}}\Big)G = \Big(\prod_{r\in\lceil 1,k\rceil}^{[n]^\pm}\prod_{}^{[n+k]^\pm}(\sigma_{i_r}^{\pm,n+r-1})^{\mathrm{op}}\Big)G\\ &= \prod_{r\in\lceil 1,k\rceil}^{G_n}\prod_{}^{G_{n+k}}\big((\sigma_{i_r}^{\pm,n+r-1})^{\mathrm{op}}\big)G = \prod_{r\in\lceil 1,k\rceil}^{G_n}\prod_{}^{G_{n+k}}\mathrm{s}_{i_r}^{G,n+r-1}\\ &= \mathrm{s}_{i_1}^{G,n}\blacktriangle\mathrm{s}_{i_2}^{G,n+1}\blacktriangle\ldots\blacktriangle\mathrm{s}_{i_r}^{G,n+k-1}\,.\end{aligned}$$

We often write $\mathrm{s}_I := \mathrm{s}_I^{G,n}$. Note that $\mathrm{s}_\emptyset^{G,n} = \mathrm{id}_{G_n}$.

9.4 Prestable simplicial subgroups

Definition 188 (Prestable simplicial subgroups)
Suppose given prestable simplicial groups G, H; cf. Definition 182.
We say that H is a *prestable simplicial subgroup* of G if the conditions $(1, 2)$ hold.

(1) For $z \in \mathbb{Z}$, we have $H_z \leqslant G_z$.

(2) Suppose given a morphism $\big([z] \xrightarrow{u} [w]\big) \in \mathrm{Mor}(\Delta^{\pm})$.
We consider the group morphisms $G_u \colon G_w \to G_z$ and $H_u \colon H_w \to H_z$.
We have
$$H_u = G_u \big|_{H_w}^{H_z} .$$

We write $H \leqslant G$ to indicate that H is a prestable simplicial subgroup of G.

Lemma 189 *Suppose given a prestable simplicial group $G \colon (\Delta^{\pm})^{\mathrm{op}} \to \mathcal{G}rp$.*
For $z \in \mathbb{Z}$, suppose given subgroups $H_z \leqslant G_z$.

(1) *For $\big([z]^{\pm} \xrightarrow{u} [w]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$, i.e. $\big([w]^{\pm} \xrightarrow{u^{\mathrm{op}}} [z]^{\pm}\big) \in \mathrm{Mor}((\Delta^{\pm})^{\mathrm{op}})$, suppose that we have $(H_w)G_u \leqslant G_z$.*
Then we have the prestable simplicial group
$$H \colon \Delta^{\mathrm{op}} \to \mathcal{G}rp\,,\ \big([w]^{\pm} \xrightarrow{u^{\mathrm{op}}} [z]^{\pm}\big) \mapsto \big([w]^{\pm}H \xrightarrow{(u^{\mathrm{op}})H} [z]^{\pm}H\big) := \big(H_w \xrightarrow{G_u \big|_{H_w}^{H_z}} H_z\big) .$$
Moreover, we have $H \leqslant G$, i.e. H is a prestable simplicial subgroup of G.

(2) *For $z \in \mathbb{Z}$ and $i \in \mathbb{Z}_{\geqslant 0}$, suppose that we have*
$$(H_z)\mathrm{d}_i^{G,z} \leqslant H_{z-1} \quad \text{and} \quad (H_z)\mathrm{s}_i^{G,z} \leqslant H_{z+1} .$$
Then we have the prestable simplicial group
$$\begin{array}{rccl} H \colon & (\Delta^{\pm})^{\mathrm{op}} & \to & \mathcal{G}rp \\ & \big([w]^{\pm} \xrightarrow{u^{\mathrm{op}}} [z]^{\pm}\big) & \mapsto & \big(([w]^{\pm})H \xrightarrow{(u^{\mathrm{op}})H} ([z]^{\pm})H\big) := \big(H_w \xrightarrow{G_u \big|_{H_w}^{H_z}} H_z\big) . \end{array}$$
Moreover, we have $H \leqslant G$, i.e. H is a prestable simplicial subgroup of G.

Proof. Ad (1). Suppose given $[t]^{\pm} \xrightarrow{v^{\mathrm{op}}} [w]^{\pm} \xrightarrow{u^{\mathrm{op}}} [z]^{\pm}$ in $(\Delta^{\pm})^{\mathrm{op}}$.

We have

$$(\mathrm{id}_{[z]^{\pm}})^{\mathrm{op}} H = \big((\mathrm{id}_{[z]^{\pm}})^{\mathrm{op}}\, G\big)\big|_{H_z}^{H_z} = \mathrm{id}_{G_z}\big|_{H_z}^{H_z} = \mathrm{id}_{H_z} = \mathrm{id}_{([z]^{\pm})H}\ .$$

We have

$$(v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}})H = \big((v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}})G\big)\big|_{H_t}^{H_z} = (G_v \blacktriangle G_u)\big|_{H_t}^{H_z} = G_v\big|_{H_t}^{H_w} \blacktriangle G_u\big|_{H_w}^{H_z} = (v^{\mathrm{op}})H \blacktriangle (u^{\mathrm{op}})H\,.$$

So H is a functor, i.e. H is a prestable simplicial group.

For $\big([z]^{\pm} \xrightarrow{u} [w]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$, we have, by construction,

$$H_u = (u^{\mathrm{op}})H = G_u\big|_{H_w}^{H_z}\,.$$

So $H \leqslant G$ is a prestable simplicial subgroup.

Ad (2). Suppose given a morphism $\big([z]^{\pm} \xrightarrow{u} [w]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$. By (1), it suffices to show that $(H_w)G_u \overset{!}{\leqslant} H_z$.

By Remark 186, we have

$$G_u = \left(\prod_{r\in\lfloor w-\ell,1\rfloor}^{G_w} \mathrm{d}_{i_r}^{G,\,\ell+r}\right) \blacktriangle \left(\prod_{q\in\lceil 1,z-\ell\rceil}^{G_\ell} \mathrm{s}_{j_q}^{G,\,\ell+q-1}\right),$$

for some $\ell \leqslant \min\{z,w\}$ and some $0 \leqslant j_1 < \ldots < j_{z-\ell}$ and some $0 \leqslant i_1 < \ldots < i_{w-\ell}$.

Using the assumptions iteratively, we obtain

$$(H_w)\left(\prod_{r\in\lfloor w-\ell,1\rfloor}^{G_w} \mathrm{d}_{i_r}^{G,\,\ell+r}\right) \leqslant H_\ell$$

and

$$(H_\ell)\left(\prod_{q\in\lceil 1,z-\ell\rceil}^{G_\ell} \mathrm{s}_{j_q}^{G,\,\ell+q-1}\right) \leqslant H_z\,.$$

So we have

$$(H_w)G_u = (H_w)\left(\left(\prod_{r\in\lfloor w-\ell,1\rfloor}^{[w]^{\pm}} \mathrm{d}_{i_r}^{G,\,\ell+r}\right) \blacktriangle \left(\prod_{q\in\lceil 1,z-\ell\rceil}^{[\ell]^{\pm}} \mathrm{s}_{j_q}^{G,\,\ell+q-1}\right)\right) \leqslant H_z\,.$$

Therefore, we have the group morphism $G_u\big|_{H_w}^{H_z} : H_w \to H_z$.

Moreover, by (1), H is a prestable simplicial subgroup of G. □

Remark 190 *Suppose given prestable simplicial groups G, H; cf. Definition* 182.
Suppose given prestable simplicial subgroups $\tilde{G} \leqslant G$ and $\tilde{H} \leqslant H$; cf. Definition 188.
Suppose given a prestable simplicial group morphism $\varphi\colon G \to H$.
Suppose that we have $\tilde{G}_k \, \varphi_k \leqslant \tilde{H}_k$ for $k \in \mathbb{Z}$.
Then we have the prestable simplicial group morphism

$$\tilde{\varphi} := \varphi\big|_{\tilde{G}}^{\tilde{H}} : \tilde{G} \to \tilde{H} \,,$$

where

$$\tilde{\varphi}_k = \big(\varphi\big|_{\tilde{G}}^{\tilde{H}}\big)_k := \varphi_k\big|_{\tilde{G}_k}^{\tilde{H}_k} : \tilde{G}_k \to \tilde{H}_k \,, \;\; g \mapsto g\tilde{\varphi}_k = g\varphi_k \,.$$

If $\tilde{G} = G$, we also write $\varphi\big|^{\tilde{H}} := \varphi\big|_{G}^{\tilde{H}}\colon G \to \tilde{H}$.
If $\tilde{H} = H$, we also write $\varphi\big|_{\tilde{G}} := \varphi\big|_{\tilde{G}}^{H}\colon \tilde{G} \to H$.
In particular, we have the inclusion morphism $\iota_{\tilde{G},G} := \mathrm{id}_G\,\big|_{\tilde{G}}\colon \tilde{G} \to G$.

Proof. Suppose given $\big([\ell]^{\pm} \xrightarrow{u} [k]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$.
We consider the group morphisms $\tilde{G}_u\colon \tilde{G}_k \to \tilde{G}_\ell$ and $\tilde{H}_u\colon \tilde{H}_k \to \tilde{H}_\ell$.
For $g \in \tilde{G}_k$, we have

$$g(\tilde{\varphi}_k \blacktriangle \tilde{H}_u) = g(\varphi_k \blacktriangle H_u) = g(G_u \blacktriangle \varphi_\ell) = g(\tilde{G}_u \blacktriangle \tilde{\varphi}_\ell) \,.$$

This shows that $\tilde{\varphi}_k \blacktriangle \tilde{H}_u = \tilde{G}_u \blacktriangle \tilde{\varphi}_\ell$.

$$\begin{array}{ccc} \tilde{G}_k & \xrightarrow{\tilde{\varphi}_k} & \tilde{H}_k \\ {\scriptstyle \tilde{G}_u}\downarrow & & \downarrow{\scriptstyle \tilde{H}_u} \\ \tilde{G}_\ell & \xrightarrow[\tilde{\varphi}_\ell]{} & \tilde{H}_\ell \end{array}$$

So $\tilde{\varphi}\colon \tilde{G} \to \tilde{H}$ is a morphism of prestable simplicial groups; cf. Definition 182. □

9.5 A functor from Δ to $\Delta^{\pm}$

Lemma 191 (Parallel extension)
We consider the categories Δ and $\Delta^{\pm}$; cf. Definitions 18, 172.

9.5. A FUNCTOR FROM Δ TO $\Delta^{\pm}$

(1) *Suppose given* $[n] \xrightarrow{u} [k] \in \mathrm{Mor}(\Delta)$. *We consider the map*

$$u^{\pm} \colon [n]^{\pm} \to [k]^{\pm}\,,\ (x,n) \mapsto (x,n)u^{\pm} := \begin{cases} (xu,k) & \text{if } x \leqslant n \\ (x-n+k,k) & \text{if } x \geqslant n+1\,. \end{cases}$$

We have $u^{\pm} \in \mathrm{Mor}(\Delta^{\pm})$.

(2) *We have the functor of* parallel extension

$$P^{\pm} \colon \Delta \to \Delta^{\pm}\,,\ \big([n] \xrightarrow{u} [k]\big) \mapsto \big([n]^{\pm} \xrightarrow{u^{\pm}} [k]^{\pm}\big)\,.$$

Proof. Ad (1). Suppose given $x,\ y \in \mathbb{Z}_{\geqslant 0}$ with $x \leqslant y$. Recall that we write $(x,n)u^{\pm} = (xu^{\pm},k)$ and $(y,n)u^{\pm} = (yu^{\pm},k)$. We have to show that $xu^{\pm} \overset{!}{\leqslant} yu^{\pm}$.
Case $y \leqslant n$. Then $x \leqslant n$. So we have

$$xu^{\pm} = xu \leqslant yu = yu^{\pm}\,.$$

Case $y > n$. Then $y - n > 0$.
If $x \leqslant n$, then we have

$$xu^{\pm} = xu \leqslant k < y - n + k = yu^{\pm}\,.$$

If $x > n$, then we have

$$xu^{\pm} = x - n + k \leqslant y - n + k = yu^{\pm}\,.$$

Altogether, this shows $xu^{\pm} \leqslant yu^{\pm}$.
So $u^{\pm}$ is monotone.
Moreover, for $x \in \mathbb{Z}_{\geqslant 0}$ with $x \geqslant n$, we have

$$(x,n)u^{\pm} = (x-n+k,k)$$

So $u^{\pm}$ is eventually parallel.
Therefore $u^{\pm} \in \mathrm{Mor}(\Delta^{\pm})$.
Ad (2). Suppose given $[n] \xrightarrow{u} [k] \xrightarrow{v} [\ell]$ in Δ.
We want to show that $\mathrm{id}_{[n]}\, \mathrm{P}^{\pm} \overset{!}{=} \mathrm{id}_{[n]^{\pm}}$.

For $x \in \mathbb{Z}_{\geqslant 0}$, we have

$$(x,n)(\mathrm{id}_{[n]}\,\mathrm{P}^{\pm}) = \begin{cases} (x\,\mathrm{id}_{[n]}, n) = (x,n) & \text{if } x \leqslant n \\ (x-n+n, n) = (x,n) & \text{if } x > n\,. \end{cases}$$

So $(x,n)(\mathrm{id}_{[n]}\,\mathrm{P}^{\pm}) = (x,n)$.

This shows $\mathrm{id}_{[n]}\,\mathrm{P}^{\pm} = \mathrm{id}_{[n]^{\pm}}$.

We want to show that $(u \blacktriangle v)\mathrm{P}^{\pm} \stackrel{!}{=} u\mathrm{P}^{\pm} \blacktriangle v\mathrm{P}^{\pm}$.

Suppose given $x \in \mathbb{Z}_{\geqslant 0}$.

We have

$$(x,n)\big((u \blacktriangle v)\mathrm{P}^{\pm}\big) = \begin{cases} (xuv, \ell) & \text{if } x \leqslant n \\ (x-n+\ell, \ell) & \text{if } x > n\,. \end{cases}$$

Now, we want to calculate $(x,n)(u\mathrm{P}^{\pm} \blacktriangle v\mathrm{P}^{\pm})$.

Case $x \leqslant n$. Since $xu \leqslant k$, we have

$$(x,n)(u\mathrm{P}^{\pm} \blacktriangle v\mathrm{P}^{\pm}) = (xu,k)(v\mathrm{P}^{\pm}) = (xuv,\ell)\,.$$

Case $x > n$. Then $x-n+k > k$. So we have

$$(x,n)(u\mathrm{P}^{\pm} \blacktriangle v\mathrm{P}^{\pm}) = (x-n+k,k)(v\mathrm{P}^{\pm}) = (x-n+k-k+\ell,\ell) = (x-n+\ell,\ell)\,.$$

Altogether, we have

$$(x,n)(u\mathrm{P}^{\pm} \blacktriangle v\mathrm{P}^{\pm}) = \begin{cases} (xuv, \ell) & \text{if } x \leqslant n \\ (x-n+\ell, \ell) & \text{if } x > n\,. \end{cases}$$

This shows $(u \blacktriangle v)\mathrm{P}^{\pm} = u\mathrm{P}^{\pm} \blacktriangle v\mathrm{P}^{\pm}$.

So $\mathrm{P}^{\pm}$ is a functor. □

Remark 192 *We consider the parallel extension functor* $\mathrm{P}^{\pm}\colon \Delta \to \Delta^{\pm}$ *from Lemma* 191.

(1) *Suppose given* $n \in \mathbb{Z}_{\geqslant 1}$ *and suppose given* $i \in [0,n]$. *We consider the morphisms* $\big([n+1] \xrightarrow{\sigma_i^n} [n]\big) \in \mathrm{Mor}(\Delta)$ *and* $\big([n+1]^{\pm} \xrightarrow{\sigma_i^{\pm,n}} [n]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$*; cf. Definitions* 19, 174.

We have

$$\sigma_i^n \mathrm{P}^{\pm} = \sigma_i^{\pm,n}\,.$$

(2) *Suppose given* $n \in \mathbb{Z}_{\geqslant 0}$ *and suppose given* $j \in [1, n]$. *We consider the morphisms* $\big([n-1] \xrightarrow{\delta_i^n} [n]\big) \in \mathrm{Mor}(\Delta)$ *and* $\big([n-1]^{\pm} \xrightarrow{\delta_i^{\pm,n}} [n]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$*; cf. Definitions* 19*,* 174.

We have

$$\delta_j^n \mathrm{P}^{\pm} = \delta_j^{\pm,n} .$$

Proof. Ad (1). Suppose given $x \in \mathbb{Z}_{\geqslant 0}$. For $x \leqslant n$, we have

$$\begin{aligned}
(x, n+1)(\sigma_i^n \mathrm{P}^{\pm}) &= \begin{cases} (x, \sigma_i^n, n) & \text{if } x \in [0, n] \\ \big(x - (n+1) + n, n\big) & \text{if } x \geqslant n+1 \end{cases} \\
&= \begin{cases} (x, n) & \text{if } x \in [0, i] \\ (x-1, n) & \text{if } x \in [i+1, n] \\ (x-1, n) & \text{if } x \geqslant n+1 \end{cases} \\
&= \begin{cases} (x, n) & \text{if } x \in [0, i] \\ (x-1, n) & \text{if } x \geqslant i+1 \end{cases} \\
&= (x, n+1)\sigma_i^{\pm,n} .
\end{aligned}$$

This shows that $\sigma_i^n \mathrm{P}^{\pm} = \sigma_i^{\pm,n}$.

Ad (2). Suppose given $x \in \mathbb{Z}_{\geqslant 0}$. We have

$$\begin{aligned}
(x, n-1)(\delta_j^n \mathrm{P}^{\pm}) &= \begin{cases} (x\delta_j^n, n) & \text{if } x \in [0, n] \\ \big(x - (n-1) + n, n\big) & \text{if } x \geqslant n+1 \end{cases} \\
&= \begin{cases} (x, n) & \text{if } x \in [0, j-1] \\ (x+1, n) & \text{if } x \in [j, n] \\ (x+1, n) & \text{if } x \geqslant n+1 \end{cases} \\
&= \begin{cases} (x, n) & \text{if } x \in [0, j-1] \\ (x+1, n) & \text{if } x \geqslant j \end{cases} \\
&= (x, n-1)\delta_j^{\pm,n} .
\end{aligned}$$

This shows that $\delta_j^n \mathrm{P}^{\pm} = \delta_j^{\pm,n}$. □

Remark 193 *We consider the functor* $\mathrm{P}^{\pm} \colon \Delta \to \Delta^{\pm}$ *from Lemma* 191.

We consider the categories $\mathcal{SimpGrp} = [\Delta^{\mathrm{op}}, \mathcal{Grp}]$ *and* $\mathcal{PreStSimpGrp} = [(\Delta^{\pm})^{\mathrm{op}}, \mathcal{Grp}]$*; cf. Definitions* 30*,* 182.

We have the functor

$$\begin{aligned} \mathrm{P} := [(\mathrm{P}^{\pm})^{\mathrm{op}}, \mathcal{G}rp] \colon \quad \mathcal{P}re\mathcal{S}t\mathcal{S}impGrp &\to \mathcal{S}impGrp \\ \big(F \xrightarrow{\varphi} G\big) &\mapsto \big((\mathrm{P}^{\pm})^{\mathrm{op}} * F \xrightarrow{(\mathrm{P}^{\pm})^{\mathrm{op}} * \varphi} (\mathrm{P}^{\pm})^{\mathrm{op}} * G)\big)\,; \end{aligned}$$

cf. §.1.1.3 *items* 13 *and* 15.

Let $n \in \mathbb{Z}_{\geqslant 0}$. *Suppose given a prestable simplicial group* $G \in \mathrm{Ob}\big(\mathcal{P}re\mathcal{S}t\mathcal{S}impGrp\big)$.

We have

$$(G\mathrm{P})_n = G_n\,.$$

For $j \in [0, n+1]$, *we have*

$$\mathrm{d}_j^{G\mathrm{P},\,n+1} = \mathrm{d}_j^{G,n+1} \colon G_{n+1} \to G_n\,.$$

For $i \in [0, n]$, *we have*

$$\mathrm{s}_i^{G\mathrm{P},\,n} = \mathrm{s}_i^{G,n} \colon G_n \to G_{n+1}\,.$$

Suppose given a prestable simplicial group morphism $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}\big(\mathcal{P}re\mathcal{S}t\mathcal{S}impGrp\big)$.

We have

$$(\varphi\mathrm{P})_n = \varphi_n \colon G_n \to H_n\,.$$

Proof. We have

$$(G\mathrm{P})_n = [n]\big((\mathrm{P}^{\pm})^{\mathrm{op}} * G\big) = ([n]^{\pm})G = G_n\,.$$

For $j \in [0, n+1]$, we have

$$\mathrm{d}_j^{G\mathrm{P},\,n+1} = (\delta_j^{n+1})^{\mathrm{op}}\,((\mathrm{P}^{\pm})^{\mathrm{op}} * G) = (\delta_j^{n+1}\mathrm{P}^{\pm})^{\mathrm{op}}\,G \overset{192}{=} (\delta_j^{\pm,n+1})^{\mathrm{op}}\,G = \mathrm{d}_j^{G,n+1}\,.$$

For $i \in [0, n]$, we have

$$\mathrm{s}_i^{G\mathrm{P},\,n} = (\sigma_i^{n})^{\mathrm{op}}\,((\mathrm{P}^{\pm})^{\mathrm{op}} * G) = (\sigma_i^{n}\mathrm{P}^{\pm})^{\mathrm{op}}\,G \overset{192}{=} (\sigma_i^{\pm,n})^{\mathrm{op}}\,G = \mathrm{s}_i^{G,n}\,.$$

We have

$$(\varphi\mathrm{P})_n = [n](\varphi\mathrm{P}) = [n]\big((\mathrm{P}^{\pm})^{\mathrm{op}}\,\varphi\big) = ([n]^{\pm})\varphi = \varphi_n\,.$$

□

9.6 The décalage functor $\mathrm{Dec}^{\pm,n}$ for prestable simplicial groups

Let $n \in \mathbb{Z}$.

Lemma 194 (The functor $\mathrm{dec}^{\pm,n}$)

We consider the prestable simplex category $\Delta^{\pm}$.

For a morphism $\big([k]^{\pm} \xrightarrow{u} [\ell]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$, recall that we write $(x,k)u = (xu,\ell) \in [\ell]^{\pm}$ for $x \in \mathbb{Z}_{\geqslant 0}$.

We have the autofunctor

$$\mathrm{dec}^{\pm,n} \colon \Delta^{\pm} \to \Delta^{\pm}\,,\ \big([k]^{\pm} \xrightarrow{u} [\ell]^{\pm}\big) \mapsto \big([k+n]^{\pm} \xrightarrow{u\mathrm{dec}^{\pm,n}} [\ell+n]^{\pm}\big)\,,$$

where

$$u\mathrm{dec}^{\pm,n} \colon [k+n]^{\pm} \to [\ell+n]^{\pm}\,,\ (x,k+n) \mapsto (xu,\ell+n)\,.$$

Its inverse is given by

$$(\mathrm{dec}^{\pm,n})^{-} = \mathrm{dec}^{\pm,-n} \colon \Delta^{\pm} \to \Delta^{\pm}\,.$$

Proof. Ad (1). Suppose given a morphism $\big([k] \xrightarrow{u} [\ell]\big) \in \mathrm{Mor}(\Delta^{\pm})$.

We consider the map

$$\tilde{u} \colon [k+n]^{\pm} \to [\ell+n]^{\pm}\,,\ (x,k+n) \mapsto (xu,\ell+n)\,.$$

We show that $\tilde{u} \overset{!}{\in} \mathrm{Mor}(\Delta^{\pm})$.

Suppose given x, $y \in \mathbb{Z}_{\geqslant 0}$ with $x \leqslant y$. Since u is monotone, we have $xu \leqslant yu$. So $\tilde{u}$ is monotone.

Since u is eventually parallel, there exists some $i_0 \in \mathbb{Z}_{\geqslant 0}$ such that $(x,k)u = (x-k+\ell,\ell)$ for $x \geqslant i_0$. For $x \geqslant i_0$, we have

$$(x,k+n)\tilde{u} = (xu,\ell+n) = (x-k+\ell,\ell+n) = \big(x-(k+n)+(\ell+n),\ell+n\big)\,.$$

So $\tilde{u}$ is eventually parallel.

This shows that $\tilde{u} \in \mathrm{Mor}(\Delta^{\pm})$.

We show that $\mathrm{dec}^{\pm,n}$ *is a functor.* Suppose given $[k]^{\pm} \xrightarrow{u} [\ell]^{\pm} \xrightarrow{v} [m]^{\pm}$ in $\Delta^{\pm}$.

For $x \in \mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned}(x, k+n)(\mathrm{id}_{[k]^\pm}\,\mathrm{dec}^{\pm,n}) &= (x\,\mathrm{id}_{[k]^\pm}, k+n) = (x, k+n) = (x, k+n)\,\mathrm{id}_{[k+n]^\pm} \\ &= (x, k+n)\,\mathrm{id}_{[k]^\pm \mathrm{dec}^{\pm,n}}\ .\end{aligned}$$

This shows that $\mathrm{id}_{[k]^\pm}\,\mathrm{dec}^{\pm,n} = \mathrm{id}_{[k]^\pm \mathrm{dec}^{\pm,n}}$.

For $x \in \mathbb{Z}_{\geqslant 0}$, we have

$(x, k+n)\big((u \blacktriangle v)\mathrm{dec}^{\pm,n}\big) = (xuv, m+n) = (xu, \ell+n)(v\mathrm{dec}^{\pm,n}) = (x, k+n)(u\mathrm{dec}^{\pm,n} \blacktriangle v\mathrm{dec}^{\pm,n})$.

This shows that $(u \blacktriangle v)\mathrm{dec}^{\pm,n} = u\mathrm{dec}^{\pm,n} \blacktriangle v\mathrm{dec}^{\pm,n}$.

So $\mathrm{dec}^{\pm,n}$ is a functor.

We show that $(\mathrm{dec}^{\pm,n})^- \overset{!}{=} \mathrm{dec}^{\pm,-n}$. Suppose given $\big([k]^\pm \xrightarrow{u} [\ell]^\pm\big) \in \mathrm{Mor}(\Delta^\pm)$.

Suppose given $x \in \mathbb{Z}_{\geqslant 0}$. Recall that we write

$$(x, k+n)(u\mathrm{dec}^{\pm,n}) = \big(x(u\mathrm{dec}^{\pm,n}), \ell+n\big) = (xu, \ell+n)\,.$$

So we have

$$\begin{aligned}(x,k)\big(u(\mathrm{dec}^{\pm,n} * \mathrm{dec}^{\pm,-n})\big) &= \big(x, (k+n)-n\big)\big((u\mathrm{dec}^{\pm,n})\mathrm{dec}^{\pm,-n}\big) \\ &= \big(x(u\mathrm{dec}^{\pm,n}), (\ell+n)-n\big) \\ &= (xu, \ell) = (x,k)u\,.\end{aligned}$$

This shows that $u(\mathrm{dec}^{\pm,n} * \mathrm{dec}^{\pm,-n}) = u$.

So $\mathrm{dec}^{\pm,n} * \mathrm{dec}^{\pm,-n} = \mathrm{id}_{\Delta^\pm}$. Similarly, we have $\mathrm{dec}^{\pm,-n} * \mathrm{dec}^{\pm,n} = \mathrm{id}_{\Delta^\pm}$.

This shows that $(\mathrm{dec}^{\pm,n})^- = \mathrm{dec}^{\pm,-n}$. □

Lemma 195 (The functor $\mathrm{Dec}^{\pm,n}$)

We consider the functor $\mathrm{dec}^{\pm,n}\colon \Delta^\pm \to \Delta^\pm$ *from Lemma* 194.

We have the stable décalage *functor*

$$\begin{aligned}\mathrm{Dec}^{\pm,n} := \big[(\mathrm{dec}^{\pm,n})^{\mathrm{op}}, \mathcal{G}rp\big]\colon\ \mathcal{P}re\mathcal{S}t\mathcal{S}impGrp &\to \mathcal{P}re\mathcal{S}t\mathcal{S}impGrp \\ \big(G \xrightarrow{\varphi} H\big) &\mapsto \big(G\mathrm{Dec}^{\pm,n} \xrightarrow{\varphi\mathrm{Dec}^{\pm,n}} H\mathrm{Dec}^{\pm,n}\big) \\ &= \big((\mathrm{dec}^{\pm,n})^{\mathrm{op}} * G \xrightarrow{(\mathrm{dec}^{\pm,n})^{\mathrm{op}} * \varphi} (\mathrm{dec}^{\pm,n})^{\mathrm{op}} * H\big);\end{aligned}$$

cf. §1.1.3 item 15.

Let $k \in \mathbb{Z}$. For a prestable simplicial group $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$, we have

$$(G\mathrm{Dec}^{\pm,n})_k = G_{k+n}\,.$$

For a prestable simplicial group morphism $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}\big(\mathcal{PreStSimpGrp}\big)$, we have

$$(\varphi\mathrm{Dec}^{\pm,n})_k = \varphi_{k+n}\colon G_{k+n} \to H_{k+n}\,.$$

Proof. Suppose given $\big(G \xrightarrow{\varphi} H\big)$ in $\mathcal{PreStSimpGrp}$.
We have

$$(G\mathrm{Dec}^{\pm,n})_k = ([k]^{\pm})\big((\mathrm{dec}^{\pm,n})^{\mathrm{op}} * G\big) = \big([k+n]^{\pm}\big)G = G_{k+n}\,.$$

Moreover, we have

$$(\varphi\mathrm{Dec}^{\pm,n})_k = ([k]^{\pm})\big((\mathrm{dec}^{\pm,n})^{\mathrm{op}} * \varphi\big) = \big([k+n]^{\pm}\big)\varphi = \varphi_{k+n}\,.$$

□

Remark 196 *We consider the functor $\mathrm{Dec}^{\pm,n}\colon \mathcal{PreStSimpGrp} \to \mathcal{PreStSimpGrp}$ from Lemma 195.*
Suppose given a prestable simplicial group $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$.
Let $k \in \mathbb{Z}$ and let $i \in \mathbb{Z}_{\geqslant 0}$.
We have

$$\mathrm{d}_i^{G\mathrm{Dec}^{\pm,n},k} = \mathrm{d}_i^{G,k+n}$$
$$\mathrm{s}_i^{G\mathrm{Dec}^{\pm,n},k} = \mathrm{s}_i^{G,k+n}\,.$$

Proof. We consider the functor $\mathrm{dec}^{\pm,n}\colon \Delta^{\pm} \to \Delta^{\pm}$ from Lemma 194.

We consider the morphism $\big([k-1]^{\pm} \xrightarrow{\delta_i^{\pm,k}} [k]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$; cf. Definition 174.
Let $x \in \mathbb{Z}_{\geqslant 0}$. If $x < i$, then we have

$$(x, k+n-1)(\delta_i^{\pm,k}\mathrm{dec}^{\pm,n}) = (x\delta_i^{\pm,k}, k+n) = (x, k+n) = (x, k+n-1)\delta_i^{\pm,k+n}\,.$$

If $x \geqslant i$, then we have

$$(x, k+n-1)(\delta_i^{\pm,k}\mathrm{dec}^{\pm,n}) = (x\delta_i^{\pm,k}, k+n) = (x+1, k+n) = (x, k+n-1)\delta_i^{\pm,k+n}\,.$$

This shows that $\delta_i^{\pm,k}\mathrm{dec}^{\pm,n} = \delta_i^{\pm,k+n}$.

So we have

$$\begin{aligned}\mathrm{d}_i^{G\mathrm{Dec}^{\pm,n},k} &= (\delta_i^{\pm,k})^{\mathrm{op}}\,(G\mathrm{Dec}^{\pm,n}) = (\delta_i^{\pm,k})^{\mathrm{op}}\,\big((\mathrm{dec}^{\pm,n})^{\mathrm{op}} * G\big) = (\delta_i^{\pm,k}\,\mathrm{dec}^{\pm,n})^{\mathrm{op}}\,G\\ &= (\delta_i^{\pm,k+n})G = \mathrm{d}_i^{G,\,k+n}\,.\end{aligned}$$

We consider the morphism $\big([k+1]^{\pm} \xrightarrow{\sigma_i^{\pm,k}} [k]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$; cf. Definition 174.

Let $x \in \mathbb{Z}_{\geqslant 0}$. If $x \leqslant i$, then we have

$$(x, k+n+1)(\sigma_i^{\pm,k}\mathrm{dec}^{\pm,n}) = (x\sigma_i^{\pm,k}, k+n) = (x, k+n) = (x, k+n+1)\sigma_i^{\pm,k+n}\,.$$

If $x > i$, then we have

$$(x, k+n+1)(\sigma_i^{\pm,k}\mathrm{dec}^{\pm,n}) = (x\sigma_i^{\pm,k}, k+n) = (x-1, k+n) = (x, k+n+1)\sigma_i^{\pm,k+n}\,.$$

This shows that $\sigma_i^{\pm,k}\mathrm{dec}^{\pm,n} = \sigma_i^{\pm,k+n}$.

So we have

$$\begin{aligned}\mathrm{s}_i^{G\mathrm{Dec}^{\pm,n},k} &= (\sigma_i^{\pm,k})^{\mathrm{op}}\,(G\mathrm{Dec}^{\pm,n}) = (\sigma_i^{\pm,k})^{\mathrm{op}}\,\big((\mathrm{dec}^{\pm,n})^{\mathrm{op}} * G\big) = (\sigma_i^{\pm,k}\,\mathrm{dec}^{\pm,n})^{\mathrm{op}}\,G = (\sigma_i^{\pm,k+n})G\\ &= \mathrm{s}_i^{G,k+n}\,.\end{aligned}$$

□

Remark 197 *We consider the functor* $P\colon \mathcal{PreStSimpGrp} \to \mathcal{SimpGrp}$ *from Remark* 193*, where*

$$(GP)_k = G_k \quad \text{and} \quad (\varphi P)_k = \varphi_k$$

for $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$, $\varphi \in \mathrm{Mor}\big(\mathcal{PreStSimpGrp}\big)$ *and* $k \in \mathbb{Z}_{\geqslant 0}$.

We consider the functor $\mathrm{Dec}\colon \mathcal{SimpGrp} \to \mathcal{SimpGrp}$ *from Lemma* 44*, where*

$$(G\mathrm{Dec})_k = G_{k+1} \quad \text{and} \quad (\varphi\mathrm{Dec})_k = \varphi_{k+1}$$

for $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$, $\varphi \in \mathrm{Mor}\big(\mathcal{SimpGrp}\big)$ *and* $k \in \mathbb{Z}_{\geqslant 0}$.

We consider the functor $\mathrm{Dec}^{\pm,1}\colon \mathcal{PreStSimpGrp} \to \mathcal{PreStSimpGrp}$ *from Lemma* 195*, where*

$$(G\mathrm{Dec}^{\pm,1})_k = G_{k+1} \quad \text{and} \quad (\varphi\mathrm{Dec}^{\pm,1})_k = \varphi_{k+1}$$

for $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$, $\varphi \in \mathrm{Mor}\big(\mathcal{PreStSimpGrp}\big)$ *and* $k \in \mathbb{Z}$.

We have

$$P * \mathrm{Dec} = \mathrm{Dec}^{\pm,1} * P\,.$$

$$\begin{array}{ccc} \mathcal{PreStSimpGrp} & \xrightarrow{P} & \mathcal{SimpGrp} \\ {\scriptstyle \mathrm{Dec}^{\pm,1}}\downarrow & & \downarrow{\scriptstyle \mathrm{Dec}} \\ \mathcal{PreStSimpGrp} & \xrightarrow[P]{} & \mathcal{SimpGrp} \end{array}$$

Proof. We consider the functor $\mathrm{P}^{\pm}\colon \Delta \to \Delta^{\pm}$ from Lemma 191 .
For $\big([k] \xrightarrow{u} [\ell]\big) \in \mathrm{Mor}(\Delta)$, recall that

$$u\mathrm{P}^{\pm}\colon [k]^{\pm} \to [\ell]^{\pm}\,,\ (x,k) \mapsto \begin{cases} (xu,\ell) & \text{for } x \in [0,k] \\ (x-k+\ell,\ell) & \text{for } x \in \mathbb{Z}_{\geqslant k+1}\,. \end{cases}$$

Recall that $\mathrm{P} = [(\mathrm{P}^{\pm})^{\mathrm{op}}, \mathcal{Grp}]$.
We consider the functor $\mathrm{dec}\colon \Delta \to \Delta$ from Lemma 43.
For $\big([k] \xrightarrow{u} [\ell]\big) \in \mathrm{Mor}(\Delta)$, recall that

$$u\mathrm{dec}\colon [k+1] \to [\ell+1]\,,\ x \mapsto \begin{cases} xu & \text{for } x \in [0,k] \\ \ell+1 & \text{for } x = k+1\,. \end{cases}$$

Recall that $\mathrm{Dec} = [\mathrm{dec}^{\mathrm{op}}, \mathcal{Grp}]$.
We consider the functor $\mathrm{dec}^{\pm,1}\colon \Delta^{\pm} \to \Delta^{\pm}$ from Lemma 194.
For $\big([k]^{\pm} \xrightarrow{u} [\ell]^{\pm}\big) \in \mathrm{Mor}(\Delta^{\pm})$, recall that

$$u\mathrm{dec}^{\pm,1}\colon [k+1]^{\pm} \to [\ell+1]^{\pm}\,,\ (x,k+1) \mapsto (xu,\ell+1)\,.$$

Recall that $\mathrm{Dec}^{\pm,1} = [(\mathrm{dec}^{\pm,1})^{\mathrm{op}}, \mathcal{Grp}]$.
We show that $\mathrm{P}^{\pm} * \mathrm{dec}^{\pm,1} \overset{!}{=} \mathrm{dec} * \mathrm{P}^{\pm}$.
Suppose given $\big([k] \xrightarrow{u} [\ell]\big) \in \mathrm{Mor}(\Delta)$.
We consider the morphism $u(\mathrm{P}^{\pm} * \mathrm{dec}^{\pm,1})\colon [k+1]^{\pm} \to [\ell+1]^{\pm}$. For $x \in \mathbb{Z}_{\geqslant 0}$, we have

$$(x,k+1)\big(u(\mathrm{P}^{\pm} * \mathrm{dec}^{\pm,1})\big) = (x,k+1)\big((u\mathrm{P}^{\pm})\mathrm{dec}^{\pm,1}\big) = \big(x(u\mathrm{P}^{\pm}),\ell+1\big)\,.$$

Let $x \in \mathbb{Z}_{\geqslant 0}$. If $x \in [0,k]$, then we have

$$(x,k+1)\big(u(\mathrm{P}^{\pm} * \mathrm{dec}^{\pm,1})\big) = \big(x(u\mathrm{P}^{\pm}),\ell+1\big) \overset{x\in[0,k]}{=} (xu,\ell+1)$$

and

$$(x,k+1)\big(u(\mathrm{dec}*\mathrm{P}^{\pm})\big)=(x,k+1)\big((u\mathrm{dec})\mathrm{P}^{\pm}\big)\overset{x\in[0,k+1]}{=}\big(x(u\mathrm{dec}),\ell+1\big)\overset{x\in[0,k]}{=}(xu,\ell+1)\,.$$

If $x=k+1$, then we have

$$(x,k+1)\big(u(\mathrm{P}^{\pm}*\mathrm{dec}^{\pm,1})\big)=\big(x(u\mathrm{P}^{\pm}),\ell+1\big)\overset{x\geqslant k+1}{=}(x-k+\ell,\ell+1)\overset{x=k+1}{=}(\ell+1,\ell+1)$$

and

$$(x,k+1)\big(u(\mathrm{dec}*\mathrm{P}^{\pm})\big)=(x,k+1)\big((u\mathrm{dec})\mathrm{P}^{\pm}\big)\overset{x\in[0,k+1]}{=}\big(x(u\mathrm{dec}),\ell+1\big)\overset{x=k+1}{=}(\ell+1,\ell+1)\,.$$

If $x\in\mathbb{Z}_{\geqslant k+2}$, then we have

$$(x,k+1)\big(u(\mathrm{P}^{\pm}*\mathrm{dec}^{\pm,1})\big)=\big(x(u\mathrm{P}^{\pm}),\ell+1\big)\overset{x\geqslant k+1}{=}(x-k+\ell,\ell+1)$$

and

$$\begin{aligned}(x,k+1)\big(u(\mathrm{dec}*\mathrm{P}^{\pm})\big)&=(x,k+1)\big((u\mathrm{dec})\mathrm{P}^{\pm}\big)\overset{x\geqslant k+2}{=}\big(x-(k+1)+(\ell+1),\ell+1\big)\\&=(x-k+\ell,\ell+1)\,.\end{aligned}$$

This shows that $u(\mathrm{P}^{\pm}*\mathrm{dec}^{\pm,1})=u(\mathrm{dec}*\mathrm{P}^{\pm})$. So $\mathrm{P}^{\pm}*\mathrm{dec}^{\pm,1}=\mathrm{dec}*\mathrm{P}^{\pm}$.
Therefore, we have

$$\begin{aligned}P*\mathrm{Dec}&=[(\mathrm{P}^{\pm})^{\mathrm{op}},\mathcal{Grp}]*[\mathrm{dec}^{\mathrm{op}},\mathcal{Grp}]=[\mathrm{dec}^{\mathrm{op}}*(\mathrm{P}^{\pm})^{\mathrm{op}},\mathcal{Grp}]\\&=[(\mathrm{dec}*\mathrm{P}^{\pm})^{\mathrm{op}},\mathcal{Grp}]=[(\mathrm{P}^{\pm}*\mathrm{dec}^{\pm,1})^{\mathrm{op}},\mathcal{Grp}]=[(\mathrm{P}^{\pm})^{\mathrm{op}}*(\mathrm{dec}^{\pm,1})^{\mathrm{op}},\mathcal{Grp}]\\&=[(\mathrm{dec}^{\pm,1})^{\mathrm{op}},\mathcal{Grp}]*[(\mathrm{P}^{\pm})^{\mathrm{op}},\mathcal{Grp}]=\mathrm{Dec}^{\pm,1}*P\,;\end{aligned}$$

cf. §1.1.3 item 15. □

9.7 The Moore complex of a prestable simplicial group

Definition 198 Let $G\in\mathrm{Ob}(\mathcal{PreStSimpGrp})$ be a prestable simplicial group. Let $k\in\mathbb{Z}$.

(1) For $i\in\mathbb{Z}_{\geqslant 0}$, we write

$$G_{k,i}:=\ker \mathrm{d}_i^{G,\,k}\trianglelefteqslant G_k\,.$$

(2) We write

$$G\mathrm{N}_k := \bigcap_{i \geqslant 1} G_{k,i} \trianglelefteqslant G_k \,.$$

(3) We write

$$G\mathrm{NZ}_k := G\mathrm{N}_k \cap G_{k,0} = \bigcap_{i \geqslant 0} G_{k,i} \trianglelefteqslant G_k \,.$$

(4) We write

$$G\mathrm{NB}_k := (G\mathrm{N}_{k+1})\mathrm{d}_0^{G,\,k+1} = \big(\bigcap_{i \geqslant 1} G_{k+1,i}\big)\mathrm{d}_0^{G,\,k+1} \leqslant G_k \,.$$

Remark 199 *Let $G \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$ be a prestable simplicial group. Let $k \in \mathbb{Z}$.*

(1) *We have* $G\mathrm{NB}_k \trianglelefteqslant G_k$.

(2) *We have* $G\mathrm{NB}_k \trianglelefteqslant G\mathrm{NZ}_k$.

Proof. Ad (1). Suppose given $g \in G\mathrm{NB}_k$ and suppose given $y \in G_k$. We have to show that $g^y \overset{!}{\in} G\mathrm{NB}_k$.

We have $g = x\mathrm{d}_0$ for some chosen $x \in \bigcap_{i \geqslant 1} G_{k+1,i} \trianglelefteqslant G_{k+1}$.

Let $z := x^{ys_0} \in G_{k+1}$.

For $i \in \mathbb{Z}_{\geqslant 1}$, we have

$$z\mathrm{d}_i = (x^{ys_0})\mathrm{d}_i = (x\mathrm{d}_i)^{ys_0\mathrm{d}_i} = 1^{ys_0\mathrm{d}_i} = 1 \,.$$

This shows $z \in \bigcap_{i \geqslant 1} G_{k+1,i} \leqslant G_{k+1}$.

Moreover, we have

$$z\mathrm{d}_0 = (x^{ys_0})\mathrm{d}_0 = (x\mathrm{d}_0)^{ys_0\mathrm{d}_0} \overset{185.(3)}{=} g^y \,.$$

This shows $g^y \in \big(\bigcap_{i \geqslant 1} G_{k+1,i}\big)\mathrm{d}_0 = G\mathrm{NB}_k$.

So $G\mathrm{NB}_k \trianglelefteqslant G_k$.

Ad (2). Suppose given $g \in G\mathrm{NB}_k$. We have $g = x\mathrm{d}_0$ for some $x \in \bigcap_{i \geqslant 1} G_{k+1,i} \trianglelefteqslant G_{k+1}$.

For $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$g\mathrm{d}_i = x\mathrm{d}_0\,\mathrm{d}_i \overset{185.(1)}{=} x\mathrm{d}_{i+1}\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows $g \in G\mathrm{NZ}_k$.

So $G\mathrm{NB}_k \leqslant G\mathrm{NZ}_k$. Moreover, by (1), we have $G\mathrm{NB}_k \trianglelefteqslant G\mathrm{NZ}_k$. □

Lemma 200 (Moore complex of a prestable simplicial group)
Let $G \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$ be a prestable simplicial group.

(1) *For $k \in \mathbb{Z}$, we have the group morphism*

$$\partial_k^G := \mathrm{d}_0^{G,k}\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}} \colon G\mathrm{N}_k \to G\mathrm{N}_{k-1}\,,\ x \mapsto x\mathrm{d}_0^{G,k}\,.$$

Often, we write $\partial_k := \partial_k^G$.

(2) *We have the complex in groups*

$$G\mathrm{N} := \ldots \xrightarrow{\partial_3} G\mathrm{N}_2 \xrightarrow{\partial_2} G\mathrm{N}_1 \xrightarrow{\partial_1} G\mathrm{N}_0 \xrightarrow{\partial_0} G\mathrm{N}_{-1} \xrightarrow{\partial_{-1}} \ldots\,.$$

We call $G\mathrm{N}$ the Moore complex *of G.*
For $k \in \mathbb{Z}$, we write

$$G\pi_k := G\mathrm{NH}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}\,.$$

for the k-th homotopy group *of G; cf. Lemma* 11.(1).

Proof. Ad (1). Let $k \in \mathbb{Z}$. We have to show that $(G\mathrm{N}_k)\mathrm{d}_0 \overset{!}{\subseteq} G\mathrm{N}_{k-1}$.
Suppose given $x \in G\mathrm{N}_k$. Then $x\mathrm{d}_j = 1$ for $j \geqslant 1$. For $i \in \mathbb{Z}_{\geqslant 1}$, we have

$$x\mathrm{d}_0\,\mathrm{d}_i = x\mathrm{d}_{i+1}\,\mathrm{d}_0 = 1\,.$$

This shows $x\mathrm{d}_0 \in \bigcap_{i \geqslant 1} G_{k-1,i} = G\mathrm{N}_{k-1}$.

So $(G\mathrm{N}_k)\mathrm{d}_0 \subseteq G\mathrm{N}_{k-1}$.

Ad (2). Let $k \in \mathbb{Z}$. *We show that* $\partial_k \vartriangleright \partial_{k-1} \overset{!}{=} !$.

For $x \in G\mathrm{N}_k$, we have

$$x\partial_k\partial_{k-1} = x\mathrm{d}_0\,\mathrm{d}_0 = x\mathrm{d}_1\,\mathrm{d}_0 = 1\,.$$

This shows $\partial_k \mathbin{\scriptstyle\blacktriangle} \partial_{k-1} = !$.

Moreover, we have

$$\operatorname{im}\partial_k = (G\mathrm{N}_k)\partial_k = (G\mathrm{N}_k)\mathrm{d}_0 = G\mathrm{NB}_{k-1} \overset{199.(1)}{\trianglelefteqslant} G_k\,.$$

Since $\operatorname{im}\partial_k \leqslant G\mathrm{N}_{k-1}$, we conclude that $\operatorname{im}\partial_k \trianglelefteqslant G\mathrm{N}_{k-1}$. □

Lemma 201 *Suppose given prestable simplicial groups G, $H \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$. Suppose given a prestable simplicial group morphism $\varphi\colon G \to H$; cf. Remark* 184.

(1) *For $k \in \mathbb{Z}$, we have the group morphism*

$$\varphi\mathrm{N}_k := \varphi_k\big|^{H\mathrm{N}_k}_{G\mathrm{N}_k}\colon G\mathrm{N}_k \to H\mathrm{N}_k\,,\ x \mapsto x\varphi_k\,.$$

(2) *Consider the complexes in groups $G\mathrm{N}$ and $H\mathrm{N}$; cf. Lemma* 200.(2).
We have the morphism of complexes in groups

$$\varphi\mathrm{N} := (\varphi\mathrm{N}_k)_{k\in\mathbb{Z}}\colon G\mathrm{N} \to H\mathrm{N}\,.$$

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{\partial^G_2} & G\mathrm{N}_1 & \xrightarrow{\partial^G_1} & G\mathrm{N}_0 & \xrightarrow{\partial^G_0} & G\mathrm{N}_{-1} & \xrightarrow{\partial^G_{-1}} & \ldots \\
 & & \downarrow{\scriptstyle \varphi\mathrm{N}_1} & & \downarrow{\scriptstyle \varphi\mathrm{N}_0} & & \downarrow{\scriptstyle \varphi\mathrm{N}_{-1}} & & \\
\ldots & \xrightarrow{\partial^H_2} & H\mathrm{N}_1 & \xrightarrow{\partial^H_1} & H\mathrm{N}_0 & \xrightarrow{\partial^H_0} & H\mathrm{N}_{-1} & \xrightarrow{\partial^H_{-1}} & \ldots
\end{array}$$

(3) *We have the functor*

$$\mathrm{N}\colon \mathcal{PreStSimpGrp} \to \mathcal{ComplexGrp}\,,\ \big(F \xrightarrow{\varphi} F'\big) \mapsto \big(F\mathrm{N} \xrightarrow{\varphi\mathrm{N}} F'\mathrm{N}\big)\,.$$

Proof. Ad (1). Let $k \in \mathbb{Z}$. We have to show that $(G\mathrm{N}_k)\varphi_k \overset{!}{\subseteq} H\mathrm{N}_k$.

Suppose given $x \in G\mathrm{N}_k$. For $i \in \mathbb{Z}_{\geqslant 1}$, we have

$$x\varphi_k\,\mathrm{d}_i^{H,\,k} = x\mathrm{d}_i^{G,\,k}\,\varphi_{k-1} = 1\,.$$

This shows $x\varphi_k \in \bigcap\limits_{i\geqslant 1} H_{k,i} = H\mathrm{N}_k$.

Ad (2). Let $k \in \mathbb{Z}$. We have to show that $\partial_k^G \blacktriangle \varphi \mathrm{N}_{k-1} \stackrel{!}{=} \varphi \mathrm{N}_k \blacktriangle \partial_k^H$.

For $x \in G\mathrm{N}_k$, we have

$$x\partial_k^G(\varphi \mathrm{N}_{k-1}) = x\mathrm{d}_0^{G,k}\varphi_{k-1} = x\varphi_k \mathrm{d}_0^{H,k} = x(\varphi \mathrm{N}_k)\partial_k^H .$$

This shows $\partial_k^G \blacktriangle \varphi \mathrm{N}_{k-1} = \varphi \mathrm{N}_k \blacktriangle \partial_k^H$.

So $(\varphi \mathrm{N}_k)_{k\in\mathbb{Z}}$ is a morphism of complexes in groups.

Ad (3). Suppose given $F \xrightarrow{\varphi} F' \xrightarrow{\varphi'} F''$ in $\mathcal{PreStSimpGrp}$.

We have

$$\mathrm{id}_F \mathrm{N} = (\mathrm{id}_F \mathrm{N}_k)_{k\in\mathbb{Z}} = \big(\mathrm{id}_{F_k}\big|_{F\mathrm{N}_k}^{F\mathrm{N}_k}\big)_{k\in\mathbb{Z}} = (\mathrm{id}_{F\mathrm{N}_k})_{k\in\mathbb{Z}} = \mathrm{id}_{F\mathrm{N}} .$$

Moreover, we have

$$\begin{aligned}(\varphi \blacktriangle \varphi')\mathrm{N} &= \big((\varphi \blacktriangle \varphi')\mathrm{N}_k\big)_{k\in\mathbb{Z}} = \big((\varphi_k \blacktriangle \varphi'_k)\big|_{F\mathrm{N}_k}^{F''\mathrm{N}_k}\big)_{k\in\mathbb{Z}} = \big(\varphi_k\big|_{F\mathrm{N}_k}^{F'\mathrm{N}_k} \blacktriangle \varphi'_k\big|_{F'\mathrm{N}_k}^{F''\mathrm{N}_k}\big)_{k\in\mathbb{Z}} \\ &= \big(\varphi_k\big|_{F\mathrm{N}_k}^{F'\mathrm{N}_k}\big)_{k\in\mathbb{Z}} \blacktriangle \big(\varphi'_k\big|_{F'\mathrm{N}_k}^{F''\mathrm{N}}\big)_{k\in\mathbb{Z}} = (\varphi \mathrm{N}_k)_{k\in\mathbb{Z}} \blacktriangle (\varphi' \mathrm{N}_k)_{k\in\mathbb{Z}} = \varphi\mathrm{N} \blacktriangle \varphi'\mathrm{N} .\end{aligned}$$

So N is a functor. □

Lemma 202 *Suppose given a prestable simplicial group $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$.*

Suppose given $z \in \mathbb{Z}$. For $\ell \in \mathbb{Z}_{\geqslant 0}$, we have

$$G\mathrm{NB}_z = \big(\bigcap_{i\in\mathbb{Z}_{\geqslant 0}\setminus\{\ell\}} G_{z+1,i}\big)\mathrm{d}_\ell^{G,z+1}$$

as subgroups of G_z .

Proof. For $\ell \in \mathbb{Z}_{\geqslant 0}$, we write

$$U_\ell := \big(\bigcap_{i\in\mathbb{Z}_{\geqslant 0}\setminus\{\ell\}} G_{z+1,i}\big)\mathrm{d}_\ell .$$

We show that $G\mathrm{NB}_z \stackrel{!}{=} U_\ell$ by induction on ℓ.

Suppose that $\ell = 0$.

Then we have

$$G\mathrm{NB}_z = \big(\bigcap_{i\geqslant 1} G_{z+1,i}\big)\mathrm{d}_0 = \big(\bigcap_{i\in\mathbb{Z}_{\geqslant 0}\setminus\{0\}} G_{z+1,i}\big)\mathrm{d}_0 = U_0 .$$

Suppose that $\ell \geqslant 1$.

Now we assume that this assertion holds for ℓ.

We have to show that $U_\ell \overset{!}{=} U_{\ell+1}$.

Ad $\subseteq$. Suppose given $y \in U_\ell$. We may choose some $x \in \bigcap_{i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell\}} G_{z+1,i}$ such that $y = x\mathrm{d}_\ell$.

Let $\tilde{x} := x^- \cdot x\mathrm{d}_\ell \, \mathrm{s}_\ell \in G_{z+1}$.

We show that $\tilde{x} \overset{!}{\in} \bigcap_{i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell+1\}} G_{z+1,i}$.

Suppose given $i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell+1\}$.

If $i \leqslant \ell - 1$, then we have

$$\tilde{x}\mathrm{d}_i = x^-\mathrm{d}_i \cdot x\mathrm{d}_\ell \, \mathrm{s}_\ell \, \mathrm{d}_i = 1 \cdot x\mathrm{d}_\ell \, \mathrm{d}_i \, \mathrm{s}_{\ell-1} = x\mathrm{d}_i \, \mathrm{d}_{\ell-1} \, \mathrm{s}_{\ell-1} = 1 \, .$$

If $i = \ell$, then we have

$$\tilde{x}\mathrm{d}_i = x^-\mathrm{d}_\ell \cdot x\mathrm{d}_\ell \, \mathrm{s}_\ell \, \mathrm{d}_\ell = (x\mathrm{d}_\ell)^- \cdot x\mathrm{d}_\ell = 1 \, .$$

If $i \geqslant \ell + 2$, then we have

$$\tilde{x}\mathrm{d}_i = x^-\mathrm{d}_i \cdot x\mathrm{d}_\ell \, \mathrm{s}_\ell \, \mathrm{d}_i = 1 \cdot x\mathrm{d}_\ell \, \mathrm{d}_{i-1} \, \mathrm{s}_\ell = x\mathrm{d}_i \, \mathrm{d}_\ell \, \mathrm{s}_\ell = 1 \, .$$

This shows $\tilde{x} \in G_{z+1,i}$ for $i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell+1\}$. So $\tilde{x} \in \bigcap_{i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell+1\}} G_{z+1,i}$.

We show that $\tilde{x}\mathrm{d}_{\ell+1} \overset{!}{=} y$.

We have

$$\tilde{x}\mathrm{d}_{\ell+1} = x^-\mathrm{d}_{\ell+1} \cdot x\mathrm{d}_\ell \, \mathrm{s}_\ell \, \mathrm{d}_{\ell+1} = 1 \cdot x\mathrm{d}_\ell = y \, .$$

Altogether, we have

$$y = \tilde{x}\mathrm{d}_{\ell+1} \in \big(\bigcap_{i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell+1\}} G_{z+1,i} \big)\mathrm{d}_{\ell+1} = U_{\ell+1} \, .$$

So $U_\ell \subseteq U_{\ell+1}$.

Ad $\supseteq$. Suppose given $y \in U_{\ell+1}$. Then we may choose some $x \in \bigcap_{i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell+1\}} G_{z+1,i}$ such that $y = x\mathrm{d}_{\ell+1}$.

Let $\tilde{x} := x^- \cdot x\mathrm{d}_{\ell+1} \, \mathrm{s}_\ell \in G_{z+1}$.

We show that $\tilde{x} \overset{!}{\in} \bigcap_{i \in \mathbb{Z}_{\geqslant 0} \setminus \{\ell\}} G_{z+1,i}$.

Suppose given $i \in \mathbb{Z}_{\geqslant 0}\backslash\{\ell\}$.

If $i \leqslant \ell - 1$, then we have

$$\tilde{x}\mathrm{d}_i = x^{-}\mathrm{d}_i \cdot x\mathrm{d}_{\ell+1}\,\mathrm{s}_\ell\,\mathrm{d}_i = 1 \cdot x\mathrm{d}_{\ell+1}\,\mathrm{d}_i\,\mathrm{s}_{\ell-1} = x\mathrm{d}_i\,\mathrm{d}_\ell\,\mathrm{s}_{\ell-1} = 1\,.$$

If $i = \ell + 1$, then we have

$$\tilde{x}\mathrm{d}_i = x^{-}\mathrm{d}_{\ell+1} \cdot x\mathrm{d}_{\ell+1}\,\mathrm{s}_\ell\,\mathrm{d}_{\ell+1} = (x\mathrm{d}_{\ell+1})^{-} \cdot x\mathrm{d}_{\ell+1} = 1\,.$$

If $i \geqslant \ell + 2$, then we have

$$\tilde{x}\mathrm{d}_i = x^{-}\mathrm{d}_i \cdot x\mathrm{d}_{\ell+1}\,\mathrm{s}_\ell\,\mathrm{d}_i = 1 \cdot x\mathrm{d}_{\ell+1}\,\mathrm{d}_{i-1}\,\mathrm{s}_\ell = x\mathrm{d}_i\,\mathrm{d}_{\ell+1}\,\mathrm{s}_\ell = 1\,.$$

This shows $\tilde{x} \in G_{z+1,i}$ for $i \in \mathbb{Z}_{\geqslant 0}\backslash\{\ell\}$. So $\tilde{x} \in \bigcap\limits_{i\in\mathbb{Z}_{\geqslant 0}\backslash\{\ell\}} G_{z+1,i}$.

We show that $\tilde{x}\mathrm{d}_\ell \overset{!}{=} y$.

We have

$$\tilde{x}\mathrm{d}_\ell = x^{-}\mathrm{d}_\ell \cdot x\mathrm{d}_{\ell+1}\,\mathrm{s}_\ell\,\mathrm{d}_\ell = 1 \cdot x\mathrm{d}_{\ell+1} = y\,.$$

Altogether, we have

$$y = \tilde{x}\mathrm{d}_\ell \in \big(\bigcap_{i\in\mathbb{Z}_{\geqslant 0}\backslash\{\ell\}} G_{z+1,i}\big)\mathrm{d}_\ell = U_\ell\,.$$

So $U_{\ell+1} \subseteq U_\ell$.

This shows $U_\ell = U_{\ell+1}$. □

Lemma 203 *Suppose given a prestable simplicial group* $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$.

Let $z \in \mathbb{Z}$. *Suppose given* x, $y \in G_z$.

For $i \in \mathbb{Z}_{\geqslant 0}$, *suppose that we have*

$$x\mathrm{d}_i^{G,\,z} = 1 \quad \textit{or} \quad y\mathrm{d}_i^{G,\,z} = 1\,.$$

Moreover, suppose that there exists an $\ell \in \mathbb{Z}_{\geqslant 0}$ *such that*

$$x\mathrm{d}_{\ell+1}^{G,\,z} = 1 \quad \textit{and} \quad y\mathrm{d}_\ell^{G,\,z} = 1\,.$$

Then we have $[x, y] \in G\mathrm{NB}_z$.

Proof. We choose $\ell \in \mathbb{Z}_{\geqslant 0}$ with $x\mathrm{d}_{\ell+1} = 1$ and $y\mathrm{d}_\ell = 1$. Let $c := [x\mathrm{s}_\ell,\, y\mathrm{s}_{\ell+1}] \in G_{z+1}$.

We have

$$c\,\mathrm{d}_{\ell+1} = [x\mathrm{s}_\ell\,\mathrm{d}_{\ell+1},\, y\mathrm{s}_{\ell+1}\,\mathrm{d}_{\ell+1}] = [x, y]\,.$$

Let $i \in \mathbb{Z}_{\geqslant 0}\backslash\{\ell+1\}$.

If $i \leqslant \ell - 1$ then we have

$$c\mathrm{d}_i = [x\mathrm{s}_\ell\,\mathrm{d}_i,\, y\mathrm{s}_{\ell+1}\,\mathrm{d}_i] = [x\mathrm{d}_i\,\mathrm{s}_{\ell-1},\, y\mathrm{d}_i\,\mathrm{s}_\ell] = 1$$

since $x\mathrm{d}_i = 1$ or $y\mathrm{d}_i = 1$.

If $i = \ell$ then we have

$$c\mathrm{d}_\ell = [x\mathrm{s}_\ell\,\mathrm{d}_\ell,\, y\mathrm{s}_{\ell+1}\,\mathrm{d}_\ell] = [x,\, y\mathrm{d}_\ell\,\mathrm{s}_\ell] = [x, 1] = 1\,.$$

If $i = \ell + 2$ then we have

$$c\mathrm{d}_{\ell+2} = [x\mathrm{s}_\ell\,\mathrm{d}_{\ell+2},\, y\mathrm{s}_{\ell+1}\,\mathrm{d}_{\ell+2}] = [x\mathrm{d}_{\ell+1}\,\mathrm{s}_\ell,\, y] = [1, y] = 1\,.$$

If $i \geqslant \ell + 3$ then we have

$$c\mathrm{d}_i = [x\mathrm{s}_\ell\,\mathrm{d}_i,\, y\mathrm{s}_{\ell+1}\,\mathrm{d}_i] = [x\mathrm{d}_{i-1}\,\mathrm{s}_\ell,\, y\mathrm{d}_{i-1}\,\mathrm{s}_{\ell+1}] = 1$$

since $x\mathrm{d}_{i-1} = 1$ or $y\mathrm{d}_{i-1} = 1$.

This shows $c \in \bigcap_{i\in\mathbb{Z}_{\geqslant 0}\backslash\{\ell+1\}} G_{z+1,\,i}$.

Altogether, we have

$$[x, y] = c\mathrm{d}_{\ell+1} \in \big(\bigcap_{i\in\mathbb{Z}_{\geqslant 0}\backslash\{\ell+1\}} G_{z+1,\,i}\big)\mathrm{d}_{\ell+1} \overset{202}{=} G\mathrm{NB}_z\,.$$

□

9.8 The category of stable simplicial groups

Definition 204 Suppose given a prestable simplicial group $G \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$.

Suppose given $k \in \mathbb{Z}$. Let $g \in G_k$.

We say that g is *finite* if there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$g\,\mathrm{d}_i^{G,k} = 1 \quad \text{for } i \geqslant \ell\,.$$

Definition 205 (The category of stable simplicial groups)

We consider the category of prestable simplicial groups $\mathcal{PreStSimpGrp}$; cf. Definition 182.

Suppose given a prestable simplicial group $G \in \mathrm{Ob}(\mathcal{PreStSimpGrp})$.

Then G is called a *stable simplicial group* if it satisfies the following *finiteness condition* (1).

(1) For $k \in \mathbb{Z}$, each element $g \in G_k$ is finite.

Let $\mathcal{StSimpGrp} \subseteq \mathcal{PreStSimpGrp}$ be the full subcategory defined by

$$\mathrm{Ob}(\mathcal{StSimpGrp}) := \big\{ G \in \mathrm{Ob}(\mathcal{PreStSimpGrp}) \colon G \text{ is a stable simplicial group} \big\} .$$

We call $\mathcal{StSimpGrp}$ the *category of stable simplicial groups.*

A morphism $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}\big(\mathcal{StSimpGrp}\big)$ between stable simplicial groups G, H is called a *stable simplicial group morphism.*

Remark 206 Suppose given a stable simplicial group G; cf. Definition 205.

Suppose given $k \in \mathbb{Z}$. Suppose given $g \in G_k$. Then g is finite, i.e. there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_i^{G,k} = 1$ for $i \geqslant \ell$.

Remark 207 (Stable simplicial subgroup)

Suppose given a stable simplicial group $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

Let $H \leqslant G$ be a prestable simplicial subgroup; cf. Definition 188.

Then H is a stable simplicial group.

We say that H is a stable simplicial subgroup *of G.*

Proof. Let $z \in \mathbb{Z}$. Let $h \in H_z \leqslant G_z$. We may choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $h\mathrm{d}_i^{G,z} = 1$ for $i \in \mathbb{Z}_{\geqslant \ell}$.

For $i \in \mathbb{Z}_{\geqslant \ell}$, we have

$$h\mathrm{d}_i^{H,z} = h\Big(\mathrm{d}_i^{G,z}\Big|_{H_z}^{H_{z-1}}\Big) = 1 .$$

This shows that h is finite.

So H is a stable simplicial group. □

Lemma 208

(1) *Suppose given a prestable simplicial group G; cf Definition* 182.
For $z \in \mathbb{Z}$, we have the subgroup

$$G_z^{\mathrm{fin}} := \{g \in G_z \colon g \text{ is finite}\} \leqslant G_z \,.$$

We have the prestable simplicial subgroup $G^{\mathrm{fin}} \leqslant G$ given by

$$(G^{\mathrm{fin}})_z := G_z^{\mathrm{fin}} \quad \text{for } z \in \mathbb{Z}\,;$$

cf. Definition 188.
Moreover, G^{fin} is a stable simplicial group.

(2) *Suppose given a prestable simplicial group morphism $\varphi \colon G \to H$.*
For $z \in \mathbb{Z}$, we have the group morphism

$$\varphi_z^{\mathrm{fin}} := \varphi_z\big|_{G_z^{\mathrm{fin}}}^{H_z^{\mathrm{fin}}} \colon G_z^{\mathrm{fin}} \to H_z^{\mathrm{fin}}\,,\ g \mapsto g\varphi_z \,.$$

Moreover, we have the stable simplicial group morphism $\varphi^{\mathrm{fin}} \colon G^{\mathrm{fin}} \to H^{\mathrm{fin}}$, where

$$(\varphi^{\mathrm{fin}})_z := \varphi_z^{\mathrm{fin}} \quad \text{for } z \in \mathbb{Z}\,.$$

(3) *We have the functor*

$$\Phi \colon \mathcal{PreStSimpGrp} \to \mathcal{StSimpGrp}\,,\ \big(G \xrightarrow{\varphi} H\big) \mapsto \big(G\Phi \xrightarrow{\varphi\Phi} H\Phi\big) := \big(G^{\mathrm{fin}} \xrightarrow{\varphi^{\mathrm{fin}}} H^{\mathrm{fin}}\big)\,.$$

So, for $z \in \mathbb{Z}$, we have

$$(G\Phi)_z = G_z^{\mathrm{fin}}$$
$$(\varphi\Phi)_z = \varphi_z^{\mathrm{fin}} \colon G_z^{\mathrm{fin}} \to H_z^{\mathrm{fin}}\,,\ g \mapsto g\varphi_z \,.$$

Proof. Ad (1). Let $z \in \mathbb{Z}$. We have $G_z^{\mathrm{fin}} \neq \emptyset$ since $1 \in G_z^{\mathrm{fin}}$.
Suppose given g, $h \in G_z^{\mathrm{fin}}$.
We may choose some $\ell_1 \in \mathbb{Z}_{\geqslant 0}$ such that $g\,\mathrm{d}_i = 1$ for $i \geqslant \ell_1$, and we may choose some $\ell_2 \in \mathbb{Z}_{\geqslant 0}$ such that $h\,\mathrm{d}_j = 1$ for $j \geqslant \ell_2$.
For $i \in \mathbb{Z}_{\geqslant 0}$ with $i \geqslant \max\{\ell_1, \ell_2\}$, we have

$$(g \cdot h^-)\mathrm{d}_i = g\,\mathrm{d}_i \cdot (h\,\mathrm{d}_i)^- = 1\,.$$

This shows $g \cdot h^- \in G_z^{\mathrm{fin}}$.

So $G_z^{\mathrm{fin}} \leqslant G_z$ is a subgroup.

Let $i \in \mathbb{Z}_{\geqslant 0}$. Suppose given $g \in G_z^{\mathrm{fin}}$. We may choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_j^{G,z} = 1$ for $j \in \mathbb{Z}_{\geqslant \ell}$.

For $j \geqslant \max\{\ell - 1, i\}$, we have

$$g\mathrm{d}_i^{G,z}\,\mathrm{d}_j^{G,z-1} = g\mathrm{d}_{j+1}^{G,z}\,\mathrm{d}_i^{G,z-1} = 1\,.$$

This shows that $g\mathrm{d}_i^{G,z} \in G_{z-1}^{\mathrm{fin}}$. So $(G_z^{\mathrm{fin}})\mathrm{d}_i^{G,z} \leqslant G_{z-1}^{\mathrm{fin}}$.

For $j \geqslant \max\{\ell + 1, i + 2\}$, we have

$$g\mathrm{s}_i^{G,z}\,\mathrm{d}_j^{G,z+1} = g\mathrm{d}_{j-1}^{G,z}\,\mathrm{s}_i^{z-1} = 1\,.$$

This shows that $g\mathrm{s}_i^{G,z} \in G_{z+1}^{\mathrm{fin}}$. So $(G_z^{\mathrm{fin}})\mathrm{s}_i^{G,z} \leqslant G_{z+1}$.

Therefore $G^{\mathrm{fin}} \leqslant G$ is a prestable simplicial subgroup; cf. Lemma 189.

By construction, the prestable simplicial group G^{fin} is a stable simplicial group.

Ad (2). Suppose given $z \in \mathbb{Z}$. We have to show that $(G_z^{\mathrm{fin}})\varphi_z \overset{!}{\leqslant} H_z^{\mathrm{fin}}$.

Suppose given $g \in G_z^{\mathrm{fin}}$. We may choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $g\,\mathrm{d}_i^{G,z} = 1$ for $i \in \mathbb{Z}_{\geqslant \ell}$.

For $i \in \mathbb{Z}_{\geqslant \ell}$, we have

$$g\varphi_z\,\mathrm{d}_i^{H,z} = g\,\mathrm{d}_i^{G,z}\,\varphi_{z-1} = 1\,.$$

This shows that $g\varphi_z \in H_z^{\mathrm{fin}}$. So $(G_z^{\mathrm{fin}})\varphi_z \leqslant H_z^{\mathrm{fin}}$.

Therefore, the group morphism $\varphi_z\big|_{G_z^{\mathrm{fin}}}^{H_z^{\mathrm{fin}}} \colon G_z^{\mathrm{fin}} \to H_z^{\mathrm{fin}}\,,\ g \mapsto g\varphi_z$ exists.

Note that $\varphi^{\mathrm{fin}} \colon G^{\mathrm{fin}} \to H^{\mathrm{fin}}$ is a stable simplicial group morphism; cf. Remark 190, Definition 205.

Ad (3). Suppose given $F \xrightarrow{\varphi} G \xrightarrow{\varphi'} H$ in $\mathcal{PreStSimpGrp}$.

We have

$$\mathrm{id}_G\,\Phi = (\mathrm{id}_{G_z})_{z\in\mathbb{Z}}\,\Phi = \big(\,\mathrm{id}_{G_z}\,\big|_{G_z^{\mathrm{fin}}}^{G_z^{\mathrm{fin}}}\big)_{z\in\mathbb{Z}} = (\mathrm{id}_{G_z^{\mathrm{fin}}})_{z\in\mathbb{Z}} = \mathrm{id}_{G^{\mathrm{fin}}} = \mathrm{id}_{G\Phi}\ .$$

We have

$$\begin{aligned}(\varphi \mathbin{\triangle} \varphi')\Phi &= \big((\varphi_z \mathbin{\triangle} \varphi'_z)\big|_{F_z^{\mathrm{fin}}}^{H_z^{\mathrm{fin}}}\big)_{z\in\mathbb{Z}} = \big(\varphi_z\big|_{F_z^{\mathrm{fin}}}^{G_z^{\mathrm{fin}}} \mathbin{\triangle} \varphi'_z\big|_{G_z^{\mathrm{fin}}}^{H_z^{\mathrm{fin}}}\big)_{z\in\mathbb{Z}} = \big(\varphi_z\big|_{F_z^{\mathrm{fin}}}^{G_z^{\mathrm{fin}}}\big)_{z\in\mathbb{Z}} \mathbin{\triangle} \big(\varphi'_z\big|_{G_z^{\mathrm{fin}}}^{H_z^{\mathrm{fin}}}\big)_{z\in\mathbb{Z}} \\ &= \varphi\Phi \mathbin{\triangle} \varphi'\Phi\,.\end{aligned}$$

So Φ is a functor. □

Remark 209 We consider the embedding functor

$$\mathrm{I}\colon \mathcal{StSimpGrp} \to \mathcal{PreStSimpGrp}\,,\ \big(G \xrightarrow{\varphi} G'\big) \mapsto \big(G\mathrm{I} \xrightarrow{\varphi\mathrm{I}} G'\mathrm{I}\big) = \big(G \xrightarrow{\varphi} G'\big)\,.$$

We consider the functor $\Phi\colon \mathcal{PreStSimpGrp} \to \mathcal{StSimpGrp}$ from Lemma 208.(3).

For a stable simplicial group G, note that we have $G\mathrm{I}\Phi = (G\mathrm{I})^{\mathrm{fin}} = G$; cf. Lemma 208.(1).

For a stable simplicial group morphism $\varphi\colon G \to H$, note that we have $\varphi\mathrm{I}\Phi = (\varphi\mathrm{I})^{\mathrm{fin}} = \varphi\colon G \to H$; cf. Lemma 208.(2).

So $\mathrm{I} * \Phi = \mathrm{id}_{\mathcal{StSimpGrp}}$.

Lemma 210 (The adjoint functors I and Φ)

We consider the embedding functor

$$\mathrm{I}\colon \mathcal{StSimpGrp} \to \mathcal{PreStSimpGrp}\,,\ \big(G \xrightarrow{\varphi} \tilde{G}\big) \mapsto \big(G \xrightarrow{\varphi} \tilde{G}\big)$$

from Remark 209.

We consider the functor

$$\Phi\colon \mathcal{PreStSimpGrp} \to \mathcal{StSimpGrp}\,,\ \big(H \xrightarrow{\psi} \tilde{H}\big) \mapsto \big(H^{\mathrm{fin}} \xrightarrow{\psi^{\mathrm{fin}}} \tilde{H}^{\mathrm{fin}}\big)$$

from Lemma 208.(3).

(1) *We have the transformation*

$$\varepsilon = \big(G \xrightarrow{G\varepsilon} G\big)_{G\in\mathrm{Ob}(\mathcal{StSimpGrp})}\colon\ \mathrm{id}_{\mathcal{StSimpGrp}} \to \mathrm{I} * \Phi\,;$$

where

$$G\varepsilon := \mathrm{id}_G\colon G \to G\,,$$

for $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

(2) *We have the transformation*

$$\eta = \big(H^{\mathrm{fin}} \xrightarrow{H\eta} H\big)_{H\in\mathrm{Ob}(\mathcal{PreStSimpGrp})}\colon \Phi * \mathrm{I} \to \mathrm{id}_{\mathcal{PreStSimpGrp}}\,,$$

where

$$H\eta := \iota_{H^{\mathrm{fin}},H} = \mathrm{id}_H\big|_{H^{\mathrm{fin}}}\colon H^{\mathrm{fin}} \to H\,,$$

for $H \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$.

(3) *We have the adjunction* $(\mathrm{I}, \Phi, \varepsilon, \eta)$*; cf.* §1.1.3 *item* 10. *In particular, we have* $\mathrm{I} \dashv \Phi$.

Proof. Ad (1). *We show that* ε *is a transformation.*

Suppose given $G \xrightarrow{\varphi} \tilde{G}$ in $\mathcal{StSimpGrp}$.

We have

$$G\varepsilon \blacktriangle \varphi(\mathrm{I} * \Phi) \overset{209}{=} \mathrm{id}_G \blacktriangle \varphi = \varphi = \varphi \blacktriangle \mathrm{id}_{\tilde{G}} = \varphi \blacktriangle \tilde{G}\varepsilon \,.$$

$$\begin{array}{ccc} G & \xrightarrow{G\varepsilon} & G(\mathrm{I} * \Phi) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi(\mathrm{I} * \Phi)} \\ \tilde{G} & \xrightarrow[\tilde{G}\varepsilon]{} & \tilde{G}(\mathrm{I} * \Phi) \end{array}$$

So ε is a transformation.

Ad (2). *We show that* η *is a transformation.*

Suppose given $H \xrightarrow{\varphi} \tilde{H}$ in $\mathcal{PreStSimpGrp}$. Suppose given $z \in \mathbb{Z}$.

For $h \in (H\Phi\mathrm{I})_z = H_z^{\mathrm{fin}}$, we have

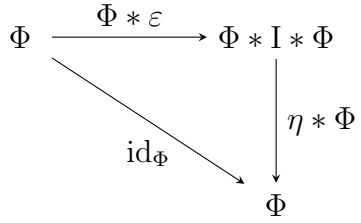
$$h(H\eta \blacktriangle \varphi)_z = h(\iota_{H^{\mathrm{fin}}, H} \blacktriangle \varphi)_z = h\varphi_z \overset{208.(2)}{=} h\big((\varphi^{\mathrm{fin}})\mathrm{I}\big)_z = h\big((\varphi^{\mathrm{fin}})\mathrm{I} \blacktriangle \iota_{\tilde{H}^{\mathrm{fin}}, \tilde{H}}\big)_z = h\big(\varphi(\Phi * \mathrm{I}) \blacktriangle \tilde{H}\eta\big)_z \,.$$

This shows $H\eta \blacktriangle \varphi = \varphi(\Phi * \mathrm{I}) \blacktriangle \tilde{H}\eta$.

$$\begin{array}{ccc} H(\Phi * \mathrm{I}) & \xrightarrow{H\eta} & H \\ {\scriptstyle\varphi(\Phi * \mathrm{I})}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \tilde{H}(\Phi * \mathrm{I}) & \xrightarrow[\tilde{H}\eta]{} & \tilde{H} \end{array}$$

So η is a transformation.

Ad (3). *We show that* $(\Phi * \varepsilon) \blacktriangle (\eta * \Phi) \overset{!}{=} \mathrm{id}_\Phi$.



Suppose given $H \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$.

We show that $H\big((\Phi * \varepsilon) \blacktriangle (\eta * \Phi)\big) \overset{!}{=} \mathrm{id}_{H\Phi}$.

Suppose given $z \in \mathbb{Z}$. Suppose given $h \in (H\Phi)_z = H_z^{\mathrm{fin}}$.

We have

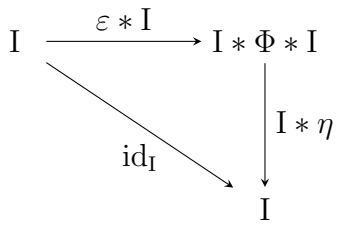
$$\begin{aligned}
h\Big(H\big((\Phi * \varepsilon) \blacktriangle (\eta * \Phi)\big)\Big)_z &= h\big(H(\Phi * \varepsilon) \blacktriangle H(\eta * \Phi)\big)_z \\
&= h\Big(\big(H(\Phi * \varepsilon)\big)_z \blacktriangle \big(H(\eta * \Phi)\big)_z\Big) \\
&\overset{(2)}{=} h\Big(\big((H^{\mathrm{fin}})\varepsilon\big)_z \blacktriangle \big((\iota_{H^{\mathrm{fin}},H})\Phi\big)_z\Big) \\
&\overset{(1)}{=} h\big(\mathrm{id}_{H_z^{\mathrm{fin}}} \blacktriangle (\iota_{H^{\mathrm{fin}},H})_z^{\mathrm{fin}}\big) \\
&= h\big((\iota_{H^{\mathrm{fin}},H})_z^{\mathrm{fin}}\big) \\
&\overset{208.(2)}{=} h \\
&= h\,\mathrm{id}_{H_z^{\mathrm{fin}}} \\
&= h\,\mathrm{id}_{(H\Phi)_z} \ .
\end{aligned}$$

This shows that $\Big(H\big((\Phi * \varepsilon) \blacktriangle (\eta * \Phi)\big)\Big)_z = \mathrm{id}_{(H\Phi)_z}$.

So we have $H\big((\Phi * \varepsilon) \blacktriangle (\eta * \Phi)\big) = \mathrm{id}_{H\Phi}$.

Therefore, we have $(\Phi * \varepsilon) \blacktriangle (\eta * \Phi) = \mathrm{id}_\Phi$.

We show that $(\varepsilon * \mathrm{I}) \blacktriangle (\mathrm{I} * \eta) \overset{!}{=} \mathrm{id}_{\mathrm{I}}$.



Suppose given $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

We show that $G\big((\varepsilon * \mathrm{I}) \blacktriangle (\mathrm{I} * \eta)\big) \overset{!}{=} \mathrm{id}_{G\mathrm{I}}$.

Suppose given $z \in \mathbb{Z}$. Suppose given $g \in (G\mathrm{I})_z$.

We have

$$\begin{aligned}
g\Big(G\big((\varepsilon * \mathrm{I}) \blacktriangle (\mathrm{I} * \eta)\big)\Big)_z &= g\big(G(\varepsilon * \mathrm{I}) \blacktriangle G(\mathrm{I} * \eta)\big)_z \\
&= g\Big(\big(G(\varepsilon * \mathrm{I})\big)_z \blacktriangle \big(G(\mathrm{I} * \eta)\big)_z\Big) \\
&\overset{(1)}{=} g\Big(\big((\mathrm{id}_G)\mathrm{I}\big)_z \blacktriangle \big((G\mathrm{I})\eta\big)_z\Big) \\
&\overset{(2)}{=} g\big(\iota_{(G\mathrm{I})^{\mathrm{fin}},G\mathrm{I}}\big)_z \\
&= g \\
&= g\,\mathrm{id}_{(G\mathrm{I})_z} \ .
\end{aligned}$$

This shows that $\Big(G\big((\varepsilon * \mathrm{I}) \blacktriangle (\mathrm{I} * \eta)\big)\Big)_z = \mathrm{id}_{(G\mathrm{I})_z}$.

So we have $G\big((\varepsilon * \mathrm{I}) \blacktriangle (\mathrm{I} * \eta)\big) = \mathrm{id}_{G\mathrm{I}}$.

Therefore, we have $(\varepsilon * \mathrm{I}) \blacktriangle (\mathrm{I} * \eta) = \mathrm{id}_{\mathrm{I}}$.

Altogether, this proves (3). □

Remark 211 *We consider the adjunction* $(\mathrm{I}, \Phi, \varepsilon, \eta)$ *from Lemma* 210.

For $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ *and* $H \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$*, we have the adjunction bijection*

$$\begin{aligned}
\gamma_{G,H} \colon \quad {}_{\mathcal{PreStSimpGrp}}(G\mathrm{I},\, H) &\overset{\sim}{\leftrightarrow} {}_{\mathcal{StSimpGrp}}(G,\, H\Phi) \\
\big(G\mathrm{I} \xrightarrow{\varphi} H\big) &\mapsto \big(G \xrightarrow{\varphi\big|^{H\Phi}} H\Phi\big) \\
\big(G\mathrm{I} \xrightarrow{\psi\mathrm{I} \blacktriangle \iota_{H\Phi,H}} H\big) &\mapsto \big(G \xrightarrow{\psi} H\Phi\big) \ .
\end{aligned}$$

So, for $\big(G\mathrm{I} \xrightarrow{\varphi} H\big) \in {}_{\mathcal{PreStSimpGrp}}(G\mathrm{I},\, H)$, $k \in \mathbb{Z}$ *and* $g \in G_k$, *we have*

$$g(\varphi\gamma_{G,H})_k = g\varphi_k\,.$$

For $\big(G \xrightarrow{\psi} H\Phi\big) \in {}_{\mathcal{StSimpGrp}}(G,\, H\Phi)$, $k \in \mathbb{Z}$ *and* $g \in (G\mathrm{I})_k = G_k$, *we have*

$$g(\psi\gamma^{-}_{G,H})_k = g\psi_k\,.$$

Proof. We use Remark 14 to show this claim.
Suppose given $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ and $H \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$.
Since $(\mathrm{I}, \Phi, \varepsilon, \eta)$ is an adjunction, Remark 14 yields the isomorphism

$$\gamma_{G,H} \colon {}_{\mathcal{PreStSimpGrp}}(G\mathrm{I},\, H) \xrightarrow[\sim]{} {}_{\mathcal{StSimpGrp}}(G,\, H\Phi)\,,\ \big(G\mathrm{I} \xrightarrow{\varphi} H\big) \mapsto \varphi\gamma_{G,H} = \big(G \xrightarrow{G\varepsilon \mathbin{\blacktriangle} \varphi\Phi} H\Phi\big)\,.$$

Moreover, we have

$$\gamma^{-}_{G,H} \colon {}_{\mathcal{StSimpGrp}}(G,\, H\Phi) \xrightarrow[\sim]{} {}_{\mathcal{PreStSimpGrp}}(G\mathrm{I},\, H)\,,\ \big(G \xrightarrow{\psi} H\Phi\big) \mapsto \psi\gamma^{-}_{G,H} = \big(G\mathrm{I} \xrightarrow{\psi\mathrm{I} \mathbin{\blacktriangle} H\eta} H\big)\,.$$

Suppose given a morphism $\big(G\mathrm{I} \xrightarrow{\varphi} H\big) \in {}_{\mathcal{PreStSimpGrp}}(G\mathrm{I},\, H)$.
We show that $\varphi\gamma_{G,H} \overset{!}{=} \varphi\big|^{H\Phi} \colon G \to H\Phi$.
Let $k \in \mathbb{Z}$. Suppose given $g \in G_k$. We have

$$g(\varphi\gamma_{G,H})_k = g(G\varepsilon \mathbin{\blacktriangle} \varphi\Phi)_k = g\big((G\varepsilon)_k \mathbin{\blacktriangle} (\varphi\Phi)_k\big) \overset{210.(1)}{=} (g\,\mathrm{id}_{G_k})\varphi_k^{\mathrm{fin}} \overset{208.(2)}{=} g\varphi_k = g\varphi_k\big|^{(H\Phi)_k}\,.$$

This shows that $(\varphi\gamma_{G,H})_k = \varphi_k\big|^{(H\Phi)_k}$, and so $\varphi\gamma_{G,H} = \varphi\big|^{H\Phi}$.

Suppose given a morphism $\big(G \xrightarrow{\psi} H\Phi\big) \in {}_{\mathcal{StSimpGrp}}(G,\, H\Phi)$.
We show that $\varphi\gamma^{-}_{G,H} \overset{!}{=} \psi \mathbin{\blacktriangle} \iota_{H,H\Phi} \colon G\mathrm{I} \to H$.
Let $k \in \mathbb{Z}$. Suppose given $g \in (G\mathrm{I})_k = G_k$. We have

$$\begin{aligned} g(\psi\gamma^{-}_{G,H})_k &= g(\psi\mathrm{I} \mathbin{\blacktriangle} H\eta)_k = g\big((\psi\mathrm{I})_k \mathbin{\blacktriangle} (H\eta)_k\big) \overset{209}{=} (g\psi_k)(H\eta)_k \overset{210.(2)}{=} (g\psi_k)\iota_{H_k,(H\Phi)_k} \\ &= g\psi_k = g(\psi\mathrm{I})_k = g(\psi\mathrm{I} \mathbin{\blacktriangle} \iota_{H,H\Phi})_k\,. \end{aligned}$$

This shows that $(\psi\gamma^{-}_{G,H})_k = (\psi\mathrm{I} \mathbin{\blacktriangle} \iota_{H,H\Phi})_k$, and so $\psi\gamma^{-}_{G,H} = \psi\mathrm{I} \mathbin{\blacktriangle} \iota_{H,H\Phi}$.
Altogether, this proves the claim. □

Remark 212 *Suppose given a stable simplicial group* $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$*; cf. Definition* 205*. Suppose given* $n \in \mathbb{Z}$*.*

We consider the functor $\mathrm{Dec}^{\pm,n} \colon \mathcal{PreStSimpGrp} \to \mathcal{PreStSimpGrp}$ *from Lemma* 195*.*

Then $G\mathrm{Dec}^{\pm,n}$ *is a stable simplicial group.*

Proof. Let $k \in \mathbb{Z}$. Suppose given $g \in (G\mathrm{Dec}^{\pm,n})_k = G_{k+n}$. Then we may choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_i^{G,\,k+n} = 1$ for $i \geqslant \ell$.

For $i \geqslant \ell$, we have

$$g\mathrm{d}_i^{G\mathrm{Dec}^{\pm,n},k} \overset{196}{=} g\mathrm{d}_i^{G,\,k+n} = 1\,.$$

This shows that g is finite.

So $G\mathrm{Dec}^{\pm,n}$ is a stable simplicial group. □

Remark 213 We given an overview of the notations used for face and degeneracy operations in various contexts.

In the simplex category $\mathbf{\Delta}$ we have the morphisms

$$\begin{aligned} &\delta_i^n \colon [n-1] \to [n] \quad \text{for } n \in \mathbb{Z}_{\geqslant 1} \text{ and } i \in [0,n] \\ &\sigma_i^n \colon [n+1] \to [n] \quad \text{for } n \in \mathbb{Z}_{\geqslant 0} \text{ and } i \in [0,n]\,; \end{aligned}$$

cf. Definitions 18, 19.

For a simplicial group $G \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$, we have

$$\begin{aligned} &\text{the face map} && \mathrm{d}_i^{G,n} = \big((\delta_i^n)^{\mathrm{op}}\big)G \colon G_n \to G_{n-1} && \text{for } n \in \mathbb{Z}_{\geqslant 1} \text{ and } i \in [0,n] \\ &\text{the degeneracy map} && \mathrm{s}_i^{G,n} = \big((\sigma_i^n)^{\mathrm{op}}\big)G \colon G_n \to G_{n+1} && \text{for } n \in \mathbb{Z}_{\geqslant 0} \text{ and } i \in [0,n]\,; \end{aligned}$$

cf. Definitions 30, 31.

In the prestable simplex category $\mathbf{\Delta}^{\pm}$ we have the morphisms

$$\begin{aligned} &\delta_i^{\pm,z} \colon [z-1]^{\pm} \to [z]^{\pm} \quad \text{for } z \in \mathbb{Z} \text{ and } i \in \mathbb{Z}_{\geqslant 0} \\ &\sigma_i^{\pm,n} \colon [z+1]^{\pm} \to [z]^{\pm} \quad \text{for } z \in \mathbb{Z} \text{ and } i \in \mathbb{Z}_{\geqslant 0}\,; \end{aligned}$$

cf. Definitions 172, 174.

For a prestable simplicial group $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$, we have

$$\begin{aligned} &\text{the face map} && \mathrm{d}_i^{G,z} = \big((\delta_i^{z,\pm})^{\mathrm{op}}\big)G \colon G_z \to G_{z-1} && \text{for } z \in \mathbb{Z} \text{ and } i \in \mathbb{Z}_{\geqslant 0} \\ &\text{the degeneracy map} && \mathrm{s}_i^{G,z} = \big((\sigma_i^{z,\pm})^{\mathrm{op}}\big)G \colon G_z \to G_{z+1} && \text{for } z \in \mathbb{Z} \text{ and } i \in \mathbb{Z}_{\geqslant 0}\,; \end{aligned}$$

cf. Definitions 182, 183.

Chapter 10

Simplicial groups and stable simplicial groups

10.1 The functor $\hat{\mathrm{Ord}}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$

Lemma 214 (The simplicial group $G\hat{\mathrm{Ord}}$)
Suppose given a stable simplicial group $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$; cf. Definition 205.
We consider the simplicial group $G\mathrm{P}$; cf. Remark 193.
We may define a simplicial subgroup $G\hat{\mathrm{Ord}} \leqslant G\mathrm{P}$ by letting, for $k \in \mathbb{Z}_{\geqslant 0}$,

$$[k]G\hat{\mathrm{Ord}} = G\hat{\mathrm{Ord}}_k := \bigcap_{i \in \mathbb{Z}_{\geqslant k+1}} G_{k,i} \leqslant G_k\,;$$

cf. Definition 198.(1). *In particular, $G\hat{\mathrm{Ord}}$ is a simplicial group.*
For $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,k]$, we have

$$\mathrm{d}_j^{G\hat{\mathrm{Ord}},k} = \mathrm{d}_j^{G,k}\big|_{G\hat{\mathrm{Ord}}_k}^{G\hat{\mathrm{Ord}}_{k-1}}\,.$$

For $k \in \mathbb{Z}_{\geqslant 0}$ and $j \in [0,k]$, we have

$$\mathrm{s}_i^{G\hat{\mathrm{Ord}},k} = \mathrm{s}_j^{G,k}\big|_{G\hat{\mathrm{Ord}}_k}^{G\hat{\mathrm{Ord}}_{k+1}}\,.$$

Proof. Suppose given $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,k]$.

We show that $(G\hat{\mathrm{Ord}}_k)\mathrm{d}_j^{G,k} \overset{!}{\leqslant} G\hat{\mathrm{Ord}}_{k-1}$.

Suppose given $g \in G\hat{\mathrm{Ord}}_k = \bigcap_{i \geqslant k+1} G_{k,i}$ and suppose given $\ell \in \mathbb{Z}_{\geqslant k}$.

Note that $j < \ell + 1$ and that $k + 1 \leqslant \ell + 1$. So we have

$$g\mathrm{d}_j^{G,k}\mathrm{d}_\ell^{G,k-1} \overset{185.(1)}{=} g\mathrm{d}_{\ell+1}^{G,k}\,\mathrm{d}_j^{G,k-1} = 1\mathrm{d}_j^{G,k-1} = 1\,.$$

This shows $g\mathrm{d}_j^{G,k} \in \bigcap_{i \geqslant k} G_{k,i} = G\hat{\mathrm{Ord}}_{k-1}$.

So $\left(G\hat{\mathrm{Ord}}_k\right)\mathrm{d}_j^{G,k} \leqslant G\hat{\mathrm{Ord}}_{k-1}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$ and $j \in [0, k]$.

We show that $\left(G\hat{\mathrm{Ord}}_k\right)\mathrm{s}_j^{G,k} \overset{!}{\leqslant} G\hat{\mathrm{Ord}}_{k+1}$.

Suppose given $g \in G\hat{\mathrm{Ord}}_k = \bigcap_{i \geqslant k+1} G_{k,i}$ and suppose given $\ell \in \mathbb{Z}_{\geqslant k+2}$.

Note that $\ell > j + 1$ and that $k + 1 \leqslant \ell - 1$. So we have

$$g\mathrm{s}_j^{G,k}\mathrm{d}_\ell^{G,k+1} \overset{185.(3)}{=} g\mathrm{d}_{\ell-1}^{G,k}\,\mathrm{s}_j^{G,k-1} = 1\mathrm{s}_j^{G,k-1} = 1\,.$$

This show $g\mathrm{s}_j^{G,k} \in \bigcap_{i \geqslant k+2} G_{k,i} = G\hat{\mathrm{Ord}}_{k+1}$.

So $\left(G\hat{\mathrm{Ord}}_k\right)\mathrm{s}_j^{G,k} \leqslant G\hat{\mathrm{Ord}}_{k+1}$.

We consider the simplicial group $G\mathrm{P}$; cf. Remark 193.

For $k \in \mathbb{Z}_{\geqslant 0}$, recall that $(G\mathrm{P})_k = G_k$, that $\mathrm{d}_j^{G\mathrm{P},k+1} = \mathrm{d}_j^{G,k+1} \colon G_{k+1} \to G_k$ for $j \in [0, k + 1]$ and that $\mathrm{s}_j^{G\mathrm{P},k} = \mathrm{s}_j^{G,k} \colon G_k \to G_{k+1}$ for $j \in [0, k]$.

We show that $G\hat{\mathrm{Ord}} \overset{!}{\leqslant} G\mathrm{P}$ *is a simplicial subgroup.*

We use Lemma 41.(2) to show this claim.

Let $k \in \mathbb{Z}_{\geqslant 1}$ and let $j \in [0, k]$. We have

$$(G\hat{\mathrm{Ord}}_k)\mathrm{d}_j^{G\mathrm{P},k} \overset{193}{=} (G\hat{\mathrm{Ord}}_k)\mathrm{d}_j^{G,k} \leqslant G\hat{\mathrm{Ord}}_{k-1}\,.$$

Let $k \in \mathbb{Z}_{\geqslant 0}$ and let $i \in [0, k]$. We have

$$(G\hat{\mathrm{Ord}}_k)\mathrm{s}_i^{G\mathrm{P},k} \overset{193}{=} (G\hat{\mathrm{Ord}}_k)\mathrm{s}_i^{G,k} \leqslant G\hat{\mathrm{Ord}}_{k-1}\,.$$

Then, by Lemma 41.(2), we have $G\hat{\mathrm{Ord}} \leqslant G\mathrm{P}$.

In particular, for $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,k]$, we have

$$\mathrm{d}_j^{G\hat{\mathrm{Ord}},k} = \mathrm{d}_j^{G\mathrm{P},k}\Big|_{(G\mathrm{P})_k}^{(G\mathrm{P})_{k-1}} \overset{193}{=} \mathrm{d}_j^{G,k}\Big|_{G_k}^{G_{k-1}},$$

and, for $k \in \mathbb{Z}_{\geqslant 0}$ and $j \in [0,k]$, we have

$$\mathrm{s}_j^{G\hat{\mathrm{Ord}},k} = \mathrm{s}_j^{G\mathrm{P},k}\Big|_{(G\mathrm{P})_k}^{(G\mathrm{P})_{k+1}} \overset{193}{=} \mathrm{s}_j^{G,k}\Big|_{G_k}^{G_{k+1}}.$$

□

Lemma 215 (The simplicial group morphism $\varphi\hat{\mathrm{Ord}}$)
Suppose given stable simplicial groups G, $H \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.
Suppose given a stable simplicial group morphism $\varphi\colon G \to H$.
We consider the simplicial groups $G\hat{\mathrm{Ord}}$ and $H\hat{\mathrm{Ord}}$ from Lemma 214.
We have the simplicial group morphism

$$\varphi\hat{\mathrm{Ord}}\colon G\hat{\mathrm{Ord}} \to H\hat{\mathrm{Ord}},$$

where

$$\varphi\hat{\mathrm{Ord}}_k := \varphi_k\big|_{G\hat{\mathrm{Ord}}_k}^{H\hat{\mathrm{Ord}}_k}\colon G\hat{\mathrm{Ord}}_k \to H\hat{\mathrm{Ord}}_k,\ g \mapsto g(\varphi\hat{\mathrm{Ord}}_k) := g\varphi_k$$

for $k \in \mathbb{Z}_{\geqslant 0}$.

Proof. We apply Remark 42 to show this claim.
Recall that $\big((G\mathrm{P})_k \xrightarrow{(\varphi\mathrm{P})_k} (H\mathrm{P})_k\big) = \big(G_k \xrightarrow{\varphi_k} H_k\big)$ for $k \in \mathbb{Z}_{\geqslant 0}$.
Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. We have to show that $(G\hat{\mathrm{Ord}}_k)\varphi_k \overset{!}{\leqslant} H\hat{\mathrm{Ord}}_k$.
Suppose given $g \in G\hat{\mathrm{Ord}}_k = \bigcap\limits_{i \geqslant k+1} G_{k,i} \leqslant G_k$.
For $\ell \in \mathbb{Z}_{\geqslant k+1}$, we have

$$(g\varphi_k)\mathrm{d}_\ell^{H,k} = g(\varphi_k \blacktriangle \mathrm{d}_\ell^{H,k}) \overset{184.(2)}{=} g(\mathrm{d}_\ell^{G,k} \blacktriangle \varphi_{k-1}) = 1\varphi_{k-1} = 1.$$

This shows $g\varphi_k \in \bigcap\limits_{i \geqslant k+1} H_{k,i} = H\hat{\mathrm{Ord}}_k$.
So we have $(G\hat{\mathrm{Ord}}_k)\varphi_k \leqslant H\hat{\mathrm{Ord}}_k$.

Therefore, by Remark 42, we have the group morphism $\varphi\hat{\mathrm{Ord}} = \varphi\mathrm{P}\big|^{H\hat{\mathrm{Ord}}}_{G\hat{\mathrm{Ord}}}$.

For $k \in \mathbb{Z}_{\geqslant 0}$, we obtain

$$\varphi\hat{\mathrm{Ord}}_k = (\varphi\mathrm{P})_k\big|^{H\hat{\mathrm{Ord}}_k}_{G\hat{\mathrm{Ord}}_k} = \varphi_k\big|^{H\hat{\mathrm{Ord}}_k}_{G\hat{\mathrm{Ord}}_k},$$

as claimed. □

Lemma 216 (The functor $\hat{\mathrm{Ord}}$)

We have the functor from the category of stable simplicial groups to the category of simplicial groups given by

$$\hat{\mathrm{Ord}}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp},\ (G \xrightarrow{\varphi} H) \mapsto (G \xrightarrow{\varphi} H)\hat{\mathrm{Ord}} := (G\hat{\mathrm{Ord}} \xrightarrow{\varphi\hat{\mathrm{Ord}}} H\hat{\mathrm{Ord}});$$

cf. Lemmas 214, 215.

Proof. Suppose given $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $g \in G\hat{\mathrm{Ord}}_k$.

We have

$$g\big((\mathrm{id}_G)\hat{\mathrm{Ord}}\big)_k = g\,\mathrm{id}_{G_k} = g\,\mathrm{id}_{(G\hat{\mathrm{Ord}})_k}.$$

This shows $\big((\mathrm{id}_G)\hat{\mathrm{Ord}}\big)_k = \mathrm{id}_{(G\hat{\mathrm{Ord}})_k} = (\mathrm{id}_{G\hat{\mathrm{Ord}}})_k$.

So $(\mathrm{id}_G)\hat{\mathrm{Ord}} = \mathrm{id}_{G\hat{\mathrm{Ord}}}$.

Suppose given $F \xrightarrow{\varphi} G \xrightarrow{\tilde{\varphi}} H$ in $\mathcal{StSimpGrp}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $x \in F\hat{\mathrm{Ord}}_k$.

We have

$$x\big(\varphi\hat{\mathrm{Ord}}_k \vartriangle \tilde{\varphi}\hat{\mathrm{Ord}}_k\big) = (x\varphi_k)(\tilde{\varphi}\hat{\mathrm{Ord}}_k) = x\varphi_k\tilde{\varphi}_k = x(\varphi \vartriangle \tilde{\varphi})_k = x\big((\varphi \vartriangle \tilde{\varphi})\hat{\mathrm{Ord}}_k\big).$$

This shows $(\varphi\hat{\mathrm{Ord}} \vartriangle \tilde{\varphi}\hat{\mathrm{Ord}})_k = (\varphi \vartriangle \tilde{\varphi})\hat{\mathrm{Ord}}_k$.

So $\varphi\hat{\mathrm{Ord}} \vartriangle \tilde{\varphi}\hat{\mathrm{Ord}} = (\varphi \vartriangle \tilde{\varphi})\hat{\mathrm{Ord}}$. □

Lemma 217 *Suppose given a stable simplicial group $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$; cf. Definition* 205.

For $k \in \mathbb{Z}_{\geqslant 0}$, we consider

$$G\mathrm{N}_k = \bigcap_{i \in \mathbb{Z}_{\geqslant 1}} G_{k,i} = \bigcap_{i \in \mathbb{Z}_{\geqslant 1}} \ker \mathrm{d}^{G,k}_i \trianglelefteqslant (G\hat{\mathrm{Ord}})_k \leqslant G_k;$$

cf. Definition 198.$(2,1)$.

We consider the simplicial group $G\hat{\mathrm{Ord}} \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ *from Lemma* 214.

For $k \in \mathbb{Z}_{\geqslant 0}$*, we consider*

$$(G\hat{\mathrm{Ord}})\mathrm{N}_k = \bigcap_{i\in[1,k]} G\hat{\mathrm{Ord}}_{k,i} = \bigcap_{i\in[1,k]} \ker \mathrm{d}_i^{G\hat{\mathrm{Ord}},k}\,;$$

cf. Definition 46.$(2,1)$.

Then, for $k \in \mathbb{Z}_{\geqslant 0}$*, we have*

$$G\mathrm{N}_k = (G\hat{\mathrm{Ord}})\mathrm{N}_k\,.$$

Proof. Ad $\subseteq$. Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $g \in G\mathrm{N}_k = \bigcap\limits_{i\geqslant 1} \ker \mathrm{d}_i^{G,k}$.

For $\ell \in \mathbb{Z}_{\geqslant k+1}$, we have

$$g\mathrm{d}_\ell^{G,k} = 1\,.$$

This shows $g \in \bigcap\limits_{i\geqslant k+1} \ker \mathrm{d}_i^{G,k} = G\hat{\mathrm{Ord}}_k$.

Moreover, for $\ell \in [1,k]$, we have

$$g\mathrm{d}_\ell^{G\hat{\mathrm{Ord}},k} = g\mathrm{d}_\ell^{G,k} = 1\,.$$

This shows $g \in \bigcap\limits_{i\in[1,k]} \ker \mathrm{d}_i^{G\hat{\mathrm{Ord}},k} = (G\hat{\mathrm{Ord}})\mathrm{N}_k$.

Ad $\supseteq$. Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $g \in (G\hat{\mathrm{Ord}})\mathrm{N}_k = \bigcap\limits_{i\in[1,k]} \ker \mathrm{d}_i^{G\hat{\mathrm{Ord}},k}$.

Suppose given $\ell \in \mathbb{Z}_{\geqslant 0}$.

If $\ell \in [1,k]$, then we have

$$g\mathrm{d}_\ell^{G,k} = g\mathrm{d}_\ell^{G\hat{\mathrm{Ord}},k} = 1\,,$$

since $g \in \bigcap\limits_{i\in[1,k]} \ker \mathrm{d}_i^{G\hat{\mathrm{Ord}},k}$.

If $\ell \geqslant k+1$, then we have

$$g\mathrm{d}_\ell^{G,k} = 1\,,$$

since $g \in G\hat{\mathrm{Ord}}_k = \bigcap\limits_{i\geqslant k+1} \ker \mathrm{d}_i^{G,k}$.

This shows $g \in \bigcap\limits_{i\geqslant 1} \ker \mathrm{d}_i^{G,k} = G\mathrm{N}_k$. □

10.2 The ordinarification functor $\mathrm{Ord}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$, preserving quasi-isomorphisms

Remark 218 *Suppose given a stable simplicial group $G \in \mathrm{Ob}(\mathcal{StSimpGrp})$; cf. Definition* 205.

(1) *Let $k \in \mathbb{Z}_{\geqslant 0}$. For $\ell \in [0,k]$, we have*

$$\mathrm{d}_{\ell}^{G,k} \blacktriangle \Big(\prod_{i \in \lfloor k-1,0 \rfloor}^{G_{k-1} \ldots G_{-1}} \mathrm{d}_0^{G,i} \Big) = \prod_{i \in \lfloor k,0 \rfloor}^{G_k \ldots G_{-1}} \mathrm{d}_0^{G,i}.$$

(2) *Let $k \in \mathbb{Z}_{\geqslant 0}$. For $\ell \in [0,k]$, we have*

$$\mathrm{s}_{\ell}^{G,k} \blacktriangle \Big(\prod_{i \in \lfloor k+1,0 \rfloor}^{G_{k+1} \ldots G_{-1}} \mathrm{d}_0^{G,i} \Big) = \prod_{i \in \lfloor k,0 \rfloor}^{G_k \ldots G_{-1}} \mathrm{d}_0^{G,i}.$$

Proof. Ad (1). We show this claim by induction on $k \in \mathbb{Z}_{\geqslant 0}$.

Let $k = 0$. Suppose given $\ell \in [0,0]$.

We have

$$\mathrm{d}_{\ell}^{G,0} \blacktriangle \Big(\prod_{i \in \lfloor -1,0 \rfloor}^{G_{-1} \ldots G_{-1}} \mathrm{d}_0^{G,i} \Big) \overset{1.1.2\,\text{item}\,9.3}{=} \mathrm{d}_0^{G,0} \blacktriangle \mathrm{id}_{G_{-1}} = \prod_{i \in \lfloor 0,0 \rfloor}^{G_0 \ldots G_{-1}} \mathrm{d}_0^{G,i}.$$

Suppose given $k \in \mathbb{Z}_{\geqslant 1}$ and $\ell \in [0,k]$.

For $j \in [0,k-1]$, suppose that we have

$$\mathrm{d}_{j}^{G,k-1} \blacktriangle \Big(\prod_{i \in \lfloor k-2,0 \rfloor}^{G_{k-2} \ldots G_{-1}} \mathrm{d}_0^{G,i} \Big) = \prod_{i \in \lfloor k-1,0 \rfloor}^{G_{k-1} \ldots G_{-1}} \mathrm{d}_0^{G,i}.$$

If $\ell = 0$, then we have

$$\mathrm{d}_{\ell}^{G,k} \blacktriangle \Big(\prod_{i \in \lfloor k-1,0 \rfloor}^{G_{k-1} \ldots G_{-1}} \mathrm{d}_0^{G,i} \Big) = \prod_{i \in \lfloor k,0 \rfloor}^{G_k \ldots G_{-1}} \mathrm{d}_0^{G,i}.$$

If $\ell \geqslant 1$, then we have

$$\begin{aligned}
\mathrm{d}_{\ell}^{G,k} \blacktriangle \Big({}^{G_{k-1}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) \quad &= \quad \mathrm{d}_{\ell}^{G,k} \blacktriangle \mathrm{d}_0^{G,k-1} \blacktriangle \Big({}^{G_{k-2}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) \\
&\overset{185.(1)}{=} \quad \mathrm{d}_0^{G,k} \blacktriangle \mathrm{d}_{\ell-1}^{G,k-1} \blacktriangle \Big({}^{G_{k-2}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) \\
&\overset{\mathrm{IH}}{=} \quad \mathrm{d}_0^{G,k} \blacktriangle \Big({}^{G_{k-1}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) = {}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} .
\end{aligned}$$

Ad (2). We show this claim by induction on $k \in \mathbb{Z}_{\geqslant 0}$.

Let $k = 0$. Suppose given $\ell \in [0, 0]$.

We have

$$\mathrm{s}_{\ell}^{G,0} \blacktriangle \Big({}^{G_1}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) = \mathrm{s}_0^{G,0} \blacktriangle \mathrm{d}_0^{G,1} \blacktriangle \mathrm{d}_0^{G,0} \overset{185.(3)}{=} = \mathrm{id}_{G_0} \blacktriangle \mathrm{d}_0^{G,0} = {}^{G_0}\prod^{G_{-1}} \mathrm{d}_0^{G,i} .$$

Suppose given $k \in \mathbb{Z}_{\geqslant 1}$ and $\ell \in [0, k]$.

For $j \in [0, k-1]$, suppose that we have

$$\mathrm{s}_j^{G,k-1} \blacktriangle \Big({}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) = {}^{G_{k-1}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} .$$

If $\ell = 0$, then we have

$$\begin{aligned}
\mathrm{s}_{\ell}^{G,k} \blacktriangle \Big({}^{G_{k+1}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) = \mathrm{s}_0^{G,k} \blacktriangle \mathrm{d}_0^{G,k+1} \blacktriangle \Big({}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) &\overset{185.(3)}{=} \mathrm{id}_{G_k} \blacktriangle \Big({}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) \\
&= {}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} .
\end{aligned}$$

If $\ell \geqslant 1$, then we have

$$\begin{aligned}
\mathrm{s}_{\ell}^{G,k} \blacktriangle \Big({}^{G_{k+1}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) = \mathrm{s}_{\ell}^{G,k} \blacktriangle \mathrm{d}_0^{G,k+1} \blacktriangle \Big({}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) &\overset{185.(3)}{=} \mathrm{d}_0^{G,k} \blacktriangle \mathrm{s}_{\ell-1}^{G,k-1} \blacktriangle \Big({}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) \\
&\overset{\mathrm{IH}}{=} \mathrm{d}_0^{G,k} \blacktriangle \Big({}^{G_{k-1}}\prod^{G_{-1}} \mathrm{d}_0^{G,i} \Big) = {}^{G_k}\prod^{G_{-1}} \mathrm{d}_0^{G,i} .
\end{aligned}$$

□

Lemma 219 (The simplicial group $G\mathrm{Ord}$)

Suppose given a stable simplicial group $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$; cf. Definition 205.

We consider the simplicial group $G\hat{\mathrm{Ord}}$; cf. Lemma 214.

We have the simplicial subgroup $G\mathrm{Ord} \leqslant G\hat{\mathrm{Ord}}$, where, for $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$G\mathrm{Ord}_k := G\hat{\mathrm{Ord}}_k \cap \ker\Big(\prod_{i\in\lfloor k,0\rfloor}^{G_k \quad G_{-1}} \mathrm{d}_0^{G,i}\Big) = \big(\bigcap_{i\geqslant k+1} G_{k,i}\big) \cap \ker(\mathrm{d}_0^{G,k} \blacktriangle \mathrm{d}_0^{G,k-1} \blacktriangle \ldots \blacktriangle \mathrm{d}_0^{G,0}) \leqslant G\hat{\mathrm{Ord}}_k \,.$$

So, for $k \in \mathbb{Z}_{\geqslant 1}$ and $j \in [0,k]$, we have

$$\mathrm{d}_j^{G\mathrm{Ord},k} = \mathrm{d}_j^{G,k}\big|_{G\mathrm{Ord}_k}^{G\mathrm{Ord}_{k-1}} \,.$$

For $k \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0,k]$, we have

$$\mathrm{s}_i^{G\mathrm{Ord},k} = \mathrm{s}_i^{G,k}\big|_{G\mathrm{Ord}_k}^{G\mathrm{Ord}_{k+1}} \,.$$

In particular, $G\mathrm{Ord}$ is a simplicial group; cf. Definition 40.

Proof. We use Lemma 41.(2) to show that the subgroups $G\mathrm{Ord}_k$ for $k \geqslant 0$ define a simplicial subgroup $G\mathrm{Ord} \leqslant G\hat{\mathrm{Ord}}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 1}$ and $\ell \in [0,k]$.

We show that $(G\mathrm{Ord}_k)\mathrm{d}_\ell^{G\hat{\mathrm{Ord}},k} \overset{!}{\leqslant} G\mathrm{Ord}_{k-1}$.

Suppose given $x \in G\mathrm{Ord}_k$.

Then we have $x\mathrm{d}_\ell^{G\hat{\mathrm{Ord}},k} \in G\hat{\mathrm{Ord}}_{k-1}$; cf. Lemma 214.

Moreover, we have

$$(x\mathrm{d}_\ell^{G\hat{\mathrm{Ord}},k})\Big(\prod_{i\in\lfloor k-1,0\rfloor}^{G_{k-1} \quad G_{-1}} \mathrm{d}_0^{G,i}\Big) = x\Big(\mathrm{d}_\ell^{G,k} \blacktriangle \Big(\prod_{i\in\lfloor k-1,0\rfloor}^{G_{k-1} \quad G_{-1}} \mathrm{d}_0^{G,i}\Big)\Big) \overset{218.(1)}{=} x\Big(\prod_{i\in\lfloor k,0\rfloor}^{G_k \quad G_{-1}} \mathrm{d}_0^{G,i}\Big) = 1\,.$$

This shows $x\mathrm{d}_\ell^{G\hat{\mathrm{Ord}},k} \in \ker\Big(\prod_{i\in\lfloor k-1,0\rfloor}^{G_{k-1} \quad G_{-1}} \mathrm{d}_0^{G,i}\Big)$.

Altogether, we have $x\mathrm{d}_\ell^{G\hat{\mathrm{Ord}},k} \in G\hat{\mathrm{Ord}}_{k-1} \cap \ker\Big(\prod_{i\in\lfloor k-1,0\rfloor}^{G_{k-1} \quad G_{-1}} \mathrm{d}_0^{G,i}\Big) = G\mathrm{Ord}_{k-1}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$ and $\ell \in [0,k]$.

10.2. THE ORDINARIFICATION FUNCTOR $\mathrm{Ord}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$, *PRESERVING QUASI-ISOMORPHISMS*

We show that $(G\mathrm{Ord}_k)\mathrm{s}_\ell^{G\hat{\mathrm{O}}\mathrm{rd},k} \overset{!}{\leqslant} G\mathrm{Ord}_{k+1}$.

Suppose given $x \in G\mathrm{Ord}_k$.

Then we have $x\mathrm{s}_\ell^{G\hat{\mathrm{O}}\mathrm{rd},k} \in G\hat{\mathrm{O}}\mathrm{rd}_{k+1}$; cf. Lemma 214.

Moreover, we have

$$(x\mathrm{s}_\ell^{G\hat{\mathrm{O}}\mathrm{rd},k})\Big(\prod_{i\in\lfloor k+1,0\rfloor}^{G_{k+1}\ G_{-1}} \mathrm{d}_0^{G,i}\Big) = x\Big(\mathrm{s}_\ell^{G,k} \blacktriangleleft \Big(\prod_{i\in\lfloor k+1,0\rfloor}^{G_{k+1}\ G_{-1}} \mathrm{d}_0^{G,i}\Big)\Big) \overset{218.(2)}{=} x\Big(\prod_{i\in\lfloor k,0\rfloor}^{G_k\ G_{-1}} \mathrm{d}_0^{G,i}\Big) = 1\,.$$

This shows $x\mathrm{s}_\ell^{G\hat{\mathrm{O}}\mathrm{rd},k} \in \ker\Big(\prod_{i\in\lfloor k+1,0\rfloor}^{G_{k+1}\ G_{-1}} \mathrm{d}_0^{G,i}\Big)$.

Altogether, we have $x\mathrm{s}_\ell^{G\hat{\mathrm{O}}\mathrm{rd},k} \in G\hat{\mathrm{O}}\mathrm{rd}_k \cap \ker\Big(\prod_{i\in\lfloor k+1,0\rfloor}^{G_{k+1}\ G_{-1}} \mathrm{d}_0^{G,i}\Big) = G\mathrm{Ord}_{k+1}$.

Then, by Lemma 41.(2), $G\mathrm{Ord} \leqslant G\hat{\mathrm{O}}\mathrm{rd}$ is a simplicial subgroup. □

Lemma 220 (The simplicial group morphism $\varphi\mathrm{Ord}$)

Suppose given stable simplicial groups G, $H \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

Suppose given a stable simplicial group morphism $\varphi\colon G \to H$.

We consider the simplicial groups $G\mathrm{Ord}$ *and* $H\mathrm{Ord}$ *from Lemma* 219.

We have the simplicial group morphism

$$\varphi\mathrm{Ord}\colon G\mathrm{Ord} \to H\mathrm{Ord}\,,$$

where

$$\varphi\mathrm{Ord}_k := \varphi_k\big|_{G\mathrm{Ord}_k}^{H\mathrm{Ord}_k}\colon G\mathrm{Ord}_k \to H\mathrm{Ord}_k\,,\ g \mapsto g(\varphi\mathrm{Ord}_k) := g\varphi_k\,,$$

for $k \in \mathbb{Z}_{\geqslant 0}$.

Proof. We apply Remark 42 to show this claim. Recall that $\varphi\hat{\mathrm{O}}\mathrm{rd}_k = \varphi_k\big|_{G\hat{\mathrm{O}}\mathrm{rd}_k}^{H\hat{\mathrm{O}}\mathrm{rd}_k}$; cf. Lemma 215.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. We have to show that $G\mathrm{Ord}_k(\varphi\hat{\mathrm{O}}\mathrm{rd}_k) \overset{!}{\leqslant} H\mathrm{Ord}_k$.

Suppose given $g \in G\mathrm{Ord}_k = G\hat{\mathrm{O}}\mathrm{rd}_k \cap \ker\Big(\prod_{i\in\lfloor k,0\rfloor}^{G_k\ G_{-1}} \mathrm{d}_0^{G,k}\Big) \leqslant G\hat{\mathrm{O}}\mathrm{rd}_k \leqslant G_k$.

Since we have $G\mathrm{Ord} \leqslant G\hat{\mathrm{O}}\mathrm{rd}$, we have $g(\varphi\hat{\mathrm{O}}\mathrm{rd}_k) \in H\hat{\mathrm{O}}\mathrm{rd}_k$.

It remains to show that $g(\varphi\mathrm{Ord}_k) \overset{!}{\in} \ker\Big(\prod_{i\in\lfloor k,0\rfloor}^{H_k}{}^{H_{-1}} \mathrm{d}_0^{H,k}\Big)$.

We have

$$g(\varphi\mathrm{Ord}_k)\Big(\prod_{i\in\lfloor k,0\rfloor}^{H_k}{}^{H_{-1}} \mathrm{d}_0^{H,k}\Big) = g\varphi_k\Big(\prod_{i\in\lfloor k,0\rfloor}^{H_k}{}^{H_{-1}} \mathrm{d}_0^{H,k}\Big) = g\Big(\Big(\prod_{i\in\lfloor k,0\rfloor}^{G_k}{}^{G_{-1}} \mathrm{d}_0^{G,k}\Big)\blacktriangle\varphi_{-1}\Big)$$
$$= 1\varphi_{-1} = 1\,.$$

This shows that $g(\varphi\mathrm{Ord}_k) \in H\hat{\mathrm{O}}\mathrm{rd}_k \cap \ker\Big(\prod_{i\in\lfloor k,0\rfloor}^{H_k}{}^{H_{-1}} \mathrm{d}_0^{H,k}\Big) = H\mathrm{Ord}_k$.

So, by Remark 42, we have the simplicial group morphism $\varphi\mathrm{Ord} = \varphi\hat{\mathrm{O}}\mathrm{rd}\,\big|_{G\mathrm{Ord}}^{H\mathrm{Ord}}$, having

$$\varphi\mathrm{Ord}_k = \varphi\hat{\mathrm{O}}\mathrm{rd}_k\,\big|_{G\mathrm{Ord}_k}^{H\mathrm{Ord}_k} = \big(\,\varphi_k\big|_{G\hat{\mathrm{O}}\mathrm{rd}_k}^{H\hat{\mathrm{O}}\mathrm{rd}_k}\big)\big|_{G\mathrm{Ord}_k}^{H\mathrm{Ord}_k} = \varphi_k\big|_{G\mathrm{Ord}_k}^{H\mathrm{Ord}_k}\,.$$

□

Lemma 221 (The functor Ord)

We have the ordinarification *functor from the category of stable simplicial groups to the category of simplicial groups given by*

$$\mathrm{Ord}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}\,,\ (G \xrightarrow{\varphi} H) \mapsto (G \xrightarrow{\varphi} H)\mathrm{Ord} := (G\mathrm{Ord} \xrightarrow{\varphi\mathrm{Ord}} H\mathrm{Ord})\,;$$

cf. Lemmas 219, 220.

Proof. Suppose given a simplicial group $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $g \in G\mathrm{Ord}_k$.

We have

$$g\big((\mathrm{id}_G)\mathrm{Ord}\,\big)_k = g\,\mathrm{id}_{G_k} = g\,\mathrm{id}_{G\mathrm{Ord}_k}\,.$$

This shows $\big((\mathrm{id}_G)\mathrm{Ord}\,\big)_k = \mathrm{id}_{G\mathrm{Ord}_k} = (\mathrm{id}_{G\mathrm{Ord}})_k$.

So $(\mathrm{id}_G)\mathrm{Ord} = \mathrm{id}_{G\mathrm{Ord}}$.

Suppose given $F \xrightarrow{\varphi} G \xrightarrow{\tilde{\varphi}} H$ in $\mathcal{StSimpGrp}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $x \in F\mathrm{Ord}_k$.

We have

$$x(\varphi\mathrm{Ord}_k \blacktriangle \tilde{\varphi}\mathrm{Ord}_k) = (x\varphi_k)(\tilde{\varphi}\mathrm{Ord}_k) = x\varphi_k\tilde{\varphi}_k = x(\varphi\blacktriangle\tilde{\varphi})_k = x\big((\varphi\blacktriangle\tilde{\varphi})\mathrm{Ord}_k\,\big)\,.$$

This shows $(\varphi\mathrm{Ord} \mathbin{\triangle} \tilde{\varphi}\mathrm{Ord})_k = (\varphi \mathbin{\triangle} \tilde{\varphi})\mathrm{Ord}_k$.

So $\varphi\mathrm{Ord} \mathbin{\triangle} \tilde{\varphi}\mathrm{Ord} = (\varphi \mathbin{\triangle} \tilde{\varphi})\mathrm{Ord}$. □

Lemma 222 *Suppose given a stable simplicial group* $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$*; cf. Definition* 205.

For $k \in \mathbb{Z}_{\geqslant 0}$*, we consider*

$$G\mathrm{N}_k = \bigcap_{i \in \mathbb{Z}_{\geqslant 1}} G_{k,i} = \bigcap_{i \in \mathbb{Z}_{\geqslant 1}} \ker \mathrm{d}_i^{G,k} \trianglelefteqslant G_k\,;$$

cf. Definition 198.$(2,1)$.

We consider the simplicial group $G\mathrm{Ord} \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$ *from Lemma* 219.

For $k \in \mathbb{Z}_{\geqslant 0}$*, we consider*

$$(G\mathrm{Ord})\mathrm{N}_k = \bigcap_{i \in [1,k]} G\mathrm{Ord}_{k,i} = \bigcap_{i \in [1,k]} \ker \mathrm{d}_i^{G\mathrm{Ord},k} \trianglelefteqslant G\mathrm{Ord}_k \leqslant G_k\,;$$

cf. Definition 46.$(2,1)$.

(1) *For* $k \in \mathbb{Z}_{\geqslant 1}$*, we have*

$$G\mathrm{N}_k = (G\mathrm{Ord})\mathrm{N}_k\,.$$

(2) *We have*

$$(G\mathrm{Ord})\mathrm{N}_0 = (G\mathrm{Ord})_0 = G\mathrm{NZ}_0 \trianglelefteqslant G\mathrm{N}_0\,;$$

cf. Definition 198.(3).

Proof. Ad (1). Ad $\subseteq$. Suppose given $k \in \mathbb{Z}_{\geqslant 1}$. Suppose given $g \in G\mathrm{N}_k = \bigcap\limits_{i \geqslant 1} \ker \mathrm{d}_i^{G,k}$.

For $\ell \in \mathbb{Z}_{\geqslant k+1}$, we have

$$g\mathrm{d}_\ell^{G,k} = 1\,.$$

This shows $g \in \bigcap\limits_{i \geqslant k+1} \ker \mathrm{d}_i^{G,k} = G\hat{\mathrm{O}}\mathrm{rd}_k$.

We have

$$g\Big(\prod_{i \in \lfloor k,0 \rfloor}^{G_k \quad G_{-1}} \mathrm{d}_0^{G,i}\Big) = g(\mathrm{d}_0^{G,k} \mathbin{\triangle} \mathrm{d}_0^{G,k-1}) \mathbin{\triangle} \Big(\prod_{i \in \lfloor k-2,0 \rfloor}^{G_{k-2} \quad G_{-1}} \mathrm{d}_0^{G,i}\Big) \overset{185.(1)}{=} g(\mathrm{d}_1^{G,k} \mathbin{\triangle} \mathrm{d}_0^{G,k-1}) \mathbin{\triangle} \Big(\prod_{i \in \lfloor k-2,0 \rfloor}^{G_{k-2} \quad G_{-1}} \mathrm{d}_0^{G,i}\Big)$$
$$= 1\Big(\mathrm{d}_0^{G,k-1} \mathbin{\triangle} \Big(\prod_{i \in \lfloor k-2,0 \rfloor}^{G_{k-2} \quad G_{-1}} \mathrm{d}_0^{G,i}\Big)\Big) = 1\,.$$

This shows $g \in \ker\Big(\prod_{i\in\lfloor k,0\rfloor}^{G_k,G_{-1}} \mathrm{d}_0^{G,i}\Big)$.

So we have $g \in G\hat{\mathrm{Ord}}_k \cap \ker\Big(\prod_{i\in\lfloor k,0\rfloor}^{G_k,G_{-1}} \mathrm{d}_0^{G,i}\Big) = G\mathrm{Ord}_k$.

Moreover, for $\ell \in [1,k]$, we have

$$g\mathrm{d}_\ell^{G\mathrm{Ord},k} = g\mathrm{d}_\ell^{G,k} = 1\,.$$

This shows $g \in \bigcap_{i\in[1,k]} \ker \mathrm{d}_i^{G\mathrm{Ord},k} = (G\mathrm{Ord})\mathrm{N}_k$.

Ad $\supseteq$. Suppose given $k \in \mathbb{Z}_{\geqslant 1}$. Suppose given $g \in (G\mathrm{Ord})\mathrm{N}_k = \bigcap_{i\in[1,k]} \ker \mathrm{d}_i^{G\mathrm{Ord},k}$.

Suppose given $\ell \in \mathbb{Z}_{\geqslant 1}$.

If $\ell \in [1,k]$, then we have

$$g\mathrm{d}_\ell^{G,k} = g\mathrm{d}_\ell^{G\mathrm{Ord},k} = 1\,,$$

since $g \in \bigcap_{i\in[1,k]} \ker \mathrm{d}_i^{G\mathrm{Ord},k}$.

If $\ell \geqslant k+1$, then we have

$$g\mathrm{d}_\ell^{G,k} = 1\,,$$

since $g \in G\mathrm{Ord}_k \leqslant G\hat{\mathrm{Ord}} = \bigcap_{i\geqslant k+1} \ker \mathrm{d}_i^{G,k}$.

This shows $g \in \bigcap_{i\geqslant 1} \ker \mathrm{d}_i^{G,k} = G\mathrm{N}_k$.

Ad (2). We have

$$G\mathrm{Ord}_0 = G\hat{\mathrm{Ord}}_0 \cap \ker \mathrm{d}_0^{G,0} = \Big(\bigcap_{i\in\mathbb{Z}_{\geqslant 1}} \ker \mathrm{d}_i^{G,0}\Big) \cap \ker \mathrm{d}_0^{G,0} \overset{198.(3)}{=} G\mathrm{NZ}_0\,.$$

□

Lemma 223 *Suppose given a stable simplicial group $G \in \mathrm{Ob}\big(\mathcal{S}t\mathcal{S}impGrp\big)$; cf. Definition* 205.

We consider its Moore complex

$$G\mathrm{N} = \ldots \xrightarrow{\partial_3^G} G\mathrm{N}_2 \xrightarrow{\partial_2^G} G\mathrm{N}_1 \xrightarrow{\partial_1^G} G\mathrm{N}_0 \xrightarrow{\partial_0^G} G\mathrm{N}_{-1} \xrightarrow{\partial_{-1}^G} \ldots\,;$$

cf. Lemma 200.(2).

10.2. THE ORDINARIFICATION FUNCTOR $\mathrm{Ord}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$, *PRESERVING QUASI-ISOMORPHISMS*

Recall that for $k \in \mathbb{Z}$, *we have*

$$G\mathrm{NH}_k = G\pi_k = \ker \partial_k / \operatorname{im} \partial_{k+1} = G\mathrm{NZ}_k / G\mathrm{NB}_k\,;$$

cf. Lemma 200.(2).

We consider the simplicial group $G\mathrm{Ord} \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$; *cf. Lemma* 219.

We consider its Moore complex

$$(G\mathrm{Ord})\mathrm{N} = \ldots \xrightarrow{\partial_3^{G\mathrm{Ord}}} (G\mathrm{Ord})\mathrm{N}_2 \xrightarrow{\partial_2^{G\mathrm{Ord}}} (G\mathrm{Ord})\mathrm{N}_1 \xrightarrow{\partial_1^{G\mathrm{Ord}}} (G\mathrm{Ord})\mathrm{N}_0 \xrightarrow{!} 1 \xrightarrow{!} \ldots\,;$$

cf. Lemma 47.(2).

(1) *We have the morphism of complexes in groups*

$$\iota^G = \iota = (\iota_k)_{k\in\mathbb{Z}}\colon (G\mathrm{Ord})\mathrm{N} \to G\mathrm{N}\,,$$

where, for $k \in \mathbb{Z}$, *we have*

$$\iota_k\colon (G\mathrm{Ord})\mathrm{N}_k \to G\mathrm{N}_k\,,\ x \mapsto x\,;$$

cf. Definition 8.

For $k \geqslant 1$, *note that* $\iota_k = \mathrm{id}_{G\mathrm{N}_k}$; *cf. Lemma* 222.(1).

For $k = 0$, *note that* ι_0 *is the inclusion morphism of* $(G\mathrm{Ord})\mathrm{N}_0 = G\mathrm{NZ}_0$ *into* $G\mathrm{N}_0$; *cf. Lemma* 222.(2).

$$\begin{array}{ccccccccccccc}
\ldots & \xrightarrow{\partial_3^{G\mathrm{Ord}}} & (G\mathrm{Ord})\mathrm{N}_2 & \xrightarrow{\partial_2^{G\mathrm{Ord}}} & (G\mathrm{Ord})\mathrm{N}_1 & \xrightarrow{\partial_1^{G\mathrm{Ord}}} & (G\mathrm{Ord})\mathrm{N}_0 & \xrightarrow{!} & 1 & \xrightarrow{!} & \ldots \\
 & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_2}} & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_1}} & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle !} & & \\
\ldots & \xrightarrow[\partial_3^G]{} & G\mathrm{N}_2 & \xrightarrow[\partial_2^G]{} & G\mathrm{N}_1 & \xrightarrow[\partial_1^G]{} & G\mathrm{N}_0 & \xrightarrow[\partial_0^G]{} & G\mathrm{N}_{-1} & \xrightarrow[\partial_{-1}^G]{} & \ldots
\end{array}$$

(2) *For* $k \in \mathbb{Z}$, *we recall that we have the homology groups*

$$\begin{array}{rrll}
 & G\mathrm{NH}_k & = \ker \partial_k^G / \operatorname{im} \partial_{k+1}^G & = G\mathrm{NZ}_k / G\mathrm{NB}_k \\
\text{and} & (G\mathrm{Ord})\mathrm{NH}_k & = \ker \partial_k^{G\mathrm{Ord}} / \operatorname{im} \partial_{k+1}^{G\mathrm{Ord}} & = (G\mathrm{Ord})\mathrm{NZ}_k / (G\mathrm{Ord})\mathrm{NB}_k\,;
\end{array}$$

cf. Lemma 11.(2).

For $k \in \mathbb{Z}_{\geqslant 0}$, we have the group isomorphism

$$\iota \mathrm{H}_k \colon (G\mathrm{Ord})\mathrm{NH}_k \to G\mathrm{NH}_k\,,\ \ x\,\mathrm{im}(G\mathrm{Ord})\mathrm{NB}_k \mapsto xG\mathrm{NB}_k\,.$$

So $\iota \mathrm{H}_k \colon (G\mathrm{Ord})\pi_k \xrightarrow{\sim} G\pi_k$.

Proof. Ad (1). Suppose given $k \in \mathbb{Z}$. Suppose given $x \in (G\mathrm{Ord})\mathrm{N}_k$.

Case $k \geqslant 1$. We have

$$\begin{aligned} x(\partial_k^{G\mathrm{Ord}} \mathbin{\blacktriangle} \iota_{k-1}) = x(\mathrm{d}_0^{G\mathrm{Ord},k} \mathbin{\blacktriangle} \iota_{k-1}) = x\mathrm{d}_0^{G\mathrm{Ord},k} = x\mathrm{d}_0^{G,k} = x(\iota_k \mathbin{\blacktriangle} \mathrm{d}_0^{G,k}) \\ = x(\iota_k \mathbin{\blacktriangle} \partial_k^G)\,. \end{aligned}$$

Case $k = 0$. Note that $x \in (G\mathrm{Ord})\mathrm{N}_0 \overset{222.(2)}{=} G\mathrm{NZ}_0$. So we have

$$x(\partial_0^{G\mathrm{Ord}} \mathbin{\blacktriangle} \iota_{-1}) = x(\,!\mathbin{\blacktriangle}!\,) = 1 = x\mathrm{d}_0^{G,0} = x(\iota_0 \mathbin{\blacktriangle} \mathrm{d}_0^{G,k}) = x(\iota_0 \mathbin{\blacktriangle} \partial_0^G)\,.$$

Case $k \leqslant -1$. We have

$$x(\partial_k^{G\mathrm{Ord}} \mathbin{\blacktriangle} \iota_{k-1}) = x(\,!\mathbin{\blacktriangle}!\,) = 1 = 1\partial_k^G = x(\,!\mathbin{\blacktriangle}\partial_k^G) = x(\iota_k \mathbin{\blacktriangle} \partial_k^G)\,.$$

Altogether, this shows $\partial_k^{G\mathrm{Ord}} \mathbin{\blacktriangle} \iota_{k-1} = \iota_k \mathbin{\blacktriangle} \partial_k^G$ for $k \in \mathbb{Z}$.

So $\iota = (\iota_k)_{k\in\mathbb{Z}}$ is a morphism of complexes in groups.

Ad (2). Suppose given $k \in \mathbb{Z}_{\geqslant 0}$.

We show that $\iota \mathrm{H}_k$ *is injective.*

Suppose given $x \in (G\mathrm{Ord})\mathrm{NZ}_k$. Suppose that $x(G\mathrm{Ord})\mathrm{NB}_k \in \ker(\iota \mathrm{H}_k)$.

We have

$$1G\mathrm{NB}_k = \big(x(G\mathrm{Ord})\mathrm{NB}_k\big)\iota \mathrm{H}_k = xG\mathrm{NB}_k\,.$$

This shows that $x \in \mathrm{im}\,\partial_{k+1}^G$.

So we we may choose $y \in G\mathrm{N}_{k+1}$ with $x = y\partial_{k+1}^G$.

Therefore, we have

$$x = y\partial_{k+1}^G = y(\iota_{k+1} \mathbin{\blacktriangle} \partial_{k+1}^G) = y(\partial_{k+1}^{G\mathrm{Ord}} \mathbin{\blacktriangle} \iota_k) = y\partial_{k+1}^{G\mathrm{Ord}}\,.$$

This shows that $x \in \mathrm{im}\,\partial_{k+1}^{G\mathrm{Ord}} = (G\mathrm{Ord})\mathrm{NB}_k$. So we have $x(G\mathrm{Ord})\mathrm{NB}_k = 1(G\mathrm{Ord})\mathrm{NB}_k$.

Therefore, $\iota\mathrm{H}_k$ is injective.

We show that $\iota\mathrm{H}_k$ *is surjective.*

Suppose given $x \in G\mathrm{NZ}_k$.

Case $k = 0$. We have $x \in G\mathrm{NZ}_0 \overset{222.(2)}{=} (G\mathrm{Ord})\mathrm{N}_0 = (G\mathrm{Ord})\mathrm{NZ}_0$.

So we have $x(G\mathrm{Ord})\mathrm{NB}_0 \in (G\mathrm{Ord})\mathrm{NH}_0$ and

$$\big(x(G\mathrm{Ord})\mathrm{NB}_0\big)\iota\mathrm{H}_k = (x\iota_0)G\mathrm{NB}_0 = xG\mathrm{NB}_0\,.$$

Case $k \geqslant 1$. Then we have $x \in G\mathrm{N}_k \overset{222.(1)}{=} (G\mathrm{Ord})\mathrm{N}_k$.

We have

$$x\partial_k^{G\mathrm{Ord}} = x(\partial_k^{G\mathrm{Ord}} \mathbin{\blacktriangle} \iota_{k-1}) = x(\iota_k \mathbin{\blacktriangle} \partial_k^G) = x\partial_k^G = 1\,.$$

This shows that $x \in \ker \partial_k^{G\mathrm{Ord}} = (G\mathrm{Ord})\mathrm{N}_k$.

So we have $x(G\mathrm{Ord})\mathrm{NB}_k \in (G\mathrm{Ord})\mathrm{NH}_k$ and

$$\big(x(G\mathrm{Ord})\mathrm{NB}_k\big)\iota\mathrm{H}_k = (x\iota_k)(G\mathrm{Ord})\mathrm{NB}_k = x(G\mathrm{Ord})\mathrm{NB}_k\,.$$

Therefore, $\iota\mathrm{H}_k$ is surjective in both cases.

Altogether, $\iota\mathrm{H}_k$ is bijective. □

Lemma 224 *Suppose given stable simplicial groups* G, H. *Suppose given a stable simplicial group morphism* $\varphi\colon G \to H$; *cf. Definition* 205.

We consider the Moore complexes $G\mathrm{N}$ *and* $H\mathrm{N}$ *and the morphism of complexes in groups* $\varphi\mathrm{N}\colon G\mathrm{N} \to H\mathrm{N}$; *cf. Lemma* 222.

Suppose that $\varphi\mathrm{N}$ *is a quasi-isomorphism; cf. Definition* 12.

We consider the functor $\mathrm{Ord}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$ *from Lemma* 221.

We have that $(\varphi\mathrm{Ord})\mathrm{N}\colon (G\mathrm{Ord})\mathrm{N} \to (H\mathrm{Ord})\mathrm{N}$ *is a quasi-isomorphism.*

Proof. Suppose given $k \in \mathbb{Z}$.

We have to show that $(\varphi\mathrm{Ord})\mathrm{NH}_k\colon (G\mathrm{Ord})\mathrm{NH}_k \to (H\mathrm{Ord})\mathrm{NH}_k$ *is a group isomorphism.*

Case $k \leqslant -1$. We have $(G\mathrm{Ord})\mathrm{N}_k = 1$ and $(H\mathrm{Ord})\mathrm{N}_k = 1$; cf. Lemma 223.(1).

So we have $(G\mathrm{Ord})\mathrm{NH}_k = 1$ and $(H\mathrm{Ord})\mathrm{NH}_k = 1$.

Therefore, $(\varphi\mathrm{Ord})\mathrm{NH}_k = ! : (G\mathrm{Ord})\mathrm{NH}_k \to (H\mathrm{Ord})\mathrm{NH}_k$ is a group isomorphism.

Case $k \geqslant 0$. We consider the morphisms of complexes in groups

$$\iota^G = (\iota^G_\ell)_{\ell\in\mathbb{Z}} \colon (G\mathrm{Ord})\mathrm{N} \to G\mathrm{N} \quad \text{and} \quad \iota^H = (\iota^H_\ell)_{\ell\in\mathbb{Z}} \colon (H\mathrm{Ord})\mathrm{N} \to H\mathrm{N}$$

from Lemma 223.(1).

For $\ell \in \mathbb{Z}$, we have $\iota^G_\ell \mathbin{\blacktriangle} \varphi\mathrm{N}_\ell = (\varphi\mathrm{Ord})\mathrm{N}_\ell \mathbin{\blacktriangle} \iota^H_\ell$, since both $(\varphi\mathrm{Ord})\mathrm{N}_\ell$ and $\varphi\mathrm{N}_\ell$ are restrictions of φ_ℓ .

$$\begin{array}{ccc} (G\mathrm{Ord})\mathrm{N}_\ell & \xrightarrow{\iota^G_\ell} & G\mathrm{N}_\ell \\ {\scriptstyle (\varphi\mathrm{Ord})\mathrm{N}_\ell}\downarrow & & \downarrow{\scriptstyle \varphi\mathrm{N}_\ell} \\ (H\mathrm{Ord})\mathrm{N}_\ell & \xrightarrow[\iota^H_\ell]{} & H\mathrm{N}_\ell \end{array}$$

So $\iota^G \mathbin{\blacktriangle} \varphi\mathrm{N} = (\varphi\mathrm{Ord})\mathrm{N} \mathbin{\blacktriangle} \iota^H$.

Since H_k is a functor, we obtain $(\iota^G)\mathrm{H}_k \mathbin{\blacktriangle} \varphi\mathrm{NH}_k = (\varphi\mathrm{Ord})\mathrm{NH}_k \mathbin{\blacktriangle} (\iota^H)\mathrm{H}_k$; cf. Lemma 11.(3).

$$\begin{array}{ccc} (G\mathrm{Ord})\mathrm{NH}_k & \xrightarrow{(\iota^G)\mathrm{H}_k} & G\mathrm{NH}_k \\ {\scriptstyle (\varphi\mathrm{Ord})\mathrm{NH}_k}\downarrow & & \downarrow{\scriptstyle \varphi\mathrm{NH}_k} \\ (H\mathrm{Ord})\mathrm{NH}_k & \xrightarrow[(\iota^H)\mathrm{H}_k]{} & H\mathrm{NH}_k \end{array}$$

By Lemma 223.(2), $(\iota^G)\mathrm{NH}_k$ and $(\iota^H)\mathrm{NH}_k$ are group isomorphisms. Moreover, $\varphi\mathrm{NH}_k$ is a group isomorphism since $\varphi\mathrm{N}$ is a quasi-isomorphism.

Therefore, $(\varphi\mathrm{Ord})\mathrm{NH}_k = (\iota^G)\mathrm{H}_k \mathbin{\blacktriangle} \varphi\mathrm{NH}_k \mathbin{\blacktriangle} \big((\iota^H)\mathrm{H}_k\big)^-$ is a group isomorphism.

Altogether, this shows that $(\varphi\mathrm{Ord})\mathrm{N}$ is a quasi-isomorphism. □

Chapter 11

$\lfloor n, -\infty \rfloor$-stable simplicial groups

11.1 Pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups

Let $n \in \mathbb{Z} \cup \{+\infty\}$. If $n = +\infty$, recall that $\mathbb{Z}_{\leqslant n} = \mathbb{Z}_{\leqslant n-1} = \mathbb{Z}$.

Definition 225 We consider the prestable simplex category $\Delta^{\pm}$; cf. Definition 172.
We have the full subcategory $\Delta^{\pm}_{\lfloor n,-\infty \rfloor} \subseteq \Delta^{\pm}$, where

$$\mathrm{Ob}\left(\Delta^{\pm}_{\lfloor n,-\infty \rfloor}\right) := \left\{ [k]^{\pm} \colon k \in \mathbb{Z}_{\leqslant n} \right\}.$$

Then note that

$$\mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n,-\infty \rfloor}\right) := \left\{ [\ell]^{\pm} \xrightarrow{u} [k]^{\pm} \colon \ell, k \in \mathbb{Z}_{\leqslant n}\,, u \text{ is monotone and essentially parallel} \right\}.$$

In particular, $\Delta^{\pm}_{\lfloor +\infty,-\infty \rfloor} = \Delta^{\pm}$.

Definition 226 (Pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups)
We consider the category of groups $\mathcal{G}rp$ and the opposite category $(\Delta^{\pm}_{\lfloor n,-\infty \rfloor})^{\mathrm{op}}$ of $\Delta^{\pm}_{\lfloor n,-\infty \rfloor}$; cf. §1.1.3 item 12.
We write

$$\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp := \left[(\Delta^{\pm}_{\lfloor n,-\infty \rfloor})^{\mathrm{op}}, \mathcal{G}rp \right]$$

for the category of functors from $(\Delta^{\pm}_{\lfloor n,-\infty \rfloor})^{\mathrm{op}}$ to $\mathcal{G}rp$ and transformations between those functors.
We call $\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp$ the *category of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups.*

A functor $\big((\Delta^{\pm}_{\lfloor n,-\infty\rfloor})^{\mathrm{op}} \xrightarrow{G} \mathcal{G}rp\big) \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$ is called a *pre*-$\lfloor n, -\infty \rfloor$-*prestable simplicial group*.

For $k \in \mathbb{Z}_{\leqslant n}$, we write $G_k := \big([k]^{\pm}\big)G$ and we write $\varphi_k := \big([k]^{\pm}\big)\varphi \colon G_k \to G'_k$.

A transformation $\big(G \xrightarrow{\varphi} G'\big) \in \mathrm{Mor}\big(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$ between pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups G, $G' \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$ is called a *pre*-$\lfloor n, -\infty \rfloor$-*prestable simplicial group morphism*.

For $\big([\ell]^{\pm} \xrightarrow{u} [k]^{\pm}\big) \in \mathrm{Mor}\,\big(\Delta^{\pm}_{\lfloor n,-\infty\rfloor}\big)$, we write $G_u := (u^{\mathrm{op}})G \colon G_k \to G_\ell$.

For $(G \xrightarrow{\varphi} G' \xrightarrow{\varphi'} G'')$ in $\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp$, their composite is given by their vertical composite of transformations

$$\varphi \mathbin{\blacktriangle} \varphi' = \big(G_k \xrightarrow{[k](\varphi \blacktriangle \varphi')} G''_k\big)_{k \in \mathbb{Z}_{\leqslant n}} := \big(G_k \xrightarrow{\varphi_k \blacktriangle \varphi'_k} G''_k\big)_{k \in \mathbb{Z}_{\leqslant n}}.$$

So $(\varphi \mathbin{\blacktriangle} \varphi')_k = \varphi_k \mathbin{\blacktriangle} \varphi'_k$ for $k \in \mathbb{Z}_{\leqslant n}$.

For $G \in \mathrm{Ob}(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp)$, its identity is given by

$$\mathrm{id}_G = \big(G_k \xrightarrow{\mathrm{id}_{G_k}} G_k\big)_{k \in \mathbb{Z}_{\leqslant n}}.$$

So $(\mathrm{id}_G)_k = \mathrm{id}_{G_k}$ for $k \in \mathbb{Z}_{\leqslant n}$.

Remark 227 In Definition 232, we will define $\lfloor n, -\infty \rfloor$-stable simplicial groups, which will be pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups that satisfy a finiteness condition (1) and a Conduché condition (2).

In the notion of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups, the first "pre" refers to the absent Conduché condition, the second "pre" refers to the absent finiteness condition.

Remark 228 (Face maps and degeneracy maps)

Suppose given a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group $G \colon (\Delta^{\pm}_{\lfloor n,-\infty\rfloor})^{\mathrm{op}} \to \mathcal{G}rp$; cf. Definition 226.

For $k \in \mathbb{Z}_{\leqslant n}$ and $j \in \mathbb{Z}_{\geqslant 0}$, we consider the morphism

$$\delta^{\pm,k}_j \colon [k-1]^{\pm} \to [k]^{\pm} \text{ in } \Delta^{\pm}_{\lfloor n,-\infty\rfloor} \subseteq \Delta^{\pm}.$$

For $k \in \mathbb{Z}_{\leqslant n-1}$ and $i \in \mathbb{Z}_{\geqslant 0}$, we consider the morphism

$$\sigma^{\pm,k}_i \colon [k+1]^{\pm} \to [k]^{\pm} \text{ in } \Delta^{\pm}_{\lfloor n,-\infty\rfloor} \subseteq \Delta^{\pm}.$$

For $k \in \mathbb{Z}_{\leqslant n}$ and $j \in \mathbb{Z}_{\geqslant 0}$, we have the face map

$$\mathrm{d}_j^{G,k} = \big((\delta_i^{\pm,k})^{\mathrm{op}}\big)G \colon G_k \to G_{k-1}\,.$$

For $k \in \mathbb{Z}_{\leqslant n-1}$ and $i \in \mathbb{Z}_{\geqslant 0}$, we have the degeneracy map

$$\mathrm{s}_i^{G,k} = \big((\sigma_i^{\pm,k})^{\mathrm{op}}\big)G \colon G_k \to G_{k+1}\,;$$

cf. also Definition 183.
Sometimes, we abbreviate $\mathrm{d}_i := \mathrm{d}_i^{G,k}$ and $\mathrm{s}_i := \mathrm{s}_i^{G,k}$.

Remark 229 Suppose given a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group G.
We consider the face maps and degeneracy maps from Remark 228.
Let $k \in \mathbb{Z}_{\leqslant n-1}$. Using Remark 179, we have the following equations (1, 2, 3).

(1) For i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$, we have

$$\mathrm{d}_j^{G,k+1} \vartriangleleft \mathrm{d}_i^{G,k} = \mathrm{d}_i^{G,k+1} \vartriangleleft \mathrm{d}_{j-1}^{G,k}\,.$$

(2) For i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$, we we have

$$\mathrm{s}_i^{G,k-1} \vartriangleleft \mathrm{s}_j^{G,k} = \mathrm{s}_{j-1}^{G,k-1} \vartriangleleft \mathrm{s}_i^{G,k}\,.$$

(3) For i, $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{s}_i^{G,k} \vartriangleleft \mathrm{d}_j^{G,k+1} = \begin{cases} \mathrm{d}_j^{G,k} \vartriangleleft \mathrm{s}_{i-1}^{G,k-1} & \text{if } j < i \\ \mathrm{id}_{G_k} & \text{if } j = i \text{ or } j = i+1 \\ \mathrm{d}_{j-1}^{G,k} \vartriangleleft \mathrm{s}_i^{G,k-1} & \text{if } j > i+1\,. \end{cases}$$

Remark 230 Let $n = +\infty$.
Then a pre-$\lfloor +\infty, -\infty \rfloor$-prestable simplicial group G is the same as a prestable simplicial group; cf. Definitions 182, 226.

11.2 $\lfloor n, -\infty \rfloor$-stable simplicial groups

Let $n \in \mathbb{Z}$.

Definition 231

Suppose given a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group $G \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$; cf. Definition 226.

Suppose given $k \in \mathbb{Z}_{\leqslant n}$. Let $g \in G_k$.

We say that g is *finite* if there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$g\,\mathrm{d}_i^{G,\,k} = 1 \quad \text{for } i \geqslant \ell\,.$$

Cf. also Definition 204.

Definition 232 ($\lfloor n, -\infty \rfloor$-stable simplicial group)

We consider the category of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups $\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp$; cf. Definition 226.

Suppose given an pre-$\lfloor n, 0 \rfloor$-prestable simplicial group $G \in \mathrm{Ob}(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp)$.

Then G is called an $\lfloor n, 0 \rfloor$-*stable simplicial group* if it satisfies the following properties $(1, 2)$.

(1) For $k \in \mathbb{Z}_{\leqslant n}$, each element $g \in G_k$ is finite; cf. Definition 231.

(2) For cofinite subsets $I,\ J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$, we have

$$\big[\bigcap_{i \in I} \ker \mathrm{d}_i^{G,n}\,,\, \bigcap_{j \in J} \ker \mathrm{d}_j^{G,n}\big] = 1\,;$$

cf. §1.1.1 item 1 and Remark 228.

We call property (1) the *finiteness condition* and we call property (2) the *Conduché condition* on $\lfloor n, -\infty \rfloor$-stable simplicial groups; cf. [**7,** Th. 1.5].

Let $\lfloor n, -\infty \rfloor\text{-}\mathcal{S}t\mathcal{S}imp\mathcal{G}rp \subseteq \mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp$ be the full subcategory defined by

$$\mathrm{Ob}\big(\lfloor n, -\infty \rfloor\text{-}\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big) := \big\{ G \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big) : G \text{ is an } \lfloor n, -\infty \rfloor\text{-stable simplicial group} \big\}\,.$$

We call $\lfloor n, -\infty \rfloor\text{-}\mathcal{S}t\mathcal{S}imp\mathcal{G}rp$ the *category of* $\lfloor n, -\infty \rfloor$-*stable simplicial groups.*

$$G_n \underset{\mathrm{s}_i^{G,n-1}}{\overset{\mathrm{d}_j^{G,n}}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i^{G,n-2}}{\overset{\mathrm{d}_j^{G,n-1}}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i^{G,0}}{\overset{\mathrm{d}_j^{G,1}}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i^{G,-1}}{\overset{\mathrm{d}_j^{G,0}}{\rightleftarrows}} G_{-1} \underset{\mathrm{s}_i^{G,-2}}{\overset{\mathrm{d}_j^{G,-1}}{\rightleftarrows}} \cdots$$

A morphism $\big(G \xrightarrow{\varphi} H\big) \in \mathrm{Mor}(\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp})$ between $\lfloor n, -\infty \rfloor$-stable simplicial groups is called an *$\lfloor n, -\infty \rfloor$-stable simplicial group morphism.*

Remark 233 *Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group G.*

Suppose given subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

Then we have

$$\big[\bigcap_{i \in I} \ker \mathrm{d}_i^{G,n}, \bigcap_{j \in J} \ker \mathrm{d}_j^{G,n}\big] = 1\,.$$

Proof. Suppose given $x \in \bigcap\limits_{i \in I} \ker \mathrm{d}_i \leqslant G_n$ and suppose given $y \in \bigcap\limits_{j \in J} \ker \mathrm{d}_j \leqslant G_n$.

We have to show that $[x, y] \overset{!}{=} 1$.

We show that there exists some cofinite subsets $\tilde{I}$, $\tilde{J} \subseteq \mathbb{Z}_{\geqslant 0}$ with $x \in \bigcap\limits_{i \in \tilde{I}} \ker \mathrm{d}_i$ and with $y \in \bigcap\limits_{j \in \tilde{J}} \ker \mathrm{d}_j$.

Since x is finite, we may choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $x\mathrm{d}_k = 1$ for $k \in \mathbb{Z}_{\geqslant \ell}$.

Let $\tilde{I} := I \cup \mathbb{Z}_{\geqslant \ell}$. Then $\tilde{I}$ is cofinite and we have $x \in \bigcap\limits_{i \in \tilde{I}} \ker \mathrm{d}_i$.

Since y is finite, we may choose some $m \in \mathbb{Z}_{\geqslant 0}$ such that $y\mathrm{d}_k = 1$ for $k \in \mathbb{Z}_{\geqslant m}$.

Let $\tilde{J} := J \cup \mathbb{Z}_{\geqslant m}$. Then $\tilde{J}$ is cofinite and we have $y \in \bigcap\limits_{j \in \tilde{J}} \ker \mathrm{d}_j$.

We have $\tilde{I} \cup \tilde{J} = \mathbb{Z}_{\geqslant 0}$.

Therefore, we have $[x, y] = 1$. □

Remark 234 *Suppose given a morphism $\big([m]^{\pm} \xrightarrow{u} [k]^{\pm}\big) \in \mathrm{Mor}\,\big(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\big)$.*

Let $\ell := k - \big|[k]^{\pm} \setminus \mathrm{im}\, u\big|$. We have $\ell \in \mathbb{Z}_{\leqslant n}$.

Let $D_u := \big\{i \in \mathbb{Z}_{\geqslant 0} \colon (i, m)u = (i{+}1, m)u\big\} = \{i_1, \ldots, i_{m-\ell}\}$, where $0 \leqslant i_1 < i_2 < \ldots < i_{m-\ell}$.

Let $C_u := \{j \in \mathbb{Z}_{\geqslant 0} \colon (j, k) \notin \mathrm{im}\, u\} = \{j_1, \ldots, j_{k-\ell}\}$, where $0 \leqslant j_1 < j_2 < \ldots < j_{k-\ell}$.

Note that $|C_u| = \big|[k]^{\pm} \setminus \mathrm{im}\, u\big|$.

(1) *Suppose that u is injective. Then we have*

$$u = \prod_{r \in \lceil 1, k-m \rceil}^{[m]^{\pm}, [k]^{\pm}} \delta_{j_r}^{\pm, m+r} = \delta_{j_1}^{\pm, m+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-m}}^{\pm, k},$$

where $\delta_{j_r}^{\pm, m+r} \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$ for $r \in [1, k-m]$.

(2) *Suppose that u is surjective. Then we have*

$$u = \prod_{r \in \lfloor m-k, 1 \rfloor}^{[m]^{\pm}, [k]^{\pm}} \sigma_{i_r}^{\pm, k+r-1} = \sigma_{i_{m-k}}^{\pm, m-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm, k},$$

where $\sigma_{i_r}^{\pm, k+r-1} \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$ for $r \in [1, m-k]$.

(3) *We have*

$$u = \left(\prod_{r \in \lfloor m-\ell, 1 \rfloor}^{[m]^{\pm}, [\ell]^{\pm}} \sigma_{i_r}^{\pm, \ell+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1, k-\ell \rceil}^{[\ell]^{\pm}, [k]^{\pm}} \delta_{j_r}^{\pm, \ell+r} \right) = \sigma_{i_{m-\ell}}^{\pm, m-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm, \ell} \blacktriangle \delta_{j_1}^{\pm, \ell+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-\ell}}^{\pm, k},$$

where $\sigma_{i_r}^{\pm, \ell+r-1} \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$ for $r \in [1, m-\ell]$ and $\delta_{j_r}^{\pm, \ell+r} \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$ for $r \in [1, k-\ell]$.

(4) *Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group G; cf. Definition 232. We have*

$$\begin{aligned} G_u = (u^{\mathrm{op}})G &= \left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k, G_\ell} \mathrm{d}_{j_r}^{G, \ell+r} \right) \blacktriangle \left(\prod_{r \in \lceil 1, m-\ell \rceil}^{G_\ell, G_m} \mathrm{s}_{i_r}^{G, \ell+r-1} \right) \\ &= \mathrm{d}_{j_{k-\ell}}^{G,k} \blacktriangle \ldots \blacktriangle \mathrm{d}_{j_1}^{G,\ell+1} \blacktriangle \mathrm{s}_{i_1}^{G,\ell} \blacktriangle \ldots \blacktriangle \mathrm{s}_{i_{m-\ell}}^{G,m-1}. \end{aligned}$$

Proof. Note that we have $u \in \mathrm{Mor}(\Delta^{\pm})$.

Ad (1). Since u is injective, we have, by Lemma 175.(1.1),

$$u = \prod_{r \in \lceil 1, k-m \rceil}^{[m]^{\pm}, [k]^{\pm}} \delta_{j_r}^{\pm, m+r} = \delta_{j_1}^{\pm, m+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-m}}^{\pm, k}.$$

We show that $\delta_{j_r}^{\pm, m+r} \overset{!}{\in} \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$ for $r \in [1, k-m]$.

We have $m + r \leqslant k \leqslant n$ for $r \in [1, k-m]$.

This shows that $\left(\delta_{j_r}^{\pm,m+r} \colon [m+r-1]^{\pm} \to [m+r]^{\pm}\right) \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n,-\infty \rfloor}\right)$ for $r \in [1, k-m]$.

Ad (2). Since u is surjective, we have, by Lemma 175.(2.1),

$$u = \prod_{r \in \lfloor m-k,1 \rfloor}^{[m]^{\pm}} {}^{[k]^{\pm}} \sigma_{i_r}^{\pm,k+r-1} = \sigma_{i_{m-k}}^{\pm,m-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm,k} .$$

We show that $\sigma_{i_r}^{\pm,k+r-1} \overset{!}{\in} \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n,-\infty \rfloor}\right)$ for $r \in [1, m-k]$.

We have $k + r \leqslant m \leqslant n$ for $r \in [1, m-k]$.

This shows that $\left(\sigma_{i_r}^{\pm,k+r-1} \colon [k+r]^{\pm} \to [k+r-1]^{\pm}\right) \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n,-\infty \rfloor}\right)$ for $r \in [1, m-k]$.

Ad (3). For $r \in [1, m-\ell]$, note that we have $\ell + r \leqslant \ell + m - \ell = m \leqslant n$.

For $r \in [1, k-\ell]$, note that we have $\ell + r \leqslant \ell + k - \ell = k \leqslant n$.

By Corollary 177.(1), we have

$$u = \left(\prod_{r \in \lfloor m-\ell,1 \rfloor}^{[m]^{\pm}} {}^{[\ell]^{\pm}} \sigma_{i_r}^{\pm,\ell+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1,k-\ell \rceil}^{[\ell]^{\pm}} {}^{[k]^{\pm}} \delta_{j_r}^{\pm,\ell+r} \right) = \sigma_{i_{m-\ell}}^{\pm,m-1} \blacktriangle \ldots \blacktriangle \sigma_{i_1}^{\pm,\ell} \blacktriangle \delta_{j_1}^{\pm,\ell+1} \blacktriangle \ldots \blacktriangle \delta_{j_{k-\ell}}^{\pm,k} .$$

Ad (4). By (3), we have $(\sigma_{i_r}^{\pm,\ell+r-1})^{\mathrm{op}} \in \mathrm{Mor}\left(\Delta^{\pm,\mathrm{op}}_{\lfloor n,-\infty \rfloor}\right)$ for $r \in [1, m-\ell]$ and we have $(\delta_{j_r}^{\pm,\ell+r})^{\mathrm{op}} \in \mathrm{Mor}\left(\Delta^{\pm,\mathrm{op}}_{\lfloor n,-\infty \rfloor}\right)$ for $r \in [1, k-\ell]$. So we may apply G to these morphisms.

We obtain

$$\begin{aligned}
(u^{\mathrm{op}})G &\overset{(3)}{=} \left(\left(\left(\prod_{r \in \lfloor m-\ell,1 \rfloor}^{[m]^{\pm}} {}^{[\ell]^{\pm}} \sigma_{i_r}^{\pm,\ell+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1,k-\ell \rceil}^{[\ell]^{\pm}} {}^{[k]^{\pm}} \delta_{j_r}^{\pm,\ell+r} \right) \right)^{\mathrm{op}} \right) G \\
&= \left(\left(\prod_{r \in \lfloor k-\ell,1 \rfloor}^{[k]^{\pm}} {}^{[\ell]^{\pm}} (\delta_{j_r}^{\pm,\ell+r})^{\mathrm{op}} \right) \blacktriangle \left(\prod_{r \in \lceil 1,m-\ell \rceil}^{[\ell]^{\pm}} {}^{[m]^{\pm}} (\sigma_{i_r}^{\pm,\ell+r-1})^{\mathrm{op}} \right) \right) G \\
&\overset{(3)}{=} \left(\prod_{r \in \lfloor k-\ell,1 \rfloor}^{G_k} {}^{G_\ell} \left((\delta_{j_r}^{\pm,\ell+r})^{\mathrm{op}} \right) G \right) \blacktriangle \left(\prod_{r \in \lceil 1,m-\ell \rceil}^{G_\ell} {}^{G_m} \left((\sigma_{i_r}^{\pm,\ell+r-1})^{\mathrm{op}} \right) G \right) \\
&\overset{228}{=} \left(\prod_{r \in \lfloor k-\ell,1 \rfloor}^{G_k} {}^{G_\ell} \mathrm{d}_{j_r}^{G,\ell+r} \right) \blacktriangle \left(\prod_{r \in \lceil 1,m-\ell \rceil}^{G_\ell} {}^{G_m} \mathrm{s}_{i_r}^{G,\ell+r-1} \right) .
\end{aligned}$$

Cf. also Remark 186. □

Definition 235 ($\lfloor n, -\infty \rfloor$-stable simplicial subgroups)
Suppose given $\lfloor n, -\infty \rfloor$-stable simplicial groups G, H; cf. Definition 232.
We say that H is an *$\lfloor n, -\infty \rfloor$-stable simplicial subgroup* of G if the condition $(1, 2)$ hold.

(1) For $k \in \mathbb{Z}_{\leqslant n}$, we have $H_k \leqslant G_k$.

