On universal properties of preadditive and additive categories

Karoubi envelope, additive envelope and tensor product

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Bachelor's Thesis

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February 2016

Contents

Chapter 0

Introduction

0.1 Envelope operations

In the construction of our envelope operations we follow Nico Stein [**7**], in that we first perform the envelope construction including its functorialities, and then use these functorialities to derive its universal properties.

0.1.1 The Karoubi envelope

We give an account of the construction of the Karoubi envelope of an additive category. This construction is due to Karoubi [**3,** II.1], see also [**4,** Theorem 6.10]. He calls it "enveloppe pseudo-abélienne" or "pseudo-abelian category" associated with the given category.

Suppose given an additive category $\mathcal{A}$. It is not necessarily idempotent complete, that is, an idempotent morphism $X \xrightarrow{e} X$ in $\mathcal{A}$ does not necessarily have an image; cf. Definition 45. We aim to endow $\mathcal{A}$ with images of all idempotents in a universal manner. More precisely, we construct an idempotent complete additive category $\operatorname{Kar}\mathcal{A}$ and an additive functor $J_\mathcal{A} : \mathcal{A} \longrightarrow \operatorname{Kar}\mathcal{A}$ such that every additive functor F from $\mathcal{A}$ to an idempotent complete additive category $\mathcal{B}$ factors uniquely, up to isomorphism, over $J_\mathcal{A}$ as $F = F' \circ J_\mathcal{A}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "J_\mathcal{A}"] \arrow[dr, "F"'] & \operatorname{Kar}\mathcal{A} \arrow[d, dotted, "F'"] \\ & \mathcal{B} \end{tikzcd}$$

Still more precisely, we have the equivalence of categories

$$\begin{array}{ccc} {}_{\text{add}}[\mathcal{A}, \mathcal{B}] & \xleftarrow{\sim} & {}_{\text{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}] \\ \Big((U \circ J_\mathcal{A}) \xrightarrow{\beta * J_\mathcal{A}} (V \circ J_\mathcal{A})\Big) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V), \end{array}$$

which is surjective on objects; cf. Theorem 78.

The category $\operatorname{Kar}\mathcal{A}$ is constructed as follows. Its objects are the pairs of the form (X, e), where $X \in \operatorname{Ob}\mathcal{A}$ and $X \xrightarrow{e} X$ is an idempotent. A morphism from (X, e) to (Y, f) in

$\operatorname{Kar}\mathcal{A}$ is given by a morphism $X \xrightarrow{\varphi} Y$ in $\mathcal{A}$ such that $e\varphi f = \varphi$. Composition is then given by the composition in $\mathcal{A}$; cf. Definition 33.

More generally, we perform this construction for arbitrary, not necessarily additive categories.

0.1.2 The additive envelope of preadditive categories

We give an account of the construction of the additive envelope of a preadditive category. In the literature, it has been mentioned for example in [**5,** VII.2, ex. 6.(a)] or in [**2,** Def. 1.1.15].

Suppose given a preadditive category $\mathcal{A}$, that is, a category whose morphism sets are abelian groups and whose composition is $\mathbf{Z}$-bilinear. Note that $\mathcal{A}$ is not necessarily additive, that is, $\mathcal{A}$ does not necessarily have direct sums. We aim to endow $\mathcal{A}$ with direct sums in a universal manner. More precisely, we construct an additive category $\operatorname{Add}\mathcal{A}$ and an additive functor $I_\mathcal{A} : \mathcal{A} \longrightarrow \operatorname{Add}\mathcal{A}$ such that every additive functor F from $\mathcal{A}$ to an additive category $\mathcal{B}$ factors uniquely, up to isomorphism, over $I_\mathcal{A}$ as $F = F' \circ I_\mathcal{A}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "I_\mathcal{A}"] \arrow[dr, "F"'] & \operatorname{Add}\mathcal{A} \arrow[d, dotted, "F'"] \\ & \mathcal{B} \end{tikzcd}$$

Still more precisely, we have the equivalence of categories

$$\begin{array}{ccc} {}_{\mathrm{add}}[\mathcal{A},\mathcal{B}] & \xleftarrow{\sim} & {}_{\mathrm{add}}[\operatorname{Add}\mathcal{A},\mathcal{B}] \\ \Big((U \circ I_\mathcal{A}) \xrightarrow{\beta * I_\mathcal{A}} (V \circ I_\mathcal{A})\Big) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V), \end{array}$$

which is surjective on objects; cf. Theorem 113.

The category $\operatorname{Add}\mathcal{A}$ is constructed as follows. Its objects are tuples of objects of $\mathcal{A}$. Its morphisms are formal matrices having as entries morphisms of $\mathcal{A}$. Composition is then given by the usual matrix multplication rule; cf. Definition 88.

More generally, we perform this construction over an arbitrary commutative ground ring.

0.2 The tensor product of categories

0.2.1 The tensor product of preadditive categories

We give an account of the construction of the tensor product of preadditive categories as mentioned in [**6,** 16.7.4], where preadditive categories in our sense are called "additive" by Schubert.

Suppose given preadditive categories $\mathcal{A}$ and $\mathcal{B}$. We construct a preadditive category $\mathcal{A}\boxtimes\mathcal{B}$ and a $\mathbf{Z}$-bilinear functor $M_{\mathcal{A},\mathcal{B}} : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A} \boxtimes \mathcal{B}$ such that every $\mathbf{Z}$-bilinear functor F

from $\mathcal{A} \times \mathcal{B}$ to a preadditive category $\mathcal{C}$ factors uniquely over $M_{\mathcal{A},\mathcal{B}}$ as $F = \overline{F} \circ M_{\mathcal{A},\mathcal{B}}$.

$$\begin{tikzcd} \mathcal{A} \times \mathcal{B} \arrow[r, "M_{\mathcal{A},\mathcal{B}}"] \arrow[dr, "F"'] & \mathcal{A} \boxtimes \mathcal{B} \arrow[d, dotted, "\overline{F}"] \\ & \mathcal{C} \end{tikzcd}$$

More precisely, we have the isomorphism of categories

$$\begin{array}{rcl} {}_{\mathbf{Z}\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}] & \xleftarrow{\sim} & {}_{\text{add}}[\mathcal{A} \boxtimes \mathcal{B}, \mathcal{C}] \\ \Big((U \circ M_{\mathcal{A},\mathcal{B}}) \xrightarrow{\beta * M_{\mathcal{A},\mathcal{B}}} (V \circ M_{\mathcal{A},\mathcal{B}})\Big) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V); \end{array}$$

cf. Theorem 130.

The category $\mathcal{A} \boxtimes \mathcal{B}$ is constructed as follows. Its objects are the pairs (A, B) with $A \in \operatorname{Ob} \mathcal{A}$ and $B \in \operatorname{Ob} \mathcal{B}$. We denote such a pair by $A \boxtimes B$. For objects $A_1 \boxtimes B_1$ and $A_2 \boxtimes B_2$ of $\mathcal{A} \boxtimes \mathcal{B}$, let

$$_{\mathcal{A}\boxtimes\mathcal{B}}(A_1 \boxtimes B_1 \,, A_2 \boxtimes B_2) := {}_{\mathcal{A}}(A_1 \,, A_2) \underset{\mathbf{Z}}{\otimes} {}_{\mathcal{B}}(B_1 \,, B_2).$$

Composition in $\mathcal{A} \boxtimes \mathcal{B}$ is then given on elementary tensors by

$$(a' \otimes b')(a'' \otimes b'') = (a'a'') \otimes (b'b'');$$

cf. Definition 123.

More generally, we perform this construction over an arbitrary commutative ground ring.

0.2.2 The tensor product of additive categories

We give an account of the construction of the tensor product of additive categories as mentioned in [**2,** Def. 1.1.15].

Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$. We construct an additive category $\mathcal{A} \overset{\text{add}}{\boxtimes} \mathcal{B}$ and a $\mathbf{Z}$-bilinear functor $M^{\text{add}}_{\mathcal{A},\mathcal{B}} : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A} \overset{\text{add}}{\boxtimes} \mathcal{B}$ such that every $\mathbf{Z}$-bilinear functor F from $\mathcal{A} \times \mathcal{B}$ to an additive category $\mathcal{C}$ factors uniquely, up to isomorphism, over $M^{\text{add}}_{\mathcal{A},\mathcal{B}}$ as $F = \overline{F}' \circ M^{\text{add}}_{\mathcal{A},\mathcal{B}}$.

$$\begin{tikzcd} \mathcal{A} \times \mathcal{B} \arrow[r, "M^{\text{add}}_{\mathcal{A},\mathcal{B}}"] \arrow[dr, "F"'] & \mathcal{A} \overset{\text{add}}{\boxtimes} \mathcal{B} \arrow[d, dotted, "\overline{F}'"] \\ & \mathcal{C} \end{tikzcd}$$

More precisely, we have the equivalence of categories

$$\begin{array}{rcl} {}_{\mathbf{Z}\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}] & \xleftarrow{\sim} & {}_{\text{add}}[\mathcal{A} \overset{\text{add}}{\boxtimes} \mathcal{B}, \mathcal{C}] \\ \Big((U \circ M^{\text{add}}_{\mathcal{A},\mathcal{B}}) \xrightarrow{\beta * M^{\text{add}}_{\mathcal{A},\mathcal{B}}} (V \circ M^{\text{add}}_{\mathcal{A},\mathcal{B}})\Big) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V), \end{array}$$

which is surjective on objects; cf. Theorem 143.

The category $\mathcal{A} \overset{\text{add}}{\boxtimes} \mathcal{B}$ is constructed by first taking the tensor product $\mathcal{A} \boxtimes \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ as preadditive categories and then taking the additive envelope of $\mathcal{A} \boxtimes \mathcal{B}$. In other words, we have $\mathcal{A} \overset{\text{add}}{\boxtimes} \mathcal{B} = \operatorname{Add}(\mathcal{A} \boxtimes \mathcal{B})$; cf. §0.1.2 and §0.2.1.

More generally, we perform this construction over an arbitrary commutative ground ring.

0.3 Counterexamples for compatibility relations

0.3.1 Karoubi envelope and additive envelope

Given a preadditive category $\mathcal{A}$, in general we have

$$\operatorname{Add}(\operatorname{Kar} \mathcal{A}) \not\simeq \operatorname{Kar}(\operatorname{Add} \mathcal{A}).$$

More precisely, consider the subring $\Lambda := \{(a,b) \in \mathbf{Z} \times \mathbf{Z} : a \equiv_5 b\}$ of $\mathbf{Z} \times \mathbf{Z}$ and the full preadditive subcategory $\mathcal{A}$ of Λ-free with $\operatorname{Ob} \mathcal{A} := \{\Lambda, 0\}$.

Then $\operatorname{Add}(\operatorname{Kar} \mathcal{A}) \not\simeq \operatorname{Kar}(\operatorname{Add} \mathcal{A})$; cf. Proposition 145.

0.3.2 Additive envelope and tensor product

Given a commutative ring R and preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R, in general we have

$$(\operatorname{Add} \mathcal{A}) \underset{R}{\boxtimes} (\operatorname{Add} \mathcal{B}) \not\simeq \operatorname{Add}(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}).$$

More precisely, consider $R = \mathbf{Q}$ and the full $\mathbf{Q}$-linear preadditive subcategory $\mathcal{A}$ of $\mathbf{Q}$-mod with

$$\operatorname{Ob} \mathcal{A} := \{V \in \operatorname{Ob} \mathbf{Q}\text{-mod} : \dim V \neq 1\};$$

cf. Remark 32. Then $(\operatorname{Add} \mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Add} \mathcal{A}) \not\simeq \operatorname{Add}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$; cf. Proposition 147.

0.3.3 Karoubi envelope and tensor product

Given a commutative ring R and preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R, in general we have

$$(\operatorname{Kar} \mathcal{A}) \underset{R}{\boxtimes} (\operatorname{Kar} \mathcal{B}) \not\simeq \operatorname{Kar}(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}).$$

More precisely, consider $R = \mathbf{Q}$ and the full $\mathbf{Q}$-linear preadditive subcategory of $\mathbf{Q}(\mathrm{i})$-mod with $\operatorname{Ob} \mathcal{A} := \{\mathbf{Q}(\mathrm{i}), 0\}$; cf. Remark 32.

Then $(\operatorname{Kar} \mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Kar} \mathcal{A}) \not\simeq \operatorname{Kar}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$; cf. Proposition 149.

0.4 Conventions

We assume the reader to be familiar with elementary category theory. An introduction to this subject can be found in [**5**] or [**6**]. Some basic definitions and notations are given below. Concerning additive categories, we essentially follow Nico Stein [**7**].

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories.

1. All categories are supposed to be small (with respect to a sufficiently big universe); cf. [**6,** §3.2 and §3.3].

2. We write $\operatorname{Ob}\mathcal{A}$ for the set of objects and $\operatorname{Mor}\mathcal{A}$ for the set of morphisms of $\mathcal{A}$. Given $A, B \in \operatorname{Ob}\mathcal{A}$, we denote the set of morphisms from A to B by ${}_{\mathcal{A}}(A, B)$. The identity morphism of $A \in \operatorname{Ob}\mathcal{A}$ is written as 1_A. If unambiguous, we often write $1 := 1_A$.

3. The composition of morphisms in $\mathcal{A}$ is written naturally:
$$\left(A \xrightarrow{f} B \xrightarrow{g} C\right) = \left(A \xrightarrow{fg} C\right) = \left(A \xrightarrow{f \cdot g} C\right).$$

4. The composition of functors is written traditionally:
$$\left(\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}\right) = \left(\mathcal{A} \xrightarrow{G \circ F} \mathcal{C}\right).$$

5. Suppose given $A, B \in \operatorname{Ob}\mathcal{A}$. If A and B are isomorphic in $\mathcal{A}$, we write $A \cong B$.
If $\varphi \in {}_{\mathcal{A}}(A, B)$ is an isomorphism, we often write $A \underset{\sim}{\xrightarrow{\varphi}} B$. Given an isomorphism $f \in {}_{\mathcal{A}}(A, B)$, we write $f^{-1} \in {}_{\mathcal{A}}(B, A)$ for its inverse.

6. Given a functor $F : \mathcal{A} \to \mathcal{B}$ and $X, Y \in \operatorname{Ob}\mathcal{A}$, we write
$$F_{X,Y} : {}_{\mathcal{A}}(X, Y) \to {}_{\mathcal{B}}(FX, FY), \varphi \mapsto F\varphi.$$

7. The opposite category (or dual category) of $\mathcal{A}$ is denoted by $\mathcal{A}^\circ$. Given $A \xrightarrow{f} B$ in $\mathcal{A}$, we write $B \xrightarrow{f^\circ} A$ for the corresponding morphism in $\mathcal{A}^\circ$.

8. We call $\mathcal{A}$ *preadditive* if it fullfills the following conditions (1, 2).

   (1) For $A, B \in \operatorname{Ob}\mathcal{A}$, the set ${}_{\mathcal{A}}(A, B)$ carries the structure of an abelian group, written additively.

   (2) For $A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \xrightarrow{h} D$ in $\mathcal{A}$, we have $f(g_1 + g_2)h = fg_1h + fg_2h$.

   Suppose $\mathcal{A}$ to be preadditive. Suppose given $A, B \in \operatorname{Ob}\mathcal{A}$. We denote the zero morphism in ${}_{\mathcal{A}}(A, B)$ by $0_{A,B}$. If unambiguous, we often write $0 := 0_{A,B}$.

   Note that full subcategories of preadditive categories are preadditive.

9. We write $\mathbf{Z}$ for the set of integers. Given $a \in \mathbf{Z}$, we write $\mathbf{Z}_{\geqslant a} := \{b \in \mathbf{Z} : b \geqslant a\}$.

10. For $a, b \in \mathbf{Z}$, we write $[a, b] := \{c \in \mathbf{Z} : a \leqslant c \leqslant b\}$.

11. Let M be a finite set. The cardinality of M is denoted by $|M|$.

12. Suppose $\mathcal{A}$ to be preadditive. We use following variant of the Kronecker delta.

    Let I be a set. Suppose given $A_i \in \operatorname{Ob}\mathcal{A}$ for $i \in I$. Let

$$\delta_{A_i, A_j} := \begin{cases} 1_{A_i} & \text{if } i = j, \\ 0_{A_i, A_j} & \text{if } i \neq j, \end{cases}$$

    for $i, j \in I$. If unambiguous, we often write $\delta_{i,j} := \delta_{A_i, A_j}$.

13. Suppose $\mathcal{A}$ to be preadditive. Suppose given $A_1, A_2 \in \operatorname{Ob}\mathcal{A}$. An object $C \in \operatorname{Ob}\mathcal{A}$ together with morphisms $A_1 \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} C \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} A_2$ is called a *direct sum* of A_1 and A_2 in $\mathcal{A}$, if $\iota_1\pi_1 = 1_{A_1}$, $\iota_2\pi_2 = 1_{A_2}$ and $\pi_1\iota_1 + \pi_2\iota_2 = 1_C$.

    In the following way this is generalized to a finite number of objects.

    Suppose given $n \in \mathbf{Z}_{\geqslant 0}$. Suppose given $A_i \in \operatorname{Ob}\mathcal{A}$ for $i \in [1, n]$. A direct sum of $A_1, \ldots, A_n$ in $\mathcal{A}$ is a tuple $(C, (\pi_i)_{i \in [1,n]}, (\iota_i)_{i \in [1,n]})$ with $C \in \operatorname{Ob}\mathcal{A}$, $\pi_i \in {}_{\mathcal{A}}(C, A_i)$ and $\iota_i \in {}_{\mathcal{A}}(A_i, C)$ for $i \in [1, n]$ such that $\iota_i\pi_j = \delta_{i,j}$ for $i, j \in [1, n]$ and $\sum_{i \in [1,n]} \pi_i\iota_i = 1_C$.

    Often the following matrix notation is used for morphisms between direct sums.

    Suppose given $m, n \in \mathbf{Z}_{\geqslant 0}$. Suppose given $A_{1,i}$ and $A_{2,j}$ in $\operatorname{Ob}\mathcal{A}$ for $i \in [1, m]$ and $j \in [1, n]$. Suppose $(C_1, (\pi_{1,i})_{i \in [1,m]}, (\iota_{1,i})_{i \in [1,m]})$ to be a direct sum of $A_{1,1}, \ldots, A_{1,m}$ and $(C_2, (\pi_{2,j})_{j \in [1,n]}, (\iota_{2,j})_{j \in [1,n]})$ to be a direct sum of $A_{2,1}, \ldots, A_{2,n}$.

    Suppose given $f \in {}_{\mathcal{A}}(C_1, C_2)$.

    Let $f_{i,j} := \iota_{1,i} f \pi_{2,j} \in {}_{\mathcal{A}}(A_{1,i}, A_{2,j})$ for $(i, j) \in [1, m] \times [1, n]$. Then

$$f = \sum_{(i,j) \in [1,m] \times [1,n]} \pi_{1,i} f_{i,j} \iota_{2,j} .$$

    We write

$$f = (f_{i,j})_{i,j} = (f_{i,j})_{i \in [1,m], j \in [1,n]} = \begin{pmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & & \vdots \\ f_{m,1} & \cdots & f_{m,n} \end{pmatrix}.$$

    Omitted matrix entries are stipulated to be zero.

    Suppose given $n \in \mathbf{Z}_{\geqslant 0}$. Suppose given $A_{1,i}$ and $A_{2,i}$ in $\operatorname{Ob}\mathcal{A}$ for $i \in [1, n]$. Suppose given $A_{1,i} \xrightarrow{f_i} A_{2,i}$ in $\mathcal{A}$ for $i \in [1, n]$. We write

$$\operatorname{diag}(f_i)_i = \operatorname{diag}(f_i)_{i \in [1,n]} = \begin{pmatrix} f_1 & & \\ & \ddots & \\ & & f_n \end{pmatrix}.$$

14. We call $A \in \operatorname{Ob}\mathcal{A}$ a *zero object* if $|{}_{\mathcal{A}}(A, B)| = 1 = |{}_{\mathcal{A}}(B, A)|$ for $B \in \operatorname{Ob}\mathcal{A}$.

15. The category $\mathcal{A}$ is called *additive* if the following conditions (1, 2, 3) hold.

    (1) The category $\mathcal{A}$ is preadditive.

    (2) There exists a zero object in $\mathcal{A}$.

(3) Every pair $(A, B) \in (\operatorname{Ob} \mathcal{A}) \times (\operatorname{Ob} \mathcal{A})$ has a direct sum in $\mathcal{A}$.

Note that in additive categories direct sums of arbitrary finite length exist.

16. Suppose $\mathcal{A}$ to be additive. We choose a zero object $0_{\mathcal{A}} \in \operatorname{Ob} \mathcal{A}$.

For $n \in \mathbf{Z}_{\geqslant 0}$ and $A_1, \ldots, A_n \in \operatorname{Ob} \mathcal{A}$, we choose a direct sum

$$\Big(\bigoplus_{i \in [1,n]} A_i \,,\, (\pi_i^{(A_j)_{j \in [1,n]}})_{i \in [1,n]} \,,\, (\iota_i^{(A_j)_{j \in [1,n]}})_{i \in [1,n]} \Big)$$

in $\mathcal{A}$.

If unambiguous, we often write $\pi_i := \pi_i^{(A_j)_{j \in [1,n]}}$ and $\iota_i := \iota_i^{(A_j)_{j \in [1,n]}}$ for $i \in [1, n]$.

In particular, we choose

$$\Big(\bigoplus_{i \in [1,1]} A_i \,,\, (\pi_i)_{i \in [1,1]} \,,\, (\iota_i)_{i \in [1,1]} \Big) = \big(A_1 \,,\, (1_{A_1})_{i \in [1,1]} \,,\, (1_{A_1})_{i \in [1,1]} \big)$$

and

$$\Big(\bigoplus_{i \in [1,0]} A_i \,,\, (\pi_i)_{i \in [1,0]} \,,\, (\iota_i)_{i \in [1,0]} \Big) = (0_{\mathcal{A}} \,,\, (\,), (\,)).$$

We often write $A_1 \oplus \cdots \oplus A_n := \bigoplus_{i \in [1,n]} A_i$.

In matrix notation, we have

$$\pi_i^{(A_j)_{j \in [1,n]}} = (\delta_{i,k})_{k \in [1,n], l \in [1,1]} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\iota_i^{(A_j)_{j \in [1,n]}} = (\delta_{i,l})_{k \in [1,1], l \in [1,n]} = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}$$

for $i \in [1, n]$; cf. Stipulation 106 below.

17. Suppose $\mathcal{A}$ and $\mathcal{B}$ to be preadditive. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *additive* if $F(\varphi + \psi) = F\varphi + F\psi$ for $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$ in $\mathcal{A}$.

18. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. Suppose given $FX \xrightarrow{\alpha_X} GX$ for $X \in \operatorname{Ob} \mathcal{A}$. The tuple $(\alpha_X)_{X \in \operatorname{Ob} \mathcal{A}}$ is called *natural* if $\alpha_X \cdot Gf = Ff \cdot \alpha_Y$ for $X \xrightarrow{f} Y$ in $\mathcal{A}$. A natural tuple is often called a *transformation*. We write $\alpha : F \to G$, $\alpha : F \Rightarrow G$ or $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$.

A transformation $\alpha : F \Rightarrow G$ is called an *isotransformation* if $FX \xrightarrow{\alpha_X} GX$ is an isomorphism in $\mathcal{B}$ for $X \in \operatorname{Ob} \mathcal{A}$. We often write $F \overset{\alpha}{\underset{\sim}{\Longrightarrow}} G$.

Suppose given functors $F, G, H : \mathcal{A} \to \mathcal{B}$. Suppose given transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$.

We have the transformation $\alpha\beta : F \Rightarrow H$ where $(\alpha\beta)_X := \alpha_X\beta_X$ for $X \in \operatorname{Ob}\mathcal{A}$.

We have the transformation $1_F : F \Rightarrow F$ where $(1_F)_X := 1_{FX}$ for $X \in \operatorname{Ob}\mathcal{A}$.

Suppose given $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, bend left, "K"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{C} \end{tikzcd}$. We have the transformation

$$(\gamma * \alpha) : (K \circ F) \Rightarrow (L \circ G)$$

where $(\gamma * \alpha)_X := K\alpha_X \cdot \gamma_{GX} = \gamma_{FX} \cdot L\alpha_X$ for $X \in \operatorname{Ob}\mathcal{A}$.

$$\begin{tikzcd} (K \circ F)X \arrow[r, "\gamma_{FX}"] \arrow[d, "K\alpha_X"'] & (L \circ F)X \arrow[d, "L\alpha_X"] \\ (K \circ G)X \arrow[r, "\gamma_{GX}"] & (L \circ G)X \end{tikzcd}$$

In particular, we have the transformations $\gamma * F := \gamma * 1_F$ and $K * \alpha := 1_K * \alpha$.

19. A functor $F : \mathcal{A} \to \mathcal{B}$ is called an *isomorphism* if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ with $F \circ G = 1_{\mathcal{B}}$ and $G \circ F = 1_{\mathcal{A}}$. In this case, we write $G = F^{-1}$.

    A functor $F : \mathcal{A} \to \mathcal{B}$ is called an *equivalence* if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ and isotransformations $G \circ F \overset{\alpha}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{A}}$ and $F \circ G \overset{\beta}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{B}}$. If there exists an equivalence $F : \mathcal{A} \to \mathcal{B}$, we call $\mathcal{A}$ and $\mathcal{B}$ *equivalent* and write $\mathcal{A} \simeq \mathcal{B}$.

    A functor $F : \mathcal{A} \to \mathcal{B}$ is called *surjective on objects*, if the map

    $$\operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B},\ A \mapsto FA$$

    is surjective.

    A functor $F : \mathcal{A} \to \mathcal{B}$ is called *dense* if given $B \in \operatorname{Ob}\mathcal{B}$, there exists $A \in \operatorname{Ob}\mathcal{A}$ with $FA \cong B$.

    A functor $F : \mathcal{A} \to \mathcal{B}$ is called *full* if $F_{X,Y}$ is surjective for $X, Y \in \operatorname{Ob}\mathcal{A}$.

    A functor $F : \mathcal{A} \to \mathcal{B}$ is called *faithful* if $F_{X,Y}$ is injective for $X, Y \in \operatorname{Ob}\mathcal{A}$.

    Note that a functor F is an equivalence if and only if it is full, faithful and dense.

20. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{A}$ be functors. Let $1_{\mathcal{A}} \xrightarrow{\eta} (G \circ F)$ and $(F \circ G) \xrightarrow{\varepsilon} 1_{\mathcal{B}}$ be transformations. We call $(F, G, \eta, \varepsilon)$ an adjunction, if the following diagrams commute.

    $$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & F \circ G \circ F \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, equal] & G \circ F \circ G \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

    In this case, F is called left adjoint to G and G is called right adjoint to F. We also write $F \dashv G$. Furthermore, we call η the unit and ε the counit of the adjunction.

21. By $[\mathcal{A}, \mathcal{B}]$ we denote the *functor category* whose objects are the functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are the transformations between such functors.

22. Suppose $\mathcal{B}$ to be preadditive. For $F \underset{\beta}{\overset{\alpha}{\rightrightarrows}} G$ in $[\mathcal{A},\mathcal{B}]$, we have the transformation

    $\alpha+\beta : F \to G$ where $(\alpha+\beta)_X := \alpha_X + \beta_X$ for $X \in \operatorname{Ob}\mathcal{A}$.

    The category $[\mathcal{A},\mathcal{B}]$ is preadditive with respect to this addition.

23. Suppose $\mathcal{A}$ and $\mathcal{B}$ to be preadditive. By ${}_{\mathrm{add}}[\mathcal{A},\mathcal{B}]$ we denote the full subcategory of $[\mathcal{A},\mathcal{B}]$ with $\operatorname{Ob}{}_{\mathrm{add}}[\mathcal{A},\mathcal{B}] := \{F \in \operatorname{Ob}[\mathcal{A},\mathcal{B}] : F \text{ is additive}\}$.

24. Given $X \in \operatorname{Ob}\mathcal{A}$, we call $X \xrightarrow{e} X$ an *idempotent* in $\mathcal{A}$ if $e^2 = e$. A tuple (Y,π,ι) with $Y \in \operatorname{Ob}\mathcal{A}$, $\pi \in {}_{\mathcal{A}}(X,Y)$ and $\iota \in {}_{\mathcal{A}}(Y,X)$ is called an *image* of $X \xrightarrow{e} X$ in $\mathcal{A}$ if the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "\pi"'] & & X \arrow[dr, "\pi"] & \\
& Y \arrow[ur, "\iota"] \arrow[rr, "1_Y"] & & Y
\end{tikzcd}$$

    The category $\mathcal{A}$ is called *idempotent complete* if every idempotent in $\mathcal{A}$ has an image in $\mathcal{A}$; cf. Definitions 1 and 45 below.

25. Suppose $\mathcal{A}$ to be idempotent complete.

    Given an idempotent $X \xrightarrow{e} X$ in $\mathcal{A}$, we choose an image $(\operatorname{Im} e, \bar{e}, \dot{e})$ of e in $\mathcal{A}$.

    For $X \in \operatorname{Ob}\mathcal{A}$, choose $(\operatorname{Im} 1_X, \overline{1_X}, (1_X)^{\cdot}) = (X, 1_X, 1_X)$; cf. Stipulation 52 below.

26. Let R be a ring. By an R-module, we understand an R-left-module. The category of R-modules is denoted by R-Mod. By R-mod we denote the category of finitely generated R-modules.

    Let R-free be the full additive subcategory of R-mod with

$$\operatorname{Ob} R\text{-free} := \{R^n : n \in \mathbf{Z}_{\geqslant 0}\}.$$

    In particular, R-free is a skeleton of the category of finitely generated free R-modules.

27. Suppose I to be a set. Suppose given categories $\mathcal{A}_i$ for $i \in I$. We denote the product category by $\prod_{i\in I}\mathcal{A}_i$. For $m \in \mathbf{Z}_{\geqslant 0}$, we also write $\mathcal{A}_1 \times \cdots \times \mathcal{A}_m := \prod_{i\in[1,m]}\mathcal{A}_i$ and $\mathcal{A}^{\times m} := \prod_{i\in[1,m]}\mathcal{A}$.

28. Let R be a commutative ring.

    Given preadditive categories $(\mathcal{D},\varphi)$ and $(\mathcal{E},\psi)$ over R, we write ${}_{R\text{-lin}}[\mathcal{D},\mathcal{E}]$ for the full subcategory of $[\mathcal{D},\mathcal{E}]$ with

$$\operatorname{Ob}{}_{R\text{-lin}}[\mathcal{D},\mathcal{E}] := \{F \in \operatorname{Ob}[\mathcal{D},\mathcal{E}] : F \text{ is } R\text{-linear}\}.$$

    Given preadditive categories $(\mathcal{D},\varphi)$, $(\mathcal{E},\psi)$ and $(\mathcal{F},\eta)$ over R, we write ${}_{R\text{-bil}}[\mathcal{D}\times\mathcal{E},\mathcal{F}]$ for the full subcategory of $[\mathcal{D}\times\mathcal{E},\mathcal{F}]$ with

$$\operatorname{Ob}{}_{R\text{-bil}}[\mathcal{D}\times\mathcal{E},\mathcal{F}] := \{F \in \operatorname{Ob}[\mathcal{D}\times\mathcal{E},\mathcal{F}] : F \text{ is } R\text{-bilinear}\}.$$

    Cf. Definitions 23 and 27 below.

29. For $m \in \mathbf{Z}_{\geqslant 0}$, the symmetric group on m symbols is denoted by S_m.

Chapter 1

Preliminaries

1.1 Idempotents

For this §1.1, let $\mathcal{A}$ be a category.

Definition 1. Define $\operatorname{Idem}\mathcal{A} := \{e \in \operatorname{Mor}\mathcal{A} : e^2 = e\}$. The elements of $\operatorname{Idem}\mathcal{A}$ are called *idempotents.*

Remark 2. *Suppose given idempotents $X \xrightarrow{e} X$ and $Y \xrightarrow{f} Y$ in $\mathcal{A}$. Suppose given $\varphi \in {}_{\mathcal{A}}(X, Y)$ with $e\varphi f = \varphi$. We have $e\varphi = \varphi$ and $\varphi f = \varphi$.*

Proof. We have $e\varphi = ee\varphi f = e\varphi f = \varphi$. Similarly, we have $\varphi f = e\varphi f f = e\varphi f = \varphi$. $\square$

Remark 3. *Suppose given an isomorphism $X \xrightarrow{\varphi} Y$ in $\mathcal{A}$. Suppose given an idempotent $X \xrightarrow{e} X$. We have the idempotent $Y \xrightarrow{\varphi^{-1}e\varphi} Y$ in $\mathcal{A}$.*

Proof. We have $(\varphi^{-1}e\varphi)^2 = \varphi^{-1}e\varphi\varphi^{-1}e\varphi = \varphi^{-1}ee\varphi = \varphi^{-1}e\varphi$. $\square$

Remark 4. *Suppose $\mathcal{A}$ to be preadditive. Suppose given an idempotent $X \xrightarrow{e} X$ in $\mathcal{A}$. Suppose given an idempotent $X \xrightarrow{\varphi} X$ with $e\varphi e = \varphi$. We have the idempotent $X \xrightarrow{e-\varphi} X$ in $\mathcal{A}$.*

Proof. We have $(e - \varphi)^2 = e^2 - e\varphi - \varphi e + \varphi^2 \overset{\text{R2}}{=} e - \varphi - \varphi + \varphi = e - \varphi$. $\square$

Remark 5. *Suppose given a category $\mathcal{B}$ and a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$. Suppose given idempotents $X \xrightarrow{e} X$ and $Y \xrightarrow{f} Y$ in $\mathcal{A}$. Suppose given $\varphi \in {}_{\mathcal{A}}(X, Y)$ with $e\varphi f = \varphi$.*

The following assertions (1, 2) *hold:*

(1) *The morphism $FX \xrightarrow{Fe} FX$ is an idempotent in $\mathcal{B}$.*

(2) *We have $Fe \cdot F\varphi \cdot Ff = F\varphi$.*

Proof. Ad (1). We have $Fe \cdot Fe = F(ee) = Fe$.

Ad (2). We have $Fe \cdot F\varphi \cdot Ff = F(e\varphi f) = F\varphi$. $\square$

1.2 A lemma on equivalences

Lemma 6. *Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$. Suppose F to be surjective on objects. Suppose given a map $\operatorname{Ob}\mathcal{D} \to \operatorname{Ob}\mathcal{C}$, $X \mapsto X'$ such that $FX' = X$ for $X \in \operatorname{Ob}\mathcal{D}$. Suppose the following assertions* (1, 2) *to hold.*

(1) *For $X, Y \in \operatorname{Ob}\mathcal{D}$, the map $F_{X',Y'} : {}_{\mathcal{C}}(X', Y') \to {}_{\mathcal{D}}(X, Y), \varphi \mapsto F\varphi$ is bijective.*

(2) *For $A, B \in \operatorname{Ob}\mathcal{C}$ with $FA = FB$, we have $A \cong B$.*

Then F is an equivalence.

Proof. Since F is surjective on objects, F is dense. Therefore, it suffices to show that F is full and faithful.

Suppose given $A, B \in \operatorname{Ob}\mathcal{C}$. We show that $F_{A,B} : {}_{\mathcal{C}}(A, B) \to {}_{\mathcal{D}}(FA, FB)$, $\varphi \mapsto F\varphi$ is a bijection.

Since (2) holds, we have isomorphisms $A \xrightarrow[\sim]{\alpha} (FA)'$ and $B \xrightarrow[\sim]{\beta} (FB)'$.

This gives a bijection

$$f : {}_{\mathcal{C}}(A, B) \to {}_{\mathcal{C}}((FA)', (FB)'),\ \varphi \mapsto \alpha^{-1}\varphi\beta.$$

Furthermore, we obtain a bijection

$$g : {}_{\mathcal{D}}(FA, FB) \to {}_{\mathcal{D}}(FA, FB),\ \psi \mapsto F\alpha \cdot \psi \cdot F\beta^{-1}.$$

Suppose given $\varphi \in {}_{\mathcal{C}}(A, B)$. We have

$$\begin{aligned} \varphi(f \cdot F_{(FA)',(FB)'} \cdot g) &= (\alpha^{-1}\varphi\beta)(F_{(FA)',(FB)'} \cdot g) \\ &= (F(\alpha^{-1}\varphi\beta))g \\ &= F\alpha \cdot F\alpha^{-1} \cdot F\varphi \cdot F\beta \cdot F\beta^{-1} \\ &= F\varphi \\ &= \varphi F_{A,B}\,. \end{aligned}$$

Because (1) holds, $F_{A,B} = f \cdot F_{(FA)',(FB)'} \cdot g$ is a bijection. $\square$

1.3 The tensor product of modules and linear maps

For this §1.3, let R be a commutative ring.

1.3.1 The tensor product of modules

In this §1.3.1 we establish the tensor product of a finite number of modules. Instead of defining the tensor product of two modules and then defining the tensor product of a finite number of modules inductively, we follow the idea of Atiyah and Macdonald [**1,** Prop. 2.12*] and do it at once. This is useful in §3.1.1.

For this §1.3.1, let $n \in \mathbf{Z}_{\geqslant 0}$, $M_i \in \operatorname{Ob} R\text{-Mod}$ for $i \in [1, n]$ and $X \in \operatorname{Ob} R\text{-Mod}$.

Definition 7. We say that a map $f : \prod_{i\in[1,n]} M_i \to X$ is *R-multilinear*, if the following property (ML) holds.

(ML) Suppose given $i \in [1, n]$. Suppose given $m'_i, m''_i \in M_i$. Suppose given $m_j \in M_j$ for $j \in [1, n]\backslash\{i\}$. Suppose given $r', r'' \in R$. Then we have

$$\begin{aligned} & (m_1, \ldots, m_{i-1}, r'm'_i + r''m''_i, m_{i+1}, \ldots, m_n)f \\ = \; & r'(m_1, \ldots, m_{i-1}, m'_i, m_{i+1}, \ldots, m_n)f + r''(m_1, \ldots, m_{i-1}, m''_i, m_{i+1}, \ldots, m_n)f. \end{aligned}$$

For $n = 2$, we often write *R-bilinear* instead of R-multilinear.

Remark 8. Let $f : \prod_{i\in[1,n]} M_i \to X$ be a map.

The following assertions (1, 2) are equivalent.

(1) The map f is R-multilinear.

(2) For $i \in [1, n]$ and $(m_j)_{j\in[1,n]\backslash\{i\}} \in \prod_{j\in[1,n]\backslash\{i\}} M_j$, the map

$$M_i \to X, a \mapsto (m_1, \ldots, m_{i-1}, a, m_{i+1}, \ldots, m_n)f$$

is R-linear.

Definition 9. Define

$$_{R\text{-mul}}\big(\Pi_{i\in[1,n]} M_i \, , X\big) := \{f : \Pi_{i\in[1,n]} M_i \to X : f \text{ is } R\text{-multilinear}\}.$$

For $f, g \in {}_{R\text{-mul}}(\Pi_{i\in[1,n]} M_i \, , X)$, we define

$$f + g : \Pi_{i\in[1,n]} M_i \to X, m \mapsto mf + mg \, .$$

For $f \in {}_{R\text{-mul}}(\Pi_{i\in[1,n]} M_i \, , X)$ and $r \in R$, we define

$$r \cdot f : \Pi_{i\in[1,n]} M_i \to X, m \mapsto r \cdot (mf) \, .$$

Remark 10. The following assertions (1, 2, 3) hold.

(1) For $f, g \in {}_{R\text{-mul}}(\Pi_{i\in[1,n]} M_i \, , X)$, we have $f + g \in {}_{R\text{-mul}}(\Pi_{i\in[1,n]} M_i \, , X)$.

(2) For $f \in {}_{R\text{-mul}}(\Pi_{i\in[1,n]} M_i \, , X)$ and $r \in R$, we have $rf \in {}_{R\text{-mul}}(\Pi_{i\in[1,n]} M_i \, , X)$.

(3) We have an R-module ${}_{R\text{-mul}}(\Pi_{i\in[1,n]} M_i \, , X)$ with respect to the operations in (1, 2).

Example 11. We examine Definition 7 in the case $n = 0$.

We have $\prod_{i\in[1,n]} M_i = \{(\,)\} =: N$, where $(\,)$ is the empty tuple. Since the condition a map $f : N \to X$ has to fulfill to be R-multilinear is empty, we find that for $m \in M$ the map $f : N \to M, (\,) \mapsto m$ is R-multilinear.

Lemma 12. *Suppose given* $(X_i)_{i\in[1,n]} \xrightarrow{(f_i)_{i\in[1,n]}} (Y_i)_{i\in[1,n]}$ *in* $(R\text{-Mod})^{\times n}$. *Suppose given* $N \in \operatorname{Ob} R\text{-Mod}$ *and* $g \in {}_{R\text{-mul}}(\Pi_{i\in[1,n]} Y_i\,, N)$. *Suppose given* $N \xrightarrow{h} P$ *in* R-Mod.

Then

$$\left(\prod_{i\in[1,n]} f_i\right) gh : \prod_{i\in[1,n]} X_i \to P$$

is R*-multilinear.*

Proof. Suppose given $r', r'' \in R$. Suppose given $i \in [1,n]$. Suppose given $x_i', x_i'' \in X_i$. Suppose given $x_j \in X_j$ for $j \in [1,n]\backslash\{i\}$. For $j \in [1,n]\backslash\{i\}$, we define $x_j'' := x_j' := x_j$.

We have

$$\begin{aligned}
&(x_1\,, \ldots, x_{i-1}\,, r'x_i' + r''x_i''\,, x_{i+1}\,, \ldots, x_n)\left(\left(\prod_{i\in[1,n]} f_i\right) gh\right)\\
&\quad = (x_1 f_1\,, \ldots, x_{i-1} f_{i-1}\,, (r'x_i' + r''x_i'') f_i\,, x_{i+1} f_{i+1}, \ldots, x_n f_n) gh\\
&\quad = (x_1 f_1\,, \ldots, x_{i-1} f_{i-1}\,, r'(x_i' f_i) + r''(x_i'' f_i), x_{i+1} f_{i+1}, \ldots, x_n f_n) gh\\
&\quad = (r'((x_j' f_j)_{j\in[1,n]} g) + r''((x_j'' f_j)_{j\in[1,n]} g))h\\
&\quad = r'(x_j' f_j)_{j\in[1,n]} gh + r''(x_j'' f_j)_{j\in[1,n]} gh\\
&\quad = r'\left((x_j')_{j\in[1,n]}\left(\left(\prod_{i\in[1,n]} f_i\right) gh\right)\right) + r''\left((x_j'')_{j\in[1,n]}\left(\left(\prod_{i\in[1,n]} f_i\right) gh\right)\right).
\end{aligned}$$

□

Definition 13. Let C be the free R-module on the set $\prod_{i\in[1,n]} M_i$. Let D be the R-submodule of C generated by the elements of the form

$$\begin{gathered}
(m_1, \ldots, m_{i-1}, r'm_i' + r''m_i'', m_{i+1}, \ldots, m_n)\\
-r'(m_1, \ldots, m_{i-1}, m_i', m_{i+1}, \ldots, m_n)\\
-r''(m_1, \ldots, m_{i-1}, m_i'', m_{i+1}, \ldots, m_n)
\end{gathered}$$

for $i \in [1,n]$, $r', r'' \in R$, m_i', $m_i'' \in M_i$, $m_j \in M_j$ for $j \in [1,n]\backslash\{i\}$.

Define the *tensor product over* R of $M_1, \ldots, M_n$ as $\bigotimes_{R\, i\in[1,n]} M_i := C/D$.

We also write $M_1 \underset{R}{\otimes} \cdots \underset{R}{\otimes} M_n := \bigotimes_{R\, i\in[1,n]} M_i$ for the tensor product.

If unambiguous, we often write $M_1 \otimes \cdots \otimes M_n := \bigotimes_{i\in[1,n]} M_i := \bigotimes_{R\, i\in[1,n]} M_i$.

For $(m_1, \ldots, m_n) \in \prod_{i\in[1,n]} M_i$, we write

$$m_1 \otimes \cdots \otimes m_n := \otimes_{i\in[1,n]} m_i := (m_1, \ldots, m_n) + C.$$

Let

$$\begin{array}{ccc}
\prod_{i\in[1,n]} M_i & \xrightarrow{\mu_{(M_1,\ldots,M_n)}} & \bigotimes_{R\, i\in[1,n]} M_i\,,\\
(m_1, \ldots, m_n) & \mapsto & \otimes_{i\in[1,n]} m_i\,.
\end{array}$$

If unambiguous, we often write $\mu := \mu_{(M_1,\ldots,M_n)}$.

Lemma 14. *The following assertions* (1, 2) *hold.*

(1) *The map μ is R-multilinear.*

(2) *The set* $\operatorname{Im}\mu = \left\{\otimes_{i\in[1,n]} m_i : (m_1,\ldots,m_n) \in \prod_{i\in[1,n]} M_i\right\}$ *is an R-linear generating set for* $\bigotimes\limits_{R}{}_{i\in[1,n]} M_i$.

Proof. *Ad* (1). Suppose given $i \in [1,n]$. Suppose given m_i', $m_i'' \in M_i$. Suppose given $r', r'' \in R$. Suppose given $m_j \in M_j$ for $j \in [1,n]\backslash\{i\}$. Let $m_j'' := m_j' := m_j$ for $j \in [1,n]\backslash\{i\}$.

We have

$$\begin{array}{rl} & (m_1,\ldots,m_{i-1}, r'm_i' + r''m_i'', m_{i+1},\ldots,m_n)\mu \\ = & m_1 \otimes \cdots \otimes m_{i-1} \otimes (r'm_i' + r''m_i'') \otimes m_{i+1} \otimes \cdots \otimes m_n \\ = & r'\left(\otimes_{j\in[1,n]} m_j'\right) + r''\left(\otimes_{j\in[1,n]} m_j''\right) \\ = & r'\left((m_j')_{j\in[1,n]}\right)\mu + r''\left((m_j'')_{j\in[1,n]}\right)\mu. \end{array}$$

Therefore, μ is R-multilinear.

Ad (2). Since $\prod_{i\in[1,n]} M_i$ is an R-linear basis of C, we obtain the R-linear generating set

$$\begin{aligned} \operatorname{Im}\mu &= \left\{ m_1 \otimes \cdots \otimes m_n : (m_1,\ldots,m_n) \in \prod_{i\in[1,n]} M_i \right\} \\ &= \left\{ (m_1,\ldots,m_n) + D : (m_1,\ldots,m_n) \in \prod_{i\in[1,n]} M_i \right\} \end{aligned}$$

for $\bigotimes_{i\in[1,n]} M_i = C/D$. □

Example 15. We examine Definition 13 in the case $n = 0$.

Then we have $C = {}_R\langle (\,) \rangle \cong R$ and $D = 0$. Therefore, we obtain $\bigotimes\limits_{R}{}_{i\in[1,0]} M_i \cong R$.

Lemma 16. *Let $Y \in \operatorname{Ob} R\text{-Mod}$. Suppose given $f \in {}_{R\text{-mul}}(\prod_{i\in[1,n]} M_i, Y)$.*

There exists a unique R-linear map $\bar{f} : \bigotimes\limits_{R}{}_{i\in[1,n]} M_i \to Y$ such that $\mu\bar{f} = f$.

Proof. Since $\prod_{i\in[1,n]} M_i$ is an R-linear basis of C, the map f extends to an R-linear map $\hat{f} : C \to Y, (m_1,\ldots,m_n) \mapsto (m_1,\ldots,m_n)f$. Since f is R-multilinear, D is contained in $\operatorname{Ker}\hat{f}$. Thus, $\hat{f}$ induces a unique R-linear map $\bar{f} : \bigotimes_{i\in[1,n]} M_i \to X$ with

$$(m_1 \otimes \cdots \otimes m_n)\bar{f} = (m_1,\ldots,m_n)\hat{f} = (m_1,\ldots,m_n)f$$

for $(m_i)_{i\in[1,n]} \in \prod_{i\in[1,n]} M_i$. Therefore, we have $\mu\bar{f} = f$.

Suppose given an R-linear map $g : \bigotimes_{i\in[1,n]} M_i \to X$ such that $\mu g = f$. Then we have $\mu g = f = \mu\bar{f}$. Since $\operatorname{Im}\mu$ is an R-linear generating set for $\bigotimes_{i\in[1,n]} M_i$ by Lemma 14.(2), we obtain $g = f$.

□

Lemma 17. *We have the R-linear isomorphism*

$$\begin{array}{rcl} {}_{R\text{-mul}}(\prod_{i\in[1,n]} M_i\,,X) & \longleftarrow & {}_{R\text{-Mod}}(\bigotimes_{R}{}_{i\in[1,n]} M_i\,,M) \\ \mu g & \longleftarrow\!\!\shortmid & g. \end{array}$$

Proof. By Lemma 14.(1) and Lemma 12, this map is welldefined. By Lemma 16, this map is bijective. We show that it is R-linear.

Suppose given $r', r'' \in R$. Suppose given $f, g \in {}_{R\text{-Mod}}(\bigotimes_{R}{}_{i\in[1,n]} M_i\,, M)$. Suppose given $m_i \in M_i$ for $i \in [1, n]$. We have

$$\begin{aligned} (m_1,\ldots,m_n)\mu(r'f + r''g) &= (m_1 \otimes \cdots \otimes m_n)(r'f + r''g) \\ &= (m_1 \otimes \cdots \otimes m_n)(r'f) + (m_1 \otimes \cdots \otimes m_n)(r''g) \\ &= r'(m_1 \otimes \cdots \otimes m_n)f + r''(m_1 \otimes \cdots \otimes m_n)g \\ &= r'(m_1,\ldots,m_n)\mu f + r''(m_1,\ldots,m_n)\mu g \\ &= (m_1,\ldots,m_n)(r'\mu f) + (m_1,\ldots,m_n)(r''\mu g) \\ &= (m_1,\ldots,m_n)(r'\mu f + r''\mu g). \end{aligned}$$

□

Remark 18. *Suppose that $n \geqslant 1$. Suppose given $x \in \bigotimes_{R}{}_{i\in[1,n]} M_i$. There exist $s \in \mathbf{Z}_{\geqslant 0}$ and $m_{i,j} \in M_i$ for $(i, j) \in [1, n] \times [1, s]$ with*

$$x = \sum_{j\in[1,s]} \otimes_{i\in[1,n]} m_{i,j}\ .$$

Proof. Since $\operatorname{Im}\mu$ is an R-linear generating set for $\bigotimes_{i\in[1,n]} M_i$, there exist $s \in \mathbf{Z}_{\geqslant 0}$, $r_j \in R$ for $j \in [1, s]$ and $m_{i,j} \in M_i$ for $(i, j) \in [1, n] \times [1, s]$ with

$$x = \sum_{j\in[1,s]} r_j \cdot (m_{1,j} \otimes \cdots \otimes m_{n,j}) = \sum_{j\in[1,s]} (r_j m_{1,j}) \otimes \cdots \otimes m_{n,j}\ .$$

□

Lemma 19. *Suppose given $j \in [2, n]$. There exists a unique R-linear isomorphism*

$$\psi : \Big(\bigotimes_{R}{}_{i\in[1,\,j-1]} M_i\Big) \underset{R}{\otimes} \Big(\bigotimes_{R}{}_{i\in[j,\,n]} M_i\Big) \xrightarrow{\sim} \bigotimes_{R}{}_{i\in[1,\,n]} M_i$$

with

$$\big((m_1 \otimes \cdots \otimes m_{j-1}) \otimes (m_j \otimes \cdots \otimes m_n)\big)\psi = m_1 \otimes \cdots \otimes m_n$$

for $(m_1\,, \ldots, m_n) \in \prod_{i\in[1,n]} M_i$.

