The minimal projective resolution of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$

Bachelor's Thesis

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Contents

Chapter 0

Introduction

0.1 Projective resolutions

Let A be a commutative ring. Let G be a finite group. Then A becomes a module over the group ring AG via the trivial group operation.

For instance, for $G = \mathrm{C}_n$, the cyclic group of order n, the $A\mathrm{C}_n$ module A has a periodic projective resolution over $A\mathrm{C}_n$; cf. Example 3.

$$\cdots \longrightarrow A\mathrm{C}_n \xrightarrow{1\mapsto(c-1)} A\mathrm{C}_n \xrightarrow{1\mapsto \sum_{i\in[0,n-1]} c^i} A\mathrm{C}_n \xrightarrow{1\mapsto(c-1)} A\mathrm{C}_n \longrightarrow 0$$

In [**1,** Theorems 5.14.2 and 5.14.5] Benson proposes a generalization to an arbitrary finite group G, where the trivial module does not have a periodic projective resolution in general. Instead, he constructs a projective resolution of the trivial module over G as a total complex of an n-fold complex in which the rows, columns, etc. are all periodic. This construction does not necessarily yield a minimal projective resolution.

0.2 Projective resolution of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$

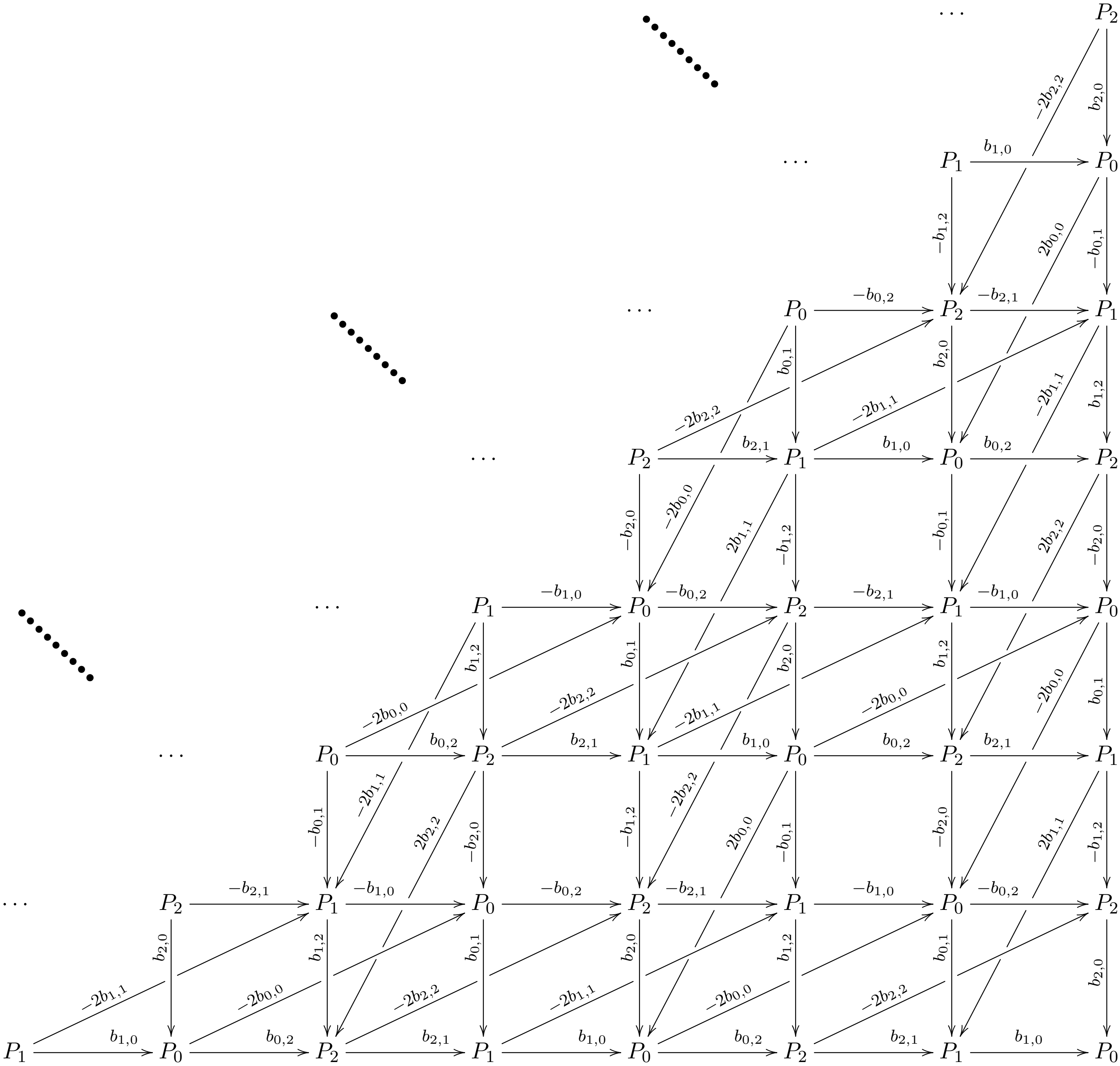
We consider the case of $G = \mathrm{A}_4$, the alternating group on four elements, and the ground ring $\mathbb{Z}_{(2)}[\zeta_3]$.

In [**5,** Example on page 50] Carlson gives a minimal projective resolution for $\mathbb{F}_4$ over $\mathbb{F}_4\mathrm{A}_4$. Note that $\mathbb{F}_4 \cong \mathbb{Z}_{(2)}[\zeta_3]/(2)$.

We construct the minimal projective resolution Q of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$, which shows regular behavior; cf. Theorem 28.

$$Q = \left(\cdots \longrightarrow Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \longrightarrow 0 \right)$$

The appearing maps can be visualized in the following diagram, where every row and column is periodic.



For $k \in \mathbb{N}$, the projective modules Q_k of the projective resolution are a direct sum of the indecomposable projective modules P_0, P_1 and P_2 of $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$; cf. Lemma 23. More precisely, Q_k is the direct sum of the entries in the k-th diagonal of the above diagram, starting with $Q_0 = P_0$; cf. Definition 25.

$$Q = \left(\cdots \xrightarrow{d_4} P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \xrightarrow{d_3} P_0 \oplus P_1 \oplus P_2 \oplus P_0 \xrightarrow{d_2} P_2 \oplus P_0 \oplus P_1 \xrightarrow{d_1} P_1 \oplus P_2 \xrightarrow{d_0} P_0 \longrightarrow 0 \right)$$

Furthermore, the differentials d_k are matrices having as entries multiples of the $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$-linear maps $b_{i,j}$ for $i, j \in \{0, 1, 2\}$ as indicated in the diagram; cf. Definition 25.

After reduction modulo 2, all diagonal maps vanish and we are in the double complex situation as described by Benson.

To construct the projective resolution Q of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$ we find the projective indecomposable modules P_0, P_1 and P_2 of $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$ as projective indecomposable modules of the Wedderburn image Λ of $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$; cf. §0.4 and Definition 23. We define the $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$-linear maps $b_{i,j}$ for $i, j \in \{0, 1, 2\}$ by multiplication with suitable basis elements of Λ; cf. Lemma 24.

To verify that Q is in fact a projective resolution of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$ we construct $\mathbb{Z}_{(2)}[\zeta_3]$-linear homotopies; cf. Definition 30.

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & Q_3 \arrow[r, "d_2"] \arrow[d, "1"] & Q_2 \arrow[r, "d_1"] \arrow[d, "1"] \arrow[dl, "h_2"'] & Q_1 \arrow[r, "d_0"] \arrow[d, "1"] \arrow[dl, "h_1"'] & Q_0 \arrow[r, "\varepsilon"] \arrow[d, "1"] \arrow[dl, "h_0"'] & \mathbb{Z}_{(2)}[\zeta_3] \arrow[d, "1"] \arrow[dl, "h_{-1}"'] \\
\cdots \arrow[r, dotted] & Q_3 \arrow[r, "d_2"] & Q_2 \arrow[r, "d_1"] & Q_1 \arrow[r, "d_0"] & Q_0 \arrow[r, "\varepsilon"] & \mathbb{Z}_{(2)}[\zeta_3]
\end{tikzcd}$$

These homotopies show a regular behavior as well.

0.3 Attempt to projectively resolve $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{A}_4$

Using the projective resolution of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$ described in §0.2, we attempt to construct a projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{A}_4$.

We find the projective indecomposable modules $\check{P}_0$ and $\check{P}_1$ of $\mathbb{Z}_{(2)}\mathrm{A}_4$ as projective indecomposable modules of the Wedderburn image $\check{\Lambda}$ of $\mathbb{Z}_{(2)}\mathrm{A}_4$; cf. §0.4 and Definition 35.

In §4.2 we give the first terms $\check{Q}$ of an augmented projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{A}_4$.

$$\check{Q} := \left(\check{Q}_6 \xrightarrow{\check{d}_5} \check{Q}_5 \xrightarrow{\check{d}_4} \check{Q}_4 \xrightarrow{\check{d}_3} \check{Q}_3 \xrightarrow{\check{d}_2} \check{Q}_2 \xrightarrow{\check{d}_1} \check{Q}_1 \xrightarrow{\check{d}_0} \check{Q}_0 \xrightarrow{\check{\varepsilon}} \mathbb{Z}_{(2)} \right) =$$

$$\left(\check{P}_0{\oplus}\check{P}_0{\oplus}\check{P}_1{\oplus}\check{P}_1{\oplus}\check{P}_0 \xrightarrow{\check{d}_5} \check{P}_1{\oplus}\check{P}_0{\oplus}\check{P}_0{\oplus}\check{P}_1 \xrightarrow{\check{d}_4} \check{P}_1{\oplus}\check{P}_1{\oplus}\check{P}_0 \xrightarrow{\check{d}_3} \check{P}_0{\oplus}\check{P}_0{\oplus}\check{P}_1 \xrightarrow{\check{d}_2} \check{P}_1{\oplus}\check{P}_0 \xrightarrow{\check{d}_1} \check{P}_1 \xrightarrow{\check{d}_0} \check{P}_0 \xrightarrow{\check{\varepsilon}} \mathbb{Z}_{(2)} \right)$$

To verify that $\check{Q}$ is in fact an exact sequence, we show that

$$\mathbb{Z}_{(2)}[\zeta_3] \underset{\mathbb{Z}_{(2)}}{\otimes} \check{Q} \cong \left(Q_6 \xrightarrow{d_5} Q_5 \xrightarrow{d_4} Q_4 \xrightarrow{d_3} Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \xrightarrow{\varepsilon} \mathbb{Z}_{(2)}[\zeta_3] \right);$$

cf. Remark 44.

In §4.3 we consider P_0, P_1 and P_2 as modules over $\check{\Lambda}$ via an injective ring homomorphism $\iota : \check{\Lambda} \to \Lambda$ to obtain an augmented projective resolution

$$\cdots \longrightarrow Q_4{\downarrow_\iota} \xrightarrow{d_3} Q_3{\downarrow_\iota} \xrightarrow{d_2} Q_2{\downarrow_\iota} \xrightarrow{d_1} Q_1{\downarrow_\iota} \xrightarrow{d_0} Q_0{\downarrow_\iota} \longrightarrow \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$$

of $\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{A}_4$; cf. Corollary 52.

We construct isomorphisms $\check{P}_0 \xrightarrow{\sim} P_0{\downarrow_\iota}$, $\check{P}_1 \xrightarrow{\sim} P_1{\downarrow_\iota}$ and $\check{P}_1 \xrightarrow{\sim} P_2{\downarrow_\iota}$. This allows us to show that

$$\left(\check{Q}_3 \xrightarrow{\check{d}_2} \check{Q}_2 \xrightarrow{\check{d}_1} \check{Q}_1 \xrightarrow{\check{d}_0} \check{Q}_0 \xrightarrow{\check{\varepsilon}} \mathbb{Z}_{(2)} \right)^{\oplus 2} \cong \left(Q_3{\downarrow_\iota} \xrightarrow{d_2} Q_2{\downarrow_\iota} \xrightarrow{d_1} Q_1{\downarrow_\iota} \xrightarrow{d_0} Q_0{\downarrow_\iota} \longrightarrow \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)} \right);$$

cf. Remark 54.

Of course, we would have liked to construct a minimal augmented projective resolution $\check{Q}'$ of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{A}_4$, not only the first few terms. But, using the augmented projective resolution Q' of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$, neither the condition

$$\mathbb{Z}_{(2)}[\zeta_3] \underset{\mathbb{Z}_{(2)}}{\otimes} \check{Q}' \overset{!}{\cong} Q'$$

nor the condition

$$Q'\!\downarrow_\iota \overset{!}{\cong} \check{Q}' \oplus \check{Q}'$$

allowed us to detect a regularly behaving $\check{Q}'$.

0.4 Wedderburn images

To perform the constructions described in §0.2 and §0.3, we construct Wedderburn isomorphisms and describe the Wedderburn images via congruences in §2.

Write $\zeta := \zeta_3$.

Let $\check{\Gamma}' := \mathbb{Z} \times \mathbb{Z}[\zeta] \times \mathbb{Z}^{3\times 3}$. We have

$$\mathbb{Z} \underset{\sim}{\xrightarrow{\check{\omega}'}} \check{\Lambda}' = \left\{ \left(u, r+s\zeta, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) \in \check{\Gamma}' : \begin{array}{c} u \equiv_3 r+s,\ u \equiv_4 i,\ g \equiv_4 0,\ h \equiv_4 0, \\ e-a \equiv_4 s \equiv_2 d,\ r-e \equiv_4 b+d \equiv_2 0 \end{array} \right\} \subseteq \check{\Gamma}'\,;$$

cf. Lemma 5.

By localization, i.e., roughly put, by retaining only the congruences modulo 2 respectively 4, we obtain $\mathbb{Z}_{(2)}\mathrm{A}_4 \underset{\sim}{\xrightarrow{\check{\omega}}} \check{\Lambda}$; cf. Corollary 7.