(2) Suppose given a morphism $\left([m]^{\pm} \xrightarrow{u} [k]^{\pm}\right) \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$.
We consider the group morphisms $G_u \colon G_k \to G_m$ and $H_u \colon H_k \to H_m$.
We have
$$H_u = G_u\big|^{H_m}_{H_k}.$$
We write $H \leqslant G$ to indicate that H is a $\lfloor n, -\infty \rfloor$-stable simplicial subgroup of G.

Remark 236 *Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group $G \colon (\Delta^{\pm}_{\lfloor n, -\infty \rfloor})^{\mathrm{op}} \to \mathcal{G}rp$.*
For $k \in \mathbb{Z}_{\leqslant n}$, suppose given subgroups $H_k \leqslant G_k$.
For $k \in \mathbb{Z}_{\leqslant n}$ and $j \in \mathbb{Z}_{\geqslant 0}$, suppose that we have $(H_k)\mathrm{d}^{G,k}_j \leqslant H_{k-1}$.
For $k \in \mathbb{Z}_{\leqslant n-1}$ and $i \in \mathbb{Z}_{\geqslant 0}$, suppose that we have $(H_k)\mathrm{s}^{G,k}_i \leqslant H_{k+1}$.
Then we have the $\lfloor n, -\infty \rfloor$-stable simplicial group
$$\begin{array}{rccl} H\colon & (\Delta^{\pm}_{\lfloor n, -\infty \rfloor})^{\mathrm{op}} & \to & \mathcal{G}rp \\ & \left([k]^{\pm} \xrightarrow{u^{\mathrm{op}}} [m]^{\pm}\right) & \mapsto & \left(([k]^{\pm})H \xrightarrow{(u^{\mathrm{op}})H} ([m]^{\pm})H\right) := \left(H_k \xrightarrow{G_u\big|^{H_m}_{H_k}} H_m\right). \end{array}$$
Moreover, $H \leqslant G$ is an $\lfloor n, -\infty \rfloor$-stable simplicial subgroup.

Proof. Suppose given a morphism $\left([m]^{\pm} \xrightarrow{u} [k]^{\pm}\right) \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$.
By Remark 234.(4), we have
$$\begin{aligned} G_u = (u^{\mathrm{op}})G &= \left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k \, G_\ell} \mathrm{d}^{G,\ell+r}_{j_r}\right) \blacktriangle \left(\prod_{r \in \lceil 1, m-\ell \rceil}^{G_\ell \, G_m} \mathrm{s}^{G,\ell+r-1}_{i_r}\right) \\ &= \mathrm{d}^{G,k}_{j_{k-\ell}} \blacktriangle \ldots \blacktriangle \mathrm{d}^{G,\ell+1}_{j_1} \blacktriangle \mathrm{s}^{G,\ell}_{i_1} \blacktriangle \ldots \blacktriangle \mathrm{s}^{G,m-1}_{i_{m-\ell}}, \end{aligned}$$
for some $\ell \in \mathbb{Z}_{\leqslant n}$, and $0 \leqslant j_1 < \ldots < j_{k-\ell}$ and $0 \leqslant i_1 < \ldots < i_{m-\ell}$.
Using the assumptions iteratively, we obtain
$$(H_k)\left(\prod_{r \in \lfloor k-\ell, 1 \rfloor}^{G_k \, G_\ell} \mathrm{d}^{G,\ell+r}_{j_r}\right) \leqslant H_\ell$$

and

$$(H_\ell)\left(\prod_{r\in\lceil 1,m-\ell\rceil}^{G_\ell}{}^{G_m} \mathrm{s}_{i_r}^{G,\ell+r-1}\right) \leqslant H_m\,.$$

So we have

$$(H_k)G_u = (H_k)\left(\left(\prod_{r\in\lfloor k-\ell,1\rfloor}^{G_k}{}^{G_\ell} \mathrm{d}_{j_r}^{G,\ell+r}\right) \blacktriangle \left(\prod_{r\in\lceil 1,m-\ell\rceil}^{G_\ell}{}^{G_m} \mathrm{s}_{i_r}^{G,\ell+r-1}\right)\right) \leqslant H_m\,.$$

Therefore, we have the group morphism $G_u\big|_{H_k}^{H_m} \colon H_k \to H_m$.

Suppose given $[\ell]^\pm \xrightarrow{v^{\mathrm{op}}} [k]^\pm \xrightarrow{u^{\mathrm{op}}} [m]^\pm$ in $(\Delta^\pm_{\lfloor n,-\infty\rfloor})^{\mathrm{op}}$.

We have

$$(\mathrm{id}_{[m]^\pm})^{\mathrm{op}}H = \big((\mathrm{id}_{[m]^\pm})^{\mathrm{op}}\, G\big)\big|_{H_m}^{H_m} = \mathrm{id}_{G_m}\big|_{H_m}^{H_m} = \mathrm{id}_{H_m} = \mathrm{id}_{([m]^\pm)H}\ .$$

We have

$$(v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}})H = \big((v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}})G\big)\big|_{H_\ell}^{H_m} = (G_v \blacktriangle G_u)\big|_{H_\ell}^{H_m} = G_v\big|_{H_\ell}^{H_k} \blacktriangle G_u\big|_{H_k}^{H_m} = (v^{\mathrm{op}})H \blacktriangle (u^{\mathrm{op}})H\,.$$

So H is a functor, i.e. H is a pre-$\lfloor n, -\infty\rfloor$-prestable simplicial group.

We show that H satisfies the finiteness condition.

Let $k \in \mathbb{Z}_{\leqslant n}$. Suppose given $h \in H_k \leqslant G_k$. Then, since h is finite, we may choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $h\mathrm{d}_i^{G,k} = 1$ for $i \in \mathbb{Z}_{\geqslant \ell}$.

So, for $i \in \mathbb{Z}_{\geqslant \ell}$, we have

$$h\mathrm{d}_i^{H,k} = h\mathrm{d}_i^{G,k}\big|_{H_k}^{H_{k-1}} = 1\,.$$

We show that H satisfies the Conduché condition.

Suppose given cofinite subsets $I,\ J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

Suppose given $x \in \bigcap_{i\in I} \ker \mathrm{d}_i^{H,n}$ and $y \in \bigcap_{j\in J} \ker \mathrm{d}_j^{H,n}$. Then we have $x \in \bigcap_{i\in I} \ker \mathrm{d}_i^{G,n}$ and $y \in \bigcap_{j\in J} \ker \mathrm{d}_j^{G,n}$.

Since we have $\big[\bigcap_{i\in I} \ker \mathrm{d}_i^{G,n}\,,\, \bigcap_{j\in J} \ker \mathrm{d}_j^{G,n}\big] = 1$, we have $[x, y] = 1$.

So H is a an $\lfloor n, -\infty\rfloor$-stable simplicial group.

Moreover, for $\big([m]^\pm \xrightarrow{u} [k]^\pm\big) \in \mathrm{Mor}(\Delta^\pm)$, we have, by construction,

$$H_u = (u^{\mathrm{op}})H = G_u\big|_{H_k}^{H_m}\,.$$

So $H \leqslant G$ is an $\lfloor n, -\infty\rfloor$-stable simplicial subgroup. □

11.3 A description using generators and relations

Let $n \in \mathbb{Z} \cup \{+\infty\}$. Suppose given m, $\tilde{m} \in \mathbb{Z}_{\leqslant n}$.

11.3.1 $(m, \tilde{m})$-index tuples

Definition 237 Let

$$R_{m,\tilde{m}} := \Big\{ \big((\varepsilon_1, i_1), (\varepsilon_2, i_2), \ldots, (\varepsilon_s, i_s)\big) : s \in \mathbb{Z}_{\geqslant 0},\ \varepsilon_k \in \{-1, +1\}, \text{ where } \sum_{j \in [1,s]} \varepsilon_j = \tilde{m} - m, \\ i_k \in \mathbb{Z}_{\geqslant 0} \text{ for } k \in [1, s] \text{ and } m + \sum_{j \in [1,k]} \varepsilon_j \leqslant n \text{ for } k \in [0, s] \Big\}.$$

An element of $R_{m,\tilde{m}}$ is called an $(m, \tilde{m})$-*index tuple*.

We consider the following relations on $R_{m,\tilde{m}}$, given as subsets of $R_{m,\tilde{m}} \times R_{m,\tilde{m}}$ consisting of pairs of $(m, \tilde{m})$-index tuples of the respective forms.

$$(\overset{++}{\rightsquigarrow}) := \Big\{ \Big(\big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_k, i_k), (\varepsilon_{k+1}, i_{k+1}), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big), \\ \big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_{k+1}, i_{k+1}), (\varepsilon_k, i_k + 1), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big) \Big) : \\ \text{for some } k \in [1, s-1] \text{ with } i_k \geqslant i_{k+1} \text{ and } \varepsilon_k = \varepsilon_{k+1} = +1 \Big\},$$

$$(\overset{--}{\rightsquigarrow}) := \Big\{ \Big(\big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_k, i_k), (\varepsilon_{k+1}, i_{k+1}), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big), \\ \big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_{k+1}, i_{k+1} + 1), (\varepsilon_k, i_k), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big) \Big) : \\ \text{for some } k \in [1, s-1] \text{ with } i_k \leqslant i_{k+1} \text{ and } \varepsilon_k = \varepsilon_{k+1} = -1 \Big\},$$

$$(\overset{+-,1}{\rightsquigarrow}) := \Big\{ \Big(\big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_k, i_k), (\varepsilon_{k+1}, i_{k+1}), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big), \\ \big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_{k+1}, i_{k+1} - 1), (\varepsilon_k, i_k), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big) \Big) : \\ \text{for some } k \in [1, s-1] \text{ with } i_k \leqslant i_{k+1} - 1 \text{ and } \varepsilon_k = +1 \text{ and } \varepsilon_{k+1} = -1 \Big\},$$

$$(\overset{+-,2}{\rightsquigarrow}) := \Big\{ \Big(\big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_k, i_k), (\varepsilon_{k+1}, i_{k+1}), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big), \\ \big((\varepsilon_1, i_1), \ldots, (\varepsilon_{k-1}, i_{k-1}), (\varepsilon_{k+2}, i_{k+2}), \ldots, (\varepsilon_s, i_s)\big) \Big) : \\ \text{for some } k \in [1, s-1] \text{ with } i_k \in \{i_{k+1}, i_{k+1} + 1\} \text{ and } \varepsilon_k = +1 \text{ and } \varepsilon_{k+1} = -1 \Big\},$$

$$\begin{aligned}
(\overset{+-,3}{\rightsquigarrow}) := \; & \Big\{ \Big(\big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_k\,,i_k),(\varepsilon_{k+1}\,,i_{k+1}),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big), \\
& \quad \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_{k-1}\,,i_{k-1}),(\varepsilon_{k+1}\,,i_{k+1}),(\varepsilon_k\,,i_k-1),(\varepsilon_{k+2}\,,i_{k+2}),\ldots,(\varepsilon_s\,,i_s)\big)\Big) : \\
& \quad \text{for some } k\in[1,s-1] \text{ with } i_k \geqslant i_{k+1}+2 \text{ and } \varepsilon_k=+1 \text{ and } \varepsilon_{k+1}=-1 \Big\} \\
\subseteq \; & R_{m,\tilde{m}} \times R_{m,\tilde{m}}\,.
\end{aligned}$$

Note that all occurring $(m,\tilde{m})$-index tuples are required to be in $R_{m,\tilde{m}}$.

We consider the relation on $R_{m,\tilde{m}}$ given by

$$(\rightsquigarrow) := (\overset{++}{\rightsquigarrow}) \;\cup\; (\overset{--}{\rightsquigarrow}) \;\cup\; (\overset{+-,1}{\rightsquigarrow}) \;\cup\; (\overset{+-,2}{\rightsquigarrow}) \;\cup\; (\overset{+-,3}{\rightsquigarrow}) \subseteq R_{m,\tilde{m}} \times R_{m,\tilde{m}}\,.$$

On $R_{m,\tilde{m}}$, we define the equivalence relation $(\sim)$ to be generated by $(\rightsquigarrow)$.

We write $R_{m,\tilde{m}}/(\sim)$ for the set of equivalence classes.

For $\underline{i} \in R_{m,\tilde{m}}$, we write $[\underline{i}] := \{\underline{j} \in R_{m,\tilde{m}} \colon \underline{j} \sim \underline{i}\} \in R_{m,\tilde{m}}/(\sim)$.

Definition 238

(1) Suppose given $\underline{i} = \big((\varepsilon_1\,,i_1),\ldots,(\varepsilon_s\,,i_s)\big) \in R_{m,\tilde{m}}$.

We write

$$\begin{aligned}
E_-(\underline{i}) &:= \{k\in[1,s] \colon \varepsilon_k=-1\} \\
E_+(\underline{i}) &:= \{k\in[1,s] \colon \varepsilon_k=+1\}\,.
\end{aligned}$$

Let

$$\begin{aligned}
S_-(\underline{i}) &:= \big\{ j\in E_-(\underline{i}) \colon \text{there exists some } \ell\in E_-(\underline{i}) \text{ such that } j<\ell \text{ and } i_j+j<i_\ell+\ell \big\} \\
S_+(\underline{i}) &:= \big\{ j\in E_+(\underline{i}) \colon \text{there exists some } \ell\in E_+(\underline{i}) \text{ such that } \ell<j \text{ and } i_\ell-\ell>i_j-j \big\}\,.
\end{aligned}$$

We write

$$\begin{aligned}
w_0(\underline{i}) &:= \big|\big\{(j,k)\in[1,s]^{\times 2} \colon j<k \text{ and } \varepsilon_j>\varepsilon_k\big\}\big| \\
w_-(\underline{i}) &:= \sum_{k\in S_-(\underline{i})} s-k \\
w_+(\underline{i}) &:= \sum_{k\in S_+(\underline{i})} k \\
w_\pm(\underline{i}) &:= w_-(\underline{i})+w_+(\underline{i})\,.
\end{aligned}$$

We call $w(\underline{i}) := \big(w_0(\underline{i}),\, w_\pm(\underline{i})\big) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ the *weight* of $\underline{i}$.

(2) The *lexicographic order* $(\leqslant)$ on $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is given as follows.

For a, b, c, $d \in \mathbb{Z}_{\geqslant 0}$, we have

$$(a, b) \leqslant (c, d) \quad :\Leftrightarrow \quad (a < c) \vee \big((a = c) \wedge (b \leqslant d)\big)\,.$$

So, for $\underline{i}$, $\underline{j} \in R_{m,\tilde{m}}$, we have

$$w(\underline{i}) \leqslant w(\underline{j}) \quad \Leftrightarrow \quad \big(w_0(\underline{i}) < w_0(\underline{j})\big) \vee \Big(\big(w_0(\underline{i}) = w_0(\underline{j})\big) \wedge \big(w_\pm(\underline{i}) \leqslant w_\pm(\underline{j})\big)\Big)\,.$$

Note that $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is linearly ordered via $(\leqslant)$ and that each nonempty subset $S \subseteq \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ has a minimal element $\min(S)$.

Definition 239 We write $R^0_{m,\tilde{m}} := \big\{\underline{i} \in R_{m,\tilde{m}} \colon w_0(\underline{i}) = 0\big\} \subseteq R_{m,\tilde{m}}$.

Note that for $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_s\,, i_s)\big) \in R^0_{m,\tilde{m}}$ we have

$$\underline{i} = \big((-1, i_1), \ldots, (-1, i_\ell), (+1, i_{\ell+1}), \ldots, (+1, i_s)\big)$$

for some $\ell \in [0, s]$.

Definition 240 Suppose given $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_s\,, i_s)\big) \in R_{m,\tilde{m}}$.

For $k \in [1, s]$, we write

$$\tau_{i_k}^{(\varepsilon_k)} := \begin{cases} \sigma_{i_k}^{\pm, r_k} & \text{if } \varepsilon_k = -1 \\ \delta_{i_k}^{\pm, r_k} & \text{if } \varepsilon_k = +1\,, \end{cases}$$

where $r_k := m + \sum\limits_{\ell \in [1,k]} \varepsilon_\ell$; cf. Definition 174. So $\tau_{i_k}^{(\varepsilon_k)} \colon [r_{k-1}]^\pm \to [r_k]^\pm$ in $\Delta^\pm_{\lfloor n, -\infty \rfloor}$.

Moreover, we write

$$\tau_{\underline{i}} := \prod_{k \in \lceil 1, s \rceil}^{[m]^\pm \; [\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} = \tau_{i_1}^{(\varepsilon_1)} \blacktriangle \tau_{i_2}^{(\varepsilon_2)} \blacktriangle \ldots \blacktriangle \tau_{i_s}^{(\varepsilon_s)} \in \mathrm{Mor}\Big(\Delta^\pm_{\lfloor n, -\infty \rfloor}\Big)\,.$$

Example 241 Let $n = -\infty$.

(1) Let $m = 5$ and $\tilde{m} = 0$. Suppose given

$$\underline{i}_0 := \big((-1, 3), (-1, 1), (-1, 5), (-1, 4), (-1, 4)\big) \in R_{m,\tilde{m}}\,.$$

We have

$$\begin{array}{rl}
 & \big((-1,3),(-1,1),(-1,5),(-1,4),(-1,4)\big) = \underline{i}_0 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,3),(-1,1),(-1,5),(-1,5),(-1,4)\big) =: \underline{i}_1 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,3),(-1,1),(-1,6),(-1,5),(-1,4)\big) =: \underline{i}_2 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,3),(-1,7),(-1,1),(-1,5),(-1,4)\big) =: \underline{i}_3 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,3),(-1,7),(-1,6),(-1,1),(-1,4)\big) =: \underline{i}_4 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,3),(-1,7),(-1,6),(-1,5),(-1,1)\big) =: \underline{i}_5 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,8),(-1,3),(-1,6),(-1,5),(-1,1)\big) =: \underline{i}_6 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,8),(-1,7),(-1,3),(-1,5),(-1,1)\big) =: \underline{i}_7 \\
\overline{\overline{\rightsquigarrow}} & \big((-1,8),(-1,7),(-1,6),(-1,3),(-1,1)\big) =: \underline{i}_8 \in R_{m,\tilde{m}}\,;
\end{array}$$

cf. Definition 237.

We have

$$\begin{array}{llllllllll}
S_-(\underline{i}_0) & = & \{1,2,3,4\}\,, & S_-(\underline{i}_1) & = & \{1,2,3\}\,, & S_-(\underline{i}_2) & = & \{1,2\}\,, & S_-(\underline{i}_3) & = & \{1,3\} \\
S_-(\underline{i}_4) & = & \{1,4\}\,, & S_-(\underline{i}_5) & = & \{1\}\,, & S_-(\underline{i}_6) & = & \{2\}\,, & S_-(\underline{i}_7) & = & \{3\} \\
S_-(\underline{i}_8) & = & \emptyset\,; & & & & & & & & &
\end{array}$$

cf. Definition 238. In fact:

	1	2	3	4	5
$\underline{i}_0$	$3+1$	$1+2$	$5+3$	$4+4$	$4+5$
$\underline{i}_1$	$3+1$	$1+2$	$5+3$	$5+4$	$4+5$
$\underline{i}_2$	$3+1$	$1+2$	$6+3$	$5+4$	$4+5$
$\underline{i}_3$	$3+1$	$7+2$	$1+3$	$5+4$	$4+5$
$\underline{i}_4$	$3+1$	$7+2$	$6+3$	$1+4$	$4+5$
$\underline{i}_5$	$3+1$	$7+2$	$6+3$	$5+4$	$1+5$
$\underline{i}_6$	$8+1$	$3+2$	$6+3$	$5+4$	$1+5$
$\underline{i}_7$	$8+1$	$7+2$	$3+3$	$5+4$	$1+5$
$\underline{i}_8$	$8+1$	$7+2$	$6+3$	$3+4$	$1+5$

$=$

	1	2	3	4	5
$\underline{i}_0$	4	3	8	8	9
$\underline{i}_1$	4	3	8	9	9
$\underline{i}_2$	4	3	9	9	9
$\underline{i}_3$	4	9	4	9	9
$\underline{i}_4$	4	9	9	5	9
$\underline{i}_5$	4	9	9	9	6
$\underline{i}_6$	9	5	9	9	6
$\underline{i}_7$	9	9	6	9	6
$\underline{i}_8$	9	9	9	7	6

So we have

$$\begin{aligned}
w_-(\underline{i}_0) &= (5-1)+(5-2)+(5-3)+(5-4) = 10\\
w_-(\underline{i}_1) &= (5-1)+(5-2)+(5-3) = 9\\
w_-(\underline{i}_2) &= (5-1)+(5-2) = 7\\
w_-(\underline{i}_3) &= (5-1)+(5-3) = 6\\
w_-(\underline{i}_4) &= (5-1)+(5-4) = 5\\
w_-(\underline{i}_5) &= 5-1 = 4\\
w_-(\underline{i}_6) &= 5-2 = 3\\
w_-(\underline{i}_7) &= 5-3 = 2\\
w_-(\underline{i}_8) &= 0\,.
\end{aligned}$$

(2) Let $m = 0$ and $\tilde{m} = 4$.

Suppose given

$$\underline{j}_0 := \big((+1,2),(+1,1),(+1,4),(+1,2)\big) \in R_{m,\tilde{m}}\,.$$

We have

$$\begin{aligned}
& \big((+1,2),(+1,1),(+1,4),(+1,2)\big) && = \underline{j}_0\\
\overset{++}{\rightsquigarrow}\ & \big((+1,1),(+1,3),(+1,4),(+1,2)\big) && =: \underline{j}_1\\
\overset{++}{\rightsquigarrow}\ & \big((+1,1),(+1,3),(+1,2),(+1,5)\big) && =: \underline{j}_2\\
\overset{++}{\rightsquigarrow}\ & \big((+1,1),(+1,2),(+1,4),(+1,5)\big) && =: \underline{j}_3 \in R_{m,\tilde{m}}\,;
\end{aligned}$$

cf. Definition 237.

We have $S_+(\underline{j}_0) = \{2,4\}$, $S_+(\underline{j}_1) = \{4\}$, $S_+(\underline{j}_2) = \{3\}$ and $S_+(\underline{j}_3) = \emptyset$; cf. Definition 238. In fact:

	1	2	3	4
$\underline{j}_0$	$2-1$	$1-2$	$4-3$	$2-4$
$\underline{j}_1$	$1-1$	$3-2$	$4-3$	$2-4$
$\underline{j}_2$	$1-1$	$3-2$	$2-3$	$5-4$
$\underline{j}_3$	$1-1$	$2-2$	$4-3$	$5-4$

$=$

	1	2	3	4
$\underline{j}_0$	1	(-1)	1	[-2]
$\underline{j}_1$	(0)	1	1	[-2]
$\underline{j}_2$	0	1	[-1]	1
$\underline{j}_3$	0	[0]	1	1

So we have

$$\begin{aligned} w_+(\underline{j}_0) &= 2+4=6 \\ w_+(\underline{j}_1) &= 4 \\ w_+(\underline{j}_2) &= 3 \\ w_+(\underline{j}_3) &= 0\,. \end{aligned}$$

(3) Let $m=3$ and $\tilde{m}=0$.

Suppose given

$$\underline{i}_0 := \big((-1,2),(-1,2),(-1,3)\big) \in R_{m,\tilde{m}}\,.$$

We have

$$\begin{array}{rl} & \big((-1,2),(-1,2),(-1,3)\big) \quad = \underline{i}_0 \\ \overset{--}{\rightsquigarrow} & \big((-1,3),(-1,2),(-1,3)\big) \quad =: \underline{i}_1 \\ \overset{--}{\rightsquigarrow} & \big((-1,3),(-1,4),(-1,2)\big) \quad =: \underline{i}_2 \\ \overset{--}{\rightsquigarrow} & \big((-1,5),(-1,3),(-1,2)\big) \quad =: \underline{i}_3 \in R_{m,\tilde{m}}\,; \end{array}$$

cf. Defnition 237.

We have

$$\begin{aligned} S_-(\underline{i}_0) &= \{1,2\} \\ S_-(\underline{i}_1) &= \{1,2\} \\ S_-(\underline{i}_2) &= \{1\} \\ S_-(\underline{i}_3) &= \emptyset\,; \end{aligned}$$

cf. Definition 238. In fact:

	1	2	3
$\underline{i}_0$	$2+1$	$2+2$	$3+3$
$\underline{i}_1$	$3+1$	$2+2$	$3+3$
$\underline{i}_2$	$3+1$	$4+2$	$2+3$
$\underline{i}_3$	$5+1$	$3+2$	$2+3$

$=$

	1	2	3
$\underline{i}_0$	(3)	[4]	6
$\underline{i}_1$	[4]	(4)	6
$\underline{i}_2$	[4]	6	(5)
$\underline{i}_3$	6	[5]	5

So we have

$$\begin{aligned} w_-(\underline{i}_0) &= (3-1)+(3-2)=3 \\ w_-(\underline{i}_1) &= (3-1)+(3-2)=3 \\ w_-(\underline{i}_2) &= 3-1=2 \\ w_-(\underline{i}_3) &= 0\,. \end{aligned}$$

Note that in this example, we have $\underline{i}_0 \overset{--}{\rightsquigarrow} \underline{i}_1$ but $w_-(\underline{i}_0) \not> w_-(\underline{i}_1)$.

Remark 242 *Suppose given* $\underline{i}$, $\underline{j} \in R_{m,\tilde{m}}$.
We have

$$\underline{i} \sim \underline{j} \;\Rightarrow\; \tau_{\underline{i}} = \tau_{\underline{j}}\,.$$

Proof. We may assume that $\underline{i} \rightsquigarrow \underline{j}$.
We write $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_s\,, i_s)\big)$ and $\underline{j} = \big((\tilde{\varepsilon}_1\,, j_1), \ldots, (\tilde{\varepsilon}_t\,, j_t)\big)$.
Case $\underline{i} \overset{++}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$(\tilde{\varepsilon}_k\,, j_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s]\backslash\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1}) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell + 1) & \text{for } k = \ell+1 \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \geqslant i_{\ell+1}$, and $\varepsilon_\ell = \varepsilon_{\ell+1} = +1$.
So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}), (\varepsilon_\ell\,, i_\ell + 1), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.
Let $r := m + \sum\limits_{k \in [1, \ell-1]} \varepsilon_k$.
Since $i_{\ell+1} < i_\ell + 1$, we have

$$\begin{aligned}
\tau_{\underline{j}} &= \prod_{k \in \lceil 1,s \rceil}^{[m]^\pm \,[\tilde{m}]^\pm} \tau_{j_k}^{(\tilde{\varepsilon}_k)} \\
&= \Big(\prod_{k \in \lceil 1, \ell-1 \rceil}^{[m]^\pm \,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle \tau_{i_{\ell+1}}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell+1}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k \in \lceil \ell+2, s \rceil}^{[r+2]^\pm \,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{240}{=} \Big(\prod_{k \in \lceil 1, \ell-1 \rceil}^{[m]^\pm \,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\delta_{i_{\ell+1}} \blacktriangle \delta_{i_\ell+1}) \blacktriangle \Big(\prod_{k \in \lceil \ell+2, s \rceil}^{[r+2]^\pm \,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{179.(1)}{=} \Big(\prod_{k \in \lceil 1, \ell-1 \rceil}^{[m]^\pm \,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\delta_{i_\ell} \blacktriangle \delta_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k \in \lceil \ell+2, s \rceil}^{[r+2]^\pm \,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{240}{=} \prod_{k \in \lceil 1,s \rceil}^{[m]^\pm \,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \\
&= \tau_{\underline{i}}\,.
\end{aligned}$$

Case $\underline{i} \overset{--}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$(\tilde{\varepsilon}_k\,, j_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s]\backslash\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1}+1) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell) & \text{for } k = \ell+1 \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \leqslant i_{\ell+1}$, and $\varepsilon_\ell = \varepsilon_{\ell+1} = -1$.

So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}+1), (\varepsilon_\ell\,, i_\ell), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.

Let $r := m + \sum\limits_{k\in[1,\ell-1]} \varepsilon_k$.

Since $i_\ell < i_{\ell+1} + 1$, we have

$$\begin{aligned}
\tau_{\underline{j}} \quad &= \quad \prod_{k\in\lceil 1,s\rceil}^{[m]^\pm \quad [\tilde{m}]^\pm} \tau_{j_k}^{(\tilde{\varepsilon}_k)} \\
&= \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm \quad [r]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle \tau_{i_{\ell+1}+1}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r-2]^\pm \quad [\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{240}{=} \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm \quad [r]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\sigma_{i_{\ell+1}+1} \blacktriangle \sigma_{i_\ell}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r-2]^\pm \quad [\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{179.(2)}{=} \quad \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm \quad [r]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \blacktriangle (\sigma_{i_\ell} \blacktriangle \sigma_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r-2]^\pm \quad [\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \Big) \\
&\overset{240}{=} \quad \prod_{k\in\lceil 1,s\rceil}^{[m]^\pm \quad [\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \\
&= \quad \tau_{\underline{i}}\,.
\end{aligned}$$

Case $\underline{i} \overset{+-,1}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$(\tilde{\varepsilon}_k\,, j_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s]\backslash\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1}-1) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell) & \text{for } k = \ell+1 \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \leqslant i_{\ell+1} - 1$, and $\varepsilon_\ell = +1$, $\varepsilon_{\ell+1} = -1$.

So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}-1), (\varepsilon_\ell\,, i_\ell), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.

Let $r := m + \sum_{k\in[1,\ell-1]} \varepsilon_k$.

Since $i_\ell < i_{\ell+1}$, we have

$$\begin{aligned}
\tau_{\underline{j}} &= \prod_{k\in\lceil 1,s\rceil}^{[m]^\pm\,[\tilde{m}]^\pm} \tau_{j_k}^{(\tilde{\varepsilon}_k)} \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm\,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \tau_{i_{\ell+1}-1}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r]^\pm\,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{240}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm\,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle (\sigma_{i_{\ell+1}-1} \blacktriangle \delta_{i_\ell}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r]^\pm\,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{179.(3)}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm\,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle (\delta_{i_\ell} \blacktriangle \sigma_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r]^\pm\,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{240}{=} \prod_{k\in\lceil 1,s\rceil}^{[m]^\pm\,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)} \\
&= \tau_{\underline{i}}\,.
\end{aligned}$$

Case $\underline{i} \overset{+-,2}{\rightsquigarrow} \underline{j}$. Then $s-2 = t$ and we have

$$(\tilde{\varepsilon}_k\,, j_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1, \ell-1] \\ (\varepsilon_{k+2}\,, i_{k+2}) & \text{for } k \in [\ell, s-2] \end{cases}$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \in \{i_{\ell+1}\,, i_{\ell+1}+1\}$, and $\varepsilon_\ell = +1\,,\ \varepsilon_{\ell+1} = -1$.

So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.

Let $r := m + \sum_{k\in[1,\ell-1]} \varepsilon_k$.

Since $i_\ell \in \{i_{\ell+1}\,, i_{\ell+1}+1\}$, we have

$$\begin{aligned}
\tau_{\underline{j}} &= \prod_{k\in\lceil 1,s-2\rceil}^{[m]^\pm\,[\tilde{m}]^\pm} \tau_{j_k}^{(\tilde{\varepsilon}_k)} \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm\,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil \ell,s-2\rceil}^{[r]^\pm\,[\tilde{m}]^\pm} \tau_{i_{k+2}}^{(\varepsilon_{k+2})}\Big) \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rceil}^{[m]^\pm\,[r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rceil}^{[r]^\pm\,[\tilde{m}]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]^\pm\ [r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \mathrm{id}_{[r]^\pm} \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]^\pm\ [\tilde m]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{179.(3)}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]^\pm\ [r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle (\delta_{i_\ell} \blacktriangle \sigma_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]^\pm\ [\tilde m]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{240}{=} \prod_{k\in\lceil 1,s\rfloor}^{[m]^\pm\ [\tilde m]^\pm} \tau_{i_k}^{(\varepsilon_k)} \\
&= \tau_{\underline{i}}\,.
\end{aligned}
$$

Case $\underline{i} \overset{+-,3}{\rightsquigarrow} \underline{j}$. Then $s = t$ and we have

$$
(\tilde\varepsilon_k\,, j_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s]\backslash\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1}) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell - 1) & \text{for } k = \ell+1 \end{cases}
$$

for some chosen $\ell \in [1, s-1]$ with $i_\ell \geqslant i_{\ell+1} + 2$, and $\varepsilon_\ell = +1$, $\varepsilon_{\ell+1} = -1$.
So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}), (\varepsilon_\ell\,, i_\ell - 1), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.
Let $r := m + \sum\limits_{k\in[1,\ell-1]} \varepsilon_k$.
Since $i_\ell > i_{\ell+1} + 1$, we have

$$
\begin{aligned}
\tau_{\underline{j}} &= \prod_{k\in\lceil 1,s\rfloor}^{[m]^\pm\ [\tilde m]^\pm} \tau_{j_k}^{(\tilde\varepsilon_k)} \\
&= \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]^\pm\ [r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \tau_{i_{\ell+1}}^{(\varepsilon_{\ell+1})} \blacktriangle \tau_{i_\ell-1}^{(\varepsilon_\ell)} \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]^\pm\ [\tilde m]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{240}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]^\pm\ [r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle (\sigma_{i_{\ell+1}} \blacktriangle \delta_{i_\ell-1}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]^\pm\ [\tilde m]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{179.(3)}{=} \Big(\prod_{k\in\lceil 1,\ell-1\rfloor}^{[m]^\pm\ [r]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle (\delta_{i_\ell} \blacktriangle \sigma_{i_{\ell+1}}) \blacktriangle \Big(\prod_{k\in\lceil \ell+2,s\rfloor}^{[r]^\pm\ [\tilde m]^\pm} \tau_{i_k}^{(\varepsilon_k)}\Big) \\
&\overset{240}{=} \prod_{k\in\lceil 1,s\rfloor}^{[m]^\pm\ [\tilde m]^\pm} \tau_{i_k}^{(\varepsilon_k)} \\
&= \tau_{\underline{i}}\,.
\end{aligned}
$$

Altogether, this shows $\tau_{\underline{i}} = \tau_{\underline{j}}$. □

Remark 243 *Suppose given $\underline{i} \in R^0_{m,\tilde{m}}$; cf. Definition* 239.

(1) *Suppose that $S_+(\underline{i}) \neq \emptyset$; cf. Definition* 238.

Then there exists some $\underline{j} \in R^0_{m,\tilde{m}}$ with $i \overset{++}{\rightsquigarrow} \underline{j}$ and $w_+(\underline{i}) > w_+(\underline{j})$ and $w_-(\underline{i}) = w_-(\underline{j})$.

(2) *Suppose that $S_-(\underline{i}) \neq \emptyset$; cf. Definition* 238.

Then there exists some $\underline{j} \in R^0_{m,\tilde{m}}$ with $i \overset{--}{\rightsquigarrow} \underline{j}$ and $w_-(\underline{i}) > w_-(\underline{j})$ and $w_+(\underline{i}) = w_+(\underline{j})$.

Proof. Since $w_0(\underline{i}) = 0$, we have

$$\underline{i} = \big((-1\,, i_1), \ldots\,, (-1,\, i_t), (+1,\, i_{t+1}), \ldots,\, (+1\,, i_s)\big)$$

for $t = \big|\{k \in [1,s] \colon \varepsilon_k = -1\}\big|$.

Ad (1). We consider the sets

$$E_+(\underline{i}) = \{k \in [1,s] \colon \varepsilon_k = +1\} = [t+1, s]$$

and

$$\begin{aligned} S_+(\underline{i}) &= \big\{j \in E_+(\underline{i}) \colon \text{there exists some } k \in E_+(\underline{i}) \text{ with } k < j \text{ and } i_k - k > i_j - j\big\} \\ &= \big\{j \in [t+1, s] \colon \text{there exists some } k \in [t+1, s] \text{ with } k < j \text{ and } i_k - k > i_j - j\big\}\,; \end{aligned}$$

cf. Definition 238.

Let $\ell := \min\big(S_+(\underline{i})\big) \geqslant t+2$. Then $\varepsilon_{\ell-1} = \varepsilon_\ell = +1$.

We show that $i_{\ell-1} \overset{!}{\geqslant} i_\ell$.

We *assume* that $i_\ell \geqslant i_{\ell-1} + 1$.

Then we have

$$i_\ell - \ell \geqslant i_{\ell-1} - (\ell - 1)\,.$$

Since $\ell \in S_+(\underline{i})$, we may choose some $k \in [t+1, \ell-1]$ with $i_k - k > i_\ell - \ell$.

Case $k = \ell - 1$. Then we have $i_{\ell-1} - (\ell-1) > i_\ell - \ell$, which *contradicts* $i_\ell - \ell \geqslant i_{\ell-1} - (\ell-1)$.

Case $k \leqslant \ell - 2$. Then we have

$$i_k - k > i_\ell - \ell \geqslant i_{\ell-1} - (\ell - 1)\,,$$

which shows $\ell - 1 \in S_+(\underline{i})$. This *contradicts* $\ell = \min\big(S_+(\underline{i})\big)$.

So we have a *contradiction* in both cases.

This shows $i_{\ell-1} \geqslant i_\ell$.

We consider the $(m, \tilde{m})$-tuple $\underline{j} = \big((\tilde{\varepsilon}_1, j_1), \dots, (\tilde{\varepsilon}_s, j_s)\big) \in R_{m,\tilde{m}}$, where

$$(\tilde{\varepsilon}_k, j_k) = \begin{cases} (\varepsilon_k, i_k) & \text{for } k \in [1,s]\backslash\{\ell-1, \ell\} \\ (\varepsilon_\ell, i_\ell) & \text{for } k = \ell - 1 \\ (\varepsilon_{\ell-1}, i_{\ell-1} + 1) & \text{for } k = \ell\,. \end{cases}$$

So

$$\underline{j} = \big((\varepsilon_1, i_1), \dots, (\varepsilon_{\ell-2}, i_{\ell-2}), (\varepsilon_\ell, i_\ell), (\varepsilon_{\ell-1}, i_{\ell-1} + 1), (\varepsilon_{\ell+1}, i_{\ell+1}), \dots, (\varepsilon_s, i_s)\big)\,.$$

Then we have $\underline{i} \overset{++}{\rightsquigarrow} \underline{j}$ and $0 = w_0(\underline{i}) = w_0(\underline{j})$, so $\underline{j} \in R^0_{m,\tilde{m}}$.

Moreover, we have $S_-(\underline{i}) = S_-(\underline{j})$ and thus $w_-(\underline{i}) = w_-(\underline{j})$.

We show that $S_+(\underline{i})\backslash\{\ell-1, \ell\} \overset{!}{=} S_+(\underline{j})\backslash\{\ell-1, \ell\}$.

Ad $\subseteq$. Suppose given $k \in S_+(\underline{i})\backslash\{\ell-1, \ell\}$. Then we may choose some $r \in [t+1, k-1]$ with $i_r - r > i_k - k$.

Case $r \in [t+1, k-1]\backslash\{\ell-1, \ell\}$. We have

$$j_r - r = i_r - r > i_k - k = j_k - k\,.$$

Case $r = \ell - 1$. Note that $\ell - 1 = r \leqslant k - 1$ and $\ell \neq k$ implies $\ell < k$.

We have

$$j_\ell - \ell = i_{\ell-1} + 1 - \ell = i_{\ell-1} - (\ell - 1) > i_k - k = j_k - k\,.$$

Case $r = \ell$. We have

$$j_{\ell-1} - (\ell - 1) = i_\ell - (\ell - 1) > i_\ell - \ell > i_k - k = j_k - k\,.$$

Altogether, this shows $k \in S_+(\underline{j})\backslash\{\ell-1, \ell\}$.

Ad $\supseteq$. Suppose given $k \in S_+(\underline{j})\backslash\{\ell-1, \ell\}$. Then we may choose some $r \in [t+1, k-1]$ with $j_r - r > j_k - k$.

Case $r \in [t+1, k-1]\backslash\{\ell-1, \ell\}$. We have

$$i_r - r = j_r - r > j_k - k = i_k - k\,.$$

Case $r = \ell - 1$. We have

$$i_{\ell-1} - (\ell-1) \geqslant i_\ell - (\ell-1) = j_{\ell-1} - (\ell-1) > j_k - k = i_k - k\,.$$

Case $r = \ell$. We have

$$i_{\ell-1} - (\ell-1) = (i_{\ell-1}+1) - \ell = j_\ell - \ell > j_k - k = i_k - k\,.$$

Altogether, this shows $k \in S_+(\underline{i})\backslash\{\ell-1, \ell\}$.

So we have $S_+(\underline{i})\backslash\{\ell-1,\ell\} = S_+(\underline{j})\backslash\{\ell-1,\ell\}$.

We show that $\ell \overset{!}{\notin} S_+(\underline{j})$.

So we have to show that $j_k - k \overset{!}{\leqslant} j_\ell - \ell$ for $k \in [t+1, \ell-1]$.

Case $k = \ell - 1$. We have

$$j_k - k = j_{\ell-1} - (\ell-1) = i_\ell - (\ell-1) \leqslant i_{\ell-1} - (\ell-1) = j_\ell - \ell\,.$$

Case $k \leqslant \ell - 2$. Since $\ell - 1 \notin S_+(\underline{i})$, note that $i_k - k \leqslant i_{\ell-1} - (\ell-1)$.

So we have

$$j_k - k = i_k - k \leqslant i_{\ell-1} - (\ell-1) = j_\ell - \ell\,.$$

This shows $\ell \notin S_+(\underline{j})$.

We show that $w_+(\underline{i}) \overset{!}{>} w_+(\underline{j})$.

Recall that $\ell \in S_+(\underline{i})$ and that $\ell \notin S_+(\underline{j})$. Recall that $\ell - 1 \notin S_+(\underline{i})$.

Case $\ell - 1 \in S_+(\underline{j})$. Then we have

$$S_+(\underline{j}) = \big(S_+(\underline{j})\backslash\{\ell-1,\ell\}\big) \cup \{\ell-1\} = \big(S_+(\underline{i})\backslash\{\ell-1,\ell\}\big) \cup \{\ell-1\} = \big(S_+(\underline{i})\backslash\{\ell\}\big) \cup \{\ell-1\}\,.$$

So we have

$$w_+(\underline{j}) = \sum_{k\in S_+(\underline{j})} k = \sum_{k\in(S_+(\underline{i})\backslash\{\ell\})\cup\{\ell-1\}} k \quad = \big(\sum_{k\in S_+(\underline{i})} k\big) - \ell + (\ell-1) = w_+(\underline{i}) - 1\,.$$

Case $\ell - 1 \notin S_+(\underline{j})$. Then we have

$$S_+(\underline{j}) = S_+(\underline{j})\backslash\{\ell-1,\ell\} = S_+(\underline{i})\backslash\{\ell-1,\ell\} = S_+(\underline{i})\backslash\{\ell\}\,.$$

So we have

$$w_+(\underline{j}) = \sum_{k\in S_+(\underline{j})} k = \sum_{k\in S_+(\underline{i})\setminus\{\ell\}} k = \big(\sum_{k\in S_+(\underline{i})} k\big) - \ell = w_+(\underline{i}) - \ell\,.$$

Note that $\ell \geqslant t+2 > 0$. This shows $w_+(\underline{i}) > w_+(\underline{j})$.

So $\underline{i} \overset{++}{\rightsquigarrow} \underline{j}$ and $w_+(\underline{i}) > w_+(\underline{j})$ and $w_-(\underline{i}) = w_-(\underline{j})$.

Ad (2). We consider the sets

$$E_-(\underline{i}) = \{k \in [1,s] \colon \varepsilon_k = -1\} = [1,t]$$

and

$$\begin{aligned} S_-(\underline{i}) &= \big\{j \in E_-(\underline{i}) \colon \text{there exists some } k \in E_-(\underline{i}) \text{ with } j<k \text{ and } i_j + j < i_k + k\big\} \\ &= \big\{j \in [1,t] \colon \text{there exists some } k \in [1,t] \text{ with } j<k \text{ and } i_j + j < i_k + k\big\}\,; \end{aligned}$$

cf. Definition 238.

Let $\ell := \max\big(S_-(\underline{i})\big) \leqslant t-1$. Then $\varepsilon_\ell = \varepsilon_{\ell+1} = -1$.

We show that $i_\ell \overset{!}{\leqslant} i_{\ell+1}$.

We *assume* that $i_{\ell+1} + 1 \leqslant i_\ell$.

Then we have

$$i_{\ell+1} + (\ell+1) \leqslant i_\ell + \ell\,.$$

Since $\ell \in S_-(\underline{i})$, we may choose some $k \in [\ell+1, t]$ with $i_\ell + \ell < i_k + k$.

Case $k = \ell+1$. Then we have $i_\ell + \ell < i_{\ell+1} + (\ell+1)$, which *contradicts* $i_{\ell+1} + (\ell+1) \leqslant i_\ell + \ell$.

Case $k \geqslant \ell+2$. Then we have

$$i_{\ell+1} + (\ell+1) \leqslant i_\ell + \ell < i_k + k\,,$$

which shows $\ell+1 \in S_-(\underline{i})$. This *contradicts* $\ell = \max\big(S_-(\underline{i})\big)$.

So we have a *contradiction* in both cases.

This shows $i_\ell \leqslant i_{\ell+1}$.

We consider the $(m,\tilde{m})$-tuple $\underline{j} = \big((\tilde{\varepsilon}_1\,, j_1), \ldots, (\tilde{\varepsilon}_s\,, j_s)\big) \in R_{m,\tilde{m}}$ where

$$(\tilde{\varepsilon}_k\,, j_k) = \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s]\setminus\{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1}+1) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell) & \text{for } k = \ell+1\,. \end{cases}$$

So

$$\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}+1), (\varepsilon_\ell\,, i_\ell), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)\,.$$

Then we have $\underline{i} \overset{--}{\rightsquigarrow} \underline{j}$ and $0 = w_0(\underline{i}) = w_0(\underline{j}) = 0$, so $\underline{j} \in R^0_{m,\tilde{m}}$.

Moreover, we have $S_+(\underline{i}) = S_+(\underline{j})$ and thus $w_+(\underline{i}) = w_+(\underline{j})$.

We show that $S_-(\underline{i})\backslash\{\ell, \ell+1\} \overset{!}{=} S_-(\underline{j})\backslash\{\ell, \ell+1\}$.

Ad $\subseteq$. Suppose given $k \in S_-(\underline{i})\backslash\{\ell, \ell+1\}$. Then we may choose some $r \in [k+1, t]$ with $i_k + k < i_r + r$.

Case $r \in [k+1, t]\backslash\{\ell, \ell+1\}$. We have

$$j_k + k = i_k + k < i_r + r = j_r + r\,.$$

Case $r = \ell$. We have

$$j_k + k = i_k + k < i_\ell + \ell = j_{\ell+1} + \ell < j_{\ell+1} + (\ell+1)\,.$$

Case $r = \ell + 1$. Note that $k + 1 \leqslant r = \ell + 1$ and $k \neq \ell$ implies $k < \ell$.

We have

$$j_k + k = i_k + k < i_{\ell+1} + (\ell+1) = j_\ell - 1 + (\ell+1) = j_\ell + \ell\,.$$

Altogether, this shows $k \in S_-(\underline{j})\backslash\{\ell, \ell+1\}$.

Ad $\supseteq$. Suppose given $k \in S_-(\underline{j})\backslash\{\ell, \ell+1\}$. Then we may choose some $r \in [k+1, t]$ with $j_k + k < j_r + r$.

Case $r \in [k+1, t]\backslash\{\ell, \ell+1\}$. We have

$$i_k + k = j_k + k < j_r + r = i_r + r\,.$$

Case $r = \ell$. We have

$$i_k + k = j_k + k < j_\ell + \ell = i_{\ell+1} + 1 + \ell = i_{\ell+1} + (\ell+1)\,.$$

Case $r = \ell + 1$. We have

$$i_k + k = j_k + k < j_{\ell+1} + (\ell+1) = i_\ell + (\ell+1) \leqslant i_{\ell+1} + (\ell+1)\,.$$

Altogether, this shows $k \in S_-(\underline{i})\backslash\{\ell, \ell+1\}$.

So we have $S_-(\underline{i})\backslash\{\ell,\ell+1\} = S_-(\underline{j})\backslash\{\ell,\ell+1\}$.

We show that $\ell \overset{!}{\notin} S_-(\underline{j})$.

So we have to show that $j_\ell + \ell \overset{!}{\geqslant} j_k + k$ for $k \in [\ell+1, t]$.

Case $k = \ell+1$. We have

$$j_\ell + \ell = i_{\ell+1} + 1 + \ell \geqslant i_\ell + (\ell+1) = j_{\ell+1} + (\ell+1) = j_k + k\,.$$

Case $k \geqslant \ell+2$. Since $\ell+1 \notin S_-(\underline{i})$, note that $i_{\ell+1} + (\ell+1) \geqslant i_k + k$.

So we have

$$j_\ell + \ell = i_{\ell+1} + 1 + \ell = i_{\ell+1} + (\ell+1) \geqslant i_k + k = j_k + k\,.$$

This shows $\ell \notin S_-(\underline{j})$.

We show that $w_-(\underline{i}) \overset{!}{>} w_-(\underline{j})$.

Recall that $\ell \in S_-(\underline{i})$ and that $\ell \notin S_-(\underline{j})$. Recall that $\ell+1 \notin S_-(\underline{i})$.

Case $\ell+1 \in S_-(\underline{j})$. Then we have

$$S_-(\underline{j}) = \big(S_-(\underline{j})\backslash\{\ell,\ell+1\}\big) \cup \{\ell+1\} = \big(S_-(\underline{i})\backslash\{\ell,\ell+1\}\big) \cup \{\ell+1\} = \big(S_-(\underline{i})\backslash\{\ell\}\big) \cup \{\ell+1\}\,.$$

So we have

$$w_-(\underline{j}) = \sum_{k\in S_-(\underline{j})} s-k = \sum_{k\in(S_-(\underline{i})\backslash\{\ell\})\cup\{\ell+1\}} s-k = \big(\sum_{k\in S_-(\underline{i})} s-k\big) - (s-\ell) + \big(s-(\ell+1)\big) = w_-(\underline{i}) - 1\,.$$

Case $\ell+1 \notin S_-(\underline{j})$. Then we have

$$S_-(\underline{j}) = S_-(\underline{j})\backslash\{\ell,\ell+1\} = S_-(\underline{i})\backslash\{\ell,\ell+1\} = S_-(\underline{i})\backslash\{\ell\}\,.$$

So we have

$$w_-(\underline{j}) = \sum_{k\in S_-(\underline{j})} s-k = \sum_{k\in S_-(\underline{i})\backslash\{\ell\}} s-k = \big(\sum_{k\in S_-(\underline{i})} s-k\big) - (s-\ell) = w_-(\underline{i}) - (s-\ell)\,.$$

Note that $\ell \leqslant t-1 < s$. So we have $0 < s-\ell$.

This shows $w_-(\underline{i}) > w_-(\underline{j})$.

So $\underline{i} \overset{--}{\rightsquigarrow} \underline{j}$ and $w_-(\underline{i}) > w_-(\underline{j})$. □

Remark 244 *Suppose given $\underline{i} \in R_{m,\tilde{m}}$ with $w(\underline{i}) > (0,0)$; cf. Definition 238. Then there exists some $\underline{j} \in [\underline{i}]$ with $\underline{i} \rightsquigarrow \underline{j}$ and $w(\underline{i}) > w(\underline{j})$.*

Proof. Case $w_0(\underline{i}) > 0$.

We consider the set

$$S_0(\underline{i}) := \big\{ j \in [1,s] \colon \varepsilon_j = +1, \text{ there exists some } k \in [1,s] \text{ with } \varepsilon_k = -1 \text{ and } j < k \big\}$$

Note that $S_0(\underline{i}) \neq \emptyset$ since $w_0(\underline{i}) > 0$.

Let $\ell := \max\big(S_0(\underline{i})\big) \leqslant s-1$. Then $\varepsilon_\ell = +1$.

We show that $\varepsilon_{\ell+1} \overset{!}{=} -1$.

We *assume* that $\varepsilon_{\ell+1} = +1$.

Since $\ell \in S_0(\underline{i})$, there exists some $k \in [\ell+1, s]$ with $\varepsilon_k = -1$.

Since $\varepsilon_{\ell+1} = +1$, we have $k \in [\ell+2, s]$.

So there exists some $k \in [\ell+2, s]$ with $\varepsilon_k = -1$. This shows $\ell+1 \in S_0(\underline{i})$, which *contradicts* $\ell = \max\big(S_0(\underline{i})\big)$.

This shows $\varepsilon_{\ell+1} = -1$.

Subcase $i_\ell \leqslant i_{\ell+1} - 1$.

We consider the $(m,\tilde{m})$-tuple $\underline{j} = \big((\tilde{\varepsilon}_1\,, j_1), \ldots, (\tilde{\varepsilon}_s\,, j_s)\big) \in R_{m,\tilde{m}}$ where

$$(\tilde{\varepsilon}_k\,, j_k) := \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s] \backslash \{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1} - 1) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell) & \text{for } k = \ell+1\,. \end{cases}$$

So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1} - 1), (\varepsilon_\ell\,, i_\ell), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.

Then we have $\underline{i} \overset{+-,1}{\rightsquigarrow} \underline{j}$ and $w_0(\underline{i}) = w_0(\underline{j}) + 1$.

So $\underline{i} \rightsquigarrow \underline{j}$ and $w(\underline{i}) > w(\underline{j})$.

Subcase $i_\ell \in \{i_{\ell+1}\,, i_{\ell+1} + 1\}$.

We consider the $(m,\tilde{m})$-tuple $\underline{j} = \big((\tilde{\varepsilon}_1\,, j_1), \ldots, (\tilde{\varepsilon}_{s-2}\,, j_{s-2})\big) \in R_{m,\tilde{m}}$ where

$$(\tilde{\varepsilon}_k\,, j_k) := \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1, \ell-1] \\ (\varepsilon_{k+2}\,, i_{k+2}) & \text{for } k \in [\ell, s-2]\,. \end{cases}$$

So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.

Then we have $\underline{i} \overset{+-}{\rightsquigarrow}{}^2 \underline{j}$ and

$$w_0(\underline{i}) = w_0(\underline{j}) + 1 + \Big|\big\{(a,b) \in [1,s]^{\times 2} \colon a < b \text{ and } \varepsilon_a > \varepsilon_b \text{ and } \big|\{a,b\} \cap \{\ell, \ell+1\}\big| = 1\big\}\Big|\,.$$

So $\underline{i} \rightsquigarrow \underline{j}$ and $w(\underline{i}) > w(\underline{j})$.

Subcase $i_\ell \geqslant i_{\ell+1} + 2$.

We consider the $(m, \tilde{m})$-tuple $\underline{j} = \big((\tilde{\varepsilon}_1\,, j_1), \ldots\,, (\tilde{\varepsilon}_s\,, j_s)\big) \in R_{m,\tilde{m}}$ where

$$(\tilde{\varepsilon}_k\,, j_k) := \begin{cases} (\varepsilon_k\,, i_k) & \text{for } k \in [1,s] \backslash \{\ell, \ell+1\} \\ (\varepsilon_{\ell+1}\,, i_{\ell+1}) & \text{for } k = \ell \\ (\varepsilon_\ell\,, i_\ell - 1) & \text{for } k = \ell + 1\,. \end{cases}$$

So $\underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_{\ell-1}\,, i_{\ell-1}), (\varepsilon_{\ell+1}\,, i_{\ell+1}), (\varepsilon_\ell\,, i_\ell - 1), (\varepsilon_{\ell+2}\,, i_{\ell+2}), \ldots, (\varepsilon_s\,, i_s)\big)$.

Then we have $\underline{i} \overset{+-}{\rightsquigarrow}{}^3 \underline{j}$ and $w_0(\underline{i}) = w_0(\underline{j}) + 1$.

So $\underline{i} \rightsquigarrow \underline{j}$ and $w(\underline{i}) > w(\underline{j})$.

Case $w_0(\underline{i}) = 0$ *and* $w_+(\underline{i}) > 0$.

Since $w_0(\underline{i}) = 0$, we have $\underline{i} \in R^0_{m,\tilde{m}}$. Moreover, we have $S_+(\underline{i}) \neq \emptyset$, since $w_+(\underline{i}) > 0$.

Therefore, by Remark 243.(1), we may choose some $\underline{j} \in R^0_{m,\tilde{m}}$ with $\underline{i} \overset{++}{\rightsquigarrow} \underline{j}$ and $w_+(\underline{i}) > w_+(\underline{j})$ and $w_-(\underline{i}) = w_-(\underline{j})$.

So we have $\underline{i} \rightsquigarrow \underline{j}$ and $w(\underline{i}) > w(\underline{j})$.

Case $w_0(\underline{i}) = 0$ *and* $w_-(\underline{i}) > 0$.

Since $w_0(\underline{i}) = 0$, we have $\underline{i} \in R^0_{m,\tilde{m}}$. Moreover, we have $S_-(\underline{i}) \neq \emptyset$, since $w_-(\underline{i}) > 0$.

Therefore, by Remark 243.(2), we may choose some $\underline{j} \in R^0_{m,\tilde{m}}$ with $\underline{i} \overset{--}{\rightsquigarrow} \underline{j}$ and $w_-(\underline{i}) > w_-(\underline{j})$ and $w_+(\underline{i}) = w_+(\underline{j})$.

So we have $\underline{i} \rightsquigarrow \underline{j}$ and $w(\underline{i}) > w(\underline{j})$. □

11.3.2 Standardization of $(m, \tilde{m})$-index tuples

Definition 245 Suppose given $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_s\,, i_s)\big) \in R_{m,\tilde{m}}$.

Let $\ell := \big|\{k \in [1,s] \colon \varepsilon_k = -1\}\big|$.

We say that $\underline{i}$ is a *standard* $(m, \tilde{m})$*-index tuple* if the conditions $(1, 2, 3)$ hold.

(1) We have
$$\varepsilon_k = \begin{cases} -1 & \text{for } k \in [1, \ell] \\ +1 & \text{for } k \in [\ell + 1, s] \,. \end{cases}$$

(2) We have
$$i_1 > i_2 > \ldots > i_\ell \geqslant 0 \,.$$

(3) We have
$$0 \leqslant i_{\ell+1} < i_{\ell+2} < \ldots < i_s \,.$$

Remark 246 *Suppose given* $\underline{i} \in R_{m,\tilde{m}}$.

We have
$$w(\underline{i}) = (0, 0) \quad \Leftrightarrow \quad \underline{i} \text{ is a standard } (m, \tilde{m})\text{-tuple} \;;$$

cf. Definitions 238*,* 245*.*

Proof. Let $\ell := \big|\{k \in [1, s] \colon \varepsilon_k = -1\}\big| \in [0, s]$.

We consider the conditions $(1, 2, 3)$ from Definition 245.

Ad $\Leftarrow$. Since $\underline{i}$ is standard, it satisfies the properties $(1, 2, 3)$.

By (1), we have $w_0(\underline{i}) = 0$; cf. Definition 238.(1),

By (2), we have $i_1 + 1 \geqslant i_2 + 2 \geqslant \ldots \geqslant i_\ell + \ell$ and thus $S_-(\underline{i}) = \emptyset$. So $w_-(\underline{i}) = 0$.

By (3), we have $i_{\ell+1} - (\ell + 1) \leqslant i_{\ell+2} - (\ell + 2) \leqslant \ldots \leqslant i_s - s$ and thus $S_+(\underline{i}) = \emptyset$. So $w_+(\underline{i}) = 0$.

This shows $w(\underline{i}) = \big(w_0(\underline{i}) \,, w_-(\underline{i}) + w_+(\underline{i})\big) = (0, 0)$.

Ad $\Rightarrow$. Since $w(\underline{i}) = (0, 0)$, note that in particular we also have $w_+(\underline{i}) = 0$ and $w_-(\underline{i}) = 0$.

We *assume* that $\underline{i}$ is a non-standard $(m, \tilde{m})$-tuple. Then we note that $\underline{i}$ satisfies
$$\neg(1) \;\vee\; \big((1) \wedge \neg(2)\big) \;\vee\; \big((1) \wedge \neg(3)\big) \,.$$

Case $\neg(1)$. Then there exists some j, $k \in [1, s]$ where $j < k$ and $\varepsilon_j = +1 > -1 = \varepsilon_k$.

This shows that $w_0(\underline{i}) > 0$ which is a *contradiction*; cf. Definition 238.

Case $(1) \wedge \neg(2)$. By condition (1), we have $\underline{i} \in R^0_{m,\tilde{m}}$.

We consider the set

$$S := \big\{k \in [1, \ell - 1] \colon i_k \leqslant i_{k+1}\big\}\,.$$

Note that $S \neq \emptyset$ since we have $\neg(2)$. Let $k \in S$. Then $k \leqslant \ell - 1 \leqslant s - 1$. We have

$$k < k + 1 \quad \text{and} \quad i_k + k \leqslant i_{k+1} + k < i_{k+1} + (k + 1)\,.$$

This shows $k \in S_-(\underline{i})$; cf. Definition 238. Therefore, we have $w_-(\underline{i}) > 0$, which *contradicts* $w_-(\underline{i}) = 0$.

Case $(1) \wedge \neg(3)$. By condition (1), we have $\underline{i} \in R^0_{m,\tilde{m}}$.

We consider the set

$$S := \big\{k \in [\ell + 1, s - 1] \colon i_k \geqslant i_{k+1}\big\}\,.$$

Note that $S \neq \emptyset$ since we have $\neg(3)$. Let $k \in S$. Then $k \geqslant \ell + 1 \geqslant 1$. We have

$$k < k + 1 \quad \text{and} \quad i_k - k \geqslant i_{k+1} - k > i_{k+1} - (k + 1)\,.$$

This shows $k+1 \in S_+(\underline{i})$; cf. Definition 238. Therefore, we have $w_+(\underline{i}) > 0$, which *contradicts* $w_+(\underline{i}) = 0$.

Altogether, we conclude that $\underline{i}$ is a standard $(m, \tilde{m})$-tuple. □

Remark 247 *For every nonempty subset $\emptyset \neq S \subseteq R_{m,\tilde{m}}$, there exists some $\underline{t} \in S$ with $w(\underline{t}) = \min\big\{w(\underline{\ell}) \colon \underline{\ell} \in S\big\}$.*

Proof. Suppose given $S \subseteq R_{m,\tilde{m}}$ with $S \neq \emptyset$.

Then $\emptyset \neq \big\{w(\underline{\ell}) \colon \underline{\ell} \in S\big\} \subseteq \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$. Therefore, $\min\big\{w(\underline{\ell}) \colon \underline{\ell} \in S\big\}$ exists; cf. Definition 238.(2). □

Remark 248 *Suppose given $\underline{i} \in R_{m,\tilde{m}}$.*

There exists some $\underline{j} \in [\underline{i}]$ where $\underline{j}$ is standard; cf. Definition 245.

Proof. We consider the set

$$T := \big\{\underline{\ell} \in R_{m,\tilde{m}} \colon\ \exists \underline{j} \in [\underline{\ell}] \text{ where } \underline{j} \text{ is standard } \big\} \subseteq R_{m,\tilde{m}}\,.$$

Let $S := R_{m,\tilde{m}} \backslash T \subseteq R_{m,\tilde{m}}$. We have to show that $S \stackrel{!}{=} \emptyset$.

We *assume* that $S \neq \emptyset$.

Then, by Remark 247, there exists some $\underline{t} \in S$ with $w(\underline{t}) = \min\big\{w(\underline{\ell}) \colon \underline{\ell} \in S\big\}$.

Case $\underline{t}$ is standard.

Then we have $\underline{t} \in [\underline{t}]$ where $\underline{t}$ is standard. This shows $\underline{t} \in T$ which *contradicts* $\underline{t} \in S$.

Case $\underline{t}$ is not standard.

Then, by Remark 246, we have $w(\underline{t}) > (0,0)$.

By Remark 244, we may choose some $\underline{j} \in [\underline{t}]$ with $\underline{t} \rightsquigarrow \underline{j}$ and $w(\underline{t}) > w(\underline{j})$.

Then, since $w(\underline{t}) = \min\big\{w(\underline{\ell})\colon \underline{\ell} \in S\big\}$, we have $\underline{j} \notin S$, i.e. $\underline{j} \in T$.

So we may choose some standard $\underline{k} \in [\underline{j}]$.

This shows $\underline{t} \rightsquigarrow \underline{j} \sim \underline{k}$, hence $\underline{t} \in T$, which *contradicts* $\underline{t} \in S$.

Altogether, this shows $S = \emptyset$. □

11.3.3 $(m, \tilde{m})$-index tuples and $\tau_{\underline{i}}$

Remark 249 *Suppose given $\underline{i}$, $\underline{j} \in R_{m,\tilde{m}}$ where $\underline{i}$ and $\underline{j}$ are standard; cf. Definition* 245. *We have*

$$\tau_{\underline{j}} = \tau_{\underline{i}} \;\Rightarrow\; \underline{j} = \underline{i}\,.$$

Proof. We write $\underline{i} := \big((\varepsilon'_1, i_1), \ldots, (\varepsilon'_s, i_s)\big)$ and $\underline{j} := \big((\varepsilon''_1, j_1), \ldots, (\varepsilon''_t, j_t)\big)$ both contained in $R_{m,\tilde{m}}$.

Let $a := |E_-(\underline{i})| = \big|\{i_k \in [1,s]\colon \varepsilon'_k = -1\}\big|$ and let $b := |E_+(\underline{j})| = \big|\{j_k \in [1,t]\colon \varepsilon''_k = -1\}\big|$.

By Definition 245, we have

$$\varepsilon'_k = \begin{cases} -1 & \text{for } k \in [1,a] \\ +1 & \text{for } k \in [a+1,s]\,, \end{cases}$$

where

$$i_1 > i_2 > \ldots > i_a \geqslant 0$$

and where

$$0 \leqslant i_{a+1} < i_{a+2} < \ldots < i_s\,.$$

Note that we have

$$\tilde{m} - m \overset{237}{=} \sum_{j \in [1,s]} \varepsilon'_j = \big(\sum_{j \in [1,a]} \varepsilon'_j\big) + \big(\sum_{j \in [a+1,s]} \varepsilon'_j\big) = -a + (s-a) = s - 2a\,.$$

Let $\ell' := m - a$. Then we have $a = m - \ell'$.

Moreover, we have

$$\tilde{m} - \ell' + a = \tilde{m} - (m - a) + a = (\tilde{m} - m) + 2a = s - 2a + 2a = s\,.$$

We have

$$\begin{aligned}
\tau_{\underline{i}} &= \prod_{k\in\lceil 1,s\rceil}^{[m]^{\pm}\;[\tilde{m}]^{\pm}} \tau_{i_k}^{(\varepsilon'_k)} \\
&= \Big(\prod_{k\in\lceil 1,a\rceil}^{[m]^{\pm}\;[m-a]^{\pm}} \tau_{i_k}^{(-1)}\Big) \blacktriangle \Big(\prod_{k\in\lceil a+1,s\rceil}^{[m-a]^{\pm}\;[\tilde{m}]^{\pm}} \tau_{i_k}^{(+1)}\Big) \\
&\overset{240}{=} \Big(\prod_{k\in\lceil 1,a\rceil}^{[m]^{\pm}\;[m-a]^{\pm}} \sigma_{i_k}\Big) \blacktriangle \Big(\prod_{k\in\lceil a+1,s\rceil}^{[m-a]^{\pm}\;[\tilde{m}]^{\pm}} \delta_{i_k}\Big) \\
&= \Big(\prod_{k\in\lceil 1,m-\ell'\rceil}^{[m]^{\pm}\;[\ell']^{\pm}} \sigma_{i_k}\Big) \blacktriangle \Big(\prod_{k\in\lceil a+1,\tilde{m}-\ell'+a\rceil}^{[\ell']^{\pm}\;[\tilde{m}]^{\pm}} \delta_{i_k}\Big) \\
&= \Big(\prod_{k\in\lfloor m-\ell',1\rfloor}^{[m]^{\pm}\;[\ell']^{\pm}} \sigma_{i_{a-k+1}}\Big) \blacktriangle \Big(\prod_{k\in\lceil 1,\tilde{m}-\ell'\rceil}^{[\ell']^{\pm}\;[\tilde{m}]^{\pm}} \delta_{i_{k+a}}\Big)\,.
\end{aligned}$$

Moreover, by Definition 245, we have

$$\varepsilon''_k = \begin{cases} -1 & \text{for } k \in [1,b] \\ +1 & \text{for } k \in [b+1,t]\,, \end{cases}$$

where

$$j_1 > j_2 > \ldots > j_b \geqslant 0$$

and where

$$0 \leqslant j_{b+1} < j_{b+2} < \ldots < j_t\,.$$

Note that we have

$$\tilde{m} - m \overset{237}{=} \sum_{j\in[1,s]} \varepsilon''_j = \big(\sum_{j\in[1,b]} \varepsilon''_j\big) + \big(\sum_{j\in[b+1,t]} \varepsilon''_j\big) = -b + (t-b) = t - 2b\,.$$

Let $\ell'' := m - b$. Then we have $b = m - \ell''$.

Moreover, we have

$$\tilde{m} - \ell'' + b = \tilde{m} - (m - b) + b = (\tilde{m} - m) + 2b = t - 2b + 2b = t\,.$$

We have

$$\begin{aligned}
\tau_{\underline{j}} &= \prod_{k\in\lceil 1,t\rceil}^{[m]^{\pm}\ [\tilde{m}]^{\pm}} \tau_{j_k}^{(\varepsilon''_k)} \\
&= \Big(\prod_{k\in\lceil 1,b\rceil}^{[m]^{\pm}\ [m-b]^{\pm}} \tau_{j_k}^{(-1)}\Big) \blacktriangle \Big(\prod_{k\in\lceil b+1,t\rceil}^{[m-b]^{\pm}\ [\tilde{m}]^{\pm}} \tau_{j_k}^{(+1)}\Big) \\
&\overset{240}{=} \Big(\prod_{k\in\lceil 1,b\rceil}^{[m]^{\pm}\ [m-b]^{\pm}} \sigma_{j_k}\Big) \blacktriangle \Big(\prod_{k\in\lceil b+1,t\rceil}^{[m-b]^{\pm}\ [\tilde{m}]^{\pm}} \delta_{j_k}\Big) \\
&= \Big(\prod_{k\in\lceil 1,m-\ell''\rceil}^{[m]^{\pm}\ [\ell'']^{\pm}} \sigma_{j_k}\Big) \blacktriangle \Big(\prod_{k\in\lceil b+1,\tilde{m}-\ell''+b\rceil}^{[\ell'']^{\pm}\ [\tilde{m}]^{\pm}} \delta_{j_k}\Big) \\
&= \Big(\prod_{k\in\lfloor m-\ell'',1\rfloor}^{[m]^{\pm}\ [\ell'']^{\pm}} \sigma_{j_{b-k+1}}\Big) \blacktriangle \Big(\prod_{k\in\lceil 1,\tilde{m}-\ell''\rceil}^{[\ell'']^{\pm}\ [\tilde{m}]^{\pm}} \delta_{j_{k+b}}\Big).
\end{aligned}$$

By Corollary 177.(3), since $\tau_{\underline{i}} = \tau_{\underline{j}}$, we have $\ell' = \ell''$ and thus $a = m - \ell' = m - \ell'' = b$. So $s = \tilde{m} - \ell' + a = \tilde{m} - \ell'' + b = t$. Moreover, we have $i_k = j_k$ for $k \in [1, m - \ell'] = [1, a]$ and $i_k = j_k$ for $k \in [a + 1, \tilde{m} - \ell' + a] = [a + 1, s]$.

So we have $(\varepsilon'_k, i_k) = (\varepsilon''_k, j_k)$ for $k \in [1, s] = [1, t]$. This shows $\underline{i} = \underline{j}$. □

Lemma 250 *Let $n \in \mathbb{Z}_{\geqslant 0}$. Suppose given m, $\tilde{m} \in [0, n]$.*

Suppose given $\underline{i}$, $\underline{j} \in R_{m,\tilde{m}}$. We have

$$\underline{i} \sim \underline{j} \ \Leftrightarrow\ \tau_{\underline{i}} = \tau_{\underline{j}}.$$

Proof. Ad $\Rightarrow$. By Remark 242, we have $\tau_{\underline{i}} = \tau_{\underline{j}}$.

Ad $\Leftarrow$. By Remark 248, we may choose a standard $\underline{k} \in [\underline{i}]$ and we may choose a standard $\underline{\ell} \in [\underline{j}]$.

By Remark 242, we have $\tau_{\underline{k}} = \tau_{\underline{i}}$ and $\tau_{\underline{j}} = \tau_{\underline{\ell}}$. So $\tau_{\underline{k}} = \tau_{\underline{\ell}}$.

Then, by Remark 249, we have $\underline{k} = \underline{\ell}$.

This shows $\underline{i} \sim \underline{k} = \underline{\ell} \sim \underline{j}$. □

11.4 A construction of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups

Let $n \in \mathbb{Z} \cup \{+\infty\}$. Suppose given m, m', $m'' \in \mathbb{Z}_{\leqslant n}$.

Remark 251 *Suppose given $\underline{i} \in R_{m,m'}$ and $\underline{j} \in R_{m',m''}$; cf. Definition 237.*

Let $\underline{i} = \big((\varepsilon_1 , i_1), \ldots, (\varepsilon_r , i_r)\big)$ and $\underline{j} = \big((\varepsilon'_1 , j_1), \ldots, (\varepsilon'_{r'} , j_{r'})\big)$.

Then we have the (m, m'')-index tuple $\underline{i} \sqcup \underline{j} = \big((\varepsilon''_1 , \ell_1) , \ldots, (\varepsilon''_{r+r'} , \ell_{r+r''})\big) \in R_{m,m''}$ defined by

$$(\varepsilon''_k , \ell_k) := \begin{cases} (\varepsilon_k , i_k) & \text{for } k \in [1, r] \\ (\varepsilon'_{k-r} , j_{k-r}) & \text{for } k \in [r+1, r+r'] . \end{cases}$$

So

$$\underline{i} \sqcup \underline{j} = \big((\varepsilon_1 , i_1), \ldots, (\varepsilon_r , i_r), (\varepsilon'_1 , j_1), \ldots, (\varepsilon'_{r'} , j_{r'})\big) .$$

Proof. We have show that $\underline{i} \sqcup \underline{j} \overset{!}{\in} R_{m,m''}$.

To that end, we show that $m + \sum\limits_{q\in[1,k]} \varepsilon''_q \overset{!}{\leqslant} n$ for $k \in [0, r+r']$ and we have to show that $\sum\limits_{q\in[1,r+r']} \varepsilon_{q''} \overset{!}{=} m'' - m$; cf. Definition 237.

Suppose given $k \in [0, r+r']$.

Case $k \in [0, r]$. We have

$$m + \sum_{q\in[1,k]} \varepsilon''_q = m + \sum_{q\in[1,k]} \varepsilon_q \leqslant n .$$

Case $k \in [r+1, r+r']$. Since $\underline{i} \in R_{m,m'}$, note that we have $m = m' - \sum\limits_{q\in[1,r]} \varepsilon_q$.

We have

$$\begin{aligned} m + \sum_{q\in[1,k]} \varepsilon''_q &= \big(m' - \sum_{q\in[1,r]} \varepsilon_q\big) + \sum_{q\in[1,k]} \varepsilon''_q = \big(m' - \sum_{q\in[1,r]} \varepsilon''_q\big) + \sum_{q\in[1,r]} \varepsilon''_q + \sum_{q\in[r+1,k]} \varepsilon''_q \\ &= m' + \sum_{q\in[r+1,k]} \varepsilon'_{q-r} = m' + \sum_{q\in[1,k-r]} \varepsilon'_q \leqslant n . \end{aligned}$$

Moreover, we have

$$\begin{aligned}\sum_{q\in[1,r+r']} \varepsilon''_q &= \big(\sum_{q\in[1,r]} \varepsilon''_q\big) + \big(\sum_{q\in[r+1,r+r']} \varepsilon''_q\big) = \big(\sum_{q\in[1,r]} \varepsilon_q\big) + \big(\sum_{q\in[r+1,r+r']} \varepsilon'_{q-r}\big) \\ &= \big(\sum_{q\in[1,r]} \varepsilon_q\big) + \big(\sum_{q\in[1,r']} \varepsilon'_q\big) = (m'-m) + (m''-m') = m''-m\,.\end{aligned}$$

Altogether, this shows $\underline{i} \sqcup \underline{j} \in R_{m,m''}$. □

Remark 252 *Suppose given $\underline{i} \in R_{m,m'}$ and $\underline{j} \in R_{m',m''}$.*
We consider the morphisms $\tau_{\underline{i}}$, $\tau_{\underline{j}} \in \mathrm{Mor}(\Delta^{\pm})$; cf. Definition 240.
Then $\tau_{\underline{i}}$ and $\tau_{\underline{j}}$ are composable and we have

$$\tau_{\underline{i}} \blacktriangle \tau_{\underline{j}} = \tau_{\underline{i}\sqcup\underline{j}}\,.$$

Proof. We write $\underline{i} = \big((\varepsilon_1, i_1), \ldots, (i_r, i_r)\big)$ and $\underline{j} = \big((\varepsilon'_1, j_1), \ldots, (\varepsilon'_{r'}, j_{r'})\big)$.
Then we have

$$\tau_{\underline{i}} = \prod_{k\in\lceil 1,r\rceil}^{[m]^{\pm}\;\;[m']^{\pm}} \tau_{i_k}^{(\varepsilon_k)} \quad \text{and} \quad \tau_{\underline{j}} = \prod_{k\in\lceil 1,r'\rceil}^{[m']^{\pm}\;\;[m'']^{\pm}} \tau_{j_k}^{(\varepsilon'_k)}\,.$$

Moreover, we have $\underline{i} \sqcup \underline{j} = \big((\varepsilon''_1, \ell_1), \ldots, (\varepsilon''_{r+r'}, \ell_{r+r'})\big)$ where

$$(\varepsilon''_k, \ell_k) = \begin{cases} (\varepsilon_k, i_k) & \text{for } k \in [1,r] \\ (\varepsilon'_{k-r}, j_{k-r}) & \text{for } k \in [r+1, r+r']\,; \end{cases}$$

cf. Remark 251.

We have

$$\begin{aligned}\tau_{\underline{i}\sqcup\underline{j}} &= \prod_{k\in\lceil 1,r+r'\rceil}^{[m]^{\pm}\;\;[m'']^{\pm}} \tau_{\ell_k}^{(\varepsilon''_k)} \\ &= \Big(\prod_{k\in\lceil 1,r\rceil}^{[m]^{\pm}\;\;[m']^{\pm}} \tau_{\ell_k}^{(\varepsilon''_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil r+1,r+r'\rceil}^{[m']^{\pm}\;\;[m'']^{\pm}} \tau_{\ell_k}^{(\varepsilon''_k)}\Big) \\ &= \Big(\prod_{k\in\lceil 1,r\rceil}^{[m]^{\pm}\;\;[m']^{\pm}} \tau_{i_k}^{(\varepsilon_k)}\Big) \blacktriangle \Big(\prod_{k\in\lceil 1,r'\rceil}^{[m']^{\pm}\;\;[m'']^{\pm}} \tau_{j_k}^{(\varepsilon'_k)}\Big) \\ &= \tau_{\underline{i}} \blacktriangle \tau_{\underline{j}}\,.\end{aligned}$$

□

Remark 253 *Suppose given* $\big([m]^{\pm} \xrightarrow{u} [m']^{\pm}\big) \in \mathrm{Mor}\big(\Delta^{\pm}_{\lfloor n,-\infty\rfloor}\big)$.
Then there exists a unique standard $\underline{a} \in R_{m,m'}$ *with* $u = \tau_{\underline{a}}$ *; cf. Definitions* 237, 240, 245.

Proof. By Corollary 177.(1, 3), we have

$$u = \left(\prod_{k\in\lfloor m-\ell,1\rfloor}^{[m]^{\pm}\ [\ell]^{\pm}} \upsigma^{\pm,\ell+k-1}_{i_k} \right) \blacktriangle \left(\prod_{k\in\lceil 1,m'-\ell\rceil}^{[\ell]^{\pm}\ [m']^{\pm}} \updelta^{\pm,\ell+k}_{j_k} \right) = \upsigma^{\pm,m-1}_{i_{m-\ell}} \blacktriangle \ldots \blacktriangle \upsigma^{\pm,\ell}_{i_1} \blacktriangle \updelta^{\pm,\ell+1}_{j_1} \blacktriangle \ldots \blacktriangle \updelta^{\pm,m'}_{j_{m'-\ell}}$$

for some unique $\ell \in \mathbb{Z}_{\leqslant \min\{m,m'\}}$, $0 \leqslant i_1 < i_2 < \ldots < i_{m-\ell}$ and $0 \leqslant j_1 < j_2 < \ldots < j_{m'-\ell}$.
Let the tuple $\underline{a} = \big((\varepsilon_1\,, a_1), \ldots, (\varepsilon_{m+m'-2\ell}\,, a_{m+m'-2\ell})\big)$ be given as follows.
Let

$$(\varepsilon_k\,, a_k) := \begin{cases} (-1, i_{m-\ell-k+1}) & \text{for } k \in [1, m-\ell] \\ (+1, j_{k-m+\ell}) & \text{for } k \in [m-\ell+1, m+m'-2\ell]\,. \end{cases}$$

Then $\underline{a} = \big((-1, i_{m-\ell}), \ldots, (-1, i_1), (+1, j_1), \ldots, (+1, j_{m'-\ell})\big)$.

We show that $m + \sum\limits_{r\in[1,k]} \varepsilon_r \overset{!}{\leqslant} n$ *for* $k \in [1, m+m'-2\ell]$.

For $k \in [1, m-\ell]$, we have

$$m + \sum_{r\in[1,k]} \varepsilon_r = m - k \leqslant n\,.$$

For $k \in [m-\ell+1, m+m'-2\ell]$, we have

$$\begin{aligned} m + \sum_{r\in[1,k]} \varepsilon_r &= m + \big(\sum_{r\in[1,m-\ell]} \varepsilon_r\big) + \big(\sum_{r\in[m-\ell+1,k]} \varepsilon_r\big) = m - (m-\ell) + \big(k - (m-\ell)\big) \\ &= k - m + 2\ell \leqslant (m+m'-2\ell) - m + 2\ell = m' \leqslant n\,. \end{aligned}$$

We show that $\sum\limits_{r\in[1,m-m'-2\ell]} \varepsilon_r \overset{!}{=} m' - m$.

We have

$$\begin{aligned} \sum_{r\in[1,m-m'-2\ell]} \varepsilon_r &= \big(\sum_{r\in[1,m-\ell]} \varepsilon_r\big) + \big(\sum_{r\in[m-\ell+1,m-m'-2\ell]} \varepsilon_r\big) \\ &= -(m-\ell) + \big(m+m'-2\ell-(m-\ell)\big) = m' - m\,. \end{aligned}$$

So $\underline{a} \in R_{m,m'}$; cf. Definition 237.

Moreover, note that $\underline{a}$ is standard; cf. Definition 245.

We have

$$\tau_{\underline{a}} = \prod_{k \in \lceil 1, m+m'-2\ell \rceil}^{[m]^{\pm}} \prod^{[m']^{\pm}} \tau_{a_k}^{(\varepsilon_k)} = \left(\prod_{k \in \lfloor m-\ell, 1 \rfloor}^{[m]^{\pm}} \prod^{[\ell]^{\pm}} \sigma_{i_k}^{\pm, \ell+k-1} \right) \blacktriangle \left(\prod_{k \in \lceil 1, m'-\ell \rceil}^{[\ell]^{\pm}} \prod^{[m']^{\pm}} \delta_{j_k}^{\pm, \ell+k} \right) = u \,;$$

cf. Definition 240.

Uniqueness of $\underline{a} \in R_{m,m'}$ standard with $\tau_{\underline{a}} = u$ follows by Remark 249. □

Lemma 254 (pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group)

For $k \in \mathbb{Z}_{\leqslant n}$, suppose given a group G_k.

For $k \in \mathbb{Z}_{\leqslant n}$ and $j \in \mathbb{Z}_{\geqslant 0}$, suppose given a group morphism $d_j^k \colon G_k \to G_{k-1}$.

For $k \in \mathbb{Z}_{\leqslant n-1}$ and $i \in \mathbb{Z}_{\geqslant 0}$, suppose given a group morphism $s_i^k \colon G_k \to G_{k+1}$.

Suppose that the following equations (i, ii, iii) *are satisfied.*

(i) *We have*

$$d_j^{k+1} \blacktriangle d_i^k = d_i^{k+1} \blacktriangle d_{j-1}^k$$

for $k \in \mathbb{Z}_{\leqslant n-1}$ and i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

(ii) *We have*

$$s_i^{k-1} \blacktriangle s_j^k = s_{j-1}^{k-1} \blacktriangle s_i^k$$

for $k \in \mathbb{Z}_{\leqslant n-1}$ and i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

(iii) *We have*

$$s_i^k \blacktriangle d_j^{k+1} = \begin{cases} d_j^k \blacktriangle s_{i-1}^{k-1} & \text{if } j < i \\ \mathrm{id}_{G_k} & \text{if } j = i \text{ or } j = i+1 \\ d_{j-1}^k \blacktriangle s_i^{k-1} & \text{if } j > i+1 \end{cases}$$

for $k \in \mathbb{Z}_{\leqslant n-1}$ and i, $j \in \mathbb{Z}_{\geqslant 0}$.

Cf. also Remark 185.

Let $k \in \mathbb{Z}_{\leqslant n}$. For $j \in \mathbb{Z}_{\geqslant 0}$, we write

$$t_j^{(+1),k} := d_j^k \colon G_k \to G_{k-1} \,.$$

We often write $t_j^{(+1)} := t_j^{(+1),k}$.

Let $k \in \mathbb{Z}_{\leqslant n-1}$. *For* $i \in \mathbb{Z}_{\geqslant 0}$, *we write*

$$t_i^{(-1),k} := s_i^k \colon G_k \to G_{k+1} \, .$$

We often write $t_i^{(-1)} := t_i^{(-1),k}$.

Moreover, for $\underline{b} = \big((\varepsilon_1 \, , b_1), \ldots, (\varepsilon_{\tilde{s}} \, , b_{\tilde{s}})\big) \in R_{m,m'}$, *we write*

$$t_{\underline{b}} := \prod_{p \in \lfloor \tilde{s}, 1 \rfloor}^{G_{m'} \quad G_m} t_{b_p}^{(\varepsilon_p)} = t_{b_{\tilde{s}}}^{(\varepsilon_{\tilde{s}})} \blacktriangle \ldots \blacktriangle t_{b_1}^{(\varepsilon_1)} \, .$$

(1) *Suppose given* $\underline{b}$, $\underline{c} \in R_{m,m'}$. *Suppose that* $b \sim c$; *cf. Definition* 237.

We have

$$t_{\underline{b}} = t_{\underline{c}} \, .$$

(2) *Suppose given* $\underline{b} \in R_{m,m'}$ *and* $\underline{c} \in R_{m',m''}$. *We consider the* (m, m'')*-index tuple* $\underline{b} \sqcup \underline{c}$; *cf. Remark* 251.

We have

$$t_{\underline{b} \sqcup \underline{c}} = t_{\underline{c}} \blacktriangle t_{\underline{b}} : G_{m''} \to G_m \, .$$

(3) *There exists a unique pre-*$\lfloor n, -\infty \rfloor$*-prestable simplicial group* G *such that* G *actually maps* $[k]^{\pm}$ *to* G_k *for* $k \in \mathbb{Z}_{\leqslant n}$ *and such that* $\mathrm{d}_j^{G,k} = d_j^k$ *for* $k \in \mathbb{Z}_{\leqslant n}$, $j \in \mathbb{Z}_{\geqslant 0}$ *and such that* $\mathrm{s}_i^{G,k} = s_i^k$ *for* $k \in \mathbb{Z}_{\leqslant n-1}$, $i \in \mathbb{Z}_{\geqslant 0}$.

The pre-$\lfloor n, -\infty \rfloor$*-prestable simplicial group* G *is given as follows.*

For $[k]^{\pm} \in \mathrm{Ob}\left(\Delta^{\pm}_{\lfloor n, -\infty \rfloor}\right)$, *we have*

$$[k]^{\pm} G = G_k \, .$$

For $\left([m]^{\pm} \xrightarrow{u} [m']^{\pm}\right) \in \mathrm{Mor}\left(\Delta^{\pm}_{\lfloor n, 0 \rfloor}\right)$, *there exists by Remark* 253 *a unique standard* $\underline{i} \in R_{m,m'}$ *with* $u = \tau_{\underline{i}}$.

We have

$$(u^{\mathrm{op}})G = (\tau_{\underline{i}}^{\mathrm{op}})G = t_{\underline{i}} \, .$$

Proof. Ad (1). We may assume that $\underline{b} \rightsquigarrow \underline{c}$; cf. Definition 237.

We write $\underline{b} = \big((\varepsilon_1 \, , b_1), \ldots, (\varepsilon_q \, , b_q)\big)$.

Case $\underline{b} \overset{++}{\rightsquigarrow} \underline{c}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_{\ell-1}\,, b_{\ell-1}), (\varepsilon_{\ell+1}\,, b_{\ell+1}), (\varepsilon_\ell\,, b_\ell + 1), (\varepsilon_{\ell+2}\,, b_{\ell+2}), \ldots, (\varepsilon_q\,, b_q)\big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \geqslant b_{\ell+1}$ and $\varepsilon_\ell = \varepsilon_{\ell+1} = +1$, which we choose.
Let $r := m + \sum\limits_{p \in [1, \ell-1]} \varepsilon_p\,.$
Since $b_{\ell+1} < b_\ell + 1\,,$ we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \; G_{r+2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle t_{b_\ell+1}^{(\varepsilon_\ell)} \blacktriangle t_{b_{\ell+1}}^{(\varepsilon_{\ell+1})} \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \; G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \; G_{r+2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle (d_{b_\ell+1}^{r+2} \blacktriangle d_{b_{\ell+1}}^{r+1}) \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \; G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&\overset{\text{(i)}}{=} \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \; G_{r+2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle (d_{b_{\ell+1}}^{r+2} \blacktriangle d_{b_\ell}^{r+1}) \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \; G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \prod_{p \in \lfloor q, 1 \rfloor}^{G_{m'} \; G_m} t_{b_p}^{(\varepsilon_p)} \\
&= t_{\underline{b}}\,.
\end{aligned}$$

Case $\underline{b} \overset{--}{\rightsquigarrow} \underline{c}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_{\ell-1}\,, b_{\ell-1}), (\varepsilon_{\ell+1}\,, b_{\ell+1} + 1), (\varepsilon_\ell\,, b_\ell), (\varepsilon_{\ell+2}\,, b_{\ell+2}), \ldots, (\varepsilon_q\,, b_q)\big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \leqslant b_{\ell+1}$ and $\varepsilon_\ell = \varepsilon_{\ell+1} = -1$.
Let $r := m + \sum\limits_{p \in [1, \ell-1]} \varepsilon_p\,.$
Since $b_\ell < b_{\ell+1} + 1\,,$ we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \; G_{r-2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle t_{b_\ell}^{(\varepsilon_\ell)} \blacktriangle t_{b_{\ell+1}+1}^{(\varepsilon_{\ell+1})} \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \; G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \Big(\prod_{p \in \lfloor q, \ell+2 \rfloor}^{G_{m'} \; G_{r-2}} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle s_{b_\ell}^{r-2} \blacktriangle s_{b_{\ell+1}+1}^{r-1} \blacktriangle \Big(\prod_{p \in \lfloor \ell-1, 1 \rfloor}^{G_r \; G_m} t_{b_p}^{(\varepsilon_p)}\Big)
\end{aligned}$$

$$\begin{aligned} &\overset{\text{(ii)}}{=} \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\ G_{r-2}} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle s_{b_{\ell+1}}^{r-2} \blacktriangle s_{b_\ell}^{r-1} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\ G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &= \prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\ G_m} t_{b_p}^{(\varepsilon_p)} \\ &= t_{\underline{b}}\,. \end{aligned}$$

Case $\underline{i} \overset{+-;1}{\rightsquigarrow} \underline{j}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_{\ell-1}\,, b_{\ell-1}), (\varepsilon_{\ell+1}\,, b_{\ell+1}-1), (\varepsilon_\ell\,, b_\ell), (\varepsilon_{\ell+2}\,, b_{\ell+2}), \ldots, (\varepsilon_q\,, b_q)\big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \leqslant b_{\ell+1}-1$ and $\varepsilon_\ell = +1\,,\ \varepsilon_{\ell+1} = -1$, which we choose.
Let $r := m + \sum\limits_{k\in[1,\ell-1]} \varepsilon_k\,.$
Since $b_\ell < b_{\ell+1}\,,$ we have

$$\begin{aligned} t_{\underline{c}} &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\ G_r} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle t_{b_\ell}^{(\varepsilon_\ell)} \blacktriangle t_{b_{\ell+1}-1}^{(\varepsilon_{\ell+1})} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\ G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\ G_r} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle d_{b_\ell}^{r} \blacktriangle s_{b_{\ell+1}-1}^{r-1} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\ G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &\overset{\text{(iii)}}{=} \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\ G_r} t_{b_p}^{(\varepsilon_p)} \Big) \blacktriangle s_{b_{\ell+1}}^{r} \blacktriangle d_{b_\ell}^{r+1} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\ G_m} t_{b_p}^{(\varepsilon_p)} \Big) \\ &= \prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\ G_m} t_{b_p}^{(\varepsilon_p)} \\ &= t_{\underline{b}}\,. \end{aligned}$$

Case $\underline{i} \overset{+-;2}{\rightsquigarrow} \underline{j}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_{\ell-1}\,, b_{\ell-1}), (\varepsilon_{\ell+2}\,, b_{\ell+2}), \ldots, (\varepsilon_q\,, b_q)\big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \in \{b_{\ell+1}\,, b_{\ell+1}+1\}$ and $\varepsilon_\ell = +1\,,\ \varepsilon_{\ell+1} = -1$, which we choose.
Let $r := m + \sum\limits_{p\in[1,\ell-1]} \varepsilon_p\,.$

Since $b_\ell \in \{b_{\ell+1}\,, b_{\ell+1}+1\}$, we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_r} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_r} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle \mathrm{id}_{G_r} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&\overset{\text{(iii)}}{=} \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_r} t_{b_p}^{(\varepsilon_p)}\Big) \blacktriangle s^r_{b_{\ell+1}} \blacktriangle d^{r+1}_{b_\ell} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_p}^{(\varepsilon_p)}\Big) \\
&= \prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\;\;G_m} t_{b_p}^{(\varepsilon_p)} \\
&= t_{\underline{b}}\,.
\end{aligned}$$

Case $\underline{i} \overset{+-,3}{\rightsquigarrow} \underline{j}$. Then we have

$$\underline{c} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_{\ell-1}\,, b_{\ell-1}), (\varepsilon_{\ell+1}\,, b_{\ell+1}), (\varepsilon_\ell\,, b_\ell - 1), (\varepsilon_{\ell+2}\,, b_{\ell+2}), \ldots, (\varepsilon_q\,, b_q)\big)$$

for some $\ell \in [1, q-1]$ with $b_\ell \geqslant b_{\ell+1}+2$ and $\varepsilon_\ell = +1\,,\ \varepsilon_{\ell+1} = -1$, which we choose.
Let $r := m + \sum\limits_{q\in[1,\ell-1]} \varepsilon_q\,.$

Since $b_\ell > b_{\ell+1}+1\,,$ we have

$$\begin{aligned}
t_{\underline{c}} &= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_r} t_{b_q}^{(\varepsilon_q)}\Big) \blacktriangle t^{(\varepsilon_\ell)}_{b_\ell-1} \blacktriangle t^{(\varepsilon_{\ell+1})}_{b_{\ell+1}} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_q}^{(\varepsilon_q)}\Big) \\
&= \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_r} t_{b_q}^{(\varepsilon_q)}\Big) \blacktriangle d^r_{b_\ell-1} \blacktriangle s^{r-1}_{b_{\ell+1}} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_q}^{(\varepsilon_q)}\Big) \\
&\overset{\text{(iii)}}{=} \Big(\prod_{p\in\lfloor q,\ell+2\rfloor}^{G_{m'}\;\;G_r} t_{b_q}^{(\varepsilon_q)}\Big) \blacktriangle s^r_{b_{\ell+1}} \blacktriangle d^{r+1}_{b_\ell} \blacktriangle \Big(\prod_{p\in\lfloor \ell-1,1\rfloor}^{G_r\;\;G_m} t_{b_q}^{(\varepsilon_q)}\Big) \\
&= \prod_{p\in\lfloor q,1\rfloor}^{G_{m'}\;\;G_m} t_{b_p}^{(\varepsilon_p)} \\
&= t_{\underline{b}}\,.
\end{aligned}$$

Altogether, this shows $t_{\underline{b}} = t_{\underline{c}}\,.$

Ad (2). We write $\underline{b} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_q\,, b_q)\big)$ and $\underline{c} = \big((\varepsilon'_1\,, c_1), \ldots, (\varepsilon'_{q'}\,, c_{q'})\big)$.

Then we have $\underline{b} \sqcup \underline{c} = \big((\varepsilon_1\,, b_1), \ldots, (\varepsilon_q\,, b_q), (\varepsilon'_1\,, c_1), \ldots, (\varepsilon'_{q'}\,, c_{q'})\big)$; cf. Remark 251.

We have

$$t_{\underline{b} \sqcup \underline{c}} = \Big(\prod_{p \in \lfloor q', 1 \rfloor}^{G_{m''} \quad G_{m'}} t_{c_p}^{(\varepsilon'_p)} \Big) \blacktriangle \Big(\prod_{p \in \lfloor q, 1 \rfloor}^{G_{m'} \quad G_m} t_{b_p}^{(\varepsilon_p)} \Big) = t_{\underline{c}} \blacktriangle t_{\underline{b}}\,.$$

Ad (3). Suppose given $[m]^{\pm} \xrightarrow{u} [m']^{\pm} \xrightarrow{v} [m'']^{\pm}$ in $\Delta^{\pm}_{\lfloor n, -\infty \rfloor}$.

We have to show that $\big((\mathrm{id}_{[m]^{\pm}})^{\mathrm{op}}\big)G \overset{!}{=} \mathrm{id}_{G_m}$.

By Remark 253, we have $\mathrm{id}_{[m]} = \tau_{\underline{a}}$, for a unique $\underline{a} \in R_{m,m}$. Note that $\underline{a} = (\ )$ is the empty tuple.

We have

$$\big((\mathrm{id}_{[m]^{\pm}})^{\mathrm{op}}\big)G = (\tau_{\underline{a}}^{\mathrm{op}})G = t_{\underline{a}} = \prod_{p \in \lfloor 0, 1 \rfloor}^{G_m \quad G_m} t_{a_p}^{(\varepsilon_p)} = \mathrm{id}_{G_m}\,.$$

We have to show that $(u \blacktriangle v)^{\mathrm{op}}\, G \overset{!}{=} (v^{\mathrm{op}})G \blacktriangle (u^{\mathrm{op}})G$.

By Remark 253, there exists a unique standard $\underline{i} \in R_{m,m'}$ with $u = \tau_{\underline{i}}$ and there exists a unique standard $\underline{j} \in R_{m',m''}$ with $v = \tau_{\underline{j}}$.

We write $\underline{i} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_r\,, i_r)\big)$ and $\underline{j} = \big((\varepsilon'_1\,, j_1), \ldots, (\varepsilon'_{r'}\,, j_{r'})\big)$.

We consider $\underline{i} \sqcup \underline{j} = \big((\varepsilon_1\,, i_1), \ldots, (\varepsilon_r\,, i_r), (\varepsilon'_1\,, j_1), \ldots, (\varepsilon'_{r'}\,, j_{r'})\big) \in R_{m,m''}$; cf. Remark 251.

By Remark 248, we may choose a standard $\underline{a} \in [\underline{i} \sqcup \underline{j}]$. By Lemma 250, we have $\tau_{\underline{i} \sqcup \underline{j}} = \tau_{\underline{a}}$.

We have

$$\begin{aligned}(u \blacktriangle v)^{\mathrm{op}}\, G = (\tau_{\underline{i}} \blacktriangle \tau_{\underline{j}})^{\mathrm{op}}\, G &\overset{252}{=} (\tau_{\underline{i} \sqcup \underline{j}})^{\mathrm{op}}\, G = \tau_{\underline{a}}^{\mathrm{op}}\, G = t_{\underline{a}} \overset{(1)}{=} t_{\underline{i} \sqcup \underline{j}} \overset{(2)}{=} t_{\underline{j}} \blacktriangle t_{\underline{i}} \\ &= (\tau_{\underline{j}}^{\mathrm{op}})G \blacktriangle (\tau_{\underline{i}}^{\mathrm{op}})G = (v^{\mathrm{op}})G \blacktriangle (u^{\mathrm{op}})G\,.\end{aligned}$$

□

Lemma 255 (pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphism)
Suppose given pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups G and $\tilde{G}$.
For $k \in \mathbb{Z}_{\leqslant n}$, suppose given group morphisms $\varphi_k \colon G_k \to \tilde{G}_k$.
Suppose that the following properties (1, 2) *hold.*

(1) *For $k \in \mathbb{Z}_{\leqslant n}$ and $j \in \mathbb{Z}_{\geqslant 0}$, we have* $\varphi_k \blacktriangle \mathrm{d}_j^{\tilde{G},k} = \mathrm{d}_j^{G,k} \blacktriangle \varphi_{k-1}$.

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & \tilde{G}_k \\ {\scriptstyle \mathrm{d}_j^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_j^{\tilde{G},k}} \\ G_{k-1} & \xrightarrow[\varphi_{k-1}]{} & \tilde{G}_{k-1} \end{array}$$

(2) *For* $k \in \mathbb{Z}_{\leqslant n-1}$ *and* $i \in \mathbb{Z}_{\geqslant 0}$, *we have* $\varphi_k \blacktriangle \mathrm{s}_i^{\tilde{G},k} = \mathrm{s}_i^{G,k} \blacktriangle \varphi_{k+1}$.

$$\begin{array}{ccc} G_k & \xrightarrow{\varphi_k} & \tilde{G}_k \\ {\scriptstyle \mathrm{s}_i^{G,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{s}_i^{\tilde{G},k}} \\ G_{k+1} & \xrightarrow[\varphi_{k+1}]{} & \tilde{G}_{k+1} \end{array}$$

Then $\varphi := \left(G_k \xrightarrow{\varphi_k} \tilde{G}_k\right)_{k \in \mathbb{Z}_{\leqslant n}}$ *is a pre-*$\lfloor n, -\infty \rfloor$*-prestable simplicial group morphism from* G *to* $\tilde{G}$.

In particular, if G *and* $\tilde{G}$ *are* $\lfloor n, -\infty \rfloor$*-stable simplicial groups, then* φ *is a* $\lfloor n, -\infty \rfloor$*-stable simplicial group morphism; cf. Definition* 232.

Proof. Suppose given $\left([m]^{\pm} \xrightarrow{u} [m']^{\pm}\right)$ in $\Delta^{\pm}_{\lfloor n, -\infty \rfloor}$.

Recall that by Corollary 177.(1, 3), we have

$$u = \left(\prod_{t \in \lfloor m-\ell, 1 \rfloor}^{[m]^{\pm}} \sigma_{i_t}^{\pm, \ell+t-1} \right) \blacktriangle \left(\prod_{t \in \lceil 1, m'-\ell \rceil}^{[\ell]^{\pm}} \delta_{j_t}^{\pm, \ell+t} \right).$$

for some unique $\ell \in \mathbb{Z}_{\leqslant \min\{m,m'\}}$ and $0 \leqslant i_1 < i_2 < \ldots < i_{n-\ell}$ and $0 \leqslant j_1 < j_2 < \ldots < j_{m'-\ell}$.

We have

$$\begin{aligned}
\varphi_{m'} \blacktriangle (u^{\mathrm{op}})\tilde{G} &= \varphi_{m'} \blacktriangle \Bigg(\Big(\prod_{t \in \lfloor m'-\ell, 1 \rfloor}^{[m']^{\pm}} [\ell]^{\pm} (\delta_{j_t}^{\pm,\ell+t})^{\mathrm{op}} \Big) \blacktriangle \Big(\prod_{t \in \lceil 1, m-\ell \rceil}^{[\ell]^{\pm}} [m]^{\pm} (\sigma_{i_t}^{\pm,\ell+t-1})^{\mathrm{op}} \Big) \Bigg) \tilde{G} \\
&\overset{183}{=} \varphi_{m'} \blacktriangle \Big(\prod_{t \in \lfloor m'-\ell, 1 \rfloor}^{[m']^{\pm}} [\ell]^{\pm} \mathrm{d}_{j_t}^{\tilde{G},\ell+t} \Big) \blacktriangle \Big(\prod_{t \in \lceil 1, m-\ell \rceil}^{[\ell]^{\pm}} [m]^{\pm} \mathrm{s}_{i_t}^{\tilde{G},\ell+t-1} \Big) \\
&\overset{(1)}{=} \Big(\prod_{t \in \lfloor m'-\ell, 1 \rfloor}^{[m']^{\pm}} [\ell]^{\pm} \mathrm{d}_{j_t}^{G,\ell+t} \Big) \blacktriangle \varphi_{m'-(m'-\ell)} \blacktriangle \Big(\prod_{t \in \lceil 1, m-\ell \rceil}^{[\ell]^{\pm}} [m]^{\pm} \mathrm{s}_{i_t}^{\tilde{G},\ell+t-1} \Big) \\
&\overset{(2)}{=} \Big(\prod_{t \in \lfloor m'-\ell, 1 \rfloor}^{[m']^{\pm}} [\ell]^{\pm} \mathrm{d}_{j_t}^{G,\ell+t} \Big) \blacktriangle \Big(\prod_{t \in \lceil 1, m-\ell \rceil}^{[\ell]^{\pm}} [m]^{\pm} \mathrm{s}_{i_t}^{G,\ell+t-1} \Big) \blacktriangle \varphi_{\ell+(m-\ell)} \\
&\overset{183}{=} \Bigg(\Big(\prod_{t \in \lfloor m'-\ell, 1 \rfloor}^{[m']^{\pm}} [\ell]^{\pm} (\delta_{j_t}^{\pm,\ell+t})^{\mathrm{op}} \Big) \blacktriangle \Big(\prod_{t \in \lceil 1, m-\ell \rceil}^{[\ell]^{\pm}} [m]^{\pm} (\sigma_{i_t}^{\pm,\ell+t-1})^{\mathrm{op}} \Big) \Bigg) G \blacktriangle \varphi_m \\
&= (u^{\mathrm{op}})G \blacktriangle \varphi_m \, .
\end{aligned}$$

$$\begin{array}{ccc}
G_{m'} & \xrightarrow{\varphi_{m'}} & \tilde{G}_{m'} \\
{\scriptstyle (u^{\mathrm{op}})G}\downarrow & & \downarrow{\scriptstyle (u^{\mathrm{op}})\tilde{G}} \\
G_m & \xrightarrow[\varphi_m]{} & \tilde{G}_m
\end{array}$$

□

Remark 256 *Suppose given an abelian group A.*

(1) *We have the pre-$\lfloor 0, -\infty \rfloor$-prestable simplicial group G that has*

$$G_k = \begin{cases} A & \text{for } k = 0 \\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1} \, , \end{cases}$$

that has

$$\mathrm{d}_j^{G,k} = ! : G_k \to G_{k-1}$$

for $k \in \mathbb{Z}_{\leqslant 0}$ and $j \in \mathbb{Z}_{\geqslant 0}$, and that has

$$\mathrm{s}_i^{G,k} = ! : G_k \to G_{k+1}$$

for $k \in \mathbb{Z}_{\leqslant -1}$ and $i \in \mathbb{Z}_{\geqslant 0}$.

(2) *The pre-$\lfloor 0, -\infty \rfloor$-prestable simplicial group G from* (1) *is a $\lfloor 0, -\infty \rfloor$-stable simplicial group; cf. Definition* 232.

Since we have $G_k = 1$ for $k \in \mathbb{Z}_{\geqslant -1}$, note that G is in fact a $\lfloor 0, 0 \rfloor$-stable simplicial group; cf. Definition 307.

From Remark 309*, we know that for a $\lfloor 0, 0 \rfloor$-stable simplicial group H we have that H_0 is abelian.*

So every $\lfloor 0, 0 \rfloor$-stable simplicial group is of the form given in (1).

Proof. Ad (1). Letting all required group morphisms equal $!$, we may use Lemma 254.(3) to construct the pre-$\lfloor 0, -\infty \rfloor$-prestable simplicial group G as required.

Ad (2). We have to verify the the finiteness condition (1) and the Conduché condition (2) from Definition 232.

We show that (1) *holds.*

Suppose given $x \in G_0$. Let $\ell = 0$.

Then, for $k \in \mathbb{Z}_{\geqslant \ell}$, we have

$$x \mathrm{d}_k^{G,0} = x\,! = 1\,.$$

We show that (2) *holds.*

Suppose given I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

Then, since $G_0 = A$ is abelian, we have

$$[\bigcap_{i \in I} \ker \mathrm{d}_i^{G,0}\,, \bigcap_{j \in J} \ker \mathrm{d}_j^{G,0}] = 1\,.$$

□

11.5 The Moore complex of an $\lfloor n, -\infty \rfloor$-stable simplicial group

Let $n \in \mathbb{Z}$.

Definition 257 Let G be a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group; cf. Definition 226.

Let $k \in \mathbb{Z}_{\leqslant n}$ and let $j \in \mathbb{Z}_{\geqslant 0}$.

(1) We write

$$G_{k,j} := \ker \mathrm{d}_j^{G,k} \trianglelefteqslant G_k \,.$$

(2) We write

$$G\mathrm{N}_k := \bigcap_{i \in \mathbb{Z}_{\geqslant 1}} G_{k,i} \trianglelefteqslant G_k \,.$$

(3) We write

$$G\mathrm{NZ}_k := G\mathrm{N}_k \cap G_{k,0} = \bigcap_{i \in \mathbb{Z}_{\geqslant 0}} G_{k,i} \trianglelefteqslant G_k \,.$$

(4) Suppose that $k \in \mathbb{Z}_{\leqslant n-1}$. We write

$$G\mathrm{NB}_k := (G\mathrm{N}_{k+1})\mathrm{d}_0^{G,k+1} = \big(\bigcap_{i \in \mathbb{Z}_{\geqslant 1}} G_{k+1,i} \big)\mathrm{d}_0^{G,k+1} \leqslant G_k \,.$$

Here, in §11.5, we will apply this Definition 257 to an $\lfloor n, -\infty \rfloor$-stable simplicial group.

Lemma 258 (The Moore complex of an $\lfloor n, -\infty \rfloor$-stable simplicial group)
Let G be an $\lfloor n, -\infty \rfloor$-stable simplicial group; cf. Definition 232.

(1) *For $k \in \mathbb{Z}_{\leqslant n}$, we have the group morphism*

$$\partial_k^G := \mathrm{d}_0^{G,k}\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}} \colon G\mathrm{N}_k \to G\mathrm{N}_{k-1} \,,\ x \mapsto x\mathrm{d}_0^{G,k} \,.$$

Sometimes, we write $\partial_k := \partial_k^G$.

(2) *We have the complex in groups*

$$G\mathrm{N} := \big(\ldots \xrightarrow{!} 1 \xrightarrow{!} G\mathrm{N}_n \xrightarrow{\partial_n} G\mathrm{N}_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} G\mathrm{N}_0 \xrightarrow{\partial_0} G\mathrm{N}_{-1} \xrightarrow{\partial_{-1}} \ldots \big);$$

cf. Definition 7.
In particular, for $k \in [0, n-1]$, we have $G\mathrm{NB}_k \trianglelefteqslant G_k$; cf. Definition 257.(4).
We call $G\mathrm{N}$ the Moore complex *of G.*
For $k \in \mathbb{Z}_{\leqslant n-1}$, we write

$$G\pi_k := G\mathrm{NH}_k = \ker \partial_k / \operatorname{im} \partial_{k+1} \,.$$

for the k-th homotopy group of G; cf. Lemma 11.(1).

For $k \in [0, n-1]$, note that we have

$$G\pi_k = G\mathrm{NZ}_k / G\mathrm{NB}_k \,.$$

Moreover, we have $G\pi_n = G\mathrm{NZ}_n$.

Proof. Ad (1). Let $k \in \mathbb{Z}_{\leqslant n}$. We have to show that $(G\mathrm{N}_k)\mathrm{d}_0 \overset{!}{\subseteq} G\mathrm{N}_{k-1}$.
Suppose given $x \in G\mathrm{N}_k$. Then $x\mathrm{d}_i = 1$ for $i \in \mathbb{Z}_{\geqslant 1}$. Suppose given $j \in \mathbb{Z}_{\geqslant 1}$.
We have

$$x\mathrm{d}_0\,\mathrm{d}_j = x\mathrm{d}_{j+1}\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows $x\mathrm{d}_0 \in G_{k-1,j}$.
So $x\mathrm{d}_0 \in \bigcap\limits_{j \in \mathbb{Z}_{\geqslant 1}} G_{k-1,\,j} = G\mathrm{N}_{k-1}$.

Ad (2). Let $k \in \mathbb{Z}_{\leqslant n}$. *We show that* $\partial_k \mathbin{\blacktriangle} \partial_{k-1} \overset{!}{=} !$.
For $x \in G\mathrm{N}_k$, we have

$$x\partial_k\partial_{k-1} = x\mathrm{d}_0\,\mathrm{d}_0 = x\mathrm{d}_1\,\mathrm{d}_0 = 1\mathrm{d}_0 = 1\,.$$

This shows $\partial_k \mathbin{\blacktriangle} \partial_{k-1} = !$.
We show that $\operatorname{im}\partial_k \overset{!}{\trianglelefteqslant} G_{k-1}$.
Suppose given $x \in \operatorname{im}\partial_k$ and suppose given $y \in G_{k-1}$. We have to show that $x^y \overset{!}{\in} \operatorname{im}\partial_k$.
We may write $x = z\partial_k = z\mathrm{d}_0$ for some chosen $z \in G\mathrm{N}_k$.
Let $\tilde{z} := z^{y\mathrm{s}_0} = y^-\mathrm{s}_0 \cdot z \cdot y\mathrm{s}_0 \in G_k$.
Since $z \in G\mathrm{N}_k \trianglelefteqslant G_k$, we also have $\tilde{z} \in G\mathrm{N}_k$.
So we have

$$x^y = (z\mathrm{d}_0)^{y\mathrm{s}_0\mathrm{d}_0} = (z^{y\mathrm{s}_0})\mathrm{d}_0 = \tilde{z}\partial_k\,.$$

This shows that $x^y \in \operatorname{im}\partial_k$.
Therefore, we have $\operatorname{im}\partial_k \trianglelefteqslant G_{k-1}$. □

Lemma 259 *Suppose given $\lfloor n, -\infty \rfloor$-stable simplicial groups G, H.*
Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group morphism $\varphi\colon G \to H$.

(1) *For $k \in \mathbb{Z}_{\leqslant n}$, we have the group morphism*

$$\varphi \mathrm{N}_k := \varphi_k \big|_{G\mathrm{N}_k}^{H\mathrm{N}_k} \colon G\mathrm{N}_k \to H\mathrm{N}_k\,,\ x \mapsto x\varphi_k\,.$$

(2) *We consider the complexes in groups $G\mathrm{N}$ and $H\mathrm{N}$; cf. Lemma* 258.(2).
We have the morphism of complexes in groups

$$\varphi \mathrm{N} := (\varphi \mathrm{N}_k)_{k\in\mathbb{Z}} \colon G\mathrm{N} \to H\mathrm{N}\,,$$

where $\varphi \mathrm{N}_k :=\, !$ for $k \in \mathbb{Z}_{\geqslant n+1}$.

(3) *We have the functor*

$$\mathrm{N} \colon \lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp} \to \mathcal{ComplexGrp}\,,\ \big(F \xrightarrow{\varphi} F'\big) \mapsto \big(F\mathrm{N} \xrightarrow{\varphi \mathrm{N}} F'\mathrm{N}\big)\,.$$

Proof. Ad (1). Let $k \in \mathbb{Z}_{\leqslant n}$. We have to show that $(G\mathrm{N}_k)\varphi_k \overset{!}{\leqslant} H\mathrm{N}_k$.
Suppose given $x \in G\mathrm{N}_k$. For $i \in \mathbb{Z}_{\geqslant 1}$, we have

$$x\varphi_k\, \mathrm{d}_i^{H,k} = x\mathrm{d}_i^{G,k}\, \varphi_{k-1} = 1\varphi_{k-1} = 1\,.$$

This shows $x\varphi_k \in H_{k,i}$ for $i \in \mathbb{Z}_{\geqslant 1}$.
So $x\varphi_k \in H\mathrm{N}_k$.
Ad (2). Let $k \in \mathbb{Z}$. We have to show that $\partial_k^G \mathbin{\blacktriangle} \varphi \mathrm{N}_{k-1} \overset{!}{=} \varphi \mathrm{N}_k \mathbin{\blacktriangle} \partial_k^H$.
We may assume that $k \in \mathbb{Z}_{\leqslant n}$.
For $x \in G\mathrm{N}_k$, we have

$$x\partial_k^G\, (\varphi \mathrm{N}_{k-1}) = x\mathrm{d}_0^{G,k}\, \varphi_{k-1} = x\varphi_k\, \mathrm{d}_0^{H,k} = x(\varphi \mathrm{N}_k)\partial_k^H\,.$$

This shows $\partial_k^G \mathbin{\blacktriangle} \varphi \mathrm{N}_{k-1} = \varphi \mathrm{N}_k \mathbin{\blacktriangle} \partial_k^H$.
So $(\varphi \mathrm{N}_k)_{k\in\mathbb{Z}}$ is a morphism of complexes in groups.
Ad (3). Suppose given $F \xrightarrow{\varphi} F' \xrightarrow{\varphi'} F''$ in $\lfloor n, -\infty \rfloor$-$\mathcal{StSimpGrp}$.
For $k \in \mathbb{Z}_{\leqslant n}$, we have

$$\mathrm{id}_F\, \mathrm{N}_k = \mathrm{id}_{F_k} \big|_{F\mathrm{N}_k}^{F\mathrm{N}_k} = \mathrm{id}_{F\mathrm{N}_k}\,.$$

For $k \in \mathbb{Z}_{\geqslant n+1}$, we have

$$\mathrm{id}_F\, \mathrm{N}_k =\, ! = \mathrm{id}_{F\mathrm{N}_k}\,.$$

Thus, we have

$$\mathrm{id}_F \mathrm{N} = (\mathrm{id}_F \mathrm{N}_k)_{k\in\mathbb{Z}} = (\mathrm{id}_{F\mathrm{N}_k})_{k\in\mathbb{Z}} = \mathrm{id}_{F\mathrm{N}} \ .$$

Moreover, for $k \in \mathbb{Z}_{\leqslant n}$, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N}_k = (\varphi_k \blacktriangle \varphi'_k)\big|_{F\mathrm{N}_k}^{F''\mathrm{N}_k} = \big(\varphi_k\big|_{F\mathrm{N}_k}^{F'\mathrm{N}_k}\big) \blacktriangle \big(\varphi'_k\big|_{F'\mathrm{N}_k}^{F''\mathrm{N}_k}\big) = \varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k \, .$$

For $k \in \mathbb{Z}_{\geqslant n+1}$, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N}_k = ! = \varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k \, .$$

Thus, we have

$$(\varphi \blacktriangle \varphi')\mathrm{N} = \big((\varphi \blacktriangle \varphi')\mathrm{N}_k\big)_{k\in\mathbb{Z}} = (\varphi\mathrm{N}_k \blacktriangle \varphi'\mathrm{N}_k)_{k\in\mathbb{Z}} = (\varphi\mathrm{N}_k)_{k\in\mathbb{Z}} \blacktriangle (\varphi'\mathrm{N}_k)_{k\in\mathbb{Z}} = \varphi\mathrm{N} \blacktriangle \varphi'\mathrm{N} \, .$$

So N is a functor. □

Chapter 12

Stable simplicial groups and pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups: the coskeleton functor

Let $n \in \mathbb{Z}$.

12.1 The cut functor $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} : \mathcal{StSimpGrp} \to \mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{PreStSimpGrp}$

Remark 260 (The functor $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}$)

Let $\mathrm{I}_n : \left(\Delta^{\pm}_{\lfloor n,-\infty\rfloor}\right)^{\mathrm{op}} \to (\Delta^{\pm})^{\mathrm{op}}$, $\left([k] \xrightarrow{u} [\ell]\right) \mapsto \left([k] \xrightarrow{u} [\ell]\right)$ be the inclusion functor.

We have the functor

$$\begin{array}{rccl} \mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} : & \mathcal{PreStSimpGrp} & \to & \mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{PreStSimpGrp} \\ & \left(G \xrightarrow{\varphi} \tilde{G}\right) & \mapsto & \left(G\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} \xrightarrow{\varphi\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}} \tilde{G}\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}\right) \\ & & & := \left(I_n * G \xrightarrow{\mathrm{I}_n * \varphi} I_n * \tilde{G}\right). \end{array}$$

So we have

$$\begin{array}{rcl} G\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} & = & G\big|_{\left(\Delta^{\pm}_{\lfloor n,-\infty\rfloor}\right)^{\mathrm{op}}} \\ \varphi\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} & = & \left(\left(G_k \xrightarrow{\varphi_k} \tilde{G}_k\right)_{k\in\mathbb{Z}}\right)\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} = \left(G_k \xrightarrow{\varphi_k} \tilde{G}_k\right)_{k\in\mathbb{Z}_{\leqslant n}}. \end{array}$$

We write $\mathrm{Cut}^{\lfloor n,-\infty\rfloor} := \mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}$ if there is no risk of confusion with the functor $\mathrm{Cut}^{\lfloor n,0\rfloor} : \mathcal{SimpGrp} \to \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$; cf. Remark 79.

Remark 261 (The cut functor $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$)

We consider the functors

$$\begin{array}{rrcll} \mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor}\colon & \mathcal{PreStSimpGrp} & \to & \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}; & \text{cf. Remark 260} \\ \mathrm{I}\colon & \mathcal{StSimpGrp} & \to & \mathcal{PreStSimpGrp}; & \text{cf. Remark 209}\,. \end{array}$$

Let

$$\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} := \mathrm{I} * \mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor} : \mathcal{StSimpGrp} \to \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}$$

be the *cut functor*.

We often write $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} := \mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$.

Let G be a stable simplicial group. For $k \in \mathbb{Z}_{\leqslant n}$, we have

$$(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k = G_k\,.$$

For $[k]^{\pm} \xrightarrow{u} [\ell]^{\pm}$ in $\Delta^{\pm}_{\lfloor n,-\infty\rfloor}$, we have

$$(G\,\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}})_u = G_u\colon G_\ell \to G_k\,.$$

In particular, for $k \in \mathbb{Z}_{\leqslant n}$ and $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\begin{array}{rcll} \mathrm{d}_i^{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},k} & = & \mathrm{d}_i^{G,k}\colon & G_k \to G_{k-1} \\ \mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},k-1} & = & \mathrm{s}_i^{G,k-1}\colon & G_{k-1} \to G_k\,. \end{array}$$

Let $\varphi\colon G \to H$ be a stable simplicial group morphism. For $k \in \mathbb{Z}_{\leqslant n}$, we have

$$\begin{array}{rrcl} (\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k\colon & (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k & \to & (H\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k \\ & g & \mapsto & g(\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k = g\varphi_k\,. \end{array}$$

12.2 The coskeleton functor $\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} : \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp} \to \mathcal{StSimpGrp}$

Definition 262 Suppose given a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group G; cf. Definition 226.

12.2. THE COSKELETON FUNCTOR

$\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\colon \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp \to \mathcal{S}t\mathcal{S}imp\mathcal{G}rp$

For $k \in \mathbb{Z}$, we consider the set

$$\begin{aligned}{}_{\Delta^\pm}([n]^\pm,[k]^\pm) &= \left\{a \in \mathrm{Mor}\left(\Delta^\pm\right) : a\boldsymbol{s} = [n]^\pm, a\boldsymbol{t} = [k]^\pm\right\}\\ &= \left\{[n]^\pm \xrightarrow{a} [k]^\pm : a \text{ is monotone and eventually parallel}\right\};\end{aligned}$$

cf. Definition 172.

Let $k \in \mathbb{Z}$. Suppose given a tuple $(g_a)_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in \prod_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)} G_n$.

We call $(g_a)_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)}$ *coherent* if we have

$$g_a G_b = g_{a'} G_{b'}$$

for $m \in \mathbb{Z}_{\leqslant n}$ and for $a,\, a' \in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)$, $b,\, b' \in {}_{\Delta^\pm}\left([m]^\pm,[n]^\pm\right)$ with $b \blacktriangle a = b' \blacktriangle a'$.

$$\begin{array}{ccc}[n]^\pm & \xrightarrow{\;a\;} & [k]^\pm\\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle a'}\\ [m]^\pm & \xrightarrow[\;b'\;]{} & [n]^\pm\end{array}$$

Sometimes, we abbreviate $(g_a)_a := (g_a)_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)}$. Note that $g_a \in G_n$ for each $[n]^\pm \xrightarrow{a} [k]^\pm$.

Lemma 263 (The prestable simplicial group $G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}$)

Suppose given a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group G; cf. Definition 226.

(1) *For $k \in \mathbb{Z}$, we have the subgroup*

$$\begin{aligned}(G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k &:= \left\{(g_a)_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in \prod_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)} G_n \colon (g_a)_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)} \text{ is coherent } \right\}\\ &\leqslant \prod_{a\in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)} G_n\,;\end{aligned}$$

cf. Definition 262.

We often write $(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k := (G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k$.

(2) *Suppose given* $\left([k]^\pm \xrightarrow{u} [\ell]^\pm\right) \in \mathrm{Mor}\left(\Delta^\pm_{\lfloor n,-\infty\rfloor}\right)$*; cf. Definition* 225.

We consider the groups $(G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k$ *and* $(G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_\ell$*; cf.* (1).

We have the group morphism

$$\begin{aligned}(G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_u\colon\ (G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_\ell &\to (G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k\\ (g_a)_{a\in_\Delta{}^\pm([n]^\pm,[\ell]^\pm)} &\mapsto (g_{b\blacktriangle u})_{b\in_\Delta{}^\pm([n]^\pm,[k]^\pm)}\,.\end{aligned}$$

We often write $(G\operatorname{Cosk}^{\lfloor n,-\infty\rfloor})_u := (G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_u$.

(3) *We have the pre-*$\lfloor n, -\infty \rfloor$*-prestable simplicial group*

$$\begin{aligned}G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor}\colon\quad \left(\Delta^\pm_{\lfloor n,-\infty\rfloor}\right)^{\mathrm{op}} &\to \mathcal{G}rp\\ \left([\ell]^\pm \xrightarrow{u^{\mathrm{op}}} [k]^\pm\right) &\mapsto \left((G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_\ell \xrightarrow{(G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_u} (G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k\right);\end{aligned}$$

cf. Definition 182.

We often write $G\operatorname{Cosk}^{\lfloor n,-\infty\rfloor} := G\operatorname{Cosk}^{\pm,\lfloor n,-\infty\rfloor}$.

Proof. Ad (1). Suppose given $(g_a)_{a\in_\Delta([n]^\pm,[k]^\pm)}$, $(h_a)_{a\in_\Delta([n]^\pm,[k]^\pm)} \in (G\operatorname{Cosk}^{\lfloor n,-\infty\rfloor})_k$.

Note that $\left((h_a)_{a\in_\Delta([n]^\pm,[k]^\pm)}\right)^- = (h_a^-)_{a\in_\Delta([n]^\pm,[k]^\pm)}$.

We have

$$(g_a)_{a\in_\Delta([n]^\pm,[k]^\pm)}\cdot\left((h_a)_{a\in_\Delta([n]^\pm,[k]^\pm)}\right)^- = (g_a)_{a\in_\Delta([n]^\pm,[k]^\pm)}\cdot(h_a^-)_{a\in_\Delta([n]^\pm,[k]^\pm)} = (g_a\cdot h_a^-)_{a\in_\Delta([n]^\pm,[k]^\pm)}\,.$$

We have to show that $(g_a \cdot h_a^-)_{a\in_\Delta([n]^\pm,[k]^\pm)}$ is coherent.

Suppose given $m \in \mathbb{Z}_{\leqslant n}$ and suppose given a, $a' \in {}_\Delta([n]^\pm,[k]^\pm)$, b, $b' \in {}_\Delta([m]^\pm,[n]^\pm)$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have

$$(g_a \cdot h_a^-)G_b = g_aG_b \cdot (h_aG_b)^- = g_{a'}G_{b'} \cdot (h_{a'}G_{b'})^- = (g_{a'} \cdot h_{a'}^-)G_{b'}\,.$$

This shows $(g_a \cdot h_a^-)_{a\in_\Delta([n]^\pm,[k]^\pm)} \in (G\operatorname{Cosk}^{\lfloor n,-\infty\rfloor})_k$.

Ad (2). *We show that* $(G\operatorname{Cosk}^{\lfloor n,-\infty\rfloor})_u$ *is a well-defined map.*

Suppose given a coherent tuple $(g_a)_{a\in_\Delta([n]^\pm,[\ell]^\pm)} \in (G\operatorname{Cosk}^{\lfloor n,-\infty\rfloor})_\ell$. We have to show that $(g_{b\blacktriangle u})_{b\in_\Delta([n]^\pm,[k]^\pm)}$ is coherent.

Suppose given b, $b' \in {}_\Delta([n]^\pm,[k]^\pm)$ and suppose given $m \in \mathbb{Z}_{\leqslant n}$, c, $c' \in {}_\Delta([m]^\pm,[n]^\pm)$, with $c \blacktriangle b = c' \blacktriangle b'$.

Then we have

$$c \blacktriangle (b \blacktriangle u) = c' \blacktriangle (b' \blacktriangle u)\,.$$

Since $(b \blacktriangle u)$, $(b' \blacktriangle u) \in {}_\Delta([n]^\pm,[\ell]^\pm)$ and since $(g_a)_{\in\Delta([n]^\pm,[\ell]^\pm)}$ is coherent, we have

$$(g_{b\blacktriangle u})G_c = (g_{b'\blacktriangle u})G_{c'}\,.$$

So $(g_{b\blacktriangle u})_{b\in\Delta([n]^\pm,[k]^\pm)}$ is coherent.

We show that $(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u$ *is a group morphism.*

Suppose given $(g_a)_{\in\Delta([n]^\pm,[\ell]^\pm)}$, $(h_a)_{\in\Delta([n]^\pm,[\ell]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_\ell$.

We have

$$\begin{aligned}
&= \big((g_a)_{a\in\Delta([n]^\pm,[\ell]^\pm)} \cdot (h_a)_{a\in\Delta([n]^\pm,[\ell]^\pm)}\big)(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u\\
&= \big((g_a \cdot h_a)_{a\in\Delta([n]^\pm,[\ell]^\pm)}\big)(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u\\
&= (g_{b\blacktriangle u} \cdot h_{b\blacktriangle u})_{b\in\Delta([n]^\pm,[k]^\pm)}\\
&= (g_{b\blacktriangle u})_{b\in\Delta([n]^\pm,[k]^\pm)} \cdot (h_{b\blacktriangle u})_{b\in\Delta([n]^\pm,[k]^\pm)}\\
&= \big((g_a)_{a\in\Delta([n]^\pm,[\ell]^\pm)}\big)(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u \cdot \big((h_a)_{a\in\Delta([n]^\pm,[\ell]^\pm)}\big)(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u\,.
\end{aligned}$$

Ad (3). Suppose given $m \in \mathbb{Z}_{\leqslant n}$. We consider $[m]^\pm \in \mathrm{Ob}\,\big(\Delta^\pm_{\lfloor n,-\infty\rfloor}\big)^{\mathrm{op}}$.

For $(g_a)_{a\in\Delta([n]^\pm,[m]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_m$, we have

$$\begin{aligned}
\big((g_a)_a\big)\big((\mathrm{id}_{[m]^\pm})G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) &= \big((g_a)_a\big)(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_{\mathrm{id}_{[m]^\pm}} = (g_{a\blacktriangle\,\mathrm{id}_{[m]^\pm}})_a = (g_a)_a\\
&= \big((g_a)_a\big)\,\mathrm{id}_{(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_m}\,.
\end{aligned}$$

This shows $(\mathrm{id}_{[m]^\pm})G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} = \mathrm{id}_{(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_m}$.

Suppose given $[k]^\pm \xrightarrow{u} [\ell]^\pm \xrightarrow{v} [m]^\pm$ in $\Delta^\pm_{\lfloor n,-\infty\rfloor}$.

We consider $[m]^\pm \xrightarrow{v^{\mathrm{op}}} [\ell]^\pm \xrightarrow{u^{\mathrm{op}}} [k]^\pm$ in $\big(\Delta^\pm_{\lfloor n,-\infty\rfloor}\big)^{\mathrm{op}}$.

For $(g_a)_{a\in\Delta([n]^\pm,[m]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_m$, we have

$$\begin{aligned}
&\big((g_a)_a\big)\big((v^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} \blacktriangle (u^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\\
=\ &\big((g_a)_a\big)\big((G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_v \blacktriangle (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u\big)\\
=\ &\big((g_{b\blacktriangle v})_{b\in\Delta([n]^\pm,[\ell]^\pm)}\big)(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u\\
=\ &(g_{(c\blacktriangle u)\blacktriangle v})_{c\in\Delta([n]^\pm,[k]^\pm)}\\
=\ &(g_{c\blacktriangle(u\blacktriangle v)})_{c\in\Delta([n]^\pm,[k]^\pm)}\\
=\ &\big((g_a)_a\big)(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_{(u\blacktriangle v)}\\
=\ &\big((g_a)_a\big)\Big(\big((u\blacktriangle v)^{\mathrm{op}}\big)G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\Big)\\
=\ &\big((g_a)_a\big)\big((v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}})G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\,.
\end{aligned}$$

This shows $(v^{\mathrm{op}} \blacktriangle u^{\mathrm{op}}) G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} = (v^{\mathrm{op}}) G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} \blacktriangle (u^{\mathrm{op}}) G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$.
So $G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} \colon \big(\Delta^{\pm}_{\lfloor n,-\infty\rfloor}\big)^{\mathrm{op}} \to \mathcal{G}rp$ is a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group. □

Lemma 264 (The functor $\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}$)

For a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group $G \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$, we consider the prestable simplicial group $G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor} \in \mathrm{Ob}\big(\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$; cf. Lemma 263.(3).

We have the coskeleton functor

$$\begin{array}{rccl} \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}\colon & \mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp & \to & \mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp \\ & \big(G \xrightarrow{\varphi} \tilde{G}\big) & \mapsto & \big(G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor} \xrightarrow{\varphi\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}} \tilde{G}\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}\big)\,, \end{array}$$

where, for $k \in \mathbb{Z}$, we have

$$(\varphi\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k \colon (G\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k \to (\tilde{G}\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_k\,,\ (g_a)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \mapsto (g_a\varphi_n)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\,.$$

We write $\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} := \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor}$ if there is no risk of confusion with the functor $\mathrm{Cosk}^{\lfloor n,0\rfloor}\colon \mathcal{P}re\text{-}\lfloor n, 0 \rfloor\text{-}\mathcal{S}imp\mathcal{G}rp \to \mathcal{S}imp\mathcal{G}rp$; cf. Lemma 82.

Proof. Suppose given $G \xrightarrow{\varphi} \tilde{G}$ in $\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp$. Let $k \in \mathbb{Z}_{\leqslant n}$.

We show that $\varphi(\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$ is a well-defined map.

Suppose given $(g_a)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$.

We have to show that $(g_a\varphi_n)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \overset{!}{\in} (\tilde{G}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$, i.e. that $(g_a\varphi_n)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}$ is coherent.

For $a,\, a' \in {}_{\Delta^{\pm}}([n]^{\pm}, [k]^{\pm})$, suppose given $m \in \mathbb{Z}_{\leqslant n}$ and $b,\, b' \in {}_{\Delta^{\pm}}([m]^{\pm}, [n]^{\pm})$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have to show that $(g_a\varphi_n)\tilde{G}_b \overset{!}{=} (g_{a'}\varphi_n)\tilde{G}_{b'}$.

We have

$$(g_a\varphi_n)\tilde{G}_b = (g_aG_b)\varphi_m = (g_{a'}G_{b'})\varphi_m = (g_{a'}\varphi_n)\tilde{G}_{b'}\,.$$

$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & \tilde{G}_n \\ {\scriptstyle G_b}\downarrow & & \downarrow{\scriptstyle \tilde{G}_b} \\ G_m & \xrightarrow[\varphi_m]{} & \tilde{G}_m \end{array} \qquad\qquad \begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & \tilde{G}_n \\ {\scriptstyle G_{b'}}\downarrow & & \downarrow{\scriptstyle \tilde{G}_{b'}} \\ G_n & \xrightarrow[\varphi_m]{} & \tilde{G}_n \end{array}$$

$\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\colon \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}impGrp \to \mathcal{S}t\mathcal{S}impGrp$

We show that $\varphi(\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$ *is a group morphism.*

Suppose given $(g_a)_{a\in\Delta([n]^\pm,[k]^\pm)}$, $(h_a)_{a\in\Delta([n]^\pm,[k]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$.

We have

$$\begin{aligned}\big((g_a)_a\cdot(h_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k &= \big((g_a\cdot h_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k = \big((g_a\cdot h_a)\varphi_n\big)_a\\ &= (g_a\varphi_n\cdot h_a\varphi_n)_a = (g_a\varphi_n)_a\cdot(h_a\varphi_n)_a\\ &= \big((g_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\cdot\big((h_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\,.\end{aligned}$$

We show that $\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$ *is a functor.*

Suppose given $G \in \mathrm{Ob}(\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}impGrp)$.

For $k\in\mathbb{Z}_{\leqslant n}$ and for $(g_a)_{a\in\Delta([n]^\pm,[k]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$, we have

$$(g_a)_a\big((\mathrm{id}_G)\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)_k = \big(g_a(\mathrm{id}_G)_n\big)_a = (g_a)_a = (g_a)_a\,\mathrm{id}_{(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k}\,.$$

This shows $(\mathrm{id}_G)\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} = \mathrm{id}_{G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}}$.

Suppose given $G \xrightarrow{\varphi} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{\tilde{G}}$ in $\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}impGrp$.

For $k\in\mathbb{Z}_{\leqslant n}$ and for $(g_a)_{a\in\Delta([n]^\pm,[k]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$, we have

$$\begin{aligned}(g_a)_a\big((\varphi\mathbin{\triangle}\tilde{\varphi})\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)_k &= \big(g_a(\varphi\mathbin{\triangle}\tilde{\varphi})_n\big)_a = \big(g_a(\varphi_n\mathbin{\triangle}\tilde{\varphi}_n)\big)_a = \big((g_a\varphi_n)_a\big)(\tilde{\varphi}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\\ &= (g_a)_a(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k(\tilde{\varphi}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\\ &= (g_a)_a(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\mathbin{\triangle}\tilde{\varphi}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\,.\end{aligned}$$

This shows $(\varphi\mathbin{\triangle}\tilde{\varphi})\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} = \varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\mathbin{\triangle}\tilde{\varphi}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$. □

Remark 265 *Let* $m\in\mathbb{Z}_{\geqslant n+1}$. *Suppose given* $a\in{}_{\Delta^\pm}([n]^\pm,[m]^\pm)$.
There exists some $b\in{}_{\Delta^\pm}([n]^\pm,[m-1]^\pm)$ *and some* $j\in\mathbb{Z}_{\geqslant m-n-1}$ *with* $a = b\mathbin{\triangle}\delta_j^{\pm,m}$.

Proof. By Corollary 177.(1), we have

$$a = \left(\prod_{r\in\lfloor n-\ell,1\rfloor}^{[n]^\pm}\sigma_{i_r}^{\pm,\ell+r-1}\right)\mathbin{\triangle}\left(\prod_{r\in\lceil 1,m-\ell\rceil}^{[\ell]^\pm}\delta_{j_r}^{\pm,\ell+r}\right)\colon[n]^\pm\to[m]^\pm\,,$$

where $\ell = m - \big|[m]^\pm\setminus\mathrm{im}\,a\big|$, where $0\leqslant i_1<i_2<\ldots<i_{n-\ell}$ and where

$$\{j_1,\ldots,j_{m-\ell}\} = \{j\in\mathbb{Z}_{\geqslant 0}\colon j\notin\mathrm{im}\,u\}\text{ and }0\leqslant j_1<j_2<\ldots<j_{m-\ell}\,.$$

Note that $\ell \leqslant n \leqslant m-1$. Let $j := j_{m-\ell}$. Note that $0 \leqslant m-n-1 \leqslant m-\ell-1 \leqslant j$.

Let

$$b := \left(\prod_{r \in \lfloor n-\ell, 1 \rfloor}^{[n]^{\pm}} \sigma_{i_r}^{\pm, \ell+r-1} \right) \blacktriangle \left(\prod_{r \in \lceil 1, m-\ell-1 \rceil}^{[\ell]^{\pm}} \delta_{j_r}^{\pm, \ell+r} \right) \colon [n]^{\pm} \to [m-1]^{\pm},$$

Then we have $a = b \blacktriangle \delta_j^{\pm, m}$, where $j \in \mathbb{Z}_{\geqslant m-n-1}$. □

Lemma 266 *Let H be a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group.*

We consider the simplicial group $H\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor}$; cf. Lemma 263.(3).

For $k \in \mathbb{Z}_{\geqslant n+2}$, we have $(H\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})\mathrm{N}_k = 1$; cf. Definition 198.(2).

Proof. Suppose given $k \in \mathbb{Z}_{\geqslant n+2}$. Suppose given $(h_a)_{a \in {}_{\Delta^{\pm}}([n]^{\pm}, [k]^{\pm})} \in (H\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})\mathrm{N}_k$.

We have to show that $(h_a)_{a \in {}_{\Delta^{\pm}}([n]^{\pm}, [k]^{\pm})} \overset{!}{=} 1$.

Suppose given $\tilde{a} \in {}_{\Delta^{\pm}}([n]^{\pm}, [k]^{\pm})$. We have to show that $h_{\tilde{a}} \overset{!}{=} 1 \in H_n$.

By Remark 265, we have $\tilde{a} = b \blacktriangle \delta_j^{\pm, k}$ for some chosen $b \in {}_{\Delta^{\pm}}([n]^{\pm}, [k-1]^{\pm})$ and for some chosen $j \in \mathbb{Z}_{\geqslant k-n-1}$.

Note that $j \in \mathbb{Z}_{\geqslant 1}$ since $k-n-1 \geqslant n+2-n-1 = 1$.

We have

$$1 = \big((h_a)_{a \in {}_{\Delta^{\pm}}([n]^{\pm}, [k]^{\pm})}\big) \mathrm{d}_j^{H\mathrm{Cosk}^{\lfloor n, -\infty \rfloor}, k} \overset{263.(2)}{=} (h_{c \blacktriangle \delta_j})_{c \in {}_{\Delta^{\pm}}([n]^{\pm}, [k-1]^{\pm})}.$$

In particular, we have $1 = h_{b \blacktriangle \delta_j} = h_{\tilde{a}}$.

So we have $(h_a)_{a \in {}_{\Delta}([n]^{\pm}, [k]^{\pm})} = 1$. This shows that $(H\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})\mathrm{N}_k = 1$. □

Remark 267 (The cokeleton functor $\mathrm{Cosk}^{\pm, \lfloor n, -\infty \rfloor, \mathrm{fin}}$)

We consider the functors

$$\begin{aligned} \mathrm{Cosk}^{\pm, \lfloor n, -\infty \rfloor} \colon \ & \mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp \ \to \ \mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp; \quad \text{cf. Lemma 264} \\ \Phi \colon \ & \mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp \ \to \ \mathcal{S}t\mathcal{S}imp\mathcal{G}rp; \quad \text{cf. Lemma 208}. \end{aligned}$$

Let

$$\mathrm{Cosk}^{\pm, \lfloor n, -\infty \rfloor, \mathrm{fin}} := \mathrm{Cosk}^{\pm, \lfloor n, -\infty \rfloor} * \Phi \colon \mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp \to \mathcal{S}t\mathcal{S}imp\mathcal{G}rp$$

be the *coskeleton functor*.

We often write $\mathrm{Cosk}^{\lfloor n, -\infty \rfloor, \mathrm{fin}} := \mathrm{Cosk}^{\pm, \lfloor n, -\infty \rfloor, \mathrm{fin}}$.

Let H be a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group.

Note that $H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \leqslant H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$; cf. Lemma 208. For $k\in\mathbb{Z}$, we have

$$\begin{aligned}
&(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k = (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k^{\mathrm{fin}}\\
&= \Big\{(h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k : (h_a)_a \text{ is finite } \Big\}\\
&\overset{204}{=} \Big\{(h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k :\\
&\qquad \exists \ell\in\mathbb{Z}_{\geqslant 0} \text{ such that } 1 = \big((h_a)_a\big)\mathrm{d}_j^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor},k} = (h_{b\blacktriangleleft\delta_j})_{b\in_{\Delta^\pm}([n]^\pm,[k-1]^\pm)} \text{ for } j\in\mathbb{Z}_{\geqslant\ell} \Big\}
\end{aligned}$$

For $[k]^\pm \xrightarrow{u} [\ell]^\pm$ in $\Delta^\pm$, and $(h_a)_{\Delta^\pm([n]^\pm,[\ell]^\pm)} \in (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_\ell$, we have

$$\big((h_a)_{a\in_{\Delta^\pm}([n]^\pm,[\ell]^\pm)}\big)(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_u = (h_{b\blacktriangleleft u})_{b\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\,.$$

Let $\varphi\colon G\to H$ be a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group morphism.

For $k\in\mathbb{Z}$, we have

$$\begin{array}{rrcl}
(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k\colon & (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k & \to & (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k\\
& (g_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} & \mapsto & \big((g_a)_a\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k = (g_a\varphi_n)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\,.
\end{array}$$

12.3 The adjunction $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$

We consider the functors

$$\begin{array}{rrcll}
\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\colon & \mathcal{PreStSimpGrp} & \to & \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}; & \text{cf. Remark 260}\\
\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\colon & \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp} & \to & \mathcal{PreStSimpGrp}; & \text{cf. Lemma 264.}
\end{array}$$

Lemma 268 (The counit η)

Suppose given a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group G; cf. Definition 226.

We consider the pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group $G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}$.

For $k\in\mathbb{Z}_{\leqslant n}$, recall that we have

$$\begin{aligned}
(G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})_k &= (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\\
&= \big\{(g_a)_a \in \prod_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} G_n\colon (g_a)_a \text{is coherent } \big\}\,;
\end{aligned}$$

cf. Lemma 263.(1).

For $[k]^{\pm} \xrightarrow{u} [\ell]^{\pm}$ *in* $\left(\Delta^{\pm}_{\lfloor n,-\infty \rfloor}\right)^{\mathrm{op}}$, *recall that we have*

$$\begin{aligned}(G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty \rfloor})_u = (G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_u \colon \quad (G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_\ell &\to (G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_k \\ (g_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[\ell]^{\pm})} &\mapsto (g_{b \blacktriangle u})_{b \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\,;\end{aligned}$$

cf. Lemma 263.(2).

(1) *For* $k \in \mathbb{Z}_{\leqslant n}$, *we have the group isomorphism*

$$(G\vartheta)_k \colon G_k \to (G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_k\,,\ x \mapsto (xG_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\,.$$

(2) *We have the pre-*$\lfloor n, -\infty \rfloor$*-prestable simplicial group isomorphism*

$$G\vartheta := \left(G_k \xrightarrow{(G\vartheta)_k} (G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_k\right)_{k \in \mathbb{Z}_{\leqslant n}} \colon G \to G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty \rfloor}\,;$$

cf. (1).

(3) *We have the isotransformation*

$$\vartheta = \left(\tilde{G} \xrightarrow{\tilde{G}\vartheta} \tilde{G}\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty \rfloor}\right)_{\tilde{G} \in \mathrm{Ob}(\mathcal{P}re\text{-}\lfloor n,-\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp)} \colon \mathrm{Id}_{\mathcal{P}re\text{-}\lfloor n,-\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp} \to \mathrm{Cosk}^{\lfloor n,-\infty \rfloor} * \mathrm{Cut}^{\lfloor n,-\infty \rfloor}\,;$$

cf. (2).

Moreover, we have the isotransformation

$$\eta := \vartheta^{-} \colon\ \mathrm{Cosk}^{\lfloor n,-\infty \rfloor} * \mathrm{Cut}^{\lfloor n,-\infty \rfloor} \to \mathrm{Id}_{\mathcal{P}re\text{-}\lfloor n,-\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}\,.$$

In particular, for $k \in \mathbb{Z}_{\leqslant n}$, *we have the group isomorphism*

$$(G\eta)_k = \left((G\vartheta)_k\right)^{-} \colon (G\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_k \to G_k\,;$$

cf. (1).

Proof. Ad (1). Suppose given $k \in \mathbb{Z}_{\leqslant n}$.

We show that $(G\vartheta)_k$ *is a well-defined map.*

Suppose given $x \in G$. We have to show that $(xG_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}$ is coherent.

Suppose given a, $a' \in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)$. Suppose given $m \in \mathbb{Z}_{\leqslant n}$ and b, $b' \in {}_{\Delta^\pm}([m]^\pm,[n]^\pm)$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have

$$(xG_a)G_b = xG_{b\blacktriangle a} = xG_{b'\blacktriangle a'} = (xG_{b'})G_{a'}\,.$$

We show that $(G\vartheta)_k$ *is a group morphism.*

For x, $y \in G_k$, we have

$$\begin{aligned}(x\cdot y)(G\vartheta)_k &= \big((x\cdot y)G_a\big)_{a\in{}_{\Delta^\pm}([n]^\pm,[k]^\pm)} = (xG_a\cdot yG_a)_{a\in{}_{\Delta^\pm}([n]^\pm,[k]^\pm)}\\ &= (xG_a)_{a\in{}_{\Delta^\pm}([n]^\pm,[k]^\pm)}\cdot(yG_a)_{a\in{}_{\Delta}([n]^\pm,[k]^\pm)} = x(G\vartheta)_k\cdot y(G\vartheta)_k\,.\end{aligned}$$

We show that $(G\vartheta)_k$ *is injective.*

Suppose given $x \in \ker\big((G\vartheta)_k\big)$. Then we have $xG_a = 1$ for $a \in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)$.

We choose $\hat{a} \in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)$ to be surjective. We choose $\hat{b} \in {}_{\Delta^\pm}([k]^\pm,[n]^\pm)$ such that $\hat{b}\blacktriangle\hat{a} = \mathrm{id}_{[k]^\pm}$.

Then we have

$$1 = x(G_{\hat{a}}\blacktriangle G_{\hat{b}}) = x(G_{\hat{b}\blacktriangle\hat{a}}) = xG_{\mathrm{id}_{[k]^\pm}} = x\,.$$

So $(G\vartheta)_k$ is injective.

We show that $(G\vartheta)_k$ *is surjective.*

Suppose given $(g_a)_{a\in{}_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$.

We choose $\hat{a} \in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)$ to be surjective. We choose $\hat{b} \in {}_{\Delta^\pm}([k]^\pm,[n]^\pm)$ such that $\hat{b}\blacktriangle\hat{a} = \mathrm{id}_{[k]^\pm}$.

For $a \in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)$, note that we have

$$(a\blacktriangle\hat{b})\blacktriangle\hat{a} = a = \mathrm{id}_{[n]^\pm}\blacktriangle a\,.$$

$$\begin{array}{ccc}[n]^\pm & \xrightarrow{\ \hat{a}\ } & [k]^\pm\\ {\scriptstyle a\blacktriangle\hat{b}}\uparrow & & \uparrow{\scriptstyle a}\\ [n]^\pm & \xrightarrow[\mathrm{id}_{[n]^\pm}]{} & [n]^\pm\end{array}$$

Since $(g_a)_a$ is coherent, we have

$$(g_{\hat{a}})G_{a\blacktriangle\hat{b}} = (g_a)G_{\mathrm{id}_{[n]^\pm}} = g_a$$

for $a \in {}_{\Delta^\pm}([n]^\pm, [k]^\pm)$.

Let $x := g_{\hat{a}} G_{\hat{b}} \in G_k$. Then we have

$$x(G\vartheta)_k = (xG_a)_a = \big((g_{\hat{a}} G_{\hat{b}}) G_a\big)_a = \big((g_{\hat{a}}) G_{a \blacktriangle \hat{b}}\big)_a = (g_a)_a \,.$$

So $(G\vartheta)_k$ is surjective.

Ad (2). *We show that $G\vartheta$ is a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphism.*

Suppose given $[k]^\pm \xrightarrow{u} [\ell]^\pm$ in $\Delta^\pm_{\lfloor n, -\infty \rfloor}$.

We consider the group isomorphisms $(G\vartheta)_k$ and $(G\vartheta)_\ell$ from (1).

For $g \in G_\ell$, we have

$$\begin{aligned} g\big((G\vartheta)_\ell \blacktriangle (G\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})_u\big) &\overset{(1)}{=} \big((gG_a)_{a \in {}_{\Delta^\pm}([n]^\pm, [\ell]^\pm)}\big)(G\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})_u \overset{263.(2)}{=} (gG_{b \blacktriangle u})_{b \in {}_{\Delta^\pm}([n]^\pm, [k]^\pm)} \\ &= (gG_u G_b)_{b \in {}_{\Delta^\pm}([n]^\pm, [k]^\pm)} = (gG_u)(G\vartheta)_k = g(G_u \blacktriangle (G\vartheta)_k) \,. \end{aligned}$$

This shows $(G\vartheta)_\ell \blacktriangle (G\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})_u = G_u \blacktriangle (G\vartheta)_k$.

$$\begin{array}{ccc} G_\ell & \xrightarrow{(G\vartheta)_\ell} & (G\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})_\ell \\ {\scriptstyle G_u}\downarrow & & \downarrow{\scriptstyle (G\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})_u} \\ G_k & \xrightarrow[(G\vartheta)_k]{} & (G\,\mathrm{Cosk}^{\lfloor n, -\infty \rfloor})_k \end{array}$$

So $G\vartheta$ is a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphism.

Moreover, by (1), $G\vartheta$ is a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group isomorphism.

Ad (3). Suppose given $\tilde{G} \xrightarrow{\varphi} \tilde{\tilde{G}}$ in $\mathcal{Pre}$-$\lfloor n, -\infty \rfloor$-$\mathcal{PreStSimpGrp}$.

Suppose given $k \in \mathbb{Z}_{\leqslant n}$. We consider the group morphisms $(\tilde{G}\vartheta)_k$ and $(\tilde{\tilde{G}}\vartheta)_k$ from (1).

For $g \in \tilde{G}_k$, we have

$$\begin{aligned} g\big((\tilde{G}\vartheta)_k \blacktriangle (\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})_k\big) &\overset{(1)}{=} \big((g\tilde{G}_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}}\big)(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})_k \\ &\overset{264}{=} \big((g\tilde{G}_a)\varphi_n\big)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \\ &= \big((g\varphi_k)\tilde{\tilde{G}}_a\big)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \\ &\overset{(1)}{=} (g\varphi_k)(\tilde{\tilde{G}}\vartheta)_k \\ &= g\big(\varphi_k \blacktriangle (\tilde{\tilde{G}}\vartheta)_k\big)\,. \end{aligned}$$

This shows $(\tilde{G}\vartheta)_k \blacktriangle (\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})_k = \varphi_k \blacktriangle (\tilde{\tilde{G}}\vartheta)_k$.

Therefore, we have $\tilde{G}\vartheta \blacktriangle (\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}) = \varphi \blacktriangle \tilde{\tilde{G}}\vartheta$.

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{G}\vartheta} & \tilde{G}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}} \\ \tilde{\tilde{G}} & \xrightarrow[\tilde{\tilde{G}}\vartheta]{} & \tilde{\tilde{G}}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \end{array}$$

So ϑ is a transformation.

Since by (2), $G\vartheta$ is an isomorphism, we have the isotransformation

$$\vartheta\colon\ \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp} \xrightarrow{\sim} \mathrm{Cosk}^{\lfloor n,-\infty\rfloor} * \mathrm{Cut}^{\lfloor n,-\infty\rfloor}\ .$$

Therefore, we also have the isotransformation $\eta := \vartheta^-$. □

Lemma 269 (The unit ε)

Suppose given a prestable simplicial group G; cf. Definition 182.

We consider the prestable simplicial group $G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$.

For $k \in \mathbb{Z}$, note that we have

$$(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k = \Big\{(g_a)_{a\in_{\Delta}([n]^\pm,[k]^\pm)} \in \prod_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} G_n\colon (g_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \text{ is coherent } \Big\};$$

cf. Lemma 263.(1).

For $[k]^{\pm} \xrightarrow{u} [\ell]^{\pm}$ in $\mathbf{\Delta}^{\pm}$, note that we have

$$\begin{aligned}(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u\colon\ (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_\ell &\to (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\\ (g_a)_{a\in{}_{\Delta}([n]^{\pm},[\ell]^{\pm})} &\mapsto (g_{b\blacktriangle u})_{b\in{}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\,;\end{aligned}$$

cf. Lemma 263.(2).

(1) *For $k \in \mathbb{Z}$, we have the group morphism*

$$(G\varepsilon)_k\colon G_k \to (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\,,\ g \mapsto (gG_a)_{a\in{}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\,;$$

cf. also Lemma 268.(1).

(2) *We have the prestable simplicial group morphism*

$$G\varepsilon := \big(G_k \xrightarrow{(G\varepsilon)_k} (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\big)_{k\in\mathbb{Z}}\colon G \to G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,;$$

cf. (1).

(3) *We have the transformation*

$$\varepsilon = \big(\tilde{G} \xrightarrow{\tilde{G}\varepsilon} \tilde{G}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)_{\tilde{G}\in\mathrm{Ob}(\mathcal{PreStSimpGrp})}\colon\ \mathrm{id}_{\mathcal{PreStSimpGrp}} \to \mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,;$$

cf. (2).

Proof. Ad (1). Suppose given $k \in \mathbb{Z}$.

We show that $(G\varepsilon)_k$ is a well-defined map.

Suppose given $x \in G$. We have to show that $(xG_a)_{a\in{}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}$ coherent; cf. Definition 262.

Suppose given a, $a' \in {}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})$. Suppose given $m \in \mathbb{Z}_{\leqslant n}$ and b, $b' \in {}_{\Delta^{\pm}}([m]^{\pm},[n]^{\pm})$ with $b \blacktriangle a = b' \blacktriangle a'$.

We have

$$(xG_a)G_b = xG_{b\blacktriangle a} = xG_{b'\blacktriangle a'} = (xG_{b'})G_{a'}\,.$$

We show that $(G\varepsilon)_k$ is a group morphism.

For x, $y \in G_k$, we have

$$\begin{aligned}(x\cdot y)(G\varepsilon)_k &= \big((x\cdot y)G_a\big)_{a\in{}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} = (xG_a \cdot yG_a)_{a\in{}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\\ &= (xG_a)_{a\in{}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \cdot (yG_a)_{a\in{}_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} = x(G\varepsilon)_k \cdot y(G\varepsilon)_k\,.\end{aligned}$$

Ad (2). *We show that $G\varepsilon$ is a prestable simplicial group morphism.*

Suppose given $[k]^{\pm} \xrightarrow{u} [\ell]^{\pm}$ in $\Delta^{\pm}$.

We consider the group morphisms $(G\varepsilon)_k$ and $(G\varepsilon)_\ell$ from (1).

For $g \in G_\ell$, we have

$$\begin{aligned} g\big((G\varepsilon)_\ell \blacktriangle (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u\big) &\overset{(1)}{=} \big((gG_a)_{a\in\Delta^{\pm}([n]^{\pm},[\ell]^{\pm})}\big)(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u \\ &= (gG_{b\blacktriangle u})_{b\in\Delta^{\pm}([n]^{\pm},[k]^{\pm})} \\ &= (gG_uG_b)_{b\in\Delta^{\pm}([n]^{\pm},[k]^{\pm})} \\ &\overset{(1)}{=} (gG_u)(G\varepsilon)_k = g(G_u \blacktriangle (G\varepsilon)_k)\,. \end{aligned}$$

This shows $(G\varepsilon)_\ell \blacktriangle (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u = G_u \blacktriangle (G\varepsilon)_k$.

$$\begin{array}{ccc} G_\ell & \xrightarrow{(G\varepsilon)_\ell} & (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_\ell \\ {\scriptstyle G_u}\downarrow & & \downarrow{\scriptstyle (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_u} \\ G_k & \xrightarrow[(G\varepsilon)_k]{} & (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k \end{array}$$

So $G\varepsilon$ is a prestable simplicial group morphism.

Ad (3). Suppose given $\tilde{G} \xrightarrow{\varphi} \tilde{\tilde{G}}$ in $\mathcal{PreStSimpGrp}$.

Suppose given $k \in \mathbb{Z}$. For $g \in \tilde{G}_k$, we have

$$\begin{aligned} g\big((\tilde{G}\varepsilon)_k \blacktriangle (\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\big) &\overset{(1)}{=} \big((g\tilde{G}_a)_{a\in\Delta^{\pm}([n]^{\pm},[k]^{\pm})}\big)(\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k \\ &\overset{264}{=} \big((g\tilde{G}_a)\varphi_n\big)_{a\in\Delta^{\pm}([n]^{\pm},[k]^{\pm})} \\ &= \big((g\varphi_k)\tilde{\tilde{G}}_a\big)_{a\in\Delta^{\pm}([n]^{\pm},[k]^{\pm})} \\ &\overset{(1)}{=} (g\varphi_k)(\tilde{\tilde{G}}\varepsilon)_k \\ &= g\big(\varphi_k \blacktriangle (\tilde{\tilde{G}}\varepsilon)_k\big)\,. \end{aligned}$$

This shows $(\tilde{G}\varepsilon)_k \blacktriangle (\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k = \varphi_k \blacktriangle (\tilde{\tilde{G}}\varepsilon)_k$.

Therefore, we have $\tilde{G}\varepsilon \blacktriangle (\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}) = \varphi \blacktriangle \tilde{\tilde{G}}\varepsilon$.

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{G}\varepsilon} & \tilde{G}\operatorname{Cut}^{\lfloor n,-\infty\rfloor}\operatorname{Cosk}^{\lfloor n,-\infty\rfloor} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi\operatorname{Cut}^{\lfloor n,-\infty\rfloor}\operatorname{Cosk}^{\lfloor n,-\infty\rfloor}} \\ \tilde{\tilde{G}} & \xrightarrow[\tilde{\tilde{G}}\varepsilon]{} & \tilde{\tilde{G}}\operatorname{Cut}^{\lfloor n,-\infty\rfloor}\operatorname{Cosk}^{\lfloor n,-\infty\rfloor} \end{array}$$

So ε is a transformation. □

Lemma 270 (The adjoint functors $\operatorname{Cut}^{\lfloor n,-\infty\rfloor}$ and $\operatorname{Cosk}^{\lfloor n,-\infty\rfloor}$)
We consider the isotransformation $\eta\colon \operatorname{Cosk}^{\lfloor n,-\infty\rfloor} * \operatorname{Cut}^{\lfloor n,-\infty\rfloor} \to \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}$ *from Lemma* 268.(3).
We consider the transformation $\varepsilon\colon \mathrm{id}_{\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp} \to \operatorname{Cut}^{\lfloor n,-\infty\rfloor} * \operatorname{Cosk}^{\lfloor n,-\infty\rfloor}$ *from Lemma* 269.(3).
Then $(\operatorname{Cut}^{\lfloor n,-\infty\rfloor}, \operatorname{Cosk}^{\lfloor n,-\infty\rfloor}, \varepsilon, \eta)$ *is an adjunction; cf.* §1.1.3 *item* 10.
In particular, we have $\operatorname{Cut}^{\lfloor n,-\infty\rfloor} \dashv \operatorname{Cosk}^{\lfloor n,-\infty\rfloor}$.

Proof. We show that $(\varepsilon * \operatorname{Cut}^{\lfloor n,-\infty\rfloor}) \blacktriangle (\operatorname{Cut}^{\lfloor n,-\infty\rfloor} * \eta) \overset{!}{=} \mathrm{id}_{\operatorname{Cut}^{\lfloor n,-\infty\rfloor}}$.

$$\begin{array}{ccc} \operatorname{Cut}^{\lfloor n,-\infty\rfloor} & \xrightarrow{\varepsilon * \operatorname{Cut}^{\lfloor n,-\infty\rfloor}} & \operatorname{Cut}^{\lfloor n,-\infty\rfloor} * \operatorname{Cosk}^{\lfloor n,-\infty\rfloor} * \operatorname{Cut}^{\lfloor n,-\infty\rfloor} \\ & \searrow^{\mathrm{id}_{\operatorname{Cut}^{\lfloor n,-\infty\rfloor}}} & \downarrow{\scriptstyle \operatorname{Cut}^{\lfloor n,-\infty\rfloor} * \eta} \\ & & \operatorname{Cut}^{\lfloor n,-\infty\rfloor} \end{array}$$

Suppose given $G \in \mathrm{Ob}\big(\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$.
We show that $G\big((\varepsilon * \operatorname{Cut}^{\lfloor n,-\infty\rfloor}) \blacktriangle (\operatorname{Cut}^{\lfloor n,-\infty\rfloor} * \eta)\big) \overset{!}{=} \mathrm{id}_{G\operatorname{Cut}^{\lfloor n,-\infty\rfloor}}$.
Suppose given $k \in \mathbb{Z}_{\leqslant n}$. Suppose given $x \in G_k$.

We have

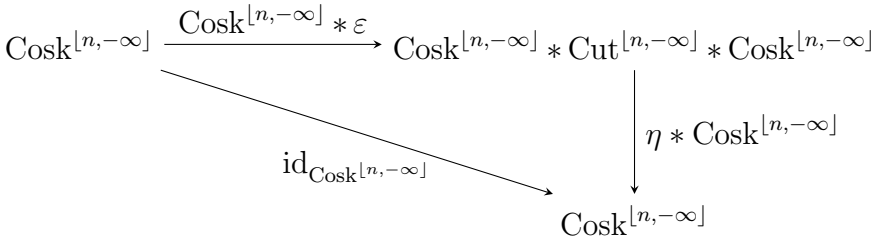
$$\begin{aligned}
& x\Big(G\big((\varepsilon * \mathrm{Cut}^{\lfloor n,-\infty\rfloor}) \blacktriangle (\mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \eta)\big)\Big)_k \\
= \quad & x\big(G(\varepsilon * \mathrm{Cut}^{\lfloor n,-\infty\rfloor}) \blacktriangle G(\mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \eta)\big)_k \\
= \quad & x\Big(\big((G\varepsilon)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}\big)_k \blacktriangle \big((G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})\eta\big)_k\Big) \\
\overset{269.(2)}{=} \quad & \big((xG_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\big)\big((G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})\eta\big)_k \\
\overset{268.(3,1)}{=} \quad & x \\
= \quad & x\,\mathrm{id}_{G_k} \\
= \quad & x(\mathrm{id}_{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}})_k\,.
\end{aligned}$$

This shows $\Big(G\big((\varepsilon * \mathrm{Cut}^{\lfloor n,-\infty\rfloor}) \blacktriangle (\mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \eta)\big)\Big)_k = (\mathrm{id}_{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}})_k$.

So we have $G\big((\varepsilon * \mathrm{Cut}^{\lfloor n,-\infty\rfloor}) \blacktriangle (\mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \eta)\big) = \mathrm{id}_{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}} = G\,\mathrm{id}_{\mathrm{Cut}^{\lfloor n,-\infty\rfloor}}$.

Therefore, we have $(\varepsilon * \mathrm{Cut}^{\lfloor n,-\infty\rfloor}) \blacktriangle (\mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \eta) = \mathrm{id}_{\mathrm{Cut}^{\lfloor n,-\infty\rfloor}}$.

We show that $(\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}) \overset{!}{=} \mathrm{id}_{\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}}$.



Suppose given $G \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$.

We show that $G\big((\mathrm{Cosk}^{\lfloor n,-\infty\rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,-\infty\rfloor})\big) \overset{!}{=} \mathrm{id}_{G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}}$.

Suppose given $k \in \mathbb{Z}$. Suppose given $(x_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$; cf. Lemma 263.(1).