Proof. Define

$$T_1 := \bigotimes_{i\in[1,j-1]} M_i \quad,\quad T_2 := \bigotimes_{i\in[j,n]} M_i \quad,\quad T := \bigotimes_{i\in[1,n]} M_i\,.$$

Let

$$\begin{aligned}
\mu &:= \mu_{M_1,\ldots,M_n} : \textstyle\prod_{i\in[1,n]} M_i \to T, \\
\mu_1 &:= \mu_{M_1,\ldots,M_{j-1}} : \textstyle\prod_{i\in[1,j-1]} M_i \to T_1\,, \\
\mu_2 &:= \mu_{M_j,\ldots,M_n} : \textstyle\prod_{i\in[j,n]} M_i \to T_2\,, \\
\mu_3 &:= \mu_{T_1,T_2} : T_1 \times T_2 \to T_1 \otimes T_2\,, \\
\tilde{\mu} &:= (\mu_1 \times \mu_2)\mu_3 : \textstyle\prod_{i\in[1,n]} M_i \to T_1 \otimes T_2\,;
\end{aligned}$$

cf. Definition 13.

We show that $\tilde{\mu}$ is R-multilinear.

Suppose given $r', r'' \in R$. Suppose given $k \in [1,n]$. Suppose given $m'_k, m''_k \in M_k$. Suppose given $m_i \in M_i$ for $i \in [1,n]\backslash\{k\}$. For $i \in [1,n]\backslash\{k\}$ we define $m'_i := m''_i := m_i$.

Case $k \in [1, j-1]$. We have

$$\begin{aligned}
(m_1, \ldots, &m_{k-1}, r'm'_k + r''m''_k, m_{k+1}, \ldots, m_n)\tilde{\mu} \\
&= ((m_1, \ldots, m_{k-1}, r'm'_k + r''m''_k, m_{k-1}, \ldots, m_{j-1})\mu_1, (m_j, \ldots, m_n)\mu_2)\,\mu_3 \\
&= (m_1 \otimes \cdots \otimes m_{k-1} \otimes (r'm'_k + r''m''_k) \otimes m_{k+1} \cdots \otimes m_{j-1}) \otimes (\otimes_{i\in[j,n]} m_i) \\
&= \big(r' \cdot (\otimes_{i\in[1,j-1]} m'_i) + r'' \cdot (\otimes_{i\in[1,j-1]} m''_i)\big) \otimes (\otimes_{i\in[j,n]} m_i) \\
&= r'\big((\otimes_{i\in[1,j-1]} m'_i) \otimes (\otimes_{i\in[j,n]} m_i)\big) + r''\big((\otimes_{i\in[1,j-1]} m''_i) \otimes (\otimes_{i\in[j,n]} m_i)\big) \\
&= r' \cdot \big((m'_i)_{i\in[1,n]}\big)\,\tilde{\mu} + r'' \cdot \big((m''_i)_{i\in[1,n]}\big)\,\tilde{\mu}\,.
\end{aligned}$$

Case $k \in [j, n]$. This follows analoguosly to the first case.

Therefore, $\tilde{\mu}$ is R-multilinear.

By Lemma 16, there exists a unique R-linear map $\varphi : T \to T_1 \otimes T_2$ with

$$(\otimes_{i\in[1,n]} m_i)\varphi = (m_1, \ldots, m_n)\tilde{\mu} = (\otimes_{i\in[1,j-1]} m_i) \otimes (\otimes_{i\in[j,n]} m_i)$$

for $(m_1, \ldots, m_n) \in \prod_{i\in[1,n]} M_i$.

For $m = (m_1, \ldots, m_{j-1}) \in \prod_{i\in[1,j-1]} M_i$, we define

$$mf : \prod_{i\in[j,n]} M_i \to T, (a_j, \ldots, a_n) \mapsto (m_1, \ldots, m_{j-1}, a_j, \ldots, a_n)\mu\,.$$

First we show that mf is R-multilinear. Suppose given $r', r'' \in R$. Suppose given $k \in [j, n]$. Suppose given $a'_k, a''_k \in M_k$. Suppose given $a_i \in M_i$ for $i \in [j,n]\backslash\{k\}$. Define $a'_i := a''_i := a_i$ for $i \in [j,n]\backslash\{k\}$. Let $a' = (a'_j, \ldots, a'_n)$ and $a'' := (a''_j, \ldots, a''_n)$. We have

$$\begin{aligned}
(a_j, \ldots, a_{k-1}, &r'a''_k + r''a''_k, a_{k+1}, \ldots, a_n)(mf) \\
&= (m_1, \ldots, m_{j-1}, a_j, \ldots, a_{k-1}, r'a'_k + r''a''_k, a_{k+1}, \ldots, a_n)\mu \\
&= r'(m, a')\mu + r''(m, a'')\mu \\
&= r'(a'(mf)) + r''(a''(mf))\,.
\end{aligned}$$

Thus, mf is R-multilinear. Therefore, we have the map

$$f : \prod_{i\in[1,j-1]} M_i \to {}_{R\text{-mul}}\Big(\prod_{i\in[j,n]} M_i, T\Big), m \mapsto mf;$$

cf. Definition 9. Now we show that f is R-multilinear.

Suppose given $r', r'' \in R$. Suppose given $k \in [1, j-1]$. Suppose given $m'_k, m''_k \in M_k$. Suppose given $m_i \in M_i$ for $i \in [1, j-1]\backslash\{k\}$. For $i \in [1, j-1]\backslash\{k\}$ define $m'_i := m''_i := m_i$. Furthermore, we define $m' = (m'_1, \ldots, m'_{j-1})$ and $m'' := (m''_1, \ldots, m''_{j-1})$.

Suppose given $a \in \prod_{i \in [j,n]} M_i$. We have

$$\begin{aligned}
a\big((m_1, \ldots, m_{k-1}, &r'm'_k + r''m''_k, m_{k+1}, \ldots, m_{j-1})f\big) \\
&= (m_1, \ldots, m_{k-1}, r'm'_k + r''m''_k, m_{k+1}, \ldots, m_{j-1}, a)\mu \\
&= r'(m', a)\mu + r''(m'', a)\mu \\
&= r'(a(m'f)) + r''(a(m''f)) \\
&= a(r'(m'f) + r''(m''f))\,.
\end{aligned}$$

Thus, we have

$$(m_1, \ldots, m_{k-1}, r'm'_k + r''m''_k, m_{k+1}, \ldots, m_k)f = r'(m'f) + r''(m''f).$$

Therefore, f is R-multilinear.

By Lemma 16, there exists a unique R-linear map $\hat{f} : T_1 \to {}_{R\text{-mul}}(\Pi_{i \in [j,n]} M_i, T)$ with $\mu_1 \hat{f} = f$. So,

$$(\otimes_{i \in [1,j-1]} m_i)\hat{f} : (m_i)_{i \in [j,n]} \mapsto \otimes_{i \in [1,n]} m_i.$$

Suppose given $x \in T_1$. Since $x\hat{f} \in {}_{R\text{-mul}}(\Pi_{i \in [j,n]} M_i\,, T)$, there exists a unique R-linear map $\overline{x\hat{f}} : T_2 \to T$ with $\mu_2 \overline{x\hat{f}} = x\hat{f}$; cf. Lemma 16.(1).

Define

$$g : T_1 \times T_2 \to T, (x, y) \mapsto y\overline{\left(x\hat{f}\right)}.$$

We show that g is R-bilinear.

Suppose given $r', r'' \in R$. Suppose given $x, x', x'' \in T_1$. Suppose given $y, y', y'' \in T_2$.

Since $\hat{f}$ is R-linear, we have

$$\begin{aligned}
(r'x' + r''x'', y)g &= y\overline{\left((r'x' + r''x'')\hat{f}\right)} \\
&= y\overline{\left(r'(x'\hat{f}) + r''(x''\hat{f})\right)} \\
&\overset{\text{L17}}{=} y(r' \cdot \overline{x'\hat{f}} + r'' \cdot \overline{x''\hat{f}}) \\
&\overset{\text{D9}}{=} r' \cdot y\overline{\left(x'\hat{f}\right)} + r'' \cdot y\overline{\left(x''\hat{f}\right)} \\
&= r' \cdot (x', y)g + r'' \cdot (x'', y)g.
\end{aligned}$$

Since $\overline{x\hat{f}}$ is R-linear, we have

$$(x, r'y' + r''y'')g = (r'y' + r''y'')\overline{x\hat{f}} = r' \cdot y'\overline{\left(x\hat{f}\right)} + r'' \cdot y''\overline{\left(x\hat{f}\right)} = r' \cdot (x, y')g + r'' \cdot (x, y'')g.$$

Thus, g is R-bilinear.

Therefore, there exists a unique R-linear map $\psi : T_1 \otimes T_2 \to T$ with $\mu_3 \psi = g$; cf. Lemma 16.

Suppose given $m_i \in M_i$ for $i \in [1, n]$. We have

$$\begin{aligned}((\otimes_{i\in[1,j-1]}m_i) \otimes (\otimes_{i\in[j,n]}m_i))\psi &= ((\otimes_{i\in[1,j-1]}m_i), (\otimes_{i\in[j,n]}m_i))g \\ &= (\otimes_{i\in[j,n]}m_i)\overline{(\otimes_{i\in[1,j-1]}m_i)\hat{f}} \\ &= (m_j, \ldots, m_n)((\otimes_{i\in[1,j-1]}m_i)\hat{f}) \\ &= (m_j, \ldots, m_n)((m_1, \ldots, m_{j-1})f) \\ &= (m_1, \ldots, m_{j-1}, m_j, \ldots, m_n)\mu \\ &= \otimes_{i\in[1,n]}m_i\,.\end{aligned}$$

Thus, we have

$$(\otimes_{i\in[1,n]}m_i)(\varphi\psi) = ((\otimes_{i\in[1,j-1]}m_i) \otimes (\otimes_{i\in[j,n]}m_i))\psi = \otimes_{i\in[1,n]}m_i$$

and

$$((\otimes_{i\in[1,j-1]}m_i) \otimes (\otimes_{i\in[j,n]}m_i))(\psi\varphi) = (\otimes_{i\in[1,n]}m_i)\varphi = (\otimes_{i\in[1,j-1]}m_i) \otimes (\otimes_{i\in[j,n]}m_i).$$

Since $\operatorname{Im}\mu$ is an R-linear generating set of T and $\operatorname{Im}\mu_3$ is an R-linear generating set of $T_1 \otimes T_2$; cf. Lemma 14.(2); we have $\varphi\psi = 1_T$ and $\psi\varphi = 1_{T_1\otimes T_2}$.

Thus, φ and ψ are mutually inverse R-linear isomorphisms. $\square$

1.3.2 The tensor product of linear maps

For this §1.3.2, let $n \in \mathbf{Z}_{\geqslant 0}$.

Definition 20 (and Lemma)**.** Suppose given $(M_i)_{i\in[1,n]} \xrightarrow{(f_i)_{i\in[1,n]}} (N_i)_{i\in[1,n]}$ in $(R\text{-Mod})^{\times n}$.

We have the R-linear map

$$\begin{array}{ccc} \bigotimes\limits_R{}_{i\in[1,n]}\, M_i & \xrightarrow{\otimes_{i\in[1,n]}f_i} & \bigotimes\limits_R{}_{i\in[1,n]}\, N_i\,, \\ \sum\limits_{j\in[1,m]} \otimes_{i\in[1,n]}m_{i,j} & \longmapsto & \sum\limits_{j\in[1,m]} \otimes_{i\in[1,n]}(m_{i,j}f_i)\,. \end{array}$$

We often write $f_1 \otimes \ldots \otimes f_n := \otimes_i f_i := \otimes_{i\in[1,n]} f_i$.

Proof. By Lemma 12,

$$\begin{array}{ccc} \prod\limits_{i\in[1,n]} M_i & \xrightarrow{(\prod_{i\in[1,n]} f_i)\mu_{N_1,\ldots,N_n}} & \bigotimes\limits_{i\in[1,n]} N_i\,, \\ (m_i)_{i\in[1,n]} & \longmapsto & \otimes_{i\in[1,n]}(m_i f_i), \end{array}$$

is R-multilinear. For brevity, we write $f := \left(\prod_{i\in[1,n]} f_i\right) \mu_{N_1,\ldots,N_n}$.

By Lemma 16, there exists a unique R-linear map

$$\bar{f} : \bigotimes_{i\in[1,n]} M_i \to \bigotimes_{i\in[1,n]} N_i$$

with $(\otimes_{i\in[1,n]} m_i)\bar{f} = (m_i)_{i\in[1,n]} f = \otimes_{i\in[1,n]}(m_i f_i)$ for $(m_i)_{i\in[1,n]} \in \prod_{i\in[1,n]} M_i$.

Suppose given $\sum_{j\in[1,m]}(\otimes_{i\in[1,n]} m_{i,j}) \in \bigotimes_{i\in[1,n]} M_i$. We have

$$\Bigg(\sum_{j\in[1,m]} \otimes_{i\in[1,n]} m_{i,j}\Bigg)\bar{f} = \sum_{j\in[1,m]} (\otimes_{i\in[1,n]} m_{i,j})\bar{f} = \sum_{j\in[1,m]} \otimes_{i\in[1,n]}(m_{i,j} f_i).$$

Thus, the assertion follows with $\otimes_{i\in[1,n]} f_i := \bar{f}$. $\square$

Definition 21 (and Lemma)**.** *We have the functor*

$$\begin{aligned} (R\text{-Mod})^{\times n} &\xrightarrow{\bigotimes_{R}{}_{i\in[1,n]}} R\text{-Mod}, \\ \Big((M_i)_{i\in[1,n]} \xrightarrow{(f_i)_{i\in[1,n]}} (N_i)_{i\in[1,n]}\Big) &\mapsto \Big(\underset{R}{\bigotimes}_{i\in[1,n]} M_i \xrightarrow{\otimes_{i\in[1,n]} f_i} \underset{R}{\bigotimes}_{i\in[1,n]} N_i\Big). \end{aligned}$$

If unambiguous, we often write $\bigotimes := \bigotimes_{i\in[1,n]} := \underset{R}{\bigotimes}_{i\in[1,n]}$.

Proof. Suppose given

$$(M_i)_{i\in[1,n]} \xrightarrow{(f_i)_{i\in[1,n]}} (N_i)_{i\in[1,n]} \xrightarrow{(g_i)_{i\in[1,n]}} (P_i)_{i\in[1,n]}$$

in $(R\text{-Mod})^{\times n}$.

We have

$$1_{(M_i)_{i\in[1,n]}} = (1_{M_i})_{i\in[1,n]} \overset{\otimes}{\longmapsto} \otimes_{i\in[1,n]} 1_{M_i} .$$

Suppose given $\otimes_{i\in[1,n]} m_i \in \bigotimes_{i\in[1,n]} M_i$. We have

$$(\otimes_{i\in[1,n]} m_i)(\otimes_{i\in[1,n]} 1_{M_i}) = \otimes_{i\in[1,n]}(m_i 1_{M_i}) = \otimes_{i\in[1,n]} m_i = (\otimes_{i\in[1,n]} m_i) 1_{\bigotimes_{i\in[1,n]} M_i} .$$

By Lemma 14.(2), we have

$$1_{(M_i)_{i\in[1,n]}} \overset{\otimes}{\longmapsto} \otimes_{i\in[1,n]} 1_{M_i} = 1_{\bigotimes_{i\in[1,n]} M_i} .$$

Suppose given $\otimes_{i\in[1,n]} m_i \in \bigotimes_{i\in[1,n]} M_i$. We have

$$\begin{aligned} (\otimes_{i\in[1,n]} m_i)((\otimes_{i\in[1,n]} f_i)(\otimes_{i\in[1,n]} g_i)) &= ((\otimes_{i\in[1,n]} m_i)(\otimes_{i\in[1,n]} f_i))(\otimes_{i\in[1,n]} g_i) \\ &= (\otimes_{i\in[1,n]}(m_i f_i))(\otimes_{i\in[1,n]} g_i) \\ &= \otimes_{i\in[1,n]}((m_i f_i) g_i) \\ &= \otimes_{i\in[1,n]}((m_i)(f_i g_i)) \\ &= (\otimes_{i\in[1,n]} m_i)(\otimes_{i\in[1,n]}(f_i g_i)). \end{aligned}$$

By Lemma 14.(2), we have

$$\begin{aligned} ((f_i)_{i\in[1,n]} \cdot (g_i)_{i\in[1,n]}) &\overset{\otimes}{\longmapsto} (\otimes_{i\in[1,n]} f_i)(\otimes_{i\in[1,n]} g_i) \\ &= \otimes_{i\in[1,n]}(f_i g_i) \overset{\otimes}{\longleftarrow\!\shortmid} (f_i g_i)_{i\in[1,n]} . \end{aligned}$$

Thus, we have a functor $\bigotimes$ indeed. $\square$

1.4 Preadditive categories over a commutative ring

In this §1.4 we establish the notion of a preadditive category over a commutative ring. For this type of categories we shall define the tensor product below; cf. §3.1.

For this §1.4, let R be a commutative ring.

Remark 22. *Let $\mathcal{A}$ be a preadditive category. Then* $\operatorname{End} 1_{\mathcal{A}} = {}_{[\mathcal{A},\mathcal{A}]}(1_{\mathcal{A}}, 1_{\mathcal{A}})$ *is a commutative ring.*

Proof. Recall that $\operatorname{End} 1_{\mathcal{A}}$ is a ring. We have to prove commutativity.

Suppose given $\alpha, \beta \in \operatorname{End} 1_{\mathcal{A}}$. Suppose given $X \in \operatorname{Ob} \mathcal{A}$. We have $\beta_X \in {}_{\mathcal{A}}(X, X)$. Since α is natural, we have the following commutative diagram.

$$\begin{tikzcd} 1_{\mathcal{A}}X \arrow[r, "1_{\mathcal{A}}\beta_X"] \arrow[d, "\alpha_X"'] & 1_{\mathcal{A}}X \arrow[d, "\alpha_X"] \\ 1_{\mathcal{A}}X \arrow[r, "1_{\mathcal{A}}\beta_X"] & 1_{\mathcal{A}}X \end{tikzcd}$$

Thus, we have $\alpha_X \beta_X = \beta_X \alpha_X$. Since $X \in \operatorname{Ob} \mathcal{A}$ was arbitrary, we have $\alpha\beta = \beta\alpha$. Therefore, $\operatorname{End} 1_{\mathcal{A}}$ is commutative.

$\square$

Definition 23.

(1) Let $\mathcal{A}$ be a preadditive category. Let $\varphi : R \to \operatorname{End} 1_{\mathcal{A}}$ be a ring morphism.

We call $(\mathcal{A}, \varphi)$ a preadditive category *over* R or an R*-linear* preadditive category. We often refer to just $\mathcal{A}$ as a preadditive category over R.

Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$. Suppose given $r \in R$. We often write

$$r \cdot f := rf := (r\varphi)_X f.$$

We have

$$\begin{aligned} r(fg) &= (r\varphi)_X(fg) = ((r\varphi)_X f)g = (rf)g \\ &= f(r\varphi)_Y g = f((r\varphi_Y)g) = f(rg). \end{aligned}$$

(2) Let $(\mathcal{A}, \varphi)$ be a preadditive category over R. Let $\mathcal{A}$ be additive. We call $(\mathcal{A}, \varphi)$ an additive category *over* R or an R*-linear* additive category.

(3) Let $(\mathcal{C}, \varphi)$ and $(\mathcal{D}, \psi)$ be preadditive categories over R. Let $F \in {}_{\text{add}}[\mathcal{C}, \mathcal{D}]$. We call F an R*-linear* functor if

$$F(rf) = F((r\varphi)_X \cdot f) = (r\psi)_{FX} \cdot Ff = r(Ff)$$

for $X \xrightarrow{f} Y$ in $\mathcal{C}$ and $r \in R$.

Remark 24. Let $(\mathcal{A}, \varphi)$ be an additive category over R. Suppose given

$$\bigoplus_{i \in [1,m]} A_{1,i} \xrightarrow{(f_{i,j})_{i,j}} \bigoplus_{j \in [1,n]} A_{2,j}$$

in $\mathcal{A}$. Suppose given $r \in R$. We have

$$r \cdot (f_{i,j})_{i,j} = (r \cdot f_{i,j})_{i,j} \,.$$

Proof. We have

$$\begin{aligned}
(r \cdot f_{i,j})_{i,j} &= (r \cdot \iota_{1,i} f \pi_{2,j})_{i,j} \\
&= ((r\varphi)_{A_{1,i}} \iota_{1,i} f \pi_{2,j})_{i,j} \\
&= (\iota_{1,i} (r\varphi)_{A_{1,1} \oplus \cdots \oplus A_{1,m}} f \pi_{2,j})_{i,j} \\
&= (\iota_{1,i} (r \cdot f) \pi_{2,j})_{i,j} \\
&= ((r \cdot f)_{i,j})_{i,j} \\
&= r \cdot (f_{i,j})_{i,j}
\end{aligned}$$

□

Remark 25. Let $(\mathcal{C}, \varphi)$ and $(\mathcal{D}, \psi)$ be preadditive categories over R. Let $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$. The following assertions (1, 2) are equivalent.

(1) The functor F is R-linear.

(2) For $C' \underset{f_2}{\overset{f_1}{\rightrightarrows}} C''$ in $\mathcal{C}$ and $r_1, r_2 \in R$, we have $F(r_1 f_1 + r_2 f_2) = r_1 F f_1 + r_2 F f_2$.

Remark 26. *Let $(\mathcal{A}, \varphi)$ be a preadditive category over R. Suppose given $X, Y \in \mathrm{Ob}\,\mathcal{A}$. We have the ring morphism*

$$\begin{aligned}
\varepsilon : \mathrm{End}\, 1_{\mathcal{A}} &\to \mathrm{End}_{\mathbb{Z}}\left({}_{\mathcal{A}}(X,Y) \right) \\
\alpha &\mapsto \Big((X \xrightarrow{f} Y) \mapsto (X \xrightarrow{\alpha_X f} Y) = (X \xrightarrow{f \alpha_Y} Y) \Big).
\end{aligned}$$

Therefore, we have the R-module $({}_{\mathcal{A}}(X,Y), \varphi\varepsilon)$, i.e. for $r \in R$ and $f \in {}_{\mathcal{A}}(X,Y)$, we have a module operation

$$r \cdot f := (r\varphi)_X f = f (r\varphi)_Y \,.$$

Proof. Suppose given $\alpha \in \mathrm{End}\, 1_{\mathcal{A}}$. Since α is natural, we have $\alpha_X f = f \alpha_Y$ for $f \in {}_{\mathcal{A}}(X,Y)$.

The map $f \mapsto \alpha_X f$ is a $\mathbf{Z}$-linear endomorphism of ${}_{\mathcal{A}}(X,Y)$ since $\mathcal{A}$ is preadditive. Thus, ε is a welldefined map. We show that ε is a ring morphism.

We have $f(1_{\mathrm{End}\, 1_{\mathcal{A}}} \varepsilon) = 1_X f = f$ for $f \in {}_{\mathcal{A}}(X,Y)$. Thus, we have $1_{\mathrm{End}\, 1_{\mathcal{A}}} \varepsilon = 1_{\mathrm{End}_{\mathbb{Z}}({}_{\mathcal{A}}(X,Y))}$.

Suppose given $\alpha, \beta \in \mathrm{End}\, 1_{\mathcal{A}}$. We have

$$f((\alpha + \beta)\varepsilon) = (\alpha + \beta)_X f = (\alpha_X + \beta_X) f = \alpha_X f + \beta_X f = f(\alpha\varepsilon) + f(\beta\varepsilon) = f(\alpha\varepsilon + \beta\varepsilon)$$

and

$$f((\alpha\beta)\varepsilon) = (\alpha\beta)_X f = \alpha_X \beta_X f = \alpha_X f \beta_Y = (f(\alpha\varepsilon))\beta_Y = f((\alpha\varepsilon)(\beta\varepsilon))$$

for $f \in {}_{\mathcal{A}}(X,Y)$. Thus, ε is a ring morphism.

Therefore, we have a ring morphism $\varphi\varepsilon : R \to \mathrm{End}_{\mathbb{Z}}({}_{\mathcal{A}}(X,Y))$.

Thus, we have an R-module $({}_{\mathcal{A}}(X,Y), \varphi\varepsilon)$. □

Definition 27. Let $(\mathcal{A},\varphi), (\mathcal{B},\psi)$ and $(\mathcal{C},\tau)$ be preadditive categories over R.
Let $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ be a functor. The functor F is called *R-bilinear* if

$$\begin{aligned} F(r_1 f_1 + r_2 f_2, g) &= r_1 F(f_1, g) + r_2 F(f_2, g), \\ F(f, r_1 g_1 + r_2 g_2) &= r_1 F(f, g_1) + r_2 F(f, g_2) \end{aligned}$$

for $r_1, r_2 \in R$, $A' \underset{f_2}{\overset{f_1}{\rightrightarrows}} A''$ and $A''' \xrightarrow{f} A''''$ in $\mathcal{A}$ and for $B' \underset{g_2}{\overset{g_1}{\rightrightarrows}} B''$ and $B''' \xrightarrow{g} B''''$ in $\mathcal{B}$.

Remark 28. *Let $(\mathcal{A},\varphi), (\mathcal{B},\psi)$ and $(\mathcal{C},\tau)$ be preadditive categories over R.
Let $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ be a functor. The following assertions* (1, 2) *are equivalent.*

(1) *The functor F is R-bilinear.*

(2) *Suppose given $A \in \mathrm{Ob}\,\mathcal{A}$ and $B \in \mathrm{Ob}\,\mathcal{B}$. Then*

$$\begin{aligned} F_A : \mathcal{B} \to \mathcal{C}, \big(B' \xrightarrow{g} B''\big) &\mapsto \big(F(A,B') \xrightarrow{F(A,g) = F(1_A, g)} F(A,B'')\big), \\ F_B : \mathcal{A} \to \mathcal{C}, \big(A' \xrightarrow{f} A''\big) &\mapsto \big(F(A',B) \xrightarrow{F(f,B) = F(f, 1_B)} F(A'',B)\big) \end{aligned}$$

are R-linear functors.

Proof. *Ad* (1) ⇒ (2). Because of the symmetry of the situation, we only have to show that F_B is an R-linear functor.

First of all, we show that F_B is a functor. Suppose given $A' \xrightarrow{f_1} A'' \xrightarrow{f_2} A'''$ in $\mathcal{A}$. We have

$$F_B(f_1 f_2) = F(f_1 f_2, 1_B) = F((f_1, 1_B)\cdot(f_2, 1_B)) = F(f_1, 1_B)\cdot F(f_2, 1_B) = F_B f_1 \cdot F_B f_2\,.$$

Furthermore, we have

$$F_B 1_{A'} = F(1_{A'}, 1_B) = 1_{F(A',B)} = 1_{F_B A'}\,.$$

Thus, F_B is a functor.

Now we show that F_B is R-linear. Suppose given $A' \underset{f_2}{\overset{f_1}{\rightrightarrows}} A''$ in $\mathcal{A}$ and $r_1, r_2 \in R$. We have

$$F_B(r_1 f_1 + r_2 f_2) = F(r_1 f_1 + r_2 f_2, 1_B) = r_1 F(f_1, 1_B) + r_2 F(f_2, 1_B) = r_1 F_B f_1 + r_2 F_B f_2\,.$$

Thus, F_B is R-linear by Remark 25.

Ad (2) ⇒ (1). Suppose given $r_1, r_2 \in R$, $A' \underset{f_2}{\overset{f_1}{\rightrightarrows}} A''$ and $A''' \xrightarrow{f} A''''$ in $\mathcal{A}$, $B' \underset{g_2}{\overset{g_1}{\rightrightarrows}} B''$ and $B''' \xrightarrow{g} B''''$ in $\mathcal{B}$.

We have

$$\begin{aligned}
F(r_1 f_1 + r_2 f_2, g) &= F((r_1 f_1 + r_2 f_2, 1_{B'''}) \cdot (1_{A''}, g)) \\
&= F(r_1 f_1 + r_2 f_2, 1_{B'''}) \cdot F(1_{A''}, g) \\
&= F_{B'''}(r_1 f_1 + r_2 f_2) \cdot F(1_{A''}, g) \\
&= (r_1 F_{B'''} f_1 + r_2 F_{B'''} f_2) \cdot F(1_{A''}, g) \\
&= (r_1 F(f_1, 1_{B'''}) + r_2 F(f_2, 1_{B'''})) \cdot F(1_{A''}, g) \\
&= r_1 F(f_1, 1_{B'''}) \cdot F(1_{A''}, g) + r_2 F(f_2, 1_{B'''}) \cdot F(1_{A''}, g) \\
&= r_1 F(f_1, g) + r_2 F(f_2, g)\,.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
F(f, r_1 g_1 + r_2 g_2) &= F((1_{A'''}, r_1 g_1 + r_2 g_2) \cdot (f, 1_{B''})) \\
&= F(1_{A'''}, r_1 g_1 + r_2 g_2) \cdot F(f, 1_{B''}) \\
&= F_{A'''}(r_1 g_1 + r_2 g_2) \cdot F(f, 1_{B''}) \\
&= (r_1 F_{A'''} g_1 + r_2 F_{A'''} g_2) \cdot F(f, 1_{B''}) \\
&= (r_1 F(1_{A'''}, g_1) + r_2 F(1_{A'''}, g_2)) \cdot F(f, 1_{B''}) \\
&= r_1 F(1_{A'''}, g_1) \cdot F(f, 1_{B''}) + r_2 F(1_{A'''}, g_2) \cdot F(f, 1_{B''}) \\
&= r_1 F(f, g_1) + r_2 F(f, g_2)\,.
\end{aligned}$$

Thus, F is R-bilinear. □

Remark 29. *Let $(\mathcal{A}, \varphi), (\mathcal{B}, \psi), (\mathcal{C}, \tau), (\mathcal{D}, \delta), (\mathcal{E}, \eta)$ and $(\mathcal{F}, \zeta)$ be preadditive categories over R. Suppose given R-linear functors $\mathcal{A} \xrightarrow{F} \mathcal{C}$ and $\mathcal{B} \xrightarrow{G} \mathcal{D}$. Suppose given an R-bilinear functor $\mathcal{C} \times \mathcal{D} \xrightarrow{H} \mathcal{E}$. Suppose given an R-linear functor $\mathcal{E} \xrightarrow{K} \mathcal{F}$.*

Then

$$\mathcal{A} \times \mathcal{B} \xrightarrow{K \circ H \circ (F \times G)} \mathcal{F}$$

is R-bilinear.

Proof. Suppose given $r_1, r_2 \in R$, $A' \underset{f_2}{\overset{f_1}{\rightrightarrows}} A''$ and $A''' \xrightarrow{f} A''''$ in $\mathcal{A}$, $B' \underset{g_2}{\overset{g_1}{\rightrightarrows}} B''$ and $B''' \xrightarrow{g} B''''$ in $\mathcal{B}$.

We have

$$\begin{array}{rl}
& (K \circ H \circ (F \times G))(f, r_1 g_1 + r_2 g_2) \\
= & (K \circ H)(Ff, G(r_1 g_1 + r_2 g_2)) \\
= & (K \circ H)(Ff, r_1 G g_1 + r_2 G g_2) \\
= & K(r_1 H(Ff, Gg_1) + r_2 H(Ff, Gg_2)) \\
= & r_1 (K \circ H)(Ff, Gg_1) + r_2 (K \circ H)(Ff, Gg_2) \\
= & r_1 (K \circ H \circ (F \times G))(f, g_1) + r_2 (K \circ H \circ (F \times G))(f, g_2)
\end{array}$$

and

$$\begin{array}{rl}
& (K \circ H \circ (F \times G))(r_1 f_1 + r_2 f_2, g) \\
= & (K \circ H)(F(r_1 f_1 + r_2 f_2), Gg) \\
= & (K \circ H)(r_1 F f_1 + r_2 F f_2, Gg) \\
= & K(r_1 H(Ff_1, Gg) + r_2 H(Ff_2, Gg)) \\
= & r_1 (K \circ H)(Ff_1, Gg) + r_2 (K \circ H)(Ff_2, Gg) \\
= & r_1 (K \circ H \circ (F \times G))(f_1, g) + r_2 (K \circ H \circ (F \times G))(f_2, g).
\end{array}$$

□

Lemma 30. *Suppose given preadditive categories* $(\mathcal{A}, \varphi_{\mathcal{A}})$ *and* $(\mathcal{B}, \varphi_{\mathcal{B}})$ *over* R. *Suppose given an* R*-linear functor* $F : \mathcal{A} \to \mathcal{B}$.

Suppose given an adjunction $(F, G, \eta, \varepsilon)$. *Then* G *is* R*-linear.*

Proof. We show that

$$\begin{array}{ccc} {}_{\mathcal{A}}(GB, GB) & \xrightarrow{\psi_B} & {}_{\mathcal{B}}(FGB, B), \\ u & \mapsto & Fu \cdot \varepsilon_B \end{array}$$

is an R-linear map for $B \in \operatorname{Ob} \mathcal{B}$.

Suppose given $B \in \operatorname{Ob} \mathcal{B}$. Suppose given $u, u' \in {}_{\mathcal{A}}(GB, GB)$. Suppose given $r, r' \in R$.

Since F is R-linear, we have

$$\begin{aligned} (ru + r'u')\psi_B &= F(ru + r'u') \cdot \varepsilon_B \\ &= (rFu + r'Fu')\varepsilon_B \\ &= rFu \cdot \varepsilon_B + r'Fu' \cdot \varepsilon_B \\ &= r(u\psi_B) + r'(u'\psi_B). \end{aligned}$$

Thus, ψ_B is R-linear.

For $B \in \operatorname{Ob} \mathcal{B}$, let

$$\begin{array}{ccc} {}_{\mathcal{A}}(GB, GB) & \xleftarrow{\tilde{\psi}_B} & {}_{\mathcal{B}}(FGB, B), \\ \eta_{GB} \cdot Gv & \leftarrow\!\shortmid & v\,. \end{array}$$

We show that we have mutually inverse R-linear isomorphisms

$$\begin{tikzcd} {}_{\mathcal{A}}(GB, GB) \arrow[r, shift left, "\psi_B"] \arrow[r, phantom, "\sim"] & {}_{\mathcal{B}}(FGB, B) \arrow[l, shift left, "\tilde{\psi}_B"] \end{tikzcd}$$

for $B \in \operatorname{Ob} \mathcal{B}$.

Suppose given $B \in \operatorname{Ob} \mathcal{B}$. Suppose given $u \in {}_{\mathcal{A}}(GB, GB)$ and $v \in {}_{\mathcal{B}}(FGB, B)$.

We have

$$\begin{aligned} u\psi_B\tilde{\psi}_B &= (Fu \cdot \varepsilon_B)\tilde{\psi}_B \\ &= \eta_{GB} \cdot G(Fu \cdot \varepsilon_B) \\ &= \eta_{GB} \cdot GFu \cdot G\varepsilon_B \\ &= u \cdot \eta_{GB} \cdot G\varepsilon_B \\ &= u \end{aligned}$$

and

$$\begin{aligned} v\tilde{\psi}_B\psi_B &= (\eta_{GB} \cdot Gv)\psi_B \\ &= F(\eta_{GB} \cdot Gv) \cdot \varepsilon_B \\ &= F\eta_{GB} \cdot FGv \cdot \varepsilon_B \\ &= F\eta_{GB} \cdot \varepsilon_{FGB} \cdot v \\ &= v. \end{aligned}$$

Thus, we have mutually inverse bijections ψ_B and $\tilde{\psi}_B$. Since ψ_B is R-linear, so is $\tilde{\psi}_B$.

For $B \in \operatorname{Ob}\mathcal{B}$, we now write $\psi_B^- = \tilde{\psi}_B$.

Suppose given $B \overset{b}{\underset{b'}{\rightrightarrows}} \tilde{B}$ in $\mathcal{B}$. Suppose given $r, r' \in R$.

We have to show that $G(rb + r'b') \overset{!}{=} rGb + r'Gb'$; cf. Remark 25.

We have

$$
\begin{aligned}
G(rb + r'b') &= \eta_{GB} \cdot G\varepsilon_B \cdot G(rb + r'b') \\
&= (\varepsilon_B \cdot (rb + r'b'))\psi_B^- \\
&= (r\varepsilon_B b + r'\varepsilon_B b')\psi_B^- \\
&= r(\varepsilon_B b)\psi^- + r'(\varepsilon_B b')\psi_B^- \\
&= r \cdot \eta_{GB} \cdot G\varepsilon_B \cdot Gb + r' \cdot \eta_{GB} \cdot G\varepsilon_B \cdot Gb' \\
&= rGb + r'Gb'.
\end{aligned}
$$

□

Corollary 31. *Suppose given preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. Suppose given an R-linear functor $F : \mathcal{A} \to \mathcal{B}$. Suppose given a functor $G : \mathcal{B} \to \mathcal{A}$. Suppose given isotransformations $(F \circ G) \overset{\alpha}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{B}}$ and $(G \circ F) \overset{\beta}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{A}}$. Then G is R-linear.*

Proof. By [**6,** Remark 16.5.9], we have $F \dashv G$.

Thus, the claim follows from Lemma 30. □

Remark 32. Suppose $(\mathcal{A}, \varphi_{\mathcal{A}})$ to be a preadditive category over R.

Suppose $\mathcal{B}$ to be a full subcategory of $\mathcal{A}$. Then $\mathcal{B}$ is a preadditive category by restriction.

We have the ring morphism

$$
\begin{array}{rcl}
\operatorname{End} 1_{\mathcal{A}} & \overset{\psi}{\to} & \operatorname{End} 1_{\mathcal{B}}\,, \\
(\alpha_X)_{X \in \operatorname{Ob}\mathcal{A}} & \mapsto & (\alpha_X)_{X \in \operatorname{Ob}\mathcal{B}}\,.
\end{array}
$$

Thus, $(\mathcal{B}, \varphi_{\mathcal{A}} \cdot \psi)$ is a preadditive category over R.

Chapter 2

Envelope operations

2.1 The Karoubi envelope

2.1.1 Definition and duality

For this §2.1.1, let $\mathcal{A}$ be a category.

Definition 33 (and Lemma)**.** We shall define a category $\operatorname{Kar}\mathcal{A}$ as follows.

Let
$$\operatorname{Ob}\operatorname{Kar}\mathcal{A} := \{(X,e) : X \in \operatorname{Ob}\mathcal{A},\, e \in {}_{\mathcal{A}}(X,X) \text{ with } e^2 = e\}.$$

For (X,e) and (Y,f) in $\operatorname{Ob}\operatorname{Kar}\mathcal{A}$, we define

$${}_{\operatorname{Kar}\mathcal{A}}((X,e),(Y,f)) := \{\, {}_{e}(\ \varphi\ )_{f} : \varphi \in {}_{\mathcal{A}}(X,Y) \text{ and } \varphi = e\varphi f\}.$$

If unambiguous, we often write $\varphi := {}_{e}(\ \varphi\ )_{f}$ for ${}_{e}(\ \varphi\ )_{f} \in {}_{\operatorname{Kar}\mathcal{A}}((X,e),(Y,f))$.

For (X,e), (Y,f) and (Z,g) in $\operatorname{Ob}\operatorname{Kar}\mathcal{A}$,

$${}_{e}(\ \varphi\ )_{f} \in {}_{\operatorname{Kar}\mathcal{A}}((X,e),(Y,f)) \quad \text{and} \quad {}_{f}(\ \psi\ )_{g} \in {}_{\operatorname{Kar}\mathcal{A}}((Y,f),(Z,g)),$$

we define composition by
$${}_{e}(\ \varphi\ )_{f}\, {}_{f}(\ \psi\ )_{g} := {}_{e}(\ \varphi\psi\ )_{g}\,.$$

For $(X,e) \in \operatorname{Ob}\operatorname{Kar}\mathcal{A}$, we define

$$1_{(X,e)} := {}_{e}(\ e\ )_{e}\,.$$

We call $\operatorname{Kar}\mathcal{A}$ the *Karoubi envelope* of $\mathcal{A}$.

This defines a category $\operatorname{Kar}\mathcal{A}$.

Proof. Suppose given $(X,e) \xrightarrow{{}_{e}(\ \varphi\ )_{f}} (Y,f) \xrightarrow{{}_{f}(\ \psi\ )_{g}} (Z,g) \xrightarrow{{}_{g}(\ \rho\ )_{h}} (W,h)$ in $\operatorname{Kar}\mathcal{A}$.

Thus, we have $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\rho} W$ in $\mathcal{A}$ with

$$e\varphi f = \varphi, \quad f\psi g = \psi \quad \text{and} \quad g\rho h = \rho.$$

Using these equalities, we obtain $e\varphi\psi g \overset{\mathrm{R2}}{=} \varphi\psi$. Therefore, we have

$$ {}_{e}(\,\varphi\psi\,)_{g} \in {}_{\operatorname{Kar}\mathcal{A}}((X,e),(Z,g)). $$

Moreover, we obtain

$$ \begin{aligned} (\,{}_{e}(\,\varphi\,)_{f}\,{}_{f}(\,\psi\,)_{g})\,{}_{g}(\,\rho\,)_{h} &= {}_{e}(\,\varphi\psi\,)_{g}\,{}_{g}(\,\rho\,)_{h} = {}_{e}(\,(\varphi\psi)\rho\,)_{h} \\ &= {}_{e}(\,\varphi(\psi\rho)\,)_{h} = {}_{e}(\,\varphi\,)_{f}\,{}_{f}(\,\psi\rho\,)_{h} = {}_{e}(\,\varphi\,)_{f}\,(\,{}_{f}(\,\psi\,)_{g}\,{}_{g}(\,\rho\,)_{h}). \end{aligned} $$

We have $1_{(Y,f)} = {}_{f}(\,f\,)_{f} \in {}_{\operatorname{Kar}\mathcal{A}}((Y,f),(Y,f))$, because $f = fff$.
Furthermore, we have

$$ 1_{(Y,f)}\,{}_{f}(\,\psi\,)_{g} = {}_{f}(\,f\,)_{f}\,{}_{f}(\,\psi\,)_{g} = {}_{f}(\,f\psi\,)_{g} \overset{\mathrm{R2}}{=} {}_{f}(\,\psi\,)_{g}\,. $$

Similarily, we obtain

$$ {}_{e}(\,\varphi\,)_{f}1_{(Y,f)} = {}_{e}(\,\varphi\,)_{f}\,{}_{f}(\,f\,)_{f} = {}_{e}(\,\varphi f\,)_{f} \overset{\mathrm{R2}}{=} {}_{e}(\,\varphi\,)_{f}\,. $$

□

Lemma 34. *We have the isomorphism of categories*

$$ \begin{array}{ccc} (\operatorname{Kar}(\mathcal{A}))^{\circ} & \longrightarrow & \operatorname{Kar}(\mathcal{A}^{\circ}) \\ \left((X,e) \xleftarrow{{}_{e}(\,\varphi\,)_{f}^{\circ}} (Y,f)\right) & \longmapsto & \left((X,e^{\circ}) \xleftarrow{{}_{f^{\circ}}(\,\varphi^{\circ}\,)_{e^{\circ}}} (Y,f^{\circ})\right) \\ \left((X,e) \xleftarrow{{}_{e}(\,\varphi\,)_{f}^{\circ}} (Y,f)\right) & \longleftarrow\!\shortmid & \left((X,e^{\circ}) \xleftarrow{{}_{f^{\circ}}(\,\varphi^{\circ}\,)_{e^{\circ}}} (Y,f^{\circ})\right). \end{array} $$

Proof. Suppose given idempotents $X \xrightarrow{e} X$ and $Y \xrightarrow{f} Y$ in $\mathcal{A}$. Suppose given $\varphi \in {}_{\mathcal{A}}(X,Y)$.
Assume $e\varphi f = \varphi$. Then we have $f^{\circ}\varphi^{\circ}e^{\circ} = (e\varphi f)^{\circ} = \varphi^{\circ}$.
Assume $f^{\circ}\varphi^{\circ}e^{\circ} = \varphi^{\circ}$. This implies

$$ e\varphi f = ((e\varphi f)^{\circ})^{\circ} = (f^{\circ}\varphi^{\circ}e^{\circ})^{\circ} = (\varphi^{\circ})^{\circ} = \varphi. $$

Thus, we have

$$ e\varphi f = \varphi \quad \text{if and only if} \quad f^{\circ}\varphi^{\circ}e^{\circ} = \varphi^{\circ}. $$

Therefore, we obtain welldefined maps on objects and morphisms in both directions.
Suppose given $(W,d) \xrightarrow{{}_{d}(\,\psi\,)_{e}} (X,e) \xrightarrow{{}_{e}(\,\varphi\,)_{f}} (Y,f)$ in $\operatorname{Kar}\mathcal{A}$.
We have

$$ \begin{aligned} {}_{e}(\,\varphi\,)_{f}^{\circ}\,{}_{d}(\,\psi\,)_{e}^{\circ} = (\,{}_{d}(\,\psi\,)_{e}\,{}_{e}(\,\varphi\,)_{f})^{\circ} &\overset{\mathrm{D33}}{=} {}_{d}(\,\psi\varphi\,)_{f}^{\circ} \\ &\mapsto {}_{f^{\circ}}(\,(\psi\varphi)^{\circ}\,)_{d^{\circ}} = {}_{f^{\circ}}(\,\varphi^{\circ}\psi^{\circ}\,)_{d^{\circ}} \overset{\mathrm{D33}}{=} {}_{f^{\circ}}(\,\varphi^{\circ}\,)_{e^{\circ}}\,{}_{e^{\circ}}(\,\psi^{\circ}\,)_{d^{\circ}} \end{aligned} $$

and

$$\begin{aligned} {}_{f^\circ}(\,\varphi^\circ\,)_{e^\circ}\,{}_{e^\circ}(\,\psi^\circ\,)_{d^\circ} = {}_{f^\circ}(\,\varphi^\circ\psi^\circ\,)_{d^\circ} &\overset{\text{D33}}{=} {}_{f^\circ}(\,(\psi\varphi)^\circ\,)_{d^\circ} \\ &\mapsto {}_{d}(\,\psi\varphi\,)_f^\circ = (\,{}_d(\,\psi\,)_e\,{}_e(\,\varphi\,)_f)^\circ \overset{\text{D33}}{=} {}_e(\,\varphi\,)_f^\circ\,{}_d(\,\psi\,)_e^\circ . \end{aligned}$$

In $(\operatorname{Kar}\mathcal{A})^\circ$ we have $1_{(X,e)} \overset{\text{D33}}{=} {}_e(\,e\,)_e^\circ$.

In $\operatorname{Kar}(\mathcal{A}^\circ)$ we have $1_{(X,e^\circ)} \overset{\text{D33}}{=} {}_{e^\circ}(\,e^\circ\,)_{e^\circ}$.

We obtain

$$1_{(X,e)} = {}_e(\,e\,)_e^\circ \mapsto {}_{e^\circ}(\,e^\circ\,)_{e^\circ} = 1_{(X,e^\circ)}$$

and

$$1_{(X,e^\circ)} = {}_{e^\circ}(\,e^\circ\,)_{e^\circ} \mapsto {}_e(\,e\,)_e^\circ = 1_{(X,e)}\,.$$

Therefore, we have functors indeed. They are mutual inverses by definition. Thus, we have isomorphisms of categories. □

Remark 35. *Suppose given an isomorphism $X \xrightarrow[\sim]{\varphi} Y$ in $\mathcal{A}$. Suppose given idempotents $e \in {}_{\mathcal{A}}(X,X)$ and $f \in {}_{\mathcal{A}}(Y,Y)$ with $e\varphi = \varphi f$. We have mutually inverse isomorphisms*

$$\begin{tikzcd} (X,e) \arrow[r, shift left, "{}_e(\,e\varphi f\,)_f"] \arrow[r, phantom, "\sim"] & (Y,f) \arrow[l, shift left, "{}_f(\,f\varphi^{-1}e\,)_e"] \end{tikzcd}$$

in $\operatorname{Kar}\mathcal{A}$.

Proof. We have

$$ee\varphi ff = e\varphi f \quad \text{and} \quad ff\varphi^{-1}ee = f\varphi^{-1}e.$$

Therefore, we have ${}_e(\,e\varphi f\,)_f \in {}_{\operatorname{Kar}\mathcal{A}}((X,e),(Y,f))$ and ${}_f(\,f\varphi^{-1}e\,)_e \in {}_{\operatorname{Kar}\mathcal{A}}((Y,f),(X,e))$.

Furthermore, we have

$$\begin{aligned} {}_e(\,e\varphi f\,)_f\,{}_f(\,f\varphi^{-1}e\,)_e &\overset{\text{D33}}{=} {}_e(\,e\varphi ff\varphi^{-1}e\,)_e = {}_e(\,e\varphi f\varphi^{-1}e\,)_e \\ &= {}_e(\,ee\varphi\varphi^{-1}e\,)_e = {}_e(\,e\,)_e \overset{\text{D33}}{=} 1_{(X,e)}\,. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} {}_f(\,f\varphi^{-1}e\,)_e\,{}_e(\,e\varphi f\,)_f &\overset{\text{D33}}{=} {}_f(\,f\varphi^{-1}ee\varphi f\,)_f = {}_f(\,f\varphi^{-1}e\varphi f\,)_f \\ &= {}_f(\,f\varphi^{-1}\varphi ff\,)_f = {}_f(\,f\,)_f \overset{\text{D33}}{=} 1_{(Y,f)}\,. \end{aligned}$$

Therefore, we have mutually inverse isomorphisms. □

Remark 36. *Suppose given an isomorphism $X \xrightarrow[\sim]{\varphi} Y$ in $\mathcal{A}$. Suppose given an idempotent $e \in {}_{\mathcal{A}}(X,X)$. Define $f := \varphi^{-1}e\varphi$. Then we have mutually inverse isomorphisms*

$$\begin{tikzcd} (X,e) \arrow[r, shift left, "{}_e(\,e\varphi\,)_f"] \arrow[r, phantom, "\sim"] & (Y,f) \arrow[l, shift left, "{}_f(\,\varphi^{-1}e\,)_e"] \end{tikzcd}$$

in $\operatorname{Kar}\mathcal{A}$.

Proof. By Remark 3, f is an idempotent. By applying φ from the left to $f = \varphi^{-1}e\varphi$, we obtain $\varphi f = e\varphi$. Thus, the claim follows from Remark 35. Note that

$$e\varphi f = e\varphi\varphi^{-}-1e\varphi = e\varphi \qquad \text{and} \qquad f\varphi^{-1}e = \varphi^{-1}e\varphi\varphi-1e = \varphi^{-}-1e.$$

□

2.1.2 The Karoubi envelope respects additivity

The main purpose of the Karoubi envelope construction as presented in §0.1.1 is to complete an additive category with respect to its idempotents. Thus, the Karoubi envelope of an additive category should be additive. In this §2.1.2, we prove that this is the case.

For this §2.1.2, let $\mathcal{A}$ be a category.

Lemma 37. *Let $\mathcal{A}$ have a zero object A. Then $(A, 1_A)$ is a zero object in $\operatorname{Kar}\mathcal{A}$.*

Proof. Suppose given $(B, f) \in \operatorname{Ob}(\operatorname{Kar}\mathcal{A})$. We have to show that

$$|\,{}_{\operatorname{Kar}\mathcal{A}}((B,f),(A,1_A))| = 1 = |\,{}_{\operatorname{Kar}\mathcal{A}}((A,1_A),(B,f))|\,.$$

By assumption, there exists exactly one morphism φ in ${}_{\mathcal{A}}(B,A)$. We have $f\varphi 1_A \in {}_{\mathcal{A}}(B,A)$. Thus, we have $f\varphi 1_A = \varphi$, i.e. ${}_f(\,\varphi\,)_{1_A} \in {}_{\operatorname{Kar}\mathcal{A}}((B,f),(A,1_A))$. In consequence, we have $|\,{}_{\operatorname{Kar}\mathcal{A}}((B,f),(A,1_A))| \geqslant 1$.