Let $\Gamma' := \mathbb{Z}[\zeta] \times \mathbb{Z}[\zeta] \times \mathbb{Z}[\zeta] \times \mathbb{Z}[\zeta]^{3\times 3}$. We have

$$\mathbb{Z}[\zeta_3]\mathrm{A}_4 \underset{\sim}{\xrightarrow{\omega'}} \Lambda' = \left\{ \left(u, r, s, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) \in \Gamma' : \begin{array}{c} b \equiv_2 c \equiv_2 d \equiv_2 f \equiv_2 g \equiv_2 h \equiv_2 0 \\ u \equiv_4 i,\ r \equiv_4 e,\ s \equiv_4 a \\ u+r+s \equiv_3 0,\ r \equiv_{1-\zeta} s \end{array} \right\} \subseteq \Gamma'\,;$$

cf. Lemma 13.

By localization and further conjugation, we obtain $\mathbb{Z}_{(2)}[\zeta]\mathrm{A}_4 \underset{\sim}{\xrightarrow{\omega}} \Lambda$; cf. Corollary 15.

In addition to a direct proof we also give a construction for ω based on $\check{\omega}$ via scalar extension and conjugation; cf. Lemma 10. The isomorphism obtained this way accomplishes a compatibility with Galois conjugation in sense of Lemma 18.

Let $R := \mathbb{Z}_{(2)}$. Let $\otimes := \underset{R}{\otimes}$.

The overall situation can be depicted as follows; cf. Lemma 10.

$$\begin{tikzcd}
R\mathrm{A}_4 \arrow[r, "\breve{\omega}", "\sim"', very thick] \arrow[d] & \breve{\Lambda} \arrow[rr, hook] \arrow[dd, bend left] & & \breve{\Gamma} = R \times R[\zeta] \times R^{3\times 3} \arrow[d] & \\
R[\zeta] \otimes R\mathrm{A}_4 \arrow[rrr, "R[\zeta]\otimes\breve{\omega}_+"] \arrow[dr, "\sim", "R[\zeta]\otimes\breve{\omega}"'] \arrow[ddd, "\psi"', "\wr"] & & & R[\zeta] \otimes \big(R \times R[\zeta] \times R^{3\times 3}\big) \arrow[dd, "\psi_1'"', "\wr"] & \\
 & R[\zeta] \otimes \breve{\Lambda} \arrow[urr, hook, "R[\zeta]\otimes\iota_1"'] \arrow[ddd, "\psi'"', "\wr"] & & & \\
 & & & R[\zeta] \times (R[\zeta] \otimes R[\zeta]) \times R[\zeta]^{3\times 3} \arrow[d, "\psi_2'"', "\wr"] & \\
R[\zeta]\mathrm{A}_4 \arrow[rrr, "\widetilde{\omega}"'] \arrow[dr, "\sim", "\widetilde{\omega}|^{\widetilde{\Lambda}}"'] \arrow[ddd, "1"'] & & & \Gamma = R[\zeta] \times R[\zeta] \times R[\zeta] \times R[\zeta]^{3\times 3} \arrow[r, hook] \arrow[dd, "\kappa|_{\Gamma}^{\Gamma\kappa}"', "\wr"] & \mathbb{Q}\Gamma \arrow[ddd, "\kappa"', "\wr"] \\
 & \widetilde{\Lambda} \arrow[urr, hook] \arrow[d, "\kappa|_{\widetilde{\Lambda}}^{\widetilde{\Lambda}\kappa}"', "\wr"] & & & \\
 & \Lambda \arrow[rr, hook] \arrow[drr, "\iota_2"] & & \Gamma\kappa \arrow[dr, hook] & \\
R[\zeta]\mathrm{A}_4 \arrow[ur, "\omega", "\sim"', very thick] \arrow[rrr, "\omega_+"] & & & \Gamma \arrow[r, hook] & \mathbb{Q}\Gamma
\end{tikzcd}$$

0.5 Conventions

Let X, Y and Z be sets. Let A be a commutative ring and let B be a ring.

- In the case of $X \cap Y = \emptyset$, we often write $X \sqcup Y$ for $X \cup Y$.
- Let $a, b \in \mathbb{Z}$. We write $[a, b] := \{i \in \mathbb{Z} : a \leq i \leq b\}$.
- Given $x, y \in X$, let $\delta_{x,y} = 1$ for $x = y$ and $\delta_{x,y} = 0$ for $x \neq y$.
- We write maps on the right. This means, given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $x \in X$ we denote the image of x under f by xf and the composite of f and g by $X \xrightarrow{fg} Z$.
- Let $X \xrightarrow{f} Y$ be a map. Given $X' \subseteq X$ and $Y' \subseteq Y$ such that $X'f \subseteq Y'$, we write $f|_{X'}^{Y'}$ for the restriction of f to X' and Y'. In the case of $Y = Y'$ we also write $f|_{X'} := f|_{X'}^{Y}$. In the case of $X = X'$ we also write $f|^{Y'} := f|_{X}^{Y'}$.
- Let $X' \subseteq X$ be a subset. We often write $X' \hookrightarrow X$ for the embedding.
- We denote the symmetric group on the set X by S_{X}.
- Let $a, b, z \in A$. We write $a \equiv_z b$ if there exists a $c \in A$ such that $a - b = cz$.
- We write $\mathrm{U}(B)$ for the group of units of B.

- If not specified otherwise, by a B-module we understand a left B-module.
- Let M be a B-module. We write
$$\mathrm{rad}(M) := \bigcap \{N : N \text{ is a maximal submodule of } M\}$$
for the radical of M.
- Let e and f be two idempotent elements of B. We identify along
$$\begin{aligned} \mathrm{Hom}_B(Be, Bf) &\xrightarrow{\sim} eBf \\ \varphi &\longmapsto e\varphi \\ (\mu e \mapsto \mu e\lambda f) &\longleftarrow e\lambda f. \end{aligned}$$
- When writing a complex of B-modules, we often omit to denote zero objects therein.
- Let $n, m \in \mathbb{N}$. We denote the A-module of $n \times m$ matrices over A by $A^{n\times m}$. In the case of $n = m$ we denote the identity matrix of $A^{n\times n}$ by I_n.
- Let $n, m \in \mathbb{N}$. The standard A-linear basis $(e_{i,j})_{i\in[1,n],\, j\in[1,m]}$ of $A^{n\times m}$ consists of the $n \times m$ matrices $e_{i,j}$ having the entry 1 at position (i, j) and 0 elsewhere.
- Let $n \in \mathbb{N}$ and $k_r \in \mathbb{N}$ for $r \in [1, n]$. The standard A-linear basis of $\prod\limits_{r\in[1,n]} A^{k_r\times k_r}$ consists of $(e_{i,j;r})_{i,j\in[1,k_r],\, r\in[1,n]}$, where $e_{i,j;r}$ is the tuple of matrices, whose matrix at position r has the entry 1 at position (i, j), and whose other matrix entries are 0.
- The standard $\mathbb{Z}$-linear basis of $\mathbb{Z}[\zeta_3]$ consists of $(1, \zeta_3)$.
- Let $n \in \mathbb{N}$ and $M, T \in A^{n\times n}$ with T being invertible. We write $A^T := T^{-1}AT$ for the conjugation of A with T.
- For $a \in A$ we write $(a) := \{\lambda a : \lambda \in A\} \subseteq A$, for the ideal generated by this element.
- By an A-order, we understand a finitely generated free A-algebra.

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Chapter 1

Preliminaries

Remark 1 Let A be a ring. Let G be a group. We recall the following facts.

(1) For every $m \in M$, where M is an A-module, the map

$$\varphi : A \to M, \quad a \mapsto a \cdot m$$

is an A-linear map.

(2) Let M be an A-module and F a free A-module with basis $(b_1, \ldots, b_n)$ where $n \in \mathbb{N}$. For every $m_1, \ldots, m_n \in M$ the map

$$\psi : F \to M, \quad \sum_{i \in [1,n]} a_i b_i \mapsto \sum_{i \in [1,n]} a_i m_i$$

is a homomorphism of A-modules.

(3) Suppose that A is commutative. Let M be an A-module. For every group homomorphism $\varphi : G \to \operatorname{Aut}_A(M)$ the map

$$\tilde{\varphi} : AG \to \operatorname{End}_A(M), \quad \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g \varphi(g)$$

is a ring homomorphism.

(4) Suppose that A is commutative. Then A together with

$$AG \times A \to A, \quad \left(\sum_{g \in G} a_g g, s \right) \mapsto \sum_{g \in G} a_g s$$

is an AG-module, called the trivial module.

Lemma 2 Let A be a ring. Let Λ be an A-algebra.

A complex of Λ-modules

$$\cdots \to X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{d_{-1}} X_{-1} \to \cdots$$

is acyclic if there exist A-linear maps $h_i : X_i \to X_{i+1}$ such that $h_i d_i + d_{i-1} h_{i-1} = 1$ for every X_i .

$$\begin{tikzcd}
{} \arrow[r, dotted] & X_2 \arrow[r, "d_1"] \arrow[d, "1"'] & X_1 \arrow[r, "d_0"] \arrow[d, "1"'] \arrow[dl, "h_1"'] & X_0 \arrow[r, "d_{-1}"] \arrow[d, "1"'] \arrow[dl, "h_0"'] & X_{-1} \arrow[r, dotted] \arrow[d, "1"'] \arrow[dl, "h_{-1}"'] & {} \\
{} \arrow[r, dotted] & X_2 \arrow[r, "d_1"] & X_1 \arrow[r, "d_0"] & X_0 \arrow[r, "d_{-1}"] & X_{-1} \arrow[r, dotted] & {}
\end{tikzcd}$$

Proof. We show that $\mathrm{Ker}(d_i) = \mathrm{Im}(d_{i+1})$ for every $i \in \mathbb{Z}$.

(1) Let $x \in \mathrm{Ker}(d_i) \subseteq X_{i+1}$ so that $xd_i = 0$. Using that h_i is A-linear we have

$$x = x\,\mathrm{id} = xh_{i+1}d_{i+1} + xd_i h_i = xh_{i+1}d_{i+1} + 0h_i = xh_{i+1}d_{i+1} \in X_{i+2}d_{i+1}\,.$$

Thus $x \in \mathrm{Im}(d_{i+1})$.

(2) Let $x \in \mathrm{Im}(d_{i+1})$. Hence there exists a $y \in X_{i+2}$ with $yd_{i+1} = x$.
So $xd_i = yd_{i+1}d_i = 0$ since $d_{i+1}d_i = 0$. Thus $x \in \mathrm{Ker}(d_i)$.

$\square$

Example 3 Let A be a commutative ring. Let $\mathrm{C}_n = \langle c : c^n = 1 \rangle$ be the cyclic group of order n.

Consider the maps

$$\begin{aligned}
\varepsilon &: A\mathrm{C}_n \to A, \quad\;\; 1 \mapsto 1 \\
d_1 &: A\mathrm{C}_n \to A\mathrm{C}_n,\ 1 \mapsto (c-1) \\
d_2 &: A\mathrm{C}_n \to A\mathrm{C}_n,\ 1 \mapsto \sum_{i \in [0,n-1]} c^i\,.
\end{aligned}$$

Then

$$X := \left(\begin{tikzcd}[column sep=large] {} \arrow[r, dotted, "d_1"] & A\mathrm{C}_n \arrow[r, "d_2", "1 \mapsto \sum_{i\in[0,n-1]} c^i"'] & A\mathrm{C}_n \arrow[r, "d_1", "1\mapsto(c-1)"'] & A\mathrm{C}_n \arrow[r, "\varepsilon", "1\mapsto 1"'] & A \arrow[r, dotted] & 0 \end{tikzcd} \right)$$

is an augmented projective resolution of A as an $A\mathrm{C}_n$-module. Note that it is periodic.

Proof. By Remark 1 the maps ε, d_1 and d_2 are all well defined homomorphisms of $A\mathrm{C}_n$-modules. In addition $A\mathrm{C}_n$ is projective as a free module.

We show that X is a periodic complex.

We have

$1d_1\varepsilon = (c-1)\varepsilon = 1 - 1 = 0$

$$1d_2d_1 = \left(\sum_{i\in[0,n-1]} c^i\right) d_1 = \left(\sum_{i\in[0,n-1]} c^i\right)(c-1) = \left(\sum_{i\in[0,n-1]} (c^{i+1}-c^i)\right) = c^n - c^0 = 0$$

$$1d_1d_2 = (c-1)d_2 = \left(\sum_{i\in[0,n-1]} c^{i+1}\right) - \left(\sum_{i\in[0,n-1]} c^i\right) = c^n - c^0 = 0$$

and X is a complex since ε, d_1 and d_2 are $A\mathrm{C}_n$-linear maps.

We show that X is acyclic by using Lemma 2. Therefore we have to find A-linear maps $h_i : A\mathrm{C}_n \to A\mathrm{C}_n$ for $i \in \{1,2\}$ and $h_0 : A \to A\mathrm{C}_n$ where $h_0\varepsilon = 1$, $h_1d_1 + \varepsilon h_0 = 1$, $h_2d_2 + d_1h_1 = 1$ and $h_1d_1 + d_2h_2 = 1$.