For $c \in {}_{\Delta^\pm}([n]^\pm,[n]^\pm)$ and $b \in {}_{\Delta^\pm}([n]^\pm,[k]^\pm)$ note that we have

$$c \blacktriangle b = \mathrm{id}_{[n]^\pm} \blacktriangle (c \blacktriangle b)\,.$$

$$\begin{array}{ccc} [n]^{\pm} & \xrightarrow{\ b\ } & [k]^{\pm} \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle c \blacktriangle b} \\ [n]^{\pm} & \xrightarrow[\mathrm{id}_{[n^{\pm}]}]{} & [n]^{\pm} \end{array}$$

So, since $(x_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}$ is coherent, we have

$$x_b \, G_c = x_{c \blacktriangle b} \, G_{\mathrm{id}_{[n]^{\pm}}} = x_{c \blacktriangle b}$$

for $c \in_{\Delta^{\pm}}([n]^{\pm}, [n]^{\pm})$ and $b \in_{\Delta^{\pm}}([n]^{\pm}, [k]^{\pm})$; cf. Definition 262.

We have

$$\begin{aligned}
& \big((x_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\big) \Big(G\big((\mathrm{Cosk}^{\lfloor n,-\infty \rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,-\infty \rfloor})\big)\Big)_k \\
= \quad & \big((x_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\big) \big(G(\mathrm{Cosk}^{\lfloor n,-\infty \rfloor} * \varepsilon) \blacktriangle G(\eta * \mathrm{Cosk}^{\lfloor n,-\infty \rfloor})\big)_k \\
= \quad & \big((x_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\big) \Big(\big((G \,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})\varepsilon\big)_k \blacktriangle \big((G\eta)\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}\big)_k \Big) \\
\overset{269.(1)}{=} \quad & \Big(\Big(\big((x_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\big) (G \,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_b \Big)_{b \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \Big) \big((G\eta)\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}\big)_k \\
\overset{263.(2)}{=} \quad & \Big(\big((x_{c \blacktriangle b})_{c \in_{\Delta^{\pm}}([n]^{\pm},[n]^{\pm})}\big)_{b \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \Big) \big((G\eta)\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}\big)_k \\
= \quad & \Big(\big((x_b \, G_c)_{c \in_{\Delta^{\pm}}([n]^{\pm},[n]^{\pm})}\big)_{b \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \Big) \big((G\eta)\,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}\big)_k \\
\overset{264}{=} \quad & \Big(\big((x_b \, G_c)_{c \in_{\Delta^{\pm}}([n]^{\pm},[n]^{\pm})}\big)(G\eta)_n \Big)_{b \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \\
\overset{268.(3,1)}{=} \quad & (x_b)_{b \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \\
= \quad & \big((x_a)_{a \in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\big) \, \mathrm{id}_{(G \,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_k} \ .
\end{aligned}$$

This shows $\Big(G\big((\mathrm{Cosk}^{\lfloor n,-\infty \rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,-\infty \rfloor})\big)\Big)_k = \mathrm{id}_{(G \,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor})_k}$.

So we have $G\big((\mathrm{Cosk}^{\lfloor n,-\infty \rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,-\infty \rfloor})\big) = \mathrm{id}_{G \,\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}} = G \,\mathrm{id}_{\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}}$.

Therefore, we have $(\mathrm{Cosk}^{\lfloor n,-\infty \rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Cosk}^{\lfloor n,-\infty \rfloor}) = \mathrm{id}_{\mathrm{Cosk}^{\lfloor n,-\infty \rfloor}}$.

Altogether, this proves the proposition. □

Remark 271 We consider the adjunction $\big(\mathrm{Cut}^{\lfloor n,-\infty\rfloor}, \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}, \varepsilon, \eta\big)$; cf. Lemma 270.
For $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$ and $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}\big)$, we have the adjunction bijection

$$\begin{array}{rcl}\tilde{\alpha}_{G,H}\colon\ {}_{\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}, H\big) & \overset{\sim}{\leftrightarrow} & {}_{\mathcal{PreStSimpGrp}}\big(G, H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\\ \big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \xrightarrow{\varphi} H\big) & \mapsto & \big(G \xrightarrow{G\varepsilon \,\blacktriangle\, \varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\\ \big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \xrightarrow{\psi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \,\blacktriangle\, H\eta} H\big) & \leftarrow\!\shortmid & \big(G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\,;\end{array}$$

cf. Remark 14.
Suppose given $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$ and suppose given $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,0\rfloor\text{-}\mathcal{PreStSimpGrp}\big)$.

(1) Suppose given a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group morphism $G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \xrightarrow{\varphi} H$.
Then we have the prestable simplicial group morphism

$$\varphi\tilde{\alpha}_{G,H} = G\varepsilon \,\blacktriangle\, \varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\colon G \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,,$$

where, for $k \in \mathbb{Z}$, we have

$$(\varphi\tilde{\alpha}_{G,H})_k = (G\varepsilon)_k \,\blacktriangle\, (\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\colon G_k \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k\,,\ x \mapsto \big((xG_a)\varphi_n\big)_{a\in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}}\,.$$

(2) Suppose given a prestable simplicial group morphism $G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$.
Then we have the pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group morphism

$$\psi\tilde{\alpha}^{-}_{G,H} = \psi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \,\blacktriangle\, H\eta : G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \to H\,,$$

where, for $k \in \mathbb{Z}_{\leqslant n}$, we have

$$(\psi\tilde{\alpha}^{-}_{G,H})_k = (\psi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})_k \,\blacktriangle\, (H\eta)_k\colon (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor})_k \to H_k\,,\ x \mapsto (x\psi_k)(H\eta)_k =: \tilde{x}$$

such that $x\psi_k = (\tilde{x}H_a)_{a\in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}}$.

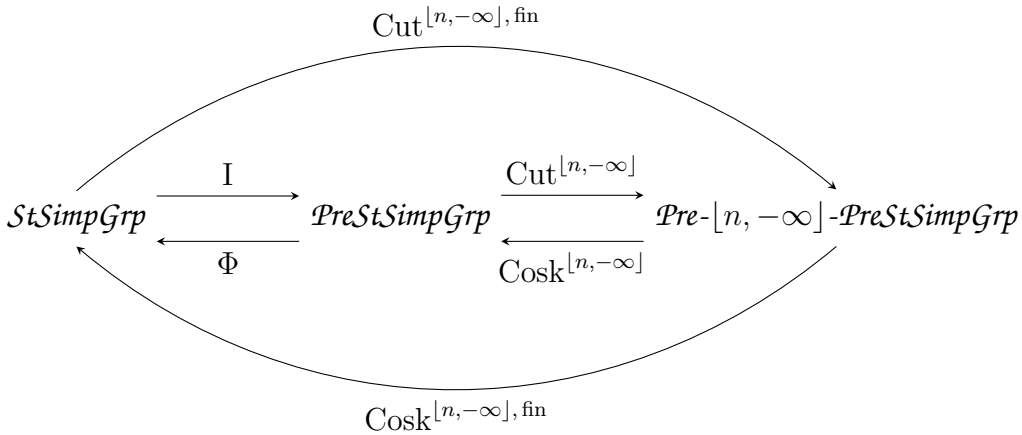
Remark 272 (The adjunction $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$)
We consider the functors

$\mathrm{Cut}^{\lfloor n,-\infty\rfloor}$:	$\mathcal{PreStSimpGrp}$	$\to$	$\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}$;	cf. Remark 260
$\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$:	$\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}$	$\to$	$\mathcal{PreStSimpGrp}$;	cf. Lemma 264
I:	$\mathcal{StSimpGrp}$	$\to$	$\mathcal{PreStSimpGrp}$;	cf. Remark 209
Φ:	$\mathcal{PreStSimpGrp}$	$\to$	$\mathcal{StSimpGrp}$;	cf. Lemma 208.

We consider the functors

$$\begin{array}{rcl} \mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} = \mathrm{I} * \mathrm{Cut}^{\lfloor n,-\infty\rfloor}\colon & \mathcal{StSimpGrp} & \to \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp} \\ \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} = \mathrm{Cosk}^{\lfloor n,-\infty\rfloor} * \Phi\colon & \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp} & \to \mathcal{StSimpGrp}\,; \end{array}$$

cf. Remarks 261, 267.



Recall that $\mathrm{I} \dashv \Phi$ *and that* $\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$*; cf. Lemmas* 210, 270.

So, by Remark 16, *we have* $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$.

For $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ *and* $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}\big)$, *we have the adjunction bijection*

$$\alpha_{G,H}\colon \quad {}_{\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},\, H\big) \quad \widetilde{\leftrightarrow} \quad {}_{\mathcal{StSimpGrp}}\big(G,\, H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\big)$$

given as follows.

For a pre-$\lfloor n,-\infty\rfloor$*-prestable simplicial group morphism* $G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \xrightarrow{\varphi} H$, *we have the the stable simplicial group morphism*

$$\varphi\alpha_{G,H} = \big((G\mathrm{I})\varepsilon \blacktriangle \varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\big|^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}} : G \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},$$

where, for $k \in \mathbb{Z}$, *we have*

$$\begin{array}{rcl} (\varphi\alpha_{G,H})_k\colon \; G_k & \to & (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k \\ g & \mapsto & g(\varphi\alpha_{G,H})_k = \big((gG_a)\varphi_n\big)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\,. \end{array}$$

Concerning the transformation ε*; cf. Lemma* 269.

12.3. THE ADJUNCTION $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$

For a stable simplicial group morphism $G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$*, we have the pre-*$\lfloor n,-\infty\rfloor$*-prestable simplicial group morphism*

$$\psi\alpha^{-}_{G,H} \colon G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to H\,,$$

where, for $k \in \mathbb{Z}_{\leqslant n}$*, we have*

$$\begin{array}{rcl} (\psi\alpha^{-}_{G,H})_k \colon \ (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k & \to & H_k \\ g & \mapsto & g(\psi\alpha^{-}_{G,H})_k = (g\psi_k)(H\eta)_k\,; \end{array}$$

cf. Lemma 268*.*

So, for $k \in \mathbb{Z}_{\leqslant n}$ *and* $g \in (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k$*, we have* $g(\psi\alpha^{-}_{G,H})_k =: \tilde{h} \in H_k$ *such that* $(\tilde{h}H_a)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} = g\psi_k$*.*

Proof. Suppose given $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ and $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}\big)$.

We consider the prestable simplicial group $G\mathrm{I}$ and the pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group H.

Since $(\mathrm{Cut}^{\lfloor n,-\infty\rfloor}, \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}, \varepsilon, \eta)$ is an adjunction, recall that we have the adjunction bijection

$$\begin{array}{rcl} \tilde{\alpha}_{G\mathrm{I},H} \colon \ {}_{\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}}\big((GI)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor}, H\big) & \tilde{\leftrightarrow} & {}_{\mathcal{PreStSimpGrp}}\big(GI, H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) \\ \big((G\mathrm{I})\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \xrightarrow{\varphi} H\big) & \mapsto & \big(G \xrightarrow{(GI)\varepsilon \,\vartriangleright\, \varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) \\ \big((G\mathrm{I})\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \xrightarrow{\psi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \,\vartriangleright\, H\eta} H\big) & \leftarrow\!\shortmid & \big(G\mathrm{I} \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\,; \end{array}$$

cf. Remark 271.

Recall from Lemma 210.(3) that we have $\mathrm{I} \dashv \Phi$.

For the sake for this proof, we shall write ε' for the unit from Lemma 210.(1) and we shall write η' for the counit from Lemma 210.(2). So we have the adjunction $(\mathrm{I}, \Phi, \varepsilon', \eta')$.

We consider the stable simplicial group G and the prestable simplicial group $H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$. Then, we have the adjunction bijection

$$\begin{array}{rcl} \gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}} \colon \ {}_{\mathcal{PreStSimpGrp}}\big(G\mathrm{I},\, H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) & \tilde{\leftrightarrow} & {}_{\mathcal{StSimpGrp}}\big(G,\, (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})\Phi\big) \\ \big(G\mathrm{I} \xrightarrow{\varphi} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) & \mapsto & \big(G \xrightarrow{\varphi\big|^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor},\mathrm{fin}}} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})\Phi\big) \\ \big(G\mathrm{I} \xrightarrow{\psi\mathrm{I} \,\vartriangleright\, \iota} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) & \leftarrow & \big(G \xrightarrow{\psi} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})\Phi\big)\,, \end{array}$$

where $\iota := \iota_{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}}$; cf. Remark 211.
By Remark 16, we may let

$$\alpha_{G,H} := \tilde{\alpha}_{G\mathrm{I},H} \blacktriangle \gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}} .$$

Suppose given a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphism $G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \xrightarrow{\varphi} H$. We obtain

$$\begin{array}{rcl} \varphi\alpha_{G,H} & = & (\varphi\tilde{\alpha}_{G\mathrm{I},H})\gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}} \overset{211}{=} (\varphi\tilde{\alpha}_{G\mathrm{I},H})\big|^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}} \\ & \overset{271.(1)}{=} & \big((G\mathrm{I})\varepsilon \blacktriangle \varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\big|^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}} . \end{array}$$

So, for $k \in \mathbb{Z}$ and $g \in G_k$, we have

$$\begin{array}{c} g(\varphi\alpha_{G,H})_k = g\big((\varphi(\tilde{\alpha}_{G\mathrm{I},H} \blacktriangle \gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}})\big)_k = g\big((\varphi\tilde{\alpha}_{G\mathrm{I},H})\gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}}\big)_k \\ \overset{211}{=} g(\varphi\tilde{\alpha}_{G,H})_k \overset{271.(1)}{=} \big((gG_a)\varphi_n\big)_{a\in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} . \end{array}$$

Moreover, we have

$$\alpha^{-}_{G,H} = (\gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}})^{-} \blacktriangle (\tilde{\alpha}_{G\mathrm{I},H})^{-} .$$

Suppose given a stable simplicial group morphism $G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$.
For $k \in \mathbb{Z}_{\leqslant n}$ and $g \in (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k = G_k$, we have

$$\begin{array}{rcl} g(\psi\alpha_{G,H})^{-}_k & = & g\Big(\psi\big((\gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}})^{-} \blacktriangle (\tilde{\alpha}_{G\mathrm{I},H})^{-}\big)\Big)_k \\ & = & g\Big(\big((\psi\gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}})^{-}\big)(\tilde{\alpha}_{G\mathrm{I},H})^{-}\Big)_k \\ & \overset{271.(2)}{=} & \big(g(\psi\gamma_{G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}})^{-}_k\big)(H\eta)_k \\ & \overset{211}{=} & (g\psi_k)(H\eta)_k =: \tilde{h} , \end{array}$$

where $(\tilde{h}H_a)_{a\in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} = g\psi_k$; cf. Lemma 268.(3, 1). □

Remark 273 (The counit η)
We consider the adjoint functors $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ *from Remark* 272.
A counit

$$\eta\colon\ \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} * \mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}$$

12.3. THE ADJUNCTION $\mathrm{Cut}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor,\mathrm{fin}}$

is given as follows.

For $H \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$*, we have the pre-*$\lfloor n,-\infty\rfloor$*-prestable simplicial group morphism*

$$H\eta\colon H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to H\,,$$

where, for $k \in \mathbb{Z}_{\leqslant n}$*, we have the injective group morphism*

$$(H\eta)_k\colon \big(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\big)_k \to H_k\,,\ (h_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \mapsto \big((h_a)_a\big)(H\eta)_k =: \tilde{h}$$

such that $(\tilde{h}H_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} = (h_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}}$*.*

Proof. For $G \in \mathrm{Ob}\big(\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$ and $H \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$, we consider the adjunction bijection from Remark 272.

$$\alpha_{G,H}\colon\ {}_{\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},\ H\big)\ \ \widetilde{\leftrightarrow}\ \ {}_{\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}\big(G,\ H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\big)$$

By Remark 15, the counit $\eta\colon \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} * \mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}$ of the adjunction $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$ is given as follows.

Suppose given $H \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$. We have the pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group morphism

$$H\eta \overset{15}{=} (\mathrm{id}_{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}})(\alpha_{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},H})^-\colon G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to H\,.$$

For $k \in \mathbb{Z}_{\leqslant n}$, note that we have

$$\big(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\big)_k = \big(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\big)_k\,.$$

Recall from Lemma 270 that we have $\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$.

For the sake for this proof, the counit from Lemma 268 shall be called $\tilde{\eta}$ here. So we have

$$\tilde{\eta}\colon \mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \mathrm{Cosk}^{\lfloor n,-\infty\rfloor} \to \mathrm{id}_{\mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}\,.$$

Then, for $k \in \mathbb{Z}_{\leqslant n}$ and $(h_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \in (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k$, we have

$$\begin{aligned}\big((h_a)_a\big)(H\eta)_k &= \big((h_a)_a\big)\big((\mathrm{id}_{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}})(\alpha_{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},H})^-\big)_k\\ &\overset{272}{=} \Big(\big((h_a)_a\big)(\mathrm{id}_{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}})_k\Big)(H\tilde{\eta})_k = \big((h_a)_a\big)(H\tilde{\eta})_k =: \tilde{h}\,,\end{aligned}$$

where $(\tilde{h}H_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} = (h_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}}$; cf. Lemma 268.

Suppose given $H \in \mathrm{Ob}\big(\mathcal{P}re\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big)$ and suppose given $k \in \mathbb{Z}_{\leqslant n}$.

We show that $(H\eta)_k$ is injective.

Suppose given $(h_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \in (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k$ such that $\big((h_a)_a\big)(H\eta)_k = 1$.

We have

$$1 = \big((h_a)_a\big)(H\eta)_k =: \tilde{h} \in H_k$$

such that $(\tilde{h}H_a)_a = (h_a)_a$. So

$$(h_a)_a = (\tilde{h}H_a)_a = (1H_a)_a = (1)_a\,.$$

This shows that $(H\eta)_k$ is injective. □

Remark 274 *Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group H; cf. Definition* 232.

We have that

$$H\eta\colon H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to H$$

is a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group isomorphism; cf. Remark 273.

For $k \in \mathbb{Z}_{\leqslant n}$, we have

$$(H\eta)_k^-\colon H_k \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k\,,\ h \mapsto (hH_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}}\,.$$

In particular, we have that $H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$ is an $\lfloor n, -\infty \rfloor$-stable simplicial group; cf. Definition 232.

Proof. Suppose given $k \in \mathbb{Z}_{\leqslant n}$. We have to show that $(H\eta)_k\colon (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k \to H_k$ is a group isomorphism.

By Remark 273, $(H\eta)_k$ is an injective group morphism. It remains to show that $(H\eta)_k$ is surjective.

Suppose given $h \in H_k$. We consider $x := (hH_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \overset{268.(1)}{\in} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_k$.

Then we have

$$x(H\eta)_k = \big((hH_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}}\big)(H\eta)_k = h$$

by construction of $(H\eta)_k$.

We have to show that $x \overset{!}{\in} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k$.

Since h is finite, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $h\mathrm{d}_j^{H,k} = 1$ for $j \in \mathbb{Z}_{\geqslant \ell}$; cf. Definition 204. Then, for $j \in \mathbb{Z}_{\geqslant \ell}$, we have

$$\begin{aligned}
x\mathrm{d}_j^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor},k} &= \big((hH_a)_{a\,\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\big)(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_{\delta_j} \\
&\overset{263.(2)}{=} (hH_{b\blacktriangle\delta_j})_{b\in_{\Delta^\pm}([n]^\pm,[k-1]^\pm)} \\
&= (hH_{\delta_j}H_b)_{b\in_{\Delta^\pm}([n]^\pm,[k-1]^\pm)} \\
&= (h\mathrm{d}_j^{H,k}H_b)_{b\in_{\Delta^\pm}([n]^\pm,[k-1]^\pm)} \\
&= (1H_b)_{b\in_{\Delta^\pm}([n]^\pm,[k-1]^\pm)} \\
&= (1)_{b\in_{\Delta^\pm}([n]^\pm,[k-1]^\pm)}\,.
\end{aligned}$$

This shows that x is finite. So we have $x \in (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k$.
This proves the claim. □

Chapter 13

$\lfloor n, -\infty \rfloor$-coskeletal stable simplicial groups

13.1 $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial groups

Let $n \in \mathbb{Z}$.

Definition 275 ($\lfloor n, -\infty \rfloor$-coskeletal stable simplicial groups)
Suppose given a stable simplicial simplicial group G; cf. Definition 205.
We say that G is $\lfloor n, -\infty \rfloor$-*coskeletal* if the properties $(1, 2)$ hold.

(1) We have $G\mathrm{NZ}_{n+1} = 1$; cf. Definition 198.(3).

(2) For I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$ we have

$$\big[\bigcap_{i \in I} G_{n,i}\,,\bigcap_{j \in J} G_{n,j}\big] = 1\,;$$

cf. property (2) from Definition 232.

Prpoerty (2) is also called the *Conduché condition* on $\lfloor n, -\infty \rfloor$-coskeletal simplicial groups; cf. [**7,** Thm. 1.5].

Remark 276 *Let $k \in \mathbb{Z}_{\geqslant 1}$. Suppose given a stable simplicial group G; cf. Definition* 205.

(1) *Suppose that G is $\lfloor n, -\infty \rfloor$-coskeletal; cf. Definition* 275.
We consider the stable simplicial group $G\mathrm{Dec}^{\pm,n-k}$; cf. Lemma 195*, Remark* 212.
Recall that $(G\mathrm{Dec}^{\pm,n-k})_i = G_{n-k+i}$ for $i \in \mathbb{Z}$.
We have that $G\mathrm{Dec}^{\pm,n-k}$ is a $\lfloor k, -\infty \rfloor$-coskeletal stable simplicial group.

(2) *Suppose that G is $\lfloor k, -\infty \rfloor$-coskeletal; cf. Definition* 275.
We consider the simplicial group $G\mathrm{Ord}$; cf. Lemma 219.
Recall that $(G\mathrm{Ord})_i = \big(\bigcap_{j \in \mathbb{Z}_{\geqslant i+1}} G_{i,j} \big) \cap \ker \Big(\prod_{j \in \lfloor i,0 \rfloor}^{G_i} \prod^{G_{-1}} \mathrm{d}_0^{G,j} \Big) \leqslant G_i$ *for $i \in \mathbb{Z}_{\geqslant 0}$.*
We have that $G\mathrm{Ord}$ is a $\lfloor k, 0 \rfloor$-coskeletal simplicial group; cf. Definition 90.
Moreover, $(G\mathrm{Ord})\mathrm{NZ}_k$ is contained in the centre of $(G\mathrm{Ord})_k$; cf. Definition 76.(3).

(3) *Suppose that G is $\lfloor n, -\infty \rfloor$-coskeletal; cf. Definition* 275.
We consider the simplicial group $(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}$.
Then, for $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\big((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}\,\big)_i = \big(\bigcap_{j \in \mathbb{Z}_{\geqslant i+1}} G_{n-k+i,j} \big) \cap \ker \Big(\prod_{j \in \lfloor i,0 \rfloor}^{G_{n-k+i}} \prod^{G_{n-k-1}} \mathrm{d}_0^{G,n-k+j} \Big) \leqslant G_{n-k+i}\,.$$

We have that $(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}$ is a $\lfloor k, 0 \rfloor$-coskeletal simplicial group.

Proof. Ad (1). Since G is $\lfloor n, -\infty \rfloor$-coskeletal, we have

$$(G\mathrm{Dec}^{\pm,n-k})\mathrm{NZ}_{k+1} = \bigcap_{i \in \mathbb{Z}_{\geqslant 0}} (G\mathrm{Dec}^{\pm,n-k})_{k+1,i} = \bigcap_{i \in \mathbb{Z}_{\geqslant 0}} G_{n+1,i} = G\mathrm{NZ}_{n+1} = 1\,.$$

Suppose given I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$. Since G is $\lfloor n, -\infty \rfloor$-coskeletal, we

$$\big[\bigcap_{i \in I} (G\mathrm{Dec}^{\pm,n-k})_{k,i}\,, \bigcap_{j \in J} (G\mathrm{Dec}^{\pm,n-k})_{k,j} \big] = \big[\bigcap_{i \in I} G_{n,i}\,, \bigcap_{j \in J} G_{n,j} \big] = 1\,.$$

This shows that $G\mathrm{Dec}^{\pm,n-k}$ is $\lfloor k, -\infty \rfloor$-coskeletal.

Ad (2). *We show that* $(G\mathrm{Ord})\mathrm{NZ}_{k+1} \stackrel{!}{=} 1$.

Suppose given $g \in (G\mathrm{Ord})\mathrm{NZ}_{k+1} = \bigcap_{i \in [0,k+1]} (G\mathrm{Ord})_{k+1,i} \leqslant G_{k+1}$.

We show that $g\mathrm{d}_j^{G,k+1} \stackrel{!}{=} 1$ for $j \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $j \in \mathbb{Z}_{\geqslant 0}$. If $j \in [0, k+1]$, then we have

$$g\mathrm{d}_j^{G,k+1} \stackrel{219}{=} g\mathrm{d}_j^{G\mathrm{Ord},k+1} = 1\,.$$

Note that $g \in (G\mathrm{Ord})_{k+1} \leqslant \bigcap_{i \in \mathbb{Z}_{\geqslant k+2}} G_{k+1,i}$. So, if $j \geqslant k+2$, then we have

$$g\mathrm{d}_j^{G,k+1} = 1\,.$$

This shows that $g \in G\mathrm{NZ}_{k+1} = 1$. So $g = 1$, and therefore, $(G\mathrm{Ord})\mathrm{NZ}_{k+1} = 1$.

Suppose given I, $J \subseteq [0,k]$, where $I \cup J = [0,k]$.

We show that $\big[\bigcap_{i \in I} (G\mathrm{Ord})_{k,i}\,, \bigcap_{j \in J} (G\mathrm{Ord})_{k,j}\big] \stackrel{!}{=} 1$.

Suppose given $x \in \bigcap_{i \in I} (G\mathrm{Ord})_{k,i}$ and suppose given $y \in \bigcap_{j \in J} (G\mathrm{Ord})_{k,j}$.

Then we have $x \in \bigcap_{i \in I} G_{k,i}$ and $y \in \bigcap_{j \in J} G_{k,j}$; cf. Lemma 219.

Note that x, $y \in (G\mathrm{Ord})_k \leqslant \bigcap_{i \in \mathbb{Z}_{\geqslant k+1}} G_{k,i}$.

Let $\tilde{I} := I \cup \mathbb{Z}_{\geqslant k+1}$ and let $\tilde{J} := J \cup \mathbb{Z}_{\geqslant k+1}$.

Then we have $\tilde{I} \cup \tilde{J} = \mathbb{Z}_{\geqslant 0}$ and we have $x \in \bigcap_{i \in \tilde{I}} G_{k,i}$ and $y \in \bigcap_{j \in \tilde{J}} G_{k,j}$.

This shows that

$$[x,y] \in \big[\bigcap_{i \in \tilde{I}} G_{k,i}\,, \bigcap_{j \in \tilde{J}} G_{k,j}\big] = 1\,.$$

So $[x,y] = 1$. This shows that $\big[\bigcap_{i \in I} (G\mathrm{Ord})_{k,i}\,, \bigcap_{j \in J} (G\mathrm{Ord})_{k,j}\big] = 1$.

In particular, for $I = \emptyset$ and $J = [0,k]$, we obtain

$$1 = \big[\bigcap_{i \in \emptyset} (G\mathrm{Ord})_{k,i}\,, \bigcap_{j \in [0,k]} (G\mathrm{Ord})_{k,j}\big] = \big[(G\mathrm{Ord})_k\,, (G\mathrm{Ord})\mathrm{NZ}_k\big]\,.$$

This shows that $(G\mathrm{Ord})\mathrm{NZ}_k$ is contained in the centre of $(G\mathrm{Ord})_k$.

Ad (3). For $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned}\big((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}\,\big)_i &= \big(\bigcap_{j\in\mathbb{Z}_{\geqslant i+1}} (G\mathrm{Dec}^{\pm,n-k})_{i,j}\big) \cap \ker\Big(\prod_{j\in\lfloor i,0\rfloor}^{(G\mathrm{Dec}^{\pm,n-k})_i} \prod^{(G\mathrm{Dec}^{\pm,n-k})_{-1}} \mathrm{d}_0^{G\mathrm{Dec}^{\pm,n-k},j}\Big)\\ &= \big(\bigcap_{j\in\mathbb{Z}_{\geqslant i+1}} G_{n-k+i,j}\big) \cap \ker\Big(\prod_{j\in\lfloor i,0\rfloor}^{G_{n-k+j}} \prod^{G_{n-k-1}} \mathrm{d}_0^{G,n-k+j}\Big)\\ &\leqslant (G\mathrm{Dec}^{\pm,n-k})_i = G_{n-k+i}\,.\end{aligned}$$

By (1), we have that $G\mathrm{Dec}^{\pm,n-k}$ is a $\lfloor k, -\infty \rfloor$-coskeletal stable simplicial group.
Then, by (2), we have that $(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}$ is a $\lfloor k, 0\rfloor$-coskeletal simplicial group. □

Remark 277 *Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group H; cf. Definition 232.*
We consider the stable simplicial group $H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$; cf. Remark 267.
Then $H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$ is $\lfloor n, -\infty \rfloor$-coskeletal; cf. Definition 275.

Proof. We show the properties $(1, 2)$ from Definition 275.
Ad (1). Suppose given

$$x := (x_a)_{a\in_{\Delta^\pm}([n],[n+1])} \in (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})\mathrm{NZ}_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_{n+1}\,.$$

Then, for $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned}(1)_{b\in_{\Delta^\pm}([n]^\pm,[n]^\pm)} &= 1 = x\mathrm{d}_i^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},n+1} = \big((x_a)_{a\in_{\Delta^\pm}([n]^\pm,[n+1]^\pm)}\big)(H\,\mathrm{Cosk}^{\pm,\lfloor n,-\infty\rfloor})_{\delta_i}\\ &\overset{263.(2)}{=} (x_{b\blacktriangle\delta_i})_{b\in_{\Delta^\pm}([n]^\pm,[n]^\pm)}\,.\end{aligned}$$

So we have

$$1 = x_{b\blacktriangle\delta_i}$$

for $b \in {}_{\Delta^\pm}([n]^\pm, [n]^\pm)$ and $i \in \mathbb{Z}_{\geqslant 0}$.
Suppose given $a \in {}_{\Delta^\pm}([n]^\pm, [n+1]^\pm)$.
By Remark 265, we may choose some $b \in {}_{\Delta^\pm}([n]^\pm, [n]^\pm)$ and some $k \in \mathbb{Z}_{\geqslant 0}$ with $a = b \blacktriangle \delta_k^{\pm,n+1}$.
Therefore, we have

$$x_a = x_{b\blacktriangle\delta_k} = 1\,.$$

This shows that

$$x = (x_a)_{a\in_{\Delta^\pm}([n]^\pm,[n+1]^\pm)} = (1)_{a\in_{\Delta^\pm}([n]^\pm,[n+1]^\pm)} = 1\,.$$

So $(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})\mathrm{NZ}_{n+1} = 1$. This shows (1).

Ad (2). Suppose given I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

We have to show that

$$\big[\bigcap_{i\in I}(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_{n,i}\,,\bigcap_{j\in J}(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_{n,j}\big] \overset{!}{=} 1\,.$$

We consider the pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group $H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$; cf. Remark 272. Recall that we have

$$(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_n = (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_n\,.$$

From Remark 273, we consider the pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphism

$$H\eta\colon H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to H$$

and the injective group morphism

$$(H\eta)_n\colon (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_n \to H_n\,.$$

Suppose given $x \in \bigcap_{i\in I}(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_{n,i}$ and suppose given $y \in \bigcap_{j\in J}(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_{n,j}$.

We show that $[x, y] \overset{!}{=} 1$. Suppose given $i \in I$. We have

$$\big(x(H\eta_n)\big)\mathrm{d}_i^{H,n} \overset{184.(2)}{=} (x\mathrm{d}_i^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor},n})(H\eta)_{n-1} = 1(H\eta)_{n-1} = 1\,.$$

This shows $x(H\eta)_n \in H_{n,i}$ for $i \in I$.

Similarly, we have $y(H\eta)_n \in H_{n,j}$ for $j \in J$.

So

$$[x, y](H\eta)_n = \big[x(H\eta)_n\,, y(H\eta)_n\big] \overset{233}{=} 1\,.$$

Since $(H\eta)_n$ is injective, we have $[x, y] = 1$.

This shows (2). □

13.2 The decomposition of G_n in subgroups

Definition 278 Suppose given a stable simplicial group G; cf. Definition 205.

Suppose given $I \subseteq \mathbb{Z}_{\geqslant 0}$. We write

$$G_{n;I} := \bigcap_{i \in I} G_{n,i} \trianglelefteqslant G_n ;$$

cf. Definition 198.(1).

Remark 279 Suppose given a stable simplicial group G; cf. Definition 205.

For $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$G_{n;\{k\}} = G_{n,k} \quad \text{and} \quad G_{n;\emptyset} = G_n ;$$

cf. Definition 198.(1) and §1.1.1 item 2.

Moreover, we have

$$G_{n;\mathbb{Z}_{\geqslant 1}} = G\mathrm{N}_n \quad \text{and} \quad G_{n;\mathbb{Z}_{\geqslant 0}} = G\mathrm{NZ}_n ;$$

cf. Definition 198.(2, 3).

Remark 280 *Suppose given stable simplicial groups G, $\tilde{H}$, H, where $\tilde{H} \leqslant H$; cf. Definition* 205.

Suppose given a stable simplicial group morphism $\varphi \colon G \to H$.

Let $k \in \mathbb{Z}$. For $\ell \in \mathbb{Z}_{\geqslant 0}$, suppose that we have $(G_{k;\mathbb{Z}_{\geqslant \ell}})\varphi_k \leqslant \tilde{H}_k$.

Then we have $G_k \varphi_k \leqslant \tilde{H}_k$.

Proof. Suppose given $g \in G_k$.

Since g is finite, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_j^{G,k} = 1$ for $j \in \mathbb{Z}_{\geqslant \ell}$; cf. Definition 204.

So $g \in G_{k;\mathbb{Z}_{\geqslant \ell}}$. This shows that $g\varphi_k \in (G_{k;\mathbb{Z}_{\geqslant \ell}})\varphi_k \leqslant \tilde{H}_k$. □

Remark 281 *Suppose given a stable simplicial group G; cf. Definition* 205.

Suppose given $k \in \mathbb{Z}_{\geqslant 2}$.

(1) *We have the surjective map*

$$p^G_{n,k} \colon G_{n;\mathbb{Z}_{\geqslant k}} \to G_{n;\mathbb{Z}_{\geqslant k-1}}, \ g \mapsto gp^G_{n,k} := g^- \mathrm{d}^{G,n}_{k-1} \mathrm{s}^{G,n-1}_{k-2} \cdot g .$$

(2) *For $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$, we have*

$$g = (g\mathrm{d}_{k-1}^{G,n})\mathrm{s}_{k-2}^{G,n-1} \cdot (g^{-}\mathrm{d}_{k-1}^{G,n}\,\mathrm{s}_{k-2}^{G,n-1} \cdot g) = (g\mathrm{d}_{k-1}^{G,n})\mathrm{s}_{k-2}^{G,n-1} \cdot gp_{n,k}^{G}\,,$$

where $g\mathrm{d}_{k-1}^{G,n} \in G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}$ and where $g^{-}\mathrm{d}_{k-1}^{G,n}\,\mathrm{s}_{k-2}^{G,n-1} \cdot g = gp_{n,k}^{G} \in G_{n;\,\mathbb{Z}_{\geqslant k-1}}$.
In particular, we have

$$G_{n;\mathbb{Z}_{\geqslant k}} = \big(G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}\big)\mathrm{s}_{k-2}^{G,n-1} \cdot G_{n;\,\mathbb{Z}_{\geqslant k-1}}\,.$$

(3) *We have*

$$\big(G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}\big)\mathrm{s}_{k-2}^{G,n-1} \;\cap\; G_{n;\,\mathbb{Z}_{\geqslant k-1}} = 1\,.$$

(4) *For $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$ there exists a unique $y \in G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}$ and a unique $z \in G_{n;\,\mathbb{Z}_{\geqslant k-1}}$ such that*

$$g = y\mathrm{s}_{k-2}^{G,n-1} \cdot z\,.$$

Proof. Ad (1). *We show that $p_{n,k}^{G}$ is a well-defined map.*
Suppose given $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$. Suppose given $j \in \mathbb{Z}_{\geqslant k-1}$.
If $j = k-1$, then we have

$$(g^{-}\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g)\mathrm{d}_{k-1} = g^{-}\mathrm{d}_{k-1} \cdot g\mathrm{d}_{k-1} = 1\,.$$

If $j \in \mathbb{Z}_{\geqslant k}$, then we have

$$(g^{-}\,\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g)\mathrm{d}_j = g^{-}\mathrm{d}_{k-1}\mathrm{d}_{j-1}\mathrm{s}_{k-2} \cdot g\mathrm{d}_j = g^{-}\,\mathrm{d}_j\,\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g\mathrm{d}_j = 1\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot 1 = 1\,.$$

This shows that $gp_{n,k}^{G} = g^{-}\,\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g \in G_{n;\,\mathbb{Z}_{\geqslant k-1}}$.
So $p_{n,k}^{G}$ is a well-defined map.
We show that $p_{n,k}^{G}$ is surjective.
Suppose given $x \in G_{n;\,\mathbb{Z}_{\geqslant k-1}} \leqslant G_{n;\,\mathbb{Z}_{\geqslant k}}$.
We have

$$xp_{n,k}^{G} = x\mathrm{d}_{k-1}\mathrm{s}_{k-2} \cdot x = 1\mathrm{s}_{k-2} \cdot x = x\,.$$

So $p_{n,k}^{G}$ is surjective.
Ad (2). Suppose given $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$.
We have

$$g = (g\mathrm{d}_{k-1}\,\mathrm{s}_{k-2}) \cdot (g^{-}\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g)\,.$$

We consider $g\mathrm{d}_{k-1} \in G_{n-1}$. For $j \in \mathbb{Z}_{\geqslant k-1}$, we have

$$(g\mathrm{d}_{k-1})\mathrm{d}_j = g\mathrm{d}_{j+1}\,\mathrm{d}_{k-1} = 1\mathrm{d}_{k-1} = 1\,.$$

This shows that $g\mathrm{d}_{k-1} \in G_{n;\,\mathbb{Z}_{\geqslant k-1}}$. So $g\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \in (G_{n;\,\mathbb{Z}_{\geqslant k-1}})\mathrm{s}_{k-2}$.

We consider $g^{-}\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g \in G_n$.

By (1), we have $g^{-}\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g = gp^{G}_{n,k} \in G_{n;\,\mathbb{Z}_{\geqslant k-1}}$.

So we have

$$g = (g\mathrm{d}_{k-1}\mathrm{s}_{k-2}) \cdot (g^{-}\mathrm{d}_{k-1}\,\mathrm{s}_{k-2} \cdot g) \in (G_{n;\,\mathbb{Z}_{\geqslant k-1}})\mathrm{s}_{k-2} \cdot G_{n;\,\mathbb{Z}_{\geqslant k-1}}\,.$$

Ad (3). Suppose given $x \in \big(G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}\big)\mathrm{s}_{k-2} \cap G_{n;\,\mathbb{Z}_{\geqslant k-1}}$.

Then we have $x = y\,\mathrm{s}_{k-2}$ for some chosen $y \in G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}$.

We have

$$1 = x\mathrm{d}_{k-1} = y\mathrm{s}_{k-2}\,\mathrm{d}_{k-1} = y\,.$$

So

$$x = y\mathrm{s}_{k-2} = 1\mathrm{s}_{k-2} = 1\,.$$

This shows $\big(G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}\big)\mathrm{s}_{k-2} \cap G_{n;\,\mathbb{Z}_{\geqslant k-1}} = 1$.

Ad (4). Suppose given $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$.

We write $Y := G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}$ and we write $Z := G_{n;\,\mathbb{Z}_{\geqslant k-1}}$.

By (2), we have $g = y\mathrm{s}_{k-2} \cdot z$, where $y := g\mathrm{d}_{k-1} \in Y$ and $z := g^{-}\mathrm{d}_{k-1}\mathrm{s}_{k-2} \cdot g \in Z$.

It remains to show uniqueness.

Suppose given $\tilde{y} \in Y$ and $\tilde{z} \in Z$ such that $g = \tilde{y}\mathrm{s}_{k-2} \cdot \tilde{z}$.

We have to show that $y \overset{!}{=} \tilde{y}$ and that $z \overset{!}{=} \tilde{z}$.

We have

$$y\mathrm{s}_{k-2} \cdot z = g = \tilde{y}\mathrm{s}_{k-2} \cdot \tilde{z}\,.$$

So we have

$$\underbrace{(\tilde{y}^{-} \cdot y)}_{\in Y}\,\mathrm{s}_{k-2} = \underbrace{\tilde{z} \cdot z^{-}}_{\in Z} \in (Y\mathrm{s}_{k-2}) \cap Z\,.$$

By (3), we have $(Y\mathrm{s}_{k-2}) \cap Z = 1$. Therefore, we have $z = \tilde{z}$.

Moreover, we have

$$\tilde{y}^- \cdot y = (\tilde{y} \cdot y)\mathrm{s}_{k-2}\,\mathrm{d}_{k-2} = 1\mathrm{d}_{k-2} = 1\,,$$

and thus $y = \tilde{y}$. □

Example 282 Suppose given a stable simplicial group G; cf. Definition 205.

We give some examples for Remark 281.(1).

Let $k = 2$. We have

$$\begin{aligned} G_{n;\,\mathbb{Z}_{\geqslant 2}} = (G_{n-1;\,\mathbb{Z}_{\geqslant 1}})\mathrm{s}_0 \cdot G_{n;\,\mathbb{Z}_{\geqslant 1}} &\overset{279}{=} (G_{n-1;\,\mathbb{Z}_{\geqslant 1}})\mathrm{s}_0 \cdot G\mathrm{N}_n \\ &\overset{279}{=} (G\mathrm{N}_{n-1})\mathrm{s}_0 \cdot G\mathrm{N}_n\,; \end{aligned}$$

cf. Definition 198.(2).

Let $k = 3$. We have

$$\begin{aligned} G_{n;\,\mathbb{Z}_{\geqslant 3}} &= (G_{n-1;\,\mathbb{Z}_{\geqslant 2}})\mathrm{s}_1 \cdot G_{n;\,\mathbb{Z}_{\geqslant 2}} \\ &\overset{279}{=} (G_{n-1;\,\mathbb{Z}_{\geqslant 2}})\mathrm{s}_1 \cdot (G_{n-1;\,\mathbb{Z}_{\geqslant 1}})\mathrm{s}_0 \cdot G\mathrm{N}_n \\ &= \big((G_{n-2;\,\mathbb{Z}_{\geqslant 1}})\mathrm{s}_0 \cdot G_{n-1;\,\mathbb{Z}_{\geqslant 1}}\big)\mathrm{s}_1 \cdot (G_{n-1;\,\mathbb{Z}_{\geqslant 1}})\mathrm{s}_0 \cdot G\mathrm{N}_n \\ &\overset{279}{=} (G\mathrm{N}_{n-2})\mathrm{s}_0\mathrm{s}_1 \cdot (G\mathrm{N}_{n-1})\mathrm{s}_1 \cdot (G\mathrm{N}_{n-1})\mathrm{s}_0 \cdot G\mathrm{N}_n\,. \end{aligned}$$

Here, in the stable case, we will not need the decomposition into Moore complex terms. It will be sufficient to decompose G into factors of the form $(G_{n-1;\,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i$. Cf. also Example 102.

Lemma 283 *Let $k \in \mathbb{Z}_{\geqslant 1}$.*

Suppose given a stable simplicial group G; cf. Definition 205.

(1) *We have*

$$G_{n;\,\mathbb{Z}_{\geqslant k}} = \Big(\prod_{i \in \lfloor k-2,0 \rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i^{G,n-1}\Big) \cdot G\mathrm{N}_n\,.$$

(2) *Suppose given $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$.*

Then there exist unique elements $x_j \in G_{n-1;\,\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$ and a unique $\tilde{x} \in G\mathrm{N}_n$ such that

$$g = \Big(\prod_{i \in \lfloor k-2,0 \rfloor} x_{i+1}\,\mathrm{s}_i^{G,n-1}\Big) \cdot \tilde{x} = x_{k-1}\,\mathrm{s}_{k-2}^{G,n-1} \cdot x_{k-2}\,\mathrm{s}_{k-3}^{G,n-1} \cdot \ldots \cdot x_1\,\mathrm{s}_0^{G,n-1} \cdot \tilde{x}\,.$$

(3) *For $j \in [2,k]$, we consider the surjective map from Remark* 281.(1)

$$p^G_{n,j} \colon G_{n;\,\mathbb{Z}_{\geqslant j}} \to G_{n;\,\mathbb{Z}_{\geqslant j-1}}\,,\ g \mapsto g p^G_{n,j} = g^- \mathrm{d}^{G,n}_{j-1}\,\mathrm{s}^{G,n-1}_{j-2} \cdot g\,.$$

We have the surjective map

$$\pi^G_{n,k} := \prod_{j\in\lfloor k,2\rfloor}^{G_{n;\,\mathbb{Z}_{\geqslant k}}\ G\mathrm{N}_n} p^G_{n,j} = p^G_{n,k} \blacktriangle p^G_{n,k-1} \blacktriangle \ldots \blacktriangle p^G_{n,2} \,:\, G_{n;\,\mathbb{Z}_{\geqslant k}} \to G\mathrm{N}_n\,.$$

Suppose given $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$. We consider its unique decomposition from (2) :

$$g = \Big(\prod_{i\in\lfloor k-2,0\rfloor} x_{i+1}\,\mathrm{s}^{G,n-1}_i \Big) \cdot \tilde{x} = x_{k-1}\,\mathrm{s}^{G,n-1}_{k-2} \cdot x_{k-2}\,\mathrm{s}^{G,n-1}_{k-3} \cdot \ldots \cdot x_1\,\mathrm{s}^{G,n-1}_0 \cdot \tilde{x}$$

We have $g\pi^G_{n,k} = \tilde{x}$.

Proof. Ad (1). We show this claim by induction on $k \in \mathbb{Z}_{\geqslant 1}$.

Let $k = 1$. We have

$$G_{n;\,\mathbb{Z}_{\geqslant 1}} \overset{281}{=} G\mathrm{N}_n = 1 \cdot G\mathrm{N}_n = \Big(\prod_{i\in\lfloor -1,0\rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i \Big) \cdot G\mathrm{N}_n\,.$$

Now suppose that $k \geqslant 2$. We have

$$\begin{aligned} G_{n;\,\mathbb{Z}_{\geqslant k}} &\overset{281.(2)}{=} (G_{n-1;\,\mathbb{Z}_{\geqslant k-1}})\mathrm{s}_{k-2} \cdot G_{n;\,\mathbb{Z}_{\geqslant k-1}} \\ &\overset{\text{IH.}}{=} (G_{n-1;\,\mathbb{Z}_{\geqslant k-1}})\mathrm{s}_{k-2} \cdot \Big(\prod_{i\in\lfloor k-3,0\rfloor} (G_{n-1;\,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i \Big) \cdot G\mathrm{N}_n \\ &= \Big(\prod_{i\in\lfloor k-2,0\rfloor} (G_{n-1;\,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i \Big) \cdot G\mathrm{N}_n\,. \end{aligned}$$

This proves (1).

Ad (2). By (1), it remains to show uniqueness.

Suppose that we have $g = \Big(\prod_{i\in\lfloor k-2,0\rfloor} x_{i+1}\,\mathrm{s}_i \Big) \cdot \tilde{x}$, where $x_j \in G_{n-1;\,\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$, and where $\tilde{x} \in G\mathrm{N}_n$.

Suppose that we have $g = \Big(\prod_{i\in\lfloor k-2,0\rfloor} y_{i+1}\,\mathrm{s}_i \Big) \cdot \tilde{y}$ for some $y_j \in G_{n-1;\,\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$, and for some $\tilde{y} \in G\mathrm{N}_n$.

We have to show that $x_j \overset{!}{=} y_j$ for $j \in [1, k-1]$ and that $\tilde{x} \overset{!}{=} \tilde{y}$.

We show this claim by induction on $k \in \mathbb{Z}_{\geqslant 1}$.

Let $k = 1$. We have

$$g = \Big(\prod_{i \in \lfloor -1, 0 \rfloor} x_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{x} = 1 \cdot \tilde{x} = \tilde{x}$$

and

$$g = \Big(\prod_{i \in \lfloor -1, 0 \rfloor} y_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{y} = 1 \cdot \tilde{y} = \tilde{y}\,.$$

This shows that $\tilde{x} = \tilde{y}$.

Now suppose that $k \geqslant 2$.

We write $\tilde{g} := \Big(\prod_{i \in \lfloor k-3, 0 \rfloor} x_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{x} \in G_n$ and we write $\tilde{h} := \Big(\prod_{i \in \lfloor k-3, 0 \rfloor} y_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{y} \in G_n$.

We have

$$g = \Big(\prod_{i \in \lfloor k-2, 0 \rfloor} x_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{x} = x_{k-1}\, \mathrm{s}_{k-2} \cdot \Big(\prod_{i \in \lfloor k-3, 0 \rfloor} x_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{x} = x_{k-1}\, \mathrm{s}_{k-2} \cdot \tilde{g}$$

and

$$g = \Big(\prod_{i \in \lfloor k-2, 0 \rfloor} y_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{y} = y_{k-1}\, \mathrm{s}_{k-2} \cdot \Big(\prod_{i \in \lfloor k-3, 0 \rfloor} y_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{y} = y_{k-1}\, \mathrm{s}_{k-2} \cdot \tilde{h}\,.$$

Recall that $x_{k-1} \in G_{n-1;\, \mathbb{Z}_{\geqslant k-1}}$.

We show that $\tilde{g} \overset{!}{\in} G_{n;\, \mathbb{Z}_{\geqslant k-1}}$. For $j \in \mathbb{Z}_{\geqslant k-1}$, we have

$$\begin{aligned} \tilde{g}\mathrm{d}_j &= \Big(\Big(\prod_{i \in \lfloor k-3, 0 \rfloor} x_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{x} \Big) \mathrm{d}_j = \Big(\prod_{i \in \lfloor k-3, 0 \rfloor} x_{i+1}\, \mathrm{s}_i\, \mathrm{d}_j \Big) \cdot \tilde{x} \mathrm{d}_j \\ &\overset{j \geqslant 1}{=} \Big(\prod_{i \in \lfloor k-3, 0 \rfloor} x_{i+1}\, \mathrm{d}_{j-1}\, \mathrm{s}_i \Big) \cdot 1 \overset{j-1 \geqslant i+1}{=} \prod_{i \in \lfloor k-3, 0 \rfloor} 1\, \mathrm{s}_i = 1\,. \end{aligned}$$

This shows that $\tilde{g} \in G_{n;\, \mathbb{Z}_{\geqslant k-1}}$.

Similarly, have $y_{k-1} \in G_{n-1;\, \mathbb{Z}_{\geqslant k-1}}$ and $\tilde{h} \in G_{n;\, \mathbb{Z}_{\geqslant k-1}}$.

So $x_{k-1}\, \mathrm{s}_{k-2} \cdot \tilde{g} = g = y_{k-1}\, \mathrm{s}_{k-2} \cdot \tilde{h}$.

Then, by Remark 281.(4), it follows that $x_{k-1} = y_{k-1} \in G_{n-1;\, \mathbb{Z}_{\geqslant k-1}}$ and that $\tilde{g} = \tilde{h} \in G_{n;\, \mathbb{Z}_{\geqslant k-1}}$. So

$$\Big(\prod_{i \in \lfloor k-3, 0 \rfloor} x_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{x} = \tilde{g} = \tilde{h} = \Big(\prod_{i \in \lfloor k-3, 0 \rfloor} y_{i+1}\, \mathrm{s}_i \Big) \cdot \tilde{y}\,.$$

Then, by induction hypothesis, we have $x_j = y_j$ for $j \in [1, k-2]$ and $\tilde{x} = \tilde{y}$.

Altogether, this shows (2).

Ad (3). We show that $g\pi^G_{n,k} \overset{!}{=} \tilde{x}$ by induction on $k \in \mathbb{Z}_{\geqslant 1}$.

Let $k = 1$. We have

$$g\pi^G_{n,1} = g\Big(\prod_{j\in\lfloor 1,2\rfloor}^{G_{n;\,\mathbb{Z}_{\geqslant 1}}} p^G_{n,j}\Big) = g\,\mathrm{id}_{G\mathrm{N}_n} = g = \Big(\prod_{i\in\lfloor -1,0\rfloor} x_{i+1}\,\mathrm{s}_i\Big)\cdot\tilde{x} = 1\cdot\tilde{x} = \tilde{x}\,.$$

Now suppose that $k \geqslant 2$. We write $\tilde{g} := g^-\mathrm{d}_{k-1}\,\mathrm{s}_{k-2}\cdot g = gp^G_{n,k} \overset{281.(1)}{\in} G_{n;\,\mathbb{Z}_{\geqslant k-1}}$.

We consider its unique decomposition

$$\tilde{g} = \Big(\prod_{i\in\lfloor k-3,0\rfloor} y_{i+1}\,\mathrm{s}_i\Big)\cdot\tilde{y}\,,$$

where $y_j \in G_{n;\,\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-2]$ and where $\tilde{y} \in G\mathrm{N}_n$; cf. (2).

Then, by induction hypothesis, we have $\tilde{g}\pi^G_{n,k-1} = \tilde{y}$.

We have

$$g\pi^G_{n,k} = g(p^G_{n,k} \blacktriangle \pi^G_{n,k-1}) = (g^-\mathrm{d}_{k-1}\,\mathrm{s}_{k-2}\cdot g)\pi^G_{n,k-1} = \tilde{g}\pi^G_{n,k-1} \overset{\text{IH.}}{=} \tilde{y}\,.$$

Moreover, we have

$$g = g\mathrm{d}_{k-1}\,\mathrm{s}_{k-1}\cdot\tilde{g}\,,$$

where $g\mathrm{d}_{k-1} \in G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}$ and $\tilde{g} \in G_{n;\,\mathbb{Z}_{\geqslant k-1}}$; cf. Remark 281.(2).

We have

$$g\mathrm{d}_{k-1}\,\mathrm{s}_{k-2}\cdot\tilde{g} = g = \Big(\prod_{i\in\lfloor k-2,0\rfloor} x_{i+1}\,\mathrm{s}_i\Big)\cdot\tilde{x} = x_{k-1}\,\mathrm{s}_{k-2}\cdot\Big(\prod_{i\in\lfloor k-3,0\rfloor} x_{i+1}\,\mathrm{s}_i\Big)\cdot\tilde{x}\,,$$

where $x_{k-1} \in G_{n-1;\,\mathbb{Z}_{\geqslant k-1}}$ and $\Big(\prod_{i\in\lfloor k-3,0\rfloor} x_{i+1}\,\mathrm{s}_i\Big)\cdot\tilde{x} \in G_{n;\,\mathbb{Z}_{\geqslant k-1}}$.

Then, by Remark 281.(4), we have $g\mathrm{d}_{k-1} = x_{k-1}$ and

$$\Big(\prod_{i\in\lfloor k-3,0\rfloor} y_{i+1}\,\mathrm{s}_i\Big)\cdot\tilde{y} = \tilde{g} = \Big(\prod_{i\in\lfloor k-3,0\rfloor} x_{i+1}\,\mathrm{s}_i\Big)\cdot\tilde{x}\,.$$

Then, by (2), we have $x_j = y_j$ for $j \in [1, k-2]$ and $\tilde{x} = \tilde{y}$.

Therefore, we have $g\pi^G_{n,k} = \tilde{y} = \tilde{x}$.

This shows (3). □

Lemma 284 *Suppose given a stable simplicial group G; cf. Definition* 205.

(1) *We have*

$$G_n = \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}}) \mathrm{s}_i^{G,n-1} \Big) \cdot G\mathrm{N}_n \, ;$$

cf. §1.1.2 *item* 10.2.

(2) *Suppose given $g \in G_n$.*

Then there exists some $k \in \mathbb{Z}_{\geqslant 1}$ such that there exists a unique element $x_j \in G_{n;\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$ and a unique element $\tilde{x} \in G\mathrm{N}_n$ such that

$$g = \Big(\prod_{i \in \lfloor k-2, 0 \rfloor} x_{i+1} \, \mathrm{s}_i^{G,n} \Big) \cdot \tilde{x} \, .$$

Proof. Ad (1). It suffices to show that $G_n \overset{!}{\subseteq} \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}}) \mathrm{s}_i^{G,n-1} \Big) \cdot G\mathrm{N}_n$.

Suppose given $g \in G_n$. Since g is finite, we may choose some $k \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_j = 1$ for $j \geqslant k$; cf. Definitions 205, 204.

We may assume that $k \geqslant 1$.

Then we have

$$g \in G_{n;\mathbb{Z}_{\geqslant k}} \overset{283.(1)}{=} \Big(\prod_{i \in \lfloor k-2, 0 \rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}}) \mathrm{s}_i \Big) \cdot G\mathrm{N}_n \subseteq \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}}) \mathrm{s}_i \Big) \cdot G\mathrm{N}_n \, .$$

This proves (1).

Ad (2). Suppose given $g \in G_n$. Since g is finite, we may choose some $k \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_j = 1$ for $j \geqslant k$; cf. Definitions 205, 204.

We may assume that $k \geqslant 1$.

We have

$$g \in G_{n;\mathbb{Z}_{\geqslant k}} \overset{283.(1)}{=} \Big(\prod_{i \in \lfloor k-2, 0 \rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}}) \mathrm{s}_i \Big) \cdot G\mathrm{N}_n$$

Then, by Lemma 283.(2), there exist unique elements $x_j \in G_{n-1;\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$ a unique $\tilde{x} \in G\mathrm{N}_n$ such that

$$g = \Big(\prod_{i \in \lfloor k-2, 0 \rfloor} x_{i+1} \, \mathrm{s}_i \Big) \cdot \tilde{x} \, .$$

This proves (2). □

13.3 The degenerate subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$

Remark 285 *Let $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $\ell \in [1, k+1]$.*
Suppose given a stable simplicial group G; cf. Definition 205.
We consider the simplicial group $(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}$; cf. Remark 276.(3).
We consider the surjective map from Lemma 283.(3)

$$\pi^{G}_{n+1,\ell} \colon G_{n+1;\,\mathbb{Z}_{\geqslant \ell}} \to G\mathrm{N}_{n+1}\,.$$

We consider the surjective map from Lemma 100.(3)

$$\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1,\ell} \colon ((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord})_{k+1;[\ell,k+1]} \to ((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord})\mathrm{N}_{k+1}\,.$$

(1) *We have* $G_{n+1;\,\mathbb{Z}_{\geqslant \ell}} \leqslant ((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord})_{k+1;[\ell,k+1]}$.

(2) *We have*
$$\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1,\ell}\big|^{G\mathrm{N}_{n+1}}_{G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}} = \pi^{G}_{n+1,\ell} \colon G_{n+1;\,\mathbb{Z}_{\geqslant \ell}} \to G\mathrm{N}_{n+1}\,.$$

(3) *Supppose given $g \in G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$.*
For $j \in \mathbb{Z}_{\geqslant \ell}$, we have $g\pi^{G}_{n+1,j} = g\pi^{G}_{n+1,\ell}$.
For $j \in [\ell, k+2]$, we have $g\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1,j} = g\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1,\ell}$.

(4) *We have*
$$\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1}\big|^{G\mathrm{N}_{n+1}}_{G_{n+1;\,\mathbb{Z}_{\geqslant k}}} = \pi^{G}_{n+1,k} \colon G_{n+1;\,\mathbb{Z}_{\geqslant k}} \to G\mathrm{N}_{n+1}\,;$$
cf. Corollary 101.(3).

Proof. We write $\tilde{G} := (G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord} \in \mathrm{Ob}\big(\mathcal{SimpGrp}\big)$.

Ad (1). Suppose given $g \in G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$. *We show that* $g \overset{!}{\in} \tilde{G}_{k+1} = ((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord})_{k+1}$.
From Remark 276.(3), note that we have

$$((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord})_{k+1} = \Big(\bigcap_{j\in\mathbb{Z}_{\geqslant k+2}} G_{n+1,j}\Big) \cap \ker\Big(\prod_{j\in\lfloor k+1,0\rfloor}^{G_{n+1}} \mathrm{d}_0^{G,n-k+j}\Big) \leqslant G_{n+1}\,.$$

For $j \in \mathbb{Z}_{\geqslant k+2}$, we have $j \geqslant \ell$, and so $g\mathrm{d}^{G,n+1}_{j} = 1$. This shows that $g \in \bigcap_{j\in\mathbb{Z}_{\geqslant k+2}} G_{n+1,j}$.

Moreover, since $k+1 \geqslant \ell$, we have

$$\begin{aligned} g\big(\prod_{j\in\lfloor k+1,0\rfloor}^{G_{n+1}\,G_{n-k-1}} \mathrm{d}_0^{G,n-k+j}\big) &= g\Big(\big(\prod_{j\in\lfloor k+1,1\rfloor}^{G_{n+1}\,G_{n-k}} \mathrm{d}_0^{G,n-k+j}\big) \blacktriangleleft \mathrm{d}_0^{G,n-k}\Big) \\ &= g\Big(\mathrm{d}_{k+1}^{G,n+1} \blacktriangleleft \big(\prod_{j\in\lfloor k,0\rfloor}^{G_n\,G_{n-k-1}} \mathrm{d}_0^{G,n-k+j}\big)\Big) \\ &= 1\big(\prod_{j\in\lfloor k,0\rfloor}^{G_{n+1}\,G_{n-k-1}} \mathrm{d}_0^{G,n-k+j}\big) \\ &= 1\,. \end{aligned}$$

So $g \in \ker\Big(\prod_{j\in\lfloor k+1,0\rfloor}^{G_{n+1}\,G_{n-k-1}} \mathrm{d}_0^{G,n-k+j}\Big)$.

This shows that $g \in ((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord})_{k+1}$. Moreover, since $g\mathrm{d}_j^{G,n+1} = 1$ for $j \in \mathbb{Z}_{\geqslant\ell}$, we have $g \in ((G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord})_{k+1;[\ell,k+1]} = \tilde{G}_{k+1;[\ell,k+1]}$; cf. Remark 196, Lemma 219.

Ad (2). We have

$$G_{n+1;\,\mathbb{Z}_{\geqslant\ell}} \overset{(1)}{\leqslant} \tilde{G}_{k+1;[\ell,k+1]}\,.$$

So the map $\pi^{\tilde{G}}_{k+1,\ell}\big|_{G_{n+1;\,\mathbb{Z}_{\geqslant\ell}}} : G_{n+1;\,\mathbb{Z}_{\geqslant\ell}} \to \tilde{G}\mathrm{N}_{k+1}$ is well-defined.

Suppose given $g \in G_{n+1;\,\mathbb{Z}_{\geqslant\ell}}$.

We show that $g\pi^G_{n+1,\ell} \overset{!}{=} g\pi^{\tilde{G}}_{k+1,\ell}$ by induction on $\ell \in [1, k+1]$.

Let $\ell = 1$. Then $g \in G_{n+1;\,\mathbb{Z}_{\geqslant 1}} = G\mathrm{N}_{n+1} \overset{(1)}{\leqslant} \tilde{G}\mathrm{N}_{k+1}$. We have

$$g\pi^G_{n+1,1} \overset{283.(3)}{=} g\big(\prod_{j\in\lfloor 1,2\rfloor}^{G\mathrm{N}_{n+1}\,G\mathrm{N}_{n+1}} p^G_{n+1,j}\big) = g\,\mathrm{id}_{G\mathrm{N}_{n+1}} = g\,.$$

Moreover, we have

$$g\pi^{\tilde{G}}_{k+1,1} \overset{100.(3)}{=} g\big(\prod_{j\in\lfloor 1,2\rfloor}^{\tilde{G}\mathrm{N}_{k+1}\,\tilde{G}\mathrm{N}_{k+1}} p^{\tilde{G}}_{k+1,j}\big) = g\,\mathrm{id}_{\tilde{G}\mathrm{N}_{k+1}} = g\,.$$

This shows that $g\pi^G_{n+1,1} = g\pi^{\tilde{G}}_{k+1,1}$

Now suppose that $\ell \in [2, k+1]$. Let $g' := g^-\mathrm{d}^{G,n+1}_{\ell-1}\,\mathrm{s}^{G,n}_{\ell-2} \cdot g = gp^G_{n+1,\ell} \overset{281.(1)}{\in} G_{n+1;\,\mathbb{Z}_{\geqslant\ell-1}}$.

We have

$$\begin{aligned} g\pi^G_{n+1,\ell} &\overset{283.(3)}{=} g\big(p^G_{n+1,\ell} \blacktriangleleft \pi^G_{n+1,\ell-1}\big) = (g^- \mathrm{d}^{G,n+1}_{\ell-1}\, \mathrm{s}^{G,n}_{\ell-2} \cdot g)\pi^G_{n+1,\ell-1} = g'\pi^G_{n+1,\ell-1} \overset{\text{IH.}}{=} g'\pi^{\tilde{G}}_{k+1,\ell-1} \\ &= (g^- \mathrm{d}^{G,n+1}_{\ell-1}\, \mathrm{s}^{G,n}_{\ell-2} \cdot g)\pi^{\tilde{G}}_{k+1,\ell-1} \overset{196,219}{=} (g^- \mathrm{d}^{\tilde{G},k+1}_{\ell-1}\, \mathrm{s}^{\tilde{G},k}_{\ell-2} \cdot g)\pi^{\tilde{G}}_{k+1,\ell-1} \\ &\overset{97.(1)}{=} g\big(p^{\tilde{G}}_{k+1,\ell} \blacktriangleleft \pi^{\tilde{G}}_{k+1,\ell-1}\big) \overset{100.(3)}{=} g\pi^{\tilde{G}}_{k+1,\ell}\,. \end{aligned}$$

This shows that $g\pi^G_{n+1,\ell} = g\pi^{\tilde{G}}_{k+1,\ell}$.

Therefore, we have

$$G\mathrm{N}_{n+1} \overset{283.(3)}{=} \big(G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}\big)\pi^G_{n+1,\ell} = \big(G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}\big)\pi^{\tilde{G}}_{k+1,\ell}\,.$$

So the map $\pi^{\tilde{G}}_{k+1,\ell}\big|^{G\mathrm{N}_{n+1}}_{G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}}$ is well-defined.

Altogether, we have $\pi^{\tilde{G}}_{k+1,\ell}\big|^{G\mathrm{N}_{n+1}}_{G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}} = \pi^G_{n+1,\ell} : G_{n+1;\,\mathbb{Z}_{\geqslant \ell}} \to G\mathrm{N}_{n+1}$.

Ad (3). Suppose given $j \in \mathbb{Z}_{\geqslant \ell}$.

We show that $g\pi^G_{n+1,j} \overset{!}{=} g\pi^G_{n+1,\ell}$ by induction on $t := j - \ell \in \mathbb{Z}_{\geqslant 0}$.

Let $t = 0$. Then $j = \ell$, and we have $g\pi^G_{n+1,j} = g\pi^G_{n+1,\ell}$.

Now suppose that $t \geqslant 1$. Then $j - 1 \geqslant \ell$, and we have

$$\begin{aligned} g\pi^G_{n+1,j} &\overset{283.(3)}{=} g\big(p^G_{n+1,j} \blacktriangleleft \pi^G_{n+1,j-1}\big) \overset{281.(1)}{=} (g^- \mathrm{d}^{G,n+1}_{j-1}\, \mathrm{s}^{G,n}_{j-2} \cdot g)\pi^G_{n+1,j-1} = (1\mathrm{s}^G_{j-2} \cdot g)\pi^G_{n+1,j-1} \\ &= g\pi^G_{n+1,j-1} \overset{\text{IH.}}{=} g\pi^G_{n+1,\ell}\,. \end{aligned}$$

Suppose given $j \in [\ell, k+2]$.

We show that $g\pi^{\tilde{G}}_{k+1,j} \overset{!}{=} g\pi^{\tilde{G}}_{k+1,\ell}$ by induction on $t := j - \ell \in [0, k+2-\ell]$. Note that $k+2-\ell \in [1, k+1]$, since $\ell \in [1, k+1]$.

Let $t = 0$. Then $j = \ell$, and we have $g\pi^{\tilde{G}}_{k+1,j} = g\pi^{\tilde{G}}_{k+1,\ell}$.

Now suppose that $t \geqslant 1$. Then $j - 1 \geqslant \ell$, and we have

$$\begin{aligned} g\pi^{\tilde{G}}_{k+1,j} &\overset{100.(3)}{=} g\big(p^{\tilde{G}}_{k+1,j} \blacktriangleleft \pi^{\tilde{G}}_{k+1,j-1}\big) \overset{97.(1)}{=} (g^- \mathrm{d}^{\tilde{G},k+1}_{j-1}\, \mathrm{s}^{\tilde{G},k}_{j-2} \cdot g)\pi^{\tilde{G}}_{k+1,j-1} \overset{196,219}{=} (1\mathrm{s}^G_{j-2} \cdot g)\pi^{\tilde{G}}_{k+1,j-1} \\ &= g\pi^{\tilde{G}}_{k+1,j-1} \overset{\text{IH.}}{=} g\pi^{\tilde{G}}_{k+1,\ell}\,. \end{aligned}$$

Ad (4). Recall from Corollary 101.(3) that we have

$$\pi^{\tilde{G}}_{k+1} = \pi^{\tilde{G}}_{k+1,k+2} = \prod_{j \in \lfloor k+2, 2 \rfloor}^{\tilde{G}_{k+1}} p^{\tilde{G}}_{k+1,j} : \tilde{G}_{k+1} \to \tilde{G}\mathrm{N}_{k+1}\,.$$

We have $G_{n+1;\,\mathbb{Z}_{\geqslant k}} \overset{(1)}{\leqslant} \tilde{G}_{k+1;\,[k,k+1]} \leqslant \tilde{G}_{k+1}$.

So the map $\pi^{\tilde{G}}_{k+1}\big|_{G_{n+1;\,\mathbb{Z}_{\geqslant k}}} : G_{n+1;\,\mathbb{Z}_{\geqslant k}} \to \tilde{G}\mathrm{N}_{k+1}$ is well-defined.

Suppose given $g \in G_{n+1,\,\mathbb{Z}_{\geqslant k}}$. We have

$$g\pi^{G}_{n+1,k} \overset{(2)}{=} g\pi^{\tilde{G}}_{k+1,k} \overset{(3)}{=} g\pi^{\tilde{G}}_{k+1,k+2} \overset{101}{=} g\pi^{\tilde{G}}_{k+1} .$$

This shows that $g\pi^{G}_{n+1,k} = g\pi^{\tilde{G}}_{k+1}$.

Therefore, we have

$$G\mathrm{N}_{n+1} \overset{283.(3)}{=} \big(G_{n+1;\,\mathbb{Z}_{\geqslant k}}\big)\pi^{G}_{n+1,k} = \big(G_{n+1;\,\mathbb{Z}_{\geqslant k}}\big)\pi^{\tilde{G}}_{k+1} .$$

So the map $\pi^{\tilde{G}}_{k+1}\big|^{G\mathrm{N}_{n+1}}_{G_{n+1;\,\mathbb{Z}_{\geqslant k}}}$ is well-defined.

Altogether, we have $\pi^{\tilde{G}}_{k+1}\big|^{G\mathrm{N}_{n+1}}_{G_{n+1;\,\mathbb{Z}_{\geqslant k}}} = \pi^{G}_{n+1,k} : G_{n+1;\,\mathbb{Z}_{\geqslant k}} \to G\mathrm{N}_{n+1}$. □

Lemma 286 *Let $k \in \mathbb{Z}_{\geqslant 1}$. Suppose given a stable simplicial group G; cf. Definition* 205. *We consider the decomposition of G_n from Lemma* 284.(1) :

$$G_n = \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} (G_{n-1;\mathbb{Z}_{\geqslant i+1}})\mathrm{s}^{G,n-1}_i \Big) \cdot G\mathrm{N}_n$$

We write

$$G\mathrm{D}_n := \prod_{i \in \rfloor \infty, 0 \rfloor} (G_{n-1;\,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}^{G,n-1}_i \ \subseteq G_n .$$

(1) *We have*

$$G\mathrm{D}_n \cap G_{n;\,\mathbb{Z}_{\geqslant k}} = \prod_{i \in \lfloor k-2, 0 \rfloor} (G_{n-1;\,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}^{G,n-1}_i \ \subseteq G_n .$$

(2) *Suppose given $g \in G_{n;\,\mathbb{Z}_{\geqslant k}}$. We have*

$$g \in G\mathrm{D}_n \quad \Leftrightarrow \quad g\pi^{G}_{n,k} = 1 ;$$

cf. Lemma 283.(3).

Proof. Ad (1). Ad $\subseteq$. Suppose given $g \in G\mathrm{D}_n \cap G_{n;\mathbb{Z}_{\geqslant k}}$.

Since $g \in G\mathrm{D}_n$, we have

$$g = \prod_{i \in \lfloor \ell-2,0 \rfloor} y_{i+1}\,\mathrm{s}_i = \Big(\prod_{i \in \lfloor \ell-2,0 \rfloor} y_{i+1}\,\mathrm{s}_i \Big) \cdot 1$$

for some chosen $\ell \in \mathbb{Z}_{\geqslant 1}$, and $y_j \in G_{n-1;\mathbb{Z}_{\geqslant j}}$ for $j \in [1, \ell-1]$; cf. §1.1.2 item 9.4.

Moreover, since $g \in G_{n;\mathbb{Z}_{\geqslant k}}$, we have, by Lemma 283.(1),

$$g = \Big(\prod_{i \in \lfloor k-2,0 \rfloor} x_{i+1}\,\mathrm{s}_i \Big) \cdot \tilde{x}\,,$$

where $x_j \in G_{n-1;\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$ and where $\tilde{x} \in G\mathrm{N}_n$.

Let $m := \max\{\ell, k\}$. Then we have

$$\Big(\prod_{i \in \lfloor m-2,0 \rfloor} y_{i+1}\,\mathrm{s}_i \Big) \cdot 1 = g = \Big(\prod_{i \in \lfloor m-2,0 \rfloor} x_{i+1}\,\mathrm{s}_i \Big) \cdot \tilde{x}\,,$$

where we let $y_j := 1 \in G_{n-1;\mathbb{Z}_{\geqslant j}}$ for $j \in [\ell, m-1]$ and $x_j := 1 \in G_{n-1;\mathbb{Z}_{\geqslant j}}$ for $j \in [k, m-1]$.

Moreover, by Lemma 283.(2), we have $y_j = x_j$ for $j \in [1, m-1]$ and $1 = \tilde{x}$.

This shows that

$$g = \prod_{i \in \lfloor k-2,0 \rfloor} x_{i+1}\,\mathrm{s}_i \;\in\; \prod_{i \in \lfloor k-2,0 \rfloor} (G_{n-1,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i\,.$$

Ad $\supseteq$. It suffices to show that $G_{n;\mathbb{Z}_{\geqslant k}} \overset{!}{\supseteq} (G_{n-1;\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i$ for $i \in [0, k-2]$.

Let $i \in [0, k-2]$. Suppose given $g \in (G_{n-1;\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i$. Then we may write $g = x\mathrm{s}_i$ for some chosen $x \in G_{n-1;\mathbb{Z}_{\geqslant i+1}}$. For $j \in \mathbb{Z}_{\geqslant k}$, we have

$$g\mathrm{d}_j = x\mathrm{s}_i\,\mathrm{d}_j = x\mathrm{d}_{j-1}\,\mathrm{s}_i = 1\mathrm{s}_i = 1\,.$$

This shows that $g \in G_{n;\mathbb{Z}_{\geqslant k}}$.

Ad (2). By Lemma 283.(1, 2), we have

$$g = \Big(\prod_{i \in \lfloor k-2,0 \rfloor} x_{i+1}\,\mathrm{s}_i \Big) \cdot \tilde{x}\,,$$

for some unique $x_j \in G_{n-1;\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$ and for a unique $\tilde{x} \in G\mathrm{N}_n$.

Ad $\Rightarrow$. Suppose that $g \in G\mathrm{D}_n$. Then we have $g \in G\mathrm{D}_n \cap G_{n;\mathbb{Z}_{\geqslant k}}$.
By (1), we have

$$g = \prod_{i \in \lfloor k-2,0 \rfloor} y_{i+1}\,\mathrm{s}_i = \Big(\prod_{i \in \lfloor k-2,0 \rfloor} y_{i+1}\,\mathrm{s}_i \Big) \cdot 1$$

for some $y_j \in G_{n-1,\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k-1]$.
B Lemma 283.(2), it follows that $1 = \tilde{x}$.
Then, by Lemma 283.(3), we have

$$g\pi^G_{n,k} = \tilde{x} = 1\,.$$

Ad $\Leftarrow$. Suppose that $g\pi^G_{n,k} = 1$. By Lemma 283.(3), we have

$$1 = g\pi^G_{n,k} = \tilde{x}\,.$$

So

$$g = \Big(\prod_{i \in \lfloor k-2,0 \rfloor} x_{i+1}\,\mathrm{s}_i \Big) \cdot \tilde{x} \;=\; \prod_{i \in \lfloor k-2,0 \rfloor} x_{i+1}\,\mathrm{s}_i \;\in G\mathrm{D}_n\,.$$

□

Lemma 287 (The degenerate subgroup $G\mathrm{D}_{n+1}$)
Suppose given an $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group G; cf. Definition 275.
We have the subgroup

$$G\mathrm{D}_{n+1} = \prod_{i \in \rfloor \infty,0 \rfloor} (G_{n,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i^{G,n} \;\leqslant G_{n+1}\,;$$

cf. Lemma 286. *We call $G\mathrm{D}_{n+1}$ the* degenerate subgroup *of G_{n+1}.*

Proof. Note that $1 \in G\mathrm{D}_n$.

Suppose given g, $h \in G\mathrm{D}_{n+1}$. We have to show that $g^- \cdot h \overset{!}{\in} G\mathrm{D}_{n+1}$.
Since G is finite, we may choose some $k \in \mathbb{Z}_{\geqslant 1}$ such that $g\mathrm{d}_j = h\mathrm{d}_j = 1$ for $j \in \mathbb{Z}_{\geqslant k}$.
Then g, $h \in G_{n+1,\mathbb{Z}_{\geqslant k}}$.
Moreover, since G is an $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group, we have the $\lfloor k, 0 \rfloor$-coskeleletal simplicial group $(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}$; cf. Remark 276.(3).

We have

$$\begin{aligned}
g,\, h \in G\mathrm{D}_{n+1} &\overset{286.(2)}{\Leftrightarrow} g\pi^G_{n+1,k} = h\pi^G_{n+1,k} = 1 \\
&\overset{285.(4)}{\Leftrightarrow} g\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1} = h\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1} = 1 \\
&\overset{116.(1)}{\Leftrightarrow} g\,, h \in (G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}\,\mathrm{D}_{k+1} \\
&\overset{113}{\Rightarrow} g^- \cdot h \in (G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}\,\mathrm{D}_{k+1} \\
&\overset{116.(1)}{\Leftrightarrow} (g^- \cdot h)\pi^{(G\mathrm{Dec}^{\pm,n-k})\mathrm{Ord}}_{k+1} = 1 \\
&\overset{285.(4)}{\Leftrightarrow} (g^- \cdot h)\pi^G_{n+1,k} = 1 \\
&\overset{286.(2)}{\Leftrightarrow} g^- \cdot h \in G\mathrm{D}_{n+1}\,.
\end{aligned}$$

This shows that $G\mathrm{D}_{n+1} \leqslant G_{n+1}$. □

Lemma 288 *Suppose given an $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group G; cf. Definition 275.*

We consider the subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$; cf. Lemma 287.

Let $\ell \in \mathbb{Z}_{\geqslant 0}$. Suppose given $g \in G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$; cf. Definition 278.

We consider the surjective map $\pi^G_{n+1,\ell} \colon G_{n+1;\,\mathbb{Z}_{\geqslant \ell}} \to G\mathrm{N}_{n+1}$ from Lemma 283.(3).

(1) *We have* $g \in G\mathrm{D}_{n+1} \Leftrightarrow g\pi^G_{n+1,\ell} = 1$.

(2) *We have* $g \in G\mathrm{N}_{n+1} \Leftrightarrow g\pi^G_{n+1,\ell} = g$.

(3) *We have* $g \in G\mathrm{D}_{n+1} \Leftrightarrow g\pi^G_{n+1,\ell}\ \mathrm{d}^{G,n+1}_0 = 1$.

Proof. Ad (1). Cf. Lemma 286.(2).

Ad (2). Ad $\Rightarrow$. Suppose that $g \in G\mathrm{N}_{n+1} \leqslant G_{n+1}$.

By Lemma 283.(2), since $g \in G\mathrm{N}_{n+1} = G_{n+1;\,\mathbb{Z}_{\geqslant 1}}$, we have

$$g = \Big(\prod_{i \in [1-2,0]} x_i\, \mathrm{s}^{G,n}_i \Big) \cdot \tilde{x} = \tilde{x}$$

where $x_j \in G_{n;\,\mathbb{Z}_{\geqslant j}}$ for $j \in [1, \ell-1]$ and where $\tilde{x} \in G\mathrm{N}_{n+1}$. So $g\pi^G_{n+1,\ell} \overset{283.(3)}{=} \tilde{x} = g$.

Ad $\Leftarrow$. Suppose that $g = g\pi^G_{n+1,\ell}$. We have $g = g\pi^G_{n+1,\ell} \in G\mathrm{N}_{n+1}$.

Ad (3). Ad $\Rightarrow$. Suppose that $g \in G\mathrm{D}_{n+1}$. By (1), we have $g\pi^G_{n+1,\ell} = 1$.

So we have $g\pi^G_{n+1,\ell}\ \mathrm{d}_0^{G,n+1} = 1$.

Ad $\Leftarrow$. Suppose that $g\pi^G_{n+1,\ell}\ \mathrm{d}_0^{G,n+1} = 1$. Recall that $g\pi^G_{n+1,\ell} \in G\mathrm{N}_{n+1}$; cf. Definition 275.

So we have $g\pi^G_{n+1} \in G\mathrm{N}_{n+1} \cap \ker \mathrm{d}_0^{G,n+1} \overset{198.(3)}{=} G\mathrm{NZ}_{n+1}$.

Since G is $\lfloor n, -\infty \rfloor$-coskeletal, we have that $G\mathrm{NZ}_{n+1} = 1$.

This shows that $g\pi^G_{n+1,\ell} = 1$. So, by (1), we have $g \in G\mathrm{D}_{n+1}$. □

Remark 289 *Suppose given stable simplicial groups G, H; cf. Definition* 205.
Suppose given a stable simplicial group morphism $\varphi\colon G \to H$.
We consider the Moore complex functor $\mathrm{N}\colon \mathcal{PreStSimpGrp} \to \mathcal{ComplexGrp}$*; cf. Lemma* 201.(3).
We consider the embedding functor $\mathrm{I}\colon \mathcal{StSimpGrp} \to \mathcal{PreStSimpGrp}$*; cf. Remark* 209.
We consider the functor $\mathrm{I} * \mathrm{N}\colon \mathcal{StSimpGrp} \to \mathcal{ComplexGrp}$.
Let $\ell \in \mathbb{Z}_{\geqslant 0}$.

(1) *We have $(G_{n+1;\,\mathbb{Z}_{\geqslant \ell}})\varphi_{n+1} \leqslant H_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$; cf. Definition* 278. *So we have the group morphism*
$$\varphi_{n+1,\ell} := \varphi_{n+1}\big|^{H_{n+1;\,\mathbb{Z}_{\geqslant \ell}}}_{G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}} : G_{n+1;\,\mathbb{Z}_{\geqslant \ell}} \to H_{n+1;\,\mathbb{Z}_{\geqslant \ell}}\,.$$

(2) *We have*
$$\pi^G_{n+1,\ell} \blacktriangleleft \varphi(\mathrm{I} * \mathrm{N}_{n+1}) = \varphi_{n+1,\ell} \blacktriangleleft \pi^H_{n+1,\ell} : G_{n+1,\,\mathbb{Z}_{\geqslant \ell}} \to H\mathrm{N}_{n+1}\,.$$

Proof. Ad (1). Suppose given $g \in G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$. For $j \in \mathbb{Z}_{\geqslant \ell}$, we have

$$(g\varphi_{n+1})\mathrm{d}_j^{H,n+1} \overset{184.(2)}{=} (g\mathrm{d}_j^{G,n+1})\varphi_n = 1\varphi_n = 1\,.$$

This shows that $g\varphi_{n+1} \in H_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$. So $(G_{n+1;\,\mathbb{Z}_{\geqslant \ell}})\varphi_{n+1} \leqslant H_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$.

Therefore, we have the group morphism $\varphi_{n+1,\ell} := \varphi_{n+1}\big|^{H_{n+1;\,\mathbb{Z}_{\geqslant \ell}}}_{G_{n+1;\,\mathbb{Z}_{\geqslant \ell}}}$.

Ad (2). Suppose given $j \in [1, \ell]$. We consider the maps

$$p^G_{n+1,j}\colon G_{n+1;\,\mathbb{Z}_{\geqslant j}} \to G_{n+1,\,\mathbb{Z}_{\geqslant j-1}} \quad \text{and} \quad p^H_{n+1,j}\colon H_{n+1;\,\mathbb{Z}_{\geqslant j}} \to H_{n+1,\,\mathbb{Z}_{\geqslant j-1}}\,;$$

cf. Remark 281.(1).

For $g \in G_{n+1;\,\mathbb{Z}_{\geqslant j}}$, we have

$$(gp^G_{n+1,j})\varphi_{n+1} \overset{281.(1)}{=} (g^- \mathrm{d}^{G,n+1}_{j-1}\mathrm{s}^{G,n}_{j-2} \cdot g)\varphi_{n+1} = (g\varphi_{n+1})^- \mathrm{d}^{H,n+1}_{j-1}\mathrm{s}^{H,n}_{j-2} \cdot g\varphi_{n+1} \overset{281.(1)}{=} (g\varphi_{n+1})p^H_{n+1,j}\,.$$

So, for $g \in G_{n+1,\,\mathbb{Z}_{\geqslant \ell}}$, we have

$$\begin{aligned}
g\big(\pi^G_{n+1,\ell} \blacktriangle \varphi(\mathrm{I} * \mathrm{N}_{n+1})\big) &\overset{283.(3)}{=} g\Big(\Big(\prod_{j\in\lfloor \ell,2\rfloor}^{{}^{G_{n+1,\,\mathbb{Z}_{\geqslant \ell}}}G\mathrm{N}_{n+1}} p^G_{n+1,j}\Big) \blacktriangle \varphi(I * \mathrm{N}_{n+1})\Big) \\
&= \Big(g\Big(\prod_{j\in\lfloor \ell,2\rfloor}^{{}^{G_{n+1,\,\mathbb{Z}_{\geqslant \ell}}}G\mathrm{N}_{n+1}} p^G_{n+1,j}\Big)\Big)\varphi_{n+1} \\
&= (g\varphi_{n+1})\Big(\prod_{j\in\lfloor \ell,2\rfloor}^{{}^{H_{n+1,\,\mathbb{Z}_{\geqslant \ell}}}H\mathrm{N}_{n+1}} p^H_{n+1,j}\Big) \\
&\overset{283.(3)}{=} g(\varphi_{n+1} \blacktriangle \pi^H_{n+1,\ell}) \\
&= g(\varphi_{n+1,\ell} \blacktriangle \pi^H_{n+1,\ell})\,.
\end{aligned}$$

This shows that $\pi^G_{n+1,\ell} \blacktriangle \varphi(\mathrm{I} * \mathrm{N}_{n+1}) = \varphi_{n+1} \blacktriangle \pi^H_{n+1,\ell}$. □

Chapter 14

Stable simplicial groups and $\lfloor n, -\infty \rfloor$-stable simplicial groups: the Conduché functor

14.1 The truncation functor $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} : \mathit{StSimpGrp} \to \lfloor n, -\infty \rfloor\text{-}\mathit{StSimpGrp}$

Let $n \in \mathbb{Z}$.

Lemma 290 (The $\lfloor n, -\infty \rfloor$-stable simplicial group $G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$)

Suppose given a stable simplicial group G; cf. Definition 205*. Note that in particular, G is a prestable simplicial group; cf. Definition* 182*.*

For $k \in \mathbb{Z}_{\leqslant n-1}$, we consider the group G_k.

Moreover, we consider the group $G_n/G\mathrm{NB}_n$; cf. Definition 198.(4)*, Lemma* 200.(2)*.*

(1) *For $j \in \mathbb{Z}_{\geqslant 0}$, we have the group morphism*

$$\bar{\mathrm{d}}_j \colon G_n/G\mathrm{NB}_n \to G_{n-1}\,,\ gG\mathrm{NB}_n \mapsto g\mathrm{d}_j^{G,n}\,.$$

For $i \in \mathbb{Z}_{\geqslant 0}$, we have the group morphism

$$\bar{\mathrm{s}}_i \colon G_{n-1} \to G_n/G\mathrm{NB}_n\,,\ g \mapsto (g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\,.$$

(2) *Using Lemma* 254.(3)*, we may define a pre-*$\lfloor n, -\infty \rfloor$*-prestable simplicial group* $G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ *as follows.*

We have

$$(G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor})_k = \big([k]^{\pm}\big)G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} := \begin{cases} G_n/G\mathrm{NB}_n & \text{for } k = n \\ G_k & \text{for } k \in \mathbb{Z}_{\leqslant -1}\,. \end{cases}$$

We have

$$\mathrm{d}_j^{G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor},k} := \begin{cases} \bar{\mathrm{d}}_j\colon & G_n/G\mathrm{NB}_n \to G_{n-1} & \text{for } k = n \\ \mathrm{d}_j^{G,k}\colon & G_k \to G_{k-1} & \text{for } k \in \mathbb{Z}_{\leqslant n-1}\,; \end{cases}$$

cf. (1).

We have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor},k} := \begin{cases} \bar{\mathrm{s}}_i\colon & G_{n-1} \to G_n/G\mathrm{NB}_n & \text{for } k = n-1 \\ \mathrm{s}_i^{G,k}\colon & G_k \to G_{k+1} & \text{for } k \in \mathbb{Z}_{\leqslant n-2}\,; \end{cases}$$

cf. (1).

$$G_n/G\mathrm{NB}_n \underset{\bar{\mathrm{s}}_i}{\overset{\bar{\mathrm{d}}_j}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i^{G,n-2}}{\overset{\mathrm{d}_j^{G,n-1}}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i^{G,0}}{\overset{\mathrm{d}_j^{G,1}}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i^{G,-1}}{\overset{\mathrm{d}_j^{G,0}}{\rightleftarrows}} G_{-1} \underset{\mathrm{s}_i^{G,-2}}{\overset{\mathrm{d}_j^{G,-1}}{\rightleftarrows}} \cdots$$

(3) *The pre-*$\lfloor n, -\infty \rfloor$*-prestable simplicial group* $G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ *given in* (2) *is an* $\lfloor n, -\infty \rfloor$*-stable simplicial group; cf. Definition* 232.

We call $G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ *the* truncation of G at $\lfloor n, -\infty \rfloor$.

Often, we also write $G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} := G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$.

Proof. Ad (1). Suppose given $j \in \mathbb{Z}_{\geqslant 0}$.

We show that $\bar{\mathrm{d}}_j$ is a well-defined group morphism. We have to show that $(G\mathrm{NB}_n)\mathrm{d}_j^{G,n} \overset{!}{=} 1$.

Suppose given $x \in G\mathrm{NB}_n$. We choose some $y \in G\mathrm{N}_{n+1}$ such that $x = y\mathrm{d}_0^{G,n+1}$; cf. Definition 198.(4).

We have

$$x\mathrm{d}_j^{G,n} = y\mathrm{d}_0^{G,n+1}\mathrm{d}_j^{G,n} = y\mathrm{d}_{j+1}^{G,n+1}\mathrm{d}_0^{G,n} = 1\,.$$

This shows $(G\mathrm{NB}_n)\mathrm{d}_j^{G,n} = 1$.

Therefore, we have the group morphism

$$\bar{\mathrm{d}}_j \colon G_n/G\mathrm{NB}_n \to G_{n-1}\,,\ gG\mathrm{NB}_n \mapsto g\mathrm{d}_j^{G,n}\,.$$

Ad (2). We use Lemma 254.(3) to show that $G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$ is a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group.

To that end, we have to verify the properties (i, ii, iii) from Lemma 254.

Ad (i). Suppose given $k \in \mathbb{Z}_{\leqslant n-1}$. Suppose given $i,\, j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

Case $k \leqslant n-2$. We have

$$\begin{aligned}
\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} &= \mathrm{d}_j^{G,k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G,k} \\
&\overset{185}{=} \mathrm{d}_i^{G,k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G,k} \\
&= \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}\,.
\end{aligned}$$

Case $k = n-1$. For $g \in G_n$, we have

$$\begin{aligned}
(gG\mathrm{NB}_n)(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}) &= (gG\mathrm{NB}_n)(\bar{\mathrm{d}}_j \mathbin{\blacktriangle} \mathrm{d}_i^{G,n-1}) \\
&= (g\mathrm{d}_j^{G,n})\mathrm{d}_i^{G,n-1} \\
&\overset{185}{=} (g\mathrm{d}_i^{G,n})\mathrm{d}_{j-1}^{G,n-1} \\
&= (gG\mathrm{NB}_n)(\bar{\mathrm{d}}_i \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G,n-1}) \\
&= (gG\mathrm{NB}_n)(\mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k})\,.
\end{aligned}$$

This shows $\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} = \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} \mathbin{\blacktriangle} \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}$.

Ad (ii). Suppose given $k \in \mathbb{Z}_{\leqslant n-1}$. Suppose given $i,\, j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

Case $k \leqslant n-2$. We have

$$\begin{aligned}
\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} &= \mathrm{s}_i^{G,k-1} \mathbin{\blacktriangle} \mathrm{s}_j^{G,k} \\
&\overset{185}{=} \mathrm{s}_{j-1}^{G,k-1} \mathbin{\blacktriangle} \mathrm{s}_i^{G,k} \\
&= \mathrm{s}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \mathbin{\blacktriangle} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}\,.
\end{aligned}$$

Case $k = n-1$. For $g \in G_{n-1}$, we have

$$\begin{aligned} g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \blacktriangle \mathrm{s}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}) &= g(\mathrm{s}_i^{G,n-2} \blacktriangle \bar{\mathrm{s}}_j) \\ &= (g\mathrm{s}_i^{G,n-2}\mathrm{s}_j^{G,n-1})G\mathrm{NB}_n \\ &\overset{185}{=} (g\mathrm{s}_{j-1}^{G,n-2}\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n \\ &= g(\mathrm{s}_{j-1}^{G,n-2} \blacktriangle \bar{\mathrm{s}}_i) \\ &= g(\mathrm{s}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}) \,. \end{aligned}$$

This shows $\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \blacktriangle \mathrm{s}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} = \mathrm{s}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}$.

Ad (iii). Suppose given $k \in \mathbb{Z}_{\leqslant n-1}$. Suppose given i, $j \in \mathbb{Z}_{\geqslant 0}$.

Case $k \leqslant n-2$.

If $j < i$, then we have

$$\begin{aligned} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} &= \mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} \\ &\overset{185}{=} \mathrm{d}_j^{G,k} \blacktriangle \mathrm{s}_{i-1}^{G,k-1} \\ &= \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{s}_{i-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \,. \end{aligned}$$

If $j \in \{i, i+1\}$, then we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} = \mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} \overset{185}{=} \mathrm{id}_{G_k} = \mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k} \,.$$

If $j > i+1$, then we have

$$\begin{aligned} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} &= \mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} \\ &\overset{185}{=} \mathrm{d}_{j-1}^{G,k} \blacktriangle \mathrm{s}_i^{G,k-1} \\ &= \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} \,. \end{aligned}$$

Case $k = n-1$.

Suppose given $g \in G_{n-1}$. We have

$$g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1}) = g(\bar{\mathrm{s}}_i \blacktriangle \bar{\mathrm{d}}_j) = \big((g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\big)\bar{\mathrm{d}}_j = g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n} \,.$$

If $j < i$, then we have

$$\begin{aligned} g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1}) &= g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n} \\ &\overset{185}{=} g\mathrm{d}_j^{G,n-1}\mathrm{s}_{i-1}^{G,n-2} \\ &= g(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{s}_{i-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1}) \,. \end{aligned}$$

If $j \in \{i, i+1\}$, then we have

$$g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1}) = g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n} \overset{185}{=} g\,\mathrm{id}_{G_k} = g\,\mathrm{id}_{G\,\mathrm{Trunc}_k^{\lfloor n,-\infty\rfloor}} \,.$$

If $j > i+1$, then we have

$$\begin{aligned} g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1}) &= g\mathrm{s}_i^{G,n-1}\mathrm{d}_j^{G,n} \\ &\overset{185}{=} g\mathrm{d}_{j-1}^{G,n-1}\mathrm{s}_i^{G,n-2} \\ &= g(\mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1}) \,. \end{aligned}$$

Altogether, we have

$$\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k+1} = \begin{cases} \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{s}_{i-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} & \text{if } j < i \\ \mathrm{id}_{G\,\mathrm{Trunc}_k^{\lfloor n,-\infty\rfloor}} & \text{if } j \in \{i, i+1\} \\ \mathrm{d}_{j-1}^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k-1} & \text{if } j > i+1 \,. \end{cases}$$

Ad (3). *Suppose given $k \in \mathbb{Z}_{\leqslant n}$. Suppose given $x \in (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k$. We have to show that x is finite; cf. Definition* 232.

Case $k \leqslant n-1$. We have $x \in (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k = G_k$. Then there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ with

$$1 = x\mathrm{d}_j^{G,k} = x\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}$$

for $j \in \mathbb{Z}_{\geqslant \ell}$.

Case $k = n$. Then we have $x = gG\mathrm{NB}_n$ for some $g \in G_n$.

Since g is finite, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ with $g\mathrm{d}_j^{G,n} = 1$ for $j \in \mathbb{Z}_{\geqslant \ell}$.

So, for $j \in \mathbb{Z}_{\geqslant \ell}$, we have

$$x\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} = x\bar{\mathrm{d}}_j = (gG\mathrm{NB}_n)\bar{\mathrm{d}}_j = (g\mathrm{d}_j^{G,n})G\mathrm{NB}_n = 1 \,.$$

This shows that x is finite.

Suppose given cofinite subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

We have to show that $\big[\bigcap_{i\in I} \ker \bar{\mathrm{d}}_i \,,\, \bigcap_{j\in J} \ker \bar{\mathrm{d}}_j\big] \overset{!}{=} 1$ *; cf.* (2) *and Definition* 232.

Suppose given $xG\mathrm{NB}_n \in \bigcap_{i\in I} \ker \bar{\mathrm{d}}_i$ with $x \in G_n$. Then we have $x\mathrm{d}_i^{G,n} = 1$ for $i \in I$; cf. (1).

Suppose given $yG\mathrm{NB}_n \in \bigcap_{j\in J} \ker \bar{\mathrm{d}}_j$ with $y \in G_n$. Then we have $y\mathrm{d}_j^{G,n} = 1$ for $j \in J$; cf. (1).

We use Lemma 203 to show that $[x, y] \overset{!}{\in} G\mathrm{NB}_n$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Since $I \cup J = \mathbb{Z}_{\geqslant 0}$, we have that $x\mathrm{d}_k^{G,n} = 1$ or $y\mathrm{d}_k^{G,n} = 1$.

Case $0 \in I \cap J$. Since $I \cup J = \mathbb{Z}_{\geqslant 0}$, we may assume that $1 \in I$.

Let $\ell := 0$. Then we have $x\mathrm{d}_{\ell+1}^{G,n} = x\mathrm{d}_1^{G,n} = 1$ and $y\mathrm{d}_\ell^{G,n} = y\mathrm{d}_0^{G,n} = 1$.

Case $0 \notin I \cap J$. Since $I \cup J = \mathbb{Z}_{\geqslant 0}$, we may assume that $0 \in J$. Then $0 \notin I$.

Using $I \neq \emptyset$, we let $\ell := (\min I) - 1 \geqslant 0$. Then we have $\ell + 1 = \min I \in I$ and $\ell \in J$.

So we have $x\mathrm{d}_{\ell+1}^{G,n} = 1$ and $y\mathrm{d}_\ell^{G,n} = 1$.

Therefore, by Lemma 203, we have $[x, y] \in G\mathrm{NB}_n$.

This shows $\big[xG\mathrm{NB}_n \,, yG\mathrm{NB}_n\big] = 1$.

Therefore, we have $\big[\bigcap_{i\in I} \ker \bar{\mathrm{d}}_i \,,\, \bigcap_{j\in J} \ker \bar{\mathrm{d}}_j\big] = 1$. □

Lemma 291 (The $\lfloor n, -\infty \rfloor$-stable simplicial group morphism $\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$)

Suppose given stable simplicial groups G and H; cf. Definition 205.

We consider the $\lfloor n, -\infty \rfloor$-stable simplicial groups $G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$ and $H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$; cf. Lemma 290.(3).

Suppose given a stable simplicial group morphism $\varphi\colon G \to H$.

For $k \in \mathbb{Z}_{\leqslant n}$, we consider the group morphism $\varphi_k\colon G_k \to H_k$.

(1) *We have the group morphism*

$$(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n\colon G_n/G\mathrm{NB}_n \to H_n/H\mathrm{NB}_n\,,\ xG\mathrm{NB}_n \mapsto (x\varphi_n)H\mathrm{NB}_n\,.$$

Moreover, for $k \in \mathbb{Z}_{\leqslant n-1}$, we let

$$(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k := \varphi_k\colon G_k \to H_k\,.$$

(2) *We have the* $\lfloor n,-\infty\rfloor$*-stable simplicial group morphism*

$$\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} = \big((G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k \xrightarrow{(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k} (H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k\big)_{k\in\mathbb{Z}_{\leqslant n}} : G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,;$$

cf. (1).

Proof. Ad (1). *We show that* $(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n$ *is a well-defined group morphism.*

It suffices to show that $(G\mathrm{NB}_n)\varphi_n \overset{!}{\subseteq} H\mathrm{NB}_n$.

Suppose given $x \in G\mathrm{NB}_n$. We choose some $y \in G\mathrm{N}_{n+1}$ such that $x = y\mathrm{d}_0^{G,n+1}$.

Let $z := y\varphi_{n+1} \in H_{n+1}$.

For $k \in \mathbb{Z}_{\geqslant 1}$, we have

$$z\mathrm{d}_k^{H,n+1} = y\varphi_{n+1}\mathrm{d}_k^{H,n+1} \overset{184}{=} y\mathrm{d}_k^{G,n+1}\varphi_n = 1\,.$$

This shows $z \in H\mathrm{N}_{n+1}$.

Moreover, we have

$$z\mathrm{d}_0^{H,n+1} = y\varphi_{n+1}\mathrm{d}_0^{H,n+1} \overset{184}{=} y\mathrm{d}_0^{G,n+1}\varphi_n = x\varphi_n\,.$$

This shows $x\varphi_n = z\mathrm{d}_0^{H,n+1} \in (H\mathrm{N}_{n+1})\mathrm{d}_0^{H,n+1} = H\mathrm{NB}_n$.

So we have $(G\mathrm{NB}_n)\varphi_n \subseteq H\mathrm{NB}_n$.

Ad (2). We use Lemma 255 to show that $\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$ is an $\lfloor n,-\infty\rfloor$-simplicial group morphism.

Suppose given $k \in \mathbb{Z}_{\leqslant n}$ and $j \in \mathbb{Z}_{\geqslant 0}$.

Case $k \leqslant n-1$. We have

$$\begin{aligned}(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k \blacktriangleleft \mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} &= \varphi_k \blacktriangleleft \mathrm{d}_j^{H,k} \overset{184}{=} \mathrm{d}_j^{G,k} \blacktriangleleft \varphi_{k-1}\\ &= \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} \blacktriangleleft (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{k-1}\,.\end{aligned}$$

Case $k = n$. For $g \in G_n$, we have

$$\begin{aligned}&(gG\mathrm{NB}_n)\big((\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k \blacktriangleleft \mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}\big)\\ =\;&\big((g\varphi_n)H\mathrm{NB}_n\big)\mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n}\\ \overset{290.(2,1)}{=}\;& g\varphi_n\mathrm{d}_j^{H,n}\end{aligned}$$

$$\begin{aligned}
&\overset{184}{=} && g\mathrm{d}_j^{G,n}\varphi_{n-1}\\
&= && (g\mathrm{d}_j^{G,n})(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{n-1}\\
&\overset{290.(2,1)}{=} && (gG\mathrm{NB}_n)\big(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}{}_{\blacktriangle}(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{k-1}\big)\,.
\end{aligned}$$

This shows $(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k{}_{\blacktriangle}\,\mathrm{d}_j^{H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} = \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}{}_{\blacktriangle}(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{k-1}$.

Suppose given $k\in\mathbb{Z}_{\leqslant n-1}$ and $i\in\mathbb{Z}_{\geqslant 0}$.

Case $k\leqslant n-2$. We have

$(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k{}_{\blacktriangle}\,\mathrm{s}_i^{H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} = \varphi_k{}_{\blacktriangle}\,\mathrm{s}_i^{H,k} \overset{184}{=} \mathrm{s}_i^{G,k}{}_{\blacktriangle}\,\varphi_{k+1} = \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}{}_{\blacktriangle}(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{k+1}$.

Case $k=n-1$. For $g\in G_{n-1}$, we have

$$\begin{aligned}
g\big((\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k{}_{\blacktriangle}\,\mathrm{s}_i^{\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}\big) &= (g\varphi_{n-1})\mathrm{s}_i^{H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n-1}\\
&\overset{290.(1)}{=} (g\varphi_{n-1}\mathrm{s}_i^{H,n-1})H\mathrm{NB}_n\\
&\overset{184}{=} (g\mathrm{s}_i^{G,n-1}\varphi_n)H\mathrm{NB}_n\\
&\overset{(1)}{=} \big((g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\big)(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n\\
&\overset{290.(1)}{=} g\big(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}{}_{\blacktriangle}(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{k+1}\big)\,.
\end{aligned}$$

This shows $(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k{}_{\blacktriangle}\,\mathrm{s}_i^{\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k} = \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},k}{}_{\blacktriangle}(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{k+1}$.

So, by Lemma 255, $\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\colon G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\to H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$ is an $\lfloor n,-\infty\rfloor$-stable simplicial group morphism. □

Lemma 292 (The truncation functor $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$)

We have the truncation functor

$$\begin{aligned}
\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}\colon\ \mathcal{StSimpGrp} &\to \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\\
\big(G\xrightarrow{\varphi}H\big) &\mapsto \big(G\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}\xrightarrow{\varphi\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}} H\,\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}\big)\,;
\end{aligned}$$

cf. Lemmas 290.(3), 291.(2).

We often write $\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} := \mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}$ *provided that there is no risk of confusion with the functor* $\mathrm{Trunc}^{\lfloor n,0\rfloor}\colon \mathcal{SimpGrp}\to\lfloor n,0\rfloor\text{-}\mathcal{SimpGrp}$*; cf. Lemma* 120.

Proof. Suppose given $G\in\mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

We show that $(\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\overset{!}{=}\mathrm{id}_{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}}$.

Suppose given $k \in \mathbb{Z}_{\leqslant n}$.

Case $k \leqslant n-1$. We have

$$\big((\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\big)_k \overset{291.(1)}{=} \mathrm{id}_{G_k} \overset{290.(2)}{=} \mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k}\,.$$

Case $k = n$. For $g \in G_n$, we have

$$\begin{aligned}(gG\mathrm{NB}_n)\big((\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\big)_k &\overset{291.(1)}{=} (g\,\mathrm{id}_{G_n})G\mathrm{NB}_n = gG\mathrm{NB}_n = (gG\mathrm{NB}_n)\,\mathrm{id}_{G/G\mathrm{NB}_n}\\ &\overset{290.(2)}{=} (gG\mathrm{NB}_n)\,\mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k}\,.\end{aligned}$$

This shows $\big((\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\big)_k = \mathrm{id}_{(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k}$.

So we have $(\mathrm{id}_G)\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} = \mathrm{id}_{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}}$.

Suppose given $F \xrightarrow{\varphi} G \xrightarrow{\tilde{\varphi}} H$ in $\mathcal{StSimpGrp}$.

We show that $(\varphi \mathbin{\triangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \overset{!}{=} \varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \mathbin{\triangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$.

Suppose given $k \in \mathbb{Z}_{\leqslant n}$.

Let $\hat{\varphi} := \varphi \mathbin{\triangle} \tilde{\varphi} \colon F \to H$. Then note that we have $\hat{\varphi}_k = \varphi_k \mathbin{\triangle} \tilde{\varphi}_k \colon F_k \to H_k$.

Case $k \leqslant n-1$. We have

$$\begin{aligned}\big((\varphi \mathbin{\triangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\big)_k &= (\hat{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k\\ &\overset{291.(1)}{=} \hat{\varphi}_k\\ &= \varphi_k \mathbin{\triangle} \tilde{\varphi}_k\\ &\overset{291.(1)}{=} (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k \mathbin{\triangle} (\tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k\\ &= (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \mathbin{\triangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k\,.\end{aligned}$$

Case $k = n$. For $x \in F_n$, we have

$$\begin{aligned}(xF\mathrm{NB}_n)\big((\varphi \mathbin{\triangle} \tilde{\varphi})\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\big)_k &= (xF\mathrm{NB}_n)(\hat{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n\\ &\overset{291.(1)}{=} (x\hat{\varphi}_n)H\mathrm{NB}_n\\ &= (x\varphi_n\tilde{\varphi}_n)H\mathrm{NB}_n\\ &\overset{291.(1)}{=} \big((x\varphi_n)G\mathrm{NB}_n\big)(\tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n\\ &\overset{291.(1)}{=} (xF\mathrm{NB}_n)\big((\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n \mathbin{\triangle} (\tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n\big)\\ &= (xF\mathrm{NB}_n)(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \mathbin{\triangle} \tilde{\varphi}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k\,.\end{aligned}$$

This shows $\big((\varphi \mathbin{\triangle} \tilde{\varphi})\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}\big)_k = (\varphi \operatorname{Trunc}^{\lfloor n,-\infty\rfloor} \mathbin{\triangle} \tilde{\varphi} \operatorname{Trunc}^{\lfloor n,-\infty\rfloor})_k$.

So we have $(\varphi \mathbin{\triangle} \tilde{\varphi})\operatorname{Trunc}^{\lfloor n,-\infty\rfloor} = \varphi \operatorname{Trunc}^{\lfloor n,-\infty\rfloor} \mathbin{\triangle} \tilde{\varphi} \operatorname{Trunc}^{\lfloor n,-\infty\rfloor}$. □

Lemma 293 (The Moore complex of $G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}$)
For a stable simplicial group G, we consider its Moore complex

$$G\mathrm{N} = \big(\ldots \xrightarrow{\partial_3^G} G\mathrm{N}_2 \xrightarrow{\partial_2^G} G\mathrm{N}_1 \xrightarrow{\partial_1^G} G\mathrm{N}_0 \xrightarrow{\partial_0^G} G\mathrm{N}_{-1} \xrightarrow{!} \ldots\big);$$

cf. Lemma 200.(2).

(1) *Suppose given a stable simplicial group G; cf. Definition* 205.
We consider the $\lfloor n, -\infty \rfloor$-stable simplicial group $G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}$; cf. Lemma 290.
Recall that

$$(G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor})_k = \begin{cases} G_n/G\mathrm{NB}_n & \text{for } k = n \\ G_k & \text{for } k \in \mathbb{Z}_{\leqslant n-1}\,. \end{cases}$$

We consider its Moore complex $(G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}$; cf. Lemma 258.(2).
We have

$$(G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k = \begin{cases} G\mathrm{N}_n/G\mathrm{NB}_n & \text{for } k = n \\ G\mathrm{N}_k & \text{for } k \in \mathbb{Z}_{\leqslant n-1}\,. \end{cases}$$

For $k \in \mathbb{Z}$, we often write $\bar{\partial}_k := \partial_k^{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}}$.
Then we have

$$\bar{\partial}_n \colon G\mathrm{N}_n/G\mathrm{NB}_n \to G\mathrm{N}_{n-1}\,,\ gG\mathrm{NB}_n \mapsto (gG\mathrm{NB}_n)\bar{\partial}_n = (gG\mathrm{NB}_n)\mathrm{d}_0^{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor},n} = g\mathrm{d}_0^{G,n}\,.$$

Note that $\operatorname{im}\bar{\partial}_n = (G\mathrm{N}_n)\mathrm{d}_0^G = \operatorname{im}\partial_n^G$.
For $k \in \mathbb{Z}_{\leqslant n-1}$, we have

$$\bar{\partial}_k = \mathrm{d}_0^{G,k}\big|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}} \colon G\mathrm{N}_k \to G\mathrm{N}_{k-1}\,,\ g \mapsto g\bar{\partial}_k = g\mathrm{d}_0^{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor},k} = g\mathrm{d}_0^{G,k}\,.$$

Note that $\bar{\partial}_k = \partial_k^G$ for $k \in \mathbb{Z}_{\leqslant n-1}$.
So, we have $\operatorname{im}\bar{\partial}_k = \operatorname{im}\partial_k^G$ for $k \in \mathbb{Z}_{\leqslant n}$.
We have

$$(G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N} = \big(\ldots \xrightarrow{!} 1 \xrightarrow{!} G\mathrm{N}_n/G\mathrm{NB}_n \xrightarrow{\bar{\partial}_n} G\mathrm{N}_{n-1} \xrightarrow{\bar{\partial}_{n-1}} \ldots \xrightarrow{\bar{\partial}_1} G\mathrm{N}_0 \xrightarrow{\bar{\partial}_0} G\mathrm{N}_{-1} \xrightarrow{\bar{\partial}_{-1}} \ldots\big)$$

(2) *Suppose given a stable simplicial group G; cf. Definition* 205.
We have the morphism of complexes in groups

$$G\psi \colon G\mathrm{N} \to (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}$$

that is given as follows.
For $k = n$, we have

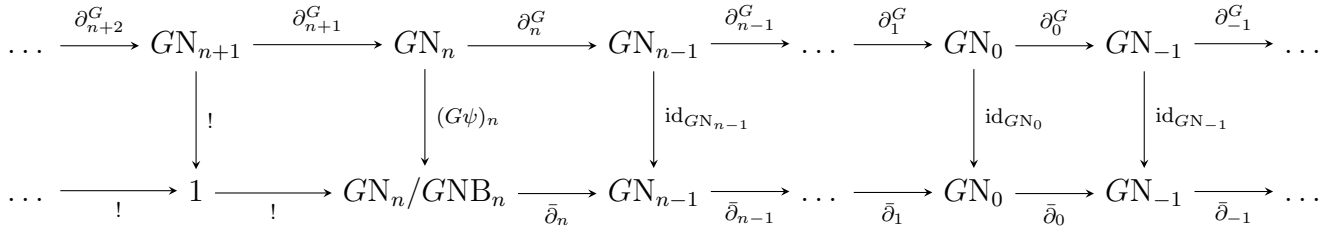
$$(G\psi)_n \colon G\mathrm{N}_n \to (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n\,,\ g \mapsto g G\mathrm{NB}_n\,.$$

For $k \in \mathbb{Z}_{\leqslant n-1}$, we have

$$(G\psi)_k = \mathrm{id}_{G\mathrm{N}_k} \colon G\mathrm{N}_k \to G\mathrm{N}_k\,,\ g \mapsto g\,.$$

Moreover, for $k \in \mathbb{Z}_{\geqslant n+1}$, we have

$$(G\psi)_k = ! : G\mathrm{N}_k \to 1\,,\ g \mapsto 1\,.$$



(3) *We consider the functors*

$$\begin{array}{rrcll}
\mathrm{N}\colon & \mathcal{StSimpGrp} & \to & \mathcal{ComplexGrp}\,; & \text{cf. Lemma 201.(3)} \\
\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\colon & \mathcal{StSimpGrp} & \to & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\,; & \text{cf. Lemma 292} \\
\mathrm{N}\colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \mathcal{ComplexGrp}\,; & \text{cf. Lemma 259.(3)}\ .
\end{array}$$

We have the transformation

$$\psi = \big(G\mathrm{N} \xrightarrow{G\psi} (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}\big)_{G\in\mathrm{Ob}(\mathcal{StSimpGrp})} : \mathrm{N} \to \mathrm{Trunc}^{\lfloor n,-\infty\rfloor} * \mathrm{N};$$

cf. (2).

(4) *Suppose given a stable simplicial group G; cf. Definition* 205.
Suppose that we have $G\mathrm{N}_k = 1$ for $k \in \mathbb{Z}_{\geqslant n+1}$; cf. Definition 198.(2).
In particular, we have $G\mathrm{NB}_n = (G\mathrm{N}_{n+1})\mathrm{d}_0^{G,n+1} = 1$; cf. Definition 198.(4).
Then the morphism of complexes in groups $G\psi$ given in (2) *is an isomorphism.*

$$\begin{array}{ccccccccccccccc}
\dots & \xrightarrow{!} & 1 & \xrightarrow{!} & G\mathrm{N}_n & \xrightarrow{\partial_n^G} & G\mathrm{N}_{n-1} & \xrightarrow{\partial_{n-1}^G} & \dots & \xrightarrow{\partial_1^G} & G\mathrm{N}_0 & \xrightarrow{\partial_0^G} & G\mathrm{N}_{-1} & \xrightarrow{\partial_{-1}^G} & \dots \\
 & & \downarrow{\scriptstyle !} & & \downarrow{\scriptstyle (G\psi)_n} & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_{n-1}}} & & & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_0}} & & \downarrow{\scriptstyle \mathrm{id}_{G\mathrm{N}_{-1}}} & & \\
\dots & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & G\mathrm{N}_n/1 & \xrightarrow{\bar\partial_n} & G\mathrm{N}_{n-1} & \xrightarrow{\bar\partial_{n-1}} & \dots & \xrightarrow{\bar\partial_1} & G\mathrm{N}_0 & \xrightarrow{\bar\partial_0} & G\mathrm{N}_{-1} & \xrightarrow{\bar\partial_{-1}} & \dots
\end{array}$$

(5) *Suppose given a stable simplicial group G; cf. Definition* 205.
We have $\ker \bar\partial_n = G\mathrm{NZ}_n/G\mathrm{NB}_n$ *as subgroups of* $G\mathrm{N}_n/G\mathrm{NB}_n$.

(6) *Suppose given a stable simplicial group G; cf. Definition* 205.
For $k \in \mathbb{Z}_{\leqslant n}$, we have the group isomorphism $(G\psi)\mathrm{H}_k \colon G\mathrm{NH}_k \xrightarrow{\sim} (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{NH}_k$ *; cf. Lemma* 11. *So in particular, we have*

$$(G\psi)\mathrm{H}_k \colon G\pi_k \xrightarrow{\sim} (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\pi_k\,;$$

cf. Lemmas 200.(2), 258.(2).

Proof. Ad (1). It suffices to show that $(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n \overset{!}{=} G\mathrm{N}_n/G\mathrm{NB}_n$ as subgroups of $G_n/G\mathrm{NB}_n$.

Ad $\leqslant$. Suppose given $g \in G_n$ such that

$$gG\mathrm{NB}_n \in (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n = \bigcap_{i\in\mathbb{Z}_{\geqslant 1}} (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_{n,i} \leqslant (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n = G_n/G\mathrm{NB}_n\,.$$

Then, for $j \in \mathbb{Z}_{\geqslant 1}$, we have

$$1 = (gG\mathrm{NB}_n)\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n} \overset{290.(2)}{=} g\mathrm{d}_j^{G,n}\,.$$

So $g \in G\mathrm{N}_n$. This shows $(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n \leqslant G\mathrm{N}_n/G\mathrm{NB}_n$.

Ad $\geqslant$. Suppose given $g \in G\mathrm{N}_n$. Then, for $j \in \mathbb{Z}_{\geqslant 1}$, we have

$$1 = g\mathrm{d}_j^{G,n} \overset{290.(2)}{=} (gG\mathrm{NB}_n)\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n}.$$

So $gG\mathrm{NB}_n \in (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n$. This shows $G\mathrm{N}_n/G\mathrm{NB}_n \leqslant (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n$.
Altogether, we have $(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n = G\mathrm{N}_n/G\mathrm{NB}_n$.

Ad (2). Suppose given $k \in \mathbb{Z}$. We have to show that $\partial_k^G \mathbin{\blacktriangle} (G\psi)_{k-1} \overset{!}{=} (G\psi)_k \mathbin{\blacktriangle} \bar{\partial}_k$.
It suffices to consider $k \in \mathbb{Z}_{\leqslant n+1}$.
Case $k \in \mathbb{Z}_{\leqslant n-1}$. For $g \in G\mathrm{N}_k$, we have

$$g\big(\partial_k^G \mathbin{\blacktriangle} (G\psi)_{k-1}\big) = (g\mathrm{d}_0^{G,k})\,\mathrm{id}_{G\mathrm{N}_{k-1}} = (g\,\mathrm{id}_{G\mathrm{N}_k})\mathrm{d}_0^{G,k} = g\big((G\psi)_k \mathbin{\blacktriangle} \bar{\partial}_k\big).$$

Case $k = n$. For $g \in G\mathrm{N}_n$, we have

$$g\big(\partial_n^G \mathbin{\blacktriangle} (G\psi)_{n-1}\big) = (g\mathrm{d}_0^{G,n})\,\mathrm{id}_{G\mathrm{N}_{n-1}} = g\mathrm{d}_0^{G,n} \overset{(1)}{=} (gG\mathrm{NB}_n)\bar{\partial}_n = g\big((G\psi)_n \mathbin{\blacktriangle} \bar{\partial}_n\big).$$

Case $k = n+1$. Suppose given $g \in G\mathrm{N}_{n+1}$. Note that $g\mathrm{d}_0^{G,n+1} \in G\mathrm{NB}_n$. We have

$$g\big(\partial_{n+1}^G \mathbin{\blacktriangle} (G\psi)_n\big) = (g\mathrm{d}_0^{G,n+1})G\mathrm{NB}_n = 1G\mathrm{NB}_n = \big(g(G\psi)_{n+1}\big)\,! = g\big((G\psi)_{n+1} \mathbin{\blacktriangle} \bar{\partial}_{n+1}\big).$$

Ad (3). Suppose given a stable simplicial group morphism $\varphi\colon G \to H$.
We have to show that $G\psi \mathbin{\blacktriangle} (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N} \overset{!}{=} \varphi\mathrm{N} \mathbin{\blacktriangle} H\psi$.

$$\begin{array}{ccc} G\mathrm{N} & \xrightarrow{G\psi} & (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N} \\ {\scriptstyle \varphi\mathrm{N}}\downarrow & & \downarrow{\scriptstyle (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}} \\ H\mathrm{N} & \xrightarrow[H\psi]{} & (H\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N} \end{array}$$

Suppose given $k \in \mathbb{Z}$. We have to show that $(G\psi)_k \mathbin{\blacktriangle} (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k \overset{!}{=} \varphi\mathrm{N}_k \mathbin{\blacktriangle} (H\psi)_k$.
Case $k \in \mathbb{Z}_{\leqslant n-1}$. For $g \in G\mathrm{N}_k$, we have

$$\begin{aligned} g\big((G\psi)_k \mathbin{\blacktriangle} (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k\big) &= (g\,\mathrm{id}_{G\mathrm{N}_k})\big((\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k\big) = g\varphi_k \\ &= \big(g(\varphi\mathrm{N}_k)\big)\,\mathrm{id}_{H\mathrm{N}_k} = g\big(\varphi\mathrm{N}_k \mathbin{\blacktriangle} (H\psi)_k\big). \end{aligned}$$

Case $k = n$. For $g \in G\mathrm{N}_n$, we have

$$g\big((G\psi)_n \blacktriangleleft (\varphi \operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n\big) = (gG\mathrm{NB}_n)\big((\varphi \operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n\big) = (g\varphi_n)H\mathrm{NB}_n = (g\varphi_n)(H\psi)_n$$
$$= g\big(\varphi\mathrm{N}_n \blacktriangleleft (H\psi)_n\big)\,.$$

Case $k \in \mathbb{Z}_{\geqslant n+1}$. We have

$$(G\psi)_k \blacktriangleleft (\varphi \operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k = ! \blacktriangleleft (\varphi \operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k = ! = \varphi\mathrm{N}_k \blacktriangleleft ! = \varphi\mathrm{N}_k \blacktriangleleft (H\psi)_k\,.$$

Ad (4). For $k \in \mathbb{Z}_{\leqslant n-1}$, note that we have

$$(G\psi)_k = \mathrm{id}_{G\mathrm{N}_k} \colon G\mathrm{N}_k \to G\mathrm{N}_k\,.$$

For $k \in \mathbb{Z}_{\geqslant n+1}$, note that we have

$$(G\psi)_k = ! : 1 \to 1\,.$$

Moreover, note that

$$(G\psi)_n \colon G\mathrm{N}_n \to G\mathrm{N}_n/G\mathrm{NB}_n\,,\ g \mapsto gG\mathrm{NB}_n$$

is a group isomorphism since we have $G\mathrm{NB}_n = (G\mathrm{N}_{n+1})\mathrm{d}_0^{G,n+1} = 1\mathrm{d}_0^{G\mathrm{N}_{n+1}} = 1$.

This shows that $G\psi$ is an isomorphism of complexes in groups.

Ad (5). Ad $\leqslant$. Suppose given $x \in G\mathrm{N}_n$ such that $xG\mathrm{NB}_n \in \ker \bar{\partial}_n$.

We have

$$1 = (xG\mathrm{NB}_n)\bar{\partial}_n \overset{(1)}{=} x\mathrm{d}_0^{G,n} = x\partial_n^G\,.$$

This shows that $x \in G\mathrm{NZ}_n$.

So we have $\ker \bar{\partial}_n \leqslant G\mathrm{NZ}_n/G\mathrm{NB}_n$.

Ad $\geqslant$. Suppose given $y \in G\mathrm{NZ}_n$. In particular, we have $y \in G\mathrm{N}_n$.

We consider $yG\mathrm{NB}_n \in G\mathrm{NZ}_n/G\mathrm{NB}_n$. By (1), we have $yG\mathrm{NB}_n \in (G \operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n$.

Moreover, we have

$$(yG\mathrm{NB}_n)\bar{\partial}_n \overset{(1)}{=} y\mathrm{d}_0^{G,n} = y\partial_n^G = 1\,.$$

This shows that $yG\mathrm{NB}_n \in \ker \bar{\partial}_n$.

So we have $G\mathrm{NZ}_n/G\mathrm{NB}_n \leqslant \ker \bar{\partial}_n$.

Altogether, this shows that $\ker \bar{\partial}_n = G\mathrm{NZ}_n/G\mathrm{NB}_n$.

Ad (6). Suppose given $k \in \mathbb{Z}_{\leqslant n}$.

Case $k \in \mathbb{Z}_{\leqslant n-1}$. We have

$$\begin{array}{rrcccl} (G\psi)\mathrm{H}_k\colon & \ker \partial^G_k / \operatorname{im} \partial^G_{k+1} & \to & \ker \bar\partial_k / \operatorname{im} \bar\partial_{k+1} & \overset{(1)}{=} & \ker \partial^G_k / \ker \partial^G_{k+1} \\ & g \operatorname{im} \partial^G_{k+1} & \mapsto & \big(g(G\psi)_k\big) \operatorname{im} \bar\partial_{k+1} & \overset{(2)}{=} & g \operatorname{im} \bar\partial_{k+1} \overset{(1)}{=} g \operatorname{im} \partial^G_{k+1}\,. \end{array}$$

So $(G\psi)\mathrm{H}_k$ is an isomorphism for $k \in \mathbb{Z}_{\leqslant n-1}$.

Case $k = n$. We have

$$(G\psi)\mathrm{H}_n\colon \ker \partial^G_n / \operatorname{im} \partial^G_{n+1} \to \ker \bar\partial_n / \operatorname{im} \bar\partial_{n+1}\,,\ g \operatorname{im} \partial^G_{n+1} \mapsto \big(g(G\psi)_n\big) \operatorname{im} \bar\partial_{n+1} = (g G\mathrm{NB}_n) \operatorname{im} \bar\partial_{n+1}\,.$$

We show that $(G\psi)\mathrm{H}_n$ *is injective.*

Suppose given $g \in \ker \partial^G_n$ such that $(g \operatorname{im} \partial^G_n)\big((G\psi)\mathrm{H}_n\big) = 1$.

I.e. we have $(gG\mathrm{NB}_n) \operatorname{im} \bar\partial_{n+1} = 1$.

Note that we have $\bar\partial_{n+1} = ! : 1 \to G\mathrm{N}_n/G\mathrm{NB}_n$. So we have $gG\mathrm{NB}_n = 1$, i.e. we have

$$g \in G\mathrm{NB}_n = (G\mathrm{N}_{n+1})\mathrm{d}_0^{G,n+1} = \operatorname{im} \partial^G_n\,.$$

This shows that $g \operatorname{im} \partial^G_n = 1 \operatorname{im} \partial^G_n$.

So $(G\psi)\mathrm{H}_n$ is injective.

We show that $(G\psi)\mathrm{H}_n$ *is surjective.*

Suppose given $g \in G\mathrm{N}_n$ such that $gG\mathrm{NB}_n \in \ker \bar\partial_n \overset{(5)}{=} G\mathrm{NZ}_n/G\mathrm{NB}_n$.

Then we have $g \in G\mathrm{NZ}_n = \ker \partial^G_n$.

We consider the element $(gG\mathrm{NB}_n) \operatorname{im} \bar\partial_{n+1} \in (G \operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{H}_n$.

We have

$$g \operatorname{im} \partial^G_{n+1} \in G\mathrm{NZ}_n/G\mathrm{NB}_n = G\mathrm{NH}_n$$

and

$$(g \operatorname{im} \partial^G_{n+1})\big((G\psi)\mathrm{H}_n\big) = \big(g(G\psi)_n\big) \operatorname{im} \bar\partial_{n+1} = \big(gG\mathrm{NB}_n\big) \operatorname{im} \bar\partial_{n+1}\,.$$

So $(G\psi)\mathrm{H}_n$ is surjective.

This shows that $(G\psi)\mathrm{H}_n$ is a group isomorphism. □

14.2 The Conduché functor $\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor} : \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \to \mathcal{StSimpGrp}$

Let $n \in \mathbb{Z}$.

Remark 294 *Suppose given an $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group G; cf. Definition 275.*

Suppose given $g \in G_n$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. We have $g\,\mathrm{s}_\ell^{G,n} \in G\mathrm{D}_{n+1}$; cf. Lemma 287.

Proof. Recall from Lemma 284.(1) that we have

$$G_n = \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} (G_{n-1;\,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i^{G,n-1} \Big) \cdot G\mathrm{N}_n \,;$$

cf. §1.1.2 item 9.4. By Lemma 284.(2), we may write

$$g = \Big(\prod_{i \in \lfloor k, 0 \rfloor} x_{i+1}\, \mathrm{s}_i^{G,n-1} \Big) \cdot \tilde{x} \,,$$

where $k \in \mathbb{Z}_{\geqslant -1}$, where $x_j \in G_{n-1;\,\mathbb{Z}_{\geqslant j}}$ for $j \in [1, k+1]$ and where $\tilde{x} \in G\mathrm{N}_n$.

We have to show that

$$g\mathrm{s}_\ell^{G,n} \overset{!}{\in} G\mathrm{D}_{n+1} \overset{287}{=} \prod_{i \in \rfloor \infty, 0 \rfloor} (G_{n;\,\mathbb{Z}_{\geqslant i+1}})\mathrm{s}_i^{G,n} \,.$$

We have

$$g\mathrm{s}_\ell^{G,n} = \Big(\prod_{i \in \lfloor k, 0 \rfloor} x_{i+1}\, \mathrm{s}_i^{G,n-1} \Big)\mathrm{s}_\ell^{G,n} \cdot \tilde{x}\mathrm{s}_\ell^{G,n} = \Big(\prod_{i \in \lfloor k, 0 \rfloor} (x_{i+1}\, \mathrm{s}_i^{G,n-1})\mathrm{s}_\ell^{G,n} \Big) \cdot \tilde{x}\mathrm{s}_\ell^{G,n} \,.$$

We have $\tilde{x} \in G\mathrm{N}_n \leqslant G_{n;\,\mathbb{Z}_{\geqslant \ell+1}}$. So $\tilde{x}\mathrm{s}_\ell^{G,n} \in (G_{n;\,\mathbb{Z}_{\geqslant \ell+1}})\mathrm{s}_\ell^{G,n}$.

Let $i \in [0, k]$.

Case $i \in [\ell, k]$. Then $\ell + 2 \leqslant i + 2$. We have

$$(x_{i+1}\, \mathrm{s}_i^{G,n-1})\mathrm{s}_\ell^{G,n} = (x_{i+1}\, \mathrm{s}_\ell^{G,n-1})\mathrm{s}_{i+1}^{G,n} \,.$$

Suppose given $j \in \mathbb{Z}_{\geqslant i+2}$. Then $\ell + 1 < j$ and $i + 1 \leqslant j - 1$.

So we have

$$(x_{i+1}\,\mathrm{s}_\ell^{G,n-1})\mathrm{d}_j^{G,n} = (x_{i+1}\,\mathrm{d}_{j-1}^{G,n-1})\mathrm{s}_\ell^{G,n} = 1\mathrm{s}_\ell^{G,n} = 1\,.$$

This shows that $x_{i+1}\,\mathrm{s}_\ell^{G,n-1} \in G_{n;\mathbb{Z}_{\geqslant i+2}}$.

So

$$(x_{i+1}\,\mathrm{s}_i^{G,n-1})\mathrm{s}_\ell^{G,n} = (x_{i+1}\,\mathrm{s}_\ell^{G,n-1})\mathrm{s}_{i+1}^{G,n} \in (G_{n;\mathbb{Z}_{\geqslant i+2}})\mathrm{s}_{i+1}^{G,n}\,.$$

Case $i \in [0, \ell-1]$. Then $i+2 \leqslant \ell+1$.

For $j \in \mathbb{Z}_{\geqslant i+2}$, we have

$$(x_{i+1}\,\mathrm{s}_i^{G,n-1})\mathrm{d}_j^{G,n} = (x_{i+1}\,\mathrm{d}_{j-1}^{G,n-1})\mathrm{s}_i^{G,n} = 1\mathrm{s}_i^{G,n} = 1\,.$$

This shows that $x_{i+1}\,\mathrm{s}_i^{G,n-1} \in G_{n;\mathbb{Z}_{\geqslant i+2}}$.

So we have $x_{i+1}\,\mathrm{s}_i^{G,n-1} \in G_{n-1;\mathbb{Z}_{\geqslant i+2}} \leqslant G_{n-1;\mathbb{Z}_{\geqslant \ell+1}}$.

Therefore, we have

$$(x_{i+1}\,\mathrm{s}_i^{G,n-1})\mathrm{s}_\ell^{G,n} \in (G_{n;\mathbb{Z}_{\geqslant \ell+1}})\mathrm{s}_\ell^{G,n}\,.$$

Altogether, we have

$$\begin{aligned} g\mathrm{s}_\ell^{G,n} &= \Big(\prod_{i\in\lfloor k,0\rfloor} (x_{i+1}\,\mathrm{s}_i^{G,n-1})\mathrm{s}_\ell^{G,n})\Big)\cdot \tilde{x}\mathrm{s}_\ell^{G,n} \\ &= \Big(\prod_{i\in\lfloor k,\ell\rfloor} \underbrace{(x_{i+1}\,\mathrm{s}_i^{G,n-1})\mathrm{s}_\ell^{G,n}}_{\in (G_{n;\mathbb{Z}_{\geqslant i+2}})\mathrm{s}_{i+1}^{G,n}}\Big)\cdot \underbrace{\Big(\prod_{i\in\lfloor \ell-1,0\rfloor} \underbrace{(x_{i+1}\,\mathrm{s}_i^{G,n-1})\mathrm{s}_\ell^{G,n}}_{\in (G_{n;\mathbb{Z}_{\geqslant \ell+1}})\mathrm{s}_\ell^{G,n}}\Big)\cdot \underbrace{\tilde{x}\mathrm{s}_\ell^{G,n}}_{\in (G_{n;\mathbb{Z}_{\geqslant \ell+1}})\mathrm{s}_\ell^{G,n}}}_{\in (G_{n;\mathbb{Z}_{\geqslant \ell+1}})\mathrm{s}_\ell^{G,n}} \in G\mathrm{D}_{n+1}\,. \end{aligned}$$

□

Lemma 295 (The stable simplicial subgroup $G^\vee$)

Suppose given an $\lfloor n,-\infty\rfloor$*-coskeletal stable simplicial group* G*; cf. Definition* 275.

We consider the degenerate subgroup $G\mathrm{D}_{n+1} \leqslant G_{n+1}$ *from Lemma* 286*; cf. Lemma* 287.

Suppose given $k \in \mathbb{Z}$.

Let $A_k := \Big\{[n+1]^\pm \xrightarrow{a} [k]^\pm\colon\ a \text{ is injective, monotone and eventually parallel}\Big\}$*; cf. Definition* 172.

We write

$$G_k^\vee := \bigcap_{a\in A_k} G_a^{-1}(G\mathrm{D}_{n+1}) \leqslant G_k\,.$$

If $k \in \mathbb{Z}_{\leqslant n}$*, then we have* $A_k = \emptyset$*. Moreover, we have* $A_{n+1} = \{\mathrm{id}_{[n+1]^\pm}\}$*.*
So we have

$$G^\vee_k = \begin{cases} G_k & \text{if } \ k \in \mathbb{Z}_{\leqslant n} \\ G\mathrm{D}_{n+1} & \text{if } \ k = n+1 \\ \{g \in G_k \colon gG_a \in G\mathrm{D}_{n+1} \text{ for } a \in A_k\} & \text{if } \ k \in \mathbb{Z}_{\geqslant n+2}\,. \end{cases}$$

We have that $G^\vee \leqslant G$ *is a stable simplicial subgroup; cf. Remark* 207*.*

Proof. Suppose given $[\ell]^\pm \xrightarrow{u} [k]^\pm \in \mathrm{Mor}(\Delta^\pm)$. We have to show that $(G^\vee_k)G_u \overset{!}{\leqslant} G^\vee_\ell$.

Suppose given $g \in G^\vee_k$. We have to show that $gG_u \overset{!}{\in} G^\vee_\ell$.

Suppose given $b \in A_\ell$. We have to show that $(gG_u)G_b \overset{!}{\in} G\mathrm{D}_{n+1}$.

We consider the monotone map $\big([n+1]^\pm \xrightarrow{b \blacktriangle u} [k]^\pm\big) \in \mathrm{Mor}(\Delta^\pm)$.
Let $m := k - \big|[k]^\pm \setminus \mathrm{im}(b \blacktriangle u)\big| \in \mathbb{Z}$. By Corollary 177.(1), we have

$$b \blacktriangle u = \bar{v} \blacktriangle \dot{v} \colon [n+1]^\pm \to [k]^\pm,$$

where $\bar{v} \colon [n+1]^\pm \to [m]^\pm$ is surjective, and where $\dot{v} \colon [m]^\pm \to [k]^\pm$ is injective. Note that $m \leqslant n+1$ since $\bar{v}$ is surjective.

Case $m = n+1$. Then we have $\bar{v} = \mathrm{id}_{[n+1]^\pm}$.

So $b \blacktriangle u = \mathrm{id}_{[n+1]^\pm} \blacktriangle \dot{v} = \dot{v} : [n+1]^\pm \to [k]^\pm$ is injective.

This shows that $b \blacktriangle u \in A_k$. Therefore, we have

$$(gG_u)G_b = gG_{b \blacktriangle u} \in (G^\vee_k)G_{b \blacktriangle u} = \Big(\bigcap_{a \in A_k} G_a^{-1}(G\mathrm{D}_{n+1}) \Big) G_{b \blacktriangle u} \leqslant G\mathrm{D}_{n+1}\,.$$

Case $m \leqslant n$. By Lemma 175.(2.1), we have

$$\bar{v} = \prod_{r \in \lfloor n+1-m, 1 \rfloor}^{[n+1]^\pm \quad [m]^\pm} \sigma^{\pm, m+r-1}_{i_r}\,,$$

where $I := \{i_1\,, i_2\,, \ldots, i_{n+1-m}\} = \{i \in [0,n] \colon i\bar{v} = (i+1)\bar{v}\}$, and where $0 \leqslant i_1 < i_2 < \ldots < i_{n+1-m} \leqslant n$.
Note that $\bar{v} = \sigma^{\pm,m}_I \colon [n+1]^\pm \to [m]^\pm$; cf. Definition 180.(2). So we have

$$G_{\bar{v}} = G_{\sigma_I} \overset{187.(2)}{=} \mathrm{s}_I : G_m \to G_{n+1}\,.$$

Note that $I \neq \emptyset$ since $|I| = n+1-m \geqslant 1$. Let $\tilde{I} := I\backslash\{\max I\} \subset I$. Then $I = \tilde{I} \dot\cup \{\max I\}$.
We have

$$G_{\bar{v}} = \mathrm{s}_I = \mathrm{s}_{\tilde{I}\dot\cup\{\max I\}} = \mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_{\max I} \colon G_m \to G_{n+1}\,.$$

Note that we have $G_{\dot{v}} \colon G_k \to G_m$ and $\mathrm{s}_{\tilde{I}} \colon G_m \to G_n$. So $(gG_{\dot{v}})\mathrm{s}_{\tilde{I}} \in G_n$.
We have

$$(gG_u)G_b = gG_{b\blacktriangle u} = gG_{\bar{v}\blacktriangle\dot{v}} = (gG_{\dot{v}})G_{\bar{v}} = (gG_{\dot{v}})(\mathrm{s}_{\tilde{I}} \blacktriangle \mathrm{s}_{\max I}) = \big((gG_{\dot{v}})\mathrm{s}_{\tilde{I}}\big)\mathrm{s}_{\max I} \overset{294}{\in} G\mathrm{D}_{n+1}\,.$$

Altogether, this shows that $(gG_u)G_b \in G\mathrm{D}_{n+1}$.
So we have $(G_k^\vee)G_u \leqslant G_\ell^\vee$.
This shows that $G^\vee \leqslant G$ is a stable simplicial subgroup; cf. Lemma 189.(1), Remark 207. □

Lemma 296 (The stable simplicial group morphism $\varphi^\vee$)
Suppose given $\lfloor n,-\infty\rfloor$-coskeletal stable simplicial groups G, H; cf. Definition 275.
Suppose given a stable simplicial group morphism $\varphi \colon G \to H$.
We consider the stable simplicial subgroups $G^\vee \leqslant G$ and $H^\vee \leqslant H$ from Lemma 295.
Then we have the stable simplicial group morphism

$$\varphi^\vee := \varphi\big|_{G^\vee}^{H^\vee} : G^\vee \to H^\vee\,,$$

where

$$\varphi_k^\vee = \big(\varphi\big|_{G^\vee}^{H^\vee}\big)_k = \varphi_k\big|_{G_k^\vee}^{H_k^\vee} \colon G_k^\vee \to H_k^\vee\,,\ g \mapsto g\varphi_k^\vee = g\varphi_k\,;$$

cf. Remark 190.

Proof. By Definition 205 and Remark 190, it suffices to show that $(G_k^\vee)\varphi_k \overset{!}{\leqslant} H_k^\vee$ for $k \in \mathbb{Z}$.
We show that $(G_{k;\mathbb{Z}_{\geqslant j}})\varphi_k \overset{!}{\leqslant} H_{k;\mathbb{Z}_{\geqslant j}}$ for $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\geqslant 0}$.
Suppose given $k \in \mathbb{Z}$, $j \in \mathbb{Z}_{\geqslant 0}$ and suppose given $x \in G_{k;\mathbb{Z}_{\geqslant j}}$.
For $\ell \in \mathbb{Z}_{\geqslant j}$, we have

$$(x\varphi_k)\mathrm{d}_\ell^{H,n} = (x\mathrm{d}_\ell^{G,n})\varphi_{k-1} = 1\varphi_{k-1} = 1\,.$$

This shows that $x\varphi_k \in H_{n;\mathbb{Z}_{\geqslant j}}$. So $(G_{k;\mathbb{Z}_{\geqslant j}})\varphi_k \leqslant H_{k;\mathbb{Z}_{\geqslant j}}$.
We show that $(G\mathrm{D}_{n+1})\varphi_{n+1} \overset{!}{\leqslant} H\mathrm{D}_{n+1}$.

Suppose given $x \in G\mathrm{D}_{n+1}$. Then we may write

$$x = \prod_{i \in \lfloor \ell-2,0 \rfloor} y_i \, \mathrm{s}_i^{G,n} ,$$

where $\ell \in \mathbb{Z}_{\geqslant 1}$ and where $y_j \in G_{n;\mathbb{Z}_{\geqslant j}}$ for $j \in [1, \ell-1]$; cf. Lemma 286.

For $j \in [1, \ell-1]$, recall that we have $y_j \, \varphi_n \in H_{n;\mathbb{Z}_{\geqslant j}}$.

We have

$$\begin{aligned} x\varphi_{n+1} &= \Big(\prod_{i \in \lfloor \ell-2,0 \rfloor} y_{i+1} \, \mathrm{s}_i^{G,n} \Big) \varphi_{n+1} = \prod_{i \in \lfloor \ell-2,0 \rfloor} y_{i+1} (\mathrm{s}_i^{G,n} \blacktriangle \varphi_{n+1}) = \prod_{i \in \lfloor \ell-2,0 \rfloor} y_{i+1} (\varphi_n \blacktriangle \mathrm{s}_i^{H,n}) \\ &= \prod_{i \in \lfloor \ell-2,0 \rfloor} (y_{i+1} \, \varphi_n) \mathrm{s}_i^{H,n} \overset{286}{\in} H\mathrm{D}_{n+1} . \end{aligned}$$

This shows that $(G\mathrm{D}_{n+1})\varphi_{n+1} \leqslant H\mathrm{D}_{n+1}$.

Suppose given $k \in \mathbb{Z}_{\geqslant 0}$ and $g \in G_k^\vee$.

Suppose given $a \in A_k = \Big\{ [n+1]^\pm \xrightarrow{u} [k]^\pm : \ u \text{ is injective, monotone and eventually parallel} \Big\}$; cf. Lemma 295.

Recall that $g \, G_a \in G\mathrm{D}_{n+1}$. So we have

$$(g\varphi_k) H_a = (g \, G_a) \varphi_{n+1} \in (G\mathrm{D}_{n+1})\varphi_{n+1} \leqslant H\mathrm{D}_{n+1} .$$

This shows that $g\varphi_k \in H_a^{-1}(H\mathrm{D}_{n+1})$.

So $g\varphi_k \in H_k^\vee$. This shows that $(G_k^\vee)\varphi_k \leqslant H_k^\vee$.

Therefore, we get the is a stable simplicial group morphism $\varphi^\vee = \varphi\big|_{G^\vee}^{H^\vee}$. □

Remark 297 (The stable simplicial group $H \, \mathrm{Cond}^{\pm, \lfloor n, -\infty \rfloor}$)

(1) Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group H; cf. Definition 232.

We consider the $\lfloor n, -\infty \rfloor$-coskeletal stable simplicial group $H \, \mathrm{Cosk}^{\lfloor n, -\infty \rfloor, \mathrm{fin}}$; cf. Remarks 272, 277.

We consider the degenerate subgroup $(H \, \mathrm{Cosk}^{\lfloor n, -\infty \rfloor, \mathrm{fin}})\mathrm{D}_{n+1} \leqslant (H \, \mathrm{Cosk}^{\lfloor n, -\infty \rfloor, \mathrm{fin}})_{n+1}$; cf. Lemmas 286, 287.

We consider the simplicial subgroup $(H \, \mathrm{Cosk}^{\lfloor n, -\infty \rfloor, \mathrm{fin}})^\vee \leqslant H \, \mathrm{Cosk}^{\lfloor n, -\infty \rfloor, \mathrm{fin}}$; cf. Lemma 295.

We write

$$H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor} := (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee}\,.$$

So, for $k \in \mathbb{Z}$, we have

$$\begin{aligned}(H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k &= \bigcap_{\substack{a\in\,_{\Delta^\pm}([n+1]^\pm,[k]^\pm)\\ a\text{ is injective}}} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_a^{-1}\big((H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}\big)\\ &= \begin{cases} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k & \text{if } k\in\mathbb{Z}_{\leqslant n}\\ (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1} & \text{if } k=n+1\\ \bigcap\limits_{\substack{a\in\,_{\Delta^\pm}([n+1]^\pm,[k]^\pm)\\ a\text{ is injective}}} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_a^{-1}\big((H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}\big) & \text{if } k\in\mathbb{Z}_{\geqslant n+2}\,. \end{cases}\end{aligned}$$

So, given $[k]^\pm \xrightarrow{u} [\ell]^\pm$ in $\Delta^\pm$, we have

$$\begin{aligned}(H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_u\colon\ (H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_\ell &\to (H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k\\ (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[\ell]^\pm)} &\mapsto (h_{b\blacktriangle u})_{b\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\,.\end{aligned}$$

In particular, for $k \in \mathbb{Z}$ and $i \in \mathbb{Z}_{\geqslant 0}$, we have the face maps and the degeneracy maps

$$\begin{aligned}\mathrm{d}_j^{H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor},k}\colon (H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k &\to (H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_{k-1}\\ (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} &\mapsto \big((h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\mathrm{d}_j^{H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor},k} = (h_{b\blacktriangle\delta_j^{\pm,k}})_{b\in_{\Delta^\pm}([n]^\pm,[k-1]^\pm)}\end{aligned}$$

$$\begin{aligned}\mathrm{s}_i^{H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor},k}\colon (H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k &\to (H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor})_{k+1}\\ (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} &\mapsto \big((h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\mathrm{s}_i^{H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor},k} = (h_{b\blacktriangle\sigma_i^{\pm,k}})_{b\in_{\Delta^\pm}([n]^\pm,[k+1]^\pm)}\,.\end{aligned}$$

We often write $H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} := H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$.

(2) Suppose given $\lfloor n,-\infty\rfloor$-stable simplicial groups G, H; cf. Definition 232.
Suppose given a stable simplicial group morphism $\varphi\colon G \to H$.
From (1), we consider the stable simplicial groups

$$G\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor} = (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee} \quad\text{and}\quad H\,\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor} = (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee}.$$

From Lemma 296, we consider the stable simplicial group morphism

$$(\varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee}\colon (G\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee} \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee}\,.$$

We write

$$\varphi \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor} := (\varphi \operatorname{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee} \colon G \operatorname{Cond}^{\lfloor n,-\infty\rfloor} \to H \operatorname{Cond}^{\lfloor n,-\infty\rfloor} .$$

Suppose given $k \in \mathbb{Z}$ and suppose given $(g_a)_{a \in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} \in (G \operatorname{Cond}^{\lfloor n,-\infty\rfloor})_k$.

Then we have

$$\big((g_a)_a\big)(\varphi \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k \underset{\overset{267}{=}}{=} \big((g_a)_a\big)(\varphi \operatorname{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})^{\vee}_k = \big((g_a)_a\big)(\varphi \operatorname{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k \\ (g_a \varphi_n)_a .$$

So, for $k \in \mathbb{Z}$, we have

$$\begin{array}{rccl} (\varphi \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k \colon & (G \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k & \to & (H \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor})_k \\ & (g_a)_{a \in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} & \mapsto & (g_a \varphi_n)_{a \in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} . \end{array}$$

Often, we also write $\varphi \operatorname{Cond}^{\lfloor n,-\infty\rfloor} := \varphi \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor}$.

Lemma 298 (The Conduché functor $\operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor}$)

We have the Conduché functor

$$\begin{array}{rccl} \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor} \colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \mathcal{StSimpGrp} , \\ & \big(G \xrightarrow{\varphi} H\big) & \mapsto & \big(G \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor} \xrightarrow{\varphi \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor}} H \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor}\big) ; \end{array}$$

cf. Remark 297.(1, 2).

We often write $\operatorname{Cond}^{\lfloor n,-\infty\rfloor} := \operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ *provided that there is no risk of confusion with the functor* $\operatorname{Cond}^{\lfloor n,0\rfloor} \colon \lfloor n,0\rfloor\text{-}\mathcal{SimpGrp} \to \mathcal{SimpGrp}$*; cf. Lemma* 126.

Proof. Suppose given $G \in \operatorname{Ob}\big(\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\big)$.

Let $k \in \mathbb{Z}$. Suppose given $(g_a)_{a \in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} \in (G \operatorname{Cond}^{\lfloor n,-\infty\rfloor})_k$.

We have

$$(g_a)_{a \in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} \big((\mathrm{id}_G) \operatorname{Cond}^{\lfloor n,-\infty\rfloor}\big)_k = (g_a \,\mathrm{id}_{G_n})_{a \in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} = (g_a)_{a \in_{\Delta^{\pm}([n]^{\pm},[k]^{\pm})}} .$$

This shows that $\big((\mathrm{id}_G) \operatorname{Cond}^{\lfloor n,-\infty\rfloor}\big)_k = \mathrm{id}_{(G \operatorname{Cond}^{\lfloor n,-\infty\rfloor})_k}$.

So we have $(\mathrm{id}_G) \operatorname{Cond}^{\lfloor n,-\infty\rfloor} = \mathrm{id}_{G \operatorname{Cond}^{\lfloor n,-\infty\rfloor}}$.

Suppose given $G \xrightarrow{\varphi} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{\tilde{G}}$ in $\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$.

Let $k \in \mathbb{Z}$. Suppose given $(g_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \in (G\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k$.
We have

$$\begin{aligned}(g_a)_a\big((\varphi \blacktriangle \tilde{\varphi})\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)_k &= \big(g_a(\varphi \blacktriangle \tilde{\varphi})_n\big)\\ &= \big(g_a(\varphi_n \blacktriangle \tilde{\varphi}_n)\big)_a\\ &= \big((g_a\varphi_n)\tilde{\varphi}_n\big)_a\\ &= (g_a\varphi_n)_a(\tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k\\ &= (g_a)_a\big((\varphi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \blacktriangle (\tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k\big)\,.\end{aligned}$$

This shows that $\big((\varphi \blacktriangle \tilde{\varphi})\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)_k = (\varphi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \blacktriangle (\tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k$.
So we have $(\varphi \blacktriangle \tilde{\varphi})\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} = \varphi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \blacktriangle \tilde{\varphi}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$. □

Remark 299 *Suppose given an $\lfloor n,-\infty\rfloor$-stable simplicial group H; cf. Definition* 232.
We consider the stable simplicial group $H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$; cf. Remark 297.(1).
For $k \in \mathbb{Z}_{\geqslant n+1}$, we have

$$(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k = 1\,;$$

cf. Definition 198.(2).

Proof. Suppose given $k \in \mathbb{Z}_{\geqslant n+1}$.
Case $k = n+1$.
Suppose given $h \in (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_{n+1} \leqslant (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_{n+1} \overset{297.(1)}{=} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})\mathrm{D}_{n+1}$.
We choose some $\ell \in \mathbb{Z}_{\geqslant 1}$ such that $h \in (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_{n+1;\,\mathbb{Z}_{\geqslant \ell}}$. Then we have

$$h = \prod_{i\in\lfloor \ell-2,0\rfloor} x_{i+1}\,\mathrm{s}_i^{H,n} = \Big(\prod_{i\in\lfloor \ell-2,0\rfloor} x_{i+1}\,\mathrm{s}_i^{H,n}\Big)\cdot 1\,,$$

where $x_j \in (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_{n;\,\mathbb{Z}_{\geqslant j}}$ for $j \in [1,\ell-1]$ and where $1 \in (H\,\mathrm{Cond}^{\lfloor n,0\rfloor})\mathrm{N}_{n+1}$; cf. Lemma 286.
Moreover, since $h \in (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_{n+1}$, we have

$$h = 1\cdot h = \Big(\prod_{i\in\lfloor \ell-2,0\rfloor} 1\mathrm{s}_i^{H,n}\Big)\cdot h\,.$$

Then, by Lemma 283.(2), we have $x_j = 1$ for $j \in [1,\ell-1]$ and we have $1 = h$.

This shows that $(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_{n+1} = 1$.

Case $k \geqslant n+2$. We have

$$H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \overset{297.(1)}{\leqslant} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \overset{267}{\leqslant} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,.$$

So

$$(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k \leqslant (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k \overset{266}{=} 1\,.$$

This shows that $(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k = 1$ for $k \in \mathbb{Z}_{\geqslant n+2}$. □

14.3 The adjunction $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$

Let $n \in \mathbb{Z}$.

Remark 300 *Recall that we have* $\lfloor n,-\infty\rfloor$-*StSimpGrp* $\subseteq$ *Pre-*$\lfloor n,-\infty\rfloor$-*PreStSimpGrp as a full subcategory; cf. Definition* 232.

Let $\mathrm{I}_n \colon \lfloor n,-\infty\rfloor$-*StSimpGrp* $\to$ *Pre-*$\lfloor n,-\infty\rfloor$-*PreStSimpGrp be the embedding functor.*

(1) *We consider the functor* $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \colon$ *StSimpGrp* $\to$ *Pre-*$\lfloor n,-\infty\rfloor$-*PreStSimpGrp; cf. Remark* 261.

For a stable simplicial group G*, recall that* $(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k = G_k$ *for* $k \in \mathbb{Z}_{\leqslant n}$.

We consider the functor $\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \colon$ *StSimpGrp* $\to \lfloor n,-\infty\rfloor$-*StSimpGrp; cf. Lemma* 292.

For a stable simplicial group G*, recall that*

$$(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k = \begin{cases} G_k & \text{for } k \in \mathbb{Z}_{\leqslant n-1} \\ G_n/G\mathrm{NB}_n & \text{for } k = n\,; \end{cases}$$

cf. Lemma 290.(2).

We have the transformation

$$\rho = \left(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \xrightarrow{G\rho} G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\right)_{G\in\mathrm{Ob}(\mathcal{StSimpGrp})} \colon \mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to \mathrm{Trunc}^{\lfloor n,-\infty\rfloor} * \mathrm{I}_n\,,$$

where the pre-$\lfloor n,-\infty\rfloor$*-prestable simplicial group morphism*

$$G\rho \colon G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$$

is given as follows. For $k \in \mathbb{Z}_{\leqslant n-1}$*, we have*

$$(G\rho)_k := \mathrm{id}_{G_k} \colon G_k \to G_k \,.$$

Moreover, we have

$$(G\rho)_n \colon G_n \to G_n/G\mathrm{NB}_n \,,\ g \mapsto g(G\rho)_n := gG\mathrm{NB}_n \,.$$

(2) *We consider the functors*

$$\begin{array}{rlll} \mathrm{Cond}^{\lfloor n,-\infty\rfloor} \colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \mathcal{StSimpGrp};\ \text{cf. Lemma 298} \\ \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \colon & \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp} & \to & \mathcal{StSimpGrp};\ \text{cf. Remark 267} \,. \end{array}$$

For an $\lfloor n,-\infty\rfloor$*-stable simplicial group* H*, recall that* $H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \leqslant H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$*. Cf. Remark* 297.(1).

We have the transformation

$$\iota = \big(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \xrightarrow{H\iota} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\big)_{H\in\mathrm{Ob}(\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp})} \colon\ \mathrm{Cond}^{\lfloor n,-\infty\rfloor} \to \mathrm{I}_n * \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \,,$$

where the simplicial group morphism $H\iota \colon H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ *is given as follows.*

For $k \in \mathbb{Z}$*, we have the embedding group morphism*

$$(H\iota)_k \colon (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k \,,\ (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \mapsto (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \,.$$

Note that we have $(H\iota)_k = \mathrm{id}_{(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k}$ *for* $k \in \mathbb{Z}_{\leqslant n}$*; cf. Remark* 297.(1).

(3) *Suppose given a stable simplicial group* G *and suppose given an* $\lfloor n,-\infty\rfloor$*-stable simplicial group* H*; cf. Definitions* 205, 232.

Suppose given a stable simplicial group morphism $c \colon G \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$*.*

Suppose that we have $(G_{n+1})c_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}$*.*

Then we have the stable simplicial group morphism

$$c^\vee := c\Big|^{H\mathrm{Cond}^{\lfloor n,-\infty\rfloor}} \colon G \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \,;$$

cf. Remark 190.

Moreover, we have $c^\vee \centerdot H\iota = c : G \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$*; cf.* (2).

(4) *Suppose given a stable simplicial group G and suppose given an $\lfloor n, -\infty \rfloor$-simplicial group H; cf. Definitions* 30, 232.

Suppose given a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphism

$$b \colon G\,\mathrm{Cut}^{\lfloor n, -\infty \rfloor,\,\mathrm{fin}} \to H\,.$$

Suppose that we have $(G\mathrm{NB}_n)b_n = 1$.

Then we have the $\lfloor n, -\infty \rfloor$-stable simplicial group morphism

$$b^{\wedge} \colon G\,\mathrm{Trunc}^{\lfloor n, -\infty \rfloor} \to H\,,$$

where

$$b^{\wedge}_k := b_k \colon G_k \to H_k$$

for $k \in \mathbb{Z}_{\leqslant n-1}$, and where

$$b^{\wedge}_n \colon G_n/G\mathrm{NB}_n \to H_n\,,\ \ gG\mathrm{NB}_n \mapsto (gG\mathrm{NB}_n)\,b^{\wedge}_n := gb_n\,.$$

Moreover, we have $b = G\rho \mathbin{\blacktriangle} b^{\wedge} \colon G\,\mathrm{Cut}^{\lfloor n, -\infty \rfloor,\,\mathrm{fin}} \to H$; cf. (1).

Proof. Ad (1). Suppose given a stable simplicial group G.

We show that $G\rho = \big((G\rho)_k\big)_{k \in \mathbb{Z}_{\leqslant n}}$ is a pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphism.

To that end, we use Lemma 255.

It suffices to show that

$$(G\rho)_n \mathbin{\blacktriangle} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n, -\infty \rfloor}, n} \overset{!}{=} \mathrm{d}_i^{G\,\mathrm{Cut}^{\lfloor n, -\infty \rfloor,\,\mathrm{fin}}, n} \mathbin{\blacktriangle} (G\rho)_{n-1}$$

and that

$$(G\rho)_{n-1} \mathbin{\blacktriangle} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n, -\infty \rfloor}, n-1} \overset{!}{=} \mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n, -\infty \rfloor,\,\mathrm{fin}}, n-1} \mathbin{\blacktriangle} (G\rho)_n$$

for $i \in \mathbb{Z}_{\geqslant 0}$; cf. Lemma 290.(2).

Suppose given $i \in \mathbb{Z}_{\geqslant 0}$. For $g \in G_n$, we have

$$\begin{aligned} g\big((G\rho)_n \mathbin{\blacktriangle} \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n, -\infty \rfloor}, n}\big) &= (gG\mathrm{NB}_n)\mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n, -\infty \rfloor}, n} \overset{290}{=} g\mathrm{d}_j^{G, n} \overset{261}{=} (g\mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n, -\infty \rfloor,\,\mathrm{fin}}, n})\,\mathrm{id}_{G_{n-1}} \\ &= g\big(\mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n, -\infty \rfloor,\,\mathrm{fin}}, n} \mathbin{\blacktriangle} (G\rho)_{n-1}\big)\,. \end{aligned}$$

This shows that $(G\rho)_n \mathbin{\blacktriangle} \mathrm{d}_j^{G\,\mathrm{Trunc}^{\lfloor n, -\infty \rfloor}, n} = \mathrm{d}_j^{G\,\mathrm{Cut}^{\lfloor n, -\infty \rfloor,\,\mathrm{fin}}, n} \mathbin{\blacktriangle} (G\rho)_{n-1}$.

14.3. THE ADJUNCTION $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$

Suppose given $i \in \mathbb{Z}_{\geqslant 0}$. For $g \in G_n$, we have

$$\begin{array}{rcl} g\big((G\rho)_{n-1} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n-1}\big) & = & (g(\mathrm{id}_{G_{n-1}})\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n-1}) \\ & \overset{290}{=} & (g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n \\ & \overset{261}{=} & (g\mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},n-1})(G\rho)_n \\ & = & g\big(\mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},n-1} \blacktriangle (G\rho)_n\big)\,. \end{array}$$

This shows that $(G\rho)_{n-1} \blacktriangle \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n-1} = \mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},n-1} \blacktriangle (G\rho)_n$.
So, by Lemma 255, $G\rho$ is a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group morphism.
We show that ρ *is a transformation from* $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ *to* $\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} * \mathrm{I}_n$.
Suppose given a stable simplicial group morphism $\varphi\colon G \to \tilde{G}$.
We have to show that $G\rho \blacktriangle \varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \overset{!}{=} \varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \blacktriangle \tilde{G}\rho$.

$$\begin{array}{ccc} G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} & \xrightarrow{G\rho} & G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \\ \Big\downarrow{\scriptstyle \varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}} & & \Big\downarrow{\scriptstyle \varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}} \\ \tilde{G}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} & \xrightarrow[\tilde{G}\rho]{} & \tilde{G}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \end{array}$$

Suppose given $k \in \mathbb{Z}_{\leqslant n}$.
Case $k \in \mathbb{Z}_{\leqslant n-1}$. For $g \in (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k = G_k$, we have

$$\begin{array}{rcl} g\big((G\rho)_k \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k\big) & = & g(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_k \overset{291.(1)}{=} g\varphi_k = (g\varphi_k)(\tilde{G}\rho)_k \\ & \overset{261}{=} & g\big((\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_k \blacktriangle (\tilde{G}\rho)_k\big)\,. \end{array}$$

Case $k = n$. For $g \in (G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_n = G_n$, we have

$$\begin{array}{rcl} g\big((G\rho)_n \blacktriangle (\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n\big) & = & (gG\mathrm{NB}_n)(\varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_n \overset{291.(1)}{=} (g\varphi_n)\tilde{G}\mathrm{NB}_n \\ & = & (g\varphi_n)(\tilde{G}\rho)_n \overset{261}{=} g\big((\varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_n \blacktriangle (\tilde{G}\rho)_n\big)\,. \end{array}$$

This shows that $G\rho \blacktriangle \varphi\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} = \varphi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \blacktriangle \tilde{G}\rho$.
So ρ is a transformation.

Ad (2). *We show that ι is a transformation from* $\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$ *to* $\mathrm{I}_n * \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$.
Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group morphism $\psi\colon H \to \tilde{H}$.
We have to show that $H\iota \mathbin{\blacktriangle} \psi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \overset{!}{=} \psi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \mathbin{\blacktriangle} \tilde{H}\iota$.

$$\begin{array}{ccc} H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} & \xrightarrow{H\iota} & H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \\ \Big\downarrow{\scriptstyle \psi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}} & & \Big\downarrow{\scriptstyle \psi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}} \\ \tilde{H}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} & \xrightarrow[\tilde{H}\iota]{} & \tilde{H}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \end{array}$$

Suppose given $(h_a)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \in (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k$. We have

$$\begin{aligned}(h_a)_a\left((H\iota)_k \mathbin{\blacktriangle} (\psi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k\right) &= (h_a)_a\,(\psi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k \overset{297.(2)}{=} (h_a)_a\,(\psi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \\ &= (h_a)_a\left((\psi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \mathbin{\blacktriangle} (\tilde{H}\iota)_k\right).\end{aligned}$$

This shows that $H\iota \mathbin{\blacktriangle} \psi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} = \psi\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \mathbin{\blacktriangle} \tilde{H}\iota$.
So ι is a transformation.

Ad (3). By Remark 190, it suffices to show that $(G_k)c_k \overset{!}{\leqslant} (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k$ for $k \in \mathbb{Z}$.
Let $k \in \mathbb{Z}$. For $a \in {}_{\Delta^\pm}([n+1]^\pm, [k]^\pm)$, we have

$$(G_k\,c_k)(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_a = (G_k\,G_a)c_{n+1} \leqslant (G_{n+1})c_{n+1} \leqslant (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})\mathrm{D}_{n+1}\,.$$

This shows that $(G_k)c_k \leqslant (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k$; cf. Remark 297.
Then, by Remark 190, we have the stable simplicial group morphism

$$c^\vee = c\big|^{H\mathrm{Cond}^{\lfloor n,-\infty\rfloor}}\colon G \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,.$$

Let $k \in \mathbb{Z}$. For $g \in G_k$, we have

$$g(c^\vee \mathbin{\blacktriangle} H\iota)_k = g\big(c_k^\vee \mathbin{\blacktriangle} (H\iota)_k\big) = (gc_k)(H\iota)_k = gc_k\,.$$

This shows that $(c^\vee \mathbin{\blacktriangle} H\iota)_k = c_k$.
So we have $c^\vee \mathbin{\blacktriangle} H\iota = c$.
Ad (4). The group morphism $b_n^\wedge\colon G_n/G\mathrm{NB}_n \to H_n$, $(gb_n)G\mathrm{NB}_n \mapsto gb_n$ is well-defined since we have $(G\mathrm{NB}_n)b_n = 1$.