Suppose given $\rho \in {}_{\operatorname{Kar}\mathcal{A}}((B,f),(A,1_A))$. By Definition 33, there exists $\psi \in {}_{\mathcal{A}}(B,A)$ with $\rho = {}_f(\,\psi\,)_{1_A}$. Since ${}_{\mathcal{A}}(B,A) = \{\varphi\}$, we have $\varphi = \psi$. Therefore, we obtain $\rho = {}_f(\,\psi\,)_{1_A} = {}_f(\,\varphi\,)_{1_A}$. In consequence, we have $|\,{}_{\operatorname{Kar}\mathcal{A}}((B,f),(A,1_A)| \leqslant 1$.

Thus, we have $|\,{}_{\operatorname{Kar}\mathcal{A}}((B,f),(A,1_A))| = 1$.

Dually, we obtain $|\,{}_{\operatorname{Kar}\mathcal{A}}((A,1_A),(B,f))| = 1$. □

Definition 38 (and Lemma)**.** Suppose $\mathcal{A}$ to be preadditive. Suppose given (X,e) and (Y,f) in $\operatorname{Ob}\operatorname{Kar}\mathcal{A}$.

For ${}_e(\,\varphi\,)_f$ and ${}_e(\,\psi\,)_f$ in ${}_{\operatorname{Kar}\mathcal{A}}((X,e),(Y,f))$, we define

$${}_e(\,\varphi\,)_f + {}_e(\,\psi\,)_f := {}_e(\,\varphi+\psi\,)_f\,.$$

The Karoubi envelope $\operatorname{Kar}\mathcal{A}$ *is preadditive with respect to this addition.*

In particular, we have

$$-\,{}_e(\,\varphi\,)_f = {}_e(\,-\varphi\,)_f\,,$$

for ${}_e(\,\varphi\,)_f \in \operatorname{Mor}\operatorname{Kar}\mathcal{A}$.

Proof. First we show that ${}_{\operatorname{Kar}\mathcal{A}}((X,e),(Y,f))$ is an abelian group carrying the addition defined above.

It suffices to show that $\mathcal{U} := \{\varphi \in {}_{\mathcal{A}}(X,Y) : e\varphi f = \varphi\}$ is a subgroup of ${}_{\mathcal{A}}(X,Y)$.

We have $e0_{X,Y}f = 0_{X,Y}$, i.e. $0_{X,Y} \in \mathcal{U}$.

Suppose given $\varphi, \psi \in \mathcal{U}$. We have

$$\varphi - \psi = e\varphi f - e\psi f = e(\varphi - \psi)f.$$

Thus, we have $\varphi - \psi \in \mathcal{U}$. In consequence, ${}_{\mathrm{Kar}\,\mathcal{A}}((X,e),(Y,f))$ is an abelian group.

Suppose given $\begin{tikzcd}(X,e) \arrow[r, "{}_e(\,\varphi\,)_f"] & (Y,f) \arrow[r, shift left, "{}_f(\,\psi_1\,)_g"] \arrow[r, shift right, "{}_f(\,\psi_2\,)_g"'] & (Z,g) \arrow[r, "{}_g(\,\rho\,)_h"] & (W,h)\end{tikzcd}$ in $\mathrm{Kar}\,\mathcal{A}$. We have

$$\begin{aligned}
{}_e(\,\varphi\,)_f\,(\,{}_f(\,\psi_1\,)_g + {}_f(\,\psi_2\,)_g)\,{}_g(\,\rho\,)_h &= {}_e(\,\varphi\,)_f\,{}_f(\,\psi_1 + \psi_2\,)_g\,{}_g(\,\rho\,)_h \\
&= {}_e(\,\varphi(\psi_1 + \psi_2)\rho\,)_h \\
&= {}_e(\,\varphi\psi_1\rho + \varphi\psi_2\rho\,)_h \\
&= {}_e(\,\varphi\psi_1\rho\,)_h + {}_e(\,\varphi\psi_2\rho\,)_h \\
&= {}_e(\,\varphi\,)_f\,{}_f(\,\psi_1\,)_g\,{}_g(\,\rho\,)_h + {}_e(\,\varphi\,)_f\,{}_f(\,\psi_2\,)_g\,{}_g(\,\rho\,)_h\,.
\end{aligned}$$

□

Lemma 39. *Suppose $\mathcal{A}$ to be preadditive. Suppose given $A_1\,, A_2 \in \mathrm{Ob}\,\mathcal{A}$. Let A_1 and A_2 have a direct sum*

$$\begin{tikzcd}A_1 \arrow[r, shift left, "\iota_1"] & C \arrow[l, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A_2 \arrow[l, shift right, "\iota_2"']\end{tikzcd}$$

in $\mathcal{A}$.

Suppose given idempotents $A_1 \xrightarrow{e_1} A_1$ and $A_2 \xrightarrow{e_2} A_2$ in $\mathcal{A}$.

Define $e := \pi_1 e_1 \iota_1 + \pi_2 e_2 \iota_2 \in {}_{\mathcal{A}}(C,C)$. Then

$$\begin{tikzcd}(A_1\,, e_1) \arrow[r, shift left, "{}_{e_1}(\,e_1\iota_1\,)_e"] & (C,e) \arrow[l, shift left, "{}_e(\,\pi_1 e_1\,)_{e_1}"] \arrow[r, shift right, "{}_e(\,\pi_2 e_2\,)_{e_2}"'] & (A_2\,, e_2) \arrow[l, shift right, "{}_{e_2}(\,e_2\iota_2\,)_e"']\end{tikzcd}$$

is a direct sum of $(A_1\,, e_1)$ and $(A_2\,, e_2)$ in $\mathrm{Kar}\,\mathcal{A}$.

Proof. We have

$$\begin{aligned}
\iota_1\pi_1 &= 1_{A_1} \\
\iota_2\pi_2 &= 1_{A_2} \\
\iota_1\pi_2 &= 0_{A_2} \\
\iota_2\pi_1 &= 0_{A_1}\,.
\end{aligned}$$

We will use these equalities throughout this proof without further notice.

We calculate

$$\begin{aligned}
e^2 &= \pi_1 e_1 \iota_1 \pi_1 e_1 \iota_1 + \pi_1 e_1 \iota_1 \pi_2 e_2 \iota_2 + \pi_2 e_2 \iota_2 \pi_1 e_2 \iota_1 + \pi_2 e_2 \iota_2 \pi_2 e_2 \iota_2 \\
&= \pi_1 e_1 e_1 \iota_1 + 0 + 0 + \pi_2 f_2 f_2 \iota_2 \\
&= e.
\end{aligned}$$

Thus, we have $(C,e) \in \mathrm{Ob}\,\mathrm{Kar}\,\mathcal{A}$.

We show that

$${}_{e_i}(\,e_i\iota_i\,)_e \in {}_{\mathrm{Kar}\,\mathcal{A}}((A_i\,, e_i),(C,e)) \quad \text{and} \quad {}_e(\,\pi_i e_i\,)_{e_i} \in {}_{\mathrm{Kar}\,\mathcal{A}}((C,e),(A_i\,, e_i))$$

for $i \in [1, 2]$.

Pars pro toto, consider $i = 1$. We have

$$e_1 \cdot e_1\iota_1 \cdot e = e_1 e_1 \iota_1 \pi_1 e_1 \iota_1 + e_1 e_1 \iota_1 \pi_2 e_2 \iota_2 = e_1 e_1 \iota_1 + 0 = e_1 \iota_1$$

and

$$e \cdot \pi_1 e_1 \cdot e_1 = \pi_1 e_1 \iota_1 \pi_1 e_1 e_1 + \pi_2 e_2 \iota_2 \pi_1 e_1 e_1 = \pi_1 e_1 e_1 + 0 = \pi_1 e_1 \, .$$

Furthermore, we have

$$\begin{aligned} {}_{e}(\ \pi_1 e_1\ )_{e_1}\ {}_{e_1}(\ e_1 \iota_1\ )_{e} + {}_{e}(\ \pi_2 e_2\ )_{e_2}\ {}_{e_2}(\ e_2 \iota_2\ )_{e} &= {}_{e}(\ \pi_1 e_1 e_1 \iota_1\ )_{e} + {}_{e}(\ \pi_2 e_2 e_2 \iota_2\ )_{e} \\ &= {}_{e}(\ \pi_1 e_1 \iota_1 + \pi_2 e_2 \iota_2\ )_{e} \\ &= {}_{e}(\ e\ )_{e} \\ &= 1_{(C,e)} \, . \end{aligned}$$

Finally, we have

$${}_{e_1}(\ e_1 \iota_1\ )_{e}\ {}_{e}(\ \pi_1 e_1\ )_{e_1} \;=\; {}_{e_1}(\ e_1 \iota_1 \pi_1 e_1\ )_{e_1} \;=\; {}_{e_1}(\ e_1\ )_{e_1} = 1_{(A_1\,, e_1)} \, .$$

and

$${}_{e_2}(\ e_2 \iota_2\ )_{e}\ {}_{e}(\ \pi_2 e_2\ )_{e_2} \;=\; {}_{e_2}(\ e_2 \iota_2 \pi_2 e_2\ )_{e_2} \;=\; {}_{e_2}(\ e_2\ )_{e_2} = 1_{(A_2\,, e_2)} \, .$$

□

Corollary 40. *Suppose $\mathcal{A}$ to be additive. Suppose given idempotents $A_1 \xrightarrow{e_1} A_1$ and $A_2 \xrightarrow{e_2} A_2$ in $\mathcal{A}$.*

Define $e := \left(\begin{smallmatrix} e_1 & 0 \\ 0 & e_2 \end{smallmatrix}\right) \in {}_{\mathcal{A}}(A_1 \oplus A_2\,, A_1 \oplus A_2)$. Then

$$\begin{tikzcd} (A_1\,, e_1) \arrow[r, shift left, "{{}_{e_1}(\ (e_1\ 0)\ )_{e}}"] & (A_1 \oplus A_2, e) \arrow[l, shift left, "{{}_{e}(\ \left(\begin{smallmatrix} e_1 \\ 0 \end{smallmatrix}\right)\ )_{e_1}}"] \arrow[r, shift right, "{{}_{e}(\ \left(\begin{smallmatrix} 0 \\ e_2 \end{smallmatrix}\right)\ )_{e_2}}"'] & (A_2\,, e_2) \arrow[l, shift right, "{{}_{e_2}(\ (0\ e_2)\ )_{e}}"'] \end{tikzcd}$$

is a direct sum of $(A_1\,, e_1)$ and $(A_2\,, e_2)$ in $\operatorname{Kar}\mathcal{A}$.

Proof. Since

$$\begin{tikzcd} A_1 \arrow[r, shift left, "{(1\ 0)}"] & A_1 \oplus A_2 \arrow[l, shift left, "{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}"] \arrow[r, shift right, "{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}"'] & A_2 \arrow[l, shift right, "{(0\ 1)}"'] \end{tikzcd}$$

is a direct sum of A_1 and A_2 in $\mathcal{A}$, this follows from Lemma 39. □

Proposition 41. *Suppose $\mathcal{A}$ to be an additive. Then* $\operatorname{Kar}\mathcal{A}$ *is additive.*

Proof. Since $\mathcal{A}$ is additive, it has a zero object. Thus, $\operatorname{Kar}\mathcal{A}$ has a zero object; cf. Lemma 37.

Since $\mathcal{A}$ is additive, it is preadditive. Thus, $\operatorname{Kar}\mathcal{A}$ is preadditive; cf. Definition 38.

Suppose given (X, e) and (Y, f) in $\operatorname{Ob}\operatorname{Kar}\mathcal{A}$. By Corollary 40, (X, e) and (Y, f) have a direct sum in $\operatorname{Kar}\mathcal{A}$.

Thus, $\operatorname{Kar}\mathcal{A}$ is additive. □

2.1.3 The inclusion functor

The inclusion functor will play the role of the universal functor of the Karoubi envelope construction; cf. §2.1.8.

For this §2.1.3, let $\mathcal{A}$ be a category.

Definition 42 (and Lemma)**.** We have the functor

$$\begin{array}{rcl} \mathcal{A} & \xrightarrow{J_{\mathcal{A}}} & \operatorname{Kar}\mathcal{A} \\ (X \xrightarrow{\varphi} Y) & \mapsto & \Big((X, 1_X) \xrightarrow{{}_{1_X}(\,\varphi\,)_{1_X}} (Y, 1_Y) \Big) . \end{array}$$

We call $J_{\mathcal{A}}$ the *inclusion functor* of $\mathcal{A}$ in $\operatorname{Kar}\mathcal{A}$.

If unambiguous, we often write $J := J_{\mathcal{A}}$.

Proof. Suppose given $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ in $\mathcal{A}$.

Since $1_X^2 = 1_X$, we obtain a welldefined map on objects.

Since $1_X \varphi 1_Y = \varphi$, we obtain a welldefined map on morphisms.

Furthermore, we have

$$J(\varphi\psi) = {}_{1_X}(\,\varphi\psi\,)_{1_Z} \overset{\text{D33}}{=} {}_{1_X}(\,\varphi\,)_{1_Y}\;{}_{1_Y}(\,\psi\,)_{1_Z} = J\varphi \cdot J\psi .$$

Finally, we have

$$J\,(1_X) = {}_{1_X}(\,1_X\,)_{1_X} \overset{\text{D33}}{=} 1_{(X,1_X)} = 1_{JX} .$$

□

Lemma 43. *The following assertions* (1, 2) *hold.*

(1) *The inclusion functor $J_{\mathcal{A}}$ is full.*

(2) *The inclusion functor $J_{\mathcal{A}}$ is faithful.*

Proof. Suppose given $A, B \in \operatorname{Ob}\mathcal{A}$.

Ad (1). Suppose given $\rho \in {}_{\operatorname{Kar}\mathcal{A}}((A, 1_A), (B, 1_B))$. By Definition 33, there exists $\varphi \in {}_{\mathcal{A}}(A, B)$ with $\rho = {}_{1_A}(\,\varphi\,)_{1_B}$. In particular, we have $J\varphi \overset{\text{D42}}{=} {}_{1_A}(\,\varphi\,)_{1_B} = \rho$. Therefore, J is full.

Ad (2). Suppose given $\varphi, \psi \in {}_{\mathcal{A}}(A, B)$ with $J\varphi = {}_{1_A}(\,\varphi\,)_{1_B} = {}_{1_A}(\,\psi\,)_{1_B} = J\psi$. By Definition 33, we have $\varphi = \psi$. Therefore, J is faithful. □

Lemma 44. *Suppose $\mathcal{A}$ to be preadditive. Then $J_{\mathcal{A}}$ is additive.*

Proof. Since $\mathcal{A}$ is preadditive, so is $\operatorname{Kar}\mathcal{A}$; cf. Definition 38.

Suppose given $A \underset{\psi}{\overset{\varphi}{\rightrightarrows}} B$ in $\mathcal{A}$. We have

$$J(\varphi + \psi) \overset{\text{D42}}{=} {}_{1_A}(\,\varphi + \psi\,)_{1_B} \overset{\text{D38}}{=} {}_{1_A}(\,\varphi\,)_{1_B} + {}_{1_A}(\,\psi\,)_{1_B} \overset{\text{D42}}{=} J\varphi + J\psi ,$$

□

2.1.4 Idempotent complete categories

In §0.1.1 we stated that the Karoubi envelope completes a category by endowing it with images of its idempotents. In this §2.1.4, we specify this by defining *images* of idempotents and introducing *idempotent complete* categories.

For this §2.1.4, let $\mathcal{A}$ be a category.

Definition 45. Let $X \xrightarrow{e} X$ be an idempotent in $\mathcal{A}$. A tuple (Y, π, ι) with $Y \in \operatorname{Ob}\mathcal{A}$, $\pi \in {}_{\mathcal{A}}(X, Y)$ and $\iota \in {}_{\mathcal{A}}(Y, X)$, is called an *image* of e if the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "\pi"] & & X \arrow[dr, "\pi"] & \\
& Y \arrow[ur, "\iota"] \arrow[rr, "1_Y"] & & Y
\end{tikzcd}$$

If every idempotent in $\mathcal{A}$ has an image, we call $\mathcal{A}$ *idempotent complete.*

Remark 46. *Suppose given an idempotent $X \xrightarrow{e} X$ in $\mathcal{A}$ with image (Y, π, ι). Suppose given $Z \in \operatorname{Ob}\mathcal{A}$ and an isomorphism $X \xrightarrow[\sim]{\varphi} Z$. Then $Z \xrightarrow{\varphi^{-1}e\varphi} Z$ is an idempotent with image $(Y, \varphi^{-1}\pi, \iota\varphi)$.*

Proof. By Remark 3, $\varphi^{-1}e\varphi$ is an idempotent.

Since (Y, π, ι) is an image of e, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "\pi"] & & X \arrow[dr, "\pi"] & \\
& Y \arrow[ur, "\iota"] \arrow[rr, "1_Y"] & & Y
\end{tikzcd}$$

Since φ is an isomorphism, we obtain the following commutative diagram.

$$\begin{tikzcd}
Z \arrow[rr, "\varphi^{-1}e\varphi"] \arrow[dr, "\varphi^{-1}\pi"'] & & Z \arrow[dr, "\varphi^{-1}\pi"] & \\
& Y \arrow[ur, "\iota\varphi"] \arrow[rr, "1_Y"] & & Y
\end{tikzcd}$$

Thus, $(Y, \varphi^{-1}\pi, \iota\varphi)$ is an image of $\varphi^{-1}e\varphi$.

□

Remark 47. *Suppose given categories $\mathcal{B}$ and $\mathcal{C}$. Suppose given a functor $F : \mathcal{B} \to \mathcal{C}$. Suppose given an idempotent $X \xrightarrow{e} X$ in $\mathcal{B}$ with image (Y, π, ι). Then $FX \xrightarrow{Fe} FX$ is an idempotent in $\mathcal{C}$ with image $(FY, F\pi, F\iota)$.*

Proof. By Remark 5.(1), Fe is an idempotent in $\mathcal{C}$.

Since (Y, π, ι) is an image of e, we have the following commutative diagram in $\mathcal{B}$.

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "\pi"] & & X \arrow[dr, "\pi"] & \\
& Y \arrow[ur, "\iota"] \arrow[rr, "1_Y"] & & Y
\end{tikzcd}$$

Since F is a functor, we have the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
FX \arrow[rr, "Fe"] \arrow[dr, "F\pi"] & & FX \arrow[dr, "F\pi"] & \\
& FY \arrow[ur, "F\iota"] \arrow[rr, "1_{FY}"] & & FY
\end{tikzcd}$$

Thus, $(FY, F\pi, F\iota)$ is an image of Fe.

□

Remark 48. *Suppose given categories $\mathcal{B}$ and $\mathcal{C}$. Suppose given an equivalence $F : \mathcal{B} \to \mathcal{C}$. Then $\mathcal{B}$ is idempotent complete if and only if $\mathcal{C}$ is idempotent complete.*

Proof. Because of the symmetry of the situation, it suffices to show that if $\mathcal{B}$ is idempotent complete, so is $\mathcal{C}$.

Assume $\mathcal{B}$ to be idempotent complete.

Suppose given an idempotent $X \xrightarrow{e} X$ in $\mathcal{C}$.

Since F is dense, there exists $A \in \operatorname{Ob} \mathcal{B}$ with $FA \cong X$. Let $FA \xrightarrow[\sim]{\varphi} X$ be an isomorphism.

By Remark 3, we have an idempotent $FA \xrightarrow{\varphi e \varphi^{-1}} FA$ in $\mathcal{C}$.

Since F is full, there exists $f \in {}_{\mathcal{B}}(A, A)$ with $Ff = \varphi e \varphi^{-1}$.

We have $F(f^2) = (Ff)^2 = \varphi e \varphi^{-1} \cdot \varphi e \varphi^{-1} = \varphi e \varphi^{-1} = Ff$. Since F is faithful, we obtain $f^2 = f$.

Therefore, we have an idempotent $A \xrightarrow{f} A$ in $\mathcal{B}$.

Since $\mathcal{B}$ is idempotent complete, there exists an image (B, π, ι) of f in $\mathcal{B}$.

By Remark 47, $(FB, F\pi, F\iota)$ is an image of $Ff = \varphi e \varphi^{-1}$ in $\mathcal{C}$.

By Remark 46, $(FB, \varphi^{-1} \cdot F\pi, F\iota \cdot \varphi)$ is an image of $e = \varphi^{-1} \cdot Ff \cdot \varphi$ in $\mathcal{C}$. □

Lemma 49. *Let $X \xrightarrow{e} X$ be an idempotent in $\mathcal{A}$ with image (Y, π, ι). The following assertions* (1, 2) *hold.*

(1) *The morphism $X \xrightarrow{\pi} Y$ is a retraction, in particular π is epic.*

(2) *The morphism $Y \xrightarrow{\iota} X$ is a coretraction, in particular ι is monic.*

Proof. Recall that $\iota\pi = 1_Y$; cf. Definition 45. Therefore, π is a retraction of ι. From the same equation we obtain that ι is a coretraction of π. □

Lemma 50. *Suppose given idempotents $X \xrightarrow{e} X$ and $X' \xrightarrow{e'} X'$ in $\mathcal{A}$. Suppose given an image (Y, π, ι) of e. Suppose given an image (Y', π', ι') of e'. Suppose given $f \in {}_{\mathcal{A}}(X, X')$ with $efe' = f$. Define $g := \iota f \pi'$.*

The following assertions (1, 2, 3) *hold:*

(1) *The following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "f"'] & Y \arrow[r, "\iota"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "\pi'"] & Y' \arrow[r, "\iota'"] & X' \end{tikzcd}$$

(2) *Suppose given $\tilde{g} \in {}_{\mathcal{A}}(Y, Y')$ such that the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "f"'] & Y \arrow[r, "\iota"] \arrow[d, "\tilde{g}"] & X \arrow[d, "f"] \\ X' \arrow[r, "\pi'"] & Y' \arrow[r, "\iota'"] & X' \end{tikzcd}$$

We have $\tilde{g} = g$.

(3) *Suppose f to be an isomorphism. Then g is an isomorphism with $g^{-1} = \iota' f^{-1} \pi$. Furthermore, the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\pi"] & Y \arrow[r, "\iota"] & X \\ X' \arrow[r, "\pi'"] \arrow[u, "f^{-1}"] & Y' \arrow[r, "\iota'"] \arrow[u, "g^{-1}"'] & X' \arrow[u, "f^{-1}"'] \end{tikzcd}$$

Proof. *Ad* (1). We have

$$\pi g = \pi \iota f \pi' = e f \pi' = f e' \pi' = f \pi' \iota' \pi' = f \pi' 1_{Y'} = f \pi'.$$

Therefore, the left quadrangle commutes.

Furthermore, we have

$$g \iota' = \iota f \pi' \iota' = \iota f e' = \iota e f = \iota \pi \iota f = 1_Y \iota f = \iota f.$$

Therefore, the right quadrangle commutes.

Ad (2). We have

$$\pi g \iota' = \pi \iota f = \pi \tilde{g} \iota'.$$

Since π is epic and ι' is monic; cf. Lemma 49; we obtain $g = \tilde{g}$.

Ad (3). We have

$$g \iota' f^{-1} \pi = \iota f \pi' \iota' f^{-1} \pi = \iota f e' f^{-1} \pi = \iota e f f^{-1} \pi = \iota \pi \iota 1_X \pi = 1_Y \iota \pi = 1_Y$$

and

$$\iota' f^{-1} \pi g = \iota' f^{-1} \pi \iota f \pi' = \iota' f^{-1} e f \pi' = \iota' f^{-1} f e' \pi' = \iota' 1_{X'} \pi' \iota' \pi' = 1_{Y'} 1_{Y'} = 1_{Y'}.$$

Therefore, g is an isomorphism with inverse $g^{-1} = \iota' f^{-1} \pi$.

The commutativity of the stated diagram follows from (1). $\square$

Corollary 51. *Let $X \xrightarrow{e} X$ be an idempotent in $\mathcal{A}$. Suppose given images (Y, π, ι) and (Y', π', ι') of e. The following assertions* (1, 2) *hold.*

(1) *We have mutually inverse isomorphisms* $Y \underset{\iota'\pi}{\overset{\iota\pi'}{\underset{\sim}{\rightleftarrows}}} Y'$.

(2) *The following diagram commutes.*

$$\begin{tikzcd}
 & Y \arrow[dd, shift left, "\iota\pi'", "\wr"'] \arrow[dr, "\iota"] & \\
X \arrow[ur, "\pi"] \arrow[dr, "\pi'"'] & & X \\
 & Y' \arrow[uu, shift left, "\iota'\pi"] \arrow[ur, "\iota'"'] &
\end{tikzcd}$$

Proof. *Ad* (1). By Lemma 50, we obtain the following commutative diagrams with mutually inverse isomorphisms $\iota\pi'$ and $\iota'\pi$.

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[d, "1_X"'] & Y \arrow[r, "\iota"] \arrow[d, "\iota\pi'"] & X \arrow[d, "1_X"] \\
X \arrow[r, "\pi'"] & Y' \arrow[r, "\iota'"] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\pi"] & Y \arrow[r, "\iota"] & X \\
X \arrow[r, "\pi'"] \arrow[u, "1_X"] & Y' \arrow[r, "\iota'"] \arrow[u, "\iota'\pi"'] & X \arrow[u, "1_X"']
\end{tikzcd}$$

Ad (2). Combining these commutative diagrams, we obtain the commutative diagram stated above. □

Stipulation 52. Suppose given an idempotent complete category $\mathcal{B}$.

Suppose given an idempotent $X \xrightarrow{e} X$ in $\mathcal{B}$. Recall that we have chosen an image $(\operatorname{Im} e, \bar{e}, \dot{e})$ of e in $\mathcal{B}$. Thus, the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "\bar{e}"'] & & X \arrow[dr, "\bar{e}"] & \\
 & \operatorname{Im} e \arrow[ur, "\dot{e}"] \arrow[rr, "1_{\operatorname{Im} e}"] & & \operatorname{Im} e
\end{tikzcd}$$

Recall that we have chosen

$$(\operatorname{Im} 1_X, \overline{1_X}, \dot{1_X}) = (X, 1_X, 1_X)$$

for $X \in \operatorname{Ob} \mathcal{B}$; cf. Convention no. 25.

If unambiguous, we also refer to just $\operatorname{Im} e$ as the *image of* e .

Lemma 53. *Let $\mathcal{A}$ be additive and idempotent complete. Suppose given an idempotent $X \xrightarrow{e} X$. The following assertions* (1, 2, 3) *hold.*

(1) *We have* $\dot{e}\overline{(1-e)} = 0$ *and* $(1-e)^{\cdot}\,\bar{e} = 0$.

(2) *We have mutually inverse isomorphisms*

$$\operatorname{Im}(e) \oplus \operatorname{Im}(1-e) \underset{\begin{pmatrix} \bar{e} & \overline{(1-e)} \end{pmatrix}}{\overset{\begin{pmatrix} \dot{e} \\ (1-e)^{\cdot} \end{pmatrix}}{\rightleftarrows}} X \; .$$

(3) *The following quadrangle is commutative.*

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathrm{Im}(e)\oplus\mathrm{Im}(1-e) \arrow[r, "\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)"] \arrow[d, shift left, "\left(\begin{smallmatrix}\dot e\\(1-e)^{\cdot}\end{smallmatrix}\right)"] & \mathrm{Im}(e)\oplus\mathrm{Im}(1-e) \arrow[d, shift left, "\left(\begin{smallmatrix}\dot e\\(1-e)^{\cdot}\end{smallmatrix}\right)"] \\
X \arrow[r, "e"'] \arrow[u, shift left, "\left(\begin{smallmatrix}\bar e & \overline{(1-e)}\end{smallmatrix}\right)", "\wr"'] & X \arrow[u, shift left, "\left(\begin{smallmatrix}\bar e & \overline{(1-e)}\end{smallmatrix}\right)", "\wr"']
\end{tikzcd}$$

Proof. *Ad* (1). We have $0 = e(1-e) = \bar e\dot e\overline{(1-e)}(1-e)^{\cdot}$. By Lemma 49, $\bar e$ is epic and $(1-e)^{\cdot}$ is monic. Therefore, $0 = \dot e\overline{(1-e)}$.

The other equality follows from the symmetry of the situation.

Ad (2). Since (1) holds, we have

$$\left(\begin{smallmatrix}\dot e\\(1-e)^{\cdot}\end{smallmatrix}\right)\left(\begin{smallmatrix}\bar e & \overline{(1-e)}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\dot e\bar e & \dot e\overline{(1-e)}\\(1-e)^{\cdot}\bar e & (1-e)^{\cdot}\overline{(1-e)}\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right).$$

Moreover, we obtain

$$\left(\begin{smallmatrix}\bar e & \overline{(1-e)}\end{smallmatrix}\right)\left(\begin{smallmatrix}\dot e\\(1-e)^{\cdot}\end{smallmatrix}\right) = \bar e\dot e + \overline{(1-e)}(1-e)^{\cdot} = e + (1-e) = 1.$$

Ad (3). Using (1), we compute $(1-e)^{\cdot}e = (1-e)^{\cdot}\bar e\dot e = 0$.

Therefore, we have

$$\left(\begin{smallmatrix}\dot e\\(1-e)^{\cdot}\end{smallmatrix}\right)e = \left(\begin{smallmatrix}\dot e e\\(1-e)^{\cdot}e\end{smallmatrix}\right) = \left(\begin{smallmatrix}\dot e\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\dot e\\(1-e)^{\cdot}\end{smallmatrix}\right).$$

Thus, the claim follows from (2). □

2.1.5 The Karoubi envelope is idempotent complete

After introducing idempotent complete categories in §2.1.4, we can now show that the Karoubi envelope is idempotent complete. Furthermore, in Proposition 56 we give a necessary and sufficient condition for a category $\mathcal{C}$ to be idempotent complete. This condition justifies the notion Karoubi *envelope* for $\operatorname{Kar}\mathcal{C}$.

For this §2.1.5, let $\mathcal{A}$ be a category.

Lemma 54. *Suppose given an idempotent* $(A,e) \xrightarrow{{}_e(\,\varphi\,)_e} (A,e)$ *in* $\operatorname{Kar}\mathcal{A}$.

Then $((A,\varphi),\ {}_e(\,\varphi\,)_\varphi,\ {}_\varphi(\,\varphi\,)_e)$ *is an image of* ${}_e(\,\varphi\,)_e$.

In particular, $\operatorname{Kar}\mathcal{A}$ *is idempotent complete.*

Proof. Since ${}_e(\,\varphi^2\,)_e \overset{\text{D33}}{=} {}_e(\,\varphi\,)_e^2 = {}_e(\,\varphi\,)_e$, we have $\varphi^2 = \varphi$; cf. Definition 33. Therefore, we have $(A,\varphi) \in \operatorname{Ob}\operatorname{Kar}\mathcal{A}$.

Moreover, we have $\varphi\varphi e = \varphi e \overset{\text{R2}}{=} \varphi$. Thus, ${}_\varphi(\,\varphi\,)_e \in {}_{\operatorname{Kar}\mathcal{A}}((A,\varphi),(A,e))$.

We also have $e\varphi\varphi = e\varphi \overset{\text{R2}}{=} \varphi$. Thus, ${}_{e}(\,\varphi\,)_{\varphi} \in {}_{\operatorname{Kar}\mathcal{A}}((A,e),(A,\varphi))$.

Therefore, we can consider the following diagram in $\operatorname{Kar}\mathcal{A}$.

$$\begin{tikzcd}
(A,e) \arrow[rr, "{}_{e}(\,\varphi\,)_{e}"] \arrow[dr, "{}_{e}(\,\varphi\,)_{\varphi}"'] & & (A,e) \arrow[dr, "{}_{e}(\,\varphi\,)_{\varphi}"] & \\
& (A,\varphi) \arrow[ur, "{}_{\varphi}(\,\varphi\,)_{e}"] \arrow[rr, "{}_{\varphi}(\,\varphi\,)_{\varphi}=1_{(A,\varphi)}"] & & (A,\varphi)
\end{tikzcd}$$

We have

$$ {}_{e}(\,\varphi\,)_{\varphi}\,{}_{\varphi}(\,\varphi\,)_{e} \overset{\text{D33}}{=} {}_{e}(\,\varphi\varphi\,)_{e} = {}_{e}(\,\varphi\,)_{e}\,. $$

Furthermore, we have

$$ {}_{\varphi}(\,\varphi\,)_{e}\,{}_{e}(\,\varphi\,)_{\varphi} \overset{\text{D33}}{=} {}_{\varphi}(\,\varphi\varphi\,)_{\varphi} = {}_{\varphi}(\,\varphi\,)_{\varphi} \overset{\text{D33}}{=} 1_{(A,\varphi)}\,. $$

Thus, the diagram stated above is commutative.

Therefore, $((A,\varphi),\ {}_{e}(\,\varphi\,)_{\varphi}\,,\ {}_{\varphi}(\,\varphi\,)_{e})$ is an image of ${}_{e}(\,\varphi\,)_{e}$; cf. Definition 45. □

Remark 55. *Suppose given an idempotent $X \xrightarrow{e} X$ in $\mathcal{A}$. Suppose e to have an image (Y,π,ι) in $\mathcal{A}$. We have mutually inverse isomorphisms*

$$\begin{tikzcd}
(X,e) \arrow[r, shift left, "{}_{e}(\,\pi\,)_{1_Y}"] & (Y,1_Y) \arrow[l, shift left, "{}_{1_Y}(\,\iota\,)_{e}"]
\end{tikzcd}$$

in $\operatorname{Kar}\mathcal{A}$.

Proof. Since (Y,π,ι) is an image of e in $\mathcal{A}$, $(JY, J\pi, J\iota) \overset{\text{D42}}{=} ((Y,1_Y),\ {}_{1_X}(\,\pi\,)_{1_Y},\ {}_{1_Y}(\,\iota\,)_{1_X})$ is an image of $Je \overset{\text{D42}}{=} {}_{1_X}(\,e\,)_{1_X}$ in $\operatorname{Kar}\mathcal{A}$; cf. Remark 47.

By Lemma 54, $((X,e),\ {}_{1_X}(\,e\,)_{e},\ {}_{e}(\,e\,)_{1_X})$ is an image of ${}_{1_X}(\,e\,)_{1_X}$ in $\operatorname{Kar}\mathcal{A}$.

By Corollary 51, we have mutually inverse isomorphisms

$$\begin{tikzcd}
(X,e) \arrow[r, shift left, "{}_{e}(\,e\,)_{1_X}\,{}_{1_X}(\,\pi\,)_{1_Y}"] & (Y,1_Y). \arrow[l, shift left, "{}_{1_Y}(\,\iota\,)_{1_X}\,{}_{1_X}(\,e\,)_{e}"]
\end{tikzcd}$$

Since we have

$$ {}_{e}(\,e\,)_{1_X}\,{}_{1_X}(\,\pi\,)_{1_Y} \overset{\text{D33}}{=} {}_{e}(\,e\pi\,)_{1_Y} = {}_{e}(\,\pi\,)_{1_Y} \quad\text{and}\quad {}_{1_Y}(\,\iota\,)_{1_X}\,{}_{1_X}(\,e\,)_{e} \overset{\text{D33}}{=} {}_{1_Y}(\,\iota e\,)_{e} = {}_{1_Y}(\,\iota\,)_{e}\,, $$

the claim follows. □

Proposition 56. *Suppose given a category $\mathcal{B}$. The following assertions* (1, 2) *are equivalent.*

(1) *The category $\mathcal{B}$ is idempotent complete.*

(2) *The inclusion functor $J_{\mathcal{B}} : \mathcal{B} \to \operatorname{Kar} \mathcal{B}$ is an equivalence.*

Proof. *Ad* (1)⇒(2). By Lemma 43, it suffices to show that J is dense. Suppose given $(X, e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{B}$. Since $\mathcal{B}$ is idempotent complete, there exists an image $\operatorname{Im} e$ of e in $\mathcal{B}$. We have $J(\operatorname{Im} e) \overset{\text{D42}}{=} (\operatorname{Im} e, 1_{\operatorname{Im} e}) \overset{\text{R55}}{\cong} (X, e)$. Thus, J is dense.

Ad (2)⇒(1). By Lemma 54, $\operatorname{Kar} \mathcal{B}$ is idempotent complete. Thus, $\mathcal{B}$ is idempotent complete; cf. Remark 48. □

2.1.6 Functoriality

In this §2.1.6 we define the Karoubi envelope construction Kar for functors and transformations. Furthermore, we establish functoriality properties of Kar, which could be expressed by saying that it is turned into a 2-functor.

For this §2.1.6, let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories.

Definition 57 (and Lemma)**.** Suppose given a functor $F : \mathcal{A} \to \mathcal{B}$. Let

$$\begin{array}{ccc} \operatorname{Kar} \mathcal{A} & \xrightarrow{\operatorname{Kar} F} & \operatorname{Kar} \mathcal{B} \\ \Big((X, e) \xrightarrow{{}_{e}(\,\varphi\,)_{f}} (Y, f) \Big) & \mapsto & \Big((FX, Fe) \xrightarrow{{}_{Fe}(\,F\varphi\,)_{Ff}} (FY, Ff) \Big). \end{array}$$

This defines a functor $\operatorname{Kar} F : \operatorname{Kar} \mathcal{A} \to \operatorname{Kar} \mathcal{B}$.

So, we have
$$(\operatorname{Kar} F)(X, e) = (FX, Fe)$$
for $(X, e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}$.

Furthermore, we have
$$(\operatorname{Kar} F)\, {}_{e}(\,\varphi\,)_{f} = {}_{Fe}(\,F\varphi\,)_{Ff}$$
for ${}_{e}(\,\varphi\,)_{f} \in \operatorname{Mor} \operatorname{Kar} \mathcal{A}$.

Proof. By Remark 5, we obtain welldefined maps on objects and morphisms; cf. Definition 33.

Suppose given $(X, e) \xrightarrow{{}_{e}(\,\varphi\,)_{f}} (Y, f) \xrightarrow{{}_{f}(\,\psi\,)_{g}} (Z, g)$ in $\operatorname{Kar} \mathcal{A}$.

We have
$$(\operatorname{Kar} F) 1_{(X,e)} \overset{\text{D33}}{=} (\operatorname{Kar} F)\, {}_{e}(\,e\,)_{e} = {}_{Fe}(\,Fe\,)_{Fe} \overset{\text{D33}}{=} 1_{(FX,Fe)} = 1_{(\operatorname{Kar} F)(X,e)} \,.$$

Furthermore, we have
$$\begin{aligned} (\operatorname{Kar} F)\, \big({}_{e}(\,\varphi\,)_{f}\, {}_{f}(\,\psi\,)_{g} \big) &\overset{\text{D33}}{=} (\operatorname{Kar} F)\, {}_{e}(\,\varphi\psi\,)_{g} = {}_{Fe}(\,F(\varphi\psi)\,)_{Fg} = {}_{Fe}(\,F\varphi \cdot F\psi\,)_{Fg} \\ &\overset{\text{D33}}{=} {}_{Fe}(\,F\varphi\,)_{Ff}\, {}_{Ff}(\,F\psi\,)_{Fg} = (\operatorname{Kar} F)\, {}_{e}(\,\varphi\,)_{f} \cdot (\operatorname{Kar} F)\, {}_{f}(\,\psi\,)_{g} \,. \end{aligned}$$

Therefore, $\operatorname{Kar} F$ is a functor indeed. □

Lemma 58. *Suppose given a functor $F : \mathcal{A} \to \mathcal{B}$. The following assertions* (1, 2, 3) *hold.*

(1) *Suppose F to be faithful. Then* $\operatorname{Kar} F$ *is faithful.*

(2) *Suppose F to be full. Then* $\operatorname{Kar} F$ *is full.*

(3) *Suppose F to be full, faithful and dense. Then* $\operatorname{Kar} F$ *is full, faithful and dense. I.e. if F is an equivalence, so is* $\operatorname{Kar} F$.

Proof. Suppose given $(X, e), (Y, f) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}$.

Ad (1). Suppose given ${}_e(\,\varphi\,)_f$ and ${}_e(\,\psi\,)_f$ in ${}_{\operatorname{Kar}\mathcal{A}}((X, e), (Y, f))$ with

$${}_{Fe}(\,F\varphi\,)_{Ff} = (\operatorname{Kar} F)\,{}_e(\,\varphi\,)_f = (\operatorname{Kar} F)\,{}_e(\,\psi\,)_f = {}_{Fe}(\,F\psi\,)_{Ff}\ .$$

By Definition 33, we obtain $F\varphi = F\psi$. Since F is faithful, we obtain $\varphi = \psi$. Therefore, we have ${}_e(\,\varphi\,)_f = {}_e(\,\psi\,)_f$. Thus, $\operatorname{Kar} F$ is faithful.

Ad (2). Suppose given ${}_{Fe}(\,\rho\,)_{Ff} \in {}_{\operatorname{Kar}\mathcal{B}}((FX, Fe), (FY, Ff))$.

Then we have $\rho \in {}_{\mathcal{B}}(FX, FY)$. Since F is full, there exists $\varphi \in {}_{\mathcal{A}}(X, Y)$ with $F\varphi = \rho$. Define $\tilde{\varphi} := e\varphi f$. Then we have

$$e\tilde{\varphi}f = ee\varphi ff = e\varphi f = \tilde{\varphi}.$$

Therefore, we have ${}_e(\,\varphi'\,)_f \in {}_{\operatorname{Kar}\mathcal{A}}((X, e), (Y, f))$.

Since ${}_{Fe}(\,\rho\,)_{Ff} \in {}_{\operatorname{Kar}\mathcal{B}}((FX, Fe), (FY, Ff))$, we have

$$F\tilde{\varphi} = F(e\varphi f) = Fe \cdot F\varphi \cdot Ff = Fe \cdot \rho \cdot Ff = \rho.$$

In consequence, we have

$$(\operatorname{Kar} F)\,{}_e(\,\tilde{\varphi}\,)_f \overset{\text{D57}}{=} {}_{Fe}(\,F\tilde{\varphi}\,)_{Ff} = {}_{Fe}(\,\rho\,)_{Ff}\ .$$

Thus, $\operatorname{Kar} F$ is full.

Ad (3). Since (1) and (2) hold, it suffices to show that $\operatorname{Kar} F$ is dense.

Suppose given $(A, f) \in \operatorname{Kar} \mathcal{B}$. Then we have $A \in \operatorname{Ob} \mathcal{B}$.

Since F is dense, there exists $X \in \operatorname{Ob} \mathcal{A}$ with $FX \cong A$. Let $FX \xrightarrow[\sim]{\rho} A$ be an isomorphism.

Define $f' := \rho f \rho^{-1}$. By Remark 3, we have $(FX, f') \in \operatorname{Ob} \operatorname{Kar} \mathcal{B}$. By Remark 36, we have an isomorphism $(FX, f') \xrightarrow[\sim]{{}_{f'}(\,f'\rho f\,)_f} (A, f)$.

Since F is full, there exists $e \in {}_{\mathcal{A}}(X, X)$ with $Fe = f'$.

We have $F(e^2) = Fe \cdot Fe = Fe$. Since F is faithful, we obtain $e^2 = e$. Therefore, we have $(X, e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}$.

In consequence, we have

$$(\operatorname{Kar} F)(X, e) \overset{\text{D57}}{=} (FX, Fe) = (FX, f') \cong (A, f)\ .$$

Thus, $\operatorname{Kar} F$ is dense.

$\square$

Lemma 59. *The following assertions* (1, 2) *hold.*

(1) *We have* $\operatorname{Kar} 1_{\mathcal{A}} = 1_{\operatorname{Kar}\mathcal{A}}$.

(2) *Suppose given functors* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$. *We have* $(\operatorname{Kar} G) \circ (\operatorname{Kar} F) = \operatorname{Kar}(G \circ F)$.

Proof. Suppose given $(X, e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y, f)$ in $\operatorname{Kar}\mathcal{A}$.

Ad (1). We have

$$\begin{aligned}
(\operatorname{Kar} 1_{\mathcal{A}})\Big((X, e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y, f)\Big) &\overset{\text{D57}}{=} \Big((1_{\mathcal{A}}X, 1_{\mathcal{A}}e) \xrightarrow{{}_{1_{\mathcal{A}}(e)}(\,1_{\mathcal{A}}\varphi\,)_{1_{\mathcal{A}}(f)}} (1_{\mathcal{A}}Y, 1_{\mathcal{A}}f)\Big) \\
&= \Big((X, e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y, f)\Big) \\
&= 1_{\operatorname{Kar}\mathcal{A}}\Big((X, e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y, f)\Big).
\end{aligned}$$

Ad (2). We have

$$\begin{aligned}
&((\operatorname{Kar} G) \circ (\operatorname{Kar} F))\Big((X, e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y, f)\Big) \\
\overset{\text{D57}}{=}\;& (\operatorname{Kar} G)\Big((FX, Fe) \xrightarrow{{}_{Fe}(\,F\varphi\,)_{Ff}} (FY, Ff)\Big) \\
\overset{\text{D57}}{=}\;& \Big((G(FX), G(Fe)) \xrightarrow{{}_{G(Fe)}(\,G(F\varphi)\,)_{G(Ff)}} (G(FY), G(Ff))\Big) \\
=\;& \Big(((G \circ F)X, (G \circ F)e) \xrightarrow{{}_{(G\circ F)e}(\,(G\circ F)\varphi\,)_{(G\circ F)f}} ((G \circ F)Y, (G \circ F)f)\Big) \\
\overset{\text{D57}}{=}\;& (\operatorname{Kar}(G \circ F))\Big((X, e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y, f)\Big).
\end{aligned}$$

□

Definition 60 (and Lemma)**.** Suppose given $F \xrightarrow{\alpha} \tilde{F}$ in $[\mathcal{A}, \mathcal{B}]$.

For $(X, e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}$, we define

$$\bar{\alpha}_{(X,e)} := Fe \cdot \alpha_X \cdot \tilde{F}e.$$

For $(X, e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}$*, we have*

$${}_{Fe}(\ \bar{\alpha}_{(X,e)}\ )_{\tilde{F}e} \in {}_{\operatorname{Kar} \mathcal{B}}\Big((FX, Fe), (\tilde{F}X, \tilde{F}e)\Big).$$

Suppose given (Y, f) *and* (Z, g) *in* $\operatorname{Ob} \operatorname{Kar} \mathcal{A}$ *and* ${}_{f}(\ \varphi\ )_{g} \in {}_{\operatorname{Kar} \mathcal{A}}((Y, f), (Z, g))$.

The following diagram is commutative.

$$\begin{tikzcd}
(FY, Ff) \arrow[r, "{}_{Ff}(\ F\varphi\ )_{Fg}"] \arrow[d, "{}_{Ff}(\ \bar{\alpha}_{(Y,f)}\ )_{\tilde{F}f}"'] & (FZ, Fg) \arrow[d, "{}_{Fg}(\ \bar{\alpha}_{(Z,g)}\ )_{\tilde{F}g}"] \\
(\tilde{F}Y, \tilde{F}f) \arrow[r, "{}_{\tilde{F}f}(\ \tilde{F}\varphi\ )_{\tilde{F}g}"] & (\tilde{F}Z, \tilde{F}g)
\end{tikzcd}$$

Define

$$\begin{aligned}
\operatorname{Kar} \alpha \quad &:= \quad \big({}_{Fe}(\ \bar{\alpha}_{(X,e)}\ )_{\tilde{F}e} \big)_{(X,e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}} \\
&= \quad \Big({}_{Fe}(\ Fe \cdot \alpha_X \cdot \tilde{F}e\ )_{\tilde{F}e} \Big)_{(X,e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}} .
\end{aligned}$$

We have the transformation $\operatorname{Kar} F \xRightarrow{\operatorname{Kar} \alpha} \operatorname{Kar} \tilde{F}$.

Proof. Suppose given $(X, e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}$. We have

$$Fe \cdot \bar{\alpha}_{(X,e)} \cdot \tilde{F}e = Fe \cdot Fe \cdot \alpha_X \cdot \tilde{F}e \cdot \tilde{F}e = Fe \cdot \alpha_X \cdot \tilde{F}e = \bar{\alpha}_{(X,e)} .$$

Therefore, we have ${}_{Fe}(\ \bar{\alpha}_{(X,e)}\ )_{\tilde{F}e} \in {}_{\operatorname{Kar} \mathcal{B}}\Big((FX, Fe), (\tilde{F}X, \tilde{F}e)\Big)$.

Furthermore, we have

$$\begin{aligned}
{}_{Ff}(\ F\varphi\ )_{Fg}\ {}_{Fg}(\ \bar{\alpha}_{(Z,g)}\ )_{\tilde{F}g} \quad &\overset{\text{D33}}{=} \quad {}_{Ff}(\ F\varphi \cdot Fg \cdot \alpha_Z \cdot \tilde{F}g\ )_{\tilde{F}g} \\
&= \quad {}_{Ff}(\ Ff \cdot F\varphi \cdot \alpha_Z \cdot \tilde{F}g\ )_{\tilde{F}g} \\
&= \quad {}_{Ff}(\ Ff \cdot \alpha_Y \cdot \tilde{F}\varphi \cdot \tilde{F}g\ )_{\tilde{F}g} \\
&= \quad {}_{Ff}(\ Ff \cdot \alpha_Y \cdot \tilde{F}f \cdot \tilde{F}\varphi\ )_{\tilde{F}g} \\
&\overset{\text{D33}}{=} \quad {}_{Ff}(\ \bar{\alpha}_{(Y,f)}\ )_{\tilde{F}f} \cdot {}_{\tilde{F}f}(\ \tilde{F}\varphi\ )_{\tilde{F}g} .
\end{aligned}$$

Therefore, the diagram stated above is commutative. Thus, $\operatorname{Kar} \alpha$ is a transformation. $\square$

Lemma 61. *Suppose given* $F \in \operatorname{Ob}[\mathcal{A}, \mathcal{B}]$*. Suppose given* $F_0 \xrightarrow{\alpha_0} F_1 \xrightarrow{\alpha_1} F_2$ *in* $[\mathcal{A}, \mathcal{B}]$.

The following assertions (1, 2) *hold.*

(1) *We have* $\operatorname{Kar} 1_F = 1_{\operatorname{Kar} F}$.

(2) *We have* $\operatorname{Kar}(\alpha_0 \alpha_1) = (\operatorname{Kar} \alpha_0)(\operatorname{Kar} \alpha_1)$.

Proof. Suppose given $(X, e) \in \operatorname{Ob}\operatorname{Kar}\mathcal{A}$.

Ad (1). We have

$$(\operatorname{Kar} 1_F)_{(X,e)} \overset{\text{D60}}{=} {}_{Fe}(\, Fe \cdot (1_F)_X \cdot Fe \,)_{Fe} = {}_{Fe}(\, Fe \,)_{Fe} \overset{\text{D33}}{=} 1_{(FX,Fe)} \,.$$

Ad (2). We have

$$\begin{aligned}
\overline{\alpha_0\alpha_1}_{(X,e)} &\overset{\text{D60}}{=} F_0e \cdot (\alpha_0)_X \cdot (\alpha_1)_X \cdot F_2e \\
&= F_0e \cdot F_0e \cdot (\alpha_0)_X \cdot (\alpha_1)_X \cdot F_2e \cdot F_2e \\
&= Fe \cdot (\alpha_0)_X \cdot F_1e \cdot F_1e \cdot (\alpha_1)_X \cdot F_2e \\
&\overset{\text{D60}}{=} \overline{\alpha_0}_{(X,e)}\overline{\alpha_1}_{(X,e)} \,.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\operatorname{Kar}(\alpha_0\alpha_1)_{(X,e)} &\overset{\text{D60}}{=} {}_{F_0e}(\, \overline{\alpha_0\alpha_1}_{(X,e)} \,)_{F_2e} = {}_{F_0e}(\, \overline{\alpha_0}_{(X,e)}\overline{\alpha_1}_{(X,e)} \,)_{F_2e} \\
&\overset{\text{D33}}{=} {}_{F_0e}(\, \overline{\alpha_0}_{(X,e)} \,)_{F_1e}\; {}_{F_1e}(\, \overline{\alpha_1}_{(X,e)} \,)_{F_2e} \overset{\text{D60}}{=} (\operatorname{Kar}\alpha_0)_{(X,e)}(\operatorname{Kar}\alpha_1)_{(X,e)} \,.
\end{aligned}$$

$\square$

Remark 62. We have the functor

$$\begin{array}{rcl}
[\mathcal{A}, \mathcal{B}] & \xrightarrow{\operatorname{Kar}} & [\operatorname{Kar}\mathcal{A}, \operatorname{Kar}\mathcal{B}] \\
\left(F \xrightarrow{\alpha} G\right) & \mapsto & \left(\operatorname{Kar} F \xrightarrow{\operatorname{Kar}\alpha} \operatorname{Kar} G\right);
\end{array}$$

cf. Lemma 61.