Using Remark 1 we define

$$\begin{aligned}
&h_0 : A \to A\mathrm{C}_n,\ 1 \mapsto 1\\
&h_1 : A\mathrm{C}_n \to A\mathrm{C}_n,\ c^i \mapsto \sum_{j\in[0,i-1]} c^j,\ \text{for } i \in [0,n-1]\\
&h_2 : A\mathrm{C}_n \to A\mathrm{C}_n,\ c^i \mapsto \delta_{i,n-1},\ \text{for } i \in [0,n-1]
\end{aligned}$$

as shown in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & A\mathrm{C}_n \arrow[r, "d_1", "1\mapsto(c-1)"'] \arrow[d, "1"'] & A\mathrm{C}_n \arrow[r, "d_2", "1\mapsto \sum_{i\in[0,n-1]} c^i"'] \arrow[d, "1"] \arrow[dl, "h_1 : c^i \mapsto \sum_{j\in[0,i-1]} c^j"'] & A\mathrm{C}_n \arrow[r, "d_1", "1\mapsto(c-1)"'] \arrow[d, "1"] \arrow[dl, "h_2 : c^i \mapsto \delta_{i,n-1}"'] & A\mathrm{C}_n \arrow[r, "\varepsilon", "1\mapsto 1"'] \arrow[d, "1"] \arrow[dl, "h_1 : c^i \mapsto \sum_{j\in[0,i-1]} c^j"'] & A \arrow[r] \arrow[d, "1"] \arrow[dl, "h_0 : 1 \mapsto 1"'] & 0 \arrow[r, dotted] \arrow[d, "1"] \arrow[dl] & \cdots \\
\cdots \arrow[r, dotted] & A\mathrm{C}_n \arrow[r, "d_1", "1\mapsto(c-1)"'] & A\mathrm{C}_n \arrow[r, "d_2", "1\mapsto \sum_{i\in[0,n-1]} c^i"'] & A\mathrm{C}_n \arrow[r, "d_1", "1\mapsto(c-1)"'] & A\mathrm{C}_n \arrow[r, "\varepsilon", "1\mapsto 1"'] & A \arrow[r] & 0 \arrow[r, dotted] & \cdots
\end{tikzcd}$$

Since $\varepsilon, d_1, d_2, h_0, h_1$ and h_2 are A-linear maps it suffices to examine c^i for all $i \in [0, n-1]$. Given $i \in \{0, \dots, n-1\}$ we have

$$1h_0\varepsilon = 1\varepsilon = 1$$

$$\begin{aligned}
c^ih_1d_1 + c^i\varepsilon h_0 &= \left(\sum_{j\in[0,i-1]} c^j\right) d_1 + 1\,h_0 = \left(\sum_{j\in[0,i-1]} (c^{j+1}-c^j)\right) + 1\\
&= c^i - c^0 + 1 = c^i
\end{aligned}$$

$$\begin{aligned}
c^ih_2d_2 + c^id_1h_1 &= (\delta_{i,n-1})d_2 + \left(c^{i+1} - c^i\right) h_1\\
&= \delta_{i,n-1} \sum_{j\in[0,n-1]} c^j + (1-\delta_{i,n-1}) \left(\sum_{j\in[0,i]} c^j\right) - \sum_{j\in[0,i-1]} c^j\\
&= \begin{cases} 0 + \sum_{j\in[0,i]} c^j - \sum_{j\in[0,i-1]} c^j = c^i & , i \neq n-1\\ \sum_{j\in[0,n-1]} c^j + 0 - \sum_{j\in[0,n-2]} c^j = c^{n-1} & , i = n-1 \end{cases}
\end{aligned}$$

$$\begin{aligned} c^i h_1 d_1 + c^i d_2 h_2 &= \left(\sum_{j \in [0, i-1]} c^j \right) d_1 + \left(\sum_{j \in [0, n-1]} c^{i+j} \right) h_2 \\ &= \sum_{j \in [0, i-1]} (c^{j+1} - c^j) + 1 = c^i - c^0 + 1 = c^i \,. \end{aligned}$$

So by Lemma 2 the complex X is acyclic and an augmented projective resolution of A over $A\mathrm{C}_n$. □

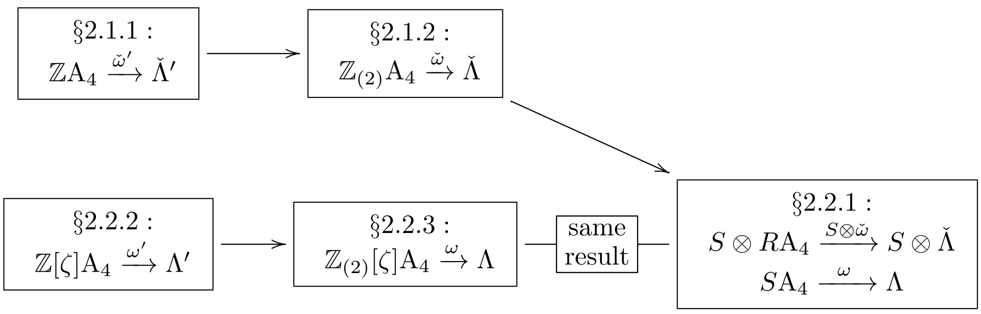
Chapter 2

Wedderburn images

Let $\zeta := \zeta_3$ be a third primitive root of unity over $\mathbb{Q}$.

Let $R := \mathbb{Z}_{(2)}$ and $S := \mathbb{Z}_{(2)}[\zeta]$ so that $R \subseteq S$. Let $\otimes := \underset{R}{\otimes}$.

> We aim to construct the Wedderburn images of $R\mathrm{A}_4$ and $S\mathrm{A}_4$.
>
> In §2.1.1 we determine the Wedderburn isomorphism $\mathbb{Z}\mathrm{A}_4 \xrightarrow{\breve{\omega}'} \breve{\Lambda}'$ and its image $\breve{\Lambda}'$.
>
> In §2.1.2 we restrict $\breve{\omega}'$ to obtain the Wedderburn isomorphism $R\mathrm{A}_4 \xrightarrow{\breve{\omega}} \breve{\Lambda}$ and its image $\breve{\Lambda}$.
>
> In §2.2.1 we construct the Wedderburn isomorphism $S\mathrm{A}_4 \xrightarrow{\omega} \Lambda$ from $\breve{\omega}$.
>
> In §2.2.2 we determine independently of §2.2.1 the Wedderburn isomorphism $\mathbb{Z}[\zeta]\mathrm{A}_4 \xrightarrow{\omega'} \Lambda'$ and its image Λ'.
>
> In §2.2.3 we restrict and conjugate ω' to obtain the same Wedderburn isomorphism $S\mathrm{A}_4 \xrightarrow{\omega} \Lambda$ as in §2.2.1.
>
> This proceeding can be outlined as follows.



Remark 4 We have the following presentation of A_4.

$$\begin{aligned} \mathrm{A}_4 &\xleftarrow{\sim} \langle \alpha, \beta : \alpha^2, \beta^3, (\alpha\beta)^3 \rangle \\ (1,2)(3,4) &\leftarrow\!\shortmid \alpha \\ (1,2,3) &\leftarrow\!\shortmid \beta \end{aligned}$$

Proof. Note that

$$((1,2)(3,4))^2 = \mathrm{id}$$

$$\begin{aligned}(1,2,3)^3 &= \mathrm{id}\\ ((1,2)(3,4)(1,2,3))^3 &= \mathrm{id}\end{aligned}$$

so that

$$\begin{aligned}\mathrm{A}_4 &\overset{\rho}{\leftarrow} \langle \alpha, \beta : \alpha^2, \beta^3, (\alpha\beta)^3\rangle\\ (1,2)(3,4) &\leftarrow\!\shortmid\ \alpha\\ (1,2,3) &\leftarrow\!\shortmid\ \beta\end{aligned}$$

is a surjective group homomorphism. Using the computer algebra system Magma, cf. [2], we calculate $\left|\langle \alpha, \beta : \alpha^2, \beta^3, (\alpha\beta)^3\rangle\right| = 12 = |\mathrm{A}_4|$. Therefore ρ must be bijective. ▫

2.1 Wedderburn image of $\mathbb{Z}_{(2)}\mathrm{A}_4$

2.1.1 Wedderburn image of $\mathbb{Z}\mathrm{A}_4$

Lemma 5 Let $\check{\Gamma}' := \mathbb{Z} \times \mathbb{Z}[\zeta] \times \mathbb{Z}^{3\times 3}$ and

$$\check{\Lambda}' := \left\{ \left(u, r+s\zeta, \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\right) \in \check{\Gamma}' : \begin{matrix} u \equiv_3 r+s,\ u \equiv_4 i,\ g \equiv_4 0,\ h \equiv_4 0,\\ e-a \equiv_4 s \equiv_2 d,\ r-e \equiv_4 b+d \equiv_2 0\end{matrix} \right\} \subseteq \check{\Gamma}'.$$

We have the isomorphism of $\mathbb{Z}$-algebras

$$\begin{aligned}\mathbb{Z}\mathrm{A}_4 &\xrightarrow[\sim]{\check{\omega}'} \check{\Lambda}'\\ (1,2)(3,4) &\mapsto \left(1, 1, \begin{pmatrix} -1 & 0 & 0\\ -2 & -1 & 1\\ -4 & 0 & 1\end{pmatrix}\right)\\ (1,2,3) &\mapsto \left(1, \zeta, \begin{pmatrix} -1 & 1 & 0\\ -1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\right).\end{aligned}$$

Proof. By Remark 4, the group A_4 is generated by $(1,2)(3,4)$ and $(1,2,3)$. We define $\check{\omega}'_+ : \mathbb{Z}\mathrm{A}_4 \to \check{\Gamma}'$ by

$$(1,2)(3,4) \mapsto \left(1, 1, \begin{pmatrix} -1 & 0 & 0\\ -2 & -1 & 1\\ -4 & 0 & 1\end{pmatrix}\right) \qquad (1,2,3) \mapsto \left(1, \zeta, \begin{pmatrix} -1 & 1 & 0\\ -1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\right).$$

We have

$$\left(1, 1, \begin{pmatrix} -1 & 0 & 0\\ -2 & -1 & 1\\ -4 & 0 & 1\end{pmatrix}\right)^2 = \left(1, 1, \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

$$\left(1, \zeta, \begin{pmatrix} -1 & 1 & 0\\ -1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\right)^3 = \left(1, 1, \begin{pmatrix} -1 & 1 & 0\\ -1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 & 1\\ 1 & -1 & 1\\ 0 & 0 & 1\end{pmatrix}\right) = \left(1, 1, \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

$$\left(\left(1,1,\begin{pmatrix}-1&0&0\\-2&-1&1\\-4&0&1\end{pmatrix}\right)\left(1,\zeta,\begin{pmatrix}-1&1&0\\-1&0&1\\0&0&1\end{pmatrix}\right)\right)^3=\left(1,\zeta\begin{pmatrix}1&-1&0\\3&-2&0\\4&-4&1\end{pmatrix}\right)^3$$
$$=\left(1,1,\begin{pmatrix}1&-1&0\\3&-2&0\\4&-4&1\end{pmatrix}\begin{pmatrix}-2&1&0\\-3&1&0\\-4&0&1\end{pmatrix}\right)=\left(1,1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$$

so that the images of the generators of A_4 fulfill the relations of Remark 4. Hence $\check{\omega}'_+$ is a well-defined $\mathbb{Z}$-algebra homomorphism.

Now we calculate the images under $\check{\omega}'_+$ for all elements in A_4.

$$\begin{aligned}
\mathrm{id}&\mapsto\left(1,1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right) & (1,2)(3,4)&\mapsto\left(1,1,\begin{pmatrix}-1&0&0\\-2&-1&1\\-4&0&1\end{pmatrix}\right)\\
(1,4)(2,3)&\mapsto\left(1,1,\begin{pmatrix}1&-2&1\\2&-3&1\\4&-4&1\end{pmatrix}\right) & (1,3)(2,4)&\mapsto\left(1,1,\begin{pmatrix}-1&2&-1\\0&3&-2\\0&4&-3\end{pmatrix}\right)\\
(1,4,3)&\mapsto\left(1,-1-\zeta,\begin{pmatrix}-2&1&0\\-3&1&0\\-4&0&1\end{pmatrix}\right) & (1,3,4)&\mapsto\left(1,\zeta,\begin{pmatrix}1&-1&0\\3&-2&0\\4&-4&1\end{pmatrix}\right)\\
(2,4,3)&\mapsto\left(1,\zeta,\begin{pmatrix}-1&-1&1\\-3&0&1\\-4&0&1\end{pmatrix}\right) & (2,3,4)&\mapsto\left(1,-1-\zeta,\begin{pmatrix}0&1&-1\\-1&3&-2\\0&4&-3\end{pmatrix}\right)\\
(1,2,4)&\mapsto\left(1,-1-\zeta,\begin{pmatrix}2&-1&0\\3&-3&1\\4&-4&1\end{pmatrix}\right) & (1,4,2)&\mapsto\left(1,\zeta,\begin{pmatrix}1&1&-1\\1&2&-2\\0&4&-3\end{pmatrix}\right)\\
(1,2,3)&\mapsto\left(1,\zeta,\begin{pmatrix}-1&1&0\\-1&0&1\\0&0&1\end{pmatrix}\right) & (1,3,2)&\mapsto\left(1,-1-\zeta,\begin{pmatrix}0&-1&1\\1&-1&1\\0&0&1\end{pmatrix}\right)
\end{aligned}$$

Note that every image fulfills the congruences of $\check{\Lambda}'$ so that the restriction

$$\check{\omega}'_+|_{\mathbb{Z}\mathrm{A}_4}^{\check{\Lambda}'}=:\check{\omega}':\mathbb{Z}\mathrm{A}_4\to\check{\Lambda}'$$

is a well-defined $\mathbb{Z}$-algebra homomorphism.

Using the standard $\mathbb{Z}$-linear basis of $\check{\Gamma}'$ and
$(\mathrm{id},(1,2)(3,4),(1,4)(2,3),(1,3)(2,4),(1,4,3),(1,3,4),(2,4,3),(2,3,4),(1,2,4),(1,4,2),(1,2,3),(1,3,2))$

as a basis for $\mathbb{Z}\mathrm{A}_4$, the map $\breve{\omega}'_+$ can be described with the following matrix.

$$\check{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -2 & 1 & -1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & -2 & 2 & 0 & -3 & 3 & -3 & -1 & 3 & 1 & -1 & 1 \\ 1 & -1 & -3 & 3 & 1 & -2 & 0 & 3 & -3 & 2 & 0 & -1 \\ 0 & 1 & 1 & -2 & 0 & 0 & 1 & -2 & 1 & -2 & 1 & 1 \\ 0 & -4 & 4 & 0 & -4 & 4 & -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 4 & 0 & -4 & 0 & 4 & -4 & 4 & 0 & 0 \\ 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & -3 & 1 & 1 \end{pmatrix}$$

We have $\det(\check{A}) = -12288 \neq 0$ so that $\breve{\omega}'_+$ and its restriction $\breve{\omega}'$ are injective.