We use Lemma 255 to show that $b^\wedge = (b_k^\wedge)_{k\in\mathbb{Z}_{\leqslant n}}$ is a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group morphism.

It suffices to shows that

$$b_n^\wedge \blacktriangleleft \mathrm{d}_i^{H,n} \overset{!}{=} \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n} \blacktriangleleft b_{n-1}^\wedge$$

and that

$$b_{n-1}^\wedge \blacktriangleleft \mathrm{s}_i^{H,n-1} \overset{!}{=} \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n-1} \blacktriangleleft b_n^\wedge$$

for $i \in \mathbb{Z}_{\geqslant 0}$; cf. Lemma 290.(2).

Suppose given $i \in \mathbb{Z}_{\geqslant 0}$. For $gG\mathrm{NB}_n \in G_n/G\mathrm{NB}_n$, we have

$$\begin{aligned} gG\mathrm{NB}_n(b_n^\wedge \blacktriangleleft \mathrm{d}_i^{H,n}) &= (gb_n)\mathrm{d}_i^{H,n} = (g\mathrm{d}_i^{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},n})b_{n-1} = (g\mathrm{d}_i^{G,n})b_{n-1} \\ &\overset{290}{=} \big((gG\mathrm{NB}_n)\mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n}\big)b_{n-1} = gG\mathrm{NB}_n(\mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n} \blacktriangleleft b_{n-1}^\wedge)\,. \end{aligned}$$

This shows that $b_n^\wedge \blacktriangleleft \mathrm{d}_i^{H,n} = \mathrm{d}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n} \blacktriangleleft b_{n-1}^\wedge$.

For $g \in G_{n-1}$, we have

$$\begin{aligned} g(b_{n-1}^\wedge \blacktriangleleft \mathrm{s}_i^{H,n-1}) = (gb_{n-1})\mathrm{s}_i^{H,n-1} &\overset{261}{=} (g\mathrm{s}_i^{G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},n-1})b_n = (g\mathrm{s}_i^{G,n-1})b_n = \big((g\mathrm{s}_i^{G,n-1})G\mathrm{NB}_n\big)b_n^\wedge \\ &\overset{290}{=} g(\mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n-1} \blacktriangleleft b_n^\wedge)\,. \end{aligned}$$

This shows that $b_{n-1}^\wedge \blacktriangleleft \mathrm{s}_i^{H,n-1} = \mathrm{s}_i^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},n-1} \blacktriangleleft b_n^\wedge$.

Then, by Lemma 255, $b^\wedge \colon G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H$ is a pre-$\lfloor n,-\infty\rfloor$-prestable simplicial group morphism.

Let $k \in \mathbb{Z}_{\leqslant n-1}$. For $g \in G_k$, we have

$$g(G\rho \blacktriangleleft b^\wedge)_k = g\big((G\rho)_k \blacktriangleleft b_k^\wedge\big) = (g\,\mathrm{id}_{G_k})b_k = gb_k\,.$$

Let $k = n$. For $g \in G_n$, we have

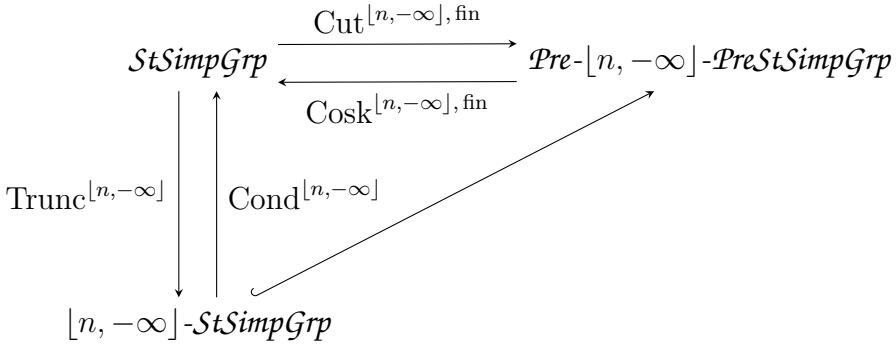
$$g(G\rho \blacktriangleleft b^\wedge)_n = g\big((G\rho)_n \blacktriangleleft b_n^\wedge\big) = (gG\mathrm{NB}_n)b_n^\wedge = gb_n\,.$$

This shows that $(G\rho \blacktriangleleft b^\wedge)_k = b_k$ for $k \in \mathbb{Z}_{\leqslant n}$.

So we have $G\rho \blacktriangleleft b^\wedge = b$. □

Lemma 301 *Suppose given* $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ *and* $H \in \mathrm{Ob}\big(\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}\big)$.
We consider the following functors.

$\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$:	$\mathcal{StSimpGrp}$	$\to$	$\mathcal{Pre}\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{PreStSimpGrp}$;	cf. Remark 261
$\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$:	$\mathcal{Pre}\text{-}\lfloor n, -\infty \rfloor\text{-}\mathcal{PreStSimpGrp}$	$\to$	$\mathcal{StSimpGrp}$;	cf. Remark 267
$\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$:	$\mathcal{StSimpGrp}$	$\to$	$\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}$;	cf. Lemma 292
$\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$:	$\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}$	$\to$	$\mathcal{StSimpGrp}$;	cf. Lemma 298



We consider the adjunction bijection concerning $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$ *from Remark* 272*,*

$$\alpha_{G,H} : \ {}_{\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}},\, H\big) \ \overset{\sim}{\leftrightarrow} \ {}_{\mathcal{StSimpGrp}}\big(G,\, H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\big)\,.$$

We consider the pre-$\lfloor n, -\infty \rfloor$*-prestable simplicial group morphism from Remark* 300.(1)*,*

$$G\rho\colon G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\ .$$

We consider the simplicial group morphism from Remark 300.(2)*,*

$$H\iota\colon H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\ .$$

We have have the bijection

$$\beta_{G,H} : {}_{\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}}\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\, H\big) \overset{\sim}{\to} {}_{\mathcal{StSimpGrp}}\big(G,\, H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)$$

given as follows.

14.3. THE ADJUNCTION $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$

For an $\lfloor n,-\infty\rfloor$*-stable simplicial group morphism* $\varphi\colon G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H$*, we have the stable simplicial group morphism*

$$\varphi\beta_{G,H} := \big((G\rho \mathbin{\triangle} \varphi)\alpha_{G,H}\big)\big|^{H\mathrm{Cond}^{\lfloor n,-\infty\rfloor}} : G \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,,$$

where, for $k \in \mathbb{Z}$*, we have*

$$\begin{array}{rccl}(\varphi\beta_{G,H})_k\colon & G_k & \to & (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \\ & g & \mapsto & g(\varphi\beta_{G,H})_k = g\big((G\rho \mathbin{\triangle} \varphi)\alpha_{G,H}\big)_k \overset{272}{=} \Big(\big((gG_a)G\mathrm{NB}_n\big)\varphi_n\Big)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}}\,.\end{array}$$

For a stable simplicial group morphism $\psi\colon G \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$*, we have the* $\lfloor n,-\infty\rfloor$*-stable simplicial group morphism*

$$\psi\beta^-_{G,H}\colon G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H\,,$$

where, for $k \in \mathbb{Z}_{\leqslant n-1}$*, we have*

$$(\psi\beta^-_{G,H})_k := \big((\psi \mathbin{\triangle} H\iota)\alpha^-_{G,H}\big)_k : G_k \to H_k\,,\ g \mapsto g\big((\psi \mathbin{\triangle} H\iota)\alpha^-_{G,H}\big)_k =: \tilde{h}\,,$$

such that $(\tilde{h}H_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} = g\psi_k$*; cf. Remark* 272*.*

For $k = n$*, we have*

$$\begin{array}{rccl}(\psi\beta^-_{G,H})_n\colon & G_n/G\mathrm{NB}_n & \to & H_n \\ & gG\mathrm{NB}_n & \mapsto & (gG\mathrm{NB}_n)(\psi\beta^-_{G,H})_n \ := g\big((\psi \mathbin{\triangle} H\iota)\alpha^-_{G,H}\big)_n =: \tilde{h}\,,\end{array}$$

such that $(\tilde{h}H_a)_{a\in_{\Delta^\pm([n]^\pm,[n]^\pm)}} = g\psi_n$*.*

Moreover, the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{P}re\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{P}re\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},\,H\big) & \xrightarrow[\sim]{\alpha_{G,H}} & \mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big(G,\,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\big) \\ \Big\uparrow{\scriptstyle G\rho \mathbin{\triangle} (-)} & & \Big\uparrow{\scriptstyle (-) \mathbin{\triangle} H\iota} \\ \lfloor n,-\infty\rfloor\text{-}\mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\,H\big) & \xrightarrow[\beta_{G,H}]{\sim} & \mathcal{S}t\mathcal{S}imp\mathcal{G}rp\big(G,\,H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big) \end{array}$$

So, for an $\lfloor n,-\infty\rfloor$*-stable simplicial group morphism* $\varphi\colon G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H$*, we have*

$$\varphi\beta_{G,H} \mathbin{\triangle} H\iota = (G\rho \mathbin{\triangle} \varphi)\alpha_{G,H}\,.$$

For a stable simplicial group morphism $\psi\colon G \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$*, we have*

$$G\rho \mathbin{\triangle} \psi\beta^-_{G,H} = (\psi \mathbin{\triangle} H\iota)\alpha^-_{G,H}\,.$$

Proof. We construct the map $\beta_{G,H}$.

We consider the map

$$\begin{array}{rccl} \tilde{\beta}_{G,H}\colon & {}_{\lfloor n,-\infty\rfloor}\text{-}\mathcal{StSimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\, H\big) & \to & \mathcal{StSimpGrp}\big(G,\, H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\big) \\ & \big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \xrightarrow{\varphi} H\big) & \mapsto & \varphi\tilde{\beta}_{G,H} := \big(G \xrightarrow{(G\rho_{\blacktriangle}\varphi)\alpha_{G,H}} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\big)\,. \end{array}$$

Suppose given $\ell \in \mathbb{Z}_{\geqslant 0}$. We write

$$\begin{array}{lll} \pi'_{n+1,\ell} := \pi^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}}_{n+1,\ell}\colon & (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_{n+1;\,\mathbb{Z}_{\geqslant\ell}} \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{N}_{n+1} \\ \partial'_{n+1} := \partial^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}}_{n+1}\colon & (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{N}_{n+1} \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{N}_{n} \\ \iota' := \iota_{(H\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{N}_n,(H\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_n}\colon & (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{N}_{n} \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_{n}\,; \end{array}$$

cf. Lemma 283.(3).

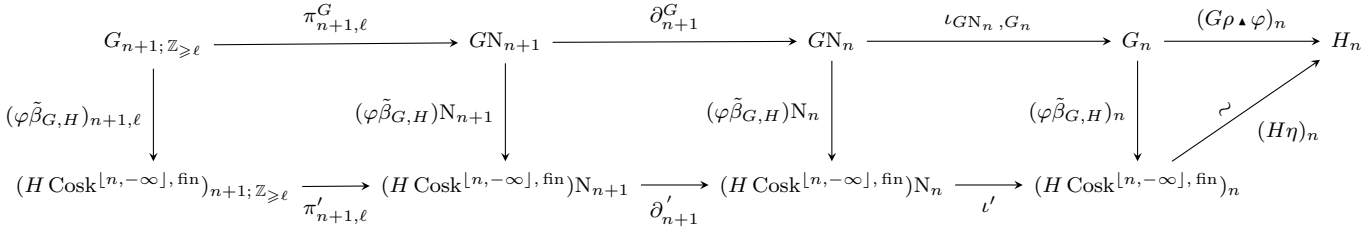
Suppose given $\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \xrightarrow{\varphi} H\big) \in {}_{\lfloor n,-\infty\rfloor}\text{-}\mathcal{StSimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\, H\big)$.

We consider the stable simplicial group morphism $\varphi\tilde{\beta}_{G,H} = (G\rho_{\blacktriangle}\varphi)\alpha_{G,H}\colon G \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$.

We write

$$(\varphi\tilde{\beta}_{G,H})_{n+1,\ell} := (\varphi\tilde{\beta}_{G,H})_{n+1}\Big|^{H_{n+1;\,\mathbb{Z}_{\geqslant\ell}}}_{G_{n+1;\,\mathbb{Z}_{\geqslant\ell}}}\colon G_{n+1;\,\mathbb{Z}_{\geqslant\ell}} \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_{n+1;\,\mathbb{Z}_{\geqslant\ell}}\,.$$

We show that the following diagram is commutative.



Note that $\pi^G_{n+1,\ell}$ and $\pi'_{n+1,\ell}$ are just maps; cf. Lemma 283.(3). Concerning the group isomorphism $(H\eta)_n$; cf. Lemma 268.

By Remark 289.(2), we have

$$\begin{aligned} \pi^G_{n+1,\ell} \blacktriangle (\varphi\tilde{\beta}_{G,H})\mathrm{N}_{n+1} &= \pi^G_{n+1,\ell} \blacktriangle (\varphi\tilde{\beta}_{G,H})(I * \mathrm{N}_{n+1}) \\ &= (\varphi\tilde{\beta}_{G,H})_{n+1} \blacktriangle \pi'_{n+1,\ell}\ : G_{n+1;\,\mathbb{Z}_{\geqslant\ell}} \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{N}_{n+1}\,. \end{aligned}$$

Moreover, we have

$$\partial^G_{n+1} \blacktriangle (\varphi\tilde{\beta}_{G,H})\mathrm{N}_n = (\varphi\tilde{\beta}_{G,H})\mathrm{N}_{n+1} \blacktriangle \partial'_{n+1} : G\mathrm{N}_{n+1} \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{N}_n\,,$$

and we have

$$\iota_{G\mathrm{N}_n,G_n} \blacktriangle (\varphi\tilde{\beta}_{G,H})_n = (\varphi\tilde{\beta}_{G,H})\mathrm{N}_n \blacktriangle \iota' : G\mathrm{N}_n \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})_n\,.$$

We consider the group isomorphism from Remark 274

$$(H\eta)^-_n \colon H_n \to (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor})_n\,,\ h \mapsto (hH_a)_{a\in_{\Delta^\pm}([n]^\pm,[n]^\pm)}\,.$$

Suppose given $g \in G_n$. Note that for $a \in {}_{\Delta^\pm}\big([n]^\pm,[n]^\pm\big)$, we have

$$G_a \blacktriangle (G\rho \blacktriangle \varphi)_n = (G\rho)_n \blacktriangle (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor})_a \blacktriangle \varphi_n = (G\rho \blacktriangle \varphi)_n \blacktriangle H_a\,.$$

So we have

$$\begin{aligned} g(\varphi\tilde{\beta}_{G,H})_n &= g\big((G\rho \blacktriangle \varphi)\alpha_{G,H}\big)_n \overset{272}{=} \big((gG_a)(G\rho \blacktriangle \varphi)_n\big)_{a\in_{\Delta^\pm}([n]^\pm,[n]^\pm)} = \big(g(G\rho \blacktriangle \varphi)_n\ H_a\big)_{a\in_{\Delta^\pm}([n]^\pm,[n]^\pm)} \\ &= \big(g(G\rho \blacktriangle \varphi)_n\big)(H\eta)^-_n = g\big((G\rho \blacktriangle \varphi)_n \blacktriangle (H\eta)^-_n\big)\,. \end{aligned}$$

This shows that

$$(\varphi\tilde{\beta}_{G,H})_n \blacktriangle (H\eta)_n = (G\rho \blacktriangle \varphi)_n\,.$$

Altogether, this shows that the diagram above is commutative.

We show that $(G_{n+1})(\varphi\tilde{\beta}_{G,H})_{n+1} \overset{!}{\leqslant} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}$.

By Remark 280, it suffices to show that we have $(G_{n+1;\,\mathbb{Z}_{\geqslant\ell}})(\varphi\tilde{\beta}_{G,H})_{n+1} \overset{!}{\leqslant} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}$ for a given $\ell \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $x \in G_{n+1;\,\mathbb{Z}_{\geqslant\ell}}$.

We have to show that $x(\varphi\tilde{\beta}_{G,H})_{n+1} \overset{!}{\in} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}$.

Recall that we have $x\pi^G_{n+1,\ell} \in G\mathrm{N}_{n+1}$. So we have $x\pi^G_{n+1,\ell}\,\mathrm{d}_0^{G,n+1} \in G\mathrm{NB}_n$.

So we have

$$(x\pi^G_{n+1,\ell}\,\mathrm{d}_0^{G,n+1})(G\rho)_n = (x\pi^G_{n+1,\ell}\,\mathrm{d}_0^{G,n+1})G\mathrm{NB}_n = 1G\mathrm{NB}_n\,.$$

We have

$$\begin{aligned}
&\big(x(\varphi\tilde{\beta}_{G,H})_{n+1}\big)\pi^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}}_{n+1,\ell}\,\mathrm{d}_0^{H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},n+1}\\
&= x\big((\varphi\tilde{\beta}_{G,H})_{n+1,\ell}\blacktriangle\pi'_{n+1,\ell}\blacktriangle\partial'_{n+1}\blacktriangle\iota'\big)\\
&= x\Big(\pi^{G}_{n+1,\ell}\blacktriangle\partial^{G}_{n+1}\blacktriangle\iota_{G\mathrm{N}_n,G_n}\blacktriangle(G\rho\blacktriangle\varphi)_n\blacktriangle(H\eta)^-_n\Big)\\
&= (x\pi^{G}_{n+1,\ell}\,\mathrm{d}_0^{G,n+1})(G\rho)_n(\varphi_n\blacktriangle(H\eta)^-_n)\\
&= 1G\mathrm{NB}_n\big(\varphi_n\blacktriangle(H\eta)^-_n\big)\\
&= 1\,.
\end{aligned}$$

Recall that $H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ is $\lfloor n,-\infty\rfloor$-coskeletal; cf. Remark 277.

Then, by Lemma 288.(3), we have $x(\varphi\tilde{\beta}_{G,H})_{n+1}\in(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}$.

This shows that $(G_{n+1})(\varphi\tilde{\beta}_{G,H})_{n+1}\leqslant(H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}})\mathrm{D}_{n+1}$.

So, by Remark 300.(3), we have the stable simplicial group morphism

$$(\varphi\tilde{\beta}_{G,H})^\vee=(\varphi\tilde{\beta}_{G,H})\big|^{H\mathrm{Cond}^{\lfloor n,-\infty\rfloor}}:G\to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,,$$

where

$$(\varphi\tilde{\beta}_{G,H})^\vee\blacktriangle H\iota=\varphi\tilde{\beta}_{G,H}:G\to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\,.$$

Let

$$\beta_{G,H}:{}_{\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}}\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\,H\big)\to{}_{\mathcal{StSimpGrp}}\big(G,\,H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)\,,\ \varphi\mapsto\varphi\beta_{G,H}:=(\varphi\tilde{\beta}_{G,H})^\vee\,.$$

Then we have

$$\varphi\beta_{G,H}\blacktriangle H\iota=\varphi\tilde{\beta}_{G,H}=(G\rho\blacktriangle\varphi)\alpha_{G,H}\,.$$

We construct the map $\beta^-_{G,H}$.

We consider the map

$$\begin{aligned}
\tilde{\beta}^{\#}_{G,H}:\ {}_{\mathcal{StSimpGrp}}\big(G,\,H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)\ &\to\ {}_{\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},\,H\big)\\
\big(G\xrightarrow{\psi}H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)\ &\mapsto\ \psi\tilde{\beta}^{\#}_{G,H}:=\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\xrightarrow{(\psi\blacktriangle H\iota)\alpha^-_{G,H}}H\big)\,.
\end{aligned}$$

Suppose given $\big(G\xrightarrow{\psi}H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)\in{}_{\mathcal{StSimpGrp}}\big(G,\,H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)$.

We show that $(G\mathrm{NB}_n)(\psi\tilde{\beta}^{\#}_{G,H})_n\overset{!}{=}1$.

14.3. THE ADJUNCTION $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$

Suppose given $x \in G\mathrm{NB}_n$. Then we have $x = y\mathrm{d}_0^{G,n+1} = y\mathrm{d}_0$ for some chosen $y \in G\mathrm{N}_{n+1}$.

Note that $y\psi_{n+1} \in (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_{n+1} \overset{299}{=} 1$.

So we have

$$\begin{aligned} x(\psi\tilde{\beta}^{\#}_{G,H})_n &= (y\mathrm{d}_0)\big((\psi \blacktriangle H\iota)\alpha^{-}_{G,H}\big)_n \overset{272}{=} (y\mathrm{d}_0)(\psi \blacktriangle H\iota)_n\,(H\eta)_n = (y\mathrm{d}_0)\psi_n\,(H\eta)_n \\ &= (y\psi_{n+1})\mathrm{d}_0\,(H\eta)_n = 1\mathrm{d}_0\,(H\eta)_n = 1\,. \end{aligned}$$

This shows that $(G\mathrm{NB}_n)(\psi\tilde{\beta}^{\#}_{G,H})_n = 1$.

So, by Remark 300.(4), we have the $\lfloor n,-\infty\rfloor$-stable simplicial group morphism

$$(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}\colon G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H\,,$$

where, for $k \in \mathbb{Z}_{\leqslant n-1}$, we have

$$(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_k := (\psi\tilde{\beta}^{\#}_{G,H})_k : G_k \to H_k\,,\ g \mapsto g(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_k = g(\psi\tilde{\beta}^{\#}_{G,H})_k$$

and where

$$(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_n\colon G_n/G\mathrm{NB}_n \to H_n\,,\ gG\mathrm{NB}_n \mapsto (gG\mathrm{NB}_n)(\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}_n = g(\psi\tilde{\beta}^{\#}_{G,H})_n\,.$$

Moreover, we have

$$\psi\tilde{\beta}^{\#}_{G,H} = G\rho \blacktriangle (\psi\tilde{\beta}^{\#}_{G,H})^{\wedge} : G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to H\,.$$

Let

$$\beta^{\#}_{G,H}\colon {}_{\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}\big(G,\, H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big) \to {}_{\lfloor n,-\infty\rfloor\text{-}\mathcal{S}t\mathcal{S}imp\mathcal{G}rp}\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,,\, H\big)\,,\ \psi \mapsto \psi\beta^{\#}_{G,H} := (\psi\tilde{\beta}^{\#}_{G,H})^{\wedge}\,.$$

Then we have

$$(\psi \blacktriangle H\iota)\alpha^{-}_{G,H} = \psi\tilde{\beta}^{\#}_{G,H} = G\rho \blacktriangle (\psi\tilde{\beta}^{\#}_{G,H})^{\wedge} = G\rho \blacktriangle \psi\beta^{\#}_{G,H}\,.$$

We want to show that $\beta^{\#}_{G,H}$ is the inverse bijection of $\beta_{G,H}$.

We show that $\beta_{G,H} \blacktriangle \beta^{\#}_{G,H} \overset{!}{=} \mathrm{id}$.

Suppose given an $\lfloor n,-\infty\rfloor$-stable simplicial group morphism $\varphi\colon G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H$.

We have to show that $\varphi(\beta_{G,H} \blacktriangle \beta^{\#}_{G,H}) \overset{!}{=} \varphi$.

If suffices to show that $G\rho \blacktriangle \big(\varphi(\beta_{G,H} \blacktriangle \beta^{\#}_{G,H})\big) \overset{!}{=} G\rho \blacktriangle \varphi$.

We have

$$G\rho \mathbin{\blacktriangle} \big(\varphi(\beta_{G,H} \mathbin{\blacktriangle} \beta^{\#}_{G,H})\big) = G\rho \mathbin{\blacktriangle} (\varphi\beta_{G,H})\beta^{\#}_{G,H} = (\varphi\beta_{G,H} \mathbin{\blacktriangle} H\iota)\alpha^{-}_{G,H} = \big((G\rho \mathbin{\blacktriangle} \varphi)\alpha_{G,H}\big)\alpha^{-}_{G,H}$$
$$= G\rho \mathbin{\blacktriangle} \varphi\,.$$

So we have $\varphi(\beta_{G,H} \mathbin{\blacktriangle} \beta^{\#}_{G,H}) = \varphi$.

This shows that $\beta_{G,H} \mathbin{\blacktriangle} \beta^{\#}_{G,H} = \mathrm{id}$.

We show that $\beta^{\#}_{G,H} \mathbin{\blacktriangle} \beta_{G,H} \overset{!}{=} \mathrm{id}$.

Suppose given a stable simplicial group morphism $\psi\colon G \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$.

We have to show that $\psi(\beta^{\#}_{G,H} \mathbin{\blacktriangle} \beta_{G,H}) \overset{!}{=} \psi$.

If suffices to show that $\big(\psi(\beta^{\#}_{G,H} \mathbin{\blacktriangle} \beta_{G,H})\big) \mathbin{\blacktriangle} H\iota \overset{!}{=} \psi \mathbin{\blacktriangle} H\iota$.

We have

$$\big(\psi(\beta^{\#}_{G,H} \mathbin{\blacktriangle} \beta_{G,H})\big) \mathbin{\blacktriangle} H\iota = (\psi\beta^{\#}_{G,H})\beta_{G,H} \mathbin{\blacktriangle} H\iota = (G\rho \mathbin{\blacktriangle} \psi\beta^{\#}_{G,H})\alpha_{G,H} = \big((\psi \mathbin{\blacktriangle} H\iota)\alpha^{-}_{G,H}\big)\alpha_{G,H}$$
$$= \psi \mathbin{\blacktriangle} H\iota\,.$$

So we have $\psi(\beta^{\#}_{G,H} \mathbin{\blacktriangle} \beta_{G,H}) = \psi$.

This shows that $\beta^{\#}_{G,H} \mathbin{\blacktriangle} \beta_{G,H} = \mathrm{id}$.

Altogether, this shows that $\beta_{G,H}$ is a bijection, where $\beta^{-}_{G,H} = \beta^{\#}_{G,H}$. □

Theorem 302 *We have the isotransformation*

$$\beta = (\beta_{G,H})_{(G,H)\in \mathrm{Ob}(\mathcal{StSimpGrp}^{\mathrm{op}} \times \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp})}\colon$$
$$\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\big((-)\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}, =\big) \xrightarrow{\sim} \mathcal{StSimpGrp}\big(-, (=)\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big)\,;$$

cf. Lemma 301.

In particular, we have $\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\lfloor n,-\infty\rfloor}$*; cf.* §1.1.3 *item* 10 *and Remark* 13.

$$\mathcal{StSimpGrp} \underset{\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}}{\overset{\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$$

Proof. Suppose given $\big(G \overset{u}{\longleftarrow} \tilde{G}\big)$ in $\mathcal{StSimpGrp}$ and $\big(H \overset{v}{\longrightarrow} \tilde{H}\big)$ in $\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$.

14.3. THE ADJUNCTION $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$

We have to show that the following diagram is commutative.

$$\begin{array}{ccc} {}_{\lfloor n,-\infty\rfloor}\text{-}\mathcal{StSimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\, H\big) & \xrightarrow[\sim]{\beta_{G,H}} & \mathcal{StSimpGrp}\big(G,\, H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big) \\ \Big\downarrow{\scriptstyle u\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\blacktriangle\,(-)\,\blacktriangle\,v} & & \Big\downarrow{\scriptstyle u\,\blacktriangle\,(-)\,\blacktriangle\,v\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}} \\ {}_{\lfloor n,-\infty\rfloor}\text{-}\mathcal{StSimpGrp}\big(\tilde{G}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\, \tilde{H}\big) & \xrightarrow[\beta_{\tilde{G},\tilde{H}}]{\sim} & \mathcal{StSimpGrp}\big(\tilde{G},\, \tilde{H}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\big) \end{array}$$

Suppose given $\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \xrightarrow{\varphi} H\big) \in {}_{\lfloor n,-\infty\rfloor}\text{-}\mathcal{StSimpGrp}\big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor},\, H\big)$.

We have to show that $(u\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\blacktriangle\varphi\blacktriangle\, v)\beta_{\tilde{G},\tilde{H}} \overset{!}{=} u\,\blacktriangle\,\varphi\beta_{G,H}\,\blacktriangle\, v\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$.

Let $k \in \mathbb{Z}$. Suppose given $\tilde{g} \in \tilde{G}_k$.

We have

$$\begin{aligned} \tilde{g}\big((u\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\blacktriangle\varphi\blacktriangle\, v)\beta_{\tilde{G},\tilde{H}}\big)_k &\overset{301}{=} \Big(\big((\tilde{g}\tilde{G}_a)\tilde{G}\mathrm{NB}_n\big)(u\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\blacktriangle\varphi\blacktriangle\, v)_n\Big)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \\ &\overset{291}{=} \Big(\Big(\big((\tilde{g}\tilde{G}_a)u_n\big)G\mathrm{NB}_n\Big)\varphi_n v_n\Big)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \tilde{g}(u\,\blacktriangle\,\varphi\beta_{G,H}\,\blacktriangle\, v\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k &= \big((\tilde{g}u_k)(\varphi\beta_{G,H})_k\big)(v\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \\ &\overset{301}{=} \Big(\Big(\big((\,(\tilde{g}u_k)G_a)G\mathrm{NB}_n\big)\varphi_n\Big)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}\Big)(v\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \\ &\overset{297.(2)}{=} \Big(\big((\,(\tilde{g}u_k)G_a)G\mathrm{NB}_n\big)\varphi_n v_n\Big)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)} \\ &= \Big(\big((\,(\tilde{g}\tilde{G}_a)u_n)G\mathrm{NB}_n\big)\varphi_n v_n\Big)_{a\in_{\Delta^\pm}([n]^\pm,[k]^\pm)}. \end{aligned}$$

This shows that $\big((u\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\blacktriangle\varphi\blacktriangle\, v)\beta_{\tilde{G},\tilde{H}}\big)_k = (u\,\blacktriangle\,\varphi\beta_{G,H}\,\blacktriangle\, v\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k$.

So we have $(u\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\blacktriangle\varphi\blacktriangle\, v)\beta_{\tilde{G},\tilde{H}} = u\,\blacktriangle\,\varphi\beta_{G,H}\,\blacktriangle\, v\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}$.

The claim can also be shown using the isotransformation α.

□

Lemma 303 (The counit $\hat{\eta}$)

(1) *Suppose given an $\lfloor n, -\infty \rfloor$-stable simplicial group H; cf. Definition* 232.

We consider the pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group morphisms

$$(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\colon H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$$

and

$$(H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\colon H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\,;$$

cf. Remark 300.(1, 2).

We consider the isomorphism of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups

$$H\eta\colon H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to H\,;$$

cf. Remarks 273, 274.

We have that $(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho$ and $(H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$ are isomorphisms of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups.

So we have the isomorphism of $\lfloor n, -\infty \rfloor$-stable simplicial groups

$$\big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\big)^{-} \vartriangle (H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \vartriangle H\eta \;:\; H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H\,;$$

cf. Definition 232.

(2) *A counit $\hat{\eta}$ of the adjunction $\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\lfloor n,-\infty\rfloor}$ is given by*

$$\hat{\eta} = \big(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \xrightarrow{H\hat{\eta}} H\big)_{H\in\mathrm{Ob}(\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp})}\colon \mathrm{Cond}^{\lfloor n,-\infty\rfloor} * \mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to \mathrm{id}_{\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}}\,,$$

where

$$H\hat{\eta} = \big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\big)^{-} \vartriangle (H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \vartriangle H\eta \;:\; H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \to H$$

for $H \in \mathrm{Ob}\big(\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}\big)$; cf. (1).

The counit $\hat{\eta}$ is an isotransformation.

$$\begin{array}{ccc}
H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} & \xrightarrow[\sim]{H\hat{\eta}} & H \\
{\scriptstyle (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho}\uparrow\wr & & \wr\uparrow{\scriptstyle H\eta} \\
H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} & \xrightarrow[(H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}]{\sim} & H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}
\end{array}$$

Proof. Ad (1). *We show that*

$$(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\colon H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}$$

is an isomorphism of pre-$\lfloor n,-\infty\rfloor$*-prestable simplicial groups. Cf. Remark* 300.(1).

For $k \in \mathbb{Z}_{\leqslant n-1}$, recall that we have $\big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\big)_k = \mathrm{id}_{(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k}$.

So it remains to show that

$$\begin{array}{rccl}
\big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\big)_n\colon & (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_n & \to & (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_n/(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{NB}_n \\
& (h_a)_{a\in_{\Delta^\pm}([n]^\pm,[n]^\pm)} & \mapsto & \big((h_a)_{a\in_{\Delta^\pm}([n]^\pm,[n]^\pm)}\big)(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{NB}_n
\end{array}$$

is a group isomorphism; cf. Remark 297.(1).

To that end, it suffices to show that $(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{NB}_n \overset{!}{=} 1$.

We have

$$(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{NB}_n \overset{257.(4)}{=} \big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_{n+1}\big)\mathrm{d}_0^{H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor},n+1} \overset{299}{=} 1\mathrm{d}_0^{H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor},n+1} = 1\,.$$

So $\big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\big)_n$ is a group isomorphism.

This shows that $(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho$ is an isomorphism of pre-$\lfloor n,-\infty\rfloor$-prestable simplicial groups.

We show that

$$(H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\colon H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}} \to H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}$$

is an isomorphism of pre-$\lfloor n,-\infty\rfloor$*-prestable simplicial groups. Cf. Remark* 300.(2).

Let $k \in \mathbb{Z}_{\leqslant n}$. Recall that we have

$$\begin{array}{rcl}
(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k & \overset{261}{=} & (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k \overset{297.(1)}{=} (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k \\
& \overset{261}{=} & (H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\mathrm{fin}}\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\mathrm{fin}})_k\,.
\end{array}$$

Moreover, for $(h_a)_{a \in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \in (H \operatorname{Cond}^{\lfloor n,-\infty \rfloor} \operatorname{Cut}^{\lfloor n,-\infty \rfloor, \mathrm{fin}})_k$, note that we have

$$\big((h_a)_a\big)\big((H\iota) \operatorname{Cut}^{\lfloor n,-\infty \rfloor, \mathrm{fin}}\big)_k \overset{261}{=} \big((h_a)_a\big)(H\iota)_k \overset{300.(2)}{=} (h_a)_a \,.$$

So

$$\begin{aligned} \big((H\iota) \operatorname{Cut}^{\lfloor n,-\infty \rfloor, \mathrm{fin}}\big)_k = (H\iota)_k \colon \quad (H \operatorname{Cond}^{\lfloor n,-\infty \rfloor})_k &\to (H \operatorname{Cosk}^{\lfloor n,-\infty \rfloor, \mathrm{fin}})_k \\ (h_a)_{a \in_{\Delta^\pm([n]^\pm,[k]^\pm)}} &\mapsto (h_a)_{a \in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \end{aligned}$$

is a group isomorphism.

Therefore $(H\iota) \operatorname{Cut}^{\lfloor n,-\infty \rfloor, \mathrm{fin}}$ is an isomorphism of pre-$\lfloor n, -\infty \rfloor$-prestable simplicial groups.

Ad (2). For an $\lfloor n, -\infty \rfloor$-stable simplicial group H, we consider the bijective map from Lemma 301,

$$\begin{aligned} \beta_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}, H} \colon {}_{\lfloor n,-\infty \rfloor}\text{-}\mathcal{StSimpGrp}\big(H \operatorname{Cond}^{\lfloor n,-\infty \rfloor} \operatorname{Trunc}^{\lfloor n,-\infty \rfloor}, H\big) \\ \xrightarrow{\sim} {}_{\mathcal{StSimpGrp}}(H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}, H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}) \,. \end{aligned}$$

By Remark 15, we have that the counit $\hat{\eta} \colon \operatorname{Cond}^{\lfloor n,-\infty \rfloor} * \operatorname{Trunc}^{\lfloor n,-\infty \rfloor} \to \mathrm{id}_{\lfloor n,-\infty \rfloor\text{-}\mathcal{StSimpGrp}}$ of $\operatorname{Trunc}^{\lfloor n,-\infty \rfloor} \dashv \operatorname{Cond}^{\lfloor n,-\infty \rfloor}$ with respect to β is obtained as

$$H\hat{\eta} := (\mathrm{id}_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}}) \beta^-_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}, H}$$

for $H \in \operatorname{Ob}\big(\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}\big)$.

So, for $H \in \operatorname{Ob}\big(\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}\big)$, we have to show that

$$(\mathrm{id}_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}}) \beta^-_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}, H} \overset{!}{=} \big((H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}) \rho\big)^- \mathbin{\triangle} (H\iota) \operatorname{Cut}^{\lfloor n,-\infty \rfloor, \mathrm{fin}} \mathbin{\triangle} H\eta \,.$$

Suppose given $H \in \operatorname{Ob}\big(\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}\big)$.

Suppose given $k \in \mathbb{Z}_{\leqslant n}$ and $(h_a)_{a \in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \in (H\mathrm{Cond}^{\lfloor n,-\infty \rfloor} \operatorname{Cut}^{\lfloor n,-\infty \rfloor, \mathrm{fin}})_k \overset{261}{=} (H\mathrm{Cond}^{\lfloor n,-\infty \rfloor})_k$.

We have

$$\begin{aligned} & \big((h_a)_a\big)\big((H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}) \rho \mathbin{\triangle} (\mathrm{id}_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}}) \beta^-_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}, H}\big)_k \\ \overset{301}{=} \; & \big((h_a)_a\big)\big((\mathrm{id}_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}} \mathbin{\triangle} H\iota) \alpha^-_{H \operatorname{Cond}^{\lfloor n,-\infty \rfloor}, H}\big)_k =: \tilde{h} \in H_k \end{aligned}$$

such that

$$(\tilde{h} H_a)_a = \big((h_a)_a\big)\big(\mathrm{id}_{(H \operatorname{Cond}^{\lfloor n,-\infty \rfloor})_k} \mathbin{\triangle} (H\iota)_k\big) = (h_a)_a \,;$$

cf. Remark 272.

Moreover, we have

$$\begin{aligned}\big((h_a)_a\big)\big((H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \blacktriangleleft H\eta\big)_k &= \Big(\big((h_a)_a\big)\big((H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\big)_k\Big)(H\eta)_k\\ &\overset{261}{=} \Big(\big(h_a)_a\big)(H\iota)_k\Big)(H\eta)_k\\ &\overset{300.(1)}{=} \big((h_a)_a\big)(H\eta)_k =: \tilde{\tilde{h}} \in H_k\,,\end{aligned}$$

such that $(\tilde{\tilde{h}}H_a)_a = (h_a)_a$; cf. Remark 273.

We have

$$\tilde{h}(H\eta)^-_k \overset{274}{=} (\tilde{h}H_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} = (h_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} = (\tilde{\tilde{h}}H_a)_{a\in_{\Delta^\pm([n]^\pm,[k]^\pm)}} \overset{274}{=} \tilde{\tilde{h}}(H\eta)^-_k\,.$$

This shows that $\tilde{h} = \tilde{\tilde{h}}$.

So $\big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho \blacktriangleleft (\mathrm{id}_{H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}})\beta^-_{H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor},H}\big)_k = \big((H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \blacktriangleleft H\eta\big)_k$.

Therefore, we have

$$\big((H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\rho\big)^- \blacktriangleleft (H\iota)\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \blacktriangleleft H\eta = (\mathrm{id}_{H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}})\beta^-_{H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor},H} = H\hat{\eta}\,.$$

By (1), $H\hat{\eta}$ is an isomorphism. □

Remark 304 (The unit $\hat{\varepsilon}$)

We consider the unit

$$\varepsilon\colon\ \mathrm{id}_{\mathcal{PreStSimpGrp}} \to \mathrm{Cut}^{\lfloor n,-\infty\rfloor} * \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$$

of the adjunction $\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$*; cf. Lemma* 269.

We consider the embedding functors

$$\begin{array}{lllll}\mathrm{I}\colon & \mathcal{StSimpGrp} & \to & \mathcal{PreStSimpGrp}; & \text{cf. Remark 209}\\ \mathrm{I}_n\colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}; & \text{cf. Remark 300}\,.\end{array}$$

We consider the transformation

$$\rho = \big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \xrightarrow{G\rho} G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\big)_{G\in\mathrm{Ob}(\mathcal{StSimpGrp})}\colon\ \mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \to \mathrm{Trunc}^{\lfloor n,-\infty\rfloor} * \mathrm{I}_n\,;$$

cf. Remark 300.(1).

Then, a unit $\hat{\varepsilon}$ *for* $\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\lfloor n,-\infty\rfloor}$ *is given by*

$$\hat{\varepsilon} = \big(G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor} \xrightarrow{G\hat{\varepsilon}} G\big)_{G\in\mathrm{Ob}(\mathcal{StSimpGrp})}\colon\ \mathrm{id}_{\mathcal{StSimpGrp}} \to \mathrm{Trunc}^{\lfloor n,-\infty\rfloor} * \mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,,$$

where, for $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$*, we have*

$$G\hat{\varepsilon} = \big((G\mathrm{I})\varepsilon \mathbin{\blacktriangle} (G\rho)\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\Big|^{G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}} : G \to G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\ .$$

So, for $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ *and* $k \in \mathbb{Z}$*, we have*

$$(G\hat{\varepsilon})_k \colon G_k \to (G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})_k\,,\ g \mapsto \big((gG_a)G\mathrm{NB}_n\big)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}\,.$$

Proof. Recall from Remark 211 that we have the adjunction $\mathrm{I} \dashv \Phi$ via the adjunction bijection

$$\begin{array}{rccl}
\gamma_{G,H}\colon & {\scriptstyle \mathcal{PreStSimpGrp}}(G\mathrm{I},\,H) & \xleftrightarrow{\sim} & {\scriptstyle \mathcal{StSimpGrp}}(G,\,H\Phi) \\
& \big(G\mathrm{I} \xrightarrow{\varphi} H\big) & \mapsto & \big(G \xrightarrow{\varphi\big|^{H\Phi}} H\Phi\big) \\
& \big(G\mathrm{I} \xrightarrow{\psi\mathrm{I} \mathbin{\blacktriangle} \iota_{H\Phi,H}} H\big) & \leftarrow\!\shortmid & \big(G \xrightarrow{\psi} H\Phi\big)
\end{array}$$

where $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ and $H \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$.

Recall from Remark 271 that we have the adjunction $\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor}$ via the adjunction bijection

$$\begin{array}{rccl}
\tilde{\alpha}_{G,H}\colon & {\scriptstyle \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor},H\big) & \xleftrightarrow{\sim} & {\scriptstyle \mathcal{PreStSimpGrp}}\big(G,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) \\
& \big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \xrightarrow{\varphi} H\big) & \mapsto & \big(G \xrightarrow{G\varepsilon \mathbin{\blacktriangle} \varphi\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big) \\
& \big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \xrightarrow{\psi\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor} \mathbin{\blacktriangle} H\eta} H\big) & \leftarrow\!\shortmid & \big(G \xrightarrow{\psi} H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}\big)\,,
\end{array}$$

where $G \in \mathrm{Ob}\big(\mathcal{PreStSimpGrp}\big)$ and $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}\big)$.

Recall from Remark 272 that

$$\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} = \mathrm{I} * \mathrm{Cut}^{\lfloor n,-\infty\rfloor} \quad \text{and that} \quad \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} = \mathrm{Cosk}^{\lfloor n,-\infty\rfloor} * \Phi$$

Moreover, recall that $\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}} \dashv \mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}$, and that we have a adjunction bijection

$$\alpha_{G,H}\colon\ {\scriptstyle \mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}}\big(G\,\mathrm{Cut}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}},\,H\big) \ \xleftrightarrow{\sim}\ {\scriptstyle \mathcal{StSimpGrp}}\big(G,\,H\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor,\,\mathrm{fin}}\big)\,,$$

where $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$ and $H \in \mathrm{Ob}\big(\mathcal{Pre}\text{-}\lfloor n,-\infty\rfloor\text{-}\mathcal{PreStSimpGrp}\big)$.

Then, for $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$, note that by Remark 16, we have

$$\alpha_{G,G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}} = \tilde{\alpha}_{GI,G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}} \mathbin{\blacktriangle} \gamma_{G,G\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor}\,\mathrm{Cosk}^{\lfloor n,-\infty\rfloor}}\,.$$

By Remark 15, a unit of $\operatorname{Trunc}^{\lfloor n,-\infty\rfloor} \dashv \operatorname{Cond}^{\lfloor n,-\infty\rfloor}$ is given by $\hat{\varepsilon}$, where

$$\begin{aligned} G\hat{\varepsilon} &\overset{302}{=} (\mathrm{id}_{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}})\beta_{G,G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}} \\ &\overset{301}{=} (G\rho \mathbin{\blacktriangle} \mathrm{id}_{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}})\alpha_{G,G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}}\Big|^{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}\operatorname{Cond}^{\lfloor n,-\infty\rfloor}} \\ &\overset{272}{=} \Big(\big((G\mathrm{I})\varepsilon \mathbin{\blacktriangle} (G\rho)\operatorname{Cosk}^{\lfloor n,-\infty\rfloor}\big)\Big|^{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}\operatorname{Cosk}^{\lfloor n,-\infty\rfloor},\mathrm{fin}}\Big)\Big|^{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}\operatorname{Cond}^{\lfloor n,-\infty\rfloor}} \\ &= \big((G\mathrm{I})\varepsilon \mathbin{\blacktriangle} (G\rho)\operatorname{Cosk}^{\lfloor n,-\infty\rfloor}\big)\Big|^{G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}\operatorname{Cond}^{\lfloor n,-\infty\rfloor}}, \end{aligned}$$

for $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$.

Moreover, for $G \in \mathrm{Ob}\big(\mathcal{StSimpGrp}\big)$, $k \in \mathbb{Z}$ and $g \in G_k$, we have

$$\begin{aligned} g(G\hat{\varepsilon})_k &= g\big((G\rho)\alpha_{G,G\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}}\big)_k \overset{272}{=} \big((gG_a)(G\rho)_n\big)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})} \\ &\overset{300.(1)}{=} \big((gG_a)G\mathrm{NB}_n\big)_{a\in_{\Delta^{\pm}}([n]^{\pm},[k]^{\pm})}. \end{aligned}$$

□

Remark 305 *We have the adjunction* $(\operatorname{Trunc}^{\lfloor n,-\infty\rfloor}, \operatorname{Cond}^{\lfloor n,-\infty\rfloor}, \hat{\varepsilon}, \hat{\eta})$*; cf. Lemma* 303*, Remark* 304*.*

Proof. Both $\hat{\varepsilon}$ and $\hat{\eta}$ are derived from the isotransformation β from Theorem 302 as in Remark 15. □

14.4 The Moore complex $(H\operatorname{Cond}^{\pm,\lfloor n,-\infty\rfloor})\mathrm{N}$

Lemma 306 *Let* $n \in \mathbb{Z}$*.*

Suppose given an $\lfloor n,-\infty\rfloor$*-stable simplicial group* H*; cf. Definition* 232*.*

We consider its Moore complex $H\mathrm{N}$*; cf. Lemma* 258*.*(2)*.*

We consider the stable simplicial group $H\operatorname{Cond}^{\lfloor n,-\infty\rfloor}$*; cf. Remark* 297*. We consider its Moore complex* $(H\operatorname{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}$*; cf. Lemma* 200*.*(2)*.*

We consider the morphism of complexes in groups

$$(H\operatorname{Cond}^{\lfloor n,-\infty\rfloor})\psi\colon (H\operatorname{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N} \to (H\operatorname{Cond}^{\lfloor n,-\infty\rfloor}\operatorname{Trunc}^{\lfloor n,-\infty\rfloor})\mathrm{N}$$

from Lemma 293*.*(2)*. It is an isomorphism.*

We consider the isomorphism of $\lfloor n, -\infty \rfloor$-stable simplicial groups

$$H\hat{\eta}\colon H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor}\,\mathrm{Trunc}^{\lfloor n,-\infty\rfloor} \xrightarrow{\sim} H$$

from Lemma 303.(2)*.*

We have the isomorphism of complexes in groups

$$\varphi := (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\psi \mathbin{\blacktriangle} (H\hat{\eta})\mathrm{N}\colon (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N} \xrightarrow{\sim} H\mathrm{N}\,.$$

$$\begin{array}{ccccccccccccccc}
\dots & \xrightarrow{!} & 1 & \xrightarrow{!} & (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_n & \xrightarrow{\partial_n} & \dots & \xrightarrow{\partial_1} & (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_0 & \xrightarrow{\partial_0} & (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_{-1} & \xrightarrow{\partial_{-1}} & \dots \\
 & & \downarrow{\scriptstyle !} & & \downarrow{\scriptstyle \wr\ \varphi_n} & & & & \downarrow{\scriptstyle \wr\ \varphi_0} & & \downarrow{\scriptstyle \wr\ \varphi_{-1}} & & \\
\dots & \xrightarrow[!]{} & 1 & \xrightarrow[!]{} & H\mathrm{N}_n & \xrightarrow[\partial_n]{} & \dots & \xrightarrow[\partial_1]{} & H\mathrm{N}_0 & \xrightarrow[\partial_0]{} & H\mathrm{N}_{-1} & \xrightarrow[\partial_{-1}]{} & \dots
\end{array}$$

Proof. By Remark 299, we have $(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\mathrm{N}_k = 1$ for $k \in \mathbb{Z}_{\geqslant n+1}$.

So, by Lemma 293.(4), we have that $(H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\psi$ is an isomorphism of complexes in groups.

Moreover, $(H\hat{\eta})\mathrm{N}$ is isomorphism of complexes in groups, since $H\hat{\eta}$ is an isomorphism of $\lfloor n, -\infty \rfloor$-stable simplicial groups.

Therefore, $\varphi = (H\,\mathrm{Cond}^{\lfloor n,-\infty\rfloor})\psi \mathbin{\blacktriangle} (H\hat{\eta})\mathrm{N}$ is an isomorphism of complexes in groups. □

Chapter 15

$\lfloor n, 0 \rfloor$-stable simplicial groups

15.1 The category of $\lfloor n, 0 \rfloor$-stable simplicial groups

Let $n \in \mathbb{Z}_{\geqslant 0}$.

Definition 307 ($\lfloor n, 0 \rfloor$-stable simplicial group)

We consider the category of $\lfloor n, -\infty \rfloor$-stable simplicial groups $\lfloor n, -\infty \rfloor$-$\mathcal{StSimpGrp}$; cf. Definition 232.

We have the full subcategory $\lfloor n, 0 \rfloor$-$\mathcal{StSimpGrp} \subseteq \lfloor n, -\infty \rfloor$-$\mathcal{StSimpGrp}$ where

$$\mathrm{Ob}\big(\lfloor n, 0 \rfloor\text{-}\mathcal{StSimpGrp}\big) := \big\{ G \in \mathrm{Ob}\big(\lfloor n, -\infty \rfloor\text{-}\mathcal{StSimpGrp}\big) : \ G_i = 1 \text{ for } i \in \mathbb{Z}_{\leqslant -1} \big\} .$$

$$G_n \underset{\mathrm{s}_i^{G,n-1}}{\overset{\mathrm{d}_j^{G,n}}{\rightleftarrows}} G_{n-1} \underset{\mathrm{s}_i^{G,n-2}}{\overset{\mathrm{d}_j^{G,n-1}}{\rightleftarrows}} \cdots \underset{\mathrm{s}_i^{G,0}}{\overset{\mathrm{d}_j^{G,1}}{\rightleftarrows}} G_0 \underset{\mathrm{s}_i^{G,-1}}{\overset{\mathrm{d}_j^{G,0}}{\rightleftarrows}} 1 \underset{\mathrm{s}_i^{G,-2}}{\overset{\mathrm{d}_j^{G,-1}}{\rightleftarrows}} \cdots$$

An object $G \in \mathrm{Ob}\big(\lfloor n, 0 \rfloor$-$\mathcal{StSimpGrp}\big)$ is called an *$\lfloor n, 0 \rfloor$-stable simplicial group*.

A morphism between $\lfloor n, 0 \rfloor$-stable simplicial groups is called an *$\lfloor n, 0 \rfloor$-stable simplicial group morphism*.

We call $\lfloor n, 0 \rfloor$-$\mathcal{StSimpGrp}$ the *category of $\lfloor n, 0 \rfloor$-stable simplicial groups*.

Remark 308 An $\lfloor n, 0 \rfloor$-stable simplicial group G has the following properties.

(1) Suppose given $k \in [1, n]$ and suppose given $x \in G_k$.

There exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $x \mathrm{d}_i^{G,k} = 1$ for $i \in \mathbb{Z}_{\geqslant \ell}$.

(2) For subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$, we have

$$\big[\bigcap_{i \in I} \ker \mathrm{d}_i^{G,n}, \bigcap_{j \in J} \ker \mathrm{d}_j^{G,n}\big] = 1;$$

cf. Remark 233.

(3) Using Remark 185, we have the following statements (i, ii, iii).

(i) We have

$$\mathrm{d}_j^{G,k+1} \blacktriangle \mathrm{d}_i^{G,k} = \mathrm{d}_i^{G,k+1} \blacktriangle \mathrm{d}_{j-1}^{G,k}$$

for $k \in [1, n-1]$ and i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

(ii) We have

$$\mathrm{s}_i^{G,k-1} \blacktriangle \mathrm{s}_j^{G,k} = \mathrm{s}_{j-1}^{G,k-1} \blacktriangle \mathrm{s}_i^{G,k}$$

for $k \in [1, n-1]$ and i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

(iii) Suppose given $k \in [0, n-1]$. For i, $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} = \begin{cases} \mathrm{d}_j^{G,k} \blacktriangle \mathrm{s}_{i-1}^{G,k-1} & \text{if } k \geqslant 1 \text{ and } j < i \\ \mathrm{id}_{G_k} & \text{if } k \geqslant 0 \text{ and } j \in \{i, i+1\} \\ ! & \text{if } k = 0 \text{ and } j \in \mathbb{Z}_{\geqslant 0} \backslash \{i, i+1\} \\ \mathrm{d}_{j-1}^{G,k} \blacktriangle \mathrm{s}_i^{G,k-1} & \text{if } k \geqslant 1 \text{ and } j > i+1. \end{cases}$$

Remark 309 *Let $n = 0$. Suppose given a $\lfloor 0, 0 \rfloor$-stable simplicial group G. Then G_0 is an abelian group.*

Proof. Since $G_{-1} = 1$, we have $\mathrm{d}_j^{G,0} = ! : G_0 \to 1$ for $j \in \mathbb{Z}_{\geqslant 0}$.
Let $I = J = \mathbb{Z}_{\geqslant 0}$.
Then, by the Conduché condition, we have

$$1 = \big[\bigcap_{i \in I} \ker \mathrm{d}_i^{G,0}, \bigcap_{j \in J} \ker \mathrm{d}_j^{G,0}\big] = \big[\bigcap_{i \in I} \ker !, \bigcap_{j \in J} \ker !\big] = [G_0, G_0].$$

This shows that G_0 is abelian. □

Remark 310 ($\lfloor n, 0 \rfloor$-stable simplicial subgroups)
Suppose given an $\lfloor n, 0 \rfloor$-stable simplicial group G.
Let $H \leqslant G$ be a $\lfloor n, -\infty \rfloor$-stable simplicial subgroup; cf. Definition 235.
Then H is an $\lfloor n, 0 \rfloor$-stable simplicial group, called an $\lfloor n, 0 \rfloor$-stable simplicial subgroup of G.

Proof. Let $k \in \mathbb{Z}_{\leqslant -1}$. We have $H_k \leqslant G_k = 1$. This shows that $H_k = 1$.
So H is an $\lfloor n, 0 \rfloor$-stable simplicial group. □

15.2 A construction of $\lfloor n, 0 \rfloor$-stable simplicial groups

Let $n \in \mathbb{Z}_{\geqslant 1}$.

Lemma 311 (Construction of an $\lfloor n, 0 \rfloor$-stable simplicial group)
For $k \in [0, n]$, suppose given a group G_k.
For $k \in [1, n]$ and $j \in \mathbb{Z}_{\geqslant 0}$, suppose given a group morphism $d_j^k \colon G_k \to G_{k-1}$.
For $k \in [0, n-1]$ and $i \in \mathbb{Z}_{\geqslant 0}$, suppose given a group morphism $s_i^k \colon G_k \to G_{k+1}$.
Suppose that the properties (1, 2, 3) are satisfied.

(1) *Suppose given $k \in [1, n]$ and suppose given $x \in G_k$.*
There exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $x d_j^k = 1$ for $\in \mathbb{Z}_{\geqslant \ell}$.

(2) *For subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$, we have*

$$\big[\bigcap_{i \in I} \ker d_i^n \,, \bigcap_{j \in J} \ker d_j^n\big] = 1 \,.$$

(3) *We have the following equations (i′, ii′, iii′).*

(i′) *We have*

$$d_j^{k+1} \blacktriangleleft d_i^k = d_i^{k+1} \blacktriangleleft d_{j-1}^k$$

for $k \in [1, n-1]$ and i, $j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

(ii′) *We have*

$$s_i^{k-1} \blacktriangle s_j^k = s_{j-1}^{k-1} \blacktriangle s_i^k$$

for $k \in [1, n-1]$ *and* $i,\ j \in \mathbb{Z}_{\geqslant 0}$ *with* $i < j$.

(iii′) *We have*

$$s_i^k \blacktriangle d_j^{k+1} = \begin{cases} d_j^k \blacktriangle s_{i-1}^{k-1} & \text{if } k \geqslant 1 \text{ and } j < i \\ \mathrm{id}_{G_k} & \text{if } k \geqslant 0 \text{ and } j \in \{i, i+1\} \\ ! & \text{if } k = 0 \text{ and } j \in \mathbb{Z}_{\geqslant 0} \backslash \{i, i+1\} \\ d_{j-1}^k \blacktriangle s_i^{k-1} & \text{if } k \geqslant 1 \text{ and } j > i+1 \end{cases}$$

for $k \in [0, n-1]$ *and* $i,\ j \in \mathbb{Z}_{\geqslant 0}$.

Cf. also Remark 308.

Then we have a $\lfloor n, 0 \rfloor$*-stable simplicial group* G *given as follows; cf. Definition* 307.

For $k \in \mathbb{Z}_{\leqslant n}$ *, we have*

$$[k]^{\pm} G = \begin{cases} G_k & \text{if } k \in [0, n] \\ 1 & \text{if } k \in \mathbb{Z}_{\leqslant -1} \,. \end{cases}$$

For $k \in \mathbb{Z}_{\leqslant n}$ *, we have*

$$\mathrm{d}_j^{G,k} = \begin{cases} d_j^k & \text{if } k \in [1, n] \\ ! & \text{if } k \in \mathbb{Z}_{\leqslant 0} \,. \end{cases}$$

For $k \in \mathbb{Z}_{\leqslant n-1}$ *, we have*

$$\mathrm{s}_i^{G,k} = \begin{cases} s_i^k & \text{if } k \in [0, n-1] \\ ! & \text{if } k \in \mathbb{Z}_{\leqslant -1} \,. \end{cases}$$

$$G_n \underset{s_i^{n-1}}{\overset{d_j^n}{\rightleftarrows}} G_{n-1} \underset{s_i^{n-2}}{\overset{d_j^{n-1}}{\rightleftarrows}} \cdots \underset{s_i^0}{\overset{d_j^1}{\rightleftarrows}} G_0 \underset{!}{\overset{!}{\rightleftarrows}} 1 \underset{!}{\overset{!}{\rightleftarrows}} \cdots$$

Proof. We show that G *is a pre-*$\lfloor n, -\infty \rfloor$*-prestable simplicial group using Lemma* 254.

To that end, we have to verify the properties (i, ii, iii) in Lemma 254 for

$$G_k := \begin{cases} G_k & \text{for } k \in [0, n] \\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1} \,, \end{cases}$$

$$\mathrm{d}_j^k := \begin{cases} d_j^k & \text{for } k \in [1, n] \\ ! & \text{for } k \in \mathbb{Z}_{\leqslant 0}\,, \end{cases}$$

$$\mathrm{s}_i^k := \begin{cases} s_i^k & \text{for } k \in [0, n-1] \\ ! & \text{for } k \in \mathbb{Z}_{\leqslant -1}\,. \end{cases}$$

Ad (i). Suppose given $k \in \mathbb{Z}_{\leqslant n-1}$ and $i,\, j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

If $k \leqslant 0$, then we have

$$d_j^{k+1} \blacktriangle d_i^k = d_j^{k+1} \blacktriangle\, ! = \;\; ! \;= d_i^{k+1} \blacktriangle\, ! = d_i^{k+1} \blacktriangle d_{j-1}^k\,.$$

If $k \in [1, n-1]$, then we have

$$d_j^{k+1} \blacktriangle d_i^k \overset{(\mathrm{i}')}{=} d_i^{k+1} \blacktriangle d_{j-1}^k\,.$$

This shows $d_j^{k+1} \blacktriangle d_i^k = d_i^{k+1} \blacktriangle d_{j-1}^k$.

Ad (ii). Suppose given $k \in \mathbb{Z}_{\leqslant n-1}$ and $i,\, j \in \mathbb{Z}_{\geqslant 0}$ with $i < j$.

If $k \leqslant 0$, then we have

$$s_i^{k-1} \blacktriangle s_j^k = !\, \blacktriangle s_j^k = \;\; ! \;= !\, \blacktriangle s_i^k = s_{j-1}^{k-1} \blacktriangle s_i^k\,.$$

If $k \in [1, n-1]$, then we have

$$s_i^{k-1} \blacktriangle s_j^k = s_i^{k-1} \blacktriangle s_j^k \overset{(\mathrm{ii}')}{=} s_{j-1}^{k-1} \blacktriangle s_i^k = s_{j-1}^{k-1} \blacktriangle s_i^k\,.$$

This shows $s_i^{k-1} \blacktriangle s_j^k = s_{j-1}^{k-1} \blacktriangle s_i^k$.

Ad (iii). Suppose given $k \in \mathbb{Z}_{\leqslant n-1}$ and $i,\, j \in \mathbb{Z}_{\geqslant 0}$.

Case $j < i$. Note that $j \in \mathbb{Z}_{\geqslant 0} \backslash \{i, i+1\}$.

If $k \leqslant -1$, then we have

$$s_i^k \blacktriangle d_j^{k+1} = !\, \blacktriangle\, !\; = d_j^k \blacktriangle s_{i-1}^{k-1}\,.$$

If $k = 0$, then we have

$$\mathrm{s}_i^k \blacktriangle d_j^{k+1} = s_i^0 \blacktriangle d_j^1 \overset{(\mathrm{iii}')}{=} \;\; ! \;= !\, \blacktriangle\, !\; = d_j^k \blacktriangle \mathrm{s}_{i-1}^{k-1}\,.$$

If $k \in [1, n-1]$, then we have

$$s_i^k \blacktriangle d_j^{k+1} \overset{(\mathrm{iii}')}{=} d_j^k \blacktriangle s_{i-1}^{k-1}\,.$$

Case $j \in \{i, i+1\}$.

If $k \leqslant -1$, then we have

$$s_i^k \blacktriangle d_j^{k+1} = ! \blacktriangle ! = \mathrm{id}_1 = \mathrm{id}_{G_k} .$$

If $k \in [0, n-1]$, then we have

$$s_i^k \blacktriangle d_j^{k+1} \overset{(\mathrm{iii}')}{=} \mathrm{id}_{G_k} .$$

Case $j > i+1$. Note that $j \in \mathbb{Z}_{\geqslant 0} \backslash \{i, i+1\}$.

If $k \leqslant -1$, then we have

$$s_i^k \blacktriangle d_j^{k+1} = ! \blacktriangle ! = d_{j-1}^k \blacktriangle s_i^{k-1} .$$

If $k = 0$, then we have

$$s_i^{G,k} \blacktriangle d_j^{k+1} = s_i^0 \blacktriangle d_j^1 \overset{(\mathrm{iii}')}{=} \quad ! = ! \blacktriangle ! = d_{j-1}^k \blacktriangle s_i^{k-1} .$$

If $k \in [1, n-1]$, then we have

$$s_i^k \blacktriangle d_j^{k+1} \overset{(\mathrm{iii}')}{=} d_{j-1}^k \blacktriangle s_i^{k-1} .$$

This shows

$$\mathrm{s}_i^{G,k} \blacktriangle \mathrm{d}_j^{G,k+1} = \begin{cases} \mathrm{d}_j^{G,k} \blacktriangle \mathrm{s}_{i-1}^{G,k-1} & \text{if } j < i \\ \mathrm{id}_{G_k} & \text{if } j = i \text{ or } j = i+1 \\ \mathrm{d}_{j-1}^{G,k} \blacktriangle \mathrm{s}_i^{G,k-1} & \text{if } j > i+1 . \end{cases}$$

So, by Lemma 254, we have the pre-$\lfloor n, -\infty \rfloor$-prestable simplicial group G as stated.

We show that G *is an* $\lfloor n, -\infty \rfloor$*-stable simplicial group.*

We show that G satisfies the finiteness condition.

Suppose given $k \in \mathbb{Z}_{\leqslant n}$.

Case $k \leqslant 0$. Suppose given $g \in G_k$. Then we have $g\mathrm{d}_j^{G,k} \in G_{k-1} = 1$ for $j \in \mathbb{Z}_{\geqslant 0}$.

Case $k \in [1, n]$. Suppose given $g \in G_k$. Then, by (1), there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $g\mathrm{d}_j^{G,k} = gd_j^k = 1$ for $j \in \mathbb{Z}_{\geqslant \ell}$.

We show that G satisfies the Conduché condition.

Suppose given cofinite subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

Then, by (2), we have

$$\big[\bigcap_{i\in I}\ker \mathrm{d}_i^{G,n}\,,\bigcap_{j\in J}\ker \mathrm{d}_j^{G,n}\big] = \big[\bigcap_{i\in I}\ker d_i^n\,,\bigcap_{j\in J}\ker d_j^n\big] = 1\,.$$

We show that G is an $\lfloor n,0\rfloor$-stable simplicial group.

Since we have $G_k = 1$ for $k \in \mathbb{Z}_{\leqslant -1}$, we have that G is a $\lfloor n,0\rfloor$-stable simplicial group; cf. Definition 307. □

15.3 The inclusion functor $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} \to \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$

Let $n \in \mathbb{Z}_{\geqslant 0}$.

Remark 312 (The inclusion functor $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}$)

We consider the category $\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$; cf. Definition 226.

We consider the subcategory $\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} \subseteq \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}$; cf. Definition 307.

We have the inclusion functor

$$\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} \to \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\,,\ \big(G \xrightarrow{\varphi} H\big) \mapsto \big(G \xrightarrow{\varphi} H\big)\,.$$

We often write $\mathrm{Inc}^{\lfloor n,0\rfloor} := \mathrm{Inc}^{\pm,\lfloor n,0\rfloor}$.

Remark 313 For an $\lfloor n,0\rfloor$-stable simplicial group G, note that we have

$$(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\pi_k = G\pi_k$$

for $k \in \mathbb{Z}_{\leqslant n}$; cf. Lemma 258.(2).

In particular, we have $(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\pi_k = 1$ for $k \in \mathbb{Z}_{\leqslant -1}$.

15.4 The shrinking functor $\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \to \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}$

Let $n \in \mathbb{Z}_{\geqslant 0}$.

Remark 314 (The $\lfloor n,0\rfloor$-stable simplicial group $G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$)
Suppose given an $\lfloor n,-\infty\rfloor$-stable simplicial group G; cf. Definition 232.
For $k \in \mathbb{Z}_{\leqslant n}$, let $A_k := \big\{a \in {}_{\Delta^\pm}\big([-1]^\pm,[k]^\pm\big)\colon a \text{ is injective}\big\}$.

(1) *We have the $\lfloor n,-\infty\rfloor$-stable simplicial subgroup $G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor} \leqslant G$, where*

$$(G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor})_k := \begin{cases} \bigcap\limits_{a\in A_k} \ker G_a & \text{for } k \in [0,n] \\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1}\,; \end{cases}$$

cf. Definition 235.

(2) *We have that $G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$ is an $\lfloor n,0\rfloor$-stable simplicial group; cf. Definition* 307.

We often write $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor} := G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$.

Proof. Ad (1). We use Remark 236 to show that $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}$ is an $\lfloor n,-\infty\rfloor$-stable simplicial subgroup of G.
Note that $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k \leqslant G_k$ for $k \in [0,n]$, and that $1 \leqslant G_k$ for $k \in \mathbb{Z}_{\leqslant -1}$.
Suppose given $k \in \mathbb{Z}_{\leqslant n}$ and $j \in \mathbb{Z}_{\geqslant 0}$. *We show that* $\big((G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k\big)\mathrm{d}_j^{G,k} \overset{!}{\leqslant} (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_{k-1}$.
We may assume that $k \in [0,n]$.
Suppose given $g \in (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$ and suppose given $\big([-1]^\pm \xrightarrow{b} [k-1]^\pm\big) \in A_{k-1}$.
Then the map $b \blacktriangle \delta_j^{\pm,k-1}\colon [-1]^\pm \to [k]^\pm$ is injective and so $b \blacktriangle \delta_j^{\pm,k-1} \in A_k$.
Therefore, we have

$$(g\mathrm{d}_j^{G,k})G_b = (gG_{\delta_j})G_b = gG_{b\blacktriangle\delta_j} = 1\,.$$

If $k \in [1,n-1]$, this shows that $g\mathrm{d}_j^{G,k} \in (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_{k-1}$.
If $k = 0$, then $b = \mathrm{id}_{[-1]^\pm}$ and so this shows $g\mathrm{d}_j^{G,k} = 1 \in 1 = (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_{-1}$.
Suppose given $k \in \mathbb{Z}_{\leqslant n-1}$ and $i \in \mathbb{Z}_{\geqslant 0}$. *We show that* $\big((G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k\big)\mathrm{s}_i^{G,k} \overset{!}{\leqslant} (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_{k+1}$.

15.4. THE SHRINKING FUNCTOR

$\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \to \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}$

We may assume that $k \in [0, n-1]$.

Suppose given $g \in (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$ and suppose given $\big([-1]^\pm \xrightarrow{b} [k+1]^\pm\big) \in A_{k+1}$.

We consider the map $b \blacktriangle \sigma_i^{\pm,k}\colon [-1]^\pm \to [k]^\pm$.

Let $C_b := \{j \in \mathbb{Z}_{\geqslant 0}\colon (j, k+1) \notin \mathrm{im}\, b\}$. Let $\ell := |C_b|$. We write $C_b =: \{j_1, j_2, \ldots, j_\ell\}$, where $0 \leqslant j_1 < j_2 < \ldots < j_\ell$.

By Remark 234.(1), since $b\colon [-1]^\pm \to [k+1]^\pm$ is injective, we have

$$b = \prod_{r\in\lceil 1,k+2\rceil}^{[-1]^\pm \to [k+1]^\pm} \delta_{j_r}^{\pm,r-1} = \delta_{j_1}^{\pm,0} \blacktriangle \ldots \blacktriangle \delta_{j_{k+2}}^{\pm,k+1}.$$

In particular, note that $|C_b| = \ell = k+2$.

Case $i \in C_b \cup \{j-1\colon j \in C_b\}$.

If $i \in \{j-1\colon j \in C_b\}$, then there exists a unique $t \in [1, k+2]$ such that $i = j_t - 1$. Note that if $i = j_t - 1$, then we have $i+1 = j_t < j_r$ for $r \in [t+1, k+2]$ and therefore

$$\delta_{j_r}^{\pm,r-1} \blacktriangle \sigma_i^{\pm,r-2} \overset{179.(3)}{=} \sigma_i^{\pm,r-3} \blacktriangle \delta_{j_r-1}^{\pm,r-2}\colon [r-2]^\pm \to [r-2]^\pm.$$

If $i \in C_b \backslash \{j-1\colon j \in C_b\}$, then there exists a unique $t \in [1, k+2]$ such that $i = j_t$. Note that if $i = j_t$, then we have $i+1 = j_t + 1 < j_r$ for $r \in [t+1, k+2]$ and therefore

$$\delta_{j_r}^{\pm,r-1} \blacktriangle \sigma_i^{\pm,r-2} \overset{179.(3)}{=} \sigma_i^{\pm,r-3} \blacktriangle \delta_{j_r-1}^{\pm,r-2}\colon [r-2]^\pm \to [r-2]^\pm.$$

We have

$$\begin{aligned}
b \blacktriangle \sigma_i^{\pm,k} &= \Big(\prod_{r\in\lceil 1,k+2\rceil}^{[-1]^\pm \to [k+1]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \sigma_i^{\pm,k} \\
&= \Big(\prod_{r\in\lceil 1,t-1\rceil}^{[-1]^\pm \to [t-2]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \delta_{j_t}^{\pm,t-1} \blacktriangle \Big(\prod_{r\in\lceil t+1,k+2\rceil}^{[t-1]^\pm \to [k+1]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \sigma_i^{\pm,k} \\
&= \Big(\prod_{r\in\lceil 1,t-1\rceil}^{[-1]^\pm \to [t-2]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \delta_{j_t}^{\pm,t-1} \blacktriangle \sigma_i^{\pm,t-2} \blacktriangle \Big(\prod_{r\in\lceil t+1,k+2\rceil}^{[t-2]^\pm \to [k]^\pm} \delta_{j_r-1}^{\pm,r-2}\Big) \\
&\overset{179.(3)}{=} \Big(\prod_{r\in\lceil 1,t-1\rceil}^{[-1]^\pm \to [t-2]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \mathrm{id}_{[t-2]^\pm} \blacktriangle \Big(\prod_{r\in\lceil t+1,k+2\rceil}^{[t-2]^\pm \to [k]^\pm} \delta_{j_r-1}^{\pm,r-2}\Big) \\
&= \Big(\prod_{r\in\lceil 1,t-1\rceil}^{[-1]^\pm \to [t-2]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \Big(\prod_{r\in\lceil t+1,k+2\rceil}^{[t-2]^\pm \to [k]^\pm} \delta_{j_r-1}^{\pm,r-2}\Big).
\end{aligned}$$

So $b \blacktriangle \sigma_i^{\pm,k}$ is injective, and therefore $b \blacktriangle \sigma_i^{\pm,k} \in A_k$.

We have

$$(g\mathrm{s}_i^{G,k})G_b = gG_{\sigma_i}G_b = gG_{b \blacktriangle \sigma_i} = 1\,.$$

Case $i \notin C_b \cup \{j-1 \colon j \in C_b\}$. Recall that $k \geqslant 0$. So $|C_b| = k+2 \neq 0$ and therefore $C_b \neq \emptyset$.

Let $t \in [0, k+2]$ such that $j_r + 1 \leqslant i$ for $r \in [1, t]$ and $i \leqslant j_r - 2$ for $r \in [t+1, k+2]$.

So

$$0 \leqslant j_1 < j_2 < \ldots < j_t < i < i+1 < j_{t+1} < \ldots < j_{k+2}\,.$$

We have

$$\begin{aligned}
b \blacktriangle \sigma_i^{\pm,k} &= \Big(\prod_{r \in \lceil 1, k+2 \rceil}^{[-1]^\pm \,\cdots\, [k+1]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \sigma_i^{\pm,k} \\
&= \Big(\prod_{r \in \lceil 1, t \rceil}^{[-1]^\pm \,\cdots\, [t-1]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \Big(\prod_{r \in \lceil t+1, k+2 \rceil}^{[t-1]^\pm \,\cdots\, [k+1]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \sigma_i^{\pm,k} \\
&\overset{179.(3)}{=} \Big(\prod_{r \in \lceil 1, t \rceil}^{[-1]^\pm \,\cdots\, [t-1]^\pm} \delta_{j_r}^{\pm,r-1}\Big) \blacktriangle \sigma_i^{\pm,t-2} \blacktriangle \Big(\prod_{r \in \lceil t+1, k+2 \rceil}^{[t-2]^\pm \,\cdots\, [k]^\pm} \delta_{j_r-1}^{\pm,r-2}\Big) \\
&\overset{179.(3)}{=} \sigma_{i-t}^{\pm,-2} \blacktriangle \Big(\prod_{r \in \lceil 1, t \rceil}^{[-2]^\pm \,\cdots\, [t-2]^\pm} \delta_{j_r}^{\pm,r-2}\Big) \blacktriangle \Big(\prod_{r \in \lceil t+1, k+2 \rceil}^{[t-2]^\pm \,\cdots\, [k]^\pm} \delta_{j_r-1}^{\pm,r-2}\Big)\,.
\end{aligned}$$

If $t = 0$, let $c := \Big(\prod_{r \in \lceil 2, k+2 \rceil}^{[-1]^\pm \,\cdots\, [k]^\pm} \delta_{j_r-1}^{\pm,r-2}\Big) \colon [-1]^\pm \to [k]^\pm$.

Then we have $c \in A_k$ and $b \blacktriangle \sigma_i^{\pm,k} = \sigma_i^{\pm,-2} \blacktriangle \delta_{j_1-1}^{\pm,-1} \blacktriangle c$.

If $t \in [1, k+2]$, let $c := \Big(\prod_{r \in \lceil 2, t \rceil}^{[-1]^\pm \,\cdots\, [t-2]^\pm} \delta_{j_r}^{\pm,r-2}\Big) \blacktriangle \Big(\prod_{r \in \lceil t+1, k+2 \rceil}^{[t-2]^\pm \,\cdots\, [k]^\pm} \delta_{j_r-1}^{\pm,r-2}\Big) \colon [-1]^\pm \to [k]^\pm$.

Then we have $c \in A_k$ and $b \blacktriangle \sigma_i^{\pm,k} = \sigma_{i-t}^{\pm,-2} \blacktriangle \delta_{j_1}^{\pm,-1} \blacktriangle c$.

So in both cases we have

$$b \blacktriangle \sigma_i^{\pm,k} = \sigma_{i-t}^{\pm,-2} \blacktriangle \delta_{j'}^{\pm,-1} \blacktriangle c$$

with $j' \in \{j_1 - 1\,, j_1\}$ and $c \in A_k$.

We have

$$(g\mathrm{s}_i^{G,k})G_b = (gG_{\sigma_i})G_b = gG_{b \blacktriangle \sigma_i} = gG_{\sigma_{i-t} \blacktriangle \delta_{j'} \blacktriangle c} = gG_c\, G_{\sigma_{i-t} \blacktriangle \delta_{j'}} = 1G_{\sigma_{i-t} \blacktriangle \delta_{j'}} = 1\,.$$

Altogether, this shows that $gs_i^{G,k} \in (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_{k+1}$.

Then, by Remark 236, we have that $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}$ is an $\lfloor n,-\infty\rfloor$-stable simplicial subgroup of G.

Ad (2). By construction, we have $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k = 1$ for $k \in \mathbb{Z}_{\leqslant -1}$. So $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}$ is an $\lfloor n,0\rfloor$-stable simplicial group. □

Remark 315 (The $\lfloor n,0\rfloor$-stable simplicial group morphism $\varphi\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$)

Suppose given an $\lfloor n,-\infty\rfloor$-stable simplicial group morphism $\varphi\colon G \to H$.

We consider the $\lfloor n,0\rfloor$-stable simplicial groups $G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$ and $H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$; cf. Remark 314.(1, 2).

We have the $\lfloor n,0\rfloor$-stable simplicial group morphism

$$\varphi\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor} := \varphi\Big|^{H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}}_{G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}} \colon G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor} \to H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\ .$$

So, for $k \in [0,n]$, we have

$$(\varphi\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor})_k \colon (G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor})_k \to (H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor})_k\,,\ g \mapsto g(\varphi\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor})_k = g\varphi_k\,.$$

We often write $\varphi\,\mathrm{Shrink}^{\lfloor n,0\rfloor} := \varphi\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$.

Proof. Let $k \in [0,n]$. Suppose given $g \in (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$.

We have to show that $g\varphi_k \overset{!}{\in} (H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$.

For an injective map $a \in {}_{\Delta^\pm}([-1]^\pm,[k]^\pm)$, we have

$$(g\varphi_k)H_a = (gG_a)\varphi_{-1} = 1\varphi_{-1} = 1\,.$$

This shows that $g\varphi_k \in (H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$. □

Remark 316 (The shrinking functor $\mathrm{Shrink}^{\lfloor n,0\rfloor}$)

We have the shrinking functor

$$\begin{array}{rccl} \mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} \\ & \big(G \xrightarrow{\varphi} H\big) & \mapsto & \big(G\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor} \xrightarrow{\varphi\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}} H\,\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\big)\,; \end{array}$$

cf. Remarks 314, 315.

We often write $\mathrm{Shrink}^{\lfloor n,0\rfloor} := \mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$.

Proof. Suppose given an $\lfloor n,-\infty\rfloor$-stable simplicial group G.

We have

$$(\mathrm{id}_G)\,\mathrm{Shrink}^{\lfloor n,0\rfloor} = \mathrm{id}_G\Big|^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}_{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}} = \mathrm{id}_{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}\ .$$

Suppose given $G \xrightarrow{\varphi} H \xrightarrow{\psi} F$ in $\lfloor n,-\infty\rfloor$-$\mathcal{StSimpGrp}$.

We have

$$\begin{aligned}\varphi\,\mathrm{Shrink}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} \psi\,\mathrm{Shrink}^{\lfloor n,0\rfloor} &= \Big(\varphi\Big|^{H\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}_{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}\Big) \mathbin{\blacktriangle} \Big(\psi\Big|^{F\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}_{H\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}\Big)\\ &= (\varphi \mathbin{\blacktriangle} \psi)\Big|^{F\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}_{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}}\\ &= (\varphi \mathbin{\blacktriangle} \psi)\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\ .\end{aligned}$$

This shows that $\mathrm{Shrink}^{\lfloor n,0\rfloor}$ is a functor. □

Lemma 317 (The Moore complex of $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}$)

Suppose given an $\lfloor n,-\infty\rfloor$-stable simplicial group G; cf. Definition 232. *We consider its Moore complex*

$$G\mathrm{N} = \big(\ldots \xrightarrow{!} 1 \xrightarrow{!} G\mathrm{N}_n \xrightarrow{\partial^G_n} G\mathrm{N}_{n-1} \xrightarrow{\partial^G_{n-1}} \ldots \xrightarrow{\partial^G_1} G\mathrm{N}_0 \xrightarrow{\partial^G_0} G\mathrm{N}_{-1} \xrightarrow{\partial^G_{-1}} \ldots\big);$$

cf. Lemma 258.(2).

(1) *We consider the $\lfloor n,0\rfloor$-stable simplicial group $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}$; cf. Remark* 314. *We consider its Moore complex* $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}$.

We have

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_k = \begin{cases} 1 & \text{for } k \in \mathbb{Z}_{\geqslant n+1}\\ G\mathrm{N}_k & \text{for } k \in [1,n]\\ G\mathrm{NZ}_0 & \text{for } k = 0\\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1}\,.\end{cases}$$

For $k \in [2,n]$, we have

$$\partial_k^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}} = \partial_k^G \colon G\mathrm{N}_k \to G\mathrm{N}_{k-1} \quad \text{and} \quad \partial_1^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}} = \partial_1^G\big|^{G\mathrm{NZ}_0} \colon G\mathrm{N}_1 \to G\mathrm{NZ}_0\,.$$

So

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N} = \big(\ldots \xrightarrow{!} 1 \xrightarrow{!} G\mathrm{N}_n \xrightarrow{\partial^G_n} G\mathrm{N}_{n-1} \xrightarrow{\partial^G_{n-1}} \ldots \xrightarrow{\partial^G_2} G\mathrm{N}_1 \xrightarrow{\partial^G_1\big|^{G\mathrm{NZ}_0}} G\mathrm{NZ}_0 \xrightarrow{!} 1 \xrightarrow{!} \ldots\big).$$

$\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\colon \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \to \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}$

In particular, for $k \in [1,n]$, we have

$$(G\mathrm{N}_k)\mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},k} = (G\mathrm{N}_k)\mathrm{d}_0^{G,k} \quad \text{and} \quad G\mathrm{N}_k \cap \ker \mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},k} = G\mathrm{N}_k \cap \ker \mathrm{d}_0^{G,k}\,.$$

(2) *Let $k \in [0,n]$. We have*

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NZ}_k = G\mathrm{NZ}_k \quad \text{and} \quad (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NB}_k = G\mathrm{NB}_k\,;$$

cf. Definition 257.(3, 4)*. In particular, we have*

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\pi_k = G\pi_k\,;$$

cf. Lemma 258.(2)*.*

Proof. Ad (1). Let $k \in \mathbb{Z}$.

Case $k \geqslant n+1$. Then we have $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_k = 1$ by definition of the Moore complex of an $\lfloor n,-\infty\rfloor$-stable simplicial group; cf. Lemma 258.(2).

Case $k \in [1,n]$. Since $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor} \leqslant G$ is an $\lfloor n,-\infty\rfloor$-stable simplicial subgroup, we have

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_k = (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k \cap G\mathrm{N}_k \leqslant G\mathrm{N}_k\,;$$

cf. Remark 314.(1). We show that $G\mathrm{N}_k \overset{!}{\leqslant} (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$.

Suppose given $g \in G\mathrm{N}_k$. Suppose given an injective morphism $a \in {}_{\Delta^\pm}\big([-1]^\pm,[k]^\pm\big)$.

By Remark 265, we may choose some $b \in {}_{\Delta^\pm}\big([-1]^\pm,[k-1]^\pm\big)$ and some $j \in \mathbb{Z}_{\geqslant k-(-1)-1} = \mathbb{Z}_{\geqslant k}$ such that $a = b \blacktriangle \delta_j^{\pm,k}$. Note that $j \geqslant 1$ since $k \geqslant 1$. We have

$$gG_a = gG_{b\blacktriangle\delta_j^\pm} = g(G_{\delta_j^{\pm,k}}G_b) = (g\mathrm{d}_j^{G,k})G_b = 1G_b = 1\,.$$

This shows $g \in \ker G_a$.

So $g \in \bigcap_{\substack{a\in{}_{\Delta^\pm}([-1]^\pm,[k]^\pm)\\ a \text{ is injective}}} \ker G_a = (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$.

This shows $G\mathrm{N}_k \leqslant (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$. Altogether, this shows $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_k = G\mathrm{N}_k$.

Case $k = 0$. Suppose given an injective morphism $a \in {}_{\Delta^\pm}\big([-1]^\pm,[0]^\pm\big)$. Then we have $a = \delta_j^{\pm,0}\colon [-1]^\pm \to [0]^\pm$ for some $j \in \mathbb{Z}_{\geqslant 0}$; cf. Lemma 175.(1.1). So $G_a = G_{\delta_j^{\pm,0}} = \mathrm{d}_j^{G,0}$.

This shows

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_0 = \bigcap_{\substack{a\in\,_{\Delta^\pm}([-1]^\pm,[0]^\pm)\\ a \text{ is injective}}} \ker G_a = \bigcap_{j\geqslant 0} \ker \mathrm{d}_j^{G,0} \overset{257.(3)}{=} G\mathrm{NZ}_0\,.$$

Case $k \leqslant -1$. Then we have $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_k = 1$ by definition of the $\lfloor n,-\infty\rfloor$-stable simplicial group $G\,\mathrm{Shrink}^{\lfloor n,0\rfloor}$; cf. Remark 314.(1).

This shows (1).

Ad (2). Note that

$$\ker \mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},0} = (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_0$$

since $\mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},0}$ maps from $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_0$ to $(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_{-1} = 1$.

So, if $k = 0$, we have

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NZ}_0 \overset{257.(3)}{=} (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_0 \cap \ker \mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},0} = (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_0 \overset{(1)}{=} G\mathrm{NZ}_0\,.$$

If $k \in [1,n]$, we have

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NZ}_k \overset{257.(3)}{=} (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_k \cap \ker \mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},k} \overset{(1)}{=} G\mathrm{N}_k \cap \ker \mathrm{d}_0^{G,k} \overset{257.(3)}{=} G\mathrm{NZ}_k\,.$$

If $k \in [0,n-1]$, we have

$$\begin{aligned}(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NB}_k &\overset{257.(4)}{=} \big((G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{N}_{k+1}\big)\mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},k} \overset{(1)}{=} (G\mathrm{N}_{k+1})\mathrm{d}_0^{G\,\mathrm{Shrink}^{\lfloor n,0\rfloor},k}\\ &\overset{(1)}{=} (G\mathrm{N}_{k+1})\mathrm{d}_0^{G,k} \overset{257.(4)}{=} G\mathrm{NB}_k\,.\end{aligned}$$

If $k = n$, we have

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NB}_n = 1 = G\mathrm{NB}_n\,.$$

So, in particular, we have

$$(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\pi_k \overset{258.(2)}{=} (G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NZ}_k/(G\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\mathrm{NB}_k = G\mathrm{NZ}_k/G\mathrm{NB}_k \overset{258.(2)}{=} G\pi_k\,.$$

□

15.5 The adjunction $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor} \dashv \mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$

Let $n \in \mathbb{Z}_{\geqslant 0}$.

Remark 318 (The counit η)

For an $\lfloor n,-\infty\rfloor$-stable simplicial group H, note that we have

$$\big((H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\,\mathrm{Inc}^{\lfloor n,0\rfloor}\big)_k \overset{312}{=} (H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k = \begin{cases} \bigcap\limits_{a\in A_k} \ker H_a & \text{for } k \in [0,n] \\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1}\,, \end{cases}$$

where $A_k = \big\{a \in {}_{\Delta^\pm}([-1]^\pm,[k]^\pm) \colon a \text{ is injective}\big\}$; cf. Remark 314.

We have the transformation

$$\eta \colon \mathrm{Shrink}^{\lfloor n,0\rfloor} * \mathrm{Inc}^{\lfloor n,0\rfloor} \to \mathrm{id}_{\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}}$$

given as follows.

For $H \in \mathrm{Ob}\big(\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\big)$ and $k \in \mathbb{Z}_{\leqslant n}$, we have the embedding group morphism

$$(H\eta)_k := \iota_{(H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k,\,H_k} \colon (H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k \to H_k\,,\ \ h \mapsto h\,.$$

Proof. Suppose given $\big(H \xrightarrow{\psi} \tilde{H}\big)$ in $\lfloor n,-\infty\rfloor$-$\mathcal{StSimpGrp}$.

For $k \in \mathbb{Z}_{\leqslant n}$, and $h \in (H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$, we have

$$\begin{aligned} h\big((H\eta)_k \blacktriangle \psi_k\big) = h\psi_k = (h\psi_k)(\tilde{H}\eta)_k &\overset{315}{=} h\big((\psi\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k \blacktriangle (\tilde{H}\eta)_k\big) \\ &\overset{312}{=} h\Big(\big((\psi\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\,\mathrm{Inc}^{\lfloor n,0\rfloor}\big)_k \blacktriangle (\tilde{H}\eta)_k\Big)\,. \end{aligned}$$

This shows that $(H\eta)_k \blacktriangle \psi_k = \big((\psi\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\,\mathrm{Inc}^{\lfloor n,0\rfloor}\big)_k \blacktriangle (\tilde{H}\eta)_k$.

So we have $H\eta \blacktriangle \psi = (\varphi\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\,\mathrm{Inc}^{\lfloor n,0\rfloor} \blacktriangle \tilde{H}\eta$.

$$\begin{array}{ccc} (H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\,\mathrm{Inc}^{\lfloor n,0\rfloor} & \xrightarrow{H\eta} & H \\ \Big\downarrow{\scriptstyle (\psi\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\,\mathrm{Inc}^{\lfloor n,0\rfloor}} & & \Big\downarrow{\scriptstyle \psi} \\ (\tilde{H}\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\,\mathrm{Inc}^{\lfloor n,0\rfloor} & \xrightarrow[\tilde{H}\eta]{} & \tilde{H} \end{array}$$

So η is a transformation. □

Remark 319 (The unit ε)
For an $\lfloor n,0\rfloor$-stable simplicial group G, note that, for $k\in[0,n]$, we have

$$\big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k=\bigcap_{a\in A_k}\ker(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_a\leqslant G_k\,,$$

where $A_k=\big\{a\in{}_{\Delta^{\pm}}([-1]^{\pm},[k]^{\pm})\colon a\text{ is injective}\big\}$; cf. Remark 314.
We have the transformation

$$\varepsilon\colon\ \mathrm{id}_{\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}}\to\mathrm{Inc}^{\lfloor n,0\rfloor}*\mathrm{Shrink}^{\lfloor n,0\rfloor}$$

given as follows.
For $G\in\mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}\big)$ and $k\in[0,n]$, we have the group morphism

$$(G\varepsilon)_k\colon G_k\to\big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k\,,\ \ g\mapsto g\,.$$

Moreover, we remark that $(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}=G$.

Proof. Suppose given $G\in\mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}\big)$. Let $k\in[0,n]$.
We show that $(G\varepsilon)_k$ is well-defined.

Suppose given $g\in G_k$. We have to show that $g\overset{!}{\in}\big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k$.
Suppose given an injective map $a\in{}_{\Delta^{\pm}}\big([-1]^{\pm},[k]^{\pm}\big)$.
We consider the group morphism $(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_a\colon(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_k\to(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_{-1}$.
Then we have

$$g(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_a\in(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_{-1}=1\,.$$

This shows that $g\in\big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k$.
So we have $(G_k)(G\varepsilon)_k\leqslant\big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k$.
We show that ε is a transformation.
Suppose given $\big(G\xrightarrow{\varphi}\tilde{G}\big)$ in $\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}$.
For $k\in[0,n]$, and $g\in G_k$, we have

$$\begin{aligned}g\Big((G\varepsilon)_k\blacktriangle\big((\varphi\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k\Big)&=g\big((\varphi\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k\\&\overset{315}{=}g(\varphi\,\mathrm{Inc}^{\lfloor n,0\rfloor})_k\overset{312}{=}g\varphi_k=g\big(\varphi_k\blacktriangle(\tilde{G}\varepsilon)_k\big)\,.\end{aligned}$$

This shows that $(G\varepsilon)_k \blacktriangle \big((\varphi\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k = \varphi_k \blacktriangle (\tilde{G}\varepsilon)_k$.
So we have $G\varepsilon \blacktriangle (\varphi\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor} = \varphi \blacktriangle \tilde{G}\varepsilon$.

$$\begin{array}{ccc} G & \xrightarrow{G\varepsilon} & (G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle(\varphi\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor}} \\ \tilde{G} & \xrightarrow[\tilde{G}\varepsilon]{} & (\tilde{G}\,\mathrm{Inc}^{\lfloor n,0\rfloor})\,\mathrm{Shrink}^{\lfloor n,0\rfloor} \end{array}$$

So ε is a transformation. □

Lemma 320 (The adjunction $\mathrm{Inc}^{\lfloor n,0\rfloor} \dashv \mathrm{Shrink}^{\lfloor n,0\rfloor}$)
We consider the functors

$$\begin{array}{lrcll} \mathrm{Inc}^{\lfloor n,0\rfloor}\colon & \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} & \to & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}; & \text{cf. Remark 312} \\ \mathrm{Shrink}^{\lfloor n,0\rfloor}\colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}; & \text{cf. Remark 316}. \end{array}$$

We consider the transformations

$$\begin{array}{lrcll} \eta\colon & \mathrm{Shrink}^{\lfloor n,0\rfloor} * \mathrm{Inc}^{\lfloor n,0\rfloor} & \to & \mathrm{id}_{\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}}; & \text{cf. Remark 318} \\ \varepsilon\colon & \mathrm{id}_{\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}} & \to & \mathrm{Inc}^{\lfloor n,0\rfloor} * \mathrm{Shrink}^{\lfloor n,0\rfloor}; & \text{cf. Remark 319.} \end{array}$$

Then $(\mathrm{Inc}^{\lfloor n,0\rfloor}, \mathrm{Shrink}^{\lfloor n,0\rfloor}, \varepsilon, \eta)$ *is an adjunction; cf.* §1.1.3 *item* 10.

Proof. We show that $(\mathrm{Shrink}^{\lfloor n,0\rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Shrink}^{\lfloor n,0\rfloor}) \overset{!}{=} \mathrm{id}_{\mathrm{Shrink}^{\lfloor n,0\rfloor}}$.
Suppose given $H \in \mathrm{Ob}\big(\lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}\big)$. Let $k \in [0,n]$ and let $h \in (H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k$.
We have

$$\begin{array}{rcll} h\Big(H\big((\mathrm{Shrink}^{\lfloor n,0\rfloor} * \varepsilon) \blacktriangle (\eta * \mathrm{Shrink}^{\lfloor n,0\rfloor})\big)\Big)_k & = & h\big((H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\varepsilon \blacktriangle (H\eta)\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k & \\ & = & h\Big(\big((H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})\varepsilon\big)_k \blacktriangle \big((H\eta)\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k\Big) & \\ & \overset{319}{=} & h\big((H\eta)\,\mathrm{Shrink}^{\lfloor n,0\rfloor}\big)_k & \\ & \overset{315}{=} & h(H\eta)_k & \\ & \overset{318}{=} & h & \\ & = & h\,\mathrm{id}_{(H\,\mathrm{Shrink}^{\lfloor n,0\rfloor})_k} & \\ & = & h(H\,\mathrm{id}_{\mathrm{Shrink}^{\lfloor n,0\rfloor}})_k. & \end{array}$$

This shows that $\Big(H\big((\mathrm{Shrink}^{\lfloor n,0\rfloor} * \varepsilon) \mathbin{\blacktriangle} (\eta * \mathrm{Shrink}^{\lfloor n,0\rfloor})\big)\Big)_k = (H\,\mathrm{id}_{\mathrm{Shrink}^{\lfloor n,0\rfloor}})_k$.

So we have $(\mathrm{Shrink}^{\lfloor n,0\rfloor} * \varepsilon) \mathbin{\blacktriangle} (\eta * \mathrm{Shrink}^{\lfloor n,0\rfloor}) = \mathrm{id}_{\mathrm{Shrink}^{\lfloor n,0\rfloor}}$.

We show that $(\varepsilon * \mathrm{Inc}^{\lfloor n,0\rfloor}) \mathbin{\blacktriangle} (\mathrm{Inc}^{\lfloor n,0\rfloor} * \eta) \overset{!}{=} \mathrm{id}_{\mathrm{Inc}^{\lfloor n,0\rfloor}}$.

Suppose given $G \in \mathrm{Ob}\big(\lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}\big)$. Let $k \in \mathbb{Z}_{\leqslant n}$ and let $g \in (G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_k$.

We have

$$\begin{aligned}
g\Big(G\big((\varepsilon * \mathrm{Inc}^{\lfloor n,0\rfloor}) \mathbin{\blacktriangle} (\mathrm{Inc}^{\lfloor n,0\rfloor} * \eta)\big)\Big)_k &= g\big((G\varepsilon)\,\mathrm{Inc}^{\lfloor n,0\rfloor} \mathbin{\blacktriangle} (G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\eta\big)_k \\
&= g\Big(\big((G\varepsilon)\,\mathrm{Inc}^{\lfloor n,0\rfloor}\big)_k \mathbin{\blacktriangle} \big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\eta\big)_k\Big) \\
&\overset{312}{=} \big(g(G\varepsilon)_k\big)\big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\eta\big)_k \\
&\overset{319}{=} g\big((G\,\mathrm{Inc}^{\lfloor n,0\rfloor})\eta\big)_k \\
&\overset{318}{=} g \\
&= g\,\mathrm{id}_{(G\,\mathrm{Inc}^{\lfloor n,0\rfloor})_k} \\
&= g(G\,\mathrm{id}_{\mathrm{Inc}^{\lfloor n,0\rfloor}})_k\,.
\end{aligned}$$

This shows that $\Big(G\big((\varepsilon * \mathrm{Inc}^{\lfloor n,0\rfloor}) \mathbin{\blacktriangle} (\mathrm{Inc}^{\lfloor n,0\rfloor} * \eta)\big)\Big)_k = (G\,\mathrm{id}_{\mathrm{Inc}^{\lfloor n,0\rfloor}})_k$.

So we have $(\varepsilon * \mathrm{Inc}^{\lfloor n,0\rfloor}) \mathbin{\blacktriangle} (\mathrm{Inc}^{\lfloor n,0\rfloor} * \eta) = \mathrm{id}_{\mathrm{Inc}^{\lfloor n,0\rfloor}}$.

Altogether, this shows that $(\mathrm{Inc}^{\lfloor n,0\rfloor}, \mathrm{Shrink}^{\lfloor n,0\rfloor}, \varepsilon, \eta)$ is an adjunction. □

Chapter 16

Stable crossed modules

16.1 The category of stable crossed modules

Definition 321 (Stable crossed modules)

Suppose given groups M_S and B_S. Suppose given a map

$$\kappa_S \colon \mathrm{B}_S \times \mathrm{B}_S \to \mathrm{M}_S\,,\ (b, c) \mapsto (b, c)\kappa_S =: \llbracket b,\, c \rrbracket\,.$$

Suppose given a group morphism $\mathrm{f}_S \colon \mathrm{M}_S \to \mathrm{B}_S$.

Suppose that the following conditions hold.

(SCM1) For b, $c \in \mathrm{B}_S$, we have $\llbracket b,\, c \rrbracket = \llbracket c,\, b \rrbracket^{-}$.

(SCM2) For b, $c \in \mathrm{B}_S$, we have $\llbracket b,\, c \rrbracket \mathrm{f}_S = [b, c]$.

(SCM3) For m, $n \in \mathrm{M}_S$, we have $\llbracket m\mathrm{f}_S,\, n\mathrm{f}_S \rrbracket = [m, n]$.

(SCM4) For a, b, $c \in \mathrm{B}_S$, we have $\llbracket a,\, b \cdot c \rrbracket = \llbracket a,\, b \rrbracket \cdot \llbracket a^b,\, c \rrbracket$.

Then we call $S := (\mathrm{M}_S\,, \mathrm{B}_S\,, \kappa_S\,, \mathrm{f}_S)$ a *stable crossed module*.

The group M_S is called the *module* of S, the group B_S is called the *base* of S and the group morphism f_S is called the *differential* of S.

For b, $c \in \mathrm{B}_S$, we call $\llbracket b,\, c \rrbracket$ the *Conduché bracket* of b and c.

Remark 322 *Suppose given a stable crossed module* $S = (\mathrm{M}_S, \mathrm{B}_S, \kappa_S, \mathrm{f}_S)$.

(1) *For* $b \in \mathrm{B}_S$, *we have*

$$\llbracket b,\, 1 \rrbracket = 1 \quad \textit{and} \quad \llbracket 1,\, b \rrbracket = 1\,.$$

(2) *For* $a,\, b,\, c \in \mathrm{B}_S$, *we have*

$$\llbracket a \cdot b,\, c \rrbracket = \llbracket b,\, c^a \rrbracket \cdot \llbracket a,\, c \rrbracket = \llbracket a^b,\, c^b \rrbracket \cdot \llbracket b,\, c \rrbracket\,.$$

(3) *For* $a,\, b,\, c \in \mathrm{B}_S$, *we have*

$$\llbracket a,\, b \cdot c \rrbracket = \llbracket a,\, b \rrbracket \cdot \llbracket a^b,\, c \rrbracket = \llbracket a,\, c \rrbracket \cdot \llbracket a^c,\, b^c \rrbracket\,.$$

(4) *For* $b,\, c \in \mathrm{B}_S$, *we have*

$$\llbracket b,\, c \rrbracket = \llbracket c^b,\, b^- \rrbracket = \llbracket c^-,\, b^c \rrbracket\,.$$

Proof. Ad (1). Suppose given $b \in \mathrm{B}_S$. We have

$$\llbracket b,\, 1 \rrbracket = \llbracket b,\, 1 \cdot 1 \rrbracket \overset{\text{(SCM4)}}{=} \llbracket b,\, 1 \rrbracket \cdot \llbracket b^1,\, 1 \rrbracket = \llbracket b,\, 1 \rrbracket \cdot \llbracket b,\, 1 \rrbracket\,.$$

This shows $1 = \llbracket b,\, 1 \rrbracket$.

Moreover, we have

$$1 = 1^- = \llbracket b,\, 1 \rrbracket^- \overset{\text{(SCM1)}}{=} \llbracket 1,\, b \rrbracket\,.$$

Ad (2). Suppose given $a,\, b,\, c \in \mathrm{B}_S$. We have

$$\begin{aligned} \llbracket a \cdot b,\, c \rrbracket &\overset{\text{(SCM1)}}{=} \llbracket c,\, a \cdot b \rrbracket^- \overset{\text{(SCM4)}}{=} (\llbracket c,\, a \rrbracket \cdot \llbracket c^a,\, b \rrbracket)^- = \llbracket c^a,\, b \rrbracket^- \cdot \llbracket c,\, a \rrbracket^- \\ &\overset{\text{(SCM1)}}{=} \llbracket b,\, c^a \rrbracket \cdot \llbracket a,\, c \rrbracket\,. \end{aligned}$$

Moreover, we have

$$\llbracket a \cdot b,\, c \rrbracket = \llbracket b \cdot a^b,\, c \rrbracket = \llbracket a^b,\, c^b \rrbracket \cdot \llbracket b,\, c \rrbracket\,.$$

Ad (3). Suppose given $a,\, b,\, c \in \mathrm{B}_S$. By (SCM4), we have

$$\llbracket a,\, b \cdot c \rrbracket = \llbracket a,\, b \rrbracket \cdot \llbracket a^b,\, c \rrbracket\,.$$

Moreover, we have

$$\llbracket a,\, b \cdot c \rrbracket = \llbracket a,\, c \cdot b^c \rrbracket = \llbracket a,\, c \rrbracket \cdot \llbracket a^c,\, b^c \rrbracket\,.$$

Ad (4). Suppose given b, $c \in \mathrm{B}_S$. We have

$$1 \overset{(1)}{=} \llbracket 1,\, c \rrbracket = \llbracket b \cdot b^-,\, c \rrbracket \overset{(2)}{=} \llbracket b^-,\, c^b \rrbracket \cdot \llbracket b,\, c \rrbracket \overset{(\mathrm{SCM1})}{=} \llbracket c^b,\, b^- \rrbracket^- \cdot \llbracket b,\, c \rrbracket .$$

This shows $\llbracket b,\, c \rrbracket = \llbracket c^b,\, b^- \rrbracket$.

Moreover, we have

$$\llbracket b,\, c \rrbracket \overset{(\mathrm{SCM1})}{=} \llbracket c,\, b \rrbracket^- = \llbracket b^c,\, c^- \rrbracket^- \overset{(\mathrm{SCM1})}{=} \llbracket c^-,\, b^c \rrbracket .$$

□

Lemma 323 *Suppose given a stable crossed module $S = (\mathrm{M}_S, \mathrm{B}_S, \kappa_S, \mathrm{f}_S)$.*
For $b \in \mathrm{B}_S$ and $m \in \mathrm{M}_S$, let $m^b := m \cdot \llbracket m\mathrm{f}_S,\, b \rrbracket$.
Suppose given a, b, $c \in \mathrm{B}_S$ and m, $n \in \mathrm{M}_S$.

(1) *We have* $(m^b)\mathrm{f}_S = (m\mathrm{f}_S)^b$.

(2) *We have* $m^{n\mathrm{f}_S} = m^n$.

(3) *We have* $m^{\llbracket b,\, c \rrbracket} = m^{[b,c]}$.

(4) *For $x \in \ker \mathrm{f}_S$, we have $x^c = x$. In particular, we have $\llbracket b,\, b \rrbracket^c = \llbracket b,\, b \rrbracket$.*

(5) *We have* $\big[m, \llbracket b,\, c \rrbracket\big] = \big\llbracket m\mathrm{f}_S,\, [b, c] \big\rrbracket$.

(6) *We have* $m^1 = m$ *and* $m^{b \cdot c} = (m^b)^c$.

(7) *We have* $(m \cdot n)^b = m^b \cdot n^b$ *and* $(m^b)^- = (m^-)^b$.

(8) *We have* $m^b = m \cdot \llbracket m\mathrm{f}_S,\, b \rrbracket = \llbracket b,\, m^-\mathrm{f}_S \rrbracket \cdot m$.

(9) *We have* $\big\llbracket [a, b],\, c \big\rrbracket = \llbracket b,\, a \rrbracket \cdot \llbracket c,\, a \rrbracket \cdot \llbracket a,\, b \cdot c \rrbracket$.

(10) *We have* $\llbracket a,\, b \rrbracket^c = \llbracket a^c,\, b^c \rrbracket$.

(11) *We have* $\big\llbracket [a, b],\, c \big\rrbracket = \llbracket b,\, a \rrbracket \cdot \llbracket a,\, b \rrbracket^c$ *and* $\big\llbracket c,\, [a, b] \big\rrbracket = \llbracket b,\, a \rrbracket^c \cdot \llbracket a,\, b \rrbracket$.

(12) *We have* $\big\llbracket [a, b],\, [c, d] \big\rrbracket = \big[\llbracket a,\, b \rrbracket, \llbracket c,\, d \rrbracket\big]$.

(13) *We have* $\llbracket a \cdot b,\, c \rrbracket = \llbracket a,\, c \rrbracket^b \cdot \llbracket b,\, c \rrbracket$ *and* $\llbracket a,\, b \cdot c \rrbracket = \llbracket a,\, c \rrbracket \cdot \llbracket a,\, b \rrbracket^c$.

(14) *We have* $\big\llbracket [b, c],\, b^- \big\rrbracket = \llbracket c,\, b \rrbracket \cdot \llbracket c,\, b^- \rrbracket$ *and* $\big\llbracket b^-,\, [b, c] \big\rrbracket = \llbracket b^-,\, c \rrbracket \cdot \llbracket b,\, c \rrbracket$.

Proof. We write $f := \mathrm{f}_S$.

Ad (1). We have

$$\begin{aligned}(m^b)f &= \big(m \cdot \llbracket mf,\, b \rrbracket\big)f = mf \cdot \llbracket mf,\, b \rrbracket f \overset{(\mathrm{SCM2})}{=} mf \cdot [mf, b] = mf \cdot m^- f \cdot b^- \cdot mf \cdot b \\ &= (mf)^b\,.\end{aligned}$$

Ad (2). We have

$$m^{nf} = m \cdot \llbracket mf,\, nf \rrbracket \overset{(\mathrm{SCM3})}{=} m \cdot [m, n] = m \cdot m^- \cdot n^- \cdot m \cdot n = m^n\,.$$

Ad (3). We have

$$m^{\llbracket b,\, c \rrbracket} \overset{(2)}{=} m^{\llbracket b,\, c \rrbracket f} \overset{(\mathrm{SCM2})}{=} m^{[b,c]}\,.$$

Ad (4). For $x \in \ker f$, we have

$$x^b = x \cdot \llbracket xf,\, b \rrbracket = x \cdot \llbracket 1,\, b \rrbracket \overset{322.(1)}{=} x \cdot 1 = x\,.$$

Moreover, we have

$$\llbracket b,\, b \rrbracket f \overset{(\mathrm{SCM2})}{=} [b, b] = 1\,.$$

Ad (5). We have

$$\llbracket b,\, mf \rrbracket \overset{(\mathrm{SCM1})}{=} \llbracket mf,\, b \rrbracket^- = (m^- \cdot m^b)^- = (m^b)^- \cdot m\,.$$

So we have

$$\begin{aligned}\Big\llbracket mf,\, [b, c] \Big\rrbracket &\overset{(\mathrm{SCM2})}{=} \Big\llbracket mf,\, \llbracket b,\, c \rrbracket f \Big\rrbracket = \big(\llbracket b,\, c \rrbracket^{mf}\big)^- \cdot \llbracket b,\, c \rrbracket \overset{(2)}{=} \big(\llbracket b,\, c \rrbracket^m\big)^- \cdot \llbracket b,\, c \rrbracket \\ &= m^- \cdot \llbracket b,\, c \rrbracket^- \cdot m \cdot \llbracket b,\, c \rrbracket = \big[m, \llbracket b,\, c \rrbracket\big]\,.\end{aligned}$$

Ad (6). We have

$$m^1 = m \cdot \llbracket mf,\, 1 \rrbracket \overset{322.(1)}{=} m \cdot 1 = m\,.$$

Moreover, we have

$$m^{b \cdot c} = m \cdot \llbracket mf,\, b \cdot c \rrbracket \overset{(\mathrm{SCM4})}{=} m \cdot \llbracket mf,\, b \rrbracket \cdot \Big\llbracket (mf)^b,\, c \Big\rrbracket \overset{(1)}{=} m^b \cdot \Big\llbracket (m^b)f,\, c \Big\rrbracket = (m^b)^c\,.$$

Ad (7). We have

$$\begin{aligned}
(m\cdot n)^b &= m\cdot n\cdot \big\llbracket (m\cdot n)f,\, b\big\rrbracket = m\cdot n\cdot \llbracket mf\cdot nf,\, b\rrbracket \\
&\overset{322.(2)}{=} m\cdot n\cdot \llbracket nf,\, b^{mf}\rrbracket\cdot \llbracket mf,\, b\rrbracket = m\cdot n^{(b^{mf})}\cdot \llbracket mf,\, b\rrbracket \\
&= m\cdot n^{m^-f\cdot b\cdot mf}\cdot \llbracket mf,\, b\rrbracket = m\cdot n^{b\cdot (b^-\cdot m^-f\cdot b\cdot mf)}\cdot \llbracket mf,\, b\rrbracket \\
&\overset{(6)}{=} m\cdot (n^b)^{[b,mf]}\cdot \llbracket mf,\, b\rrbracket \overset{(3)}{=} m\cdot (n^b)^{\llbracket b,\, mf\rrbracket}\cdot \llbracket mf,\, b\rrbracket \\
&\overset{(SCM1)}{=} m\cdot \llbracket mf,\, b\rrbracket\cdot n^b = m^b\cdot n^b\,.
\end{aligned}$$

Moreover, we have

$$m^b\cdot (m^-)^b = (m\cdot m^-)^b = 1^b = 1\cdot \llbracket 1f,\, b\rrbracket = 1\cdot \llbracket 1,\, b\rrbracket \overset{322.(1)}{=} 1\,.$$

Ad (8). We have

$$m^b \overset{(7)}{=} \big((m^-)^b\big)^- = \big(m^-\cdot \llbracket m^-f,\, b\rrbracket\big)^- \overset{(SCM1)}{=} \llbracket b,\, m^-f\rrbracket\cdot m\,.$$

Ad (9). We have

$$\begin{aligned}
\big\llbracket [a,b],\, c\big\rrbracket &= \llbracket a^-\cdot a^b,\, c\rrbracket \\
&\overset{322.(2)}{=} \big\llbracket (a^-)^{(a^b)},\, c^{(a^b)}\big\rrbracket\cdot \llbracket a^b,\, c\rrbracket \\
&\overset{322.(4)}{=} \Big\llbracket \big(c^{(a^b)}\big)^{\left((a^-)^{(a^b)}\right)},\, \big((a^-)^{(a^b)}\big)^-\Big\rrbracket\cdot \llbracket a^b,\, c\rrbracket \\
&= \Big\llbracket c^{\left(a^b\cdot (a^-)^{(a^b)}\right)},\, a^{(a^b)}\Big\rrbracket\cdot \llbracket a^b,\, c\rrbracket \\
&= \Big\llbracket c^{\left(a^-\cdot (a^b)\right)},\, a^{\left(a^-\cdot (a^b)\right)}\Big\rrbracket\cdot \llbracket a^b,\, c\rrbracket \\
&= \llbracket c^{[a,b]},\, a^{[a,b]}\rrbracket\cdot \llbracket a^b,\, c\rrbracket \\
&\overset{(SCM2)}{=} \big\llbracket c^{\llbracket a,\, b\rrbracket f},\, a^{\llbracket a,\, b\rrbracket f}\big\rrbracket\cdot \llbracket a^b,\, c\rrbracket \\
&\overset{322.(2)}{=} \big\llbracket c\cdot \llbracket a,\, b\rrbracket f,\, a\big\rrbracket\cdot \big\llbracket \llbracket a,\, b\rrbracket f,\, a\big\rrbracket^-\cdot \llbracket a^b,\, c\rrbracket \\
&\overset{322.(2)}{=} \big\llbracket \llbracket a,\, b\rrbracket f,\, a^c\big\rrbracket\cdot \llbracket c,\, a\rrbracket\cdot \big\llbracket \llbracket a,\, b\rrbracket f,\, a\big\rrbracket^-\cdot \llbracket a^b,\, c\rrbracket \\
&= \llbracket a,\, b\rrbracket^-\cdot \llbracket a,\, b\rrbracket^{a^c}\cdot \llbracket c,\, a\rrbracket\cdot \big\llbracket \llbracket a,\, b\rrbracket f,\, a\big\rrbracket^-\cdot \llbracket a^b,\, c\rrbracket \\
&\overset{(SCM1)}{=} \llbracket b,\, a\rrbracket\cdot \llbracket a,\, b\rrbracket^{a^c}\cdot \llbracket c,\, a\rrbracket\cdot \big\llbracket \llbracket a,\, b\rrbracket f,\, a\big\rrbracket^-\cdot \llbracket a^b,\, c\rrbracket \\
&= \llbracket b,\, a\rrbracket\cdot \llbracket c,\, a\rrbracket\cdot \big(\llbracket a,\, b\rrbracket^{(a^c)}\big)^{\llbracket c,\, a\rrbracket}\cdot \big\llbracket \llbracket a,\, b\rrbracket f,\, a\big\rrbracket^-\cdot \llbracket a^b,\, c\rrbracket \\
&\overset{(3)}{=} \llbracket b,\, a\rrbracket\cdot \llbracket c,\, a\rrbracket\cdot \big(\llbracket a,\, b\rrbracket^{(a^c)}\big)^{[c,a]}\cdot \big\llbracket \llbracket a,\, b\rrbracket f,\, a\big\rrbracket^-\cdot \llbracket a^b,\, c\rrbracket
\end{aligned}$$

$$\begin{aligned}
&\overset{(6)}{=} && \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, b\rrbracket^{a} \cdot \left\llbracket \llbracket a,\, b\rrbracket f,\, a\right\rrbracket^{-} \cdot \llbracket a^{b},\, c\rrbracket \\
&= && \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, b\rrbracket \cdot \left\llbracket \llbracket a,\, b\rrbracket f,\, a\right\rrbracket \cdot \left\llbracket \llbracket a,\, b\rrbracket f,\, a\right\rrbracket^{-} \cdot \llbracket a^{b},\, c\rrbracket \\
&= && \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, b\rrbracket \cdot \llbracket a^{b},\, c\rrbracket \\
&\overset{(\mathrm{SCM4})}{=} && \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, b\cdot c\rrbracket\,.
\end{aligned}$$

Ad (10). We have

$$\begin{aligned}
\llbracket a,\, b\rrbracket^{c} \quad &= && \llbracket a,\, b\rrbracket \cdot \left\llbracket \llbracket a,\, b\rrbracket f,\, c\right\rrbracket \\
&\overset{(\mathrm{SCM2})}{=} && \llbracket a,\, b\rrbracket \cdot \left\llbracket [a,b],\, c\right\rrbracket \\
&\overset{(9)}{=} && \llbracket a,\, b\rrbracket \cdot \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, b\cdot c\rrbracket \\
&\overset{322.(3)}{=} && \llbracket a,\, b\rrbracket \cdot \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, c\rrbracket \cdot \llbracket a^{c},\, b^{c}\rrbracket \\
&\overset{(\mathrm{SCM1})}{=} && \llbracket a^{c},\, b^{c}\rrbracket\,.
\end{aligned}$$

Ad (11). We have

$$\begin{aligned}
\left\llbracket [a,b],\, c\right\rrbracket \quad &\overset{(9)}{=} && \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, b\cdot c\rrbracket \\
&\overset{322.(3)}{=} && \llbracket b,\, a\rrbracket \cdot \llbracket c,\, a\rrbracket \cdot \llbracket a,\, c\rrbracket \cdot \llbracket a^{c},\, b^{c}\rrbracket \\
&\overset{(\mathrm{SCM1})}{=} && \llbracket b,\, a\rrbracket \cdot \llbracket a^{c},\, b^{c}\rrbracket \\
&\overset{(10)}{=} && \llbracket b,\, a\rrbracket \cdot \llbracket a,\, b\rrbracket^{c}\,.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\left\llbracket c,\, [a,b]\right\rrbracket \quad &\overset{(\mathrm{SCM1})}{=} && \left\llbracket [a,b],\, c\right\rrbracket^{-} = \left(\llbracket b,\, a\rrbracket \cdot \llbracket a,\, b\rrbracket^{c}\right)^{-} = \left(\llbracket a,\, b\rrbracket^{c}\right)^{-} \cdot \llbracket b,\, a\rrbracket^{-} \\
&\overset{(7)}{=} && \left(\llbracket a,\, b\rrbracket^{-}\right)^{c} \cdot \llbracket b,\, a\rrbracket^{-} \overset{(\mathrm{SCM1})}{=} \llbracket b,\, a\rrbracket^{c} \cdot \llbracket a,\, b\rrbracket\,.
\end{aligned}$$

Ad (12). We have

$$\left\llbracket [a,b],\, [c,d]\right\rrbracket \overset{(11)}{=} \llbracket b,\, a\rrbracket \cdot \llbracket a,\, b\rrbracket^{[c,d]} \overset{(3)}{=} \llbracket b,\, a\rrbracket \cdot \llbracket a,\, b\rrbracket^{\llbracket c,\, d\rrbracket} \overset{(\mathrm{SCM1})}{=} \left[\llbracket a,\, b\rrbracket, \llbracket c,\, d\rrbracket\right].$$

Alternatively, we can argue as follows. We have

$$\left\llbracket [a,b],\, [c,d]\right\rrbracket \overset{(\mathrm{SCM2})}{=} \left\llbracket \llbracket a,\, b\rrbracket f,\, [c,d]\right\rrbracket \overset{(5)}{=} \left[\llbracket a,\, b\rrbracket, \llbracket c,\, d\rrbracket\right].$$

Ad (13). We have

$$\llbracket a \cdot b,\, c \rrbracket \overset{322.(2)}{=} \llbracket a^b,\, c^b \rrbracket \cdot \llbracket b,\, c \rrbracket \overset{(10)}{=} \llbracket a,\, c \rrbracket^b \cdot \llbracket b,\, c \rrbracket .$$

Moreover, we have

$$\llbracket a,\, b \cdot c \rrbracket \overset{322.(3)}{=} \llbracket a,\, c \rrbracket \cdot \llbracket a^c,\, b^c \rrbracket \overset{(10)}{=} \llbracket a,\, c \rrbracket \cdot \llbracket a,\, b \rrbracket^c .$$

Ad (14). We have

$$\big\llbracket [b, c],\, b^- \big\rrbracket \overset{(11)}{=} \llbracket c,\, b \rrbracket \cdot \llbracket b,\, c \rrbracket^{b^-} \overset{(10)}{=} \llbracket c,\, b \rrbracket \cdot \llbracket b,\, c^{b^-} \rrbracket \overset{322.(4)}{=} \llbracket c,\, b \rrbracket \cdot \llbracket c,\, b^- \rrbracket .$$

Moreover, we have

$$\big\llbracket b^-,\, [b, c] \big\rrbracket \overset{(\mathrm{SCM1})}{=} \big\llbracket [b, c],\, b^- \big\rrbracket^- = \big(\llbracket c,\, b \rrbracket \cdot \llbracket c,\, b^- \rrbracket \big)^- \overset{(\mathrm{SCM1})}{=} \llbracket b^-,\, c \rrbracket \cdot \llbracket b,\, c \rrbracket .$$

□

Definition 324 (Stable crossed module morphism)
Suppose given stable crossed modules $S = (\mathrm{M}_S, \mathrm{B}_S, \kappa_S, \mathrm{f}_S)$ and $\tilde{S} = (\mathrm{M}_{\tilde{S}}, \mathrm{B}_{\tilde{S}}, \kappa_{\tilde{S}}, \mathrm{f}_{\tilde{S}})$.
Suppose given group morphisms $\mu\colon \mathrm{M}_S \to \mathrm{M}_{\tilde{S}}$ and $\beta\colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}$ that satisfy (1, 2).

(1) We have $\quad \mu \triangleleft \mathrm{f}_{\tilde{S}} = \mathrm{f}_S \triangleleft \beta$.

$$\begin{array}{ccc} \mathrm{M}_S & \xrightarrow{\mu} & \mathrm{M}_{\tilde{S}} \\ {\scriptstyle \mathrm{f}_S}\downarrow & & \downarrow{\scriptstyle \mathrm{f}_{\tilde{S}}} \\ \mathrm{B}_S & \xrightarrow[\beta]{} & \mathrm{B}_{\tilde{S}} \end{array}$$

(2) For $b, c \in \mathrm{B}_S$, we have $\quad \big((b, c)\kappa_S\big)\mu = (b\beta, c\beta)\kappa_{\tilde{S}}$, i.e. we have $\quad \llbracket b,\, c \rrbracket \mu = \llbracket b\beta,\, c\beta \rrbracket$.

$$\begin{array}{ccc} \mathrm{M}_S & \xrightarrow{\mu} & \mathrm{M}_{\tilde{S}} \\ {\scriptstyle \kappa_S}\uparrow & & \uparrow{\scriptstyle \kappa_{\tilde{S}}} \\ \mathrm{B}_S \times \mathrm{B}_S & \xrightarrow[\beta \times \beta]{} & \mathrm{B}_{\tilde{S}} \times \mathrm{B}_{\tilde{S}} \end{array}$$

Then $\rho := (\mu, \beta)$ is called a *stable crossed module morphism.*

Definition 325 Suppose given stable crossed modules S and $\tilde{S}$. Suppose given a stable crossed module morphism $(\mu, \beta)\colon S \to \tilde{S}$.

(1) We say that (μ, β) is *injective* if the group morphisms $\mu\colon \mathrm{M}_S \to \mathrm{M}_{\tilde{S}}$ and $\beta\colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}$ are injective.

(2) We say that (μ, β) is *surjective* if the group morphisms $\mu\colon \mathrm{M}_S \to \mathrm{M}_{\tilde{S}}$ and $\beta\colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}$ are surjective.

(3) We say that (μ, β) is *bijective* if the group morphisms $\mu\colon \mathrm{M}_S \to \mathrm{M}_{\tilde{S}}$ and $\beta\colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}$ are bijective.

Remark 326 (The category of stable crossed modules)

We have the category $\mathcal{StCrMod}$ where the objects are given by stable crossed modules and where the morphisms are given by stable crossed module morphisms; cf. Definitions 321, 324.

We call $\mathcal{StCrMod}$ the category of stable crossed modules.

(1) *For $S \xrightarrow{(\mu,\beta)} \tilde{S} \xrightarrow{(\mu',\beta')} \tilde{S}'$ in $\mathcal{StCrMod}$, their composite is given by*

$$(\mu, \beta) \mathbin{\triangle} (\mu', \beta') := (\mu \mathbin{\triangle} \mu', \beta \mathbin{\triangle} \beta')\,.$$

For $S \in \mathrm{Ob}(\mathcal{StCrMod})$, its identity is given by

$$\mathrm{id}_S = (\mathrm{id}_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})\,.$$

(2) *Suppose given stable crossed modules S and $\tilde{S}$. Suppose given a bijective stable crossed module morphism $(\mu, \beta)\colon S \to \tilde{S}$.*

Then (μ, β) is an isomorphism in $\mathcal{StCrMod}$. Its inverse is given by

$$(\mu, \beta)^- = (\mu^-, \beta^-)\colon \tilde{S} \to S\,.$$

Proof. Ad (1). Suppose given $S \xrightarrow{(\mu,\beta)} \tilde{S} \xrightarrow{(\mu',\beta')} \tilde{S}'$ in $\mathcal{StCrMod}$.

We show that $(\mu \mathbin{\triangle} \mu', \beta \mathbin{\triangle} \beta')$ is a stable crossed module morphism.

We have

$$(\mu \mathbin{\triangle} \mu') \mathbin{\triangle} \mathrm{f}_{\tilde{S}'} = \mu \mathbin{\triangle} \mathrm{f}_{\tilde{S}} \mathbin{\triangle} \beta' = \mathrm{f}_S \mathbin{\triangle} (\beta \mathbin{\triangle} \beta')\,.$$

For b, $c \in \mathrm{B}_S$, we have

$$[\![b,\, c]\!](\mu \mathbin{\triangle} \mu') = [\![b\beta,\, c\beta]\!]\mu' = [\![b(\beta \mathbin{\triangle} \beta'),\, c(\beta \mathbin{\triangle} \beta')]\!]\,.$$

So $(\mu \blacktriangle \mu', \beta \blacktriangle \beta')$ is a stable crossed module morphism; cf. Definition 324.

Suppose given $S \in \mathrm{Ob}(\mathcal{StCrMod})$.

We show that $(\mathrm{id}_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})$ *is a stable crossed module morphism.*

We have

$$\mathrm{id}_{\mathrm{M}_S} \blacktriangle \mathrm{f}_S = \mathrm{f}_S = \mathrm{f}_S \blacktriangle \mathrm{id}_{\mathrm{B}_S} .$$

For $b,\, c \in \mathrm{B}_S$, we have

$$[\![b,\, c]\!]\, \mathrm{id}_{\mathrm{B}_S} = [\![b,\, c]\!] = [\![b\, \mathrm{id}_{\mathrm{B}_S},\, c\, \mathrm{id}_{\mathrm{B}_S}]\!] .$$

So $(\mathrm{id}_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})$ is a stable crossed module morphism.

Ad (2). *We show that* (μ^-, β^-) *is stable crossed module morphism.*

We have

$$\mu^- \blacktriangle \mathrm{f}_S = \mu^- \blacktriangle (\mathrm{f}_S \blacktriangle \beta) \blacktriangle \beta^- = \mu^- \blacktriangle (\mu \blacktriangle \mathrm{f}_{\tilde{S}}) \blacktriangle \beta^- = \mathrm{f}_{\tilde{S}} \blacktriangle \beta^- .$$

For $b,\, c \in \mathrm{B}_{\tilde{S}}$, we have

$$[\![b,\, c]\!]\mu^- = [\![b\beta^-\beta,\, c\beta^-\beta]\!]\mu^- = \big([\![b\beta^-,\, c\beta^-]\!]\mu\big)\mu^- = [\![b\beta^-,\, c\beta^-]\!] .$$

So (μ, β) is a stable crossed module isomorphism. □

16.2 The ordinarification functor $\mathrm{Ord}^{\mathrm{mod}}\colon \mathcal{StCrMod} \to \mathcal{CrMod}$

Lemma 327 (The crossed module $S\mathrm{Ord}^{\mathrm{mod}}$)

Suppose given a stable crossed module $S = (\mathrm{M}_S, \mathrm{B}_S, \kappa_S, \mathrm{f}_S)$.

(1) *We have an action of* B_S *on* M_S *given by*

$$\gamma_S\colon \mathrm{B}_S \to \mathrm{Aut}(\mathrm{M}_S),\ b \mapsto \big(m \mapsto m^b = m \cdot [\![m\mathrm{f}_S,\, b]\!]\big) ;$$

cf. Lemma 323 *and cf. also Lemma* 323.(8). *In particular, we have*

$$(m^b)^c = m^{b\cdot c} \quad \textit{and} \quad (m \cdot n)^b = m^b \cdot n^b$$

for $b,\, c \in \mathrm{B}_S$ *and* $m,\, n \in \mathrm{M}_S$.

(2) *We have the crossed module*

$$S\mathrm{Ord}^{\mathrm{mod}} = (\mathrm{M}_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\boldsymbol{\gamma}_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\mathrm{f}_{S\mathrm{Ord}^{\mathrm{mod}}}) := (\mathrm{M}_S\,,\mathrm{B}_S\,,\boldsymbol{\gamma}_S\,,\mathrm{f}_S).$$

Proof. Ad (1). *We show that $\boldsymbol{\gamma}_S$ is a well-defined group morphism.*

For b, $c \in \mathrm{B}_S$ and $m \in \mathrm{M}_S$, we have

$$m^1 \overset{323.(6)}{=} m \qquad \text{and} \qquad (m^b)^c \overset{323.(6)}{=} m^{b\cdot c}\,.$$

In particular, the map $\mathrm{M}_S \to \mathrm{M}_S$, $x \mapsto x^b$ has the inverse $\mathrm{M}_S \to \mathrm{M}_S$, $x \mapsto x^{b^-}$ and is thus bijective.

Moreover, we have

$$(m \cdot n)^b \overset{323.(7)}{=} m^b \cdot n^b\,.$$

So $\boldsymbol{\gamma}_S$ is a well-defined group morphism.

Ad (2). For $b \in \mathrm{B}_S$ and $m \in \mathrm{M}_S$, we have

$$(m^b)\mathrm{f}_S \overset{323.(1)}{=} (m\mathrm{f}_S)^b\,.$$

This shows (CM1).

For m, $n \in \mathrm{M}_S$, we have

$$m^{n\mathrm{f}_S} \overset{323.(2)}{=} m^n\,.$$

This shows (CM2).

So $S\mathrm{Ord} = (\mathrm{M}_S\,,\mathrm{B}_S\,,\boldsymbol{\gamma}_S\,,\mathrm{f}_S)$ is a crossed module. □

Lemma 328 (The crossed module morphism $\rho\mathrm{Ord}^{\mathrm{mod}}$)

Suppose given a stable crossed module morphism $\rho = (\mu, \beta)\colon S \to \tilde{S}$.

(1) *For $m \in \mathrm{M}_S$ and $b \in \mathrm{B}_S$, we have*

$$(m^b)\mu = (m\mu)^{b\beta}\,;$$

cf. Lemma 327.(1).

In particular, for a, b, $c \in \mathrm{B}_S$, we have

$$(\llbracket b,\, c \rrbracket^a)\mu = \llbracket b\beta,\, c\beta \rrbracket^{a\beta}\,.$$

(2) *We consider the crossed modules* $S\mathrm{Ord}^{\mathrm{mod}}$ *and* $\tilde{S}\mathrm{Ord}^{\mathrm{mod}}$ *from Lemma* 327.
Then $\rho\mathrm{Ord}^{\mathrm{mod}} := \rho = (\mu, \beta)\colon S\mathrm{Ord}^{\mathrm{mod}} \to \tilde{S}\mathrm{Ord}^{\mathrm{mod}}$ *is a crossed module morphism.*

Sometimes, we write $\rho\mathrm{Ord} := \rho\mathrm{Ord}^{\mathrm{mod}}$.

Proof. Ad (1). For $m \in \mathrm{M}_S$ and $b \in \mathrm{B}_S$, we have

$$(m^b)\mu \overset{327.(1)}{=} (m \cdot \llbracket m\mathrm{f}_S,\, b \rrbracket)\mu = m\mu \cdot \llbracket m\mathrm{f}_S,\, b \rrbracket\mu \overset{324.(2)}{=} m\mu \cdot \big\llbracket (m\mathrm{f}_S)\beta,\, b\beta \big\rrbracket$$
$$\overset{324.(1)}{=} m\mu \cdot \big\llbracket (m\mu)\mathrm{f}_{\tilde{S}},\, b\beta \big\rrbracket = (m\mu)^{b\beta}\,.$$

In particular, for a, b, $c \in \mathrm{B}_S$, we have

$$\big(\llbracket b,\, c \rrbracket^a\big)\mu = \big(\llbracket b,\, c \rrbracket\mu\big)^{a\beta} \overset{324.(1)}{=} \llbracket b\beta,\, c\beta \rrbracket^{a\beta}\,.$$

Ad (2). We have $\mu \blacktriangleleft \mathrm{f}_{\tilde{S}} = \mathrm{f}_S \blacktriangleleft \beta$.
Moreover, for $b \in \mathrm{B}_S$ and $m \in \mathrm{M}_S$, we have, by (1),

$$(m^b)\mu = (m\mu)^{b\beta}\,.$$

This shows that $\rho\mathrm{Ord} = (\mu, \beta)$ is a crossed module morphism. □

Lemma 329 (The ordinarification functor $\mathrm{Ord}^{\mathrm{mod}}$)
We have the faithful ordinarification functor

$$\mathrm{Ord}^{\mathrm{mod}}\colon \mathcal{StCrMod} \to \mathcal{CrMod}\,,\ \big(S \xrightarrow{\rho} \tilde{S}\big) \mapsto \big(S\mathrm{Ord}^{\mathrm{mod}} \xrightarrow{\rho\mathrm{Ord}^{\mathrm{mod}}} \tilde{S}\mathrm{Ord}^{\mathrm{mod}}\big);$$

cf. Lemmas 327, 328.

Proof. We show that $\mathrm{Ord}^{\mathrm{mod}}$ *is a functor.*

Suppose given $S \xrightarrow{(\mu,\beta)} \tilde{S} \xrightarrow{(\mu',\beta')} \tilde{S}'$ in $\mathcal{StCrMod}$.
We have

$$\mathrm{id}_S = (\mathrm{id}_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S}) = \mathrm{id}_{S\mathrm{Ord}^{\mathrm{mod}}}\,.$$

We have

$$\big((\mu, \beta) \blacktriangleleft (\mu', \beta')\big)\mathrm{Ord}^{\mathrm{mod}} = (\mu \blacktriangleleft \mu', \beta \blacktriangleleft \beta')\mathrm{Ord}^{\mathrm{mod}} = (\mu \blacktriangleleft \mu', \beta \blacktriangleleft \beta') = (\mu, \beta) \blacktriangleleft (\mu', \beta')$$
$$= (\mu, \beta)\mathrm{Ord} \blacktriangleleft (\mu', \beta')\mathrm{Ord}^{\mathrm{mod}}\,.$$

So $\mathrm{Ord}^{\mathrm{mod}}$ is a functor.

We show that $\mathrm{Ord}^{\mathrm{mod}}$ *is faithful.*

Suppose given $S \underset{(\mu',\beta')}{\overset{(\mu,\beta)}{\longrightarrow}} \tilde{S}$ in $\mathcal{StCrMod}$. Suppose that we have $(\mu,\beta)\mathrm{Ord}^{\mathrm{mod}} = (\mu',\beta')\mathrm{Ord}^{\mathrm{mod}}$.

Then

$$(\mu,\beta) = (\mu,\beta)\mathrm{Ord}^{\mathrm{mod}} = (\mu',\beta')\mathrm{Ord}^{\mathrm{mod}} = (\mu',\beta')\,.$$

So $\mathrm{Ord}^{\mathrm{mod}}$ is faithful. □

Definition 330 Suppose given a stable crossed module $S = (\mathrm{M}_S\,, \mathrm{B}_S\,, \kappa_S\,, \mathrm{f}_S)$.

We consider the crossed module $S\mathrm{Ord}^{\mathrm{mod}}$; cf. Lemma 327.

We write

$$S\pi_0 := (S\mathrm{Ord}^{\mathrm{mod}})\pi_0 = \mathrm{B}_S/(\mathrm{M}_S\,\mathrm{f}_S) \quad \text{and} \quad S\pi_1 := (S\mathrm{Ord}^{\mathrm{mod}})\pi_1 = \ker \mathrm{f}_S\,;$$

cf. §1.1.4 item 2.

Remark 331 *Suppose given a stable crossed module* $S = (\mathrm{M}_S\,, \mathrm{B}_S\,, \kappa_S\,, \mathrm{f}_S)$.

We consider the groups $S\pi_0 = \mathrm{B}_S/(\mathrm{M}_S\,\mathrm{f}_S)$ *and* $S\pi_1 = \ker \mathrm{f}_S$.

Then $S\pi_0$ *and* $S\pi_1$ *are abelian groups.*

Proof. Suppose given $b,\, c \in B$. We have

$$\big[b(\mathrm{M}_S\,\mathrm{f}_S), c(\mathrm{M}_S\,\mathrm{f}_S)\big] = [b,c](\mathrm{M}_S\,\mathrm{f}_S) \overset{(\mathrm{SCM2})}{=} \llbracket b,\, c \rrbracket \mathrm{f}_S(\mathrm{M}_S\,\mathrm{f}_S) = 1(\mathrm{M}_S\,\mathrm{f}_S)\,.$$

So $S\pi_0$ is abelian.

Suppose given $m,\, n \in S\pi_1$. We have

$$[m,n] \overset{(\mathrm{SCM3})}{=} \llbracket m\mathrm{f}_s,\, n\mathrm{f}_s \rrbracket = \llbracket 1,\, 1 \rrbracket \overset{322.(1)}{=} 1\,.$$

So $S\pi_1$ is abelian; cf. also **[28,** Lem. 7.(1)**]**. □

Example 332 Suppose given a stable crossed module S.

We consider the abelian groups

$$\begin{aligned} S\pi_1 &= \ker \mathrm{f}_S \\ S\pi_0 &= \mathrm{B}_S/(\mathrm{M}_S\,\mathrm{f}_S) \\ (S\pi_0)^2 &= \{\, b^2(\mathrm{M}_S\,\mathrm{f}_S) \colon b \in \mathrm{B}_S \,\} \leqslant S\pi_0\,; \end{aligned}$$

cf. Remark 331.

(1) *We have the group morphism*

$$h''\colon \mathrm{B}_S \to S\pi_1\,,\ b \mapsto [\![b,\, b]\!]\,.$$

For $b \in \mathrm{B}_S$, we have

$$(bh'')\mathrm{f}_S = [\![b,\, b]\!]\mathrm{f}_S \overset{(\mathrm{SCM2})}{=} [b, b] = 1\,.$$

This shows $bh'' \in \ker \mathrm{f}_S = S\pi_1$.

So h'' is a well-defined map.

For $b,\, c \in \mathrm{B}_S$, we have

$$\begin{aligned}
(b \cdot c)h'' &= [\![b \cdot c,\, b \cdot c]\!] \\
&\overset{(\mathrm{SCM4})}{=} [\![b \cdot c,\, b]\!] \cdot [\![(b \cdot c)^b,\, c]\!] \\
&= [\![b \cdot c,\, b]\!] \cdot [\![c \cdot b,\, c]\!] \\
&\overset{322.(2)}{=} [\![b^c,\, b^c]\!] \cdot [\![c,\, b]\!] \cdot [\![c^b,\, c^b]\!] \cdot [\![b,\, c]\!] \\
&\overset{323.(10)}{=} [\![b,\, b]\!]^c \cdot [\![c,\, b]\!] \cdot [\![c,\, c]\!]^b \cdot [\![b,\, c]\!] \\
&\overset{(\mathrm{SCM1})}{=} [\![b,\, b]\!]^c \cdot \big([\![c,\, c]\!]^b\big)^{[\![b,\, c]\!]} \\
&\overset{323.(3)}{=} [\![b,\, b]\!]^c \cdot \big([\![c,\, c]\!]^b\big)^{[b,c]} \\
&\overset{323.(6)}{=} [\![b,\, b]\!]^c \cdot [\![c,\, c]\!]^{b\cdot[b,c]} \\
&\overset{323.(4)}{=} [\![b,\, b]\!] \cdot [\![c,\, c]\!] \\
&= bh'' \cdot ch''\,.
\end{aligned}$$

This shows that h'' is a group morphism.

(2) *We have the group morphism*

$$h'\colon S\pi_0 \to S\pi_1\,,\ b(\mathrm{M}_S\,\mathrm{f}_S) \mapsto [\![b,\, b]\!]\,.$$

It suffices to show that $(\mathrm{M}_S\,\mathrm{f}_S)h'' \overset{!}{=} 1$.

For $m \in \mathrm{M}_S$, we have

$$(m\mathrm{f}_S)h'' = [\![m\mathrm{f}_S,\, m\mathrm{f}_S]\!] \overset{(\mathrm{SCM3})}{=} [m, m] = 1\,.$$

(3) *We have the group morphism*

$$h\colon S\pi_0/(S\pi_0)^2 \to S\pi_1\,,\ \big(b(\mathrm{M}_S\,\mathrm{f}_S)\big)(S\pi_0)^2 \mapsto \llbracket b,\, b\rrbracket\,.$$

Cf. [**7,** Prop. 3.8, Rem. 3.9], [**12,** p. 1235].

It suffices to show that $\big((S\pi_0)^2\big)h' \stackrel{!}{=} 1$.

For $b \in \mathrm{B}_S$, we have

$$\begin{aligned}\big(b^2(\mathrm{M}_S\,\mathrm{f}_S)\big)h' \quad &= \quad \llbracket b^2,\, b^2\rrbracket = \llbracket b^2,\, b\cdot b\rrbracket \stackrel{(\mathrm{SCM4})}{=} \llbracket b^2,\, b\rrbracket \cdot \big\llbracket (b^2)^b,\, b\big\rrbracket = \llbracket b\cdot b,\, b\rrbracket^2\\ &\stackrel{322.(2)}{=} \big(\llbracket b^b,\, b^b\rrbracket \cdot \llbracket b,\, b\rrbracket\big)^2 = \llbracket b,\, b\rrbracket^4 \stackrel{(\mathrm{SCM1})}{=} 1\,.\end{aligned}$$

For examples of stable crossed modules with a nontrivial morphism h, see Examples 336 and 340 below.

16.3 Examples for stable crossed modules

Example 333 Suppose given a group G. Suppose given a subgroup $N \leqslant G$ with $N \leqslant \mathrm{Z}(G)$. Suppose given a subgroup $M \leqslant G$ with $G^{(1)} \leqslant M \leqslant G$.

Note that we have $M \trianglelefteqslant G$ and $N \trianglelefteqslant G$.

Consider the group G/N. Consider the group morphism

$$f\colon M \to G/N\,,\ m \mapsto mN\,.$$

Consider the map

$$\kappa\colon G/N \times G/N \to M\,,\ (gN, hN) \mapsto [g,h] =: \llbracket gN,\, hN\rrbracket\,.$$

We show that κ is a well-defined map.

For g, $h \in G$ and $n \in N$, we have

$$[g\cdot n, h] = n^-\cdot g^-\cdot h^-\cdot g\cdot n\cdot h = n^-\cdot n\cdot g^-\cdot h^-\cdot g\cdot h = [g,h]\,,$$

and we have

$$[g, h\cdot n] = g^-\cdot n^-\cdot h^-\cdot g\cdot h\cdot n = g^-\cdot h^-\cdot g\cdot h\cdot n^-\cdot n = [g,h]\,.$$

Moreover, for g, $h \in G$, we have $[g,h] \in G^{(1)} \leqslant M$.

So κ is well-defined.

We show that $(M, G/N, \kappa, f)$ is a stable crossed module.

Ad (SCM1). For g, $h \in G$, we have

$$\llbracket gN,\, hN \rrbracket = [g,h] = [h,g]^- = \llbracket hN,\, gN \rrbracket^- .$$

Ad (SCM2). For g, $h \in G$, we have

$$\llbracket gN,\, hN \rrbracket f = [g,h]f = [g,h]N = [gN\,, hN] .$$

Ad (SCM3). For m, $m' \in M$, we have

$$\llbracket mf,\, m'f \rrbracket = \llbracket mN,\, m'N \rrbracket = [m,m'] .$$

Ad (SCM4). For g, h, $k \in G$, we have

$$\begin{aligned}
\llbracket gN,\, hN \rrbracket \cdot \left\llbracket (gN)^{hN},\, kN \right\rrbracket &= \llbracket gN,\, hN \rrbracket \cdot \left\llbracket (g^h)N,\, kN \right\rrbracket = [g,h] \cdot [g^h,k] \\
&= (g^- \cdot h^- \cdot g \cdot h) \cdot (h^- \cdot g^- \cdot h \cdot k^- \cdot h^- \cdot g \cdot h \cdot k) \\
&= g^- \cdot k^- \cdot h^- \cdot g \cdot h \cdot k = [g, h \cdot k] \\
&= \left\llbracket gN,\, (h \cdot k)N \right\rrbracket = \llbracket gN,\, hN \cdot kN \rrbracket .
\end{aligned}$$

So $(M, G/N, \kappa, f)$ is a stable crossed module.

Example 334 Suppose given a group G. Suppose given a subgroup $M \leqslant G$ with $G^{(1)} \leqslant M \leqslant G$. Note that we have $M \trianglelefteqslant G$.

Consider the map

$$\lambda \colon G \times G \to M\,,\ (g,h) \mapsto [g,h] =: \llbracket g,\, h \rrbracket .$$

Consider the group morphism

$$\iota \colon M \to G\,,\ m \mapsto m .$$

We show that (M, G, λ, ι) is a stable crossed module.

In order to see that (M, G, λ, ι) is a stable crossed module, we consider the stable crossed module $(M, G/N, \kappa, f)$ from Example 333 letting $N := 1$.

Then

$$\kappa \colon G \times G \to M\,,\ (g,h) \mapsto [g,h] .$$

So $\kappa = \lambda$.

Moreover, we have

$$f\colon M \to G\,,\ m \mapsto m\,.$$

So $f = \iota$.

This shows that (M, G, λ, ι) is a stable crossed module.

Example 335 Suppose given abelian groups M and B, written multiplicatively. Suppose given a group morphism $f\colon M \to B$.

In other words, $M \xrightarrow{f} B$ can be seen as a complex in abelian groups concentrated in positions 1 and 0.

We consider the map

$$!\colon B \times B \to M\,,\ (b, c) \mapsto (b, c)! := 1\,.$$

Then $(M, B, !, f)$ *is a stable crossed module.* Note that $\llbracket b,\, c \rrbracket = (b, c)! = 1$ for $b,\, c \in B$.

Ad (SCM1). For $b,\, c \in B$, we have

$$\llbracket b,\, c \rrbracket = 1 = \llbracket c,\, b \rrbracket^{-}\,.$$

Ad (SCM2). For $b,\, c \in B$, we have

$$\llbracket b,\, c \rrbracket f = 1f = 1 = [b, c]\,.$$

Ad (SCM3). For $m,\, n \in M$, we have

$$\llbracket mf,\, nf \rrbracket = 1 = [m, n]\,.$$

Ad (SCM4). For $a,\, b,\, c \in B$, we have

$$\llbracket a,\, b \cdot c \rrbracket = 1 = 1 \cdot 1 = \llbracket a,\, b \rrbracket \cdot \llbracket a^b,\, c \rrbracket\,.$$

So $(M, B, !, f)$ is a stable crossed module.

Example 336 Suppose given abelian groups M and B written multiplicatively. Suppose given a skew-symmetric $\mathbb{Z}$-bilinear map

$$\kappa\colon B \times B \to M\,,\ (b, c) \mapsto (b, c)\kappa =: \llbracket b,\, c \rrbracket\,.$$

Consider the group morphism

$$!\colon M \to B\,,\ m \mapsto 1\,.$$

(1) *We show that* $(M, B, \kappa, !)$ *is a stable crossed module.*

Ad (SCM1). For b, $c \in B$, we have

$$\llbracket b,\, c \rrbracket = (b,c)\kappa = \big((c,b)\kappa\big)^{-} = \llbracket c,\, b \rrbracket^{-}$$

since κ is skew-symmetric.

Ad (SCM2). For b, $c \in B$, we have

$$\llbracket b,\, c \rrbracket ! = 1 = [b,c]$$

since B is abelian.

Ad (SCM3). For m, $n \in M$, we have

$$\llbracket m!,\, n! \rrbracket = \llbracket 1,\, 1 \rrbracket = (1,1)\kappa = 1 = [m,n]$$

since κ is $\mathbb{Z}$-bilinear and since M is abelian.

Ad (SCM4). For a, b, $c \in B$, we have

$$\llbracket a,\, b \cdot c \rrbracket = (a, b\cdot c)\kappa = (a,b)\kappa \cdot (a,c)\kappa = (a,b)\kappa \cdot (a^b, c)\kappa = \llbracket a,\, b \rrbracket \cdot \llbracket a^b,\, c \rrbracket$$

since κ is $\mathbb{Z}$-bilinear and since B is abelian.

So $(M, B, \kappa, !)$ is a stable crossed module.

(2) Let

$$\begin{aligned} M &= \langle m \colon m^2 = 1 \rangle \\ B &= \langle b \colon b^2 = 1 \rangle \\ \kappa &\colon B \times B \to M\,,\ (b^i, b^j) \mapsto m^{i\cdot j}\,. \end{aligned}$$

Then, by (1), we have the stable crossed module $(M, B, \kappa, !)$.

(3) We consider the group morphism h from Example 332.(3) with $S := (M, B, \kappa, !)$ from (2) given as follows.

$$h \colon S\pi_0/(S\pi_0)^2 \to S\pi_1\,,\ \big(b^i(M!)\big)(S\pi_0)^2 \mapsto \llbracket b^i,\, b^i \rrbracket\,.$$

Then h is nontrivial since we have

$$\Big(\big(b(M!)\big)(S\pi_0)^2\Big)h = \llbracket b,\, b \rrbracket = m^{1\cdot 1} = m \neq 1\,.$$

Remark 337 *Suppose given a group G.*

We consider the crossed module $V := (1, G, \gamma, !)$ with

$$\gamma\colon G \to \mathrm{Aut}(1)\,,\ g \mapsto \mathrm{id}_1 \quad \text{and} \quad !\colon 1 \to G\,,\ 1 \mapsto 1\,;$$

cf. e.g. [**28,** Rem. 10.(1)].

We consider the functor $\mathrm{Ord}^{\mathrm{mod}}\colon \mathcal{StCrMod} \to \mathcal{CrMod}$ *from Lemma* 329.

There exists some stable crossed module $S \in \mathrm{Ob}(\mathcal{StCrMod})$ with $S\mathrm{Ord}^{\mathrm{mod}} \simeq V$ if and only if G is abelian.

In particular, the functor $\mathrm{Ord}^{\mathrm{mod}}$ *is not dense.*

Proof. Ad $\Rightarrow$. Suppose that we may choose some $S = (\mathrm{M}_S\,, \mathrm{B}_S\,, \kappa_S\,, \mathrm{f}_S) \in \mathrm{Ob}(\mathcal{StCrMod})$ with $S\mathrm{Ord}^{\mathrm{mod}} \simeq V$.

Then we have

$$\mathrm{M}_S = 1\,, \quad \mathrm{B}_S \simeq G\,, \quad \kappa_S\colon \mathrm{B}_S \times \mathrm{B}_S \to 1\,,\ (b, c) \mapsto 1 \quad \text{and} \quad \mathrm{f}_S = !\,.$$

For b, $c \in \mathrm{B}_S$, we have

$$[b, c] \overset{(\mathrm{SCM2})}{=} \llbracket b,\ c \rrbracket \mathrm{f}_S = 1\,.$$

This shows that B_S and thus G is abelian.

Ad $\Leftarrow$. Suppose that G is abelian.

Then, by Example 334 or by Example 336, we have the stable crossed module $S := (1, G, \kappa, !)$ with

$$\kappa\colon G \times G \to 1\,,\ (g, h) \mapsto 1\,.$$

We consider the crossed module $S\mathrm{Ord}^{\mathrm{mod}} = (\mathrm{M}_{S\mathrm{Ord}^{\mathrm{mod}}}\,, \mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}}\,, \boldsymbol{\gamma}_{S\mathrm{Ord}^{\mathrm{mod}}}\,, \mathrm{f}_{S\mathrm{Ord}^{\mathrm{mod}}})$.

We have

$$\mathrm{M}_{S\mathrm{Ord}^{\mathrm{mod}}} = 1\,, \quad \mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}} = G \quad \text{and} \quad \mathrm{f}_{S\mathrm{Ord}^{\mathrm{mod}}} = !\,.$$

Moreover, we have

$$\boldsymbol{\gamma}_{S\mathrm{Ord}^{\mathrm{mod}}}\colon G \to \mathrm{Aut}(1)\,,\ g \mapsto \big(1 \mapsto 1 \cdot (1\mathrm{f}_{S\mathrm{Ord}^{\mathrm{mod}}}\,, g)\kappa = 1\big)\,.$$

So $\boldsymbol{\gamma}_{S\mathrm{Ord}^{\mathrm{mod}}} = \gamma$.

This shows $S\mathrm{Ord}^{\mathrm{mod}} = V$. □

Example 338 Let $M := \langle m \rangle$ be the free abelian group generated by the element m.
Let $B := \langle x, y \colon [x, y] = 1 \rangle$ be the free abelian group generated by the elements x and y.
Consider the group morphism

$$! \colon M \to B\,,\ m \mapsto 1 = x^0 \cdot y^0\,.$$

Consider the skew-symmetric $\mathbb{Z}$-bilinear maps

$$\kappa_1 \colon B \times B \to M\,,\ (x^{a_1} \cdot y^{a_2}, x^{b_1} \cdot y^{b_2}) \mapsto m^{a_1 \cdot b_2 - a_2 \cdot b_1}$$
$$\kappa_2 \colon B \times B \to M\,,\ (x^{a_1} \cdot y^{a_2}, x^{b_1} \cdot y^{b_2}) \mapsto 1\,,$$

where $a_1\,,\ a_2\,,\ b_1\,,\ b_2 \in \mathbb{Z}$.
By Example 336, we have the stable crossed modules $S_1 := (M, B, \kappa_1, !)$ and $S_2 := (M, B, \kappa_2, !)$.
We show that $S_1 \overset{!}{\ncong} S_2$ *.*
We *assume* that there exists an isomorphism $(\mu, \beta) \colon S_1 \to S_2$; cf. Remark 326.(2).
Then we have

$$[\![x,\, y]\!]\mu = \big((x, y)\kappa_1\big)\mu = (m^{1 \cdot 1 - 0 \cdot 0})\mu = m\mu \neq 1\,,$$

and

$$[\![x,\, y]\!]\mu = [\![x\beta,\, y\beta]\!] = (x\beta, y\beta)\kappa_2 = 1\,,$$

which is a *contradiction.*
This shows $S_1 \ncong S_2$.
We consider the functor $\mathrm{Ord}^{\mathrm{mod}} \colon \mathcal{StCrMod} \to \mathcal{CrMod}$ from Lemma 329.
Consider the crossed modules

$$S_1\mathrm{Ord}^{\mathrm{mod}} =: (M_1\,, B_1\,, \gamma_1\,, f_1)$$
$$S_2\mathrm{Ord}^{\mathrm{mod}} =: (M_2\,, B_2\,, \gamma_2\,, f_2)\,.$$

We show that $S_1\mathrm{Ord}^{\mathrm{mod}} \overset{!}{=} S_2\mathrm{Ord}^{\mathrm{mod}}$ *.*
We have

$$M_1 = M_2 = M$$
$$B_1 = B_2 = B$$
$$f_1 = f_2 = !\,.$$

For b_1 , $b_2 \in \mathbb{Z}$, we have

$$m\big((x^{b_1} \cdot y^{b_2})\gamma_1\big) = m \cdot (m!, x^{b_1} \cdot y^{b_2})\kappa_1 = m \cdot (1, x^{b_1} \cdot y^{b_2})\kappa_1 = m \cdot 1 = m\,.$$

So $\gamma_1 = !$.

Moreover, for b_1 , $b_2 \in \mathbb{Z}$, we have

$$m\big((x^{b_1} \cdot y^{b_2})\gamma_2\big) = m \cdot (m!, x^{b_1} \cdot y^{b_2})\kappa_2 = m \cdot 1 = m\,.$$

So $\gamma_2 = !$.

So we have $\gamma_1 = ! \ = \gamma_2$.

This shows $S_1\mathrm{Ord}^{\mathrm{mod}} = S_2\mathrm{Ord}^{\mathrm{mod}}$.

Remark 339 *The functor* $\mathrm{Ord}^{\mathrm{mod}}\colon \mathcal{StCrMod} \to \mathcal{CrMod}$ *from Lemma* 329 *is faithful, but not full.*

Proof. The functor $\mathrm{Ord}^{\mathrm{mod}}$ is faithful by construction; cf. Lemma 329.

In Example 338, we have stable crossed modules S_1 and S_2 where $S_1 \not\cong S_2$ and where $S_1\mathrm{Ord}^{\mathrm{mod}} = S_2\mathrm{Ord}^{\mathrm{mod}}$.

This shows that $\mathrm{Ord}^{\mathrm{mod}}$ is not full and faithful. Hence, $\mathrm{Ord}^{\mathrm{mod}}$ is not full. □

Example 340 Consider the groups $M := \langle m\colon m^4 = 1\rangle$ and $B := \langle b\colon b^4 = 1\rangle$.

Consider the group morphism $f\colon M \to B\,,\ m \mapsto b^2$.

Consider the map

$$\kappa\colon B \times B \to M\,,\ (b^i, b^j) \mapsto \llbracket b^i,\, b^j \rrbracket := m^{2\cdot i\cdot j}\,.$$

(1) *We show that* (M, B, κ, f) *is a stable crossed module.*

Suppose given i, j, $\ell \in [0, 3]$.

Ad (SCM1). We have

$$\llbracket b^i,\, b^j \rrbracket = m^{2\cdot i\cdot j} = m^{-2\cdot i\cdot j} = (m^{2\cdot j\cdot i})^- = \llbracket b^j,\, b^i \rrbracket^-\,.$$

Ad (SCM2). We have

$$\llbracket b^i,\, b^j \rrbracket f = (m^{2\cdot i\cdot j})f = (mf)^{2\cdot i\cdot j} = (b^2)^{2\cdot i\cdot j} = (b^4)^{i\cdot j} = 1 = [b^i, b^j]\,.$$

Ad (SCM3). We have

$$\big\llbracket (m^i)f,\, (m^j)f \big\rrbracket = \big\llbracket (mf)^i,\, (mf)^j \big\rrbracket = \llbracket b^{2i},\, b^{2j} \rrbracket = m^{2\cdot 2i\cdot 2j} = (m^4)^{2\cdot i\cdot j} = 1 = [m^i, m^j]\,.$$

Ad (SCM4). We have

$$\begin{aligned}\left\llbracket b^i,\, b^j \cdot b^\ell \right\rrbracket &= \left\llbracket b^i,\, b^{j+\ell} \right\rrbracket = m^{2\cdot i\cdot (j+\ell)} = m^{2\cdot i\cdot j + 2\cdot i\cdot \ell} = m^{2\cdot i\cdot j} \cdot m^{2\cdot i\cdot \ell} = \left\llbracket b^i,\, b^j \right\rrbracket \cdot \left\llbracket b^i,\, b^\ell \right\rrbracket \\ &= \left\llbracket b^i,\, b^j \right\rrbracket \cdot \left\llbracket (b^i)^{b^j},\, b^\ell \right\rrbracket .\end{aligned}$$

So (M, B, κ, f) is stable crossed module.

(2) Consider the functor $\mathrm{Ord}^{\mathrm{mod}} \colon \mathcal{StCrMod} \to \mathcal{CrMod}$ from Lemma 329.

We consider the crossed module $V = (\mathrm{M}_V\,, \mathrm{B}_V\,, \gamma_V\,, \mathrm{f}_V) := (M, B, \kappa, f)\mathrm{Ord}^{\mathrm{mod}}$.

Then we have

$$\begin{aligned}\mathrm{M}_V &= M \\ \mathrm{B}_V &= B \\ \gamma_V &\colon B \to \mathrm{Aut}(M)\,,\ b \mapsto \left(m \mapsto m^b = m \cdot \left\llbracket m\mathrm{f}_V,\, b \right\rrbracket\right) \\ \mathrm{f}_V &= f\,.\end{aligned}$$

Note that we have

$$m(b\gamma_V) = m^b = m \cdot \left\llbracket m\mathrm{f}_V,\, b \right\rrbracket = m \cdot \left\llbracket b^2,\, b \right\rrbracket = m \cdot m^{2\cdot 2\cdot 1} = m \cdot 1 = m\,.$$

This shows $b\gamma_V = \mathrm{id}_M$.

Therefore, we have $\gamma_V \colon B \to \mathrm{Aut}(M)\,,\ b \mapsto \mathrm{id}_M$, i.e. $\gamma_V = !$.

(3) We consider the crossed module $\tilde{V} = (\tilde{M}, \tilde{B}, \tilde{\gamma}, \tilde{f})$ defined in [**27,** Ch.V, §1, (5.6)] with

$$\begin{aligned}\tilde{M} &= \langle \tilde{m} \colon \tilde{m}^4 = 1 \rangle \\ \tilde{B} &= \langle \tilde{b} \colon \tilde{b}^4 = 1 \rangle \\ \tilde{\gamma} &\colon \tilde{B} \to \tilde{M}\,,\ \tilde{b} \mapsto \left(\tilde{m} \mapsto \tilde{m}^{\tilde{b}} := \tilde{m}^-\right) \\ \tilde{f} &\colon \tilde{M} \to \tilde{B}\,,\ \tilde{m} \mapsto \tilde{b}^2\,.\end{aligned}$$

We show that $V \not\cong \tilde{V}$.

We *assume* that there exists an isomorphism $(\mu, \beta) \colon V \to \tilde{V}$; cf. §1.1.4 item 3.

Note that we have

$$m \cdot m^b = m \cdot m = m^2\,.$$

Moreover, we have

$$\beta \colon B \to \tilde{B}\,,\ b \mapsto \tilde{b}^j$$

for either $j = 1$ or $j = 3$ since β is a group isomorphism.

So we have

$$\begin{aligned} 1 = 1\mu \neq m^2\mu = (m \cdot m^b)\mu = m\mu \cdot (m^b)\mu &\overset{\text{(CM1)}}{=} m\mu \cdot (m\mu)^{b\beta} \\ = m\mu \cdot (m\mu)^{\tilde{b}^j} &\overset{j=1 \text{ or } j=3}{=} m\mu \cdot (m\mu)^{-} = 1\,, \end{aligned}$$

which is a *contradiction*.

This shows $V \not\cong \tilde{V}$.

(4) Consider the following group morphism h from Example 332.(3) with $S := (M, B, \kappa, f)$; cf. (1).

$$h\colon S\pi_0/(S\pi_0)^2 \to S\pi_1\,,\ \big(b^i(Mf)\big)(S\pi_0)^2 \mapsto \llbracket b^i,\, b^i \rrbracket$$

Then h is nontrivial since we have

$$\Big(\big(b(Mf)\big)(S\pi_0)^2\Big)h = \llbracket b,\, b \rrbracket = m^{2\cdot 1\cdot 1} = m^2 \neq 1\,.$$

Chapter 17

Construction of the group H_∞ from a stable crossed module

Suppose given a stable crossed module $S = (\mathrm{M}_S\,,\, \mathrm{B}_S\,,\, \kappa_S\,,\, \mathrm{f}_S)$.
We often write $M := \mathrm{M}_S\,,\, B := \mathrm{B}_S\,.$

17.1 The group H_n

17.1.1 The construction of the iterated semidirect product H_n

Lemma 341

Let G and H be groups. Suppose given an action of G on H as follows.

$$\beta\colon G \to \mathrm{Aut}(H)\,,\ x \mapsto (h \mapsto h^x)$$

Suppose given a map $u\colon G \times G \to H$ such that (1, 2, 3) are satisfied.

(1) *For x, y, $z \in G$, we have*

$$(x \cdot y, z)u = (y, z^x)u \cdot (x, z)u\,.$$

(2) *For x, y, $z \in G$, we have*

$$(x, y \cdot z)u = (x, z)u \cdot \big((x, y)u\big)^z\,.$$

(3) *For x, $y \in G$ and $h \in H$, we have*

$$h \cdot (x,y)u = (x,y)u \cdot h^{[x,y]}\,,$$

i.e., we have

$$h^{(x,y)u} = h^{[x,y]}\,.$$

Then we have an action of G on $G \ltimes_\beta H$ given by the following group morphism.

$$\alpha\colon G \to \mathrm{Aut}(G \ltimes_\beta H)\,,\ x \mapsto \Big((g,h) \mapsto (g,h)^x := \big(g^x, (x,g)u \cdot h\big)\Big)$$

Proof. For $z \in G$, note that we have

$$(1 \cdot 1, z)u \overset{(1)}{=} (1, z^1)u \cdot (1,z)u\,.$$

So $(1,z)u = 1$ for $z \in G$.

Suppose given g, $\tilde{g} \in G$ and h, $\tilde{h} \in H$.

We have

$$(g,h)^1 = \big(g^1, (1,g)u \cdot h\big) = (g, 1 \cdot h) = (g,h)\,.$$

This shows $1\alpha = \mathrm{id}_{G \ltimes_\beta H}$.