Lemma 63. *Suppose $\mathcal{A}$ and $\mathcal{B}$ to be preadditive. The following assertions* (1, 2) *hold.*

(1) *Suppose given $F \in \operatorname{Ob} {}_{\text{add}}[\mathcal{A}, \mathcal{B}]$. Then $\operatorname{Kar} F \in \operatorname{Ob} {}_{\text{add}}[\operatorname{Kar}\mathcal{A}, \operatorname{Kar}\mathcal{B}]$.*

(2) *Suppose $F \underset{\tilde{\alpha}}{\overset{\alpha}{\rightrightarrows}} \tilde{F}$ in ${}_{\text{add}}[\mathcal{A}, \mathcal{B}]$. We have $\operatorname{Kar}(\alpha + \tilde{\alpha}) = \operatorname{Kar}\alpha + \operatorname{Kar}\tilde{\alpha}$.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are preadditive, $\operatorname{Kar}\mathcal{A}$ and $\operatorname{Kar}\mathcal{B}$ are preadditive by Definition 38.

Ad (1). Suppose given $(X, e), (Y, f) \in \operatorname{Ob}\operatorname{Kar}\mathcal{A}$. Suppose given ${}_e(\, \varphi \,)_f$ and ${}_e(\, \psi \,)_f$ in ${}_{\operatorname{Kar}\mathcal{A}}((X, e), (Y, f))$. We have

$$\begin{aligned}
(\operatorname{Kar} F)\,(\, {}_e(\, \varphi \,)_f + {}_e(\, \psi \,)_f) &\overset{\text{D38}}{=} (\operatorname{Kar} F)\, {}_e(\, \varphi + \psi \,)_f \\
&\overset{\text{D57}}{=} {}_{Fe}(\, F(\varphi + \psi) \,)_{Ff} \\
&= {}_{Fe}(\, F\varphi + F\psi \,)_{Ff} \\
&\overset{\text{D38}}{=} {}_{Fe}(\, F\varphi \,)_{Ff} + {}_{Fe}(\, F\psi \,)_{Ff} \\
&\overset{\text{D57}}{=} (\operatorname{Kar} F)\, {}_e(\, \varphi \,)_f + (\operatorname{Kar} F)\, {}_e(\, \psi \,)_f \,.
\end{aligned}$$

Ad (2). Suppose given $(X, e) \in \operatorname{Ob}\operatorname{Kar}\mathcal{A}$. We have

$$\begin{aligned}
(\operatorname{Kar}(\alpha + \tilde{\alpha}))_{(X,e)} &\overset{\text{D60}}{=} {}_{Fe}(\, Fe \cdot (\alpha + \tilde{\alpha})_X \cdot \tilde{F}e \,)_{\tilde{F}e} \\
&= {}_{Fe}(\, Fe \cdot (\alpha_X + \tilde{\alpha}_X) \cdot \tilde{F}e \,)_{\tilde{F}e} \\
&= {}_{Fe}(\, Fe \cdot \alpha_X \cdot \tilde{F}e + Fe \cdot \tilde{\alpha}_X \cdot \tilde{F}e \,)_{\tilde{F}e} \\
&\overset{\text{D38}}{=} {}_{Fe}(\, Fe \cdot \alpha_X \cdot \tilde{F}e \,)_{\tilde{F}e} + {}_{Fe}(\, Fe \cdot \tilde{\alpha}_X \cdot \tilde{F}e \,)_{\tilde{F}e} \\
&\overset{\text{D60}}{=} (\operatorname{Kar}\alpha)_{(X,e)} + (\operatorname{Kar}\tilde{\alpha})_{(X,e)} \\
&= (\operatorname{Kar}\alpha + \operatorname{Kar}\tilde{\alpha})_{(X,e)} \,.
\end{aligned}$$

$\square$

Remark 64. *Suppose $\mathcal{A}$ and $\mathcal{B}$ to be preadditive.*

We have the additive functor

$$\begin{array}{ccc} {}_{\mathrm{add}}[\mathcal{A},\mathcal{B}] & \xrightarrow{\mathrm{Kar}} & {}_{\mathrm{add}}[\mathrm{Kar}\,\mathcal{A},\mathrm{Kar}\,\mathcal{B}] \\ \Big(F \overset{\alpha}{\to} G\Big) & \mapsto & \Big(\mathrm{Kar}\,F \xrightarrow{\mathrm{Kar}\,\alpha} \mathrm{Kar}\,G\Big); \end{array}$$

cf. Remark 62 and Lemma 63.

Remark 65. *Suppose given an isotransformation $F \xrightarrow[\sim]{\alpha} F'$ in $[\mathcal{A},\mathcal{B}]$.*

Then $\mathrm{Kar}\,F \xrightarrow[\sim]{\mathrm{Kar}\,\alpha} \mathrm{Kar}\,F'$ is an isotransformation in $[\mathrm{Kar}\,\mathcal{A},\mathrm{Kar}\,\mathcal{B}]$.

Proof. We have $(\mathrm{Kar}\,\alpha)(\mathrm{Kar}\,\alpha^{-1}) \overset{\mathrm{L61}}{=} \mathrm{Kar}(\alpha\alpha^{-1}) = \mathrm{Kar}\,1_F \overset{\mathrm{L59}}{=} 1_{\mathrm{Kar}\,F}$. $\square$

Remark 66. We can now give a new proof for Lemma 58.(3).

Suppose given an equivalence $F : \mathcal{A} \to \mathcal{B}$. Then $\mathrm{Kar}\,F : \mathrm{Kar}\,\mathcal{A} \to \mathrm{Kar}\,\mathcal{B}$ is an equivalence.

Proof. Since F is an equivalence, there exists a functor $G : \mathcal{B} \to \mathcal{A}$ and isotransformations $(G \circ F) \overset{\alpha}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{A}}$ and $(F \circ G) \overset{\beta}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{B}}$.

By Remark 65, we obtain the isotransformation

$$(\mathrm{Kar}\,G \circ \mathrm{Kar}\,F) \overset{\mathrm{L59}}{=} \mathrm{Kar}(G \circ F) \overset{\mathrm{Kar}\,\alpha}{\underset{\sim}{\Longrightarrow}} 1_{\mathrm{Kar}\,\mathcal{A}} .$$

Similarily, we have the isotransformation

$$(\mathrm{Kar}\,F \circ \mathrm{Kar}\,G) \overset{\mathrm{L59}}{=} \mathrm{Kar}(F \circ G) \overset{\mathrm{Kar}\,\beta}{\underset{\sim}{\Longrightarrow}} 1_{\mathrm{Kar}\,\mathcal{B}} ;$$

cf. Remark 65.

Thus, $\mathrm{Kar}\,F$ is an equivalence. $\square$

Lemma 67. *Suppose given functors $F, \tilde{F} : \mathcal{A} \to \mathcal{B}$ and $G, \tilde{G} : \mathcal{B} \to \mathcal{C}$. Suppose given transformations $\alpha : F \Rightarrow \tilde{F}$ and $\beta : G \Rightarrow \tilde{G}$. We have* $\mathrm{Kar}(\beta * \alpha) = (\mathrm{Kar}\,\beta) * (\mathrm{Kar}\,\alpha)$.

Proof. Suppose given $(X, e) \in \mathrm{Ob}\,\mathrm{Kar}\,\mathcal{A}$. We have

$$
\begin{aligned}
& ((\mathrm{Kar}\,\beta) * (\mathrm{Kar}\,\alpha))_{(X,e)} \\
= \; & (\mathrm{Kar}\,G)(\mathrm{Kar}\,\alpha)_{(X,e)} \cdot (\mathrm{Kar}\,\beta)_{(\mathrm{Kar}\,\tilde{F})(X,e)} \\
\overset{\mathrm{D60}}{\underset{\mathrm{D57}}{=}} \; & (\mathrm{Kar}\,G)\; {}_{Fe}(\; \bar{\alpha}_{(X,e)} \;)_{\tilde{F}e} \cdot {}_{(G\circ\tilde{F})e}(\; \bar{\beta}_{(\tilde{F}X, \tilde{F}e)} \;)_{(\tilde{G}\circ\tilde{F})e} \\
= \; & (\mathrm{Kar}\,G)\; {}_{Fe}(\; Fe \cdot \alpha_X \cdot \tilde{F}e \;)_{\tilde{F}e} \cdot {}_{(G\circ\tilde{F})e}(\; (G \circ \tilde{F})e \cdot \beta_{\tilde{F}X} \cdot (\tilde{G} \circ \tilde{F})e \;)_{(\tilde{G}\circ\tilde{F})e} \\
\overset{\mathrm{D57}}{=} \; & {}_{(G\circ F)e}(\; (G \circ F)e \cdot G\alpha_X \cdot (G \circ \tilde{F})e \;)_{(G\circ\tilde{F})e} \cdot {}_{(G\circ\tilde{F})e}(\; (G \circ \tilde{F})e \cdot \beta_{\tilde{F}X} \cdot (\tilde{G} \circ \tilde{F})e \;)_{(\tilde{G}\circ\tilde{F})e} \\
= \; & {}_{(G\circ F)e}(\; (G \circ F)e \cdot G\alpha_X \cdot (G \circ \tilde{F})e \cdot (G \circ \tilde{F})e \cdot \beta_{\tilde{F}X} \cdot (\tilde{G} \circ \tilde{F})e \;)_{(\tilde{G}\circ\tilde{F})e} \\
= \; & {}_{(G\circ F)e}(\; (G \circ F)e \cdot G\alpha_X \cdot (G \circ \tilde{F})e \cdot \beta_{\tilde{F}X} \cdot (\tilde{G} \circ \tilde{F})e \;)_{(\tilde{G}\circ\tilde{F})e} \\
= \; & {}_{(G\circ F)e}(\; (G \circ F)e \cdot (G \circ F)e \cdot G\alpha_X \cdot \beta_{\tilde{F}X} \cdot (\tilde{G} \circ \tilde{F})e \;)_{(\tilde{G}\circ\tilde{F})e} \\
= \; & {}_{(G\circ F)e}(\; (G \circ F)e \cdot G\alpha_X \cdot \beta_{\tilde{F}X} \cdot (\tilde{G} \circ \tilde{F})e \;)_{(\tilde{G}\circ\tilde{F})e} \\
= \; & {}_{(G\circ F)e}(\; (G \circ F)e \cdot (\beta * \alpha)_X \cdot (\tilde{G} \circ \tilde{F})e \;)_{(\tilde{G}\circ\tilde{F})e} \\
\overset{\mathrm{D60}}{=} \; & {}_{(G\circ F)e}(\; \overline{(\beta * \alpha)}_{(X,e)} \;)_{(\tilde{G}\circ\tilde{F})e} \\
\overset{\mathrm{D60}}{=} \; & (\mathrm{Kar}(\beta * \alpha))_{(X,e)} \; .
\end{aligned}
$$

$\square$

2.1.7 The image functor

For this §2.1.7, let $\mathcal{A}$ be an idempotent complete category.

Definition 68 (and Lemma)**.** Recall that in Stipulation 52 we assigned to every idempotent $X \xrightarrow{e} X$ in $\mathcal{A}$ an image $(\mathrm{Im}\,e, \bar{e}, \dot{e})$.

We have the functor

$$
\begin{aligned}
\mathrm{Kar}\,\mathcal{A} & \xrightarrow{I_{\mathcal{A}}} \mathcal{A} \\
((X, e) \xrightarrow{{}_{e}(\,\varphi\,)_{f}} (Y, f)) & \;\mapsto\; (\mathrm{Im}\,e \xrightarrow{\dot{e}\varphi\bar{f}} \mathrm{Im}\,f).
\end{aligned}
$$

We call $I_{\mathcal{A}}$ the *image functor* of $\mathcal{A}$.

If unambiguous, we often write $I := I_{\mathcal{A}}$.

Proof. Since $\mathcal{A}$ is idempotent complete, we obtain a welldefined map on objects. By Definition 33 and Stipulation 52, we obtain a welldefined map on morphisms.

Suppose given $(X, e) \xrightarrow{{}_{e}(\,\varphi\,)_{f}} (Y, f) \xrightarrow{{}_{f}(\,\psi\,)_{g}} (Z, g)$ in $\mathrm{Kar}\,\mathcal{A}$.

We have $I 1_{(X,e)} = I\,{}_{e}(\,e\,)_{e} = \dot{e}e\bar{e} = \dot{e}\bar{e}\dot{e}\bar{e} = 1_{\mathrm{Im}\,e} 1_{\mathrm{Im}\,e} = 1_{\mathrm{Im}\,e} = 1_{I(X,e)}$.

Furthermore, we have

$$I(\,{}_{e}(\,\varphi\,)_{f}\;{}_{f}(\,\psi\,)_{g}) = I\,{}_{e}(\,\varphi\psi\,)_{g} = \dot{e}\varphi\psi\bar{g} = \dot{e}\varphi f \psi\bar{g} = \dot{e}\varphi\bar{f}\dot{f}\psi\bar{g} = I\,{}_{e}(\,\varphi\,)_{f} \cdot I\,{}_{f}(\,\psi\,)_{g} \; . \qquad \square$$

Remark 69. We have $I_{\mathcal{A}} \circ J_{\mathcal{A}} = 1_{\mathcal{A}}$; cf. Definitions 42 and 68 and Stipulation 52.

Lemma 70. *The following assertions* (1, 2, 3) *hold.*

(1) *The image functor $I_{\mathcal{A}}$ is faithful.*

(2) *The image functor $I_{\mathcal{A}}$ is full.*

(3) *The image functor $I_{\mathcal{A}}$ is surjective on objects. In particular, $I_{\mathcal{A}}$ is dense.*

Thus, $I_{\mathcal{A}}$ is an equivalence.

Proof. Suppose given (X, e) and (Y, f) in $\operatorname{Ob}\operatorname{Kar}\mathcal{A}$.

Ad (1). Suppose given ${}_e(\,\varphi\,)_f$ and ${}_e(\,\psi\,)_f$ in ${}_{\operatorname{Kar}\mathcal{A}}((X, e), (Y, f))$ with

$$\dot{e}\varphi\bar{f} = I\,{}_e(\,\varphi\,)_f = I\,{}_e(\,\psi\,)_f = \dot{e}\psi\bar{f}.$$

Applying $\bar{e}$ from the left and $\dot{f}$ from the right to this equation, we obtain

$$\varphi = e\varphi f = \bar{e}\dot{e}\varphi\bar{f}\dot{f} = \bar{e}\dot{e}\psi\bar{f}\dot{f} = e\psi f = \psi.$$

Thus, I is faithful.

Ad (2). Suppose given $\varphi \in {}_{\mathcal{A}}(I(X, e), I(Y, f)) \overset{\text{D68}}{=} {}_{\mathcal{A}}(\operatorname{Im} e, \operatorname{Im} f)$.

Consider $\tilde{\rho} := \bar{e}\rho\dot{f} \in {}_{\mathcal{A}}(X, Y)$. We have

$$e\tilde{\rho}f = e\bar{e}\rho\dot{f}f = \bar{e}\dot{e}\bar{e}\rho\dot{f}\bar{f}\dot{f} = \bar{e} \cdot 1_{\operatorname{Im} e} \cdot \rho \cdot 1_{\operatorname{Im} f} \cdot \dot{f} = \bar{e}\rho\dot{f} = \tilde{\rho}.$$

Therefore, we have ${}_e(\,\tilde{\rho}\,)_f \in {}_{\operatorname{Kar}\mathcal{A}}((X, e), (Y, f))$. Furthermore, we have

$$I\,{}_e(\,\tilde{\rho}\,)_f \overset{\text{D68}}{=} \dot{e}\tilde{\rho}\bar{f} = \dot{e}\bar{e}\rho\dot{f}\bar{f} = 1_{\operatorname{Im} e} \cdot \rho \cdot 1_{\operatorname{Im} f} = \rho.$$

Thus, I is full.

Ad (3). Suppose given $Z \in \operatorname{Ob}\mathcal{A}$. We have $I(Z, 1_Z) = \operatorname{Im} 1_Z = Z$; cf. Stipluation Stipulation 52. Thus, I is surjective on objects. $\square$

Lemma 71. *Suppose $\mathcal{A}$ to be preadditive. Then $I_{\mathcal{A}}$ is additive.*

Proof. Since $\mathcal{A}$ is preadditive, so is $\operatorname{Kar}\mathcal{A}$; cf. Definition 38.

Suppose given $(X, e) \underset{{}_e(\,\psi\,)_f}{\overset{{}_e(\,\varphi\,)_f}{\rightrightarrows}} (Y, f)$ in $\operatorname{Kar}\mathcal{A}$.

We have

$$\begin{aligned} I\left({}_e(\,\varphi\,)_f + {}_e(\,\psi\,)_f\right) &\overset{\text{D38}}{=} I\,{}_e(\,\varphi + \psi\,)_f \\ &\overset{\text{D68}}{=} \dot{e}(\varphi + \psi)\bar{f} \\ &= \dot{e}\varphi\bar{f} + \dot{e}\psi\bar{f} \\ &\overset{\text{D68}}{=} I\,{}_e(\,\varphi\,)_f + I\,{}_e(\,\psi\,)_f\,. \end{aligned}$$

$\square$

2.1.8 Universal property

For this §2.1.8, let $\mathcal{A}$ and $\mathcal{B}$ be categories. Let $\mathcal{B}$ be idempotent complete.

Definition 72 (and Lemma)**.** *We have the functor*

$$\begin{array}{rcl} [\mathcal{A},\mathcal{B}] & \to & [\operatorname{Kar}\mathcal{A},\mathcal{B}] \\ (F \xrightarrow{\alpha} G) & \mapsto & \Big((I_{\mathcal{B}} \circ \operatorname{Kar} F) \xrightarrow{I_{\mathcal{B}} * \operatorname{Kar}\alpha} (I_{\mathcal{B}} \circ \operatorname{Kar} G)\Big). \end{array}$$

Cf. Definitions 57, 60 and 68.

For $F \xrightarrow{\alpha} G$ in $[\mathcal{A},\mathcal{B}]$, we write $F' := I_{\mathcal{B}} \circ \operatorname{Kar} F$ and $\alpha' := I_{\mathcal{B}} * \operatorname{Kar}\alpha$.

Suppose given $F \xrightarrow{\alpha} G$ *in* $[\mathcal{A},\mathcal{B}]$*. We have*

$$\begin{array}{rcl} \operatorname{Kar}\mathcal{A} & \xrightarrow{F'} & \mathcal{B} \\ \Big((X,e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y,f)\Big) & \mapsto & \Big(\operatorname{Im} Fe \xrightarrow{(Fe)^{\cdot}\,\cdot\, F\varphi\,\cdot\,\overline{Ff}} \operatorname{Im} Ff\Big). \end{array}$$

Suppose given $(X,e) \in \operatorname{Ob}\operatorname{Kar}\mathcal{A}$*. We have*

$$\alpha'_{(X,e)} = (Fe)^{\cdot} \cdot \alpha_X \cdot \overline{Ge}.$$

Proof. Suppose given $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ in $[\mathcal{A},\mathcal{B}]$.

Suppose given $(X,e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y,f)$ in $\operatorname{Kar}\mathcal{A}$. We have

$$\begin{array}{rcl} F'\Big((X,e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y,f)\Big) & = & (I_{\mathcal{B}} \circ \operatorname{Kar} F)\Big((X,e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y,f)\Big) \\ & \overset{\text{D57}}{=} & I_{\mathcal{B}}\Big((FX,Fe) \xrightarrow{{}_{Fe}(\,F\varphi\,)_{Ff}} (FY,Ff)\Big) \\ & \overset{\text{D68}}{=} & \Big(\operatorname{Im} Fe \xrightarrow{(Fe)^{\cdot}\,\cdot\, F\varphi\,\cdot\,\overline{Ff}} \operatorname{Im} Ff\Big). \end{array}$$

Furthermore, we have

$$\begin{array}{rcl} \alpha'_{(X,e)} & = & (I_{\mathcal{B}} * \operatorname{Kar}\alpha)_{(X,e)} \\ & = & I_{\mathcal{B}}\big((\operatorname{Kar}\alpha)_{(X,e)}\big) \\ & \overset{\text{D60}}{=} & I_{\mathcal{B}}\big(\,{}_{Fe}(\,Fe \cdot \alpha_X \cdot Ge\,)_{Ge}\big) \\ & \overset{\text{D68}}{=} & (Fe)^{\cdot} \cdot Fe \cdot \alpha_X \cdot Ge \cdot \overline{Ge} \\ & = & (Fe)^{\cdot} \cdot \alpha_X \cdot \overline{Ge}. \end{array}$$

Thus, we have

$$\begin{aligned} (1_F)'_{(X,e)} &= (Fe)^{\cdot} \cdot (1_F)_X \cdot \overline{Fe} \\ &= (Fe)^{\cdot} \cdot \overline{Fe} \\ &= 1_{\operatorname{Im}(Fe)} \\ &= 1_{F'X}\,. \end{aligned}$$

Therefore, we have $(1_F)' = 1_{F'}$.

Moreover, we have

$$\begin{aligned}
(\alpha\beta)'_{(X,e)} &= (Fe)^{\cdot} \cdot (\alpha\beta)_X \cdot \overline{He} \\
&= (Fe)^{\cdot} \cdot \alpha_X \cdot \beta_X \cdot \overline{He} \\
&= (Fe)^{\cdot} \cdot Fe \cdot \alpha_X \cdot \beta_X \cdot \overline{He} \\
&= (Fe)^{\cdot} \cdot \alpha_X \cdot Ge \cdot \beta_X \cdot \overline{He} \\
&= (Fe)^{\cdot} \cdot \alpha_X \cdot \overline{Ge} \cdot (Ge)^{\cdot} \cdot \beta_X \cdot \overline{He} \\
&= \alpha'_{(X,e)} \cdot \beta'_{(X,e)} \,.
\end{aligned}$$

In consequence, we have $(\alpha\beta)' = \alpha'\beta'$.

Thus, we have a functor indeed. □

Lemma 73. *Suppose given $F \xrightarrow{\alpha} G$ in $[\mathcal{A}, \mathcal{B}]$. The following assertions* (1, 2) *hold.*

(1) *We have $F' \circ J_{\mathcal{A}} = F$.*

(2) *We have $\alpha' * J_{\mathcal{A}} = \alpha$.*

Proof. *Ad* (1). Suppose given $X \xrightarrow{\varphi} Y$ in $\mathcal{A}$. We have

$$\begin{aligned}
(F' \circ J_{\mathcal{A}})\Big(X \xrightarrow{\varphi} Y\Big) &\overset{\text{D42}}{=} F'\Big((X, 1_X) \xrightarrow{1_X(\,\varphi\,)1_Y} (Y, 1_Y)\Big) \\
&\overset{\text{D72}}{=} \Big(\operatorname{Im}(F1_X) \xrightarrow{(F1_X)^{\cdot}\,\cdot\, F\varphi\,\cdot\,\overline{F1_Y}} \operatorname{Im}(F1_Y)\Big) \\
&\overset{\text{S52}}{=} \Big(FX \xrightarrow{F1_X\,\cdot\, F\varphi\,\cdot\, F1_Y} FY\Big) \\
&= F\Big(X \xrightarrow{\varphi} Y\Big).
\end{aligned}$$

Ad (2). Suppose given $X \in \operatorname{Ob}\mathcal{A}$. We have

$$(\alpha' * J_{\mathcal{A}})_X \overset{\text{D42}}{=} \alpha'_{(X,1_X)} \overset{\text{D72}}{=} (F1_X)^{\cdot} \cdot \alpha_X \cdot \overline{G1_X} \overset{\text{S52}}{=} F1_X \cdot \alpha_X \cdot G1_X = \alpha_X \,.$$

□

Lemma 74. *Suppose given $F \in \operatorname{Ob}[\mathcal{A}, \mathcal{B}]$. The following assertions* (1, 2, 3) *hold.*

(1) *If F is faithful, so is F'.*

(2) *If F is full, so is F'.*

(3) *If F is dense, so is F'.*

Proof. *Ad* (1). Since F is faithful, so is $\operatorname{Kar} F$; cf. Lemma 58.(1). By Lemma 70.(1), $I_{\mathcal{B}}$ is faithful. Thus, $F' = I_{\mathcal{B}} \circ \operatorname{Kar} F$ is faithful.

Ad (2). Since F is full, so is $\operatorname{Kar} F$; cf. Lemma 58.(2). By Lemma 70.(2), $I_{\mathcal{B}}$ is full. Thus, $F' = I_{\mathcal{B}} \circ \operatorname{Kar} F$ is full.

Ad (3). Suppose given $B \in \operatorname{Ob} \mathcal{B}$. Since F is dense, there exists $A \in \operatorname{Ob} \mathcal{A}$ with $FA \cong B$. We have

$$F'(A, 1_A) = (F' \circ J_{\mathcal{A}})A \overset{\text{L73.(1)}}{=} FA \cong B.$$

Thus, F' is dense. □

Proposition 75. Recall that $\mathcal{A}$ and $\mathcal{B}$ are categories and that $\mathcal{B}$ is idempotent complete; cf. Definition 45. Recall that $\operatorname{Kar} \mathcal{A}$ is idempotent complete; cf. Lemma 54.

The following assertions (1, 2, 3) *hold.*

(1) *We have* $J_{\mathcal{A}} \in \operatorname{Ob}[\mathcal{A}, \operatorname{Kar} \mathcal{A}]$.

Suppose given $F \xrightarrow{\alpha} G$ *in* $[\mathcal{A}, \mathcal{B}]$. *We have* $F' \xrightarrow{\alpha'} G'$ *in* $[\operatorname{Kar} \mathcal{A}, \mathcal{B}]$ *with* $F' \circ J_{\mathcal{A}} = F$, $G' \circ J_{\mathcal{A}} = G$ *and* $\alpha' * J_{\mathcal{A}} = \alpha$*;* cf. Definition 72 and Lemma 73.

Suppose given $\beta \in {}_{[\operatorname{Kar} \mathcal{A}, \mathcal{B}]}(F', G')$ *with* $\beta * J_{\mathcal{A}} = \alpha$. *Then we have* $\beta = \alpha'$.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "J_{\mathcal{A}}"] \arrow[drr, bend right, "F"'] \arrow[drr, bend left, "G"] & & \operatorname{Kar} \mathcal{A} \arrow[d, bend right, "F'"'] \arrow[d, bend left, "G'"] \arrow[d, phantom, "\overset{\alpha'}{\Rightarrow}"] \\
& & \mathcal{B}
\end{tikzcd}$$

(2) *Suppose given* $U, V \in \operatorname{Ob}[\operatorname{Kar} \mathcal{A}, \mathcal{B}]$ *with* $U \circ J_{\mathcal{A}} = V \circ J_{\mathcal{A}}$. *Then* $U \cong V$.

(3) *We have the equivalence of categories*

$$\begin{array}{ccc}
[\mathcal{A}, \mathcal{B}] & \xleftarrow{\Lambda_{\mathcal{A},\mathcal{B}}} & [\operatorname{Kar} \mathcal{A}, \mathcal{B}] \\
\Big((U \circ J_{\mathcal{A}}) \xrightarrow{\beta * J_{\mathcal{A}}} (V \circ J_{\mathcal{A}})\Big) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V),
\end{array}$$

that is surjective on objects.

If unambiguous, we often write $\Lambda := \Lambda_{\mathcal{A},\mathcal{B}}$.

Proof. *Ad* (1). Suppose given $(X, e) \in \operatorname{Ob} \operatorname{Kar} \mathcal{A}$. We show that $\alpha'_{(X,e)} = \beta_{(X,e)}$.

By assumption, we have

$$\beta_{(X,1_X)} = (\beta * J_{\mathcal{A}})_X = \alpha_X = (\alpha' * J_{\mathcal{A}})_X = \alpha'_{(X,1_X)}\,.$$

Since α' is natural, the following diagram commutes.

$$\begin{tikzcd}
F'(X, 1_X) \arrow[r, "F'\,{}_{1_X}(\,e\,)_e"] \arrow[d, shift right, "\beta_{(X,1_X)}"'] \arrow[d, shift left, "\alpha'_{(X,1_X)}"] \arrow[d, phantom, "="] & F'(X, e) \arrow[r, "F'\,{}_e(\,e\,)_{1_X}"] \arrow[d, "\alpha'_{(X,e)}"] & F'(X, 1_X) \arrow[d, shift right, "\beta_{(X,1_X)}"'] \arrow[d, shift left, "\alpha'_{(X,1_X)}"] \arrow[d, phantom, "="] \\
G'(X, 1_X) \arrow[r, "G'\,{}_{1_X}(\,e\,)_e"] & G'(X, e) \arrow[r, "G'\,{}_e(\,e\,)_{1_X}"] & G'(X, 1_X)
\end{tikzcd}$$

Since β is natural, the following diagram commutes.

$$\begin{tikzcd}
F'(X,1_X) \arrow[r, "F'\,{}_{1_X}(\,e\,)_e"] \arrow[d, shift right=1.5, "\beta_{(X,1_X)}"'] \arrow[d, shift left=1.5, "\alpha'_{(X,1_X)}"] \arrow[d, phantom, "="] & F'(X,e) \arrow[r, "F'\,{}_e(\,e\,)_{1_X}"] \arrow[d, "\beta_{(X,e)}"] & F'(X,1_X) \arrow[d, shift right=1.5, "\beta_{(X,1_X)}"'] \arrow[d, shift left=1.5, "\alpha'_{(X,1_X)}"] \arrow[d, phantom, "="] \\
G'(X,1_X) \arrow[r, "G'\,{}_{1_X}(\,e\,)_e"] & G'(X,e) \arrow[r, "G'\,{}_e(\,e\,)_{1_X}"] & G'(X,1_X)
\end{tikzcd}$$

By Lemma 54, $((X,e),\ {}_{1_X}(\,e\,)_e,\ {}_e(\,e\,)_{1_X})$ is an image of $(X,1_X) \xrightarrow{{}_{1_X}(\,e\,)_{1_X}} (X,1_X)$ in $\operatorname{Kar}\mathcal{A}$.

Thus, $(F'(X,e), F'\,{}_{1_X}(\,e\,)_e\,, F'\,{}_e(\,e\,)_{1_X})$ is an image of

$$\left(F'(X,1_X) \xrightarrow{F'\,{}_{1_X}(\,e\,)_{1_X}} F'(X,1_X)\right)$$

in $\mathcal{B}$; cf. Remark 47.

Similarily, $(G'(X,e), G'\,{}_{1_X}(\,e\,)_e\,, G'\,{}_e(\,e\,)_{1_X})$ is an image of

$$\left(G'(X,1_X) \xrightarrow{G'\,{}_{1_X}(\,e\,)_{1_X}} G'(X,1_X)\right)$$

in $\mathcal{B}$; cf. Remark 47.

In consequence, we obtain $\alpha'_{(X,e)} = \beta_{(X,e)}$; cf. Lemma 50.(2).

Ad (2). Suppose we have shown that $(W \circ J_{\mathcal{A}})' \cong W$ for $W \in \operatorname{Ob}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$. Then we have

$$U \cong (U \circ J_{\mathcal{A}})' = (V \circ J_{\mathcal{A}})' \cong V.$$

Therefore, it suffices to show that $(W \circ J_{\mathcal{A}})' \cong W$ for $W \in \operatorname{Ob}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$.

Suppose given $W \in \operatorname{Ob}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$.

Suppose given $(X,e) \in \operatorname{Ob}\operatorname{Kar}\mathcal{A}$.

By Lemma 54 and Remark 47, $(W(X,e), W\,{}_{1_X}(\,e\,)_e\,, W\,{}_e(\,e\,)_{1_X})$ is an image of $W\,{}_{1_X}(\,e\,)_{1_X} = (W \circ J_{\mathcal{A}})e$ in $\mathcal{B}$.

By Corollary 51, we have mutually inverse isomorphisms

$$(W \circ J_{\mathcal{A}})'(X,e) \overset{\text{D72}}{=} \operatorname{Im}((W \circ J_{\mathcal{A}})e) \underset{W\,{}_e(\,e\,)_{1_X} \,\cdot\, \overline{(W\circ J_{\mathcal{A}})e}}{\overset{((W\circ J_{\mathcal{A}})e)^{\boldsymbol{\cdot}} \,\cdot\, W\,{}_{1_X}(\,e\,)_e}{\rightleftarrows}} W(X,e)$$

in $\mathcal{B}$.

Define

$$\delta := \left(W\,{}_e(\,e\,)_{1_X} \cdot \overline{(W \circ J_{\mathcal{A}})e}\right)_{(X,e)\in \operatorname{Ob}\operatorname{Kar}\mathcal{A}}.$$

We show that δ is natural. Suppose given $(X,e) \xrightarrow{{}_e(\,\varphi\,)_f} (Y,f)$ in $\operatorname{Kar}\mathcal{A}$.

We have

$$
\begin{aligned}
& \delta_{(X,e)} \cdot (W \circ J_{\mathcal{A}})'\,{}_e(\,\varphi\,)_f \\
\overset{\text{D72}}{=}\;& W\,{}_e(\,e\,)_{1_X} \cdot \overline{(W \circ J_{\mathcal{A}})e} \cdot ((W \circ J_{\mathcal{A}})e)^{\cdot} \cdot (W \circ J_{\mathcal{A}})\varphi \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W\,{}_e(\,e\,)_{1_X} \cdot (W \circ J_{\mathcal{A}})e \cdot (W \circ J_{\mathcal{A}})\varphi \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W\,{}_e(\,e\,)_{1_X} \cdot W\,{}_{1_X}(\,e\,)_{1_X} \cdot W\,{}_{1_X}(\,\varphi\,)_{1_Y} \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W\,(\,{}_e(\,e\,)_{1_X} \cdot {}_{1_X}(\,e\,)_{1_X} \cdot {}_{1_X}(\,\varphi\,)_{1_Y}\,) \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W\,{}_e(\,ee\varphi\,)_{1_Y} \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W\,{}_e(\,\varphi f\,)_{1_Y} \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W(\,{}_e(\,\varphi\,)_f \cdot {}_f(\,f\,)_{1_Y}\,) \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W\,{}_e(\,\varphi\,)_f \cdot W\,{}_f(\,f\,)_{1_Y} \cdot \overline{(W \circ J_{\mathcal{A}})f} \\
=\;& W\,{}_e(\,\varphi\,)_f \cdot \delta_{(Y,f)}\,.
\end{aligned}
$$

Thus, we have $W \overset{\delta}{\underset{\sim}{\Longrightarrow}} (W \circ J_{\mathcal{A}})'$.

Ad (3). Suppose given $F \in \operatorname{Ob}[\mathcal{A},\mathcal{B}]$. Because (1) holds, we have $F' \in [\operatorname{Kar}\mathcal{A},\mathcal{B}]$ and $\Lambda F' = F' \circ J_{\mathcal{A}} = F$. Therefore, Λ is surjective on objects.

Suppose given $F, G \in \operatorname{Ob}[\mathcal{A},\mathcal{B}]$. Since (1) holds, we have a bijection

$$ {}_{[\operatorname{Kar}\mathcal{A},\mathcal{B}]}(F', G') \to {}_{[\mathcal{A},\mathcal{B}]}(F, G),\ \beta \mapsto \Lambda\beta. $$

Suppose given $U, V \in \operatorname{Ob}[\operatorname{Kar}\mathcal{A},\mathcal{B}]$ with $\Lambda U = \Lambda V$. Since (2) holds, we have $U \cong V$.

Therefore, Λ is an equivalence by Lemma 6. □

Lemma 76. *Suppose $\mathcal{A}$ and $\mathcal{B}$ to be preadditive. Suppose given $F \in \operatorname{Ob}{}_{\text{add}}[\mathcal{A},\mathcal{B}]$. We have $F' \in \operatorname{Ob}{}_{\text{add}}[\operatorname{Kar}\mathcal{A},\mathcal{B}]$.*

Proof. Since $\mathcal{A}$ is preadditive, so is $\operatorname{Kar}\mathcal{A}$; cf. Definition 38.

Since F is additive, so is $\operatorname{Kar} F$; cf. Lemma 63. By Lemma 71, $I_{\mathcal{B}}$ is additive. Thus, $F' \overset{\text{D72}}{=} I_{\mathcal{B}} \circ \operatorname{Kar} F$ is additive. □

Proposition 77. *Suppose $\mathcal{A}$ and $\mathcal{B}$ to be preadditive categories.*

Recall that $\mathcal{B}$ is idempotent complete; cf. Definition 45. Recall that $\operatorname{Kar}\mathcal{A}$ is preadditive; cf. Definition 38; and idempotent complete; cf. Lemma 54.

The following assertions (1, 2, 3) *hold.*

(1) *We have $J_{\mathcal{A}} \in {}_{\text{add}}[\mathcal{A}, \operatorname{Kar}\mathcal{A}]$;* cf. Lemma 44.

Suppose given $F \xrightarrow{\alpha} G$ in ${}_{\text{add}}[\mathcal{A},\mathcal{B}]$. We have $F' \xrightarrow{\alpha'} G'$ in ${}_{\text{add}}[\operatorname{Kar}\mathcal{A},\mathcal{B}]$ with $F' \circ J_{\mathcal{A}} = F$, $G' \circ J_{\mathcal{A}} = G$ and $\alpha' * J_{\mathcal{A}} = \alpha$; cf. Definition 72, Lemmas 73 and 76.

Suppose given $\beta \in {}_{\text{add}[\operatorname{Kar}\mathcal{A},\mathcal{B}]}(F', G')$ with $\beta * J_{\mathcal{A}} = \alpha$. Then we have $\beta = \alpha'$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "J_{\mathcal{A}}"] \arrow[drr, bend left=15, "G"] \arrow[drr, bend right=15, "F"'] \arrow[drr, phantom, "\overset{\alpha}{\Uparrow}"] & & \operatorname{Kar}\mathcal{A} \arrow[d, bend right=30, "F'"'] \arrow[d, bend left=30, "G'"] \arrow[d, phantom, "\overset{\alpha'}{\Rightarrow}"] \\
& & \mathcal{B}
\end{tikzcd}
$$

(2) *Suppose given* $U, V \in \operatorname{Ob} {}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ *with* $U \circ J_{\mathcal{A}} = V \circ J_{\mathcal{A}}$. *Then* $U \cong V$.

(3) *We have the equivalence of categories*

$$\begin{array}{ccc} {}_{\mathrm{add}}[\mathcal{A}, \mathcal{B}] & \xleftarrow{\Lambda_{\mathcal{A},\mathcal{B}}} & {}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}] \\ \Big((U \circ J_{\mathcal{A}}) \xrightarrow{\beta * J_{\mathcal{A}}} (V \circ J_{\mathcal{A}})\Big) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V)\,, \end{array}$$

that is surjective on objects.

If unambiguous, we often write $\Lambda := \Lambda_{\mathcal{A},\mathcal{B}}$.

Proof. *Ad* (1). This follows from Proposition 75.(1).

Ad (2). This follows from Proposition 75.(2).

Ad (3). Suppose given $F \in \operatorname{Ob} {}_{\mathrm{add}}[\mathcal{A}, \mathcal{B}]$. Because (1) holds, we have $F' \in {}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ and $\Lambda F' = F' \circ J_{\mathcal{A}} = F$. Therefore, Λ is surjective on objects.

Suppose given $F, G \in \operatorname{Ob} {}_{\mathrm{add}}[\mathcal{A}, \mathcal{B}]$. Since (1) holds, we have a bijection

$${}_{\mathrm{add}[\operatorname{Kar}\mathcal{A},\mathcal{B}]}(F', G') \to {}_{\mathrm{add}[\mathcal{A},\mathcal{B}]}(F, G), \beta \mapsto \Lambda\beta.$$

Suppose given $U, V \in \operatorname{Ob} {}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ with $\Lambda U = \Lambda V$. Since (2) holds, we have $U \cong V$.

Therefore, Λ is an equivalence by Lemma 6. $\square$

Theorem 78. *Suppose* $\mathcal{A}$ *and* $\mathcal{B}$ *to be additive categories.*

Recall that $\mathcal{B}$ is idempotent complete; cf. Definition 45. Recall that $\operatorname{Kar}\mathcal{A}$ is additive; cf. Proposition 41; and idempotent complete; cf. Lemma 54.

The following assertions (1, 2, 3) *hold.*

(1) *We have* $J_{\mathcal{A}} \in {}_{\mathrm{add}}[\mathcal{A}, \operatorname{Kar}\mathcal{A}]$*;* cf. Lemma 44.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{\mathrm{add}}[\mathcal{A}, \mathcal{B}]$. *We have* $F' \xrightarrow{\alpha'} G'$ *in* ${}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ *with* $F' \circ J_{\mathcal{A}} = F$, $G' \circ J_{\mathcal{A}} = G$ *and* $\alpha' * J_{\mathcal{A}} = \alpha$*;* cf. Definition 72, Lemmas 73 and 76.

Suppose given $\beta \in {}_{[\operatorname{Kar}\mathcal{A},\mathcal{B}]}(F', G')$ *with* $\beta * J_{\mathcal{A}} = \alpha$. *Then we have* $\beta = \alpha'$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "J_{\mathcal{A}}"] \arrow[dr, bend right=40, "F"'] \arrow[dr, bend left=10, "G"] \arrow[dr, phantom, "\alpha\Uparrow"] & \operatorname{Kar}\mathcal{A} \arrow[d, bend right=30, "F'"'] \arrow[d, bend left=30, "G'"] \arrow[d, phantom, "\overset{\alpha'}{\Rightarrow}"] \\
& \mathcal{B}
\end{tikzcd}$$

(2) *Suppose given* $U, V \in \operatorname{Ob} {}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ *with* $U \circ J_{\mathcal{A}} = V \circ J_{\mathcal{A}}$. *Then* $U \cong V$.

(3) *We have the equivalence of categories*

$$\begin{array}{ccc} {}_{\mathrm{add}}[\mathcal{A}, \mathcal{B}] & \xleftarrow{\Lambda_{\mathcal{A},\mathcal{B}}} & {}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}] \\ \Big((U \circ J_{\mathcal{A}}) \xrightarrow{\beta * J_{\mathcal{A}}} (V \circ J_{\mathcal{A}})\Big) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V)\,, \end{array}$$

that is surjective on objects.

If unambiguous, we often write $\Lambda := \Lambda_{\mathcal{A},\mathcal{B}}$.

Proof. Since additive categories are preadditive, this follows from Proposition 77. $\square$

Proposition 79. *Suppose given an idempotent complete additive category* $\mathcal{C}$. *Suppose given a full subcategory* $\mathcal{D}$ *of* $\mathcal{C}$. *Let* $\mathcal{D}'$ *be the full subcategory of* $\mathcal{C}$ *with*

$$\operatorname{Ob}\mathcal{D}' := \{A \in \operatorname{Ob}\mathcal{C} : \exists B \in \operatorname{Ob}\mathcal{C}\ \exists D \in \operatorname{Ob}\mathcal{D} : A \oplus B \cong D\}.$$

The following assertions (1, 2, 3) *hold.*

(1) *The category* $\mathcal{D}$ *is a full subcategory of* $\mathcal{D}'$.

(2) *The category* $\mathcal{D}'$ *is idempotent complete.*

(3) *Consider the additive functor*

$$\begin{array}{rcl} \mathcal{D} & \xrightarrow{F} & \mathcal{D}' \\ \left(A \xrightarrow{f} B\right) & \mapsto & \left(A \xrightarrow{f} B\right). \end{array}$$

Then

$$\begin{array}{rcl} \operatorname{Kar}\mathcal{D} & \xrightarrow{F'} & \mathcal{D}' \\ \left((A,e) \xrightarrow{e(\ \varphi\ )_f} (B,f)\right) & \mapsto & \left(\operatorname{Im} e \xrightarrow{\dot{e}\varphi\bar{f}} \operatorname{Im} f\right), \end{array}$$

is an additive equivalence; cf. Definition 72.

Proof. Ad (1). Since $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, it suffices to show that $\operatorname{Ob}\mathcal{D} \subseteq \operatorname{Ob}\mathcal{D}'$.

Suppose given $D \in \operatorname{Ob}\mathcal{D}$. Since $\mathcal{C}$ is additive, there exists a zero object $N \in \operatorname{Ob}\mathcal{C}$. We have $D \oplus N \cong D$. Thus, we have $D \in \operatorname{Ob}\mathcal{D}'$.

Ad (2). Suppose given an idempotent $A \xrightarrow{e} A$ in $\mathcal{D}'$.

Since $\mathcal{C}$ is idempotent complete, we have an image $(\operatorname{Im} e, \bar{e}, \dot{e})$ of e in $\mathcal{C}$. Since $\mathcal{D}'$ is a full subcategory of $\mathcal{C}$, it suffices to show that $\operatorname{Im} e \in \operatorname{Ob}\mathcal{D}'$.

By Lemma 53.(2), we have $\operatorname{Im} e \oplus \operatorname{Im}(1-e) \cong A$ in $\mathcal{C}$. Since $A \in \operatorname{Ob}\mathcal{D}'$, there exist $B \in \operatorname{Ob}\mathcal{C}$ and $D \in \operatorname{Ob}\mathcal{D}$ with $A \oplus B \cong D$. Thus, we have $\operatorname{Im} e \oplus (\operatorname{Im}(1-e) \oplus B) \cong D$. Therefore, we have $\operatorname{Im} e \in \operatorname{Ob}\mathcal{D}'$.

Ad (3). Since (1) holds, F is additive, full and faithful.

Since (2) holds, $\mathcal{D}'$ is idempotent complete. Thus, F' exists; cf. Definition 72.

By Lemma 76, F' is additive. By Lemma 74.(1, 2), F' is full and faithful.

Thus, it suffices to show that F' is dense.

Suppose given $A_1 \in \operatorname{Ob}\mathcal{D}'$.

There exist $A_2 \in \operatorname{Ob}\mathcal{C}$, $D \in \operatorname{Ob}\mathcal{D}$ and an isomorphism $A_1 \oplus A_2 \xrightarrow[\sim]{\varphi} D$ in $\mathcal{C}$. So, we have $A_1\,, A_2 \in \operatorname{Ob}\mathcal{D}'$.

Consider $e := \varphi^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \varphi \in {}_{\mathcal{D}}(D, D)$. We have

$$e^2 = \varphi^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \varphi\varphi^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \varphi = \varphi^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \varphi = \varphi^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \varphi = e.$$

Thus, e is an idempotent in $\mathcal{D}$. Since (1) holds, e is an idempotent in $\mathcal{D}'$. Since $\mathcal{D}'$ is idempotent complete; cf. (2); e has an image $(\operatorname{Im} e, \bar{e}, \dot{e})$ in $\mathcal{D}'$.

We have the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
D \arrow[rr, "e"] \arrow[dr, "\varphi^{-1}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & & D \arrow[dr, "\varphi^{-1}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \\
& A_1 \arrow[ur, "(1\ 0)\varphi"] \arrow[rr, "1_{A_1}"] & & A_1
\end{tikzcd}$$

Since $\mathcal{D}'$ is a full subcategory of $\mathcal{C}$, this diagram is in $\mathcal{D}'$. Thus, A_1 is an image of e in $\mathcal{D}'$. By Corollary 51, we have $A_1 \cong \operatorname{Im} e$ in $\mathcal{D}'$.

By Definition 72, we have

$$F'(D, e) = \operatorname{Im}(Fe) = \operatorname{Im} e \cong A_1\,.$$

Thus, F' is dense. $\square$

2.1.9 Karoubi envelope for preadditive categories over a commutative ring

For this §2.1.9, let R be a commutative ring.

Definition 80 (and Lemma)**.** Let $\mathcal{A}$ be a preadditive category. We have the ring morphism

$$\begin{array}{rcl} \operatorname{End} 1_{\mathcal{A}} & \xrightarrow{\psi_{\mathcal{A}}} & \operatorname{End} 1_{\operatorname{Kar}\mathcal{A}} \\ \alpha & \mapsto & \operatorname{Kar}\alpha. \end{array}$$

Proof. Since $\mathcal{A}$ is preadditive, so is $\operatorname{Kar}\mathcal{A}$; cf. Definition 38.

Suppose given $\alpha, \alpha' \in \operatorname{End} 1_{\mathcal{A}}$. For brevity, we write $\psi := \psi_{\mathcal{A}}\,$.

We have

$$(\alpha + \alpha')\psi = \operatorname{Kar}(\alpha + \alpha') \overset{\text{L63.(2)}}{=} \operatorname{Kar}\alpha + \operatorname{Kar}\alpha' = \alpha\psi + \alpha'\psi$$

and

$$(\alpha\alpha')\psi = \operatorname{Kar}(\alpha\alpha') \overset{\text{L61.(2)}}{=} (\operatorname{Kar}\alpha)(\operatorname{Kar}\alpha') = (\alpha\psi)(\alpha'\psi).$$

Furthermore, we have

$$(1_{\operatorname{End} 1_{\mathcal{A}}})\psi = (1_{1_{\mathcal{A}}})\psi = \operatorname{Kar} 1_{1_{\mathcal{A}}} \overset{\text{L61.(1)}}{=} 1_{\operatorname{Kar} 1_{\mathcal{A}}} \overset{\text{L59.(1)}}{=} 1_{1_{\operatorname{Kar}\mathcal{A}}} = 1_{\operatorname{End} 1_{\operatorname{Kar}\mathcal{A}}}\,.$$

Thus, ψ is a ring morphism. $\square$

Lemma 81. *The following assertions* (1, 2) *hold.*

(1) *Let* $(\mathcal{A}, \varphi_\mathcal{A})$ *be a preadditive category over* R. *Then* $(\operatorname{Kar}\mathcal{A}, \varphi_\mathcal{A}\psi_\mathcal{A})$ *is a preadditive category over* R*;* cf. Definition 80.

(2) *Let* $(\mathcal{A}, \varphi_\mathcal{A})$ *be an additive category over* R. *Then* $(\operatorname{Kar}\mathcal{A}, \varphi_\mathcal{A}\psi_\mathcal{A})$ *is an additive category over* R*;* cf. Definition 80.

Proof. Ad (1). By Definition 80, $\psi_\mathcal{A} : \operatorname{End} 1_\mathcal{A} \to \operatorname{End} 1_{\operatorname{Kar}\mathcal{A}}$ is a ring morphism. Thus, $\varphi_\mathcal{A}\psi_\mathcal{A} : R \to \operatorname{End} 1_{\operatorname{Kar}\mathcal{A}}$ is a ring morphism. Therefore, $(\operatorname{Kar}\mathcal{A}, \varphi_\mathcal{A}\psi_\mathcal{A})$ is a preadditive category over R; cf. Definition 23.(1).

Ad (2). By Proposition 41, $\operatorname{Kar}\mathcal{A}$ is additive. Thus, the assertions follows from (1) and Definition 23.(2). $\square$

Remark 82. *Suppose given a preadditive category* $(\mathcal{A}, \varphi_\mathcal{A})$ *over* R. *Suppose given* $(X, e) \xrightarrow{{}_e(\,\alpha\,)_f} (Y, f)$ *in* $\operatorname{Kar}\mathcal{A}$. *Suppose given* $r \in R$. *We have* $r \cdot {}_e(\,\alpha\,)_f = {}_e(\,r \cdot \alpha\,)_f$.