We already know that $(\mathbb{Z}\mathrm{A}_4)\breve{\omega}' \subseteq \check{\Lambda}'$. To show the equality we calculate the index of $(\mathbb{Z}\mathrm{A}_4)\breve{\omega}'$ in $\check{\Gamma}'$.

$$[\check{\Gamma}' : (\mathbb{Z}\mathrm{A}_4)\breve{\omega}'] = \left|\check{\Gamma}'/(\mathbb{Z}\mathrm{A}_4)\breve{\omega}'\right| = \left|\det(\check{A})\right| = 12288 = 3 \cdot 2^{12}$$

By using a $\mathbb{Z}$-linear basis of $\check{\Lambda}'$ and the standard $\mathbb{Z}$-linear basis of $\check{\Gamma}'$, the embedding $\check{\Lambda}' \hookrightarrow \check{\Gamma}'$ can be described by the following matrix.

$$\check{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

We have $\det(\check{B}) = 12288$ and calculate

$$[\check{\Gamma}' : \check{\Lambda}'] = \left|\check{\Gamma}'/\check{\Lambda}'\right| = \left|\det(\check{B})\right| = 12288 = 3 \cdot 2^{12}.$$

Therefore we obtain

$$[\check{\Gamma}' : \check{\Lambda}'] \cdot [\check{\Lambda}' : (\mathbb{Z}\mathrm{A}_4)\breve{\omega}'] = [\check{\Gamma}' : (\mathbb{Z}\mathrm{A}_4)\breve{\omega}'] = 3 \cdot 2^{12} = [\check{\Gamma}' : \check{\Lambda}']$$

so that the equality $(\mathbb{Z}[\zeta]\mathrm{A}_4)\breve{\omega}' = \check{\Lambda}'$ holds.

In conclusion, $\breve{\omega}' : \mathbb{Z}\mathrm{A}_4 \to \check{\Lambda}'$ is an isomorphism of $\mathbb{Z}$-algebras. □

Remark 6 Independently, we may calculate the index of $(\mathbb{Z}\mathrm{A}_4)\,\breve{\omega}'$ in $\check{\Gamma}'$ to be

$$[\check{\Gamma}' : (\mathbb{Z}\mathrm{A}_4)\breve{\omega}'] = \left|\check{\Gamma}'/(\mathbb{Z}\mathrm{A}_4)\breve{\omega}'\right| = \sqrt{\left|\frac{12^{12}}{1 \cdot (-3) \cdot (3^9)}\right|} = \left(\frac{3^{12} \cdot 2^{24}}{3 \cdot 3^9}\right)^{1/2} = 3 \cdot 2^{12}\,;$$

cf. [**6,** Proposition 1.1.5].

2.1.2 Wedderburn image of $\mathbb{Z}_{(2)}\mathrm{A}_4$

Corollary 7 Recall that $R = \mathbb{Z}_{(2)}$.

Let $\check{\Gamma} := R \times R[\zeta] \times R^{3\times 3}$ and

$$\check{\Lambda} := \left\{ \left(u, r+s\zeta, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) \in \check{\Gamma} : \begin{array}{c} u \equiv_4 i,\ g \equiv_4 0,\ h \equiv_4 0, \\ e-a \equiv_4 s \equiv_2 d,\ r-e \equiv_4 b+d \equiv_2 0 \end{array} \right\} \subseteq \check{\Gamma}.$$

We have the isomorphism of R-algebras

$$\begin{aligned} R\mathrm{A}_4 &\xrightarrow[\sim]{\check{\omega}} \check{\Lambda} \\ (1,2)(3,4) &\mapsto \left(1, 1, \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 1 \\ -4 & 0 & 1 \end{pmatrix}\right) \\ (1,2,3) &\mapsto \left(1, \zeta, \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right). \end{aligned}$$

Remark 8

- Let X be an $R\mathrm{A}_4$-module.

  Given $\xi \in \check{\Lambda}$ and $x \in X$ we define

  $$\xi \cdot x := \xi\check{\omega}^{-1} \cdot x.$$

  Thus X becomes a $\check{\Lambda}$-module.

  We identify $R\mathrm{A}_4$-modules and $\check{\Lambda}$-modules in this way.

- Note that R with the action given by

  $$\begin{aligned} \check{\Lambda} &\longrightarrow R^{1\times 1} \\ (u, v, N) &\longmapsto u \end{aligned}$$

  is the trivial $R\mathrm{A}_4$-module. We also refer to this module as the trivial $\check{\Lambda}$-module.

  So given $x \in R$ and $(u, v, N) \in \check{\Lambda}$ we have

  $$(u, v, N) \cdot x = u \cdot x.$$

2.2 Wedderburn image of $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$

2.2.1 Scalar extension

Recall that $R = \mathbb{Z}_{(2)}$ and $S = R[\zeta] = \mathbb{Z}_{(2)}[\zeta]$. Recall that $\otimes := \underset{R}{\otimes}$.

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$ with $\zeta\sigma = \zeta^2$. We have $\mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q}) = \{\mathrm{id}, \sigma\}$. Given $\xi \in S$, we write $\bar{\xi} := \xi\sigma$.

We aim to construct the Wedderburn isomorphism $\omega : S\mathrm{A}_4 \to \Lambda$.
We also obtain an isomorphism $\varphi : S \otimes \check{\Lambda} \to \Lambda$.

Lemma 9 We have the following isomorphism of S-algebras.

$$\delta \; : \; S \otimes S \xrightarrow{\sim} S \times S, \; \xi \otimes v \mapsto (\xi v, \xi \bar{v})$$

Proof. Let $\eta \in \mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$. We have the following S-algebra homomorphism.

$$S \times S \to S, \; (a, b) \mapsto a(b\eta)$$

Using the universal property of the tensor product we obtain a well defined S-algebra homomorphism

$$S \otimes S \to S, \; a \otimes b \mapsto a(b\eta).$$

Using the above homomorphism for each component we obtain a well defined S-algebra homomorphism

$$\delta \; : \; S \otimes S \to S \times S, \; \xi \otimes v \mapsto (\xi(v\,\mathrm{id}), \xi(v\sigma)) = (\xi v, \xi \bar{v}),$$

also known as the Dedekind embedding.

Using $(1 \otimes 1, 1 \otimes \zeta)$ as an S-linear basis for $S \otimes S$ and $((1,0),(0,1))$ as an S-linear basis for $S \times S$, the S-algebra homomorphism δ can be described with the following matrix.

$$D = \begin{pmatrix} 1 & 1 \\ \zeta & \zeta^2 \end{pmatrix}$$

We have that $\mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}(\det(D)) = \mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}(\zeta^2 - \zeta) = 3 \in \mathrm{U}(S)$ so that δ is bijective and an isomorphism. □

Lemma 10 Recall that $\check{\omega} : R\mathrm{A}_4 \xrightarrow{\sim} \check{\Lambda}$; cf. Corollary 7.

Let $\iota_1 : \check{\Lambda} \hookrightarrow \check{\Gamma}$ be the natural embedding and $\check{\omega}_+ := \check{\omega}\,\iota_1 : R\mathrm{A}_4 \to \check{\Gamma}$.

We define the R-algebra homomorphisms

$$\begin{aligned} \tau \; &: \; R\mathrm{A}_4 \longrightarrow R[\zeta] \otimes R\mathrm{A}_4, \; x \longmapsto 1 \otimes x \\ \tau' \; &: \; \check{\Lambda} \longrightarrow R[\zeta] \otimes \check{\Lambda}, \; \lambda \longmapsto 1 \otimes \lambda \\ \tau'_0 \; &: \; \check{\Gamma} \longrightarrow R[\zeta] \otimes \check{\Gamma}, \; x \longmapsto 1 \otimes x \, . \end{aligned}$$

Furthermore we define the S-algebra isomorphisms

$$\psi \; : \; R[\zeta] \otimes R\mathrm{A}_4 \xrightarrow{\sim} R[\zeta]\mathrm{A}_4, \; \xi \otimes r\sigma \mapsto \xi r\sigma$$

$$\psi'_1 \; : \; R[\zeta] \otimes \left(R \times R[\zeta] \times R^{3\times 3}\right) \xrightarrow{\sim} R[\zeta] \times (R[\zeta] \otimes R[\zeta]) \times R[\zeta]^{3\times 3}, \; \xi \otimes (u, v, N) \mapsto (\xi u, \xi \otimes v, \xi N)$$

and using Lemma 9

$$\psi'_2 \; : \; R[\zeta] \times (R[\zeta] \otimes R[\zeta]) \times R[\zeta]^{3\times 3} \xrightarrow{\sim} R[\zeta] \times R[\zeta] \times R[\zeta] \times R[\zeta]^{3\times 3}, (u, \xi \otimes v, N) \mapsto (u, \xi v, \xi \bar{v}, N).$$

Let $\widetilde{\omega} := \psi^{-1}(\mathbb{R}[\zeta] \otimes \check{\omega})\psi'_1\psi'_2 : R[\zeta]\mathrm{A}_4 \to R[\zeta] \times R[\zeta] \times R[\zeta] \times R[\zeta]^{3\times 3}$ and $\widetilde{\Lambda} := (R[\zeta]\mathrm{A}_4)\widetilde{\omega}$.

We define the S-algebra isomorphism $\psi' := (\psi_1'\psi_2')|_{R[\zeta]\otimes\check{\Lambda}}^{\widetilde{\Lambda}}$,

$$\begin{aligned} R[\zeta]\otimes\check{\Lambda} &\xrightarrow{\psi'} \widetilde{\Lambda}, \\ (\xi\otimes(u,v,N)) &\longmapsto (\xi u, \xi v, \xi\bar{v}, \xi N)\,. \end{aligned}$$

Let $\Gamma = R[\zeta]\times R[\zeta]\times R[\zeta]\times R[\zeta]^{3\times 3}$ and define $x := (1,1,1,T)\in\Gamma$ with

$$T := \begin{pmatrix}1&\zeta&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}1-\zeta^2&0&0\\-2&1&0\\0&0&\zeta^2\end{pmatrix} = \begin{pmatrix}1&\zeta&0\\\zeta&1&0\\0&0&2\zeta^2\end{pmatrix}.$$

Using this we define the $\mathbb{Q}(\zeta)$-algebra isomorphism

$$\kappa\ :\ \mathbb{Q}\Gamma \xrightarrow{\sim} \mathbb{Q}\Gamma,\ (u,r,s,N)\mapsto x^{-1}(u,r,s,N)x = (u,r,s,N^T).$$

Let $\Lambda := \widetilde{\Lambda}\kappa$ and we obtain the isomorphism of S-algebras

$$\omega := \left(\widetilde{\omega}|^{\widetilde{\Lambda}}\right)\left(\kappa|_{\widetilde{\Lambda}}^{\widetilde{\Lambda}\kappa}\right):\ R[\zeta]\mathrm{A}_4 \xrightarrow{\sim} \Lambda\,.$$

Let $\iota_2 : \Lambda \hookrightarrow \Gamma$ be the natural embedding and $\omega_+ := \omega\iota_2$.
We have an isomorphism of S-algebras $\varphi := \psi'\kappa|_{\widetilde{\Lambda}}^{\Lambda}$,

$$\varphi\ :\ R[\zeta]\otimes\check{\Lambda} \xrightarrow{\sim} \Lambda,\ \xi\otimes(u,v,N)\longmapsto \left(\xi u,\xi v,\xi\bar{v},\xi N^T\right).$$

The following diagram commutes.

$$\begin{tikzcd}
R\mathrm{A}_4 \arrow[r, "\check{\omega}", "\sim"'] \arrow[d, "\tau"'] & \check{\Lambda} \arrow[r, hook, "\iota_1"] \arrow[dd, bend left, "\tau'"] & \check{\Gamma} = R\times R[\zeta]\times R^{3\times 3} \arrow[d, "\tau_0'"] & \\
R[\zeta]\otimes R\mathrm{A}_4 \arrow[rr, "R[\zeta]\otimes\check{\omega}_+"] \arrow[dr, "\sim", "R[\zeta]\otimes\check{\omega}"'] \arrow[ddd, "\psi", "\wr"'] & & R[\zeta]\otimes\left(R\times R[\zeta]\times R^{3\times 3}\right) \arrow[dd, "\psi_1'", "\wr"'] & \\
 & R[\zeta]\otimes\check{\Lambda} \arrow[ur, hook, "R[\zeta]\otimes\iota_1"'] \arrow[ddd, "\psi'", "\wr"'] \arrow[dddd, bend left, "\varphi", "\wr"'] & & \\
 & & R[\zeta]\times(R[\zeta]\otimes R[\zeta])\times R[\zeta]^{3\times 3} \arrow[d, "\psi_2'", "\wr"'] & \\
R[\zeta]\mathrm{A}_4 \arrow[rr, "\widetilde{\omega}"] \arrow[dr, "\sim", "\widetilde{\omega}|^{\widetilde{\Lambda}}"'] \arrow[ddd, "1"'] & & \Gamma = R[\zeta]\times R[\zeta]\times R[\zeta]\times R[\zeta]^{3\times 3} \arrow[r, hook] \arrow[dd, "\kappa|_{\Gamma}^{\Gamma\kappa}", "\wr"'] & \mathbb{Q}\Gamma \arrow[ddd, "\kappa", "\wr"'] \\
 & \widetilde{\Lambda} \arrow[ur, hook] \arrow[d, "\kappa|_{\widetilde{\Lambda}}^{\widetilde{\Lambda}\kappa}", "\wr"'] & & \\
 & \Lambda \arrow[r, hook] \arrow[dr, hook, "\iota_2"] & \Gamma\kappa \arrow[dr, hook] & \\
S\mathrm{A}_4 \arrow[ur, "\omega", "\sim"'] \arrow[rr, "\omega_+"] & & \Gamma \arrow[r, hook] & \mathbb{Q}\Gamma
\end{tikzcd}$$

We calculate the images of the S-linear generators of $S\mathrm{A}_4$ under ω.