For $x \in G$, we have

$$\begin{aligned}
(g,h)^x \cdot (\tilde{g},\tilde{h})^x &= \big(g^x, (x,g)u \cdot h\big) \cdot \big(\tilde{g}^x, (x,\tilde{g})u \cdot \tilde{h}\big) \\
&= \Big(g^x \cdot \tilde{g}^x, \big((x,g)u \cdot h\big)^{(\tilde{g}^x)} \cdot (x,\tilde{g})u \cdot \tilde{h}\Big) \\
&\overset{(3)}{=} \Big((g \cdot \tilde{g})^x, (x,\tilde{g})u \cdot \big((x,g)u \cdot h\big)^{(\tilde{g}^x \cdot [x,\tilde{g}])} \cdot \tilde{h}\Big) \\
&= \Big((g \cdot \tilde{g})^x, (x,\tilde{g})u \cdot \big((x,g)u \cdot h\big)^{\tilde{g}} \cdot \tilde{h}\Big) \\
&= \Big((g \cdot \tilde{g})^x, (x,\tilde{g})u \cdot \big((x,g)u\big)^{\tilde{g}} \cdot h^{\tilde{g}} \cdot \tilde{h}\Big) \\
&\overset{(2)}{=} \big((g \cdot \tilde{g})^x, (x, g \cdot \tilde{g})u \cdot h^{\tilde{g}} \cdot \tilde{h}\big) \\
&= (g \cdot \tilde{g}, h^{\tilde{g}} \cdot \tilde{h})^x \\
&= \big((g,h) \cdot (\tilde{g},\tilde{h})\big)^x\,.
\end{aligned}$$

So given $x \in G$, the map $x\alpha\colon G \ltimes_\beta H \to G \ltimes_\beta H$, $(g,h) \mapsto (g,h)^x$ is a group morphism.

For x, $y \in G$, we have

$$\begin{aligned}\big((g,h)^x\big)^y &= \big(g^x, (x,g)u\cdot h\big)^y = \big((g^x)^y, (y,g^x)u\cdot(x,g)u\cdot h\big) \overset{(1)}{=} \big(g^{x\cdot y}, (x\cdot y, g)u\cdot h\big)\\ &= (g,h)^{x\cdot y}\,.\end{aligned}$$

This shows $x\alpha \blacktriangle y\alpha = (x\cdot y)\alpha$.

We conclude that, for $x \in G$, the map $x\alpha$ has the inverse $x^-\alpha$.

So we obtain $x\alpha \in \mathrm{Aut}(G \ltimes_\beta H)$. We can conclude that $\alpha\colon G \to \mathrm{Aut}(G \ltimes_\beta H)$ is a group morphism. □

Lemma 342 *For $n \in \mathbb{Z}_{\geqslant -1}$, there exist uniquely*

- *a group* H_n
- *a map* $u_n\colon B \times B \to H_n$
- *a group morphism* $\alpha_n\colon B \to \mathrm{Aut}(H_n)$

that satisfy the following conditions (i_n - ix_n).

(i_n) *We have*

$$H_n = \begin{cases} M & \text{if } n = -1\\ B \ltimes_{\alpha_{n-1}} H_{n-1} & \text{if } n \geqslant 0\,.\end{cases}$$

(ii_n) *We have*

$$u_n\colon B\times B \to H_n\,,\ (x,y) \mapsto \begin{cases} (x,y)\kappa_S = [\![x,\, y]\!] & \text{if } n = -1\\ \big([x,y],(y,x)u_{n-1}\big) & \text{if } n \geqslant 0\,.\end{cases}$$

(iii_n) *We have*

$$\alpha_n = B \to \mathrm{Aut}(H_n)\,,\ x \mapsto \begin{cases} x\alpha_{-1} = x\gamma_S = \big(m \mapsto m^x \overset{327.(1)}{=} m\cdot[\![m\mathrm{f}_S,\, x]\!]\big) & \text{if } n = -1\\ x\alpha_n = \Big((b,h) \mapsto \big(b^x, (x,b)u_{n-1}\cdot h\big)\Big) & \text{if } n \geqslant 0\,.\end{cases}$$

(iv_n) *For x, $y \in B$, we have*

$$\big([x,y],[y,x]\big)u_n = 1\,.$$

(v_n) *For x, $y \in B$, we have*

$$\big((x,y)u_n\big)\big([y,x]\alpha_n\big) = (x,y)u_n\,.$$

(vi_n) *For x, $y \in B$, we have*

$$\big((x,y)u_n\big)^- = (y,x)u_n\,.$$

(vii_n) *For x, y, $z \in B$, we have*

$$(x\cdot y,z)u_n = (y,z^x)u_n \cdot (x,z)u_n\,.$$

(viii_n) *For x, y, $z \in B$, we have*

$$(x,y\cdot z)u_n = (x,z)u_n \cdot \big((x,y)u_n\big)(z\alpha_n)\,.$$

(ix_n) *For x, $y \in B$ and $h \in H_n$, we have*

$$h\cdot(x,y)u_n = (x,y)u_n \cdot \Big(h\big([x,y]\alpha_n\big)\Big)\,,$$

i.e., *we have*

$$h^{(x,y)u_n} = h\big([x,y]\alpha_n\big)\,.$$

Of course, (v_n) is a particular case of (ix_n), using (vi_n). In the proof of the induction step, this redundancy will turn out to be convenient.

If $n=-1$, for $h \in H_{-1}$, we write $h_{\{-1\}} := h$ and we write $h_{\{k\}} := 1$ for $k \in \mathbb{Z}_{\geqslant 0}$.

If $n \geqslant 0$, for $h = (b,h') \in H_n = B \ltimes_{\alpha_{n-1}} H_{n-1}$ and $k \in [-1,n]$, we write

$$h_{\{k\}} := \begin{cases} 1 & \text{for } k \geqslant n+1 \\ b & \text{for } k = n \\ h'_{\{k\}} & \text{for } k \in [-1,n-1]\,. \end{cases}$$

So, for $h \in H_n$, we have $h = (h_{\{n\}}, h_{\{n-1\}}, \ldots, h_{\{-1\}})$, omitting inner brackets in notation.

Proof. The conditions uniquely determine H_n, u_n and α_n for $n \in \mathbb{Z}_{\geqslant -1}$.

We have to show existence.

By induction on $n \in \mathbb{Z}_{\geqslant -1}$, we show that α_n is a group morphism and that (i_n – ix_n) hold.

Let $n=-1$. We have $H_{-1} = M$. For x, $y \in B$, we have $(x,y)u_{-1} = \llbracket x, y \rrbracket$. Moreover, for $m \in M$ and $x \in B$, we have $(m)(x\alpha_{-1}) = m^x = m \cdot \llbracket m\mathrm{f}_S, x \rrbracket$.

It suffices to show (iv_{-1} – ix_{-1}).

Ad (iv_{-1}). For $x,\, y \in B$, we have

$$\big([x,y],[y,x]\big)u_{-1} = \big[\!\big[[x,y],\,[y,x]\big]\!\big] \overset{323.(12)}{=} \big[[\![x,\,y]\!],[\![y,\,x]\!]\big] \overset{(\mathrm{SCM1})}{=} 1\,.$$

Ad (v_{-1}). For $x,\, y \in B$, we have

$$\big((x,y)u_{-1}\big)\big([y,x]\alpha_{-1}\big) = [\![x,\,y]\!]^{[y,x]} \overset{323.(3)}{=} [\![x,\,y]\!]^{[\![y,\,x]\!]} \overset{(\mathrm{SCM1})}{=} [\![x,\,y]\!] = (x,y)u_{-1}\,.$$

Ad (vi_{-1}). For $x,\, y \in B$, we have

$$\big((x,y)u_{-1}\big)^- = [\![x,\,y]\!]^- \overset{(\mathrm{SCM1})}{=} [\![y,\,x]\!] = (y,x)u_{-1}\,.$$

Ad (vii_{-1}). For $x,\, y,\, z \in B$, we have

$$(x\cdot y,z)u_{-1} = [\![x\cdot y,\,z]\!] \overset{322.(2)}{=} [\![y,\,z^x]\!]\cdot[\![x,\,z]\!] = (y,z^x)u_{-1}\cdot(x,z)u_{-1}\,.$$

Ad (viii_{-1}). For $x,\, y,\, z \in B$, we have

$$(x,y\cdot z)u_{-1} = [\![x,\,y\cdot z]\!] \overset{323.(13)}{=} [\![x,\,z]\!]\cdot[\![x,\,y]\!]^z = (x,z)u_{-1}\cdot\big((x,y)u_{-1}\big)(z\alpha_{-1})\,.$$

Ad (ix_{-1}). For $x,\, y \in B$ and $h \in H_{-1} = M$, we have

$$h^{(x,y)u_{-1}} = h^{[\![x,\,y]\!]} \overset{323.(3)}{=} = h^{[x,y]} = h\big([x,y]\alpha_{-1}\big)\,.$$

Suppose given $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given groups H_k, maps $u_k\colon B\times B \to H_k$ and group morphisms $\alpha_k\colon B \to \mathrm{Aut}(H_k)$ for $k \in [-1,n]$ that satisfy (i_k – ix_k) for $k \in [-1,n]$.

We have to show that there exist a group H_{n+1}, a map $u_{n+1}\colon B\times B \to \mathrm{Aut}(H_{n+1})$ and a group morphism $\alpha_{n+1}\colon B \to \mathrm{Aut}(H_{n+1})$ such that (i_{n+1} – ix_{n+1}) hold.

We may take the group

$$H_{n+1} := B \ltimes_{\alpha_n} H_n\,,$$

so that (i_{n+1}) holds.

We may take the map

$$u_{n+1}\colon B\times B \to H_{n+1}\,,\ (x,y) \mapsto \big([x,y],(y,x)u_n\big)\,,$$

so that (ii_{n+1}) holds.

Since (vii_n – ix_n) hold, Lemma 341 yields the group morphism

$$\alpha_{n+1}\colon B \to \mathrm{Aut}(H_{n+1}) = \mathrm{Aut}(B \ltimes_\beta H_n)\,,\ x \mapsto \Big((b,h) \mapsto \big(b^x, (x,b)u_n \cdot h\big)\Big)\,.$$

So (iii_{n+1}) holds.

Ad (iv_{n+1}). For x, $y \in B$, we have

$$\big([x,y],[y,x]\big)u_{n+1} = \Big(\big[[x,y],[y,x]\big], \big([y,x],[x,y]\big)u_n\Big) \overset{(\mathrm{iv}_n)}{=} (1,1) = 1\,.$$

Ad (v_{n+1}). For x, $y \in B$, we have

$$\begin{aligned}\big((x,y)u_{n+1}\big)\big([y,x]\alpha_{n+1}\big) = \big([x,y],(y,x)u_n\big)\big([y,x]\alpha_{n+1}\big) &= \Big([x,y]^{[y,x]}, \big([y,x],[x,y]\big)u_n \cdot (y,x)u_n\Big)\\ &\overset{(\mathrm{iv}_n)}{=} \big([x,y], 1\cdot(y,x)u_n\big) = (x,y)u_{n+1}\,.\end{aligned}$$

Ad (vi_{n+1}). For x, $y \in B$, we have

$$\begin{aligned}\big((x,y)u_{n+1}\big)^- &= \big([x,y],(y,x)u_n\big)^-\\ &= \Big([x,y]^-, \big(\big((y,x)u_n\big)^-\big)\big(\big([x,y]^-\big)\alpha_n\big)\Big)\\ &\overset{(\mathrm{vi}_n)}{=} \Big([y,x], \big((x,y)u_n\big)\big(\big([y,x]\big)\alpha_n\big)\Big)\\ &\overset{(\mathrm{v}_n)}{=} \big([y,x],(x,y)u_n\big)\\ &= (y,x)u_{n+1}\,.\end{aligned}$$

Ad (vii_{n+1}). For x, y, $z \in B$, we have

$$\begin{aligned}(y,z^x)u_{n+1}\cdot(x,z)u_{n+1} &= \big([y,z^x],(z^x,y)u_n\big)\cdot\big([x,z],(z,x)u_n\big)\\ &= \Big(\big[y,z^x\big]\cdot[x,z], \big((z^x,y)u_n\big)\big([x,z]\alpha_n\big)\cdot(z,x)u_n\Big)\\ &\overset{(\mathrm{ix}_n)}{=} \Big([x\cdot y,z], \big((z^x,y)u_n\big)^{(x,z)u_n}\cdot(z,x)u_n\Big)\\ &\overset{(\mathrm{vi}_n)}{=} \big([x\cdot y,z],(z,x)u_n\cdot(z^x,y)u_n\big)\\ &\overset{(\mathrm{vi}_n)}{=} \Big([x\cdot y,z],\big((y,z^x)u_n\cdot(x,z)u_n\big)^-\Big)\end{aligned}$$

$$\begin{aligned}
&\overset{(\mathrm{vii}_n)}{=} && \Big([x\cdot y,z],\big((x\cdot y,z)u_n\big)^-\Big)\\
&\overset{(\mathrm{vi}_n)}{=} && \big([x\cdot y,z],(z,x\cdot y)u_n\big)\\
&= && (x\cdot y,z)u_{n+1}\,.
\end{aligned}$$

Ad (viii_{n+1}). For x, y, $z\in B$, we have

$$\begin{aligned}
& && (x,z)u_{n+1}\cdot\big((x,y)u_{n+1}\big)(z\alpha_{n+1})\\
&= && \big([x,z],(z,x)u_n\big)\cdot\big([x,y],(y,x)u_n\big)(z\alpha_{n+1})\\
&= && \big([x,z],(z,x)u_n\big)\cdot\Big([x,y]^z,\big(z,[x,y]\big)u_n\cdot(y,x)u_n\Big)\\
&= && \Big([x,z]\cdot[x,y]^z,\big((z,x)u_n\big)\big([x,y]^z\big)\alpha_n\cdot\big(z,[x,y]\big)u_n\cdot(y,x)u_n\Big)\\
&\overset{(\mathrm{ix}_n)}{=} && \Big([x,y\cdot z],\big(z,[x,y]\big)u_n\cdot\Big(\big((z,x)u_n\big)\Big(\big([x,y]^z\big)\alpha_n\Big)\Big)\Big(\big[z,[x,y]\big]\alpha_n\Big)\cdot(y,x)u_n\Big)\\
&= && \Big([x,y\cdot z],\big(z,[x,y]\big)u_n\cdot\big((z,x)u_n\big)\Big(\Big([x,y]^z\cdot\big[z,[x,y]\big]\Big)\alpha_n\Big)\cdot(y,x)u_n\Big)\\
&= && \Big([x,y\cdot z],\big(z,[x,y]\big)u_n\cdot\big((z,x)u_n\big)\big([x,y]\alpha_n\big)\cdot(y,x)u_n\Big)\\
&\overset{(\mathrm{viii}_n)}{=} && \Big([x,y\cdot z],\big(z,x\cdot[x,y]\big)u_n\cdot(y,x)u_n\Big)\\
&= && \big([x,y\cdot z],(z,x^y)u_n\cdot(y,x)u_n\big)\\
&\overset{(\mathrm{vii}_n)}{=} && \big([x,y\cdot z],(y\cdot z,x)u_n\big)\\
&= && (x,y\cdot z)u_{n+1}\,.
\end{aligned}$$

Ad (ix_{n+1}). Suppose given x, $y\in B$ and $h=:(b,\tilde h)\in H_{n+1}$ with $b\in B$, $\tilde h\in H_n$. Note that we have

$$(1,b)u_n=(1\cdot 1,b)u_n\overset{(\mathrm{vii}_n)}{=}(1,b^1)u_n\cdot(1,b)u_n\,.$$

Therefore, $(1,b)u_n=1$.

So we have

$$\begin{aligned}
\big((x,y)u_n\big)(b\alpha_n) &= (1,b)u_n \cdot \big((x,y)u_n\big)(b\alpha_n)\\
&= (x^- \cdot x, b)u_n \cdot \big((x,y)u_n\big)(b\alpha_n)\\
&\overset{(\mathrm{vii}_n)}{=} (x, b^{x^-})u_n \cdot (x^-, b)u_n \cdot \big((x,y)u_n\big)(b\alpha_n)\\
&= (x, b^{x^-})u_n \cdot (x^-, b)u_n \cdot \big((x,y)u_n\big)\Big(\big(b^{x^-} \cdot [x^-, b]\big)\alpha_n\Big)\\
&= (x, b^{x^-})u_n \cdot (x^-, b)u_n \cdot \Big(\big((x,y)u_n\big)\big((b^{x^-})\alpha_n\big)\Big)\big([x^-, b]\alpha_n\big)\\
&\overset{(\mathrm{ix}_n)}{=} (x, b^{x^-})u_n \cdot \big((x,y)u_n\big)\big((b^{x^-})\alpha_n\big) \cdot (x^-, b)u_n\\
&\overset{(\mathrm{viii}_n)}{=} (x, y \cdot b^{x^-})u_n \cdot (x^-, b)u_n\\
&\overset{(\mathrm{vi}_n)}{=} \big((y \cdot b^{x^-}, x)u_n\big)^- \cdot (x^-, b)u_n\\
&\overset{(\mathrm{vii}_n)}{=} \big((b^{x^-}, x^y)u_n \cdot (y,x)u_n\big)^- \cdot (x^-, b)u_n\\
&\overset{(\mathrm{vi}_n)}{=} (x,y)u_n \cdot (x^y, b^{x^-})u_n \cdot (x^-, b)u_n\\
&\overset{(\mathrm{vii}_n)}{=} (x,y)u_n \cdot (x^- \cdot x^y, b)u_n\\
&= (x,y)u_n \cdot \big([x,y], b\big)u_n\,.
\end{aligned}$$

We have

$$\begin{aligned}
h^{(x,y)u_{n+1}} &\overset{(\mathrm{vi}_n)}{=} (y,x)u_{n+1} \cdot h \cdot (x,y)u_{n+1}\\
&= \big([y,x], (x,y)u_n\big) \cdot (b, \tilde{h}) \cdot \big([x,y], (y,x)u_n\big)\\
&= \Big([y,x] \cdot b, \big((x,y)u_n\big)(b\alpha_n) \cdot \tilde{h}\Big) \cdot \big([x,y], (y,x)u_n\big)\\
&= \Big([y,x] \cdot b \cdot [x,y], \Big(\big((x,y)u_n\big)(b\alpha_n) \cdot \tilde{h}\Big)\big([x,y]\alpha_n\big) \cdot (y,x)u_n\Big)\\
&\overset{(\mathrm{ix}_n)}{=} \Big(b^{[x,y]}, (y,x)u_n \cdot \Big(\Big(\big((x,y)u_n\big)(b\alpha_n) \cdot \tilde{h}\Big)\big([x,y]\alpha_n\big)\Big)\big([y,x]\alpha_n\big)\Big)\\
&= \Big(b^{[x,y]}, (y,x)u_n \cdot \Big(\big((x,y)u_n\big)(b\alpha_n) \cdot \tilde{h}\Big)\big([x,y] \cdot [y,x]\big)\alpha_n\Big)\\
&= \Big(b^{[x,y]}, (y,x)u_n \cdot \big((x,y)u_n\big)(b\alpha_n) \cdot \tilde{h}\Big)
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{see above}}{=} && \Big(b^{[x,y]}, (y,x)u_n \cdot (x,y)u_n \cdot \big([x,y],b\big)u_n \cdot \tilde{h}\Big) \\
&\overset{(\mathrm{vi}_n)}{=} && \Big(b^{[x,y]}, \big([x,y],b\big)u_n \cdot \tilde{h}\Big) \\
&= && (b,\tilde{h})\big([x,y]\alpha_{n+1}\big) \\
&= && h\big([x,y]\alpha_{n+1}\big)\,.
\end{aligned}$$

□

17.1.2 The action on H_n

Stipulation 343 In the sequel of this section §17.1.1, we consider the data constructed in Lemma 342.

Notation 344 Let $n \in \mathbb{Z}_{\geqslant -1}$.

Suppose given $h \in H_n$. Suppose given $b \in B$.

(1) We write

$$h^b := h(b\alpha_n)\,.$$

(2) Recall from Lemma 342 that we write $h = (h_{\{n\}}, h_{\{n-1\}}, \ldots, h_{\{-1\}})$, where $h_{\{i\}} \in B$ for $i \in [0,n]$ and where $h_{\{-1\}} \in M$.

In other words, we may write

$$h = (b_n, b_{n-1}, \ldots, b_0, m)$$

with $b_i \in B$ for $i \in [0,n]$ and with $m \in M$.

Recall that $h_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant n+1}$.

Then

$$h_{\{k\}} = \begin{cases} 1 & \text{for } k \in \mathbb{Z}_{\geqslant n+1} \\ b_k & \text{for } k \in [0,n] \\ m & \text{for } k = -1\,. \end{cases}$$

In particular, if $n = -1$, then we may write

$$h_{\{-1\}} =: m \in M$$

and get

$$h = (h_{\{-1\}}) = (m)\,.$$

In this case, we often omit the brackets and write $h = h_{\{-1\}} = m$.

(3) For $\ell \in [-1, n]$, we have the group element $h_{\lfloor \ell, -1 \rfloor} \in H_\ell$, where

$$(h_{\lfloor \ell, -1 \rfloor})_{\{k\}} := \begin{cases} 1 & \text{for } k \geqslant \ell + 1 \\ h_{\{k\}} & \text{for } k \in [-1, \ell]\,. \end{cases}$$

So, if $h = (b_n\,, b_{n-1}\,, \ldots\,, b_0\,, m) \in H_n$, then, for $\ell \in [-1, n]$, we have

$$h_{\lfloor \ell, -1 \rfloor} = (b_\ell\,, b_{\ell-1}\,, \ldots\, b_0\,, m) \in H_\ell\,.$$

For $h = (b_n\,, b_{n-1}\,, \ldots\,, b_0\,, m) \in H_n$, note that we have

$$h = h_{\lfloor n, -1 \rfloor} = (b_n\,, b_{n-1}\,, \ldots, b_0\,, m) \in H_n\,.$$

Remark 345 Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given h, $\tilde{h} \in H_{n+1} = B \ltimes_{\alpha_n} H_n$.
Write

$$h := (b_{n+1}\,, b_n\,, \ldots\,, b_0\,, m) \quad \text{and} \quad \tilde{h} := (\tilde{b}_{n+1}\,, \tilde{b}_n\,, \ldots\,, \tilde{b}\,, \tilde{m})\,.$$

By construction, we have

$$h = \big(b_{n+1}\,, (b_n\,, \ldots\,, b_0\,, m)\big) \quad \text{and} \quad \tilde{h} = \big(\tilde{b}_{n+1}\,, (\tilde{b}_n\,, \ldots\,, \tilde{b}\,, \tilde{m})\big) \quad \text{in } B \ltimes_{\alpha_n} H_n\,.$$

So in particular, we have

$$h \cdot \tilde{h} = \big(b_{n+1} \cdot \tilde{b}_{n+1}\,, (b_n\,, \ldots\,, b_0\,, m)^{\tilde{b}_{n+1}} \cdot (\tilde{b}_n\,, \ldots\,, \tilde{b}\,, \tilde{m})\big)\,.$$

Remark 346 *Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given x, y, $z \in B$.*

(1) *We have*

$$(1, z)u_n = 1 \quad \text{and} \quad (z, 1)u_n = 1\,.$$

(2) *We have*

$$(x \cdot y, z)u_n = \big((x, z)u_n\big)^y \cdot (y, z)u_n\,.$$

(3) *We have*

$$(x, y \cdot z)u_n = (x, y)u_n \cdot (x^y, z)u_n\,.$$

(4) *We have*

$$\big((y,x)u_n\big)^z = \big(z,[x,y]\big)u_n \cdot (y,x)u_n\,.$$

(5) *We have*

$$\big((x,y)u_n\big)^z = (x^z,y^z)u_n\,.$$

Proof. Ad (1). We have

$$(1,z)u_n = (1\cdot 1,z)u_n \overset{342.(\mathrm{vii}_n)}{=} (1,z^1)u_n\cdot(1,z)u_n = (1,z)u_n\cdot(1,z)u_n\,.$$

This shows $(1,z)u_n = 1$.

Moreover, we have

$$(z,1)u_n \overset{342.(\mathrm{vi}_n)}{=} \big((1,z)u_n\big)^- = 1^- = 1\,.$$

Ad (2). We have

$$(x\cdot y,z)u_n \overset{342.(\mathrm{vi}_n)}{=} \big((z,x\cdot y)u_n\big)^- \overset{342.(\mathrm{viii}_n)}{=} \Big((z,y)u_n\cdot\big((z,x)u_n\big)^y\Big)^- \overset{342.(\mathrm{vi}_n)}{=} \big((x,z)u_n\big)^y\cdot(y,z)u_n\,.$$

Ad (3). We have

$$(x,y\cdot z)u_n \overset{342.(\mathrm{vi}_n)}{=} \big((y\cdot z,x)u_n\big)^- \overset{342.(\mathrm{vii}_n)}{=} \big((z,x^y)u_n\cdot(y,x)u_n\big)^- \overset{342.(\mathrm{vi}_n)}{=} (x,y)u_n\cdot(x^y,z)u_n\,.$$

Ad (4). We have

$$\begin{aligned}
\big(z,[x,y]\big)u_n\cdot(y,x)u_n &= (z,x^-\cdot x^y)u_n\cdot(y,x)u_n\\
&\overset{342.(\mathrm{viii}_n)}{=} (z,x^y)u_n\cdot\big((z,x^-)u_n\big)^{(x^y)}\cdot(y,x)u_n\\
&= (z,x^y)u_n\cdot\Big(\big((z,x^-)u_n\big)^x\Big)^{[x,y]}\cdot(y,x)u_n\\
&\overset{342.(\mathrm{ix}_n)}{=} (z,x^y)u_n\cdot\Big(\big((z,x^-)u_n\big)^x\Big)^{(x,y)u_n}\cdot(y,x)u_n\\
&\overset{342.(\mathrm{vi}_n)}{=} (z,x^y)u_n\cdot(y,x)u_n\cdot\big((z,x^-)u_n\big)^x\\
&\overset{342.(\mathrm{vii}_n)}{=} (y\cdot z,x)u_n\cdot\big((z,x^-)u_n\big)^x\\
&\overset{(2)}{=} \big((y,x)u_n\big)^z\cdot(z,x)u_n\cdot\big((z,x^-)u_n\big)^x\\
&\overset{342.(\mathrm{viii}_n)}{=} \big((y,x)u_n\big)^z\cdot(z,x^-\cdot x)u_n\\
&= \big((y,x)u_n\big)^z\cdot(z,1)u_n\\
&\overset{(1)}{=} \big((y,x)u_n\big)^z\,.
\end{aligned}$$

Ad (5). We show this claim by induction on $n \in \mathbb{Z}_{\geqslant -1}$.

Let $n = -1$.

We have

$$\big((x,y)u_{-1}\big)^z = \llbracket x,\, y \rrbracket^z \overset{323.(10)}{=} \llbracket x^z,\, y^z \rrbracket = (x^z, y^z)u_{-1}\,.$$

Let $n \in \mathbb{Z}_{\geqslant -1}$.

Suppose that (5) holds for n.

We have

$$\begin{aligned}\big((x,y)u_{n+1}\big)^z &= \big([x,y],(y,x)u_n\big)^z = \Big([x,y]^z, \big(z,[x,y]\big)u_n \cdot (y,x)u_n\Big) \\ &\overset{(4)}{=} \Big([x^z,y^z], \big((y,x)u_n\big)^z\Big) \overset{\text{IH}}{=} \big([x^z,y^z],(y^z,x^z)u_n\big) = (x^z,y^z)u_{n+1}\,.\end{aligned}$$

□

Lemma 347 *Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given $h \in H_n$. Write $b_i := h_{\{i\}} \in B$ for $i \in [0,n]$ and $m := h_{\{-1\}} \in M$. So $h = (b_n\,, b_{n-1}\,, \ldots, b_0\,, m)$. Suppose given $x,\, y \in B$.*

(1) *We have*

$$\big((x,y)u_n \cdot h\big)_{\{k\}} = \begin{cases} \big([x,y]^{(-1)^{n-k}}\big)^{\left(\prod\limits_{j \in \lfloor n,k+1 \rfloor} b_j\right)} \cdot b_k & \text{for } k \in [0,n] \\ \big(\llbracket x,\, y \rrbracket^{(-1)^{n+1}}\big)^{\left(\prod\limits_{j \in \lfloor n,0 \rfloor} b_j\right)} \cdot m & \text{for } k = -1\,. \end{cases}$$

(2) *If $n \geqslant 0$, then we have*

$$(h^x)_{\{k\}} = \begin{cases} \big([b_n\,, x]^{(-1)^{n-k}}\big)^{\left(\prod\limits_{j \in \lfloor n-1,k+1 \rfloor} b_j\right)} \cdot b_k & \text{for } k \in [0,n] \\ \big(\llbracket b_n,\, x \rrbracket^{(-1)^{n+1}}\big)^{\left(\prod\limits_{j \in \lfloor n-1,0 \rfloor} b_j\right)} \cdot m & \text{for } k = -1\,. \end{cases}$$

Proof. Ad (1). We show this assertion by induction on $n \in \mathbb{Z}_{\geqslant -1}$.

Let $n = -1$.

Suppose given $h \in H_{-1} = M$. Note that $m = h_{\{-1\}} = h$.

We have

$$\big((x,y)u_{-1} \cdot h\big)_{\{-1\}} = (x,y)u_{-1} \cdot h = \llbracket x,\, y \rrbracket \cdot h = \big(\llbracket x,\, y \rrbracket^{(-1)^0}\big)^1 \cdot m\,.$$

Let $n \in \mathbb{Z}_{\geqslant -1}$. Now suppose that (1) holds for n.

We have to show that (1) *holds for* $n+1$.

Let $h \in H_{n+1}$. Write $b_i := h_{\{i\}} \in B$ for $i \in [0, n+1]$ and $m := h_{\{-1\}} \in M$.

So $h = (b_{n+1}, b_n, b_{n-1}, \ldots, b_0, m)$.

We have

$$\begin{aligned}(x,y)u_{n+1} \cdot h &= \big([x,y], (y,x)u_n\big) \cdot (b_{n+1}, b_n, \ldots, b_0, m) \\ &= \Big([x,y] \cdot b_{n+1}, \big((y,x)u_n\big)^{b_{n+1}} \cdot (b_n, \ldots, b_0, m)\Big) \\ &\overset{346.(5)}{=} \Big([x,y] \cdot b_{n+1}, (y^{b_{n+1}}, x^{b_{n+1}})u_n \cdot (b_n, \ldots, b_0, m)\Big).\end{aligned}$$

First, for $k = n+1$, we have

$$\big([x,y]^{(-1)^{(n+1)-(n+1)}}\big)^{\left(\prod\limits_{j \in \lfloor n+1, (n+1)+1 \rfloor} b_j\right)} \cdot b_{n+1} = \big([x,y]^{(-1)^0}\big)^{\left(\prod\limits_{j \in \lfloor n+1, n+2 \rfloor} b_j\right)} \cdot b_{n+1} = [x,y] \cdot b_{n+1}.$$

By induction hypothesis, we have

$$\begin{aligned}&\big((y^{b_{n+1}}, x^{b_{n+1}})u_n \cdot (b_n, \ldots, b_0, m)\big)_{\{k\}} \\ = \quad &\begin{cases} \big([y^{b_{n+1}}, x^{b_{n+1}}]^{(-1)^{n-k}}\big)^{\left(\prod\limits_{j \in \lfloor n, k+1 \rfloor} b_j\right)} \cdot b_k & \text{for } k \in [0,n] \\ \big(\llbracket y^{b_{n+1}},\, x^{b_{n+1}} \rrbracket^{(-1)^{n+1}}\big)^{\left(\prod\limits_{j \in \lfloor n, 0 \rfloor} b_j\right)} \cdot m & \text{for } k = -1 \end{cases} \\ \overset{323.(10)}{=} \quad &\begin{cases} \big([y,x]^{(-1)^{n-k}}\big)^{\left(b_{n+1} \cdot \prod\limits_{j \in \lfloor n, k+1 \rfloor} b_j\right)} \cdot b_k & \text{for } k \in [0,n] \\ \big(\llbracket y,\, x \rrbracket^{(-1)^{n+1}}\big)^{\left(b_{n+1} \cdot \prod\limits_{j \in \lfloor n, 0 \rfloor} b_j\right)} \cdot m & \text{for } k = -1 \end{cases} \\ \overset{\text{(SCM1)}}{=} \quad &\begin{cases} \big([x,y]^{(-1)^{n+1-k}}\big)^{\left(\prod\limits_{j \in \lfloor n+1, k+1 \rfloor} b_j\right)} \cdot b_k & \text{for } k \in [0,n] \\ \big(\llbracket x,\, y \rrbracket^{(-1)^{n+2}}\big)^{\left(\prod\limits_{j \in \lfloor n+1, 0 \rfloor} b_j\right)} \cdot m & \text{for } k = -1. \end{cases}\end{aligned}$$

Altogether, we have

$$\big((x,y)u_{n+1} \cdot h\big)_{\{k\}} = \begin{cases} \big([x,y]^{(-1)^{n+1-k}}\big)^{\left(\prod\limits_{j \in \lfloor n+1, k+1 \rfloor} b_j\right)} \cdot b_k & \text{for } k \in [0, n+1] \\ \big(\llbracket x,\, y \rrbracket^{(-1)^{n+2}}\big)^{\left(\prod\limits_{j \in \lfloor n+1, 0 \rfloor} b_j\right)} \cdot m & \text{for } k = -1. \end{cases}$$

This proves (1).

Ad (2). We have

$$h^x = \big(b_n\,,(b_{n-1}\,,\dots,b_0\,,m)\big)^x = \big(b_n^x\,,(x,b_n)u_{n-1}\cdot(b_{n-1}\,,\dots,b_0\,,m)\big)\,.$$

First, for $k=n$, we have

$$\big([b_n\,,x]^{(-1)^{n-n}}\big)^{\left(\prod\limits_{j\in\lfloor n-1,n+1\rfloor} b_j\right)}\cdot b_n = \big([b_n\,,x]^{(-1)^0}\big)^{b_n^-}\cdot b_n = b_n\cdot[b_n\,,x]\cdot b_n^-\cdot b_n = b_n^x = (h^x)_{\{n\}}\,.$$

Moreover, by (1), we have

$$\begin{aligned}
&\big((x,b_n)u_{n-1}\cdot(b_{n-1}\,,\dots\,,b_0\,,m)\big)_{\{k\}}\\
\overset{(1)}{=}\quad &\begin{cases}\big([x,b_n]^{(-1)^{n-1-k}}\big)^{\left(\prod\limits_{j\in\lfloor n-1,k+1\rfloor} b_j\right)}\cdot b_k & \text{for } k\in[0,n-1]\\ \big(\llbracket x,\,b_n\rrbracket^{(-1)^n}\big)^{\left(\prod\limits_{j\in\lfloor n-1,0\rfloor} b_j\right)}\cdot m & \text{for } k=-1\end{cases}\\
\overset{\text{(SCM1)}}{=}\quad &\begin{cases}\big([b_n\,,x]^{(-1)^{n-k}}\big)^{\left(\prod\limits_{j\in\lfloor n-1,k+1\rfloor} b_j\right)}\cdot b_k & \text{for } k\in[0,n-1]\\ \big(\llbracket b_n,\,x\rrbracket^{(-1)^{n+1}}\big)^{\left(\prod\limits_{j\in\lfloor n-1,0\rfloor} b_j\right)}\cdot m & \text{for } k=-1\,.\end{cases}
\end{aligned}$$

Altogether, we have

$$(h^x)_{\{k\}} = \begin{cases}\big([b_n\,,x]^{(-1)^{n-k}}\big)^{\left(\prod\limits_{j\in\lfloor n-1,k+1\rfloor} b_j\right)}\cdot b_k & \text{for } k\in[0,n]\\ \big(\llbracket b_n,\,x\rrbracket^{(-1)^{n+1}}\big)^{\left(\prod\limits_{j\in\lfloor n-1,0\rfloor} b_j\right)}\cdot m & \text{for } k=-1\,.\end{cases}$$

This proves (2). □

17.1.3 The multiplication in H_n

We consider the data constructed in Lemma 342.

Remark 348

(0) *Suppose given a, $b\in\mathbb{Z}$ with $b\geqslant a-1$. Suppose given y_j , $z_j\in B$ for $j\in\lfloor b,a\rfloor$. We have*

$$\prod_{j\in\lfloor b,a\rfloor} y_j\cdot z_j = \Big(\prod_{t\in\lfloor b,a\rfloor} z_t\Big)\cdot\Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\Big)\,.$$

(1) *Suppose given x, y, z, v, $w \in B$. Let $k \in \mathbb{Z}$.*

We have

$$\Big[x, \big([y,z]^{(-1)^k}\big)^v \cdot w\Big] = [x,w] \cdot \big([z,y]^{(-1)^k}\big)^{v\cdot x\cdot w} \cdot \big([y,z]^{(-1)^k}\big)^{v\cdot w} \in B\,.$$

(2) *Suppose given a, $b \in \mathbb{Z}$ with $b \geqslant a-1$. Suppose given y_j , $z_j \in B$ for $j \in \lfloor b,a \rfloor$.*

For $x \in B$, we have

$$x^{\left(\prod\limits_{j\in\lfloor b,a\rfloor} y_j\cdot z_j\right)} = \Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\Big)^- \cdot x^{\left(\prod\limits_{t\in\lfloor b,a\rfloor} z_t\right)} \cdot \Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\Big)\,.$$

(3) *Suppose given x, y, z, v, $w \in B$. Let $k \in \mathbb{Z}$.*

We have

$$\Big\llbracket x,\ \big([y,z]^{(-1)^k}\big)^v \cdot w\Big\rrbracket = \llbracket x,\, w\rrbracket \cdot \big(\llbracket z,\, y\rrbracket^{(-1)^k}\big)^{v\cdot x\cdot w} \cdot \big(\llbracket y,\, z\rrbracket^{(-1)^k}\big)^{v\cdot w} \in M\,.$$

(4) *Suppose given a, $b \in \mathbb{Z}$ with $b \geqslant a-1$. Suppose given $y_j \in M$ and $z_j \in B$ for $j \in \lfloor b,a \rfloor$. We consider $y_j \mathrm{f}_S \in B$ for $j \in \lfloor b,a \rfloor$.*

For $x \in M$, we have

$$x^{\left(\prod\limits_{j\in\lfloor b,a\rfloor} y_j\mathrm{f}_S\cdot z_j\right)} = \Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\Big)^- \cdot x^{\left(\prod\limits_{t\in\lfloor b,a\rfloor} z_t\right)} \cdot \Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\Big)\,.$$

Proof. Ad (0). If $b = a-1$, then we have

$$\Big(\prod_{t\in\lfloor a-1,a\rfloor} z_t\Big) \cdot \Big(\prod_{j\in\lfloor a-1,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\Big) = 1\cdot 1 = 1 = \prod_{j\in\lfloor a-1,a\rfloor} y_j\cdot z_j\,;$$

cf. §1.1.2 item 9.3.

If $b \geqslant a$, then we have

$$\begin{aligned}
&\Big(\prod_{t\in\lfloor b,a\rfloor} z_t\Big) \cdot \Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\Big)\\
=\ &\Big(\prod_{t\in\lfloor b,a\rfloor} z_t\Big) \cdot \Big(\prod_{j\in\lfloor b,a\rfloor} \Big(\prod_{t\in\lfloor j,a\rfloor} z_t\Big)^- \cdot y_j \cdot \Big(\prod_{t\in\lfloor j,a\rfloor} z_t\Big)\Big)\\
=\ &\Big(\prod_{t\in\lfloor b,a\rfloor} z_t\Big) \cdot \Big(\prod_{t\in\lfloor b,a\rfloor} z_t\Big)^- \cdot \Big(\prod_{j\in\lfloor b,a\rfloor} y_j \cdot \Big(\prod_{t\in\lfloor j,a\rfloor} z_t\Big) \cdot \Big(\prod_{t\in\lfloor j-1,a\rfloor} z_t\Big)^-\Big) \cdot \Big(\prod_{t\in\lfloor a-1,a\rfloor} z_t\Big)
\end{aligned}$$

$$\begin{aligned} &= \Big(\prod_{j\in\lfloor b,a\rfloor} y_j \cdot z_j \cdot \big(\prod_{t\in\lfloor j-1,a\rfloor} z_t\big) \cdot \big(\prod_{t\in\lfloor j-1,a\rfloor} z_t\big)^- \Big) \cdot 1 \\ &= \prod_{j\in\lfloor b,a\rfloor} y_j \cdot z_j \,. \end{aligned}$$

Ad (1). We write $u := [y,z]^{(-1)^k}$. Then we have

$$\begin{aligned} & [x,w] \cdot \big([z,y]^{(-1)^k}\big)^{v\cdot x\cdot w} \cdot \big([y,z]^{(-1)^k}\big)^{v\cdot w} \\ = \;& [x,w] \cdot (u^-)^{v\cdot x\cdot w} \cdot u^{v\cdot w} \\ = \;& x^- \cdot w^- \cdot x \cdot w \cdot w^- \cdot x^- \cdot (u^-)^v \cdot x \cdot w \cdot w^- \cdot u^v \cdot w \\ = \;& x^- \cdot w^- \cdot (u^-)^v \cdot x \cdot u^v \cdot w \\ = \;& [x, u^v \cdot w] \\ = \;& \Big[x, \big([y,z]^{(-1)^k}\big)^v \cdot w\Big] \,. \end{aligned}$$

Ad (2). For $x \in G$, we have

$$\begin{aligned} x^{\left(\prod_{j\in\lfloor b,a\rfloor} y_j\cdot z_j\right)} &\overset{(0)}{=} \Big(x^{\left(\prod_{t\in\lfloor b,a\rfloor} z_t\right)}\Big)^{\left(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod_{t\in\lfloor j,a\rfloor} z_t\right)}\right)} \\ &= \Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod_{t\in\lfloor j,a\rfloor} z_t\right)}\Big)^- \cdot x^{\left(\prod_{t\in\lfloor b,a\rfloor} z_t\right)} \cdot \Big(\prod_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod_{t\in\lfloor j,a\rfloor} z_t\right)}\Big) \,. \end{aligned}$$

Ad (3). If $k \equiv_2 0$, then we have

$$\begin{aligned} \big\llbracket x,\, \big([y,z]^{(-1)^k}\big)^v \cdot w \big\rrbracket &= \big\llbracket x,\, [y,z]^v \cdot w \big\rrbracket \\ &\overset{322.(3)}{=} \llbracket x,\, w\rrbracket \cdot \big\llbracket x^w,\, [y,z]^{v\cdot w}\big\rrbracket \\ &\overset{323.(10)}{=} \llbracket x,\, w\rrbracket \cdot \big\llbracket x,\, [y^v, z^v]\big\rrbracket^w \\ &\overset{323.(11)}{=} \llbracket x,\, w\rrbracket \cdot \big(\llbracket z^v,\, y^v\rrbracket^x \cdot \llbracket y^v,\, z^v\rrbracket\big)^w \\ &\overset{323.(10)}{=} \llbracket x,\, w\rrbracket \cdot \llbracket z,\, y\rrbracket^{v\cdot x\cdot w} \cdot \llbracket y,\, z\rrbracket^{v\cdot w} \\ &= \llbracket x,\, w\rrbracket \cdot \big(\llbracket z,\, y\rrbracket^{(-1)^k}\big)^{v\cdot x\cdot w} \cdot \big(\llbracket y,\, z\rrbracket^{(-1)^k}\big)^{v\cdot w} \,. \end{aligned}$$

If $k \equiv_2 1$, then we have

$$\begin{array}{rcl}
\left\llbracket x,\, \left([y,z]^{(-1)^k}\right)^v \cdot w \right\rrbracket & = & \left\llbracket x,\, [z,y]^v \cdot w \right\rrbracket \\
& \overset{322.(3)}{=} & \left\llbracket x,\, w \right\rrbracket \cdot \left\llbracket x^w,\, [z,y]^{v\cdot w} \right\rrbracket \\
& \overset{323.(10)}{=} & \left\llbracket x,\, w \right\rrbracket \cdot \left\llbracket x,\, [z^v, y^v] \right\rrbracket^w \\
& \overset{323.(11)}{=} & \left\llbracket x,\, w \right\rrbracket \cdot \left(\left\llbracket y^v,\, z^v \right\rrbracket^x \cdot \left\llbracket z^v,\, y^v \right\rrbracket\right)^w \\
& \overset{323.(10)}{=} & \left\llbracket x,\, w \right\rrbracket \cdot \left\llbracket y,\, z \right\rrbracket^{v\cdot x\cdot w} \cdot \left\llbracket z,\, y \right\rrbracket^{v\cdot w} \\
& \overset{(\mathrm{SCM1})}{=} & \left\llbracket x,\, w \right\rrbracket \cdot \left(\left\llbracket z,\, y \right\rrbracket^{(-1)^k}\right)^{v\cdot x\cdot w} \cdot \left(\left\llbracket y,\, z \right\rrbracket^{(-1)^k}\right)^{v\cdot w}.
\end{array}$$

Ad (4). For $x \in M$, we have

$$\begin{array}{rcl}
x^{\left(\prod\limits_{j\in\lfloor b,a\rfloor} y_j \mathrm{f}_S \cdot z_j\right)} & \overset{(0)}{=} & \left(x^{\left(\prod\limits_{t\in\lfloor b,a\rfloor} z_t\right)}\right)^{\left(\prod\limits_{j\in\lfloor b,a\rfloor} (y_j\mathrm{f}_S)^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\right)} \\
& \overset{323.(1)}{=} & \left(x^{\left(\prod\limits_{t\in\lfloor b,a\rfloor} z_t\right)}\right)^{\left(\left(\prod\limits_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\right)\mathrm{f}_S\right)} \\
& \overset{323.(2)}{=} & \left(x^{\left(\prod\limits_{t\in\lfloor b,a\rfloor} z_t\right)}\right)^{\left(\prod\limits_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\right)} \\
& = & \left(\prod\limits_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\right)^- \cdot x^{\left(\prod\limits_{t\in\lfloor b,a\rfloor} z_t\right)} \cdot \left(\prod\limits_{j\in\lfloor b,a\rfloor} y_j^{\left(\prod\limits_{t\in\lfloor j,a\rfloor} z_t\right)}\right).
\end{array}$$

□

Lemma 349 *Let $n \in \mathbb{Z}_{\geqslant -1}$.*
Suppose given $h \in H_n$. Write $b_i := h_{\{i\}}$ for $i \in [0,n]$ and $m := h_{\{-1\}} \in M$.
So $h = (b_n, b_{n-1}, \ldots, b_0, m)$. Write $b_i := 1$ for $i \in \mathbb{Z}_{\geqslant n+1}$.
Suppose given $\tilde{h} \in H_n$. Write $\tilde{b}_i := \tilde{h}_{\{i\}}$ for $i \in [0,n]$ and $\tilde{m} := \tilde{h}_{\{-1\}} \in M$.
So $\tilde{h} = (\tilde{b}_n, \tilde{b}_{n-1}, \ldots, \tilde{b}_0, \tilde{m})$. Write $\tilde{b}_i := 1$ for $i \in \mathbb{Z}_{\geqslant n+1}$.

The product $h \cdot \tilde{h} \in H_n$ is determined by

$$(h \cdot \tilde{h})_{\{k\}} =$$
$$\begin{cases} \left(\prod_{\substack{i \in \lceil k,n \rceil \\ i \equiv_2 k}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1} \cdot \left(\prod_{j \in \lfloor i,k+1 \rfloor} b_j \cdot \tilde{b}_j \right)} \right) \cdot \left(\prod_{\substack{i \in \lceil k,n \rceil \\ i \equiv_2 k+1}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1} \cdot \left(\prod_{j \in \lfloor i,k+1 \rfloor} b_j \cdot \tilde{b}_j \right)} \right)^{-} \cdot b_k^{\tilde{b}_{k+1}} \cdot \tilde{b}_k \quad \textit{for } k \in [0,n] \\ \left(\prod_{\substack{i \in \lceil -1,n \rceil \\ i \equiv_2 -1}} \llbracket \tilde{b}_{i+2},\, b_{i+1} \rrbracket^{\tilde{b}_{i+1} \cdot \left(\prod_{j \in \lfloor i,0 \rfloor} b_j \cdot \tilde{b}_j \right)} \right) \cdot \left(\prod_{\substack{i \in \lceil -1,n \rceil \\ i \equiv_2 0}} \llbracket \tilde{b}_{i+2},\, b_{i+1} \rrbracket^{\tilde{b}_{i+1} \cdot \left(\prod_{j \in \lfloor i,0 \rfloor} b_j \cdot \tilde{b}_j \right)} \right)^{-} \cdot m^{\tilde{b}_0} \cdot \tilde{m} \quad \textit{for } k = -1\,. \end{cases}$$

Proof. We show this claim by induction on $n \in \mathbb{Z}_{\geqslant -1}$.

Let $n = -1$.

Suppose given $h \in H_{-1} = M$. Write $m := h_{\{-1\}} = h$ and write $b_i := 1$ for $i \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $\tilde{h} \in H_{-1} = M$. Write $\tilde{m} := \tilde{h}_{\{-1\}} = \tilde{h}$ and write $\tilde{b}_i := 1$ for $i \in \mathbb{Z}_{\geqslant 0}$.

We have

$$h' \;\; := h \cdot \tilde{h} = m \cdot \tilde{m}\,.$$

So we have

$$\begin{aligned} & h'_{\{-1\}} = m \cdot \tilde{m} = 1 \cdot 1 \cdot m^1 \cdot \tilde{m} \\ = \;& \left(\prod_{\substack{i \in \lceil -1,-1 \rceil \\ i \equiv_2 -1}} \llbracket \tilde{b}_{i+2},\, b_{i+1} \rrbracket^{\tilde{b}_{i+1} \cdot \left(\prod_{j \in \lfloor i,0 \rfloor} b_j \cdot \tilde{b}_j \right)} \right) \cdot \left(\prod_{\substack{i \in \lceil -1,-1 \rceil \\ i \equiv_2 0}} \llbracket \tilde{b}_{i+2},\, b_{i+1} \rrbracket^{\tilde{b}_{i+1} \cdot \left(\prod_{j \in \lfloor i,0 \rfloor} b_j \cdot \tilde{b}_j \right)} \right)^{-} \cdot m^{\tilde{b}_0} \cdot \tilde{m}\,. \end{aligned}$$

This proves the claim for $n = -1$.

Let $n = 0$.

Suppose given $h \in H_0$. Write $b_0 := h_{\{0\}}$ and $m := h_{\{-1\}} \in M$.

So $h = (b_0\,, m)$. Write $b_i := 1$ for $i \in \mathbb{Z}_{\geqslant 1}$.

Suppose given $\tilde{h} \in H_0$. Write $\tilde{b} := \tilde{h}_{\{0\}}$ and $\tilde{m} := \tilde{h}_{\{-1\}} \in M$.

So $\tilde{h} = (\tilde{b}\,, \tilde{m})$. Write $\tilde{b}_i := 1$ for $i \in \mathbb{Z}_{\geqslant 1}$.

We have

$$h' \;\; := h \cdot \tilde{h} = (b_0\,, m) \cdot (\tilde{b}_0\,, \tilde{m}) = (b_0 \cdot \tilde{b}_0\,, m^{\tilde{b}_0} \cdot \tilde{m})\,.$$

So we have

$$\begin{aligned}
h'_{\{0\}} &= b_0 \cdot \tilde{b}_0 \\
&= 1 \cdot 1 \cdot b_0^1 \cdot \tilde{b}_0 \\
&= \Bigg(\prod_{\substack{i \in \lceil 0,0 \rceil \\ i \equiv_2 0}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i,1 \rfloor} b_j \cdot \tilde{b}_j \right)} \Bigg) \cdot \Bigg(\prod_{\substack{i \in \lceil 0,0 \rceil \\ i \equiv_2 1}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i,1 \rfloor} b_j \cdot \tilde{b}_j \right)} \Bigg)^{-} \cdot b_0^{\tilde{b}_1} \cdot \tilde{b}_0 \\
h'_{\{-1\}} &= m^{\tilde{b}_0} \cdot \tilde{m} \\
&= 1 \cdot 1 \cdot m^{\tilde{b}_0} \cdot \tilde{m} \\
&= \Bigg(\prod_{\substack{i \in \lceil -1,0 \rceil \\ i \equiv_2 -1}} \llbracket \tilde{b}_{i+2},\, b_{i+1} \rrbracket^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i,0 \rfloor} b_j \cdot \tilde{b}_j \right)} \Bigg) \cdot \Bigg(\prod_{\substack{i \in \lceil -1,0 \rceil \\ i \equiv_2 0}} \llbracket \tilde{b}_{i+2},\, b_{i+1} \rrbracket^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i,0 \rfloor} b_j \cdot \tilde{b}_j \right)} \Bigg)^{-} \cdot m^{\tilde{b}_0} \cdot \tilde{m}\,.
\end{aligned}$$

This proves the claim for $n = 0$.

Let $n \in \mathbb{Z}_{\geqslant 0}$. Now suppose that the claim holds for n.

We have to show that the claim holds for $n+1$.

Suppose given $h \in H_{n+1}$. Write $b_i := h_{\{i\}} \in B$ for $i \in [0, n+1]$ and $m := h_{\{-1\}} \in M$.

So $h = (b_{n+1}\,, b_n\,, b_{n-1}\,, \ldots, b_0\,, m)$. Write $b_i := 1$ for $i \in \mathbb{Z}_{\geqslant n+2}$.

Suppose given $\tilde{h} \in H_{n+1}$. Write $\tilde{b}_i := \tilde{h}_{\{i\}} \in B$ for $i \in [0, n+1]$ and $\tilde{m} := \tilde{h}_{\{-1\}} \in M$.

So $\tilde{h} = (\tilde{b}_{n+1}\,, \tilde{b}_n\,, \tilde{b}_{n-1}\,, \ldots, \tilde{b}_0\,, \tilde{m})$. Write $\tilde{b}_i := 1$ for $i \in \mathbb{Z}_{\geqslant n+2}$.

Let $h' := h_{\lfloor n,-1 \rfloor}^{\tilde{b}_{n+1}} \in H_n$; cf. Notation 344.(3). Write $b'_i := h'_{\{i\}} \in B$ for $i \in [0, n]$ and $m' := h'_{\{-1\}} \in M$.

So $h' = (b'_n\,, b'_{n-1}\,, \ldots, b'_0\,, m') = (b_n\,, b_{n-1}\,, \ldots, b_0\,, m)^{\tilde{b}_{n+1}}$. Write $b'_i := 1$ for $i \in \mathbb{Z}_{\geqslant n+1}$.

By Lemma 347.(2), we have

$$h'_{\{s\}} = \begin{cases} b'_s = \left([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-s}} \right)^{\left(\prod\limits_{j \in \lfloor n-1, s+1 \rfloor} b_j \right)} \cdot b_s & \text{for } s \in [0, n] \\ m' = \left(\llbracket b_n,\, \tilde{b}_{n+1} \rrbracket^{(-1)^{n+1}} \right)^{\left(\prod\limits_{j \in \lfloor n-1, 0 \rfloor} b_j \right)} \cdot m & \text{for } s = -1\,. \end{cases}$$

Let $\tilde{h}' := \tilde{h}_{[n,-1]} \in H_n$. Write $\tilde{b}'_i := \tilde{h}'_{\{i\}} \in B$ for $i \in [0, n]$ and $\tilde{m}' := \tilde{h}'_{\{-1\}} \in M$.

So $\tilde{h}' = (\tilde{b}'_n\,, \tilde{b}'_{n-1}\,, \ldots, \tilde{b}'_0\,, \tilde{m}') = (\tilde{b}_n\,, \tilde{b}_{n-1}\,, \ldots, \tilde{b}_0\,, \tilde{m})$. Write $\tilde{b}'_i := 1$ for $i \in \mathbb{Z}_{\geqslant n+1}$.

Note that we have $\tilde{b}_i = \tilde{b}'_i$ for $i \in [0, n]$ and that $\tilde{m} = \tilde{m}'$, but that in general $\tilde{b}_{n+1}$ does not equal $\tilde{b}'_{n+1} = 1$.

We need some auxiliary elements.

Suppose given r, $\ell \in \mathbb{Z}_{\geqslant 0}$ with $r \geqslant \ell$.

We write

$$\begin{aligned} x_r &:= [\tilde{b}'_{r+2}\,, b'_{r+1}]^{\tilde{b}'_{r+1}} \in B \\ y_r &:= \left([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-r}}\right)^{\left(\prod_{\ell \in \lfloor n-1, r+1 \rfloor} b_\ell\right)} \in B \\ z_{r,\ell} &:= \prod_{t \in \lfloor r, \ell+1 \rfloor} b_t \cdot \tilde{b}_t \in B \\ w_{r,\ell} &:= \prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{z_{j,\ell}} \in B\,. \end{aligned}$$

We *claim* that the assertions (i – iv) hold.

(i) *For $0 \leqslant \ell$, we have*

$$w_{\ell,\ell} = 1\,.$$

For $0 \leqslant \ell \leqslant n-2$, we have

$$w_{\ell+1,\ell} = b_\ell^{\tilde{b}_{\ell+1}} \cdot \left((b'_\ell)^{\tilde{b}_{\ell+1}}\right)^-\,.$$

(ii) *For $0 \leqslant \ell \leqslant r \leqslant n-2$, we have*

$$w_{r,\ell}^- \cdot x_r^{z_{r,\ell}} \cdot w_{r,\ell} = [\tilde{b}'_{r+2}\,, b'_{r+1}]^{\tilde{b}'_{r+1} \cdot \left(\prod_{j \in \lfloor r, \ell+1 \rfloor} b'_j \cdot \tilde{b}'_j\right)}\,.$$

(iii) *For $0 \leqslant \ell \leqslant r \leqslant n-2$, we have*

$$x_r^{z_{r,\ell}} = [\tilde{b}_{r+2}\,, b_{r+1}]^{\tilde{b}_{r+1} \cdot z_{r,\ell}} \cdot y_{r+2}^{z_{r+2,\ell}} \cdot y_{r+1}^{z_{r+1,\ell}}\,,$$

and

$$x_r^{z_{r,\ell}} \cdot w_{r,\ell} \cdot w_{r+2,\ell}^- = [\tilde{b}_{r+2}\,, b_{r+1}]^{\tilde{b}_{r+1} \cdot z_{r,\ell}}\,.$$

(iv) *For $0 \leqslant \ell \leqslant n-3$, we have*

$$\begin{aligned} & x_{n-2}^{z_{n-2,\ell}} \cdot w_{n-2,\ell} \cdot w_{n-3,\ell}^- \cdot (x_{n-3}^{z_{n-3,\ell}})^- \\ &= [\tilde{b}_n\,, b_{n-1}]^{\tilde{b}_{n-1} \cdot z_{n-2,\ell}} \cdot [b_n\,, \tilde{b}_{n+1}]^{\tilde{b}_n \cdot z_{n-1,\ell}} \cdot [b_{n-2}\,, \tilde{b}_{n-1}]^{\tilde{b}_{n-2} \cdot z_{n-3,\ell}}\,. \end{aligned}$$

Ad (i). For $0 \leqslant \ell$, we have

$$w_{\ell,\ell} = \prod_{j \in \lfloor \ell, \ell+1 \rfloor} y_j^{z_{j,\ell}} = 1\,.$$

For $\ell \in [0, n-2]$, we have

$$\begin{aligned}
w_{\ell+1,\ell} &= \prod_{j \in \lfloor \ell+1, \ell+1 \rfloor} y_j^{z_{j,\ell}} \\
&= y_{\ell+1}^{z_{\ell+1,\ell}} \\
&= \left([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(\ell+1)}}\right)^{\left(\prod_{j \in \lfloor n-1, \ell+2 \rfloor} b_j\right) \cdot \left(\prod_{t \in \lfloor \ell+1, \ell+1 \rfloor} b_t \cdot \tilde{b}_t\right)} \\
&= \left([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-\ell-1}}\right)^{\left(\prod_{j \in \lfloor n-1, \ell+1 \rfloor} b_j\right) \cdot \tilde{b}_{\ell+1}} \\
&= b_\ell^{\tilde{b}_{\ell+1}} \cdot \left(b_\ell^- \cdot \left([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-\ell-1}}\right)^{\left(\prod_{j \in \lfloor n-1, \ell+1 \rfloor} b_j\right)}\right)^{\tilde{b}_{\ell+1}} \\
&= b_\ell^{\tilde{b}_{\ell+1}} \cdot \left(\left(\left([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-\ell}}\right)^{\left(\prod_{j \in \lfloor n-1, \ell+1 \rfloor} b_j\right)} \cdot b_\ell\right)^{\tilde{b}_{\ell+1}}\right)^- \\
&= b_\ell^{\tilde{b}_{\ell+1}} \cdot \left((b'_\ell)^{\tilde{b}_{\ell+1}}\right)^-\,.
\end{aligned}$$

Ad (ii). For $0 \leqslant \ell \leqslant r \leqslant n-2$, we have

$$\begin{aligned}
& [\tilde{b}'_{r+2}\,, b'_{r+1}]^{\tilde{b}'_{r+1} \cdot \left(\prod_{j \in \lfloor r, \ell+1 \rfloor} b'_j \cdot \tilde{b}'_j\right)} \\
= \quad & x_r^{\left(\prod_{j \in \lfloor r, \ell+1 \rfloor} b'_j \cdot \tilde{b}'_j\right)} \\
= \quad & x_r^{\left(\prod_{j \in \lfloor r, \ell+1 \rfloor} \left([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-j}}\right)^{\left(\prod_{i \in \lfloor n-1, j+1 \rfloor} b_i\right)} \cdot b_j \cdot \tilde{b}'_j\right)} \\
= \quad & x_r^{\left(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j \cdot (b_j \cdot \tilde{b}'_j)\right)} \\
\overset{348.(2)}{=} \quad & \left(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{\left(\prod_{t \in \lfloor j, \ell+1 \rfloor} b_t \cdot \tilde{b}'_t\right)}\right)^- \cdot x_r^{\left(\prod_{t \in \lfloor r, \ell+1 \rfloor} b_t \cdot \tilde{b}'_t\right)} \cdot \left(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{\left(\prod_{t \in \lfloor j, \ell+1 \rfloor} b_t \cdot \tilde{b}'_t\right)}\right) \\
\overset{r \leqslant n}{=} \quad & \left(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{\left(\prod_{t \in \lfloor j, \ell+1 \rfloor} b_t \cdot \tilde{b}_t\right)}\right)^- \cdot x_r^{\left(\prod_{t \in \lfloor r, \ell+1 \rfloor} b_t \cdot \tilde{b}_t\right)} \cdot \left(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{\left(\prod_{t \in \lfloor j, \ell+1 \rfloor} b_t \cdot \tilde{b}_t\right)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \big(\prod_{j\in\lfloor r,\ell+1\rfloor} y_j^{z_{j,\ell}}\big)^- \cdot x_r^{z_{r,\ell}} \cdot \big(\prod_{j\in\lfloor r,\ell+1\rfloor} y_j^{z_{j,\ell}}\big)\\
&= w_{r,\ell}^- \cdot x_r^{z_{r,\ell}} \cdot w_{r,\ell}\,.
\end{aligned}
$$

Ad (iii). Suppose given $0 \leqslant \ell \leqslant r \leqslant n-2$.

We have

$$
\begin{aligned}
& x_r^{z_{r,\ell}}\\
= \quad & [\tilde{b}'_{r+2}\,, b'_{r+1}]^{\tilde{b}'_{r+1}\cdot z_{r,\ell}}\\
= \quad & \Big[\tilde{b}'_{r+2}\,, \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+1)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+2\rfloor} b_j\right)} \cdot b_{r+1}\Big]^{\tilde{b}'_{r+1}\cdot z_{r,\ell}}\\
\overset{348.(1)}{=} \quad & \Big([\tilde{b}'_{r+2}\,, b_{r+1}] \cdot \big([\tilde{b}_{n+1}\,, b_n]^{(-1)^{n-(r+1)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+2\rfloor} b_j\right)\cdot \tilde{b}'_{r+2}\cdot b_{r+1}}\\
& \qquad\qquad \cdot \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+1)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+2\rfloor} b_j\right)\cdot b_{r+1}}\Big)^{\tilde{b}'_{r+1}\cdot z_{r,\ell}}\\
= \quad & [\tilde{b}'_{r+2}\,, b_{r+1}]^{\tilde{b}'_{r+1}\cdot z_{r,\ell}}\\
& \cdot \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+2)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+2\rfloor} b_j\right)\cdot \tilde{b}'_{r+2}\cdot b_{r+1}\cdot \tilde{b}'_{r+1}\cdot\left(\prod_{t\in\lfloor r,\ell+1\rfloor} b_t\cdot\tilde{b}_t\right)}\\
& \cdot \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+1)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+2\rfloor} b_j\right)\cdot b_{r+1}\cdot \tilde{b}'_{r+1}\cdot\left(\prod_{t\in\lfloor r,\ell+1\rfloor} b_t\cdot\tilde{b}_t\right)}\\
\overset{\substack{r+2\leqslant n\\ \S1.1.2\,\text{item}\,9.3}}{=} \quad & [\tilde{b}_{r+2}\,, b_{r+1}]^{\tilde{b}_{r+1}\cdot z_{r,\ell}}\\
& \cdot \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+2)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+3\rfloor} b_j\right)\cdot b_{r+2}\cdot\tilde{b}_{r+2}\cdot b_{r+1}\cdot\tilde{b}_{r+1}\cdot\left(\prod_{t\in\lfloor r,\ell+1\rfloor} b_t\cdot\tilde{b}_t\right)}\\
& \cdot \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+1)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+2\rfloor} b_j\right)\cdot b_{r+1}\cdot\tilde{b}_{r+1}\cdot\left(\prod_{t\in\lfloor r,\ell+1\rfloor} b_t\cdot\tilde{b}_t\right)}\\
= \quad & [\tilde{b}_{r+2}\,, b_{r+1}]^{\tilde{b}_{r+1}\cdot z_{r,\ell}}\\
& \cdot \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+2)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+3\rfloor} b_j\right)\cdot\left(\prod_{t\in\lfloor r+2,\ell+1\rfloor} b_t\cdot\tilde{b}_t\right)}\\
& \cdot \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+1)}}\big)^{\left(\prod_{j\in\lfloor n-1,r+2\rfloor} b_j\right)\cdot\left(\prod_{t\in\lfloor r+1,\ell+1\rfloor} b_t\cdot\tilde{b}_t\right)}\\
= \quad & [\tilde{b}_{r+2}\,, b_{r+1}]^{\tilde{b}_{r+1}\cdot z_{r,\ell}} \cdot y_{r+2}^{z_{r+2,\ell}} \cdot y_{r+1}^{z_{r+1,\ell}}\,.
\end{aligned}
$$

Moreover, we have

$$\begin{aligned} w_{r,\ell} \cdot w_{r+2,\ell}^{-} &= \Big(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{z_{j,\ell}} \Big) \cdot \Big(\prod_{j \in \lfloor r+2, \ell+1 \rfloor} y_j^{z_{j,\ell}} \Big)^{-} \\ &= \Big(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{z_{j,\ell}} \Big) \cdot \Big(\prod_{j \in \lfloor r, \ell+1 \rfloor} y_j^{z_{j,\ell}} \Big)^{-} \cdot (y_{r+1}^{z_{r+1,\ell}})^{-} \cdot (y_{r+2}^{z_{r+2,\ell}})^{-} \\ &= (y_{r+1}^{z_{r+1,\ell}})^{-} \cdot (y_{r+2}^{z_{r+2,\ell}})^{-} \,. \end{aligned}$$

So we have

$$x_r^{z_{r,\ell}} \cdot w_{r,\ell} \cdot w_{r+2,\ell}^{-} = [\tilde{b}_{r+2} \,, b_{r+1}]^{\tilde{b}_{r+1} \cdot z_{r,\ell}} \,.$$

Ad (iv). For $0 \leqslant \ell \leqslant n-3$, we have

$$\begin{aligned} & x_{n-2}^{z_{n-2,\ell}} \cdot w_{n-2,\ell} \cdot w_{n-3,\ell}^{-} \cdot (x_{n-3}^{z_{n-3,\ell}})^{-} \\ \overset{\text{(iii)}}{=} \quad & [\tilde{b}_n \,, b_{n-1}]^{\tilde{b}_{n-1} \cdot z_{n-2,\ell}} \cdot y_n^{z_{n,\ell}} \cdot y_{n-1}^{z_{n-1,\ell}} \\ & \cdot \big(\prod_{j \in \lfloor n-2, \ell+1 \rfloor} y_j^{z_{j,\ell}} \big) \cdot \big(\prod_{j \in \lfloor n-3, \ell+1 \rfloor} y_j^{z_{j,\ell}} \big)^{-} \\ & \cdot \Big([\tilde{b}_{n-1} \,, b_{n-2}]^{\tilde{b}_{n-2} \cdot z_{n-3,\ell}} \cdot y_{n-1}^{z_{n-1,\ell}} \cdot y_{n-2}^{z_{n-2,\ell}} \Big)^{-} \\ = \quad & [\tilde{b}_n \,, b_{n-1}]^{\tilde{b}_{n-1} \cdot z_{n-2,\ell}} \cdot \big([b_n \,, \tilde{b}_{n+1}]^{(-1)^{n-n}} \big)^{\big(\prod_{\ell \in \lfloor n-1, n+1 \rfloor} b_\ell \big) \cdot \big(\prod_{t \in \lfloor n, \ell+1 \rfloor} b_t \cdot \tilde{b}_t \big)} \cdot y_{n-1}^{z_{n-1,\ell}} \\ & \cdot y_{n-2}^{z_{n-2,\ell}} \cdot \Big(\prod_{j \in \lfloor n-3, \ell+1 \rfloor} y_j^{z_{j,\ell}} \Big) \cdot \Big(\prod_{j \in \lfloor n-3, \ell+1 \rfloor} y_j^{z_{j,\ell}} \Big)^{-} \\ & \cdot (y_{n-2}^{z_{n-2,\ell}})^{-} \cdot (y_{n-1}^{z_{n-1,\ell}})^{-} \cdot [b_{n-2} \,, \tilde{b}_{n-1}]^{\tilde{b}_{n-2} \cdot z_{n-3,\ell}} \\ \overset{\S 1.1.2 \text{ item } 9.3}{=} \quad & [\tilde{b}_n \,, b_{n-1}]^{\tilde{b}_{n-1} \cdot z_{n-2,\ell}} \cdot [b_n \,, \tilde{b}_{n+1}]^{b_n^{-} \cdot b_n \cdot \tilde{b}_n \cdot \big(\prod_{t \in \lfloor n-1, \ell+1 \rfloor} b_t \cdot \tilde{b}_t \big)} \cdot [b_{n-2} \,, \tilde{b}_{n-1}]^{\tilde{b}_{n-2} \cdot z_{n-3,\ell}} \\ = \quad & [\tilde{b}_n \,, b_{n-1}]^{\tilde{b}_{n-1} \cdot z_{n-2,\ell}} \cdot [b_n \,, \tilde{b}_{n+1}]^{\tilde{b}_n \cdot z_{n-1,\ell}} \cdot [b_{n-2} \,, \tilde{b}_{n-1}]^{\tilde{b}_{n-2} \cdot z_{n-3,\ell}} \,. \end{aligned}$$

This shows (i – iv) and proves the *claim*.

We have

$$\begin{aligned} h \cdot \tilde{h} &= (b_{n+1}\,, b_n\,, b_{n-1}\,, \ldots\,, b_0\,, m) \cdot (\tilde{b}_{n+1}\,, \tilde{b}_n\,, \tilde{b}_{n-1}\,, \ldots\,, \tilde{b}_0\,, \tilde{m}) \\ &= \big(b_{n+1} \cdot \tilde{b}_{n+1}\,, (b_n\,, b_{n-1}\,, \ldots\,, b_0\,, m)^{\tilde{b}_{n+1}} \cdot (\tilde{b}_n\,, \tilde{b}_{n-1}\,, \ldots\,, \tilde{b}_0\,, \tilde{m})\big) \\ &= (b_{n+1} \cdot \tilde{b}_{n+1}\,, h' \cdot \tilde{h}')\,. \end{aligned}$$

By induction hypothesis we have, for $k \in [0, n]$,

$$\begin{aligned} & (h' \cdot \tilde{h}')_{\{k\}} \\ = \;& \big((b'_n\,, \ldots\,, b'_0\,, m') \cdot (\tilde{b}'_n\,, \ldots\,, \tilde{b}'_0\,, \tilde{m}')\big)_{\{k\}} \\ = \;& \Bigg(\prod_{\substack{i \in \lceil k,n \rceil \\ i \equiv_2 k}} [\tilde{b}'_{i+2}\,, b'_{i+1}]^{\tilde{b}'_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, k+1 \rfloor} b'_j \cdot \tilde{b}'_j\right)}\Bigg) \cdot \Bigg(\prod_{\substack{i \in \lceil k,n \rceil \\ i \equiv_2 k+1}} [\tilde{b}'_{i+2}\,, b'_{i+1}]^{\tilde{b}'_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, k+1 \rfloor} b'_j \cdot \tilde{b}'_j\right)}\Bigg)^{-} \cdot (b'_k)^{\tilde{b}'_{k+1}} \cdot \tilde{b}'_k\,. \end{aligned}$$

To perform the inductive step, we now calculate $(h \cdot \tilde{h})_{\{k\}}$ for $k \in [-1, n+1]$.

Case $k = n+1$. We have

$$\begin{aligned} & (h \cdot \tilde{h})_{\{n+1\}} = b_{n+1} \cdot \tilde{b}_{n+1} = 1 \cdot 1 \cdot b_{n+1}^1 \cdot \tilde{b}_{n+1} \\ = \;& \Bigg(\prod_{\substack{i \in \lceil n+1, n+1 \rceil \\ i \equiv_2 n+1}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\left(\tilde{b}_{i+1} \cdot \prod\limits_{j \in \lfloor i, n+2 \rfloor} b_j \cdot \tilde{b}_j\right)}\Bigg) \cdot \Bigg(\prod_{\substack{i \in \lceil n+1, n+1 \rceil \\ i \equiv_2 n+2}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\left(\tilde{b}_{i+1} \cdot \prod\limits_{j \in \lfloor i, n+2 \rfloor} b_j \cdot \tilde{b}_j\right)}\Bigg)^{-} \cdot b_{n+1}^{\tilde{b}_{n+2}} \cdot \tilde{b}_{n+1}\,. \end{aligned}$$

Case $k = n$. We have

$$\begin{aligned} & (h \cdot \tilde{h})_{\{n\}} \\ = \;& (h' \cdot \tilde{h}')_{\{n\}} \\ = \;& \big((b'_n\,, \ldots\,, b'_0\,, m') \cdot (\tilde{b}'_n\,, \ldots\,, \tilde{b}'_0\,, \tilde{m}')\big)_{\{n\}} \\ \overset{\text{IH}}{=} \;& \Bigg(\prod_{\substack{i \in \lceil n,n \rceil \\ i \equiv_2 n}} [\tilde{b}'_{i+2}\,, b'_{i+1}]^{\tilde{b}'_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, n+1 \rfloor} b'_j \cdot \tilde{b}'_j\right)}\Bigg) \cdot \Bigg(\prod_{\substack{i \in \lceil n,n \rceil \\ i \equiv_2 n+1}} [\tilde{b}'_{i+2}\,, b'_{i+1}]^{\tilde{b}'_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, n+1 \rfloor} b'_j \cdot \tilde{b}'_j\right)}\Bigg)^{-} \cdot (b'_n)^{\tilde{b}'_{n+1}} \cdot \tilde{b}'_n \\ = \;& 1 \cdot 1 \cdot (b'_n)^1 \cdot \tilde{b}_n \\ = \;& \Big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-n}}\Big)^{\left(\prod\limits_{j \in \lfloor n-1, n+1 \rfloor} b_j\right)} \cdot b_n \cdot \tilde{b}_n \end{aligned}$$

$$
\begin{aligned}
&\overset{\S 1.1.2\,\text{item } 9.3}{=} && [b_n\,,\tilde{b}_{n+1}]^{b_n^-}\cdot b_n\cdot\tilde{b}_n\\
&= && b_n\cdot(b_n^-\cdot\tilde{b}_{n+1}^-\cdot b_n\cdot\tilde{b}_{n+1})\cdot b_n^-\cdot b_n\cdot\tilde{b}_n\\
&= && b_n^{\tilde{b}_{n+1}}\cdot\tilde{b}_n\\
&= && 1\cdot 1\cdot b_n^{\tilde{b}_{n+1}}\cdot\tilde{b}_n\\
&= && \left(\prod_{\substack{i\in\lceil n,n+1\rceil\\ i\equiv_2 n}}[\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n+1\rfloor}b_j\cdot\tilde{b}_j\right)}\right)\cdot\left(\prod_{\substack{i\in\lceil n,n+1\rceil\\ i\equiv_2 n+1}}[\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n+1\rfloor}b_j\cdot\tilde{b}_j\right)}\right)^-\cdot b_n^{\tilde{b}_{n+1}}\cdot\tilde{b}_n\,.
\end{aligned}
$$

Case $n\geqslant 1$ and $k=n-1$. We have

$$
\begin{aligned}
&&& (h\cdot\tilde{h})_{\{n-1\}}\\
&= && (h'\cdot\tilde{h}')_{\{n-1\}}\\
&= && \big((b'_n\,,\dots\,,b'_0\,,m')\cdot(\tilde{b}'_n\,,\dots\,,\tilde{b}'_0\,,\tilde{m}')\big)_{\{n-1\}}\\
&\overset{\text{IH}}{=} && \left(\prod_{\substack{i\in\lceil n-1,n\rceil\\ i\equiv_2 n-1}}[\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n\rfloor}b'_j\cdot\tilde{b}'_j\right)}\right)\cdot\left(\prod_{\substack{i\in\lceil n-1,n\rceil\\ i\equiv_2 n}}[\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n\rfloor}b'_j\cdot\tilde{b}'_j\right)}\right)^-\cdot(b'_{n-1})^{\tilde{b}'_n}\cdot\tilde{b}'_{n-1}\\
&\overset{\tilde{b}'_{n+1}=1}{=} && 1\cdot 1\cdot(b'_{n-1})^{\tilde{b}_n}\cdot\tilde{b}_{n-1}\\
&= && \left(\left([b_n\,,\tilde{b}_{n+1}]^{(-1)^{n-(n-1)}}\right)^{\left(\prod_{j\in\lfloor n-1,n\rfloor}b_j\right)}\cdot b_{n-1}\right)^{\tilde{b}_n}\cdot\tilde{b}_{n-1}\\
&= && [\tilde{b}_{n+1}\,,b_n]^{1\cdot\tilde{b}_n}\cdot b_{n-1}^{\tilde{b}_n}\cdot\tilde{b}_{n-1}\\
&= && [\tilde{b}_{n+1}\,,b_n]^{\tilde{b}_n\cdot 1}\cdot 1\cdot b_{n-1}^{\tilde{b}_n}\cdot\tilde{b}_{n-1}\\
&= && \left(\prod_{\substack{i\in\lceil n-1,n+1\rceil\\ i\equiv_2 n-1}}[\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n\rfloor}b_j\cdot\tilde{b}_j\right)}\right)\cdot\left(\prod_{\substack{i\in\lceil n-1,n+1\rceil\\ i\equiv_2 n}}[\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n\rfloor}b_j\cdot\tilde{b}_j\right)}\right)^-\cdot b_{n-1}^{\tilde{b}_n}\cdot\tilde{b}_{n-1}\,.
\end{aligned}
$$

Case $n\geqslant 2$ and $k=n-2$. We have

$$
\begin{aligned}
&&& (h\cdot\tilde{h})_{\{n-2\}}\\
&= && (h'\cdot\tilde{h}')_{\{n-2\}}\\
&= && \big((b'_n\,,\dots\,,b'_0\,,m')\cdot(\tilde{b}'_n\,,\dots\,,\tilde{b}'_0\,,\tilde{m}')\big)_{\{n-2\}}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{IH}}{=} \Bigg(\prod_{\substack{i\in\lceil n-2,n\rceil \\ i\equiv_2 n-2}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-1\rfloor} b'_j\cdot\tilde{b}'_j\right)} \Bigg)\cdot \Bigg(\prod_{\substack{i\in\lceil n-2,n\rceil \\ i\equiv_2 n-1}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-1\rfloor} b'_j\cdot\tilde{b}'_j\right)} \Bigg)^{-} \\
&\qquad \cdot (b'_{n-2})^{\tilde{b}'_{n-1}}\cdot\tilde{b}'_{n-2} \\
&\overset{\substack{\tilde{b}'_{n+1}=1\\ \tilde{b}'_{n+2}=1}}{=} [\tilde{b}'_n\,,b'_{n-1}]^{\tilde{b}'_{n-1}\cdot\left(\prod_{j\in\lfloor n-2,n-1\rfloor} b'_j\cdot\tilde{b}'_j\right)}\cdot 1\cdot(b'_{n-2})^{\tilde{b}'_{n-1}}\cdot\tilde{b}'_{n-2} \\
&\overset{\text{(ii)}}{=} w^{-}_{n-2,n-2}\cdot x_{n-2}^{z_{n-2,n-2}}\cdot w_{n-2,n-2}\cdot(b'_{n-2})^{\tilde{b}_{n-1}}\cdot\tilde{b}_{n-2} \\
&\overset{\text{(i)}}{=} 1^{-}\cdot x_{n-2}^{z_{n-2,n-2}}\cdot 1\cdot\Big(\big([b_n\,,\tilde{b}_{n+1}]^{(-1)^{n-(n-2)}}\big)^{\left(\prod_{j\in\lfloor n-1,n-1\rfloor} b_j\right)}\cdot b_{n-2}\Big)^{\tilde{b}_{n-1}}\cdot\tilde{b}_{n-2} \\
&\overset{\text{(iii)}}{=} \big([\tilde{b}_n\,,b_{n-1}]^{\tilde{b}_{n-1}\cdot z_{n-2,n-2}}\cdot y_n^{z_{n,n-2}}\cdot y_{n-1}^{z_{n-1,n-2}}\big)\cdot[b_n\,,\tilde{b}_{n+1}]^{b_{n-1}\cdot\tilde{b}_{n-1}}\cdot b_{n-2}^{\tilde{b}_{n-1}}\cdot\tilde{b}_{n-2} \\
&= [\tilde{b}_n\,,b_{n-1}]^{\tilde{b}_{n-1}\cdot z_{n-2,n-2}}\cdot\big([b_n\,,\tilde{b}_{n+1}]^{(-1)^{n-n}}\big)^{\left(\prod_{j\in\lfloor n-1,n+1\rfloor} b_j\right)\cdot\left(\prod_{t\in\lfloor n,n-1\rfloor} b_t\cdot\tilde{b}_t\right)} \\
&\qquad \cdot\big([b_n\,,\tilde{b}_{n+1}]^{(-1)^{n-(n-1)}}\big)^{\left(\prod_{j\in\lfloor n-1,n\rfloor} b_j\right)\cdot\left(\prod_{t\in\lfloor n-1,n-1\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot[b_n\,,\tilde{b}_{n+1}]^{b_{n-1}\cdot\tilde{b}_{n-1}}\cdot b_{n-2}^{\tilde{b}_{n-1}}\cdot\tilde{b}_{n-2} \\
&\overset{\text{§1.1.2 item 9.3}}{=} [\tilde{b}_n\,,b_{n-1}]^{\tilde{b}_{n-1}\cdot\left(\prod_{t\in\lfloor n-2,n-1\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot[b_n\,,\tilde{b}_{n+1}]^{b_n^{-}\cdot b_n\cdot\tilde{b}_n\cdot\left(\prod_{t\in\lfloor n-1,n-1\rfloor} b_t\cdot\tilde{b}_t\right)} \\
&\qquad \cdot[\tilde{b}_{n+1}\,,b_n]^{1\cdot b_{n-1}\cdot\tilde{b}_{n-1}}\cdot[b_n\,,\tilde{b}_{n+1}]^{b_{n-1}\cdot\tilde{b}_{n-1}}\cdot b_{n-2}^{\tilde{b}_{n-1}}\cdot\tilde{b}_{n-2} \\
&= [\tilde{b}_n\,,b_{n-1}]^{\tilde{b}_{n-1}\cdot\left(\prod_{t\in\lfloor n-2,n-1\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot\Big([\tilde{b}_{n+1}\,,b_n]^{\tilde{b}_n\cdot\left(\prod_{t\in\lfloor n-1,n-1\rfloor} b_t\cdot\tilde{b}_t\right)}\Big)^{-}\cdot b_{n-2}^{\tilde{b}_{n-1}}\cdot\tilde{b}_{n-2} \\
&\overset{\substack{\tilde{b}_{n+2}=1\\ \tilde{b}_{n+3}=1}}{=} \Bigg(\prod_{\substack{i\in\lceil n-2,n+1\rceil \\ i\equiv_2 n-2}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-1\rfloor} b_j\cdot\tilde{b}_j\right)} \Bigg)\cdot \Bigg(\prod_{\substack{i\in\lceil n-2,n+1\rceil \\ i\equiv_2 n-1}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-1\rfloor} b_j\cdot\tilde{b}_j\right)} \Bigg)^{-} \\
&\qquad \cdot b_{n-2}^{\tilde{b}_{n-1}}\cdot\tilde{b}_{n-2}\,.
\end{aligned}
$$

Case $n\geqslant 3$ and $k=n-3$. We have

$$
\begin{aligned}
&(h\cdot\tilde{h})_{\{n-3\}} \\
=\;& (h'\cdot\tilde{h}')_{\{n-3\}} \\
=\;& \big((b'_n\,,\dots\,,b'_0\,,m')\cdot(\tilde{b}'_n\,,\dots\,,\tilde{b}'_0\,,\tilde{m}')\big)_{\{n-3\}}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{IH}}{=} \Bigg(\prod_{\substack{i\in\lceil n-3,n\rceil\\ i\equiv_2 n-3}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-2\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)\cdot\Bigg(\prod_{\substack{i\in\lceil n-3,n\rceil\\ i\equiv_2 n-2}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-2\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)^{-}\\
&\quad\cdot (b'_{n-3})^{\tilde{b}'_{n-2}}\cdot\tilde{b}'_{n-3}\\
&\overset{\substack{\tilde{b}'_{n+1}=1\\ \tilde{b}'_{n+2}=1}}{=} \Bigg(\prod_{\substack{i\in\lceil n-3,n-3\rceil\\ i\equiv_2 n-3}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-2\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)\cdot\Bigg(\prod_{\substack{i\in\lceil n-2,n-2\rceil\\ i\equiv_2 n-2}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-2\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)^{-}\\
&\quad\cdot (b'_{n-3})^{\tilde{b}_{n-2}}\cdot\tilde{b}_{n-3}\\
&\overset{\text{(ii)}}{=} (w^{-}_{n-3,n-3}\cdot x_{n-3}^{z_{n-3,n-3}}\cdot w_{n-3,n-3})\cdot(w^{-}_{n-2,n-3}\cdot x_{n-2}^{z_{n-2,n-3}}\cdot w_{n-2,n-3})^{-}\cdot(b'_{n-3})^{\tilde{b}_{n-2}}\cdot\tilde{b}_{n-3}\\
&= w^{-}_{n-3,n-3}\cdot\big(x_{n-3}^{z_{n-3,n-3}}\cdot w_{n-3,n-3}\cdot w^{-}_{n-2,n-3}\cdot(x_{n-2}^{z_{n-2,n-3}})^{-}\big)\cdot w_{n-2,n-3}\cdot(b'_{n-3})^{\tilde{b}_{n-2}}\cdot\tilde{b}_{n-3}\\
&\overset{\text{(i)}}{=} 1^{-}\cdot\big(x_{n-2}^{z_{n-2,n-3}}\cdot w_{n-2,n-3}\cdot w^{-}_{n-3,n-3}\cdot(x_{n-3}^{z_{n-3,n-3}})^{-}\big)^{-}\cdot b_{n-3}^{\tilde{b}_{n-2}}\cdot\big((b'_{n-3})^{\tilde{b}_{n-2}}\big)^{-}\cdot(b'_{n-3})^{\tilde{b}_{n-2}}\cdot\tilde{b}_{n-3}\\
&\overset{\text{(iv)}}{=} \big([\tilde{b}_n\,,b_{n-1}]^{\tilde{b}_{n-1}\cdot z_{n-2,n-3}}\cdot[b_n\,,\tilde{b}_{n+1}]^{\tilde{b}_n\cdot z_{n-1,n-3}}\cdot[b_{n-2}\,,\tilde{b}_{n-1}]^{\tilde{b}_{n-2}\cdot z_{n-3,n-3}}\big)^{-}\cdot b_{n-3}^{\tilde{b}_{n-2}}\cdot\tilde{b}_{n-3}\\
&= [\tilde{b}_{n-1}\,,b_{n-2}]^{\tilde{b}_{n-2}\cdot\left(\prod_{t\in\lfloor n-3,n-2\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot[\tilde{b}_{n+1}\,,b_n]^{\tilde{b}_n\cdot\left(\prod_{t\in\lfloor n-1,n-2\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot\Bigg([\tilde{b}_n\,,b_{n-1}]^{\tilde{b}_{n-1}\cdot\left(\prod_{t\in\lfloor n-2,n-2\rfloor} b_t\cdot\tilde{b}_t\right)}\Bigg)^{-}\\
&\quad\cdot b_{n-3}^{\tilde{b}_{n-2}}\cdot\tilde{b}_{n-3}\\
&\overset{\substack{\tilde{b}_{n+2}=1\\ \tilde{b}_{n+3}=1}}{=} \Bigg(\prod_{\substack{i\in\lceil n-3,n+1\rceil\\ i\equiv_2 n-3}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-2\rfloor} b_j\cdot\tilde{b}_j\right)}\Bigg)\cdot\Bigg(\prod_{\substack{i\in\lceil n-3,n+1\rceil\\ i\equiv_2 n-2}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,n-2\rfloor} b_j\cdot\tilde{b}_j\right)}\Bigg)^{-}\\
&\quad\cdot b_{n-3}^{\tilde{b}_{n-2}}\cdot\tilde{b}_{n-3}\,.
\end{aligned}
$$

Case $n\geqslant 4$ and $k\in[0,n-4]$. We have

$$
\begin{aligned}
&(h\cdot\tilde{h})_{\{k\}}\\
&= (h'\cdot\tilde{h}')_{\{k\}}\\
&= \big((b'_n,\dots,b'_0\,,m')\cdot(\tilde{b}'_n,\dots,\tilde{b}'_0\,,\tilde{m}')\big)_{\{k\}}\\
&\overset{\text{IH}}{=} \Bigg(\prod_{\substack{i\in\lceil k,n\rceil\\ i\equiv_2 k}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)\cdot\Bigg(\prod_{\substack{i\in\lceil k,n\rceil\\ i\equiv_2 k+1}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)^{-}\cdot(b'_k)^{\tilde{b}'_{k+1}}\cdot\tilde{b}'_k
\end{aligned}
$$

$$\begin{array}{rl}
\overset{\tilde{b}'_{n+1}=1}{\underset{\tilde{b}'_{n+2}=1}{=}} & \left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b'_j\cdot\tilde{b}'_j\right)}\right)\cdot\left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k+1}} [\tilde{b}'_{i+2}\,,b'_{i+1}]^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b'_j\cdot\tilde{b}'_j\right)}\right)^{-}\cdot (b'_k)^{\tilde{b}'_{k+1}}\cdot\tilde{b}'_k\\
\overset{\text{(ii)}}{=} & \left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k}} w_{i,k}^{-}\cdot x_i^{z_{i,k}}\cdot w_{i,k}\right)\cdot\left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k+1}} w_{i,k}^{-}\cdot x_i^{z_{i,k}}\cdot w_{i,k}\right)^{-}\cdot (b'_k)^{\tilde{b}'_{k+1}}\cdot\tilde{b}'_k\\
\overset{k+1\leqslant n}{=} & \left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k}} w_{i,k}^{-}\cdot x_i^{z_{i,k}}\cdot w_{i,k}\right)\cdot\left(\prod_{\substack{i\in\lfloor n-2,k\rfloor\\ i\equiv_2 k+1}} w_{i,k}^{-}\cdot (x_i^{z_{i,k}})^{-}\cdot w_{i,k}\right)\cdot (b'_k)^{\tilde{b}_{k+1}}\cdot\tilde{b}_k
\end{array}$$

If $n-k\equiv_2 0$, then we have

$$\begin{array}{rl}
& (h\cdot\tilde{h})_{\{k\}}\\
= & \left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k}} w_{i,k}^{-}\cdot x_i^{z_{i,k}}\cdot w_{i,k}\right)\cdot\left(\prod_{\substack{i\in\lfloor n-2,k\rfloor\\ i\equiv_2 k+1}} w_{i,k}^{-}\cdot (x_i^{z_{i,k}})^{-}\cdot w_{i,k}\right)\cdot (b'_k)^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\\
= & w_{k,k}^{-}\cdot\left(\prod_{\substack{i\in\lceil k,n-4\rceil\\ i\equiv_2 k}} x_i^{z_{i,k}}\cdot w_{i,k}\cdot w_{i+2,k}^{-}\right)\\
& \cdot\, x_{n-2}^{z_{n-2,k}}\cdot w_{n-2,k}\cdot w_{n-3,k}^{-}\cdot (x_{n-3}^{z_{n-3,k}})^{-}\\
& \cdot\left(\prod_{\substack{i\in\lfloor n-4,k+1\rfloor\\ i\equiv_2 k+1}} w_{i+2,k}\cdot w_{i,k}^{-}\cdot (x_i^{z_{i,k}})^{-}\right)\cdot w_{k+1,k}\cdot (b'_k)^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\\
\overset{\text{(i,iii,iv)}}{=} & 1^{-}\cdot\left(\prod_{\substack{i\in\lceil k,n-4\rceil\\ i\equiv_2 k}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\\
& \cdot\,[\tilde{b}_n\,,b_{n-1}]^{\tilde{b}_{n-1}\cdot z_{n-2,k}}\cdot[b_n\,,\tilde{b}_{n+1}]^{\tilde{b}_n\cdot z_{n-1,k}}\cdot[b_{n-2}\,,\tilde{b}_{n-1}]^{\tilde{b}_{n-2}\cdot z_{n-3,k}}\\
& \cdot\left(\prod_{\substack{i\in\lfloor n-4,k+1\rfloor\\ i\equiv_2 k+1}} [b_{i+1}\,,\tilde{b}_{i+2}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\cdot b_k^{\tilde{b}_{k+1}}\cdot\left((b'_k)^{\tilde{b}_{k+1}}\right)^{-}\cdot (b'_k)^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\\
= & \left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\cdot\left(\prod_{\substack{i\in\lfloor n-1,k+1\rfloor\\ i\equiv_2 k+1}} [b_{i+1}\,,\tilde{b}_{i+2}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\cdot b_k^{\tilde{b}_{k+1}}\cdot\tilde{b}_k
\end{array}$$

$$\overset{\tilde{b}_{n+2}=1}{\underset{\tilde{b}_{n+3}=1}{=}} \left(\prod_{\substack{i\in\lceil k,n+1\rceil\\ i\equiv_2 k}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)}\right)\cdot\left(\prod_{\substack{i\in\lceil k,n+1\rceil\\ i\equiv_2 k+1}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)}\right)^{-}\cdot b_k^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\,.$$

If $n-k\equiv_2 1$, then we have

$$\begin{aligned}
&(h\cdot\tilde{h})_{\{k\}}\\
=\;& \left(\prod_{\substack{i\in\lceil k,n-2\rceil\\ i\equiv_2 k}} w_{i,k}^{-}\cdot x_i^{z_{i,k}}\cdot w_{i,k}\right)\cdot\left(\prod_{\substack{i\in\lfloor n-2,k\rfloor\\ i\equiv_2 k+1}} w_{i,k}^{-}\cdot (x_i^{z_{i,k}})^{-}\cdot w_{i,k}\right)\cdot (b'_k)^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\\
=\;& w_{k,k}^{-}\cdot\left(\prod_{\substack{i\in\lceil k,n-5\rceil\\ i\equiv_2 k}} x_i^{z_{i,k}}\cdot w_{i,k}\cdot w_{i+2,k}^{-}\right)\\
&\cdot x_{n-3}^{z_{n-3,k}}\cdot w_{n-3,k}\cdot w_{n-2,k}^{-}\cdot (x_{n-2}^{z_{n-2,k}})^{-}\\
&\cdot\left(\prod_{\substack{i\in\lfloor n-4,k+1\rfloor\\ i\equiv_2 k+1}} w_{i+2,k}\cdot w_{i,k}^{-}\cdot (x_i^{z_{i,k}})^{-}\right)\cdot w_{k+1,k}\cdot (b'_k)^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\\
\overset{\text{(i,iii,iv)}}{=}\;& 1^{-}\cdot\left(\prod_{\substack{i\in\lceil k,n-5\rceil\\ i\equiv_2 k}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\\
&\cdot[\tilde{b}_{n-1}\,,b_{n-2}]^{\tilde{b}_{n-2}\cdot z_{n-3,k}}\cdot[\tilde{b}_{n+1}\,,b_n]^{\tilde{b}_n\cdot z_{n-1,k}}\cdot[b_{n-1}\,,\tilde{b}_n]^{\tilde{b}_{n-1}\cdot z_{n-2,k}}\\
&\cdot\left(\prod_{\substack{i\in\lfloor n-4,k+1\rfloor\\ i\equiv_2 k+1}} [b_{i+1}\,,\tilde{b}_{i+2}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\cdot b_k^{\tilde{b}_{k+1}}\cdot\left((b'_k)^{\tilde{b}_{k+1}}\right)^{-}\cdot (b'_k)^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\\
=\;& \left(\prod_{\substack{i\in\lceil k,n-1\rceil\\ i\equiv_2 k}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\cdot\left(\prod_{\substack{i\in\lfloor n-2,k+1\rfloor\\ i\equiv_2 k+1}} [b_{i+1}\,,\tilde{b}_{i+2}]^{\tilde{b}_{i+1}\cdot z_{i,k}}\right)\cdot b_k^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\\
\overset{\tilde{b}_{n+2}=1}{\underset{\tilde{b}_{n+3}=1}{=}}\;& \left(\prod_{\substack{i\in\lceil k,n+1\rceil\\ i\equiv_2 k}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)}\right)\cdot\left(\prod_{\substack{i\in\lceil k,n+1\rceil\\ i\equiv_2 k+1}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)}\right)^{-}\cdot b_k^{\tilde{b}_{k+1}}\cdot\tilde{b}_k\,.
\end{aligned}$$

Case $k=-1$. In this case $k=-1$, we need variants of the formerly defined auxiliary elements.

Suppose given $r\in\mathbb{Z}_{\geqslant -1}$.

We write

$$\hat{x}_r := \llbracket \tilde{b}'_{r+2},\, b'_{r+1} \rrbracket^{\tilde{b}'_{r+1}} \in M$$

$$\hat{y}_r := \Big(\llbracket b_n,\, \tilde{b}_{n+1} \rrbracket^{(-1)^{n-r}} \Big)^{\left(\prod_{\ell \in \lfloor n-1, r+1 \rfloor} b_\ell\right)} \in M$$

$$\hat{z}_r := \prod_{t \in \lfloor r, 0 \rfloor} b_t \cdot \tilde{b}_t \in B$$

$$\hat{w}_r := \prod_{j \in \lfloor r, 0 \rfloor} \hat{y}_j^{\hat{z}_j} \in M\,.$$

Moreover, we have

$$\hat{y}_r \mathrm{f}_S \overset{323.(1)}{=} \Big(\big(\llbracket b_n,\, \tilde{b}_{n+1} \rrbracket \mathrm{f}_S \big)^{(-1)^{n-r}} \Big)^{\left(\prod_{\ell \in \lfloor n-1, r+1 \rfloor} b_\ell\right)} \overset{\text{(SCM2)}}{=} \Big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-r}} \Big)^{\left(\prod_{\ell \in \lfloor n-1, r+1 \rfloor} b_\ell\right)} = y_r \in B\,.$$

We *claim* that the assertions (v – viii) hold.

(v) *For $r \in \mathbb{Z}_{\geqslant -1}$, we have*

$$\hat{w}_{-1} = 1 \quad \text{and} \quad \hat{w}_0 \cdot (m')^{\tilde{b}_0} = m^{\tilde{b}_0}\,.$$

(vi) *For $r \in [-1, n-2]$, we have*

$$\hat{w}_r^- \cdot \hat{x}_r^{\hat{z}_r} \cdot \hat{w}_r = \llbracket \tilde{b}'_{r+2}\,, b'_{r+1} \rrbracket^{\tilde{b}'_{r+1} \cdot \left(\prod_{j \in \lfloor r, 0 \rfloor} b'_j \cdot \tilde{b}'_j\right)}\,.$$

(vii) *For $r \in [-1, n-2]$, we have*

$$\hat{x}_r^{\hat{z}_r} = \llbracket \tilde{b}_{r+2}\,, b_{r+1} \rrbracket^{\tilde{b}_{r+1} \cdot \hat{z}_r} \cdot \hat{y}_{r+2}^{\hat{z}_{r+2}} \cdot \hat{y}_{r+1}^{\hat{z}_{r+1}}$$

and

$$\hat{x}_r^{\hat{z}_r} \cdot \hat{w}_r \cdot \hat{w}_{r+2}^- = \llbracket \tilde{b}_{r+2}\,, b_{r+1} \rrbracket^{\tilde{b}_{r+1} \cdot \hat{z}_r}\,.$$

(viii) *For $n \geqslant 2$, we have*

$$\begin{aligned} &\hat{x}_{n-2}^{\hat{z}_{n-2}} \cdot \hat{w}_{n-2} \cdot \hat{w}_{n-3}^- \cdot (\hat{x}_{n-3}^{\hat{z}_{n-3}})^- \\ &= \llbracket \tilde{b}_n\,, b_{n-1} \rrbracket^{\tilde{b}_{n-1} \cdot \hat{z}_{n-2}} \cdot \llbracket b_n\,, \tilde{b}_{n+1} \rrbracket^{\tilde{b}_n \cdot \hat{z}_{n-1}} \cdot \llbracket b_{n-2}\,, \tilde{b}_{n-1} \rrbracket^{\tilde{b}_{n-2} \cdot \hat{z}_{n-3}}\,. \end{aligned}$$

Ad (v). We have

$$\hat{w}_{-1} = \prod_{j\in\lfloor -1,0\rfloor} \hat{y}_j^{\hat{z}_j} = 1\,,$$

and

$$\begin{aligned}
\hat{w}_0 &= \prod_{j\in\lfloor 0,0\rfloor} \hat{y}_j^{\hat{z}_j} \\
&= \hat{y}_0^{\hat{z}_0} \\
&= \big(\llbracket b_n,\, \tilde{b}_{n+1}\rrbracket^{(-1)^{n-0}}\big)^{\left(\prod\limits_{\ell\in\lfloor n-1,1\rfloor} b_\ell\right)\cdot\left(\prod\limits_{t\in\lfloor 0,0\rfloor} b_t\cdot\tilde{b}_t\right)} \\
&= m^{\tilde{b}_0}\cdot\Big(m^-\cdot\big(\llbracket b_n,\, \tilde{b}_{n+1}\rrbracket^{(-1)^n}\big)^{\left(\prod\limits_{\ell\in\lfloor n-1,1\rfloor} b_\ell\right)\cdot b_0}\Big)^{\tilde{b}_0} \\
&\overset{\text{(SCM1)}}{=} m^{\tilde{b}_0}\cdot\Big(\Big(\big(\llbracket b_n,\, \tilde{b}_{n+1}\rrbracket^{(-1)^{n+1}}\big)^{\left(\prod\limits_{\ell\in\lfloor n-1,0\rfloor} b_\ell\right)}\cdot m\Big)^{\tilde{b}_0}\Big)^-\\
&= m^{\tilde{b}_0}\cdot\big((m')^{\tilde{b}_0}\big)^-\,.
\end{aligned}$$

Ad (vi). For $r\in[-1,n-2]$, we have

$$\begin{aligned}
\llbracket\tilde{b}'_{r+2},\, b'_{r+1}\rrbracket^{\tilde{b}'_{r+1}\cdot\left(\prod\limits_{j\in\lfloor r,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} &= \hat{x}_r^{\left(\prod\limits_{j\in\lfloor r,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \\
&= \hat{x}_r^{\left(\prod\limits_{j\in\lfloor r,0\rfloor}\left([b_n\,,\tilde{b}_{n+1}]^{(-1)^{n-j}}\right)^{\left(\prod\limits_{\ell\in\lfloor n-1,j+1\rfloor} b_\ell\right)}\cdot b_j\cdot\tilde{b}'_j\right)} \\
&= \hat{x}_r^{\left(\prod\limits_{j\in\lfloor r,0\rfloor} y_j\cdot(b_j\cdot\tilde{b}'_j)\right)} \\
&\overset{r\leqslant n}{=} \hat{x}_r^{\left(\prod\limits_{j\in\lfloor r,0\rfloor}\hat{y}_j\mathfrak{f}_S\cdot(b_j\cdot\tilde{b}_j)\right)} \\
&\overset{348.(4)}{=} \Big(\prod_{j\in\lfloor r,0\rfloor}\hat{y}_j^{\left(\prod\limits_{t\in\lfloor j,0\rfloor} b_t\cdot\tilde{b}_t\right)}\Big)^-\cdot\hat{x}_r^{\left(\prod\limits_{t\in\lfloor r,0\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot\Big(\prod_{j\in\lfloor r,0\rfloor}\hat{y}_j^{\left(\prod\limits_{t\in\lfloor j,0\rfloor} b_t\cdot\tilde{b}_t\right)}\Big) \\
&= \Big(\prod_{j\in\lfloor r,0\rfloor}\hat{y}_j^{\hat{z}_j}\Big)^-\cdot\hat{x}_r^{\hat{z}_r}\cdot\Big(\prod_{j\in\lfloor r,0\rfloor}\hat{y}_j^{\hat{z}_j}\Big) \\
&= \hat{w}_r^-\cdot\hat{x}_r^{\hat{z}_r}\cdot\hat{w}_r\,.
\end{aligned}$$

Ad (vii). Suppose given $r \in [-1, n-2]$. We have

$$
\begin{aligned}
& \hat{x}_r^{\hat{z}_r} \\
= \quad & \llbracket \tilde{b}'_{r+2}\,, b'_{r+1} \rrbracket^{\tilde{b}'_{r+1}\cdot\,\hat{z}_r} \\
= \quad & \Big\llbracket \tilde{b}'_{r+2},\ \big([b_n\,, \tilde{b}_{n+1}]^{(-1)^{n-(r+1)}}\big)^{\left(\prod_{\ell\in\lfloor n-1,r+2\rfloor} b_\ell\right)} \cdot b_{r+1} \Big\rrbracket^{\tilde{b}'_{r+1}\cdot\,\hat{z}_r} \\
\overset{348.(3)}{=} \quad & \Big(\llbracket \tilde{b}'_{r+2},\ b_{r+1} \rrbracket \cdot \big(\llbracket \tilde{b}_{n+1},\ b_n \rrbracket^{(-1)^{n-(r+1)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+2\rfloor} b_\ell\right)\tilde{b}'_{r+2}\cdot b_{r+1}} \\
& \qquad\qquad \cdot \big(\llbracket b_n,\ \tilde{b}_{n+1} \rrbracket^{(-1)^{n-(r+1)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+2\rfloor} b_\ell\right)\cdot b_{r+1}} \Big)^{\tilde{b}'_{r+1}\cdot\,\hat{z}_r} \\
\overset{r+2\leqslant n}{=} \quad & \Big(\llbracket \tilde{b}_{r+2},\ b_{r+1} \rrbracket \cdot \big(\llbracket \tilde{b}_{n+1},\ b_n \rrbracket^{(-1)^{n-(r+1)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+2\rfloor} b_\ell\right)\cdot \tilde{b}_{r+2}\cdot b_{r+1}} \\
& \qquad\qquad \cdot \big(\llbracket b_n,\ \tilde{b}_{n+1} \rrbracket^{(-1)^{n-(r+1)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+2\rfloor} b_\ell\right)\cdot b_{r+1}} \Big)^{\tilde{b}_{r+1}\cdot\,\hat{z}_r} \\
\overset{\S 1.1.2\,\text{item } 9.3}{=} \quad & \llbracket \tilde{b}_{r+2},\ b_{r+1} \rrbracket^{\tilde{b}_{r+1}\cdot\,\hat{z}_r} \\
& \cdot \big(\llbracket b_n,\ \tilde{b}_{n+1} \rrbracket^{(-1)^{n-(r+2)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+3\rfloor} b_\ell\right)\cdot b_{r+2}\cdot\tilde{b}_{r+2}\cdot b_{r+1}\cdot\tilde{b}_{r+1}\cdot\left(\prod_{t\in\lfloor r,0\rfloor} b_t\tilde{b}_t\right)} \\
& \cdot \big(\llbracket b_n,\ \tilde{b}_{n+1} \rrbracket^{(-1)^{n-(r+1)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+2\rfloor} b_\ell\right)\cdot b_{r+1}\cdot\tilde{b}_{r+1}\cdot\left(\prod_{t\in\lfloor r,0\rfloor} b_t\tilde{b}_t\right)} \\
= \quad & \llbracket \tilde{b}_{r+2},\ b_{r+1} \rrbracket^{\tilde{b}_{r+1}\cdot\hat{z}_r} \\
& \cdot \big(\llbracket b_n,\ \tilde{b}_{n+1} \rrbracket^{(-1)^{n-(r+2)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+3\rfloor} b_\ell\right)\cdot\left(\prod_{t\in\lfloor r+2,0\rfloor} b_t\cdot\tilde{b}_t\right)} \\
& \cdot \big(\llbracket b_n,\ \tilde{b}_{n+1} \rrbracket^{(-1)^{n-(r+1)}} \big)^{\left(\prod_{\ell\in\lfloor n-1,r+2\rfloor} b_\ell\right)\cdot\left(\prod_{t\in\lfloor r+1,0\rfloor} b_t\cdot\tilde{b}_t\right)} \\
= \quad & \llbracket \tilde{b}_{r+2},\ b_{r+1} \rrbracket^{\tilde{b}_{r+1}\cdot\hat{z}_r} \cdot \hat{y}_{r+2}^{\hat{z}_{r+2}} \cdot \hat{y}_{r+1}^{\hat{z}_{r+1}}\,.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\hat{w}_r \cdot \hat{w}_{r+2}^{-} &= \left(\prod_{j\in\lfloor r,0\rfloor} \hat{y}_j^{\hat{z}_j}\right)\cdot\left(\prod_{j\in\lfloor r+2,0\rfloor} \hat{y}_j^{\hat{z}_j}\right)^{-} = \left(\prod_{j\in\lfloor r,0\rfloor} \hat{y}_j^{\hat{z}_j}\right)\cdot\left(\prod_{j\in\lfloor r,0\rfloor} \hat{y}_j^{\hat{z}_j}\right)^{-}\cdot(\hat{y}_{r+1}^{\hat{z}_{r+1}})^{-}\cdot(\hat{y}_{r+2}^{\hat{z}_{r+2}})^{-} \\
&= (\hat{y}_{r+1}^{\hat{z}_{r+1}})^{-}\cdot(\hat{y}_{r+2}^{\hat{z}_{r+2}})^{-}\,.
\end{aligned}
$$

So we have

$$\hat{x}_r^{\hat{z}_r} \cdot \hat{w}_r \cdot \hat{w}_{r+2}^- = \llbracket \tilde{b}_{r+2},\, b_{r+1} \rrbracket^{\tilde{b}_{r+1}\cdot\hat{z}_r}\,.$$

Ad (viii). For $n \geqslant 2$, we have

$$\begin{aligned}
& \hat{x}_{n-2}^{\hat{z}_{n-2}} \cdot \hat{w}_{n-2} \cdot \hat{w}_{n-3}^- \cdot (\hat{x}_{n-3}^{\hat{z}_{n-3}})^- \\
\overset{\text{(vii)}}{=}\ & \llbracket \tilde{b}_n,\, b_{n-1} \rrbracket^{\tilde{b}_{n-1}\cdot\hat{z}_{n-2}} \cdot \hat{y}_n^{\hat{z}_n} \cdot \hat{y}_{n-1}^{\hat{z}_{n-1}} \\
& \cdot \Big(\prod_{j\in\lfloor n-2,0\rfloor} \hat{y}_j^{\hat{z}_j} \Big) \cdot \Big(\prod_{j\in\lfloor n-3,0\rfloor} \hat{y}_j^{\hat{z}_j} \Big)^- \\
& \cdot \Big(\llbracket \tilde{b}_{n-1},\, b_{n-2} \rrbracket^{\tilde{b}_{n-2}\cdot\hat{z}_{n-3}} \cdot \hat{y}_{n-1}^{\hat{z}_{n-1}} \cdot \hat{y}_{n-2}^{\hat{z}_{n-2}} \Big)^- \\
\overset{\text{(SCM1)}}{=}\ & \llbracket \tilde{b}_n,\, b_{n-1} \rrbracket^{\tilde{b}_{n-1}\cdot\hat{z}_{n-2}} \cdot \Big(\llbracket b_n,\, \tilde{b}_{n+1} \rrbracket^{(-1)^{n-n}} \Big)^{\left(\prod_{\ell\in\lfloor n-1,n+1\rfloor} b_\ell\right)\cdot\left(\prod_{t\in\lfloor n,0\rfloor} b_t\cdot\tilde{b}_t\right)} \cdot \hat{y}_{n-1}^{\hat{z}_{n-1}} \\
& \cdot \hat{y}_{n-2}^{\hat{z}_{n-2}} \cdot \Big(\prod_{j\in\lfloor n-3,0\rfloor} \hat{y}_j^{\hat{z}_j} \Big) \cdot \Big(\prod_{j\in\lfloor n-3,0\rfloor} \hat{y}_j^{\hat{z}_j} \Big)^- \\
& \cdot (\hat{y}_{n-2}^{\hat{z}_{n-2}})^- \cdot (\hat{y}_{n-1}^{\hat{z}_{n-1}})^- \cdot \llbracket b_{n-2},\, \tilde{b}_{n-1} \rrbracket^{\tilde{b}_{n-2}\cdot\hat{z}_{n-3}} \\
\overset{\S1.1.2\,\text{item}\,9.3}{=}\ & \llbracket \tilde{b}_n,\, b_{n-1} \rrbracket^{\tilde{b}_{n-1}\cdot\hat{z}_{n-2}} \cdot \llbracket b_n,\, \tilde{b}_{n+1} \rrbracket^{b_n^-\cdot b_n\cdot\tilde{b}_n\cdot\left(\prod_{t\in\lfloor n-1,0\rfloor} b_t\cdot\tilde{b}_t\right)} \cdot \llbracket b_{n-2},\, \tilde{b}_{n-1} \rrbracket^{\tilde{b}_{n-2}\cdot\hat{z}_{n-3}} \\
=\ & \llbracket \tilde{b}_n,\, b_{n-1} \rrbracket^{\tilde{b}_{n-1}\cdot\hat{z}_{n-2}} \cdot \llbracket b_n,\, \tilde{b}_{n+1} \rrbracket^{\tilde{b}_n\cdot\hat{z}_{n-1}} \cdot \llbracket b_{n-2},\, \tilde{b}_{n-1} \rrbracket^{\tilde{b}_{n-2}\cdot\hat{z}_{n-3}}\,.
\end{aligned}$$

This shows (v – viii) and proves the *claim*.

By induction hypothesis, we have

$$\begin{aligned}
& (h'\cdot\tilde{h}')_{\{-1\}} \\
=\ & \big((b'_n\,,\ldots,b'_0\,,m')\cdot(\tilde{b}'_n\,,\ldots,\tilde{b}'_0\,,\tilde{m}')\big)_{\{-1\}} \\
\overset{\text{IH}}{=}\ & \Bigg(\prod_{\substack{i\in\lceil -1,n\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1} \rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \Bigg) \cdot \Bigg(\prod_{\substack{i\in\lceil -1,n\rceil \\ i\equiv_2 0}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1} \rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \Bigg)^- \cdot (m')^{\tilde{b}'_0}\cdot\tilde{m}' \\
\overset{0\leqslant n}{=}\ & \Bigg(\prod_{\substack{i\in\lceil -1,n\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1} \rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \Bigg) \cdot \Bigg(\prod_{\substack{i\in\lceil -1,n\rceil \\ i\equiv_2 0}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1} \rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \Bigg)^- \cdot (m')^{\tilde{b}_0}\cdot\tilde{m}\,.
\end{aligned}$$

Subcase $k=-1$ and $n=0$. We have

$$
\begin{aligned}
& (h\cdot\tilde{h})_{\{-1\}} \\
= \quad & (h'\cdot\tilde{h}')_{\{-1\}} \\
= \quad & \Bigg(\prod_{\substack{i\in\lceil -1,0\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1}\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)\cdot\Bigg(\prod_{\substack{i\in\lceil -1,0\rceil \\ i\equiv_2 0}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1}\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)^{-}\cdot(m')^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\substack{\tilde{b}'_1=1\\ \tilde{b}'_2=1}}{=} \quad & 1\cdot 1\cdot(m')^{\tilde{b}_0}\cdot\tilde{m} \\
= \quad & \Big(\big(\llbracket b_0,\,\tilde{b}_1\rrbracket^{(-1)^{0+1}}\big)^{\left(\prod_{j\in\lfloor -1,0\rfloor} b_j\right)}\cdot m\Big)^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(SCM1)}}{=} \quad & \llbracket \tilde{b}_1,\, b_0\rrbracket^{\tilde{b}_0}\cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\substack{\tilde{b}'_2=1\\ \tilde{b}'_3=1}}{=} \quad & \Bigg(\prod_{\substack{i\in\lceil -1,1\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}_{i+2},\, b_{i+1}\rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\Bigg)\cdot\Bigg(\prod_{\substack{i\in\lceil -1,1\rceil \\ i\equiv_2 0}} \llbracket \tilde{b}_{i+2},\, b_{i+1}\rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\Bigg)^{-}\cdot m^{\tilde{b}_0}\cdot\tilde{m}\,.
\end{aligned}
$$

Subcase $k=-1$ and $n=1$. We have

$$
\begin{aligned}
& (h\cdot\tilde{h})_{\{-1\}} \\
= \quad & (h'\cdot\tilde{h}')_{\{-1\}} \\
= \quad & \Bigg(\prod_{\substack{i\in\lceil -1,1\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1}\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)\cdot\Bigg(\prod_{\substack{i\in\lceil -1,1\rceil \\ i\equiv_2 0}} \llbracket \tilde{b}'_{i+2},\, b'_{i+1}\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\Bigg)^{-}\cdot(m')^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\substack{\tilde{b}'_2=1\\ \tilde{b}'_3=1}}{=} \quad & \llbracket \tilde{b}'_1\,,b'_0\rrbracket^{\tilde{b}'_0\cdot\left(\prod_{\lfloor -1,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\cdot 1\cdot(m')^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(vi)}}{=} \quad & \hat{w}_{-1}^{-}\cdot\hat{x}_{-1}^{\hat{z}_{-1}}\cdot\hat{w}_{-1}\cdot\Big(\big(\llbracket b_1\,,\tilde{b}_2\rrbracket^{(-1)^{1+1}}\big)^{\left(\prod_{j\in\lfloor 0,0\rfloor} b_j\right)}\cdot m\Big)^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(v)}}{=} \quad & 1^{-}\cdot\hat{x}_{-1}^{\hat{z}_{-1}}\cdot 1\cdot\llbracket b_1\,,\tilde{b}_2\rrbracket^{b_0\cdot\tilde{b}_0}\cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(vii)}}{=} \quad & \llbracket \tilde{b}_1\,,b_0\rrbracket^{\tilde{b}_0\cdot\hat{z}_{-1}}\cdot\hat{y}_1^{\hat{z}_1}\cdot\hat{y}_0^{\hat{z}_0}\cdot\llbracket b_1\,,\tilde{b}_2\rrbracket^{b_0\cdot\tilde{b}_0}\cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
= \quad & \llbracket \tilde{b}_1\,,b_0\rrbracket^{\tilde{b}_0\cdot\hat{z}_{-1}}\cdot\big(\llbracket b_1\,,\tilde{b}_2\rrbracket^{(-1)^{1-1}}\big)^{\left(\prod_{\ell\in\lfloor 0,2\rfloor} b_\ell\right)\cdot\hat{z}_1}\cdot\big(\llbracket b_1\,,\tilde{b}_2\rrbracket^{(-1)^{1-0}}\big)^{\left(\prod_{\ell\in\lfloor 0,1\rfloor} b_\ell\right)\cdot\hat{z}_0}\cdot\llbracket b_1\,,\tilde{b}_2\rrbracket^{b_0\cdot\tilde{b}_0}\cdot m^{\tilde{b}_0}\cdot\tilde{m}
\end{aligned}
$$

$$\begin{aligned}
&\overset{\text{§1.1.2 item 9.3}}{=} \quad \left\llbracket \tilde{b}_1\,,b_0 \right\rrbracket^{\tilde{b}_0\cdot\left(\prod\limits_{t\in\lfloor -1,0\rfloor} b_t\cdot\tilde{b}_t\right)} \cdot \left\llbracket b_1\,,\tilde{b}_2 \right\rrbracket^{b_1^-\cdot b_1\cdot\tilde{b}_1\cdot\left(\prod\limits_{t\in\lfloor 0,0\rfloor} b_t\cdot\tilde{b}_t\right)} \cdot \left(\left\llbracket b_1,\,\tilde{b}_2 \right\rrbracket^{1\cdot b_0\cdot\tilde{b}_0}\right)^- \cdot \left\llbracket b_1\,,\tilde{b}_2 \right\rrbracket^{b_0\cdot\tilde{b}_0} \cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
&\overset{\text{(SCM1)}}{=} \quad \left\llbracket \tilde{b}_1\,,b_0 \right\rrbracket^{\tilde{b}_0\cdot\left(\prod\limits_{t\in\lfloor -1,0\rfloor} b_t\cdot\tilde{b}_t\right)} \cdot \left(\left\llbracket \tilde{b}_2\,,b_1 \right\rrbracket^{\tilde{b}_1\cdot\left(\prod\limits_{t\in\lfloor 0,0\rfloor} b_t\cdot\tilde{b}_t\right)}\right)^- \cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
&\overset{\substack{\tilde{b}_3=1\\ \tilde{b}_4=1}}{=} \quad \left(\prod_{\substack{i\in\lceil -1,2\rceil\\ i\equiv_2 -1}} \left\llbracket \tilde{b}_{i+2},\,b_{i+1} \right\rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\right) \cdot \left(\prod_{\substack{i\in\lceil -1,2\rceil\\ i\equiv_2 0}} \left\llbracket \tilde{b}_{i+2},\,b_{i+1} \right\rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\right)^- \cdot m^{\tilde{b}_0}\cdot\tilde{m}\,.
\end{aligned}$$

Subcase $k=-1$ and $n=2$. We have

$$\begin{aligned}
&(h\cdot\tilde{h})_{\{-1\}} \\
= \quad &(h'\cdot\tilde{h}')_{\{-1\}} \\
= \quad &\left(\prod_{\substack{i\in\lceil -1,2\rceil\\ i\equiv_2 -1}} \left\llbracket \tilde{b}'_{i+2},\,b'_{i+1} \right\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\right) \cdot \left(\prod_{\substack{i\in\lceil -1,2\rceil\\ i\equiv_2 0}} \left\llbracket \tilde{b}'_{i+2},\,b'_{i+1} \right\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\right)^- \cdot (m')^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\substack{\tilde{b}'_3=1\\ \tilde{b}'_4=1}}{=} \quad &\left\llbracket \tilde{b}'_1,\,b'_0 \right\rrbracket^{\tilde{b}'_0\cdot\left(\prod\limits_{j\in\lfloor -1,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \cdot \left(\left\llbracket \tilde{b}'_2,\,b'_1 \right\rrbracket^{\tilde{b}'_1\cdot\left(\prod\limits_{j\in\lfloor 0,0\rfloor} b'_j\cdot\tilde{b}'_j\right)}\right)^- \cdot (m')^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(vi)}}{=} \quad &\hat{w}_{-1}^-\cdot\hat{x}_{-1}^{\hat{z}_{-1}}\cdot\hat{w}_{-1}\cdot(\hat{w}_0^-\cdot\hat{x}_0^{\hat{z}_0}\cdot\hat{w}_0)^-\cdot(m')^{\tilde{b}_0}\cdot\tilde{m} \\
= \quad &\hat{w}_{-1}^-\cdot\left(\hat{x}_{-1}^{\hat{z}_{-1}}\cdot\hat{w}_{-1}\cdot\hat{w}_0^-\cdot(\hat{x}_0^{\hat{z}_0})^-\right)\cdot\hat{w}_0\cdot(m')^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(v)}}{=} \quad &1^-\cdot\left(\hat{x}_0^{\hat{z}_0}\cdot\hat{w}_0\cdot\hat{w}_{-1}^-\cdot(\hat{x}_{-1}^{\hat{z}_{-1}})^-\right)^-\cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(viii)}}{=} \quad &\left(\left\llbracket \tilde{b}_2\,,b_1 \right\rrbracket^{\tilde{b}_1\cdot\hat{z}_0}\cdot\left\llbracket b_2,\,\tilde{b}_3 \right\rrbracket^{\tilde{b}_2\cdot\hat{z}_1}\cdot\left\llbracket b_0\,,\tilde{b}_1 \right\rrbracket^{\tilde{b}_0\cdot\hat{z}_{-1}}\right)^-\cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\text{(SCM1)}}{=} \quad &\left\llbracket \tilde{b}_1\,,b_0 \right\rrbracket^{\tilde{b}_0\cdot\left(\prod\limits_{t\in\lfloor -1,0\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot\left\llbracket \tilde{b}_3\,,b_2 \right\rrbracket^{\tilde{b}_2\cdot\left(\prod\limits_{t\in\lfloor 1,0\rfloor} b_t\cdot\tilde{b}_t\right)}\cdot\left(\left\llbracket \tilde{b}_2\,,b_1 \right\rrbracket^{\tilde{b}_1\cdot\left(\prod\limits_{t\in\lfloor 0,0\rfloor} b_t\cdot\tilde{b}_t\right)}\right)^-\cdot m^{\tilde{b}_0}\cdot\tilde{m} \\
\overset{\substack{\tilde{b}_4=1\\ \tilde{b}_5=1}}{=} \quad &\left(\prod_{\substack{i\in\lceil -1,3\rceil\\ i\equiv_2 -1}} \left\llbracket \tilde{b}_{i+2},\,b_{i+1} \right\rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\right) \cdot \left(\prod_{\substack{i\in\lceil -1,3\rceil\\ i\equiv_2 0}} \left\llbracket \tilde{b}_{i+2},\,b_{i+1} \right\rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\right)^- \cdot m^{\tilde{b}_0}\cdot\tilde{m}\,.
\end{aligned}$$

Subcase $k=-1$ and $n\geqslant 3$. We have

$$\begin{aligned}
&(h\cdot\tilde{h})_{\{-1\}} \\
= \quad &(h'\cdot\tilde{h}')_{\{-1\}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{\substack{i\in\lceil -1,n\rceil \\ i\equiv_2 -1}} \left\llbracket \tilde{b}'_{i+2}\,,\, b'_{i+1} \right\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \right) \cdot \left(\prod_{\substack{i\in\lceil -1,n\rceil \\ i\equiv_2 0}} \left\llbracket \tilde{b}'_{i+2}\,,\, b'_{i+1} \right\rrbracket^{\tilde{b}'_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \right)^{-} \cdot (m')^{\tilde{b}_0}\cdot\tilde{m} \\
&\overset{\substack{\tilde{b}'_{n+2}=1 \\ \tilde{b}'_{n+1}=1}}{=} \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 -1}} \left\llbracket \tilde{b}'_{i+2}\,,\, b'_{i+1} \right\rrbracket^{\left(\tilde{b}'_{i+1}\cdot\prod\limits_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \right) \cdot \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 0}} \left\llbracket \tilde{b}'_{i+2}\,,\, b'_{i+1} \right\rrbracket^{\left(\tilde{b}'_{i+1}\cdot\prod\limits_{j\in\lfloor i,0\rfloor} b'_j\cdot\tilde{b}'_j\right)} \right)^{-} \cdot (m')^{\tilde{b}_0}\cdot\tilde{m} \\
&\overset{\text{(vi)}}{=} \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 -1}} \hat{w}_i^{-}\cdot\hat{x}_i^{\hat{z}_i}\cdot\hat{w}_i \right) \cdot \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 0}} \hat{w}_i^{-}\cdot\hat{x}_i^{\hat{z}_i}\cdot\hat{w}_i \right)^{-} \cdot (m')^{\tilde{b}_0}\cdot\tilde{m} \\
&= \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 -1}} \hat{w}_i^{-}\cdot\hat{x}_i^{\hat{z}_i}\cdot\hat{w}_i \right) \cdot \left(\prod_{\substack{i\in\lfloor n-2,-1\rfloor \\ i\equiv_2 0}} \hat{w}_i^{-}\cdot(\hat{x}_i^{\hat{z}_i})^{-}\cdot\hat{w}_i \right) \cdot (m')^{\tilde{b}_0}\cdot\tilde{m}\,.
\end{aligned}
$$

If $n \equiv_2 1$, then we have

$$
\begin{aligned}
&(h\cdot\tilde{h})_{\{-1\}} \\
&= \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 -1}} \hat{w}_i^{-}\cdot\hat{x}_i^{\hat{z}_i}\cdot\hat{w}_i \right) \cdot \left(\prod_{\substack{i\in\lfloor n-2,-1\rfloor \\ i\equiv_2 0}} \hat{w}_i^{-}\cdot(\hat{x}_i^{\hat{z}_i})^{-}\cdot\hat{w}_i \right) \cdot (m')^{\tilde{b}_0}\cdot\tilde{m} \\
&= \hat{w}_{-1}^{-}\cdot \left(\prod_{\substack{i\in\lceil -1,n-4\rceil \\ i\equiv_2 -1}} \hat{x}_i^{\hat{z}_i}\cdot\hat{w}_i\cdot\hat{w}_{i+2}^{-} \right) \\
&\quad \cdot \hat{x}_{n-2}^{\hat{z}_{n-2}}\cdot\hat{w}_{n-2}\cdot\hat{w}_{n-3}^{-}\cdot(\hat{x}_{n-3}^{\hat{z}_{n-3}})^{-} \\
&\quad \cdot \left(\prod_{\substack{i\in\lfloor n-4,0\rfloor \\ i\equiv_2 0}} \hat{w}_{i+2}\cdot\hat{w}_i^{-}\cdot(\hat{x}_i^{\hat{z}_i})^{-} \right) \cdot \hat{w}_0\cdot(m')^{\tilde{b}_0}\cdot\tilde{m} \\
&\overset{\substack{\text{(v,vii,viii)} \\ \text{(SCM1)}}}{=} 1^{-}\cdot \left(\prod_{\substack{i\in\lceil -1,n-4\rceil \\ i\equiv_2 -1}} \left\llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \right\rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right) \\
&\quad \cdot \left\llbracket \tilde{b}_n\,,\, b_{n-1} \right\rrbracket^{\tilde{b}_{n-1}\cdot\hat{z}_{n-2}} \cdot \left\llbracket b_n\,,\, \tilde{b}_{n+1} \right\rrbracket^{\tilde{b}_n\cdot\hat{z}_{n-1}} \cdot \left\llbracket b_{n-2}\,,\, \tilde{b}_{n-1} \right\rrbracket^{\tilde{b}_{n-2}\cdot\hat{z}_{n-3}} \\
&\quad \cdot \left(\prod_{\substack{i\in\lfloor n-4,0\rfloor \\ i\equiv_2 0}} \left\llbracket b_{i+1}\,,\, \tilde{b}_{i+2} \right\rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right) \cdot m^{\tilde{b}_0}\cdot\tilde{m}
\end{aligned}
$$

$$= \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right) \cdot \left(\prod_{\substack{i\in\lfloor n-1,0\rfloor \\ i\equiv_2 0}} \llbracket b_{i+1}\,,\, \tilde{b}_{i+2} \rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right) m^{\tilde{b}_0}\cdot\tilde{m}$$

$$\overset{\substack{\tilde{b}_{n+2}=1=\tilde{b}_{n+3} \\ \text{(SCM1)}}}{=} \left(\prod_{\substack{i\in\lceil -1,n+1\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)} \right) \cdot \left(\prod_{\substack{i\in\lceil -1,n+1\rceil \\ i\equiv_2 0}} \llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)} \right)^{-} \cdot m^{\tilde{b}_0}\cdot\tilde{m}\,.$$

If $n \equiv_2 0$, then we have

$$(h\cdot\tilde{h})_{\{-1\}}$$

$$= \left(\prod_{\substack{i\in\lceil -1,n-2\rceil \\ i\equiv_2 -1}} \hat{w}_i^{-}\cdot\hat{x}_i^{\hat{z}_i}\cdot\hat{w}_i \right) \cdot \left(\prod_{\substack{i\in\lfloor n-2,-1\rfloor \\ i\equiv_2 0}} \hat{w}_i^{-}\cdot(\hat{x}_i^{\hat{z}_i})^{-}\cdot\hat{w}_i \right) \cdot (m')^{\tilde{b}_0}\cdot\tilde{m}$$

$$= \hat{w}_{-1}^{-}\cdot \left(\prod_{\substack{i\in\lceil -1,n-5\rceil \\ i\equiv_2 -1}} \hat{x}_i^{\hat{z}_i}\cdot\hat{w}_i\cdot\hat{w}_{i+2}^{-} \right)$$

$$\cdot\,\hat{x}_{n-3}^{\hat{z}_{n-3}}\cdot\hat{w}_{n-3}\cdot\hat{w}_{n-2}^{-}\cdot(\hat{x}_{n-2}^{\hat{z}_{n-2}})^{-}$$

$$\cdot \left(\prod_{\substack{i\in\lfloor n-4,0\rfloor \\ i\equiv_2 0}} \hat{w}_{i+2}\cdot\hat{w}_i^{-}\cdot(\hat{x}_i^{\hat{z}_i})^{-} \right) \cdot \hat{w}_0\cdot(m')^{\tilde{b}_0}\cdot\tilde{m}$$

$$\overset{\substack{\text{(v,vii,viii)} \\ \text{(SCM1)}}}{=} 1^{-}\cdot \left(\prod_{\substack{i\in\lceil -1,n-5\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right)$$

$$\cdot\,\llbracket \tilde{b}_{n-1}\,,\, b_{n-2} \rrbracket^{\tilde{b}_{n-2}\cdot\hat{z}_{n-3}} \cdot \llbracket \tilde{b}_{n+1}\,,\, b_n \rrbracket^{\tilde{b}_n\cdot\hat{z}_{n-1}} \cdot \llbracket b_{n-1}\,,\, \tilde{b}_n \rrbracket^{\tilde{b}_{n-1}\cdot\hat{z}_{n-2}}$$

$$\cdot \left(\prod_{\substack{i\in\lfloor n-4,0\rfloor \\ i\equiv_2 0}} \llbracket b_{i+1}\,,\, \tilde{b}_{i+2} \rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right) \cdot m^{\tilde{b}_0}\cdot\tilde{m}$$

$$= \left(\prod_{\substack{i\in\lceil -1,n-1\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right) \cdot \left(\prod_{\substack{i\in\lfloor n-2,0\rfloor \\ i\equiv_2 0}} \llbracket b_{i+1}\,,\, \tilde{b}_{i+2} \rrbracket^{\tilde{b}_{i+1}\cdot\hat{z}_i} \right) \cdot m^{\tilde{b}_0}\cdot\tilde{m}$$

$$\overset{\substack{\tilde{b}_{n+3}=1=\tilde{b}_{n+2} \\ \text{(SCM1)}}}{=} \left(\prod_{\substack{i\in\lceil -1,n+1\rceil \\ i\equiv_2 -1}} \llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)} \right) \cdot \left(\prod_{\substack{i\in\lceil -1,n+1\rceil \\ i\equiv_2 0}} \llbracket \tilde{b}_{i+2}\,,\, b_{i+1} \rrbracket^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)} \right)^{-} \cdot m^{\tilde{b}_0}\cdot\tilde{m}\,.$$

□

Example 350 Suppose given $x \in B$. Suppose given $b_i\,,\ \tilde{b}_i \in B$ for $i \in [0,4]$, and suppose given $m,\ \tilde{m} \in M$.

(0) In H_0, we have

$$(b_0\,, m)^x \overset{347}{=} \left([b_0, x]^{b_0^-} \cdot b_0\,, \llbracket x,\ b_0 \rrbracket \cdot m\right) = \left(b_0^x\,, \llbracket x,\ b_0 \rrbracket \cdot m\right).$$

Moreover, we have

$$(b_0\,, m) \cdot (\tilde{b}_0\,, \tilde{m}) \overset{349}{=} (b_0 \cdot \tilde{b}_0\,, m^{\tilde{b}_0} \cdot \tilde{m})\,.$$

(1) In H_1, we have

$$(b_1\,, b_0\,, m)^x \overset{347}{=} \left(b_1^x\,, [x, b_1] \cdot b_0\,, \llbracket b_1,\ x \rrbracket^{b_0} \cdot m\right).$$

Moreover, we have

$$(b_1\,, b_0\,, m) \cdot (\tilde{b}_1\,, \tilde{b}_0\,, \tilde{m}) \overset{349}{=} \left(b_1 \cdot \tilde{b}_1\,, b_0^{\tilde{b}_1} \cdot \tilde{b}_0\,, \llbracket \tilde{b}_1,\ b_0 \rrbracket^{\tilde{b}_0} \cdot m^{\tilde{b}_0} \cdot \tilde{m}\right).$$

(2) In H_2, we have

$$(b_2\,, b_1\,, b_0\,, m)^x \overset{347}{=} \left(b_2^x\,, [x, b_2] \cdot b_1\,, [b_2\,, x]^{b_1} \cdot b_0\,, \llbracket x,\ b_2 \rrbracket^{b_1 \cdot b_0} \cdot m\right).$$

Moreover, we have

$$\begin{aligned}
&(b_2\,, b_1\,, b_0\,, m) \cdot (\tilde{b}_2\,, \tilde{b}_1\,, \tilde{b}_0\,, \tilde{m}) \\
&\overset{349}{=} \left(b_2 \cdot \tilde{b}_2\,, b_1^{\tilde{b}_2} \cdot \tilde{b}_1\,, [\tilde{b}_2\,, b_1]^{\tilde{b}_1} \cdot b_0^{\tilde{b}_1} \cdot \tilde{b}_0\,, \llbracket \tilde{b}_1,\ b_0 \rrbracket^{\tilde{b}_0} \cdot \llbracket b_1,\ \tilde{b}_2 \rrbracket^{\tilde{b}_1 \cdot b_0 \cdot \tilde{b}_0} \cdot m^{\tilde{b}_0} \cdot \tilde{m}\right).
\end{aligned}$$

(3) In H_3, we have

$$(b_3\,, b_2\,, b_1\,, b_0\,, m)^x \overset{347}{=} \left(b_3^x\,, [x, b_3] \cdot b_2\,, [b_3\,, x]^{b_2} \cdot b_1\,, [x, b_3]^{b_2 \cdot b_1} \cdot b_0\,, \llbracket b_3,\ x \rrbracket^{b_2 \cdot b_1 \cdot b_0} \cdot m\right).$$

Moreover, we have

$$\begin{aligned}
&(b_3\,, b_2\,, b_1\,, b_0\,, m) \cdot (\tilde{b}_3\,, \tilde{b}_2\,, \tilde{b}_1\,, \tilde{b}_0\,, \tilde{m}) \\
&\overset{349}{=} \left(b_3 \cdot \tilde{b}_3\,, b_2^{\tilde{b}_3} \cdot \tilde{b}_2\,, [\tilde{b}_3\,, b_2]^{\tilde{b}_2} \cdot b_1^{\tilde{b}_2} \cdot \tilde{b}_1\,, [\tilde{b}_2\,, b_1]^{\tilde{b}_1} \cdot [b_2\,, \tilde{b}_3]^{\tilde{b}_2 \cdot b_1 \cdot \tilde{b}_1} \cdot b_0^{\tilde{b}_1} \cdot \tilde{b}_0\,,\right. \\
&\left. \llbracket \tilde{b}_1,\ b_0 \rrbracket^{\tilde{b}_0} \cdot \llbracket \tilde{b}_3,\ b_2 \rrbracket^{\tilde{b}_2 \cdot b_1 \cdot \tilde{b}_1 \cdot b_0 \cdot \tilde{b}_0} \cdot \llbracket b_1,\ \tilde{b}_2 \rrbracket^{\tilde{b}_1 \cdot b_0 \cdot \tilde{b}_0} \cdot m^{\tilde{b}_0} \cdot \tilde{m}\right).
\end{aligned}$$

(4) In H_4, we have

$$\begin{aligned}
&(b_4\,, b_3\,, b_2\,, b_1\,, b_0\,, m)^x \\
&\overset{347}{=} \left(b_4^x\,, [x, b_4] \cdot b_3\,, [b_4\,, x]^{b_3} \cdot b_2\,, [x, b_4]^{b_3 \cdot b_2} \cdot b_1\,, [b_4\,, x]^{b_3 \cdot b_2 \cdot b_1} \cdot b_0\,, \llbracket x,\ b_4 \rrbracket^{b_3 \cdot b_2 \cdot b_1 \cdot b_0} \cdot m\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&(b_4\,,b_3\,,b_2\,,b_1\,,b_0\,,m)\cdot(\tilde{b}_4\,,\tilde{b}_3\,,\tilde{b}_2\,,\tilde{b}_1\,,\tilde{b}_0\,,\tilde{m})\\
&\overset{349}{=}\big(b_4\cdot\tilde{b}_4\,,\\
&b_3^{\tilde{b}_4}\cdot\tilde{b}_3\,,\\
&[\tilde{b}_4\,,b_3]^{\tilde{b}_3}\cdot b_2^{\tilde{b}_3}\cdot\tilde{b}_2\,,\\
&[\tilde{b}_3\,,b_2]^{\tilde{b}_2}\cdot[b_3\,,\tilde{b}_4]^{\tilde{b}_3\cdot b_2\cdot\tilde{b}_2}\cdot b_1^{\tilde{b}_2}\cdot\tilde{b}_1\,,\\
&[\tilde{b}_2\,,b_1]^{\tilde{b}_1}\cdot[\tilde{b}_4\,,b_3]^{\tilde{b}_3\cdot b_2\cdot\tilde{b}_2\cdot b_1\cdot\tilde{b}_1}\cdot[b_2\,,\tilde{b}_3]^{\tilde{b}_2\cdot b_1\cdot\tilde{b}_1}\cdot b_0^{\tilde{b}_1}\cdot\tilde{b}_0\,,\\
&\llbracket\tilde{b}_1,\,b_0\rrbracket^{\tilde{b}_0}\cdot\llbracket\tilde{b}_3,\,b_2\rrbracket^{\tilde{b}_2\cdot b_1\cdot\tilde{b}_1\cdot b_0\cdot\tilde{b}_0}\cdot\llbracket b_3,\,\tilde{b}_4\rrbracket^{\tilde{b}_3\cdot b_2\cdot\tilde{b}_2\cdot b_1\cdot\tilde{b}_1\cdot b_0\cdot\tilde{b}_0}\cdot\llbracket b_1,\,\tilde{b}_2\rrbracket^{\tilde{b}_1\cdot b_0\cdot\tilde{b}_0}\cdot m^{\tilde{b}_0}\cdot\tilde{m}\big)\,.
\end{aligned}$$

17.1.4 The group morphisms $\iota_{n,m}$ and s_i in H_n

We consider the data constructed in Lemma 342.

Remark 351 (The group morphism $\iota_{n,m}$)
Suppose given $k\in\mathbb{Z}_{\geqslant -1}$. We consider the groups H_k and $H_{k+1}=B\ltimes H_k$.

(1) *We have the injective group morphism*

$$\iota_{k,k+1}\colon H_k\to H_{k+1}\,,\ h\mapsto h\iota_{k,k+1}:=(1,h)\,.$$

(2) *Suppose given $\ell\in\mathbb{Z}_{\geqslant k}$.*

We have the injective group morphism

$$\iota_{k,\ell}:=\prod_{j\in[k,\ell-1]}^{H_k}{}^{H_\ell}\,\iota_{j,\,j+1}\quad:H_k\to H_\ell\,,\ h\mapsto h\iota_{k,\ell}\,,$$

where

$$(h\iota_{k,\ell})_{\{i\}}=\begin{cases}1 & \text{for } i\in[k+1,\ell]\\ h_{\{i\}} & \text{for } i\in[-1,k]\,;\end{cases}$$

cf. §1.1.3 *item* 4.

So, for $h\in H_k$, we have

$$h\iota_{k,\ell}=(1,\dots,1,h_{\{k\}}\,,h_{\{k-1\}}\,,\dots,h_{\{-1\}})\in H_\ell\,.$$

(3) *Suppose given* $\ell \in \mathbb{Z}_{\geqslant k}$ *. Suppose given* $h \in H_k$ *.*
Recall that we write $h_{\{j\}} = 1$ *for* $j \in \mathbb{Z}_{\geqslant k+1}$ *; cf. Lemma* 342*.*
For $i \in \mathbb{Z}_{\geqslant -1}$ *, we have*

$$(h\iota_{k,\ell})_{\{i\}} = h_{\{i\}} .$$

Proof. Ad (1). Suppose given h, $\tilde{h} \in H_k$.

We have

$$(h \cdot \tilde{h})\iota_{k,k+1} = (1, h \cdot \tilde{h}) = (1,h) \cdot (1,\tilde{h}) = h\iota_{k,k+1} \cdot \tilde{h}\iota_{k,k+1} .$$

So $\iota_{k,k+1}$ is a group morphism.

Suppose given $h \in H_k$ with $h\iota_{k,k+1} = 1$.

Then we have

$$(1,h) = h\iota_{k,k+1} = 1 = (1,1) .$$

This shows $h = 1$.

So $\iota_{k,k+1}$ is injective.

Ad (2). By (1), we have the injective group morphism $\iota_{j,\,j+1} \colon H_j \to H_{j+1}$ for $j \in [k, \ell - 1]$.

So

$$\iota_{k,\ell} = \prod_{j \in [k,\ell-1]}^{H_k \quad H_\ell} \iota_{j,\,j+1} \;= \iota_{k,k+1} \mathbin{\blacktriangle} \iota_{k+1,k+2} \mathbin{\blacktriangle} \ldots \mathbin{\blacktriangle} \iota_{\ell-1,\ell} \;: H_k \to H_\ell$$

is an injective group morphism.

Ad (3). We have

$$\begin{aligned}
(h\iota_{k,\ell})_{\{i\}} &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant \ell+1} \\ h_{\{i\}} & \text{for } i \in [-1, \ell] \end{cases} \\
&= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant \ell+1} \\ 1 & \text{for } i \in [k+1, \ell] \\ h_{\{i\}} & \text{for } i \in [-1, k] \end{cases} \\
&= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+1} \\ h_{\{i\}} & \text{for } i \in [-1, k] . \end{cases} \\
&= h_{\{i\}} .
\end{aligned}$$

□

Lemma 352 (The group morphism s_i^n)

(1) *Suppose given n, $i \in \mathbb{Z}_{\geqslant 0}$ with $i \leqslant n$.*
We have the group morphism

$$s_i^n \colon B \to H_n\,,\ b \mapsto bs_i^n\,,$$

where

$$(bs_i^n)_{\{k\}} = \begin{cases} 1 & \text{for } k \in [-1,n]\backslash\{i\} \\ b & \text{for } k = i\,. \end{cases}$$

(2) *Suppose given $n \in \mathbb{Z}_{\geqslant -1}$.*
We have the group morphism

$$s_{-1}^n \colon M \to H_n\,,\ m \mapsto ms_{-1}^n\,,$$

where

$$(ms_{-1}^n)_{\{k\}} = \begin{cases} 1 & \text{for } k \in [0,n] \\ m & \text{for } k = -1\,. \end{cases}$$

Proof. Ad (1). *We show that $s_j^j \colon B \to H_j = B \ltimes H_{j-1}$ is a group morphism for $j \in \mathbb{Z}_{\geqslant 0}$.*
Suppose given b, $\tilde{b} \in B$.
We have

$$(b \cdot \tilde{b})s_j^j = (b \cdot \tilde{b}, 1) = (b,1) \cdot (\tilde{b},1) = bs_j^j \cdot \tilde{b}s_j^j\,.$$

This shows that s_j^j is a group morphism.
For n, $i \in \mathbb{Z}_{\geqslant 0}$, with $i \leqslant n$, we have the group morphism

$$\begin{aligned} s_i^n := s_i^i \blacktriangleleft \iota_{i,n} \colon B \to H_n\,,\ b \mapsto bs_i^n &= \big(1,\dots,1,(b,1,\dots,1)\big) \\ &= \big(1,\dots,1,\,b,1,\dots,1\big)\,, \end{aligned}$$

where

$$(bs_i^n)_{\{k\}} = \begin{cases} b & \text{for } k = i \\ 1 & \text{for } k \neq i\,; \end{cases}$$

cf. Remark 351.(2).

This shows that s_i^n is a group morphism.

Ad (2). For m, $\tilde{m} \in M$, we have

$$(m \cdot \tilde{m})s_{-1}^0 = (1, m \cdot \tilde{m}) = (1, m) \cdot (1, \tilde{m}) = ms_{-1}^0 \cdot \tilde{m}s_{-1}^0 \,.$$

This shows that s_{-1}^0 is a group morphism.

Moreover, we have

$$s_{-1}^n = s_{-1}^0 \triangleleft \iota_{0,n} \colon M \to H_n \,,\ m \mapsto (1, \ldots, 1, m) \,;$$

cf. Remark 351.(2).

This shows that s_{-1}^n is a group morphism. □

Remark 353 *Let $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given $h \in H_n$.*

We write $b_i := h_{\{i\}} \in B$ for $i \in [0, n]$ and $m := h_{\{-1\}} \in M$. So $h = (b_n, \ldots, b_0, m)$. Moreover, we write $b_i := 1$ for $i \in \mathbb{Z}_{\geqslant n+1}$.

(1) *For $t \in [0, n]$, we consider*

$$(h_{\{t\}})s_t^n = (1, \ldots, 1, b_t, 1, \ldots, 1) \in H_n$$
$$(h_{\{-1\}})s_{-1}^n = (1, \ldots, 1, m) \in H_n \,.$$

We have

$$h = \prod_{t \in \lfloor n, -1 \rfloor} (h_{\{t\}})s_t^n \,.$$

In particular, we have

$$h = (h_{\lfloor n, r+1 \rfloor}, 1, \ldots, 1) \cdot (1, \ldots, 1, h_{\lfloor r, -1 \rfloor}) = (h_{\lfloor n, r+1 \rfloor}, 1) \cdot (h_{\lfloor r, -1 \rfloor})\iota_{r,n}$$

for $r \in [-1, n]$; cf. Remark 351.(2).

(2) *Suppose given $t \in [0, n]$. Suppose given $\tilde{b} \in B$. Suppose given $\tilde{m} \in M$.*

For $t \in [0, n]$, we consider

$$\tilde{b}s_t^n = (1, \ldots, 1, \tilde{b}, 1, \ldots, 1) \in H_n$$
$$\tilde{m}s_{-1}^n = (1, \ldots, 1, \tilde{m}) \in H_n \,.$$

If $t \in [1,n]$, then we have

$$(h \cdot \tilde{b} s_t^n)_{\{i\}} = \begin{cases} b_i & \text{for } i \in [t+1,n] \\ b_t \cdot \tilde{b} & \text{for } i = t \\ \left([b_{t-1}\,, \tilde{b}]^{(-1)^{t-1-i}}\right)^{\left(\prod\limits_{j \in \lfloor t-2, i+1 \rfloor} b_j\right)} \cdot b_i & \text{for } i \in [0, t-1] \\ \left(\llbracket b_{t-1}\,, \tilde{b} \rrbracket^{(-1)^t}\right)^{\left(\prod\limits_{j \in \lfloor t-2, 0 \rfloor} b_j\right)} \cdot m & \text{for } i = -1\,. \end{cases}$$

If $t = 0$, then we have

$$(h \cdot \tilde{b} s_0^n)_{\{i\}} = \begin{cases} b_i & \text{for } i \in [1,n] \\ b_0 \cdot \tilde{b} & \text{for } i = 0 \\ m^{\tilde{b}} & \text{for } i = -1\,. \end{cases}$$

If $t = -1$, then we have

$$(h \cdot \tilde{m} s_{-1}^n)_{\{i\}} = \begin{cases} b_i & \text{for } i \in [0,n] \\ m \cdot \tilde{m} & \text{for } i = -1\,. \end{cases}$$

We shall prove these statements directly, without using Lemma 349. We will make use of Lemma 347, however.

Proof. Ad (1). We show this claim by induction on n.

Let $n = -1$. Suppose given $h \in H_{-1} = M$.

Then we have

$$h = h_{\{-1\}} = (h_{\{-1\}}) s_{-1}^{-1} = \prod_{t \in \lfloor -1, -1 \rfloor} (h_{\{t\}}) s_t^{-1}\,.$$

Now suppose that $g = \prod\limits_{t \in \lfloor n, -1 \rfloor} (g_{\{t\}}) s_t^n$ holds for $g \in H_n$.

Suppose given $h \in H_{n+1}$.

We have to show that $h \stackrel{!}{=} \prod\limits_{t \in \lfloor n+1, -1 \rfloor} (h_{\{t\}}) s_t^{n+1}$.

We consider

$$h_{\lfloor n, -1 \rfloor} = (h_{\{n\}}\,, \ldots, h_{\{-1\}}) \in H_n\,;$$

cf. Notation 344.(3).

By induction hypothesis, we have

$$h_{\lfloor n,-1\rfloor} = \prod_{t\in\lfloor n,-1\rfloor} \big((h_{\lfloor n,-1\rfloor})_{\{t\}}\big) s_t^n = \prod_{t\in\lfloor n,-1\rfloor} (h_{\{t\}}) s_t^n \,.$$

Note that we have

$$(h_{\{n+1\}}) s_{n+1}^{n+1} = (h_{\{n+1\}}\,, 1) \in H_{n+1} = B \ltimes H_n \,.$$

For $t \in [-1, n]$, note that we have

$$(h_{\{t\}}) s_t^{n+1} = \big(1, (h_{\{t\}}) s_t^n\big) \in H_n \,.$$

We have

$$\begin{aligned}
\prod_{t\in\lfloor n+1,-1\rfloor} (h_{\{t\}}) s_t^{n+1} &= (h_{\{n+1\}}) s_{n+1}^{n+1} \cdot \big(\prod_{t\in\lfloor n,-1\rfloor} (h_{\{t\}}) s_t^{n+1}\big) \\
&= \big(h_{\{n+1\}}\,, 1\big) \cdot \Big(\prod_{t\in\lfloor n,-1\rfloor} \big(1, (h_{\{t\}}) s_t^n\big)\Big) \\
&\overset{351.(1)}{=} \big(h_{\{n+1\}}\,, 1\big) \cdot \Big(1, \prod_{t\in\lfloor n,-1\rfloor} (h_{\{t\}}) s_t^n\Big) \\
&= \big(h_{\{n+1\}}\,, 1\big) \cdot \big(1, h_{\lfloor n,-1\rfloor}\big) \\
&= \big(h_{\{n+1\}}\,, h_{\lfloor n,-1\rfloor}\big) \\
&= h \,.
\end{aligned}$$

In particular, for $r \in [-1, n]$, we have

$$\begin{aligned}
h &= \prod_{t\in\lfloor n,-1\rfloor} (h_{\{t\}}) s_t^n \\
&= \Big(\big(\prod_{t\in\lfloor n,r+1\rfloor} (h_{\{t\}}) s_t^n\big) \cdot \big(\prod_{t\in\lfloor r,-1\rfloor} (1) s_t^n\big)\Big) \cdot \big(\prod_{t\in\lfloor r,-1\rfloor} (h_{\{t\}}) s_t^n\big) \\
&= (h_{\lfloor n,r+1\rfloor}\,, 1, \ldots, 1) \cdot (1, \ldots, 1, h_{\lfloor r,-1\rfloor}) \\
&= (h_{\lfloor n,r+1\rfloor}\,, 1) \cdot (h_{\lfloor r,-1\rfloor}) \iota_{r,n} \,.
\end{aligned}$$

Ad (2). Suppose given $t \in [1, n]$.

Note that we have

$$(\tilde{b} s_t^n)_{\lfloor t,-1\rfloor} = (\tilde{b}\,, 1, \ldots, 1) = \tilde{b}\, s_t^t = (\tilde{b}\,, 1) \in H_t \,.$$

We have

$$\begin{aligned}
h \cdot \tilde{b}\, s_t^n &= \big(b_n\,,\ldots,b_{t+1}\,,(b_t\,,\ldots,b_0\,,m)\big)\cdot\big(1,\ldots,1,(\tilde{b}\,,1,\ldots,1)\big)\\
&\overset{(1)}{=} (h_{\lfloor n,t+1\rfloor}\,,1)\cdot(h_{\lfloor t,-1\rfloor})\iota_{t,n}\cdot(\tilde{b}\,s_t^t)\iota_{t,n}\\
&\overset{351.(2)}{=} (h_{\lfloor n,t+1\rfloor}\,,1)\cdot\big(h_{\lfloor t,-1\rfloor}\cdot(\tilde{b}\,s_t^t)\big)\iota_{t,n}\\
&\overset{(1)}{=} \big(h_{\lfloor n,t+1\rfloor}\,,h_{\lfloor t,-1\rfloor}\cdot(\tilde{b}\,s_t^t)\big)\\
&= \big(h_{\lfloor n,t+1\rfloor}\,,(b_t\,,h_{\lfloor t-1,-1\rfloor})\cdot(\tilde{b}\,,1)\big)\\
&= \big(h_{\lfloor n,t+1\rfloor}\,,(b_t\cdot\tilde{b}\,,h^{\tilde{b}}_{\lfloor t-1,-1\rfloor})\big)\\
&= \big(b_n\,,\ldots,b_{t+1}\,,b_t\cdot\tilde{b}\,,(b_{t-1}\,,\ldots,b_0\,,m)^{\tilde{b}}\big)\,.
\end{aligned}$$

So, by Lemma 347.(2), we have

$$(h\cdot\tilde{b}\,s_t^t)_{\{i\}} = \begin{cases} b_i & \text{if } i\in[t+1,n]\\ b_t\cdot\tilde{b} & \text{if } i=t\\ \big([b_{t-1}\,,\tilde{b}]^{(-1)^{t-1-i}}\big)^{\left(\prod_{j\in\lfloor t-2,i+1\rfloor} b_j\right)}\cdot b_i & \text{if } i\in[0,t-1]\\ \big(\llbracket b_{t-1}\,,\tilde{b}\rrbracket^{(-1)^t}\big)^{\left(\prod_{j\in\lfloor t-2,0\rfloor} b_j\right)}\cdot m & \text{if } i=-1\,. \end{cases}$$

Now let $t=0$. We have

$$\begin{aligned}
h\cdot\tilde{b}s_0^n &= (b_n\,,\ldots,b_0\,,m)\cdot(1,\ldots,1,\tilde{b}\,,1)\\
&\overset{(1)}{=} (h_{\lfloor n,1\rfloor}\,,1)\cdot(b_0\,,m)\iota_{0,n}\cdot(\tilde{b}\,,1)\iota_{0,n}\\
&\overset{351.(2)}{=} (h_{\lfloor n,1\rfloor}\,,1)\cdot\big((b_0\,,m)\cdot(\tilde{b}\,,1)\big)\iota_{0,n}\\
&= (h_{\lfloor n,1\rfloor}\,,1)\cdot(b_0\cdot\tilde{b}\,,m^{\tilde{b}})\iota_{0,n}\\
&\overset{(1)}{=} \big(h_{\lfloor n,1\rfloor}\,,(b_0\cdot\tilde{b}\,,m^{\tilde{b}})\big)\\
&= (b_n\,,\ldots,b_1\,,b_0\cdot\tilde{b}\,,m^{\tilde{b}})\,.
\end{aligned}$$

So we have

$$(h\cdot\tilde{b}s_0^n)_{\{i\}} = \begin{cases} b_i & \text{if } i\in[1,n]\\ b_0\cdot\tilde{b} & \text{if } i=0\\ m^{\tilde{b}} & \text{if } i=-1\,. \end{cases}$$

Now let $t = -1$. We have

$$\begin{aligned} h \cdot \tilde{m} s_{-1}^n &= (b_n \,, \ldots, b_0 \,, m) \cdot (1, \ldots, 1, \tilde{m}) \\ &\overset{(1)}{=} (h_{\lfloor n,0 \rfloor} \,, 1) \cdot m s_{-1}^n \cdot \tilde{m} s_{-1}^n \\ &\overset{351.(2)}{=} (h_{\lfloor n,0 \rfloor} \,, 1) \cdot (m \cdot \tilde{m}) s_{-1}^n \\ &\overset{(1)}{=} (h_{\lfloor n,0 \rfloor} \,, m \cdot \tilde{m}) \\ &= (b_n \,, \ldots, b_0 \,, m \cdot \tilde{m}) \,. \end{aligned}$$

So we have

$$(h \cdot \tilde{m} s_{-1}^n)_{\{i\}} = \begin{cases} b_i & \text{if } i \in [0, n] \\ m \cdot \tilde{m} & \text{if } i = -1 \,. \end{cases}$$

□

17.2 The group H_∞

Recall that we have the stable crossed module $S = (\mathrm{M}_S \,, \mathrm{B}_S \,, \kappa_S \,, \mathrm{f}_S)$ and that we often write $M = \mathrm{M}_S$ and $B = \mathrm{B}_S$.

17.2.1 Construction of the group H_∞

We consider the data constructed in Lemma 342.

Definition 354

We consider the linearly ordered set $\mathbb{Z}_{\geqslant -1}$.

For $n \in \mathbb{Z}_{\geqslant -1}$, we consider the group H_n.

For i, $j \in \mathbb{Z}_{\geqslant -1}$ with $i \leqslant j$, we consider the injective group morphism

$$\iota_{i,j} \colon H_i \to H_j \,, \;\; h \mapsto h\iota_{i,j} = (1, \ldots, 1, h) \,;$$

cf. Remark 351.(2).

We consider the colimit $\varinjlim_n H_n$ of $\big((H_n)_{n \geqslant -1} \,, (\iota_{i,j})_{-1 \leqslant i \leqslant j}\big)$; cf. Definition 3.(2).

For $[k, h] \in \varinjlim_n H_n$, recall that we have $k \in \mathbb{Z}_{\geqslant -1}$ and $h \in H_k$.

(1) As a set, let

$$H_\infty^S := \{(\dots, b_3, b_2, b_1, b_0, m) \in (\prod_{i\geqslant 0} B) \times M \colon \exists k \in \mathbb{Z}_{\geqslant 0} \text{ with } b_\ell = 1 \text{ for } \ell \geqslant k\}\,.$$

We often write $H_\infty := H_\infty^S$.

(2) Suppose given $h \in H_\infty$. Then we have $h = (\dots, 1, b_k, b_{k-1}, \dots, b_0, m)$ for some $k \in \mathbb{Z}_{\geqslant 0}$, where $b_i \in B$ for $i \in [0, k]$ and where $m \in M$.

We write

$$h_{\{i\}} := \begin{cases} 1 & \text{for } i \geqslant k+1 \\ b_i & \text{for } i \in [0,k] \\ m & \text{for } i = -1\,. \end{cases}$$

Cf. also Lemma 342.

For h, $\tilde{h} \in H_\infty$, note that we have

$$h = \tilde{h} \;\Leftrightarrow\; h_{\{i\}} = \tilde{h}_{\{i\}} \text{ for } i \in \mathbb{Z}_{\geqslant -1}\,.$$

(3) Suppose given $h \in H_\infty$. Suppose given ℓ, $r \in \mathbb{Z}_{\geqslant -1}$ with $r \leqslant \ell$.

We write $h_{\lfloor \ell, r \rfloor} \in H_\ell$ for the element with

$$(h_{\lfloor \ell, r \rfloor})_{\{i\}} := \begin{cases} h_{\{i\}} & \text{for } i \in [r, \ell] \\ 1 & \text{for } i \in [-1, r-1]\,. \end{cases}$$

So $h_{\lfloor \ell, r \rfloor} = (h_{\{\ell\}}, h_{\{\ell-1\}}, \dots, h_{\{r\}}, 1, \dots, 1) \in H_\ell$.

(4) Suppose given $h \in H_\infty$. Suppose given $r \in \mathbb{Z}_{\geqslant -1}$.

We write $h_{\lfloor \infty, r \rfloor} \in H_\infty$ for the element in H_∞ with

$$(h_{\lfloor \infty, r \rfloor})_{\{i\}} := \begin{cases} h_{\{i\}} & \text{for } i \in \mathbb{Z}_{\geqslant r} \\ 1 & \text{for } i \in [-1, r-1]\,; \end{cases}$$

cf. §1.1.2 item 9.4.

So $h_{\lfloor \infty, r \rfloor} = (\dots, h_{\{r+1\}}, h_{\{r\}}, 1, \dots, 1) \in H_\infty$.

Remark 355 *We have the bijection*

$$\rho\colon \varinjlim_n H_n \to H_\infty\,,\ [k,h] \mapsto [k,h]\rho\,,$$

where

$$\big([k,h]\rho\big)_{\{i\}} := \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+1} \\ h_{\{i\}} & \text{for } i \in [-1,k]\,. \end{cases}$$

Its inverse is given by

$$\rho^-\colon H_\infty \to \varinjlim_n H_n\,,\ h \mapsto h\rho^- = [k, h_{\lfloor k,-1\rfloor}]\,,$$

where $k \in \mathbb{Z}_{\geqslant -1}$ is chosen such that $h_{\{i\}} = 1$ for $i \geqslant k+1$.

Proof. We show that ρ is well-defined.

Suppose given $[k,h]$, $[\tilde{k},\tilde{h}] \in \varinjlim_n H_n$ with $[k,h] = [\tilde{k},\tilde{h}]$.

We may assume that $k \leqslant \tilde{k}$.

Then, by Remark 5.(1), we have $h\iota_{k,\tilde{k}} = \tilde{h} \in H_{\tilde{k}}$, since $\iota_{n,m}$ is injective for n, $m \in \mathbb{Z}_{\geqslant -1}$ with $n \leqslant m$; cf. Remark 351.(2).

We have

$$(h\iota_{k,\tilde{k}})_{\{i\}} = \begin{cases} 1 & \text{for } i \in [k+1,\tilde{k}] \\ h_{\{i\}} & \text{for } i \in [-1,k]\,. \end{cases}$$

Therefore, we have

$$\tilde{h}_{\{i\}} = \begin{cases} 1 & \text{for } i \in [k+1,\tilde{k}] \\ h_{\{i\}} & \text{for } i \in [-1,k]\,. \end{cases}$$

We have

$$\left\{ \begin{array}{ll} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+1} \\ h_{\{i\}} & \text{for } i \in [-1,k] \end{array} \right\} = \left\{ \begin{array}{ll} 1 & \text{for } i \in \mathbb{Z}_{\geqslant \tilde{k}+1} \\ 1 & \text{for } i \in [k+1,\tilde{k}] \\ h_{\{i\}} & \text{for } i \in [-1,k] \end{array} \right\} = \left\{ \begin{array}{ll} 1 & \text{for } i \in \mathbb{Z}_{\geqslant \tilde{k}+1} \\ \tilde{h}_{\{i\}} & \text{for } i \in [-1,\tilde{k}] \end{array} \right\}.$$

This shows $[k,h]\rho = [\tilde{k},\tilde{h}]\rho$. So the map ρ is well-defined.

We claim to have the map

$$\tilde{\rho}\colon H_\infty \to \varinjlim_n H_n\,,\ h \mapsto [k, h_{\lfloor k,-1\rfloor}]\,,$$

where $k \in \mathbb{Z}_{\geqslant -1}$ is chosen such that $h_{\{i\}} = 1$ for $i \geqslant k+1$.

We show that $\tilde{\rho}$ is well-defined.

Suppose given $h \in \varinjlim_n H_n$.

Suppose given $k \in \mathbb{Z}_{\geqslant -1}$ such that $h_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant k+1}$.

Suppose given $\ell \in \mathbb{Z}_{\geqslant -1}$ such that $h_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant \ell+1}$.

We may assume that $k \leqslant \ell$.

Then we have $h_{\{j\}} = 1$ for $j \in [k+1, \ell]$.