Proof. For brevity, we write $\varphi := \varphi_\mathcal{A}$ and $\psi := \psi_\mathcal{A}$. We have

$$\begin{aligned}
r \cdot {}_e(\,\alpha\,)_f &\overset{\text{D23.(1)}}{=} ((r\varphi)\psi)_{(X,e)} \cdot {}_e(\,\alpha\,)_f \\
&\overset{\text{D80}}{=} (\operatorname{Kar}(r\varphi))_{(X,e)} \cdot {}_e(\,\alpha\,)_f \\
&\overset{\text{D60}}{=} {}_e(\,e \cdot (r\varphi)_X \cdot e\,)_e \cdot {}_e(\,\alpha\,)_f \\
&= {}_e(\,(r\varphi)_X \cdot ee \cdot \alpha\,)_f \\
&= {}_e(\,(r\varphi)_X \cdot \alpha\,)_f \\
&\overset{\text{D23.(1)}}{=} {}_e(\,r \cdot \alpha\,)_f\,.
\end{aligned}$$

$\square$

Lemma 83. *Suppose given a preadditive category* $(\mathcal{A}, \varphi_\mathcal{A})$ *over* R. *The inclusion functor* $J_\mathcal{A} : \mathcal{A} \to \operatorname{Kar}\mathcal{A}$ *is* R*-linear.*

Proof. By Lemma 81.(1), $(\operatorname{Kar}\mathcal{A}, \varphi_\mathcal{A}\psi_\mathcal{A})$ is a preadditive category over R.

By Lemma 44, J is additive.

Suppose given $r \in R$. Suppose given $X \xrightarrow{\alpha} Y$ in $\mathcal{A}$. We have

$$J(r \cdot \alpha) \overset{\text{D42}}{=} {}_{1_X}(\,r \cdot \alpha\,)_{1_Y} \overset{\text{R82}}{=} r \cdot {}_{1_X}(\,\alpha\,)_{1_Y} \overset{\text{D42}}{=} r \cdot J\alpha.$$

Thus, J is R-linear; cf. Definition 23.(3). $\square$

Lemma 84. *Suppose given preadditive categories* $(\mathcal{A}, \varphi_\mathcal{A})$ *and* $(\mathcal{B}, \varphi_\mathcal{B})$ *over* R. *Suppose given* $F \in \operatorname{Ob}\ {}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$. *Then* $\operatorname{Kar} F \in \operatorname{Ob}\ {}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \operatorname{Kar}\mathcal{B}]$.

Proof. By Lemma 81.(1), $(\operatorname{Kar}\mathcal{A}, \varphi_\mathcal{A}\psi_\mathcal{A})$ and $(\operatorname{Kar}\mathcal{B}, \varphi_\mathcal{B}\psi_\mathcal{B})$ are preadditive categories over R.

By Lemma 63.(1), $\operatorname{Kar} F$ is additive.

Suppose given $r \in R$. Suppose given $(X, e) \xrightarrow{{}_e(\,\alpha\,)_f} (Y, f)$ in $\operatorname{Kar}\mathcal{A}$. We have

$$\begin{aligned}
(\operatorname{Kar}F)(r \cdot {}_e(\,\alpha\,)_f) &\overset{\text{R82}}{=} (\operatorname{Kar}F)\,{}_e(\,r \cdot \alpha\,)_f \\
&\overset{\text{D57}}{=} {}_{Fe}(\,F(r \cdot \alpha)\,)_{Ff} \\
&= {}_{Fe}(\,r \cdot F\alpha\,)_{Ff} \\
&\overset{\text{R82}}{=} r \cdot {}_{Fe}(\,F\alpha\,)_{Ff} \\
&\overset{\text{D57}}{=} r \cdot (\operatorname{Kar}F)\,{}_e(\,\alpha\,)_f\,.
\end{aligned}$$

Thus, $\operatorname{Kar}F$ is R-linear; cf. Definition 23.(3). □

Lemma 85. *Suppose given a preadditive category $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. Suppose $\mathcal{B}$ to be idempotent complete. The image functor $I_{\mathcal{B}} : \operatorname{Kar}\mathcal{B} \to \mathcal{B}$ is R-linear;* cf. Definition 68.

Proof. By Lemma 81.(1), $(\operatorname{Kar}\mathcal{B}, \varphi_{\mathcal{B}}\psi_{\mathcal{B}})$ is a preadditive category over R.

By Lemma 71, I is additive.

Suppose given $r \in R$. Suppose given $(X, e) \xrightarrow{{}_e(\,\alpha\,)_f} (Y, f)$ in $\operatorname{Kar}\mathcal{B}$. We have

$$\begin{aligned}
I(r \cdot {}_e(\,\alpha\,)_f) &\overset{\text{R82}}{=} I\,{}_e(\,r \cdot \alpha\,)_f \\
&\overset{\text{D68}}{=} \dot{e} \cdot (r \cdot \alpha) \cdot \bar{f} \\
&\overset{\text{D23.(1)}}{=} \dot{e} \cdot (r\varphi)_X \cdot \alpha \cdot \bar{f} \\
&= (r\varphi)_X \cdot \dot{e} \cdot \alpha \cdot \bar{f} \\
&\overset{\text{R82}}{=} r \cdot I\,{}_e(\,\alpha\,)_f\,.
\end{aligned}$$

Thus, I is R-linear; cf. Definition 23.(3). □

Lemma 86. *Suppose given preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. Suppose $\mathcal{B}$ to be idempotent complete. Suppose given $F \in \operatorname{Ob}\ {}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$.*

We have $F' \in \operatorname{Ob}\ {}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$; cf. Definition 72.

Proof. By Lemma 81.(1), $(\operatorname{Kar}\mathcal{A}, \varphi_{\mathcal{A}}\psi_{\mathcal{A}})$ and $(\operatorname{Kar}\mathcal{B}, \varphi_{\mathcal{B}}\psi_{\mathcal{B}})$ are preadditive categories over R.

Recall that $F' = I_{\mathcal{B}} \circ \operatorname{Kar}F$; cf. Definition 72.

Since F is R-linear, so is $\operatorname{Kar}F$; cf. Lemma 84. Furthermore, $I_{\mathcal{B}}$ is R-linear; cf. Lemma 85. Thus, $F' = I_{\mathcal{B}} \circ \operatorname{Kar}F$ is R-linear. □

Proposition 87. *Suppose given preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. Suppose $\mathcal{B}$ to be idempotent complete;* cf. Definition 45.

Recall that $(\operatorname{Kar}\mathcal{A}, \varphi_{\mathcal{A}}\psi_{\mathcal{A}})$ is a preadditive category over R; cf. Lemma 81.(1). Recall that $\operatorname{Kar}\mathcal{A}$ is idempotent complete; cf. Lemma 54.

The following assertions (1, 2, 3) *hold.*

(1) *We have* $J_{\mathcal{A}} \in \operatorname{Ob} {}_{R\text{-lin}}[\mathcal{A}, \operatorname{Kar}\mathcal{A}]$. *Suppose given* $F \xrightarrow{\alpha} G$ *in* ${}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$. *We have* $F' \xrightarrow{\alpha'} G'$ *in* ${}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ *with* $F' \circ J_{\mathcal{A}} = F$, $G' \circ J_{\mathcal{A}} = G$ *and* $\alpha' * J_{\mathcal{A}} = \alpha$*;* cf. Definition 72, Lemmas 73 and 86.

Suppose given $\beta \in {}_{{}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]}(F', G')$ *with* $\beta * J_{\mathcal{A}} = \alpha$. *Then we have* $\beta = \alpha'$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "J_{\mathcal{A}}"] \arrow[dr, bend right=30, "F"'] \arrow[dr, bend left=15, "G"] \arrow[dr, phantom, "\alpha \Uparrow"] & \operatorname{Kar}\mathcal{A} \arrow[d, bend right=30, "F'"'] \arrow[d, bend left=30, "G'"] \arrow[d, phantom, "\overset{\alpha'}{\Rightarrow}"] \\
& \mathcal{B}
\end{tikzcd}$$

(2) *Suppose given* $U, V \in \operatorname{Ob} {}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ *with* $U \circ J_{\mathcal{A}} = V \circ J_{\mathcal{A}}$. *Then* $U \cong V$.

(3) *We have the equivalence of categories*

$$\begin{array}{ccc} {}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}] & \xleftarrow{\Lambda_{\mathcal{A},\mathcal{B}}} & {}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}] \\ \Big((U \circ J_{\mathcal{A}}) \xrightarrow{\beta * J_{\mathcal{A}}} (V \circ J_{\mathcal{A}})\Big) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V)\,, \end{array}$$

that is surjective on objects.

If unambiguous, we often write $\Lambda := \Lambda_{\mathcal{A},\mathcal{B}}$.

Proof. *Ad* (1). This follows from Lemma 83 and Proposition 75.(1).

Ad (2). This follows from Proposition 75.(2).

Ad (3). Suppose given $F \in \operatorname{Ob} {}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$. Because (1) holds, we have $F' \in {}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ and $\Lambda F' = F' \circ J_{\mathcal{A}} = F$. Therefore, Λ is surjective on objects.

Suppose given $F, G \in \operatorname{Ob} {}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$. Since (1) holds, we have a bijection

$${}_{{}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]}(F', G') \to {}_{{}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]}(F, G),\, \beta \mapsto \Lambda\beta.$$

Suppose given $U, V \in \operatorname{Ob} {}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ with $\Lambda U = \Lambda V$. Since (2) holds, we have $U \cong V$.

Therefore, Λ is an equivalence by Lemma 6. $\square$

2.2 The additive envelope of preadditive categories

2.2.1 Definition and additivity

For this §2.2.1, let $\mathcal{A}$ be a preadditive category.

Definition 88 (and Lemma)**.** We shall write tuples of objects and tuples of morphisms of $\mathcal{A}$ in square brackets.

We shall define a category $\operatorname{Add}\mathcal{A}$ as follows.

Let
$$\operatorname{Ob}\operatorname{Add}\mathcal{A} := \{[A_1, \dots, A_m] : m \in \mathbf{Z}_{\geqslant 0}\,,\ A_i \in \operatorname{Ob}\mathcal{A} \text{ for } i \in [1, m]\}.$$

We often write $A_1 \boxplus \dots \boxplus A_m := \underset{i \in [1,m]}{\boxplus} A_i := [A_1, \dots, A_m] \in \operatorname{Ob}\operatorname{Add}\mathcal{A}$.

We write $N_{\operatorname{Add}\mathcal{A}} := [\,]$ for the empty tuple of objects; cf. Lemma 89 below.

For $A_{1,1} \boxplus \dots \boxplus A_{1,m}$ and $A_{2,1} \boxplus \dots \boxplus A_{2,n}$ in $\operatorname{Ob}\operatorname{Add}\mathcal{A}$, let

$$\begin{aligned} &{}_{\operatorname{Add}\mathcal{A}}(A_{1,1} \boxplus \dots \boxplus A_{1,m}\,, A_{2,1} \boxplus \dots \boxplus A_{2,n}) \\ &\quad := \big\{[f_{i,j}]_{i\in[1,m], j\in[1,n]} \colon f_{i,j} \in {}_{\mathcal{A}}(A_{1,i}\,, A_{2,j}) \text{ for } (i, j) \in [1, m] \times [1, n]\big\}\,. \end{aligned}$$

For $[f_{i,j}]_{i\in[1,m], j\in[1,n]} \in \operatorname{Mor}\operatorname{Add}\mathcal{A}$, we often write

$$f := \begin{bmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & & \vdots \\ f_{m,1} & \cdots & f_{m,n} \end{bmatrix} := [f_{i,j}]_{i,j} := [f_{i,j}]_{i\in[1,m], j\in[1,n]}\,.$$

Omitted entries are stipulated to be zero.

For $A_{1,1} \boxplus \dots \boxplus A_{1,m}$ and $A_{2,1} \boxplus \dots \boxplus A_{2,m}$ in $\operatorname{Ob}\operatorname{Add}\mathcal{A}$ and $f_i \in {}_{\mathcal{A}}(A_{1,i}\,, A_{2,i})$ for $i \in [1, m]$, we often write
$$\operatorname{diag}[f_i]_i := \operatorname{diag}[f_i]_{i\in[1,m]} := \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_m \end{bmatrix}.$$

For $A_1 \xrightarrow{\varphi} A_2$ in $\mathcal{A}$, we often write

$$[\varphi] := [\varphi]_{i\in[1,1], j\in[1,1]} \in {}_{\operatorname{Add}\mathcal{A}}([A_1], [A_2]).$$

For $A_{1,1} \boxplus \dots \boxplus A_{1,m}\,,\ A_{2,1} \boxplus \dots \boxplus A_{2,n}$ and $A_{3,1} \boxplus \dots \boxplus A_{3,p}$ in $\operatorname{Ob}\operatorname{Add}\mathcal{A}$,

$$f = [f_{i,j}]_{i,j} \in {}_{\operatorname{Add}\mathcal{A}}(A_{1,1} \boxplus \dots \boxplus A_{1,m}\,, A_{2,1} \boxplus \dots \boxplus A_{2,n})$$

and
$$g = [g_{j,k}]_{j,k} \in {}_{\operatorname{Add}\mathcal{A}}(A_{2,1} \boxplus \dots \boxplus A_{2,n}\,, A_{3,1} \boxplus \dots \boxplus A_{3,p}),$$

we define composition by

$$fg = f \cdot g = [f_{i,j}]_{i,j} \cdot [g_{j,k}]_{j,k} := \left[\sum_{j\in[1,n]} f_{i,j} g_{j,k}\right]_{i\in[1,m], k\in[1,p]}.$$

For $A_1 \boxplus \cdots \boxplus A_m \in \operatorname{Ob} \operatorname{Add} \mathcal{A}$, let

$$1_{A_1 \boxplus \cdots \boxplus A_m} := [\delta_{i,j}]_{i,j} = \operatorname{diag}[1_{A_i}]_{i \in [1,n]} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

For $A_{1,1} \boxplus \cdots \boxplus A_{1,m}$ and $A_{2,1} \boxplus \cdots \boxplus A_{2,n}$ in $\operatorname{Ob} \operatorname{Add} \mathcal{A}$, $[f_{i,j}]_{i,j}$ and $[g_{i,j}]_{i,j}$ in ${}_{\operatorname{Add} \mathcal{A}}(A_{1,1} \boxplus \cdots \boxplus A_{1,m} \, , A_{2,1} \boxplus \cdot \cdot$
we define

$$[f_{i,j}]_{i,j} + [g_{i,j}]_{i,j} := [f_{i,j} + g_{i,j}]_{i,j} \, .$$

We call $\operatorname{Add} \mathcal{A}$ the *additive envelope* of $\mathcal{A}$.

This defines a preadditive category $\operatorname{Add} \mathcal{A}$.

Proof. First, we show that $\operatorname{Add} \mathcal{A}$ is a category.

Consider

$$\mathop{\boxplus}_{i \in [1,m]} A_{1,i} \xrightarrow{f} \mathop{\boxplus}_{j \in [1,n]} A_{2,j} \xrightarrow{g} \mathop{\boxplus}_{k \in [1,p]} A_{3,k} \xrightarrow{h} \mathop{\boxplus}_{l \in [1,q]} A_{4,l}$$

in $\operatorname{Add} \mathcal{A}$.

We calculate

$$1_{A_{2,1} \boxplus \cdots \boxplus A_{2,n}} \cdot g = [\delta_{i,j}]_{i,j} \cdot [g_{j,k}]_{j,k} = \left[\sum_{j \in [1,m]} \delta_{i,j} g_{j,k} \right]_{i,k} = [g_{i,k}]_{i,k} = g.$$

Similarily, we have

$$f \cdot 1_{A_{2,1} \boxplus \cdots \boxplus A_{2,n}} = [f_{i,j}]_{i,j} \cdot [\delta_{j,k}]_{j,k} = \left[\sum_{j \in [1,n]} f_{i,j} \delta_{j,k} \right]_{i,k} = [f_{i,k}]_{i,k} = f.$$

Furthermore, we obtain

$$\begin{aligned}
f(gh) &= [f_{i,j}]_{i,j} \cdot ([g_{j,k}]_{j,k} \cdot [h_{k,l}]_{k,l}) \\
&= [f_{i,j}]_{i,j} \cdot \left[\sum_{k \in [1,p]} g_{j,k} h_{k,l} \right]_{j,l} \\
&= \left[\sum_{j \in [1,n]} \left(f_{i,j} \sum_{k \in [1,p]} g_{j,k} h_{k,l} \right) \right]_{i,l} \\
&= \left[\sum_{(j,k) \in [1,n] \times [1,p]} f_{i,j} (g_{j,k} h_{k,l}) \right]_{i,l} \\
&= \left[\sum_{(j,k) \in [1,n] \times [1,p]} (f_{i,j} g_{j,k}) h_{k,l} \right]_{i,l}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{k\in[1,p]} \left(\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right) h_{k,l}\right]_{i,l} \\
&= \left[\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right]_{i,k} \cdot [h_{k,l}]_{k,l} \\
&= ([f_{i,j}]_{i,j} \cdot [g_{j,k}]_{j,k}) \cdot [h_{k,l}]_{k,l} \\
&= (fg)h.
\end{aligned}
$$

Thus, $\operatorname{Add}\mathcal{A}$ is a category.

Now we show that $\operatorname{Add}\mathcal{A}$ is preadditive.

Suppose given $\boxplus_{i\in[1,m]} A_{1,i}$ and $\boxplus_{j\in[1,n]} A_{2,j}$ in $\operatorname{Ob}\operatorname{Add}\mathcal{A}$. Since $\mathcal{A}$ is preadditive, ${}_{\mathcal{A}}(A_{1,i}\,, A_{2,j})$ is an abelian group for $(i,j) \in [1,m] \times [1,n]$. By definition of the addition in $\operatorname{Add}\mathcal{A}$, we have

$$
{}_{\operatorname{Add}\mathcal{A}}(\boxplus_{i\in[1,m]} A_{1,i}\,, \boxplus_{j\in[1,n]} A_{2,j}) = \bigoplus_{(i,j)\in[1,m]\times[1,n]} {}_{\mathcal{A}}(A_{1,i}, A_{2,j})
$$

as abelian groups, so that ${}_{\operatorname{Add}\mathcal{A}}(\boxplus_{i\in[1,m]} A_{1,i}\,, \boxplus_{j\in[1,n]} A_{2,j})$ is an abelian group.

Suppose given

$$
\begin{tikzcd}
\boxplus_{i\in[1,m]} A_{1,i} \arrow[r, "f"] & \boxplus_{j\in[1,n]} A_{2,j} \arrow[r, shift left, "g"] \arrow[r, shift right, "\tilde{g}"'] & \boxplus_{k\in[1,p]} A_{3,k} \arrow[r, "h"] & \boxplus_{l\in[1,q]} A_{4,l}
\end{tikzcd}
$$

in $\operatorname{Add}\mathcal{A}$.

We calculate

$$
\begin{aligned}
f(g+\tilde{g})h &= [f_{i,j}]_{i,j} \cdot ([g_{j,k}]_{j,k} + [\tilde{g}_{j,k}]_{j,k}) \cdot [h_{k,l}]_{k,l} \\
&= [f_{i,j}]_{i,j} \cdot [g_{j,k} + \tilde{g}_{j,k}]_{j,k} \cdot [h_{k,l}]_{k,l} \\
&= \left[\sum_{(j,k)\in[1,n]\times[1,p]} f_{i,j}(g_{j,k} + \tilde{g}_{j,k})h_{k,l}\right]_{i,l} \\
&= \left[\sum_{(j,k)\in[1,n]\times[1,p]} (f_{i,j}g_{j,k}h_{k,l} + f_{i,j}\tilde{g}_{j,k}h_{k,l})\right]_{i,l} \\
&= \left[\sum_{(j,k)\in[1,n]\times[1,p]} f_{i,j}g_{j,k}h_{k,l}\right]_{i,l} + \left[\sum_{(j,k)\in[1,n]\times[1,p]} f_{i,j}\tilde{g}_{j,k}h_{k,l}\right]_{i,l} \\
&= [f_{i,j}]_{i,j} \cdot [g_{j,k}]_{j,k} \cdot [h_{k,l}]_{k,l} + [f_{i,j}]_{i,j} \cdot [\tilde{g}_{j,k}]_{j,k} \cdot [h_{k,l}]_{k,l} \\
&= fgh + f\tilde{g}h.
\end{aligned}
$$

Thus, $\operatorname{Add}\mathcal{A}$ is preadditive. $\square$

Lemma 89. *We have a zero object $N_{\mathrm{Add}\,\mathcal{A}}$ in $\mathrm{Add}\,\mathcal{A}$;* cf. Definition 88.

Proof. Suppose given $A_1 \boxplus \cdots \boxplus A_n \in \mathrm{Ob}\,\mathrm{Add}\,\mathcal{A}$. We have to show that

$$|\,{}_{\mathrm{Add}\,\mathcal{A}}(A_1 \boxplus \cdots \boxplus A_n, N_{\mathrm{Add}\,\mathcal{A}})| = 1 = |\,{}_{\mathrm{Add}\,\mathcal{A}}(N_{\mathrm{Add}\,\mathcal{A}}, A_1 \boxplus \cdots \boxplus A_n)|\,.$$

We have

$${}_{\mathrm{Add}\,\mathcal{A}}(A_1 \boxplus \cdots \boxplus A_n, N_{\mathrm{Add}\,\mathcal{A}}) = \{[\;]\}; \text{cf. Definition 88;}$$

and

$${}_{\mathrm{Add}\,\mathcal{A}}(N_{\mathrm{Add}\,\mathcal{A}}, A_1 \boxplus \cdots \boxplus A_n) = \{[\;]\}; \text{cf. Definition 88.}$$

$\square$

Remark 90. Let $\mathcal{F}$ and $\mathcal{G}$ be finite totally ordered sets. For $m := |\mathcal{F}|$, $n := |\mathcal{G}|$, we have unique monotone bijections $[1,m] \xrightarrow{\sigma} \mathcal{F}$, $[1,n] \xrightarrow{\tau} \mathcal{G}$. Given $A_{1,\zeta} \in \mathrm{Ob}\,\mathcal{A}$ for $\zeta \in \mathcal{F}$ and $A_{2,\eta} \in \mathrm{Ob}\,\mathcal{A}$ for $\eta \in \mathcal{G}$, we write $\underset{\zeta\in\mathcal{F}}{\boxplus} A_{1,\zeta} := \underset{i\in[1,m]}{\boxplus} A_{i\sigma}$ and $\underset{\eta\in\mathcal{G}}{\boxplus} A_{2,\eta} := \underset{j\in[1,n]}{\boxplus} A_{j\tau}$.

Given $f_{\zeta,\eta} \in {}_{\mathcal{A}}(A_{1,\zeta}, A_{2,\eta})$ for $(\zeta,\eta) \in \mathcal{F} \times \mathcal{G}$, we write $[f_{\zeta,\eta}]_{\zeta,\eta} := [f_{i\sigma,j\tau}]_{i,j}$.

Lemma 91. *Suppose given $m, n \in \mathbf{Z}_{\geqslant 0}$. Define*

$$\begin{aligned} I_1 &:= \{(1,i) : \;\; i \in [1,m]\}\,, \\ I_2 &:= \{(2,j) : \;\; j \in [1,n]\} \text{ and} \\ I &:= I_1 \cup I_2\,, \end{aligned}$$

ordered lexicographically.

Suppose given $A_{1,1} \boxplus \cdots \boxplus A_{1,m}$ and $A_{2,1} \boxplus \cdots \boxplus A_{2,n}$ in $\mathrm{Ob}\,\mathrm{Add}\,\mathcal{A}$. Then

$$\begin{tikzcd}
A_{1,1} \boxplus \cdots \boxplus A_{1,m} \arrow[d, shift left, "{\begin{bmatrix} 1 & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & 1 & 0 & \dots & 0 \end{bmatrix} = [\delta_{\theta,\zeta}]_{\theta\in I_1,\zeta\in I}}"] \\
A_{1,1} \boxplus \cdots \boxplus A_{1,m} \boxplus A_{2,1} \boxplus \cdots \boxplus A_{2,n} \arrow[u, shift left, "{[\delta_{\zeta,\theta}]_{\zeta\in I,\theta\in I_1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}}"] \arrow[d, shift right, "{[\delta_{\zeta,\theta}]_{\zeta\in I,\theta\in I_2} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}"'] \\
A_{2,1} \boxplus \cdots \boxplus A_{2,n} \arrow[u, shift right, "{\begin{bmatrix} 0 & \dots & 0 & 1 & & \\ \vdots & & \vdots & & \ddots & \\ 0 & \dots & 0 & & & 1 \end{bmatrix} = [\delta_{\theta,\zeta}]_{\theta\in I_2,\zeta\in I}}"']
\end{tikzcd}$$

is a direct sum of $A_{1,1} \boxplus \cdots \boxplus A_{1,m}$ and $A_{2,1} \boxplus \cdots \boxplus A_{2,n}$ in $\mathrm{Add}\,\mathcal{A}$.

Proof. We calculate

$$[\delta_{\theta,\zeta}]_{\theta\in I_1,\zeta\in I} \cdot [\delta_{\zeta,\eta}]_{\zeta\in I,\eta\in I_1} = \left[\sum_{\zeta\in I} \delta_{\theta,\zeta}\delta_{\zeta,\eta}\right]_{\theta\in I_1,\eta\in I_1} = [\delta_{\theta,\eta}]_{\theta\in I_1,\eta\in I_1} = 1_{A_{1,1}\boxplus\cdots\boxplus A_{1,m}}\,.$$

Similarily, we obtain

$$[\delta_{\theta,\zeta}]_{\theta\in I_2,\zeta\in I}\cdot[\delta_{\zeta,\eta}]_{\zeta\in I,\eta\in I_2} = \left[\sum_{\zeta\in I}\delta_{\theta,\zeta}\delta_{\zeta,\eta}\right]_{\theta\in I_2,\eta\in I_2} = [\delta_{\theta,\eta}]_{\theta\in I_2,\eta\in I_2} = 1_{A_{2,1}\boxplus\cdots\boxplus A_{2,n}} .$$

Furthermore, we have

$$\begin{aligned}
& [\delta_{\zeta,\eta}]_{\zeta\in I,\eta\in I_1}\cdot[\delta_{\eta,\xi}]_{\eta\in I_1,\xi\in I} + [\delta_{\zeta,\eta}]_{\zeta\in I,\eta\in I_2}\cdot[\delta_{\eta,\xi}]_{\eta\in I_2,\xi\in I} \\
= \;& \left[\sum_{\eta\in I_1}\delta_{\zeta,\eta}\delta_{\eta,\xi}\right]_{\zeta\in I,\xi\in I} + \left[\sum_{\eta\in I_2}\delta_{\zeta,\eta}\delta_{\eta,\xi}\right]_{\zeta\in I,\xi\in I} \\
= \;& \left[\sum_{\eta\in I}\delta_{\zeta,\eta}\delta_{\eta,\xi}\right]_{\zeta\in I,\xi\in I} \\
= \;& [\delta_{\zeta,\xi}]_{\zeta\in I,\xi\in I} \\
= \;& 1_{A_{1,1}\boxplus\cdots\boxplus A_{1,m}\boxplus A_{2,1}\boxplus\cdots\boxplus A_{2,n}} .
\end{aligned}$$

□

Proposition 92. *The additive envelope* $\operatorname{Add}\mathcal{A}$ *of* $\mathcal{A}$ *is additive.*

Proof. By Definition 88, $\operatorname{Add}\mathcal{A}$ is preadditive. By Lemma 89, $\operatorname{Add}\mathcal{A}$ has a zero object. By Lemma 91, all objects $A, B \in \operatorname{Ob}\operatorname{Add}\mathcal{A}$ have a direct sum in $\operatorname{Add}\mathcal{A}$. Thus, $\operatorname{Add}\mathcal{A}$ is additive. □

Remark 93. *Suppose given* $A_1, \ldots, A_m \in \operatorname{Ob}\mathcal{A}$. *Let*

$$[A_i] \underset{\pi_i^{(A_j)_{j\in[1,m]}} := [\delta_{k,i}]_{k\in[1,m],l\in[1,1]} = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}}{\overset{\iota_i^{(A_j)_{j\in[1,m]}} := [\delta_{i,l}]_{k\in[1,1],l\in[1,m]} = [0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]}{\rightleftarrows}} A_1 \boxplus \cdots \boxplus A_m$$

for $i \in [1,m]$. *If unambiguous, we often write*

$$\iota_i := \iota_i^{(A_j)_{j\in[1,m]}} \quad \textit{and} \quad \pi_i := \pi_i^{(A_j)_{j\in[1,m]}}$$

for $i \in [1,m]$.

Then $(A_1 \boxplus \cdots \boxplus A_m\, ,\, (\pi_i)_{i\in[1,m]}\, ,\, (\iota_i)_{i\in[1,m]})$ *is a direct sum of* $[A_1], \ldots, [A_m]$ *in* $\operatorname{Add}\mathcal{A}$.

Proof. We have

$$\begin{aligned}
\sum_{i\in[1,m]} \pi_i\cdot\iota_i &= \sum_{i\in[1,m]} [\delta_{k,i}]_{k\in[1,m],l\in[1,1]}\cdot[\delta_{i,j}]_{l\in[1,1],j\in[1,m]} \\
&= \sum_{i\in[1,m]} [\delta_{k,i}\cdot\delta_{i,j}]_{k\in[1,m],j\in[1,m]} \\
&= \left[\sum_{i\in[1,m]} \delta_{k,i}\delta_{i,j}\right]_{k\in[1,m],j\in[1,m]} \\
&= [\delta_{k,j}]_{k\in[1,m],j\in[1,m]} \\
&= 1_{A_1\boxplus\cdots\boxplus A_m} .
\end{aligned}$$

Suppose given $i, j \in [1, m]$. We have

$$\iota_i \cdot \pi_j = [\delta_{i,l}]_{k\in[1,1],l\in[1,m]} \cdot [\delta_{l,j}]_{l\in[1,m],n\in[1,1]} = \left[\sum_{l\in[1,m]} \delta_{i,l}\delta_{l,j} \right] = [\delta_{i,j}].$$

Thus, we have

$$\iota_i \cdot \pi_i = [1_{A_i}] = 1_{[A_i]}$$

and

$$\iota_i \cdot \pi_j = [0_{A_i,A_j}] = 0_{[A_i],[A_j]}$$

for $j \in [1, m]\backslash\{i\}$. □

Remark 94. Suppose given $m \in \mathbf{Z}_{\geqslant 0}$. Suppose given $A_i \in \operatorname{Ob}\mathcal{A}$ for $i \in [1, m]$.

The following assertions (1, 2, 3) hold.

(1) Suppose given $\sigma \in \mathrm{S}_m$. Then $\underset{i\in[1,m]}{\boxplus} A_i \underset{[\delta_{k\sigma,l}]_{k,l}}{\overset{[\delta_{i,j\sigma}]_{i,j}}{\rightleftarrows}} \underset{j\in[1,m]}{\boxplus} A_{j\sigma}$ are mutually inverse isomorphisms in $\operatorname{Add}\mathcal{A}$.

(2) Suppose A_m to be a zero object in $\mathcal{A}$. Then $\underset{i\in[1,m]}{\boxplus} A_i \underset{[\delta_{k,l}]_{k,l}}{\overset{[\delta_{i,j}]_{i,j}}{\rightleftarrows}} \underset{j\in[1,m-1]}{\boxplus} A_j$ are mutually inverse isomorphisms in $\operatorname{Add}\mathcal{A}$.

(3) Suppose $A_1, \ldots, A_m$ to have a direct sum $(C, (\pi_i)_{i\in[1,m]}, (\iota_i)_{i\in[1,m]})$ in $\mathcal{A}$.

Then $[C] \underset{\left[\begin{smallmatrix}\iota_1\\ \vdots \\ \iota_m\end{smallmatrix}\right]}{\overset{[\pi_1 \;\ldots\; \pi_m]}{\rightleftarrows}} \underset{i\in[1,m]}{\boxplus} A_i$ are mutually inverse isomorphisms in $\operatorname{Add}\mathcal{A}$.

Remark 95. *Suppose given a ring R. Suppose $\mathcal{B}$ to be a full preadditive subcategory of* R-free *with $R \in \operatorname{Ob}\mathcal{B}$. Then*

$$\begin{array}{ccc} R\text{-free} & \xrightarrow{F} & \operatorname{Add}\mathcal{B} \\ \left(R^m \xrightarrow{(f_{i,j})_{i,j}} R^n\right) & \mapsto & \left(\underset{i\in[1,m]}{\boxplus} R \xrightarrow{[f_{i,j}]_{i,j}} \underset{j\in[1,n]}{\boxplus} R\right) \end{array}$$

is an equivalence.

Proof. For $x \in R$ we also write x for the map $R \to R, y \mapsto yx$.

First we show that F is a functor.

Suppose given $R^m \xrightarrow{(f_{i,j})_{i,j}} R^n \xrightarrow{(g_{j,k})_{j,k}} R^p$ in R-free.

We have

$$F(1_{R^m}) = F\operatorname{diag}(1)_i = \operatorname{diag}[1]_i = 1_{\boxplus_{i\in[1,m]} R} = 1_{F(R^m)}\,.$$

Furthermore, we have

$$\begin{aligned} F((f_{i,j})_{i,j} \cdot (g_{j,k})_{j,k}) &= F\left(\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right)_{j,k} \\ &= \left[\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right]_{j,k} \\ &= [f_{i,j}]_{i,j} \cdot [g_{j,k}]_{j,k} \\ &= F(f_{i,j})_{i,j} \cdot F(g_{j,k})_{j,k}\,. \end{aligned}$$

Thus, F is a functor.

We show that F is full.

Suppose given $R^m, R^n \in \operatorname{Ob} R\text{-free}$ and $f = [f_{i,j}]_{i,j} \in {}_{\operatorname{Add}\mathcal{B}}(F(R^m), F(R^n))$. By Definition 88, we have $f_{i,j} \in {}_{\mathcal{B}}(R,R) = R$ for $(i,j) \in [1,m] \times [1,n]$. Thus, we have $f = [f_{i,j}]_{i,j} = F(f_{i,j})_{i,j}$. Therefore, F is full.

We show that F is faithful.

Suppose give R^m and R^n in $\operatorname{Ob} R$-free, $(f_{i,j})_{i,j}$ and $(g_{i,j})_{i,j}$ in ${}_{R\text{-free}}(R^m, R^n)$ with

$$[f_{i,j}]_{i,j} = F(f_{i,j})_{i,j} = F(g_{i,j})_{i,j} = [g_{i,j}]_{i,j}\,.$$

By Definition 88, we obtain $f_{i,j} = g_{i,j}$ for $(i,j) \in [1,m] \times [1,n]$. Thus, we have $(f_{i,j})_{i,j} = (g_{i,j})_{i,j}$. Therefore, F is faithful.

We show that F is dense.

Suppose given $B_1 \boxplus \cdots \boxplus B_m \in \operatorname{Ob}\operatorname{Add}\mathcal{B}$. Since $\mathcal{B}$ is a subcategory of R-free, there exist $k_i \in \mathbf{Z}_{\geqslant 0}$ with $B_i = R^{k_i}$ for $i \in [1,m]$. Define $n := \sum_{i\in[1,m]} k_i$. We have

$$B_1 \boxplus \cdots \boxplus B_m = R^{k_1} \boxplus \cdots \boxplus R^{k_m} \underset{\text{R94.(3)}}{\cong} [R^n] \underset{\text{R94.(3)}}{\cong} \bigboxplus_{i\in[1,n]} R.$$

Thus, we have $B_1 \boxplus \cdots \boxplus B_m \cong \bigboxplus_{i\in[1,n]} R = F(R^n)$. Therefore, F is dense. □

2.2.2 The inclusion functor

The inclusion functor will play the role of the universal functor of the additive envelope construction; cf. §2.2.5. In Proposition 99 we give a necessary and sufficient condition for a preadditive category $\mathcal{C}$ to be additive. This condition justifies the notion additive *envelope* for $\operatorname{Add}\mathcal{C}$.

For this §2.2.2, let $\mathcal{A}$ be a preadditive category.

Definition 96 (and Lemma)**.** We have the additive functor

$$\begin{aligned} \mathcal{A} &\xrightarrow{I_{\mathcal{A}}} \operatorname{Add}\mathcal{A} \\ (A \xrightarrow{\varphi} B) &\mapsto \Big([A] \xrightarrow{[\varphi]} [B]\Big). \end{aligned}$$

We call $I_{\mathcal{A}}$ the *inclusion functor* of $\mathcal{A}$ in $\operatorname{Add}\mathcal{A}$.

If unambiguous, we often write $I := I_{\mathcal{A}}$.

Cf. also Remark 94.(3).

Proof. Suppose given $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ in $\mathcal{A}$. We have

$$I(\varphi\psi) = [\varphi\psi] = [\varphi][\psi] = I\varphi \cdot I\psi.$$

Furthermore, we have

$$I(1_A) = [1_A] = 1_{IA} \;.$$

Thus, I is a functor.

Suppose given $A \overset{\varphi}{\underset{\psi}{\rightrightarrows}} B$ in $\mathcal{A}$. We have

$$I(\varphi + \psi) = [\varphi + \psi] = [\varphi] + [\psi] = I\varphi + I\psi \;.$$

Thus, I is additive. $\square$

Lemma 97. *The following assertions* (1, 2) *hold.*

(1) *The inclusion functor $I_{\mathcal{A}}$ is full.*

(2) *The inclusion functor $I_{\mathcal{A}}$ is faithful.*

Proof. Suppose given $A, B \in \operatorname{Ob}\mathcal{A}$.

Ad (1). Suppose given $f \in {}_{\operatorname{Add}\mathcal{A}}([A],[B])$. By Definition 88, there exists $\varphi \in {}_{\mathcal{A}}(A, B)$ with $f = [\varphi] = I\varphi$. Thus, I is full.

Ad (2). Suppose given $\varphi, \psi \in {}_{\mathcal{A}}(A, B)$ with $I\varphi = I\psi$. We have

$$[\varphi] = I\varphi = I\psi = [\psi].$$

Therefore, we obtain $\varphi = \psi$. Thus, I is faithful. $\square$

Lemma 98. *Suppose $\mathcal{A}$ to be additive. Then $I_{\mathcal{A}} : \mathcal{A} \to \operatorname{Add}\mathcal{A}$ is dense.*

Proof. Suppose given $A_1 \boxplus \cdots \boxplus A_n \in \operatorname{Ob}\operatorname{Add}\mathcal{A}$.

Since $\mathcal{A}$ is additive, we have a direct sum

$$\left(\bigoplus_{i\in[1,n]} A_i \,,\, (\pi_i)_{i\in[1,n]} \,,\, (\iota_i)_{i\in[1,n]} \right)$$

of $A_1 \,, \ldots, A_n$ in $\mathcal{A}$.

Since $I_{\mathcal{A}}$ is an additive functor, we have a direct sum

$$\left(I\left(\bigoplus_{i\in[1,n]} A_i \right) , (I\pi_i)_{i\in[1,n]} \,,\, (I\iota_i)_{i\in[1,n]} \right)$$

of $IA_1 = [A_1], \ldots, IA_1 = [A_n]$ in $\operatorname{Add}\mathcal{A}$.

By Remark 93, $A_1 \boxplus \cdots \boxplus A_n$ is also a direct sum of $[A_1], \ldots, [A_n]$ in $\operatorname{Add}\mathcal{A}$.

Therefore, $I(A_1 \oplus \cdots \oplus A_n) \cong A_1 \boxplus \cdots \boxplus A_n$. Thus, $I_{\mathcal{A}}$ is dense. $\square$

Proposition 99. *Suppose given a preadditive category* $\mathcal{C}$.

The following assertions (1, 2) *are equivalent.*

(1) *The preadditive category* $\mathcal{C}$ *is additive.*

(2) *The inclusion functor* $\mathcal{C} \xrightarrow{I_{\mathcal{C}}} \operatorname{Add}\mathcal{C}$ *is an equivalence.*

Proof. *Ad* (1)⇒(2). By Lemma 97, $I_{\mathcal{C}}$ is full and faithful. Since $\mathcal{C}$ is additive, $I_{\mathcal{C}}$ is dense; cf. Lemma 98. Thus, $I_{\mathcal{C}}$ is an equivalence.

Ad (2)⇒(1). By Proposition 92, $\operatorname{Add}\mathcal{C}$ is additive. Thus, $\mathcal{C} \simeq \operatorname{Add}\mathcal{C}$ is additive. $\square$

2.2.3 Functoriality

In this §2.2.3 we define the additive envelope construction Add for additive functors and transformations between them. Furthermore, we establish functoriality properties of Add, which could be expressed by saying that it is turned into a 2-functor.

For this §2.2.3, let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be preadditive categories.

Definition 100 (and Lemma)**.** Let $F \in \operatorname{Ob}{}_{\text{add}}[\mathcal{A}, \mathcal{B}]$. *We have the additive functor*

$$\begin{array}{ccc} \operatorname{Add}\mathcal{A} & \xrightarrow{\operatorname{Add}F} & \operatorname{Add}\mathcal{B} \\ \left(\underset{i\in[1,m]}{\boxplus} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \underset{j\in[1,n]}{\boxplus} A_{2,j} \right) & \mapsto & \left(\underset{i\in[1,m]}{\boxplus} FA_{1,i} \xrightarrow{[Ff_{i,j}]_{i,j}} \underset{j\in[1,n]}{\boxplus} FA_{2,j} \right). \end{array}$$

So, we have

$$(\operatorname{Add}F)\left(\underset{i\in[1,m]}{\boxplus} A_i \right) = \underset{i\in[1,m]}{\boxplus} FA_i$$

for $\underset{i\in[1,m]}{\boxplus} A_i \in \operatorname{Ob}\operatorname{Add}\mathcal{A}$.

Furthermore, we have

$$(\operatorname{Add}F)[f_{i,j}]_{i,j} = [Ff_{i,j}]_{i,j}$$

for $[f_{i,j}]_{i,j} \in \operatorname{Mor}\operatorname{Add}\mathcal{A}$.

Proof. Suppose given $\underset{i\in[1,m]}{\boxplus} A_{1,i} \xrightarrow{f} \underset{j\in[1,n]}{\boxplus} A_{2,j} \xrightarrow{g} \underset{k\in[1,p]}{\boxplus} A_{3,k}$ in $\operatorname{Add}\mathcal{A}$. We have

$$\begin{aligned}
(\operatorname{Add}F)(fg) &= (\operatorname{Add}F)\left(\left[\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right]_{i,k}\right) \\
&= \left[F\left(\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right)\right]_{i,k} \\
&= \left[\sum_{j\in[1,n]} F(f_{i,j}g_{j,k})\right]_{i,k} \\
&= \left[\sum_{j\in[1,n]} Ff_{i,j}\cdot Fg_{j,k}\right]_{i,k} \\
&= [Ff_{i,j}]_{i,j}\cdot[Fg_{j,k}]_{j,k} \\
&= (\operatorname{Add}F)f\cdot(\operatorname{Add}F)g.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
(\operatorname{Add}F)1_{A_{1,1}\boxplus\cdots\boxplus A_{1,m}} &= (\operatorname{Add}F)[\delta_{i,j}]_{i,j} \\
&= [F\delta_{i,j}]_{i,j} \\
&= [\delta_{i,j}]_{i,j} \\
&= 1_{FA_{1,1}\boxplus\cdots\boxplus FA_{1,m}} \\
&= 1_{(\operatorname{Add}F)(A_{1,1}\boxplus\cdots\boxplus A_{1,m})}\,.
\end{aligned}$$

Thus, $\operatorname{Add}F$ is a functor.

Consider $\underset{i\in[1,m]}{\boxplus} A_{1,i} \underset{g}{\overset{f}{\rightrightarrows}} \underset{j\in[1,n]}{\boxplus} A_{2,j}$ in $\operatorname{Add}\mathcal{A}$. We calculate

$$\begin{aligned}
(\operatorname{Add}F)(f+g) &= (\operatorname{Add}F)\left([f_{i,j}+g_{i,j}]_{i,j}\right) \\
&= [F(f_{i,j}+g_{i,j})]_{i,j} \\
&= [Ff_{i,j}+Fg_{i,j}]_{i,j} \\
&= [Ff_{i,j}]_{i,j}+[Fg_{i,j}]_{i,j} \\
&= (\operatorname{Add}F)f+(\operatorname{Add}F)g.
\end{aligned}$$

Thus, $\operatorname{Add}F$ is additive. $\square$

Lemma 101. *Let* $F\in\operatorname{Ob}{}_{\operatorname{add}}[\mathcal{A},\mathcal{B}]$. *The following assertions* (1, 2, 3) *hold.*

(1) *Suppose* F *to be full. Then* $\operatorname{Add}F$ *is full.*

(2) *Suppose* F *to be faithful. Then* $\operatorname{Add}F$ *is faithful.*

(3) *Suppose* F *to be dense. Then* $\operatorname{Add}F$ *is dense.*

Proof. Suppose given $\underset{i\in[1,m]}{\boxplus} A_{1,i}\ ,\ \underset{j\in[1,n]}{\boxplus} A_{2,j} \in \operatorname{Ob}\operatorname{Add}\mathcal{A}$.

Ad (1). Suppose given

$$\begin{aligned}
[\varphi_{i,j}]_{i,j} \quad &\in \quad {}_{\operatorname{Add}\mathcal{B}}\Big((\operatorname{Add}F)\Big(\underset{i\in[1,m]}{\boxplus} A_{1,i}\Big)\ ,\ (\operatorname{Add}F)\Big(\underset{j\in[1,n]}{\boxplus} A_{1,j}\Big)\Big) \\
&= \quad {}_{\operatorname{Add}\mathcal{B}}\Big(\underset{i\in[1,m]}{\boxplus} FA_{1,i}\ ,\ \underset{j\in[1,n]}{\boxplus} FA_{1,j}\Big).
\end{aligned}$$

Since F is full, there exists $f_{i,j} \in {}_{\mathcal{A}}(A_{1,i}, A_{2,j})$ with $\varphi_{i,j} = Ff_{i,j}$ for $(i,j) \in [1,m] \times [1,n]$. With $f = [f_{i,j}]_{i,j} \in {}_{\operatorname{Add}\mathcal{A}}(\underset{i\in[1,m]}{\boxplus} A_{1,i}\ ,\ \underset{j\in[1,n]}{\boxplus} A_{2,j})$, we obtain

$$(\operatorname{Add}F)f = (\operatorname{Add}F)[f_{i,j}]_{i,j} = [Ff_{i,j}]_{i,j} = [\varphi_{i,j}]_{i,j}\,.$$

Thus, $\operatorname{Add}F$ is full.

Ad (2). Suppose given $f, g \in {}_{\operatorname{Add}\mathcal{A}}\Big(\underset{i\in[1,m]}{\boxplus} A_{1,i}\ ,\ \underset{j\in[1,n]}{\boxplus} A_{2,j}\Big)$ with $(\operatorname{Add}F)f = (\operatorname{Add}F)g$. We have $Ff_{i,j} = Fg_{i,j}$ for $(i,j) \in [1,m] \times [1,n]$. Since F is faithful, we obtain $f_{i,j} = g_{i,j}$ for $(i,j) \in [1,m] \times [1,n]$. Therefore, we have $f = g$. Thus, $\operatorname{Add}F$ is faithful.

Ad (3). Suppose given $B_1 \boxplus \cdots \boxplus B_n \in \operatorname{Ob}\operatorname{Add}\mathcal{B}$. Since F is dense, there exists $A_i \in \operatorname{Ob}\mathcal{A}$ with $FA_i \cong B_i$ for $i \in [1,n]$. Let $FA_i \xrightarrow[\sim]{\varphi_i} B_i$ be an isomorphism for $i \in [1,n]$.

We have mutually inverse isomorphisms

$$\begin{tikzcd} FA_1 \boxplus \cdots \boxplus FA_n \arrow[r, shift left, "{\operatorname{diag}[\varphi_i]_i}"] & B_1 \boxplus \cdots \boxplus B_n \arrow[l, shift left, "{\operatorname{diag}[(\varphi_i)^{-1}]_i}"] \end{tikzcd}.$$

Therefore, we have

$$(\operatorname{Add}F)(A_1 \boxplus \cdots \boxplus A_n) = FA_1 \boxplus \cdots \boxplus FA_n \cong B_1 \boxplus \cdots \boxplus B_n\,.$$

Thus, $\operatorname{Add}F$ is dense.

□

Lemma 102. *Suppose given additive functors* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$.

The following assertions (1, 2) *hold.*

(1) *We have* $\operatorname{Add}1_{\mathcal{A}} = 1_{\operatorname{Add}\mathcal{A}}$.

(2) *We have* $\operatorname{Add}(G \circ F) = (\operatorname{Add}G) \circ (\operatorname{Add}F)$.

Proof. Suppose given $\boxplus_{i\in[1,m]} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \boxplus_{j\in[1,n]} A_{2,j}$ in $\operatorname{Add}\mathcal{A}$.

Ad (1). We have

$$(\mathrm{Add}\,1_{\mathcal{A}})\left(\bigboxplus_{i\in[1,m]} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} A_{2,j}\right) = \left(\bigboxplus_{i\in[1,m]} 1_{\mathcal{A}}A_{1,i} \xrightarrow{[1_{\mathcal{A}}f_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} 1_{\mathcal{A}}A_{2,j}\right)$$
$$= \left(\bigboxplus_{i\in[1,m]} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} A_{2,j}\right)$$
$$= 1_{\mathrm{Add}\,\mathcal{A}}\left(\bigboxplus_{i\in[1,m]} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} A_{2,j}\right).$$

Ad (2). We have

$$(\mathrm{Add}(G\circ F))\left(\bigboxplus_{i\in[1,m]} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} A_{2,j}\right)$$
$$= \left(\bigboxplus_{i\in[1,m]} (G\circ F)A_{1,i} \xrightarrow{[(G\circ F)f_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} (G\circ F)A_{2,j}\right)$$
$$= (\mathrm{Add}\,G)\left(\bigboxplus_{i\in[1,m]} FA_{1,i} \xrightarrow{[Ff_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} FA_{2,j}\right)$$
$$= ((\mathrm{Add}\,G)\circ(\mathrm{Add}\,F))\left(\bigboxplus_{i\in[1,m]} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \bigboxplus_{j\in[1,n]} A_{2,j}\right).$$

□

Definition 103 (and Lemma)**.** Suppose given $F \xrightarrow{\alpha} \tilde{F}$ in ${}_{\mathrm{add}}[\mathcal{A},\mathcal{B}]$. Let

$$(\mathrm{Add}\,\alpha)_{A_1\boxplus\cdots\boxplus A_n} := \mathrm{diag}[\alpha_{A_i}]_{i\in[1,n]}$$

for $A_1 \boxplus \cdots \boxplus A_n \in \mathrm{Ob}\,\mathrm{Add}\,\mathcal{A}$, being a morphism from

$$\bigboxplus_{i\in[1,n]} FA_i \overset{\mathrm{D100}}{=} (\mathrm{Add}\,F)\left(\bigboxplus_{i\in[1,n]} A_i\right) \quad \text{to} \quad \bigboxplus_{i\in[1,n]} \tilde{F}A_i \overset{\mathrm{D100}}{=} (\mathrm{Add}\,\tilde{F})\left(\bigboxplus_{i\in[1,n]} A_i\right).$$

Let $\mathrm{Add}\,\alpha := ((\mathrm{Add}\,\alpha)_{A_1\boxplus\cdots\boxplus A_n})_{A_1\boxplus\cdots\boxplus A_n \in \mathrm{Ob}\,\mathrm{Add}\,\mathcal{A}}$.