$$(1,2)(3,4)\widetilde{\omega}\kappa = \left(1,1,1,\begin{pmatrix}1&0&0\\-2&-1&1\\-4&0&1\end{pmatrix}\right)\kappa = \left(1,1,1,\begin{pmatrix}\frac{1}{3}(4\zeta-1)&\frac{1}{3}(2\zeta^2-2)&\frac{1}{3}(2\zeta-2)\\\frac{1}{3}(2\zeta-2)&\frac{1}{3}(4\zeta^2-1)&\frac{1}{3}(2\zeta^2-2)\\-2\zeta&-2\zeta^2&1\end{pmatrix}\right)$$

$$(1,2,3)\widetilde{\omega}\kappa = \left(1,\zeta,\zeta^2,\begin{pmatrix}-1&1&0\\-1&0&1\\0&0&1\end{pmatrix}\right)\kappa = \left(1,\zeta,\zeta^2,\begin{pmatrix}\frac{1}{3}(4\zeta-\zeta^2)&\frac{1}{3}(2\zeta^2-2\zeta)&\frac{1}{3}(2\zeta-2)\\\frac{1}{3}(2\zeta-2\zeta^2)&\frac{1}{3}(4\zeta^2-\zeta)&\frac{1}{3}(2\zeta^2-2)\\0&0&1\end{pmatrix}\right)$$

In conclusion, we have an isomorphism of S-algebras

$$\begin{aligned} S\mathrm{A}_4 &\xrightarrow[\sim]{\omega} \Lambda \\ (1,2)(3,4) &\mapsto \left(1,1,1,\begin{pmatrix}\frac{1}{3}(4\zeta-1)&\frac{1}{3}(2\zeta^2-2)&\frac{1}{3}(2\zeta-2)\\\frac{1}{3}(2\zeta-2)&\frac{1}{3}(4\zeta^2-1)&\frac{1}{3}(2\zeta^2-2)\\-2\zeta&-2\zeta^2&1\end{pmatrix}\right) \\ (1,2,3) &\mapsto \left(1,\zeta,\zeta^2,\begin{pmatrix}\frac{1}{3}(4\zeta-\zeta^2)&\frac{1}{3}(2\zeta^2-2\zeta)&\frac{1}{3}(2\zeta-2)\\\frac{1}{3}(2\zeta-2\zeta^2)&\frac{1}{3}(4\zeta^2-\zeta)&\frac{1}{3}(2\zeta^2-2)\\0&0&1\end{pmatrix}\right). \end{aligned}$$

Remark 11 The conjugating element

$$x := \left(1,1,1,\begin{pmatrix}1&\zeta&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}1-\zeta^2&0&0\\-2&1&0\\0&0&\zeta^2\end{pmatrix}\right) \in \Gamma$$

was constructed using the computer algebra system Magma, cf. [2].

The first three matrix factors have the purpose of simplifying the congruences of Λ. The last matrix then accomplishes a compatibility with the Galois conjugation in sense of Lemma 18.

Corollary 12 Consider the following maps from Lemma 10

$$\begin{aligned} \psi &: R[\zeta]\otimes\mathrm{A}_4 \longrightarrow R[\zeta]\mathrm{A}_4,\ \xi\otimes r\sigma \longmapsto \xi r\sigma \\ \varphi &: R[\zeta]\otimes\check{\Lambda} \longrightarrow \Lambda,\ \xi\otimes(u,v,N) \longmapsto (\xi u,\xi v,\xi\bar{v},\xi N^T) \\ \tau &: R\mathrm{A}_4 \longrightarrow R[\zeta]\otimes R\mathrm{A}_4,\ x \longmapsto 1\otimes x \\ \tau' &: \check{\Lambda} \longrightarrow R[\zeta]\otimes\check{\Lambda},\ \lambda \longmapsto 1\otimes\lambda\,. \end{aligned}$$

Then

$$\begin{tikzcd} R\mathrm{A}_4 \arrow[r, "\check{\omega}", "\sim"'] \arrow[d, "\tau"] & \check{\Lambda} \arrow[d, "\tau'"'] \\ R[\zeta]\otimes\mathrm{A}_4 \arrow[r, "R[\zeta]\otimes\check{\omega}", "\sim"'] & R[\zeta]\otimes\check{\Lambda} \end{tikzcd}$$

is a commutative diagram of R-algebras and

$$\begin{tikzcd} R[\zeta]\otimes \mathrm{A}_4 \arrow[r, "R[\zeta]\otimes\check{\omega}", "\sim"'] \arrow[d, "\psi", "\wr"'] & R[\zeta]\otimes\check{\Lambda} \arrow[d, "\varphi"', "\wr"] \\ R[\zeta]\mathrm{A}_4 \arrow[r, "\omega", "\sim"'] & \Lambda \end{tikzcd}$$

is a commutative diagram of S-algebras.

We obtain an injective ring homomorphism $\iota := \tau'\varphi$,

$$\begin{array}{c} \check{\Lambda} \xrightarrow{\iota} \Lambda \\ (u, v, N) \mapsto (u, v, \bar{v}, N^T). \end{array}$$

$$\begin{tikzcd} R\mathrm{A}_4 \arrow[r, "\check{\omega}", "\sim"'] \arrow[d, "\tau"] \arrow[dd, hook, bend right] & \check{\Lambda} \arrow[d, "\tau'"'] \arrow[dd, bend left, "\iota"] \\ R[\zeta]\otimes \mathrm{A}_4 \arrow[r, "R[\zeta]\otimes\check{\omega}", "\sim"'] \arrow[d, "\psi", "\wr"'] & R[\zeta]\otimes\check{\Lambda} \arrow[d, "\varphi"', "\wr"] \\ R[\zeta]\mathrm{A}_4 \arrow[r, "\omega", "\sim"'] & \Lambda \end{tikzcd}$$

2.2.2 Wedderburn image of $\mathbb{Z}[\zeta_3]\mathrm{A}_4$

We verify independently of § 2.2.1 that $\omega : S\mathrm{A}_4 \to \Lambda$ is an isomorphism of S-algebras. This can be done for the more general ground ring $\mathbb{Z}[\zeta]$.

Lemma 13 Let $\Gamma' := \mathbb{Z}[\zeta]\times\mathbb{Z}[\zeta]\times\mathbb{Z}[\zeta]\times\mathbb{Z}[\zeta]^{3\times 3}$ and

$$\Lambda' := \left\{ \left(u, r, s, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) \in \Gamma' : \begin{array}{c} b \equiv_2 c \equiv_2 d \equiv_2 f \equiv_2 g \equiv_2 h \equiv_2 0 \\ u \equiv_4 i,\ r \equiv_4 e,\ s \equiv_4 a \\ u + r + s \equiv_3 0,\ r \equiv_{1-\zeta} s \end{array} \right\} \subseteq \Gamma'.$$

We have the isomorphism of $\mathbb{Z}[\zeta]$-algebras

$$\begin{array}{rl} \mathbb{Z}[\zeta]\mathrm{A}_4 \underset{\sim}{\xrightarrow{\omega'}} & \Lambda' \\ (1,2)(3,4) \mapsto & \left(1, 1, 1, \begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix} \right) \\ (1,2,3) \mapsto & \left(1, \zeta, \zeta^2, \begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix} \right). \end{array}$$

Proof. Recall that A_4 is generated by $(1,2)(3,4)$ and $(1,2,3)$; cf. Remark 4. We define $\omega'_+ : \mathbb{Z}[\zeta]\mathrm{A}_4 \to \Gamma'$ by

$$(1,2)(3,4) \mapsto \left(1,1,1,\begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix}\right) \quad (1,2,3) \mapsto \left(1,\zeta,\zeta^2,\begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix}\right).$$

We have

$$\left(1,1,1,\begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix}\right)^2 = \left(1,1,1,\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$\left(1,\zeta,\zeta^2,\begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix}\right)^3 = \left(1,1,1,\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$\left(\left(1,1,1,\begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix}\right)\left(1,\zeta,\zeta^2,\begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix}\right)\right)^3$$

$$= \left(1,\zeta,\zeta^2,\begin{pmatrix} -3\zeta^2 & 4\zeta-2\zeta^2 & 0 \\ -2 & 4\zeta^2+\zeta & 0 \\ 2\zeta & 2\zeta-2 & 1 \end{pmatrix}\right)^3 = \left(1,1,1,\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

so that the images of the generators of A_4 fulfill the relations of Remark 4. Hence ω'_+ is a well-defined $\mathbb{Z}[\zeta]$-algebra homomorphism.

Now we calculate the images under ω'_+ for all elements in A_4.

$$\mathrm{id} \mapsto \left(1,1,1,\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \qquad (1,2)(3,4) \mapsto \left(1,1,1,\begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix}\right)$$

$$(1,4)(2,3) \mapsto \left(1,1,1,\begin{pmatrix} 4\zeta+1 & 6\zeta & 2-2\zeta \\ 2\zeta^2 & 4\zeta^2+1 & 2\zeta \\ 2\zeta & 2\zeta-2 & 1 \end{pmatrix}\right) \qquad (1,3)(2,4) \mapsto \left(1,1,1,\begin{pmatrix} 1-4\zeta & 2-4\zeta & 4\zeta-2 \\ 2\zeta-2\zeta^2 & 4\zeta+1 & 2\zeta^2-2\zeta \\ 2 & 2 & -3 \end{pmatrix}\right)$$

$$(1,4,3) \mapsto \left(1,\zeta^2,\zeta,\begin{pmatrix} 4\zeta^2+\zeta & 2\zeta^2-4\zeta & 0 \\ 2 & -3\zeta^2 & 0 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix}\right) \qquad (1,3,4) \mapsto \left(1,\zeta,\zeta^2,\begin{pmatrix} -3\zeta^2 & 4\zeta-2\zeta^2 & 0 \\ -2 & 4\zeta^2+\zeta & 0 \\ 2\zeta & 2\zeta-2 & 1 \end{pmatrix}\right)$$

$$(2,4,3) \mapsto \left(1,\zeta,\zeta^2,\begin{pmatrix} \zeta^2-4 & 4\zeta^2 & 2-2\zeta \\ 2-2\zeta & 4+\zeta & 2\zeta \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix}\right) \qquad (2,3,4) \mapsto \left(1,\zeta^2,\zeta,\begin{pmatrix} -3\zeta & -4\zeta & 4\zeta-2 \\ -2\zeta^2 & -3\zeta^2 & 2\zeta^2-2\zeta \\ 2 & 2 & -3 \end{pmatrix}\right)$$

$$(1,2,4) \mapsto \left(1,\zeta^2,\zeta,\begin{pmatrix} \zeta-4\zeta^2 & 6\zeta-2\zeta^2 & -2\zeta \\ 2\zeta^2-2 & 5\zeta^2 & -2\zeta^2 \\ 2\zeta & 2\zeta-2 & 1 \end{pmatrix}\right) \qquad (1,4,2) \mapsto \left(1,\zeta,\zeta^2,\begin{pmatrix} 4+\zeta^2 & 2 & 4\zeta-2 \\ 2\zeta & \zeta & 2\zeta^2-2\zeta \\ 2 & 2 & -3 \end{pmatrix}\right)$$

$$(1,2,3) \mapsto \left(1,\zeta,\zeta^2,\begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix}\right) \qquad (1,3,2) \mapsto \left(1,\zeta^2,\zeta,\begin{pmatrix} \zeta & 2\zeta & 2-2\zeta \\ 0 & \zeta^2 & 2\zeta \\ 0 & 0 & 1 \end{pmatrix}\right)$$

Note that every image fulfills the congruences of Λ' so that the restriction

$$\omega'_+|^{\Lambda'}_{\mathbb{Z}[\zeta]\mathrm{A}_4} =: \omega' : \mathbb{Z}[\zeta]\mathrm{A}_4 \to \Lambda'$$

is a well-defined $\mathbb{Z}[\zeta]$-algebra homomorphism.

Using the standard $\mathbb{Z}[\zeta]$-linear basis of Γ' and $(\mathrm{id},(1,2)(3,4),(1,4)(2,3),(1,3)(2,4),(1,4,3),(1,3,4),(2,4,3),(2,3,4),(1,2,4),(1,4,2),(1,2,3),(1,3,2))$ as a basis for $\mathbb{Z}[\zeta]\mathrm{A}_4$, the map ω'_+ can be described with the following matrix.

$$A := \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & \zeta^2 & \zeta & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta^2 \\
1 & 1 & 1 & 1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta^2 & \zeta^2 & \zeta \\
1 & -3 & 4\zeta+1 & 1-4\zeta & 4\zeta^2+\zeta & -3\zeta^2 & \zeta^2-4 & -3\zeta & \zeta-4\zeta^2 & 4+\zeta^2 & \zeta^2 & \zeta \\
0 & 2\zeta^2 & 6\zeta & 2-4\zeta & 2\zeta^2-4\zeta & 4\zeta-2\zeta^2 & 4\zeta^2 & -4\zeta & 6\zeta-2\zeta^2 & 2 & -2\zeta & 2\zeta \\
0 & -2\zeta & 2-2\zeta & 4\zeta-2 & 0 & 0 & 2-2\zeta & 4\zeta-2 & -2\zeta & 4\zeta-2 & -2\zeta & 2-2\zeta \\
0 & -2\zeta & 2\zeta^2 & 2\zeta-2\zeta^2 & 2 & -2 & 2-2\zeta & -2\zeta^2 & 2\zeta^2-2 & 2\zeta & 0 & 0 \\
1 & 1 & 4\zeta^2+1 & 4\zeta+1 & -3\zeta^2 & 4\zeta^2+\zeta & 4+\zeta & -3\zeta^2 & 5\zeta^2 & \zeta & \zeta & \zeta^2 \\
0 & -2\zeta^2 & 2\zeta & 2\zeta^2-2\zeta & 0 & 0 & 2\zeta & 2\zeta^2-2\zeta & -2\zeta^2 & 2\zeta^2-2\zeta & -2\zeta^2 & 2\zeta \\
0 & 2\zeta^2 & 2\zeta & 2 & 2\zeta^2 & 2\zeta & 2\zeta^2 & 2 & 2\zeta & 2 & 0 & 0 \\
0 & -2\zeta & 2\zeta-2 & 2 & -2\zeta & 2\zeta-2 & -2\zeta & 2 & 2\zeta-2 & 2 & 0 & 0 \\
1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & -3 & 1 & 1
\end{pmatrix}$$

We have $\det(A) = 24576\zeta + 12288 \neq 0$ so that ω'_+ and its restriction ω' are injective.

We already know that $(\mathbb{Z}[\zeta]\mathrm{A}_4)\omega' \subseteq \Lambda'$. To show the equality we calculate the index of $(\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'$ in Γ'.