So

$$(h_{\lfloor \ell,-1\rfloor})_{\{i\}} = \begin{cases} 1 & \text{for } i \in [k+1, \ell] \\ h_{\{i\}} & \text{for } i \in [-1, k]\,. \end{cases}$$

We consider the group element $(h_{\lfloor k,-1\rfloor})\iota_{k,\ell} \in H_\ell$.

We have

$$\big((h_{\lfloor k,-1\rfloor})\iota_{k,\ell}\big)_{\{i\}} = \begin{cases} 1 & \text{for } i \in [k+1, \ell] \\ h_{\{i\}} & \text{for } i \in [-1, k]\,. \end{cases}$$

This shows $(h_{\lfloor k,-1\rfloor})\iota_{k,\ell} = h_{\lfloor \ell,-1\rfloor}$.

Therefore, by Remark 5.(1), we have $[k, h_{\lfloor k,-1\rfloor}] = [\ell, h_{\lfloor \ell,-1\rfloor}]$.

So $\tilde{\rho}$ is well-defined.

We show that $\rho \blacktriangle \tilde{\rho} \stackrel{!}{=} \mathrm{id}_{\varinjlim H_n}$.

Suppose given $[k, h] \in \varinjlim_n H_n$.

We consider $[k, h]\rho \in H_\infty$. We have

$$\big([k, h]\rho\big)_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+1} \\ h_{\{i\}} & \text{for } i \in [-1, k]\,. \end{cases}$$

We have

$$[k,h](\rho \blacktriangle \tilde{\rho}) = \big([k,h]\rho\big)\tilde{\rho} = \Big[\ell, \big([k,h]\rho\big)_{\lfloor \ell,-1\rfloor}\Big]$$

for some $\ell \in \mathbb{Z}_{\geqslant -1}$ with $\Big(\big([k,h]\rho\big)_{\lfloor \ell,-1\rfloor}\Big)_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant \ell+1}$.

We may choose $\ell := k$ since $\Big(\big([k,h]\rho\big)_{\lfloor \ell,-1\rfloor}\Big)_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant k+1}$.

Moreover, we have $\big([k,h]\rho\big)_{\lfloor k,-1\rfloor} = h$.

This shows $[k,h](\rho \blacktriangle \tilde{\rho}) = [k,h]$.

So $\rho \blacktriangle \tilde{\rho} = \mathrm{id}_{\varinjlim_n H_n}$.

We show that $\tilde{\rho} \blacktriangle \rho \overset{!}{=} \mathrm{id}_{H_\infty}$.

Suppose given $h \in H_\infty$.

Then we have $h\tilde{\rho} = [k, h_{\lfloor k,-1\rfloor}] \in \varinjlim_n H_n$ for some $k \in \mathbb{Z}_{\geqslant -1}$ with $h_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant k+1}$.

We consider the element $[k, h_{\lfloor k,-1\rfloor}]\rho \in H_\infty$.

We have

$$\begin{aligned}\big([k, h_{\lfloor k,-1\rfloor}]\rho\big)_{\{i\}} &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+1} \\ (h_{\lfloor k,-1\rfloor})_{\{i\}} & \text{for } i \in [-1,k] \end{cases} \\ &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+1} \\ h_{\{i\}} & \text{for } i \in [-1,k] \end{cases} \\ &= h_{\{i\}}\,.\end{aligned}$$

So

$$h(\tilde{\rho} \blacktriangle \rho) = [k, h_{\lfloor k,-1\rfloor}]\rho = h\,.$$

This shows $\tilde{\rho} \blacktriangle \rho = \mathrm{id}_{H_\infty}$.

Altogether, ρ is a bijection and its inverse is given by $\rho^- = \tilde{\rho}$. □

Definition 356 (The group H_∞)

We consider the set

$$H_\infty^S = H_\infty = \{h \in (\prod_{i \geqslant 0} B) \times M \colon \exists k \in \mathbb{Z}_{\geqslant 0} \text{ with } h_{\{\ell\}} = 1 \text{ for } \ell \geqslant k\}$$
$$= \{(\dots, b_3, b_2, b_1, b_0, m) \in (\prod_{i \geqslant 0} B) \times M \colon \exists k \in \mathbb{Z}_{\geqslant 0} \text{ with } b_\ell = 1 \text{ for } \ell \geqslant k\}$$

from Definition 354.

Moreover, we consider the bijection $\rho \colon \varinjlim_n H_n \to H_\infty$ from Remark 355.

Then, by transport of structure along ρ, the set H_∞ is a group.

I.e., for $h, \tilde{h} \in H_\infty$, we let

$$h \cdot \tilde{h} := \left(h\rho^- \cdot \tilde{h}\rho^-\right)\rho\,.$$

Then $\rho \colon \varinjlim_n H_n \to H_\infty$ is a group isomorphism.

For an explicit formula for the multiplication in H_∞ cf. Proposition 361 below, or, for the partial formula, cf. Lemma 362.(2) below.

Moreover, recall that we have the action

$$\gamma_S \colon B \to \mathrm{Aut}(M)\,,\ b \mapsto \left(m \mapsto m^b = m \cdot [\![m\mathrm{f}_S, b]\!]\right);$$

cf. Lemma 327.(1).

Remark 357 (The group morphism $\iota_{n,\infty}$)

Suppose given $n \in \mathbb{Z}_{\geqslant -1}$.

(1) *We have the injective group morphism*

$$\iota_{n,\infty} \colon H_n \to H_\infty\,,\ h \mapsto h\iota_{n,\infty}\,,$$

where $(h\iota_{n,\infty})_{\{i\}} = h_{\{i\}}$ *for* $i \in \mathbb{Z}_{\geqslant -1}$.

So

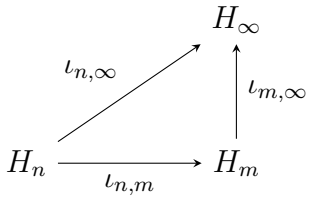
$$h\iota_{n,\infty} = [n, h]\rho = (\dots, 1, h_{\{n\}}, h_{\{n-1\}}, \dots, h_{\{-1\}}) \in H_\infty\,.$$

Cf. also Remark 351.

(2) *Suppose given* $m \in \mathbb{Z}_{\geqslant n}$.

We have

$$\iota_{n,\infty} = \iota_{n,m} \blacktriangle \iota_{m,\infty} \colon H_n \to H_\infty \,.$$



(3) *Suppose given* $h \in H_\infty$. *Suppose given* $k \in \mathbb{Z}_{\geqslant -1}$ *such that* $h_{\{j\}} = 1$ *for* $j \in \mathbb{Z}_{\geqslant k+1}$.

We have

$$h = (h_{\lfloor k,-1 \rfloor})\iota_{k,\infty} \in H_\infty \,.$$

(4) *Suppose given* $k \in \mathbb{Z}_{\geqslant -1}$. *Suppose given* h, $\tilde{h} \in H_\infty$ *such that* $h_{\{j\}} = 1$ *and* $\tilde{h}_{\{j\}} = 1$ *for* $j \in \mathbb{Z}_{\geqslant k+1}$.

Then $(h \cdot \tilde{h})_{\{j\}} = 1$ *and* $(h^-)_{\{j\}} = 1$ *for* $j \in \mathbb{Z}_{\geqslant k+1}$.

Proof. Ad (1). We consider the injective group morphism

$$j_{n,\infty} \colon H_n \to \varinjlim_n H_n \,,\ h \mapsto [n,h] \,;$$

cf. Remark 4.(1) and Remark 5.(2).

Moreover, we consider the group isomorphism $\rho \colon \varinjlim_n H_n \to H_\infty$; cf. Remark 355 and Definition 356.

Let $\iota_{n,\infty} := j_{n,\infty} \blacktriangle \rho \colon H_n \to H_\infty$, which is an injective group morphism by composition.

For $h \in H_n$, we have

$$(h\iota_{n,\infty})_{\{i\}} = \big(h(j_{n,\infty} \blacktriangle \rho)\big)_{\{i\}} = \big([n,h]\rho\big)_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant n+1} \\ h_{\{i\}} & \text{for } i \in [-1,n] \,. \end{cases}$$

This shows the claim.

Ad (2). For $h \in H_n$, we have

$$\begin{aligned}\big(h(\iota_{n,m} \blacktriangle \iota_{m,\infty})\big)_{\{i\}} &= \big((h\iota_{n,m})\iota_{m,\infty}\big)_{\{i\}} \\ &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant m+1} \\ (h\iota_{n,m})_{\{i\}} & \text{for } i \in [-1,m] \end{cases} \\ &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant m+1} \\ 1 & \text{for } i \in [n+1,m] \\ h_{\{i\}} & \text{for } i \in [-1,n] \end{cases} \\ &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant n+1} \\ h_{\{i\}} & \text{for } i \in [-1,n] \end{cases} \\ &= (h\iota_{n,\infty})_{\{i\}}\,. \end{aligned}$$

This shows $h(\iota_{n,m} \blacktriangle \iota_{m,\infty}) = h\iota_{n,\infty}$.

So we have $\iota_{n,m} \blacktriangle \iota_{m,\infty} = \iota_{n,\infty}$.

Ad (3). For $i \in \mathbb{Z}_{\geqslant -1}$, we have

$$\big((h_{\lfloor k,-1\rfloor})\iota_{k,\infty}\big)_{\{i\}} = \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant k+1} \\ (h_{\lfloor k,-1\rfloor})_{\{i\}} & \text{if } i \in [-1,k] \end{array} \right\} = \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant k+1} \\ h_{\{i\}} & \text{if } i \in [-1,k] \end{array} \right\} = h_{\{i\}}\,.$$

This shows $h = (h_{\lfloor k,-1\rfloor})\iota_{k,\infty}$.

Ad (4). We have

$$h \cdot \tilde{h} \overset{(3)}{=} (h_{\lfloor k,-1\rfloor})\iota_{k,\infty} \cdot (\tilde{h}_{\lfloor k,-1\rfloor})\iota_{k,\infty} = (h_{\lfloor k,-1\rfloor} \cdot \tilde{h}_{\lfloor k,-1\rfloor})\iota_{k,\infty}$$

and

$$h^- = \big((h_{\lfloor k,-1\rfloor})\iota_{k,\infty}\big)^- \overset{(3)}{=} \big((h_{\lfloor k,-1\rfloor})^-\big)\iota_{k,\infty}\,.$$

Using (1), this shows (4). □

The universal property of $\varinjlim_n H_n$ carries over to H_∞, which we shall make explicit.

Lemma 358 (The universal property of H_∞)

Suppose given a group T. For $z \in \mathbb{Z}_{\geqslant -1}$, suppose given a group morphism $\omega_z \colon H_z \to T$. For n, $m \in \mathbb{Z}_{\geqslant -1}$ with $n \leqslant m$, suppose that we have

$$\omega_n = \iota_{n,m} \blacktriangle \omega_m \colon H_n \to T\,.$$

Then there exists a unique group morphism $\omega_\infty \colon H_\infty \to T$ *such that*

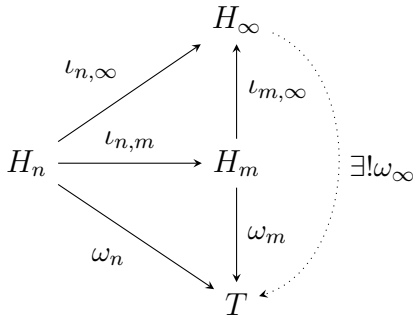
$$\omega_z = \iota_{z,\infty} \vartriangleleft \omega_\infty$$

for $z \in \mathbb{Z}_{\geqslant -1}$.

Then ω_∞ *is given as follows. We have*

$$\omega_\infty \colon H_\infty \to T\,,\ h \mapsto (h_{\lfloor k,-1 \rfloor})\omega_k\,,$$

where $k \in \mathbb{Z}_{\geqslant -1}$ *is chosen such that* $h_{\{i\}} = 1$ *for* $i \in \mathbb{Z}_{\geqslant k+1}$.



Proof. We consider the colimit $\varinjlim_z H_z$; cf. Definition 354.

We consider the group isomorphism $\rho \colon \varinjlim_z H_z \to H_\infty$; cf. Remark 355 and Definition 356.

For $k \in \mathbb{Z}_{\geqslant -1}$, we consider the group morphism

$$j_{k,\infty} \colon H_k \to \varinjlim_z H_z\,,\ h \mapsto [k,h]\,;$$

cf. Remark 4.(1).

We have

$$j_{k,\infty} \vartriangleleft \rho = \iota_{k,\infty}\,,$$

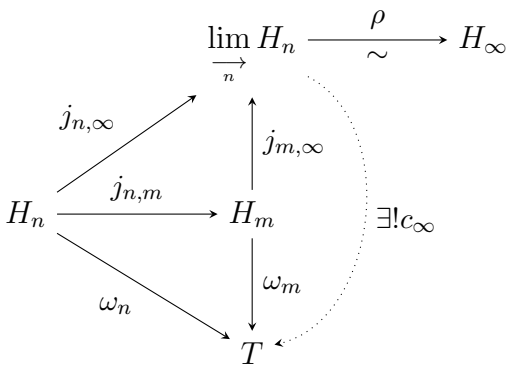
as both map $h \in H_k$ to $[k,h]\rho$; cf. Remark 357.(1).

Then, by the universal property of the colimit $\varinjlim_z H_z$ from Remark 4.(3) , there exists a unique morphism $c_\infty \colon \varinjlim_z H_z \to T$ with

$$j_{n,\infty} \vartriangleleft c_\infty = \omega_n \colon H_n \to T\,,$$

where

$$c_\infty \colon \varinjlim_z H_z \to T\,,\ [k,h] \mapsto h\omega_k\,.$$



We show existence and the formula.

We let $\omega_\infty := \rho^- \vartriangleleft c_\infty \colon H_\infty \to T$. We have

$$\omega_n = j_{n,\infty} \vartriangleleft c_\infty = (j_{n,\infty} \vartriangleleft \rho) \vartriangleleft (\rho^- \vartriangleleft c_\infty) = \iota_{n,\infty} \vartriangleleft \omega_\infty\,.$$

Moreover, for $h \in H_\infty$, we have

$$h\omega_\infty = h(\rho^- \vartriangleleft c_\infty) \overset{355}{=} [k, h_{\lfloor k,-1\rfloor}]c_\infty = (h_{\lfloor k,-1\rfloor})\omega_k\,,$$

where $k \in \mathbb{Z}_{\geqslant -1}$ is chosen such that $h_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant k+1}$.

We show uniqueness.

Suppose given a group morphism $\tilde{\omega}_\infty \colon H_\infty \to T$ such that $\iota_{n,\infty} \vartriangleleft \tilde{\omega}_\infty = \omega_n$ for $n \in \mathbb{Z}_{\geqslant -1}$.

Then, for $n \in \mathbb{Z}_{\geqslant 1}$, we have

$$j_{n,\infty} \vartriangleleft (\rho \vartriangleleft \tilde{\omega}_\infty) = \iota_{n,\infty} \vartriangleleft \tilde{\omega}_\infty = \omega_n\,.$$

So $\rho \vartriangleleft \tilde{\omega}_\infty = c_\infty$; cf. Remark 4.(3).

This shows $\tilde{\omega}_\infty = \rho^- \vartriangleleft c_\infty = \omega_\infty$. □

Lemma 359 (The group morphism s_i^∞)

(1) *For $i \in \mathbb{Z}_{\geqslant 0}$, we have the group morphism*

$$s_i^\infty \colon B \to H_\infty\,,\ b \mapsto bs_i^\infty\,,$$

where

$$(bs_i^\infty)_{\{k\}} := \begin{cases} 1 & \text{for } k \in \mathbb{Z}_{\geqslant -1}\backslash\{i\} \\ b & \text{for } k = i\,. \end{cases}$$

(2) *We have the group morphism*

$$s_{-1}^\infty \colon M \to H_\infty\,,\ m \mapsto ms_{-1}^\infty\,,$$

where

$$(ms_{-1}^\infty)_{\{k\}} := \begin{cases} 1 & \text{for } k \in \mathbb{Z}_{\geqslant 0} \\ m & \text{for } k = -1\,. \end{cases}$$

Proof. Ad (1). Suppose given $i \in \mathbb{Z}_{\geqslant 0}$.

We consider the group morphism

$$s_i^i \colon B \to H_i\,,\ b \mapsto bs_i^i\,,$$

where

$$(bs_i^i)_{\{k\}} = \begin{cases} 1 & \text{for } k \in [-1, i-1] \\ b & \text{for } k = i\,; \end{cases}$$

cf. Lemma 352.(1).

We let $s_i^\infty := s_i^i \mathbin{\blacktriangle} \iota_{i,\infty} \colon B \to H_\infty$.

Then, for $b \in B$ and $k \in \mathbb{Z}_{\geqslant -1}$, we have

$$(bs_i^\infty)_{\{k\}} = \big(b(s_i^i \mathbin{\blacktriangle} \iota_{i,\infty})\big)_{\{k\}} = \big((bs_i^i)\iota_{i,\infty}\big)_{\{k\}} \overset{357.(1)}{=} \begin{cases} 1 & \text{if } k \in \mathbb{Z}_{\geqslant i+1} \\ (bs_i^i)_{\{k\}} & \text{if } k \in [-1, i] \end{cases}$$

$$= \begin{cases} 1 & \text{if } k \in \mathbb{Z}_{\geqslant i+1} \\ 1 & \text{if } k \in [-1, i-1] \\ b & \text{if } k = i \end{cases}$$

$$= \begin{cases} 1 & \text{if } k \in \mathbb{Z}_{\geqslant -1}\backslash\{i\} \\ b & \text{if } k = i\,. \end{cases}$$

Ad (2). We consider the group morphism

$$s_{-1}^{-1} \colon M \to H_{-1}\,,\ m \mapsto ms_{-1}^{-1} := m\,;$$

cf. Lemma 352.(2). So $s_{-1}^{-1} = \mathrm{id}_M$. We let

$$s_{-1}^{\infty} := s_{-1}^{-1} \blacktriangle \iota_{-1,\infty} \colon M \to H_\infty .$$

Then, for $m \in M$ and $k \in \mathbb{Z}_{\geqslant -1}$, we have

$$(ms_{-1}^{\infty})_{\{k\}} = \big(m(s_{-1}^{-1} \blacktriangle \iota_{i,\infty})\big)_{\{k\}} = (m\iota_{-1,\infty})_{\{k\}} \overset{357.(1)}{=} \begin{cases} 1 & \text{if } k \in \mathbb{Z}_{\geqslant 0} \\ m & \text{if } k = -1 . \end{cases}$$

□

Remark 360 *Suppose given $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given $k \in [-1, n]$.*
We have

$$s_k^{\infty} = s_k^n \blacktriangle \iota_{n,\infty} .$$

Proof. Case $k \geqslant 0$. Suppose given $b \in B$.

For $i \in \mathbb{Z}_{\geqslant -1}$, we have

$$\begin{aligned} \big(b(s_k^n \blacktriangle \iota_{n,\infty})\big)_{\{i\}} &= \big((bs_k^n)\iota_{n,\infty}\big)_{\{i\}} \\ &= \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \\ (bs_k^n)_{\{i\}} & \text{if } i \in [-1, n] \end{array} \right\} = \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \\ 1 & \text{if } i \in [-1, n] \backslash \{k\} \\ b & \text{if } i = k \end{array} \right\} \\ &= \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \backslash \{k\} \\ b & \text{if } i = k \end{array} \right\} = (bs_k^{\infty})_{\{i\}} . \end{aligned}$$

This shows $b(s_k^n \blacktriangle \iota_{n,\infty}) = bs_k^{\infty}$.

Case $k = -1$. Suppose given $m \in M$.

For $i \in \mathbb{Z}_{\geqslant -1}$, we have

$$\begin{aligned} \big(m(s_{-1}^n \blacktriangle \iota_{n,\infty})\big)_{\{i\}} &= \big((ms_{-1}^n)\iota_{n,\infty}\big)_{\{i\}} \\ &= \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \\ (ms_{-1}^n)_{\{i\}} & \text{if } i \in [-1, n] \end{array} \right\} = \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \\ 1 & \text{if } i \in [0, n] \\ m & \text{if } i = -1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \backslash \{-1\} \\ m & \text{if } i = -1 \end{array} \right\} = (ms_{-1}^{\infty})_{\{i\}} . \end{aligned}$$

This shows $m(s_{-1}^n \blacktriangle \iota_{n,\infty}) = ms_{-1}^{\infty}$.

So we have $s_k^n \blacktriangle \iota_{n,\infty} = s_k^{\infty}$ for $k \in [-1, n]$. □

17.2.2 The multiplication in H_∞

Proposition 361 *The multiplication in H_∞ is given as follows.*

Suppose given h, $\tilde{h} \in H_\infty$.

We write

$$h_{\{k\}} =: \begin{cases} b_k & \text{for } k \in \mathbb{Z}_{\geqslant 0} \\ m & \text{for } k = -1\,, \end{cases}$$

where $b_k \in B$, for $k \in \mathbb{Z}_{\geqslant 0}$ and $m \in M$, and we write

$$\tilde{h}_{\{k\}} =: \begin{cases} \tilde{b}_k & \text{for } k \in \mathbb{Z}_{\geqslant 0} \\ \tilde{m} & \text{for } k = -1\,, \end{cases}$$

where $\tilde{b}_k \in B$, for $k \in \mathbb{Z}_{\geqslant 0}$ and $\tilde{m} \in M$.

We have

$$(h \cdot \tilde{h})_{\{k\}} =$$

$$\begin{cases} \left(\prod\limits_{\substack{i \in \lceil k, \infty \rceil \\ i \equiv_2 k}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, k+1 \rfloor} b_j \cdot \tilde{b}_j \right)} \right) \cdot \left(\prod\limits_{\substack{i \in \lceil k, \infty \rceil \\ i \equiv_2 k+1}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, k+1 \rfloor} b_j \cdot \tilde{b}_j \right)} \right)^{-} \cdot b_k^{\tilde{b}_{k+1}} \cdot \tilde{b}_k & \textit{for } k \in \mathbb{Z}_{\geqslant 0} \\ \left(\prod\limits_{\substack{i \in \lceil -1, \infty \rceil \\ i \equiv_2 -1}} [\![\tilde{b}_{i+2},\, b_{i+1}]\!]^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, 0 \rfloor} b_j \cdot \tilde{b}_j \right)} \right) \cdot \left(\prod\limits_{\substack{i \in \lceil -1, \infty \rceil \\ i \equiv_2 0}} [\![\tilde{b}_{i+2},\, b_{i+1}]\!]^{\tilde{b}_{i+1} \cdot \left(\prod\limits_{j \in \lfloor i, 0 \rfloor} b_j \cdot \tilde{b}_j \right)} \right)^{-} \cdot m^{\tilde{b}_0} \cdot \tilde{m} & \textit{for } k = -1\,; \end{cases}$$

cf. also Lemma 349.

Proof. We choose $\ell \in \mathbb{Z}_{\geqslant -1}$ such that $b_i = 1$ for $i \in \mathbb{Z}_{\geqslant \ell+1}$.

We choose $\tilde{\ell} \in \mathbb{Z}_{\geqslant -1}$ such that $\tilde{b}_i = 1$ for $i \in \mathbb{Z}_{\geqslant \tilde{\ell}+1}$.

Let $n := \max\{\ell, \tilde{\ell}\}$.

We have

$$h_{\{k\}} = \begin{cases} 1 & \text{for } k \in \mathbb{Z}_{\geqslant n+1} \\ b_k & \text{for } k \in [0, n] \\ m & \text{for } k = -1 \end{cases}$$

and we have

$$\tilde{h}_{\{k\}} = \begin{cases} 1 & \text{for } k \in \mathbb{Z}_{\geqslant n+1} \\ \tilde{b}_k & \text{for } k \in [0,n] \\ \tilde{m} & \text{for } k = -1\,. \end{cases}$$

We consider the group elements $h_{\lfloor n,-1\rfloor}\,,\ \tilde{h}_{\lfloor n,-1\rfloor} \in H_n$; cf. Definition 354.(3).
By Remark 357.(3), we have $h = (h_{\lfloor n,-1\rfloor})\iota_{n,\infty}$ and $\tilde{h} = (\tilde{h}_{\lfloor n,-1\rfloor})\iota_{n,\infty}\,.$
For $i \in \mathbb{Z}_{\geqslant -1}$, note that we have

$$(h_{\lfloor n,-1\rfloor})_{\{i\}} = h_{\{i\}} = \begin{cases} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \\ b_i & \text{if } i \in [0,n] \\ m & \text{if } i = -1\,, \end{cases}$$

$$(\tilde{h}_{\lfloor n,-1\rfloor})_{\{i\}} = \tilde{h}_{\{i\}} = \begin{cases} 1 & \text{if } i \in \mathbb{Z}_{\geqslant n+1} \\ \tilde{b}_i & \text{if } i \in [0,n] \\ \tilde{m} & \text{if } i = -1\,; \end{cases}$$

cf. Notation 344.(2).
For $k \in \mathbb{Z}_{\geqslant -1}$, we have

$$(h\cdot\tilde{h})_{\{k\}} \;=\; \big((h_{\lfloor n,-1\rfloor})\iota_{n,\infty}\cdot(\tilde{h}_{\lfloor n,-1\rfloor})\iota_{n,\infty}\big)_{\{k\}} \overset{357.(1)}{=} \big((h_{\lfloor n,-1\rfloor}\cdot\tilde{h}_{\lfloor n,-1\rfloor})\iota_{n,\infty}\big)_{\{k\}}\,.$$

So, if $k \in [-1,n]$, then, by Lemma 349, we have

$$(h\cdot\tilde{h})_{\{k\}} = \big((h_{\lfloor n,-1\rfloor}\cdot\tilde{h}_{\lfloor n,-1\rfloor})\iota_{n,\infty}\big)_{\{k\}} \overset{357.(1)}{=} (h_{\lfloor n,-1\rfloor}\cdot\tilde{h}_{\lfloor n,-1\rfloor})_{\{k\}}$$

$$\begin{cases} \left(\prod\limits_{\substack{i\in\lceil k,\infty\rceil\\ i\equiv_2 k}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)}\right)\cdot\left(\prod\limits_{\substack{i\in\lceil k,\infty\rceil\\ i\equiv_2 k+1}} [\tilde{b}_{i+2}\,,b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)}\right)^{-}\cdot b_k^{\tilde{b}_{k+1}}\cdot\tilde{b}_k & \text{if } k\in[0,n] \\ \left(\prod\limits_{\substack{i\in\lceil -1,\infty\rceil\\ i\equiv_2 -1}} [\![\tilde{b}_{i+2},\, b_{i+1}]\!]^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\right)\cdot\left(\prod\limits_{\substack{i\in\lceil -1,\infty\rceil\\ i\equiv_2 0}} [\![\tilde{b}_{i+2},\, b_{i+1}]\!]^{\tilde{b}_{i+1}\cdot\left(\prod\limits_{j\in\lfloor i,0\rfloor} b_j\cdot\tilde{b}_j\right)}\right)^{-}\cdot m^{\tilde{b}_0}\cdot\tilde{m} & \text{if } k=-1\,. \end{cases}$$

If $k \in \mathbb{Z}_{\geqslant n+1}$, then

$$\begin{aligned} & (h \cdot \tilde{h})_{\{k\}} \\ = \ & \big((h_{\lfloor n,-1\rfloor} \cdot \tilde{h}_{\lfloor n,-1\rfloor})\iota_{n,\infty}\big)_{\{k\}} \overset{357.(1)}{=} 1 \\ = \ & \Bigg(\prod_{\substack{i \in \lceil k,\infty\lceil \\ i \equiv_2 k}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)} \Bigg) \cdot \Bigg(\prod_{\substack{i \in \lceil k,\infty\lceil \\ i \equiv_2 k+1}} [\tilde{b}_{i+2}\,, b_{i+1}]^{\tilde{b}_{i+1}\cdot\left(\prod_{j\in\lfloor i,k+1\rfloor} b_j\cdot\tilde{b}_j\right)} \Bigg)^{-} \cdot b_k^{\tilde{b}_{k+1}} \cdot \tilde{b}_k\,. \end{aligned}$$

This shows the claim. □

To prove Lemma 362, we will make use of Remark 353, but not of Lemma 349 or Proposition 361.

Lemma 362 *Suppose given $h \in H_\infty$. Suppose given $t \in \mathbb{Z}_{\geqslant -1}$.*

We write

$$h_{\{i\}} =: \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ m & \text{for } i = -1\,, \end{cases}$$

where $b_i \in B$ for $i \in \mathbb{Z}_{\geqslant 0}$ and $m \in M$.

(1) *If $t \in \mathbb{Z}_{\geqslant 0}$, we consider*

$$(h_{\{t\}})s_t^\infty = (\ldots, 1, \underbrace{b_t}_{\text{pos. } t}, 1, \ldots, \underbrace{1}_{\text{pos. } -1}) \in H_\infty\,;$$

cf. Lemma 359.(1). *Moreover, we consider*

$$(h_{\{-1\}})s_{-1}^\infty = (\ldots, 1, \underbrace{m}_{\text{pos. } -1}) \in H_\infty\,;$$

cf. Lemma 359.(2). *We have*

$$h = \prod_{j\in\rfloor\infty,-1\rfloor} (h_{\{j\}})s_t^\infty\,;$$

cf. §1.1.2 *item* 9.4.

(2) *Suppose given $\tilde{b} \in B$. Suppose given $\tilde{m} \in M$.*

If $t \in \mathbb{Z}_{\geqslant 0}$, we consider

$$\tilde{b}s_t^\infty \overset{359.(1)}{=} (\ldots, 1, \underbrace{\tilde{b}}_{\text{pos. } t}, 1, \ldots, \underbrace{1}_{\text{pos. } -1}) \in H_\infty\,,$$

and we consider

$$\tilde{m}s_{-1}^{\infty} \overset{359.(2)}{=} (\dots, 1, \underbrace{\tilde{m}}_{\text{pos. } -1}) \in H_\infty \,.$$

If $t \in \mathbb{Z}_{\geqslant 1}$ *, then we have*

$$(h \cdot \tilde{b}s_t^{\infty})_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant t+1} \\ b_t \cdot \tilde{b} & \text{for } i = t \\ \big([b_{t-1}\,, \tilde{b}]^{(-1)^{t-1-i}}\big)^{\left(\prod\limits_{j \in \lfloor t-2, i+1 \rfloor} b_j\right)} \cdot b_i & \text{for } i \in [0, t-1] \\ \big(\llbracket b_{t-1}\,, \tilde{b} \rrbracket^{(-1)^t}\big)^{\left(\prod\limits_{j \in \lfloor t-2, 0 \rfloor} b_j\right)} \cdot m & \text{for } i = -1\,, \end{cases}$$

where, in particular, we have

$$(h \cdot \tilde{b}s_t^{\infty})_{\{t-1\}} = (b_{t-1})^{\tilde{b}} \,.$$

If $t = 0$*, then we have*

$$(h \cdot \tilde{b}s_0^{\infty})_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ b_0 \cdot \tilde{b} & \text{for } i = 0 \\ m^{\tilde{b}} & \text{for } i = -1\,. \end{cases}$$

If $t = -1$*, then we have*

$$(h \cdot \tilde{m}s_{-1}^{\infty})_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ m \cdot \tilde{m} & \text{for } i = -1\,. \end{cases}$$

Cf. also Remark 353.

Proof. We choose $\ell \in \mathbb{Z}_{\geqslant -1}$ such that $b_j = 1$ for $j \in \mathbb{Z}_{\geqslant \ell+1}$.

Ad (1). We have

$$h \overset{357.(3)}{=} (h_{\lfloor \ell, -1 \rfloor})\iota_{\ell,\infty} \overset{353.(1)}{=} \Big(\prod_{j \in \lfloor \ell, -1 \rfloor} \big((h_{\lfloor \ell, -1 \rfloor})_{\{j\}}\big) s_j^{\ell} \Big) \iota_{\ell,\infty} \overset{360}{=} \prod_{j \in \lfloor \ell, -1 \rfloor} \big((h_{\lfloor \ell, -1 \rfloor})_{\{j\}}\big) s_j^{\infty}$$

$$\overset{354.(3)}{=} \prod_{j \in \lfloor \ell, -1 \rfloor} (h_{\{j\}}) s_j^{\infty} = \prod_{j \in \rfloor \infty, -1 \rfloor} (h_{\{j\}}) s_j^{\infty} \,.$$

Ad (2). Let $n := \max\{\ell, t\}$.

Then we have $b_j = 1$ for $j \in \mathbb{Z}_{\geqslant n+1}$. Moreover, we have $t \leqslant n$.

If $t \geqslant 0$, then we have

$$\begin{aligned}(h \cdot \tilde{b}\, s_t^\infty)_{\{i\}} &\overset{357.(3)}{=} \left((h_{\lfloor n,-1\rfloor})\iota_{n,\infty} \cdot \tilde{b} s_t^\infty\right)_{\{i\}} \overset{360}{=} \left((h_{\lfloor n,-1\rfloor})\iota_{n,\infty} \cdot (\tilde{b}\, s_t^n)\iota_{n,\infty}\right)_{\{i\}} \\ &\overset{357.(1)}{=} \left(\left(h_{\lfloor n,-1\rfloor} \cdot \tilde{b} s_t^n\right)\iota_{n,\infty}\right)_{\{i\}} \overset{357.(1)}{=} \left(h_{\lfloor n,-1\rfloor} \cdot \tilde{b}\, s_t^n\right)_{\{i\}} \\ &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant n+1} \\ \left(h_{\lfloor n,-1\rfloor} \cdot \tilde{b} s_t^n\right)_{\{i\}} & \text{for } i \in [-1, n]\,. \end{cases}\end{aligned}$$

Case $t \in \mathbb{Z}_{\geqslant 1}$. We have

$$\begin{aligned}&\left(h_{\lfloor n,-1\rfloor} \cdot \tilde{b}\, s_t^n\right)_{\{i\}} \\ &\overset{353.(2)}{=} \begin{cases} (h_{\lfloor n,-1\rfloor})_{\{i\}} & \text{for } i \in [t+1, n] \\ (h_{\lfloor n,-1\rfloor})_{\{t\}} \cdot \tilde{b} & \text{for } i = t \\ \left(\left[(h_{\lfloor n,-1\rfloor})_{\{t-1\}}, \tilde{b}\right]^{(-1)^{t-1-i}}\right)^{\left(\prod_{j \in \lfloor t-2, i+1\rfloor} (h_{\lfloor n,-1\rfloor})_{\{j\}}\right)} \cdot (h_{\lfloor n,-1\rfloor})_{\{i\}} & \text{for } i \in [0, t-1] \\ \left(\left[\!\left[(h_{\lfloor n,-1\rfloor})_{\{t-1\}}, \tilde{b}\right]\!\right]^{(-1)^t}\right)^{\left(\prod_{j \in \lfloor t-2,0\rfloor} b_j\right)} \cdot (h_{\lfloor n,-1\rfloor})_{\{-1\}} & \text{for } i = -1 \end{cases} \\ &\overset{354.(3)}{=} \begin{cases} b_i & \text{for } i \in [t+1, n] \\ b_t \cdot \tilde{b} & \text{for } i = t \\ \left([b_{t-1}, \tilde{b}]^{(-1)^{t-1-i}}\right)^{\left(\prod_{j \in \lfloor t-2, i+1\rfloor} b_j\right)} \cdot b_i & \text{for } i \in [0, t-1] \\ \left(\left[\!\left[b_{t-1}, \tilde{b}\right]\!\right]^{(-1)^t}\right)^{\left(\prod_{j \in \lfloor t-2,0\rfloor} b_j\right)} \cdot m & \text{for } i = -1\,. \end{cases}\end{aligned}$$

This shows

$$(h \cdot \tilde{b} s_t^\infty)_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant t+1} \\ b_t \cdot \tilde{b} & \text{for } i = t \\ \left([b_{t-1}, \tilde{b}]^{(-1)^{t-1-i}}\right)^{\left(\prod_{j \in \lfloor t-2, i+1\rfloor} b_j\right)} \cdot b_i & \text{for } i \in [0, t-1] \\ \left(\left[\!\left[b_{t-1}, \tilde{b}\right]\!\right]^{(-1)^t}\right)^{\left(\prod_{j \in \lfloor t-2,0\rfloor} b_j\right)} \cdot m & \text{for } i = -1\,. \end{cases}$$

Moreover, note that, for $i = t-1$, we have

$$\begin{aligned}(h \cdot \tilde{b} s_t^\infty)_{\{t-1\}} &= \left([b_{t-1}\,, \tilde{b}]^{t-1-(t-1)}\right)^{\left(\prod_{j \in \lfloor t-2, t\rfloor} b_j\right)} \cdot b_{t-1} \overset{\S 1.1.2 \text{item } 9.3}{=} [b_{t-1}\,, \tilde{b}]^{(b_{t-1}^-)} \cdot b_{t-1} \\ &= b_{t-1} \cdot (b_{t-1}^- \cdot \tilde{b}^- \cdot b_{t-1} \cdot \tilde{b}) = (b_{t-1})^{\tilde{b}}\,.\end{aligned}$$

Case $t = 0$. We have

$$\begin{aligned}\left(h_{\lfloor n,-1\rfloor} \cdot \tilde{b} s_0^n\right)_{\{i\}} &\overset{353.(2)}{=} \begin{cases} (h_{\lfloor n,-1\rfloor})_{\{i\}} & \text{for } i \in [1,n] \\ (h_{\lfloor n,-1\rfloor})_{\{0\}} \cdot \tilde{b} & \text{for } i = 0 \\ \left((h_{\lfloor n,-1\rfloor})_{\{-1\}}\right)^{\tilde{b}} & \text{for } i = -1 \end{cases} \\ &\overset{354.(3)}{=} \begin{cases} b_i & \text{for } i \in [1,n] \\ b_0 \cdot \tilde{b} & \text{for } i = 0 \\ m^{\tilde{b}} & \text{for } i = -1\,. \end{cases}\end{aligned}$$

This shows

$$(h \cdot \tilde{b} s_0^\infty)_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ b_0 \cdot \tilde{b} & \text{for } i = 0 \\ m^{\tilde{b}} & \text{for } i = -1\,. \end{cases}$$

Case $t = -1$. We have

$$\begin{aligned}(h \cdot \tilde{m} s_{-1}^\infty)_{\{i\}} &\overset{357.(3)}{=} \left((h_{\lfloor n,-1\rfloor})\iota_{n,\infty} \cdot \tilde{m} s_{-1}^\infty\right)_{\{i\}} \overset{360}{=} \left((h_{\lfloor n,-1\rfloor})\iota_{n,\infty} \cdot (\tilde{m} s_{-1}^n)\iota_{n,\infty}\right)_{\{i\}} \\ &\overset{357.(1)}{=} \left((h_{\lfloor n,-1\rfloor} \cdot \tilde{m} s_{-1}^n)\iota_{n,\infty}\right)_{\{i\}} \overset{357.(1)}{=} (h_{\lfloor n,-1\rfloor} \cdot \tilde{m} s_{-1}^n)_{\{i\}} \\ &= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant n+1} \\ (h_{\lfloor n,-1\rfloor} \cdot \tilde{m} s_{-1}^n)_{\{i\}} & \text{for } i \in [-1,n]\,. \end{cases}\end{aligned}$$

We have

$$\begin{aligned}\left(h_{\lfloor n,-1\rfloor} \cdot \tilde{m} s_{-1}^n\right)_{\{i\}} &\overset{353.(2)}{=} \begin{cases} (h_{\lfloor n,-1\rfloor})_{\{i\}} & \text{for } i \in [0,n] \\ (h_{\lfloor n,-1\rfloor})_{\{-1\}} \cdot \tilde{m} & \text{for } i = -1 \end{cases} \\ &\overset{354.(3)}{=} \begin{cases} b_i & \text{for } i \in [0,n] \\ m \cdot \tilde{m} & \text{for } i = -1\,. \end{cases}\end{aligned}$$

This shows

$$(h \cdot \tilde{m} s^\infty_{-1})_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ m \cdot \tilde{m} & \text{for } i = -1\,. \end{cases}$$

□

17.2.3 The group morphisms d_j and s_i in H_∞

Remark 363 *Suppose given a group A. Suppose given a map $\varphi\colon H_\infty \to A$. Then φ is a group morphism if and only if we have*

$$(h \cdot \tilde{b} s^\infty_t)\varphi = h\varphi \cdot (\tilde{b} s^\infty_t)\varphi\,,$$
$$(h \cdot \tilde{m} s^\infty_{-1})\varphi = h\varphi \cdot (\tilde{m} s^\infty_{-1})\varphi$$

for $h \in H_\infty$, $t \in \mathbb{Z}_{\geqslant 0}$, $\tilde{b} \in B$, $\tilde{m} \in M$.

Proof. Ad $\Rightarrow$. Since φ is a group morphism, we have

$$(h \cdot \tilde{b} s^\infty_t)\varphi = h\varphi \cdot (\tilde{b} s^\infty_t)\varphi$$

and

$$(h \cdot \tilde{m} s^\infty_{-1})\varphi = h\varphi \cdot (\tilde{m} s^\infty_{-1})\varphi$$

for $h \in H_\infty$, $t \in \mathbb{Z}_{\geqslant 0}$, $\tilde{b} \in B$, $\tilde{m} \in M$.

Ad $\Leftarrow$. Suppose given h, $\tilde{h} \in H_n$.

Note that, by Lemma 362.(1), we have

$$\tilde{h} = \prod_{t \in \rfloor \infty, -1 \rfloor} \tilde{h}_{\{t\}} s^\infty_t\,.$$

Moreover, note that we may choose some $n \in \mathbb{Z}_{\geqslant -1}$ such that $\tilde{h}_{\{k\}} = 1$ for $k \in \mathbb{Z}_{\geqslant n+1}$.

So we have

$$\tilde{h} = \prod_{t \in \rfloor \infty, -1 \rfloor} \tilde{h}_{\{t\}} s^\infty_t = \prod_{t \in \lfloor n, -1 \rfloor} \tilde{h}_{\{t\}} s^\infty_t\,.$$

We have

$$\begin{aligned}(h\cdot\tilde{h})\varphi &= \Big(h\cdot\big(\prod_{t\in\rfloor\infty,-1\rfloor}\tilde{h}_{\{t\}}s_t^\infty\big)\Big)\varphi = \Big(h\cdot\big(\prod_{t\in\lfloor n,-1\rfloor}\tilde{h}_{\{t\}}s_t^\infty\big)\Big)\varphi\\ &= h\varphi\cdot\Big(\prod_{t\in\lfloor n,-1\rfloor}(\tilde{h}_{\{t\}}s_t^\infty)\varphi\Big) = h\varphi\cdot\Big(\prod_{t\in\lfloor n,-1\rfloor}\tilde{h}_{\{t\}}s_t^\infty\Big)\varphi\\ &= h\varphi\cdot\Big(\prod_{t\in\rfloor\infty,-1\rfloor}\tilde{h}_{\{t\}}s_t^\infty\Big)\varphi = h\varphi\cdot\tilde{h}\varphi\,.\end{aligned}$$

□

Lemma 364 (The group morphism d_j^∞)

(1) *For $j\in\mathbb{Z}_{\geqslant 1}$, we have the group morphism*

$$d_j^\infty\colon H_\infty\to B\,,\ h\mapsto hd_j^\infty := h_{\{j\}}\cdot h_{\{j-1\}}\,.$$

(2) *We have the group morphism*

$$d_0^\infty\colon H_\infty\to B\,,\ h\mapsto hd_0^\infty := h_{\{0\}}\cdot h_{\{-1\}}\mathrm{f}_S\,.$$

Proof. Suppose given $j\in\mathbb{Z}_{\geqslant 0}$. We write $d_j := d_j^\infty$.

We use Remark 363 to show that d_j is a group morphism.

Suppose given $h\in H_\infty$, $\tilde{b}\in B$ and $\tilde{m}\in M$.

We write $b_i := h_{\{i\}}\in B$ for $i\in\mathbb{Z}_{\geqslant 0}$ and $m := h_{\{-1\}}\in M$. So $h = (\dots b_1, b_0, m)$.

Suppose given $t\in\mathbb{Z}_{\geqslant 0}$.

Case $t\in\mathbb{Z}_{\geqslant 1}$.

By Lemma 362.(2), we have

$$(h\cdot\tilde{b}s_t^\infty)_{\{i\}} = \begin{cases} b_i & \text{for } i\in\mathbb{Z}_{\geqslant t+1}\\ b_t\cdot\tilde{b} & \text{for } i=t\\ \big([b_{t-1},\tilde{b}]^{(-1)^{t-1-i}}\big)^{\left(\prod_{k\in\lfloor t-2,i+1\rfloor}b_k\right)}\cdot b_i & \text{for } i\in[0,t-1]\\ \big([\![b_{t-1},\tilde{b}]\!]^{(-1)^t}\big)^{\left(\prod_{k\in\lfloor t-2,0\rfloor}b_k\right)}\cdot m & \text{for } i=-1\,.\end{cases}$$

If $j\geqslant t+2$, then we have

$$(h\cdot\tilde{b}s_t^\infty)d_j = (h\cdot\tilde{b}s_t^\infty)_{\{j\}}\cdot(h\cdot\tilde{b}s_t^\infty)_{\{j-1\}} = b_j\cdot b_{j-1}\overset{j-1\geqslant t+1}{=} hd_j\cdot(\tilde{b}s_t^\infty)d_j\,.$$

If $j = t+1$, then we have

$$(h \cdot \tilde{b}s_t^\infty)d_{t+1} = (h \cdot \tilde{b}s_t^\infty)_{\{t+1\}} \cdot (h \cdot \tilde{b}s_t^\infty)_{\{t\}} = b_{t+1} \cdot b_t \cdot \tilde{b}^{\,(\tilde{b}s_t^\infty)_{\{t+1\}}=1} \overset{}{=} hd_{t+1} \cdot (\tilde{b}s_t^\infty)d_{t+1}\,.$$

If $j = t$, then we have

$$(h \cdot \tilde{b}s_t^\infty)d_t \;=\; (h \cdot \tilde{b}s_t^\infty)_{\{t\}} \cdot (h \cdot \tilde{b}s_t^\infty)_{\{t-1\}} = b_t \cdot \tilde{b} \cdot b_{t-1}^{\tilde{b}} = b_t \cdot b_{t-1} \cdot \tilde{b}^{\,(\tilde{b}s_t^\infty)_{\{t-1\}}=1} \overset{}{=} hd_t \cdot (\tilde{b}s_t^\infty)d_t\,.$$

If $j \in [1, t-1]$, then we have

$$\begin{aligned}
& (h \cdot \tilde{b}s_t^\infty)d_j \\
= \;& (h \cdot \tilde{b}s_t^\infty)_{\{j\}} \cdot (h \cdot \tilde{b}s_t^\infty)_{\{j-1\}} \\
= \;& \big([b_{t-1}\,, \tilde{b}]^{(-1)^{t-1-j}}\big)^{\left(\prod_{k\in\lfloor t-2,j+1\rfloor} b_k\right)} \cdot b_j \cdot \big([b_{t-1}\,, \tilde{b}]^{(-1)^{t-1-(j-1)}}\big)^{\left(\prod_{k\in\lfloor t-2,j\rfloor} b_k\right)} \cdot b_{j-1} \\
= \;& b_j \cdot b_{j-1} \cdot \big([\tilde{b}\,, b_{t-1}]^{(-1)^{t-j}}\big)^{\left(\prod_{k\in\lfloor t-2,j+1\rfloor} b_k\right)\cdot b_j \cdot b_{j-1}} \cdot \big([b_{t-1}\,, \tilde{b}]^{(-1)^{t-j}}\big)^{\left(\prod_{k\in\lfloor t-2,j\rfloor} b_k\right)\cdot b_{j-1}} \\
= \;& b_j \cdot b_{j-1} \cdot \big([\tilde{b}\,, b_{t-1}]^{(-1)^{t-j}}\big)^{\left(\prod_{k\in\lfloor t-2,j-1\rfloor} b_k\right)} \cdot \big([b_{t-1}\,, \tilde{b}]^{(-1)^{t-j}}\big)^{\left(\prod_{k\in\lfloor t-2,j-1\rfloor} b_k\right)} \\
= \;& b_j \cdot b_{j-1} \\
\overset{t-1\geqslant j}{=} \;& hd_j \cdot (\tilde{b}s_t^\infty)d_j\,.
\end{aligned}$$

If $j = 0$, then we have

$$\begin{aligned}
& (h \cdot \tilde{b}s_t^\infty)d_0 \\
= \;& (h \cdot \tilde{b}s_t^\infty)_{\{0\}} \cdot \big((h \cdot \tilde{b}s_t^\infty)_{\{-1\}}\big)\mathrm{f}_S \\
= \;& \big([b_{t-1}\,, \tilde{b}]^{(-1)^{t-1-0}}\big)^{\left(\prod_{k\in\lfloor t-2,1\rfloor} b_k\right)} \cdot b_0 \cdot \Big(\big(\llbracket b_{t-1}\,, \tilde{b}\rrbracket^{(-1)^t}\big)^{\left(\prod_{k\in\lfloor t-2,0\rfloor} b_k\right)} \cdot m\Big)\mathrm{f}_S \\
\overset{323.(1)}{=} \;& b_0 \cdot \big([\tilde{b}\,, b_{t-1}]^{(-1)^t}\big)^{\left(\prod_{k\in\lfloor t-2,1\rfloor} b_k\right)\cdot b_0} \cdot \Big(\big(\llbracket b_{t-1}\,, \tilde{b}\rrbracket^{(-1)^t}\big)\mathrm{f}_S\Big)^{\left(\prod_{k\in\lfloor t-2,0\rfloor} b_k\right)} \cdot m\mathrm{f}_S \\
\overset{(\mathrm{SCM2})}{=} \;& b_0 \cdot \big([\tilde{b}\,, b_{t-1}]^{(-1)^t}\big)^{\left(\prod_{k\in\lfloor t-2,0\rfloor} b_k\right)} \cdot \big([b_{t-1}\,, \tilde{b}]^{(-1)^t}\big)^{\left(\prod_{k\in\lfloor t-2,0\rfloor} b_k\right)} \cdot m\mathrm{f}_S \\
= \;& b_0 \cdot m\mathrm{f}_S \\
\overset{t\geqslant 1>0}{=} \;& hd_0 \cdot (\tilde{b}s_t^\infty)d_0\,.
\end{aligned}$$

Case $t = 0$. By Lemma 362.(2), we have

$$(h \cdot \tilde{b} s_0^\infty)_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ b_0 \cdot \tilde{b} & \text{for } i = 0 \\ m^{\tilde{b}} & \text{for } i = -1\,. \end{cases}$$

If $j \geqslant 2$, then we have

$$(h \cdot \tilde{b} s_0^\infty) d_j = (h \cdot \tilde{b} s_0^\infty)_{\{j\}} \cdot (h \cdot \tilde{b} s_0^\infty)_{\{j-1\}} = b_j \cdot b_{j-1} \overset{j-1 \geqslant 1}{=} h d_j \cdot (\tilde{b} s_0^\infty) d_j\,.$$

If $j = 1$, then we have

$$(h \cdot \tilde{b} s_0^\infty) d_1 = (h \cdot \tilde{b} s_0^\infty)_{\{1\}} \cdot (h \cdot \tilde{b} s_0^\infty)_{\{0\}} = b_1 \cdot b_0 \cdot \tilde{b} \overset{(\tilde{b} s_0^\infty)_{\{1\}} = 1}{=} h d_1 \cdot (\tilde{b} s_0^\infty) d_1\,.$$

If $j = 0$ then we have

$$\begin{aligned} (h \cdot \tilde{b} s_0^\infty) d_0 &= (h \cdot \tilde{b} s_0^\infty)_{\{0\}} \cdot \big((h \cdot \tilde{b} s_0^\infty)_{\{-1\}}\big) \mathrm{f}_S = b_0 \cdot \tilde{b} \cdot (m^{\tilde{b}}) \mathrm{f}_S \overset{323.(1)}{=} b_0 \cdot \tilde{b} \cdot \tilde{b}^- \cdot m \mathrm{f}_S \cdot \tilde{b} \\ &= b_0 \cdot m \mathrm{f}_S \cdot \tilde{b} \cdot 1 \mathrm{f}_S \overset{(\tilde{b} s_0^\infty)_{\{-1\}} = 1}{=} h d_0 \cdot (\tilde{b} s_0^\infty) d_0\,. \end{aligned}$$

Case $t = -1$. By Lemma 362.(2), we have

$$(h \cdot \tilde{m} s_{-1}^\infty)_{\{i\}} = \begin{cases} b_i & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ m \cdot \tilde{m} & \text{for } i = -1\,. \end{cases}$$

If $j \geqslant 1$, then we have

$$(h \cdot \tilde{m} s_{-1}^\infty) d_j = (h \cdot \tilde{m} s_{-1}^\infty)_{\{j\}} \cdot (h \cdot \tilde{m} s_{-1}^\infty)_{\{j-1\}} = b_j \cdot b_{j-1} \overset{i-1 \geqslant 0}{=} h d_j \cdot (\tilde{m} s_{-1}^\infty) d_j\,.$$

If $j = 0$, then we have

$$\begin{aligned} (h \cdot \tilde{m} s_{-1}^\infty) d_0 &= (h \cdot \tilde{m} s_{-1}^\infty)_{\{0\}} \cdot \big((h \cdot \tilde{m} s_{-1}^\infty)_{\{-1\}}\big) \mathrm{f}_S = b_0 \cdot (m \cdot \tilde{m}) \mathrm{f}_S \\ &= b_0 \cdot m \mathrm{f}_S \cdot 1 \cdot \tilde{m} \mathrm{f}_S \overset{(\tilde{m} s_{-1}^\infty)_{\{0\}} = 1}{=} h d_0 \cdot (\tilde{m} s_{-1}^\infty) d_0\,. \end{aligned}$$

Altogether, by Remark 363, d_j is a group morphism. □

Remark 365 *Suppose given* $i \in \mathbb{Z}_{\geqslant 0}$ *and* $j \in \mathbb{Z}_{\geqslant 0}$.

For $k \in \mathbb{Z}_{\geqslant 0}$, *we consider the group morphisms* $s_k^\infty \colon B \to H_\infty$ *and* $d_k^\infty \colon H_\infty \to B$; *cf. Lemmas* 359, 364.

We have

$$s_i^\infty \blacktriangleleft d_j^\infty = \begin{cases} ! & \text{if } j \in \mathbb{Z}_{\geqslant 0} \backslash \{i, i+1\} \\ \mathrm{id}_B & \text{if } j \in \{i, i+1\}\,. \end{cases}$$

Proof. Suppose given $b \in B$.

Case $i = 0$.

Subcase $j = 0$. We have

$$b(s_0^\infty \blacktriangleleft d_0^\infty) = (bs_0^\infty)d_0^\infty = (bs_0^\infty)_{\{0\}} \cdot \big((bs_0^\infty)_{\{-1\}}\big)\mathrm{f}_S = b \cdot 1\mathrm{f}_S = b\,.$$

Subcase $j = 1$. We have

$$b(s_0^\infty \blacktriangleleft d_1^\infty) = (bs_0^\infty)d_1^\infty = (bs_0^\infty)_{\{1\}} \cdot (bs_0^\infty)_{\{0\}} = 1 \cdot b = b\,.$$

Subcase $j \geqslant 2$. We have

$$b(s_0^\infty \blacktriangleleft d_j^\infty) = (bs_0^\infty)d_j^\infty = (bs_0^\infty)_{\{j\}} \cdot (bs_0^\infty)_{\{j-1\}} = 1 \cdot 1 = 1\,.$$

This shows

$$s_0^\infty \blacktriangleleft d_j^\infty = \begin{cases} \mathrm{id}_{\mathrm{B}_S} & \text{if } j \in \{0, 1\} \\ ! & \text{if } j \in \mathbb{Z}_{\geqslant 0} \backslash \{0, 1\}\,. \end{cases}$$

Case $i \geqslant 1$.

Subcase $j = 0$. We have

$$b(s_i^\infty \blacktriangleleft d_0^\infty) = (bs_i^\infty)d_0^\infty = (bs_i^\infty)_{\{0\}} \cdot \big((bs_i^\infty)_{\{-1\}}\big)\mathrm{f}_S = 1 \cdot 1\mathrm{f}_S = 1\,.$$

Subcase $j = i$. We have

$$b(s_i^\infty \blacktriangleleft d_i^\infty) = (bs_i^\infty)d_i^\infty = (bs_i^\infty)_{\{i\}} \cdot (bs_i^\infty)_{\{i-1\}} = b \cdot 1 = b\,.$$

Subcase $j = i + 1$. We have

$$b(s_i^\infty \blacktriangleleft d_{i+1}^\infty) = (bs_i^\infty)d_{i+1}^\infty = (bs_i^\infty)_{\{i+1\}} \cdot (bs_i^\infty)_{\{i\}} = 1 \cdot b = b\,.$$

Subcase $j \in \mathbb{Z}_{\geqslant 1} \backslash \{i, i+1\}$. We have

$$b(s_i^\infty \blacktriangleleft d_j^\infty) = (bs_i^\infty)d_j^\infty = (bs_i^\infty)_{\{j\}} \cdot (bs_i^\infty)_{\{j-1\}} = 1 \cdot 1 = 1\,.$$

This shows

$$s_i^\infty \mathbin{\blacktriangle} d_j^\infty = \begin{cases} \mathrm{id}_{\mathrm{B}S} & \text{if } j \in \{i, i+1\} \\ ! & \text{if } j \in \mathbb{Z}_{\geqslant 1} \backslash \{i, i+1\} \end{cases}$$

for $i \in \mathbb{Z}_{\geqslant 1}$.

Altogether, this proves the claim. □

Remark 366

We consider the group morphism $d_j^\infty \colon H_\infty \to \mathrm{B}S$ for $j \in \mathbb{Z}_{\geqslant 1}$; cf. Definition 364.(1).

Suppose given $n \in \mathbb{Z}_{\geqslant 1}$. We consider the group $\bigcap_{j \geqslant n} \ker d_j^\infty \leqslant H_\infty$.

Suppose given $x \in \bigcap_{j \geqslant n} \ker d_j^\infty$.

Then we have $x_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant n-1}$.

Proof. Since $x \in H_\infty$, we may choose some $k \in \mathbb{Z}_{\geqslant 0}$ such that $x_{\{\ell\}} = 1$ for $\ell \in \mathbb{Z}_{\geqslant k}$; cf. Definition 356.

We may assume that $n \leqslant k$.

It remains to show that $x_{\{\ell\}} \overset{!}{=} 1$ for $\ell \in [n-1, k-1]$.

We show this claim by downwards induction on ℓ.

Let $\ell = k - 1$. Then $\ell + 1 = k \geqslant n$, and we have

$$1 = x d_k^\infty = x_{\{k\}} \cdot x_{\{k-1\}} = 1 \cdot x_{\{\ell\}} = x_{\{\ell\}}\,.$$

Now we assume that $\ell \in [n-1, k-2]$ and that $x_{\{\ell+1\}} = 1$.

We have to show $x_{\{\ell\}} \overset{!}{=} 1$.

We have $\ell + 1 \geqslant 1$ and thus

$$1 = x d_{\ell+1}^\infty = x_{\{\ell+1\}} \cdot x_{\{\ell\}} \overset{\text{IH}}{=} 1 \cdot x_{\{\ell\}} = x_{\{\ell\}}\,.$$

This proves the *claim*. □

17.2.4 The group elements $b\zeta_{\lfloor \ell, k \rfloor}$ and $m\zeta_{\lfloor \ell, k \rfloor}$ in H_∞

Definition 367

(1) Suppose given $k \in \mathbb{Z}_{\geqslant 0}$. Suppose given $\ell \in \mathbb{Z}_{\geqslant k-1}$.

We have the map

$$\zeta_{\lfloor \ell, k \rfloor} \colon B \to H_\infty\,,\ b \mapsto b\zeta_{\lfloor \ell, k \rfloor}\,,$$

where

$$(b\zeta_{\lfloor \ell, k \rfloor})_{\{i\}} := \begin{cases} b^{(-1)^{i-k}} & \text{for } i \in [k, \ell] \\ 1 & \text{for } i \in \mathbb{Z}_{\geqslant 0} \backslash [k, \ell]\,. \end{cases}$$

So, for $b \in B$, we have

$$\begin{aligned} b\zeta_{\lfloor \ell, k \rfloor} &= (\dots, 1, \underbrace{b^{(-1)^{\ell-k}}}_{\text{pos. } \ell}, \dots, \underbrace{b^{-}}_{\text{pos. } k+1}, \underbrace{b}_{\text{pos. } k}, 1, \dots, \underbrace{1}_{\text{pos. } -1}) \\ &= \prod_{i \in \lfloor \ell, k \rfloor} (b^{(-1)^{i-k}}) \zeta_{\lfloor i, i \rfloor} \\ &\overset{362.(1)}{=} \prod_{i \in \lfloor \ell, k \rfloor} (b^{(-1)^{i-k}}) s_i^\infty \ \in H_\infty\,. \end{aligned}$$

Note that for $b \in B$, we have $b\zeta_{\lfloor k-1, k \rfloor} = 1$.

(2) Suppose given $\ell \in \mathbb{Z}_{\geqslant -1}$.

We have the map

$$\zeta_{\lfloor \ell, -1 \rfloor} \colon M \to H_\infty\,,\ m \mapsto m\zeta_{\lfloor \ell, -1 \rfloor}\,,$$

where

$$(m\zeta_{\lfloor \ell, -1 \rfloor})_{\{i\}} := \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant \ell+1} \\ (m\mathrm{f}_S)^{(-1)^{i+1}} & \text{for } i \in [0, \ell] \\ m & \text{for } i = -1\,. \end{cases}$$

So, for $m \in M$, we have

$$\begin{aligned} m\zeta_{\lfloor \ell, -1 \rfloor} &= \big(\dots, 1, \underbrace{(m\mathrm{f}_S)^{(-1)^{\ell+1}}}_{\text{pos. } \ell}, \dots, \underbrace{(m\mathrm{f}_S)^{-}}_{\text{pos. } 0}, \underbrace{m}_{\text{pos. } -1}\big) \\ &= \Big(\prod_{i \in \lfloor \ell, 0 \rfloor} \big((m\mathrm{f}_S)^{(-1)^{i+1}}\big) \zeta_{\lfloor i, i \rfloor} \Big) \cdot m\zeta_{\lfloor -1, -1 \rfloor} \\ &\overset{362.(1)}{=} \Big(\prod_{i \in \lfloor \ell, 0 \rfloor} \big((m\mathrm{f}_S)^{(-1)^{i+1}}\big) s_i^\infty \Big) \cdot m s_{-1}^\infty \ \in H_\infty\,. \end{aligned}$$

Note that for $m \in M$, we have $m\zeta_{\lfloor -1,\,-1\rfloor} = (\ldots, 1, 1, m)$.

(3) For $\ell,\, k \in \mathbb{Z}_{\geqslant 0}$ with $\ell \geqslant k$, we often write

$$\zeta_{\rfloor \ell,\, k\rfloor} = \zeta_{\lfloor \ell-1,\, k\rfloor} \colon B \to H_\infty \,.$$

So, for $b \in B$, we have

$$\begin{aligned} b\zeta_{\rfloor \ell,\, k\rfloor} \quad &= \quad (\ldots, 1, \underbrace{b^{(-1)^{\ell-1+k}}}_{\text{pos. } \ell-1}, \ldots, \underbrace{b^-}_{\text{pos. } k+1}, \underbrace{b}_{\text{pos. } k}, 1, \ldots, \underbrace{1}_{\text{pos. } -1}) \\ &\overset{362.(1)}{=} \quad \prod_{i \in \rfloor \ell, k\rfloor} (b^{(-1)^{i-k}}) s_i^\infty \ \in H_\infty \,. \end{aligned}$$

For $\ell \in \mathbb{Z}_{\geqslant 0}$, we often write

$$\zeta_{\rfloor \ell,\, -1\rfloor} = \zeta_{\lfloor \ell-1,\, -1\rfloor} \colon M \to H_\infty \,.$$

So, for $m \in M$, we have

$$\begin{aligned} m\zeta_{\rfloor \ell,\, -1\rfloor} \quad &= \quad \big(\ldots, 1, \underbrace{(m\mathrm{f}_S)^{(-1)^\ell}}_{\text{pos. } \ell-1}, \ldots, \underbrace{(m\mathrm{f}_S)^-}_{\text{pos. } 0}, \underbrace{m}_{\text{pos. } -1}\big) \\ &\overset{362.(1)}{=} \quad \Big(\prod_{i \in \rfloor \ell, 0\rfloor} \big((m\mathrm{f}_S)^{(-1)^{i+1}}\big) s_i^\infty \Big) \cdot m s_{-1}^\infty \ \in H_\infty \,. \end{aligned}$$

Note that in general, $\zeta_{\lfloor \ell,\, k\rfloor}$ is not a group morphism; cf. Example 368.

Example 368

(1) For $b \in B$, we have

$$b\zeta_{\lfloor 5,\, 2\rfloor} = (\ldots, 1, 1, \underbrace{b^-}_{\text{pos. } 5}, b, b^-, \underbrace{b}_{\text{pos. } 2}, 1, 1, \underbrace{1}_{\text{pos. } -1}) \,.$$

For $m \in M$, we have

$$m\zeta_{\lfloor 3,\, -1\rfloor} = (\ldots, 1, 1, \underbrace{m\mathrm{f}_S}_{\text{pos. } 3}, m^-\mathrm{f}_S, m\mathrm{f}_S, m^-\mathrm{f}_S, \underbrace{m}_{\text{pos. } -1}) \,.$$

(2) Suppose given $b \in B$. We consider the group elements

$$\begin{aligned}(b^-)\zeta_{\lfloor 1,0\rfloor} &= (\dots, 1, b, b^-, 1)\,,\\ b\zeta_{\lfloor 1,0\rfloor} &= (\dots, 1, b^-, b, 1) \in H_\infty\,;\end{aligned}$$

cf. Definition 367.(1).

We consider the group elements $(b, b^-, 1)$, $(b^-, b, 1) \in H_1$.

Note that we have $(b, b^-, 1)\iota_{1,\infty} = (b^-)\zeta_{\lfloor 1,0\rfloor}$ and $(b^-, b, 1)\iota_{1,\infty} = b\zeta_{\lfloor 1,0\rfloor}$; cf. Remark 357.(1).

We have

$$\begin{aligned}(b\zeta_{\lfloor 1,0\rfloor}) \cdot \big((b^-)\zeta_{\lfloor 1,0\rfloor}\big) &= (b^-, b, 1)\iota_{1,\infty} \cdot (b, b^-, 1)\iota_{1,\infty}\\ &= \big((b^-, b, 1) \cdot (b, b^-, 1)\big)\iota_{1,\infty}\\ &\overset{350.(1)}{=} \big(b^- \cdot b, b^b \cdot b^-, \llbracket b,\, b\rrbracket^{b^-} \cdot 1^{b^-} \cdot 1\big)\iota_{1,\infty}\\ &\overset{323.(4)}{=} \big(1, 1, \llbracket b,\, b\rrbracket\big)\iota_{1,\infty}\,.\end{aligned}$$

For an example where $\llbracket b,\, b\rrbracket \neq 1$, cf. Example 340.(4).

This example also shows that $(b\zeta_{\lfloor 1,0\rfloor})^-$ is not equal to $(b^-)\zeta_{\lfloor 1,0\rfloor}$ in general.

17.2.5 Certain commuting elements in H_∞

Lemma 369 *Suppose given b, $x \in B$.*

(1) *For ℓ, $k \in \mathbb{Z}_{\geqslant 0}$ with $|\ell - k| \geqslant 2$, we have* $[b\zeta_{\lfloor \ell,\ell\rfloor}, x\zeta_{\lfloor k,k\rfloor}] = 1$.

(2) *For $k \in \mathbb{Z}_{\geqslant 1}$, we have* $[b\zeta_{\lfloor k+1,k-1\rfloor}, x\zeta_{\lfloor k,k\rfloor}] = 1$.

(3) *For $k \in \mathbb{Z}_{\geqslant 1}$, we have* $[b\zeta_{\lfloor k+1,k\rfloor}, x\zeta_{\lfloor k,k-1\rfloor}] = 1$.

Cf. Definition 367.(1).

Proof. Ad (1). Suppose given ℓ, $k \in \mathbb{Z}_{\geqslant 0}$ with $|\ell - k| \geqslant 2$. We may assume that $\ell > k$.
We calculate $b\zeta_{\lfloor \ell,\ell\rfloor} \cdot x\zeta_{\lfloor k,k\rfloor}$.
To this end, we use Remark 362.(2) with $t := k$.

Case $k = 0$. For $i \geqslant 1$, we have

$$(b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{i\}} = (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{i\}} = \begin{cases} b & \text{if } i = \ell \\ 1 & \text{if } i \in \mathbb{Z}_{\geqslant 1} \backslash \{\ell\} \,. \end{cases}$$

For $i = 0$, we have

$$(b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{0\}} = (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{0\}} \cdot (x\zeta_{\lfloor k, k \rfloor})_{\{0\}} = 1 \cdot x = x \,.$$

For $i = -1$, we have

$$(b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{-1\}} = \big((b\zeta_{\lfloor \ell, \ell \rfloor})_{\{-1\}}\big)^{(x\zeta_{\lfloor k, k \rfloor})_{\{0\}}} = 1^x = 1 \,.$$

Case $k \geqslant 1$. For $i \geqslant k+1$, we have

$$(b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{i\}} = (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{i\}} = \begin{cases} b & \text{if } i = \ell \\ 1 & \text{if } i \in \mathbb{Z}_{\geqslant k+1} \backslash \{\ell\} \,. \end{cases}$$

For $i = k$, we have

$$(b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{k\}} = (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{k\}} \cdot (x\zeta_{\lfloor k, k \rfloor})_{\{k\}} = 1 \cdot x = x \,.$$

For $i \in [0, k-1]$, we have

$$\begin{aligned} (b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{i\}} &= \Big(\big[(b\zeta_{\lfloor \ell, \ell \rfloor})_{\{k-1\}},\, (x\zeta_{\lfloor k, k \rfloor})_{\{k\}} \big]^{(-1)^{k-1-i}} \Big)^{\left(\prod\limits_{j \in \lfloor k-2, i+1 \rfloor} (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{j\}} \right)} \cdot (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{i\}} \\ &= \big([1, x]^{(-1)^{k-1-i}} \big)^1 \cdot 1 \\ &= 1 \,. \end{aligned}$$

For $i = -1$, we have

$$\begin{aligned} (b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{-1\}} &= \Big(\big\llbracket (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{k-1\}},\, (x\zeta_{\lfloor k, k \rfloor})_{\{k\}} \big\rrbracket^{(-1)^k} \Big)^{\left(\prod\limits_{j \in \lfloor k-2, 0 \rfloor} (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{j\}} \right)} \cdot (b\zeta_{\lfloor \ell, \ell \rfloor})_{\{-1\}} \\ &= \big(\llbracket 1, x \rrbracket^{(-1)^k} \big)^1 \cdot 1 \\ &\overset{322.(1)}{=} 1 \,. \end{aligned}$$

So, for $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$(b\zeta_{\lfloor \ell, \ell \rfloor} \cdot x\zeta_{\lfloor k, k \rfloor})_{\{i\}} = \begin{cases} b & \text{for } i = \ell \\ x & \text{for } i = k \\ 1 & \text{for } i \in \mathbb{Z}_{\geqslant -1} \backslash \{k, \ell\} \,. \end{cases}$$

We calculate $x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor \ell,\ell\rfloor}$.

To this end, we use Lemma 362.(2) with $t:=\ell$. Note that $\ell\geqslant 2$.

For $i\geqslant \ell+1$, we have

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor \ell,\ell\rfloor})_{\{i\}}=(x\zeta_{\lfloor k,k\rfloor})_{\{i\}}=1\,.$$

For $i=\ell$, we have

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor \ell,\ell\rfloor})_{\{\ell\}}=(x\zeta_{\lfloor k,k\rfloor})_{\{\ell\}}\cdot(b\zeta_{\lfloor \ell,\ell\rfloor})_{\{\ell\}}=1\cdot b=b\,.$$

For $i\in[0,\ell-1]$, we have

$$\begin{aligned}(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor \ell,\ell\rfloor})_{\{i\}}&=\Big(\big[(x\zeta_{\lfloor k,k\rfloor})_{\{\ell-1\}}\,,(b\zeta_{\lfloor \ell,\ell\rfloor})_{\{\ell\}}\big]^{(-1)^{\ell-1-i}}\Big)^{\left(\prod\limits_{j\in\lfloor \ell-2,i+1\rfloor}(x\zeta_{\lfloor k,k\rfloor})_{\{j\}}\right)}\cdot(x\zeta_{\lfloor k,k\rfloor})_{\{i\}}\\&=\big([1,b]^{(-1)^{\ell-1-i}}\big)^{\left(\prod\limits_{j\in\lfloor \ell-2,i+1\rfloor}(x\zeta_{\lfloor k,k\rfloor})_{\{j\}}\right)}\cdot(x\zeta_{\lfloor k,k\rfloor})_{\{i\}}\\&=(x\zeta_{\lfloor k,k\rfloor})_{\{i\}}\\&=\begin{cases}x & \text{if } i=k\\ 1 & \text{if } i\in[0,\ell-1]\backslash\{k\}\,.\end{cases}\end{aligned}$$

For $i=-1$, we have

$$\begin{aligned}(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor \ell,\ell\rfloor})_{\{-1\}}&=\Big(\big[\!\big[(x\zeta_{\lfloor k,k\rfloor})_{\{\ell-1\}}\,,(b\zeta_{\lfloor \ell,\ell\rfloor})_{\{\ell\}}\big]\!\big]^{(-1)^{\ell}}\Big)^{\left(\prod\limits_{j\in\lfloor \ell-2,0\rfloor}(x\zeta_{\lfloor k,k\rfloor})_{\{j\}}\right)}\cdot(x\zeta_{\lfloor k,k\rfloor})_{\{-1\}}\\&=\big([\![1,b]\!]^{(-1)^{\ell}}\big)^{\left(\prod\limits_{j\in\lfloor \ell-2,i+1\rfloor}(x\zeta_{\lfloor k,k\rfloor})_{\{j\}}\right)}\cdot 1\overset{322.(1)}{=}1\,.\end{aligned}$$

So, for $k\in\mathbb{Z}_{\geqslant 0}$, we have

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor \ell,\ell\rfloor})_{\{i\}}=\begin{cases}b & \text{for } i=\ell\\ x & \text{for } i=k\\ 1 & \text{for } i\in\mathbb{Z}_{\geqslant -1}\backslash\{k,\ell\}\,.\end{cases}$$

Altogether, this shows $[b\zeta_{\lfloor \ell,\ell\rfloor}\,,x\zeta_{\lfloor k,k\rfloor}]=1$.

Ad (2). We calculate $b\zeta_{\lfloor k+1,k-1\rfloor}\cdot x\zeta_{\lfloor k,k\rfloor}$.

To this end, we use Lemma 362.(2) with $t:=k$.

For $i\geqslant k+1$, we have

$$(b\zeta_{\lfloor k+1,k-1\rfloor}\cdot x\zeta_{\lfloor k,k\rfloor})_{\{i\}}=(b\zeta_{\lfloor k+1,k-1\rfloor})_{\{i\}}=\begin{cases}1 & \text{if } i\geqslant k+2\\ b & \text{if } i=k+1\,.\end{cases}$$

For $i = k$, we have

$$(b\zeta_{\lfloor k+1,k-1\rfloor} \cdot x\zeta_{\lfloor k,k\rfloor})_{\{k\}} = (b\zeta_{\lfloor k+1,k-1\rfloor})_{\{k\}} \cdot (x\zeta_{\lfloor k,k\rfloor})_{\{k\}} = b^- \cdot x\,.$$

For $i = k-1$, we have

$$(b\zeta_{\lfloor k+1,k-1\rfloor} \cdot x\zeta_{\lfloor k,k\rfloor})_{\{k-1\}} = \big((b\zeta_{\lfloor k+1,k-1\rfloor})_{\{k-1\}}\big)^{(x\zeta_{\lfloor k,k\rfloor})_{\{k\}}} = b^x\,.$$

For $i \in [0, k-2]$, we have

$$\begin{aligned}
&(b\zeta_{\lfloor k+1,k-1\rfloor} \cdot x\zeta_{\lfloor k,k\rfloor})_{\{i\}} \\
&= \Big(\big[(b\zeta_{\lfloor k+1,k-1\rfloor})_{\{k-1\}}\,, (x\zeta_{\lfloor k,k\rfloor})_{\{k\}}\big]^{(-1)^{k-1-i}}\Big)^{\left(\prod\limits_{j\in\lfloor k-2,i+1\rfloor}^{(b\zeta_{\lfloor k+1,k-1\rfloor})_{\{j\}}}\right)} \cdot (b\zeta_{\lfloor k+1,k-1\rfloor})_{\{i\}} \\
&= \big([b,x]^{(-1)^{k-1-i}}\big)^1 \cdot 1 \\
&= [x,b]^{(-1)^{k-i}}\,.
\end{aligned}$$

For $i = -1$, we have

$$\begin{aligned}
&(b\zeta_{\lfloor k+1,k-1\rfloor} \cdot x\zeta_{\lfloor k,k\rfloor})_{\{-1\}} \\
= \quad &\Big(\big\|(b\zeta_{\lfloor k+1,k-1\rfloor})_{\{k-1\}}\,, (x\zeta_{\lfloor k,k\rfloor})_{\{k\}}\big\|^{(-1)^k}\Big)^{\left(\prod\limits_{j\in\lfloor k-2,0\rfloor}^{(b\zeta_{\lfloor k+1,k-1\rfloor})_{\{j\}}}\right)} \cdot (b\zeta_{\lfloor k+1,k-1\rfloor})_{\{-1\}} \\
= \quad &\big(\|b,x\|^{(-1)^k}\big)^1 \cdot 1 \\
\overset{\text{(SCM1)}}{=} \quad &\|x,b\|^{(-1)^{k+1}}\,.
\end{aligned}$$

So

$$(b\zeta_{\lfloor k+1,k-1\rfloor} \cdot x\zeta_{\lfloor k,k\rfloor})_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+2} \\ b & \text{for } i = k+1 \\ b^- \cdot x & \text{for } i = k \\ b^x & \text{for } i = k-1 \\ [x,b]^{(-1)^{k-i}} & \text{for } i \in [0, k-2] \\ \|x,\, b\|^{(-1)^{k+1}} & \text{for } i = -1\,. \end{cases}$$

We calculate $x\zeta_{\lfloor k,k\rfloor} \cdot b\zeta_{\lfloor k+1,k-1\rfloor}$.

Note that

$$b\zeta_{\lfloor k+1,k-1\rfloor} = (\ldots, 1, \underbrace{b}_{\text{pos. } k+1}, \underbrace{b^-}_{\text{pos. } k}, \underbrace{b}_{\text{pos. } k-1}, 1, \ldots, \underbrace{1}_{\text{pos. } -1}) \overset{362.(1)}{=} b\zeta_{\lfloor k+1,k+1\rfloor} \cdot (b^-)\zeta_{\lfloor k,k\rfloor} \cdot b\zeta_{\lfloor k-1,k-1\rfloor}\,.$$

So $(x\zeta_{\lfloor k,k\rfloor})\cdot(b\zeta_{\lfloor k+1,k-1\rfloor}) = \big((x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor})\cdot(b^-)\zeta_{\lfloor k,k\rfloor}\big)\cdot b\zeta_{\lfloor k-1,k-1\rfloor}$.
We calculate $x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor}$ using Lemma 362.(2) with $t := k+1$.
For $i \geqslant k+2$, we have

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor})_{\{i\}} = (x\zeta_{\lfloor k,k\rfloor})_{\{i\}} = 1\,.$$

For $i = k+1$, we have

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor})_{\{k+1\}} = (x\zeta_{\lfloor k,k\rfloor})_{\{k+1\}}\cdot(b\zeta_{\lfloor k+1,k+1\rfloor})_{\{k+1\}} = 1\cdot b = b\,.$$

For $i = k$, we have

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor})_{\{k\}} = \big((x\zeta_{\lfloor k,k\rfloor})_{\{k\}}\big)^{(b\zeta_{\lfloor k+1,k+1\rfloor})_{\{k+1\}}} = x^b\,.$$

For $i \in [0,k-1]$, we have

$$\begin{aligned}
&(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor})_{\{i\}}\\
&= \Big(\big[(x\zeta_{\lfloor k,k\rfloor})_{\{k\}}\,,(b\zeta_{\lfloor k+1,k+1\rfloor})_{\{k+1\}}\big]^{(-1)^{k-i}}\Big)^{\left(\prod\limits_{j\in\lfloor k-1,i+1\rfloor}(x\zeta_{\lfloor k,k\rfloor})_{\{j\}}\right)}\cdot(x\zeta_{\lfloor k,k\rfloor})_{\{i\}}\\
&= \big([x,b]^{(-1)^{k-i}}\big)^1\cdot 1\\
&= [x,b]^{(-1)^{k-i}}\,.
\end{aligned}$$

For $i = -1$, we have

$$\begin{aligned}
&(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor})_{\{-1\}}\\
&= \Big(\big[\!\big[(x\zeta_{\lfloor k,k\rfloor})_{\{k\}}\,,(b\zeta_{\lfloor k+1,k+1\rfloor})_{\{k+1\}}\big]\!\big]^{(-1)^{k+1}}\Big)^{\left(\prod\limits_{j\in\lfloor k-1,0\rfloor}(x\zeta_{\lfloor k,k\rfloor})_{\{j\}}\right)}\cdot(x\zeta_{\lfloor k,k\rfloor})_{\{-1\}}\\
&= \big([\![x,b]\!]^{(-1)^{k+1}}\big)^1\cdot 1\\
&= [\![x,b]\!]^{(-1)^{k+1}}\,.
\end{aligned}$$

So

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor})_{\{i\}} = \begin{cases} 1 & \text{for } i\in\mathbb{Z}_{\geqslant k+2}\\ b & \text{for } i = k+1\\ x^b & \text{for } i = k\\ [x,b]^{(-1)^{k-i}} & \text{for } i\in[0,k-1]\\ [\![x,\,b]\!]^{(-1)^{k+1}} & \text{for } i = -1\,. \end{cases}$$

Let $\tilde{x} := x\zeta_{\lfloor k,k\rfloor} \cdot b\zeta_{\lfloor k+1,k+1\rfloor}$.

Note that, for $v \in [0, k-1]$ and $u \in [v-1, k-1]$, we have

$$\prod_{j\in\lfloor u,v\rfloor} \tilde{x}_{\{j\}} = \prod_{j\in\lfloor u,v\rfloor} [x,b]^{(-1)^{k-j}} = \begin{cases} [x,b]^{(-1)^{k-u}} & \text{if } u-v \equiv_2 0 \\ 1 & \text{if } u-v \equiv_2 1\,. \end{cases}$$

We calculate $\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k\rfloor}$ using Lemma 362.(2) with $t := k$.

For $i \geqslant k+1$, we have

$$(\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k\rfloor})_{\{i\}} = \tilde{x}_{\{i\}} = \begin{cases} 1 & \text{if } i \geqslant k+2 \\ b & \text{if } i = k+1\,. \end{cases}$$

For $i = k$, we have

$$(\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k\rfloor})_{\{k\}} = \tilde{x}_{\{k\}} \cdot \left((b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{k\}} = x^b \cdot b^- = b^- \cdot x\,.$$

For $i = k-1$, we have

$$(\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k\rfloor})_{\{k-1\}} = (\tilde{x}_{\{k-1\}})^{\left((b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{k\}}} = \left([x,b]^{(-1)^{k-(k-1)}}\right)^{b^-} = [b,x]^{b^-} = [x,b^-]\,.$$

For $i \in [0, k-2]$, we obtain the following.

Suppose that $k - i \equiv_2 1$. Then $(k-2)-(i+1) \equiv_2 0$, and we have

$$\begin{aligned} &\left(\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{i\}} \\ &= \left(\left[\tilde{x}_{\{k-1\}}\,, \left((b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{k\}}\right]^{(-1)^{k-1-i}}\right)^{\left(\prod_{j\in\lfloor k-2,i+1\rfloor} \tilde{x}_{\{j\}}\right)} \cdot \tilde{x}_{\{i\}} \\ &= \left(\left[[x,b]^{(-1)^{k-(k-1)}}, b^-\right]^{(-1)^0}\right)^{[x,b]^{(-1)^{k-(k-2)}}} \cdot [x,b]^{(-1)^{k-i}} \\ &= \left[[b,x], b^-\right]^{[x,b]} \cdot [b,x] \\ &= [b,x] \cdot \left([x,b] \cdot b \cdot [b,x] \cdot b^-\right) \cdot [x,b] \cdot [b,x] \\ &= b \cdot (b^- \cdot x^- \cdot b \cdot x) \cdot b^- \\ &= [x,b^-]\,. \end{aligned}$$

Suppose that $k-i \equiv_2 0$. Then $(k-2)-(i+1) \equiv_2 1$ and we have

$$\begin{aligned}
&\left(\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k \rfloor}\right)_{\{i\}} \\
&= \left(\left[\tilde{x}_{\{k-1\}}, \left((b^-)\zeta_{\lfloor k,k \rfloor}\right)_{\{k\}} \right]^{(-1)^{k-1-i}} \right)^{\left(\prod\limits_{j \in \lfloor k-2, i+1 \rfloor} \tilde{x}_{\{j\}}\right)} \cdot \tilde{x}_{\{i\}} \\
&= \left(\left[[x,b]^{(-1)^{k-(k-1)}}, b^- \right]^{(-1)^1} \right)^1 \cdot [x,b]^{(-1)^{k-i}} \\
&= \left[b^-, [b,x] \right] \cdot [x,b] \\
&= b \cdot [x,b] \cdot b^- \cdot [b,x] \cdot [x,b] \\
&= b \cdot (x^- \cdot b^- \cdot x \cdot b) \cdot b^- \\
&= [b^-, x] \,.
\end{aligned}$$

So, for $i \in [0, k-1]$, we have

$$\left(\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k \rfloor}\right)_{\{i\}} = [b^-, x]^{(-1)^{k-i}} \,.$$

For $i = -1$, we obtain the following.

Suppose that $k \equiv_2 0$. Then $(k-2)-0 \equiv_2 0$, and we have

$$\begin{aligned}
& \left(\tilde{x} \cdot (b^-)\zeta_{\lfloor k,k \rfloor}\right)_{\{-1\}} \\
= \quad & \left(\left\| \tilde{x}_{\{k-1\}}, \left((b^-)\zeta_{\lfloor k,k \rfloor}\right)_{\{k\}} \right\|^{(-1)^k} \right)^{\left(\prod\limits_{j \in \lfloor k-2, 0 \rfloor} \tilde{x}_{\{j\}}\right)} \cdot \tilde{x}_{\{-1\}} \\
= \quad & \left(\left\| [x,b]^{(-1)^{k-(k-1)}}, b^- \right\|^{(-1)^0} \right)^{[x,b]^{(-1)^{k-(k-2)}}} \cdot \left\| x, b \right\|^{(-1)^{k+1}} \\
\overset{\text{(SCM1)}}{=} \quad & \left(\left\| [b,x],\, b^- \right\| \right)^{[x,b]} \cdot \left\| b, x \right\|
\end{aligned}$$

$$\begin{aligned}
\overset{323.(3)}{=} \quad & \left\| x, b \right\|^- \cdot \left\| [b,x],\, b^- \right\| \cdot \left\| x, b \right\| \cdot \left\| b, x \right\| \\
\overset{\text{(SCM1)}}{=} \quad & \left\| x, b \right\|^- \cdot \left\| [b,x],\, b^- \right\| \\
\overset{323.(14)}{=} \quad & \left\| x, b \right\|^- \cdot \left\| x, b \right\| \cdot \left\| x, b^- \right\| \\
= \quad & \left\| x, b^- \right\| .
\end{aligned}$$

Suppose that $k \equiv_2 1$. Then $(k-2)-0 \equiv_2 1$, and we have

$$\begin{aligned}
& \left(\tilde{x}\cdot(b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{-1\}} \\
= \quad & \left(\left\llbracket \tilde{x}_{\{k-1\}}\,,\left((b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{k\}}\right\rrbracket^{(-1)^k}\right)^{\left(\prod\limits_{j\in\lfloor k-2,0\rfloor}\tilde{x}_{\{j\}}\right)}\cdot\tilde{x}_{\{-1\}} \\
= \quad & \left(\left\llbracket [x,b]^{(-1)^{k-(k-1)}},b^-\right\rrbracket^{(-1)^1}\right)^1\cdot\left\llbracket x,b\right\rrbracket^{(-1)^{k+1}} \\
\overset{\text{(SCM1)}}{=} \quad & \left\llbracket b^-,\,[b,x]\right\rrbracket\cdot\left\llbracket x,b\right\rrbracket \\
\overset{323.(14)}{=} \quad & \left\llbracket b^-,x\right\rrbracket\cdot\left\llbracket b,x\right\rrbracket\cdot\left\llbracket x,b\right\rrbracket \\
\overset{\text{(SCM1)}}{=} \quad & \left\llbracket b^-,x\right\rrbracket.
\end{aligned}$$

Therefore, for $i=-1$, we have, using (SCM1),

$$\left(\tilde{x}\cdot(b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{-1\}} = \left\llbracket b^-,x\right\rrbracket^{(-1)^{k+1}}.$$

So

$$\left(\tilde{x}\cdot(b^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{i\}} = \begin{cases} 1 & \text{for } i\in\mathbb{Z}_{\geqslant k+2} \\ b & \text{for } i=k+1 \\ b^-\cdot x & \text{for } i=k \\ [b^-,x]^{(-1)^{k-i}} & \text{for } i\in[0,k-1] \\ \left\llbracket b^-,\,x\right\rrbracket^{(-1)^{k+1}} & \text{for } i=-1\,. \end{cases}$$

Let $\tilde{\tilde{x}} := \tilde{x}\cdot(b^-)\zeta_{\lfloor k,k\rfloor} = x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k+1\rfloor}\cdot(b^-)\zeta_{\lfloor k,k\rfloor}$.

Note that, for $v\in[0,k-1]$ and $u\in[v-1,k-1]$, we have

$$\prod_{j\in\lfloor u,v\rfloor}\tilde{\tilde{x}}_{\{j\}} = \prod_{j\in\lfloor u,v\rfloor}[b^-,x]^{(-1)^{k-j}} = \begin{cases} [b^-,x]^{(-1)^{k-u}} & \text{if } u-v\equiv_2 0 \\ 1 & \text{if } u-v\equiv_2 1\,. \end{cases}$$

We calculate $x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k-1\rfloor} = \tilde{\tilde{x}}\cdot b\zeta_{\lfloor k-1,k-1\rfloor}$ using Lemma 362.(2) with $t:=k-1$.

For $i\geqslant k$, we have

$$(\tilde{\tilde{x}}\cdot b\zeta_{\lfloor k-1,k-1\rfloor})_{\{i\}} = \tilde{\tilde{x}}_{\{i\}} = \begin{cases} 1 & \text{if } i\geqslant k+2 \\ b & \text{if } i=k+1 \\ b^-\cdot x & \text{if } i=k\,. \end{cases}$$

For $i = k-1$, we have, both in case $k-1 \geqslant 1$ and in case $k-1 = 0$,

$$(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{k-1\}} = \tilde{\tilde{x}}_{\{k-1\}} \cdot (b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{k-1\}} = [x, b^-] \cdot b = b^x \,.$$

Let $i \in [0, k-2]$. Note that $k-1 \geqslant 1$.

For $i = k-2$, we have

$$(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{k-2\}} = (\tilde{\tilde{x}}_{\{k-2\}})^{(b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{k-1\}}} = [b^-, x]^b = [x, b] \,.$$

Suppose that $i \in [0, k-3]$.

Suppose that $k - i \equiv_2 0$. Then $(k-3) - (i+1) \equiv_2 0$, and we have

$$\begin{aligned}
(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{i\}} &= \Big(\big[\tilde{\tilde{x}}_{\{k-2\}} \,, (b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{k-1\}} \big]^{(-1)^{k-2-i}} \Big)^{\left(\prod\limits_{j \in \lfloor k-3, i+1 \rfloor} \tilde{\tilde{x}}_{\{j\}} \right)} \cdot \tilde{\tilde{x}}_{\{i\}} \\
&= \Big(\big[[b^-, x], b \big]^{(-1)^0} \Big)^{[b^-,x]^{(-1)^{k-(k-3)}}} \cdot [b^-, x]^{(-1)^{k-i}} \\
&= \big[[b^-, x], b \big]^{[x, b^-]} \cdot [b^-, x] \\
&= [b^-, x] \cdot \big([x, b^-] \cdot b^- \cdot [b^-, x] \cdot b \big) \cdot [x, b^-] \cdot [b^-, x] \\
&= b^- \cdot (b \cdot x^- \cdot b^- \cdot x) \cdot b \\
&= [x, b] \,.
\end{aligned}$$

Suppose that $k - i \equiv_2 1$. Then $(k-3) - (i+1) \equiv_2 1$, and we have

$$\begin{aligned}
(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{i\}} &= \Big(\big[\tilde{\tilde{x}}_{\{k-2\}} \,, (b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{k-1\}} \big]^{(-1)^{k-2-i}} \Big)^{\left(\prod\limits_{j \in \lfloor k-3, i+1 \rfloor} \tilde{\tilde{x}}_{\{j\}} \right)} \cdot \tilde{\tilde{x}}_{\{i\}} \\
&= \Big(\big[[b^-, x], b \big]^{(-1)^1} \Big)^1 \cdot [b^-, x]^{(-1)^{k-i}} \\
&= \big[b, [b^-, x] \big] \cdot [x, b^-] \\
&= \big(b^- \cdot [x, b^-] \cdot b \cdot [b^-, x] \big) \cdot [x, b^-] \\
&= b^- \cdot (x^- \cdot b \cdot x \cdot b^-) \cdot b \\
&= [b, x] \,.
\end{aligned}$$

Therefore, for $i \in [0, k-2]$, we have

$$(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1 \rfloor})_{\{i\}} = [x, b]^{(-1)^{k-i}}$$

Let $i = -1$.

Suppose that $k - 1 = 0$.

Then we obtain

$$\begin{aligned}(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1\rfloor})_{\{-1\}} &= (\tilde{\tilde{x}}_{\{-1\}})^{(b\zeta_{\lfloor k-1,k-1\rfloor})_{\{0\}}} = \left(\left\llbracket b^-,\, x\right\rrbracket^{(-1)^{k+1}}\right)^{(b\zeta_{\lfloor 0,0\rfloor})_{\{0\}}} \\ &= \left\llbracket b^-,\, x\right\rrbracket^b \overset{323.(10)}{=} \left\llbracket b^-,\, x^b\right\rrbracket \overset{322.(4)}{=} \left\llbracket x,\, b\right\rrbracket.\end{aligned}$$

Suppose that $k-1 \geqslant 1$.

Suppose that $k \equiv_2 1$. Then $(k-3)-0 \equiv_2 0$, and we have

$$\begin{aligned}(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1\rfloor})_{\{-1\}} &= \left(\left\llbracket \tilde{\tilde{x}}_{\{k-2\}},\, (b\zeta_{\lfloor k-1,k-1\rfloor})_{\{k-1\}}\right\rrbracket^{(-1)^{k-1}}\right)^{\left(\prod\limits_{j\in\lfloor k-3,0\rfloor} \tilde{\tilde{x}}_{\{j\}}\right)} \cdot \tilde{\tilde{x}}_{\{-1\}} \\ &= \left(\left\llbracket [b^-,x],\, b\right\rrbracket^{(-1)^0}\right)^{[b^-,x]^{(-1)^{k-(k-3)}}} \cdot \left\llbracket b^-,x\right\rrbracket^{(-1)^{k+1}} \\ &= \left\llbracket [b^-,x],\, b\right\rrbracket^{[x,b^-]} \cdot \left\llbracket b^-,x\right\rrbracket \\ &\overset{323.(3)}{=} \left\llbracket x,b^-\right\rrbracket^- \cdot \left\llbracket [b^-,x],\, b\right\rrbracket \cdot \left\llbracket x,b^-\right\rrbracket \cdot \left\llbracket b^-,x\right\rrbracket \\ &\overset{(SCM1)}{=} \left\llbracket x,b^-\right\rrbracket^- \cdot \left\llbracket [b^-,x],\, b\right\rrbracket \\ &\overset{323.(14)}{=} \left\llbracket x,b^-\right\rrbracket^- \cdot \left\llbracket x,b^-\right\rrbracket \cdot \left\llbracket x,b\right\rrbracket \\ &= \left\llbracket x,b\right\rrbracket.\end{aligned}$$

Suppose that $k \equiv_2 0$. Then $(k-3)-0 \equiv_2 1$, and we have

$$\begin{aligned}(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1\rfloor})_{\{-1\}} &= \left(\left\llbracket \tilde{\tilde{x}}_{\{k-2\}},\, (b\zeta_{\lfloor k-1,k-1\rfloor})_{\{k-1\}}\right\rrbracket^{(-1)^{k-1}}\right)^{\left(\prod\limits_{j\in\lfloor k-3,0\rfloor} \tilde{\tilde{x}}_{\{j\}}\right)} \cdot \tilde{\tilde{x}}_{\{-1\}} \\ &= \left(\left\llbracket [b^-,x],\, b\right\rrbracket^{(-1)^1}\right)^1 \cdot \left\llbracket b^-,x\right\rrbracket^{(-1)^{k+1}} \\ &\overset{(SCM1)}{=} \left\llbracket b,\, [b^-,x]\right\rrbracket \cdot \left\llbracket b^-,x\right\rrbracket^- \\ &\overset{323.(14)}{=} \left\llbracket b,x\right\rrbracket \cdot \left\llbracket b^-,x\right\rrbracket \cdot \left\llbracket b^-,x\right\rrbracket^- \\ &= \left\llbracket b,x\right\rrbracket.\end{aligned}$$

Therefore, for $i=-1$, we have

$$(\tilde{\tilde{x}} \cdot b\zeta_{\lfloor k-1,k-1\rfloor})_{\{-1\}} = \left\llbracket x,b\right\rrbracket^{(-1)^{k+1}}.$$

So

$$(x\zeta_{\lfloor k,k\rfloor}\cdot b\zeta_{\lfloor k+1,k-1\rfloor})_{\{i\}}=(\tilde{\tilde{x}}\cdot b\zeta_{\lfloor k-1,k-1\rfloor})_{\{i\}}=\begin{cases}1 & \text{for } i\in\mathbb{Z}_{\geqslant k+2}\\ b & \text{for } i=k+1\\ b^-\cdot x & \text{for } i=k\\ b^x & \text{for } i=k-1\\ [x,b]^{(-1)^{k-i}} & \text{for } i\in[0,k-2]\\ \llbracket x,\,b\rrbracket^{(-1)^{k+1}} & \text{for } i=-1\,.\end{cases}$$

Altogether, this shows $[b\zeta_{\lfloor k+1,k-1\rfloor}\,,x\zeta_{\lfloor k,k\rfloor}]=1$.

Ad (3). We calculate $b\zeta_{\lfloor k+1,k\rfloor}\cdot x\zeta_{\lfloor k,k-1\rfloor}$.

Note that

$$x\zeta_{\lfloor k,k-1\rfloor}=(\dots,1,\underbrace{x^-}_{\text{pos. }k},\underbrace{x}_{\text{pos. }k-1},1\dots,1)\overset{362.(1)}{=}(x^-)\zeta_{\lfloor k,k\rfloor}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\,.$$

We calculate $b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}$ using Lemma 362.(2) with $t:=k$.

For $i\geqslant k+1$, we have

$$\big(b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}\big)_{\{i\}}=(b\zeta_{\lfloor k+1,k\rfloor})_{\{i\}}=\begin{cases}1 & \text{if } i\geqslant k+2\\ b^- & \text{if } i=k+1\,.\end{cases}$$

For $i=k$, we have

$$\big(b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}\big)_{\{k\}}=(b\zeta_{\lfloor k+1,k\rfloor})_{\{k\}}\cdot\big((x^-)\zeta_{\lfloor k,k\rfloor}\big)_{\{k\}}=b\cdot x^-\,.$$

For $i=k-1$, we have

$$\big(b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}\big)_{\{k-1\}}=\big((b\zeta_{\lfloor k+1,k\rfloor})_{\{k-1\}}\big)^{((x^-)\zeta_{\lfloor k,k\rfloor})_{\{k\}}}=1^{x^-}=1\,.$$

For $i\in[0,k-2]$, we have

$$\begin{aligned}&\big(b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}\big)_{\{i\}}\\ =\;&\Big(\big[(b\zeta_{\lfloor k+1,k\rfloor})_{\{k-1\}}\,,\big((x^-)\zeta_{\lfloor k,k\rfloor}\big)_{\{k\}}\big]^{(-1)^{k-1-i}}\Big)^{\left(\prod_{j\in\lfloor k-2,i+1\rfloor}(b\zeta_{\lfloor k+1,k\rfloor})_{\{j\}}\right)}\cdot(b\zeta_{\lfloor k+1,k\rfloor})_{\{i\}}\\ =\;&\big([1,x^-]^{(-1)^{k-1-i}}\big)^1\cdot 1\\ =\;&1\,.\end{aligned}$$

For $i=-1$, we have

$$\begin{aligned}
&\left(b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{-1\}}\\
=\;&\left(\left[\!\left[(b\zeta_{\lfloor k+1,k\rfloor})_{\{k-1\}}\,,\left((x^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{k\}}\right]\!\right]^{(-1)^k}\right)^{\left(\prod\limits_{j\in\lfloor k-2,0\rfloor}(b\zeta_{\lfloor k+1,k\rfloor})_{\{j\}}\right)}\cdot(b\zeta_{\lfloor k+1,k\rfloor})_{\{-1\}}\\
=\;&\left(\left[\!\left[1,x^-\right]\!\right]^{(-1)^k}\right)^1\cdot 1\\
\overset{322.(1)}{=}\;&1\,.
\end{aligned}$$

So

$$\left(b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}\right)_{\{i\}}=\begin{cases}1 & \text{for } i\in\mathbb{Z}_{\geqslant k+2}\\ b^- & \text{for } i=k+1\\ b\cdot x^- & \text{for } i=k\\ 1 & \text{for } i\in[-1,k-1]\,.\end{cases}$$

Let $\tilde{b}:=b\zeta_{\lfloor k+1,k\rfloor}\cdot(x^-)\zeta_{\lfloor k,k\rfloor}$.

We calculate $b\zeta_{\lfloor k+1,k\rfloor}\cdot x\zeta_{\lfloor k,k-1\rfloor}=\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}$ using Lemma 362.(2) with $t:=k-1$.

Case $k\geqslant 2$. For $i\geqslant k$, we have

$$\left(\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{i\}}=\tilde{b}_{\{i\}}=\begin{cases}1 & \text{if } i\geqslant k+2\\ b^- & \text{if } i=k+1\\ b\cdot x^- & \text{if } i=k\,.\end{cases}$$

For $i=k-1$, we have

$$\left(\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{k-1\}}=\tilde{b}_{\{k-1\}}\cdot\left(x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{k-1\}}=1\cdot x=x\,.$$

For $i=k-2$, we have

$$\left(\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{k-2\}}=(\tilde{b}_{\{k-2\}})^{(x\zeta_{\lfloor k-1,k-1\rfloor})_{\{k-1\}}}=1^x=1\,.$$

For $i\in[0,k-3]$, we have

$$\begin{aligned}
\left(\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{i\}}&=\left(\left[\tilde{b}_{\{k-2\}}\,,\left(x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{k-1\}}\right]^{(-1)^{k-2-i}}\right)^{\left(\prod\limits_{j\in\lfloor k-3,i+1\rfloor}\tilde{b}_{\{j\}}\right)}\cdot\tilde{b}_{\{i\}}\\
&=\left([1,x]^{(-1)^{k-i}}\right)^1\cdot 1\\
&=1\,.
\end{aligned}$$

For $i=-1$, we have

$$\begin{aligned}\left(\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{-1\}} &= \left(\llbracket \tilde{b}_{\{k-2\}}\,,\left(x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{k-1\}}\rrbracket^{(-1)^{k-1}}\right)^{\left(\prod\limits_{j\in\lfloor k-3,0\rfloor}\tilde{b}_{\{j\}}\right)}\cdot\tilde{b}_{\{-1\}}\\ &= \left(\llbracket 1,x\rrbracket^{(-1)^{k-1}}\right)^1\cdot 1\\ &\overset{322.(1)}{=} 1\,.\end{aligned}$$

Case $k=1$. We have

$$\left(\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{i\}} = \left\{\begin{array}{ll}\tilde{b}_{\{i\}} & \text{for } i\in\mathbb{Z}_{\geqslant 1}\\ \tilde{b}_{\{0\}}\cdot c & \text{for } i=0\\ \tilde{b}^x_{\{-1\}} & \text{for } i=-1\end{array}\right\} = \begin{cases}1 & \text{for } i\in\mathbb{Z}_{\geqslant 3}\\ b^- & \text{for } i=2\\ b\cdot x^- & \text{for } i=1\\ x & \text{for } i=0\\ 1 & \text{for } i=-1\end{cases}.$$

So

$$(b\zeta_{\lfloor k+1,k\rfloor}\cdot x\zeta_{\lfloor k,k-1\rfloor})_{\{i\}} = \left(\tilde{b}\cdot x\zeta_{\lfloor k-1,k-1\rfloor}\right)_{\{i\}} = \begin{cases}1 & \text{for } i\in\mathbb{Z}_{\geqslant k+2}\\ b^- & \text{for } i=k+1\\ b\cdot x^- & \text{for } i=k\\ x & \text{for } i=k-1\\ 1 & \text{for } i\in[-1,k-2]\,.\end{cases}$$

We calculate $x\zeta_{\lfloor k,k-1\rfloor}\cdot b\zeta_{\lfloor k+1,k\rfloor}$.

Note that

$$b\zeta_{\lfloor k+1,k\rfloor} = (\ldots,\,1,\,\underbrace{b^-}_{\text{pos. } k+1}\,,\underbrace{b}_{\text{pos. } k},1\ldots,1) \overset{362.(1)}{=} (b^-)\zeta_{\lfloor k+1,k+1\rfloor}\cdot b\zeta_{\lfloor k,k\rfloor}\,.$$

We calculate $x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}$ using Lemma 362.(2) with $t:=k+1$.

For $i\geqslant k+2$, we have

$$\left(x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{i\}} = (x\zeta_{\lfloor k,k-1\rfloor})_{\{i\}} = 1\,.$$

For $i=k+1$, we have

$$\left(x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{k+1\}} = (x\zeta_{\lfloor k,k-1\rfloor})_{\{k+1\}}\cdot\left((b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{k+1\}} = 1\cdot b^- = b^-\,.$$

For $i = k$, we have

$$\left(x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{k\}} = (x\zeta_{\lfloor k,k-1\rfloor})_{\{k\}})^{((b^-)\zeta_{\lfloor k+1,k+1\rfloor})_{\{k+1\}}} = (x^-)^{b^-}\,.$$

For $i = k-1$, we have

$$\begin{aligned}
&\left(x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{k-1\}}\\
&=\left(\left[(x\zeta_{\lfloor k,k-1\rfloor})_{\{k\}}\,,\left((b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{k+1\}}\right]^{(-1)^{k-(k-1)}}\right)^{\left(\prod\limits_{j\in\lfloor k-1,k\rfloor}(x\zeta_{\lfloor k,k-1\rfloor})_{\{j\}}\right)}\cdot\left(x\zeta_{\lfloor k,k-1\rfloor}\right)_{\{k-1\}}\\
&=\left([x^-,b^-]^{(-1)^1}\right)^1\cdot x\\
&=[b^-,x^-]\cdot x\\
&=x^{b^-}\,.
\end{aligned}$$

For $i\in[0,k-2]$, we have

$$\begin{aligned}
&\left(x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{i\}}\\
&=\left(\left[(x\zeta_{\lfloor k,k-1\rfloor})_{\{k\}}\,,\left((b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{k+1\}}\right]^{(-1)^{k-i}}\right)^{\left(\prod\limits_{j\in\lfloor k-1,i+1\rfloor}(x\zeta_{\lfloor k,k-1\rfloor})_{\{j\}}\right)}\cdot\left(x\zeta_{\lfloor k,k-1\rfloor}\right)_{\{i\}}\\
&=\left([x^-,b^-]^{(-1)^{k-i}}\right)^x\cdot 1\\
&=\left(x^-\cdot(x\cdot b\cdot x^-\cdot b^-)\cdot x\right)^{(-1)^{k-i}}\\
&=[b^-,x]^{(-1)^{k-i}}\,.
\end{aligned}$$

For $i=-1$, we have

$$\begin{aligned}
&\left(x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{-1\}}\\
=\;&\left(\left\|(x\zeta_{\lfloor k,k-1\rfloor})_{\{k\}}\,,\left((b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{k+1\}}\right\|^{(-1)^{k+1}}\right)^{\left(\prod\limits_{j\in\lfloor k-1,0\rfloor}(x\zeta_{\lfloor k,k-1\rfloor})_{\{j\}}\right)}\cdot\left(x\zeta_{\lfloor k,k-1\rfloor}\right)_{\{-1\}}\\
=\;&\left(\left\|x^-,b^-\right\|^{(-1)^{k+1}}\right)^x\cdot 1\\
=\;&\left(\left\|x^-,b^-\right\|^x\right)^{(-1)^{k+1}}\\
\overset{323.(10)}{=}\;&\left\|x^-,(b^-)^x\right\|^{(-1)^{k+1}}\\
\overset{322.(4)}{=}\;&\left\|b^-,x\right\|^{(-1)^{k+1}}\,.
\end{aligned}$$

So

$$\left(x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}\right)_{\{i\}}=\begin{cases}1 & \text{for } i\in\mathbb{Z}_{\geqslant k+2}\\ b^- & \text{for } i=k+1\\ (x^-)^{b^-} & \text{for } i=k\\ x^{b^-} & \text{for } i=k-1\\ [b^-,x]^{(-1)^{k-i}} & \text{for } i\in[0,k-2]\\ \llbracket b^-,\,x\rrbracket^{(-1)^{k+1}} & \text{for } i=-1\,.\end{cases}$$

Let $\tilde{x}:=x\zeta_{\lfloor k,k-1\rfloor}\cdot(b^-)\zeta_{\lfloor k+1,k+1\rfloor}$.

We calculate $x\zeta_{\lfloor k,k-1\rfloor}\cdot b\zeta_{\lfloor k+1,k\rfloor}=\tilde{x}\cdot b\zeta_{\lfloor k,k\rfloor}$ using Lemma 362.(2) with $t:=k$.

Note that, for $v\in[0,k-2]$ and $u\in[v-1,k-2]$, we have

$$\prod_{j\in\lfloor u,v\rfloor}\tilde{x}_{\{j\}}=\prod_{j\in\lfloor u,v\rfloor}[b,x^-]^{(-1)^{k-j}}=\begin{cases}[b^-,x]^{(-1)^{k-u}} & \text{if } u-v\equiv_2 0\\ 1 & \text{if } u-v\equiv_2 1\,.\end{cases}$$

For $i\geqslant k+1$, we have

$$\left(\tilde{x}\cdot b\zeta_{\lfloor k,k\rfloor}\right)_{\{i\}}=\tilde{x}_{\{i\}}=\begin{cases}1 & \text{if } i\geqslant k+2\\ b^- & \text{if } i=k+1\,.\end{cases}$$

For $i=k$, we have

$$\left(\tilde{x}\cdot b\zeta_{\lfloor k,k\rfloor}\right)_{\{k\}}=\tilde{x}_{\{k\}}\cdot\left(b\zeta_{\lfloor k,k\rfloor}\right)_{\{k\}}=(x^-)^{b^-}\cdot b=b\cdot x^-\,.$$

For $i=k-1$, we have

$$\left(\tilde{x}\cdot b\zeta_{\lfloor k,k\rfloor}\right)_{\{k-1\}}=(\tilde{x}_{\{k-1\}})^{(b\zeta_{\lfloor k,k\rfloor})_{\{k\}}}=\left(x^{b^-}\right)^b=x\,.$$

Let $i \in [0, k-2]$. Suppose that $k - i \equiv_2 1$. Then $(k-2) - (i+1) \equiv_2 0$, and we have

$$\begin{aligned}
\left(\tilde{x} \cdot b\zeta_{\lfloor k,k \rfloor}\right)_{\{i\}} &= \left(\left[\tilde{x}_{\{k-1\}}\,, (b\zeta_{\lfloor k,k \rfloor})_{\{k\}}\right]^{(-1)^{k-1-i}}\right)^{\left(\prod\limits_{j \in \lfloor k-2, i+1 \rfloor} \tilde{x}_{\{j\}}\right)} \cdot \tilde{x}_{\{i\}} \\
&= \left(\left[x^{b^-}, b\right]^{(-1)^0}\right)^{[b^-,x]^{(-1)^{k-(k-2)}}} \cdot [b^-, x]^{(-1)^{k-i}} \\
&= \left[x^{b^-}, b\right]^{[b^-,x]} \cdot [x, b^-] \\
&= [x, b^-] \cdot \left[x^{b^-}, b\right] \\
&= x^- \cdot x^{b^-} \cdot \left((x^-)^{b^-} \cdot b^- \cdot x^{b^-} \cdot b\right) \\
&= 1\,.
\end{aligned}$$

Suppose that $k - i \equiv_2 0$. Then $(k-2) - (i+1) \equiv_2 1$, and we have

$$\begin{aligned}
\left(\tilde{x} \cdot b\zeta_{\lfloor k,k \rfloor}\right)_{\{i\}} &= \left(\left[\tilde{x}_{\{k-1\}}\,, (b\zeta_{\lfloor k,k \rfloor})_{\{k\}}\right]^{(-1)^{k-1-i}}\right)^{\left(\prod\limits_{j \in \lfloor k-2, i+1 \rfloor} \tilde{x}_{\{j\}}\right)} \cdot \tilde{x}_{\{i\}} \\
&= \left(\left[x^{b^-}, b\right]^{(-1)^1}\right)^1 \cdot [b^-, x]^{(-1)^{k-i}} \\
&= \left[b, x^{b^-}\right] \cdot [b^-, x] \\
&= \left(b^- \cdot (x^-)^{b^-} \cdot b \cdot x^{b^-}\right) \cdot (x^-)^{b^-} \cdot x \\
&= 1\,.
\end{aligned}$$

So, for $i \in [0, k-2]$, we have

$$\left(\tilde{x} \cdot b\zeta_{\lfloor k,k \rfloor}\right)_{\{i\}} = 1\,.$$

Let $i = -1$. Suppose that $k \equiv_2 0$. Then $(k-2) - 0 \equiv_2 0$, and we have

$$\begin{aligned}
\left(\tilde{x} \cdot b\zeta_{\lfloor k,k \rfloor}\right)_{\{-1\}} &= \left(\left\llbracket \tilde{x}_{\{k-1\}}\,, (b\zeta_{\lfloor k,k \rfloor})_{\{k\}} \right\rrbracket^{(-1)^k}\right)^{\left(\prod\limits_{j \in \lfloor k-2, 0 \rfloor} \tilde{x}_{\{j\}}\right)} \cdot \tilde{x}_{\{-1\}} \\
&= \left(\left\llbracket x^{b^-}, b \right\rrbracket^{(-1)^0}\right)^{[b^-,x]^{(-1)^{k-(k-2)}}} \cdot \left\llbracket b^-, x \right\rrbracket^{(-1)^{k+1}} \\
&= \left\llbracket x^{b^-},\, b \right\rrbracket^{[b^-,x]} \cdot \left\llbracket b^-, x \right\rrbracket^- \\
&\overset{323.(3)}{=} \left\llbracket b^-,\, x \right\rrbracket^- \cdot \left\llbracket x^{b^-},\, b \right\rrbracket \cdot \left\llbracket b^-,\, x \right\rrbracket \cdot \left\llbracket b^-, x \right\rrbracket^- \\
&\overset{322.(4)}{=} \left\llbracket b^-,\, x \right\rrbracket^- \cdot \left\llbracket b^-,\, x \right\rrbracket \\
&= 1\,.
\end{aligned}$$

Suppose that $k \equiv_2 1$. Then $(k-2)-0 \equiv_2 1$, and we have

$$\begin{aligned}
\big(\tilde{x} \cdot b\zeta_{\lfloor k,k \rfloor}\big)_{\{-1\}} &= \Big(\big\llbracket \tilde{x}_{\{k-1\}}\,, (b\zeta_{\lfloor k,k \rfloor})_{\{k\}} \big\rrbracket^{(-1)^k}\Big)^{\left(\prod\limits_{j \in \lfloor k-2,0 \rfloor} \tilde{x}_{\{j\}}\right)} \cdot \tilde{x}_{\{-1\}} \\
&= \Big(\big\llbracket x^{b^-}, b \big\rrbracket^{(-1)^1}\Big)^1 \cdot \big\llbracket b^-, x \big\rrbracket^{(-1)^{k+1}} \\
&\overset{\text{(SCM1)}}{=} \big\llbracket b,\, x^{b^-} \big\rrbracket \cdot \big\llbracket b^-, x \big\rrbracket \\
&\overset{322.(4)}{=} \big\llbracket x,\, b^- \big\rrbracket \cdot \big\llbracket b^-,\, x \big\rrbracket \\
&\overset{\text{(SCM1)}}{=} 1\,.
\end{aligned}$$

So, for $i = -1$, we have

$$\big(\tilde{x} \cdot b\zeta_{\lfloor k,k \rfloor}\big)_{\{-1\}} = 1\,.$$

So

$$(x\zeta_{\lfloor k,k-1 \rfloor} \cdot b\zeta_{\lfloor k+1,k \rfloor})_{\{i\}} = \big(\tilde{x} \cdot b\zeta_{\lfloor k,k \rfloor}\big)_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant k+2} \\ b^- & \text{for } i = k+1 \\ b \cdot x^- & \text{for } i = k \\ x & \text{for } i = k-1 \\ 1 & \text{for } i \in [-1, k-2]\,. \end{cases}$$

Altogether, this shows $[b\zeta_{\lfloor k+1,k \rfloor}\,, x\zeta_{\lfloor k,k-1 \rfloor}] = 1\,.$ □

Example 370 We give examples for some commuting elements; cf. Lemma 369.
Suppose given b, $x \in B$.

(1) Let $k = 2$ and $\ell = 0$. We consider the elements

$$\begin{aligned}
b\zeta_{\lfloor 2,2 \rfloor} &= (\ldots, 1, b, 1, 1, 1) = (\ldots, 1, \underbrace{b}_{\text{pos. } 2}, 1, 1, \underbrace{1}_{\text{pos. } -1}) \\
x\zeta_{\lfloor 0,0 \rfloor} &= (\ldots, 1, 1, 1, x, 1) = (\ldots, 1, 1, 1, \underbrace{x}_{\text{pos. } 0}, \underbrace{1}_{\text{pos. } -1}) \in H_\infty\,.
\end{aligned}$$

By Lemma 369.(1), since $|k-\ell| = 2 \geqslant 2$, we have $\big[b\zeta_{\lfloor 2,2 \rfloor}\,, x\zeta_{\lfloor 0,0 \rfloor}\big] = 1\,.$
Using Lemma 362.(2) directly, we obtain

$$b\zeta_{\lfloor 2,2 \rfloor} \cdot x\zeta_{\lfloor 0,0 \rfloor} = (\ldots, 1, b, 1, 1, 1) \cdot (\ldots, 1, 1, 1, x, 1) \overset{362.(2)}{=} (\ldots, 1, b, 1, x, 1)\,.$$

and

$$\begin{array}{rcl} x\zeta_{\lfloor 0,0\rfloor}\cdot b\zeta_{\lfloor 2,2\rfloor} & = & (\dots,1,1,1,x,1)\cdot(\dots,1,b,1,1,1) \\ & \overset{362.(2)}{=} & \Big(\dots,1,1\cdot b,1^b,\big([1,b]^{(-1)}\big)^1\cdot x,\big(\llbracket 1,\ b\rrbracket^{(-1)^2}\big)^x\cdot 1\Big) \\ & = & \big(\dots,1,b,1,x,\llbracket 1,\ b\rrbracket^x\big) \\ & \overset{322.(1)}{=} & (\dots,1,b,1,x,1)\,. \end{array}$$

This comfirms that $b\zeta_{\lfloor 2,2\rfloor}\cdot x\zeta_{\lfloor 0,0\rfloor} = x\zeta_{\lfloor 0,0\rfloor}\cdot b\zeta_{\lfloor 2,2\rfloor}$.

(2) Let $k=1$. We consider the elements

$$\begin{array}{rcccl} b\zeta_{\lfloor 2,0\rfloor} & = & (\dots,1,b,b^-,b,1) & = & (\dots,1,\underbrace{b}_{\text{pos. 2}},\underbrace{b^-}_{\text{pos. 1}},\underbrace{b}_{\text{pos. 0}},1) \\ x\zeta_{\lfloor 1,1\rfloor} & = & (\dots,1,1,x,1,1) & = & (\dots,\ 1,\ 1,\underbrace{x}_{\text{pos. 1}},1,1) \qquad\qquad \in H_\infty\,. \end{array}$$

By Lemma 369.(2), we have $\left[b\zeta_{\lfloor 2,0\rfloor}\,,x\zeta_{\lfloor 1,1\rfloor}\right]=1$.

In particular, we have

$$\begin{array}{rcl} x\zeta_{\lfloor 1,1\rfloor}\cdot b\zeta_{\lfloor 2,0\rfloor} & = & (\dots,1,1,x,1,1)\cdot(\dots,1,b,b^-,b,1) \\ & = & b\zeta_{\lfloor 2,0\rfloor}\cdot x\zeta_{\lfloor 1,1\rfloor} \\ & = & (\dots,1,b,b^-,b,1)\cdot(\dots,1,1,x,1,1) \\ & \overset{362.(2)}{=} & (\dots,1,b,b^-\cdot x,b^x,1)\,. \end{array}$$

(3) Let $k=1$. We consider the elements

$$\begin{array}{rcccl} b\zeta_{\lfloor 2,1\rfloor} & = & (\dots,1,b^-,b,1,1) & = & (\dots,1\,,\underbrace{b^-}_{\text{pos. 2}},\underbrace{b}_{\text{pos. 1}},1,1) \\ x\zeta_{\lfloor 1,0\rfloor} & = & (\dots,1,1,x^-,x,1) & = & (\dots,\ 1,\ 1,\underbrace{x^-}_{\text{pos. 1}},\underbrace{x}_{\text{pos. 0}},1) \quad \in H_\infty\,. \end{array}$$

By Lemma 369.(3), we have $\left[b\zeta_{\lfloor 2,1\rfloor}\,,x\zeta_{\lfloor 1,0\rfloor}\right]=1$.

In particular, we have

$$\begin{array}{rcl} x\zeta_{\lfloor 1,0\rfloor}\cdot b\zeta_{\lfloor 2,1\rfloor} & = & (\dots,1,1,x^-,x,1)\cdot(\dots,1,b^-,b,1,1) \\ & = & b\zeta_{\lfloor 2,1\rfloor}\cdot x\zeta_{\lfloor 1,0\rfloor} \\ & = & (\dots,1,b^-,b,1,1)\cdot(\dots,1,1,x^-,x,1) \\ & \overset{362.(2)}{=} & (\dots,1,b^-,b\cdot x^-,x,1)\,. \end{array}$$

Remark 371 *Suppose given b, $x \in B$.*
For ℓ, $k \in \mathbb{Z}_{\geqslant 0}$ with $|\ell - k| \geqslant 2$, we have

$$[b\mathrm{s}_\ell^\infty, x\mathrm{s}_\ell^\infty] = 1.$$

Proof. This is a reformulation of Lemma 369.(1), using Lemma 359.(1). □

17.2.6 Certain commuting elements in H_∞, involving position -1

Lemma 372 *Suppose given $b \in B$ and m, $\tilde{m} \in M$.*

(1) *For $k \in \mathbb{Z}_{\geqslant 1}$, we have* $[b\zeta_{\lfloor k,k \rfloor}, m\zeta_{\lfloor -1,-1 \rfloor}] = 1$.

(2) *We have* $[b\zeta_{\lfloor 0,0 \rfloor}, m\zeta_{\lfloor 1,-1 \rfloor}] = 1$.

(3) *We have* $[b\zeta_{\lfloor 1,0 \rfloor}, m\zeta_{\lfloor 0,-1 \rfloor}] = 1$.

(4) *We have* $[m\zeta_{\lfloor 0,-1 \rfloor}, \tilde{m}\zeta_{\lfloor -1,-1 \rfloor}] = 1$.

Cf. Definition 367.(1, 2).

Proof. Ad (1). Suppose given $k \in \mathbb{Z}_{\geqslant 1}$.
We calculate $b\zeta_{\lfloor k,k \rfloor} \cdot m\zeta_{\lfloor -1,-1 \rfloor}$ using Lemma 362.(2) with $t := -1$.
For $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$(b\zeta_{\lfloor k,k \rfloor} \cdot m\zeta_{\lfloor -1,-1 \rfloor})_{\{i\}} = (b\zeta_{\lfloor k,k \rfloor})_{\{i\}} = \begin{cases} 1 & \text{if } i \in \mathbb{Z}_{\geqslant 0} \backslash \{k\} \\ b & \text{if } i = k. \end{cases}$$

For $i = -1$, we have

$$(b\zeta_{\lfloor k,k \rfloor} \cdot m\zeta_{\lfloor -1,-1 \rfloor})_{\{-1\}} = (b\zeta_{\lfloor k,k \rfloor})_{\{-1\}} \cdot (m\zeta_{\lfloor -1,-1 \rfloor})_{\{-1\}} = 1 \cdot m = m.$$

So

$$(b\zeta_{\lfloor k,k \rfloor} \cdot m\zeta_{\lfloor -1,-1 \rfloor})_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 0} \backslash \{k\} \\ b & \text{for } i = k \\ m & \text{for } i = -1. \end{cases}$$

We calculate $m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor k,k\rfloor}$ using Lemma 362.(2) with $t:=k$.
For $i\geqslant k+1$, we have

$$(m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor k,k\rfloor})_{\{i\}}=(m\zeta_{\lfloor -1,-1\rfloor})_{\{i\}}=1\,.$$

For $i=k$, we have

$$(m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor k,k\rfloor})_{\{k\}}=(m\zeta_{\lfloor -1,-1\rfloor})_{\{k\}}\cdot(b\zeta_{\lfloor k,k\rfloor})_{\{k\}}=1\cdot b=b\,.$$

For $i\in[0,k-1]$, we have

$$\begin{aligned}&(m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor k,k\rfloor})_{\{i\}}\\ &=\Big(\big[(m\zeta_{\lfloor -1,-1\rfloor})_{\{k-1\}}\,,(b\zeta_{\lfloor k,k\rfloor})_{\{k\}}\big]^{(-1)^{k-1-i}}\Big)^{\left(\prod\limits_{j\in\lfloor k-2,i+1\rfloor}(m\zeta_{\lfloor -1,-1\rfloor})_{\{j\}}\right)}\cdot(m\zeta_{\lfloor -1,-1\rfloor})_{\{i\}}\\ &=\big([1,b]^{(-1)^{k-1-i}}\big)^1\cdot 1\\ &=1\,.\end{aligned}$$

For $i=-1$, we have

$$\begin{aligned}&(m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor k,k\rfloor})_{\{-1\}}\\ =\;&\Big(\big[\!\big[(m\zeta_{\lfloor -1,-1\rfloor})_{\{k-1\}},\,(b\zeta_{\lfloor k,k\rfloor})_{\{k\}}\big]\!\big]^{(-1)^k}\Big)^{\left(\prod\limits_{j\in\lfloor k-2,0\rfloor}(m\zeta_{\lfloor -1,-1\rfloor})_{\{j\}}\right)}\cdot(m\zeta_{\lfloor -1,-1\rfloor})_{\{-1\}}\\ =\;&\big([\![1,\,b]\!]^{(-1)^k}\big)^1\cdot m\\ \overset{322.(1)}{=}\;&m\,.\end{aligned}$$

So

$$(m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor k,k\rfloor})_{\{i\}}=\begin{cases}1 & \text{for } i\in\mathbb{Z}_{\geqslant 0}\backslash\{k\}\\ b & \text{for } i=k\\ m & \text{for } i=-1\,.\end{cases}$$

Altogether, this shows $[b\zeta_{\lfloor k,k\rfloor}\,,m\zeta_{\lfloor -1,-1\rfloor}]=1$ for $k\in\mathbb{Z}_{\geqslant 1}$.
Ad (2). We calculate $b\zeta_{\lfloor 0,0\rfloor}\cdot m\zeta_{\lfloor 1,-1\rfloor}$.
Note that we have

$$m\zeta_{\lfloor 1,-1\rfloor}\overset{367.(2)}{=}\big(\ldots,1,\underbrace{mf}_{\text{pos. }1},\underbrace{(mf)^-}_{\text{pos. }0},\underbrace{m}_{\text{pos. }-1}\big)\overset{362.(1)}{=}(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\cdot m\zeta_{\lfloor -1,-1\rfloor}\,.$$

We calculate $b\zeta_{\lfloor 0,0\rfloor}\cdot(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}$ using Lemma 362.(2) with $t:=1$.
For $i\geqslant 2$, we have

$$\big(b\zeta_{\lfloor 0,0\rfloor}\cdot(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{i\}}=(b\zeta_{\lfloor 0,0\rfloor})_{\{i\}}=1\,.$$

For $i=1$, we have

$$\big(b\zeta_{\lfloor 0,0\rfloor}\cdot(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{1\}}=(b\zeta_{\lfloor 0,0\rfloor})_{\{1\}}\cdot\big((m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{1\}}=1\cdot m\mathrm{f}_S=m\mathrm{f}_S\,.$$

For $i=0$, we have

$$\big(b\zeta_{\lfloor 0,0\rfloor}\cdot(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{0\}}=\big((b\zeta_{\lfloor 0,0\rfloor})_{\{0\}}\big)^{((m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor})_{\{1\}}}=b^{m\mathrm{f}_S}\,.$$

For $i=-1$, we have

$$\begin{array}{rl}
& \big(b\zeta_{\lfloor 0,0\rfloor}\cdot(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{-1\}}\\
= & \Big(\big\llbracket(b\zeta_{\lfloor 0,0\rfloor})_{\{j\}},\,\big((m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{1\}}\big\rrbracket^{(-1)^1}\Big)^{\left(\prod\limits_{j\in\lfloor -1,0\rfloor}(b\zeta_{\lfloor 0,0\rfloor})_{\{0\}}\right)}\cdot\big((m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{-1\}}\\
= & \big(\big\llbracket b,\,m\mathrm{f}_S\big\rrbracket^-\big)^1\cdot 1\\
\overset{\text{(SCM1)}}{=} & \big\llbracket m\mathrm{f}_S,\,b\big\rrbracket\,.
\end{array}$$

So

$$\big(b\zeta_{\lfloor 0,0\rfloor}\cdot(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}\big)_{\{i\}}=\begin{cases}1 & \text{for } i\geqslant 2\\ m\mathrm{f}_S & \text{for } i=1\\ b^{m\mathrm{f}_S} & \text{for } i=0\\ \big\llbracket m\mathrm{f}_S,\,b\big\rrbracket & \text{for } i=-1\,.\end{cases}$$

Let $\tilde{b}:=b\zeta_{\lfloor 0,0\rfloor}\cdot(m\mathrm{f}_S)\zeta_{\lfloor 1,1\rfloor}$.
We calculate $\tilde{b}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}$ using Lemma 362.(2) with $t:=0$.
For $i\geqslant 1$, we have

$$\Big(\tilde{b}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{i\}}=\tilde{b}_{\{i\}}=\begin{cases}1 & \text{if } i\geqslant 2\\ m\mathrm{f}_S & \text{if } i=1\,.\end{cases}$$

For $i=0$, we have

$$\Big(\tilde{b}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{0\}}=\tilde{b}_{\{0\}}\cdot\Big(\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{0\}}=b^{m\mathrm{f}_S}\cdot(m\mathrm{f}_S)^-=(m\mathrm{f}_S)^-\cdot b\,.$$

For $i = -1$, we have

$$\begin{aligned}\Big(\tilde{b}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{-1\}} &= (\tilde{b}_{\{-1\}})^{\left(((m\mathrm{f}_S)^-)\zeta_{\lfloor 0,0\rfloor}\right)_{\{0\}}}\\ &= \big\llbracket m\mathrm{f}_S,\, b\big\rrbracket^{(m\mathrm{f}_S)^-}\\ &\overset{323.(10)}{=} \big\llbracket m\mathrm{f}_S,\, b^{(m\mathrm{f}_S)^-}\big\rrbracket\\ &\overset{322.(4)}{=} \big\llbracket b,\, (m\mathrm{f}_S)^-\big\rrbracket\,.\end{aligned}$$

So

$$\Big(\tilde{b}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{i\}} = \begin{cases}1 & \text{for } i\geqslant 2\\ m\mathrm{f}_S & \text{for } i=1\\ (m\mathrm{f}_S)^-\cdot b & \text{for } i=0\\ \big\llbracket b,\,(m\mathrm{f}_S)^-\big\rrbracket & \text{for } i=-1\,.\end{cases}$$

Let $\tilde{\tilde{b}} := \tilde{b}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}$.

We calculate $\tilde{\tilde{b}}\cdot m\zeta_{\lfloor -1,-1\rfloor} = b\zeta_{\lfloor 0,0\rfloor}\cdot m\zeta_{\lfloor 1,-1\rfloor}$ using Lemma 362.(2) with $t := -1$.

For $i \geqslant 0$, we have

$$\big(\tilde{\tilde{b}}\cdot m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{i\}} = \tilde{b}_{\{i\}} = \begin{cases}1 & \text{if } i\geqslant 2\\ m\mathrm{f}_S & \text{if } i=1\\ (m\mathrm{f}_S)^-\cdot b & \text{if } i=0\,.\end{cases}$$

For $i = -1$, we have

$$\big(\tilde{\tilde{b}}\cdot m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{-1\}} = \tilde{\tilde{b}}_{\{-1\}}\cdot(m\zeta_{\lfloor -1,-1\rfloor})_{\{-1\}} = \big\llbracket b,\,(m\mathrm{f}_S)^-\big\rrbracket\cdot m \overset{323.(8)}{=} m^b\,.$$

So

$$(b\zeta_{\lfloor 0,0\rfloor}\cdot m\zeta_{\lfloor 1,-1\rfloor})_{\{i\}} = \big(\tilde{\tilde{b}}\cdot m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{i\}} = \begin{cases}1 & \text{for } i\geqslant 2\\ m\mathrm{f}_S & \text{for } i=1\\ (m\mathrm{f}_S)^-\cdot b & \text{for } i=0\\ m^b & \text{for } i=-1\,.\end{cases}$$

We calculate $m\zeta_{\lfloor 1,-1\rfloor}\cdot b\zeta_{\lfloor 0,0\rfloor}$ using Lemma 362.(2) with $t := 0$.

For $i \geqslant 1$, we have

$$(m\zeta_{\lfloor 1,-1\rfloor} \cdot b\zeta_{\lfloor 0,0\rfloor})_{\{i\}} = (m\zeta_{\lfloor 1,-1\rfloor})_{\{i\}} = \begin{cases} 1 & \text{if } i \geqslant 2 \\ m\mathrm{f}_S & \text{if } i = 1\,. \end{cases}$$

For $i = 0$, we have

$$(m\zeta_{\lfloor 1,-1\rfloor} \cdot b\zeta_{\lfloor 0,0\rfloor})_{\{0\}} = (m\zeta_{\lfloor 1,-1\rfloor})_{\{0\}} \cdot (b\zeta_{\lfloor 0,0\rfloor})_{\{0\}} = (m\mathrm{f}_S)^- \cdot b\,.$$

For $i = -1$, we have

$$(m\zeta_{\lfloor 1,-1\rfloor} \cdot b\zeta_{\lfloor 0,0\rfloor})_{\{-1\}} = \big((m\zeta_{\lfloor 1,-1\rfloor})_{\{-1\}}\big)^{(b\zeta_{\lfloor 0,0\rfloor})_{\{0\}}} = m^b\,.$$

So

$$(m\zeta_{\lfloor 1,-1\rfloor} \cdot b\zeta_{\lfloor 0,0\rfloor})_{\{i\}} = \begin{cases} 1 & \text{for } i \geqslant 2 \\ m\mathrm{f}_S & \text{for } i = 1 \\ (m\mathrm{f}_S)^- \cdot b & \text{for } i = 0 \\ m^b & \text{for } i = -1\,. \end{cases}$$

Altogether, this shows $[b\zeta_{\lfloor 0,0\rfloor}, m\zeta_{\lfloor 1,-1\rfloor}] = 1\,.$

Ad (3). We calculate $b\zeta_{\lfloor 1,0\rfloor} \cdot m\zeta_{\lfloor 0,-1\rfloor}\,.$

Note that

$$m\zeta_{\lfloor 0,-1\rfloor} \overset{367.(2)}{=} (\ldots, 1, \underbrace{(m\mathrm{f}_S)^-}_{\text{pos. } 0}, \underbrace{m}_{\text{pos. } -1}) \overset{362.(1)}{=} \big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor} \cdot m\zeta_{\lfloor -1,-1\rfloor}\,.$$

We calculate $b\zeta_{\lfloor 1,0\rfloor} \cdot \big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}$ using Lemma 362.(2) with $t := 0$.

For $i \geqslant 1$, we have

$$\big(b\zeta_{\lfloor 1,0\rfloor} \cdot \big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\big)_{\{i\}} = (b\zeta_{\lfloor 1,0\rfloor})_{\{i\}} = \begin{cases} 1 & \text{if } i \geqslant 2 \\ b^- & \text{if } i = 1\,. \end{cases}$$

For $i = 0$, we have

$$\Big(b\zeta_{\lfloor 1,0\rfloor} \cdot \big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{0\}} = (b\zeta_{\lfloor 1,0\rfloor})_{\{0\}} \cdot \Big(\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{0\}} = b \cdot (m\mathrm{f}_S)^-\,.$$

For $i = -1$, we have

$$\big(b\zeta_{\lfloor 1,0\rfloor} \cdot \big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\big)_{\{-1\}} = \big((b\zeta_{\lfloor 1,0\rfloor})_{\{-1\}}\big)^{\left(((m\mathrm{f}_S)^-)\zeta_{\lfloor 0,0\rfloor}\right)_{\{0\}}} = 1^{(m\mathrm{f}_S)^-} = 1\,.$$

So

$$\big(b\zeta_{\lfloor 1,0\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\big)_{\{i\}}=\begin{cases}1 & \text{for } i\geqslant 2\\ b^- & \text{for } i=1\\ b\cdot(m\mathrm{f}_S)^- & \text{for } i=0\\ 1 & \text{for } i=-1\,.\end{cases}$$

Let $\tilde{b}:=b\zeta_{\lfloor 1,0\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}$.

We calculate $\tilde{b}\cdot m\zeta_{\lfloor -1,-1\rfloor}=b\zeta_{\lfloor 1,0\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor}$ using Lemma 362.(2) with $t:=-1$.

For $i\geqslant 0$, we have

$$\big(\tilde{b}\cdot m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{i\}}=\tilde{b}_{\{i\}}=\begin{cases}1 & \text{if } i\geqslant 2\\ b^- & \text{if } i=1\\ b\cdot(m\mathrm{f}_S)^- & \text{if } i=0\,.\end{cases}$$

For $i=-1$, we have

$$\big(\tilde{b}\cdot m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{-1\}}=\tilde{b}_{\{-1\}}\cdot\big(m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{-1\}}=1\cdot m=m\,.$$

So

$$(b\zeta_{\lfloor 1,0\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor})_{\{i\}}=\big(\tilde{b}\cdot m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{i\}}=\begin{cases}1 & \text{for } i\geqslant 2\\ b^- & \text{for } i=1\\ b\cdot(m\mathrm{f}_S)^- & \text{for } i=0\\ m & \text{for } i=-1\,.\end{cases}$$

We calculate $m\zeta_{\lfloor 0,-1\rfloor}\cdot b\zeta_{\lfloor 1,0\rfloor}$.

Note that

$$b\zeta_{\lfloor 1,0\rfloor}\overset{367.(1)}{=}(\ldots,1,\underbrace{b^-}_{\text{pos. }1},\underbrace{b}_{\text{pos. }0},1)\overset{362.(1)}{=}(b^-)\zeta_{\lfloor 1,1\rfloor}\cdot b\zeta_{\lfloor 0,0\rfloor}\,.$$

We calculate $m\zeta_{\lfloor 0,-1\rfloor}\cdot(b^-)\zeta_{\lfloor 1,1\rfloor}$ using Remark 362.(2) with $t:=1$.

For $i\geqslant 2$, we have

$$(m\zeta_{\lfloor 0,-1\rfloor}\cdot(b^-)\zeta_{\lfloor 1,1\rfloor})_{\{i\}}=(m\zeta_{\lfloor 0,-1\rfloor})_{\{i\}}=1\,.$$

For $i=1$, we have

$$\big(m\zeta_{\lfloor 0,-1\rfloor}\cdot(b^-)\zeta_{\lfloor 1,1\rfloor}\big)_{\{1\}}=(m\zeta_{\lfloor 0,-1\rfloor})_{\{1\}}\cdot\big((b^-)\zeta_{\lfloor 1,1\rfloor}\big)_{\{1\}}=1\cdot b^-=b^-\,.$$

For $i = 0$, we have

$$\big(m\zeta_{\lfloor 0,-1\rfloor} \cdot (b^-)\zeta_{\lfloor 1,1\rfloor}\big)_{\{0\}} = \big((m\zeta_{\lfloor 0,-1\rfloor})_{\{0\}}\big)^{((b^-)\zeta_{\lfloor 1,1\rfloor})_{\{1\}}} = \big((m\mathrm{f}_S)^-\big)^{b^-}.$$

For $i = -1$, we have

$$\begin{array}{rl} & \big(m\zeta_{\lfloor 0,-1\rfloor} \cdot (b^-)\zeta_{\lfloor 1,1\rfloor}\big)_{\{-1\}} \\ = & \Big(\big\llbracket (m\zeta_{\lfloor 0,-1\rfloor})_{\{0\}},\, ((b^-)\zeta_{\lfloor 1,1\rfloor})_{\{1\}} \big\rrbracket^{(-1)^1}\Big)^{\left(\prod_{j\in\lfloor -1,0\rfloor} (m\zeta_{\lfloor 0,-1\rfloor})_{\{j\}}\right)} \cdot (m\zeta_{\lfloor 0,-1\rfloor})_{\{-1\}} \\ = & \big(\big\llbracket (m\mathrm{f}_S)^-,\, b^- \big\rrbracket^-\big)^1 \cdot m \\ \overset{\text{(SCM1)}}{=} & \big\llbracket b^-,\, (m\mathrm{f}_S)^- \big\rrbracket \cdot m \\ \overset{323.(8)}{=} & m^{b^-}. \end{array}$$

So

$$\big(m\zeta_{\lfloor 0,-1\rfloor} \cdot (b^-)\zeta_{\lfloor 1,1\rfloor}\big)_{\{i\}} = \begin{cases} 1 & \text{for } i \geqslant 2 \\ b^- & \text{for } i = 1 \\ \big((m\mathrm{f}_S)^-\big)^{b^-} & \text{for } i = 0 \\ m^{b^-} & \text{for } i = -1\,. \end{cases}$$

Let $n := m\zeta_{\lfloor 0,-1\rfloor} \cdot (b^-)\zeta_{\lfloor 1,1\rfloor}$.

We calculate $n \cdot b\zeta_{\lfloor 0,0\rfloor} = m\zeta_{\lfloor 0,-1\rfloor} \cdot b\zeta_{\lfloor 1,0\rfloor}$ using Lemma 362.(2) with $t := 0$.

For $i \geqslant 1$, we have

$$\big(n \cdot b\zeta_{\lfloor 0,0\rfloor}\big)_{\{i\}} = n_{\{i\}} = \begin{cases} 1 & \text{if } i \geqslant 2 \\ b^- & \text{if } i = 1\,. \end{cases}$$

For $i = 0$, we have

$$\big(n \cdot b\zeta_{\lfloor 0,0\rfloor}\big)_{\{0\}} = n_{\{0\}} \cdot (b\zeta_{\lfloor 0,0\rfloor})_{\{0\}} = \big((m\mathrm{f}_S)^-\big)^{b^-} \cdot b = b \cdot (m\mathrm{f}_S)^-.$$

For $i = -1$, we have

$$(n \cdot b\zeta_{\lfloor 0,0\rfloor})_{\{-1\}} = (n_{\{-1\}})^{(b\zeta_{\lfloor 0,0\rfloor})_{\{0\}}} = (m^{b^-})^b = m\,.$$

So

$$(m\zeta_{\lfloor 0,-1\rfloor}\cdot b\zeta_{\lfloor 1,0\rfloor})_{\{i\}} = (n\cdot b\zeta_{\lfloor 0,0\rfloor})_{\{i\}} = \begin{cases} 1 & \text{for } i\geqslant 2 \\ b^- & \text{for } i=1 \\ b\cdot(m\mathrm{f}_S)^- & \text{for } i=0 \\ m & \text{for } i=-1\,. \end{cases}$$

Altogether, this shows $[b\zeta_{\lfloor 1,0\rfloor}\,, m\zeta_{\lfloor 0,-1\rfloor}] = 1\,.$

Ad (4). We calculate $m\zeta_{\lfloor 0,-1\rfloor}\cdot\tilde{m}\zeta_{\lfloor -1,-1\rfloor}$ using Lemma 362.(2) with $t := -1$.

For $i\in\mathbb{Z}_{\geqslant 0}$, we have

$$\big(m\zeta_{\lfloor 0,-1\rfloor}\cdot\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\big)_{\{i\}} = (m\zeta_{\lfloor 0,-1\rfloor})_{\{i\}} = \begin{cases} 1 & \text{if } \ i\geqslant 1 \\ (m\mathrm{f}_S)^- & \text{if } \ i=0\,. \end{cases}$$

For $i=-1$, we have

$$\big(m\zeta_{\lfloor 0,-1\rfloor}\cdot\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\big)_{\{-1\}} = (m\zeta_{\lfloor 0,-1\rfloor})_{\{-1\}}\cdot(\tilde{m}\zeta_{\lfloor -1,-1\rfloor})_{\{-1\}} = m\cdot\tilde{m}\,.$$

So

$$\big(m\zeta_{\lfloor 0,-1\rfloor}\cdot\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\big)_{\{i\}} = \begin{cases} 1 & \text{if } \ i\geqslant 1 \\ (m\mathrm{f}_S)^- & \text{if } \ i=0 \\ m\cdot\tilde{m} & \text{if } \ i=-1\,. \end{cases}$$

We calculate $\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor}$.

Note that

$$m\zeta_{\lfloor 0,-1\rfloor} \overset{367.(1)}{=} (\ldots,1,\underbrace{(m\mathrm{f}_S)^-}_{\text{pos. }0},\underbrace{m}_{\text{pos. }-1}) \overset{367.(2)}{=} \big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\cdot m\zeta_{\lfloor -1,-1\rfloor}\,.$$

We calculate $\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}$, using Lemma 362.(2) with $t := 0$.

For $i\in\mathbb{Z}_{\geqslant 1}$, we have

$$\Big(\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{i\}} = (\tilde{m}\zeta_{\lfloor -1,-1\rfloor})_{\{i\}} = 1\,.$$

For $i=0$, we have

$$\Big(\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{0\}} = (\tilde{m}\zeta_{\lfloor -1,-1\rfloor})_{\{0\}}\cdot\Big(\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{0\}} = 1\cdot(m\mathrm{f}_S)^- = (m\mathrm{f}_S)^-\,.$$

For $i=-1$, we have

$$\Big(\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{-1\}}=\big((\tilde{m}\zeta_{\lfloor -1,-1\rfloor})_{\{-1\}}\big)^{\left(((m\mathrm{f}_S)^-)\zeta_{\lfloor 0,0\rfloor}\right)_{\{0\}}}=\tilde{m}^{(m\mathrm{f}_S)^-}\overset{323.(2)}{=}\tilde{m}^{m^-}\,.$$

So for $i\in\mathbb{Z}_{\geqslant -1}$, we have

$$\Big(\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot\big((m\mathrm{f}_S)^-\Big)\zeta_{\lfloor 0,0\rfloor}\Big)_{\{i\}}=\begin{cases}1 & \text{if } \ i\geqslant 1\\ (m\mathrm{f}_S)^- & \text{if } \ i=0\\ \tilde{m}^{m^-} & \text{if } \ i=-1\,.\end{cases}$$

Let $n:=\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}$.

We calculate $n\cdot m\zeta_{\lfloor -1,-1\rfloor}=\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor}$ using Lemma 362.(2) with $t:=-1$.

For $i\in\mathbb{Z}_{\geqslant 0}$, we have

$$\big(n\cdot m\zeta_{\lfloor -1,-1\rfloor}\big)_{\{i\}}=n_{\{i\}}=\begin{cases}1 & \text{if } i\geqslant 1\\ (m\mathrm{f}_S)^- & \text{if } i=0\,.\end{cases}$$

For $i=-1$, we have

$$(n\cdot m\zeta_{\lfloor -1,-1\rfloor})_{\{-1\}}=n_{\{-1\}}\cdot(m\zeta_{\lfloor -1,-1\rfloor})_{\{-1\}}=\tilde{m}^{m^-}\cdot m=m\cdot\tilde{m}\,.$$

So for $i\in\mathbb{Z}_{\geqslant -1}$, we have

$$\big(\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor}\big)_{\{i\}}=(n\cdot m\zeta_{\lfloor -1,-1\rfloor})_{\{i\}}=\begin{cases}1 & \text{if } \ i\geqslant 1\\ (m\mathrm{f}_S)^- & \text{if } \ i=0\\ m\cdot\tilde{m} & \text{if } \ i=-1\,.\end{cases}$$

This shows $\big[m\zeta_{\lfloor 0,-1\rfloor}\,,\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\big]=1$. □

Example 373 Suppose given $b\in B$ and m, $\tilde{m}\in M$.

(1) We consider the elements

$$\begin{aligned}b\zeta_{\lfloor 1,1\rfloor} &= (\ldots,1,\underbrace{b}_{\text{pos. }1},1,1)\\ m\zeta_{\lfloor -1,-1\rfloor} &= (\ldots,1,1,1,1,\underbrace{m}_{\text{pos. }-1}) \ \in H_\infty\,.\end{aligned}$$

By Lemma 372.(1), we have $\big[b\zeta_{\lfloor 1,1\rfloor}\,,m\zeta_{\lfloor -1,-1\rfloor}\big]=1\,.$

Using Lemma 362.(2) directly, we obtain

$$b\zeta_{\lfloor 1,1\rfloor}\cdot m\zeta_{\lfloor -1,-1\rfloor}=(\dots,1,b,1,1)\cdot(\dots,1,1,1,m)\overset{362.(2)}{=}(\dots,1,b,1,m)$$

and

$$\begin{array}{rcl} m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor 1,1\rfloor} & = & (\dots,1,1,1,m)\cdot(\dots,1,b,1,1)\\ & \overset{362.(2)}{=} & \left(\dots,1,1\cdot b,1^b,\left(\llbracket 1,\ b\rrbracket^{(-1)^1}\right)^1\cdot m\right)\\ & = & (\dots,1,b,1,\llbracket 1,\ b\rrbracket\cdot m)\\ & \overset{322.(1)}{=} & (\dots,1,b,1,m)\,. \end{array}$$

This confirms that $b\zeta_{\lfloor 1,1\rfloor}\cdot m\zeta_{\lfloor -1,-1\rfloor}=m\zeta_{\lfloor -1,-1\rfloor}\cdot b\zeta_{\lfloor 1,1\rfloor}$.

(2) We consider the elements

$$\begin{array}{rcl} b\zeta_{\lfloor 0,0\rfloor} & = & (\dots,1,1,b,1)\\ m\zeta_{\lfloor 1,-1\rfloor} & = & \left(\dots,1,m\mathrm{f}_S\,,(m\mathrm{f}_S)^-,m\right)\ \in H_\infty\,. \end{array}$$

By Lemma 372.(2), we have $\left[b\zeta_{\lfloor 0,0\rfloor}\,,m\zeta_{\lfloor 1,-1\rfloor}\right]=1$.

In particular, we have

$$\begin{array}{rcl} m\zeta_{\lfloor 1,-1\rfloor}\cdot b\zeta_{\lfloor 0,0\rfloor} & = & \left(\dots,1,m\mathrm{f}_S\,,(m\mathrm{f}_S)^-\,,m\right)\cdot(\dots,1,1,b,1)\\ & = & b\zeta_{\lfloor 0,0\rfloor}\cdot m\zeta_{\lfloor 1,-1\rfloor}\\ & = & (\dots,1,1,b,1)\cdot\left(\dots,1,m\mathrm{f}_S\,,(m\mathrm{f}_S)^-\,,m\right)\\ & \overset{362.(2)}{=} & \left(\dots,1,m\mathrm{f}_S\,,(m\mathrm{f}_S)^-\cdot b\,,m^b\right). \end{array}$$

(3) We consider the elements

$$\begin{array}{rcl} b\zeta_{\lfloor 1,0\rfloor} & = & (\dots,1,b^-,b,1)\\ m\zeta_{\lfloor 0,-1\rfloor} & = & \left(\dots,1,1,(m\mathrm{f}_S)^-,m\right)\ \in H_\infty\,. \end{array}$$

By Lemma 372.(3), we have $\left[b\zeta_{\lfloor 1,0\rfloor}\,,m\zeta_{\lfloor 0,-1\rfloor}\right]=1$.

In particular, we have

$$\begin{array}{rcl} m\zeta_{\lfloor 0,-1\rfloor}\cdot b\zeta_{\lfloor 1,0\rfloor} & = & \left(\dots,1,1,(m\mathrm{f}_S)^-,m\right)\cdot(\dots,1,b^-,b,1)\\ & = & b\zeta_{\lfloor 1,0\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor}\\ & = & (\dots,1,b^-,b,1)\cdot\left(\dots,1,1,(m\mathrm{f}_S)^-,m\right)\\ & \overset{362.(2)}{=} & \left(\dots,1,b^-,b\cdot(m\mathrm{f}_S)^-,m\right). \end{array}$$

(4) We consider the elements

$$\begin{array}{rcl} m\zeta_{\lfloor 0,-1\rfloor} & = & \big(\dots,1,1,(m\mathrm{f}_S)^-,m\big) \\ \tilde{m}\zeta_{\lfloor -1,-1\rfloor} & = & (\dots,1,1,1,\tilde{m}) \quad \in H_\infty\,. \end{array}$$

By Lemma 372.(4), we have $\big[m\zeta_{\lfloor 0,-1\rfloor}\,,\tilde{m}\zeta_{\lfloor -1,-1\rfloor}\big]=1\,.$

In particular, we have

$$\begin{array}{rcl} \tilde{m}\zeta_{\lfloor -1,-1\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor} & = & (\dots,1,1,1,\tilde{m})\cdot\big(\dots,1,1,(m\mathrm{f}_S)^-,m\big) \\ & = & m\zeta_{\lfloor 0,-1\rfloor}\cdot\tilde{m}\zeta_{\lfloor -1,-1\rfloor} \\ & = & \big(\dots,1,1,(m\mathrm{f}_S)^-,m\big)\cdot(\dots,1,1,1,\tilde{m}) \\ & \overset{362.(2)}{=} & \big(\dots,1,1,(m\mathrm{f}_S)^-,m\cdot\tilde{m}\big)\,. \end{array}$$

Remark 374 *Suppose given $b\in B$ and $m\in M$.*

For $k\in\mathbb{Z}_{\geqslant 1}$, we have

$$[b\mathrm{s}^\infty_k, m\mathrm{s}^\infty_{-1}]=1.$$

Proof. This is a reformulation of Lemma 372.(1), using Lemma 359.(2). □

17.2.7 The Conduché condition

Lemma 375 *Suppose given b, $c\in B$. Suppose given ℓ, $\tilde{\ell}$, k, $\tilde{k}\in\mathbb{Z}_{\geqslant 0}$ such that $\ell\geqslant k$ and $\tilde{\ell}\geqslant\tilde{k}$. We consider the elements $b\zeta_{\rfloor\ell,k\rfloor}$ and $c\,\zeta_{\rfloor\tilde{\ell},\tilde{k}\rfloor}\in H_\infty$; cf. Definition* 367.(3).

If $\{\ell,\,k\}\cap\{\tilde{\ell},\,\tilde{k}\}=\emptyset$, then we have

$$[b\zeta_{\rfloor\ell,k\rfloor}\,,c\,\zeta_{\rfloor\tilde{\ell},\tilde{k}\rfloor}]=1\,.$$

Proof. Note that if $\ell=k$, then $b\zeta_{\rfloor\ell,k\rfloor}=1$. If $\tilde{\ell}=\tilde{k}$, then $c\,\zeta_{\rfloor\tilde{\ell},\tilde{k}\rfloor}=1$. So we may assume that $\ell>k$ and $\tilde{\ell}>\tilde{k}$.

It suffices to show the following assertions $(1, 2.1, 2.2, 3)$.

(1) If $\ell>k>\tilde{\ell}>\tilde{k}$, then we have $[b\zeta_{\rfloor\ell,k\rfloor}\,,c\,\zeta_{\rfloor\tilde{\ell},\tilde{k}\rfloor}]=1\,.$

(2.1) If $\ell>\tilde{\ell}>\tilde{k}>k$ and if $\tilde{\ell}-\tilde{k}\equiv_2 0$, then we have $[b\zeta_{\rfloor\ell,k\rfloor}\,,c\,\zeta_{\rfloor\tilde{\ell},\tilde{k}\rfloor}]=1\,.$

(2.2) If $\ell > \tilde{\ell} > \tilde{k} > k$ and if $\tilde{\ell} - \tilde{k} \equiv_2 1$, then we have $\quad [b\zeta_{\rfloor \ell,k \rfloor}\,, c\,\zeta_{\rfloor \tilde{\ell},\tilde{k} \rfloor}] = 1\,.$

(3) If $\ell > \tilde{\ell} > k > \tilde{k}$, then we have $\quad [b\zeta_{\rfloor \ell,k \rfloor}\,, c\,\zeta_{\rfloor \tilde{\ell},\tilde{k} \rfloor}] = 1\,.$

Ad (1). We have $\ell > k > \tilde{\ell} > \tilde{k}$. We have

$$b\zeta_{\rfloor \ell,k \rfloor} \overset{367.(1)}{=} \prod_{i \in \rfloor \ell,k \rfloor} (b^{(-1)^{k-i}})\zeta_{\lfloor i,i \rfloor}$$

and we have

$$c\,\zeta_{\rfloor \tilde{\ell},\tilde{k} \rfloor} \overset{367.(1)}{=} \prod_{j \in \rfloor \tilde{\ell},\tilde{k} \rfloor} (c^{(-1)^{\tilde{k}-j}})\zeta_{\lfloor j,j \rfloor}\,.$$

Let $i \in [k, \ell - 1]$ and let $j \in [\tilde{k}, \tilde{\ell} - 1]$. Then, we have $|i - j| \geqslant 2$.
So by Lemma 369.(1), we have

$$\Big[(b^{(-1)^{k-i}})\zeta_{\lfloor i,i \rfloor}\,, (c^{(-1)^{\tilde{k}-j}})\zeta_{\lfloor j,j \rfloor}\Big] = 1\,.$$

Therefore, we also have

$$\big[b\zeta_{\rfloor \ell,k \rfloor}\,, c\,\zeta_{\rfloor \tilde{\ell},\tilde{k} \rfloor}\big] = 1\,.$$

Ad (2.1). We have $\ell > \tilde{\ell} > \tilde{k} > k$ and $\tilde{\ell} - \tilde{k} \equiv_2 0$. We have

$$b\zeta_{\rfloor \ell,k \rfloor} \overset{367.(1)}{=} b\zeta_{\rfloor \ell,\tilde{\ell}+1 \rfloor} \cdot \Big(\prod_{\substack{i \in \lfloor \tilde{\ell},\tilde{k} \rfloor \\ i \equiv_2 \tilde{\ell}}} (b^{(-1)^{i-k+1}})\zeta_{\lfloor i,i-1 \rfloor} \Big) \cdot b\zeta_{\lfloor \tilde{k}-2,k \rfloor}\,,$$

and we have

$$c\,\zeta_{\rfloor \tilde{\ell},\tilde{k} \rfloor} = c\,\zeta_{\lfloor \tilde{\ell}-1\,,\tilde{k} \rfloor} \overset{367.(1)}{=} \prod_{\substack{j \in \lfloor \tilde{\ell}-1,\,\tilde{k}+1 \rfloor \\ j \equiv_2 \tilde{\ell}-1}} (c^{(-1)^{j-\tilde{k}+1}})\zeta_{\lfloor j,\,j-1 \rfloor}\,.$$

Let $j \in [\tilde{k}+1, \tilde{\ell}-1]$ with $j \equiv_2 \tilde{\ell} - 1$. We write $\tilde{c} := c^{(-1)^{j-\tilde{k}+1}}$.
We show that $\big[b\zeta_{\rfloor \ell,\,\tilde{\ell}+1 \rfloor}\,, \tilde{c}\,\zeta_{\lfloor j,\,j-1 \rfloor}\big] \overset{!}{=} 1\,.$
If $\ell = \tilde{\ell} + 1$, then we have $b\zeta_{\rfloor \ell,\tilde{\ell}+1 \rfloor} = 1$.
If $\ell > \tilde{\ell} + 1$, then we have $\ell > \tilde{\ell} + 1 > j + 1 > j > j - 1$, and by (1), we have

$$\big[b\zeta_{\rfloor \ell,\tilde{\ell}+1 \rfloor}\,, \tilde{c}\,\zeta_{\lfloor j,j-1 \rfloor}\big] = \big[b\zeta_{\rfloor \ell,\tilde{\ell}+1 \rfloor}\,, \tilde{c}\,\zeta_{\rfloor j+1,j-1 \rfloor}\big] = 1\,.$$

Let $i \in [\tilde{k}, \tilde{\ell}]$ with $i \equiv_2 \tilde{\ell}$. We write $\tilde{b} := b^{(-1)^{i-k+1}}$. Note that $i \equiv_2 \tilde{\ell} \not\equiv_2 \tilde{\ell} - 1 \equiv_2 j$.

We show that $\left[\tilde{b}\zeta_{\lfloor i,i-1\rfloor}, \tilde{c}\,\zeta_{\lfloor j,j-1\rfloor}\right] \overset{!}{=} 1$.

If $i \in \{j+1, j-1\}$, then we have

$$\left[\tilde{b}\zeta_{\lfloor i,i-1\rfloor}, \tilde{c}\,\zeta_{\lfloor j,j-1\rfloor}\right] = 1$$

by Lemma 369.(3).

If $i < j-1$, then $j+1 > j-1 > i+1 > i-1$, and by (1), we have

$$\left[\tilde{b}\zeta_{\lfloor i,i-1\rfloor}, \tilde{c}\,\zeta_{\lfloor j,j-1\rfloor}\right] = \left[\tilde{b}\zeta_{\rfloor i+1,i-1\rfloor}, \tilde{c}\,\zeta_{\rfloor j+1,j-1\rfloor}\right] = 1.$$

If $i > j+1$, then $i+1 > i-1 > j+1 > j-1$, and by (1), we have

$$\left[\tilde{b}\zeta_{\lfloor i,i-1\rfloor}, \tilde{c}\,\zeta_{\lfloor j,j-1\rfloor}\right] = \left[\tilde{b}\zeta_{\rfloor i+1,i-1\rfloor}, \tilde{c}\,\zeta_{\rfloor j+1,j-1\rfloor}\right] = 1.$$

This shows $\left[\tilde{b}\zeta_{\lfloor i,i-1\rfloor}, \tilde{c}\,\zeta_{\lfloor j,j-1\rfloor}\right] = 1$.

We show that $\left[\tilde{c}\,\zeta_{\lfloor j,j-1\rfloor}, b\zeta_{\lfloor \tilde{k}-2,k\rfloor}\right] \overset{!}{=} 1$.

If $\tilde{k} = k+1$, then we have $b\zeta_{\lfloor \tilde{k}-2,k\rfloor} = 1$.

If $\tilde{k} > k+1$, then we have $j+1 > j-1 > \tilde{k}-1 > k$, and by (1), we have

$$\left[\tilde{c}\,\zeta_{\lfloor j,j-1\rfloor}, b\zeta_{\lfloor \tilde{k}-2,k\rfloor}\right] = \left[\tilde{c}\,\zeta_{\rfloor j+1,j-1\rfloor}, b\zeta_{\rfloor \tilde{k}-1,k\rfloor}\right] = 1.$$

Altogether, this shows $[b\zeta_{\rfloor \ell,k\rfloor}, c\,\zeta_{\rfloor \tilde{\ell},\tilde{k}\rfloor}] = 1$.

Ad (2.2). We have $\ell > \tilde{\ell} > \tilde{k} > k$ and $\tilde{\ell} - \tilde{k} \equiv_2 1$.

We have

$$b\zeta_{\rfloor \ell,k\rfloor} \overset{367.(1)}{=} (b^{(-1)^{\tilde{k}-k}})\zeta_{\rfloor \ell,\tilde{k}+2\rfloor} \cdot (b^{(-1)^{\tilde{k}-k+1}})\zeta_{\lfloor \tilde{k}+1,\tilde{k}-1\rfloor} \cdot b\zeta_{\lfloor \tilde{k}-2,k\rfloor},$$

and we have

$$c\,\zeta_{\rfloor \tilde{\ell},\tilde{k}\rfloor} \overset{367.(1)}{=} (c^-)\zeta_{\rfloor \tilde{\ell},\tilde{k}+1\rfloor} \cdot c\,\zeta_{\lfloor \tilde{k},\tilde{k}\rfloor}.$$

We show that $\left[b\zeta_{\rfloor \ell,k\rfloor}, c^-\,\zeta_{\rfloor \tilde{\ell},\tilde{k}+1\rfloor}\right] \overset{!}{=} 1$.

If $\tilde{\ell} = \tilde{k}+1$, then we have $c\,\zeta_{\rfloor \tilde{\ell},\tilde{k}+1\rfloor} = 1$.