This defines a transformation $\mathrm{Add}\,\alpha : \mathrm{Add}\,F \Rightarrow \mathrm{Add}\,\tilde{F}$*.*

Proof. Suppose given $\bigboxplus_{i\in[1,m]} A_{1,i} \xrightarrow{f} \bigboxplus_{j\in[1,n]} A_{2,j}$ in $\mathrm{Add}\,\mathcal{A}$.

We have

$$\begin{aligned}(\mathrm{Add}\,\alpha)_{A_{1,1}\boxplus\cdots\boxplus A_{1,m}} \cdot (\mathrm{Add}\,\tilde{F})f &= \mathrm{diag}[\alpha_{A_{1,i}}]_{i\in[1,m]} \cdot [\tilde{F}f_{i,j}]_{i,j} \\ &= [\alpha_{A_{1,i}} \cdot \tilde{F}f_{i,j}]_{i,j} \\ &= [Ff_{i,j} \cdot \alpha_{A_{2,j}}]_{i,j} \\ &= [Ff_{i,j}]_{i,j} \cdot \mathrm{diag}[\alpha_{A_{2,j}}]_{j\in[1,n]} \\ &= (\mathrm{Add}\,F)f \cdot (\mathrm{Add}\,\alpha)_{A_{2,1}\boxplus\cdots\boxplus A_{2,n}} .\end{aligned}$$

Thus, $\mathrm{Add}\,\alpha$ is natural. □

Lemma 104. *Suppose given additive functors F, F_0, F_1, $F_2 : \mathcal{A} \to \mathcal{B}$.*
The following assertions (1, 2, 3) *hold.*

(1) *We have* $\operatorname{Add} 1_F = 1_{\operatorname{Add} F}$.

(2) *Suppose given transformations $\alpha, \tilde{\alpha} : F_0 \Rightarrow F_1$. We have*

$$\operatorname{Add}(\alpha + \tilde{\alpha}) = \operatorname{Add} \alpha + \operatorname{Add} \tilde{\alpha}.$$

(3) *Suppose given transformations $\alpha_0 : F_0 \Rightarrow F_1$ and $\alpha_1 : F_1 \Rightarrow F_2$. We have*

$$\operatorname{Add}(\alpha_0 \alpha_1) = (\operatorname{Add} \alpha_0)(\operatorname{Add} \alpha_1).$$

Proof. Suppose given $A_1 \boxplus \cdots \boxplus A_m \in \operatorname{Ob} \operatorname{Add} \mathcal{A}$.

Ad (1). We compute

$$\begin{array}{rcl} (\operatorname{Add} 1_F)_{A_1 \boxplus \cdots \boxplus A_m} & = & \operatorname{diag}[(1_F)_{A_i}]_{i \in [1.m]} \\ & = & \operatorname{diag}[1_{F A_i}]_{i \in [1,m]} \\ & = & 1_{F A_1 \boxplus \cdots \boxplus F A_m} \\ & = & 1_{(\operatorname{Add} F)(A_1 \boxplus \cdots \boxplus F A_m)} \\ & = & (1_{\operatorname{Add} F})_{A_1 \boxplus \cdots \boxplus A_m} . \end{array}$$

Ad (2). We have

$$\begin{aligned} (\operatorname{Add}(\alpha + \tilde{\alpha}))_{A_1 \boxplus \cdots \boxplus A_m} &= \operatorname{diag}[(\alpha + \tilde{\alpha})_{A_i}]_{i \in [1,m]} \\ &= \operatorname{diag}[\alpha_{A_i} + \tilde{\alpha}_{A_i}]_{i \in [1,m]} \\ &= \operatorname{diag}[\alpha_{A_i}]_{i \in [1,m]} + \operatorname{diag}[\tilde{\alpha}_{A_i}]_{i \in [1,m]} \\ &= (\operatorname{Add} \alpha)_{A_1 \boxplus \cdots \boxplus A_m} + (\operatorname{Add} \tilde{\alpha})_{A_1 \boxplus \cdots \boxplus A_m} . \end{aligned}$$

Ad (3). We have

$$\begin{aligned} (\operatorname{Add}(\alpha_0 \alpha_1))_{A_1 \boxplus \cdots \boxplus A_m} &= \operatorname{diag}[(\alpha_0 \alpha_1)_{A_i}]_{i \in [1,m]} \\ &= \operatorname{diag}[(\alpha_0)_{A_i} (\alpha_1)_{A_i}]_{i \in [1,m]} \\ &= \operatorname{diag}[(\alpha_0)_{A_i}]_{i \in [1,m]} \cdot \operatorname{diag}[(\alpha_1)_{A_i}]_{i \in [1,m]} \\ &= (\operatorname{Add} \alpha_0)_{A_1 \boxplus \cdots \boxplus A_m} \cdot (\operatorname{Add} \alpha_1)_{A_1 \boxplus \cdots \boxplus A_m} \\ &= ((\operatorname{Add} \alpha_0)(\operatorname{Add} \alpha_1))_{A_1 \boxplus \cdots \boxplus A_m} . \end{aligned}$$

□

Lemma 105. *Suppose given additive functors $F, \tilde{F} : \mathcal{A} \to \mathcal{B}$ and $G, \tilde{G} : \mathcal{B} \to \mathcal{C}$. Suppose given transformations $\alpha : F \Rightarrow \tilde{F}$ and $\beta : G \Rightarrow \tilde{G}$. Then we have $\operatorname{Add}(\beta * \alpha) = (\operatorname{Add} \beta) * (\operatorname{Add} \alpha)$.*

Proof. Suppose given $A_1 \boxplus \cdots \boxplus A_m \in \operatorname{Ob} \operatorname{Add} \mathcal{A}$.

We calculate

$$\begin{aligned}(\mathrm{Add}(\beta * \alpha))_{A_1 \boxplus \cdots \boxplus A_m} &= \mathrm{diag}[(\beta * \alpha)_{A_i}]_{i \in [1,m]} \\ &= \mathrm{diag}[G\alpha_{A_i} \cdot \beta_{\tilde{F} A_i}]_{i \in [1,m]} \\ &= \mathrm{diag}[G\alpha_{A_i}]_{i \in [1,m]} \cdot \mathrm{diag}[\beta_{\tilde{F} A_i}]_{i \in [1,m]} \\ &= (\mathrm{Add}\, G)(\mathrm{Add}\, \alpha)_{A_1 \boxplus \cdots \boxplus A_m} \cdot (\mathrm{Add}\, \beta)_{(\mathrm{Add}\, \tilde{F})(A_1 \boxplus \cdots \boxplus A_m)} \\ &= ((\mathrm{Add}\, \beta) * (\mathrm{Add}\, \alpha))_{A_1 \boxplus \cdots \boxplus A_m} \,.\end{aligned}$$

□

2.2.4 The realisation functor

In this §2.2.4 we introduce the realisation functor, mapping formal direct sums as constructed in §2.2.1 to direct sums, provided the latter exist. In Definition 110, the realisation functor is used for the construction of the induced functors and transformations.

For this §2.2.4, let $\mathcal{A}$ be an additive category.

Stipulation 106. We recall no. 16. Recall that we have chosen a zero object $0_{\mathcal{A}}$ in $\mathcal{A}$.

Suppose given $m \in \mathbf{Z}_{\geqslant 0}$. Suppose given $A_i \in \mathrm{Ob}\,\mathcal{A}$ for $i \in [1,m]$. Recall that we have chosen a direct sum

$$\Big(\bigoplus_{i \in [1,m]} A_i \,,\, (\pi_i^{(A_j)_{j \in [1,m]}})_{i \in [1,m]} \,,\, (\iota_i^{(A_j)_{j \in [1,m]}})_{i \in [1,m]}\Big)$$

of $A_1, \ldots, A_m$ in $\mathcal{A}$.

Recall that we have chosen

$$\Big(\bigoplus_{i \in [1,1]} A_i \,,\, (\pi_i)_{i \in [1,1]} \,,\, (\iota_i)_{i \in [1,1]}\Big) = \big(A_1 \,,\, (1_{A_1})_{i \in [1,1]} \,,\, (1_{A_1})_{i \in [1,1]}\big)$$

and

$$\Big(\bigoplus_{i \in [1,0]} A_i \,,\, (\pi_i)_{i \in [1,0]} \,,\, (\iota_i)_{i \in [1,0]}\Big) = (0_{\mathcal{A}} \,,\, (\,) , (\,))\,.$$

Definition 107 (and Lemma)**.** We have the additive functor

$$\begin{array}{rcl} \mathrm{Add}\,\mathcal{A} & \xrightarrow{R_{\mathcal{A}}} & \mathcal{A} \\ \Big(\underset{i \in [1,m]}{\boxplus} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \underset{j \in [1,n]}{\boxplus} A_{2,j}\Big) & \mapsto & \Big(\bigoplus_{i \in [1,m]} A_{1,i} \xrightarrow{(f_{i,j})_{i,j}} \bigoplus_{j \in [1,n]} A_{2,j}\Big). \end{array}$$

We call $R_{\mathcal{A}}$ the *realisation functor* from $\mathrm{Add}\,\mathcal{A}$ to $\mathcal{A}$.

If unambiguous, we often write $R := R_{\mathcal{A}}$.

Proof. Suppose given

$$\bigboxplus_{i\in[1,m]} A_{1,i} \xrightarrow{f} \bigboxplus_{j\in[1,n]} A_{2,j} \xrightarrow{g} \bigboxplus_{k\in[1,p]} A_{3,k}$$

in $\operatorname{Add}\mathcal{A}$. We have

$$\begin{aligned}
R(fg) &= R\left([f_{i,j}]_{i,j} \cdot [g_{j,k}]_{j,k}\right) \\
&= R\left(\left[\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right]_{i,k}\right) \\
&= \left(\sum_{j\in[1,n]} f_{i,j}g_{j,k}\right)_{i,k} \\
&= (f_{i,j})_{i,j} \cdot (g_{j,k})_{j,k} \\
&= R\left([f_{i,j}]_{i,j}\right) \cdot R\left([g_{j,k}]_{j,k}\right) \\
&= Rf \cdot Rg.
\end{aligned}$$

Furthermore, we obtain

$$R(1_{A_1\boxplus\cdots\boxplus A_m}) = R\big(\operatorname{diag}[1_{A_i}]_{i\in[1,m]}\big) = \operatorname{diag}(1_{A_i})_{i\in[1,m]} = 1_{A_1\oplus\cdots\oplus A_m} = 1_{R(A_1\boxplus\cdots\boxplus A_m)}\ .$$

Thus, R is a functor.

Suppose given $\bigboxplus_{i\in[1,m]} A_{1,i} \overset{f}{\underset{\tilde{f}}{\rightrightarrows}} \bigboxplus_{j\in[1,n]} A_{2,j}$ in $\operatorname{Add}\mathcal{A}$.

We have

$$R(f+\tilde{f}) = R\Big([f_{i,j}+\tilde{f}_{i,j}]_{i,j}\Big) = (f_{i,j}+\tilde{f}_{i,j})_{i,j} = (f_{i,j})_{i,j} + (\tilde{f}_{i,j})_{i,j} = Rf + R\tilde{f}.$$

Thus, R is additive. □

Lemma 108. *The following assertions* (1, 2, 3) *hold.*

(1) *The realisation functor $R_{\mathcal{A}}$ is full.*

(2) *The realisation functor $R_{\mathcal{A}}$ is faithful.*

(3) *The realisation functor $R_{\mathcal{A}}$ is surjective on objects. In particular, $R_{\mathcal{A}}$ is dense.*

Thus, $R_{\mathcal{A}}$ is an equivalence.

Proof. Suppose given $A_{1,1}\boxplus\cdots\boxplus A_{1,m}$ and $A_{2,1}\boxplus\cdots\boxplus A_{2,n}$ in $\operatorname{Ob}\operatorname{Add}\mathcal{A}$.
Ad (1). Suppose given $(f_{i,j})_{i,j} \in {}_{\mathcal{A}}(A_{1,1}\oplus\cdots\oplus A_{1,m}\ ,\ A_{2,1}\oplus\cdots\oplus A_{2,n})$. We have

$$R\left([f_{i,j}]_{i,j}\right) = (f_{i,j})_{i,j}\ .$$

Thus, R is full.

Ad (2). Suppose given $f, g \in {}_{\operatorname{Add}\mathcal{A}}(A_{1,1} \boxplus \cdots \boxplus A_{1,m}\, ,\ A_{2,1} \boxplus \cdots \boxplus A_{2,n})$ with $Rf = Rg$. We have $(f_{i,j})_{i,j} = Rf = Rg = (g_{i,j})_{i,j}$. We obtain $f_{i,j} = g_{i,j}$ for $(i,j) \in [1,m] \times [1,n]$. Therefore, we have $f = [f_{i,j}]_{i,j} = [g_{i,j}]_{i,j} = g$. Thus, R is faithful.

Ad (3). Suppose given $A \in \operatorname{Ob}\mathcal{A}$. We have $[A] \in \operatorname{Ob}\operatorname{Add}\mathcal{A}$; cf. Definition 88. We obtain $R[A] = A$; cf. Stipulation 106. Thus, R is surjective on objects. $\square$

Remark 109. *We have* $R_{\mathcal{A}} \circ I_{\mathcal{A}} = 1_{\mathcal{A}}$.

Proof. Suppose given $A \xrightarrow{\varphi} B$ in $\mathcal{A}$. We have

$$(R \circ I)(A \xrightarrow{\varphi} B) \overset{\text{D96}}{=} R\Big([A] \xrightarrow{[\varphi]} [B]\Big) \underset{\text{S106}}{\overset{\text{D107}}{=}} (A \xrightarrow{\varphi} B) = 1_{\mathcal{A}}(A \xrightarrow{\varphi} B).$$

$\square$

2.2.5 Universal property

For this §2.2.5, let $\mathcal{A}$ be a preadditive category. Let $\mathcal{B}$ be an additive category.

Definition 110 (and Lemma)**.** We have the functor

$$\begin{array}{rcl} {}_{\text{add}}[\mathcal{A},\mathcal{B}] & \to & {}_{\text{add}}[\operatorname{Add}\mathcal{A},\mathcal{B}] \\ (F \xrightarrow{\alpha} G) & \mapsto & \Big((R_{\mathcal{B}} \circ \operatorname{Add} F) \xrightarrow{R_{\mathcal{B}} * \operatorname{Add}\alpha} (R_{\mathcal{B}} \circ \operatorname{Add} G)\Big). \end{array}$$

For $F \xrightarrow{\alpha} G$ in ${}_{\text{add}}[\mathcal{A},\mathcal{B}]$, we write $F' := R_{\mathcal{B}} \circ \operatorname{Add} F$ and $\alpha' := R_{\mathcal{B}} * \operatorname{Add}\alpha$.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{\text{add}}[\mathcal{A},\mathcal{B}]$. *We have*

$$\begin{array}{ccc} \operatorname{Add}\mathcal{A} & \xrightarrow{F'} & \mathcal{B} \\ \left(\underset{i\in[1,m]}{\boxplus} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \underset{j\in[1,n]}{\boxplus} A_{2,j}\right) & \mapsto & \left(\bigoplus_{i\in[1,m]} FA_{1,i} \xrightarrow{(Ff_{i,j})_{i,j}} \bigoplus_{j\in[1,n]} FA_{2,j}\right). \end{array}$$

Suppose given $A_1 \boxplus \cdots \boxplus A_m \in \operatorname{Ob}\operatorname{Add}\mathcal{A}$. *We have*

$$\alpha'_{A_1 \boxplus \cdots \boxplus A_m} = \operatorname{diag}(\alpha_{A_i})_{i\in[1,m]} .$$

Proof. Since $R_{\mathcal{B}}$ and $\operatorname{Add} F$ are additive, cf. Definitions 107 and 100, $F' = R_{\mathcal{B}} \circ \operatorname{Add} F$ is additive.

Suppose given $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ in ${}_{\text{add}}[\mathcal{A},\mathcal{B}]$.

We have

$$\begin{aligned} \alpha'_{A_1 \boxplus \cdots \boxplus A_m} &= (R_{\mathcal{B}} * \operatorname{Add}\alpha)_{A_1 \boxplus \cdots \boxplus A_m} \\ &= R_{\mathcal{B}}(\operatorname{Add}\alpha)_{A_1 \boxplus \cdots \boxplus A_m} \\ &= R_{\mathcal{B}}(\operatorname{diag}[\alpha_{A_i}]_i) \\ &= \operatorname{diag}(\alpha_{A_i})_i . \end{aligned}$$

Thus, we have

$$(1_F)'_{A_1 \boxplus \cdots \boxplus A_m} = \operatorname{diag}((1_F)_{A_i})_i = \operatorname{diag}(1_{FA_i})_i = 1_{FA_1 \oplus \cdots \oplus FA_m} = 1_{F'(A_1 \boxplus \cdots \boxplus A_m)} .$$

Furthermore, we have

$$\begin{array}{rcl} (\alpha\beta)'_{A_1 \boxplus \cdots \boxplus A_m} & = & \operatorname{diag}((\alpha\beta)_{A_i})_i \\ & = & \operatorname{diag}(\alpha_{A_i} \cdot \beta_{A_i})_i \\ & = & \operatorname{diag}(\alpha_{A_i})_i \cdot \operatorname{diag}(\beta_{A_i})_i \\ & = & \alpha'_{A_1 \boxplus \cdots \boxplus A_m} \cdot \beta'_{A_1 \boxplus \cdots \boxplus A_m} . \end{array}$$

Thus, we have a functor indeed.

Furthermore, we have

$$\begin{aligned} F'\left(\underset{i\in[1,m]}{\boxplus} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \underset{j\in[1,n]}{\boxplus} A_{2,j}\right) &= (R_{\mathcal{B}} \circ \operatorname{Add} F)\left(\underset{i\in[1,m]}{\boxplus} A_{1,i} \xrightarrow{[f_{i,j}]_{i,j}} \underset{j\in[1,n]}{\boxplus} A_{2,j}\right) \\ &= R_{\mathcal{B}}\left(\underset{i\in[1,m]}{\boxplus} FA_{1,i} \xrightarrow{[Ff_{i,j}]_{i,j}} \underset{j\in[1,n]}{\boxplus} FA_{2,j}\right) \\ &= \left(\bigoplus_{i\in[1,m]} FA_{1,i} \xrightarrow{(Ff_{i,j})_{i,j}} \bigoplus_{j\in[1,n]} FA_{2,j}\right). \end{aligned}$$

□

Lemma 111. *Suppose given $F \in \operatorname{Ob}\ {}_{\mathrm{add}}[\mathcal{A}, \mathcal{B}]$. The following assertions* (1, 2, 3) *hold.*

(1) *If F is full, so is F'.*

(2) *If F is faithful, so is F'.*

(3) *If F is dense, so is F'.*

Proof. Ad (1). Since F is full, so is $\operatorname{Add} F$; cf. Lemma 101.(1). By Lemma 108.(1), $R_{\mathcal{B}}$ is full. Thus, $F' = R_{\mathcal{B}} \circ \operatorname{Add} F$ is full.

Ad (2). Since F is faithful, so is $\operatorname{Add} F$; cf. Lemma 101.(2). By Lemma 108.(2), $R_{\mathcal{B}}$ is faithful. Thus, $F' = R_{\mathcal{B}} \circ \operatorname{Add} F$ is faithful.

Ad (3). Since F is dense, so is $\operatorname{Add} F$; cf. Lemma 101.(3). By Lemma 108.(3), $R_{\mathcal{B}}$ is dense. Thus, $F' = R_{\mathcal{B}} \circ \operatorname{Add} F$ is dense. □

Lemma 112. *Suppose given $F \xrightarrow{\alpha} G$ in ${}_{\mathrm{add}}[\mathcal{A}, \mathcal{B}]$. The following assertions* (1, 2) *hold.*

(1) *We have $F' \circ I_{\mathcal{A}} = F$.*

(2) *We have $\alpha' * I_{\mathcal{A}} = \alpha$.*

Cf. Definition 96.

Proof. *Ad* (1). Suppose given $X \xrightarrow{\varphi} Y$ in $\mathcal{A}$. We have

$$
\begin{array}{rcl}
(F' \circ I_{\mathcal{A}})(X \xrightarrow{\varphi} Y) & = & F'([X] \xrightarrow{[\varphi]} [Y]) \\
& \overset{\text{D110}}{=} & \left(\bigoplus\limits_{i \in [1,1]} FX \xrightarrow{(F\varphi)_{i,j}} \bigoplus\limits_{j \in [1,1]} FY \right) \\
& \overset{\text{S106}}{=} & \left(FX \xrightarrow{F\varphi} FY \right) \\
& = & F(X \xrightarrow{\varphi} Y).
\end{array}
$$

Ad (2). Suppose given $X \in \operatorname{Ob} \mathcal{A}$. We have

$$
\begin{array}{rcl}
(\alpha' * I_{\mathcal{A}})_X & = & \alpha'_{I_{\mathcal{A}} X} \\
& = & \alpha'_{[X]} \\
& \overset{\text{D110}}{=} & \operatorname{diag}(\alpha_X)_{i \in [1,1]} \\
& \overset{\text{S106}}{=} & \alpha_X \, .
\end{array}
$$

$\square$

Theorem 113. Recall that $\mathcal{A}$ is a preadditive category and that $\mathcal{B}$ is an additive category. Recall that $\operatorname{Add} \mathcal{A}$ is additive; cf. Proposition 92.

The following assertions (1, 2, 3) *hold.*

(1) *We have* $I_{\mathcal{A}} \in {}_{\text{add}}[\mathcal{A}, \operatorname{Add} \mathcal{A}]$*;* cf. Definition 96.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{\text{add}}[\mathcal{A}, \mathcal{B}]$. *We have* $F' \xrightarrow{\alpha'} G'$ *in* ${}_{\text{add}}[\operatorname{Add} \mathcal{A}, \mathcal{B}]$ *with* $F' \circ I_{\mathcal{A}} = F$, $G' \circ I_{\mathcal{A}} = G$ *and* $\alpha' * I_{\mathcal{A}} = \alpha$*;* cf. Definition 110 and Lemma 112.

Suppose given $\beta \in {}_{\text{add}[\operatorname{Add} \mathcal{A}, \mathcal{B}]}(F', G')$ *with* $\beta * I_{\mathcal{A}} = \alpha$. *Then we have* $\beta = \alpha'$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "I_{\mathcal{A}}"] \arrow[drr, bend right=30, "F"'] \arrow[drr, bend left=10, "G"] \arrow[drr, phantom, "\alpha \Uparrow"] & & \operatorname{Add} \mathcal{A} \arrow[d, bend right, "F'"'] \arrow[d, bend left, "G'"] \arrow[d, phantom, "\overset{\alpha'}{\Rightarrow}"] \\
& & \mathcal{B}
\end{tikzcd}
$$

(2) *Suppose given* $U, V \in \operatorname{Ob} {}_{\text{add}}[\operatorname{Add} \mathcal{A}, \mathcal{B}]$ *with* $U \circ I_{\mathcal{A}} = V \circ I_{\mathcal{A}}$. *Then* $U \cong V$.

(3) *We have the equivalence of categories*

$$
\begin{array}{ccc}
{}_{\text{add}}[\mathcal{A}, \mathcal{B}] & \xleftarrow{\Phi_{\mathcal{A},\mathcal{B}}} & {}_{\text{add}}[\operatorname{Add} \mathcal{A}, \mathcal{B}] \\
(U \circ I_{\mathcal{A}} \xrightarrow{\beta * I_{\mathcal{A}}} V \circ I_{\mathcal{A}}) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V),
\end{array}
$$

that is surjective on objects.

If unambiguous, we often write $\Phi := \Phi_{\mathcal{A},\mathcal{B}}$.

Proof. *Ad* (1). Suppose given $X_1 \boxplus \cdots \boxplus X_m \in \operatorname{Ob} \operatorname{Add} \mathcal{A}$.

We show that $\alpha'_{X_1 \boxplus \cdots \boxplus X_m} = \beta_{X_1 \boxplus \cdots \boxplus X_m}$.

Let $\beta_{i,j} := (\beta_{X_1 \boxplus \cdots \boxplus X_m})_{i,j}$ and $\alpha'_{i,j} := (\alpha'_{X_1 \boxplus \cdots \boxplus X_m})_{i,j}$ for $i, j \in [1, m]$.

Suppose given $e, f \in [1, m]$. We show that $\beta_{e,f} = \alpha'_{e,f}$.

We have

$$\alpha'_{e,f} = (\alpha'_{X_1 \boxplus \cdots \boxplus X_m})_{e,f} = (\operatorname{diag}(\alpha_{X_i})_{i \in [1,m]})_{e,f} = \alpha_{X_e} \delta_{e,f} \,.$$

We have $F'(X_1 \boxplus \cdots \boxplus X_m) = FX_1 \oplus \cdots \oplus FX_m$ and $G'(X_1 \boxplus \cdots \boxplus X_m) = GX_1 \oplus \cdots \oplus GX_m$.

Since β is natural, the following diagram commutes.

$$\begin{tikzcd}
F'[X_e] = FX_e \arrow[r, "\alpha'_{e,e} = \alpha_{X_e} = \beta_{[X_e]}"] \arrow[d, "(\delta_{e,j})_{i \in [1,1], j \in [1,m]}"', "= F'([\delta_{e,j}]_{i \in [1,1], j \in [1,m]})"] & G'[X_e] = GX_e \arrow[d, "G'([\delta_{e,j}]_{i \in [1,1], j \in [1,m]})"', "= (\delta_{e,j})_{i \in [1,1], j \in [1,m]}"] \\
FX_1 \oplus \cdots \oplus FX_m \arrow[r, "\beta_{X_1 \boxplus \cdots \boxplus X_m} = (\beta_{j,k})_{j \in [1,m], k \in [1,m]}"] & GX_1 \oplus \cdots \oplus X_m \arrow[d, "(\delta_{k,f})_{k \in [1,m], l \in [1,1]}"] \\
& G'[X_f] = GX_f
\end{tikzcd}$$

Therefore, we obtain

$$\begin{aligned}
\beta_{e,f} &= (\delta_{e,j})_{i \in [1,1], j \in [1,m]} \cdot (\beta_{j,k})_{j \in [1,m], k \in [1,m]} \cdot (\delta_{k,f})_{k \in [1,m], l \in [1,1]} \\
&= \alpha'_{e,e} \cdot (\delta_{e,j})_{i \in [1,1], j \in [1,m]} \cdot (\delta_{j,f})_{j \in [1,m], l \in [1,1]} \\
&= \alpha'_{e,e} \cdot \left(\sum_{j \in [1,m]} \delta_{e,j} \cdot \delta_{j,f} \right)_{i \in [1,1], l \in [1,1]} \\
&= \alpha'_{e,e} \cdot \delta_{e,f} \\
&= \alpha_{X_e} \cdot \delta_{e,f} \\
&= \alpha'_{e,f} \,.
\end{aligned}$$

Thus, $\beta = \alpha'$.

Ad (2). Suppose we have shown that $(W \circ I_{\mathcal{A}})' \cong W$ for $W \in \operatorname{Ob}\,_{\text{add}}[\operatorname{Add} \mathcal{A}, \mathcal{B}]$. Then we have

$$U \cong (U \circ I_{\mathcal{A}})' = (V \circ I_{\mathcal{A}})' \cong V.$$

Therefore, it suffices to show that $(W \circ I_{\mathcal{A}})' \cong W$ for $W \in \operatorname{Ob}\,_{\text{add}}[\operatorname{Add} \mathcal{A}, \mathcal{B}]$.

Suppose given $W \in \operatorname{Ob}\,_{\text{add}}[\operatorname{Add} \mathcal{A}, \mathcal{B}]$.

Suppose given $A_1 \boxplus \cdots \boxplus A_m \in \operatorname{Ob} \operatorname{Add} \mathcal{A}$.

By Definition 110, we have

$$(W \circ I_{\mathcal{A}})'(A_1 \boxplus \cdots \boxplus A_m) = ((W \circ I_{\mathcal{A}})A_1) \oplus \cdots \oplus ((W \circ I_{\mathcal{A}})A_m) = W[A_1] \oplus \cdots \oplus W[A_m].$$

By Remark 93 and additivity of W,

$$(W(A_1 \boxplus \cdots \boxplus A_m)\, ,\, (W\pi_i)_{i\in[1,m]}\, ,\, (W\iota_i)_{i\in[1,m]})$$

is a direct sum of $W[A_1], \ldots, W[A_m]$ in $\mathcal{B}$.

With the notation of Remark 93, we have mutually inverse isomorphisms

$$\begin{tikzcd} W(A_1 \boxplus \cdots \boxplus A_m) \arrow[r, shift left, "(W\pi_i)_i = (W\pi_1\ \ldots\ W\pi_m)"] & W[A_1] \oplus \cdots \oplus W[A_m]. \arrow[l, shift left, "(W\iota_i)_i = \begin{pmatrix} W\iota_1 \\ \vdots \\ W\iota_m \end{pmatrix}"] \end{tikzcd}$$

Define

$$(\delta_{A_1\boxplus\cdots\boxplus A_m})_{A_1\boxplus\cdots\boxplus A_m\in \operatorname{Ob}\operatorname{Add}\mathcal{A}} := \left(\left(W\iota_i^{(A_j)_{j\in[1,m]}}\right)_{i\in[1,m],k\in[1,1]}\right)_{A_1\boxplus\cdots\boxplus A_m\in\operatorname{Ob}\operatorname{Add}\mathcal{A}}\ ;$$

cf. Remark 93.

Suppose given $A_{1,1} \boxplus \cdots \boxplus A_{1,m} \xrightarrow{[f_{i,j}]_{i,j}} A_{2,1} \boxplus \cdots \boxplus A_{2,n}$ in $\operatorname{Add}\mathcal{A}$.

We have

$$\delta_{A_{1,1}\boxplus\cdots\boxplus A_{1,m}} = (W\iota_k^{(A_{1,x})_{x\in[1,m]}})_{k\in[1,m],l\in[1,1]} = \big(W[\delta_{k,j}]_{i\in[1,1],j\in[1,m]}\big)_{k\in[1,m],l\in[1,1]}$$

and

$$\delta_{A_{2,1}\boxplus\cdots\boxplus A_{2,n}} = (W\iota_k^{(A_{2,x})_{x\in[1,n]}})_{k\in[1,n],l\in[1,1]} = \big(W[\delta_{k,j}]_{i\in[1,1],j\in[1,n]}\big)_{k\in[1,n],l\in[1,1]}\ .$$

Furthermore, we have

$$(W\circ I_{\mathcal{A}})'[f_{i,j}]_{i,j} \overset{\text{D110}}{=} ((W\circ I_{\mathcal{A}})f_{i,j})_{i,j} = (W[f_{i,j}])_{i,j}\ .$$

We show that the following quadrangle is commutative.

$$\begin{tikzcd} W[A_{1,1}] \oplus \cdots \oplus W[A_{1,m}] \arrow[r, "(W[f_{i,j}])_{i\in[1,m],j\in[1,n]}"] \arrow[d, "(W\iota_k^{(A_{1,x})_{x\in[1,m]}})_{k\in[1,m],l\in[1,1]}"'] & W[A_{2,1}] \oplus \cdots \oplus W[A_{2,n}] \arrow[d, "(W\iota_k^{(A_{2,y})_{y\in[1,n]}})_{k\in[1,n],l\in[1,1]}"] \\ W(A_{1,1} \boxplus \cdots \boxplus A_{1,m}) \arrow[r, "W\left([f_{i,j}]_{i\in[1,m],j\in[1,n]}\right)"] & W(A_{2,1} \boxplus \cdots \boxplus A_{2,n}) \end{tikzcd}$$

We have

$$\begin{aligned}
& (W\iota_k^{(A_{1,x})_{x\in[1,m]}})_{k\in[1,m],l\in[1,1]} \cdot W\left([f_{i,j}]_{i\in[1,m],j\in[1,n]}\right) \\
= \ & \left(W\iota_k^{(A_{1,x})_{x\in[1,m]}} \cdot W\left([f_{i,j}]_{i\in[1,m],j\in[1,n]}\right)\right)_{k\in[1,m],l\in[1,1]} \\
= \ & \left(W\left(\iota_k^{(A_{1,x})_{x\in[1,m]}} \cdot [f_{i,j}]_{i\in[1,m],j\in[1,n]}\right)\right)_{k\in[1,m],l\in[1,1]} \\
= \ & \left(W\left([\delta_{k,i}]_{p\in[1,1],i\in[1,m]} \cdot [f_{i,j}]_{i\in[1,m],j\in[1,n]}\right)\right)_{k\in[1,m],l\in[1,1]} \\
= \ & \left(W\left(\left[\sum_{i\in[1,m]} \delta_{k,i}\cdot f_{i,j}\right]_{p\in[1,1],j\in[1,n]}\right)\right)_{k\in[1,m],l\in[1,1]} \\
= \ & \left(W[f_{k,j}]_{p\in[1,1],j\in[1,n]}\right)_{k\in[1,m],l\in[1,1]}
\end{aligned}$$

and

$$
\begin{aligned}
& (W[f_{i,j}])_{i\in[1,m],j\in[1,n]} \cdot (W\iota_j^{(A_{2,y})_{y\in[1,n]}})_{j\in[1,n],l\in[1,1]} \\
= & \left(W\left(\sum_{j\in[1,n]} [f_{i,j}]\cdot \iota_j^{(A_{2,y})_{y\in[1,n]}}\right)\right)_{i\in[1,m],l\in[1,1]} \\
= & \left(W\left(\sum_{j\in[1,n]} [f_{i,j}]\cdot [\delta_{j,k}]_{p\in[1,1],k\in[1,n]}\right)\right)_{i\in[1,m],l\in[1,1]} \\
= & \left(W\left(\sum_{j\in[1,n]} [f_{i,j}\cdot \delta_{j,k}]_{p\in[1,1],k\in[1,n]}\right)\right)_{i\in[1,m],l\in[1,1]} \\
= & \left(W\left[\sum_{j\in[1,n]} f_{i,j}\cdot \delta_{j,k}\right]_{p\in[1,1],k\in[1,n]}\right)_{i\in[1,m],l\in[1,1]} \\
= & (W[f_{i,k}]_{p\in[1,1],k\in[1,n]})_{i\in[1,m],l\in[1,1]}\,.
\end{aligned}
$$

Thus, δ is natural. Therefore, $(W\circ I_{\mathcal{A}})' \overset{\delta}{\underset{\sim}{\Longrightarrow}} W$ is an isotransformation.

In consequence, we have $W \cong (W\circ I_{\mathcal{A}})'$.

Ad (3). Suppose given $F \in \operatorname{Ob}\, {}_{\text{add}}[\mathcal{A},\mathcal{B}]$. Because (1) holds, we have $F' \in {}_{\text{add}}[\operatorname{Add}\mathcal{A},\mathcal{B}]$ with $\Phi F' = F'\circ I_{\mathcal{A}} = F$. Therefore, Φ is surjective on objects.

Suppose given $F, G \in \operatorname{Ob}\, {}_{\text{add}}[\mathcal{A},\mathcal{B}]$. Since (1) holds, we have a bijection

$$
{}_{\text{add}}[\operatorname{Add}\mathcal{A},\mathcal{B}](F',G') \to {}_{\text{add}}[\mathcal{A},\mathcal{B}](F,G),\, \beta \mapsto \Phi\beta.
$$

Suppose given $U, V \in \operatorname{Ob}\, {}_{\text{add}}[\operatorname{Add}\mathcal{A},\mathcal{B}]$ with $\Phi U = \Phi V$. Since (2) holds, we have $U \cong V$.

Therefore, Φ is an equivalence by Lemma 6.

□

2.2.6 Additive envelope for preadditive categories over a commutative ring

For this §2.2.6, let R be a commutative ring.

Definition 114 (and Lemma)**.** Suppose given a preadditive category $\mathcal{A}$. We have the ring morphism

$$
\begin{array}{rcl}
\operatorname{End} 1_{\mathcal{A}} & \xrightarrow{\psi_{\mathcal{A}}} & \operatorname{End} 1_{\operatorname{Add}\mathcal{A}} \\
\alpha & \mapsto & \operatorname{Add}\alpha.
\end{array}
$$

Proof. Suppose given $\alpha, \tilde{\alpha} \in \operatorname{End} 1_{\mathcal{A}}$. For brevity, we write $\psi := \psi_{\mathcal{A}}$.

We have

$$(\alpha + \tilde{\alpha})\psi = \operatorname{Add}(\alpha + \tilde{\alpha}) \overset{\text{L104.(2)}}{=} \operatorname{Add}\alpha + \operatorname{Add}\tilde{\alpha} = \alpha\psi + \tilde{\alpha}\psi$$

and

$$(\alpha\tilde{\alpha})\psi = \operatorname{Add}(\alpha\tilde{\alpha}) \overset{\text{L104.(3)}}{=} (\operatorname{Add}\alpha)(\operatorname{Add}\tilde{\alpha}) = (\alpha\psi)(\tilde{\alpha}\psi).$$

Furthermore, we have

$$(1_{\operatorname{End} 1_{\mathcal{A}}})\psi = (1_{1_{\mathcal{A}}})\psi = \operatorname{Add} 1_{1_{\mathcal{A}}} \overset{\text{L104.(1)}}{=} 1_{\operatorname{Add} 1_{\mathcal{A}}} \overset{\text{L102.(1)}}{=} 1_{1_{\operatorname{Add}\mathcal{A}}} = 1_{\operatorname{End} 1_{\operatorname{Add}\mathcal{A}}}.$$

Thus, ψ is a ring morphism. □

Lemma 115. *Suppose given a preadditive category* $(\mathcal{A}, \varphi_{\mathcal{A}})$ *over* R*. Then* $(\operatorname{Add}\mathcal{A}, \varphi_{\mathcal{A}}\psi_{\mathcal{A}})$ *is an additive category over* R*; cf. Definition 114.*

Proof. By Proposition 92, $\operatorname{Add}\mathcal{A}$ is additive. By Definition 114, we have a ring morphism $\varphi_{\mathcal{A}}\psi_{\mathcal{A}} : R \to \operatorname{End} 1_{\operatorname{Add}\mathcal{A}}$.

Thus, $(\operatorname{Add}\mathcal{A}, \varphi_{\mathcal{A}}\psi_{\mathcal{A}})$ is an additive category over R. □

Remark 116. *Suppose given a preadditive category* $(\mathcal{A}, \varphi_{\mathcal{A}})$ *over* R*. Suppose given*

$$A_{1,1} \boxplus \cdots \boxplus A_{1,m} \xrightarrow{[f_{i,j}]_{i,j}} A_{2,1} \boxplus \cdots \boxplus A_{2,n}$$

in $\operatorname{Add}\mathcal{A}$*. Suppose given* $r \in R$*. We have* $r \cdot [f_{i,j}]_{i,j} = [r \cdot f_{i,j}]_{i,j}$*.*

Proof. For brevity, we write $\varphi := \varphi_{\mathcal{A}}$ and $\psi := \psi_{\mathcal{A}}$. We have

$$\begin{aligned} r \cdot [f_{i,j}]_{i,j} &= (r(\varphi\psi))_{A_{1,1}\boxplus\cdots\boxplus A_{1,m}} \cdot [f_{i,j}]_{i,j} \\ &= (\operatorname{Add}(r\varphi))_{A_{1,1}\boxplus\cdots\boxplus A_{1,m}} \cdot [f_{i,j}]_{i,j} \\ &= \operatorname{diag}[(r\varphi)_{A_{1,i}}]_i \cdot [f_{i,j}]_{i,j} \\ &= [(r\varphi)_{A_{1,i}} \cdot f_{i,j}]_{i,j} \\ &= [r \cdot f_{i,j}]_{i,j}\,. \end{aligned}$$

□

Lemma 117. *Suppose given a preadditive category $(\mathcal{A}, \varphi_{\mathcal{A}})$ over R. The inclusion functor $I_{\mathcal{A}} : \mathcal{A} \to \operatorname{Add}\mathcal{A}$ is R-linear.*

Proof. By Definition 96, I is additive.

Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}$. Suppose given $r \in R$. We have

$$I(r \cdot f) = [r \cdot f] \overset{\text{R116}}{=} r \cdot [f] = r \cdot If.$$

Thus, I is R-linear; cf. Definition 23.(3). □

Lemma 118. *Suppose given preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. Suppose given $F \in \operatorname{Ob}\ _{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$. Then $\operatorname{Add} F \in \operatorname{Ob}\ _{R\text{-lin}}[\operatorname{Add}\mathcal{A}, \operatorname{Add}\mathcal{B}]$.*

Proof. By Definition 100, $\operatorname{Add} F$ is additive.

Suppose given $A_{1,1} \boxplus \cdots \boxplus A_{1,m} \xrightarrow{[f_{i,j}]_{i,j}} A_{2,1} \boxplus \cdots \boxplus A_{2,n}$ in $\operatorname{Add}\mathcal{A}$. Suppose given $r \in R$. We have

$$\begin{array}{rcl} (\operatorname{Add} F)(r \cdot [f_{i,j}]_{i,j}) & \overset{\text{R116}}{=} & (\operatorname{Add} F)[r \cdot f_{i,j}]_{i,j} \\ & = & [F(r \cdot f_{i,j})]_{i,j} \\ & = & [r \cdot F f_{i,j}]_{i,j} \\ & \overset{\text{R116}}{=} & r \cdot [F f_{i,j}]_{i,j} \\ & = & r \cdot (\operatorname{Add} F)[f_{i,j}]_{i,j} \,. \end{array}$$

Thus, $\operatorname{Add} F$ is R-linear; cf. Definition 23.(3). □

Lemma 119. *Suppose given an additive category $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. The realisation functor $R_{\mathcal{B}} : \operatorname{Add}\mathcal{B} \to \mathcal{B}$ is R-linear.*

Proof. By Definition 107, $R_{\mathcal{B}}$ is additive.

Suppose given $B_{1,1} \boxplus \cdots \boxplus B_{1,m} \xrightarrow{[f_{i,j}]_{i,j}} B_{2,1} \boxplus \cdots \boxplus B_{2,n}$ in $\operatorname{Add}\mathcal{B}$. Suppose given $r \in R$. We have

$$R(r \cdot [f_{i,j}]_{i,j}) \overset{\text{R116}}{=} R[r \cdot f_{i,j}]_{i,j} = (r \cdot f_{i,j})_{i,j} \overset{\text{R24}}{=} r \cdot (f_{i,j})_{i,j} = r \cdot R[f_{i,j}]_{i,j} \,.$$

Thus, R is R-linear; cf. Definition 23.(3). □

Lemma 120. *Suppose given a preadditive category $(\mathcal{A}, \varphi_{\mathcal{A}})$ over R. Suppose given an additive category $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. Suppose given an R-linear functor $F : \mathcal{A} \to \mathcal{B}$. Then $F' : \operatorname{Add}\mathcal{A} \to \mathcal{B}$ is R-linear.*

Proof. By Lemma 115, $(\operatorname{Add}\mathcal{A}, \varphi_{\mathcal{A}}\psi_{\mathcal{A}})$ and $(\operatorname{Add}\mathcal{B}, \varphi_{\mathcal{B}}\psi_{\mathcal{B}})$ are additive categories over R.

Recall that $F' = R_{\mathcal{B}} \circ \operatorname{Add} F$; cf. Definition 110.

Since F is R-linear, so is $\operatorname{Add} F$; cf. Lemma 118. Furthermore, $R_{\mathcal{B}}$ is R-linear; cf. Lemma 119. Thus, $F' = R_{\mathcal{B}} \circ \operatorname{Add} F$ is R-linear. □

Proposition 121. Suppose given a preadditive category $(\mathcal{A}, \varphi_{\mathcal{A}})$ over R. Suppose given an additive category $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R. Recall that $(\operatorname{Add}\mathcal{A}, \varphi_{\mathcal{A}}\psi_{\mathcal{A}})$ is an additive category over R; cf. Lemma 115.

The following assertions (1, 2, 3) *hold.*

(1) *We have* $I_{\mathcal{A}} \in \operatorname{Ob}\, {}_{R\text{-lin}}[\mathcal{A}, \operatorname{Add}\mathcal{A}]$*;* cf. Lemma 117.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$*. We have* $F' \xrightarrow{\alpha'} G'$ *in* ${}_{R\text{-lin}}[\operatorname{Add}\mathcal{A}, \mathcal{B}]$ *with* $F' \circ I_{\mathcal{A}} = F$*,* $G' \circ I_{\mathcal{A}} = G$ *and* $\alpha' * I_{\mathcal{A}} = \alpha$*;* cf. Definition 110 and Lemmas 112 and 120.

Suppose given $\beta \in {}_{{}_{R\text{-lin}}[\operatorname{Add}\mathcal{A},\mathcal{B}]}(F', G')$ *with* $\beta * I_{\mathcal{A}} = \alpha$*. Then we have* $\beta = \alpha'$*.*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "I_{\mathcal{A}}"] \arrow[dr, bend left=15, "G"] \arrow[dr, bend right=15, "F"'] \arrow[dr, phantom, "\alpha\Uparrow"] & \operatorname{Add}\mathcal{A} \arrow[d, bend right, "F'"'] \arrow[d, bend left, "G'"] \arrow[d, phantom, "\overset{\alpha'}{\Longrightarrow}"] \\
& \mathcal{B}
\end{tikzcd}$$

(2) *Suppose given* $U, V \in \operatorname{Ob}\, {}_{R\text{-lin}}[\operatorname{Add}\mathcal{A}, \mathcal{B}]$ *with* $U \circ I_{\mathcal{A}} = V \circ I_{\mathcal{A}}$*. Then* $U \cong V$*.*

(3) *We have the equivalence of categories*

$$\begin{array}{ccc}
{}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}] & \xleftarrow{\Phi_{\mathcal{A},\mathcal{B}}} & {}_{R\text{-lin}}[\operatorname{Add}\mathcal{A}, \mathcal{B}] \\
(U \circ I_{\mathcal{A}} \xrightarrow{\beta * I_{\mathcal{A}}} V \circ I_{\mathcal{A}}) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V)\,,
\end{array}$$

that is surjective on objects.

If unambiguous, we often write $\Phi := \Phi_{\mathcal{A},\mathcal{B}}$*.*

Proof. *Ad* (1). This follows from Theorem 113.(1).

Ad (2). This follows from Theorem 113.(2).

Ad (3). Suppose given $F \in \operatorname{Ob}\, {}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$. Because (1) holds, we have $F' \in {}_{R\text{-lin}}[\operatorname{Add}\mathcal{A}, \mathcal{B}]$ and $\Phi F' = F' \circ J_{\mathcal{A}} = F$. Therefore, Φ is surjective on objects.

Suppose given $F, G \in \operatorname{Ob}\, {}_{R\text{-lin}}[\mathcal{A}, \mathcal{B}]$. Since (1) holds, we have the bijection

$${}_{{}_{R\text{-lin}}[\operatorname{Add}\mathcal{A},\mathcal{B}]}(F', G') \to {}_{{}_{R\text{-lin}}[\mathcal{A},\mathcal{B}]}(F, G),\, \beta \mapsto \Phi\beta.$$

Suppose given $U, V \in \operatorname{Ob}\, {}_{R\text{-lin}}[\operatorname{Kar}\mathcal{A}, \mathcal{B}]$ with $\Phi U = \Phi V$. Since (2) holds, we have $U \cong V$. Therefore, Φ is an equivalence by Lemma 6. $\square$

Chapter 3

The tensor product

3.1 The tensor product of preadditive categories over a commutative ring

For this §3.1, let R be a commutative ring.

We often write $\otimes$ instead of $\underset{R}{\otimes}$; cf. §1.3.

3.1.1 Definition

For this §3.1.1, let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be preadditive categories over R.

Definition 122 (and Lemma)**.** Let $A_i \in \operatorname{Ob}\mathcal{A}$ and $B_i \in \operatorname{Ob}\mathcal{B}$ for $i \in [1, 3]$.

There exists a unique R-bilinear map

$$\Big({}_{\mathcal{A}}(A_1, A_2) \underset{R}{\otimes} {}_{\mathcal{B}}(B_1, B_2) \Big) \times \Big({}_{\mathcal{A}}(A_2, A_3) \underset{R}{\otimes} {}_{\mathcal{B}}(B_2, B_3) \Big) \xrightarrow{\kappa_{A_1,A_2,A_3,B_1,B_2,B_3}} {}_{\mathcal{A}}(A_1, A_3) \underset{R}{\otimes} {}_{\mathcal{B}}(B_1, B_3)$$

with

$$(a' \otimes b', a'' \otimes b'')\kappa_{A_1,A_2,A_3,B_1,B_2,B_3} = (a'a'') \otimes (b'b'')$$

for $a' \in {}_{\mathcal{A}}(A_1, A_2)$, $a'' \in {}_{\mathcal{A}}(A_2, A_3)$, $b' \in {}_{\mathcal{B}}(B_1, B_2)$, $b'' \in {}_{\mathcal{B}}(B_2, B_3)$.

Proof. Define

$$\begin{array}{rcl} {}_{\mathcal{A}}(A_1, A_2) \times {}_{\mathcal{B}}(B_1, B_2) \times {}_{\mathcal{A}}(A_2, A_3) \times {}_{\mathcal{B}}(B_2, B_3) & \xrightarrow{\lambda} & {}_{\mathcal{A}}(A_1, A_3) \otimes {}_{\mathcal{B}}(B_1, B_3) \\ (a', b', a'', b'') & \mapsto & (a'a'') \otimes (b'b''). \end{array}$$

We show that λ is R-multilinear.

Pars pro toto, consider the first component. Suppose given $a'_1, a'_2 \in {}_{\mathcal{A}}(A_1, A_2)$. Suppose given $r_1, r_2 \in R$. Suppose given $b' \in {}_{\mathcal{B}}(B_1, B_2)$, $a'' \in {}_{\mathcal{A}}(A_2, A_3)$ and $b'' \in {}_{\mathcal{B}}(B_2, B_3)$.

We have

$$\begin{aligned}(r_1 a_1' + r_2 a_2', b', a'', b'')\lambda &= ((r_1 a_1' + r_2 a_2')a'') \otimes (b'b'') \\ &= (r_1 a_1' a'' + r_2 a_2' a'') \otimes (b'b'') \\ &= r_1((a_1' a'') \otimes (b'b'')) + r_2((a_2' a'') \otimes (b'b'')) \\ &= r_1((a_1', a'', b', b'')\lambda) + r_2((a_2', a'', b', b'')\lambda).\end{aligned}$$

Define $\mu_1 := \mu_{{}_{\mathcal{A}}(A_1,A_2),\, {}_{\mathcal{B}}(B_1,B_2),\, {}_{\mathcal{A}}(A_2,A_3),\, {}_{\mathcal{B}}(B_2,B_3)}$; cf. Definition 13. By Lemma 16, we obtain a unique R-linear map

$$\bar{\lambda} : {}_{\mathcal{A}}(A_1, A_2) \otimes {}_{\mathcal{B}}(B_1, B_2) \otimes {}_{\mathcal{A}}(A_2, A_3) \otimes {}_{\mathcal{B}}(B_2, B_3) \to {}_{\mathcal{A}}(A_1, A_3) \otimes {}_{\mathcal{B}}(B_1, B_3)$$

with $\mu_1 \bar{\lambda} = \lambda$.

Define $\mu_2 := \mu_{{}_{\mathcal{A}}(A_1,A_2)\otimes {}_{\mathcal{B}}(B_1,B_2),\, {}_{\mathcal{A}}(A_2,A_3)\otimes {}_{\mathcal{B}}(B_2,B_3)}$; cf. Definition 13.
By Lemma 19, there exists a unique R-linear isomorphism

$$({}_{\mathcal{A}}(A_1, A_2) \otimes {}_{\mathcal{B}}(B_1, B_2)) \otimes ({}_{\mathcal{A}}(A_2, A_3) \otimes {}_{\mathcal{B}}(B_2, B_3)) \xrightarrow{\xi} {}_{\mathcal{A}}(A_1, A_2) \otimes {}_{\mathcal{B}}(B_1, B_2) \otimes {}_{\mathcal{A}}(A_2, A_3) \otimes {}_{\mathcal{B}}(B_2, B_3)$$

with

$$((a' \otimes b') \otimes (a'' \otimes b''))\xi = a' \otimes b' \otimes a'' \otimes b'',$$

for $a' \in {}_{\mathcal{A}}(A_1, A_2)$, $b' \in {}_{\mathcal{B}}(B_1, B_2)$, $a'' \in {}_{\mathcal{A}}(A_2, A_3)$, $b'' \in {}_{\mathcal{B}}(B_2, B_3)$.