$[\Gamma' : (\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'] = \left|\Gamma'/(\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'\right| = \left|\mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}(\det(A))\right| = 24576^2+12288^2-24576\cdot 12288 = 3^3\cdot 2^{24}$

By using a $\mathbb{Z}[\zeta]$-linear basis of Λ' and the standard $\mathbb{Z}[\zeta]$-linear basis of Γ', the embedding $\Lambda' \hookrightarrow \Gamma'$ can be described by the following matrix.

$$B := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \zeta-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \zeta^2-1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \zeta^2-1 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & \zeta-1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}$$

We have $\det(B) = 12288\zeta - 12288$ and calculate

$$[\Gamma' : \Lambda'] = \left|\Gamma'/\Lambda'\right| = \left|\mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}(\det(B))\right| = 12288^2 \left|\mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}(\det(\zeta-1))\right| = 3^3\cdot 2^{24}.$$

Therefore we obtain

$$[\Gamma' : \Lambda'] \cdot [\Lambda' : (\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'] = [\Gamma' : (\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'] = 3^3\cdot 2^{24} = [\Gamma' : \Lambda']$$

so that the equality $(\mathbb{Z}[\zeta]\mathrm{A}_4)\omega' = \Lambda'$ holds.

In conclusion, $\omega' : \mathbb{Z}[\zeta]\mathrm{A}_4 \to \Lambda'$ is an isomorphism of $\mathbb{Z}[\zeta]$-algebras. □

Remark 14 Independently, we may calculate the index of $(\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'$ in Γ' to be

$$\begin{aligned}
[\Gamma' : (\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'] = \left|\Gamma'/(\mathbb{Z}[\zeta]\mathrm{A}_4)\omega'\right| &= \sqrt{\left|\mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}\left(\frac{12^{12}}{1\cdot 1\cdot 1\cdot (3^9)}\right)\right|} \\
&= \left(\mathrm{N}_{\mathbb{Q}(\zeta)|\mathbb{Q}}\left(\frac{3^{12}\cdot 2^{24}}{3^9}\right)\right)^{1/2} = \left(\left(\frac{3^{12}\cdot 2^{24}}{3^9}\right)^2\right)^{1/2} = 3^3\cdot 2^{24}\,;
\end{aligned}$$

cf. [**6,** Proposition 1.1.5].

2.2.3 Wedderburn image of $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$

Corollary 15 Recall that $S = \mathbb{Z}_{(2)}[\zeta]$.

Let $\Gamma := S \times S \times S \times S^{3\times 3}$ and

$$\Lambda := \left\{ \left(u, r, s, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) \in \Gamma \ : \ \begin{array}{c} b \equiv_2 c \equiv_2 d \equiv_2 f \equiv_2 g \equiv_2 h \equiv_2 0 \\ u \equiv_4 i,\ r \equiv_4 e,\ s \equiv_4 a \end{array} \right\} \subseteq \Gamma.$$

We have the isomorphism of S-algebras

$$\begin{array}{rcl} S\mathrm{A}_4 & \underset{\sim}{\overset{\omega}{\to}} & \Lambda \\ (1,2)(3,4) & \mapsto & \left(1,1,1, \begin{pmatrix} \frac{1}{3}(4\zeta - 1) & \frac{1}{3}(2\zeta^2 - 2) & \frac{1}{3}(2\zeta - 2) \\ \frac{1}{3}(2\zeta - 2) & \frac{1}{3}(4\zeta^2 - 1) & \frac{1}{3}(2\zeta^2 - 2) \\ -2\zeta & -2\zeta^2 & 1 \end{pmatrix} \right) \\ (1,2,3) & \mapsto & \left(1,\zeta,\zeta^2, \begin{pmatrix} \frac{1}{3}(4\zeta - \zeta^2) & \frac{1}{3}(2\zeta^2 - 2\zeta) & \frac{1}{3}(2\zeta - 2) \\ \frac{1}{3}(2\zeta - 2\zeta^2) & \frac{1}{3}(4\zeta^2 - \zeta) & \frac{1}{3}(2\zeta^2 - 2) \\ 0 & 0 & 1 \end{pmatrix} \right). \end{array}$$

See also Lemma 10 and [**9,** Beispiel 2, page 47].

Proof. Let $x := \left(\zeta^2, 1, 3(\zeta^2 - 1), \begin{pmatrix} 1-\zeta^2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \right) \in \mathrm{U}(\Lambda)$.

We have the following S-algebra generators of $\Lambda'_{(2)}$, cf. Lemma 13.

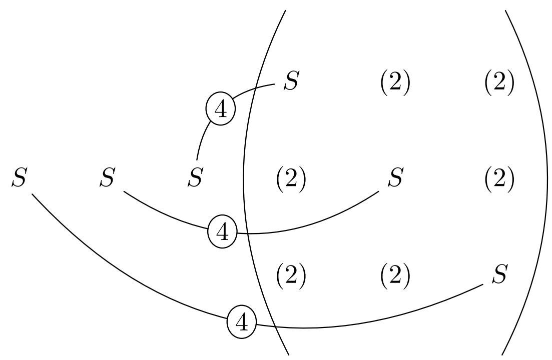
$$\begin{array}{rcl} (1,2)(3,4)\omega' & = & \left(1,1,1, \begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix} \right) \\ (1,2,3)\omega' & = & \left(1,\zeta,\zeta^2, \begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix} \right) \end{array}$$

We obtain

$$\begin{array}{rcl} x^{-1} \left(1,1,1, \begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix} \right) x & = & \left(1,1,1, \begin{pmatrix} \frac{1}{3}(4\zeta - 1) & \frac{1}{3}(2\zeta^2 - 2) & \frac{1}{3}(2\zeta - 2) \\ \frac{1}{3}(2\zeta - 2) & \frac{1}{3}(4\zeta^2 - 1) & \frac{1}{3}(2\zeta^2 - 2) \\ -2\zeta & -2\zeta^2 & 1 \end{pmatrix} \right) \\ x^{-1} \left(1,\zeta,\zeta^2, \begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix} \right) x & = & \left(1,\zeta,\zeta^2, \begin{pmatrix} \frac{1}{3}(4\zeta - \zeta^2) & \frac{1}{3}(2\zeta^2 - 2\zeta) & \frac{1}{3}(2\zeta - 2) \\ \frac{1}{3}(2\zeta - 2\zeta^2) & \frac{1}{3}(4\zeta^2 - \zeta) & \frac{1}{3}(2\zeta^2 - 2) \\ 0 & 0 & 1 \end{pmatrix} \right). \end{array}$$

$\square$

Remark 16 Λ can be visualized as follows.



Remark 17

- Let M be an $S\mathrm{A}_4$-module. Given $\xi \in \Lambda$ and $x \in M$ we define
$$\xi \cdot x := \xi\omega^{-1} \cdot x.$$
Thus M becomes a $\check{\Lambda}$-module. We identify $S\mathrm{A}_4$-modules and Λ-modules in this way.
- Note that S with the operation given by
$$\begin{aligned} \Lambda &\longrightarrow S^{1\times 1} \\ (u, r, s, N) &\longmapsto u \end{aligned}$$
is the trivial $S\mathrm{A}_4$-module. We also refer to this module as the trivial Λ-module.
So given $x \in S$ and $(u, r, s, N) \in \Lambda$ we have
$$(u, r, s, N) \cdot x = u \cdot x.$$

2.3 Wedderburn and Galois conjugation

Recall that $S = \mathbb{Z}_{(2)}[\zeta]$.
Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$ such that $\sigma(\zeta) = \zeta^2$. For $\xi \in R[\zeta] = S$ we often write $\bar{\xi} := \xi\sigma$.
We denote the R-linear expansion of $\sigma|_S^S$ to an R-algebra automorphism of $S\mathrm{A}_4$ by $S\mathrm{A}_4 \underset{\sim}{\xrightarrow{\sigma}} S\mathrm{A}_4$.

Lemma 18 We define the R-algebra automorphism
$$\sigma' : \Lambda \longrightarrow \Lambda,\ (u, r, s, N) \longmapsto (\bar{u}, \bar{s}, \bar{r}, \bar{N}^U) \text{ with } U := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U^{-1}.$$

The following diagram commutes.
$$\begin{tikzcd} S\mathrm{A}_4 \arrow[r, "\omega", "\sim"'] \arrow[d, "\sigma", "\wr"'] & \Lambda \arrow[d, "\sigma'", "\wr"'] \\ S\mathrm{A}_4 \arrow[r, "\omega", "\sim"'] & \Lambda \end{tikzcd}$$

Proof. We show $\omega\sigma' = \sigma\omega$.

Since σ' is an R-algebra homomorphism, it suffices to verify the equality for the R-algebra generators $((1,2)(3,4),(1,2,3),\zeta)$ of $S\mathrm{A}_4$.

Using that $\sigma|_{\mathrm{A}_4}^{\mathrm{A}_4} = \mathrm{id}_{\mathrm{A}_4}$ we have

$$\begin{aligned}
(1,2)(3,4)\omega\sigma' &= \left(1,1,1,\begin{pmatrix} \frac{1}{3}(4\zeta-1) & \frac{1}{3}(2\zeta^2-2) & \frac{1}{3}(2\zeta-2) \\ \frac{1}{3}(2\zeta-2) & \frac{1}{3}(4\zeta^2-1) & \frac{1}{3}(2\zeta^2-2) \\ -2\zeta & -2\zeta^2 & 1 \end{pmatrix}\right)\sigma' \\
&= \left(1,1,1,U\begin{pmatrix} \frac{1}{3}(4\zeta^2-1) & \frac{1}{3}(2\zeta-2) & \frac{1}{3}(2\zeta^2-2) \\ \frac{1}{3}(2\zeta^2-2) & \frac{1}{3}(4\zeta-1) & \frac{1}{3}(2\zeta-2) \\ -2\zeta^2 & -2\zeta & 1 \end{pmatrix}U\right) \\
&= \left(1,1,1,\begin{pmatrix} \frac{1}{3}(4\zeta-1) & \frac{1}{3}(2\zeta^2-2) & \frac{1}{3}(2\zeta-2) \\ \frac{1}{3}(2\zeta-2) & \frac{1}{3}(4\zeta^2-1) & \frac{1}{3}(2\zeta^2-2) \\ -2\zeta & -2\zeta^2 & 1 \end{pmatrix}\right) \\
&= (1,2)(3,4)\omega = (1,2)(3,4)\sigma\omega
\end{aligned}$$

$$\begin{aligned}
(1,2,3)\omega\sigma' &= \left(1,\zeta,\zeta^2,\begin{pmatrix} \frac{1}{3}(4\zeta-\zeta^2) & \frac{1}{3}(2\zeta^2-2\zeta) & \frac{1}{3}(2\zeta-2) \\ \frac{1}{3}(2\zeta-2\zeta^2) & \frac{1}{3}(4\zeta^2-\zeta) & \frac{1}{3}(2\zeta^2-2) \\ 0 & 0 & 1 \end{pmatrix}\right)\sigma' \\
&= \left(1,\zeta,\zeta^2,U\begin{pmatrix} \frac{1}{3}(4\zeta^2-\zeta) & \frac{1}{3}(2\zeta-2\zeta^2) & \frac{1}{3}(2\zeta^2-2) \\ \frac{1}{3}(2\zeta^2-2\zeta) & \frac{1}{3}(4\zeta-\zeta^2) & \frac{1}{3}(2\zeta-2) \\ 0 & 0 & 1 \end{pmatrix}U\right) \\
&= \left(1,\zeta,\zeta^2,\begin{pmatrix} \frac{1}{3}(4\zeta-\zeta^2) & \frac{1}{3}(2\zeta^2-2\zeta) & \frac{1}{3}(2\zeta-2) \\ \frac{1}{3}(2\zeta-2\zeta^2) & \frac{1}{3}(4\zeta^2-\zeta) & \frac{1}{3}(2\zeta^2-2) \\ 0 & 0 & 1 \end{pmatrix}\right) \\
&= (1,2,3)\omega = (1,2,3)\sigma\omega
\end{aligned}$$

$$\begin{aligned}
\zeta\omega\sigma' = (\zeta\,\mathrm{id})\omega\sigma' &= \left(\zeta,\zeta,\zeta,\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix}\right)\sigma' = \left(\zeta^2,\zeta^2,\zeta^2,\begin{pmatrix} \zeta^2 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}\right) \\
&= \zeta^2\omega = \zeta\sigma\omega.
\end{aligned}$$

In conclusion, we have $\omega\sigma' = \sigma\omega$ so that the diagram commutes. ▫

2.4 A remark on $\mathrm{Out}_{\mathbb{Z}_{(2)}[\zeta_3]\text{-alg}}(\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4)$

For an S-algebra A we write $\mathrm{Aut}(A) := \mathrm{Aut}_{S\text{-alg}}(A)$, $\mathrm{Inn}(A) := \mathrm{Inn}_{S\text{-alg}}(A)$ and $\mathrm{Out}(A) := \mathrm{Out}_{S\text{-alg}}(A)$.

Definition 19 We define the following idempotent elements of Λ.

$$\mathcal{E}_0 = \left(1,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right)\quad \mathcal{E}_1 = \left(0,1,0,\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)\quad \mathcal{E}_2 = \left(0,0,1,\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\right)$$

Note that $\mathcal{E}_0, \mathcal{E}_1$ and $\mathcal{E}_2$ are pairwise orthogonal.

Remark 20 For $\tau \in \mathrm{S}_{\{0,1,2\}}$ we denote the permutation matrix of τ by $P_\tau := (\delta_{i\tau,j})_{i,j}$.

Let $\pi : \Lambda \to S \times S \times S,\ (u,r,s,N) \mapsto (u,r,s)$. By Corollary 15 we have $\Lambda\pi = S \times S \times S$.