If $\tilde{\ell} > \tilde{k}+1$, then we have $\ell > \tilde{\ell} > \tilde{k}+1 > k$ and $\tilde{\ell} - (\tilde{k}+1) \equiv_2 0$. So, by (2.1), we have

$$\left[b\zeta_{\rfloor \ell,k\rfloor}\,, c^{-}\,\zeta_{\rfloor \tilde{\ell},\tilde{k}+1\rfloor}\right] = 1\,.$$

We show that $\left[b\zeta_{\rfloor \ell,k\rfloor}\,, c\,\zeta_{\lfloor \tilde{k},\tilde{k}\rfloor}\right] \stackrel{!}{=} 1\,.$

Since $\ell > \tilde{k}+2 > \tilde{k}+1 > \tilde{k}$, we have, by (1),

$$\left[(b^{(-1)^{\tilde{k}-k}})\zeta_{\rfloor \ell,\tilde{k}+2\rfloor}\,, c\,\zeta_{\lfloor \tilde{k},\tilde{k}\rfloor}\right] = \left[(b^{(-1)^{\tilde{k}-k}})\zeta_{\rfloor \ell,\tilde{k}+2\rfloor}\,, c\,\zeta_{\rfloor \tilde{k}+1,\tilde{k}\rfloor}\right] = 1\,.$$

Moreover, by Lemma 369.(2), we have

$$\left[(b^{(-1)^{\tilde{k}-k+1}})\zeta_{\lfloor \tilde{k}+1,\tilde{k}-1\rfloor}\,, c\,\zeta_{\lfloor \tilde{k},\tilde{k}\rfloor}\right] = 1\,.$$

If $\tilde{k} = k+1$, then we have $b\zeta_{\lfloor \tilde{k}-2,k\rfloor} = 1\,.$

If $\tilde{k} > k+1$, then we have $\tilde{k}+1 > \tilde{k} > \tilde{k}-1 > k\,,$ and by (1), we have

$$\left[b\zeta_{\lfloor \tilde{k}-2,k\rfloor}\,, c\,\zeta_{\lfloor \tilde{k},\tilde{k}\rfloor}\right] = \left[b\zeta_{\rfloor \tilde{k}-1,k\rfloor}\,, c\,\zeta_{\rfloor \tilde{k}+1,\tilde{k}\rfloor}\right] = 1\,.$$

This shows $\left[b\zeta_{\rfloor \ell,k\rfloor}\,, c\,\zeta_{\lfloor \tilde{k},\tilde{k}\rfloor}\right] = 1\,.$

Altogether, this shows $[b\zeta_{\rfloor \ell,k\rfloor}\,, c\,\zeta_{\rfloor \tilde{\ell},\tilde{k}\rfloor}] = 1.$

Ad (3). We have $\ell > \tilde{\ell} > k > \tilde{k}\,.$ We have

$$b\zeta_{\rfloor \ell,k\rfloor} \stackrel{367.(1)}{=} b\zeta_{\rfloor \ell,k+2\rfloor} \cdot b\zeta_{\lfloor k+1,k\rfloor}\,,$$

and we have

$$c\,\zeta_{\rfloor \tilde{\ell},\tilde{k}\rfloor} \stackrel{367.(1)}{=} (c^{(-1)^{k-\tilde{k}+1}})\zeta_{\rfloor \tilde{\ell},k+1\rfloor} \cdot (c^{(-1)^{k-\tilde{k}+1}})\zeta_{\lfloor k,k-1\rfloor} \cdot c\,\zeta_{\lfloor k-2,\tilde{k}\rfloor}\,.$$

We write $\tilde{c} := c^{(-1)^{k-\tilde{k}+1}}$. We show that $\left[b\zeta_{\rfloor \ell,k\rfloor}\,, \tilde{c}\,\zeta_{\rfloor \tilde{\ell},k+1\rfloor}\right] \stackrel{!}{=} 1.$

If $\tilde{\ell} = k+1$, then we have $\tilde{c}\,\zeta_{\rfloor \tilde{\ell},k+1\rfloor} = 1.$

If $\tilde{\ell} > k+1$, then we have $\ell > \tilde{\ell} > k+1 > k$, and by (2.1) and (2.2), we have

$$\left[b\zeta_{\rfloor \ell,k\rfloor}\,, \tilde{c}\,\zeta_{\rfloor \tilde{\ell},k+1\rfloor}\right] = 1\,.$$

We show that $\left[b\zeta_{\rfloor \ell,k\rfloor}\,, \tilde{c}\,\zeta_{\lfloor k,k-1\rfloor}\right] \stackrel{!}{=} 1.$

If $\ell = k+2$, then we have $b\zeta_{\rfloor \ell,k+2\rfloor} = 1$.

If $\ell > k+2$, then we have $\ell > k+2 > k+1 > k-1$, and by (1), we have

$$\left[b\zeta_{\rfloor \ell,k+2\rfloor}\,,\tilde{c}\,\zeta_{\lfloor k,k-1\rfloor}\right] = \left[b\zeta_{\rfloor \ell,k+2\rfloor}\,,\tilde{c}\,\zeta_{\rfloor k+1,k-1\rfloor}\right] = 1\,.$$

Moreover, by Lemma 369.(3), we have

$$\left[b\zeta_{\lfloor k+1,k\rfloor}\,,\tilde{c}\,\zeta_{\lfloor k,k-1\rfloor}\right] = 1\,.$$

So $\left[b\zeta_{\rfloor \ell,k\rfloor}\,,\tilde{c}\,\zeta_{\lfloor k,k-1\rfloor}\right] = 1$.

We show that $\left[b\zeta_{\rfloor \ell,k\rfloor}\,,c\,\zeta_{\lfloor k-2,\tilde{k}\rfloor}\right] \overset{!}{=} 1$.

If $k = \tilde{k}+1$, then we have $c\,\zeta_{\lfloor k-2,\tilde{k}\rfloor} = 1$.

If $k > \tilde{k}+1$, then we have $\ell > k > k-1 > \tilde{k}$, and by (1), we have

$$\left[b\zeta_{\rfloor \ell,k\rfloor}\,,c\,\zeta_{\lfloor k-2,\tilde{k}\rfloor}\right] = \left[b\zeta_{\rfloor \ell,k\rfloor}\,,c\,\zeta_{\rfloor k-1,\tilde{k}\rfloor}\right] = 1\,.$$

Altogether, this shows $\left[b\zeta_{\rfloor \ell,k\rfloor}\,,c\,\zeta_{\rfloor \tilde{\ell},\tilde{k}\rfloor}\right] = 1$. □

Lemma 376 *Suppose given $b \in B$ and $m \in M$. Suppose given ℓ, $\tilde{\ell}$, $k \in \mathbb{Z}_{\geqslant 0}$ such that $\ell \geqslant k$.*

We consider the elements $b\zeta_{\rfloor \ell,k\rfloor}$ and $m\zeta_{\rfloor \tilde{\ell},-1\rfloor} \in H_\infty$.

If $\{\ell,\, k\} \cap \{\tilde{\ell}\} = \emptyset$, then we have

$$[b\zeta_{\rfloor \ell,k\rfloor}\,,m\zeta_{\rfloor \tilde{\ell},-1\rfloor}] = 1\,.$$

Proof. Suppose that $\{\ell,\, k\} \cap \{\tilde{\ell}\} = \emptyset$.

Case $k \geqslant 1$. We have

$$b\zeta_{\rfloor \ell,k\rfloor} = b\zeta_{\lfloor \ell-1,k\rfloor} \overset{367.(1)}{=} \prod_{i\in\lfloor \ell-1,k\rfloor} (b^{(-1)^{k-i}})\zeta_{\lfloor i,i\rfloor}$$

and

$$m\zeta_{\rfloor \tilde{\ell},-1\rfloor} \overset{367.(2)}{=} \left((m\mathrm{f}_S)^-\right)\zeta_{\rfloor \tilde{\ell},0\rfloor}\cdot m\zeta_{\lfloor -1,-1\rfloor}\,.$$

For $i \in \lfloor \ell-1, k\rfloor$, we have

$$\left[b\zeta_{\lfloor i,i\rfloor}\,,m\zeta_{\lfloor -1,-1\rfloor}\right] = 1$$

by Lemma 372.(1).

Moreover, since $\{\ell,k\}\cap\{\tilde{\ell},0\}=\emptyset$, we have, by Lemma 375,

$$\Big[b\zeta_{\rfloor\ell,k\rfloor}\,,\ \big((m\mathrm{f}_S)^-\big)\zeta_{\rfloor\tilde{\ell},0\rfloor}\Big]=1\,.$$

This shows $[b\zeta_{\rfloor\ell,k\rfloor}\,,m\zeta_{\rfloor\tilde{\ell},-1\rfloor}]=1$ if $k\geqslant 1$.

Case $k=0$. Note that $\{\ell,0\}\cap\{\tilde{\ell}\}=\emptyset$.

Subcase $\ell=0$. We have $b\zeta_{\rfloor\ell,k\rfloor}=b\zeta_{\rfloor 0,0\rfloor}=1$. This shows $[b\zeta_{\rfloor 0,0\rfloor}\,,m\zeta_{\rfloor\tilde{\ell},-1\rfloor}]=1$.

Subcase $\ell=1$. If $\tilde{\ell}=-1$, then we have $m\zeta_{\rfloor\tilde{\ell},-1\rfloor}=1$.

Let $\tilde{\ell}\geqslant 0$. Then note that we have $\tilde{\ell}\geqslant 2$.

We have

$$b\zeta_{\rfloor\ell,k\rfloor}=b\zeta_{\rfloor 1,0\rfloor}=b\zeta_{\lfloor 0,0\rfloor}\,,$$

and we have

$$m\zeta_{\rfloor\tilde{\ell},-1\rfloor}\overset{367.(2)}{=}\big((m\mathrm{f}_S)^-\big)\zeta_{\rfloor\tilde{\ell},2\rfloor}\cdot m\zeta_{\lfloor 1,-1\rfloor}\,.$$

We show that $\big[b\zeta_{\rfloor 1,0\rfloor}\,,m\zeta_{\rfloor\tilde{\ell},-1\rfloor}\big]\overset{!}{=}1$.

If $\tilde{\ell}=2$, then we have $\big((m\mathrm{f}_S)^-\big)\zeta_{\rfloor\tilde{\ell},2\rfloor}=1$.

If $\tilde{\ell}>2$, then $\tilde{\ell}>2>1>0$, and by Lemma 375, we have

$$\Big[b\zeta_{\rfloor 1,0\rfloor}\,,\big((m\mathrm{f}_S)^-\big)\zeta_{\rfloor\tilde{\ell},2\rfloor}\Big]=1\,.$$

Moreover, by Lemma 372.(2), we have

$$\big[b\zeta_{\rfloor 1,0\rfloor}\,,m\zeta_{\lfloor 1,-1\rfloor}\big]=\big[b\zeta_{\lfloor 0,0\rfloor}\,,m\zeta_{\lfloor 1,-1\rfloor}\big]=1\,.$$

This shows $\big[b\zeta_{\rfloor 1,0\rfloor}\,,m\zeta_{\rfloor\tilde{\ell},-1\rfloor}\big]=1$.

Subcase $\ell\geqslant 2$. By Definition 367.(1), we have

$$b\zeta_{\rfloor\ell,k\rfloor}=\Big(\prod_{i\in\lfloor\ell-1,2\rfloor}(b^{(-1)^i})\zeta_{\lfloor i,i\rfloor}\Big)\cdot b\zeta_{\lfloor 1,0\rfloor}\,,$$

and we have

$$m\zeta_{\rfloor\tilde{\ell},-1\rfloor}=(m\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}\cdot m\zeta_{\lfloor 0,-1\rfloor}=(m\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}\cdot\big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\cdot m\zeta_{\lfloor -1,-1\rfloor}\,.$$

We show that $[b\zeta_{\rfloor\ell,0\rfloor}, (m\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}] \stackrel{!}{=} 1$.

If $\tilde{\ell} = 1$, then we have $(m\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor} = 1$.

If $\tilde{\ell} \geqslant 2$, then $\{\ell, 0\} \cap \{\tilde{\ell}, 1\} = \emptyset$, and by Lemma 375, we have

$$[b\zeta_{\rfloor\ell,0\rfloor}, (m\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}] = 1\,.$$

This shows $[b\zeta_{\rfloor\ell,0\rfloor}, (m\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}] = 1$.

We show that $[b\zeta_{\rfloor\ell,0\rfloor}, m\zeta_{\lfloor 0,-1\rfloor}] \stackrel{!}{=} 1$.

Let $i \in \lfloor \ell - 1, 2\rfloor$. By Lemma 369.(1), we have

$$\Big[(b^{(-1)^i})\zeta_{\lfloor i,i\rfloor}, \big((m\mathrm{f}_S)^-\big)\zeta_{\lfloor 0,0\rfloor}\Big] = 1\,.$$

By Lemma 372.(1), we have

$$\big[(b^{(-1)^i})\zeta_{\lfloor i,i\rfloor}, m\zeta_{\lfloor -1,-1\rfloor}\big] = 1\,.$$

Moreover, by Lemma 372.(3), we have

$$[b\zeta_{\lfloor 1,0\rfloor}, m\zeta_{\lfloor 0,-1\rfloor}] = 1\,.$$

This shows $[b\zeta_{\rfloor\ell,0\rfloor}, m\zeta_{\lfloor 0,-1\rfloor}] = 1$.

So we have $[b\zeta_{\rfloor\ell,0\rfloor}, m\zeta_{\rfloor\tilde{\ell},-1\rfloor}] = 1$ if $\ell \geqslant 2$. □

Lemma 377 *Suppose given m, $\tilde{m} \in M$. Suppose given ℓ, $\tilde{\ell} \in \mathbb{Z}_{\geqslant 0}$.*
We consider the elements $m\zeta_{\rfloor\ell,-1\rfloor}$ and $\tilde{m}\zeta_{\rfloor\tilde{\ell},-1\rfloor} \in H_\infty$.
If $\{\ell\} \cap \{\tilde{\ell}\} = \emptyset$, i.e. $\ell \neq \tilde{\ell}$, then we have

$$[m\zeta_{\rfloor\ell,-1\rfloor}, \tilde{m}\zeta_{\rfloor\tilde{\ell},-1\rfloor}] = 1\,.$$

Proof. Suppose that $\ell \neq \tilde{\ell}$. We may assume that $\ell > \tilde{\ell}$.
Case $\tilde{\ell} = 0$. Note that $\ell \geqslant 1$. By Definition 367.(2), we have

$$m\zeta_{\rfloor\ell,-1\rfloor} = \Big(\prod_{i\in\lfloor\ell-1,1\rfloor} \big((m\mathrm{f}_S)^{(-1)^{i+1}}\big)\zeta_{\lfloor i,i\rfloor}\Big) \cdot m\zeta_{\lfloor 0,-1\rfloor}\,,$$

and we have

$$\tilde{m}\zeta_{\rfloor\tilde{\ell},-1\rfloor} = \tilde{m}\zeta_{\lfloor -1,-1\rfloor}\,.$$

We show that $\big[m\zeta_{\rfloor\ell,-1\rfloor}\,,\,\tilde{m}\zeta_{\lfloor-1,-1\rfloor}\big] \overset{!}{=} 1$.

For $i \in \lfloor \ell-1, 1\rfloor$, we have, by Lemma 372.(1),

$$\Big[\big((m\mathrm{f}_S)^{(-1)^{i+1}}\big)\zeta_{\lfloor i,i\rfloor}\,,\,\tilde{m}\zeta_{\lfloor-1,-1\rfloor}\Big] = 1\,.$$

Moreover, by Lemma 372.(4), we have

$$\big[m\zeta_{\lfloor0,-1\rfloor}\,,\,\tilde{m}\zeta_{\lfloor-1,-1\rfloor}\big] = 1\,.$$

This shows $\big[m\zeta_{\rfloor\ell,-1\rfloor}\,,\,\tilde{m}\zeta_{\lfloor-1,-1\rfloor}\big] = 1$.

Case $\tilde{\ell} \geqslant 1$. Note that $\ell \geqslant 2$. By Definition 367.(2), we have

$$m\zeta_{\rfloor\ell,-1\rfloor} = \big((m\mathrm{f}_S)^-\big)\zeta_{\rfloor\ell,0\rfloor}\cdot m\zeta_{\lfloor-1,-1\rfloor}\,,$$

and we have

$$\tilde{m}\zeta_{\rfloor\tilde{\ell},-1\rfloor} = (\tilde{m}\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}\cdot\tilde{m}\zeta_{\lfloor0,-1\rfloor}\,.$$

We show that $\big[m\zeta_{\rfloor\ell,-1\rfloor}\,,(\tilde{m}\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}\big] \overset{!}{=} 1$.

Since $\{\tilde{\ell},1\}\cap\{\ell\} = \emptyset$, we have, by Lemma 376,

$$\big[m\zeta_{\rfloor\ell,-1\rfloor}\,,\,(\tilde{m}\mathrm{f}_S)\zeta_{\rfloor\tilde{\ell},1\rfloor}\big] = 1\,.$$

We show that $\big[m\zeta_{\rfloor\ell,-1\rfloor}\,,\tilde{m}\zeta_{\lfloor0,-1\rfloor}\big] \overset{!}{=} 1$.

Since $\{\ell,0\}\cap\{1\} = \emptyset$, we have, by Lemma 376,

$$\Big[\big((m\mathrm{f}_S)^-\big)\zeta_{\rfloor\ell,0\rfloor}\,,\tilde{m}\zeta_{\lfloor0,-1\rfloor}\Big] = \Big[\big((m\mathrm{f}_S)^-\big)\zeta_{\rfloor\ell,0\rfloor}\,,\tilde{m}\zeta_{\rfloor1,-1\rfloor}\Big] = 1\,.$$

Moreover, by Lemma 372.(4), we have

$$\big[m\zeta_{\lfloor-1,-1\rfloor}\,,\tilde{m}\zeta_{\lfloor0,-1\rfloor}\big] = 1\,.$$

This shows $\big[m\zeta_{\rfloor\ell,-1\rfloor}\,,\tilde{m}\zeta_{\lfloor0,-1\rfloor}\big] = 1$.

Altogether, we have $\big[m\zeta_{\rfloor\ell,-1\rfloor}\,,\tilde{m}\zeta_{\rfloor\tilde{\ell},-1\rfloor}\big] = 1$. □

Remark 378

Suppose given $a,\, b \in \mathbb{Z}_{\geqslant0}$ *with* $a \leqslant b$. *Suppose given* $x \in \bigcap_{i\in[a,b]} \ker d_i^\infty \trianglelefteqslant H_\infty$ *; cf. Lemma* 364. *If* $a = 0$, *then, for* $i \in [a,b]$, *we have*

$$x_{\{i\}} = \big((x_{\{-1\}})\mathrm{f}_S\big)^{(-1)^{i+1}}\,.$$

If $a \geqslant 1$, then, for $i \in [a,b]$, we have

$$x_{\{i\}} = \left(x_{\{a-1\}}\right)^{(-1)^{i-a+1}}.$$

Proof. We show this claim by induction on $n := b - a$.

Case $a \geqslant 1$. Let $n = 0$. Since $x \in \ker d_a^\infty$, we have

$$1 = x d_a^\infty \overset{364.(1)}{=} x_{\{a\}} \cdot x_{\{a-1\}}.$$

So we have $x_{\{a\}} = (x_{\{a-1\}})^-$. This proves the claim for $n = 0$.

Let $n \geqslant 1$. By induction, since $x \in \bigcap_{i\in[a,b-1]} \ker \mathrm{d}_i^\infty$, we have $x_{\{i\}} = \left(x_{\{a-1\}}\right)^{(-1)^{i-a+1}}$ for $i \in [a, b-1]$.

We have to show that $x_{\{b\}} \overset{!}{=} \left(x_{\{a-1\}}\right)^{(-1)^{b-a+1}}$.

Since $x \in \ker d_b^\infty$, we have

$$1 = x d_b^\infty \overset{364.(1)}{=} x_{\{b\}} \cdot x_{\{b-1\}} = x_{\{b\}} \cdot \left(x_{\{a-1\}}\right)^{(-1)^{b-1-a+1}} = x_{\{b\}} \cdot \left(x_{\{a-1\}}\right)^{(-1)^{b-a}}.$$

So we have $x_{\{b\}} = \left(x_{\{a-1\}}\right)^{(-1)^{b-a+1}}$.

This proves the claim if $a \geqslant 1$.

Case $a = 0$. Let $n = 0$. Since $x \in \ker d_0^\infty$, we have

$$1 = x d_0^\infty \overset{364.(2)}{=} x_{\{0\}} \cdot (x_{\{-1\}})\mathrm{f}_S.$$

So we have $x_{\{0\}} = \left((x_{\{-1\}})\mathrm{f}_S\right)^-$.

This proves the claim for $n = 0$.

Let $n \geqslant 1$. By induction, since $x \in \bigcap_{i\in[0,b-1]} \ker \mathrm{d}_i^\infty$, we have $x_{\{i\}} = \left((x_{\{-1\}})\mathrm{f}_S\right)^{(-1)^{i+1}}$ for $i \in [0, b-1]$.

We have to show that $x_{\{b\}} \overset{!}{=} \left((x_{\{-1\}})\mathrm{f}_S\right)^{(-1)^{b-a+1}}$.

Since $x \in \ker d_b^\infty$, we have

$$1 = x d_b^\infty \overset{364.(1)}{=} x_{\{b\}} \cdot x_{\{b-1\}} = x_{\{b\}} \cdot \left((x_{\{-1\}})\mathrm{f}_S\right)^{(-1)^{b-1+1}} = x_{\{b\}} \cdot \left((x_{\{-1\}})\mathrm{f}_S\right)^{(-1)^{b}}.$$

So we have $x_{\{b\}} = \left((x_{\{-1\}})\mathrm{f}_S\right)^{(-1)^{b+1}}$.

This proves the claim if $a = 0$. □

Lemma 379 *Suppose given a cofinite subset $I \subseteq \mathbb{Z}_{\geqslant 0}$; cf.* §1.1.1 *item* 1.

Let $\ell := |\mathbb{Z}_{\geqslant 0} \backslash I|$.

We write $j_0 := -1$ *and we write* $\mathbb{Z}_{\geqslant 0} \backslash I =: \{j_1 , j_2 , \dots , j_\ell\}$, *where* $j_s < j_t$ *for* $s < t$.

So $I = \big(\bigsqcup_{k \in [0,\, \ell-1]}]j_k , j_{k+1}[\big) \sqcup]j_\ell , \infty[$.

Suppose given $x \in \bigcap_{i \in I} \ker d_i^\infty \trianglelefteqslant H_\infty$.

We write $b_k := x_{\{k\}} \in B$ *for* $k \in \mathbb{Z}_{\geqslant 0}$ *and we write* $m := x_{\{-1\}} \in M$.

So $x = (\dots , b_1 , b_0 , m) \in H_\infty$.

For $k \in [1, \ell - 1]$, *recall that we have*

$$(b_{j_k})\zeta_{\lfloor j_{k+1},\, j_k \rfloor} = (\dots , 1, \underbrace{(b_{j_k})^{(-1)^{j_{k+1}-j_k+1}}}_{\text{pos. } j_{k+1}-1}, \dots , \underbrace{b_{j_k}}_{\text{pos. } j_k} , 1, \dots , 1) \in H_\infty ;$$

cf. Definition 367.(3).

Moreover, recall that we have

$$m\zeta_{\lfloor j_1,\, -1 \rfloor} = (\dots , 1, \underbrace{(m\mathrm{f}_S)^{(-1)^{j_1}}}_{\text{pos. } j_1-1}, \dots , \underbrace{(m\mathrm{f}_S)^-}_{\text{pos. } 0}, \underbrace{m}_{\text{pos. } -1}) \in H_\infty ;$$

cf. Definition 367.(2).

We have

$$x = \Big(\prod_{k \in \lfloor \ell-1, 1 \rfloor} (b_{j_k})\zeta_{\lfloor j_{k+1},\, j_k \rfloor} \Big) \cdot m\zeta_{\lfloor j_1,\, -1 \rfloor} .$$

For an example how such an element x looks like, cf. Example 380 *below.*

Proof. By construction, we have

$$x \in \big(\bigcap_{k \in [0, \ell-1]} \ \bigcap_{i \in\,]j_k\, ,\, j_{k+1}[} \ker d_i^\infty \big) \cap \big(\bigcap_{i \in\,]j_\ell\, ,\, \infty[} \ker d_i^\infty \big) .$$

Since $x \in \bigcap_{i \in\,]j_\ell\, ,\, \infty[} \ker d_i^\infty$, we have $b_i = 1$ for $i \in \mathbb{Z}_{\geqslant j_\ell}$; cf. Remark 366.

Moreover, since

$$x \in \bigcap_{i \in\,]-1,\, j_1[} \ker d_i^\infty = \bigcap_{i \in [0,\, j_1 - 1]} \ker d_i^\infty ,$$

we have, by Remark 378

$$b_i = (m\mathrm{f}_S)^{(-1)^{i+1}}$$

for $i \in \,]-1, j_1[$.

We consider the element $m\zeta_{\rfloor j_1, -1\rfloor} \in H_\infty$; cf. Definition 367.(3).

Then

$$\begin{aligned} m\zeta_{\rfloor j_1, -1\rfloor} &\overset{362.(1)}{=} \big(\prod_{i\in\, \rfloor j_1, 0\rfloor} \big((m\mathrm{f}_S)^{(-1)^{i+1}}\big) s_i^\infty\big) \cdot ms_{-1}^\infty \\ &= \big(\prod_{i\in\rfloor j_1, 0\rfloor} b_i s_i^\infty\big) \cdot ms_{-1}^\infty\,. \end{aligned}$$

Let $k \in [1, \ell-1]$. Since $x \in \bigcap_{i\in]j_k,\, j_{k+1}[} \ker d_i^\infty = \bigcap_{i\in[j_k+1,\, j_{k+1}-1]} \ker d_i^\infty$, we have, by Remark 378,

$$b_i = (b_{j_k})^{(-1)^{i-(j_k+1)-1}} = (b_{j_k})^{(-1)^{i-j_k}}$$

for $i \in [j_k,\, j_{k+1}[$.

We consider the element $(b_{j_k})\zeta_{\rfloor j_{k+1}, j_k\rfloor} \in H_\infty$; cf. Definition 367.(3).

Then

$$\begin{aligned} (b_{j_k})\zeta_{\rfloor j_{k+1}, j_k\rfloor} &\overset{362.(1)}{=} \big(\prod_{i\in\rfloor j_{k+1},\, j_k\rfloor} \big((b_{j_k})^{(-1)^{i-j_k}}\big) s_i^\infty\big) \\ &= \prod_{i\in\rfloor j_{k+1}\,,\, j_k\rfloor} b_i s_i^\infty\,. \end{aligned}$$

So we have

$$\begin{aligned} x &\overset{362.(1)}{=} \big(\prod_{i\in\rfloor\infty,\, 0\rfloor} b_i s_i^\infty\big) \cdot ms_{-1}^\infty \\ &= \big(\prod_{i\in\rfloor j_\ell\,,\, 0\rfloor} b_i s_i^\infty\big) \cdot ms_{-1}^\infty \\ &= \Big(\prod_{k\in\lfloor \ell-1,\, 1\rfloor} \big(\prod_{i\in\,\rfloor j_{k+1}\,,\, j_k\rfloor} b_i s_i^\infty\big)\Big) \cdot \big(\prod_{i\in\,\rfloor j_1\,,\, 0\rfloor} b_i s_i^\infty\big) \cdot ms_{-1}^\infty \\ &= \big(\prod_{k\in\lfloor \ell-1,\, 1\rfloor} (b_{j_k})\zeta_{\rfloor j_{k+1},\, j_k\rfloor}\big) \cdot m\zeta_{\rfloor j_1,\, -1\rfloor}\,. \end{aligned}$$

□

Example 380 We consider the situation given in Lemma 379.

Let $I := \mathbb{Z}_{\geqslant 0}\backslash\{2,3,5\}$.

Then we have $\mathbb{Z}_{\geqslant 0}\backslash I = \{2,3,5\}$ and $\ell = |\mathbb{Z}_{\geqslant 0}\backslash I| = 3$. Moreover, we have $j_0 = -1$, $j_1 = 2$, $j_2 = 3$ and $j_3 = 5$.

So we have

$$\begin{array}{rcl} I & = & \ ]j_0\,,j_1[\ \sqcup\ ]j_1\,,j_2[\ \sqcup\ ]j_2\,,j_3[\ \sqcup\ ]j_3\,,\infty[\ =\ ]-1,2[\ \sqcup\ ]2,3[\ \sqcup\ ]3,5[\ \sqcup\ ]5,\infty[\\ & = & \{0,1,4,6,7,8,9,\dots\}\,. \end{array}$$

and

$$\begin{array}{rcl} x & = & \Big(\prod\limits_{k\in\lfloor \ell-1,1\rfloor}(b_{j_k})\zeta_{\rfloor j_{k+1},\,j_k\lfloor}\Big)\cdot m\zeta_{\rfloor j_1,\,-1\rfloor} \\ & = & \Big(\prod\limits_{k\in\lfloor 2,1\rfloor}(b_{j_k})\zeta_{\rfloor j_{k+1},\,j_k\lfloor}\Big)\cdot m\zeta_{\rfloor 2,\,-1\rfloor} \\ & = & (b_{j_2})\zeta_{\rfloor j_3,\,j_2\rfloor}\cdot(b_{j_1})\zeta_{\rfloor j_2,\,j_1\rfloor}\cdot m\zeta_{\rfloor j_1,\,-1\rfloor} \\ & = & (b_3)\zeta_{\lfloor 4,\,3\rfloor}\cdot(b_2)\zeta_{\lfloor 2,\,2\rfloor}\cdot m\zeta_{\lfloor 1,\,-1\rfloor} \\ & = & (\dots,1,\underbrace{b_3^-}_{\text{pos. }4},\underbrace{b_3}_{\text{pos. }3},1,\,1\,,1,\,1)\cdot(\dots,1,1\,,1\,,\underbrace{b_2}_{\text{pos. }2},\,1\,,1,\,1) \\ & & \cdot\,\big(\dots,1,1\,,1\,,1,\,\underbrace{m\mathrm{f}_S}_{\text{pos. }1},\underbrace{(m\mathrm{f}_S)^-}_{\text{pos. }0},\,\underbrace{m}_{\text{pos. }-1}\big)\,. \\ & = & \big(\dots,b_3^-\,,b_3\,,b_2\,,m\mathrm{f}_S\,,(m\mathrm{f}_S)^-\,,m\big)\,. \end{array}$$

Proposition 381 (The Conduché condition)

Suppose given cofinite subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

We have

$$\big[\bigcap_{i\in I}\ker d_i^\infty\,,\bigcap_{j\in J}\ker d_j^\infty\big] = 1\,.$$

Proof. Suppose given $x \in \bigcap\limits_{i\in I}\ker d_i^\infty$ and suppose given $y \in \bigcap\limits_{j\in J}\ker d_j^\infty$.

We have to shows that $[x,y] \stackrel{!}{=} 1$.

We write $b_k := x_{\{k\}} \in B$ for $k \in \mathbb{Z}_{\geqslant 0}$ and we write $m := x_{\{-1\}} \in M$.

So $x = (\dots,b_1\,,b_0\,,m) \in H_\infty$.

We write $c_k := y_{\{k\}} \in B$ for $k \in \mathbb{Z}_{\geqslant 0}$ and we write $n := y_{\{-1\}} \in M$.

So $y = (\dots, c_1, c_0, n) \in H_\infty$.

We write $\mathbb{Z}_{\geqslant 0} \backslash I =: \{i_1, i_2, \dots, i_\ell\}$ where $i_s < i_t$ for $s < t$.

We write $\mathbb{Z}_{\geqslant 0} \backslash J =: \{j_1, j_2, \dots, j_r\}$ where $j_s < j_t$ for $s < t$.

Note that we have $(\mathbb{Z}_{\geqslant 0} \backslash I) \cap (\mathbb{Z}_{\geqslant 0} \backslash J) = \emptyset$ since $I \cup J = \mathbb{Z}_{\geqslant 0}$.

By Lemma 379, we have

$$x = \Big(\prod_{k \in \lfloor \ell-1, 1 \rfloor} (b_{i_k}) \zeta_{\rfloor i_{k+1}, i_k \lfloor} \Big) \cdot m \zeta_{\rfloor i_1, -1 \rfloor}$$

and

$$y = \Big(\prod_{k \in \lfloor r-1, 1 \rfloor} (c_{j_k}) \zeta_{\rfloor j_{k+1}, j_k \lfloor} \Big) \cdot n \zeta_{\rfloor j_1, -1 \rfloor} .$$

We show that $\big[m\zeta_{\rfloor i_1, -1 \rfloor}, n\zeta_{\rfloor j_1, -1 \rfloor} \big] \stackrel{!}{=} 1$.

Since $(\mathbb{Z}_{\geqslant 0} \backslash I) \cap (\mathbb{Z}_{\geqslant 0} \backslash J) = \emptyset$, we have $i_1 \neq j_1$. So, by Lemma 377, we have

$$\big[m\zeta_{\rfloor i_1, -1 \rfloor}, n\zeta_{\rfloor j_1, -1 \rfloor} \big] = 1 .$$

Let $k \in [1, r-1]$. *We show that* $\big[m\zeta_{\rfloor i_1, -1 \rfloor}, (c_{j_k}) \zeta_{\rfloor j_{k+1}, j_k \lfloor} \big] \stackrel{!}{=} 1$.

Since $(\mathbb{Z}_{\geqslant 0} \backslash I) \cap (\mathbb{Z}_{\geqslant 0} \backslash J) = \emptyset$, we have $\{j_{k+1}, j_k\} \cap \{i_1\} = \emptyset$. So, by Lemma 376, we have

$$\big[m\zeta_{\rfloor i_1, -1 \rfloor}, (c_{j_k}) \zeta_{\rfloor j_{k+1}, j_k \lfloor} \big] = 1 .$$

Let $k \in [1, \ell-1]$. *We show that* $\big[n\zeta_{\rfloor j_1, -1 \rfloor}, (b_{i_k}) \zeta_{\rfloor i_{k+1}, i_k \lfloor} \big] \stackrel{!}{=} 1$.

Since $(\mathbb{Z}_{\geqslant 0} \backslash I) \cap (\mathbb{Z}_{\geqslant 0} \backslash J) = \emptyset$, we have $\{i_{k+1}, i_k\} \cap \{j_1\} = \emptyset$. So, by Lemma 376, we have

$$\big[n\zeta_{\rfloor j_1, -1 \rfloor}, (b_{i_k}) \zeta_{\rfloor i_{k+1}, i_k \lfloor} \big] = 1 .$$

Let $k \in [1, \ell-1]$ and let $q \in [1, r-1]$. *We show that* $\big[(b_{i_k}) \zeta_{\rfloor i_{k+1}, i_k \lfloor}, (c_{q_k}) \zeta_{\rfloor j_{q+1}, j_q \lfloor} \big] \stackrel{!}{=} 1$.

Since $(\mathbb{Z}_{\geqslant 0} \backslash I) \cap (\mathbb{Z}_{\geqslant 0} \backslash J) = \emptyset$, we have $\{i_{k+1}, i_k\} \cap \{j_{q+1}, j_q\} = \emptyset$. So, by Lemma 375, we have

$$\big[(b_{i_k}) \zeta_{\rfloor i_{k+1}, i_k \lfloor}, (c_{j_q}) \zeta_{\rfloor j_{q+1}, j_q \lfloor} \big] = 1 .$$

Altogether, this shows

$$[x,y] = \Big[\big(\prod_{k\in\lfloor \ell-1,1\rfloor} (b_{i_k})\zeta_{\rfloor i_{k+1},\, i_k\rfloor} \big) \cdot m\zeta_{\rfloor i_1,\, -1\rfloor} \,,\, \big(\prod_{k\in\lfloor r-1,1\rfloor} (c_{j_k})\zeta_{\rfloor j_{k+1},\, j_k\rfloor} \big) \cdot n\zeta_{\rfloor j_1,\, -1\rfloor} \Big] = 1\,.$$

□

Corollary 382 *Suppose given subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.*

We have

$$\big[\bigcap_{i\in I} \ker d_i^\infty \,,\, \bigcap_{j\in J} \ker d_j^\infty \big] = 1\,.$$

Proof. Suppose given $x \in \bigcap\limits_{i\in I} \ker d_i^\infty$ and suppose given $y \in \bigcap\limits_{j\in J} \ker d_j^\infty$.

We may choose some $k \in \mathbb{Z}_{\geqslant 0}$ with $x_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant k}$. So, for $i \in \mathbb{Z}_{\geqslant k+1}$, we have

$$xd_i^\infty = x_{\{i\}} \cdot x_{\{i-1\}} = 1 \cdot 1 = 1\,.$$

This shows that $x \in \bigcap\limits_{i\in \tilde{I}} \ker d_i^\infty$, where $\tilde{I} := I \cup \mathbb{Z}_{\geqslant k+1}$.

We may choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ with $y_{\{j\}} = 1$ for $j \in \mathbb{Z}_{\geqslant \ell}$. So, for $j \in \mathbb{Z}_{\geqslant \ell+1}$, we have

$$yd_j^\infty = y_{\{j\}} \cdot y_{\{j-1\}} = 1 \cdot 1 = 1\,.$$

This shows that $y \in \bigcap\limits_{j\in \tilde{J}} \ker d_j^\infty$, where $\tilde{J} := J \cup \mathbb{Z}_{\geqslant \ell+1}$.

Note that $\tilde{I}$ and $\tilde{J}$ are cofinite. So, by Proposition 381, we have $[x,y] = 1$. □

17.3 An example for the group H_n via Magma

17.3.1 The group H_n for $B = \mathrm{S}_3$ and $M = \mathrm{A}_3$

We consider the symmetric group S_3 and the alternating group $\mathrm{A}_3 \trianglelefteqslant \mathrm{S}_3$.

We consider the map $\kappa_3 \colon \mathrm{S}_3 \times \mathrm{S}_3 \to \mathrm{A}_3$, $(b,c) \mapsto [b,c]$, and we consider the inclusion morphism $\iota_3 \colon \mathrm{A}_3 \to \mathrm{S}_3$, $m \mapsto m$.

Then $V_3 := (\mathrm{A}_3\,, \mathrm{S}_3\,, \kappa_3\,, \iota_3)$ is a stable crossed module; cf. Example 334.

Let $n \in [0,3]$. For V_3, we shall calculate the group H_n together with its multiplication for $n \in [-1,6]$; cf. Lemma 342.

Recall that $H_{-1} = \mathrm{A}_3$. Recall that the multiplication in $H_n = \mathrm{S}_3 \ltimes H_{n-1}$ is given as follows. For b, $\tilde{b} \in \mathrm{S}_3$ and h, $\tilde{h} \in H_{n-1}$, we have

$$(b, h) \cdot (\tilde{b}, \tilde{h}) = \left(b \cdot \tilde{b}, h(\tilde{b}\alpha_{n-1}) \cdot \tilde{h}\right),$$

using $\alpha_{n-1} \colon \mathrm{S}_3 \to \mathrm{Aut}(H_{n-1})$. To calculate α_{n-1}, we shall use Lemma 347.(2).

In H_0, for $m \in \mathrm{A}_3$ and $\tilde{b} \in \mathrm{S}_3$, We write

$$m(\tilde{b}\alpha_{-1}) = m \cdot [m\iota, \tilde{b}] = m \cdot (m^- \cdot \tilde{b}^- \cdot m \cdot \tilde{b}) = m^{\tilde{b}}.$$

```
B := SymmetricGroup(3);
M := AlternatingGroup(3);
```

The multiplication in H_0 is given as follows. We write `alpha` $= \alpha_{-1} \colon \mathrm{S}_3 \to \mathrm{Aut}(\mathrm{A}_3)$.

```
alpha := function(m,bb) // yields m^bb
 return M!(m^bb);
end function;

mult0 := function(x,xx) // x = <b0,m>,  xx = <bb0,mm> are multiplied
 b0  := x[1];
 m   := x[2];
 bb0 := xx[1];
 mm  := xx[2];
 return <b0 * bb0, alpha(m,bb0) * mm>;
end function;
```

The multiplication `mult1` in H_1 is given as follows. We write `alpha0` $= \alpha_0 \colon \mathrm{S}_3 \to \mathrm{Aut}(H_0)$.

```
alpha0 := function(x,bb) // yields x^bb , where x = <b0,m>
 b0 := x[1];
 m  := x[2];
 return <b0^bb, M!((bb,b0) * m)>;
end function;

mult1 := function(x,xx)
// x = <b1,b0,m>, xx = <bb1,bb0,mm> are multiplied
 b1  := x[1];
 b0  := x[2];
 m   := x[3];
 bb1 := xx[1];
 bb0 := xx[2];
```

```
 mm  := xx[3];
 y := mult0(alpha0(<b0,m>,bb1),<bb0,mm>);
 return <b1 * bb1, y[1], y[2]>;
end function;
```

The multiplication `mult2` in H_2 is given as follows. We write `alpha1` $= \alpha_1 \colon \mathrm{S}_3 \to \mathrm{Aut}(H_1)$.

```
alpha1 := function(x,bb) // yields x^bb , where x = <b1,b0,m>
 b1 := x[1];
 b0 := x[2];
 m  := x[3];
 return <b1^bb, (bb,b1) * b0, M!((b1,bb)^b0 * m)>;
end function;

mult2 := function(x,xx)
// x = <b2,b1,b0,m>, xx = <bb2,bb1,bb0,mm> are multiplied
 b2  := x[1];
 b1  := x[2];
 b0  := x[3];
 m   := x[4];
 bb2 := xx[1];
 bb1 := xx[2];
 bb0 := xx[3];
 mm  := xx[4];
 y := mult1(alpha1(<b1,b0,m>,bb2),<bb1,bb0,mm>);
 return <b2 * bb2, y[1], y[2], y[3]>;
end function;
```

The multiplication `mult3` in H_3 is given as follows. We write `alpha2` $= \alpha_2 \colon \mathrm{S}_3 \to \mathrm{Aut}(H_2)$.

```
alpha2 := function(x,bb) // yields x^bb , where x = <b2,b1,b0,m>
 b2 := x[1];
 b1 := x[2];
 b0 := x[3];
 m  := x[4];
 return <b2^bb, (bb,b2) * b1, (b2,bb)^b1 * b0, M!((bb,b2)^(b1*b0) * m)>;
end function;

mult3 := function(x,xx)
// x = <b3,b2,b1,b0,m>,  xx = <bb3,bb2,bb1,bb0,mm> are multiplied
 b3  := x[1];
 b2  := x[2];
 b1  := x[3];
 b0  := x[4];
```

```
 m   := x[5];
 bb3 := xx[1];
 bb2 := xx[2];
 bb1 := xx[3];
 bb0 := xx[4];
 mm  := xx[5];
 y := mult2(alpha2(<b2,b1,b0,m>,bb3),<bb2,bb1,bb0,mm>);
 return <b3 * bb3, y[1], y[2], y[3], y[4]>;
end function;
```

The multiplication `mult4` in H_4 is given as follows. We write `alpha3` $= \alpha_3 \colon \mathrm{S}_3 \to \mathrm{Aut}(H_3)$.

```
alpha3 := function(x,bb) // yields x^bb , where x = <b3,b2,b1,b0,m>
 b3 := x[1];
 b2 := x[2];
 b1 := x[3];
 b0 := x[4];
 m  := x[5];
 return <b3^bb, (bb,b3) * b2, (b3,bb)^b2 * b1, (bb,b3)^(b2*b1) * b0,
          M!((b3,bb)^(b2*b1*b0) * m)>;
end function;

mult4 := function(x,xx)
// x = <b4,b3,b2,b1,b0,m>,  xx = <bb4,bb3,bb2,bb1,bb0,mm> are multiplied
 b4 := x[1];
 b3 := x[2];
 b2 := x[3];
 b1 := x[4];
 b0 := x[5];
 m  := x[6];
 bb4 := xx[1];
 bb3 := xx[2];
 bb2 := xx[3];
 bb1 := xx[4];
 bb0 := xx[5];
 mm  := xx[6];
 y := mult3(alpha3(<b3,b2,b1,b0,m>,bb4),<bb3,bb2,bb1,bb0,mm>);
 return <b4 * bb4, y[1], y[2], y[3], y[4], y[5]>;
end function;
```

The multiplication `mult5` in H_5 is given as follows. We write `alpha4` $= \alpha_4 \colon \mathrm{S}_3 \to \mathrm{Aut}(H_4)$.

```
alpha4 := function(x,bb) // yields x^bb , where x = <b4,b3,b2,b1,b0,m>
 b4 := x[1];
```

```
 b3 := x[2];
 b2 := x[3];
 b1 := x[4];
 b0 := x[5];
 m  := x[6];
 return <b4^bb, (bb,b4) * b3, (b4,bb)^b3 * b2, (bb,b4)^(b3*b2) * b1,
         (b4,bb)^(b3*b2*b1) * b0, M!((bb,b4)^(b3*b2*b1*b0) * m)>;
end function;


mult5 := function(x,xx)
// x = <b5,b4,b3,b2,b1,b0,m>,
//xx = <bb5,bb4,bb3,bb2,bb1,bb0,mm> are multiplied
 b5 := x[1];
 b4 := x[2];
 b3 := x[3];
 b2 := x[4];
 b1 := x[5];
 b0 := x[6];
 m  := x[7];
 bb5 := xx[1];
 bb4 := xx[2];
 bb3 := xx[3];
 bb2 := xx[4];
 bb1 := xx[5];
 bb0 := xx[6];
 mm  := xx[7];
 y := mult4(alpha4(<b4,b3,b2,b1,b0,m>,bb5),<bb4,bb3,bb2,bb1,bb0,mm>);
 return <b5 * bb5, y[1], y[2], y[3], y[4], y[5], y[6]>;
end function;
```

The multiplication `mult6` in H_6 is given as follows. We write `alpha5` $= \alpha_5 \colon \mathrm{S}_3 \to \mathrm{Aut}(H_5)$.

```
alpha5 := function(x,bb) // yields x^bb , where x = <b5,b4,b3,b2,b1,b0,m>
 b5 := x[1];
 b4 := x[2];
 b3 := x[3];
 b2 := x[4];
 b1 := x[5];
 b0 := x[6];
 m  := x[7];
 return <b5^bb, (bb,b5) * b4, (b5,bb)^b4 * b3, (bb,b5)^(b4*b3) * b2,
         (b5,bb)^(b4*b3*b2) * b1, (bb,b5)^(b4*b3*b2*b1) * b0,
           M!( (b5,bb)^(b4*b3*b2*b1*b0) * m )>;
end function;
```

```
mult6 := function(x,xx)
// x = <b6,b5,b4,b3,b2,b1,b0,m>,
//xx = <bb6,bb5,bb4,bb3,bb2,bb1,bb0,mm> are multiplied
 b6 := x[1];
 b5 := x[2];
 b4 := x[3];
 b3 := x[4];
 b2 := x[5];
 b1 := x[6];
 b0 := x[7];
 m  := x[8];
 bb6 := xx[1];
 bb5 := xx[2];
 bb4 := xx[3];
 bb3 := xx[4];
 bb2 := xx[5];
 bb1 := xx[6];
 bb0 := xx[7];
 mm  := xx[8];
 y := mult5(alpha5(<b5,b4,b3,b2,b1,b0,m>,bb6),<bb5,bb4,bb3,bb2,bb1,bb0,mm>);
 return <b6 * bb6, y[1], y[2], y[3], y[4], y[5], y[6], y[7]>;
end function;
```

17.3.2 The multiplication formula for H_n

We implement the multiplication formula given in Lemma 349.

```
// The multiplication in H_n

pos := function(j,k)
// pos(j,k) yields the position of bk in the tuple <bj,..,b0,m>, where m = b_{-1}
 return j-1-k;
end function;

multoo := function(x,xx)
 //x = <bn,...,b0,m> and xx = <bbn,...,bb0,mm> are multiplied
 n := #x-2;

 prod := <B!1 : k in [0..n]>;  // the entries bi
 Append(~prod,M!1);        // the entry m
 prod[1] := x[1] * xx[1];    // Calculates the entry in position n

 if n eq -1 then   // Case n = -1
```

```
  return prod;
 else
  if n eq 0 then   // Calculates the entry in position n-1
   prod[2] := alpha(x[2],xx[1]) * xx[2];
   return prod;
  else
   prod[2] := x[2]^xx[1] * xx[2];
  end if;
 end if;

 for k in [-1..n-2] do
  for i in [t : t in [k..n-2] | (t mod 2) eq (k mod 2)] do
  // i congruent to k mod 2
   if IsEmpty([k+1..i]) then
    expo := xx[pos(#x,i+1)];
   else
    expo := xx[pos(#x,i+1)] * &*[ x[pos(#x,j)] * xx[pos(#x,j)] :
                 j in Reverse([k+1..i])];
   end if;
   prod[pos(#x,k)] *:= (xx[pos(#x,i+2)],x[pos(#x,i+1)])^expo;
  end for;

  for i in Reverse([t : t in [k..n-2] | (t mod 2) eq ( (k+1) mod 2)]) do
  // i congruent k+1 mod 2  + inverted
   if IsEmpty([k+1..i]) then
    expo := xx[pos(#x,i+1)];
   else
    expo := xx[pos(#x,i+1)] * &*[ x[pos(#x,j)] * xx[pos(#x,j)] :
                 j in Reverse([k+1..i])];
   end if;
   prod[pos(#x,k)] *:= (x[pos(#x,i+1)],xx[pos(#x,i+2)])^expo;
  end for;
  prod[pos(#x,k)] *:= x[pos(#x,k)]^xx[pos(#x,k+1)] * xx[pos(#x,k)]; // rest term
 end for;
 return prod;
end function;
```

We want to verify the formula from Lemma 349 by comparison with `mult1`, `mult2`, `mult3` and `mult4` from § 17.3.1.

```
// Test for mult1
test1 := function()
 for b1,bb1 in B do
  for b0,bb0 in B do
```

```
   for m,mm in M do
    x  := <b1,b0,m>;
    xx := <bb1,bb0,mm>;
    prod1 := mult1(x,xx);
    prod2 := multoo(x,xx);
    if not prod1 eq prod2 then
     return false;
    end if;
   end for;
  end for;
 end for;
 return true;
end function;
```

The program is called by `test1();`.

```
// Test for mult2
test2 := function()
 for b2,bb2 in B do
  for b1,bb1 in B do
   for b0,bb0 in B do
    for m,mm in M do
     x  := <b2,b1,b0,m>;
     xx := <bb2,bb1,bb0,mm>;
     prod1 := mult2(x,xx);
     prod2 := multoo(x,xx);
     if not prod1 eq prod2 then
      return false;
     end if;
    end for;
   end for;
  end for;
 end for;
 return true;
end function;
```

The program is called by `test2();`.

```
// Test for mult3
test3 := function()
 for b3,bb3 in B do
  for b2,bb2 in B do
   for b1,bb1 in B do
```

```
    for b0,bb0 in B do
     for m,mm in M do
      x  := <b3,b2,b1,b0,m>;
      xx := <bb3,bb2,bb1,bb0,mm>;
      prod1 := mult3(x,xx);
      prod2 := multoo(x,xx);
      if not prod1 eq prod2 then
       return false;
      end if;
     end for;
    end for;
   end for;
  end for;
 end for;
 return true;
end function;
```

The program is called by `test3();`.

```
// Test for mult4
test4 := function()
 for b4,bb4 in B do
  for b3,bb3 in B do
   for b2,bb2 in B do
    for b1,bb1 in B do
     for b0,bb0 in B do
      for m,mm in M do
       x  := <b4,b3,b2,b1,b0,m>;
       xx := <bb4,bb3,bb2,bb1,bb0,mm>;
       prod1 := mult4(x,xx);
       prod2 := multoo(x,xx);
       if not prod1 eq prod2 then
        return false;
       end if;
      end for;
     end for;
    end for;
   end for;
  end for;
 end for;
 return true;
end function;
```

The program is called by `test4();`.

For `B = SymmetricGroup(3)` and `M = AlternatingGroup(3)`, the test programs `test1`, `test2`, `test3` and `test4` yield `true`.

17.3.3 The group H_n for $B = \mathrm{S}_4$ and $M = \mathrm{A}_4$

We consider the symmetric group S_4 and the alternating group $\mathrm{A}_4 \trianglelefteq \mathrm{S}_4$.

Similar to the the stable crossed module V_3 in §17.3.1, we have the stable crossed module $V_4 := (\mathrm{A}_4, \mathrm{S}_4, \kappa_4, \iota_4)$ with

$$\kappa_4 \colon \mathrm{S}_4 \times \mathrm{S}_4 \to \mathrm{A}_4, \ (b, c) \mapsto [b, c],$$
$$\iota_4 \colon \mathrm{A}_4 \to \mathrm{A}_4, \ m \mapsto m.$$

We let `B := SymmetricGroup(4)` and `M := AlternatingGroup(4)`.

We use the same Magma code as in §17.3.1 to define the multiplication in $H_n = \mathrm{S}_4 \ltimes H_{n-1}$, where $H_{-1} = \mathrm{A}_4$, for $n \in \{1, 2\}$.

Moreover, we use the Magma code in §17.3.2 to compare the results the code yields in §17.3.1 and in §17.3.2.

For `B = SymmetricGroup(4)` and `M = AlternatingGroup(4)`, the test programs `test1` and `test2` yield `true`.

17.3.4 The group H_n for $B = \mathrm{A}_4$ and $M = \mathrm{V}_4$

We consider the alternating group A_4 and the Klein four-group $\mathrm{V}_4 \trianglelefteq \mathrm{A}_4$. We write $\mathrm{K}_4 := \mathrm{V}_4$ to avoid confusion with the stable crossed module V_4 from §17.3.3.

Similar to the the stable crossed module V_3 in §17.3.1, we have the stable crossed module $W_4 := (\mathrm{K}_4, \mathrm{A}_4, \tilde{\kappa}_4, \tilde{\iota}_4)$ with

$$\tilde{\kappa}_4 \colon \mathrm{A}_4 \times \mathrm{A}_4 \to \mathrm{K}_4, \ (b, c) \mapsto [b, c],$$
$$\tilde{\iota}_4 \colon \mathrm{K}_4 \to \mathrm{K}_4, \ m \mapsto m.$$

We let `B := AlternatingGroup(4)` and `M := sub<B | B!(1,2)(3,4), B!(1,3)(2,4)>`.

We use the same Magma code as in §17.3.1 to define the multiplication in $H_n = \mathrm{A}_4 \ltimes H_{n-1}$, where $H_{-1} = \mathrm{K}_4$, for $n \in \{1, 2, 3\}$.

Moreover, we use the Magma code in §17.3.2 to compare the results the code yields in §17.3.1 and in §17.3.2.

For `B = AlternatingGroup(4)` and `M := sub<B | B!(1,2)(3,4), B!(1,3)(2,4)>`, the test programs `test1`, `test2` and `test3` yield `true`.

17.3.5 The orders of elements in H_n

The calculations

We calculate the orders of the elements in H_0 for $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$ and for $\mathtt{B} = \mathrm{S}_4$, $\mathtt{M} = \mathrm{A}_4$.

```
// Order of elements in H_0

order0 := function(x) // yields the order of x = <b0,m> in H_0
 ord := 1;
 y := x;
 while not y eq <B!1,M!1> do
  ord +:= 1;
  y := mult0(x,y);
 end while;
 return ord;
end function;

orders_H0 := function();
 order_list := [];
 order_multi := [];
 for b0 in B do
  for m in M do
   ord := order0(<b0,m>);
   if not ord in order_list then
    Append(~order_list, ord);
    Append(~order_multi, <ord,1>);
   else
    order_multi[Index(order_list,ord)] +:= <0,1>;
   end if;
  end for;
 end for;
 return order_multi;
end function;
```

This yields the following.

```
     orders_H0(); // B = S3, M = A3
// [ <1, 1>, <3, 8>, <2, 9> ]
```

```
     orders_H0(); // B = S4, M = A4
// [ <1, 1>, <2, 51>, <3, 80>, <4, 108>, <6, 48> ]
```

So in the case $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$, we have 1 element of order 4, 8 elements of order 3 and 9 elements of order 2.

We calculate the order of the elements in H_1 for $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$ and for $\mathtt{B} = \mathrm{S}_4$, $\mathtt{M} = \mathrm{A}_4$.

```
// Order of elements in H_1

order1 := function(x) // yields the order of x = <b1,b0,m> in H_1
 ord := 1;
 y := x;
 while not y eq <B!1,B!1,M!1> do
  ord +:= 1;
  y := mult1(x,y);
 end while;
 return ord;
end function;

orders_H1 := function();
 order_list := [];
 order_multi := [];
 for b1 in B do
  for b0 in B do
   for m in M do
    ord := order1(<b1,b0,m>);
    if not ord in order_list then
     Append(~order_list, ord);
     Append(~order_multi, <ord,1>);
    else
     order_multi[Index(order_list,ord)] +:= <0,1>;
    end if;
   end for;
  end for;
 end for;
 return order_multi;
end function;
```

This yields the following.

```
     orders_H1();  // B = S3, M = A3
// [ <1, 1>, <3, 26>, <2, 27>, <6, 54> ]
```

```
     orders_H1(); // B = S4, M = A4
// [ <1, 1>, <2, 495>, <3, 728>, <4, 1296>, <6, 1800>, <12, 2592> ]
```

We calculate the orders of the elements in H_2 for $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$ and for $\mathtt{B} = \mathrm{S}_4$, $\mathtt{M} = \mathrm{A}_4$.

```
// Order of elements in H_2

order2 := function(x) // yields the order of x = <b2,b1,b0,m> in H_2
 ord := 1;
 y := x;
 while not y eq <B!1,B!1,B!1,M!1> do
  ord +:= 1;
  y := mult2(x,y);
 end while;
 return ord;
end function;

orders_H2 := function();
 order_list := [];
 order_multi := [];
 for b2 in B do
  for b1 in B do
   for b0 in B do
    for m in M do
     ord := order2(<b2,b1,b0,m>);
     if not ord in order_list then
      Append(~order_list, ord);
      Append(~order_multi, <ord,1>);
     else
      order_multi[Index(order_list,ord)] +:= <0,1>;
     end if;
    end for;
   end for;
  end for;
 end for;
 return order_multi;
end function;
```

This yields the following.

```
     orders_H2(); // B = S3, M = A3
// [ <1, 1>, <3, 80>, <2, 135>, <6, 432> ]
```

```
     orders_H2(); // B = S4, M = A4
// [ <1, 1>, <2, 5007>, <3, 6560>, <4, 29808>, <6, 41568>, <12, 82944> ]
```

We calculate the orders of the elements in H_3 for $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$ and for $\mathtt{B} = \mathrm{S}_4$, $\mathtt{M} = \mathrm{A}_4$.

```
// Order of elements in H_3

order3 := function(x) //yields the order of x = <b3,b2,b1,b0,m> in H_3
 ord := 1;
 y := x;
 while not y eq <B!1,B!1,B!1,B!1,M!1> do
  ord +:= 1;
  y := mult3(x,y);
 end while;
 return ord;
end function;

orders_H3 := function();
 order_list := [];
 order_multi := [];
 for b3 in B do
  for b2 in B do
   for b1 in B do
    for b0 in B do
     for m in M do
      ord := order3(<b3,b2,b1,b0,m>);
      if not ord in order_list then
       Append(~order_list, ord);
       Append(~order_multi, <ord,1>);
      else
       order_multi[Index(order_list,ord)] +:= <0,1>;
      end if;
     end for;
    end for;
   end for;
  end for;
 end for;
 return order_multi;
end function;
```

This yields the following.

```
     orders_H3(); // B = S3, M = A3
// [ <1, 1>, <3, 242>, <2, 495>, <6, 3150> ]

     orders_H3(); // B = S4, M = A4
// [ <1, 1>, <2, 49983>, <3, 59048>, <4, 457920>, <6, 839640>, <12, 2574720> ]
```

We calculate the orders of the elements in H_4 for $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$.

```
// Order of elements in H_4
order4 := function(x) //yields the order of x = <b4,b3,b2,b1,b0,m> in H_4
 ord := 1;
 y := x;
 while not y eq <B!1,B!1,B!1,B!1,B!1,M!1> do
  ord +:= 1;
  y := mult4(x,y);
 end while;
 return ord;
end function;

orders_H4 := function();
 order_list := [];
 order_multi := [];
 for b4 in B do
  for b3 in B do
   for b2 in B do
    for b1 in B do
     for b0 in B do
      for m in M do
       ord := order4(<b4,b3,b2,b1,b0,m>);
       if not ord in order_list then
        Append(~order_list, ord);
        Append(~order_multi, <ord,1>);
       else
        order_multi[Index(order_list,ord)] +:= <0,1>;
       end if;
      end for;
     end for;
    end for;
   end for;
  end for;
 end for;
 return order_multi;
end function;
```

This yields the following.

```
   orders_H4(); // B = S3, M = A3
// [ <1, 1>, <3, 728>, <2, 2079>, <6, 20520> ]
```

We calculate the orders of the elements in H_5 for $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$.

```
// Order of elements in H_5

order5 := function(x) //yields the order of x = <b5,b4,b3,b2,b1,b0,m> in H_5
 ord := 1;
 y := x;
 while not y eq <B!1,B!1,B!1,B!1,B!1,B!1,M!1> do
  ord +:= 1;
  y := mult5(x,y);
 end while;
 return ord;
end function;

orders_H5 := function();
 order_list := [];
 order_multi := [];
 for b5 in B do
  for b4 in B do
   for b3 in B do
    for b2 in B do
     for b1 in B do
      for b0 in B do
       for m in M do
        ord := order5(<b5,b4,b3,b2,b1,b0,m>);
        if not ord in order_list then
         Append(~order_list, ord);
         Append(~order_multi, <ord,1>);
        else
         order_multi[Index(order_list,ord)] +:= <0,1>;
        end if;
       end for;
      end for;
     end for;
    end for;
   end for;
  end for;
 end for;
 return order_multi;
end function;
```

This yields the following.

```
    orders_H5(); // B = S3, M = A3
//[ <1, 1>, <3, 2186>, <2, 8127>, <6, 129654> ]
```

We calculate the orders of the elements in H_6 for $\mathtt{B} = \mathrm{S}_3$, $\mathtt{M} = \mathrm{A}_3$.

```
// Order of elements in H_6
order6 := function(x) //yields the order of x = <b6,b5,b4,b3,b2,b1,b0,m> in H_6
 ord := 1;
 y := x;
 while not y eq <B!1,B!1,B!1,B!1,B!1,B!1,B!1,M!1> do
  ord +:= 1;
  y := mult6(x,y);
 end while;
 return ord;
end function;

orders_H6 := function();
 order_list := [];
 order_multi := [];
 for b6 in B do
  for b5 in B do
   for b4 in B do
    for b3 in B do
     for b2 in B do
      for b1 in B do
       for b0 in B do
        for m in M do
         ord := order6(<b6,b5,b4,b3,b2,b1,b0,m>);
         if not ord in order_list then
          Append(~order_list, ord);
          Append(~order_multi, <ord,1>);
         else
          order_multi[Index(order_list,ord)] +:= <0,1>;
         end if;
        end for;
       end for;
      end for;
     end for;
    end for;
   end for;
  end for;
 end for;
 return order_multi;
end function;
```

This yields the following.

```
    orders_H6(); // B = S3, M = A3
//[ <1, 1>, <3, 6560>, <2, 32895>, <6, 800352> ]
```

The result of the experiment

Let $\mathtt{B} = \mathrm{S}_3$ and let $\mathtt{M} = \mathrm{A}_3$. For $n \in [0, 6]$, let $\#k := \left|\{x \in H_n \colon |\langle x \rangle| = k\}\right|$ be the number of elements of order k in the group H_n.

We obtain the following.

n	#1	#2	#3	#6
0	1	9	8	-
1	1	27	26	54
2	1	135	80	432
3	1	495	242	3150
4	1	2079	728	20520
5	1	8127	2186	129654
6	1	32895	6560	800352

Let $\mathtt{B} = \mathrm{S}_4$ and let $\mathtt{M} = \mathrm{A}_4$.

For $n \in [0, 3]$, let $\#k := \left|\{x \in H_n \colon |\langle x \rangle| = k\}\right|$ be the number of elements of order k in the group H_n.

We obtain the following.

n	#1	#2	#3	#4	#6	#12
0	1	51	80	108	48	-
1	1	495	728	1296	1800	2592
2	1	5007	6560	29808	41568	82944
3	1	49983	59048	457920	839640	2574720

Chapter 18

The equivalent categories of $\lfloor 1,0 \rfloor$-stable simplicial groups and stable crossed modules

18.1 The category of $\lfloor 1,0 \rfloor$-stable simplicial groups

Let $n = 1$.

Remark 383 ($\lfloor 1,0 \rfloor$-stable simplicial groups)

Suppose given a $\lfloor 1,0 \rfloor$-stable simplicial group $G \in \mathrm{Ob}\big(\lfloor 1,0 \rfloor\text{-}\mathcal{StSimpGrp}\big)$; cf. Definition 307.

For $i \in \mathbb{Z}_{\geqslant 0}$, we have the face maps $\mathrm{d}_i^{G,1} \colon G_1 \to G_0$ and the degeneracy maps $\mathrm{s}_i^{G,0} \colon G_0 \to G_1$; cf. Remark 228.

In this context of $\lfloor 1,0 \rfloor$-stable simplicial groups, we often write $\mathrm{s}_i^G := \mathrm{s}_i^{G,0}$ and $\mathrm{d}_i^G := \mathrm{d}_i^{G,1}$ for $i \in \mathbb{Z}_{\geqslant 0}$.

$$G_1 \underset{\mathrm{s}_i^G}{\overset{\mathrm{d}_j^G}{\rightleftarrows}} G_0 \underset{!}{\overset{!}{\rightleftarrows}} 1 \underset{!}{\overset{!}{\rightleftarrows}} \cdots$$

By Remark 308, the $\lfloor 1,0 \rfloor$-stable simplicial group G has the following properties.

(1) For $x \in G_1$, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $x\mathrm{d}_i^G = 1$ for $i \in \mathbb{Z}_{\geqslant \ell}$.

(2) For subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$ we have

$$\big[\bigcap_{i\in I} \ker \mathrm{d}_i^G \,, \bigcap_{j\in J} \ker \mathrm{d}_j^G\big] = 1\,.$$

(3) For i, $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{s}_i^G \blacktriangle \mathrm{d}_j^G = \begin{cases} \mathrm{id}_{G_0} & \text{if } j \in \{i, i+1\} \\ ! & \text{if } j \in \mathbb{Z}_{\geqslant 0} \backslash \{i, i+1\}\,. \end{cases}$$

Remark 384 ($\lfloor 1,0 \rfloor$-stable simplicial group morphisms)
Suppose given a $\lfloor 1,0 \rfloor$-stable simplicial group morphism $\varphi\colon G \to H$; cf. Definition 307.
Then, for $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\varphi_0 \blacktriangle \mathrm{s}_i^H = \mathrm{s}_i^G \blacktriangle \varphi_1 \quad \text{and} \quad \mathrm{d}_i^G \blacktriangle \varphi_0 = \varphi_1 \blacktriangle \mathrm{d}_i^H\,;$$

cf. Remark 184.

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_0} & H_0 \\ {\scriptstyle \mathrm{s}_i^G}\downarrow & & \downarrow{\scriptstyle \mathrm{s}_i^H} \\ G_1 & \xrightarrow[\varphi_1]{} & H_1 \end{array} \qquad \begin{array}{ccc} G_0 & \xrightarrow{\varphi_0} & H_0 \\ {\scriptstyle \mathrm{d}_i^G}\uparrow & & \uparrow{\scriptstyle \mathrm{d}_i^H} \\ G_1 & \xrightarrow[\varphi_1]{} & H_1 \end{array}$$

Remark 385 (Construction of a $\lfloor 1,0 \rfloor$-stable simplicial group)
Suppose given groups G_0, G_1. For $i \in \mathbb{Z}_{\geqslant 0}$, suppose given group morphisms $d_i\colon : G_1 \to G_0$ and $s_i\colon G_0 \to G_1$.
Suppose that the conditions $(1, 2, 3)$ hold.

(1) For $x \in G_1$, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $xd_i = 1$ for $i \in \mathbb{Z}_{\geqslant \ell}$.

(2) For subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$ we have

$$\big[\bigcap_{i\in I} \ker d_i \,, \bigcap_{j\in J} \ker d_j\big] = 1\,.$$

(3) For i, $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$s_i \blacktriangle d_j = \begin{cases} \mathrm{id}_{G_0} & \text{if } j \in \{i, i+1\} \\ ! & \text{if } j \in \mathbb{Z}_{\geqslant 0} \backslash \{i, i+1\}\,. \end{cases}$$

Then, by Lemma 311, we have the $\lfloor 1,0 \rfloor$-stable simplicial group G given as follows.
For $k \in \mathbb{Z}_{\leqslant 1}$, we have

$$[k]^{\pm} G := \begin{cases} G_k & \text{if } k \in [0,1] \\ 1 & \text{if } k \in \mathbb{Z}_{\leqslant -1}. \end{cases}$$

For $k \in \mathbb{Z}_{\leqslant 1}$, we have

$$\mathrm{d}_j^{G,k} := \begin{cases} d_j & \text{if } k = 1 \\ ! & \text{if } k \in \mathbb{Z}_{\leqslant 0}. \end{cases}$$

For $k \in \mathbb{Z}_{\leqslant 0}$, we have

$$\mathrm{s}_i^{G,k} := \begin{cases} s_i & \text{if } k = 0 \\ ! & \text{if } k \in \mathbb{Z}_{\leqslant -1}. \end{cases}$$

$$G_1 \underset{s_i}{\overset{d_j}{\rightleftarrows}} G_0 \underset{!}{\overset{!}{\rightleftarrows}} 1 \underset{!}{\overset{!}{\rightleftarrows}} \cdots$$

Remark 386 (Construction of a $\lfloor 1,0 \rfloor$-stable simplicial group morphism)
Suppose given $\lfloor 1,0 \rfloor$-stable simplicial groups G and H.
Suppose given group morphisms $\varphi_1 \colon G_1 \to H_1$ and $\varphi_0 \colon G_0 \to H_0$ such that $(1,2)$ hold.

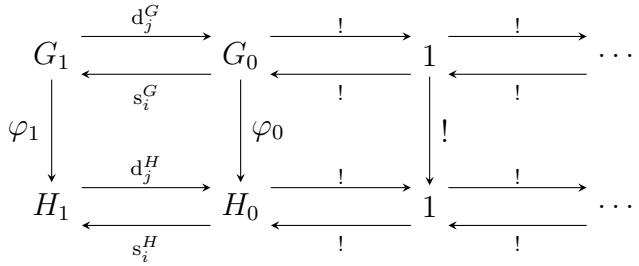
(1) For $j \in \mathbb{Z}_{\geqslant 0}$, we have $\mathrm{d}_j^G \blacktriangleleft \varphi_0 = \varphi_1 \blacktriangleleft \mathrm{d}_j^H$.

(2) For $i \in \mathbb{Z}_{\geqslant 0}$, we have $\mathrm{s}_i^G \blacktriangleleft \varphi_1 = \varphi_0 \blacktriangleleft \mathrm{s}_i^H$.

Then, by Lemma 255, $\varphi \colon G \to H$ is a pre-$\lfloor 1, -\infty \rfloor$-prestable simplicial group morphism, where

$$\varphi_k := \begin{cases} \varphi_k & \text{for } k \in [0,1] \\ ! & \text{for } k \in \mathbb{Z}_{\leqslant -1}. \end{cases}$$

Since G and H are $\lfloor 1,0 \rfloor$-stable simplicial groups, we have that φ is a $\lfloor 1,0 \rfloor$-stable simplicial group morphism; cf. Definitions 307, 232.



Lemma 387 *Suppose given a $\lfloor 1,0 \rfloor$-stable simplicial group G. Suppose given $g \in G_1$.*

Since g is finite, we may choose some $\ell \in \mathbb{Z}_{\geqslant 1}$ such that $g\mathrm{d}_j^G = 1$ for $j \in \mathbb{Z}_{\geqslant \ell}$; cf. Remark 383.(1).

There exists $x_0 \in G\mathrm{N}_1$ and $x_k \in G_1$ such that $x_k = x_k \mathrm{d}_k^G \mathrm{s}_{k-1}^G$ for $k \in [1, \ell - 1]$ and such that

$$g = \big(\prod_{k \in \lfloor \ell - 1, 1 \rfloor} x_k \big) \cdot x_0 \, .$$

Proof. We write $\mathrm{s}_i := \mathrm{s}_i^G$ and $\mathrm{d}_i := \mathrm{d}_i^G$ for $i \in \mathbb{Z}_{\geqslant 0}$.

We show this claim by induction on $\ell \in \mathbb{Z}_{\geqslant 1}$.

Let $\ell = 1$. Then $g \in G\mathrm{N}_1$. So we have

$$g = 1 \cdot g = \big(\prod_{k \in \lfloor 0, 1 \rfloor} x_k \big) \cdot g \, .$$

Now suppose that $\ell \geqslant 2$. We have $g = g\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-2} \cdot (g\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-2})^- \cdot g$.

Let $\tilde{g} := (g\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-2})^- \cdot g \in G_1$.

For $j = \ell - 1$, we have

$$\tilde{g}\mathrm{d}_j = \big((g\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-2})^- \cdot g\big)\mathrm{d}_{\ell-1} = (g\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-2}\mathrm{d}_{\ell-1})^- \cdot g\mathrm{d}_{\ell-1} = (g\mathrm{d}_{\ell-1})^- \cdot g\mathrm{d}_{\ell-1} = 1 \, .$$

For $j \geqslant \ell$, we have

$$\tilde{g}\mathrm{d}_j = (g\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-2}\,\mathrm{d}_j)^- \cdot g\mathrm{d}_j = (g\mathrm{d}_{\ell-1}\,\mathrm{d}_{j-1}\,\mathrm{s}_{\ell-2})^- \cdot 1 = (g\mathrm{d}_j\,\mathrm{d}_{\ell-2}\,\mathrm{s}_{\ell-2})^- = (1\mathrm{d}_{\ell-1}\,\mathrm{s}_{\ell-2})^- = 1 \, .$$

This shows that $\tilde{g}\mathrm{d}_j = 1$ for $j \in \mathbb{Z}_{\geqslant \ell - 1}$.

So, by induction hypothesis, we may write

$$\tilde{g} = \big(\prod_{k \in \lfloor \ell - 2, 1 \rfloor} x_k \big) \cdot x_0 \, ,$$

where $x_0 \in G\mathrm{N}_1$ and where $x_k \in G_1$ are such that $x_k = x_k \mathrm{d}_k \mathrm{s}_{k-1}$ for $k \in [1, \ell-2]$.
Moreover, let $x_{\ell-1} := g\mathrm{d}_{\ell-1}\mathrm{s}_{\ell-2} \in G_1$. We have

$$x_{\ell-1}\,\mathrm{d}_{\ell-1}\mathrm{s}_{\ell-2} = g\mathrm{d}_{\ell-1}\mathrm{s}_{\ell-2}\,\mathrm{d}_{\ell-1}\mathrm{s}_{\ell-2} = g\mathrm{d}_{\ell-1}\mathrm{s}_{\ell-2} = x_{\ell-1}\,.$$

So altogether, we have

$$g = g\mathrm{d}_{\ell-1}\mathrm{s}_{\ell-2}\cdot\big((g\mathrm{d}_{\ell-1}\mathrm{s}_{\ell-2})^{-}\cdot g\big) = x_{\ell-1}\cdot\tilde{g} = x_{\ell-1}\cdot\Big(\prod_{k\in\lfloor \ell-2,1\rfloor} x_k\Big)\cdot x_0 = \Big(\prod_{k\in\lfloor \ell-1,1\rfloor} x_k\Big)\cdot x_0\,,$$

where $x_0 \in G\mathrm{N}_1$ and where $x_k = x_k\,\mathrm{d}_k\,\mathrm{s}_{k-1}$ for $k \in [1, \ell-1]$. □

18.2 The Conduché bracket in a $\lfloor 1,0 \rfloor$-stable simplicial group

We transport the arguments of [**7,** §3] into our context.
Let G be a $\lfloor 1,0 \rfloor$-stable simplicial group; cf. Remark 383.
Let $M := G\mathrm{N}_1 \trianglelefteqslant G_1$ and let $B := G_0$.

Definition 388 (Conduché bracket)
For b, $c \in B$, we write

$$[\![b,\,c]\!] := c^{-}\mathrm{s}_1\cdot b^{-}\mathrm{s}_0\cdot c\mathrm{s}_1\cdot c^{-}\mathrm{s}_0\cdot b\mathrm{s}_0\cdot c\mathrm{s}_0 = (b^{-}\mathrm{s}_0)^{c\mathrm{s}_1}\cdot(b\mathrm{s}_0)^{c\mathrm{s}_0} \in G_1$$

for the *Conduché bracket* of b and c.

Remark 389 *Suppose given b, $c \in B$. We have $[\![b,\,c]\!] \in M$.*

Proof. Suppose given $i \in \mathbb{Z}_{\geqslant 1}$.
If $i = 1$, then we have

$$[\![b,\,c]\!]\mathrm{d}_1 = (b^{-}\mathrm{s}_0\,\mathrm{d}_1)^{c\mathrm{s}_1\mathrm{d}_1}\cdot(b\mathrm{s}_0\,\mathrm{d}_1)^{c\mathrm{s}_0\mathrm{d}_1} \overset{383.(3)}{=} (b^{c})^{-}\cdot b^{c} = 1\,.$$

If $i = 2$ then, we have

$$[\![b,\,c]\!]\mathrm{d}_2 = (b^{-}\mathrm{s}_0\,\mathrm{d}_2)^{c\mathrm{s}_1\mathrm{d}_2}\cdot(b\mathrm{s}_0\,\mathrm{d}_2)^{c\mathrm{s}_0\mathrm{d}_2} \overset{383.(3)}{=} 1^{c}\cdot 1 = 1\,.$$

If $i \geqslant 3$, then we have

$$[\![b,\,c]\!]\mathrm{d}_i = (b^{-}\mathrm{s}_0\,\mathrm{d}_i)^{c\mathrm{s}_1\mathrm{d}_i}\cdot(b\mathrm{s}_0\,\mathrm{d}_i)^{c\mathrm{s}_0\mathrm{d}_i} \overset{383.(3)}{=} 1\cdot 1 = 1\,.$$

This shows $[\![b,\,c]\!] \in G\mathrm{N}_1 = M$. □

Remark 390 *Suppose given m, $n \in M$ and $b \in B$.*

(1) *We have* $m^{n\mathrm{d}_0\mathrm{s}_0} = m^n$.

(2) *For $k \in \mathbb{Z}_{\geqslant 1}$, we have* $m^{b\mathrm{s}_k} = m$.

(3) *We have* $m \cdot \llbracket m\mathrm{d}_0, b \rrbracket = m^{b\mathrm{s}_0}$.

Proof. Ad (1). We have

$$(n \cdot n^-\mathrm{d}_0\mathrm{s}_0)\mathrm{d}_0 = n\mathrm{d}_0 \cdot n^-\mathrm{d}_0\mathrm{s}_0\mathrm{d}_0 \overset{383.(3)}{=} n\mathrm{d}_0 \cdot n^-\mathrm{d}_0 = 1\,.$$

Therefore, by Remark 383.(2), with $I := \mathbb{Z}_{\geqslant 1}$ and $J := \{0\}$, we have $[m, n \cdot n^-\mathrm{d}_0\,\mathrm{s}_0] = 1$. So we have

$$\begin{aligned} m^{n\mathrm{d}_0\mathrm{s}_0} &= n^-\mathrm{d}_0\mathrm{s}_0 \cdot m \cdot n\mathrm{d}_0\mathrm{s}_0 = n^- \cdot (n \cdot n^-\mathrm{d}_0\mathrm{s}_0) \cdot m \cdot n\mathrm{d}_0\mathrm{s}_0 \\ &= n^- \cdot m \cdot (n \cdot n^-\mathrm{d}_0\mathrm{s}_0) \cdot n\mathrm{d}_0\mathrm{s}_0 = n^- \cdot m \cdot n = m^n\,. \end{aligned}$$

Ad (2). Suppose given $k \in \mathbb{Z}_{\geqslant 1}$. We have

$$(b\mathrm{s}_k)\mathrm{d}_0 \overset{383.(3)}{=} 1\,.$$

Therefore, by Remark 383.(2), with $I := \mathbb{Z}_{\geqslant 1}$ and $J := \{0\}$, we have $[m, b\mathrm{s}_k] = 1$. So we have

$$m^{b\mathrm{s}_k} = b^-\mathrm{s}_k \cdot m \cdot b\mathrm{s}_k = b^-\mathrm{s}_k \cdot b\mathrm{s}_k \cdot m = m\,.$$

Ad (3). Let $I := \mathbb{Z}_{\geqslant 0}\backslash\{1\}$ and let $J := \{1\}$. *We show that* $m \cdot m^-\mathrm{d}_0\mathrm{s}_0 \overset{!}{\in} \bigcap_{i \in I} \ker \mathrm{d}_i$.

Suppose given $i \in I$. If $i = 0$, then we have

$$(m \cdot m^-\mathrm{d}_0\mathrm{s}_0)\mathrm{d}_0 = m\mathrm{d}_0 \cdot m^-\mathrm{d}_0\mathrm{s}_0\mathrm{d}_0 \overset{383.(3)}{=} m\mathrm{d}_0 \cdot m^-\mathrm{d}_0 = 1\,.$$

If $i \geqslant 2$, then we have

$$(m \cdot m^-\mathrm{d}_0\mathrm{s}_0)\mathrm{d}_i = m\mathrm{d}_i \cdot (m^-\mathrm{d}_0)\mathrm{s}_0\mathrm{d}_i \overset{383.(3)}{=} 1 \cdot 1 = 1\,.$$

This shows $m \cdot m^-\mathrm{d}_0\mathrm{s}_0 \in \bigcap_{i \in I} \ker \mathrm{d}_i$.

Moreover, we have

$$(b\mathrm{s}_1 \cdot b^-\mathrm{s}_0)\mathrm{d}_1 = b\mathrm{s}_1\mathrm{d}_1 \cdot b^-\mathrm{s}_0\mathrm{d}_1 \overset{383.(3)}{=} b \cdot b^- = 1\,.$$

This shows $b\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \in \bigcap_{j \in J} \ker \mathrm{d}_j$.

Then, by Remark 383.(2), we have $[m \cdot m^-\mathrm{d}_0\mathrm{s}_0\,,\, b\mathrm{s}_1 \cdot b^-\mathrm{s}_0] = 1$.

So we have

$$\begin{aligned}
m \cdot \llbracket m\mathrm{d}_0,\, b \rrbracket &= m \cdot b^-\mathrm{s}_1 \cdot m^-\mathrm{d}_0\mathrm{s}_0 \cdot b\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot m\mathrm{d}_0\mathrm{s}_0 \cdot b\mathrm{s}_0 \\
&= m \cdot b^-\mathrm{s}_1 \cdot m^- \cdot (m \cdot m^-\mathrm{d}_0\mathrm{s}_0) \cdot (b\mathrm{s}_1 \cdot b^-\mathrm{s}_0) \cdot m\mathrm{d}_0\mathrm{s}_0 \cdot b\mathrm{s}_0 \\
&= m \cdot b^-\mathrm{s}_1 \cdot m^- \cdot (b\mathrm{s}_1 \cdot b^-\mathrm{s}_0) \cdot (m \cdot m^-\mathrm{d}_0\mathrm{s}_0) \cdot m\mathrm{d}_0\mathrm{s}_0 \cdot b\mathrm{s}_0 \\
&= m \cdot b^-\mathrm{s}_1 \cdot m^- \cdot b\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot m \cdot b\mathrm{s}_0 \\
&\overset{(2)}{=} m \cdot m^- \cdot b^-\mathrm{s}_0 \cdot m \cdot b\mathrm{s}_0 \\
&= b^-\mathrm{s}_0 \cdot m \cdot b\mathrm{s}_0 \\
&= m^{b\mathrm{s}_0} .
\end{aligned}$$

□

Remark 391 *Suppose given a, b, $c \in B$.*

(1) *We have* $\llbracket b,\, 1 \rrbracket = 1$ and $\llbracket 1,\, b \rrbracket = 1$.

(2) *We have* $\llbracket a,\, b \cdot c \rrbracket = \llbracket a,\, b \rrbracket \cdot \llbracket a^b,\, c \rrbracket$.

(3) *We have* $\llbracket b,\, c \rrbracket^- = \llbracket b^c,\, c^- \rrbracket$.

Proof. Ad (1). We have

$$\llbracket b,\, 1 \rrbracket = (b^-\mathrm{s}_0)^{1\mathrm{s}_1} \cdot (b\mathrm{s}_0)^{1\mathrm{s}_0} = (b\mathrm{s}_0)^- \cdot b\mathrm{s}_0 = 1 .$$

Moreover, we have

$$\llbracket 1,\, b \rrbracket = (1^-\mathrm{s}_0)^{b\mathrm{s}_1} \cdot (1\mathrm{s}_0)^{b\mathrm{s}_0} = 1^{b\mathrm{s}_1} \cdot 1^{b\mathrm{s}_0} = 1 .$$

Ad (2). By Remark 389, we have $\llbracket a,\, b \rrbracket \in M$.

So, by Remark 390.(2), we have $\left[\llbracket a,\, b \rrbracket, c^-\mathrm{s}_1\right] = 1$. We obtain

$$\begin{aligned}
\llbracket a,\, b \rrbracket \cdot \llbracket a^b,\, c \rrbracket &= \llbracket a,\, b \rrbracket \cdot \left(c^-\mathrm{s}_1 \cdot (a^b)^-\mathrm{s}_0 \cdot c\mathrm{s}_1 \cdot c^-\mathrm{s}_0 \cdot (a^b)\mathrm{s}_0 \cdot c\mathrm{s}_0\right) \\
&= c^-\mathrm{s}_1 \cdot \llbracket a,\, b \rrbracket \cdot (a^b)^-\mathrm{s}_0 \cdot c\mathrm{s}_1 \cdot c^-\mathrm{s}_0 \cdot (a^b)\mathrm{s}_0 \cdot c\mathrm{s}_0 \\
&= c^-\mathrm{s}_1 \cdot \left(b^-\mathrm{s}_1 \cdot a^-\mathrm{s}_0 \cdot b\mathrm{s}_1 \cdot (a^b)\mathrm{s}_0\right) \cdot \left((a^b)^-\mathrm{s}_0 \cdot c\mathrm{s}_1 \cdot c^-\mathrm{s}_0 \cdot b^-\mathrm{s}_0 \cdot a\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot c\mathrm{s}_0\right) \\
&= c^-\mathrm{s}_1 \cdot b^-\mathrm{s}_1 \cdot a^-\mathrm{s}_0 \cdot b\mathrm{s}_1 \cdot c\mathrm{s}_1 \cdot c^-\mathrm{s}_0 \cdot b^-\mathrm{s}_0 \cdot a\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot c\mathrm{s}_0 \\
&= (b \cdot c)^-\mathrm{s}_1 \cdot a^-\mathrm{s}_0 \cdot (b \cdot c)\mathrm{s}_1 \cdot (b \cdot c)^-\mathrm{s}_0 \cdot a\mathrm{s}_0 \cdot (b \cdot c)\mathrm{s}_0 \\
&= \llbracket a,\, b \cdot c \rrbracket .
\end{aligned}$$

Ad (3). We have

$$\llbracket b,\, c \rrbracket \cdot \llbracket b^c,\, c^- \rrbracket \overset{(2)}{=} \llbracket b,\, c \cdot c^- \rrbracket \overset{(1)}{=} 1 \,.$$

□

Lemma 392 *Suppose given a, b, $c \in B$.*

(1) *Suppose that $[a, b] = 1$. Then we have*

$$[a\mathrm{s}_1 \cdot b\mathrm{s}_0 \,, c\mathrm{s}_0] = [b\mathrm{s}_0 \cdot a\mathrm{s}_1 \,, c\mathrm{s}_0] \,.$$

In particular, we have

$$[b^-\mathrm{s}_1 \cdot b\mathrm{s}_0 \,, c\mathrm{s}_0] = [b\mathrm{s}_0 \cdot b^-\mathrm{s}_1 \,, c\mathrm{s}_0] \,.$$

(2) *For i, $j \in \mathbb{Z}_{\geqslant 0}$ with $|i - j| \geqslant 2$, we have* $\quad [b\mathrm{s}_i \,, c\mathrm{s}_j] = 1 \,.$

(3) *For $i \in \mathbb{Z}_{\geqslant 1}$, we have* $\quad [b\mathrm{s}_i \,, c\mathrm{s}_{i+1} \cdot c^-\mathrm{s}_i \cdot c\mathrm{s}_{i-1}] = 1 \,.$

(4) *For $i \in \mathbb{Z}_{\geqslant 1}$, we have* $\quad [b^-\mathrm{s}_i \cdot b\mathrm{s}_{i-1} \,, c^-\mathrm{s}_{i+1} \cdot c\mathrm{s}_i] = 1 \,.$

Proof. Ad (1). Let $I := \mathbb{Z}_{\geqslant 0}$ and let $J := \{2\}$. *We show that* $[b\mathrm{s}_0 \,, a\mathrm{s}_1] \overset{!}{\in} \bigcap_{i \in I} \ker \mathrm{d}_i$.

For $i = 0$, we have

$$[b\mathrm{s}_0 \,, a\mathrm{s}_1]\mathrm{d}_0 \overset{383.(3)}{=} [b, 1] = 1 \,.$$

For $i = 1$, we have

$$[b\mathrm{s}_0 \,, a\mathrm{s}_1]\mathrm{d}_1 \overset{383.(3)}{=} [b, a] = 1 \,.$$

For $i = 2$, we have

$$[b\mathrm{s}_0 \,, a\mathrm{s}_1]\mathrm{d}_2 \overset{383.(3)}{=} [1, a] = 1 \,.$$

For $i \in \mathbb{Z}_{\geqslant 3}$, we have

$$[b\mathrm{s}_0 \,, a\mathrm{s}_1]\mathrm{d}_i \overset{383.(3)}{=} [1, 1] = 1 \,.$$

This shows $[b\mathrm{s}_0 \,, a\mathrm{s}_1] \in \bigcap_{i \in I} \ker \mathrm{d}_i$.

We show that $a^-\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot b\mathrm{s}_0 \overset{!}{\in} \ker \mathrm{d}_2 = \bigcap_{j\in J} \ker \mathrm{d}_j$.

We have

$$(a^-\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot b\mathrm{s}_0)\mathrm{d}_2 \overset{383.(3)}{=} a^- \cdot 1 \cdot 1 \cdot a \cdot 1 = 1\,.$$

This shows $a^-\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot b\mathrm{s}_0 \in \ker \mathrm{d}_2$.

Then, by Remark 383.(2), we have

$$\big[[b\mathrm{s}_0\,, a\mathrm{s}_1], a^-\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot b\mathrm{s}_0\big] = 1\,.$$

So we have

$$\begin{aligned}
[a\mathrm{s}_1 \cdot b\mathrm{s}_0\,, c\mathrm{s}_0] &= b^-\mathrm{s}_0 \cdot a^-\mathrm{s}_1 \cdot c^-\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot b\mathrm{s}_0 \cdot c\mathrm{s}_0 \\
&= (b^-\mathrm{s}_0 \cdot a^-\mathrm{s}_1 \cdot b\mathrm{s}_0 \cdot a\mathrm{s}_1) \cdot (a^-\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot b\mathrm{s}_0) \cdot c\mathrm{s}_0 \\
&= (a^-\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot b\mathrm{s}_0) \cdot (b^-\mathrm{s}_0 \cdot a^-\mathrm{s}_1 \cdot b\mathrm{s}_0 \cdot a\mathrm{s}_1) \cdot c\mathrm{s}_0 \\
&= a^-\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot a\mathrm{s}_1 \cdot c\mathrm{s}_0 \\
&= [b\mathrm{s}_0 \cdot a\mathrm{s}_1\,, c\mathrm{s}_0]\,.
\end{aligned}$$

Ad (2). Suppose given $i,\, j \in \mathbb{Z}_{\geqslant 0}$ with $|i-j| \geqslant 2$. We may assume that $i \leqslant j-2$.

Let $I := \mathbb{Z}_{\geqslant i+2}$ and let $J := [0, i+1]$.

For $k \in I$, we have

$$b\mathrm{s}_i\mathrm{d}_k \overset{383.(3)}{=} 1\,.$$

This shows $b\mathrm{s}_i \in \bigcap_{k\in I} \ker \mathrm{d}_k$.

Suppose given $k \in J$. Then $k \leqslant j-1$. We have

$$c\mathrm{s}_j\mathrm{d}_k \overset{383.(3)}{=} 1\,.$$

This shows $c\mathrm{s}_j \in \bigcap_{k\in J} \ker \mathrm{d}_k$.

Therefore, by Remark 383.(2), we have $[b\mathrm{s}_i\,, c\mathrm{s}_j] = 1$.

Ad (3). Suppose given $i \in \mathbb{Z}_{\geqslant 1}$. Let $I := \mathbb{Z}_{\geqslant 0}\backslash\{i, i+1\}$ and let $J := \{i, i+1\}$.

We show that $b\mathrm{s}_i \overset{!}{\in} \bigcap_{k\in I} \ker \mathrm{d}_k$.

For $k \in I$, we have

$$b\mathrm{s}_i\mathrm{d}_k \overset{383.(3)}{=} 1\,.$$

We show that $c\mathrm{s}_{i+1} \cdot c^{-}\mathrm{s}_i \cdot c\mathrm{s}_{i-1} \overset{!}{\in} \bigcap_{k \in J} \ker \mathrm{d}_k$.

For $k = i$, we have

$$(c\mathrm{s}_{i+1} \cdot c^{-}\mathrm{s}_i \cdot c\mathrm{s}_{i-1})\mathrm{d}_i \overset{383.(3)}{=} 1 \cdot c^{-} \cdot c = 1 \,.$$

For $k = i + 1$, we have

$$(c\mathrm{s}_{i+1} \cdot c^{-}\mathrm{s}_i \cdot c\mathrm{s}_{i-1})\mathrm{d}_{i+1} \overset{383.(3)}{=} c \cdot c^{-} \cdot 1 = 1 \,.$$

Therefore, by Remark 383.(2), we have $[b\mathrm{s}_i \,, c\mathrm{s}_{i+1} \cdot c^{-}\mathrm{s}_i \cdot c\mathrm{s}_{i-1}] = 1$.

Ad (4). Suppose given $i \in \mathbb{Z}_{\geqslant 1}$. Let $I := \{i\} \cup \mathbb{Z}_{\geqslant i+2}$ and let $J := [0, i-1] \cup \{i+1\}$.

We show that $b^{-}\mathrm{s}_i \cdot b\mathrm{s}_{i-1} \overset{!}{\in} \bigcap_{k \in I} \ker \mathrm{d}_k$.

Suppose given $k \in I$. If $k = i$, then we have

$$(b^{-}\mathrm{s}_i \cdot b\mathrm{s}_{i-1})\mathrm{d}_i \overset{383.(3)}{=} b^{-} \cdot b = 1 \,.$$

If $k \geqslant i + 2$, then we have

$$(b^{-}\mathrm{s}_i \cdot b\mathrm{s}_{i-1})\mathrm{d}_k \overset{383.(3)}{=} 1 \cdot 1 = 1 \,.$$

We show that $c^{-}\mathrm{s}_{i+1} \cdot c\mathrm{s}_i \overset{!}{\in} \bigcap_{k \in J} \ker \mathrm{d}_k$.

Suppose given $k \in J$. If $k \in [0, i-1]$, then we have

$$(c^{-}\mathrm{s}_{i+1} \cdot c\mathrm{s}_i)\mathrm{d}_k \overset{383.(3)}{=} 1 \cdot 1 = 1 \,.$$

If $k = i + 1$, then we have

$$(c^{-}\mathrm{s}_{i+1} \cdot c\mathrm{s}_i)\mathrm{d}_{i+1} \overset{383.(3)}{=} c^{-} \cdot c = 1 \,.$$

Therefore, by Remark 383.(2), we have $[b^{-}\mathrm{s}_i \cdot b\mathrm{s}_{i-1} \,, c^{-}\mathrm{s}_{i+1} \cdot c\mathrm{s}_i] = 1$. □

Lemma 393 *Suppose given* b, $c \in B$.

We have

$$\llbracket b, c \rrbracket = \llbracket c, b \rrbracket^{-} \,.$$

Proof. We consider the commutator $[b^-\mathrm{s}_1 \cdot b\mathrm{s}_0\,, c\mathrm{s}_2 \cdot c^-\mathrm{s}_1 \cdot c\mathrm{s}_0] \in G_1$.

We have

$$\begin{aligned}
& [b^-\mathrm{s}_1 \cdot b\mathrm{s}_0\,, (c\mathrm{s}_2 \cdot c^-\mathrm{s}_1 \cdot c\mathrm{s}_0)] \\
\overset{1.(1)}{=}\ & [b^-\mathrm{s}_1\,, c\mathrm{s}_2 \cdot c^-\mathrm{s}_1 \cdot c\mathrm{s}_0]^{b\mathrm{s}_0} \cdot [b\mathrm{s}_0\,, c\mathrm{s}_2 \cdot c^-\mathrm{s}_1 \cdot c\mathrm{s}_0] \\
\overset{392.(3)}{=}\ & [b\mathrm{s}_0\,, c\mathrm{s}_2 \cdot (c^-\mathrm{s}_1 \cdot c\mathrm{s}_0)] \\
\overset{1.(2)}{=}\ & [b\mathrm{s}_0\,, c^-\mathrm{s}_1 \cdot c\mathrm{s}_0] \cdot [b\mathrm{s}_0\,, c\mathrm{s}_2]^{c^-\mathrm{s}_1 \cdot c\mathrm{s}_0} \\
\overset{392.(2)}{=}\ & [b\mathrm{s}_0\,, c^-\mathrm{s}_1 \cdot c\mathrm{s}_0] \\
\overset{392.(1)}{=}\ & [b\mathrm{s}_0\,, c\mathrm{s}_0 \cdot c^-\mathrm{s}_1] \\
=\ & b^-\mathrm{s}_0 \cdot c\mathrm{s}_1 \cdot c^-\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot c\mathrm{s}_0 \cdot c^-\mathrm{s}_1 \\
=\ & c\mathrm{s}_0 \cdot (c^-\mathrm{s}_0 \cdot b^-\mathrm{s}_0 \cdot c\mathrm{s}_0) \cdot c^-\mathrm{s}_0 \cdot c\mathrm{s}_1 \cdot (c^-\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot c\mathrm{s}_0) \cdot c^-\mathrm{s}_1 \\
=\ & c\mathrm{s}_0 \cdot (b^c)^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot c\mathrm{s}_1 \cdot (b^c)\mathrm{s}_0 \cdot c^-\mathrm{s}_1 \\
=\ & \big(c\mathrm{s}_1 \cdot (b^c)^-\mathrm{s}_0 \cdot c^-\mathrm{s}_1 \cdot c\mathrm{s}_0 \cdot (b^c)\mathrm{s}_0 \cdot c^-\mathrm{s}_0\big)^- \\
=\ & \llbracket b^c,\, c^- \rrbracket^- \\
\overset{391.(3)}{=}\ & \llbracket b,\, c \rrbracket\,.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
& [(b^-\mathrm{s}_1 \cdot b\mathrm{s}_0)\,, (c\mathrm{s}_2 \cdot c^-\mathrm{s}_1) \cdot c\mathrm{s}_0] \\
\overset{1.(2)}{=}\ & [b^-\mathrm{s}_1 \cdot b\mathrm{s}_0\,, c\mathrm{s}_0] \cdot [b^-\mathrm{s}_1 \cdot b\mathrm{s}_0\,, c\mathrm{s}_2 \cdot c^-\mathrm{s}_1]^{c\mathrm{s}_0} \\
\overset{392.(4)}{=}\ & [b^-\mathrm{s}_1 \cdot b\mathrm{s}_0\,, c\mathrm{s}_0] \\
\overset{392.(1)}{=}\ & [b\mathrm{s}_0 \cdot b^-\mathrm{s}_1\,, c\mathrm{s}_0] \\
=\ & b\mathrm{s}_1 \cdot b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot b^-\mathrm{s}_1 \cdot c\mathrm{s}_0 \\
=\ & b\mathrm{s}_1 \cdot (b^-\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot b\mathrm{s}_0) \cdot b^-\mathrm{s}_1 \cdot b\mathrm{s}_0 \cdot (b^-\mathrm{s}_0 \cdot c\mathrm{s}_0 \cdot b\mathrm{s}_0) \cdot b^-\mathrm{s}_0 \\
=\ & b\mathrm{s}_1 \cdot (c^b)^-\mathrm{s}_0 \cdot b^-\mathrm{s}_1 \cdot b\mathrm{s}_0 \cdot (c^b)\mathrm{s}_0 \cdot b^-\mathrm{s}_0 \\
=\ & \llbracket c^b,\, b^- \rrbracket \\
\overset{391.(3)}{=}\ & \llbracket c,\, b \rrbracket^-\,.
\end{aligned}$$

This shows $\llbracket b,\, c \rrbracket = \llbracket c,\, b \rrbracket^-$. □

Lemma 394 *Suppose given $b,\, c \in B$.*

(1) *Suppose given $t \in \mathbb{Z}_{\geqslant 1}$. We have*

$$(b\mathrm{s}_t)^{c\mathrm{s}_{t+1}} = (b^c)\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t]\,.$$

(2) *Suppose given $t \in \mathbb{Z}_{\geqslant 0}$. If $t \equiv_2 1$, then we have*

$$(b\mathrm{s}_t)^{c\mathrm{s}_{t+1}} = (b^c)\mathrm{s}_t \cdot \Big(\prod_{i \in \lfloor t-1,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [c\mathrm{s}_0\,, b\mathrm{s}_1]\,.$$

If $t \equiv_2 0$, then we have

$$(b\mathrm{s}_t)^{c\mathrm{s}_{t+1}} = (b^c)\mathrm{s}_t \cdot \Big(\prod_{i \in \lfloor t-1,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot \big[b\mathrm{s}_0\,, c\mathrm{s}_1\big]\,.$$

(3) *Suppose given $t \in \mathbb{Z}_{\geqslant 1}$. We have*

$$(b\mathrm{s}_{t-1})^{c\mathrm{s}_t} = (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,0 \rfloor} \big([c,b]^{(-1)^{t-i}}\big)\mathrm{s}_i \Big) \cdot \llbracket c,\, b \rrbracket^{(-1)^{t+1}}\,.$$

Proof. Ad (1). By Lemma 392.(3), we have $[c\mathrm{s}_{t+1} \cdot c^-\mathrm{s}_t \cdot c\mathrm{s}_{t-1}\,, b^-\mathrm{s}_t] = 1$.
So we have

$$(b^-\mathrm{s}_t)^{c\mathrm{s}_{t+1}} = (b^-\mathrm{s}_t)^{(c^-\mathrm{s}_{t-1} \cdot c\mathrm{s}_t)}\,.$$

We obtain

$$\begin{aligned}
& (b^-\mathrm{s}_t)^{c\mathrm{s}_{t+1}} \cdot (b^c)\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t] \\
= \;& (b^-\mathrm{s}_t)^{(c^-\mathrm{s}_{t-1} \cdot c\mathrm{s}_t)} \cdot (b^c)\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t] \\
= \;& c^-\mathrm{s}_t \cdot c\mathrm{s}_{t-1} \cdot b^-\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1} \cdot b\mathrm{s}_t \cdot c\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t] \\
= \;& (c^-\mathrm{s}_t \cdot c\mathrm{s}_{t-1}) \cdot (b^-\mathrm{s}_{t+1} \cdot b\mathrm{s}_t)^- \cdot b^-\mathrm{s}_{t+1} \cdot c^-\mathrm{s}_{t-1} \cdot b\mathrm{s}_t \cdot c\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t] \\
\overset{392.(4)}{=} \;& (b^-\mathrm{s}_{t+1} \cdot b\mathrm{s}_t)^- \cdot c^-\mathrm{s}_t \cdot (c\mathrm{s}_{t-1} \cdot b^-\mathrm{s}_{t+1} \cdot c^-\mathrm{s}_{t-1}) \cdot b\mathrm{s}_t \cdot c\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t] \\
\overset{392.(2)}{=} \;& b^-\mathrm{s}_t \cdot b\mathrm{s}_{t+1} \cdot c^-\mathrm{s}_t \cdot b^-\mathrm{s}_{t+1} \cdot b\mathrm{s}_t \cdot c\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t] \\
= \;& b^-\mathrm{s}_t \cdot b\mathrm{s}_{t+1} \cdot c^-\mathrm{s}_t \cdot (b^-\mathrm{s}_{t+1} \cdot b\mathrm{s}_t) \cdot (c\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1}) \cdot (c\mathrm{s}_{t-1})^{b\mathrm{s}_t} \\
\overset{392.(4)}{=} \;& b^-\mathrm{s}_t \cdot b\mathrm{s}_{t+1} \cdot c^-\mathrm{s}_t \cdot (c\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1}) \cdot (b^-\mathrm{s}_{t+1} \cdot b\mathrm{s}_t) \cdot (c\mathrm{s}_{t-1})^{b\mathrm{s}_t} \\
= \;& b^-\mathrm{s}_t \cdot b\mathrm{s}_{t+1} \cdot (c^-\mathrm{s}_{t-1} \cdot b^-\mathrm{s}_{t+1} \cdot c\mathrm{s}_{t-1}) \cdot b\mathrm{s}_t \\
\overset{392.(2)}{=} \;& b^-\mathrm{s}_t \cdot b\mathrm{s}_{t+1} \cdot b^-\mathrm{s}_{t+1} \cdot b\mathrm{s}_t \\
= \;& 1\,.
\end{aligned}$$

This shows $(b\mathrm{s}_t)^{c\mathrm{s}_{t+1}} = (b^c)\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t]$.

Ad (2). We show this *claim* by induction on $t \in \mathbb{Z}_{\geqslant 0}$.

Let $t = 0$. We have

$$\begin{aligned}
& (b^c)\mathrm{s}_0 \cdot \Big(\prod_{i \in \lfloor -1,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [b\mathrm{s}_0\,, c\mathrm{s}_1] \\
\overset{\S 1.1.2 \text{ item } 9.3}{=} \; & (b^c)\mathrm{s}_0 \cdot \Big(\big([b,c]^{(-1)^0}\big)\mathrm{s}_0 \Big)^- \cdot [b\mathrm{s}_0\,, c\mathrm{s}_1] \\
= \; & (b^c)\mathrm{s}_0 \cdot [c,b]\mathrm{s}_0 \cdot [b\mathrm{s}_0\,, c\mathrm{s}_1] \\
= \; & c^-\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot c\mathrm{s}_0 \cdot c^-\mathrm{s}_0 \cdot b^-\mathrm{s}_0 \cdot c\mathrm{s}_0 \cdot b\mathrm{s}_0 \cdot b^-\mathrm{s}_0 \cdot (b\mathrm{s}_0)^{c\mathrm{s}_1} \\
= \; & (b\mathrm{s}_0)^{c\mathrm{s}_1}\,.
\end{aligned}$$

Let $t \geqslant 1$. We have

$$(b\mathrm{s}_t)^{c\mathrm{s}_{t+1}} \overset{(1)}{=} (b^c)\mathrm{s}_t \cdot [c\mathrm{s}_{t-1}\,, b\mathrm{s}_t] = (b^c)\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1} \cdot (c\mathrm{s}_{t-1})^{b\mathrm{s}_t}\,.$$

If $t \equiv_2 1$, then we have

$$\begin{aligned}
(b\mathrm{s}_t)^{c\mathrm{s}_{t+1}} \quad = \quad & (b^c)\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1} \cdot (c\mathrm{s}_{t-1})^{b\mathrm{s}_t} \\
\overset{\mathrm{IH},\, t-1 \equiv_2 0}{=} \; & (b^c)\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1} \cdot (c^b)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [c\mathrm{s}_0\,, b\mathrm{s}_1] \\
= \; & (b^c)\mathrm{s}_t \cdot [c,b]\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [c\mathrm{s}_0\,, b\mathrm{s}_1] \\
= \; & (b^c)\mathrm{s}_t \cdot \Big(\prod_{i \in \lfloor t-1,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [c\mathrm{s}_0\,, b\mathrm{s}_1]\,.
\end{aligned}$$

If $t \equiv_2 0$, then we have

$$\begin{aligned}
(b\mathrm{s}_t)^{c\mathrm{s}_{t+1}} \quad = \quad & (b^c)\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1} \cdot (c\mathrm{s}_{t-1})^{b\mathrm{s}_t} \\
\overset{\mathrm{IH},\, t-1 \equiv_2 1}{=} \; & (b^c)\mathrm{s}_t \cdot c^-\mathrm{s}_{t-1} \cdot (c^b)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [b\mathrm{s}_0\,, c\mathrm{s}_1] \\
= \; & (b^c)\mathrm{s}_t \cdot \big([b,c]\mathrm{s}_{t-1}\big)^- \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [b\mathrm{s}_0\,, c\mathrm{s}_1] \\
= \; & (b^c)\mathrm{s}_t \cdot \Big(\prod_{i \in \lfloor t-1,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [b\mathrm{s}_0\,, c\mathrm{s}_1]\,.
\end{aligned}$$

This shows the *claim*.

Ad (3). *Case* $t \equiv_2 0$. Then $t - 1 \equiv_2 1$.

So we have

$$\begin{aligned}
(b\mathrm{s}_{t-1})^{c\mathrm{s}_t} &\overset{(2)}{=} (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [c\mathrm{s}_0\,, b\mathrm{s}_1] \\
&= (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot (c\mathrm{s}_0)^- \cdot (c\mathrm{s}_0)^{b\mathrm{s}_1} \\
&= (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot (c\mathrm{s}_0)^- \cdot (c\mathrm{s}_0)^{b\mathrm{s}_0} \cdot (c^-\mathrm{s}_0)^{b\mathrm{s}_0} \cdot (c\mathrm{s}_0)^{b\mathrm{s}_1} \\
&= (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [c,b]\mathrm{s}_0 \cdot \big((c^-\mathrm{s}_0)^{b\mathrm{s}_1} \cdot (c\mathrm{s}_0)^{b\mathrm{s}_0}\big)^- \\
&\overset{388}{=} (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,0 \rfloor} \big([c,b]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [\![c,\, b]\!]^- \\
&= (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,0 \rfloor} \big([c,b]^{(-1)^{t-i}}\big)\mathrm{s}_i \Big) \cdot [\![c,\, b]\!]^{(-1)^{t+1}}\,.
\end{aligned}$$

Case $t \equiv_2 1$. Then $t-1 \equiv_2 0$.

So we have

$$\begin{aligned}
(b\mathrm{s}_{t-1})^{c\mathrm{s}_t} &\overset{(2)}{=} (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [b\mathrm{s}_0\,, c\mathrm{s}_1] \\
&= (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot (b\mathrm{s}_0)^- \cdot (b\mathrm{s}_0)^{c\mathrm{s}_1} \\
&= (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot (b\mathrm{s}_0)^- \cdot (b\mathrm{s}_0)^{c\mathrm{s}_0} \cdot (b^-\mathrm{s}_0)^{c\mathrm{s}_0} \cdot (b\mathrm{s}_0)^{c\mathrm{s}_1} \\
&= (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,1 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [b,c]\mathrm{s}_0 \cdot \big((b^-\mathrm{s}_0)^{c\mathrm{s}_1} \cdot (b\mathrm{s}_0)^{c\mathrm{s}_0}\big)^- \\
&\overset{388}{=} (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,0 \rfloor} \big([b,c]^{(-1)^i}\big)\mathrm{s}_i \Big) \cdot [\![b,\, c]\!]^- \\
&\overset{393}{=} (b^c)\mathrm{s}_{t-1} \cdot \Big(\prod_{i \in \lfloor t-2,0 \rfloor} \big([c,b]^{(-1)^{t-i}}\big)\mathrm{s}_i \Big) \cdot [\![c,\, b]\!]^{(-1)^{t+1}}\,.
\end{aligned}$$

□

18.3 The functor $\hat{\mathrm{N}}^{\pm}\colon \lfloor 1,0\rfloor$-*StSimpGrp* $\to$ *StCrMod*

Remark 395 (The stable simplicial group $G\hat{\mathrm{N}}^{\pm}$)
Suppose given a $\lfloor 1,0\rfloor$*-stable simplicial group* G *cf. Remark* 383.
We consider the groups

$$G\mathrm{N}_1 \overset{257.(2)}{=} \bigcap_{j\in\mathbb{Z}_{\geqslant 1}} \ker \mathrm{d}_j^G \leqslant G_1 \quad \text{and} \quad G_0\,.$$

We consider the map

$$\kappa_G\colon G_0\times G_0 \to G\mathrm{N}_1\,,\ (b,c)\mapsto (b,c)\kappa_G := [\![b,\,c]\!] = (b^-\mathrm{s}_0^G)^{c\mathrm{s}_1^G}\cdot (b\mathrm{s}_0^G)^{c\mathrm{s}_0^G}\,;$$

cf. Definition 388*, Remark* 389.
We consider the group morphism

$$\mathrm{d}_0^G\big|_{G\mathrm{N}_1}\colon G\mathrm{N}_1\to G_0\,,\ x\mapsto x\mathrm{d}_0^G\,.$$

Then

$$G\hat{\mathrm{N}}^{\pm} = (\mathrm{M}_{G\hat{\mathrm{N}}^{\pm}}\,,\,\mathrm{B}_{G\hat{\mathrm{N}}^{\pm}}\,,\,\kappa_{G\hat{\mathrm{N}}^{\pm}}\,,\,\mathrm{f}_{G\hat{\mathrm{N}}^{\pm}}) := \big(G\mathrm{N}_1\,,G_0\,,\kappa_G\,,\mathrm{d}_0^G\big|_{G\mathrm{N}_1}\big)$$

is a stable crossed module; cf. Definition 321.

Proof. For $k\in\mathbb{Z}_{\geqslant 0}$, we write $\mathrm{s}_k := \mathrm{s}_k^G$ and we write $\mathrm{d}_k := \mathrm{d}_k^G$.
Ad (SCM1). Suppose given $b,\,c\in G_0$.
By Lemma 393, we have

$$[\![b,\,c]\!] = [\![c,\,b]\!]^-\,.$$

Ad (SCM2). Suppose given $b,\,c\in G_0$.
We have

$$[\![b,\,c]\!]\mathrm{d}_0 = \big((b^-\mathrm{s}_0)^{c\mathrm{s}_1}\cdot(b\mathrm{s}_0)^{c\mathrm{s}_0}\big)\mathrm{d}_0 \overset{383.(3)}{=} b^-\cdot b^c = [b,c]\,.$$

Ad (SCM3). Suppose given $m,\,n\in G\mathrm{N}_1$.
We have

$$[\![m\mathrm{d}_0,\,n\mathrm{d}_0]\!] \overset{390.(3)}{=} m^-\cdot m^{n\mathrm{d}_0\mathrm{s}_0} \overset{390.(1)}{=} m^-\cdot m^n = [m,n]\,.$$

Ad (SCM4). Suppose given a, b, $c \in G_0$.

By Remark 391.(2), we have

$$\llbracket a,\, b\cdot c\rrbracket = \llbracket a,\, b\rrbracket \cdot \llbracket a^b,\, c\rrbracket .$$

This shows that $\big(G\mathrm{N}_1 , G_0 , \kappa_G , \mathrm{d}_0\big|_{G\mathrm{N}_1}\big)$ is a stable crossed module. □

Remark 396 (The stable crossed module morphism $\varphi\hat{\mathrm{N}}^{\pm}$)

Suppose given a $\lfloor 1,0 \rfloor$-simplicial group morphism $\varphi\colon G \to H$; cf. Remark 384.

We consider the stable crossed modules

$$G\hat{\mathrm{N}}^{\pm} = \big(G\mathrm{N}_1 , G_0 , \kappa_G , \mathrm{d}_0^G\big|_{G\mathrm{N}_1}\big) \quad \textit{and} \quad H\hat{\mathrm{N}}^{\pm} = \big(H\mathrm{N}_1 , H_0 , \kappa_H , \mathrm{d}_0^H\big|_{H\mathrm{N}_1}\big) ;$$

cf. Remark 395. *Then*

$$\varphi\hat{\mathrm{N}}^{\pm} := (\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} , \varphi_0)\colon G\hat{\mathrm{N}}^{\pm} \to H\hat{\mathrm{N}}^{\pm}$$

is a stable crossed module morphism; cf. Definition 324.

Proof. Suppose given $m \in G\mathrm{N}_1$. For $j \in \mathbb{Z}_{\geqslant 1}$, we have

$$m\varphi_1\mathrm{d}_j^H \overset{384}{=} m\mathrm{d}_j^G\varphi_0 = 1\varphi_0 = 1 .$$

This shows $m\varphi_1 \in H\mathrm{N}_1$.

So we have the group morphism $\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}\colon G\mathrm{N}_1 \to H\mathrm{N}_1$.

We show that $(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} , \varphi_0)$ is a stable crossed module morphism.

For $m \in G\mathrm{N}_1$, we have

$$m(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} \triangleleft \mathrm{d}_0^H\big|_{H\mathrm{N}_1}) = m\varphi_1\mathrm{d}_0^H \overset{384}{=} m\mathrm{d}_0^G\varphi_0 = m(\mathrm{d}_0^G\big|_{G\mathrm{N}_1} \triangleleft \varphi_0) .$$

This shows $\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} \triangleleft \mathrm{d}_0^H\big|_{H\mathrm{N}_1} = \mathrm{d}_0^G\big|_{G\mathrm{N}_1} \triangleleft \varphi_0$.

For b, $c \in G_0$, we have

$$\begin{aligned} \llbracket b,\, c\rrbracket\big(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}\big) &= \llbracket b,\, c\rrbracket\varphi_1 = \big((b^-\mathrm{s}_0^G)^{c\mathrm{s}_1^G}\cdot(b\mathrm{s}_0^G)^{c\mathrm{s}_0^G}\big)\varphi_1 \\ &\overset{384}{=} \big((b^-\varphi_0)\mathrm{s}_0^H\big)^{(c\varphi_0)\mathrm{s}_1^H}\cdot\big((b\varphi_0)\mathrm{s}_0^H\big)^{(c\varphi_0)\mathrm{s}_0^H} = \llbracket b\varphi_0,\, c\varphi_0\rrbracket . \end{aligned}$$

Altogether, this shows that $(\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1} , \varphi_0)$ is a stable crossed module morphism. □

Lemma 397 (The functor $\hat{\mathrm{N}}^{\pm}$)
We have the functor

$$\hat{\mathrm{N}}^{\pm}\colon \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp} \to \mathcal{StCrMod}\,,\ (G \xrightarrow{\varphi} H) \mapsto (G\hat{\mathrm{N}}^{\pm} \xrightarrow{\varphi\hat{\mathrm{N}}^{\pm}} H\hat{\mathrm{N}}^{\pm});$$

cf. Remarks 395, 396.

Proof. Suppose given $G \in \mathrm{Ob}\big(\lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp}\big)$. We have

$$(\mathrm{id}_G)\hat{\mathrm{N}}^{\pm} = (\mathrm{id}_{G_1}, \mathrm{id}_{G_0})\hat{\mathrm{N}}^{\pm} = \big(\mathrm{id}_{G_1}\big|_{G\mathrm{N}_1}^{G\mathrm{N}_1}, \mathrm{id}_{G_0}\big) = (\mathrm{id}_{G\mathrm{N}_1}, \mathrm{id}_{G_0}) = \mathrm{id}_{G\hat{\mathrm{N}}^{\pm}}\,.$$

Suppose given $G \xrightarrow{\varphi} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{\tilde{G}}$ in $\lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp}$.
Since $(G\mathrm{N}_1)\varphi_1 \leqslant \tilde{G}\mathrm{N}_1$, note that we have

$$(\varphi_1 \blacktriangle \tilde{\varphi}_1)\big|_{G\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1} = (\varphi_1\big|_{G\mathrm{N}_1}^{\tilde{G}\mathrm{N}_1}) \blacktriangle (\tilde{\varphi}_1\big|_{\tilde{G}\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1})\,.$$

So we have

$$\begin{aligned}(\varphi \blacktriangle \tilde{\varphi})\hat{\mathrm{N}}^{\pm} &= \Big((\varphi_1 \blacktriangle \tilde{\varphi}_1)\big|_{G\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1}, (\varphi \blacktriangle \tilde{\varphi})_0\Big) = \Big((\varphi_1\big|_{G\mathrm{N}_1}^{\tilde{G}\mathrm{N}_1}) \blacktriangle (\tilde{\varphi}_1\big|_{\tilde{G}\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1}), \varphi_0 \blacktriangle \tilde{\varphi}_0\Big)\\ &= (\varphi_1\big|_{G\mathrm{N}_1}^{\tilde{G}\mathrm{N}_1}, \varphi_0) \blacktriangle (\tilde{\varphi}_1\big|_{\tilde{G}\mathrm{N}_1}^{\tilde{\tilde{G}}\mathrm{N}_1}, \tilde{\varphi}_0) = \varphi\hat{\mathrm{N}}^{\pm} \blacktriangle \tilde{\varphi}\hat{\mathrm{N}}^{\pm}\,.\end{aligned}$$

This shows that $\hat{\mathrm{N}}^{\pm}$ is a functor. □

18.4 The functor $\mathrm{Rec}^{\pm}\colon \mathcal{StCrMod} \to \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp}$

Remark 398 (The $\lfloor 1,0\rfloor$-stable simplicial group $S\mathrm{Rec}^{\pm}$)
Suppose given a stable crossed module $S = (\mathrm{M}_S, \mathrm{B}_S, \kappa_S, \mathrm{f}_S)$; cf. Definition 321.
We have the $\lfloor 1,0\rfloor$-stable simplicial group $S\mathrm{Rec}^{\pm}$ given as follows.
We have

$$\begin{aligned}(S\mathrm{Rec}^{\pm})_0 &:= \mathrm{B}_V\\ (S\mathrm{Rec}^{\pm})_1 &:= H^S_\infty\\ &\overset{356}{=} \{h \in (\prod_{i\geqslant 0}\mathrm{B}_S) \times \mathrm{M}_S\colon \exists \ell \in \mathbb{Z}_{\geqslant 0} \text{ with } h_{\{k\}} = 1 \text{ for } k \geqslant \ell\}\\ &= \{(\ldots, b_3, b_2, b_1, b_0, m) \in (\prod_{i\geqslant 0}\mathrm{B}_S) \times \mathrm{M}_S\colon \exists \ell \in \mathbb{Z}_{\geqslant 0} \text{ with } b_k = 1 \text{ for } k \geqslant \ell\}\,.\end{aligned}$$

Recall that H^S_∞ is a group; cf. Definition 356*, Lemma* 362.(2) *and Proposition* 361.

For $i \in \mathbb{Z}_{\geqslant 0}$, we have the group morphisms

$$\mathrm{s}_i^{S\mathrm{Rec}^\pm} := s_i^\infty \colon \mathrm{B}_S \to H^S_\infty\,,\ b \mapsto bs_i^\infty = (\ldots, 1, \underbrace{b}_{\text{pos. } i}, 1, \ldots, 1);$$

cf. Lemma 359.(1).

For $j \in \mathbb{Z}_{\geqslant 1}$, we have the group morphism

$$\mathrm{d}_j^{S\mathrm{Rec}^\pm} := d_j^\infty \colon H^S_\infty \to \mathrm{B}_S\,,\ h \mapsto h\mathrm{d}_j^\infty = h_{\{j\}} \cdot h_{\{j-1\}}\,;$$

cf. Lemma 364.(1).

Moreover, we have the group morphism

$$\mathrm{d}_0^{S\mathrm{Rec}^\pm} := d_0^\infty \colon H^S_\infty \to \mathrm{B}_S\,,\ h \mapsto h\mathrm{d}_0^\infty = h_{\{0\}} \cdot h_{\{-1\}}\mathrm{f}_S\,;$$

cf. Lemma 364.(2).

Proof. We have to show the properties $(1, 2, 3)$ from Remark 385.

Ad (1). Suppose given $h \in H^S_\infty$. We may choose some $m \in \mathbb{Z}_{\geqslant 0}$ such that $h_{\{k\}} = 1$ for $k \in \mathbb{Z}_{\geqslant m}$; cf. Definition 356.

Let $\ell := m + 1$. Then, for $k \in \mathbb{Z}_{\geqslant \ell}$, we have

$$hd_k^\infty = h_{\{k\}} \cdot h_{\{k-1\}} = 1 \cdot 1 = 1\,.$$

This shows (1).

Ad (2). Suppose given subsets I, $J \subseteq \mathbb{Z}_{\geqslant 0}$ with $I \cup J = \mathbb{Z}_{\geqslant 0}$.

Then, by Corollary 382, we have

$$\big[\bigcap_{i \in I} \ker d_i^\infty\,, \bigcap_{j \in J} \ker d_j^\infty\big] = 1\,.$$

This shows (2).

Ad (3). Since we have $\mathrm{s}_i^{S\mathrm{Rec}^\pm} = \mathrm{s}_i^\infty$ and $\mathrm{d}_i^{S\mathrm{Rec}^\pm} = \mathrm{d}_i^\infty$ for $i \in \mathbb{Z}_{\geqslant 0}$, this follows from Remark 365.

So, by Remark 385, we have that $S\mathrm{Rec}^\pm$ is a $\lfloor 1,0 \rfloor$-stable simplicial group. □

Remark 399 (The $\lfloor 1,0\rfloor$-stable simplicial group morphism $\rho\,\mathrm{Rec}^{\pm}$)
Suppose given stable crossed modules

$$S = (\mathrm{M}_S\,, \mathrm{B}_S\,, \kappa_S\,, \mathrm{f}_S) \text{ and } \tilde{S} = (\mathrm{M}_{\tilde{S}}\,, \mathrm{B}_{\tilde{S}}\,, \kappa_{\tilde{S}}\,, \mathrm{f}_{\tilde{S}})\,;$$

cf. Definition 321.
Suppose given a stable crossed module morphism $\rho = (\mu, \beta)\colon S \to \tilde{S}$*; cf. Definition* 324.
We consider the $\lfloor 1,0\rfloor$*-stable simplicial groups* $S\mathrm{Rec}^{\pm}$ *and* $\tilde{S}\mathrm{Rec}^{\pm}$*; cf. Remark* 398.
We have the $\lfloor 1,0\rfloor$*-stable simplicial group morphism*

$$\rho\,\mathrm{Rec}^{\pm}\colon S\mathrm{Rec}^{\pm} \to \tilde{S}\mathrm{Rec}^{\pm}$$

given as follows. We have the group morphism

$$(\rho\,\mathrm{Rec}^{\pm})_1\colon H^S_\infty \to H^{\tilde{S}}_\infty\,,\ h \mapsto h(\rho\,\mathrm{Rec}^{\pm})_1\,,$$

where

$$\big(h(\rho\,\mathrm{Rec}^{\pm})_1\big)_{\{i\}} = \begin{cases} (h_{\{i\}})\beta & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ (h_{\{-1\}})\mu & \text{for } i = -1\,, \end{cases}$$

and we have the group morphism $(\rho\,\mathrm{Rec}^{\pm})_0 := \beta\colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}$*.*
So we have

$$\begin{array}{lrcl} (\rho\,\mathrm{Rec}^{\pm})_1\colon & H^S_\infty & \to & H^{\tilde{S}}_\infty\,, \quad (\ldots, b_1\,, b_0\,, m) \mapsto (\ldots, b_1\beta,\, b_0\beta,\, m\mu) \\ (\rho\,\mathrm{Rec}^{\pm})_0\colon & \mathrm{B}_S & \to & \mathrm{B}_{\tilde{S}}\,, \quad b \mapsto b\beta\,. \end{array}$$

Proof. We write $\varphi := (\rho\,\mathrm{Rec}^{\pm})_1\colon H^S_\infty \to H^{\tilde{S}}_\infty$*.*
We show that the following assertions (1, 2, 3) *hold.*

(1) We have $\mathrm{s}_t^{S\mathrm{Rec}^{\pm}} \triangleleft \varphi = \beta \triangleleft \mathrm{s}_t^{\tilde{S}\mathrm{Rec}^{\pm}}\colon \mathrm{B}_S \to H^{\tilde{S}}_\infty$ for $t \in \mathbb{Z}_{\geqslant 0}$*.*

(2) We have $\mathrm{d}_t^{S\mathrm{Rec}^{\pm}} \triangleleft \beta = \varphi \triangleleft \mathrm{d}_t^{\tilde{S}\mathrm{Rec}^{\pm}}\colon H^S_\infty \to \mathrm{B}_{\tilde{S}}$ for $t \in \mathbb{Z}_{\geqslant 0}$*.*

(3) The map φ is a group morphism.

Ad (1). Suppose given $c \in \mathrm{B}_S$ and $t \in \mathbb{Z}_{\geqslant 0}$*.*

For $i \in \mathbb{Z}_{\geqslant -1}$, we have

$$\begin{aligned}\big((c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} &= \begin{cases} \big((c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})_{\{i\}}\big)\beta & \text{if } i \in \mathbb{Z}_{\geqslant 0} \\ \big((c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})_{\{-1\}}\big)\mu & \text{if } i = -1 \end{cases} \\ &= \begin{cases} c\beta & \text{if } i = t \\ 1\beta & \text{if } i \in \mathbb{Z}_{\geqslant 0}\backslash\{t\} \\ 1\mu & \text{if } i = -1 \end{cases} \\ &= \begin{cases} c\beta & \text{if } i = t \\ 1 & \text{if } i \in \mathbb{Z}_{\geqslant -1}\backslash\{t\} \end{cases} \\ &= \big((c\beta)\mathrm{s}_t^{\tilde{S}\mathrm{Rec}^{\pm}}\big)_{\{i\}}.\end{aligned}$$

So we have $(c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi = (c\beta)\mathrm{s}_t^{\tilde{S}\mathrm{Rec}^{\pm}}$. This shows $\mathrm{s}_t^{S\mathrm{Rec}^{\pm}} \triangleleft \varphi = \beta \triangleleft \mathrm{s}_t^{\tilde{S}\mathrm{Rec}^{\pm}}$.

Ad (2). Suppose given $h \in H_\infty^S$ and $t \in \mathbb{Z}_{\geqslant 0}$.

Case $t = 0$. We have

$$\begin{aligned}h(\mathrm{d}_0^{S\mathrm{Rec}^{\pm}} \triangleleft \beta) &= \big(h_{\{0\}} \cdot (h_{\{-1\}})\mathrm{f}_S\big)\beta = (h_{\{0\}})\beta \cdot \big((h_{\{-1\}})\mathrm{f}_S\big)\beta \overset{324.(1)}{=} (h_{\{0\}})\beta \cdot \big((h_{\{-1\}})\mu\big)\mathrm{f}_{\tilde{S}} \\ &= (h\varphi)_{\{0\}} \cdot \big((h\varphi)_{\{-1\}}\big)\mathrm{f}_{\tilde{S}} = (h\varphi)\mathrm{d}_0^{\tilde{S}\mathrm{Rec}^{\pm}} = h(\varphi \triangleleft \mathrm{d}_0^{\tilde{S}\mathrm{Rec}^{\pm}}).\end{aligned}$$

Case $t \geqslant 1$. We have

$$\begin{aligned}h(\mathrm{d}_t^{S\mathrm{Rec}^{\pm}} \triangleleft \beta) &= (h_{\{t\}} \cdot h_{\{t-1\}})\beta = (h_{\{t\}})\beta \cdot (h_{\{t-1\}})\beta = (h\varphi)_{\{t\}} \cdot (h\varphi)_{\{t-1\}} \\ &= (h\varphi)\mathrm{d}_t^{\tilde{S}\mathrm{Rec}^{\pm}} = h(\varphi \triangleleft \mathrm{d}_t^{\tilde{S}\mathrm{Rec}^{\pm}}).\end{aligned}$$

This shows $\mathrm{d}_t^{S\mathrm{Rec}^{\pm}} \triangleleft \beta = \varphi \triangleleft \mathrm{d}_t^{\tilde{S}\mathrm{Rec}^{\pm}}$.

Ad (3). We use Remark 363 to show that $\varphi\colon H_\infty^S \to H_\infty^{\tilde{S}}$ is a group morphism.

Suppose given $h \in H_\infty^S$, $c \in \mathrm{B}_S$, $n \in \mathrm{M}_S$ and $t \in \mathbb{Z}_{\geqslant 0}$.

For $i \in \mathbb{Z}_{\geqslant 0}$, we write $b_i := h_{\{i\}} \in \mathrm{B}_S$ and we write $m := h_{\{-1\}} \in \mathrm{M}_S$.

We show that $(h \cdot c\,\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi \overset{!}{=} h\varphi \cdot (c\,\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi$.

Case $t \geqslant 1$. We have

$$
\begin{aligned}
& \big((h\cdot c\,\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} \\
= \quad & \begin{cases} \big((h\cdot c\,\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})_{\{i\}}\big)\beta & \text{for } i\in\mathbb{Z}_{\geqslant 0} \\ \big((h\cdot c\,\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})_{\{-1\}}\big)\mu & \text{for } i=-1 \end{cases} \\
\overset{362.(2)}{=} \quad & \begin{cases} b_i\beta & \text{for } i\in\mathbb{Z}_{\geqslant t+1} \\ (b_i\cdot c)\beta & \text{for } i=t \\ \Big(\big([b_{t-1}\,,c]^{(-1)^{t-1-i}}\big)^{\left(\prod\limits_{j\in\lfloor t-2,i+1\rfloor} b_j\right)}\cdot b_i\Big)\beta & \text{for } i\in[0,t-1] \\ \Big(\big(\llbracket b_{t-1},\,c\rrbracket^{(-1)^t}\big)^{\left(\prod\limits_{j\in\lfloor t-2,0\rfloor} b_j\right)}\cdot m\Big)\mu & \text{for } i=-1 \end{cases} \\
\overset{324,\,328.(1)}{=} \quad & \begin{cases} b_i\beta & \text{for } i\in\mathbb{Z}_{\geqslant t+1} \\ b_i\beta\cdot c\beta & \text{for } i=t \\ \Big(\big[(b_{t-1})\beta\,,c\beta\big]^{(-1)^{t-1-i}}\Big)^{\left(\prod\limits_{j\in\lfloor t-2,i+1\rfloor} b_j\beta\right)}\cdot b_i\beta & \text{for } i\in[0,t-1] \\ \Big(\big\llbracket(b_{t-1})\beta,\,c\beta\big\rrbracket^{(-1)^t}\Big)^{\left(\prod\limits_{j\in\lfloor t-2,0\rfloor} b_j\beta\right)}\cdot m\mu & \text{for } i=-1\,. \end{cases}
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
& \big(h\varphi\cdot(c\,\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} \\
\overset{(1)}{=} \quad & \big(h\varphi\cdot(c\beta)\mathrm{s}_t^{\tilde{S}\mathrm{Rec}^{\pm}}\big)_{\{i\}} \\
\overset{362.(2)}{=} \quad & \begin{cases} (h\varphi)_{\{i\}} & \text{for } i\in\mathbb{Z}_{\geqslant t+1} \\ (h\varphi)_{\{i\}}\cdot c\beta & \text{for } i=t \\ \Big(\big[(h\varphi)_{\{t-1\}}\,,c\beta\big]^{(-1)^{t-1-i}}\Big)^{\left(\prod\limits_{j\in\lfloor t-2,i+1\rfloor}(h\varphi)_{\{j\}}\right)}\cdot(h\varphi)_{\{i\}}\Big) & \text{for } i\in[0,t-1] \\ \Big(\big\llbracket(h\varphi)_{\{t-1\}},\,c\beta\big\rrbracket^{(-1)^t}\Big)^{\left(\prod\limits_{j\in\lfloor t-2,0\rfloor}(h\varphi)_{\{j\}}\right)}\cdot(h\varphi)_{\{-1\}} & \text{for } i=-1 \end{cases} \\
= \quad & \begin{cases} b_i\beta & \text{for } i\in\mathbb{Z}_{\geqslant t+1} \\ b_i\beta\cdot c\beta & \text{for } i=t \\ \Big(\big[(b_{t-1})\beta\,,c\beta\big]^{(-1)^{t-1-i}}\Big)^{\left(\prod\limits_{j\in\lfloor t-2,i+1\rfloor} b_j\beta\right)}\cdot b_i\beta & \text{for } i\in[0,t-1] \\ \Big(\big\llbracket(b_{t-1})\beta,\,c\beta\big\rrbracket^{(-1)^t}\Big)^{\left(\prod\limits_{j\in\lfloor t-2,0\rfloor} b_j\beta\right)}\cdot m\mu & \text{for } i=-1\,. \end{cases}
\end{aligned}
$$

This shows $(h \cdot c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi = h\varphi \cdot (c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi$ for $t \geqslant 1$.

Case $t = 0$. We have

$$\begin{aligned}
\big((h \cdot c\mathrm{s}_0^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} &= \begin{cases} \big((h \cdot c\mathrm{s}_0^{S\mathrm{Rec}^{\pm}})_{\{i\}}\big)\beta & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ \big((h \cdot c\mathrm{s}_0^{S\mathrm{Rec}^{\pm}})_{\{-1\}}\big)\mu & \text{for } i = -1 \end{cases} \\
&\overset{362.(2)}{=} \begin{cases} b_i\beta & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ (b_0 \cdot c)\beta & \text{for } i = 0 \\ (m^c)\mu & \text{for } i = -1 \end{cases} \\
&\overset{328.(1)}{=} \begin{cases} b_i\beta & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ b_0\beta \cdot c\beta & \text{for } i = 0 \\ (m\mu)^{c\beta} & \text{for } i = -1\,. \end{cases}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\big(h\varphi \cdot (c\mathrm{s}_0^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} &\overset{(1)}{=} \big(h\varphi \cdot (c\beta)\mathrm{s}_0^{\tilde{S}\mathrm{Rec}^{\pm}}\big)_{\{i\}} \\
&\overset{362.(2)}{=} \begin{cases} (h\varphi)_{\{i\}} & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ (h\varphi)_{\{0\}} \cdot c\beta & \text{for } i = 0 \\ \big((h\varphi)_{\{-1\}}\big)^{c\beta} & \text{for } i = -1 \end{cases} \\
&= \begin{cases} b_i\beta & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ b_0\beta \cdot c\beta & \text{for } i = 0 \\ (m\mu)^{c\beta} & \text{for } i = -1\,. \end{cases}
\end{aligned}$$

This shows $(h \cdot c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi = h\varphi \cdot (c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi$ for $t = 0$.

So we have $(h \cdot c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi = h\varphi \cdot (c\mathrm{s}_t^{S\mathrm{Rec}^{\pm}})\varphi$ for $t \in \mathbb{Z}_{\geqslant 0}$.

We show that $(h \cdot n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi \overset{!}{=} h\varphi \cdot (n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi$.

We have

$$\begin{aligned}
\big((h \cdot n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} &= \begin{cases} \big((h \cdot n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})_{\{i\}}\big)\beta & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ \big((h \cdot n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})_{\{-1\}}\big)\mu & \text{for } i = -1 \end{cases} \\
&\overset{362.(2)}{=} \begin{cases} b_i\beta & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ (m \cdot n)\mu & \text{for } i = -1 \end{cases}
\end{aligned}$$

$$= \begin{cases} b_i\beta & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ m\mu \cdot n\mu & \text{for } i = -1\,. \end{cases}$$

For $i \in \mathbb{Z}_{\geqslant -1}$, note that we have

$$\begin{aligned} \big((n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} &= \begin{cases} \big((n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})_{\{i\}}\big)\beta & \text{if } i \in \mathbb{Z}_{\geqslant 0} \\ \big((n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})_{\{-1\}}\big)\mu & \text{if } i = -1 \end{cases} \\ &\overset{359.(2)}{=} \begin{cases} 1\beta & \text{if } i \in \mathbb{Z}_{\geqslant 0} \\ n\mu & \text{if } i = -1 \end{cases} \\ &= \big((n\mu)\mathrm{s}_{-1}^{\tilde{S}\mathrm{Rec}^{\pm}}\big)_{\{i\}}\,. \end{aligned}$$

This shows $(n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi = (n\mu)\mathrm{s}_{-1}^{\tilde{S}\mathrm{Rec}^{\pm}}$.

So we have

$$\begin{aligned} \big(h\varphi \cdot (n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi\big)_{\{i\}} &= \big(h\varphi \cdot (n\mu)\mathrm{s}_{-1}^{\tilde{S}\mathrm{Rec}^{\pm}}\big)_{\{i\}} \\ &\overset{362.(2)}{=} \begin{cases} (h\varphi)_{\{i\}} & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ (h\varphi)_{\{-1\}} \cdot n\mu & \text{for } i = -1 \end{cases} \\ &= \begin{cases} b_i\beta & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ m\mu \cdot n\mu & \text{for } i = -1\,. \end{cases} \end{aligned}$$

This shows $(h \cdot n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi = h\varphi \cdot (n\mathrm{s}_{-1}^{S\mathrm{Rec}^{\pm}})\varphi$.

So, by Remark 363, φ is a group morphism.

Altogether, by $(1,2,3)$, $\rho\,\mathrm{Rec}^{\pm}$ is a $\lfloor 1,0\rfloor$-stable simplicial group morphism; cf. Remark 137. □

Lemma 400 (The functor $\mathrm{Rec}^{\pm}$)

We have the reconstruction functor

$$\mathrm{Rec}^{\pm}\colon \mathcal{StCrMod} \to \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp}\,,\ (S \xrightarrow{\rho} \tilde{S}) \mapsto (S\mathrm{Rec}^{\pm} \xrightarrow{\rho\,\mathrm{Rec}^{\pm}} \tilde{S}\mathrm{Rec}^{\pm});$$

cf. Remarks 398, 399.

Proof. Suppose given a stable crossed module $S = (\mathrm{M}_S\,,\ \mathrm{B}_S\,,\ \kappa_S\,,\ \mathrm{f}_S)$.

We consider the identity stable crossed module morphism $\mathrm{id}_S = (\mathrm{id}_{\mathrm{M}_V}, \mathrm{id}_{\mathrm{B}_V})\colon S \to S$.
We have

$$\big((\mathrm{id}_S)\,\mathrm{Rec}^{\pm}\big)_0 = \mathrm{id}_{\mathrm{B}_V}\ .$$

Moreover, for $h \in H^S_\infty$ and $i \in \mathbb{Z}_{\geqslant -1}$, we have

$$\begin{aligned}\Big(h\big((\mathrm{id}_S)\,\mathrm{Rec}^{\pm}\big)_1\Big)_{\{i\}} &= \begin{cases}(h_{\{i\}})\,\mathrm{id}_{\mathrm{B}_S} & \text{for } i \in \mathbb{Z}_{\geqslant 0}\\ (h_{\{-1\}})\,\mathrm{id}_{\mathrm{M}_S} & \text{for } i = -1\end{cases}\\ &= \begin{cases}h_{\{i\}} & \text{for } i \in \mathbb{Z}_{\geqslant 0}\\ h_{\{-1\}} & \text{for } i = -1\end{cases}\\ &= h_{\{i\}}\,.\end{aligned}$$

So $\big(h(\mathrm{id}_S)\,\mathrm{Rec}^{\pm}\big)_1 = h$ and therefore $\big((\mathrm{id}_S)\,\mathrm{Rec}^{\pm}\big)_1 = \mathrm{id}_{H^S_\infty}$.
This shows that $(\mathrm{id}_S)\,\mathrm{Rec}^{\pm} = \mathrm{id}_{S\mathrm{Rec}^{\pm}}$.

Suppose given $S \xrightarrow{\rho} \tilde{S} \xrightarrow{\tilde{\rho}} \tilde{\tilde{S}}$ in $\mathcal{S}t\mathcal{S}impGrp$.
We have $\rho = (\mu, \beta)$, where $\mu\colon \mathrm{M}_S \to \mathrm{M}_{\tilde{S}}$ and $\beta\colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}$.
We have $\tilde{\rho} = (\tilde{\mu}, \tilde{\beta})$, where $\tilde{\mu}\colon \mathrm{M}_{\tilde{S}} \to \mathrm{M}_{\tilde{\tilde{S}}}$ and $\beta\colon \mathrm{B}_{\tilde{S}} \to \mathrm{B}_{\tilde{\tilde{S}}}$.
Note that $\rho \mathbin{\triangle} \tilde{\rho} = (\mu, \beta) \mathbin{\triangle} (\tilde{\mu}, \tilde{\beta}) = (\mu \mathbin{\triangle} \tilde{\mu}, \beta \mathbin{\triangle} \tilde{\beta})$; cf. Remark 326.(1).
We have

$$\big((\rho \mathbin{\triangle} \rho)\,\mathrm{Rec}^{\pm}\big)_0 = \beta \mathbin{\triangle} \tilde{\beta} = (\rho\,\mathrm{Rec}^{\pm})_0 \mathbin{\triangle} (\tilde{\rho}\,\mathrm{Rec}^{\pm})_0\,.$$

Suppose given $h \in (S\mathrm{Rec}^{\pm})_1 = H^S_\infty$. We have

$$\begin{aligned}\Big(h\big((\rho\,\mathrm{Rec}^{\pm})_1 \mathbin{\triangle} (\tilde{\rho}\,\mathrm{Rec}^{\pm})_1\big)\Big)_{\{i\}} &= \Big(\big(h(\rho\,\mathrm{Rec}^{\pm})_1\big)(\tilde{\rho}\,\mathrm{Rec}^{\pm})_1\Big)_{\{i\}}\\ &= \begin{cases}\Big(\big(h(\rho\,\mathrm{Rec}^{\pm})_1\big)_{\{i\}}\Big)\tilde{\beta} & \text{for } i \in \mathbb{Z}_{\geqslant 0}\\ \Big(\big(h(\rho\,\mathrm{Rec}^{\pm})_1\big)_{\{-1\}}\Big)\tilde{\mu} & \text{for } i = -1\end{cases}\\ &= \begin{cases}\big((h_{\{i\}})\beta\big)\tilde{\beta} & \text{for } i \in \mathbb{Z}_{\geqslant 0}\\ \big((h_{\{-1\}})\mu\big)\tilde{\mu} & \text{for } i = -1\end{cases}\\ &= \Big(h\big((\rho \mathbin{\triangle} \tilde{\rho})\,\mathrm{Rec}^{\pm}\big)_1\Big)_{\{i\}}\,.\end{aligned}$$

So we have $h\big((\rho\,\mathrm{Rec}^{\pm})_1 \mathbin{\triangle} (\tilde{\rho}\,\mathrm{Rec}^{\pm})_1\big) = h\big((\rho \mathbin{\triangle} \tilde{\rho})\,\mathrm{Rec}^{\pm}\big)_1$.

Hence $(\rho\,\mathrm{Rec}^{\pm})_1 \blacktriangle (\tilde{\rho}\,\mathrm{Rec}^{\pm})_1 = \big((\rho \blacktriangle \tilde{\rho})\,\mathrm{Rec}^{\pm}\big)_1$.

This shows that $(\rho \blacktriangle \tilde{\rho})\,\mathrm{Rec}^{\pm} = \rho\,\mathrm{Rec}^{\pm} \blacktriangle \tilde{\rho}\,\mathrm{Rec}^{\pm}$.

Therefore, $\mathrm{Rec}^{\pm}$ is a functor. □

18.5 The equivalences $\hat{\mathrm{N}}^{\pm}$ and $\mathrm{Rec}^{\pm}$

We consider the functors

$$\begin{array}{lrcll} \hat{\mathrm{N}}^{\pm}\colon & \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp} & \to & \mathcal{StCrMod}; & \text{cf. Lemma 397} \\ \mathrm{Rec}^{\pm}\colon & \mathcal{StCrMod} & \to & \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp}; & \text{cf. Lemma 400.} \end{array}$$

Remark 401 (The stable crossed module $S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}$)

(1) *Suppose given a stable crossed module $S = (\mathrm{M}_S\,,\,\mathrm{B}_S\,,\,\kappa_S\,,\,\mathrm{f}_S)$. We consider the stable crossed module $S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}$.*

Recall that we have the $\lfloor 1,0\rfloor$-stable simplicial group $S\,\mathrm{Rec}^{\pm}$, where

$$(S\,\mathrm{Rec}^{\pm})_1 = H^S_\infty\,,\ (S\,\mathrm{Rec}^{\pm})_0 = \mathrm{B}_S\,,$$

where for $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{s}_i^{S\,\mathrm{Rec}^{\pm}}\colon \mathrm{B}_S \to H^S_\infty\,,\ b \mapsto (\dots\,1, \underbrace{b}_{\text{pos. } i}, 1\,, \dots\,, \underbrace{1}_{\text{pos. } -1})$$

and where for $j \in \mathbb{Z}_{\geqslant 1}$, we have

$$\mathrm{d}_j^{S\,\mathrm{Rec}^{\pm}}\colon H^S_\infty \to \mathrm{B}_S\,,\ (\dots\,, b_1\,, b_0\,, m) \mapsto b_j \cdot b_{j-1}\,,$$

and where

$$\mathrm{d}_0^{S\,\mathrm{Rec}^{\pm}}\colon H^S_\infty \to \mathrm{B}_S\,,\ (\dots\,, b_1\,, b_0\,, m) \mapsto b_0 \cdot m\mathrm{f}_S\,;$$

cf. Remark 398.

Then, by Remark 395*, we have the stable crossed module*

$$\begin{aligned} S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} &= (\mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\,,\,\mathrm{B}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\,,\,\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\,,\,\mathrm{f}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}) \\ &= \Big((S\,\mathrm{Rec}^{\pm})\mathrm{N}_1\,,\mathrm{B}_S\,,\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\,,\mathrm{d}_0^{S\,\mathrm{Rec}^{\pm}}\big|_{(S\,\mathrm{Rec}^{\pm})\mathrm{N}_1}\Big) \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = (S\,\mathrm{Rec}^{\pm})\mathrm{N}_1 \overset{257.(2)}{=} \bigcap_{j\in\mathbb{Z}_{\geqslant 1}} \ker \mathrm{d}_j^{S\,\mathrm{Rec}^{\pm}} &\overset{366}{=} \{(\dots,\,1,\,1,\,m)\colon m\in \mathrm{M}_S\} \\ &= \{h\in H_\infty^S\colon h_{\{i\}}=1 \text{ for } i\in\mathbb{Z}_{\geqslant 0}\} \\ &\leqslant H_\infty^S\,, \end{aligned}$$

and that therefore

$$\mathrm{f}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\colon \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} \to \mathrm{B}_S\,,\ h\mapsto h\mathrm{f}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = h\mathrm{d}_0^{S\,\mathrm{Rec}^{\pm}} = h_{\{0\}}\cdot(h_{\{-1\}})\mathrm{f}_S = (h_{\{-1\}})\mathrm{f}_S\,.$$

We have

$$\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\colon \mathrm{B}_S\times\mathrm{B}_S\to \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\,,\ (b,c)\mapsto (b,c)\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\,,$$

where

$$\big((b,c)\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\big)_{\{i\}} = \begin{cases} 1 & \text{for } i\in\mathbb{Z}_{\geqslant 0} \\ (b,c)\kappa_S & \text{for } i=-1\,. \end{cases}$$

So, for $b,\,c\in\mathrm{B}_S$, we have

$$[\![b,\,c]\!] = (b,c)\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = \big(\dots,\,1,\,1\,,(b,c)\kappa_S\big) = \big(\dots,\,1,\,1\,,[\![b,\,c]\!]\big)\in H_\infty^S\,.$$

(2) *Suppose given a stable crossed module morphism $\rho=(\mu,\beta)\colon S\to\tilde{S}$; cf. Definition* 324.
We consider the stable crossed modules $S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}$ and $\tilde{S}\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}$; cf. (1).
We consider the stable crossed module morphism $\rho\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}\colon S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}\to\tilde{S}\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}$.
Recall that we have the $\lfloor 1,0\rfloor$-stable simplicial group morphism

$$\rho\,\mathrm{Rec}^{\pm}\colon S\,\mathrm{Rec}^{\pm}\to\tilde{S}\,\mathrm{Rec}^{\pm}\,,$$

where

$$\begin{aligned} (\rho\,\mathrm{Rec}^{\pm})_1\colon&\ H_\infty^S\to H_\infty^{\tilde{S}}\,,\quad (\dots,b_1\,,b_0\,,m)\mapsto(\dots,b_1\beta\,,b_0\beta\,,m\mu) \\ (\rho\,\mathrm{Rec}^{\pm})_0\colon&\ \mathrm{B}_S\to\mathrm{B}_{\tilde{S}}\,,\quad b\mapsto b\beta\,; \end{aligned}$$

cf. Remark 399.

Then, by Remark 396*, we have the stable crossed module morphism*

$$\begin{aligned} \rho\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} &= \big((\rho\,\mathrm{Rec}^{\pm})_1\big|^{\mathrm{M}_{\tilde{S}\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}}_{\mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}}\,,(\rho\,\mathrm{Rec}^{\pm})_0\big) \\ &= \big((\rho\,\mathrm{Rec}^{\pm})_1\big|^{(\tilde{S}\,\mathrm{Rec}^{\pm})\mathrm{N}_1}_{(S\,\mathrm{Rec}^{\pm})\mathrm{N}_1}\,,\beta\big)\colon S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}\to\tilde{S}\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}\,. \end{aligned}$$

Note that

$$(\rho\,\mathrm{Rec}^{\pm})_1\big|^{(\tilde{S}\,\mathrm{Rec}^{\pm})\mathrm{N}_1}_{(S\,\mathrm{Rec}^{\pm})\mathrm{N}_1}\colon H_\infty^S\to H_\infty^{\tilde{S}}\,,\ (\dots,1\,,1\,,m)\mapsto(\dots,1\,,1\,,m\mu)\,.$$

Proof. Ad (1). Suppose given $b,\, c \in \mathrm{B}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = \mathrm{B}_S$.
We have

$$\begin{array}{rccl} \kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\colon & \mathrm{B}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} \times \mathrm{B}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} & \to & \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} \\ & (b, c) & \mapsto & (b, c)\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = (b^{-}\mathrm{s}_0)^{c\mathrm{s}_1} \cdot (b\mathrm{s}_0)^{c\mathrm{s}_0}\,, \end{array}$$

where

$$\begin{aligned} \mathrm{s}_0 &:= \mathrm{s}_0^{S\mathrm{Rec}^{\pm}} \colon \mathrm{B}_S \to H^S_\infty\,,\ b \mapsto (\ldots, 1, \underbrace{b}_{\text{pos. } 0}, 1) \\ \mathrm{s}_1 &:= \mathrm{s}_1^{S\mathrm{Rec}^{\pm}} \colon \mathrm{B}_S \to H^S_\infty\,,\ b \mapsto (\ldots, 1, \underbrace{b}_{\text{pos. } 1}, 1, 1)\,; \end{aligned}$$

cf. Remark 395.
We show that $(b^{-}\mathrm{s}_0)^{c\mathrm{s}_1} \cdot (b\mathrm{s}_0)^{c\mathrm{s}_0} \overset{!}{=} \big(\ldots,\, 1,\, 1,\, [\![b,\, c]\!]\big)$.
We calculate the product $c^{-}\mathrm{s}_1 \cdot b^{-}\mathrm{s}_0 \cdot c\mathrm{s}_1 \cdot (b^{c})\mathrm{s}_0$ using Lemma 362.(1, 2).
We have

$$(c^{-}\mathrm{s}_1 \cdot b^{-}\mathrm{s}_0)_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 2} \\ c^{-} & \text{for } i = 1 \\ b^{-} & \text{for } i = 0 \\ 1 & \text{for } i = -1\,. \end{cases}$$

So we have

$$\begin{aligned} \big((c^{-}\mathrm{s}_1 \cdot b^{-}\mathrm{s}_0) \cdot c\mathrm{s}_1\big)_{\{i\}} \quad &= \quad \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 2} \\ c^{-} \cdot c & \text{for } i = 1 \\ (b^{-})^{c} & \text{for } i = 0 \\ [\![b^{-},\, c]\!]^{-} \cdot 1 & \text{for } i = -1 \end{cases} \\ &\overset{(\mathrm{SCM1})}{=} \quad \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ (b^{-})^{c} & \text{for } i = 0 \\ [\![c,\, b^{-}]\!] & \text{for } i = -1\,. \end{cases} \end{aligned}$$

Therefore, we have

$$\big((c^- \mathrm{s}_1 \cdot b^- \mathrm{s}_0 \cdot c\mathrm{s}_1) \cdot (b^c)\mathrm{s}_0\big)_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 1} \\ (b^-)^c \cdot b^c & \text{for } i = 0 \\ \llbracket c,\, b^- \rrbracket^{(b^c)} & \text{for } i = -1 \end{cases}$$
$$= \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ \llbracket c,\, b^- \rrbracket^{(b^c)} & \text{for } i = -1\,. \end{cases}$$

It remains to show that $\llbracket c,\, b^- \rrbracket^{(b^c)} \overset{!}{=} \llbracket b,\, c \rrbracket$.

We have

$$\llbracket c,\, b^- \rrbracket^{(b^c)} = \big\llbracket c^{b \cdot b^-},\, (b^-)^{b^-} \big\rrbracket^{(b^c)} \overset{323.(10)}{=} \llbracket c^b,\, b^- \rrbracket^{[b,c]} \overset{322.(4)}{=} \llbracket b,\, c \rrbracket^{[b,c]} \overset{323.(3)}{=} \llbracket b,\, c \rrbracket^{\llbracket b,\, c \rrbracket} = \llbracket b,\, c \rrbracket\,.$$

This shows $(b, c)\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = (b^-\mathrm{s}_0)^{c\mathrm{s}_1} \cdot (b\mathrm{s}_0)^{c\mathrm{s}_0} = \big(\ldots,\, 1\,,\, 1,\, \llbracket b,\, c \rrbracket\big)$. □

Lemma 402 (The functor $\mathrm{Rec}^{\pm} * \hat{\mathrm{N}}^{\pm}$)

(1) *Suppose given a stable crossed module* $S = (\mathrm{M}_S\,,\, \mathrm{B}_S\,,\, \kappa_S\,,\, \mathrm{f}_S)$.
We have the stable crossed module isomorphism

$$S\hat{\psi} := (\pi_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})\colon S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} \underset{\sim}{\longrightarrow} S$$

where

$$\pi_{\mathrm{M}_S}\colon \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} \underset{\sim}{\longrightarrow} \mathrm{M}_S\,,\ h \mapsto h_{\{-1\}}\,.$$

(2) *We have the isotransformation*

$$\hat{\psi} = \big(S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} \xrightarrow{S\hat{\psi}} S\big)_{S \in \mathrm{Ob}(\mathcal{StCrMod})}\colon\ \mathrm{Rec}^{\pm} * \hat{\mathrm{N}}^{\pm} \to \mathrm{id}_{\mathcal{StCrMod}}\,;$$

cf. (1).

Proof. Ad (1). *We show that* π_{M_S} *is a group isomorphism.*

Suppose given $h,\, \tilde{h} \in \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}$. Note that we have $h_{\{i\}} = 1$ and $\tilde{h}_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant 0}$.

By Lemma 362.(2), we have

$$(h \cdot \tilde{h})_{\{i\}} = \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ h_{\{-1\}} \cdot \tilde{h}_{\{-1\}} & \text{for } i = -1\,. \end{cases}$$

So we have

$$(h \cdot \tilde{h})\pi_{\mathrm{M}_S} = (h \cdot \tilde{h})_{\{-1\}} = h_{\{-1\}} \cdot \tilde{h}_{\{-1\}} = h\pi_{\mathrm{M}_S} \cdot \tilde{h}\pi_{\mathrm{M}_S} .$$

Since π_{M_S} is bijective, this shows that π_{M_S} is a group isomorphism.

We show that $S\psi = (\pi_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})$ is a stable crossed module morphism.

For $h \in \mathrm{M}_V$, we have

$$h(\pi_{\mathrm{M}_S} \triangleleft \mathrm{f}_S) = (h_{\{-1\}})\mathrm{f}_S \overset{401.(1)}{=} h\mathrm{f}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = h(\mathrm{f}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} \triangleleft \mathrm{id}_{\mathrm{B}_S}) .$$

This shows $\pi_{\mathrm{M}_S} \triangleleft \mathrm{f}_S = \mathrm{f}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} \triangleleft \mathrm{id}_{\mathrm{B}_S}$.

$$\begin{array}{ccc} \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} & \xrightarrow{\pi_{\mathrm{M}_S}} & \mathrm{M}_S \\ \Big\downarrow{\scriptstyle \mathrm{f}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}} & & \Big\downarrow{\scriptstyle \mathrm{f}_S} \\ \mathrm{B}_S & \xrightarrow[\mathrm{id}_{\mathrm{B}_S}]{} & \mathrm{B}_S \end{array}$$

Moreover, for b, $c \in \mathrm{B}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = \mathrm{B}_S$, we have

$$\big((b,c)\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\big)\pi_{\mathrm{M}_S} = \big((b,c)\kappa_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}\big)_{\{-1\}} \overset{401.(1)}{=} (b,c)\kappa_S = (b\,\mathrm{id}_{\mathrm{B}_S}, c\,\mathrm{id}_{\mathrm{B}_S})\kappa_S .$$

So $(\pi_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})$ is a stable crossed module morphism; cf. Definition 324.

Since π_{M_S} and $\mathrm{id}_{\mathrm{B}_S}$ are group isomorphisms, we have that $(\pi_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})$ is a stable crossed module isomorphism; cf. Remark 326.(2).

Ad (2). Suppose given $S \xrightarrow{\rho} \tilde{S}$ in $\mathcal{StCrMod}$.

We have $\rho = (\mu, \beta)$ where $\mu\colon \mathrm{M}_S \to \mathrm{M}_{\tilde{S}}$ and $\beta\colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}$.

We write $(\hat{\mu}, \hat{\beta}) := \Big((\rho\,\mathrm{Rec}^{\pm})_1\big|^{(\tilde{S}\,\mathrm{Rec}^{\pm})\mathrm{N}_1}_{(S\,\mathrm{Rec}^{\pm})\mathrm{N}_1}, \beta\Big) \overset{401.(2)}{=} \rho\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}\colon S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} \to \tilde{S}\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}$.

For $h \in \mathrm{M}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}}$, we have

$$\begin{aligned} h(\pi_{\mathrm{M}_S} \triangleleft \mu) = (h_{\{-1\}})\mu &\overset{401.(2)}{=} \big(h(\rho\,\mathrm{Rec}^{\pm})_1\big)_{\{-1\}} = \big(h(\rho\,\mathrm{Rec}^{\pm})_1\big)\pi_{\mathrm{M}_{\tilde{S}}} \\ &= h\big((\rho\,\mathrm{Rec}^{\pm})_1 \triangleleft \pi_{\mathrm{M}_{\tilde{S}}}\big) = h(\hat{\mu} \triangleleft \pi_{\mathrm{M}_{\tilde{S}}}) . \end{aligned}$$

This shows $\pi_{\mathrm{M}_S} \triangleleft \mu = \hat{\mu} \triangleleft \pi_{\mathrm{M}_{\tilde{S}}}$.

For $b \in \mathrm{B}_{S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm}} = \mathrm{B}_S$, we have

$$b(\mathrm{id}_{\mathrm{B}_S} \triangleleft \beta) = b\beta = (b\beta)\,\mathrm{id}_{\mathrm{B}_{\tilde{S}}} = b(\hat{\beta} \triangleleft \mathrm{id}_{\mathrm{B}_{\tilde{S}}}) .$$

This shows $\mathrm{id}_{\mathrm{B}_S} \blacktriangle \beta = \hat{\beta} \blacktriangle \mathrm{id}_{\mathrm{B}_{\tilde{S}}}$.

So we have

$$S\hat{\psi} \blacktriangle \rho = (\pi_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S}) \blacktriangle \rho = (\pi_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S}) \blacktriangle (\mu, \beta) = (\pi_{\mathrm{M}_S} \blacktriangle \mu, \mathrm{id}_{\mathrm{B}_S} \blacktriangle \beta) = (\hat{\mu} \blacktriangle \pi_{\mathrm{M}_{\tilde{S}}}, \hat{\beta} \blacktriangle \mathrm{id}_{\mathrm{B}_{\tilde{S}}})$$
$$\overset{326.(1)}{=} (\hat{\mu}, \hat{\beta}) \blacktriangle (\pi_{\mathrm{M}_{\tilde{S}}}, \mathrm{id}_{\mathrm{B}_{\tilde{S}}}) = \rho\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} \blacktriangle (\pi_{\mathrm{M}_{\tilde{S}}}, \mathrm{id}_{\mathrm{B}_{\tilde{S}}}) = \rho\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} \blacktriangle \tilde{S}\hat{\psi}.$$

$$\begin{array}{ccc} S\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} & \xrightarrow{S\hat{\psi}} & S \\ \rho\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} \downarrow & & \downarrow \rho \\ \tilde{S}\,\mathrm{Rec}^{\pm}\,\hat{\mathrm{N}}^{\pm} & \xrightarrow[\tilde{S}\hat{\psi}]{} & \tilde{S} \end{array}$$

So $\hat{\psi}$ is a transformation. Using (1), we conclude that $\hat{\psi}$ is an isotransformation. □

Remark 403 (The $\lfloor 1,0 \rfloor$-stable simplicial group $G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}$)

(1) Suppose given a $\lfloor 1,0 \rfloor$-stable simplicial group G. We consider the $\lfloor 1,0 \rfloor$-stable simplicial group $G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}$.

Recall that we have the stable crossed module

$$G\hat{\mathrm{N}}^{\pm} = (G\mathrm{N}_1, G_0, \kappa_G, \mathrm{d}_0^G\big|_{G\mathrm{N}_1})$$

with

$$\kappa_G \colon G_0 \times G_0 \to G\mathrm{N}_1, \ (b,c) \mapsto [\![b, c]\!] = (b^{-}\mathrm{s}_0^G)^{c\mathrm{s}_1^G} \cdot (b\mathrm{s}_0^G)^{c\mathrm{s}_0^G};$$

cf. Remark 395.

Then, by Remark 398, we have the $\lfloor 1,0 \rfloor$-stable simplicial group $G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}$, where

$$\begin{aligned} (G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1 &= H_\infty^{G\hat{\mathrm{N}}^{\pm}} \\ &\overset{356}{=} \{h \in (\prod_{i \geqslant 0} G_0) \times G\mathrm{N}_1 \colon \exists k \in \mathbb{Z}_{\geqslant 0} \text{ with } h_{\{\ell\}} = 1 \text{ for } \ell \geqslant k\} \\ &= \{(\ldots, b_2, b_1, b_0, m) \in (\prod_{i \geqslant 0} G_0) \times G\mathrm{N}_1 \colon \exists k \in \mathbb{Z}_{\geqslant 0} \text{ with } b_\ell = 1 \text{ for } \ell \geqslant k\} \\ (G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_0 &\overset{398}{=} G_0, \end{aligned}$$

Moreover, for $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{s}_i^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} \colon G_0 \to H_\infty^{G\hat{\mathrm{N}}^{\pm}}, \ g \mapsto (\ldots, 1, \underbrace{g}_{\text{pos. } i}, 1, \ldots, 1)$$

and for $j \in \mathbb{Z}_{\geqslant 1}$, we have

$$\mathrm{d}_j^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} \colon H_{\infty}^{G\hat{\mathrm{N}}^{\pm}} \to G_0\,,\ (\dots, b_2\,, b_1\,, b_0\,, m) \mapsto b_j \cdot b_{j-1}\,,$$

and we have

$$\mathrm{d}_0^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} \colon H_{\infty}^{G\hat{\mathrm{N}}^{\pm}} \to G_0\,,\ (\dots, b_2\,, b_1\,, b_0\,, m) \mapsto b_0 \cdot m\mathrm{d}_0^G\,.$$

(2) Suppose given a $\lfloor 1,0\rfloor$-stable simplicial group morphism $\varphi\colon G \to \tilde{G}$. We consider the $\lfloor 1,0\rfloor$-stable simplicial groups $G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}$ and $\tilde{G}\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}$; cf. (1).

Recall that we have the stable crossed module morphism

$$\varphi\hat{\mathrm{N}}^{\pm} = (\varphi_1\big|_{G\mathrm{N}_1}^{H\mathrm{N}_1}\,, \varphi_0)\colon G\hat{\mathrm{N}}^{\pm} \to \tilde{G}\hat{\mathrm{N}}^{\pm}\,;$$

cf. Remark 396.

Then, by Remark 399, we have the $\lfloor 1,0\rfloor$-stable simplicial group morphism $\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}$, where

$$\begin{aligned}
&(\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1\colon\ H_{\infty}^{G\hat{\mathrm{N}}^{\pm}} \to H_{\infty}^{\tilde{G}\hat{\mathrm{N}}^{\pm}}\,,\ \ (\dots, b_1\,, b_0\,, m) \mapsto (\dots\,, b_1\varphi_0\,, b_0\varphi_0\,, m\varphi_1)\\
&(\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_0\colon\ G_0 \to \tilde{G}_0\,, \qquad\qquad g \mapsto g\varphi_0\,.
\end{aligned}$$

Lemma 404 (The functor $\hat{\mathrm{N}}^{\pm} * \mathrm{Rec}^{\pm}$)

(1) *Suppose given a $\lfloor 1,0\rfloor$-stable simplicial group G.*

We have the $\lfloor 1,0\rfloor$-stable simplicial group isomorphism $G\psi\colon G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm} \to G$, where

$$\begin{aligned}
&(G\psi)_1\colon\qquad\qquad H_{\infty}^{G\hat{\mathrm{N}}^{\pm}} \to G_1\,,\ \ h \mapsto h(G\psi)_1 := \big(\prod_{i\in\rfloor\infty,0\rfloor}(h_{\{i\}})\mathrm{s}_i^G\big)\cdot h_{\{-1\}}\\
&(G\psi)_0 := \mathrm{id}_{G_0}\colon\ G_0 \to G_0\,, \qquad g \mapsto g\,.
\end{aligned}$$

So, for $h =: (\dots, b_3\,, b_2\,, b_1\,, b_0\,, m) \in H_{\infty}^{G\hat{\mathrm{N}}^{\pm}}$, we have

$$h(G\psi)_1 \overset{362.(1)}{=} \big(\dots\cdot b_3\mathrm{s}_3^{\infty}\cdot b_2\mathrm{s}_2^{\infty}\cdot b_1\mathrm{s}_1^{\infty}\cdot b_0\mathrm{s}_0^{\infty}\cdot m\mathrm{s}_{-1}^{\infty}\big)(G\psi)_1 = \dots\cdot b_3\mathrm{s}_3^G\cdot b_2\mathrm{s}_2^G\cdot b_1\mathrm{s}_1^G\cdot b_0\mathrm{s}_0^G\cdot m\,;$$

cf. §1.1.2 *item* 9.4.