Define $\kappa := \kappa_{A_1,A_2,A_3,B_1,B_2,B_3} := \mu_2 \xi \bar{\lambda}$. We have

$$\begin{aligned}(a' \otimes b', a'' \otimes b'')\kappa &= (a' \otimes b', a'' \otimes b'')\mu_2 \xi \bar{\lambda} \\ &= ((a' \otimes b') \otimes (a'' \otimes b''))\xi \bar{\lambda} \\ &= (a' \otimes b' \otimes a'' \otimes b'')\bar{\lambda} \\ &= (a'a'') \otimes (b'b''),\end{aligned}$$

for $a' \in {}_{\mathcal{A}}(A_1, A_2)$, $b' \in {}_{\mathcal{B}}(B_1, B_2)$, $a'' \in {}_{\mathcal{A}}(A_2, A_3)$, $b'' \in {}_{\mathcal{B}}(B_2, B_3)$.

Since μ_2 is R-bilinear and $\xi, \bar{\lambda}$ are R-linear, κ is R-bilinear; cf. Lemma 12.

Suppose given an R-bilinear map

$$\kappa' : ({}_{\mathcal{A}}(A_1, A_2) \otimes {}_{\mathcal{B}}(B_1, B_2)) \times ({}_{\mathcal{A}}(A_2, A_3) \otimes {}_{\mathcal{B}}(B_2, B_3)) \to {}_{\mathcal{A}}(A_1, A_3) \otimes {}_{\mathcal{B}}(B_1, B_3)$$

with

$$(a' \otimes b', a'' \otimes b'')\kappa' = (a'a'') \otimes (b'b'') = (a' \otimes b', a'' \otimes b'')\kappa$$

for $a' \in {}_{\mathcal{A}}(A_1, A_2)$, $a'' \in {}_{\mathcal{A}}(A_2, A_3)$, $b' \in {}_{\mathcal{B}}(B_1, B_2)$, $b'' \in {}_{\mathcal{B}}(B_2, B_3)$.

Suppose given

$$\left(\sum_{i \in [1,m]} a_i' \otimes b_i' \,,\ \sum_{j \in [1,n]} a_j'' \otimes b_j'' \right) \in ({}_{\mathcal{A}}(A_1, A_2) \otimes {}_{\mathcal{B}}(B_1, B_2)) \times ({}_{\mathcal{A}}(A_2, A_3) \otimes {}_{\mathcal{B}}(B_2, B_3)) \ .$$

We have

$$\begin{aligned}\Bigg(\sum_{i\in[1,m]} a_i'\otimes b_i'\,,\ \sum_{j\in[1,n]} a_j''\otimes b_j''\Bigg)\kappa' &= \sum_{(i,j)\in[1,m]\times[1,n]} (a_i'\otimes b_i'\,,\ a_j''\otimes b_j'')\kappa' \\ &= \sum_{(i,j)\in[1,m]\times[1,n]} (a_i'\otimes b_i'\,,\ a_j''\otimes b_j'')\kappa \\ &= \Bigg(\sum_{i\in[1,m]} a_i'\otimes b_i'\,,\ \sum_{j\in[1,n]} a_j''\otimes b_j''\Bigg)\kappa\,.\end{aligned}$$

Thus, we have $\kappa' = \kappa$. □

Definition 123 (and Lemma)**.** We shall define a category $\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$ as follows.

Let

$$\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B} := \operatorname{Ob}\mathcal{A}\times\operatorname{Ob}\mathcal{B}.$$

We write $A\boxtimes B := (A,B)$ for $(A,B)\in\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$.

For $A_1\boxtimes B_1$ and $A_2\boxtimes B_2$ in $\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$, we define

$${}_{\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}}(A_1\boxtimes B_1\,, A_2\boxtimes B_2) := {}_{\mathcal{A}}(A_1\,, A_2)\underset{R}{\otimes}{}_{\mathcal{B}}(B_1\,, B_2).$$

Since this is an R-module, it is in particular an abelian group.

For $A_1\boxtimes B_1$, $A_2\boxtimes B_2$ and $A_3\boxtimes B_3$ in $\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$,

$$\zeta\in{}_{\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}}(A_1\boxtimes B_1, A_2\boxtimes B_2)\quad\text{and}\quad\eta\in{}_{\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}}(A_2\boxtimes B_2, A_3\boxtimes B_3),$$

we define composition by

$$\zeta\eta := (\zeta,\eta)\kappa_{A_1,A_2,A_3,B_1,B_2,B_3}\in{}_{\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}}(A_1\boxtimes B_1\,, A_3\boxtimes B_3);$$

cf. Definition 122. Recall that in particular

$$(a'\otimes b')(a''\otimes b'') = (a'a'')\otimes(b'b'')$$

for $a'\in{}_{\mathcal{A}}(A_1\,, A_2)$, $b'\in{}_{\mathcal{B}}(B_1\,, B_2)$, $a''\in{}_{\mathcal{A}}(A_2\,, A_3)$, $b''\in{}_{\mathcal{B}}(B_2\,, B_3)$.

For $A\boxtimes B\in\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$, we define $1_{A\boxtimes B} := 1_A\otimes 1_B$.

We call $\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$ the *tensor product over* R of $\mathcal{A}$ and $\mathcal{B}$.

If unambiguous, we often write $\mathcal{A}\boxtimes\mathcal{B} := \mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$.

This defines a preadditive category $\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}$*.*

Proof. First we show that $\mathcal{A}\boxtimes\mathcal{B}$ is a category.

Suppose given

$$A_1 \boxtimes B_1 \xrightarrow{\zeta = \sum\limits_{i \in [1,m']} a'_i \otimes b'_i} A_2 \boxtimes B_2 \xrightarrow{\eta = \sum\limits_{j \in [1,m'']} a''_j \otimes b''_j} A_3 \boxtimes B_3 \xrightarrow{\theta = \sum\limits_{k \in [1,m''']} a'''_k \otimes b'''_k} A_4 \boxtimes B_4$$

in $\mathcal{A} \boxtimes \mathcal{B}$.

For brevity, we write $\kappa_{i,j,k} := \kappa_{A_i,A_j,A_k,B_i,B_j,B_k}$; cf. Definition 122.

We have $1_{A_2 \boxtimes B_2} = 1_{A_2} \otimes 1_{B_2} \in {}_{\mathcal{A}}(A_2, A_2) \otimes {}_{\mathcal{B}}(B_2, B_2) = {}_{\mathcal{A}\boxtimes\mathcal{B}}(A_2 \boxtimes B_2, A_2 \boxtimes B_2)$.

Furthermore, we have

$$\zeta\eta = (\zeta, \eta)\kappa_{1,2,3} \in {}_{\mathcal{A}}(A_1, A_3) \otimes {}_{\mathcal{B}}(B_1, B_3) = {}_{\mathcal{A}\boxtimes\mathcal{B}}(A_1 \boxtimes B_1, A_3 \boxtimes B_3) .$$

We calculate

$$\begin{aligned}
\zeta \cdot 1_{A_2 \boxtimes B_2} &= \left(\sum_{i \in [1,m']} a'_i \otimes b'_i \, , 1_{A_2} \otimes 1_{B_2} \right) \kappa_{1,2,2} \\
&= \sum_{i \in [1,m']} (a'_i \otimes b'_i \, , 1_{A_2} \otimes 1_{B_2}) \, \kappa_{1,2,2} \\
&= \sum_{i \in [1,m']} (a'_i 1_{A_2}) \otimes (b'_i 1_{B_2}) \\
&= \sum_{i \in [1,m']} a'_i \otimes b'_i \\
&= \zeta
\end{aligned}$$

and

$$\begin{aligned}
1_{A_2 \boxtimes B_2} \cdot \eta &= \left(1_{A_2} \otimes 1_{B_2} \, , \sum_{j \in [1,m'']} a''_j \otimes b''_j \right) \kappa_{2,2,3} \\
&= \sum_{j \in [1,m'']} \big(1_{A_2} \otimes 1_{B_2} \, , a''_j \otimes b''_j \big) \, \kappa_{2,2,3} \\
&= \sum_{j \in [1,m'']} (1_{A_2} a''_j) \otimes (1_{B_2} b''_j) \\
&= \sum_{j \in [1,m'']} a''_j \otimes b''_j \\
&= \eta \, .
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
(\zeta\eta)\theta &= \big((\zeta,\eta)\kappa_{1,2,3}\,,\theta\big)\kappa_{1,3,4} \\
&= \left(\left(\sum_{i\in[1,m']} a'_i\otimes b'_i\,,\ \sum_{j\in[1,m'']} a''_j\otimes b''_j\right)\kappa_{1,2,3}\,,\theta\right)\kappa_{1,3,4} \\
&= \left(\sum_{(i,j)\in[1,m']\times[1,m'']} (a'_i\otimes b'_i\,,a''_j\otimes b''_j)\kappa_{1,2,3}\,,\theta\right)\kappa_{1,3,4} \\
&= \left(\sum_{(i,j)\in[1,m']\times[1,m'']} (a'_ia''_j)\otimes(b'_ib''_j)\,,\ \sum_{k\in[1,m''']} a'''_k\otimes b'''_k\right)\kappa_{1,3,4} \\
&= \sum_{(i,j,k)\in[1,m']\times[1,m'']\times[1,m''']} \big((a'_ia''_j)\otimes(b'_ib''_j)\,,a'''_k\otimes b'''_k\big)\,\kappa_{1,3,4}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{(i,j,k)\in[1,m']\times[1,m'']\times[1,m''']} ((a'_ia''_j)a'''_k)\otimes((b'_ib''_j)b'''_k) \\
&= \sum_{(i,j,k)\in[1,m']\times[1,m'']\times[1,m''']} (a'_i(a''_ja'''_k))\otimes(b'_i(b''_jb'''_k)) \\
&= \sum_{(i,j,k)\in[1,m']\times[1,m'']\times[1,m''']} \big(a'_i\otimes b'_i\,,(a''_ja'''_k)\otimes(b''_jb'''_k)\big)\,\kappa_{1,2,4} \\
&= \left(\sum_{i\in[1,m']} a'_i\otimes b'_i\,,\ \sum_{(j,k)\in[1,m'']\times[1,m''']} (a''_ja'''_k)\otimes(b''_jb'''_k)\right)\kappa_{1,2,4} \\
&= \left(\zeta,\ \sum_{(j,k)\in[1,m'']\times[1,m''']} (a''_j\otimes b''_j\,,a'''_k\otimes b'''_k)\kappa_{2,3,4}\right)\kappa_{1,2,4} \\
&= \left(\zeta,\ \left(\sum_{j\in[1,m'']} a''_j\otimes b''_j\,,\ \sum_{k\in[1,m''']} a'''_k\otimes b'''_k\right)\kappa_{2,3,4}\right)\kappa_{1,2,4} \\
&= (\zeta,(\eta,\theta)\kappa_{2,3,4})\kappa_{1,2,4} \\
&= \zeta(\eta\theta)\,.
\end{aligned}
$$

Now we show that $\mathcal{A}\boxtimes\mathcal{B}$ is preadditive.

Suppose given

$$
A_1\boxtimes B_1 \xrightarrow{\ \zeta\ } A_2\boxtimes B_2 \underset{\eta_2}{\overset{\eta_1}{\rightrightarrows}} A_3\boxtimes B_3 \xrightarrow{\ \theta\ } A_4\boxtimes B_4
$$

in $\mathcal{A}\boxtimes\mathcal{B}$.

We have

$$\begin{aligned}
\zeta(\eta_1+\eta_2)\theta &= ((\zeta,\eta_1+\eta_2)\kappa_{1,2,3}\,,\theta)\kappa_{1,3,4} \\
&= ((\zeta,\eta_1)\kappa_{1,2,3}+(\zeta,\eta_2)\kappa_{1,2,3},\theta)\kappa_{1,3,4} \\
&= ((\zeta,\eta_1)\kappa_{1,2,3}\,,\theta)\kappa_{1,3,4}+((\zeta,\eta_2)\kappa_{1,2,3}\,,\theta)\kappa_{1,3,4} \\
&= \zeta\eta_1\theta+\zeta\eta_2\theta\ .
\end{aligned}$$

□

Lemma 124. *We have the ring morphism*

$$\begin{array}{rcll}
R & \xrightarrow{\varphi\boxtimes\psi} & \operatorname{End} 1_{\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}} & \\
r & \mapsto & (r(\varphi\boxtimes\psi)_{A\boxtimes B})_{A\boxtimes B\in\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}} & := \quad ((r\varphi)_A\otimes 1_B)_{A\boxtimes B\in\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}} \\
 & & & = \quad (1_A\otimes(r\psi)_B)_{A\boxtimes B\in\operatorname{Ob}\mathcal{A}\underset{R}{\boxtimes}\mathcal{B}}\ .
\end{array}$$

Thus, $(\mathcal{A}\underset{R}{\boxtimes}\mathcal{B},\varphi\boxtimes\psi)$ *is a preadditive category over* R*.*

Proof. Suppose given $r\in R$. Suppose given $A\boxtimes B\in\operatorname{Ob}\mathcal{A}\boxtimes\mathcal{B}$. We have

$$\begin{aligned}
(r\varphi)_A\otimes 1_B &= ((r\varphi_A)1_A)\otimes 1_B \\
&= (r\cdot 1_A)\otimes 1_B \\
&= r\cdot(1_A\otimes 1_B) \\
&= 1_A\otimes(r\cdot 1_B) \\
&= 1_A\otimes((r\psi)_B 1_B) \\
&= 1_A\otimes(r\psi)_B \\
&\in {}_{\mathcal{A}}(A,A)\otimes{}_{\mathcal{B}}(B,B) = {}_{\mathcal{A}\boxtimes\mathcal{B}}(A\boxtimes B,A\boxtimes B).
\end{aligned}$$

We show that $r(\varphi\boxtimes\psi)$ is natural.

Suppose given $A_1 \boxtimes B_1 \xrightarrow{\zeta = \sum_{i \in [1,m]} a_i \otimes b_i} A_2 \boxtimes B_2$ in $\mathcal{A} \boxtimes \mathcal{B}$. We have

$$\begin{aligned}
(r(\varphi \boxtimes \psi))_{A_1 \boxtimes B_1} \cdot \zeta &= ((r\varphi)_{A_1} \otimes 1_{B_1}) \cdot \left(\sum_{i \in [1,m]} a_i \otimes b_i \right) \\
&= \sum_{i \in [1,m]} ((r\varphi)_{A_1} \otimes 1_{B_1})(a_i \otimes b_i) \\
&= \sum_{i \in [1,m]} ((r\varphi)_{A_1} a_i) \otimes (1_{B_1} b_i) \\
&= \sum_{i \in [1,m]} (a_i (r\varphi)_{A_2}) \otimes (b_i 1_{B_2}) \\
&= \sum_{i \in [1,m]} (a_i \otimes b_i)((r\varphi)_{A_2} \otimes 1_{B_2}) \\
&= \left(\sum_{i \in [1,m]} a_i \otimes b_i \right) \cdot ((r\varphi)_{A_2} \otimes 1_{B_2}) \\
&= \zeta \cdot (r(\varphi \boxtimes \psi))_{A_2 \boxtimes B_2} \, .
\end{aligned}$$

Thus, $r(\varphi \boxtimes \psi)$ is natural. Therefore, $\varphi \boxtimes \psi$ is a welldefined map.

Now we show that $\varphi \boxtimes \psi$ is a ring morphism.

We have

$$\begin{aligned}
(1_R(\varphi \boxtimes \psi))_{A \boxtimes B} &= (1_R \varphi)_A \otimes 1_B \\
&= 1_A \otimes 1_B \\
&= 1_{A \boxtimes B} \, ,
\end{aligned}$$

for $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$. Thus, we have $1_R(\varphi \boxtimes \psi) = 1_{\operatorname{End} 1_{\mathcal{A} \boxtimes \mathcal{B}}}$.

We have

$$\begin{aligned}
((r+s)(\varphi \boxtimes \psi))_{A \boxtimes B} &= ((r+s)\varphi)_A \otimes 1_B \\
&= ((r\varphi)_A + (s\varphi)_A) \otimes 1_B \\
&= (r\varphi)_A \otimes 1_B \ + \ (s\varphi)_A \otimes 1_B \\
&= (r(\varphi \boxtimes \psi))_{A \boxtimes B} + (s(\varphi \boxtimes \psi))_{A \boxtimes B} \, ,
\end{aligned}$$

for $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$. Thus, $(r+s)(\varphi \boxtimes \psi) = r(\varphi \boxtimes \psi) + s(\varphi \boxtimes \psi)$.

We have

$$\begin{aligned}
((rs)(\varphi \boxtimes \psi))_{A \boxtimes B} &= ((rs)\varphi)_A \otimes 1_B \\
&= ((r\varphi)_A (s\varphi)_A) \otimes (1_B 1_B) \\
&= (r\varphi)_A \otimes 1_B \ \cdot \ (s\varphi)_A \otimes 1_B \\
&= (r(\varphi \boxtimes \psi))_{A \boxtimes B} \cdot (s(\varphi \boxtimes \psi))_{A \boxtimes B} \, ,
\end{aligned}$$

for $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$. Thus, $(rs)(\varphi \boxtimes \psi) = r(\varphi \boxtimes \psi) \cdot s(\varphi \boxtimes \psi)$.

Therefore, $\varphi \boxtimes \psi$ is a ring morphism. $\square$

3.1.2 Universal property

For this §3.1.2, let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be preadditive categories over R.

Definition 125 (and Lemma)**.** We have the R-bilinear functor

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{B} & \xrightarrow{M_{\mathcal{A},\mathcal{B}}} & \mathcal{A} \underset{R}{\boxtimes} \mathcal{B} \\ ((A_1, B_1) \xrightarrow{(a,b)} (A_2, B_2)) & \mapsto & (A_1 \boxtimes B_1 \xrightarrow{a \otimes b} A_2 \boxtimes B_2). \end{array}$$

If unambiguous, we often write $M := M_{\mathcal{A},\mathcal{B}}$.

Proof. Suppose given $(A_1, B_1) \xrightarrow{(a',b')} (A_2, B_2) \xrightarrow{(a'',b'')} (A_3, B_3)$ in $\mathcal{A} \times \mathcal{B}$. We have

$$\begin{aligned} M((a', b')(a'', b'')) &= M(a'a'', b'b'') \\ &= (a'a'') \otimes (b'b'') \\ &= (a' \otimes b')(a'' \otimes b'') \\ &= M(a', b') \cdot M(a'', b'') \,. \end{aligned}$$

Furthermore, we have

$$M 1_{(A,B)} = M(1_A, 1_B) = 1_A \otimes 1_B = 1_{A \boxtimes B} = 1_{M(A,B)} \,.$$

Thus, M is a functor.

Suppose given $A_1 \overset{a'}{\underset{a''}{\rightrightarrows}} A_2$ in $\mathcal{A}$. Suppose given $B_1 \xrightarrow{b} B_2$ in $\mathcal{B}$. Suppose given $r', r'' \in R$. We have

$$\begin{aligned} M(r'a' + r''a'', b) &= (r'a' + r''a'') \otimes b \\ &= r'(a' \otimes b) \ + \ r''(a'' \otimes b) \\ &= r'M(a', b) + r''M(a'', b) \,. \end{aligned}$$

Suppose given $A_1 \xrightarrow{a} A_2$ in $\mathcal{A}$. Suppose given $B_1 \overset{b'}{\underset{b''}{\rightrightarrows}} B_2$ in $\mathcal{A}$. Suppose given $r', r'' \in R$.

$$\begin{aligned} M(a, r'b' + r''b'') &= a \otimes (r'b' + r''b'') \\ &= r'(a \otimes b') \ + \ r''(a \otimes b'') \\ &= r'M(a, b') + r''M(a, b'') \,. \end{aligned}$$

Thus, M is R-bilinear; cf. Definition 27. $\square$

Remark 126. The functor $M_{\mathcal{A},\mathcal{B}} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \underset{R}{\boxtimes} \mathcal{B}$ is bijective on objects. In particular, $M_{\mathcal{A},\mathcal{B}}$ is dense.

Definition 127 (and Lemma)**.** *Let $(\mathcal{C}, \omega)$ be an R-linear preadditive category. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be an R-bilinear functor.*

The following assertions (1, 2, 3) *hold.*

(1) *We have the R-linear functor*

$$\begin{array}{ccc} \mathcal{A} \underset{R}{\boxtimes} \mathcal{B} & \xrightarrow{\overline{F}} & \mathcal{C} \\ \left(A_1 \boxtimes B_1 \xrightarrow{\sum\limits_{i\in[1,m]} a_i\otimes b_i} A_2 \boxtimes B_2\right) & \mapsto & \left(F(A_1,B_1) \xrightarrow{\sum\limits_{i\in[1,m]} F(a_i\,,b_i)} F(A_2,B_2)\right). \end{array}$$

(2) *We have* $\overline{F} \circ M_{\mathcal{A},\mathcal{B}} = F$.

(3) *Suppose given an R-linear functor* $\mathcal{A} \underset{R}{\boxtimes} \mathcal{B} \xrightarrow{\tilde{F}} \mathcal{C}$ *with* $\tilde{F} \circ M_{\mathcal{A},\mathcal{B}} = F$. *Then* $\tilde{F} = \overline{F}$.

Proof. *Ad* (1). Suppose given $A_1 \boxtimes B_1$ and $A_2 \boxtimes B_2$ in $\mathrm{Ob}\,\mathcal{A}\boxtimes\mathcal{B}$. Since F is an R-bilinear functor, we have the R-bilinear map

$$\begin{array}{rcl} {}_{\mathcal{A}\times\mathcal{B}}((A_1,B_1),(A_2,B_2)) = {}_{\mathcal{A}}(A_1,A_2) \times {}_{\mathcal{B}}(B_1,B_2) & \xrightarrow{F_{(A_1,B_1),(A_2,B_2)}} & {}_{\mathcal{C}}(F(A_1,B_1),F(A_2,B_2)) \\ (a,b) & \mapsto & F(a,b). \end{array}$$

By Lemma 16, we obtain a unique R-linear map

$$\overline{F_{(A_1,B_1),(A_2,B_2)}} : {}_{\mathcal{A}}(A_1,A_2) \otimes {}_{\mathcal{B}}(B_1,B_2) \to {}_{\mathcal{C}}(F(A_1,B_1),F(A_2,B_2))$$

with $\mu_{{}_{\mathcal{A}}(A_1,A_2),\,{}_{\mathcal{B}}(B_1,B_2)}\overline{F_{(A_1,B_1),(A_2,B_2)}} = F_{(A_1,B_1),(A_2,B_2)}$, i.e.

$$(a\otimes b)\overline{F_{(A_1,B_1),(A_2,B_2)}} = (a,b)F_{(A_1,B_1),(A_2,B_2)} = F(a,b)$$

for $a \in {}_{\mathcal{A}}(A_1,A_2)$ and $b \in {}_{\mathcal{B}}(B_1,B_2)$.

For $\zeta = \sum\limits_{i\in[1,m]} a_i \otimes b_i \in {}_{\mathcal{A}\boxtimes\mathcal{B}}(A_1 \boxtimes B_1\,, A_2 \boxtimes B_2)$ we have

$$\zeta\overline{F_{(A_1,B_1),(A_2,B_2)}} = \sum_{i\in[1,m]} F(a_i\,,b_i)\,.$$

Thus, $\overline{F}$ is welldefined on morphisms.

Now we show that $\overline{F}$ is a functor.

Suppose given

$$A_1 \boxtimes B_1 \xrightarrow{\zeta = \sum\limits_{i\in[1,m']} a'_i\otimes b'_i} A_2 \boxtimes B_2 \xrightarrow{\eta = \sum\limits_{j\in[1,m'']} a''_j\otimes b''_j} A_3 \boxtimes B_3$$

in $\mathcal{A} \boxtimes \mathcal{B}$. We calculate

$$
\begin{aligned}
\overline{F}(\zeta\eta) &= \overline{F}\left(\sum_{(i,j)\in[1,m']\times[1,m'']} (a'_i a''_j) \otimes (b'_i b''_j)\right) \\
&= \sum_{(i,j)\in[1,m']\times[1,m'']} F(a'_i a''_j\,, b'_i b''_j) \\
&= \sum_{(i,j)\in[1,m']\times[1,m'']} F(a'_i\,, b'_i) \cdot F(a''_j\,, b''_j) \\
&= \left(\sum_{i\in[1,m']} F(a'_i\,, b'_i)\right) \cdot \left(\sum_{j\in[1,m'']} F(a''_j\,, b''_j)\right) \\
&= \overline{F}\zeta \cdot \overline{F}\eta\,.
\end{aligned}
$$

Furthermore, we have

$$
\overline{F}1_{A_1 \boxtimes B_1} = \overline{F}(1_{A_1} \otimes 1_{B_1}) = F(1_{A_1}\,, 1_{B_1}) = F1_{(A_1\,,B_1)} = 1_{F(A_1\,,B_1)} = 1_{\overline{F}(A_1 \boxtimes B_1)}\,.
$$

Thus, $\overline{F}$ is a functor.

Now we show that $\overline{F}$ is R-linear.

Suppose given $A_1 \boxtimes B_1 \overset{\zeta_1}{\underset{\zeta_2}{\rightrightarrows}} A_2 \boxtimes B_2$ in $\mathcal{A} \boxtimes \mathcal{B}$. Suppose given $r_1, r_2 \in R$. We have

$$
\begin{aligned}
\overline{F}(r_1\zeta_1 + r_2\zeta_2) &= (r_1\zeta_1 + r_2\zeta_2)\overline{F_{(A_1,B_1),(A_2,B_2)}} \\
&= r_1\left((\zeta_1)\overline{F_{(A_1,B_1),(A_2,B_2)}}\right) \;+\; r_2\left((\zeta_2)\overline{F_{(A_1,B_1),(A_2,B_2)}}\right) \\
&= r_1 \cdot \overline{F}\zeta_1 \;+\; r_2 \cdot \overline{F}\zeta_2\,.
\end{aligned}
$$

Thus, $\overline{F}$ is R-linear by Remark 25.

Ad (2). Suppose given $(A_1\,, B_1) \xrightarrow{(a,b)} (A_2\,, B_2)$ in $\mathcal{A} \times \mathcal{B}$. We have

$$
\begin{aligned}
(\overline{F} \circ M)\left((A_1\,, B_1) \xrightarrow{(a,b)} (A_2\,, B_2)\right) &= \overline{F}\left(A_1 \boxtimes B_1 \xrightarrow{a \otimes b} A_2 \boxtimes B_2\right) \\
&= \left(F(A_1\,, B_1) \xrightarrow{F(a,b)} F(A_2\, B_2)\right) \\
&= F\left((A_1\,, B_1) \xrightarrow{(a,b)} (A_2\,, B_2)\right).
\end{aligned}
$$

Ad (3). Suppose given $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$. We have

$$
\tilde{F}(A \boxtimes B) = (\tilde{F} \circ M)(A, B) = F(A, B) = \overline{F}(A \boxtimes B)\,.
$$

Suppose given $A_1 \boxtimes B_1 \xrightarrow{\zeta = \sum\limits_{i \in [1,m]} a_i \otimes b_i} A_2 \boxtimes B_2$ in $\mathcal{A} \boxtimes \mathcal{B}$. We obtain

$$\begin{aligned} \tilde{F}\zeta &= \sum_{i \in [1,m]} \tilde{F}(a_i \otimes b_i) \\ &= \sum_{i \in [1,m]} F(a_i, b_i) \\ &= \sum_{i \in [1,m]} \overline{F}(a_i \otimes b_i) \\ &= \overline{F}\zeta\ . \end{aligned}$$

Thus, we have $\tilde{F} = \overline{F}$. $\square$

Lemma 128. *Suppose given an R-linear preadditive category $(\mathcal{C}, \omega)$. Suppose given an R-bilinear functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. The following assertions* (1, 2) *hold.*

(1) *Suppose F to be full. Then $\overline{F}$ is full.*

(2) *Suppose F to be dense. Then $\overline{F}$ is dense.*

Proof. *Ad* (1). Suppose given $A_1 \boxtimes B_1$ and $A_2 \boxtimes B_2$ in $\operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$.

Suppose given $\rho \in {}_{\mathcal{C}}(\overline{F}(A_1 \boxtimes B_1), \overline{F}(A_2 \boxtimes B_2)) \overset{\text{D127.(2)}}{=} {}_{\mathcal{C}}(F(A_1\,, B_1), F(A_2\,, B_2))$.
Since F is full, there exists $(a, b) \in {}_{\mathcal{A} \times \mathcal{B}}((A_1\,, B_1), (A_2\,, B_2))$ with $F(a, b) = \rho$.

We have $a \otimes b \in {}_{\mathcal{A} \boxtimes \mathcal{B}}(A_1 \boxtimes B_1\,, A_2 \boxtimes B_2)$ and $\overline{F}(a \otimes b) \overset{\text{D127.(2)}}{=} F(a, b) = \rho$.
Thus, $\overline{F}$ is full.

Ad (2). Suppose given $C \in \operatorname{Ob} \mathcal{C}$. Since F is dense, there exists $(A, B) \in \operatorname{Ob} \mathcal{A} \times \mathcal{B}$ with $F(A, B) \cong C$. We have $\overline{F}(A \boxtimes B) \overset{\text{D/127.(2)}}{=} F(A, B) \cong C$. Thus, $\overline{F}$ is dense. $\square$

Definition 129 (and Lemma)**.** *Let $(\mathcal{C}, \omega)$ be an R-linear preadditive category. Suppose given $F \xrightarrow{\alpha} G$ in* ${}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}]$. Define $\overline{\alpha} := (\alpha_{(A,B)})_{A \boxtimes B \in \operatorname{Ob} \mathcal{A} \underset{R}{\boxtimes} \mathcal{B}}$.

The following assertions (1, 2, 3) *hold.*

(1) *We have $\overline{F} \xrightarrow{\overline{\alpha}} \overline{G}$ in* ${}_{R\text{-lin}}[\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}, \mathcal{C}]$.

(2) *We have $\overline{\alpha} * M_{\mathcal{A},\mathcal{B}} = \alpha$.*

(3) *Suppose given $\overline{F} \xrightarrow{\tilde{\alpha}} \overline{G}$ in ${}_{R\text{-lin}}[\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}, \mathcal{C}]$ with $\tilde{\alpha} * M_{\mathcal{A},\mathcal{B}} = \alpha$. Then $\tilde{\alpha} = \overline{\alpha}$.*

Proof. *Ad* (1). We have

$$\Big(F(A, B) \xrightarrow{\alpha_{(A,B)}} G(A, B)\Big) = \Big(\overline{F}(A \boxtimes B) \xrightarrow{\overline{\alpha}_{A \boxtimes B}} \overline{G}(A \boxtimes B)\Big)$$

for $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$,

It remains to show that $\overline{\alpha}$ is natural.

Suppose given $A_1 \boxtimes B_1 \xrightarrow{\zeta = \sum_{i\in[1,m]} a_i \otimes b_i} A_2 \boxtimes B_2$ in $\mathcal{A} \boxtimes \mathcal{B}$. We have

$$\begin{aligned}
\overline{\alpha}_{A_1 \boxtimes B_1} \cdot \overline{G}\zeta &= \alpha_{(A_1\,,B_1)} \cdot \sum_{i\in[1,m]} \overline{G}(a_i \otimes b_i) \\
&= \sum_{i\in[1,m]} \alpha_{(A_1\,,B_1)} G(a_i\,, b_i) \\
&= \sum_{i\in[1,m]} F(a_i\,, b_i) \alpha_{(A_2\,,B_2)} \\
&= \left(\sum_{i\in[1,m]} \overline{F}(a_i \otimes b_i) \right) \cdot \alpha_{(A_2\,,B_2)} \\
&= \overline{F}\zeta \cdot \overline{\alpha}_{A_2 \boxtimes B_2}\,.
\end{aligned}$$

Ad (2). Suppose given $(A, B) \in \operatorname{Ob} \mathcal{A} \times \mathcal{B}$. We have

$$(\overline{\alpha} * M)_{(A,B)} = \overline{\alpha}_{M(A,B)} = \overline{\alpha}_{A \boxtimes B} = \alpha_{(A,B)}\,.$$

Ad (3). Suppose given $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$. We have

$$\tilde{\alpha}_{A \boxtimes B} = \tilde{\alpha}_{M(A,B)} = (\tilde{\alpha} * M)_{(A,B)} = \alpha_{(A,B)} = \overline{\alpha}_{A \boxtimes B}\,.$$

$\square$

Theorem 130. Recall that $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ are preadditive categories over R. Recall that $(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}, \varphi \boxtimes \psi)$ is a preadditive category over R; cf. Definition 123 and Lemma 124. Suppose given an R-linear preadditive category $(\mathcal{C}, \omega)$.

The following assertions (1, 2) *hold.*

(1) *We have* $M_{\mathcal{A},\mathcal{B}} \in \operatorname{Ob}\ {}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{A} \underset{R}{\boxtimes} \mathcal{B}]$*;* cf. Definition 125.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}]$.

There exist unique R-linear functors $\overline{F}, \overline{G} : \mathcal{A} \underset{R}{\boxtimes} \mathcal{B} \to \mathcal{C}$ *with* $\overline{F} \circ M_{\mathcal{A},\mathcal{B}} = F$ *and* $\overline{G} \circ M_{\mathcal{A},\mathcal{B}} = G$*;* cf. Definition 127.

There exists a unique transformation $\overline{F} \overset{\overline{\alpha}}{\Rightarrow} \overline{G}$ *with* $\overline{\alpha} * M_{\mathcal{A},\mathcal{B}} = \alpha$*;* cf. Definition 129.

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "M_{\mathcal{A},\mathcal{B}}"] \arrow[dr, bend right=30, "F"'] \arrow[dr, bend left=15, "G"] \arrow[dr, phantom, "\alpha \Uparrow"] & \mathcal{A} \underset{R}{\boxtimes} \mathcal{B} \arrow[d, bend right=30, "\overline{F}"'] \arrow[d, bend left=30, "\overline{G}"] \arrow[d, phantom, "\overset{\overline{\alpha}}{\Rightarrow}"] \\
& \mathcal{C}
\end{tikzcd}$$

(2) *We have the isomorphism of categories*

$$\begin{array}{ccc} {}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}] & \xleftarrow{\Psi^R_{\mathcal{A},\mathcal{B},\mathcal{C}}} & {}_{R\text{-lin}}[\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}, \mathcal{C}] \\ \left(U \circ M_{\mathcal{A},\mathcal{B}} \xrightarrow{\beta * M_{\mathcal{A},\mathcal{B}}} V \circ M_{\mathcal{A},\mathcal{B}}\right) & \leftarrow\!\shortmid & \left(U \xrightarrow{\beta} V\right) \end{array}$$

with inverse

$$\begin{array}{ccc} {}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}] & \longrightarrow & {}_{R\text{-lin}}[\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}, \mathcal{C}] \\ (F \xrightarrow{\alpha} G) & \mapsto & (\overline{F} \xrightarrow{\overline{\alpha}} \overline{G}) \end{array}$$

If unambiguous, we often write $\Psi := \Psi_{\mathcal{A},\mathcal{B},\mathcal{C}} := \Psi^R_{\mathcal{A},\mathcal{B},\mathcal{C}}$.

Proof. *Ad* (1). This follows from Definition 127 and Definition 129.

Ad (2). This follows from (1). □

3.1.3 Functoriality

In this §3.1.3, we define the tensor product for R-linear functors between preadditive categories over R and for transformations between them. Furthermore, we establish functoriality properties of the tensor product, which could be expressed by saying that it is turned into a 2-bifunctor.

For this §3.1.3, let $(\mathcal{A}, \varphi_{\mathcal{A}})$, $(\mathcal{B}, \varphi_{\mathcal{B}})$, $(\mathcal{C}, \varphi_{\mathcal{C}})$, $(\mathcal{D}, \varphi_{\mathcal{D}})$, $(\mathcal{E}, \varphi_{\mathcal{E}})$ and $(\mathcal{F}, \varphi_{\mathcal{F}})$ be preadditive categories over R.

Definition 131 (and Lemma)**.** Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{D}$ be R-linear functors. *We have the R-linear functor*

$$\begin{array}{ccc} \mathcal{A} \underset{R}{\boxtimes} \mathcal{B} & \xrightarrow{F \boxtimes G} & \mathcal{C} \underset{R}{\boxtimes} \mathcal{D} \\ \left(A_1 \boxtimes B_1 \xrightarrow{\sum_{i \in [1,n]} a_i \otimes b_i} A_2 \boxtimes B_2\right) & \mapsto & \left(FA_1 \boxtimes GB_1 \xrightarrow{\sum_{i \in [1,n]} (Fa_i) \otimes (Gb_i)} FA_2 \boxtimes GB_2\right). \end{array}$$

So, we have

$$(F \boxtimes G)(A \boxtimes B) = FA \boxtimes FG$$

for $A \boxtimes B \in \mathrm{Ob}(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B})$.

Furthermore, we have

$$(F \boxtimes G)\left(\sum_{i \in [1,n]} a_i \otimes b_i\right) = \sum_{i \in [1,n]} (Fa_i) \otimes (Gb_i)$$

for $\sum_{i \in [1,n]} a_i \otimes b_i \in \mathrm{Mor}(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B})$.

Proof. Suppose given $A_1 \boxtimes B_1$ and $A_2 \boxtimes B_2$ in $\operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$.

Since F and G are R-linear functors, we have

$$\big(\,{}_{\mathcal{A}}(A_1\,,A_2)\,,\ {}_{\mathcal{B}}(B_1\,,B_2)\big) \xrightarrow{(F_{A_1,A_2}\,,G_{B_1,B_2})} {}_{\mathcal{C}}(FA_1\,,FA_2) \times {}_{\mathcal{D}}(GB_1\,GB_2)$$

in $(R\text{-Mod})^{\times 2}$.

By Definitions 20 and 123, we have

$$\begin{array}{ccc} {}_{\mathcal{A}\boxtimes\mathcal{B}}(A_1 \boxtimes B_1\,, A_2 \boxtimes B_2) & \xrightarrow{F_{A_1,A_2}\otimes G_{B_1,B_2}} & {}_{\mathcal{C}\boxtimes\mathcal{D}}(FA_1 \boxtimes GB_1\,, FA_2 \boxtimes GB_2) \\ \sum\limits_{i\in[1,n]} a_i \otimes b_i & \mapsto & \sum\limits_{i\in[1,n]} (Fa_i) \otimes (Gb_i) \end{array}$$

in R-Mod.

Thus, $F \boxtimes G$ is welldefined on morphisms.

Now we show that $F \boxtimes G$ is a functor.

Suppose given $A_1 \boxtimes B_1 \xrightarrow{\zeta=\sum_{i\in[1,m]} a_i'\otimes b_i'} A_2 \boxtimes B_2 \xrightarrow{\eta=\sum_{j\in[1,k]} a_j''\otimes b_j''} A_3 \boxtimes B_3$ in $\mathcal{A} \boxtimes \mathcal{B}$. We have

$$\begin{aligned} (F \boxtimes G)(\zeta\eta) &= (F \boxtimes G)\left(\sum_{(i,j)\in[1,m]\times[1,k]} (a_i' a_j'') \otimes (b_i' b_j'')\right) \\ &= \sum_{(i,j)\in[1,m]\times[1,k]} F(a_i' a_j'') \otimes G(b_i' b_j'') \\ &= \sum_{(i,j)\in[1,m]\times[1,k]} (Fa_i')(Fa_j'') \otimes (Gb_i')(Gb_j'') \\ &= \sum_{i\in[1,m]} (Fa_i') \otimes (Gb_i') \cdot \sum_{j\in[1,k]} (Fa_i'') \otimes (Gb_j'') \\ &= (F \boxtimes G)\zeta \cdot (F \boxtimes G)\eta. \end{aligned}$$

Suppose given $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$. We have

$$\begin{aligned} (F \boxtimes G)1_{A\boxtimes B} &= (F \boxtimes G)(1_A \otimes 1_B) \\ &= F1_A \otimes G1_B \\ &= 1_{FA} \otimes 1_{GB} \\ &= 1_{FA\boxtimes GB} \\ &= 1_{(F\boxtimes G)(A\boxtimes B)}\,. \end{aligned}$$

Thus, $F \boxtimes G$ is indeed a functor.

Since $(F \boxtimes G)_{A_1\boxtimes B_1, A_2\boxtimes B_2} = F_{A_1,A_2} \otimes G_{B_1,B_2}$ is R-linear for $A_1 \boxtimes B_1$ and $A_2 \boxtimes B_2$ in $\operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}$, we conclude that $F \boxtimes G$ is R-linear. $\square$

Lemma 132. *Suppose given R-linear functors* $\mathcal{A} \xrightarrow{F} \mathcal{C} \xrightarrow{H} \mathcal{E}$ *and* $\mathcal{B} \xrightarrow{G} \mathcal{D} \xrightarrow{I} \mathcal{F}$*. The following assertions* (1, 2) *hold.*

(1) *We have* $1_{\mathcal{A}} \boxtimes 1_{\mathcal{B}} = 1_{\mathcal{A} \boxtimes \mathcal{B}}$.

(2) *We have* $(H \boxtimes I) \circ (F \boxtimes G) = (H \circ F) \boxtimes (I \circ G)$.

Proof. *Ad* (1). Suppose given $A \boxtimes B \in \mathrm{Ob}(\mathcal{A} \boxtimes \mathcal{B})$. We have

$$(1_{\mathcal{A}} \boxtimes 1_{\mathcal{B}})(A \boxtimes B) = 1_{\mathcal{A}} A \boxtimes 1_{\mathcal{B}} B = A \boxtimes B = 1_{\mathcal{A} \boxtimes \mathcal{B}}(A \boxtimes B).$$

Suppose given $A_1 \boxtimes B_1$ and $A_2 \boxtimes B_2$ in $\mathrm{Ob}(\mathcal{A} \boxtimes \mathcal{B})$. We have

$$\begin{aligned}
(1_{\mathcal{A}} \boxtimes 1_{\mathcal{B}})_{A_1 \boxtimes B_1 \,,\, A_2 \boxtimes B_2} &= (1_{\mathcal{A}})_{A_1 \,,\, A_2} \otimes (1_{\mathcal{B}})_{B_1 \,,\, B_2} \\
&= 1_{\mathcal{A}(A_1 \,,\, A_2)} \otimes 1_{\mathcal{B}(B_1 \,,\, B_2)} \\
&\overset{\text{D21}}{=} 1_{\mathcal{A}(A_1, A_2) \otimes \mathcal{B}(B_1, B_2)} \\
&= 1_{\mathcal{A} \boxtimes \mathcal{B}(A_1 \boxtimes B_1 \,,\, A_2 \boxtimes B_2)} \\
&= (1_{\mathcal{A} \boxtimes \mathcal{B}})_{A_1 \boxtimes B_1 \,,\, A_2 \boxtimes B_2} \,.
\end{aligned}$$

Ad (2). Suppose given $A \boxtimes B \in \mathrm{Ob}(\mathcal{A} \boxtimes \mathcal{B})$. We have

$$\begin{aligned}
((H \boxtimes I) \circ (F \boxtimes G))(A \boxtimes B) &= (H \boxtimes I)(FA \boxtimes GB) \\
&= ((H \circ F)A) \boxtimes ((I \circ G)B) \\
&= ((H \circ F) \boxtimes (I \circ G))(A \boxtimes B).
\end{aligned}$$

Suppose given $A_1 \boxtimes B_1$ and $A_2 \boxtimes B_2$ in $\mathrm{Ob}(\mathcal{A} \boxtimes \mathcal{B})$. We have

$$\begin{aligned}
& ((H \boxtimes I) \circ (F \boxtimes G))_{A_1 \boxtimes B_1 \,,\, A_2 \boxtimes B_2} \\
= \;& (F \boxtimes G)_{A_1 \boxtimes B_1 \,,\, A_2 \boxtimes B_2} \cdot (H \boxtimes I)_{(FA_1) \boxtimes (GB_1) \,,\, (FA_2) \boxtimes (GB_2)} \\
= \;& (F_{A_1 \,,\, A_2} \otimes G_{B_1 \,,\, B_2})(H_{FA_1 \,,\, FA_2} \otimes I_{GB_1 \,,\, GB_2}) \\
\overset{\text{D21}}{=} \;& (F_{A_1 \,,\, A_2} \cdot H_{FA_1 \,,\, FA_2}) \otimes (G_{B_1 \,,\, B_2} \cdot I_{GB_1 \,,\, GB_2}) \\
= \;& (H \circ F)_{A_1 \,,\, A_2} \otimes (I \circ G)_{B_1 \,,\, B_2} \\
= \;& ((H \circ F) \boxtimes (I \circ G))_{A_1 \boxtimes B_1 \,,\, A_2 \boxtimes B_2} \,.
\end{aligned}$$

□

Definition 133 (and Lemma)**.** Let $F, F' : \mathcal{A} \to \mathcal{C}$ and $G, G' : \mathcal{B} \to \mathcal{D}$ be R-linear functors. Let $F \overset{\alpha}{\Rightarrow} F'$ and $G \overset{\beta}{\Rightarrow} G'$ be transformations.

We have the transformation

$$F \boxtimes G \overset{\alpha \boxtimes \beta}{\Longrightarrow} F' \boxtimes G'$$

with

$$(\alpha \boxtimes \beta)_{A \boxtimes B} = \alpha_A \otimes \beta_B \in {}_{\mathcal{C} \underset{R}{\boxtimes} \mathcal{D}}((F \boxtimes G)(A \boxtimes B), (F' \boxtimes G')(A \boxtimes B))$$

for $A \boxtimes B \in \mathrm{Ob}\, \mathcal{A} \underset{R}{\boxtimes} \mathcal{B}$.

Proof. Suppose given $A \boxtimes B \in \mathrm{Ob}(\mathcal{A} \boxtimes \mathcal{B})$. We have

$$FA \xrightarrow{\alpha_A} F'A \quad \text{and} \quad GB \xrightarrow{\beta_B} G'B.$$

Since we have $(F \boxtimes G)(A \boxtimes B) = FA \boxtimes GB$ and $(F' \boxtimes G')(A \boxtimes B) = F'A \boxtimes G'B$, we obtain

$$(F \boxtimes G)(A \boxtimes B) \xrightarrow{\alpha_A \otimes \beta_B} (F' \boxtimes G')(A \boxtimes B).$$

We have to show that $\alpha \boxtimes \beta$ is natural.

Suppose given $A_1 \boxtimes B_1 \xrightarrow{\zeta = \sum_{i \in [1,n]} a_i \otimes b_i} A_2 \boxtimes B_2$ in $\mathcal{A} \boxtimes \mathcal{B}$. We have

$$\begin{aligned}
(F \boxtimes G)\zeta \cdot (\alpha \boxtimes \beta)_{A_2 \boxtimes B_2} &= \left(\sum_{i \in [1,n]} (Fa_i) \otimes (Gb_i) \right) (\alpha_{A_2} \otimes \beta_{B_2}) \\
&= \sum_{i \in [1,n]} (Fa_i \cdot \alpha_{A_2}) \otimes (Gb_i \cdot \beta_{B_2}) \\
&= \sum_{i \in [1,n]} (\alpha_{A_1} \cdot F'a_i) \otimes (\beta_{B_1} \cdot G'b_i) \\
&= (\alpha_{A_1} \otimes \beta_{B_1}) \left(\sum_{i \in [1,n]} (F'a_i) \otimes (G'b_i) \right) \\
&= (\alpha \boxtimes \beta)_{A_1 \boxtimes B_1} \cdot (F' \boxtimes G')\zeta.
\end{aligned}$$

$\square$

Lemma 134. *Let $F, F', F'' : \mathcal{A} \to \mathcal{C}$ and $G, G', G'' : \mathcal{B} \to \mathcal{D}$ be R-linear functors. Suppose given transformations $F \overset{\alpha}{\Rightarrow} F' \overset{\alpha'}{\Rightarrow} F''$ and $G \overset{\beta}{\Rightarrow} G' \overset{\beta'}{\Rightarrow} G''$.*
The following assertions (1, 2) *hold.*

(1) *We have $1_F \boxtimes 1_G = 1_{F \boxtimes G}$.*

(2) *We have $(\alpha \boxtimes \beta)(\alpha' \boxtimes \beta') = (\alpha\alpha') \boxtimes (\beta\beta')$.*

Proof. Ad (1). Suppose given $A \boxtimes B \in \mathrm{Ob}(\mathcal{A} \boxtimes \mathcal{B})$. We have

$$(1_F \boxtimes 1_G)_{A \boxtimes B} = (1_F)_A \otimes (1_G)_B = 1_{FA} \otimes 1_{GB} = 1_{FA \boxtimes GB} = 1_{(F \boxtimes G)(A \boxtimes B)} = (1_{F \boxtimes G})_{A \boxtimes B}\,.$$

Ad (2). Suppose given $A \boxtimes B \in \mathrm{Ob}(\mathcal{A} \boxtimes \mathcal{B})$. We have

$$\begin{aligned}
((\alpha \boxtimes \beta)(\alpha' \boxtimes \beta'))_{A \boxtimes B} &= (\alpha \boxtimes \beta)_{A \boxtimes B} \cdot (\alpha' \boxtimes \beta')_{A \boxtimes B} \\
&= (\alpha_A \otimes \beta_B)(\alpha'_A \otimes \beta'_B) \\
&= (\alpha_A \alpha'_A) \otimes (\beta_B \beta'_B) \\
&= (\alpha\alpha')_A \otimes (\beta\beta')_B \\
&= ((\alpha\alpha') \boxtimes (\beta\beta'))_{A \boxtimes B}\,.
\end{aligned}$$

$\square$

Lemma 135. *Suppose given R-linear functors $F, F' : \mathcal{A} \to \mathcal{C}$ and $G, G' : \mathcal{B} \to \mathcal{D}$. Suppose given isotransformations $F \overset{\alpha}{\underset{\sim}{\Longrightarrow}} F'$ and $G \overset{\beta}{\underset{\sim}{\Longrightarrow}} G'$. Then $(F \boxtimes G) \overset{\alpha\boxtimes\beta}{\underset{\sim}{\Longrightarrow}} (F' \boxtimes G')$ is an isotransformation with inverse $(F' \boxtimes G') \overset{\alpha^{-1}\boxtimes\beta^{-1}}{\underset{\sim}{\Longrightarrow}} (F \boxtimes G)$.*

Proof. We have

$$(\alpha \boxtimes \beta)(\alpha^{-1} \boxtimes \beta^{-1}) \overset{\text{L134.(2)}}{=} (\alpha\alpha^{-1}) \boxtimes (\beta\beta^{-1}) = 1_F \boxtimes 1_G \overset{\text{L134.(1)}}{=} 1_{F\boxtimes G}$$

and

$$(\alpha^{-1} \boxtimes \beta^{-1})(\alpha \boxtimes \beta) \overset{\text{L134.(2)}}{=} (\alpha^{-1}\alpha) \boxtimes (\beta^{-1}\beta) = 1_{F'} \boxtimes 1_{G'} \overset{\text{L134.(1)}}{=} 1_{F'\boxtimes G'} .$$

□

Lemma 136. *Suppose given R-linear equivalences $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{D}$. Then $F \boxtimes G : \mathcal{A} \underset{R}{\boxtimes} \mathcal{B} \to \mathcal{C} \underset{R}{\boxtimes} \mathcal{D}$ is an R-linear equivalence.*

Proof. By Definition 131, it suffices to show that $F \boxtimes G$ is an equivalence.