Let $\rho \in \mathrm{Aut}(\Lambda)$. We show that there exists exactly one $\bar{\rho} \in \mathrm{Aut}(S \times S \times S)$ such that the following diagram commutes.

$$\begin{tikzcd} \Lambda \arrow[r, "\rho"] \arrow[d, "\pi"'] & \Lambda \arrow[d, "\pi"] \\ S \times S \times S \arrow[r, "\bar{\rho}"'] & S \times S \times S \end{tikzcd}$$

Consider $\mathbb{Q}(\zeta)\Lambda := \mathbb{Q}(\zeta) \underset{S}{\otimes} \Lambda \cong \mathbb{Q}(\zeta) \times \mathbb{Q}(\zeta) \times \mathbb{Q}(\zeta) \times \mathbb{Q}(\zeta)^{3\times 3}$. We view

$$\begin{aligned} \Lambda &\to \mathbb{Q}(\zeta) \underset{S}{\otimes} \Lambda = \mathbb{Q}(\zeta)\Lambda \\ \lambda &\mapsto 1 \otimes \lambda =: \lambda \end{aligned}$$

as an embedding of a subring.

Let $\mathcal{E}$ be a primitive central idempotent of $\mathbb{Q}(\zeta)\Lambda$. Then we have

$$\dim_{\mathbb{Q}(\zeta)}(\mathcal{E}(\mathbb{Q}(\zeta)\Lambda)) = \dim_{\mathbb{Q}(\zeta)}(\mathcal{E}(\mathbb{Q}(\zeta) \otimes \rho)(\mathbb{Q}(\zeta)\Lambda)).$$

Consider the following orthogonal decomposition of $1_{\mathbb{Q}(\zeta)\Lambda}$ into central primitive idempotents of $\mathbb{Q}(\zeta)\Lambda$.

$$1_{\mathbb{Q}(\zeta)\otimes\Lambda} = \left(1,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right) + \left(0,1,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right) + \left(0,0,1,\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right) + \left(0,0,0,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$$

Therefore $\mathcal{E} := \left(0,0,0,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$ is the only central primitive idempotent of $\mathbb{Q}(\zeta)\Lambda$ with $\dim_{\mathbb{Q}(\zeta)}(\mathcal{E}(\mathbb{Q}(\zeta)\Lambda)) = 9$. Hence $\mathcal{E}(\mathbb{Q}(\zeta) \otimes \rho) = \mathcal{E}$.

Suppose $x \in \mathrm{Ker}(\pi)$ so that $x = x\mathcal{E}$. We have

$$x\rho = (x\mathcal{E})\rho = (x\mathcal{E})(\mathbb{Q}(\zeta) \otimes \rho) = x(\mathbb{Q}(\zeta) \otimes \rho)\mathcal{E} = x\rho\mathcal{E}.$$

Thus $\mathrm{Ker}(\pi)\rho \subseteq \mathrm{Ker}(\pi)$.

We define

$$\begin{aligned} \bar{\rho}\ :\ & S \times S \times S \to S \times S \times S \\ & (a,b,c) \mapsto \left(a,b,c,\begin{pmatrix}c&0&0\\0&b&0\\0&0&a\end{pmatrix}\right)\rho\pi. \end{aligned}$$

Suppose given $(u, r, s, N) \in \Lambda$. Since

$$(u, r, s, N) - \left(u, r, s, \begin{pmatrix} s & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & u \end{pmatrix}\right) \in \mathrm{Ker}(\pi)$$
$$\Rightarrow (u, r, s, N)\rho - \left(u, r, s, \begin{pmatrix} s & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & u \end{pmatrix}\right) \rho \in \mathrm{Ker}(\pi)\rho \subseteq \mathrm{Ker}(\pi),$$

we have

$$(u, r, s, N)\pi\bar{\rho} = (u, r, s)\bar{\rho} = \left(u, r, s, \begin{pmatrix} s & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & u \end{pmatrix}\right) \rho\pi = (u, r, s, N)\rho\pi.$$

Therefore the above diagram commutes.

The uniqueness of $\bar{\rho}$ follows by surjectivity of π.

Note that $\mathcal{E}_0\pi, \mathcal{E}_1\pi$ and $\mathcal{E}_2\pi$ are the only primitive idempotents of $\Lambda\pi$; cf. Definition 19. Hence, for every $i \in \{0, 1, 2\}$ there exists a unique $j \in \{0, 1, 2\}$ such that

$$(\mathcal{E}_i\pi)\bar{\rho} = \mathcal{E}_j\pi.$$

For $\rho \in \mathrm{Aut}(\Lambda)$ and $i \in \{0, 1, 2\}$ let $k_{i,\rho} \in \{0, 1, 2\}$ be such that $(\mathcal{E}_i\pi)\bar{\rho} = \mathcal{E}_{k_{i,\rho}}\pi$.

We define the following group homomorphisms.

$$\gamma_1 \;:\; \mathrm{S}_{\{0,1,2\}} \longrightarrow \mathrm{Out}(\Lambda) \;:\; \tau \longmapsto \big((a_0, a_1, a_2, A) \mapsto (a_{0\tau}, a_{1\tau}, a_{2\tau}, A^{P_\tau})\big)\,\mathrm{Inn}(\Lambda)$$
$$\gamma_2 \;:\; \mathrm{Out}(\Lambda) \longrightarrow \mathrm{S}_{\{0,1,2\}} \;:\; \rho\,\mathrm{Inn}(\Lambda) \longmapsto \begin{cases} 0 \mapsto k_{0,\rho} \\ 1 \mapsto k_{1,\rho} \\ 2 \mapsto k_{2,\rho} \end{cases}$$

Hereby γ_2 is well defined since for $\rho \in \mathrm{Inn}(\Lambda)$ we obtain $\bar{\rho} = \mathrm{id}_{S\times S\times S}$.

We have $\gamma_1\gamma_2 = \mathrm{id}_{S_{\{0,1,2\}}}$ and therefore S_3 is isomorphic to a subgroup of $\mathrm{Out}(\Lambda)$.

In comparison, we have $\mathrm{Out}(\mathrm{A}_4) \cong \mathrm{C}_2$.

We do not know whether we also have $\gamma_2\gamma_1 = \mathrm{id}_{\mathrm{Out}(\Lambda)}$.

Chapter 3

Projective resolution over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$

Let $\zeta := \zeta_3$ be a third primitive root of unity over $\mathbb{Q}$. Let $S := \mathbb{Z}_{(2)}[\zeta]$.

3.1 Indecomposable projective modules

Recall that

$$\Lambda = \left\{ \left(u, r, s, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) \in S \times S \times S \times S^{3\times 3} \ : \ \begin{array}{c} b \equiv_2 c \equiv_2 d \equiv_2 f \equiv_2 g \equiv_2 h \equiv_2 0 \\ u \equiv_4 i, \ r \equiv_4 e, \ s \equiv_4 a \end{array} \right\},$$

cf. Corollary 15.

Lemma 21 $S = \mathbb{Z}_{(2)}[\zeta]$ is a discrete valuation ring with maximal ideal $(2) \subseteq S$.

Proof. Note that $\mathbb{Z}[\zeta]$ is the integral closure of $\mathbb{Z}$ over $\mathbb{Q}(\zeta)$ and therefore is a Dedekind domain. Hence $S = \mathbb{Z}_{(2)}[\zeta] = \mathbb{Z}[\zeta]_{(2)}$ is a principal ideal domain with prime ideals

$$\{\mathfrak{p}_{(2)} \subseteq S : \mathfrak{p} \subseteq \mathbb{Z}[\zeta] \text{ prime ideal with } (2)\mathbb{Z}[\zeta] \subseteq \mathfrak{p}\}.$$

We show that $(2) \subseteq \mathbb{Z}[\zeta]$ is maximal so that $(2) \subseteq S$ is the only maximal ideal of S.

We have that $\mu_{\zeta,\mathbb{Q}}(X) = X^2 + X + 1$ is irreducible over $\mathbb{F}_2$. Hence we obtain

$$\mathbb{Z}[\zeta]/(2) \cong \mathbb{Z}[X]/(\mu_{\zeta,\mathbb{Q}}(X), 2) \cong \mathbb{F}_2[X]/(\bar{\mu}_{\zeta,\mathbb{Q}}(X)) \cong \mathbb{F}_4\,.$$

□

Lemma 22 Recall from Definition 19, that

$$\mathcal{E}_0 = \left(1, 0, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \quad \mathcal{E}_1 = \left(0, 1, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \quad \mathcal{E}_2 = \left(0, 0, 1, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

We have the orthogonal decomposition $1_\Lambda = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2$ into primitive idempotent elements of Λ. We obtain the Peirce decomposition

$$\Lambda = \mathcal{E}_0\Lambda\mathcal{E}_0 \oplus \mathcal{E}_0\Lambda\mathcal{E}_1 \oplus \mathcal{E}_0\Lambda\mathcal{E}_2 \oplus \mathcal{E}_1\Lambda\mathcal{E}_0 \oplus \mathcal{E}_1\Lambda\mathcal{E}_1 \oplus \mathcal{E}_1\Lambda\mathcal{E}_2 \oplus \mathcal{E}_2\Lambda\mathcal{E}_0 \oplus \mathcal{E}_2\Lambda\mathcal{E}_1 \oplus \mathcal{E}_2\Lambda\mathcal{E}_2\,.$$

Proof. We show that $\mathcal{E}_0, \mathcal{E}_1$ and $\mathcal{E}_2$ are primitive. Let $j \in \{0,1,2\}$.

Suppose $\mathcal{E}_j = \alpha_j + \beta_j$ with $\alpha_j, \beta_j \in \Lambda\backslash\{0\}$ orthogonal idempotent elements. We have

$$\alpha_j = (\alpha_j + \beta_j)\alpha_j(\alpha_j + \beta_j) = \mathcal{E}_j\alpha_j\mathcal{E}_j \in \mathcal{E}_j\Lambda\mathcal{E}_j\,.$$

We write $(u,i) := \left(u,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&i\end{pmatrix}\right) \in \mathcal{E}_0\Lambda\mathcal{E}_0$, $(r,e) := \left(r,0,0,\begin{pmatrix}0&0&0\\0&e&0\\0&0&0\end{pmatrix}\right) \in \mathcal{E}_1\Lambda\mathcal{E}_1$ and $(s,a) := \left(s,0,0,\begin{pmatrix}a&0&0\\0&0&0\\0&0&0\end{pmatrix}\right) \in \mathcal{E}_2\Lambda\mathcal{E}_2$.

Then there exist $x_j, y_j \in S$ with $x_j \equiv_4 y_j$ and $\alpha_j = (x_j, y_j)$ since $\alpha_j \in \mathcal{E}_j\Lambda\mathcal{E}_j$. In addition we have $\beta_j = (1-x_j, 1-y_j)$ since $\alpha_j + \beta_j = \mathcal{E}_1 = (1,1)$.

The element α_j is idempotent, therefore x_j, y_j are idempotent as well. Since 0 and 1 are the only idempotent elements in S, we obtain $\alpha_j \in \{(0,0),(1,0),(0,1),(1,1)\}$.

However, $\alpha_j \neq 0$ and $\beta_j = (1,1) - \alpha_j \neq 0$, so $\alpha_j = (0,0)$ or $\alpha_j = (1,1)$ is not possible. Furthermore we know that $x_j \equiv_4 y_j$ so that $\alpha_j = (1,0)$ and $\alpha_j = (0,1)$ is also not possible. This is a contradiction. So we obtain that there exist no such x_j and y_j and $\mathcal{E}_j$ must be primitive. □

Definition 23 We denote the indecomposable projective modules belonging to the idempotent elements from Lemma 22 by

$$P_0 = \Lambda\mathcal{E}_0 = \left\{ \left(u,0,0,\begin{pmatrix}0&0&c\\0&0&f\\0&0&i\end{pmatrix}\right) : \begin{array}{c} u,c,f,i \in S \\ u \equiv_4 i,\ c \equiv_2 0 \equiv_2 f \end{array} \right\}$$
$$P_1 = \Lambda\mathcal{E}_1 = \left\{ \left(0,r,0,\begin{pmatrix}0&b&0\\0&e&0\\0&h&0\end{pmatrix}\right) : \begin{array}{c} r,b,e,h \in S \\ r \equiv_4 e,\ b \equiv_2 0 \equiv_2 h \end{array} \right\}$$
$$P_2 = \Lambda\mathcal{E}_2 = \left\{ \left(0,0,s,\begin{pmatrix}a&0&0\\d&0&0\\g&0&0\end{pmatrix}\right) : \begin{array}{c} s,a,d,g \in S \\ s \equiv_4 a,\ d \equiv_2 0 \equiv_2 g \end{array} \right\}.$$

By abuse of notation we often write

$$\left(u,\begin{pmatrix}c\\f\\i\end{pmatrix}\right) := \left(u,0,0,\begin{pmatrix}0&0&c\\0&0&f\\0&0&i\end{pmatrix}\right) \in P_0$$
$$\left(r,\begin{pmatrix}b\\e\\h\end{pmatrix}\right) := \left(0,r,0,\begin{pmatrix}0&b&0\\0&e&0\\0&h&0\end{pmatrix}\right) \in P_1$$
$$\left(s,\begin{pmatrix}a\\d\\g\end{pmatrix}\right) := \left(0,0,s,\begin{pmatrix}a&0&0\\d&0&0\\g&0&0\end{pmatrix}\right) \in P_2.$$

So given $\lambda := \left(u',r',s',\begin{pmatrix}a'&b'&c'\\d'&e'&f'\\g'&h'&i'\end{pmatrix}\right) \in \Lambda$ and

$x := \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$, $y := \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$ and $z := \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$ we have

$$\lambda \cdot x = \left(u'u, \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0,$$

$$\lambda \cdot y = \left(r'r, \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1,$$

$$\lambda \cdot z = \left(s's, \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2.$$

Lemma 24 Let

$$b_{0,0} = \left(1,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \quad b_{1,0} = \left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\right) \quad b_{2,0} = \left(0,0,0, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$b_{0,1} = \left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\right) \quad b_{1,1} = \left(0,1,0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \quad b_{2,1} = \left(0,0,0, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$b_{0,2} = \left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\right) \quad b_{1,2} = \left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \quad b_{2,2} = \left(0,0,1, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$\widetilde{b}_{0,0} = \left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right) \quad \widetilde{b}_{1,1} = \left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \quad \widetilde{b}_{2,2} = \left(0,0,0, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

Then $(b_{i,i}, \widetilde{b}_{i,i})$ is an S-linear basis of $\mathcal{E}_i\Lambda\mathcal{E}_i$ for $i \in \{0,1,2\}$ and $(b_{i,j})$ is an S-linear basis of $\mathcal{E}_i\Lambda\mathcal{E}_j$ for $i, j \in \{0,1,2\}$ with $i \neq j$.