(2) *We have the isotransformation*

$$\psi = \big(G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm} \xrightarrow{G\psi} G\big)_{G\in\mathrm{Ob}(\lfloor 1,0\rfloor\text{-}\mathcal{S}t\mathcal{S}impG\mathcal{r}p)}\colon \hat{\mathrm{N}}^{\pm} * \mathrm{Rec}^{\pm} \to \mathrm{id}_{\lfloor 1,0\rfloor\text{-}\mathcal{S}t\mathcal{S}impG\mathcal{r}p}\,;$$

cf. (1).

Proof. Ad (1). We write $H_\infty := H_\infty^{G\hat{\mathrm{N}}^\pm}$. For $i \in \mathbb{Z}_{\geqslant 0}$, we write $\mathrm{s}_i := \mathrm{s}_i^G$ and $\mathrm{d}_i := \mathrm{d}_i^G$.

So, for $h \in H_\infty$, we have $h(G\psi)_1 = \Big(\big(\prod_{i \in \rfloor \infty, 0 \rfloor} (h_{\{i\}})\mathrm{s}_i \big) \cdot h_{\{-1\}} \Big) \in G_1$.

We show that $(G\psi)_1$ is a group morphism.

To be able to use induction, we formulate the following *claim*.

Suppose given $n \in \mathbb{Z}_{\geqslant -1}$. Suppose given h, $\tilde{h} \in H_\infty$ with $\tilde{h}_{\{i\}} = 1$ for $i \in \mathbb{Z}_{\geqslant n+1}$.

We have

$$(h \cdot \tilde{h})(G\psi)_1 \stackrel{!}{=} h(G\psi)_1 \cdot \tilde{h}(G\psi)_1 .$$

We show this *claim* by induction on $n \in \mathbb{Z}_{\geqslant -1}$.

We write $b_i := h_{\{i\}} \in G_0$ for $i \in \mathbb{Z}_{\geqslant 0}$ and $m := h_{\{-1\}} \in G_1$.

We write $\tilde{b}_i := \tilde{h}_{\{i\}} \in G_0$ for $i \in \mathbb{Z}_{\geqslant 0}$ and $\tilde{m} := \tilde{h}_{\{-1\}} \in G_1$.

Case $n = -1$. Then we have $\tilde{b}_i = 1$ for $i \in \mathbb{Z}_{\geqslant 0}$.

We have

$$\begin{aligned}
(h \cdot \tilde{h})(G\psi)_1 &\stackrel{362.(1)}{=} (h \cdot \tilde{m}\mathrm{s}_{-1}^\infty)(G\psi)_1 \\
&= \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} \big((h \cdot \tilde{m}\mathrm{s}_{-1}^\infty)_{\{i\}} \big) \mathrm{s}_i \Big) \cdot (h \cdot \tilde{m}\mathrm{s}_{-1}^\infty)_{\{-1\}} \\
&\stackrel{362.(2)}{=} \big(\prod_{i \in \rfloor \infty, 0 \rfloor} b_i \mathrm{s}_i \big) \cdot m \cdot \tilde{m} \\
&= h(G\psi)_1 \cdot \tilde{h}(G\psi)_1 .
\end{aligned}$$

Case $n = 0$. Then we have $\tilde{b}_i = 1$ for $i \in \mathbb{Z}_{\geqslant 1}$.

For $b \in \mathrm{B}_{G\hat{\mathrm{N}}^\pm} = G_0$ and $x \in \mathrm{M}_{G\hat{\mathrm{N}}^\pm} = G\mathrm{N}_1$, recall that we have

$$x^b = x(b\gamma_{G\hat{\mathrm{N}}^\pm}) \stackrel{356}{=} x \cdot \llbracket x\mathrm{d}_0, b \rrbracket \stackrel{390.(3)}{=} x^{b\mathrm{s}_0} .$$

We have

$$\begin{aligned}
(h \cdot \tilde{h})(G\psi)_1 &\stackrel{362.(1)}{=} (h \cdot \tilde{b}_0\mathrm{s}_0^\infty \cdot \tilde{m}\mathrm{s}_{-1}^\infty)(G\psi)_1 \\
&= \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} \big((h \cdot \tilde{b}_0\mathrm{s}_0^\infty \cdot \tilde{m}\mathrm{s}_{-1}^\infty)_{\{i\}} \big) \mathrm{s}_i \Big) \cdot (h \cdot \tilde{b}_0\mathrm{s}_0^\infty \cdot \tilde{m}\mathrm{s}_{-1}^\infty)_{\{-1\}} \\
&\stackrel{362.(2)}{=} \Big(\prod_{i \in \rfloor \infty, 0 \rfloor} \big((h \cdot \tilde{b}_0\mathrm{s}_0^\infty)_{\{i\}} \big) \mathrm{s}_i \Big) \cdot (h \cdot \tilde{b}_0\mathrm{s}_0)_{\{-1\}} \cdot \tilde{m}
\end{aligned}$$

$$\begin{aligned}
&\overset{362.(2)}{=} \quad \big(\prod_{i\in\rfloor\infty,1\rfloor} b_i\mathrm{s}_i\big)\cdot(b_0\cdot\tilde{b}_0)\mathrm{s}_0\cdot m^{\tilde{b}_0}\cdot\tilde{m}\\
&= \quad \big(\prod_{i\in\rfloor\infty,0\rfloor} b_i\mathrm{s}_i\big)\cdot\tilde{b}_0\mathrm{s}_0\cdot m^{\tilde{b}_0\mathrm{s}_0}\cdot\tilde{m}\\
&= \quad \big(\prod_{i\in\rfloor\infty,0\rfloor} b_i\mathrm{s}_i\big)\cdot m\cdot\tilde{b}_0\mathrm{s}_0\cdot\tilde{m}\\
&= \quad h(G\psi)_1\cdot\tilde{h}(G\psi)_1\,.
\end{aligned}$$

Case $n\geqslant 1$. Let $\hat{h}\in H_\infty$ be defined by

$$\hat{h}_{\{i\}} := \begin{cases} 1 & \text{for } i\in\mathbb{Z}_{\geqslant n}\\ (b_{n-1})^{\tilde{b}_n} & \text{for } i=n-1\\ [\tilde{b}_n\,,b_{n-1}]^{(-1)^{n-i}} & \text{for } i\in[0,n-2]\\ \llbracket b_{n-1},\,\tilde{b}_n\rrbracket^{(-1)^n} & \text{for } i=-1\,. \end{cases}$$

We have

$$(\tilde{b}_n\mathrm{s}_n^\infty\cdot\hat{h})_{\{i\}} \overset{362.(1)}{=} \Big(\tilde{b}_n\mathrm{s}_n^\infty\cdot\big(\prod_{j\in\lfloor n-1,-1\rfloor}\hat{h}_{\{j\}}\mathrm{s}_j^\infty\big)\Big)_{\{i\}} \overset{362.(1)}{=} \begin{cases} 1 & \text{for } i\in\mathbb{Z}_{\geqslant n+1}\\ \tilde{b}_n & \text{for } i=n\\ (b_{n-1})^{\tilde{b}_n} & \text{for } i=n-1\\ [\tilde{b}_n\,,b_{n-1}]^{(-1)^{n-i}} & \text{for } i\in[0,n-2]\\ \llbracket b_{n-1},\,\tilde{b}_n\rrbracket^{(-1)^n} & \text{for } i=-1\,. \end{cases}$$

Moreover, we have

$$\begin{aligned}
(b_{n-1}\mathrm{s}_{n-1}^\infty\cdot\tilde{b}_n\mathrm{s}_n^\infty)_{\{i\}} &\overset{362.(2)}{=} \begin{cases} 1 & \text{for } i\in\mathbb{Z}_{\geqslant n+1}\\ 1\cdot\tilde{b}_n & \text{for } i=n\\ (b_{n-1})^{\tilde{b}_n} & \text{for } i=n-1\\ \big([b_{n-1}\,,\tilde{b}_n]^{(-1)^{n-1-i}}\big)^1\cdot 1 & \text{for } i\in[0,n-2]\\ \big(\llbracket b_{n-1},\,\tilde{b}_n\rrbracket^{(-1)^n}\big)^1\cdot 1 & \text{for } i=-1 \end{cases}\\
&= \begin{cases} 1 & \text{for } i\in\mathbb{Z}_{\geqslant n+1}\\ \tilde{b}_n & \text{for } i=n\\ (b_{n-1})^{\tilde{b}_n} & \text{for } i=n-1\\ [\tilde{b}_n\,,b_{n-1}]^{(-1)^{n-i}} & \text{for } i\in[0,n-2]\\ \llbracket b_{n-1},\,\tilde{b}_n\rrbracket^{(-1)^n} & \text{for } i=-1\,. \end{cases}
\end{aligned}$$

This shows $b_{n-1}\mathrm{s}^\infty_{n-1}\cdot\tilde{b}_n\mathrm{s}^\infty_n=\tilde{b}_n\mathrm{s}^\infty_n\cdot\hat{h}$.

So, we have

$$\begin{aligned}
&(h\cdot\tilde{h})(G\psi)_1\\
\overset{362.(1)}{=}\ &\Big(\big(\prod_{i\in\rfloor\infty,-1\rfloor}(h_{\{i\}})\mathrm{s}^\infty_i\big)\cdot\big(\prod_{i\in\rfloor\infty,-1\rfloor}(\tilde{h}_{\{i\}})\mathrm{s}^\infty_i\big)\Big)(G\psi)_1\\
=\ &\Big(\big(\prod_{i\in\rfloor\infty,n\rfloor}b_i\mathrm{s}^\infty_i\big)\cdot b_{n-1}\mathrm{s}^\infty_{n-1}\cdot\big(\prod_{i\in\lfloor n-2,-1\rfloor}(h_{\{i\}})\mathrm{s}^\infty_i\big)\cdot\tilde{b}_n\mathrm{s}^\infty_n\cdot\big(\prod_{i\in\lfloor n-1,-1\rfloor}(\tilde{h}_{\{i\}})\mathrm{s}^\infty_i\big)\Big)(G\psi)_1\\
\overset{371,374}{=}\ &\Big(\big(\prod_{i\in\rfloor\infty,n\rfloor}b_i\mathrm{s}^\infty_i\big)\cdot b_{n-1}\mathrm{s}^\infty_{n-1}\cdot\tilde{b}_n\mathrm{s}^\infty_n\cdot\big(\prod_{i\in\lfloor n-2,-1\rfloor}(h_{\{i\}})\mathrm{s}^\infty_i\big)\cdot\big(\prod_{i\in\lfloor n-1,-1\rfloor}(\tilde{h}_{\{i\}})\mathrm{s}^\infty_i\big)\Big)(G\psi)_1\\
=\ &\Big(\big(\prod_{i\in\rfloor\infty,n\rfloor}b_i\mathrm{s}^\infty_i\big)\cdot\tilde{b}_n\mathrm{s}^\infty_n\cdot\hat{h}\cdot\Big(\big(\prod_{i\in\lfloor n-2,-1\rfloor}(h_{\{i\}})\mathrm{s}^\infty_i\big)\cdot\big(\prod_{i\in\lfloor n-1,-1\rfloor}(\tilde{h}_{\{i\}})\mathrm{s}^\infty_i\big)\Big)(G\psi)_1\\
\overset{357.(4),\,\mathrm{IH}}{=}\ &\Big(\big(\prod_{i\in\rfloor\infty,n\rfloor}b_i\mathrm{s}^\infty_i\big)\cdot\tilde{b}_n\mathrm{s}^\infty_n\cdot\hat{h}\Big)(G\psi)_1\cdot\Big(\big(\prod_{i\in\lfloor n-2,-1\rfloor}(h_{\{i\}})\mathrm{s}^\infty_i\big)\cdot\big(\prod_{i\in\lfloor n-1,-1\rfloor}(\tilde{h}_{\{i\}})\mathrm{s}^\infty_i\big)\Big)(G\psi)_1\\
\overset{\mathrm{IH}}{=}\ &\Big(\big(\prod_{i\in\rfloor\infty,n+1\rfloor}b_i\mathrm{s}^\infty_i\big)\cdot(b_n\cdot\tilde{b}_n)\mathrm{s}^\infty_n\Big)(G\psi)_1\cdot\hat{h}(G\psi)_1\\
&\cdot\big(\prod_{i\in\lfloor n-2,-1\rfloor}(h_{\{i\}})\mathrm{s}^\infty_i\big)(G\psi)_1\cdot\big(\prod_{i\in\lfloor n-1,-1\rfloor}(\tilde{h}_{\{i\}})\mathrm{s}^\infty_i\big)(G\psi)_1\\
=\ &\big(\prod_{i\in\rfloor\infty,n+1\rfloor}b_i\mathrm{s}_i\big)\cdot(b_n\cdot\tilde{b}_n)\mathrm{s}_n\\
&\cdot\big((b_{n-1})^{\tilde{b}_n}\big)\mathrm{s}_{n-1}\cdot\Big(\prod_{i\in\lfloor n-2,0\rfloor}\big([\tilde{b}_n\,,b_{n-1}]^{(-1)^{n-i}}\big)\mathrm{s}_i\Big)\cdot[\![b_{n-1},\,\tilde{b}_n]\!]^{(-1)^n}\\
&\cdot\big(\prod_{i\in\lfloor n-2,0\rfloor}b_i\mathrm{s}_i\big)\cdot m\cdot\big(\prod_{i\in\lfloor n-1,0\rfloor}\tilde{b}_i\mathrm{s}_i\big)\cdot\tilde{m}\,.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&h(G\psi)_1\cdot\tilde{h}(G\psi)_1\\
=\ &\big(\prod_{i\in\rfloor\infty,0\rfloor}b_i\mathrm{s}_i\big)\cdot m\cdot\big(\prod_{i\in\rfloor\infty,0\rfloor}\tilde{b}_i\mathrm{s}_i\big)\cdot\tilde{m}\\
=\ &\big(\prod_{i\in\rfloor\infty,n\rfloor}b_i\mathrm{s}_i\big)\cdot\big(\prod_{i\in\lfloor n-1,0\rfloor}b_i\mathrm{s}_i\big)\cdot m\cdot\tilde{b}_n\mathrm{s}_n\cdot\big(\prod_{i\in\lfloor n-1,0\rfloor}\tilde{b}_i\mathrm{s}_i\big)\cdot\tilde{m}\\
\overset{390.(2)}{=}\ &\big(\prod_{i\in\rfloor\infty,n\rfloor}b_i\mathrm{s}_i\big)\cdot\big(\prod_{i\in\lfloor n-1,0\rfloor}b_i\mathrm{s}_i\big)\cdot\tilde{b}_n\mathrm{s}_n\cdot m\cdot\big(\prod_{i\in\lfloor n-1,0\rfloor}\tilde{b}_i\mathrm{s}_i\big)\cdot\tilde{m}\\
\overset{392.(2)}{=}\ &\big(\prod_{i\in\rfloor\infty,n\rfloor}b_i\mathrm{s}_i\big)\cdot\tilde{b}_n\mathrm{s}_n\cdot(b_{n-1}\mathrm{s}_{n-1})^{\tilde{b}_n\mathrm{s}_n}\cdot\big(\prod_{i\in\lfloor n-2,0\rfloor}b_i\mathrm{s}_i\big)\cdot m\cdot\big(\prod_{i\in\lfloor n-1,0\rfloor}\tilde{b}_i\mathrm{s}_i\big)\cdot\tilde{m}\,.
\end{aligned}$$

It remains to show that

$$(b_{n-1}\mathrm{s}_{n-1})^{\tilde{b}_n\mathrm{s}_n} \overset{!}{=} \big((b_{n-1})^{\tilde{b}_n}\big)\mathrm{s}_{n-1}\cdot\Big(\prod_{i\in\lfloor n-2,0\rfloor}\big([\tilde{b}_n\,,b_{n-1}]^{(-1)^{n-i}}\big)\mathrm{s}_i\Big)\cdot\llbracket b_{n-1},\,\tilde{b}_n\rrbracket^{(-1)^n}\,.$$

We have

$$\begin{aligned}(b_{n-1}\mathrm{s}_{n-1})^{\tilde{b}_n\mathrm{s}_n} &\overset{394.(3)}{=} \big((b_{n-1})^{\tilde{b}_n}\big)\mathrm{s}_{n-1}\cdot\Big(\prod_{i\in\lfloor n-2,0\rfloor}\big([\tilde{b}_n\,,b_{n-1}]^{(-1)^{n-i}}\big)\mathrm{s}_i\Big)\cdot\llbracket\tilde{b}_n,\,b_{n-1}\rrbracket^{(-1)^{n+1}}\\ &\overset{(\mathrm{SCM1})}{=} \big((b_{n-1})^{\tilde{b}_n}\big)\mathrm{s}_{n-1}\cdot\Big(\prod_{i\in\lfloor n-2,0\rfloor}\big([\tilde{b}_n\,,b_{n-1}]^{(-1)^{n-i}}\big)\mathrm{s}_i\Big)\cdot\llbracket b_{n-1},\,\tilde{b}_n\rrbracket^{(-1)^n}\,.\end{aligned}$$

This shows the *claim*.

Suppose given $h,\,\tilde{h}\in H_\infty$. Then there exists some $n\in\mathbb{Z}_{\geqslant 0}$ such that $\tilde{h}_{\{i\}}=1$ for $i\in\mathbb{Z}_{\geqslant n}$. So the claim yields

$$(h\cdot\tilde{h})(G\psi)_1 = h(G\psi)_1\cdot\tilde{h}(G\psi)_1\,.$$

This shows that $(G\psi)_1$ is a group morphism.

We show that $(G\psi)_1$ is injective.

Suppose given $h\in H_\infty$ with $h(G\psi)_1=1\in G_1$. We have to show that $h\overset{!}{=}1$, i.e. we have to show that $h_{\{i\}}\overset{!}{=}1$ for $i\in\mathbb{Z}_{\geqslant -1}$.

Since $h\in H_\infty$, we may choose some $k\in\mathbb{Z}_{\geqslant 0}$ such that $h_{\{\ell\}}=1$ for $\ell\in\mathbb{Z}_{\geqslant k}$; cf. Definition 356. Recall that $h_{\{-1\}}\in \mathrm{M}_{G\hat{\mathrm{N}}}=G\mathrm{N}_1$; cf. Remark 403.(1).

For $j\in[1,k]$, we have

$$\begin{aligned}1 &= h(G\psi)_1\,\mathrm{d}_j = \Big(\big(\prod_{i\in\rfloor\infty,0\rfloor}(h_{\{i\}})\mathrm{s}_i\big)\cdot h_{\{-1\}}\Big)\mathrm{d}_j = \Big(\big(\prod_{i\in\lfloor k,0\rfloor}(h_{\{i\}})\mathrm{s}_i\big)\cdot h_{\{-1\}}\Big)\mathrm{d}_j\\ &= \big(\prod_{i\in\lfloor k,0\rfloor}(h_{\{i\}})\mathrm{s}_i\mathrm{d}_j\big)\cdot(h_{\{-1\}})\mathrm{d}_j \overset{383.(3)}{=} h_{\{j\}}\cdot h_{\{j-1\}}\,.\end{aligned}$$

Since we have $h_{\{k\}}=1$, this shows that $h_{\{\ell\}}=1$ for $\ell\in[0,k-1]$.

Now, we have

$$1 = h(G\psi)_1 = \big(\prod_{i\in\rfloor\infty,0\rfloor}(h_{\{i\}})\mathrm{s}_i\big)\cdot h_{\{-1\}} = 1\cdot h_{\{-1\}} = h_{\{-1\}}\,.$$

Altogether, we have $h_{\{i\}}=1$ for $i\in\mathbb{Z}_{\geqslant -1}$. This shows $h=1$.

We show that $(G\psi)_1$ is surjective.

Suppose given $g \in G_1$. We have to show that $g \overset{!}{\in} (H_\infty)(G\psi)_1$.

By Lemma 387, we may assume that $g \in G\mathrm{N}_1$ or $g\mathrm{d}_k\mathrm{s}_{k-1} = g$ for some $k \in \mathbb{Z}_{\geqslant 1}$.

Suppose that $g \in G\mathrm{N}_1 = \mathrm{M}_{G\hat{\mathrm{N}}^\pm}$.

Let $h \in H_\infty$ be defined by

$$h_{\{i\}} := \begin{cases} 1 & \text{for } i \in \mathbb{Z}_{\geqslant 0} \\ g & \text{for } i = -1\,. \end{cases}$$

Then we have

$$h(G\psi)_1 = \big(\prod_{i \in \rfloor \infty, 0 \rfloor} (h_{\{i\}})\mathrm{s}_i \big) \cdot h_{\{-1\}} = 1 \cdot g = g\,.$$

Now suppose that we have $g\mathrm{d}_k\mathrm{s}_{k-1} = g$ for some chosen $k \in \mathbb{Z}_{\geqslant 1}$.

Let $h \in H_\infty$ be defined by

$$h_{\{i\}} := \begin{cases} g\mathrm{d}_k & \text{for } i = k-1 \\ 1 & \text{for } i \in \mathbb{Z}_{\geqslant -1} \backslash \{k-1\}\,. \end{cases}$$

So $h = (\ldots, 1, \underbrace{g\mathrm{d}_k}_{\text{pos. } k-1}, 1, \ldots, 1) \in H_\infty$.

Then we have

$$h(G\psi)_1 = \big(\prod_{i \in \rfloor \infty, 0 \rfloor} (h_{\{i\}})\mathrm{s}_i \big) \cdot h_{\{-1\}} = (g\mathrm{d}_k)\mathrm{s}_{k-1} = g\,.$$

So $(G\psi)_1$ is surjective.

We show that $G\psi$ is a $\lfloor 1,0 \rfloor$-stable simplicial group morphism.

Recall that we write $\mathrm{s}_i = \mathrm{s}_i^G$ and $\mathrm{d}_i = \mathrm{d}_i^G$ for $i \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $h \in (G\hat{\mathrm{N}}^\pm \mathrm{Rec}^\pm)_1 = H_\infty$. Note that $h_{\{-1\}} \in G\mathrm{N}_1$; cf. Remark 403.(1).

For $j \in \mathbb{Z}_{\geqslant 1}$, we have

$$\begin{aligned} h\big((G\psi)_1 \mathbin{\triangleleft} \mathrm{d}_j\big) &= \Big(\big(\prod_{i \in \rfloor \infty, 0 \rfloor} (h_{\{i\}})\mathrm{s}_i \big) \cdot h_{\{-1\}} \Big)\mathrm{d}_j = \big(\prod_{i \in \rfloor \infty, 0 \rfloor} (h_{\{i\}})\mathrm{s}_i\mathrm{d}_j \big) \cdot (h_{\{-1\}})\mathrm{d}_j \\ &\overset{383.(3)}{=} h_{\{j\}} \cdot h_{\{j-1\}} \overset{403.(1)}{=} h\mathrm{d}_j^{G\hat{\mathrm{N}}^\pm \mathrm{Rec}^\pm} = h(\mathrm{d}_j^{G\hat{\mathrm{N}}^\pm \mathrm{Rec}^\pm} \mathbin{\triangleleft} \mathrm{id}_{G_0})\,. \end{aligned}$$

Moreover, we have

$$\begin{aligned} h\big((G\psi)_1 \mathbin{\blacktriangle} \mathrm{d}_0\big) \quad &= \quad \Big(\big(\prod_{i\in\rfloor\infty,0\rfloor}(h_{\{i\}})\mathrm{s}_i\big)\cdot h_{\{-1\}}\Big)\mathrm{d}_0 = \big(\prod_{i\in\rfloor\infty,0\rfloor}(h_{\{i\}})\mathrm{s}_i\mathrm{d}_0\big)\cdot(h_{\{-1\}})\mathrm{d}_0 \\ &\overset{383.(3)}{=} \quad h_{\{0\}}\cdot(h_{\{-1\}})\mathrm{d}_0 \overset{403.(1)}{=} h\mathrm{d}_0^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} = h(\mathrm{d}_0^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} \mathbin{\blacktriangle} \mathrm{id}_{G_0})\,. \end{aligned}$$

This shows $(G\psi)_1 \mathbin{\blacktriangle} \mathrm{d}_j = \mathrm{d}_j^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} \mathbin{\blacktriangle} \mathrm{id}_{G_0}$ for $j \in \mathbb{Z}_{\geqslant 0}$.

$$\begin{array}{ccc} & \overset{\mathrm{d}_j^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}}}{} & \\ H_\infty & \longrightarrow & G_0 \\ {\scriptstyle (G\psi)_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G_0}} \\ G_1 & \underset{\mathrm{d}_j}{\longrightarrow} & G_0 \end{array}$$

Suppose given $g \in G_0$. For $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned} g\big(\mathrm{s}_i^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} \mathbin{\blacktriangle} (G\psi)_1\big) \quad &= \quad (g\mathrm{s}_i^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}})(G\psi)_1 \\ &= \quad \Big(\prod_{k\in\rfloor\infty,0\rfloor}\big((g\mathrm{s}_i^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}})_{\{k\}}\big)\mathrm{s}_k\Big)\cdot(g\mathrm{s}_i^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}})_{\{-1\}} \\ &\overset{403.(1)}{=} \quad g\mathrm{s}_i = g(\mathrm{id}_{G_0} \mathbin{\blacktriangle} \mathrm{s}_i)\,. \end{aligned}$$

This shows $\mathrm{s}_i^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}} \mathbin{\blacktriangle} (G\psi)_1 = \mathrm{id}_{G_0} \mathbin{\blacktriangle} \mathrm{s}_i$.

$$\begin{array}{ccc} & \overset{\mathrm{s}_i^{G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}}}{} & \\ H_\infty & \longleftarrow & G_0 \\ {\scriptstyle (G\psi)_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G_0}} \\ G_1 & \underset{\mathrm{s}_i}{\longleftarrow} & G_0 \end{array}$$

So, by Remark 386, $G\psi$ is a $\lfloor 1,0\rfloor$-stable simplicial group morphism.

Since $(G\psi)_0 = \mathrm{id}_{G_0}$ and $(G\psi)_1$ are group isomorphisms we have that $G\psi$ is a $\lfloor 1,0\rfloor$-simplicial group isomorphism.

Ad (2). Suppose given $G \xrightarrow{\varphi} \tilde{G}$ in $\lfloor 1,0\rfloor$-*StSimpGrp*.

We have

$$\mathrm{id}_{G_0} \mathbin{\blacktriangle} \varphi_0 = \varphi_0 \overset{403.(2)}{=} (\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_0 = (\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_0 \mathbin{\blacktriangle} \mathrm{id}_{\tilde{G}_0}\,.$$

$$\begin{array}{ccc} G_0 & \xrightarrow{\mathrm{id}_{G_0}} & G_0 \\ {\scriptstyle (\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_0}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ \tilde{G}_0 & \xrightarrow[\mathrm{id}_{\tilde{G}_0}]{} & \tilde{G}_0 \end{array}$$

Moreover, for $h \in (G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1 = H_{\infty}^{G\hat{\mathrm{N}}^{\pm}}$, we have

$$\begin{aligned} h\big((\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1 \mathbin{\blacktriangle} (\tilde{G}\psi)_1\big) &\overset{(1)}{=} \Big(\prod_{i\in\rfloor\infty,0\rfloor} \big(h(\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1\big)_{\{i\}}\Big)\mathrm{s}_i^{\tilde{G}}\Big) \cdot \big(h(\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1\big)_{\{-1\}} \\ &\overset{403.(2)}{=} \Big(\prod_{i\in\rfloor\infty,0\rfloor} \big((h_{\{i\}})\varphi_0\big)\mathrm{s}_i^{\tilde{G}}\Big) \cdot (h_{\{-1\}})\varphi_1 \\ &\overset{384}{=} \Big(\prod_{i\in\rfloor\infty,0\rfloor} (h_{\{i\}})\mathrm{s}_i^{G}\varphi_1\Big) \cdot (h_{\{-1\}})\varphi_1 \\ &\overset{403.(1)}{=} \Big(\big(\prod_{i\in\rfloor\infty,0\rfloor} (h_{\{i\}})\mathrm{s}_i^{G}\big) \cdot h_{\{-1\}}\Big)\varphi_1 \\ &= h((G\psi)_1 \mathbin{\blacktriangle} \varphi_1)\,. \end{aligned}$$

This shows $(G\psi)_1 \mathbin{\blacktriangle} \varphi_1 = (\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1 \mathbin{\blacktriangle} (\tilde{G}\psi)_1$.

$$\begin{array}{ccc} (G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1 & \xrightarrow{(G\psi)_1} & G_1 \\ {\scriptstyle (\varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ (\tilde{G}\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm})_1 & \xrightarrow[(\tilde{G}\psi)_1]{} & \tilde{G}_1 \end{array}$$

So we have $G\psi \mathbin{\blacktriangle} \varphi = \varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm} \mathbin{\blacktriangle} \tilde{G}\psi$.

$$\begin{array}{ccc} G\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm} & \xrightarrow{G\psi} & G \\ {\scriptstyle \varphi\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \tilde{G}\hat{\mathrm{N}}^{\pm}\,\mathrm{Rec}^{\pm} & \xrightarrow[\tilde{G}\psi]{} & \tilde{G} \end{array}$$

So ψ is a transformation. Using (1), we conclude that ψ is an isotransformation. □

Proposition 405 (The equivalences $\hat{\mathrm{N}}^{\pm}$ and $\mathrm{Rec}^{\pm}$)

By Lemma 404.(2), we have the isotransformation $\psi\colon \hat{\mathrm{N}}^{\pm} * \mathrm{Rec}^{\pm} \underset{\sim}{\longrightarrow} \mathrm{id}_{\lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp}}$.

By Lemma 402.(2), we have the isotransformation $\hat{\psi}\colon\ \mathrm{Rec}^{\pm} * \hat{\mathrm{N}}^{\pm} \underset{\sim}{\longrightarrow} \mathrm{id}_{\mathcal{StCrMod}}$.

So $\hat{\mathrm{N}}^{\pm}$ and $\mathrm{Rec}^{\pm}$ are equivalences. In particular, the categories $\lfloor 1,0\rfloor$-$\mathcal{StSimpGrp}$ and $\mathcal{StCrMod}$ are equivalent.

Chapter 19

Symmetric monoidal categories

19.1 The category of symmetric monoidal categories

Definition 406 (Symmetric monoidal category)

Suppose given an invertible monoidal category $(\mathcal{C}, I, \otimes)$; cf. Definition 158.

We consider the functors

$$T = (\otimes)\colon\ \mathcal{C}\times\mathcal{C}\to\mathcal{C}\,,\ \big((X,Y)\xrightarrow{(u,v)}(X',Y')\big)\mapsto\big((X\otimes Y)\xrightarrow{u\otimes v}(X'\otimes Y')\big)$$

and

$$T'\colon\ \mathcal{C}\times\mathcal{C}\to\mathcal{C}\,,\ \big((X,Y)\xrightarrow{(u,v)}(X',Y')\big)\mapsto\big((Y\otimes X)\xrightarrow{v\otimes u}(Y'\otimes X')\big)\,.$$

A transformation $\varphi\colon T\to T'$ is called a *symmetric braiding* of the invertible monoidal category $\mathcal{C}$ if the following properties $(1,2,3)$ hold.

(SB1) For X, $Y\in\mathrm{Ob}(\mathcal{C})$, we have

$$(X,Y)\varphi\vartriangleleft(Y,X)\varphi=\mathrm{id}_{X\otimes Y}\ .$$

(SB2) We have

$$(I,I)\varphi=\mathrm{id}_I\ .$$

(SB3) For X, Y, $Z\in\mathrm{Ob}(\mathcal{C})$, we have

$$(X\otimes Y,Z)\varphi=\big(X\otimes(Y,Z)\varphi\big)\vartriangleleft\big((X,Z)\varphi\otimes Y\big)$$

So we have the following commutative diagram.

$$\begin{array}{ccc} X\otimes Y\otimes Z & \xrightarrow{(X\otimes Y,Z)\varphi} & Z\otimes X\otimes Y \\ & {\scriptstyle X\otimes(Y,Z)\varphi}\searrow \quad \nearrow{\scriptstyle (X,Z)\varphi\otimes Y} & \\ & X\otimes Z\otimes Y & \end{array}$$

We often write

$$\varphi = \big((X,Y)T \xrightarrow{(X,Y)\varphi} (X,Y)T'\big)_{(X,Y)\in \mathrm{Ob}(\mathcal{C}\times\mathcal{C})} =: \big(X\otimes Y \xrightarrow{\varphi_{X,Y}} Y\otimes X\big)_{(X,Y)\in \mathrm{Ob}(\mathcal{C}\times\mathcal{C})}$$

Note that φ is an isotransformation since every morphism in $\mathcal{C}$ is an isomorphism; cf. Remark 160. The inverse to $\varphi_{X,Y}$ is given by $\varphi_{Y,X}$; cf. (SB1).

A *symmetric monoidal category* is an invertible monoidal category $(\mathcal{C}, I, \otimes)$ together with a symmetric braiding φ on $(\mathcal{C}, I, \otimes)$.

Often, we write $\mathcal{C} = (\mathcal{C}, I, \otimes, \varphi)$.

Remark 407 Suppose given a symmetric monoidal category $(\mathcal{C}, I, \otimes, \varphi)$.

For X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned} \varphi_{X,Y} \blacktriangle \varphi_{Y,X} &\overset{(\mathrm{SB1})}{=} \mathrm{id}_{X\otimes Y} \\ \varphi_{I,I} &\overset{(\mathrm{SB2})}{=} \mathrm{id}_I \\ \varphi_{X\otimes Y,Z} &\overset{(\mathrm{SB3})}{=} (X\otimes\varphi_{Y,Z}) \blacktriangle (\varphi_{X,Z}\otimes Y)\,. \end{aligned}$$

Moreover, we have

$$\varphi_{X,Y\otimes Z} \overset{(\mathrm{SB1})}{=} (\varphi_{Y\otimes Z,X})^- \overset{(\mathrm{SB3})}{=} \big((Y\otimes\varphi_{Z,X}) \blacktriangle (\varphi_{Y,X}\otimes Z)\big)^- \overset{(\mathrm{SB1})}{=} (\varphi_{X,Y}\otimes Z) \blacktriangle (Y\otimes\varphi_{X,Z})\,.$$

So we have the following commutative diagram.

$$\begin{array}{ccc} X\otimes Y\otimes Z & \xrightarrow{\varphi_{X,Y\otimes Z}} & Y\otimes Z\otimes X \\ & {\scriptstyle \varphi_{X,Y}\otimes Z}\searrow \quad \nearrow{\scriptstyle Y\otimes\varphi_{X,Z}} & \\ & Y\otimes X\otimes Z & \end{array}$$

Remark 408 *Suppose given a symmetric monoidal category* $(\mathcal{C}, I, \otimes, \varphi)$.

(1) *For* $X \in \mathrm{Ob}(\mathcal{C})$, *we have*

$$\varphi_{X,I} = \varphi_{I,X} = \mathrm{id}_X \ .$$

(2) *For* X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, *we have*

$$\varphi_{X,Y\otimes Z} = \varphi_{X,Y} \otimes X^{\otimes -} \otimes \varphi_{X,Z} \, .$$

(3) *For* X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, *we have*

$$\varphi_{X\otimes Y,Z} = \varphi_{X,Z} \otimes Z^{\otimes -} \otimes \varphi_{Y,Z} \, .$$

Proof. Suppose given X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$.
Ad (1). For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$\varphi_{X,I} = \varphi_{X,I\otimes I} \overset{407}{=} (\varphi_{X,I} \otimes I) \vartriangle (I \otimes \varphi_{X,I}) = \varphi_{X,I} \vartriangle \varphi_{X,I} \, .$$

This shows $\mathrm{id}_X = \varphi_{X,I}$.
Moreover, we have

$$\varphi_{I,X} \overset{(SB1)}{=} \varphi^-_{X,I} = \mathrm{id}^-_X = \mathrm{id}_X \ .$$

Ad (2). We have

$$\begin{array}{rcl} \varphi_{X,Y\otimes Z} & \overset{407}{=} & (\varphi_{X,Y} \otimes Z) \vartriangle (Y \otimes \varphi_{X,Z}) \\ & \overset{160.(3)}{=} & \varphi_{X,Y} \otimes Z \otimes (Y \otimes X \otimes Z)^{\otimes -} \otimes Y \otimes \varphi_{X,Z} \\ & = & \varphi_{X,Y} \otimes X^{\otimes -} \otimes \varphi_{X,Z} \, . \end{array}$$

Ad (3). We have

$$\begin{array}{rcl} \varphi_{X\otimes Y,Z} & = & (\varphi_{Z,X\otimes Y})^- \overset{(2)}{=} (\varphi_{Z,X} \otimes Z^{\otimes -} \otimes \varphi_{Z,Y})^- = \varphi^-_{Z,X} \otimes Z^{\otimes -} \otimes \varphi^-_{Z,Y} \\ & \overset{(SB1)}{=} & \varphi_{X,Z} \otimes Z^{\otimes -} \otimes \varphi_{Y,Z} \, . \end{array}$$

□

Remark 409 Suppose given a braided monoidal category $(\mathcal{C}, I, \otimes, \varphi)$.
Then $\varphi = (\varphi_{X,Y})_{(X,Y)\in \mathrm{Ob}(\mathcal{C}\times\mathcal{C})}$ being a transformation from T to T' means that

$$\begin{array}{ccc} X\otimes Y & \xrightarrow{\varphi_{X,Y}} & Y\otimes X \\ {\scriptstyle u\otimes v}\downarrow & & \downarrow{\scriptstyle v\otimes u} \\ X'\otimes Y' & \xrightarrow[\varphi_{X',Y'}]{} & Y'\otimes X' \end{array}$$

commutes for $\left(X \xrightarrow{u} X'\right)$, $\left(Y \xrightarrow{v} Y'\right)$ in $\mathcal{C}$, i.e. that

$$\varphi_{X,Y} \vartriangleleft (v\otimes u) = (u\otimes v) \vartriangleleft \varphi_{X',Y'}\,.$$

Definition 410 (Symmetric monoidal functor)
Suppose given symmetric monoidal categories $(\mathcal{C}, I_{\mathcal{C}}, \otimes, \varphi)$ and $(\mathcal{D}, I_{\mathcal{D}}, \otimes, \psi)$.
Suppose given a functor $F\colon \mathcal{C}\to\mathcal{D}$.
We say that F is a *symmetric monoidal* functor if the conditions $(1, 2)$ hold.

(1) The functor F is monoidal; cf. Definition 161.

(2) For X, $Y\in \mathrm{Ob}(\mathcal{C})$, we have

$$(\varphi_{X,Y})F = \psi_{XF,YF}\,.$$

So we have the following commutative diagram

$$\begin{array}{ccc} (X\otimes Y)F & \xrightarrow{(\varphi_{X,Y})F} & (Y\otimes X)F \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ XF\otimes YF & \xrightarrow[\psi_{XF,YF}]{} & YF\otimes XF \end{array}$$

Lemma 411 *Suppose given symmetric monoidal categories*

$$\mathcal{C} = (\mathcal{C}, I_{\mathcal{C}}, \otimes, \varphi)\,,\ \mathcal{D} = (\mathcal{D}, I_{\mathcal{D}}, \otimes, \psi) \quad \text{and} \quad \mathcal{E} = (\mathcal{E}, I_{\mathcal{E}}, \otimes, \chi)\,.$$

Suppose given symmetric monoidal functors $F\colon \mathcal{C}\to\mathcal{D}$ and $G\colon \mathcal{D}\to\mathcal{E}$.

(1) *The identity functor* $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C}\to\mathcal{C}$ *is a symmetric monoidal functor.*

(2) *The composite* $F * G \colon \mathcal{C} \to \mathcal{E}$ *is a symmetric monoidal functor.*

(3) *Suppose that* F *is an isofunctor. Then its inverse* $F^- \colon \mathcal{D} \to \mathcal{C}$ *is a symmetric monoidal functor.*

Proof. Ad (1). We have that $\mathrm{id}_{\mathcal{C}}$ is a monoidal functor; cf. [**29,** Lem. 34.(1)].

Moreover, for X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(\varphi_{X,Y})\,\mathrm{id}_{\mathcal{C}} = \varphi_{X,Y} = \varphi_{X\,\mathrm{id}_{\mathcal{C}},Y\,\mathrm{id}_{\mathcal{C}}}\,.$$

Ad (2). We have that $F * G$ is a monoidal functor; cf. [**29,** Lem. 34.(2)].

Moreover, for X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(\varphi_{X,Y})(F * G) = (\varphi_{X,Y})FG = (\psi_{XF,YF})G = \chi_{XFG,YFG} = \chi_{X(F*G),Y(F*G)}\,.$$

Ad (3). We have that F^- is a monoidal functor; cf. [**29,** Lem. 34.(3)].

Moreover, for X, $Y \in \mathrm{Ob}(\mathcal{D})$, we have

$$(\psi_{X,Y})F^- = (\psi_{XF^-F,YF^-F})F^- = (\varphi_{XF^-,YF^-})FF^- = \varphi_{XF^-,YF^-}\,.$$

□

Definition 412 (The category of symmeric monoidal categories)

We have the category $\mathcal{SymMonCat}$, called the *category of symmetric monoidal categories*, having symmetric monoidal categories as objects and having symmetric monoidal functors as morphisms.

The composition in $\mathcal{SymMonCat}$ is given by the composition $(*)$ of functors; cf. Lemma 411.(1, 2).

19.2 The functor SymCat: $\mathcal{StCrMod} \to \mathcal{SymMonCat}$

We consider the following functors.

$$\begin{array}{lrcll} \mathrm{Ord}^{\mathrm{mod}}\colon & \mathcal{StCrMod} & \to & \mathcal{CrMod}; & \text{cf. Lemma 329} \\ \mathrm{Cat}\colon & \mathcal{CrMod} & \to & \mathcal{InvMonCat}; & \text{cf. Remark 165} \end{array}$$

Lemma 413 (The symmetric monoidal category $S\,\mathrm{SymCat}$)

Suppose given a stable crossed module S; cf. Definition 321.

We consider the crossed module $S\mathrm{Ord}^{\mathrm{mod}} = (\mathrm{M}_S\,,\,\mathrm{B}_S\,,\,\gamma_S\,,\,\mathrm{f}_S)$ *; cf. Lemma* 327.

We consider the invertible monoidal category $\big((S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}, 1, (\,\cdot\,)\big)$*; cf. Remark* 165.(1).

Recall that we have

$$\mathrm{Ob}\big((S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\big) = \mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}} = \mathrm{B}_S$$
$$\mathrm{Mor}\big((S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\big) = \mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}} \ltimes_{\gamma_S} \mathrm{M}_{S\mathrm{Ord}^{\mathrm{mod}}} = \mathrm{B}_S \ltimes \mathrm{M}_S\,.$$

Recall that the tensor product $(\,\cdot\,)$ *is given by the group multiplication* $(\,\cdot\,)$ *in* B_S*, respectively given by the group multiplication* $(\,\cdot\,)$ *in* $\mathrm{B}_S \ltimes \mathrm{M}_S$*, and that the unit object is given by* $1 \in \mathrm{B}_S$*.*

Moreover, recall that, for (b,m)*,* $(c,n) \in \mathrm{B}_S \ltimes \mathrm{M}_S$*, we have*

$$(b,m)\cdot(c,n) = (b\cdot c, m^c\cdot n) \overset{327.(1)}{=} \big(b\cdot c, m\cdot [\![m\mathrm{f}_S,\, c]\!]\cdot n\big)\,.$$

Finally, recall that (b,m)*,* $(c,n) \in \mathrm{B}_S \ltimes \mathrm{M}_S$ *are composable if and only if* $b\cdot m\mathrm{f}_S = c$ *and that in this case the composite is given by*

$$(b,m) \blacktriangle (c,n) = (b, m\cdot n)\,.$$

For b*,* $c \in \mathrm{Ob}\big((S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\big) = \mathrm{B}_S$*, let*

$$\varphi_{b,c} := \big(b\cdot c, [\![c,\, b]\!]\big) \in \mathrm{Mor}\big((S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\big)\ .$$

Then

$$\varphi := \big(b\cdot c \xrightarrow{\varphi_{b,c}} c\cdot b\big)_{(b,c)\in\mathrm{Ob}((S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\,\times(S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat})}$$

is a symmetric braiding of $(S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}$*.*

So

$$S\,\mathrm{SymCat} := \big((S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}, 1, (\,\cdot\,), \varphi\big)$$

is a symmetric monoidal category; cf. Definition 406*.*

Proof. We write $\mathcal{C} := (S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}$ and we write $f := \mathrm{f}_S$.

Suppose given b, $c \in \mathrm{Ob}(\mathcal{C}) = \mathrm{B}_S$.

We have

$$(\varphi_{b,c})\boldsymbol{s} = b\cdot c$$
$$(\varphi_{b,c})\boldsymbol{t} = b\cdot c\cdot [\![c,\, b]\!] f \overset{(\mathrm{SCM2})}{=} b\cdot c\cdot [c,b] = b\cdot c\cdot c^{-}\cdot b^{-}\cdot c\cdot b = c\cdot b\,.$$

So $\varphi_{b,c} \colon b \cdot c \to c \cdot b$.

Let $\varphi := \left(b \cdot c \xrightarrow{\varphi_{b,c}} c \cdot b\right)_{(b,c) \in \mathrm{Ob}(\mathcal{C} \times \mathcal{C})}$.

We show that φ is a transformation. Suppose given $\left(b \xrightarrow{(b,m)} b \cdot mf\right)$, $\left(c \xrightarrow{(c,n)} c \cdot nf\right)$ in $\mathcal{C}$. We have to show that the following diagram is commutative.

$$\begin{array}{ccc} b \cdot c & \xrightarrow{\varphi_{b,c}} & c \cdot b \\ {\scriptstyle (b,m)\cdot(c,n)}\downarrow & & \downarrow{\scriptstyle (c,n)\cdot(b,m)} \\ b \cdot mf \cdot c \cdot nf & \xrightarrow[\varphi_{b\cdot mf, c\cdot nf}]{} & c \cdot nf \cdot b \cdot mf \end{array}$$

We have

$$\begin{array}{rl} & m^c \cdot n \cdot \llbracket c \cdot nf,\, b \cdot mf \rrbracket \\ \overset{322.(2)}{=} & m^c \cdot \left(n \cdot \llbracket nf,\, (b \cdot mf)^c \rrbracket\right) \cdot \llbracket c,\, b \cdot mf \rrbracket \\ = & m^c \cdot n^{(b\cdot mf)^c} \cdot \llbracket c,\, b \cdot mf \rrbracket \\ = & \llbracket c,\, b \cdot mf \rrbracket \cdot \left(m^c \cdot n^{(b\cdot mf)^c}\right)^{\llbracket c,\, b\cdot mf \rrbracket} \\ \overset{323.(3)}{=} & \llbracket c,\, b \cdot mf \rrbracket \cdot \left(m^c \cdot n^{(b\cdot mf)^c}\right)^{[c,b\cdot mf]} \\ = & \llbracket c,\, b \cdot mf \rrbracket \cdot m^{c\cdot[c,b\cdot mf]} \cdot n^{(b\cdot mf)^c \cdot [c,b\cdot mf]} \\ = & \llbracket c,\, b \cdot mf \rrbracket \cdot m^{c^{b\cdot mf}} \cdot n^{b\cdot mf} \\ \overset{323.(8)}{=} & \llbracket c,\, b \cdot mf \rrbracket \cdot \llbracket c^{b\cdot mf},\, m^- f \rrbracket \cdot m \cdot n^{b\cdot mf} \\ = & \llbracket c,\, b \cdot mf \rrbracket \cdot \llbracket c^{b\cdot mf},\, m^- f \rrbracket \cdot \left(n^{b\cdot mf}\right)^{m^-} \cdot m \\ \overset{323.(2)}{=} & \llbracket c,\, b \cdot mf \rrbracket \cdot \llbracket c^{b\cdot mf},\, m^- f \rrbracket \cdot \left(n^{b\cdot mf}\right)^{m^- f} \cdot m \\ \overset{(SCM4)}{=} & \llbracket c,\, b \cdot mf \cdot m^- f \rrbracket \cdot n^{b\cdot mf\cdot m^- f} \cdot m \\ = & \llbracket c,\, b \rrbracket \cdot n^b \cdot m\,. \end{array}$$

So we have

$$\begin{aligned}\varphi_{b,c} \blacktriangle \big((c,n)\cdot(b,m)\big) &= \big(b\cdot c, \llbracket c,\, b\rrbracket\big) \blacktriangle (c\cdot b, n^b\cdot m)\\ &= \big(b\cdot c, \llbracket c,\, b\rrbracket \cdot n^b \cdot m\big)\\ &= \big(b\cdot c, m^c\cdot n\cdot \llbracket c\cdot nf,\, b\cdot mf\rrbracket\big)\\ &= (b\cdot c, m^c\cdot n) \blacktriangle \big(b\cdot mf\cdot c\cdot nf, \llbracket c\cdot nf,\, b\cdot mf\rrbracket\big)\\ &= \big((b,m)\cdot(c,n)\big) \blacktriangle \varphi_{b\cdot mf, c\cdot nf}\,.\end{aligned}$$

This shows that φ is a transformation.

We show that φ is a symmetric braiding of $(S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}$. Cf. Remark 407.

Ad (SB1). For b, $c \in \mathrm{B}_S$, we have

$$\varphi_{b,c} \blacktriangle \varphi_{c,b} = \big(b\cdot c, \llbracket c,\, b\rrbracket\big) \blacktriangle \big(c\cdot b, \llbracket b,\, c\rrbracket\big) = \big(b\cdot c, \llbracket c,\, b\rrbracket \cdot \llbracket b,\, c\rrbracket\big) \overset{(\mathrm{SCM1})}{=} (b\cdot c, 1\,) = \mathrm{id}_{b\cdot c}\,.$$

Ad (SB2). We have

$$\varphi_{1,1} = \big(1\cdot 1, \llbracket 1,\, 1\rrbracket\big) \overset{322.(1)}{=} (1,1) = \mathrm{id}_1\,.$$

Ad (SB3). For a, b, $c \in \mathrm{B}_S$, we have

$$\begin{aligned}(\mathrm{id}_a\cdot\varphi_{b,c}) \blacktriangle (\varphi_{a,c}\cdot \mathrm{id}_b) \quad &= \quad \Big((a,1)\cdot\big(b\cdot c, \llbracket c,\, b\rrbracket\big)\Big) \blacktriangle \Big(\big(a\cdot c, \llbracket c,\, a\rrbracket\big)\cdot(b,1)\Big)\\ &= \quad \big(a\cdot b\cdot c, \llbracket c,\, b\rrbracket\big) \blacktriangle \big(a\cdot c\cdot b, \llbracket c,\, a\rrbracket^b\big)\\ &= \quad \big(a\cdot b\cdot c, \llbracket c,\, b\rrbracket \cdot \llbracket c,\, a\rrbracket^b\big)\\ &\overset{323.(10)}{=} \quad \big(a\cdot b\cdot c, \llbracket c,\, b\rrbracket \cdot \llbracket c^b,\, a^b\rrbracket\big)\\ &\overset{322.(3)}{=} \quad \big(a\cdot b\cdot c, \llbracket c,\, b\rrbracket \cdot \llbracket c,\, b\rrbracket^- \cdot \llbracket c,\, a\cdot b\rrbracket\big)\\ &= \quad \big(a\cdot b\cdot c, \llbracket c,\, a\cdot b\rrbracket\big)\\ &= \quad \varphi_{a\cdot b,c}\,.\end{aligned}$$

□

Lemma 414 (The symmetric monoidal functor $\rho\,\mathrm{SymCat}$)

Suppose given a stable crossed module morphism $\rho = (\mu,\beta)\colon S \to \tilde{S}$.

We consider the crossed module morphism $\rho\mathrm{Ord}^{\mathrm{mod}} = (\mu,\beta)\colon S\mathrm{Ord}^{\mathrm{mod}} \to \tilde{S}\mathrm{Ord}^{\mathrm{mod}}$; cf. Lemma 328.

Recall that we have the monoidal functor

$$(\rho\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\colon (S\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat} \to (\tilde{S}\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\,,\ \big(b \xrightarrow{(b,m)} b{\cdot}mf\big) \mapsto \big(b\beta \xrightarrow{(b\beta,m\mu)} (b{\cdot}m\mathrm{f}_S)\beta\big)\,;$$

cf. Lemma 165.(2).

We have that

$$\begin{array}{rccl} \rho\,\mathrm{SymCat} := (\rho\mathrm{Ord}^{\mathrm{mod}})\,\mathrm{Cat}\colon & S\,\mathrm{SymCat} & \to & \tilde{S}\,\mathrm{SymCat} \\ & \big(b \xrightarrow{(b,m)} b\cdot m\mathrm{f}_S\big) & \mapsto & \big(b\beta \xrightarrow{(b\beta,m\mu)} (b\cdot m\mathrm{f}_S)\beta\big) \end{array}$$

is a symmetric monoidal functor.

Proof. For $b,\ c \in \mathrm{B}_S$, we have

$$\begin{aligned} (\varphi_{b,c})(\rho\,\mathrm{SymCat}) &= \big(b\cdot c, [\![c,\ b]\!]\big)(\rho\,\mathrm{SymCat}) = \big((b\cdot c)\beta, [\![c,\ b]\!]\mu\big) \overset{324.(2)}{=} \big(b\beta\cdot c\beta, [\![c\beta,\ b\beta]\!]\big) \\ &= \varphi_{b\beta,c\beta} = \varphi_{b(\rho\,\mathrm{SymCat}),c(\rho\,\mathrm{SymCat})}\,. \end{aligned}$$

□

Lemma 415 (The functor SymCat)

We have the functor

$$\begin{array}{rccl} \mathrm{SymCat}\colon & \mathcal{StCrMod} & \to & \mathcal{SymMonCat} \\ & \big(S \xrightarrow{\rho} \tilde{S}\big) & \mapsto & \big(S\,\mathrm{SymCat} \xrightarrow{\rho\,\mathrm{SymCat}} \tilde{S}\,\mathrm{SymCat}\big); \end{array}$$

cf. Lemmas 413, 414.

In particular, for a stable crossed module S, we have

$$\mathrm{Ob}(S\,\mathrm{SymCat}) = \mathrm{B}_S \quad \textit{and} \quad \mathrm{Mor}(S\,\mathrm{SymCat}) = \mathrm{B}_S \ltimes \mathrm{M}_S\,.$$

Proof. Since the composite $\mathrm{Ord}^{\mathrm{mod}} * \mathrm{Cat}\colon \mathcal{StCrMod} \to \mathcal{InvMonCat}$ is a functor, we have that $\mathrm{SymCat}\colon \mathcal{StCrMod} \to \mathcal{SymMonCat}$ is a functor. □

19.3 The functor StCM: *SymMonCat* → *StCrMod*

Remark 416 *Suppose given a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, I, \otimes, \varphi)$; cf. Definition* 158.

For X, $Y \in \mathrm{Ob}(\mathcal{C})$, let

$$\llbracket X,\, Y \rrbracket := (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X} \in \ker \boldsymbol{s} \ .$$

Note that $\llbracket X,\, Y \rrbracket \colon I \to [X, Y]$.

(1) *For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have*

$$\llbracket X,\, Y \rrbracket = \llbracket Y,\, X \rrbracket^{\otimes -} \ .$$

(2) *For X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, we have*

$$\llbracket X,\, Y \otimes Z \rrbracket = \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket^{Z} \ .$$

(3) *For $X \in \mathrm{Ob}(\mathcal{C})$, we have*

$$\llbracket X,\, I \rrbracket = \llbracket I,\, X \rrbracket = \mathrm{id}_I \ .$$

(4) *For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have*

$$\llbracket X^{\otimes -},\, Y \rrbracket = \llbracket Y,\, X \rrbracket^{X^{\otimes -}} \ .$$

(5) *For $\left(I \xrightarrow{u} X\right) \in \mathrm{Mor}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{C})$, we have*

$$u^Y = Y^{\otimes -} \otimes u \otimes Y = u \otimes \llbracket X,\, Y \rrbracket \ .$$

(6) *For $\left(I \xrightarrow{u} X\right)$ and $\left(I \xrightarrow{v} Y\right)$ in $\mathcal{C}$, we have*

$$u^v = v^{\otimes -} \otimes u \otimes v = Y^{\otimes -} \otimes u \otimes Y = u^Y \ .$$

Proof. Ad (1). For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned}
\llbracket Y,\, X \rrbracket^{\otimes -} &= \left((X \otimes Y)^{\otimes -} \otimes \varphi_{X,Y}\right)^{\otimes -} \\
&= (\varphi_{X,Y})^{\otimes -} \otimes (X \otimes Y) \\
&\overset{160.(1)}{=} (Y \otimes X)^{\otimes -} \otimes (\varphi_{X,Y})^{-} \otimes (X \otimes Y)^{\otimes -} \otimes (X \otimes Y) \\
&\overset{(\mathrm{SB1})}{=} (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X} \\
&= \llbracket X,\, Y \rrbracket \, .
\end{aligned}$$

Ad (2). For X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned}
& \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket^Z \\
= \quad & (Z \otimes X)^{\otimes -} \otimes \varphi_{Z,X} \otimes Z^{\otimes -} \otimes (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X} \otimes Z \\
= \quad & X^{\otimes -} \otimes Z^{\otimes -} \otimes (\varphi_{Z,X} \otimes Z^{\otimes -} \otimes X^{\otimes -}) \otimes (Y^{\otimes -} \otimes \varphi_{Y,X} \otimes X^{\otimes -}) \otimes X \otimes Z \\
\overset{160.(4)}{=} \quad & X^{\otimes -} \otimes Z^{\otimes -} \otimes (Y^{\otimes -} \otimes \varphi_{Y,X} \otimes X^{\otimes -}) \otimes (\varphi_{Z,X} \otimes Z^{\otimes -} \otimes X^{\otimes -}) \otimes X \otimes Z \\
= \quad & X^{\otimes -} \otimes Z^{\otimes -} \otimes Y^{\otimes -} \otimes (\varphi_{Y,X} \otimes X^{\otimes -} \otimes \varphi_{Z,X}) \\
\overset{408.(3)}{=} \quad & (Y \otimes Z \otimes X)^{\otimes -} \otimes \varphi_{Y \otimes Z,X} \\
= \quad & \llbracket X,\, Y \otimes Z \rrbracket\,.
\end{aligned}$$

Ad (3). For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$\llbracket X,\, I \rrbracket = \llbracket X,\, I \otimes I \rrbracket \overset{(2)}{=} \llbracket X,\, I \rrbracket \otimes \llbracket X,\, I \rrbracket^I = \llbracket X,\, I \rrbracket \otimes \llbracket X,\, I \rrbracket\,.$$

Applying $\llbracket X,\, I \rrbracket^{\otimes -}$, this shows $\llbracket X,\, I \rrbracket = \mathrm{id}_I$.

Morover, we have

$$\llbracket I,\, X \rrbracket \overset{(1)}{=} \llbracket X,\, I \rrbracket^{\otimes -} = (\mathrm{id}_I)^{\otimes -} = \mathrm{id}_I\ .$$

Ad (4). For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$\llbracket Y,\, X \rrbracket^{X^{\otimes -}} \overset{(2)}{=} \llbracket Y,\, X^{\otimes -} \rrbracket^{\otimes -} \otimes \llbracket Y,\, X \otimes X^{\otimes -} \rrbracket \overset{(1)}{=} \llbracket X^{\otimes -},\, Y \rrbracket \otimes \llbracket Y,\, I \rrbracket \overset{(3)}{=} \llbracket X^{\otimes -},\, Y \rrbracket\,.$$

Ad (5). Suppose given $\big(I \xrightarrow{u} X\big) \in \mathrm{Mor}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{C})$.

We have

$$\begin{aligned}
u \otimes \llbracket X,\, Y \rrbracket \quad = \quad & u \otimes (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X} \\
= \quad & Y^{\otimes -} \otimes \big((Y \otimes u) \otimes (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X}\big) \\
\overset{160.(3)}{=} \quad & Y^{\otimes -} \otimes \big((Y \otimes u) \blacktriangle \varphi_{Y,X}\big) \\
\overset{409}{=} \quad & Y^{\otimes -} \otimes \big(\varphi_{Y,I} \blacktriangle (u \otimes Y)\big) \\
\overset{408.(1)}{=} \quad & Y^{\otimes -} \otimes \big(\mathrm{id}_Y \blacktriangle (u \otimes Y)\big) \\
= \quad & Y^{\otimes -} \otimes u \otimes Y \\
= \quad & u^Y\,.
\end{aligned}$$

Ad (6). Suppose given $\big(I \xrightarrow{u} X\big)$ and $\big(I \xrightarrow{v} Y\big)$ in $\mathcal{C}$.

We have

$$\begin{aligned}
u^v &= v^{\otimes -} \otimes u \otimes v \\
&= v^{\otimes -} \otimes \big(\mathrm{id}_I \blacktriangleleft (u \otimes v) \big) \\
&\overset{(\mathrm{SB2})}{=} v^{\otimes -} \otimes \big(\varphi_{I,I} \blacktriangleleft (u \otimes v) \big) \\
&\overset{409}{=} v^{\otimes -} \otimes \big((v \otimes u) \blacktriangleleft \varphi_{Y,X} \big) \\
&\overset{160.(3)}{=} v^{\otimes -} \otimes (v \otimes u) \otimes (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X} \\
&= u \otimes [\![X, Y]\!] \\
&\overset{(5)}{=} Y^{\otimes -} \otimes u \otimes Y \\
&= u^Y \, .
\end{aligned}$$

□

We have considered the functor CM: $\mathcal{InvMonCat} \to \mathcal{CrMod}$; cf. Remark 166.

We will give a variant of this construction for symmetric monoidal categories and stable crossed modules. In order to do so, we will not make use of the functor CM and proceed using direct arguments. Cf. also Lemma 426 below.

Lemma 417 (The functor StCM)

(1) *Suppose given a symmetric monoidal category* $\mathcal{C} = (\mathcal{C}, I, \otimes, \varphi)$*; cf. Definition* 406*.*

Let

$$\kappa_{\mathcal{C}} \colon \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \to \ker \mathcal{s} \, , \ (X, Y) \mapsto [\![X, Y]\!] := (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X} \, .$$

Then $\mathcal{C}\,\mathrm{StCM} := (\ker \mathcal{s}_{\mathcal{C}} \, , \mathrm{Ob}(\mathcal{C}) \, , \kappa_{\mathcal{C}} \, , \mathcal{t}_{\mathcal{C}} \big|_{\ker \mathcal{s}_{\mathcal{C}}})$ *is a stable crossed module.*

(2) *Suppose given symmetric monoidal categories* $(\mathcal{C}, I_{\mathcal{C}} \, , \otimes, \varphi)$ *and* $(\mathcal{D}, I_{\mathcal{D}} \, , \otimes, \psi)$*.*

Suppose given a symmetric monoidal functor $F \colon \mathcal{C} \to \mathcal{D}$*; cf. Definition* 410*.*

Using Remark 162.(1)*, we let*

$$\begin{aligned}
\mu_F &\colon \ker \mathcal{s}_{\mathcal{C}} \to \ker \mathcal{s}_{\mathcal{D}} \, , \ \big(I_{\mathcal{C}} \xrightarrow{u} X \big) \mapsto \big(I_{\mathcal{D}} \xrightarrow{uF} XF \big) \\
\beta_F &\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}) \ , \ X \mapsto XF \, .
\end{aligned}$$

We consider the stable crossed modules

$$\mathcal{C}\,\mathrm{StCM} = (\ker \mathcal{s}_{\mathcal{C}} \, , \mathrm{Ob}(\mathcal{C}) \, , \kappa_{\mathcal{C}} \, , \mathcal{t}_{\mathcal{C}} \big|_{\ker \mathcal{s}_{\mathcal{C}}}) \quad \textit{and} \quad \mathcal{D}\,\mathrm{StCM} = (\ker \mathcal{s}_{\mathcal{D}} \, , \mathrm{Ob}(\mathcal{D}) \, , \kappa_{\mathcal{D}} \, , \mathcal{t}_{\mathcal{D}} \big|_{\ker \mathcal{s}_{\mathcal{D}}}) \, .$$

Then

$$F\,\mathrm{StCM} := (\mu_F\,, \beta_F)\colon\ \mathcal{C}\,\mathrm{StCM} \to \mathcal{D}\,\mathrm{StCM}$$

is a stable crossed module morphism.

(3) *We have the functor*

$$\mathrm{StCM}\colon \mathcal{SymMonCat} \to \mathcal{StCrMod}\,,\ \big(\mathcal{C} \xrightarrow{F} \mathcal{D}\big) \mapsto \big(\mathcal{C}\,\mathrm{StCM} \xrightarrow{F\,\mathrm{StCM}} \mathcal{D}\,\mathrm{StCM}\big);$$

cf. (1, 2).

Proof. Ad (1). We write $\boldsymbol{t} := \boldsymbol{t}_{\mathcal{C}}\,\big|_{\ker \mathcal{s}_{\mathcal{C}}}$.

Ad (SCM1). For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$\llbracket Y,\, X \rrbracket^{\otimes -} = \llbracket X,\, Y \rrbracket$$

by Remark 416.(1).

Ad (SCM2). For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned}\llbracket X,\, Y \rrbracket\, \boldsymbol{t} &= \big((Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X}\big)\, \boldsymbol{t} = X^{\otimes -} \otimes Y^{\otimes -} \otimes (\varphi_{Y,X})\, \boldsymbol{t} = X^{\otimes -} \otimes Y^{\otimes -} \otimes X \otimes Y \\ &= [X, Y]\,.\end{aligned}$$

Cf. also Remark 416.

Ad (SCM3). Suppose given $\big(I \xrightarrow{u} X\big)$, $\big(I \xrightarrow{v} Y\big) \in \ker \mathcal{s}$.

We have

$$[u, v] = u^{\otimes -} \otimes u^{v} \overset{416.(6)}{=} u^{\otimes -} \otimes u^{Y} \overset{416.(5)}{=} u^{\otimes -} \otimes u \otimes \llbracket X,\, Y \rrbracket = \llbracket u\, \boldsymbol{t},\, v\, \boldsymbol{t} \rrbracket\,.$$

Ad (SCM4). Suppose given X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$.

We have

$$\begin{aligned}
& \llbracket X,\, Y \otimes Z \rrbracket \\
\overset{416.(2)}{=}\ & \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket^{Z} \\
\overset{416.(5)}{=}\ & \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \big\llbracket [X, Y],\, Z \big\rrbracket \\
\overset{416.(1)}{=}\ & \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \llbracket Z,\, X^{\otimes -} \otimes X^{Y} \rrbracket^{\otimes -} \\
\overset{416.(2)}{=}\ & \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \Big(\llbracket Z,\, X^{Y} \rrbracket \otimes \llbracket Z,\, X^{\otimes -} \rrbracket^{X^{Y}}\Big)^{\otimes -} \\
\overset{416.(1)}{=}\ & \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \llbracket X^{\otimes -},\, Z \rrbracket^{X^{Y}} \otimes \llbracket X^{Y},\, Z \rrbracket \\
\overset{416.(4)}{=}\ & \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \llbracket Z,\, X \rrbracket^{X^{\otimes -} \otimes X^{Y}} \otimes \llbracket X^{Y},\, Z \rrbracket
\end{aligned}$$

$$
\begin{aligned}
&= \quad \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \llbracket Z,\, X \rrbracket^{[X,Y]} \otimes \llbracket X^Y,\, Z \rrbracket \\
&\overset{416.(6)}{=} \quad \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \llbracket Z,\, X \rrbracket^{\llbracket X,\, Y \rrbracket} \otimes \llbracket X^Y,\, Z \rrbracket \\
&= \quad \llbracket X,\, Z \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \llbracket X,\, Y \rrbracket^{\otimes -} \otimes \llbracket Z,\, X \rrbracket \otimes \llbracket X,\, Y \rrbracket \otimes \llbracket X^Y,\, Z \rrbracket \\
&\overset{416.(1)}{=} \quad \llbracket X,\, Y \rrbracket \otimes \llbracket X^Y,\, Z \rrbracket \,.
\end{aligned}
$$

Ad (2). We write $\boldsymbol{t} := \boldsymbol{t}_{\mathcal{C}} \big|_{\ker \boldsymbol{s}_{\mathcal{C}}}$ and $\boldsymbol{t}' := \boldsymbol{t}_{\mathcal{D}} \big|_{\ker \boldsymbol{s}_{\mathcal{D}}}$.

For $u,\, v \in \ker \boldsymbol{s}_{\mathcal{C}}$, we have

$$(u \otimes v)\mu_F = (u \otimes v)F = uF \otimes vF = u\mu_F \otimes v\mu_F \,.$$

This shows that μ_F is a group morphism.

For $X,\, Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X \otimes Y)\beta_F = (X \otimes Y)F = XF \otimes YF = X\beta_F \otimes Y\beta_F \,.$$

This shows that β_F is a group morphism.

For $\big(I_{\mathcal{C}} \xrightarrow{u} X\big) \in \ker \boldsymbol{s}_{\mathcal{C}}$, we have

$$u(\mu_F \blacktriangle \boldsymbol{t}') = (uF)\, \boldsymbol{t}' = XF = X\beta_F = u(\boldsymbol{t} \blacktriangle \beta_F) \,.$$

This shows $\mu_F \blacktriangle \boldsymbol{t}' = \boldsymbol{t} \blacktriangle \beta_F$.

For $X,\, Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$
\begin{aligned}
\llbracket X,\, Y \rrbracket \mu_F &= \big((Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X}\big)F = \big((Y \otimes X)^{\otimes -}\big)F \otimes (\varphi_{Y,X})F \\
&= (YF \otimes XF)^{\otimes -} \otimes \psi_{YF,XF} = \llbracket XF,\, YF \rrbracket = \llbracket X\beta_F,\, Y\beta_F \rrbracket \,.
\end{aligned}
$$

So $(\mu_F\,, \beta_F)$ is a stable crossed module morphism.

Ad (3). Suppose given $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in $\mathcal{SymMonCat}$.

Consider the symmetric monoidal functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$.

For $u \in \ker \boldsymbol{s}_{\mathcal{C}}$, we have

$$u\, \mu_{\mathrm{id}_{\mathcal{C}}} = u\, \mathrm{id}_{\mathcal{C}} = u \,.$$

This shows $\mu_{\mathrm{id}_{\mathcal{C}}} = \mathrm{id}_{\ker \boldsymbol{s}_{\mathcal{C}}}$.

For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$X\beta_{\mathrm{id}_{\mathcal{C}}} = X\, \mathrm{id}_{\mathcal{C}} = X \,.$$

This shows $\beta_{\mathrm{id}_{\mathcal{C}}} = \mathrm{id}_{\mathrm{Ob}(\mathcal{C})}$.

So we have

$$(\mathrm{id}_{\mathcal{C}})\,\mathrm{StCM} = (\mu_{\mathrm{id}_{\mathcal{C}}}, \beta_{\mathrm{id}_{\mathcal{C}}}) = (\mathrm{id}_{\ker s_{\mathcal{C}}}, \mathrm{id}_{\mathrm{Ob}(\mathcal{C})}) = \mathrm{id}_{\mathcal{C}\,\mathrm{StCM}}\ .$$

We consider the symmetric monoidal functor $F * G\colon\ \mathcal{C} \to \mathcal{E}$.

We consider the group morphism $\mu_{F*G}\colon\ \ker s_{\mathcal{C}} \to \ker s_{\mathcal{E}}$, $u \mapsto uFG$.

For $u \in \ker s_{\mathcal{C}}$, we have

$$u\mu_{F*G} = uFG = (uF)\mu_G = u(\mu_F \triangleleft \mu_G)\,.$$

This shows $\mu_{F*G} = \mu_F \triangleleft \mu_G$.

We consider the group morphism $\beta_{F*G}\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{E})$, $X \mapsto XFG$.

For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$X\beta_{F*G} = XFG = (XF)\beta_G = X(\beta_F \triangleleft \beta_G)\,.$$

This shows $\beta_{F*G} = \beta_F \triangleleft \beta_G$.

So we have

$$\begin{aligned}(F * G)\,\mathrm{StCM} &= (\mu_{F*G}, \beta_{F*G}) = (\mu_F \triangleleft \mu_G, \beta_F \triangleleft \beta_G) = (\mu_F, \beta_F) \triangleleft (\mu_G, \beta_G)\\ &= F\,\mathrm{StCM} \triangleleft G\,\mathrm{StCM}\ .\end{aligned}$$

Therefore, StCM is a functor. □

Remark 418 *Suppose given a symmetric monoidal category $(\mathcal{C}, I, \otimes, \varphi)$.*

We consider the functors

$$\begin{array}{rccl}\mathrm{StCM}\colon & \mathcal{SymMonCat} & \to & \mathcal{StCrMod}\,;\ \text{cf. Lemma 417.(3)}\\ \mathrm{Ord}^{\mathrm{mod}}\colon & \mathcal{StCrMod} & \to & \mathcal{CrMod}\,;\ \text{cf. Lemma 329}\,.\end{array}$$

We consider the stable crossed module $\mathcal{C}\,\mathrm{StCM} = (\ker s, \mathrm{Ob}(\mathcal{C}), \kappa_{\mathcal{C}}, t\,|_{\ker s})$ where

$$\kappa_{\mathcal{C}}\colon\ \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \to \ker s\,,\ (X, Y) \mapsto [\![X, Y]\!] = (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X}\,;$$

cf. Lemma 417.(1).

We consider the stable crossed module

$$\mathcal{C}(\mathrm{StCM} * \mathrm{Ord}^{\mathrm{mod}}) = (\ker s, \mathrm{Ob}(\mathcal{C}), \gamma_{\mathcal{C}\,\mathrm{StCM}}, t\,|_{\ker s})\,,$$

where

$$\gamma_{\mathcal{C}\,\mathrm{StCM}}\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Aut}(\ker \mathcal{s})\,,\ Y \mapsto \big(\big(I \xrightarrow{u} X\big) \mapsto u^Y = u \otimes [\![X,\, Y]\!]\big)\,;$$

cf. Lemma 327.

Suppose given $\big(I \xrightarrow{u} X\big) \in \ker \mathcal{s}$ *and* $Y \in \mathrm{Ob}(\mathcal{C})$.

Then we have

$$Y^{\otimes -} \otimes u \otimes Y = u(Y\gamma_{\mathcal{C}\,\mathrm{StCM}})\,.$$

In other words, there is no ambiguity concerning u^Y*, i.e.*

$$Y^{\otimes -} \otimes u \otimes Y = u^Y = u(Y\gamma_{\mathcal{C}\,\mathrm{StCM}})\,.$$

Proof. We have

$$u(Y\gamma_{\mathcal{C}\,\mathrm{StCM}}) = u \otimes \big(u\,\mathcal{t}\,\big|_{\ker \mathcal{s}}, Y\big)\kappa_{\mathcal{C}} = u \otimes [\![X,\, Y]\!] \overset{416.(5)}{=} u^Y\,.$$

□

Remark 419 *Suppose given a symmetric monoidal category* $(\mathcal{C}, \otimes, I, \varphi)$.

We consider the stable crossed module $\mathcal{C}\,\mathrm{StCM} = (\ker \mathcal{s}_{\mathcal{C}}\,, \mathrm{Ob}(\mathcal{C})\,, \kappa_{\mathcal{C}}\,, \mathcal{t}_{\mathcal{C}}\,\big|_{\ker \mathcal{s}_{\mathcal{C}}})$*; cf. Lemma* 417.(1).

(1) *For* X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$*, we have*

$$[\![X,\, Y]\!]^Z = [\![X^Z,\, Y^Z]\!]\,.$$

(2) *Moreover, for* X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$*, we have*

$$(\varphi_{X,Y})^Z = \varphi_{X^Z,Y^Z}\,.$$

Cf. also Lemma 323 *and Remark* 418.

Proof. Ad (1). For X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, we have

$$[\![X,\, Y]\!]^Z \overset{418}{=} [\![X,\, Y]\!]\big(Z\gamma_{\mathcal{C}\,\mathrm{StCM}}\big) \overset{323.(10)}{=} [\![X^Z,\, Y^Z]\!]\,.$$

Ad (2). Suppose given X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$.

Note that we have

$$\varphi_{X,Y} \overset{416}{=} X \otimes Y \otimes [\![Y,\, X]\!]\,.$$

So we have

$$\begin{aligned}(\varphi_{X,Y})^Z &= \big(X \otimes Y \otimes [\![Y,\, X]\!]\big)^Z = X^Z \otimes Y^Z \otimes [\![Y,\, X]\!]^Z \overset{(2)}{=} X^Z \otimes Y^Z \otimes [\![Y^Z,\, X^Z]\!] \\ &= \varphi_{X^Z,Y^Z}\,.\end{aligned}$$

□

19.4 The equivalent categories *SymMonCat* and *StCrMod*

We consider the following functors.

$$\begin{array}{lrcll} \mathrm{SymCat}\colon & \mathcal{StCrMod} & \to & \mathcal{SymMonCat}; & \text{cf. Lemma 415} \\ \mathrm{StCM}\colon & \mathcal{SymMonCat} & \to & \mathcal{StCrMod}; & \text{cf. Lemma 417} \end{array}$$

Remark 420 (The stable crossed module $S\,\mathrm{SymCat}\,\mathrm{StCM}$)

(1) Let $S = (\mathrm{M}_S\,, \mathrm{B}_S\,, \kappa_S\,, \mathrm{f}_S)$ be a stable crossed module. We consider the stable crossed module $S\,\mathrm{SymCat}\,\mathrm{StCM}$.

Recall that we have the symmetric monoidal category $\big(S\,\mathrm{SymCat}, 1_{\mathrm{B}_S}\,, (\,\cdot\,), \varphi\big)$, where

$$\begin{aligned}\mathrm{Ob}(S\,\mathrm{SymCat}) &= \mathrm{B}_S \\ \mathrm{Mor}(S\,\mathrm{SymCat}) &= \mathrm{B}_S \ltimes \mathrm{M}_S\,,\end{aligned}$$

and where

$$\varphi_{b,c} = \big(b \cdot c, [\![c,\, b]\!]\big) \in \mathrm{Mor}(S\,\mathrm{SymCat})$$

for $b,\, c \in \mathrm{B}_S$; cf. Lemma 413.

Then, by Lemma 417, we have the stable crossed module

$$\begin{aligned}S\,\mathrm{SymCat}\,\mathrm{StCM} &= (\mathrm{M}_{S\,\mathrm{SymCat}\,\mathrm{StCM}}\,,\ \mathrm{B}_{S\,\mathrm{SymCat}\,\mathrm{StCM}}\,,\ \kappa_{S\,\mathrm{SymCat}\,\mathrm{StCM}}\,,\ \mathrm{f}_{S\,\mathrm{SymCat}\,\mathrm{StCM}}) \\ &= \big(\ker s\,, \mathrm{Ob}(S\,\mathrm{SymCat})\ , \kappa_{S\,\mathrm{SymCat}}\,, t\big|_{\ker s}\big) \\ &= \big(\ker s\,, \mathrm{B}_S\,, \kappa_{S\,\mathrm{SymCat}}\,, t\big|_{\ker s}\big)\,,\end{aligned}$$

where we have the group morphisms $s\colon \mathrm{B}_S \ltimes \mathrm{M}_S \to \mathrm{B}_S\,,\ (b, m) \mapsto b$ and $t\colon \mathrm{B}_S \ltimes \mathrm{M}_S \to \mathrm{B}_S\,,\ (b, m) \mapsto b \cdot m\mathrm{f}_S$.

For $b,\, c \in \mathrm{B}_S$, note that we have

$$(b, c)\kappa_{S\,\mathrm{SymCat}} \overset{417.(1)}{=} \big((c, 1) \cdot (b, 1)\big)^- \cdot \varphi_{c,b} \overset{413}{=} (c \cdot b, 1)^- \cdot \big(c \cdot b, [\![b,\, c]\!]\big) = \big(1, [\![b,\, c]\!]\big)\,.$$

So we have

$$\mathrm{M}_{S\,\mathrm{SymCat}\,\mathrm{StCM}} = \ker s = \{(1,m) \in \mathrm{B}_S \ltimes \mathrm{M}_S \colon m \in \mathrm{M}_S\} = 1 \ltimes \mathrm{M}_S \leqslant \mathrm{B}_S \ltimes \mathrm{M}_S$$
$$\mathrm{B}_{S\,\mathrm{SymCat}\,\mathrm{StCM}} = \mathrm{Ob}(S\,\mathrm{SymCat}) = \mathrm{B}_S$$

and

$$\kappa_{S\,\mathrm{SymCat}} \colon \mathrm{B}_S \times \mathrm{B}_S \to 1 \ltimes \mathrm{M}_S\,,\ (b,c) \mapsto (b,c)\kappa_{S\,\mathrm{SymCat}} = \big(1, [\![b,\, c]\!]\big) =: [\![b,\, c]\!]'$$
$$t\big|_{\ker s} \colon 1 \ltimes \mathrm{M}_S \to \mathrm{B}_S\,,\ (1,m) \mapsto m\mathrm{f}_S\,.$$

(2) Suppose given a stable crossed module morphism $\rho = (\mu, \beta) \colon S \to \tilde{S}$; cf. Definition 324.
We consider the stable crossed modules $S\,\mathrm{SymCat}\,\mathrm{StCM}$ and $\tilde{S}\,\mathrm{SymCat}\,\mathrm{StCM}$; cf. (1).
We consider the stable crossed module morphism

$$\rho\,\mathrm{SymCat}\,\mathrm{StCM} \colon S\,\mathrm{SymCat}\,\mathrm{StCM} \to \tilde{S}\,\mathrm{SymCat}\,\mathrm{StCM}\ .$$

Recall that we have the symmetric monoidal functor

$$\rho\,\mathrm{SymCat} \colon S\,\mathrm{SymCat} \to \tilde{S}\,\mathrm{SymCat}\,,\ \big(b \xrightarrow{(b,m)} b \cdot m\mathrm{f}_S\big) \mapsto \big(b\beta \xrightarrow{(b\beta, m\mu)} b\beta \cdot m\mu\,\mathrm{f}_{\tilde{S}}\big)\,;$$

cf. Lemmas 415, 414.
Then, by Lemma 417.(2), we have the stable crossed module morphism

$$\rho\,\mathrm{SymCat}\,\mathrm{StCM} = \big(\mu_{\rho\,\mathrm{SymCat}}\,, \beta_{\rho\,\mathrm{SymCat}}\big)\,,$$

where

$$\mu_{\rho\,\mathrm{SymCat}} \colon 1 \ltimes \mathrm{M}_S \to 1 \ltimes \mathrm{M}_{\tilde{S}}\,,\ (1,m) \mapsto (1,m)(\rho\,\mathrm{SymCat}) = (1, m\mu)$$
$$\beta_{\rho\,\mathrm{SymCat}} \colon \mathrm{B}_S \to \mathrm{B}_{\tilde{S}}\,,\ b \mapsto b(\rho\,\mathrm{SymCat}) = b\beta\,.$$

Lemma 421 (The functor $\mathrm{SymCat} * \mathrm{StCM}$)

(1) *Let $S = (\mathrm{M}_S\,, \mathrm{B}_S\,, \kappa_S\,, \mathrm{f}_S)$ be a stable crossed module.*
We have the stable crossed module isomorphism

$$S\hat{\psi} = (\iota_{\mathrm{M}_S}\,, \mathrm{id}_{\mathrm{B}_S}) \colon S \xrightarrow{\sim} S\,\mathrm{SymCat}\,\mathrm{StCM}\,,$$

where

$$\iota_{\mathrm{M}_S} \colon \mathrm{M}_S \xrightarrow{\sim} 1 \ltimes \mathrm{M}_S\,, m \mapsto (1,m)\,.$$

(2) *We have the isotransformation*

$$\hat{\psi} = \big(S \xrightarrow{(\iota_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})} S\,\mathrm{SymCat}\,\mathrm{StCM}\big)_{S \in \mathrm{Ob}(\mathcal{StCrMod})} \colon \mathrm{id}_{\mathcal{StCrMod}} \to \mathrm{SymCat} * \mathrm{StCM}\,;$$

cf. (1).

Proof. Ad (1). We have that ι_{M_S} and $\mathrm{id}_{\mathrm{B}_S}$ are group morphisms.

Then for $m \in \mathrm{M}_S$, we have

$$m\big(\iota_{\mathrm{M}_S} \blacktriangleleft t\big|_{\ker s}\big) = (1, m)t\big|_{\ker s} = m\mathrm{f}_S = m(\mathrm{f}_S \blacktriangleleft \mathrm{id}_{\mathrm{B}_S})\,.$$

This shows $\iota_{\mathrm{M}_S} \blacktriangleleft t\big|_{\ker s} = \mathrm{f}_S \blacktriangleleft \mathrm{id}_{\mathrm{B}_S}$.

For b, $c \in \mathrm{B}_S$, we have

$$[\![b,\, c]\!]\iota_{\mathrm{M}_S} = \big(1, [\![b,\, c]\!]\big) = [\![b,\, c]\!]' = [\![b\,\mathrm{id}_{\mathrm{B}_S},\, c\,\mathrm{id}_{\mathrm{B}_S}]\!]'\,.$$

So $(\iota_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})$ is a stable crossed module morphism.

Since ι_{M_S} and $\mathrm{id}_{\mathrm{B}_S}$ are group isomorphisms, we have that $(\iota_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})$ is a stable crossed module isomorphism; cf. Remark 326.(2).

Ad (2). Suppose given $\big(S \xrightarrow{\rho} T\big)$ in $\mathcal{StCrMod}$.

We have $\rho = (\mu, \beta)$, where $\mu \colon \mathrm{M}_S \to \mathrm{M}_T$ and $\beta \colon \mathrm{B}_S \to \mathrm{B}_T$.

We show that we have the following commutative diagram.

$$\begin{array}{ccc} S & \xrightarrow{(\iota_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S})} & S(\mathrm{SymCat} * \mathrm{StCM}) \\ {\scriptstyle (\mu,\beta)}\downarrow & & \downarrow{\scriptstyle (\mu,\beta)(\mathrm{SymCat} * \mathrm{StCM})} \\ T & \xrightarrow[(\iota_{\mathrm{M}_T}, \mathrm{id}_{\mathrm{B}_T})]{} & T(\mathrm{SymCat} * \mathrm{StCM}) \end{array}$$

Recall that we write $(\mu_{S\,\mathrm{SymCat}}, \beta_{S\,\mathrm{SymCat}}) = (\mu, \beta)(\mathrm{SymCat} * \mathrm{StCM})$.

For $m \in \mathrm{M}_S$, we have

$$m(\iota_{\mathrm{M}_S} \blacktriangleleft \mu_{\rho\,\mathrm{SymCat}}) = (1, m)\mu_{\rho\,\mathrm{SymCat}} \overset{420.(2)}{=} (1, m\mu) = (m\mu)\iota_{\mathrm{M}_T} = m(\mu \blacktriangleleft \iota_{\mathrm{M}_T})\,.$$

This shows $\iota_{\mathrm{M}_S} \blacktriangleleft \mu_{\rho\,\mathrm{SymCat}} = \mu \blacktriangleleft \iota_{\mathrm{M}_T}$.

For $b \in \mathrm{B}_S$, we have

$$b(\mathrm{id}_{\mathrm{B}_S} \triangleleft \beta_{\rho\,\mathrm{SymCat}}) = b\beta_{\rho\,\mathrm{SymCat}} \overset{420.(2)}{=} b\beta = b(\beta \triangleleft \mathrm{id}_{\mathrm{B}_T})\,.$$

This shows $\mathrm{id}_{\mathrm{B}_S} \triangleleft \beta_{\rho\,\mathrm{SymCat}} = \beta \triangleleft \mathrm{id}_{\mathrm{B}_T}$.

So we have

$$\begin{aligned}(\iota_{\mathrm{M}_S}, \mathrm{id}_{\mathrm{B}_S}) \triangleleft (\mu_{\rho\,\mathrm{SymCat}}, \beta_{\rho\,\mathrm{SymCat}}) &= (\iota_{\mathrm{M}_S} \triangleleft \mu_{\rho\,\mathrm{SymCat}}, \mathrm{id}_{\mathrm{B}_S} \triangleleft \beta_{\rho\,\mathrm{SymCat}}) = (\mu \triangleleft \iota_{\mathrm{M}_T}, \beta \triangleleft \mathrm{id}_{\mathrm{B}_T}) \\ &= (\mu, \beta) \triangleleft (\iota_{\mathrm{M}_T}, \mathrm{id}_{\mathrm{B}_T})\,.\end{aligned}$$

So $\hat{\psi}$ is a transformation. Using (1), we conclude that $\hat{\psi}$ is an isotransformation. □

Remark 422 (The symmetric monoidal category $\mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}$)

(1) Suppose given a symmetric monoidal category $(\mathcal{C}, I, \otimes, \varphi)$; cf. Definition 406.

We consider the symmetric monoidal category $\mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}$.

Recall that we have the stable crossed module

$$\mathcal{C}\,\mathrm{StCM} = (\mathrm{M}_{\mathcal{C}\,\mathrm{StCM}}, \mathrm{B}_{\mathcal{C}\,\mathrm{StCM}}, \kappa_{\mathcal{C}\,\mathrm{StCM}}, \mathrm{f}_{\mathcal{C}\,\mathrm{StCM}})\,,$$

where

$$\begin{aligned}\mathrm{M}_{\mathcal{C}\,\mathrm{StCM}} &= \ker \mathcal{s} = \left\{ \left(I \xrightarrow{u} Y\right) \in \mathrm{Mor}(\mathcal{C}) \right\} \leqslant \mathrm{Mor}(\mathcal{C}) \\ \mathrm{B}_{\mathcal{C}\,\mathrm{StCM}} &= \mathrm{Ob}(\mathcal{C})\ ,\end{aligned}$$

and where

$$\begin{aligned}\kappa_{\mathcal{C}}\colon\ \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) &\to \ker \mathcal{s}\,,\ (X, Y) \mapsto [\![X, Y]\!] = (Y \otimes X)^{\otimes -} \otimes \varphi_{Y,X} \\ \mathrm{f}_{\mathcal{C}\,\mathrm{StCM}} = \mathcal{t}\,\big|_{\ker \mathcal{s}}\colon\ \ker \mathcal{s} &\to \mathrm{Ob}(\mathcal{C})\ ,\ \left(1 \xrightarrow{u} Y\right) \mapsto Y\,;\end{aligned}$$

cf. Lemma 417.

By Lemma 413, we have the symmetric monoidal category $(\mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}, I, \otimes, \tilde{\varphi})$, where

$$\begin{aligned}\mathrm{Ob}(\mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}) &= \mathrm{B}_{\mathcal{C}\,\mathrm{StCM}} = \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}(\mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}) &= \mathrm{B}_{\mathcal{C}\,\mathrm{StCM}} \ltimes \mathrm{M}_{\mathcal{C}\,\mathrm{StCM}} = \mathrm{Ob}(\mathcal{C}) \ltimes \ker \mathcal{s}\,,\end{aligned}$$

and where

$$\tilde{\varphi}_{X,Y} = \left(X \otimes Y, [\![Y, X]\!]\right) = \left(X \otimes Y, (X \otimes Y)^{\otimes -} \otimes \varphi_{X,Y}\right)\colon X \otimes Y \to Y \otimes X$$

for X, $Y \in \mathrm{Ob}(\mathcal{C})$.

Suppose given $\big(X, \big(I \xrightarrow{u} Y\big)\big) \in \mathrm{Mor}(\mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}) = \mathrm{Ob}(\mathcal{C}) \ltimes \ker \boldsymbol{s}$.

Note that we have

$$\begin{aligned}(X,u)\,\boldsymbol{s} &= X\\ (X,u)\,\boldsymbol{t} &= X \otimes u\,\mathrm{f}_{\mathcal{C}\,\mathrm{StCM}} = X \otimes u\,\boldsymbol{t} = X \otimes Y\,.\end{aligned}$$

So we have $\big(X, \big(I \xrightarrow{u} Y\big)\big) \colon X \to X \otimes Y$.

(2) Suppose given symmetric monoidal categories $(\mathcal{C}, I_{\mathcal{C}}\,, \otimes, \varphi_{\mathcal{C}})$ and $(\mathcal{D}, I_{\mathcal{D}}\,, \otimes, \varphi_{\mathcal{D}})$. Suppose given a symmetric monoidal functor $F \colon\ \mathcal{C} \to \mathcal{D}$.

We consider the symmetric monoidal categories $\mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}$ and $\mathcal{D}\,\mathrm{StCM}\,\mathrm{SymCat}$; cf. (1). We consider the symmetric monoidal functor

$$F\,\mathrm{StCM}\,\mathrm{SymCat} \colon\ \mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat} \to \mathcal{D}\,\mathrm{StCM}\,\mathrm{SymCat}\ .$$

Recall that we have the stable crossed module morphism

$$F\,\mathrm{StCM} = (\mu_F\,, \beta_F) \colon\ \mathcal{C}\,\mathrm{StCM} \to \mathcal{D}\,\mathrm{StCM}\,,$$

where

$$\begin{aligned}&\mu_F \colon\ \ker \boldsymbol{s}_{\mathcal{C}} \to \ker \boldsymbol{s}_{\mathcal{D}}\,,\ \big(I_{\mathcal{C}} \xrightarrow{u} Y\big) \mapsto \big(I_{\mathcal{D}} \xrightarrow{uF} YF\big)\\ &\beta_F \colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})\ ,\ X \mapsto XF\,;\end{aligned}$$

cf. Lemma 417.(2).

Then, by Lemma 414, we have the symmetric monoidal functor

$$\begin{array}{rccc} F\,\mathrm{StCM}\,\mathrm{SymCat}\colon & \mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat} & \to & \mathcal{D}\,\mathrm{StCM}\,\mathrm{SymCat}\\ & \Big(X \xrightarrow{(X,\, I_{\mathcal{C}} \xrightarrow{u} Y)} X \otimes Y\Big) & \mapsto & \Big(XF \xrightarrow{(XF,\, I_{\mathcal{D}} \xrightarrow{uF} YF)} XF \otimes YF\Big)\,. \end{array}$$

Lemma 423 (The functor $\mathrm{StCM} * \mathrm{SymCat}$)

(1) *Let $\mathcal{C} = (\mathcal{C}, I, \otimes, \varphi)$ be a symmetric monoidal category.*

We have the symmetric monoidal isofunctor

$$\begin{array}{rccc} \mathcal{C}\psi\colon & \mathcal{C} & \to & \mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}\\ & \big(X \xrightarrow{u} Y\big) & \mapsto & \Big(X \xrightarrow{(X,\, I \xrightarrow{X^{\otimes -} \otimes u} X^{\otimes -} \otimes Y)} Y\Big)\,. \end{array}$$

(2) *We have the isotransformation*

$$\psi = \big(\mathcal{C} \xrightarrow{\mathcal{C}\psi} \mathcal{C}\,\mathrm{StCM}\,\mathrm{SymCat}\big)_{\mathcal{C}\in\mathrm{Ob}(\mathcal{SymMonCat})} \colon \mathrm{id}_{\mathcal{SymMonCat}} \to \mathrm{StCM} * \mathrm{SymCat}\,;$$

cf. (1).

Proof. Ad (1). *We show that $\mathcal{C}\psi$ is a functor.*

Suppose given $X \in \mathrm{Ob}(\mathcal{C})$. We have

$$\mathrm{id}_X(\mathcal{C}\psi) = (X, X^{\otimes -} \otimes \mathrm{id}_X) = (X, \mathrm{id}_{X^{\otimes -}} \otimes \mathrm{id}_X) = (X, \mathrm{id}_{X^{\otimes -}\otimes X}) = (X, \mathrm{id}_I) = \mathrm{id}_{X(\mathcal{C}\psi)}\,.$$

Suppose given $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$. We have

$$\begin{aligned}(u \mathbin{\blacktriangle} v)\,\mathcal{C}\psi &= \big(X, X^{\otimes -} \otimes (u \mathbin{\blacktriangle} v)\big) \overset{160.(3)}{=} \big(X, X^{\otimes -} \otimes (u \otimes Y^{\otimes -} \otimes v)\big)\\ &= \big(X, (X^{\otimes -} \otimes u) \otimes (Y^{\otimes -} \otimes v)\big) = (X, X^{\otimes -} \otimes u) \mathbin{\blacktriangle} (Y, Y^{\otimes -} \otimes v) = u(\mathcal{C}\psi) \mathbin{\blacktriangle} v(\mathcal{C}\psi)\,.\end{aligned}$$

This shows that $\mathcal{C}\psi$ is a functor.

We show that $\mathcal{C}\psi$ is an isofunctor.

It suffices to show that $\mathrm{Mor}(\mathcal{C}\psi)$ is bijective; cf. §1.1.3 item 7.

We consider the map

$$\psi' \colon \mathrm{Mor}\big(\mathcal{C}(\mathrm{StCM} * \mathrm{SymCat})\big) \to \mathrm{Mor}(\mathcal{C})\ ,\ \big(X \xrightarrow{(X, I \xrightarrow{u} Y)} X \otimes Y\big) \mapsto \big(X \xrightarrow{X\otimes u} X \otimes Y\big)\,.$$

For $(X, I \xrightarrow{u} Y) \in \mathrm{Mor}\big(\mathcal{C}(\mathrm{StCM} * \mathrm{SymCat})\big)$, we have

$$\begin{aligned}(X, u)(\psi' \mathbin{\blacktriangle} \mathrm{Mor}(\mathcal{C}\psi)) &= (X \otimes u)\,\mathrm{Mor}(\mathcal{C}\psi) = \big(X \otimes I, (X \otimes I)^{\otimes -} \otimes X \otimes u\big) = (X, X^{\otimes -} \otimes X \otimes u)\\ &= (X, u)\,.\end{aligned}$$

This shows $\psi' \mathbin{\blacktriangle} \mathrm{Mor}(\Psi) = \mathrm{id}_{\mathrm{Mor}(\mathcal{C}(\mathrm{StCM} * \mathrm{SymCat}))}$.

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{C})$, we have

$$u\big(\mathrm{Mor}(\mathcal{C}\psi) \mathbin{\blacktriangle} \psi'\big) = (X, X^{\otimes -} \otimes u)\psi' = X \otimes X^{\otimes -} \otimes u = u\,.$$

This shows $\mathrm{Mor}(\mathcal{C}\psi) \mathbin{\blacktriangle} \psi' = \mathrm{id}_{\mathcal{C}}$.

So the map $\mathrm{Mor}(\mathcal{C}\psi)$ is bijective. Therefore, $\mathcal{C}\psi$ is an isofunctor.

We show that $\mathcal{C}\psi$ is monoidal.

For $\big(X \xrightarrow{u} X'\big)$, $\big(Y \xrightarrow{v} Y'\big) \in \mathrm{Mor}(\mathcal{C})$, we have

$$\begin{aligned} u(\mathcal{C}\psi) \otimes v(\mathcal{C}\psi) &= (X, X^{\otimes -} \otimes u) \otimes (Y, Y^{\otimes -} \otimes v) \overset{413,418}{=} \big(X \otimes Y,\, (X^{\otimes -} \otimes u)^Y \otimes Y^{\otimes -} \otimes v\big) \\ &= (X \otimes Y,\, Y^{\otimes -} \otimes X^{\otimes -} \otimes u \otimes v) = \big(X \otimes Y,\, (X \otimes Y)^{\otimes -} \otimes u \otimes v\big) \\ &= (u \otimes v)\, \mathcal{C}\psi\,. \end{aligned}$$

We show that $\mathcal{C}\psi$ is a symmetric monoidal functor.

For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(\varphi_{X,Y})(\mathcal{C}\psi) = \big(X \otimes Y, (X \otimes Y)^{\otimes -} \otimes \varphi_{X,Y}\big) = \tilde{\varphi}_{X,Y} = \tilde{\varphi}_{X(\mathcal{C}\psi),Y(\mathcal{C}\psi)}\,.$$

Altogether, $\mathcal{C}\psi$ is a symmetric monoidal isofunctor.

Ad (2). Suppose given $\big(\mathcal{C} \xrightarrow{F} \mathcal{D}\big)$ in *SymMonCat*.

We show that we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\mathcal{C}\psi\quad} & \mathcal{C}(\mathrm{StCM} * \mathrm{SymCat}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F(\mathrm{StCM} * \mathrm{SymCat})} \\ \mathcal{D} & \xrightarrow[\quad\mathcal{D}\psi\quad]{} & \mathcal{D}(\mathrm{StCM} * \mathrm{SymCat}) \end{array}$$

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{C})$, we have

$$\begin{aligned} u(\mathcal{C}\psi * F(\mathrm{StCM} * \mathrm{SymCat})) &\overset{(1)}{=} (X, X^{\otimes -} \otimes u)\big(F(\mathrm{StCM} * \mathrm{SymCat})\big) \\ &\overset{422.(2)}{=} \big(XF, (X^{\otimes -} \otimes u)F\big) \overset{161}{=} \big(XF, (X^{\otimes -})F \otimes uF\big) \\ &\overset{162.(3)}{=} \big(XF, (XF)^{\otimes -} \otimes uF\big) \overset{(1)}{=} (uF)(\mathcal{D}\psi) = u(F * \mathcal{D}\psi)\,. \end{aligned}$$

This shows $\mathcal{C}\psi * F(\mathrm{StCM} * \mathrm{SymCat}) = F * \mathcal{D}\psi$.

So ψ is a transformation. Using (1), we conclude that ψ is an isotransformation. □

Proposition 424 (The equivalences SymCat and StCM)

By Lemma 421.(2)*. we have the isotransformation* $\hat{\psi}\colon \mathrm{id}_{\mathcal{StCrMod}} \xrightarrow{\sim} \mathrm{SymCat} * \mathrm{StCM}$.

By Lemma 423.(2)*, we have the isotransformation* $\psi\colon \mathrm{id}_{\mathcal{SymMonCat}} \xrightarrow{\sim} \mathrm{StCM} * \mathrm{SymCat}$.

So SymCat *and* StCM *are equivalences.*

In particular, the categories $\mathcal{StCrMod}$ *and* $\mathcal{SymMonCat}$ *are equivalent.*

$$\mathcal{StCrMod} \underset{\text{StCM}}{\overset{\text{SymCat}}{\underset{\sim}{\rightleftarrows}}} \mathcal{SymMonCat}$$

19.5 The ordinarification functor $\mathrm{Ord}^{\mathrm{cat}} : \mathcal{SymMonCat} \to \mathcal{InvMonCat}$

Remark 425 (The ordinarification functor $\mathrm{Ord}^{\mathrm{cat}}$)

We consider the categories $\mathcal{InvMonCat}$ *and* $\mathcal{SymMonCat}$*; cf. Definitions* 163 *and* 412*.*

We have the ordinarification functor

$$\begin{array}{rccll} \mathrm{Ord}^{\mathrm{cat}}\colon & \mathcal{SymMonCat} & \to & \mathcal{InvMonCat} & \\ & (\mathcal{C}, I, \otimes, \varphi) & \mapsto & \mathcal{C}\mathrm{Ord}^{\mathrm{cat}} := (\mathcal{C}, I, \otimes) & \text{for } \mathcal{C} \in \mathrm{Ob}\big(\mathcal{SymMonCat}\big) \\ & \big(\mathcal{C} \xrightarrow{F} \mathcal{D}\big) & \mapsto & F\mathrm{Ord}^{\mathrm{cat}} := F & \text{for } F \in \mathrm{Mor}\big(\mathcal{SymMonCat}\big)\ . \end{array}$$

Proof. Note that for a symmetric monoidal category $(\mathcal{C}, I, \otimes, \varphi)$ we have that $(\mathcal{C}, I, \otimes)$ is an invertible monoidal category; cf. Definition 406.

Moreover, note that every symmetric monoidal functor is also a monoidal functor; cf. Definition 410.

Suppose given $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in $\mathcal{SymMonCat}$.

We have

$$(\mathrm{id}_{\mathcal{C}})\mathrm{Ord}^{\mathrm{cat}} = \mathrm{id}_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}\mathrm{Ord}^{\mathrm{cat}}}\ .$$

We have

$$(F * G)\mathrm{Ord}^{\mathrm{cat}} = F * G = F\mathrm{Ord}^{\mathrm{cat}} * G\mathrm{Ord}^{\mathrm{cat}}\ .$$

So $\mathrm{Ord}^{\mathrm{cat}}$ is a functor. □

Lemma 426 *We consider the following functors.*

$$\begin{array}{rrcll}
\mathrm{Ord}^{\mathrm{mod}}: & \mathcal{StCrMod} & \to & \mathcal{CrMod}; & \textit{cf. Lemma } 329 \\
\mathrm{Ord}^{\mathrm{cat}}: & \mathcal{SymMonCat} & \to & \mathcal{InvMonCat}; & \textit{cf. Remark } 425 \\
\mathrm{Cat}: & \mathcal{CrMod} & \to & \mathcal{InvMonCat}; & \textit{cf. Remark } 165 \\
\mathrm{SymCat}: & \mathcal{StCrMod} & \to & \mathcal{SymMonCat}; & \textit{cf. Lemma } 415 \\
\mathrm{CM}: & \mathcal{InvMonCat} & \to & \mathcal{CrMod}; & \textit{cf. Remark } 166 \\
\mathrm{StCM}: & \mathcal{SymMonCat} & \to & \mathcal{StCrMod}; & \textit{cf. Lemma } 417
\end{array}$$

(1) *We have the following commutative diagram.*

$$\begin{array}{ccc}
\mathcal{StCrMod} & \xrightarrow{\mathrm{Ord}^{\mathrm{mod}}} & \mathcal{CrMod} \\
\Big\downarrow{\scriptstyle \mathrm{SymCat}} & & \Big\downarrow{\scriptstyle \mathrm{Cat}} \\
\mathcal{SymMonCat} & \xrightarrow[\mathrm{Ord}^{\mathrm{cat}}]{} & \mathcal{InvMonCat}
\end{array}$$

(2) *We have the following commutative diagram.*

$$\begin{array}{ccc}
\mathcal{SymMonCat} & \xrightarrow{\mathrm{Ord}^{\mathrm{cat}}} & \mathcal{InvMonCat} \\
\Big\downarrow{\scriptstyle \mathrm{StCM}} & & \Big\downarrow{\scriptstyle \mathrm{CM}} \\
\mathcal{StCrMod} & \xrightarrow[\mathrm{Ord}^{\mathrm{mod}}]{} & \mathcal{CrMod}
\end{array}$$

Proof. Ad (1). Suppose given a stable crossed module morphism $(\mu, \beta) \in \mathrm{Mor}(\mathcal{StCrMod})$.
We have

$$(\mu,\beta)(\mathrm{Ord}^{\mathrm{mod}} * \mathrm{Cat}) = (\mu,\beta)\,\mathrm{Cat} = (\mu,\beta)\,\mathrm{SymCat} = (\mu,\beta)(\mathrm{SymCat} * \mathrm{Ord}^{\mathrm{cat}})\,.$$

This shows $\mathrm{Ord}^{\mathrm{mod}} * \mathrm{Cat} = \mathrm{SymCat} * \mathrm{Ord}^{\mathrm{cat}}$.

Ad (2). Suppose given a symmetric monoidal functor $F \in \mathrm{Mor}\big(\mathcal{SymMonCat}\big)$.
We have

$$F(\mathrm{Ord}^{\mathrm{cat}} * \mathrm{CM}) = F\,\mathrm{CM} = (\mu_F\,, \beta_F) = (\mu_F\,, \beta_F)\mathrm{Ord}^{\mathrm{mod}} = F(\mathrm{StCM} * \mathrm{Ord}^{\mathrm{mod}})\,.$$

This shows $\mathrm{Ord}^{\mathrm{cat}} * \mathrm{CM} = \mathrm{StCM} * \mathrm{Ord}^{\mathrm{mod}}$. □

19.6 Inner automorphisms

Remark 427 *Suppose given a symmetric monoidal category* $(\mathcal{C}, I, \otimes, \varphi)$.
Suppose given $A \in \mathrm{Ob}(\mathcal{C})$.
We have the symmetric monoidal autofunctor given by

$$F_A \colon \mathcal{C} \to \mathcal{C}\,,\ \left(X \xrightarrow{u} Y\right) \mapsto \left(X^A \xrightarrow{u^A} Y^A\right) = \left(A^{\otimes -} \otimes X \otimes A \xrightarrow{\mathrm{id}_{A^{\otimes -}} \otimes u \otimes \mathrm{id}_A} A^{\otimes -} \otimes Y \otimes A\right);$$

cf. Definition 158.

Proof. We show that F_A *is a functor.*
Suppose given $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$.
We have

$$(\mathrm{id}_X)F_A = (\mathrm{id}_X)^A = \mathrm{id}_{A^{\otimes -}} \otimes \mathrm{id}_X \otimes \mathrm{id}_A = \mathrm{id}_{A^{\otimes -} \otimes X \otimes A} = \mathrm{id}_{X^A} = \mathrm{id}_{XF_A}\,.$$

We have

$$\begin{aligned}(u \mathbin{\blacktriangle} v)F_A = (u \mathbin{\blacktriangle} v)^A = A^{\otimes -} \otimes (u \mathbin{\blacktriangle} v) \otimes A &\overset{164}{=} (A^{\otimes -} \otimes u \otimes A) \mathbin{\blacktriangle} (A^{\otimes -} \otimes v \otimes A) = u^A \mathbin{\blacktriangle} v^A \\ &= uF_A \mathbin{\blacktriangle} vF_A\,.\end{aligned}$$

This shows that F_A is a functor.
We show that F_A *is an autofunctor.*
We consider the functor

$$F_{A^{\otimes -}} \colon \mathcal{C} \to \mathcal{C}\,,\ \left(X \xrightarrow{u} Y\right) \mapsto \left(X^{A^{\otimes -}} \xrightarrow{u^{A^{\otimes -}}} Y^{A^{\otimes -}}\right).$$

We show that $(F_A)^- \overset{!}{=} F_{A^{\otimes -}}$.
Suppose given $u \in \mathrm{Mor}(\mathcal{C})$.
We have

$$u(F_A * F_{A^{\otimes -}}) = (u^A)F_{A^{\otimes -}} = (u^A)^{A^{\otimes -}} = u^{A \otimes A^{\otimes -}} = u^I = u\,.$$

This shows $F_A * F_{A^{\otimes -}} = \mathrm{id}_{\mathcal{C}}$.
Similarly, we have $F_{A^{\otimes -}} * F_A = \mathrm{id}_{\mathcal{C}}$.
This shows $(F_A)^- = F_{A^{\otimes -}}$.

We show that F_A is monoidal.

For u, $v \in \mathrm{Mor}(\mathcal{C})$, we have

$$(u \otimes v)F_A = (u \otimes v)^A = A^{\otimes -} \otimes (u \otimes v) \otimes A = (A^{\otimes -} \otimes u \otimes A) \otimes (A^{\otimes -} \otimes v \otimes A) = u^A \otimes v^A$$
$$= uF_A \otimes vF_A \,.$$

We show that F_A is a symmetric monoidal functor.

For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(\varphi_{X,Y})F_A = (\varphi_{X,Y})^A \overset{419.(2)}{=} \varphi_{X^A,Y^A} = \varphi_{XF_A,YF_A} \,.$$

Altogether, F_A is a symmetric monoidal autofunctor. □

Remark 428 *Suppose given a stable crossed module S.*

Suppose given $c \in \mathrm{B}_S$.

We have the stable crossed module isomorphism $(\mu_c, \beta_c)\colon S \to S$ given by

$$\mu_c\colon \mathrm{M}_S \to \mathrm{M}_S\,,\ m \mapsto m^c = m(c\gamma_S)$$
$$\beta_c\colon \mathrm{B}_S \to \mathrm{B}_S\,,\ b \mapsto b^c = c^- \cdot b \cdot c\,.$$

Proof. We show that μ_c and β_c are group morphisms.

For m, $n \in \mathrm{M}_S$, we have

$$(m \cdot n)\mu_c = (m \cdot n)^c \overset{323.(7)}{=} m^c \cdot n^c = m\mu_c \cdot n\mu_c \,.$$

For a, $b \in \mathrm{B}_S$, we have

$$(a \cdot b)\beta_c = (a \cdot b)^c = a^c \cdot b^c = a\beta_c \cdot b\beta_c \,.$$

We show that (μ_c, β_c) is a stable crossed module morphism.

For $m \in \mathrm{M}_S$, we have

$$m(\mu_c \blacktriangle \mathrm{f}_S) = (m^c)\mathrm{f}_S \overset{323.(1)}{=} (m\mathrm{f}_S)^c = m(\mathrm{f}_S \blacktriangle \beta_c) \,.$$

This shows that $\mu_c \blacktriangle \mathrm{f}_S = \mathrm{f}_S \blacktriangle \beta_c$.

For a, $b \in \mathrm{B}_S$, we have

$$[\![a, b]\!]\mu_c = [\![a, b]\!]^c \overset{323.(10)}{=} [\![a^c, b^c]\!] = [\![a\beta_c, b\beta_c]\!] \,.$$

This shows that $(\mu_c\,,\beta_c)$ is a stable crossed module morphism.

We show that $(\mu_c\,,\beta_c)$ is a stable crossed module isomorphism.

We consider the stable crossed module morphism $(\mu_{c^-}\,,\beta_{c^-})\colon S \to S$, where

$$\mu_{c^-}\colon \mathrm{M}_S \to \mathrm{M}_S\,,\ m \mapsto m^{c^-}$$
$$\beta_{c^-}\colon \mathrm{B}_S \to \mathrm{B}_S\,,\ b \mapsto b^{c^-}\,.$$

We show that $(\mu_c\,,\beta_c)^- \overset{!}{=} (\mu_{c^-}\,,\beta_{c^-})$.

For $m \in \mathrm{M}_S$, we have

$$m(\mu_c \mathbin{\blacktriangle} \mu_{c^-}) = (m^c)\mu_{c^-} = (m^c)^{c^-} \overset{323.(6)}{=} m^{c\cdot c^-} = m^1 \overset{323.(6)}{=} m\,.$$

This shows $\mu_c \mathbin{\blacktriangle} \mu_{c^-} = \mathrm{id}_{\mathrm{M}_S}$.

Similarly, we have $\mu_{c^-} \mathbin{\blacktriangle} \mu_c = \mathrm{id}_{\mathrm{M}_S}$.

For $b \in \mathrm{B}_S$, we have

$$b(\beta_c \mathbin{\blacktriangle} \beta_{c^-}) = (b^c)\beta_{c^-} = (b^c)^{c^-} = b^{c\cdot c^-} = b^1 = b\,.$$

This shows $\beta_c \mathbin{\blacktriangle} \beta_{c^-} = \mathrm{id}_{\mathrm{B}_S}$.

Similarly, we have $\beta_{c^-} \mathbin{\blacktriangle} \beta_c = \mathrm{id}_{\mathrm{B}_S}$.

Therefore, we have

$$(\mu_c\,,\beta_c) \mathbin{\blacktriangle} (\mu_{c^-}\,,\beta_{c^-}) = (\mu_c \mathbin{\blacktriangle} \mu_{c^-}\,,\beta_c \mathbin{\blacktriangle} \beta_{c^-}) = (\mathrm{id}_{\mathrm{M}_S}\,,\mathrm{id}_{\mathrm{B}_S})\,.$$

Similarly, we have $(\mu_{c^-}\,,\beta_{c^-}) \mathbin{\blacktriangle} (\mu_c\,,\beta_c) = (\mathrm{id}_{\mathrm{M}_S}\,,\mathrm{id}_{\mathrm{B}_S})\,.$

So $(\mu_c\,,\beta_c)$ is a stable crossed module isomorphism. □

Remark 429 *Suppose given a stable crossed module S.*

Suppose given $c \in B_{\mathrm{S}}$.

We consider the stable crossed module isomorphism $(\mu_c\,,\beta_c)\colon S \to S$ from Remark 428.

We consider the functor $\mathrm{SymCat}\colon \mathcal{StCrMod} \to \mathcal{SymMonCat}$ *from Lemma* 415.

In particular, we consider the symmetric monoidal category $S\,\mathrm{SymCat}$.

We consider the symmetric monoidal isofunctor

$$F_c\colon S\,\mathrm{SymCat} \to S\,\mathrm{SymCat}\,,\ \left(b \xrightarrow{(b,m)} b\cdot m\mathrm{f}_S\right) \mapsto \left(b^c \xrightarrow{(b,m)^c} (b\cdot m\mathrm{f}_S)^c\right)$$

from Remark 427*, using* $c \in \mathrm{B}_S \overset{413}{=} \mathrm{Ob}(S\,\mathrm{SymCat})$.
We have

$$(\mu_c\,,\beta_c)\,\mathrm{SymCat} = F_c\,.$$

Proof. Suppose given $\big(b \xrightarrow{(b,m)} b \cdot m\mathrm{f}_S\big) \in \mathrm{Mor}(S\,\mathrm{SymCat})$.
We have

$$\begin{aligned}(b,m)F_c &\overset{427}{=} (b,m)^c \overset{427}{=} (c^-,1)\cdot(b,m)\cdot(c,1) = (b^c,m^c) = (b\beta_c\,,m\mu_c)\\ &\overset{414}{=} (b,m)\big((\mu_c\,,\beta_c)\,\mathrm{SymCat}\,\big)\,.\end{aligned}$$

This shows $(\mu_c\,,\beta_c)\,\mathrm{SymCat} = F_c\,.$ □

19.7 Remarks on dualisation

Remark 430 (Dualisation of invertible and symmetric monoidal categories)

(1) Suppose given an invertible monoidal category $(\mathcal{C}, I, \otimes)$.
We consider the category $\mathcal{C}^{\mathrm{op}}$; cf. §1.1.3 item 12.
We have the functor

$$(\otimes)\colon\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}\,,\ \Big(\big(X \xrightarrow{u^{\mathrm{op}}} X'\big), \big(Y \xrightarrow{v^{\mathrm{op}}} Y'\big)\Big) \mapsto \big(X \otimes Y \xrightarrow{(u\otimes v)^{\mathrm{op}}} X' \otimes Y'\big)\,.$$

So, for u^{op}, $v^{\mathrm{op}} \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$*, we have*

$$u^{\mathrm{op}} \otimes v^{\mathrm{op}} = (u \otimes v)^{\mathrm{op}}\,.$$

We have the invertible monoidal category $\mathcal{C}^{\mathrm{op}} = (\mathcal{C}^{\mathrm{op}}, I, \otimes)$.
We show that $(\otimes)$ *is a functor.*
Suppose given $X \xrightarrow{u^{\mathrm{op}}} Y \xrightarrow{v^{\mathrm{op}}} Z$ and $\tilde{X} \xrightarrow{\tilde{u}^{\mathrm{op}}} \tilde{Y} \xrightarrow{\tilde{v}^{\mathrm{op}}} \tilde{Z}$ in $\mathcal{C}^{\mathrm{op}}$.
We have

$$(\mathrm{id}_X)^{\mathrm{op}} \otimes (\mathrm{id}_{\tilde{X}})^{\mathrm{op}} = (\mathrm{id}_X \otimes \mathrm{id}_{\tilde{X}})^{\mathrm{op}} = (\mathrm{id}_{X\otimes\tilde{X}})^{\mathrm{op}}\,.$$

Moreover, we have

$$\begin{aligned}(u^{\mathrm{op}} \mathbin{\triangle} v^{\mathrm{op}}) \otimes (\tilde{u}^{\mathrm{op}} \mathbin{\triangle} \tilde{v}^{\mathrm{op}}) &= (v \mathbin{\triangle} u)^{\mathrm{op}} \otimes (\tilde{v} \mathbin{\triangle} \tilde{u})^{\mathrm{op}} = \big((v \mathbin{\triangle} u) \otimes (\tilde{v} \mathbin{\triangle} \tilde{u})\big)^{\mathrm{op}} \\ &= \big((v \otimes \tilde{v}) \mathbin{\triangle} (u \otimes \tilde{u})\big)^{\mathrm{op}} = (u \otimes \tilde{u})^{\mathrm{op}} \mathbin{\triangle} (v \otimes \tilde{v})^{\mathrm{op}} \\ &= (u^{\mathrm{op}} \otimes \tilde{u}^{\mathrm{op}}) \mathbin{\triangle} (v^{\mathrm{op}} \otimes \tilde{v}^{\mathrm{op}})\,.\end{aligned}$$

This shows that $(\otimes)$ is a functor.

We show that $(\mathcal{C}^{\mathrm{op}}, I, \otimes)$ *is an invertible monoidal category.*

For $u^{\mathrm{op}} \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$, we have

$$u^{\mathrm{op}} \otimes (\mathrm{id}_I)^{\mathrm{op}} = (u \otimes \mathrm{id}_I)^{\mathrm{op}} = u^{\mathrm{op}}$$

and

$$(\mathrm{id}_I)^{\mathrm{op}} \otimes u^{\mathrm{op}} = (\mathrm{id}_I \otimes u)^{\mathrm{op}} = u^{\mathrm{op}}\,.$$

For u^{op}, v^{op} and $w^{\mathrm{op}} \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$, we have

$$\begin{aligned}(u^{\mathrm{op}} \otimes v^{\mathrm{op}}) \otimes w^{\mathrm{op}} &= (u \otimes v)^{\mathrm{op}} \otimes w^{\mathrm{op}} = \big((u \otimes v) \otimes w\big)^{\mathrm{op}} = \big(u \otimes (v \otimes w)\big)^{\mathrm{op}} \\ &= u^{\mathrm{op}} \otimes (v \otimes w)^{\mathrm{op}} = u^{\mathrm{op}} \otimes (v^{\mathrm{op}} \otimes w^{\mathrm{op}})\,.\end{aligned}$$

Suppose given $u^{\mathrm{op}} \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$. Consider the morphism $(u^{\otimes -})^{\mathrm{op}} \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$.

We have

$$u^{\mathrm{op}} \otimes (u^{\otimes -})^{\mathrm{op}} = (u \otimes u^{\otimes -})^{\mathrm{op}} = (\mathrm{id}_I)^{\mathrm{op}}$$

and

$$(u^{\otimes -})^{\mathrm{op}} \otimes u^{\mathrm{op}} = (u^{\otimes -} \otimes u)^{\mathrm{op}} = (\mathrm{id}_I)^{\mathrm{op}}\,.$$

This shows that $(\mathcal{C}^{\mathrm{op}}, I, \otimes)$ is an invertible monoidal category.

(2) Suppose given a symmetric monoidal category $(\mathcal{C}, I, \otimes, \varphi)$.

We consider the invertible monoidal category $(\mathcal{C}^{\mathrm{op}}, I, \otimes)$ from (1).

For X, $Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, let

$$\varphi'_{X,Y} := (\varphi_{Y,X})^{\mathrm{op}} \colon X \otimes Y \to Y \otimes X\,.$$

Moreover, let

$$\varphi' := (\,\varphi'_{X,Y}\,)_{X,Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})}\,.$$

We have the symmetric monoidal category $(\mathcal{C}^{\mathrm{op}}, I, \otimes, \varphi')$.

We show that φ' is a symmetric braiding.

For $\big(X \xrightarrow{u^{\mathrm{op}}} \tilde{X}\big)$, $\big(Y \xrightarrow{v^{\mathrm{op}}} \tilde{Y}\big) \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$, we have

$$\begin{aligned}\varphi'_{X,Y} \mathbin{\blacktriangle} (v^{\mathrm{op}} \otimes u^{\mathrm{op}}) &= (\varphi_{Y,X})^{\mathrm{op}} \mathbin{\blacktriangle} (v \otimes u)^{\mathrm{op}} = \big((v \otimes u) \mathbin{\blacktriangle} \varphi_{Y,X}\big)^{\mathrm{op}} = \big(\varphi_{\tilde{Y},\tilde{X}} \mathbin{\blacktriangle} (u \otimes v)\big)^{\mathrm{op}} \\ &= (u \otimes v)^{\mathrm{op}} \mathbin{\blacktriangle} (\varphi_{\tilde{Y},\tilde{X}})^{\mathrm{op}} = (u^{\mathrm{op}} \otimes v^{\mathrm{op}}) \mathbin{\blacktriangle} \varphi'_{\tilde{X},\tilde{Y}}\,.\end{aligned}$$

This shows that φ' is a transformation.

Ad (SB1). For X, $Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, we have

$$\varphi'_{X,Y} \mathbin{\blacktriangle} \varphi'_{Y,X} = (\varphi_{Y,X})^{\mathrm{op}} \mathbin{\blacktriangle} (\varphi_{X,Y})^{\mathrm{op}} = (\varphi_{X,Y} \mathbin{\blacktriangle} \varphi_{Y,X})^{\mathrm{op}} = (\mathrm{id}_{X \otimes Y})^{\mathrm{op}}\,.$$

Ad (SB2). We have

$$\varphi'_{I,I} = (\varphi_{I,I})^{\mathrm{op}} = (\mathrm{id}_I)^{\mathrm{op}}\,.$$

Ad (SB3). For X, Y, $Z \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, we have

$$\begin{aligned}\varphi'_{X\otimes Y,Z} &= (\varphi_{Z,X\otimes Y})^{\mathrm{op}} \overset{407}{=} \big((\varphi_{Z,X} \otimes Y) \mathbin{\blacktriangle} (X \otimes \varphi_{Z,Y})\big)^{\mathrm{op}} = (X \otimes \varphi_{Z,Y})^{\mathrm{op}} \mathbin{\blacktriangle} (\varphi_{Z,X} \otimes Y)^{\mathrm{op}} \\ &= \big(X \otimes (\varphi_{Z,Y})^{\mathrm{op}}\big) \mathbin{\blacktriangle} \big((\varphi_{Z,X})^{\mathrm{op}} \otimes Y\big) = (X \otimes \varphi'_{Y,Z}) \mathbin{\blacktriangle} (\varphi'_{X,Z} \otimes Y)\,.\end{aligned}$$

This shows that φ' is a symmetric braiding.

So $(\mathcal{C}^{\mathrm{op}}, I, \otimes, \varphi')$ is a symmetric monoidal category.

Remark 431 (Dualisation of ordinary and stable crossed modules)

(1) Let V be a crossed module.

We consider the invertible monoidal category $V\mathrm{Cat}$; cf. Remark 165.(1).

We consider the invertible monoidal category $\big((V\mathrm{Cat})^{\mathrm{op}}, I, (\,\cdot\,)\big)$; cf. Remark 430.(1).

Note that for the tensor product $(\,\cdot\,)$, we have

$$(b,m)^{\mathrm{op}} \cdot (\tilde{b},\tilde{m})^{\mathrm{op}} \overset{430.(1)}{=} \big((b,m) \cdot (\tilde{b},\tilde{m})\big)^{\mathrm{op}} = (b \cdot \tilde{b}, m^{\tilde{b}} \cdot \tilde{m})^{\mathrm{op}}\,,$$

for $(b,m)^{\mathrm{op}}$, $(\tilde{b},\tilde{m})^{\mathrm{op}} \in \mathrm{Mor}\big((V\mathrm{Cat})^{\mathrm{op}}\big)$.

For $b \cdot (m \cdot n)\mathrm{f}_V \xrightarrow{(b\cdot m\mathrm{f}_V\,,\,n)^{\mathrm{op}}} b \cdot m\mathrm{f}_V \xrightarrow{(b,m)^{\mathrm{op}}} b$ in $(V\mathrm{Cat})^{\mathrm{op}}$, note that we have

$$(b \cdot m\mathrm{f}_V\,, n)^{\mathrm{op}} \mathbin{\blacktriangle} (b,m)^{\mathrm{op}} = \big((b,m) \mathbin{\blacktriangle} (b \cdot m\mathrm{f}_V\,, n)\big)^{\mathrm{op}} = (b, m \cdot n)^{\mathrm{op}}\,.$$

Consider the crossed module

$$\big((V\mathrm{Cat})^{\mathrm{op}}\big)\,\mathrm{CM} =: W = \big(\mathrm{M}_W\,,\mathrm{B}_W\,,\gamma_W\,,\mathrm{f}_W\big)\,,$$

where

$$\begin{aligned}\mathrm{M}_W &= \{(b,m)^{\mathrm{op}} \in \mathrm{B}_V \ltimes \mathrm{M}_V \colon \big((b,m)^{\mathrm{op}}\big)\,\boldsymbol{s}_{(V\mathrm{Cat})^{\mathrm{op}}} = 1\}\\ &= \{(b,m)^{\mathrm{op}} \in \mathrm{B}_S \ltimes \mathrm{M}_S \colon \big((b,m)^{\mathrm{op}}\big)\,\boldsymbol{t}_{V\mathrm{Cat}} = b\cdot m\mathrm{f}_S = 1\}\\ &= \{\big((m\mathrm{f}_V)^-,m\big)^{\mathrm{op}} \colon m \in \mathrm{M}_V\}\\ &= \{(m\mathrm{f}_V\,,m^-)^{\mathrm{op}} \colon m \in \mathrm{M}_V\}\\ \mathrm{B}_W &= \mathrm{Ob}\big((V\mathrm{Cat})^{\mathrm{op}}\big) = \mathrm{Ob}(V\mathrm{Cat}) = \mathrm{B}_V\,.\end{aligned}$$

Moreover, by Remark 166.(1), we have

$$\begin{array}{rccl}\gamma_W\colon & \mathrm{B}_W & \to & \mathrm{Aut}(\mathrm{M}_W)\\ & b & \mapsto & \Big((m\mathrm{f}_V\,,m^-) \mapsto (b^-,1)\cdot(m\mathrm{f}_V\,,m^-)\cdot(b,1) = \big((m\mathrm{f}_V)^b\,,(m^-)^b\big)\Big)\,.\end{array}$$

and we have

$$\mathrm{f}_W = \boldsymbol{t}_{(V\mathrm{Cat})^{\mathrm{op}}}\big|_{\mathrm{M}_W} = \boldsymbol{s}_{V\mathrm{Cat}}\big|_{\mathrm{M}_W} \colon \mathrm{M}_W \to \mathrm{B}_W\,,\ (m\mathrm{f}_V\,,m^-)^{\mathrm{op}} \mapsto m\mathrm{f}_V\,.$$

We have a crossed module isomorphism $(\psi,\mathrm{id}_{\mathrm{B}_V})\colon V \to W$*, where*

$$\psi\colon \mathrm{M}_V \to \mathrm{M}_W\,,\ m \mapsto (m\mathrm{f}_V\,,m^-)^{\mathrm{op}}\,.$$

We proceed in several steps.

We show that ψ *is a group isomorphism.*

For m, $n \in \mathrm{M}_V$, we have

$$\begin{aligned}m\psi\cdot n\psi &= (m\mathrm{f}_V\,,m^-)^{\mathrm{op}}\cdot(n\mathrm{f}_V\,,n^-)^{\mathrm{op}} = \big(m\mathrm{f}_V\cdot n\mathrm{f}_V\,,(m^-)^{n\mathrm{f}_V}\cdot n^-\big)^{\mathrm{op}}\\ &\overset{(\mathrm{CM2})}{=} \big((m\cdot n)\mathrm{f}_V\,,(m^-)^n\cdot n^-\big)^{\mathrm{op}} = \big((m\cdot n)\mathrm{f}_V\,,n^-\cdot m^-\big)^{\mathrm{op}}\\ &= \big((m\cdot n)\mathrm{f}_V\,,(m\cdot n)^-\big)^{\mathrm{op}} = (m\cdot n)\psi\,.\end{aligned}$$

Consider the map

$$\psi'\colon \mathrm{M}_W \to \mathrm{M}_V\colon (m\mathrm{f}_V\,,m^-)^{\mathrm{op}} \mapsto m\,.$$

For $(m\mathrm{f}_V\,,m^-)^{\mathrm{op}} \in \mathrm{M}_W$, where $m \in \mathrm{M}_V$, we have

$$(m\mathrm{f}_V\,,m^-)(\psi'\mathbin{\triangleleft}\psi) = m\psi = (m\mathrm{f}_V\,,m^-)\,.$$

For $m \in \mathrm{M}_V$, we have

$$m(\psi \triangleleft \psi') = (m\mathrm{f}_V, m^-)\psi' = m\,.$$

This shows $\psi^- = \psi'$.

So ψ is a group isomorphism.

We show that $(\psi, \mathrm{id}_{\mathrm{B}_V})$ *is a crossed module morphism.*

For $m \in \mathrm{M}_V$, we have

$$m(\psi \triangleleft \mathrm{f}_W) = (m\mathrm{f}_V, m^-)^{\mathrm{op}}\,\mathrm{f}_W = m\mathrm{f}_V = m(\mathrm{f}_V \triangleleft \mathrm{id}_{\mathrm{B}_V})\,.$$

So $\psi \triangleleft \mathrm{f}_W = \mathrm{f}_V \triangleleft \mathrm{id}_{\mathrm{B}_V}$.

For $m \in \mathrm{M}_V$ and $b \in \mathrm{B}_V$, we have

$$\begin{aligned}(m^b)\psi \quad &= \quad \big((m^b)\mathrm{f}_V, (m^b)^-\big)^{\mathrm{op}} \overset{(\mathrm{CM1})}{=} \big((m\mathrm{f}_V)^b, (m^-)^b\big)^{\mathrm{op}} = \big((m\mathrm{f}_V, m^-)^{(b,1)}\big)^{\mathrm{op}} \\ &\overset{430.(1)}{=} \big((m\mathrm{f}_V, m^-)^{\mathrm{op}}\big)^{(b,1)^{\mathrm{op}}} \overset{166.(1)}{=} (m\psi)^b\,.\end{aligned}$$

This shows that $(\psi, \mathrm{id}_{\mathrm{B}_V})$ is a crossed module morphism.

Since ψ and $\mathrm{id}_{\mathrm{B}_V}$ are group isomorphisms, we have that $(\psi, \mathrm{id}_{\mathrm{B}_V})$ is a crossed module isomorphism.

(2) Let S be a stable crossed module.

We consider the symmetric monoidal category $S\,\mathrm{SymCat}$; cf. Lemma 413.

We consider the symmetric monoidal category $\big((S\,\mathrm{SymCat})^{\mathrm{op}}, 1_{\mathrm{B}_S}, (\,\cdot\,), \varphi'\big)$; cf. Remark 430.(2).

So for the tensor product $(\,\cdot\,)$, we have

$$(b, m)^{\mathrm{op}} \cdot (\tilde{b}, \tilde{m})^{\mathrm{op}} \overset{430.(2,1)}{=} \big((b, m) \cdot (\tilde{b}, \tilde{m})\big)^{\mathrm{op}} = (b \cdot \tilde{b}, m^{\tilde{b}} \cdot \tilde{m})^{\mathrm{op}}\,,$$

for $(b, m)^{\mathrm{op}}$, $(\tilde{b}, \tilde{m})^{\mathrm{op}} \in \mathrm{Mor}\big((S\,\mathrm{SymCat})^{\mathrm{op}}\big)$.

Note that, for b, $c \in \mathrm{Ob}(S\,\mathrm{SymCat}) = \mathrm{B}_S$, we have

$$\varphi'_{b,c} = (\varphi_{c,b})^{\mathrm{op}} \overset{413}{=} \big(c \cdot b, [\![b,\, c]\!]\big)^{\mathrm{op}} \colon b \cdot c \to c \cdot b\,.$$

Consider the stable crossed module

$$\big((S\,\mathrm{SymCat})^{\mathrm{op}}\big)\,\mathrm{StCM} =: T = (\mathrm{M}_T, \mathrm{B}_T, \kappa_T, \mathrm{f}_T)$$

from Lemma 417.(1), where

$$\begin{aligned}
\mathrm{M}_T &= \{(b,m)^{\mathrm{op}} \colon (b,m) \in \mathrm{B}_S \ltimes \mathrm{M}_S \,, \big((b,m)^{\mathrm{op}}\big)\, \boldsymbol{s}_{(S\,\mathrm{SymCat})^{\mathrm{op}}} = 1\} \\
&= \{(b,m)^{\mathrm{op}} \colon (b,m) \in \mathrm{B}_S \ltimes \mathrm{M}_S \,, \big((b,m)^{\mathrm{op}}\big)\, \boldsymbol{t}_{S\,\mathrm{SymCat}} = b \cdot m\mathrm{f}_S = 1\} \\
&= \{\big((m\mathrm{f}_S)^-, m\big)^{\mathrm{op}} \colon m \in \mathrm{M}_S\} \\
&= \{(m\mathrm{f}_S \,, m^-)^{\mathrm{op}} \colon m \in \mathrm{M}_S\} \\
\mathrm{B}_T &= \mathrm{Ob}\big((S\,\mathrm{SymCat})^{\mathrm{op}}\big) = \mathrm{Ob}(S\,\mathrm{SymCat}) = \mathrm{B}_S \,.
\end{aligned}$$

Moreover, by Lemma 417.(1), we have

$$\begin{aligned}
\kappa_T \colon \mathrm{B}_T \times \mathrm{B}_T \to \mathrm{M}_T \,,\ (b,c) \mapsto \quad & \big((c,1)^{\mathrm{op}} \cdot (b,1)^{\mathrm{op}}\big)^- \cdot \varphi'_{c,b} \\
&= \big((c \cdot b)^-, 1\big)^{\mathrm{op}} \cdot (\varphi_{b,c})^{\mathrm{op}} \\
&= \big((c \cdot b)^-, 1\big)^{\mathrm{op}} \cdot \big(b \cdot c, [\![c,\, b]\!]\big)^{\mathrm{op}} \\
&= \Big(\big((c \cdot b)^-, 1\big) \cdot \big(b \cdot c, [\![c,\, b]\!]\big)\Big)^{\mathrm{op}} \\
&= \big([b,c], [\![c,\, b]\!]\big)^{\mathrm{op}} =: [\![b,\, c]\!]' \,,
\end{aligned}$$

and we have

$$\mathrm{f}_T \colon\ \boldsymbol{t}_{(S\,\mathrm{SymCat})^{\mathrm{op}}}\big|_{\mathrm{M}_T} = \boldsymbol{s}_{S\,\mathrm{SymCat}}\big|_{\mathrm{M}_T} \colon \mathrm{M}_T \to \mathrm{B}_T \,,\ (m\mathrm{f}_S \,, m^-)^{\mathrm{op}} \mapsto m\mathrm{f}_S \,.$$

We have a stable crossed module morphism

$$(\psi, \mathrm{id}_{\mathrm{B}_S}) \colon S \to T \,,$$

where

$$\psi \colon \mathrm{M}_S \to \mathrm{M}_T \,, m \mapsto (m\mathrm{f}_S \,, m^-)^{\mathrm{op}} \,;$$

cf. (1).

From (1), applicable after using $\mathrm{Ord}^{\mathrm{mod}}$, we know that ψ is a group isomorphism.

For $m \in \mathrm{M}_S$, we have, as in (1),

$$m(\psi \triangleleft \mathrm{f}_T) = (m\mathrm{f}_S \,, m^-)^{\mathrm{op}}\, \mathrm{f}_T = m\mathrm{f}_S = m(\mathrm{f}_S \triangleleft \mathrm{id}_{\mathrm{B}_S}) \,.$$

This shows $\psi \triangleleft \mathrm{f}_T = \mathrm{f}_S \triangleleft \mathrm{id}_{\mathrm{B}_S}$.

For $b,\, c \in \mathrm{B}_T$, we have

$$\begin{aligned}
[\![b,\, c]\!]\psi &= \big([\![b,\, c]\!]\mathrm{f}_S \,, [\![b,\, c]\!]^-\big)^{\mathrm{op}} \overset{(\mathrm{SCM2})}{=} \big([b,c], [\![b,\, c]\!]^-\big)^{\mathrm{op}} \overset{(\mathrm{SCM1})}{=} \big([b,c], [\![c,\, b]\!]\big)^{\mathrm{op}} \\
&= [\![b,\, c]\!]' = [\![b\,\mathrm{id}_{\mathrm{B}_S},\, c\,\mathrm{id}_{\mathrm{B}_S}]\!]' \,.
\end{aligned}$$

So $(\psi, \mathrm{id}_{\mathrm{B}_S})$ is a stable crossed module isomorphism.

Chapter 20

Stable simplicial groups, stable crossed modules, simplicial groups and crossed modules

Remark 432 Let $n \in \mathbb{Z}_{\geqslant 0}$. We consider the functors

$$\begin{array}{lrcll}
\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}\colon & \mathcal{StSimpGrp} & \to & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}; & \text{cf. Remark 292} \\
\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}\colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \mathcal{StSimpGrp}; & \text{cf. Lemma 298} \\
\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}\colon & \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp} & \to & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp}; & \text{cf. Remark 312} \\
\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}\colon & \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} & \to & \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}; & \text{cf. Remark 316}.
\end{array}$$

Recall that $\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor} \dashv \mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}$ and that $\mathrm{Inc}^{\pm,\lfloor n,0\rfloor} \dashv \mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}$; cf. Theorem 302, Lemma 320.

We have the following overview.

$$\mathcal{StSimpGrp} \underset{\mathrm{Cond}^{\pm,\lfloor n,-\infty\rfloor}}{\overset{\mathrm{Trunc}^{\pm,\lfloor n,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor n,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Inc}^{\pm,\lfloor n,0\rfloor}}{\overset{\mathrm{Shrink}^{\pm,\lfloor n,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \lfloor n,0\rfloor\text{-}\mathcal{StSimpGrp}$$

Remark 433 *Let $n = 1$. We consider the functors*

$$\begin{array}{lrcll}
\hat{\mathrm{N}}^{\pm}\colon & \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp} & \to & \mathcal{StCrMod}; & \text{cf. Lemma 397} \\
\mathrm{Rec}^{\pm}\colon & \mathcal{StCrMod} & \to & \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp}; & \text{cf. Lemma 400}.
\end{array}$$

Recall that $\hat{\mathrm{N}}^{\pm}$ *and* $\mathrm{Rec}^{\pm}$ *are equivalences; cf. Proposition* 405.

By Remark 432*, we obtain the following overview.*

$$\mathcal{StSimpGrp} \underset{\mathrm{Cond}^{\lfloor 1,-\infty\rfloor}}{\overset{\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lfloor 1,-\infty\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Inc}^{\lfloor 1,0\rfloor}}{\overset{\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \lfloor 1,0\rfloor\text{-}\mathcal{StSimpGrp} \underset{\mathrm{Rec}^{\pm}}{\overset{\hat{\mathrm{N}}^{\pm}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{StCrMod}$$

Suppose given a stable simplicial group G*; cf. Definition* 205.

We consider the stable crossed module $S := \big((G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\big)\hat{\mathrm{N}}^{\pm}$*; cf. Definition* 321.

We have

$$\mathrm{M}_S = G\mathrm{N}_1/G\mathrm{NB}_1 \quad \text{and} \quad \mathrm{B}_S = G\mathrm{NZ}_0\,.$$

Moreover, we have

$$\begin{array}{rccl} \kappa_S\colon & G\mathrm{NZ}_0 \times G\mathrm{NZ}_0 & \to & G\mathrm{N}_1/G\mathrm{NB}_1 \\ & (g,h) & \mapsto & [\![g,\,h]\!] = \big((g^{-}\mathrm{s}_0^{G,0})^{h\mathrm{s}_1^{G,0}} \cdot (g\mathrm{s}_0^{G,0})^{h\mathrm{s}_0^{G,0}}\big)G\mathrm{NB}_1 \end{array}$$

and

$$\mathrm{f}_S\colon G\mathrm{N}_1/G\mathrm{NB}_1 \to G\mathrm{NZ}_0\,,\ gG\mathrm{NB}_1 \mapsto g\mathrm{d}_0^{G,1}\,.$$

Proof. By Lemma 290, we have the $\lfloor 1,-\infty\rfloor$-stable simplicial group $G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}$, where

$$(G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_k = \begin{cases} G_1/G\mathrm{NB}_1 & \text{for } k=1 \\ G_k & \text{for } k \in \mathbb{Z}_{\leqslant 0}\,. \end{cases}$$

Moreover, recall that for $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned} &\mathrm{d}_i^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},1}\colon G_1/G\mathrm{NB}_1 \to G_0\,,\ gG\mathrm{NB}_1 \mapsto g\mathrm{d}_i^{G,1} \\ &\mathrm{s}_i^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},0}\colon G_0 \to G_1/G\mathrm{NB}_1\,,\ g \mapsto (g\mathrm{s}_i^{G,0})G\mathrm{NB}_1\,. \end{aligned}$$

For $k \in \{0,1\}$, let $A_k = \big\{a \in {}_{\Delta^{\pm}}\big([-1]^{\pm},[k]^{\pm}\big) : a \text{ is injective}\big\}$.

By Lemma 175.(1.1), note that for $a \in {}_{\Delta^{\pm}}\big([-1]^{\pm},[1]^{\pm}\big)$, we have have $a = \delta_i^{\pm,0} \blacktriangle \delta_j^{\pm,1}$ for some $i,\,j \in \mathbb{Z}_{\geqslant 0}$ with $i<j$.

So

$$\begin{aligned} (G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_a &= (G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_{\delta_i \blacktriangle \delta_j} = (G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_{\delta_j} \blacktriangle (G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_{\delta_i} \\ &= \mathrm{d}_j^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},1} \blacktriangle \mathrm{d}_i^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},0} = \mathrm{d}_j^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},1} \blacktriangle \mathrm{d}_i^{G,0}\,. \end{aligned}$$

Moreover, for $a \in {}_{\Delta^\pm}\big([-1]^\pm, [0]^\pm\big)$, we have have $a = \delta_j^{\pm,0}$ for some $j \in \mathbb{Z}_{\geqslant 0}$.
So

$$(G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_a = (G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_{\delta_j} = \mathrm{d}_j^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},0} = \mathrm{d}_j^{G,0}\,.$$

We consider the $\lfloor 1,0\rfloor$-stable simplicial group $(G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}$.
Then we have

$$\begin{aligned}\big((G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\big)_k &\overset{314}{=} \begin{cases} \bigcap\limits_{a\in A_k} \ker(G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_a & \text{for } k \in \{0,1\} \\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1} \end{cases} \\ &= \begin{cases} \bigcap\limits_{0\leqslant i<j} \ker\big(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},1} \blacktriangleleft \mathrm{d}_0^{G,i}\big) & \text{for } k = 1 \\ G\mathrm{NZ}_0 & \text{for } k = 0 \\ 1 & \text{for } k \in \mathbb{Z}_{\leqslant -1}\,, \end{cases}\end{aligned}$$

and, for $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\bar{\mathrm{d}}_i := \mathrm{d}_i^{(G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor},1} \colon \big((G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\big)_1 \to G\mathrm{NZ}_0\,,\ gG\mathrm{NB}_1 \mapsto g\mathrm{d}_i^{G,1}$$
$$\bar{\mathrm{s}}_i := \mathrm{s}_i^{(G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor},0} \colon G\mathrm{NZ}_0 \to \big((G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\big)_1\,,\ g \mapsto (g\mathrm{s}_i^{G,0})G\mathrm{NB}_1\,.$$

Note that

$$\begin{aligned}\bigcap_{j\in\mathbb{Z}_{\geqslant 1}} \ker \bar{\mathrm{d}}_j &= \Big\{ gG\mathrm{NB}_1 \in \bigcap_{0\leqslant i<j} \ker(\mathrm{d}_j^{G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor},1} \blacktriangleleft \mathrm{d}_i^{G,0}) \ :\ g \in \bigcap_{j\geqslant 1} \ker \mathrm{d}_j^{G,1} \Big\} \\ &= \Big\{ g \in G\mathrm{NB}_1 \in G/G\mathrm{NB}_1 \colon g \in \bigcap_{j\geqslant 1} \ker \mathrm{d}_j^{G,1} \Big\} \\ &= G\mathrm{N}_1/G\mathrm{NB}_1\,.\end{aligned}$$

By Remark 395, we have the stable crossed module

$$S = \big((G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\big)\hat{\mathrm{N}}^\pm = (\mathrm{M}_S\,,\,\mathrm{B}_S\,,\,\kappa_S\,,\,\mathrm{f}_S)\,,$$

where

$$\mathrm{M}_S = \big((G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\big)\mathrm{N}_1 = \bigcap_{j\in\mathbb{Z}_{\geqslant 1}} \ker \bar{\mathrm{d}}_j = \big(\bigcap_{j\in\mathbb{Z}_{\geqslant 1}} \ker \mathrm{d}_j^G\big)/G\mathrm{NB}_1 = G\mathrm{N}_1/G\mathrm{NB}_1$$
$$\mathrm{B}_S = \big((G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\big)_0 = G\mathrm{NZ}_0\,,$$

where

$$\begin{array}{rccl} \kappa_S\colon & G\mathrm{NZ}_0 \times G\mathrm{NZ}_0 & \to & G\mathrm{N}_1/G\mathrm{NB}_1 \\ & (g,h) & \mapsto & (g^-\bar{\mathrm{s}}_0)^{h\bar{\mathrm{s}}_1}\cdot(g\bar{\mathrm{s}}_0)^{h\bar{\mathrm{s}}_0} = \big((g^-\mathrm{s}_0^{G,0})^{h\mathrm{s}_1^{G,0}}\cdot(g\mathrm{s}_0^{G,0})^{h\mathrm{s}_0^{G,0}}\big)G\mathrm{NB}_1\,, \end{array}$$

and where

$$\mathrm{f}_S = \bar{\mathrm{d}}_0\big|_{G\mathrm{N}_1/G\mathrm{NB}_1}\colon G\mathrm{N}_1/G\mathrm{NB}_1 \to G\mathrm{NZ}_0\,,\ gG\mathrm{NB}_1 \mapsto g\mathrm{d}_0^{G,1}\,.$$

□

Lemma 434 *We consider the functors*

$\mathrm{Ord}\colon$	$\mathcal{StSimpGrp}$	$\to$	$\mathcal{SimpGrp}$;	cf. Lemma 221
$\mathrm{Trunc}^{\lfloor 1,0\rfloor}\colon$	$\mathcal{SimpGrp}$	$\to$	$\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$;	cf. Lemma 120
$\hat{\mathrm{N}}\colon$	$\lfloor 1,0\rfloor$-$\mathcal{SimpGrp}$	$\to$	$\mathcal{CrMod}$;	cf. Lemma 143
$\mathrm{Ord}^{\mathrm{mod}}\colon$	$\mathcal{StCrMod}$	$\to$	$\mathcal{CrMod}$;	cf. Lemma 329.

We have

$$\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor} * \mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor} * \hat{\mathrm{N}}^{\pm} * \mathrm{Ord}^{\mathrm{mod}} = \mathrm{Ord} * \mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}}\,.$$

So we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{StSimpGrp} & \xrightarrow{\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor} * \mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor} * \hat{\mathrm{N}}^{\pm}} & \mathcal{StCrMod} \\ \Big\downarrow{\scriptstyle \mathrm{Ord}} & & \Big\downarrow{\scriptstyle \mathrm{Ord}^{\mathrm{mod}}} \\ \mathcal{SimpGrp} & \xrightarrow[\mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}}]{} & \mathcal{CrMod} \end{array}$$

Proof. Suppose given a stable simplicial group G.

For $i \in \mathbb{Z}_{\geqslant 0}$, we write $\mathrm{d}_i := \mathrm{d}_i^{G,1}$ and $\mathrm{s}_i := \mathrm{s}_i^{G,0}$.

We show that $G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm}\,\mathrm{Ord}^{\mathrm{mod}} \stackrel{!}{=} G\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}$.

We construct the crossed module $G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm}\,\mathrm{Ord}^{\mathrm{mod}}$.

We consider the stable crossed module $S := G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm}$.

From Remark 433, recall that we have the stable crossed module

$$G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm} =: S = (\mathrm{M}_S\,,\,\mathrm{B}_S\,,\,\kappa_S\,,\,\mathrm{f}_S)\,,$$

where

$$\mathrm{M}_S = G\mathrm{N}_1/G\mathrm{NB}_1 \quad \text{and} \quad \mathrm{B}_S = G\mathrm{NZ}_0\,,$$

where

$$\begin{array}{rccl} \kappa_S\colon & G\mathrm{NZ}_0 \times G\mathrm{NZ}_0 & \to & G\mathrm{N}_1/G\mathrm{NB}_1 \\ & (g,h) & \mapsto & [\![g,\,h]\!] = \big((g^{-}\mathrm{s}_0^{G,0})^{h\mathrm{s}_1^{G,0}} \cdot (g\mathrm{s}_0^{G,0})^{h\mathrm{s}_0^{G,0}}\big)G\mathrm{NB}_1 \end{array}$$

and where

$$\mathrm{f}_S\colon G\mathrm{N}_1/G\mathrm{NB}_1 \to G\mathrm{NZ}_0\,,\ gG\mathrm{NB}_1 \mapsto g\mathrm{d}_0^{G,1}\,.$$

We consider the crossed module $S\mathrm{Ord}^{\mathrm{mod}} = (\mathrm{M}_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\gamma_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\mathrm{f}_{S\mathrm{Ord}^{\mathrm{mod}}})$.

By Lemmas 327, 329, we have

$$\begin{aligned} \mathrm{M}_{S\mathrm{Ord}^{\mathrm{mod}}} &= \mathrm{M}_S = G\mathrm{N}_1/G\mathrm{NB}_1 \\ \mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}} &= \mathrm{B}_S = G\mathrm{NZ}_0\,, \end{aligned}$$

and we have

$$\begin{array}{rccl} \gamma_{S\mathrm{Ord}^{\mathrm{mod}}}\colon & G\mathrm{NZ}_0 & \to & \mathrm{Aut}(G\mathrm{N}_1/G\mathrm{NB}_1) \\ & b & \mapsto & \big(mG\mathrm{NB}_1 \mapsto (mG\mathrm{NB}_1)^b = mG\mathrm{NB}_1 \cdot [\![(mG\mathrm{NB}_1)\mathrm{f}_S,\,b]\!] \overset{390.(3)}{=} (m^{b\mathrm{s}_0})G\mathrm{NB}_1\big) \end{array}$$

and

$$\mathrm{f}_{S\mathrm{Ord}^{\mathrm{mod}}} = \mathrm{f}_S\colon G\mathrm{N}_1/G\mathrm{NB}_1 \to G\mathrm{NZ}_0\,,\ mG\mathrm{NB}_1 \mapsto m\mathrm{d}_0\,.$$

We construct the crossed module $G\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}$.

We consider the $\lfloor 1,0\rfloor$-simplicial group $G\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}$. We have

$$\begin{aligned} (G\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor})_k &\overset{118}{=} \begin{cases} (G\mathrm{Ord})_1/(G\mathrm{Ord})\mathrm{NB}_1 & \text{for } k=1 \\ (G\mathrm{Ord})_0 & \text{for } k=0 \end{cases} \\ &\overset{222.(1,2)}{=} \begin{cases} (G\mathrm{Ord})_1/G\mathrm{NB}_1 & \text{for } k=1 \\ G\mathrm{NZ}_0 & \text{for } k=0\,. \end{cases} \end{aligned}$$

Recall that $(G\mathrm{Ord})_1 \overset{219}{=} \big(\bigcap\limits_{i\geqslant 2} G_{1,i}\big) \cap \ker(\mathrm{d}_0^{G,1} \blacktriangle \mathrm{d}_0^{G,0}) \leqslant G_1\,.$

We consider the crossed module $G\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}\,\hat{\mathrm{N}} =: V = (\mathrm{M}_V\,,\,\mathrm{B}_V\,,\,\gamma_V\,,\,\mathrm{f}_V)$.

By Lemma 141, we have

$$\begin{aligned} \mathrm{M}_V &= (G\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor})\mathrm{N}_1 \overset{121.(1)}{=} (G\mathrm{Ord})\mathrm{N}_1/G\mathrm{NB}_1 \overset{222.(1)}{=} G\mathrm{N}_1/G\mathrm{NB}_1 \\ \mathrm{B}_V &= (G\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor})_0 = G\mathrm{NZ}_0\,, \end{aligned}$$

and we have

$$\gamma_V \colon G\mathrm{NZ}_0 \to \mathrm{Aut}(G\mathrm{N}_1/G\mathrm{NB}_1)\,,\ b \mapsto \big(mG\mathrm{NB}_1 \mapsto (mG\mathrm{NB}_1)^b = (m^{b\mathrm{s}_0})G\mathrm{NB}_1\big)$$

and

$$\mathrm{f}_V = \mathrm{d}_0^{G\,\mathrm{Trunc}^{\lfloor 1,0\rfloor},1}\big|_{G\mathrm{N}_1/G\mathrm{NB}_1} \colon G\mathrm{N}_1/G\mathrm{NB}_1 \to G\mathrm{NZ}_0\,,\ mG\mathrm{NB}_1 \mapsto m\mathrm{d}_0\,.$$

This shows that $(\mathrm{M}_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\mathrm{B}_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\gamma_{S\mathrm{Ord}^{\mathrm{mod}}}\,,\,\mathrm{f}_{S\mathrm{Ord}^{\mathrm{mod}}}) = (\mathrm{M}_V\,,\,\mathrm{B}_V\,,\,\gamma_V\,,\,\mathrm{f}_V)$.

Suppose given a stable simplicial group morphism $\varphi\colon G \to H$.

We show that $\varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm}\,\mathrm{Ord}^{\mathrm{mod}} \overset{!}{=} \varphi\mathrm{Ord}\,\mathrm{Trunc}^{\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}$.

We construct the crossed module morphism $(\lambda,\mu) := \varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm}\,\mathrm{Ord}^{\mathrm{mod}}$.

We consider the $\lfloor 1,-\infty\rfloor$-stable simplicial group morphism $\varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}$.

By Lemma 291, we have

$$(\varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_1 \colon G_1/G\mathrm{NB}_1 \to H_1/H\mathrm{NB}_1\,,\ gG\mathrm{NB}_1 \mapsto (g\varphi_1)H\mathrm{NB}_1$$
$$(\varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor})_0 \colon G_0 \to H_0\,,\ g \mapsto g\varphi_0\,.$$

Further, we have the $\lfloor 1,0\rfloor$-stable simplicial group morphism $\varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}$.

By Lemma 315, we have

$$\begin{aligned}(\varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor})_1 \colon (G\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor})_1 &\to (H\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor})_1\\ gG\mathrm{NB}_1 &\mapsto (g\varphi_1)H\mathrm{NB}_1\end{aligned}$$

and

$$(\varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor})_0 = \varphi_0\big|^{H\mathrm{NZ}_0}_{G\mathrm{NZ}_0} \colon G\mathrm{NZ}_0 \to H\mathrm{NZ}_0\,,\ g \mapsto g\varphi_0\,.$$

By Remark 396, we have the stable crossed module morphism

$$(\lambda',\mu') := \varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm}\,,$$

where

$$\lambda' \colon G\mathrm{N}_1/G\mathrm{NB}_1 \to H\mathrm{N}_1/H\mathrm{NB}_1\,,\ gG\mathrm{NB}_1 \mapsto (g\varphi_1)H\mathrm{NB}_1$$
$$\mu' \colon G\mathrm{NZ}_0 \to H\mathrm{NZ}_0\,,\ g \mapsto g\varphi_0\,.$$

Then, by Lemmas 328, 329, we have the crossed module morphism

$$(\lambda,\mu) = \varphi\,\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor}\,\mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor}\,\hat{\mathrm{N}}^{\pm}\,\mathrm{Ord}^{\mathrm{mod}} = (\lambda',\mu')\mathrm{Ord}^{\mathrm{mod}} = (\lambda',\mu')\,.$$

So

$$\lambda\colon G\mathrm{N}_1/G\mathrm{NB}_1 \to H\mathrm{N}_1/H\mathrm{NB}_1\,,\ gG\mathrm{NB}_1 \mapsto (g\varphi_1)H\mathrm{NB}_1$$
$$\mu\colon G\mathrm{NZ}_0 \to H\mathrm{NZ}_0\,,\ g \mapsto g\varphi_0\,.$$

We construct the crossed module morphism $(\tilde{\lambda},\tilde{\mu}) := \varphi\mathrm{Ord}\ \mathrm{Trunc}^{\lfloor 1,0\rfloor}\ \hat{\mathrm{N}}$.

We consider the simplicial group morphism $\varphi\mathrm{Ord}$. By Lemma 220, we have $(\varphi\mathrm{Ord})_k = \varphi_k\big|_{G_k}^{H_k}$ for $k \in \mathbb{Z}_{\geqslant 0}$.

We consider the $\lfloor 1,0\rfloor$-simplicial group morphism $\varphi\mathrm{Ord}\ \mathrm{Trunc}^{\lfloor 1,0\rfloor}$. By Lemma 119, we have

$$(\varphi\mathrm{Ord}\ \mathrm{Trunc}^{\lfloor 1,0\rfloor})_1\colon (G\mathrm{Ord})_1/G\mathrm{NB}_1 \to (H\mathrm{Ord})_1/H\mathrm{NB}_1\,,\ gG\mathrm{NB}_1 \mapsto (g\varphi_1)H\mathrm{NB}_1$$
$$(\varphi\mathrm{Ord}\ \mathrm{Trunc}^{\lfloor 1,0\rfloor})_0 = (\varphi\mathrm{Ord})_0\colon G\mathrm{NZ}_0 \to H\mathrm{NZ}_0\,,\ g \mapsto g\varphi_0\,.$$

We consider the crossed module morphism $(\tilde{\lambda},\tilde{\mu}) = \varphi\mathrm{Ord}\ \mathrm{Trunc}^{\lfloor 1,0\rfloor}\ \hat{\mathrm{N}}$. By Lemma 142, we have

$$\tilde{\lambda}\colon G\mathrm{N}_1/G\mathrm{NB}_1 \to H\mathrm{N}_1/H\mathrm{NB}_1\,,\ gG\mathrm{NB}_1 \mapsto (g\varphi_1)H\mathrm{NB}_1$$
$$\tilde{\mu}\colon G\mathrm{NZ}_0 \to H\mathrm{NZ}_0\,,\ g \mapsto g\varphi_0\,.$$

This shows that $(\lambda,\mu) = (\tilde{\lambda},\tilde{\mu})$.

Altogether, this shows that $\mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor} * \mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor} * \hat{\mathrm{N}}^{\pm} * \mathrm{Ord}^{\mathrm{mod}} = \mathrm{Ord} * \mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}}$. □

Corollary 435 *Then functor* $\mathrm{Ord}\colon \mathcal{StSimpGrp} \to \mathcal{SimpGrp}$ *is not dense.*

Proof. We *assume* that Ord is dense.

By Corollary 132, we have that $\mathrm{Trunc}^{\lfloor 1,0\rfloor}$ is dense. From Proposition 152, we conclude that $\hat{\mathrm{N}}$ is dense. Therefore, the functor $\mathrm{Ord} * \mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}}$ is dense.

By Lemma 434, have

$$\mathrm{Ord} * \mathrm{Trunc}^{\lfloor 1,0\rfloor} * \hat{\mathrm{N}} = \mathrm{Trunc}^{\pm,\lfloor 1,-\infty\rfloor} * \mathrm{Shrink}^{\pm,\lfloor 1,0\rfloor} * \hat{\mathrm{N}}^{\pm} * \mathrm{Ord}^{\mathrm{mod}}\ .$$

In particular, it follows that functor $\mathrm{Ord}^{\mathrm{mod}}$ is dense, which *contradicts* Remark 337 that states that $\mathrm{Ord}^{\mathrm{mod}}$ is not dense. □

Bibliography

[1] M. Artin, A. Grothendieck, J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas. Tome 1–3*, SGA4, 1963–1964.

[2] H.J. Baues, D. Conduché, *On the* 2*-type of an iterated loop space*, Forum Math. 9(6), pp. 721–738, 1997.

[3] R. Brown, P.J. Higgins, R. Sivera, *Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubic Homotopy Grupoids*, EMS Tracts Math. 15, 2011.

[4] R. Brown, C.B. Spencer, *$\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group,* Indag. Math. 38(4), pp. 296–302, 1976.

[5] M. Bullejos, P. Carrasco, A.M. Cegarra, *Cohomology with coefficients in symmetric Cat-groups. An extension of Eilenberg–MacLane's classification theorem*, Math. Proc. Camb. Philos. Soc. 114, pp. 163–189, 1993.

[6] M. Bullejos, A.M. Cegarra, *A 3-dimensional Non-Abelian Cohomology of Groups With Appliciations to Homotopy Classification of Continuous Maps*, Can. J. Math. 43(2), pp. 265–296, 1991.

[7] D. Conduché, *Modules croisés genéralisés de longeur* 2, J. Pure Appl. Algebra 34, pp. 155–178, 1984.

[8] J. Dallendörfer, *Semisimplicial and simplicial homotopy and a resolution functor*, master thesis, Stuttgart, 2019.

[9] J. Duskin, *Simplicial methods and the interpretation of "triple" cohomology*, Mem. Amer. Math. Soc. 163, 1975.

[10] J.I. Extremiana Aldana, L.J. Hernández Paricio, M.T. Rivas Rodríguez, *A closed model category for* $(n-1)$*-connected spaces*, Proc. Am. Math. Soc. 124(11), pp. 3545–3553, 1996.

[11] P. G. Goerss, J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics 174, 1999.

[12] N. Gurski, N. Johnson, A.M. Osorno, *Topological Invariants from Higher Categories*, Notices Am. Math. Soc. 66(8), pp. 1225–1238, 2019.

[13] N. Gurski, N. Johnson, A.M. Osorno, *The* 2*-dimensional stable homotopy hypothesis*, J. Pure Appl. Algebra 223(10), pp. 4348–4383, 2019.

[14] A. Joyal, R. Street, *Braided Monoidal Categories*, Macquarie Math. Reports 860081, 1986.

[15] A. Joyal, R. Street, *Braided Tensor Categories*, Adv. Math. 102(1), pp. 20–78, 1993.

[16] D. Kan, *The Hurewicz theorem*, Symposium internacional de topología algebraica (International symposium on algebraic topology), Mexico City, pp. 225–231, 1958.

[17] D. Kan, *On homotopy theory and c.s.s. groups*, Ann. Math. 68(1), pp. 38–53, 1958.

[18] D. Kan, *Semisimplicial Spectra*, Ill. J. Math. 7, pp. 463–478, 1963.

[19] W. Messing, *Theory of displays I*, lecture in Summer School, 2006. (`https://youtu.be/dvUVfc916XM?t=3420`, 57:00)

[20] J.C. Moore, *Homotopie des complexes monoïdaux, I.*, Séminaire Henri Cartan 7(2), Exposé 18, pp. 1–8, 1954–1955.

[21] A. Mutlu, T. Porter, *Applications of Peiffer pairings in the Moore complex of a simplicial group*, Theory and Applications of Categories 4(7), pp. 148–173, 1998.

[22] A. Mutlu, T. Porter, *Iterated Peiffer Pairings in the Moore Complex of a Simplicial Group*, Applied Categorical Structures 9(2), pp. 111–130, 2001.

[23] D. Quillen, *Homotopical algebra*, Lecture Notes in Math 43, 1967.

[24] E. Spanier, *Algebraic topology*, Springer, New York (originally published by McGraw-Hill), 1966.

[25] E. Spanier, G. Whitehead, *A first approximation to homotopy theory*, Proc. Nat. Acad. Sci. U.S.A. 39(7), pp. 655–660, 1953.

[26] M. Stephan, *Kan spectra, group spectra and twisting structures*, doctoral thesis, Lausanne, EPFL, 2015.

[27] S. Thomas, *(Co)homology of crossed modules*, diploma thesis, Aachen, 2007.

[28] M. Truong, *On Elementary Properties of Crossed Modules*, bachelor thesis, Stuttgart, 2015.

[29] M. Truong, *Basic Representation Theory of Crossed Modules*, master thesis, Stuttgart, 2018.

[30] F. Weilandt, *Abelsche Kan-Spektren – Modellstrukturen und unbeschränkte Dold-Kan-Äquivalenz (Abelian Kan Spectra – Model Structures and Unbounded Dold-Kan Equivalence)*, diploma thesis, Bonn, 2009.

[31] J.H.C. Whitehead, *Combinatorial homotopy. II*, Bull. Am. Math. Soc. 55, pp. 213–245, 1949.

Erklärung

Hiermit erkläre ich, dass ich meine Arbeit selbstständig verfasst und keine andere als die angegebenen Quellen benutzt habe. Alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen habe ich als solche gekennzeichnet. Ich versichere, dass ich zur Erstellung dieser Arbeit keine KI-basierten Anwendungen oder Werkzeuge verwendet habe. Weiterhin versichere ich, dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, Dezember 2024

Monika Truong