Since $F : \mathcal{A} \to \mathcal{C}$ is an equivalence, there exists $F' : \mathcal{C} \to \mathcal{A}$ and isotransformations

$$(F' \circ F) \overset{\alpha}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{A}} \quad \text{and} \quad (F \circ F') \overset{\gamma}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{C}} \ .$$

Since $G : \mathcal{B} \to \mathcal{D}$ is an equivalence, there exists $G' : \mathcal{D} \to \mathcal{B}$ and isotransformations

$$(G' \circ G) \overset{\beta}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{B}} \quad \text{and} \quad (G \circ G') \overset{\delta}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{D}} \ .$$

By Corollary 31, F' and G' are R-linear.

By Lemma 135, we have isotransformations

$$(F' \boxtimes G') \circ (F \boxtimes G) \overset{\text{L132.(2)}}{=} (F' \circ F) \boxtimes (G' \circ G) \overset{\alpha\boxtimes\beta}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{A}} \boxtimes 1_{\mathcal{B}} \overset{\text{L132.(1)}}{=} 1_{\mathcal{A}\boxtimes\mathcal{B}}$$

and

$$(F \boxtimes G) \circ (F' \boxtimes G') \overset{\text{L132.(2)}}{=} (F \circ F') \boxtimes (G \circ G') \overset{\gamma\boxtimes\delta}{\underset{\sim}{\Longrightarrow}} 1_{\mathcal{C}} \boxtimes 1_{\mathcal{D}} \overset{\text{L132.(1)}}{=} 1_{\mathcal{C}\boxtimes\mathcal{D}} .$$

Thus, $F \boxtimes G$ is an equivalence. □

Lemma 137. *Suppose given*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \arrow[r, bend left, "H"] \arrow[r, bend right, "H'"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{E} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{D} \arrow[r, bend left, "I"] \arrow[r, bend right, "I'"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{F} \end{tikzcd}$$

with R-linear functors F, F', G, G', H, H', I and I'.

We have

$$(\gamma * \alpha) \boxtimes (\delta * \beta) = (\gamma \boxtimes \delta) * (\alpha \boxtimes \beta).$$

Proof. Suppose given $A \boxtimes B \in \operatorname{Ob} \mathcal{A} \underset{R}{\boxtimes} \mathcal{B}$.

We have

$$\begin{aligned}
((\gamma * \alpha) \boxtimes (\delta * \beta))_{A \boxtimes B} &\overset{\text{D133}}{=} (\gamma * \alpha)_A \otimes (\delta * \beta)_B \\
&= (\gamma_{FA} \cdot H' \alpha_A) \otimes (\delta_{GB} \cdot I' \beta_B) \\
&= (\gamma_{FA} \otimes \delta_{GB}) \cdot (H' \alpha_A \otimes I' \beta_B) \\
&\underset{\text{D133}}{\overset{\text{D131}}{=}} (\gamma \boxtimes \delta)_{FA \boxtimes GB} \cdot (H' \boxtimes I')(\alpha_A \otimes \beta_B) \\
&\underset{\text{D133}}{\overset{\text{D131}}{=}} (\gamma \boxtimes \delta)_{(F \boxtimes G)(A \boxtimes B)} \cdot (H' \boxtimes I')(\alpha \boxtimes \beta)_{A \boxtimes B} \\
&= ((\gamma \boxtimes \delta) * (\alpha \boxtimes \beta))_{A \boxtimes B} \ .
\end{aligned}$$

$\square$

3.2 The tensor product of additive categories over a commutative ring

For this § 3.2, let R be a commutative ring. Furthermore, let $(\mathcal{A}, \varphi_\mathcal{A})$ and $(\mathcal{B}, \varphi_\mathcal{B})$ be additive categories over R.

3.2.1 Definition

Definition 138 (and Lemma)**.** We call

$$\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B} := \mathrm{Add}\left(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}\right)$$

the *additive tensor product over* R of $\mathcal{A}$ and $\mathcal{B}$.

If unambiguous, we often write $\mathcal{A} \overset{\mathrm{add}}{\boxtimes} \mathcal{B} := \mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}$.

This defines an additive category $\left(\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}, (\varphi_\mathcal{A} \boxtimes \varphi_\mathcal{B})\psi_{\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}}\right)$ *over* R*;* cf. Lemmas 124 and 115.

Proof. By Lemma 124, $(\mathcal{A} \boxtimes \mathcal{B}, \varphi_\mathcal{A} \boxtimes \varphi_\mathcal{B})$ is a preadditive category over R. By Proposition 92, $\mathrm{Add}(\mathcal{A} \boxtimes \mathcal{B})$ is additive. By Lemma 115, $(\mathrm{Add}(\mathcal{A} \boxtimes \mathcal{B}), (\varphi_\mathcal{A} \boxtimes \varphi_\mathcal{B})\psi_{\mathcal{A} \boxtimes \mathcal{B}})$ is a preadditive category over R. □

3.2.2 Universal property

Definition 139 (and Lemma)**.** Define

$$M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} := I_{\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}} \circ M_{\mathcal{A},\mathcal{B}}\,;$$

cf. Definitions 96 and 125.

If unambiguous, we often write $M^{\mathrm{add}} := M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}$.

We have the R*-bilinear functor* $M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}$.

Proof. By Definition 125, $M = M_{\mathcal{A},\mathcal{B}}$ is R-bilinear. By Lemma 117, $I = I_{\mathcal{A} \boxtimes \mathcal{B}}$ is R-linear. Thus, $M^{\mathrm{add}} = I \circ M$ is R-bilinear; cf. Remark 29. □

Definition 140 (and Lemma)**.** We have the functor

$$\begin{array}{ccc} {}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}] & \xrightarrow{\Gamma^R_{\mathcal{A},\mathcal{B},\mathcal{C}}} & {}_{R\text{-lin}}[\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}, \mathcal{C}] \\ (F \xrightarrow{\alpha} G) & \mapsto & \left(\overline{F}' \xrightarrow{\overline{\alpha}'} \overline{G}'\right)\,; \end{array}$$

cf. Definitions 110, 127 and 129.

If unambiguous, we often write $\Gamma := \Gamma_{\mathcal{A},\mathcal{B},\mathcal{C}} := \Gamma^R_{\mathcal{A},\mathcal{B},\mathcal{C}}$.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}]$. *Suppose given*

$$\boxplus_{i\in[1,m]} A_{1,i} \boxtimes B_{1,i} \xrightarrow{\left[\sum_{k\in[1,m_{i,j}]} f_{k,i,j}\otimes g_{k,i,j}\right]_{i,j}} \boxplus_{j\in[1,n]} A_{2,j} \boxtimes B_{2,j}$$

in $\mathcal{A} \overset{\text{add}}{\underset{R}{\boxtimes}} \mathcal{B}$. *We have*

$$\begin{aligned}
&\overline{F}'\left(\boxplus_{i\in[1,m]} A_{1,i} \boxtimes B_{1,i} \xrightarrow{\left[\sum_{k\in[1,m_{i,j}]} f_{k,i,j}\otimes g_{k,i,j}\right]_{i,j}} \boxplus_{j\in[1,n]} A_{2,j} \boxtimes B_{2,j} \right) \\
&\quad = \left(\bigoplus_{i\in[1,m]} F(A_{1,i}\,, B_{1,i}) \xrightarrow{\left(\sum_{k\in[1,m_{i,j}]} F(f_{k,i,j}\,, g_{k,i,j})\right)_{i,j}} \bigoplus_{j\in[1,n]} F(A_{2,j}\,, B_{2,j}) \right).
\end{aligned}$$

Suppose given $\boxplus_{i\in[1,m]} A_i \boxtimes B_i \in \mathrm{Ob}(\mathcal{A} \overset{\text{add}}{\underset{R}{\boxtimes}} \mathcal{B})$. *We have*

$$\overline{\alpha}'_{(A_1\boxtimes B_1)\boxplus\cdots\boxplus(A_m\boxtimes B_m)} = \mathrm{diag}(\alpha_{(A_i\,,B_i)})_{i\in[1,m]}\,.$$

Proof. By Definition 127.(1), $\overline{F}$ is R-linear. By Lemma 120, $\overline{F}'$ is R-linear. Therefore, we obtain a welldefined map on objects.

Consider the functor $\Psi^{-1}_{\mathcal{A},\mathcal{B},\mathcal{C}}$; cf. Theorem 130. Consider the restriction of the functor from Definition 110; cf. Lemma 120. Then Γ is the composition of these functors.

We have

$$\begin{aligned}
&\overline{F}'\left(\boxplus_{i\in[1,m]} A_{1,i} \boxtimes B_{1,i} \xrightarrow{\left[\sum_{k\in[1,m_{i,j}]} f_{k,i,j}\otimes g_{k,i,j}\right]_{i,j}} \boxplus_{j\in[1,n]} A_{2,j} \boxtimes B_{2,j} \right) \\
&\overset{\text{D110}}{=} \left(\bigoplus_{i\in[1,m]} \overline{F}(A_{1,i} \boxtimes B_{1,i}) \xrightarrow{\left(\overline{F}\left(\sum_{k\in[1,m_{i,j}]} f_{k,i,j}\otimes g_{k,i,j}\right)\right)_{i,j}} \bigoplus_{j\in[1,n]} \overline{F}(A_{2,j} \boxtimes B_{2,j}) \right) \\
&\overset{\text{D127.(1)}}{=} \left(\bigoplus_{i\in[1,m]} F(A_{1,i}\,, B_{1,i}) \xrightarrow{\left(\sum_{k\in[1,m_{i,j}]} F(f_{k,i,j}\,, g_{k,i,j})\right)_{i,j}} \bigoplus_{j\in[1,n]} F(A_{2,j}\,, B_{2,j}) \right).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\overline{\alpha}'_{(A_1 \boxtimes B_1) \boxplus \cdots \boxplus (A_m \boxtimes B_m)} &\overset{\mathrm{D110}}{=} \operatorname{diag}(\overline{\alpha}_{A_i \boxtimes B_i})_i \\ &\overset{\mathrm{D129.(1)}}{=} \operatorname{diag}(\alpha_{(A_i\,,B_i)})_i\,.\end{aligned}$$

□

Lemma 141. *Suppose given $F \in \operatorname{Ob}\, {}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}]$. The following assertions* (1, 2) *hold.*

(1) *If F is full, so is $\overline{F}'$.*

(2) *If F is dense, so is $\overline{F}'$.*

Proof. *Ad* (1). This follows from Lemma 128.(1) and Lemma 111.(1).

Ad (2). This follows from Lemma 128.(2) and Lemma 111.(3). □

Lemma 142. *Suppose given $F \xrightarrow{\alpha} G$ in ${}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}]$. The following assertions* (1, 2) *hold.*

(1) *We have $\overline{F}' \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = F$.*

(2) *We have $\overline{\alpha}' * M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = \alpha$.*

Proof. Recall that $M^{\mathrm{add}} = I_{\mathcal{A} \boxtimes \mathcal{B}} \circ M_{\mathcal{A},\mathcal{B}}$; cf. Definition 139.

Ad (1). We have

$$\overline{F}' \circ M^{\mathrm{add}} = (\overline{F}' \circ I_{\mathcal{A} \boxtimes \mathcal{B}}) \circ M_{\mathcal{A},\mathcal{B}} \overset{\mathrm{L112.(1)}}{=} \overline{F} \circ M_{\mathcal{A},\mathcal{B}} \overset{\mathrm{D127.(2)}}{=} F.$$

Ad (2). We have

$$\overline{\alpha}' * M^{\mathrm{add}} = (\overline{\alpha}' * I_{\mathcal{A} \boxtimes \mathcal{B}}) * M_{\mathcal{A},\mathcal{B}} \overset{\mathrm{L112.(2)}}{=} \overline{\alpha} * M_{\mathcal{A},\mathcal{B}} \overset{\mathrm{D129.(2)}}{=} \alpha.$$

□

Theorem 143. Recall that $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ are additive categories over R. Recall that $(\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}, (\varphi_{\mathcal{A}} \boxtimes \varphi_{\mathcal{B}})\psi_{\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}})$ is an additive category over R; cf. Definition 138.

Suppose given an additive category $(\mathcal{C}, \varphi_{\mathcal{C}})$ over R.

The following assertions (1, 2, 3) *hold.*

(1) *We have $M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} \in \operatorname{Ob}\, {}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}]$; cf. Definition 139.*

Suppose given $F \xrightarrow{\alpha} G$ in ${}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}]$. We have $\overline{F}' \xrightarrow{\overline{\alpha}'} \overline{G}'$ in ${}_{R\text{-lin}}[\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}, \mathcal{C}]$ with $\overline{F}' \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = F$, $\overline{G}' \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = G$ and $\overline{\alpha}' * M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = \alpha$; cf. Definition 140 and Lemma 142.

Suppose given $\beta \in {}_{[\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}, \mathcal{C}]}(\overline{F}', \overline{G}')$ *with* $\beta * M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = \alpha$. *Then we have* $\beta = \alpha$.

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}"] \arrow[dr, bend right, "F"'] \arrow[dr, bend left, "G"] \arrow[dr, phantom, "\alpha \Uparrow"] & \mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B} \arrow[d, bend right, "\overline{F}'"'] \arrow[d, bend left, "\overline{G}'"] \arrow[d, phantom, "\overset{\overline{\alpha}'}{\Rightarrow}"] \\
& \mathcal{C}
\end{tikzcd}$$

(2) *Suppose given* $U, V \in \mathrm{Ob}\ {}_{R\text{-lin}}[\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}, \mathcal{C}]$ *with* $U \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = V \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}$. *Then* $U \cong V$.

(3) *We have the equivalence of categories*

$$\begin{array}{ccc}
{}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}] & \xleftarrow{\Omega^R_{\mathcal{A},\mathcal{B},\mathcal{C}}} & {}_{R\text{-lin}}[\mathcal{A} \overset{\mathrm{add}}{\underset{R}{\boxtimes}} \mathcal{B}, \mathcal{C}] \\
\big((U \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}) \xrightarrow{\beta * M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}} (V \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}})\big) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V)\,,
\end{array}$$

that is surjective on objects.

If unambiguous, we often write $\Omega := \Omega_{\mathcal{A},\mathcal{B},\mathcal{C}} := \Omega^R_{\mathcal{A},\mathcal{B},\mathcal{C}}$.

Proof. *Ad* (1). This follows from (3).

Ad (2). This follows from (3).

Ad (3). We have equivalences of categories

$${}_{R\text{-lin}}[\mathcal{A} \overset{\mathrm{add}}{\boxtimes} \mathcal{B}, \mathcal{C}] \xrightarrow{\Phi_{\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}}} {}_{R\text{-lin}}[\mathcal{A} \boxtimes \mathcal{B}, \mathcal{C}] \xrightarrow{\Psi_{\mathcal{A},\mathcal{B},\mathcal{C}}} {}_{R\text{-bil}}[\mathcal{A} \times \mathcal{B}, \mathcal{C}];$$

cf. Proposition 121.(3) and Theorem 130.(2).

Suppose given $U \xrightarrow{\beta} V$ in ${}_{R\text{-lin}}[\mathcal{A} \overset{\mathrm{add}}{\boxtimes} \mathcal{B}, \mathcal{C}]$.

To visualize the situation, we give the following diagram.

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "M_{\mathcal{A},\mathcal{B}}"] \arrow[rr, bend left, "M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}"] \arrow[drr, bend right, "{U \circ (I_{\mathcal{A}\boxtimes\mathcal{B}} \circ M_{\mathcal{A},\mathcal{B}}) \atop = U \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}}"'] & \mathcal{A} \boxtimes \mathcal{B} \arrow[r, "I_{\mathcal{A}\boxtimes\mathcal{B}}"] \arrow[dr, "U \circ I_{\mathcal{A}\boxtimes\mathcal{B}}"] & \mathcal{A} \overset{\mathrm{add}}{\boxtimes} \mathcal{B} \arrow[d, "U"] \\
& & \mathcal{C}
\end{tikzcd}$$

We have

$$
\begin{aligned}
(\Psi_{\mathcal{A},\mathcal{B},\mathcal{C}} \circ \Phi_{\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}})\left(U \xrightarrow{\beta} V\right) &= \Psi_{\mathcal{A},\mathcal{B},\mathcal{C}}\left((U \circ I_{\mathcal{A}\boxtimes\mathcal{B}}) \xrightarrow{\beta * I_{\mathcal{A}\boxtimes\mathcal{B}}} (V \circ I_{\mathcal{A}\boxtimes\mathcal{B}})\right) \\
&= \left(((U \circ I_{\mathcal{A}\boxtimes\mathcal{B}}) \circ M_{\mathcal{A},\mathcal{B}}) \xrightarrow{(\beta * I_{\mathcal{A}\boxtimes\mathcal{B}}) * M_{\mathcal{A},\mathcal{B}}} ((V \circ I_{\mathcal{A}\boxtimes\mathcal{B}}) \circ M_{\mathcal{A},\mathcal{B}})\right) \\
&= \left((U \circ (I_{\mathcal{A}\boxtimes\mathcal{B}} \circ M_{\mathcal{A},\mathcal{B}})) \xrightarrow{\beta * (I_{\mathcal{A}\boxtimes\mathcal{B}} \circ M_{\mathcal{A},\mathcal{B}})} (V \circ (I_{\mathcal{A}\boxtimes\mathcal{B}} \circ M_{\mathcal{A},\mathcal{B}}))\right) \\
&= \left((U \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}) \xrightarrow{\beta * M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}} (V \circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}})\right) \\
&= \Omega_{\mathcal{A},\mathcal{B},\mathcal{C}}\left(U \xrightarrow{\beta} V\right).
\end{aligned}
$$

Thus, we have the equivalence of categories $\Omega = \Omega_{\mathcal{A},\mathcal{B},\mathcal{C}} = \Psi_{\mathcal{A},\mathcal{B},\mathcal{C}} \circ \Phi_{\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}}$.

Now $\Phi = \Phi_{\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}}$ is surjective on objects; cf. Proposition 121.(3). Moreover, $\Psi = \Psi_{\mathcal{A},\mathcal{B},\mathcal{C}}$ is bijective on objects; cf. Theorem 130.(2). Thus, $\Omega = \Psi \circ \Phi$ is surjective on objects. $\square$

Chapter 4

Counterexamples for compatibility relations

4.1 Additive envelope and Karoubi envelope

Lemma 144. *There exists an idempotent complete preadditive category $\mathcal{A}$ such that* $\operatorname{Add}\mathcal{A}$ *is not idempotent complete.*

More precisely, consider the subring $\Lambda := \{(a,b) \in \mathbf{Z} \times \mathbf{Z} : a \equiv_5 b\}$ *of* $\mathbf{Z} \times \mathbf{Z}$ *and the full preadditive subcategory* $\mathcal{A}$ *of* Λ-free *with* $\operatorname{Ob}\mathcal{A} := \{\Lambda, 0\}$. *Then* $\mathcal{A}$ *is preadditive and idempotent complete, but* $\operatorname{Add}\mathcal{A}$ *is not idempotent complete.*

Proof. For $x \in \Lambda$ we also write x for the Λ-linear map $\Lambda \to \Lambda, y \mapsto yx$.

Suppose given an idempotent $(a,b) \in {}_{\mathcal{A}}(\Lambda,\Lambda)$. We have

$$(a^2, b^2) = (a,b)^2 = (a,b).$$

Since $a, b \in \mathbf{Z}$, we have $a, b \in \{0,1\}$. Thus, we have $(a,b) \in \{(1,1),(0,0)\}$ because $a \equiv_5 b$.

Therefore, we have

$$\operatorname{Idem}\mathcal{A} = \{\Lambda \xrightarrow{(1,1)} \Lambda, \Lambda \xrightarrow{(0,0)} \Lambda, 0 \xrightarrow{1_0} 0\}.$$

Furthermore,

$$\begin{array}{rcl} (\Lambda,(1,1),(1,1)) & \text{is an image of} & \Lambda \xrightarrow{(1,1)} \Lambda, \\ (0, 0_{\Lambda,0}, 0_{0,\Lambda}) & \text{is an image of} & \Lambda \xrightarrow{(0,0)} \Lambda, \\ (0, 1_0, 1_0) & \text{is an image of} & 0 \xrightarrow{1_0} 0. \end{array}$$

Thus, $\mathcal{A}$ is idempotent complete.

We want to show that $\operatorname{Add}\mathcal{A}$ is not idempotent complete. By Remarks 48 and 95, it suffices to show that Λ-free is not idempotent complete.

Consider $\Lambda^2 \xrightarrow{e := \left(\begin{smallmatrix} (1,26) & (0,10) \\ (0,-65) & (0,-25) \end{smallmatrix}\right)} \Lambda^2$ in Λ-free. We have

$$e^2 = \begin{pmatrix} (1,26) & (0,10) \\ (0,-65) & (0,-25) \end{pmatrix} \begin{pmatrix} (1,26) & (0,10) \\ (0,-65) & (0,-25) \end{pmatrix} = \begin{pmatrix} (1,26) & (0,10) \\ (0,-65) & (0,-25) \end{pmatrix} = e.$$

Thus, e is an idempotent.

Assume (X, π, ι) to be an image of e in Λ-free.

Consider

$$\Lambda\text{-free} \xrightarrow{G} \mathbf{Z}\text{-free}$$

$$\left(\Lambda^m \xrightarrow{(a_{ij}, b_{ij})_{i,j}} \Lambda^n\right) \mapsto \left(\mathbf{Z}^m \xrightarrow{(a_{ij})_{i,j}} \mathbf{Z}^n\right).$$

We show that G is a functor.

Suppose given $\Lambda^m \xrightarrow{((a_{ij}, b_{ij}))_{i,j}} \Lambda^n \xrightarrow{((c_{jk}, d_{jk}))_{j,k}} \Lambda^p$ in Λ-free. We have

$$G(1_{\Lambda^m}) = G(((\delta_{i,j}, \delta_{i,j}))_{i,j}) = (\delta_{i,j})_{i,j} = 1_{\mathbf{Z}^m} = 1_{G(\Lambda^m)}\,.$$

Furthermore, we have

$$\begin{aligned}
G\left(((a_{ij}, b_{ij}))_{i,j} \cdot ((c_{jk}, d_{jk}))_{j,k}\right) &= G\left(\sum_{j \in [1,n]} (a_{ij}, b_{ij})(c_{jk}, d_{jk})\right)_{i,k} \\
&= G\left(\left(\sum_{j \in [1,n]} a_{ij} c_{jk}\,,\ \sum_{j \in [1,n]} b_{ij} d_{jk}\right)\right)_{i,k} \\
&= \left(\sum_{j \in [1,n]} a_{ij} c_{jk}\right)_{i,k} \\
&= (a_{ij})_{i,j} \cdot (c_{jk})_{jk} \\
&= G(((a_{ij}, b_{ij}))_{i,j}) \cdot G(((c_{jk}, d_{jk}))_{j,k}).
\end{aligned}$$

Thus, G is a functor. Furthermore, G is bijective on objects.

By Remark 47, $(GX, G\pi, G\iota)$ is an image of Ge in $\mathbf{Z}$-free. We have

$$Ge = G\begin{pmatrix} (1,26) & (0,10) \\ (0,-65) & (0,-25) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have $GX = \mathbf{Z}$. Thus, we have $X = \Lambda$.

In consequence, we have

$$\pi = \begin{pmatrix} (u', v') \\ (u'', v'') \end{pmatrix} \quad \text{and} \quad \iota = \begin{pmatrix} (x', y') & (x'', y'') \end{pmatrix}$$

for some $(u', v'), (u'', v''), (x', y'), (x'', y'') \in \Lambda$.

Since (Λ, π, ι) is an image of e, the following diagram commutes.

$$\begin{tikzcd}
\Lambda^2 \arrow[rr, "\begin{pmatrix} (1,26) & (0,10) \\ (0,-65) & (0,-25) \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} (u',v') \\ (u'',v'') \end{pmatrix}"'] & & \Lambda^2 \arrow[dr, "\begin{pmatrix} (u',v') \\ (u'',v'') \end{pmatrix}"] & \\
& \Lambda \arrow[ur, "\begin{pmatrix} (x',y') & (x'',y'') \end{pmatrix}"] \arrow[rr, "(1,1)"] & & \Lambda
\end{tikzcd}$$

Therefore, we have
$$\begin{pmatrix} (u'x',v'y') & (u'x'',v'y'') \\ (u''x',v''y') & (u''x'',v''y'') \end{pmatrix} = \begin{pmatrix} (1,26) & (0,10) \\ (0,-65) & (0,-25) \end{pmatrix}$$
and
$$(x'u' + x''u'', y'v' + y''v'') = (1,1).$$
Thus, $u'x' = 1$. Since we may multiply π and ι by (-1), we may assume that $u' = x' = 1$.

We have $y' \equiv_5 x' = 1$ and $v' \equiv_5 u' = 1$.

Since we have $v'y' = 26$, we obtain $(v', y') = (1, 26)$ or $(v', y') = (26, 1)$.

Since we have $v'y'' = 10$, we obtain $v' = 1$ and therefore $y' = 26$.

But we have $v''y' = -65$, in *contradiction* to $y' = 26$.

Thus, e has no image in Λ-free. Therefore, Λ-free is not idempotent complete.

$\square$

Proposition 145. *There exists a preadditive category $\mathcal{A}$ such that*
$$\mathrm{Add}(\mathrm{Kar}\,\mathcal{A}) \not\simeq \mathrm{Kar}(\mathrm{Add}\,\mathcal{A}).$$
More precisely, consider the subring $\Lambda := \{(a,b) \in \mathbf{Z} \times \mathbf{Z} : a \equiv_5 b\}$ of $\mathbf{Z} \times \mathbf{Z}$ and the full preadditive subcategory $\mathcal{A}$ of Λ-free with $\mathrm{Ob}\,\mathcal{A} := \{\Lambda, 0\}$.

Then $\mathrm{Add}(\mathrm{Kar}\,\mathcal{A}) \not\simeq \mathrm{Kar}(\mathrm{Add}\,\mathcal{A})$.

Proof. *Assume* that $\mathrm{Add}(\mathrm{Kar}\,\mathcal{A}) \simeq \mathrm{Kar}(\mathrm{Add}\,\mathcal{A})$.

By Lemma 144, $\mathcal{A}$ is idempotent complete. Thus, $\mathcal{A} \xrightarrow{J_{\mathcal{A}}} \mathrm{Kar}\,\mathcal{A}$ is an equivalence; cf. Proposition 56.

Since $J_{\mathcal{A}}$ is an additive equivalence, cf. Lemma 44, so is $\mathrm{Add}\,\mathcal{A} \xrightarrow{\mathrm{Add}(J_{\mathcal{A}})} \mathrm{Add}(\mathrm{Kar}\,\mathcal{A})$; cf. Definition 100 and Lemma 101.

Thus, we have $\mathrm{Add}\,\mathcal{A} \simeq \mathrm{Add}(\mathrm{Kar}\,\mathcal{A}) \simeq \mathrm{Kar}(\mathrm{Add}\,\mathcal{A})$.

By Lemma 54, $\mathrm{Kar}(\mathrm{Add}\,\mathcal{A})$ is idempotent complete. Therefore, $\mathrm{Add}\,\mathcal{A}$ is idempotent complete, cf. Remark 48, in *contradiction* to Lemma 144.

Thus, $\mathrm{Add}(\mathrm{Kar}\,\mathcal{A}) \not\simeq \mathrm{Kar}(\mathrm{Add}\,\mathcal{A})$. $\square$

4.2 Additive envelope and tensor product

Lemma 146. *There exists a commutative ring R and additive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R such that $(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}, \varphi_{\mathcal{A}} \boxtimes \psi_{\mathcal{B}})$ is not additive.*

More precisely, consider $R = \mathbf{Q}$ and the full subcategory $\mathcal{A}$ of $\mathbf{Q}$-mod with
$$\mathrm{Ob}\,\mathcal{A} := \{V \in \mathrm{Ob}\,\mathbf{Q}\text{-mod} : \dim V \neq 1\}.$$
Then $\mathcal{A}$ is $\mathbf{Q}$-linear; cf. Remark 32.

Furthermore, $\mathcal{A}$ is an additive category over $\mathbf{Q}$, but $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A}$ is not additive.

Proof. Suppose given $A, B \in \operatorname{Ob} \mathcal{A}$. Then $\dim A \neq 1 \neq \dim B$. Since $\mathbf{Q}$-mod is additive, there exists a direct sum C of A and B in $\mathbf{Q}$-mod. Since $\dim A \neq 1 \neq \dim B$, we have $\dim C = \dim A + \dim B \neq 1$. Thus, we have $C \in \operatorname{Ob} \mathcal{A}$. Therefore, $\mathcal{A}$ is additive.

We have the $\mathbf{Q}$-linear functor

$$\begin{array}{rcl} \mathcal{A} & \xrightarrow{F} & \mathbf{Q}\text{-mod} \\ (A \xrightarrow{f} B) & \mapsto & (A \xrightarrow{f} B). \end{array}$$

Furthermore, we have the $\mathbf{Q}$-bilinear functor

$$\begin{array}{rcl} (\mathbf{Q}\text{-mod}) \times (\mathbf{Q}\text{-mod}) & \xrightarrow{G} & \mathbf{Q}\text{-mod} \\ \Big((A, B) \xrightarrow{(f,g)} (C, D)\Big) & \mapsto & \Big(A \underset{\mathbf{Q}}{\otimes} B \xrightarrow{f \otimes g} C \underset{\mathbf{Q}}{\otimes} D\Big). \end{array}$$

By Remark 29, we have the $\mathbf{Q}$-bilinear functor

$$\begin{array}{rcl} \mathcal{A} \times \mathcal{A} & \xrightarrow{G \circ (F \times F)} & \mathbf{Q}\text{-mod} \\ \Big((A, B) \xrightarrow{(f,g)} (C, D)\Big) & \mapsto & \Big(A \underset{\mathbf{Q}}{\otimes} B \xrightarrow{f \otimes g} C \underset{\mathbf{Q}}{\otimes} D\Big). \end{array}$$

By Definition 127, we have the R-linear functor

$$\begin{array}{rcl} \mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A} & \xrightarrow{H} & \mathbf{Q}\text{-mod} \\ \Big((A \boxtimes B) \xrightarrow{\sum_{i \in [1,n]} f_i \otimes g_i} (C \boxtimes D)\Big) & \mapsto & \Big(A \underset{\mathbf{Q}}{\otimes} B \xrightarrow{\sum_{i \in [1,n]} f_i \otimes g_i} C \underset{\mathbf{Q}}{\otimes} D\Big). \end{array}$$

Assume $\mathbf{Q}^2 \boxtimes \mathbf{Q}^2$ and $\mathbf{Q}^3 \boxtimes \mathbf{Q}^3$ to have a direct sum $X \boxtimes Y$ in $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A}$.

Since H is additive, $H(X \boxtimes Y) = X \underset{\mathbf{Q}}{\otimes} Y$ is a direct sum of

$$H(\mathbf{Q}^2 \boxtimes \mathbf{Q}^2) = \mathbf{Q}^2 \underset{\mathbf{Q}}{\otimes} \mathbf{Q}^2 \quad \text{and} \quad H(\mathbf{Q}^3 \boxtimes \mathbf{Q}^3) = \mathbf{Q}^3 \underset{\mathbf{Q}}{\otimes} \mathbf{Q}^3$$

in $\mathbf{Q}$-mod.

We have

$$\begin{aligned} (\dim X)(\dim Y) &= \dim(X \underset{\mathbf{Q}}{\otimes} Y) \\ &= \dim(\mathbf{Q}^2 \underset{\mathbf{Q}}{\otimes} \mathbf{Q}^2) + \dim(\mathbf{Q}^3 \underset{\mathbf{Q}}{\otimes} \mathbf{Q}^3) \\ &= 4 + 9 \\ &= 13. \end{aligned}$$

We obtain $\dim X = 1$ or $\dim Y = 1$, in *contradiction* to $X, Y \in \operatorname{Ob} \mathcal{A}$.

Thus, $\mathbf{Q}^2 \boxtimes \mathbf{Q}^2$ and $\mathbf{Q}^3 \boxtimes \mathbf{Q}^3$ have no direct sum in $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A}$.

Therefore, $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A}$ is not additive. $\square$

Proposition 147. *There exists a commutative ring R and preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R such that* $(\operatorname{Add}\mathcal{A}) \underset{R}{\boxtimes} (\operatorname{Add}\mathcal{B}) \not\simeq \operatorname{Add}(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B})$.

More precisely, consider $R = \mathbf{Q}$ and the full subcategory $\mathcal{A}$ of $\mathbf{Q}\text{-mod}$ *with*

$$\operatorname{Ob}\mathcal{A} := \{V \in \operatorname{Ob}\mathbf{Q}\text{-mod} : \dim V \neq 1\};$$

Then $\mathcal{A}$ is $\mathbf{Q}$-linear; cf. Remark 32.

Furthermore, $(\operatorname{Add}\mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Add}\mathcal{A}) \not\simeq \operatorname{Add}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$.

Proof. *Assume* that $(\operatorname{Add}\mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Add}\mathcal{A}) \simeq \operatorname{Add}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$.

By Lemma 146, $\mathcal{A}$ is additive. Thus, $\mathcal{A} \xrightarrow{I_{\mathcal{A}}} \operatorname{Add}\mathcal{A}$ is an equivalence; cf. Proposition 99.

By Lemma 117, $I_{\mathcal{A}}$ is $\mathbf{Q}$-linear. Therefore, we have the equivalence

$$\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A} \xrightarrow{I_{\mathcal{A}} \boxtimes I_{\mathcal{A}}} (\operatorname{Add}\mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Add}\mathcal{A});$$

cf. Lemma 136.

Thus, we have $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A} \simeq (\operatorname{Add}\mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Add}\mathcal{A}) \simeq \operatorname{Add}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$.

By Proposition 92, $\operatorname{Add}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$ is additive. Therefore, $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A}$ is additive, in *contradiction* to Lemma 146.

Thus, $(\operatorname{Add}\mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Add}\mathcal{A}) \not\simeq \operatorname{Add}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$. □

4.3 Karoubi envelope and tensor product

Lemma 148. *There exists a commutative ring R and idempotent complete preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R such that $(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}, \varphi_{\mathcal{A}} \boxtimes \psi_{\mathcal{B}})$ is not idempotent complete.*

More precisely, consider $R = \mathbf{Q}$ and the full subcategory of $\mathbf{Q}(\mathrm{i})\text{-mod}$ *with*

$$\operatorname{Ob}\mathcal{A} := \{\mathbf{Q}(\mathrm{i}), 0\}.$$

Then $\mathcal{A}$ is $\mathbf{Q}$-linear; cf. Remark 32.

Furthermore, $\mathcal{A}$ is idempotent complete, but $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A}$ is not idempotent complete.

Proof. For $x \in \mathbf{Q}(\mathrm{i})$ we also write x for the map $\mathbf{Q}(\mathrm{i}) \to \mathbf{Q}(\mathrm{i}), y \mapsto yx$.

Since $\mathbf{Q}(\mathrm{i})$ is a field, we have

$$\operatorname{Idem}\mathcal{A} = \{\mathbf{Q}(\mathrm{i}) \xrightarrow{1} \mathbf{Q}(\mathrm{i}),\ \mathbf{Q}(\mathrm{i}) \xrightarrow{0} \mathbf{Q}(\mathrm{i}),\ 0 \xrightarrow{1_0} 0\}.$$

Furthermore,

$$\begin{array}{ccc} (\mathbf{Q}(\mathrm{i}), 1, 1) & \text{is an image of} & \mathbf{Q}(\mathrm{i}) \xrightarrow{1} \mathbf{Q}(\mathrm{i}), \\ (0, 0_{\mathbf{Q}(\mathrm{i}),0}, 0_{0,\mathbf{Q}(\mathrm{i})}) & \text{is an image of} & \mathbf{Q}(\mathrm{i}) \xrightarrow{0} \mathbf{Q}(\mathrm{i}), \\ (0, 1_0, 1_0) & \text{is an image of} & 0 \xrightarrow{1_0} 0. \end{array}$$

Thus, $\mathcal{A}$ is idempotent complete.

We have
$$\operatorname{Ob}\left(\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A}\right) = \{\mathbf{Q}(\mathrm{i})\boxtimes\mathbf{Q}(\mathrm{i}), \mathbf{Q}(\mathrm{i})\boxtimes 0, 0\boxtimes\mathbf{Q}(\mathrm{i}), 0\boxtimes 0\}.$$
Since 0 is a zero object in $\mathcal{A}$, we have zero objects
$$\mathbf{Q}(\mathrm{i})\boxtimes 0 \cong 0\boxtimes\mathbf{Q}(\mathrm{i}) \cong 0\boxtimes 0$$
in $\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A}$; cf. Definition 123.

Consider $\mathbf{Q}(\mathrm{i})\boxtimes\mathbf{Q}(\mathrm{i}) \xrightarrow{e:=\frac{1}{2}(1\otimes 1+\mathrm{i}\otimes\mathrm{i})} \mathbf{Q}(\mathrm{i})\boxtimes\mathbf{Q}(\mathrm{i})$ in $\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A}$. We have
$$e^2 = \frac{1}{4}(1\otimes 1 + \mathrm{i}\otimes\mathrm{i} + \mathrm{i}\otimes\mathrm{i} + (-1)\otimes(-1)) = \frac{1}{4}(2(1\otimes 1) + 2(\mathrm{i}\otimes\mathrm{i})) = e.$$

Thus, e is an idempotent.

Since $\mathbf{Q}$ is a field and $(1, i)$ is a $\mathbf{Q}$-linear basis of $\mathbf{Q}(i)$, we have $1\otimes 1 \neq e \neq 0$.

In particular, $\mathbf{Q}(\mathrm{i})\boxtimes 0$, $0\boxtimes\mathbf{Q}(\mathrm{i})$ and $0\boxtimes 0$ cannot be images of e.

Assume $X := \mathbf{Q}(\mathrm{i})\boxtimes\mathbf{Q}(\mathrm{i})$ to be an image of e.

There exist $X \xrightarrow{\pi} X$ and $X \xrightarrow{\iota} X$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "\pi"'] & & X \arrow[dr, "\pi"] & \\
& X \arrow[ur, "\iota"] \arrow[rr, "1_X"] & & X
\end{tikzcd}$$

Since ${}_{\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A}}(X, X) = \mathbf{Q}(\mathrm{i})\underset{\mathbf{Q}}{\otimes}\mathbf{Q}(\mathrm{i})$ is commutative, we conclude
$$e = \pi\iota = \iota\pi = 1_X = 1\otimes 1,$$
a *contradiction.*

Thus, e has no image in $\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A}$. Therefore, $\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A}$ is not idempotent complete. $\square$

Proposition 149. *There exists a commutative ring R and preadditive categories $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ over R such that* $(\operatorname{Kar}\mathcal{A})\underset{R}{\boxtimes}(\operatorname{Kar}\mathcal{B}) \not\simeq \operatorname{Kar}(\mathcal{A}\underset{R}{\boxtimes}\mathcal{B})$.

More precisely, consider $R = \mathbf{Q}$ and the full subcategory of $\mathbf{Q}(\mathrm{i})$-mod *with* $\operatorname{Ob}\mathcal{A} := \{\mathbf{Q}(\mathrm{i}), 0\}$. *Then $\mathcal{A}$ is $\mathbf{Q}$-linear;* cf. Remark 32.

Furthermore, $(\operatorname{Kar}\mathcal{A})\underset{\mathbf{Q}}{\boxtimes}(\operatorname{Kar}\mathcal{A}) \not\simeq \operatorname{Kar}(\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A})$.

Proof. Assume that $(\operatorname{Kar}\mathcal{A})\underset{\mathbf{Q}}{\boxtimes}(\operatorname{Kar}\mathcal{A}) \simeq \operatorname{Kar}(\mathcal{A}\underset{\mathbf{Q}}{\boxtimes}\mathcal{A})$.

By Lemma 148, $\mathcal{A}$ is idempotent complete. Thus, $\mathcal{A} \xrightarrow{J_{\mathcal{A}}} \operatorname{Kar}\mathcal{A}$ is an equivalence; cf. Proposition 56.

By Lemma 84, $J_{\mathcal{A}}$ is $\mathbf{Q}$-linear.

By Lemma 136, we have the $\mathbf{Q}$-linear equivalence

$$\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A} \xrightarrow{J_{\mathcal{A}} \boxtimes J_{\mathcal{A}}} (\operatorname{Kar} \mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Kar} \mathcal{A}).$$

Thus, $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A} \simeq (\operatorname{Kar} \mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Kar} \mathcal{A}) \simeq \operatorname{Kar}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$.

Since $\operatorname{Kar}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$ is idempotent complete, cf. Lemma 54, so is $\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A}$; cf. Remark 48. This is a *contradiction* to Lemma 148.

Thus, $(\operatorname{Kar} \mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Kar} \mathcal{A}) \not\simeq \operatorname{Kar}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$. $\square$

Bibliography

[1] Atiyah, M.F., Macdonald, I.G., Introduction to Commutative Algebra, Addison-Wesley, 1969.

[2] Bakalov, B., Kirillov, A.A., Lectures on Tensor Categories and Modular Functors, AMS, 2001.

[3] Karoubi, M., Fondements de la K-théorie (Notes de séminaire), Faculté des Sciences d'Alger, 1966.

[4] Karoubi, M., K-Theory, Springer, 1978.

[5] Mac Lane, S., Categories for the Working Mathematician, Springer, 1971.

[6] Schubert, H., Categories, Springer, 1972.

[7] Stein, N., Adelman's abelianisation of an additive category (Bachelor's Thesis), Stuttgart, 2012.

Zusammenfassung

Karoubihülle

Wir beschreiben die Konstruktion der Karoubihülle einer additiven Kategorie, wie von Max Karoubi in [**3,** II.1] eingeführt; siehe auch [**4,** Theorem 6.10].

Sei $\mathcal{A}$ eine additive Kategorie. Wir konstruieren eine idempotentvollständige additive Kategorie $\operatorname{Kar}\mathcal{A}$ und einen additiven Funktor $J_{\mathcal{A}} : \mathcal{A} \to \operatorname{Kar}\mathcal{A}$ so, dass jeder additive Funktor F von $\mathcal{A}$ in eine idempotentvollständige additive Kategorie $\mathcal{B}$ eindeutig, bis auf Isomorphie, über $J_{\mathcal{A}}$ als $F' \circ J_{\mathcal{A}} = F$ faktorisiert.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "J_{\mathcal{A}}"] \arrow[dr, "F"'] & \operatorname{Kar}\mathcal{A} \arrow[d, dotted, "F'"] \\ & \mathcal{B} \end{tikzcd}$$

Genauer erhalten wir folgende Äquivalenz von Kategorien.

$$\begin{array}{ccc} {}_{\mathrm{add}}[\mathcal{A},\mathcal{B}] & \xleftarrow{\sim} & {}_{\mathrm{add}}[\operatorname{Kar}\mathcal{A},\mathcal{B}] \\ (U \circ J_{\mathcal{A}} \xrightarrow{\beta * J_{\mathcal{A}}} V \circ J_{\mathcal{A}}) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V). \end{array}$$

Diese Äquivalenz ist surjektiv auf Objekten; cf. Theorem 78.

Additive Hülle

Wir beschreiben die Konstruktion der additiven Hülle einer präadditiven Kategorie. Diese Konstruktion wird in [**5,** VII.2, ex. 6.(a)] und in [**2,** Def. 1.1.15] erwähnt.

Sei $\mathcal{A}$ eine präadditive, nicht notwendigerweise additive Kategorie. Wir konstruieren eine additive Kategorie $\operatorname{Add}\mathcal{A}$ und einen additiven Funktor $I_{\mathcal{A}} : \mathcal{A} \to \operatorname{Add}\mathcal{A}$ so, dass jeder additive Funktor F von $\mathcal{A}$ in eine additive Kategorie $\mathcal{B}$ eindeutig, bis auf Isomorphie, über $I_{\mathcal{A}}$ als $F' \circ I_{\mathcal{A}} = F$ faktorisiert.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "I_{\mathcal{A}}"] \arrow[dr, "F"'] & \operatorname{Add}\mathcal{A} \arrow[d, dotted, "F'"] \\ & \mathcal{B} \end{tikzcd}$$

Genauer erhalten wir folgende Äquivalenz von Kategorien.

$$\begin{array}{ccc} {}_{\mathrm{add}}[\mathcal{A},\mathcal{B}] & \xleftarrow{\sim} & {}_{\mathrm{add}}[\operatorname{Add}\mathcal{A},\mathcal{B}] \\ (U \circ I_{\mathcal{A}} \xrightarrow{\beta * I_{\mathcal{A}}} V \circ I_{\mathcal{A}}) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V). \end{array}$$

Diese Äquivalenz ist surjektiv auf Objekten; cf. Theorem 113.

Tensorprodukt präadditiver Kategorien

Wir beschreiben die Konstruktion des Tensorproduktes präadditiver Kategorien, wie in [**6,** 16.7.4] erwähnt.

Seien $\mathcal{A}$ und $\mathcal{B}$ präadditive Kategorien. Wir konstruieren eine präadditive Kategorie $\mathcal{A}\boxtimes\mathcal{B}$ und einen $\mathbf{Z}$-bilinearen Funktor $M_{\mathcal{A},\mathcal{B}} : \mathcal{A}\times\mathcal{B} \to \mathcal{A}\boxtimes\mathcal{B}$ so, dass jeder $\mathbf{Z}$-bilineare Funktor F von $\mathcal{A}\times\mathcal{B}$ in eine präadditive Kategorie $\mathcal{C}$ eindeutig über $M_{\mathcal{A},\mathcal{B}}$ als $\overline{F}\circ M_{\mathcal{A},\mathcal{B}} = F$ faktorisiert.

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B} \arrow[r, "M_{\mathcal{A},\mathcal{B}}"] \arrow[dr, "F"'] & \mathcal{A}\boxtimes\mathcal{B} \arrow[d, dotted, "\overline{F}"] \\
& \mathcal{C}
\end{tikzcd}$$

Genauer erhalten wir folgenden Isomorphismus von Kategorien.

$$\begin{array}{ccc}
{}_{\mathbf{Z}\text{-bil}}[\mathcal{A}\times\mathcal{B},\mathcal{C}] & \xleftarrow{\sim} & {}_{\mathrm{add}}[\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}] \\
(U\circ M_{\mathcal{A},\mathcal{B}} \xrightarrow{\beta * M_{\mathcal{A},\mathcal{B}}} V\circ M_{\mathcal{A},\mathcal{B}}) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V);
\end{array}$$

cf. Theorem 130.

Tensorprodukt additiver Kategorien

Wir beschreiben die Konstruktion des Tensorproduktes additiver Kategorien, wie in [**2,** Def. 1.1.15] erwähnt.

Seien $\mathcal{A}$ und $\mathcal{B}$ additive Kategorien. Wir konstuieren eine additive Kategorie $\mathcal{A}\overset{\mathrm{add}}{\boxtimes}\mathcal{B}$ und einen $\mathbf{Z}$-bilinearen Funktor $M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} : \mathcal{A}\times\mathcal{B} \to \mathcal{A}\overset{\mathrm{add}}{\boxtimes}\mathcal{B}$ so, dass jeder $\mathbf{Z}$-bilineare Funktor F von $\mathcal{A}\times\mathcal{B}$ in eine additive Kategorie $\mathcal{C}$ eindeutig, bis auf Isomorphie, über $M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}$ als $\overline{F}'\circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} = F$ faktorisiert.

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B} \arrow[r, "M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}"] \arrow[dr, "F"'] & \mathcal{A}\overset{\mathrm{add}}{\boxtimes}\mathcal{B} \arrow[d, dotted, "\overline{F}'"] \\
& \mathcal{C}
\end{tikzcd}$$

Genauer erhalten wir folgende Äquivalenz von Kategorien.

$$\begin{array}{ccc}
{}_{\mathbf{Z}\text{-bil}}[\mathcal{A}\times\mathcal{B},\mathcal{C}] & \xleftarrow{\sim} & {}_{\mathrm{add}}[\mathcal{A}\overset{\mathrm{add}}{\boxtimes}\mathcal{B},\mathcal{C}] \\
(U\circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}} \xrightarrow{\beta * M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}} V\circ M^{\mathrm{add}}_{\mathcal{A},\mathcal{B}}) & \longleftarrow\!\shortmid & (U \xrightarrow{\beta} V).
\end{array}$$

Diese Äquivalenz ist surjektiv auf Objekten; cf. Theorem 143.

Gegenbeispiele für Kompatibilitätsrelationen

Für präadditive Kategorien $\mathcal{A}$ gilt im Allgemeinen

$$\operatorname{Kar}(\operatorname{Add}\mathcal{A}) \not\simeq \operatorname{Add}(\operatorname{Kar}\mathcal{A}).$$

Genauer, betrachte den Teilring $\Lambda := \{(a,b) \in \mathbf{Z} \times \mathbf{Z} : a \equiv_5 b\}$ von $\mathbf{Z} \times \mathbf{Z}$ und die volle präadditive Teilkategorie $\mathcal{A}$ von Λ-free mit $\operatorname{Ob}\mathcal{A} := \{\Lambda, 0\}$.

Dann gilt $\operatorname{Add}(\operatorname{Kar}\mathcal{A}) \not\simeq \operatorname{Kar}(\operatorname{Add}\mathcal{A})$; cf. Proposition 145.

Für kommutative Ringe R und präadditive Kategorien $(\mathcal{A}, \varphi_{\mathcal{A}})$ und $(\mathcal{B}, \varphi_{\mathcal{B}})$ über R gilt im Allgemeinen

$$\operatorname{Add}(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}) \not\simeq (\operatorname{Add}\mathcal{A}) \underset{R}{\boxtimes} (\operatorname{Add}\mathcal{B}).$$

Genauer, betrachte $R = \mathbf{Q}$ und die volle Teilkategorie $\mathcal{A}$ von $\mathbf{Q}$-mod mit

$$\operatorname{Ob}\mathcal{A} := \{V \in \operatorname{Ob}\mathbf{Q}\text{-mod} : \dim V \neq 1\}.$$

Dann ist $\mathcal{A}$ $\mathbf{Q}$-linear; cf. Remark 32. Außerdem gilt $(\operatorname{Add}\mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Add}\mathcal{A}) \not\simeq \operatorname{Add}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$; cf. Proposition 147.

Für kommutative Ringe R und präadditive Kategorien $(\mathcal{A}, \varphi_{\mathcal{A}})$ und $(\mathcal{B}, \varphi_{\mathcal{B}})$ über R gilt im Allgemeinen

$$\operatorname{Kar}(\mathcal{A} \underset{R}{\boxtimes} \mathcal{B}) \not\simeq (\operatorname{Kar}\mathcal{A}) \underset{R}{\boxtimes} (\operatorname{Kar}\mathcal{B}).$$

Genauer, betrachte $R = \mathbf{Q}$ und die volle Teilkategorie $\mathcal{A}$ von $\mathbf{Q}(\mathrm{i})$-mod mit $\operatorname{Ob}\mathcal{A} := \{\mathbf{Q}(\mathrm{i}), 0\}$. Dann ist $\mathcal{A}$ $\mathbf{Q}$-linear; cf. Remark 32.

Außerdem gilt $(\operatorname{Kar}\mathcal{A}) \underset{\mathbf{Q}}{\boxtimes} (\operatorname{Kar}\mathcal{A}) \not\simeq \operatorname{Kar}(\mathcal{A} \underset{\mathbf{Q}}{\boxtimes} \mathcal{A})$; cf. Proposition 149.

Hiermit versichere ich,

(1) dass ich meine Arbeit selbstständig verfasst habe,

(2) dass ich keine anderen als die angegebenen Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,

(3) dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und

(4) dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, Februar 2016

Mathias Ritter