So altogether, $(b_{i,j} : i, j \in \{0,1,2\}) \sqcup (\widetilde{b}_{i,i} : i \in \{0,1,2\})$ is an S-linear basis of Λ.

Let $i, j \in \{0,1,2\}$. We identify along

$$\begin{aligned} \mathrm{Hom}_\Lambda(P_i, P_j) &\xrightarrow{\sim} \mathcal{E}_i\Lambda\mathcal{E}_j \\ \varphi &\longmapsto \mathcal{E}_i\varphi \\ (\mu\mathcal{E}_i \mapsto \mu\mathcal{E}_i\lambda\mathcal{E}_j) &\longleftarrow \mathcal{E}_i\lambda\mathcal{E}_j\,. \end{aligned}$$

This yields the following map.

$$\begin{aligned} P_i &\xrightarrow{b_{i,j}} P_j \\ \xi &\mapsto \xi b_{i,j} \end{aligned}$$

This map can also be written as an S-linear matrix; cf. Appendix, Remark 56.

We have the following multiplication table for the basis elements.

$(\cdot)$	$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$\widetilde{b}_{0,0}$	$\widetilde{b}_{1,1}$	$\widetilde{b}_{2,2}$
$b_{0,0}$	$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	0	0	0	0	0	0	$\widetilde{b}_{0,0}$	0	0
$b_{0,1}$	0	0	0	$\widetilde{b}_{0,0}$	$b_{0,1}$	$2b_{0,2}$	0	0	0	0	$4b_{0,1}$	0
$b_{0,2}$	0	0	0	0	0	0	$\widetilde{b}_{0,0}$	$2b_{0,1}$	$b_{0,2}$	0	0	$4b_{0,2}$
$b_{1,0}$	$b_{1,0}$	$\widetilde{b}_{1,1}$	$2b_{1,2}$	0	0	0	0	0	0	$4b_{1,0}$	0	0
$b_{1,1}$	0	0	0	$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	0	0	0	0	$\widetilde{b}_{1,1}$	0
$b_{1,2}$	0	0	0	0	0	0	$2b_{1,0}$	$\widetilde{b}_{1,1}$	$b_{1,2}$	0	0	$4b_{1,2}$
$b_{2,0}$	$b_{2,0}$	$2b_{2,1}$	$\widetilde{b}_{2,2}$	0	0	0	0	0	0	$4b_{2,0}$	0	0
$b_{2,1}$	0	0	0	$2b_{2,0}$	$b_{2,1}$	$\widetilde{b}_{2,2}$	0	0	0	0	$4b_{2,1}$	0
$b_{2,2}$	0	0	0	0	0	0	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	0	0	$\widetilde{b}_{2,2}$
$\widetilde{b}_{0,0}$	$\widetilde{b}_{0,0}$	$4b_{0,1}$	$4b_{0,2}$	0	0	0	0	0	0	$4\widetilde{b}_{0,0}$	0	0
$\widetilde{b}_{1,1}$	0	0	0	$4b_{1,0}$	$\widetilde{b}_{1,1}$	$4b_{1,2}$	0	0	0	0	$4\widetilde{b}_{1,1}$	0
$\widetilde{b}_{2,2}$	0	0	0	0	0	0	$4b_{2,0}$	$4b_{2,1}$	$\widetilde{b}_{2,2}$	0	0	$4\widetilde{b}_{2,2}$

For $i, j, r, s \in \{0, 1, 2\}$ the multiplication can be written as

$$b_{i,j} \cdot b_{r,s} = \begin{cases} 0 & , \quad j \neq r \\ b_{i,i} & , \quad i = j = r = s \\ \widetilde{b}_{i,i} & , \quad i = s \neq j = r \\ 2b_{i,s} & , \quad i \neq j = r \neq s \wedge i \neq s \\ b_{i,s} & , \quad i = j = r \neq s \vee i \neq j = r = s \end{cases}$$

$$\widetilde{b}_{i,i} \cdot b_{r,s} = \begin{cases} 0 & , \quad i \neq r \\ \widetilde{b}_{i,i} & , \quad i = r = s \\ 4b_{i,s} & , \quad i = r \neq s \end{cases}$$

$$b_{i,j} \cdot \widetilde{b}_{r,r} = \begin{cases} 0 & , \quad j \neq r \\ \widetilde{b}_{r,r} & , \quad i = j = r \\ 4b_{i,j} & , \quad i \neq j = r \end{cases}$$

$$\widetilde{b}_{i,i} \cdot \widetilde{b}_{r,r} = \begin{cases} 0 & , \quad i \neq r \\ 4\widetilde{b}_{i,i} & , \quad i = r\,. \end{cases}$$

3.2 Projective resolution of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$

We aim to construct the minimal projective resolution

$$\cdots \longrightarrow Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0$$

of the trivial module S over Λ.

For $l \in \mathbb{Z}$ we write $\bar{l}$ for the element in $\{0, 1, 2\}$ such that $l \equiv_3 \bar{l}$. Recall that S can be seen as a Λ-module called the trivial module over Λ; cf. Remark 17.

Definition 25 We define the following projective Λ-modules.

$$Q_{2l} := \left(\bigoplus_{i\in[0,l-1]} (P_{\overline{2l+2i}} \oplus P_{\overline{2l+2i+1}}) \right) \oplus P_{\overline{l}}\,, \quad \text{for } l \geq 0$$
$$Q_{2l+1} := \left(\bigoplus_{i\in[0,l]} (P_{\overline{2l+2i+1}} \oplus P_{\overline{2l+2i+2}}) \right), \quad \text{for } l \geq 0$$

For $l \geq 0$ let

$\pi_{2l,i} \ :\ Q_{2l} \to P_{\overline{2l+2i}} \oplus P_{\overline{2l+2i+1}},\ \big((x_j, y_j)_{j\in[0,l-1]}, z\big) \mapsto (x_i, y_i)$ for $i \in [0, l-1]$

$\pi'_{2l} \ :\ Q_{2l} \to P_{\overline{l}},\ \big((x_j, y_j)_{j\in[0,l-1]}, z\big) \mapsto z$

$\pi_{2l+1,i} \ :\ Q_{2l+1} \to P_{\overline{2l+2i+1}} \oplus P_{\overline{2l+2i+2}},\ \big((x_j, y_j)_{j\in[0,l]}\big) \mapsto (x_i, y_i)$ for $i \in [0, l]$

$\pi_{P_{\overline{l}}} \ :\ P_{\overline{l}} \oplus P_{\overline{l+1}} \to P_{\overline{l}},\ (x, y) \mapsto x$

and

$\iota_{2l,i} \ :\ P_{\overline{2l+2i}} \oplus P_{\overline{2l+2i+1}} \to Q_{2l},\ (x, y) \mapsto \big((\delta_{i,j}\ x,\ \delta_{i,j}\ y)_{j\in[0,l-1]}, 0\big)$ for $i \in [0, l-1]$

$\iota'_{2l} \ :\ P_{\overline{l}} \to Q_{2l},\ z \mapsto \big((0, 0)_{j\in[0,l-1]}, z\big)$

$\iota_{2l+1,i} \ :\ P_{\overline{2l+2i+1}} \oplus P_{\overline{2l+2i+2}} \to Q_{2l+1},\ (x, y) \mapsto \big((\delta_{i,j}\ x,\ \delta_{i,j}\ y)_{j\in[0,l]}\big)$ for $i \in [0, l]$

$\iota_{P_{\overline{l+1}}} : : P_{\overline{l+1}} \to P_{\overline{l+1}} \oplus P_{\overline{l+2}},\ x \mapsto (x, 0).$

Definition 26 Let

$$B_0^+ := \begin{pmatrix} b_{1,0} & -2b_{1,1} \\ b_{2,0} & -b_{2,1} \end{pmatrix} :\ P_1 \oplus P_2 \to P_0 \oplus P_1\,,\quad B_0^- := \begin{pmatrix} b_{2,1} & -2b_{2,2} \\ b_{0,1} & -b_{0,2} \end{pmatrix} :\ P_2 \oplus P_0 \to P_1 \oplus P_2$$
$$B_1^+ := \begin{pmatrix} b_{0,2} & -2b_{0,0} \\ b_{1,2} & -b_{1,0} \end{pmatrix} :\ P_0 \oplus P_1 \to P_2 \oplus P_0\,,\quad B_1^- := \begin{pmatrix} b_{1,0} & -2b_{1,1} \\ b_{2,0} & -b_{2,1} \end{pmatrix} :\ P_1 \oplus P_2 \to P_0 \oplus P_1$$
$$B_2^+ := \begin{pmatrix} b_{2,1} & -2b_{2,2} \\ b_{0,1} & -b_{0,2} \end{pmatrix} :\ P_2 \oplus P_0 \to P_1 \oplus P_2\,,\quad B_2^- := \begin{pmatrix} b_{0,2} & -2b_{0,0} \\ b_{1,2} & -b_{1,0} \end{pmatrix} :\ P_0 \oplus P_1 \to P_2 \oplus P_0$$

$$C_0^+ := \begin{pmatrix} 0 & -b_{0,1} \\ 0 & -2b_{1,1} \end{pmatrix} :\ P_0 \oplus P_1 \to P_0 \oplus P_1\,,\quad C_0^- := \begin{pmatrix} 2b_{1,1} & -b_{1,2} \\ 0 & 0 \end{pmatrix} :\ P_1 \oplus P_2 \to P_1 \oplus P_2$$
$$C_1^+ := \begin{pmatrix} 0 & -b_{2,0} \\ 0 & -2b_{0,0} \end{pmatrix} :\ P_2 \oplus P_0 \to P_2 \oplus P_0\,,\quad C_1^- := \begin{pmatrix} 2b_{0,0} & -b_{0,1} \\ 0 & 0 \end{pmatrix} :\ P_0 \oplus P_1 \to P_0 \oplus P_1$$
$$C_2^+ := \begin{pmatrix} 0 & -b_{1,2} \\ 0 & -2b_{2,2} \end{pmatrix} :\ P_1 \oplus P_2 \to P_1 \oplus P_2\,,\quad C_2^- := \begin{pmatrix} 2b_{2,2} & -b_{2,0} \\ 0 & 0 \end{pmatrix} :\ P_2 \oplus P_0 \to P_2 \oplus P_0\,.$$

So for $i \in \{0, 1, 2\}$ we have $B_{\overline{i}}^+ = B_{\overline{i+1}}^-$ and

$$B_i^+ = \begin{pmatrix} b_{\overline{-i+1},\overline{-i}} & -2b_{\overline{-i+1},\overline{-i+1}} \\ b_{\overline{-i+2},\overline{-i}} & -b_{\overline{-i+2},\overline{-i+1}} \end{pmatrix} :\ P_{\overline{-i+1}} \oplus P_{\overline{-i+2}} \to P_{\overline{-i}} \oplus P_{\overline{-i+1}}$$
$$B_i^- = \begin{pmatrix} b_{\overline{-i+2},\overline{-i+1}} & -2b_{\overline{-i+2},\overline{-i+2}} \\ b_{\overline{-i},\overline{-i+1}} & -b_{\overline{-i},\overline{-i+2}} \end{pmatrix} :\ P_{\overline{-i+2}} \oplus P_{\overline{-i}} \to P_{\overline{-i+1}} \oplus P_{\overline{-i+2}}$$
$$C_i^+ = \begin{pmatrix} 0 & -b_{\overline{-i},\overline{-i+1}} \\ 0 & -2b_{\overline{-i+1},\overline{-i+1}} \end{pmatrix} :\ P_{\overline{-i}} \oplus P_{\overline{-i+1}} \to P_{\overline{-i}} \oplus P_{\overline{-i+1}}$$
$$C_i^- = \begin{pmatrix} 2b_{\overline{-i+1},\overline{-i+1}} & -b_{\overline{-i+1},\overline{-i+2}} \\ 0 & 0 \end{pmatrix} :\ P_{\overline{-i+1}} \oplus P_{\overline{-i+2}} \to P_{\overline{-i+1}} \oplus P_{\overline{-i+2}}\,.$$

Using this, we define the following Λ-linear maps for $l \geq 0$.

$$\varepsilon \;:\; Q_0 \to S,\ \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \mapsto u$$

$$\begin{aligned} d_{2l} \;:=\; & \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i}\, B^{+}_{\overline{l+i}}\, \iota_{2l,i}\right) + \left(\pi_{2l+1,l}\ (B^{+}_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right) \\ + \; & \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i+1}\, C^{+}_{\overline{l+i}}\, \iota_{2l,i}\right) :\ Q_{2l+1} \to Q_{2l} \end{aligned}$$

$$\begin{aligned} d_{2l+1} \;:=\; & \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\, B^{-}_{\overline{l+i}}\, \iota_{2l+1,i}\right) \\ + \; & \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\, C^{-}_{\overline{l+i}}\, \iota_{2l+1,i}\right) + \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}})\ \iota_{2l+1,l}\right) :\ Q_{2l+2} \to Q_{2l+1} \end{aligned}$$

Remark 27 The differentials from Definition 26 can be written as follows.

$$d_0 = \begin{pmatrix} b_{1,0} \\ b_{2,0} \end{pmatrix} :\ Q_1 = P_1 \oplus P_2 \longrightarrow P_0 = Q_0$$

$$d_1 = \begin{pmatrix} b_{2,1} & -2b_{2,2} \\ b_{0,1} & -b_{0,2} \\ 2b_{1,1} & -b_{1,2} \end{pmatrix} :\ Q_2 = P_2 \oplus P_0 \oplus P_1 \longrightarrow P_1 \oplus P_2 = Q_1$$

$$d_2 = \begin{pmatrix} b_{0,2} & -2b_{0,0} & 0 \\ b_{1,2} & -b_{1,0} & 0 \\ 0 & -b_{2,0} & b_{2,1} \\ 0 & -2b_{0,0} & b_{0,1} \end{pmatrix} :\ Q_3 = P_0 \oplus P_1 \oplus P_2 \oplus P_0 \longrightarrow P_2 \oplus P_0 \oplus P_1 = Q_2$$

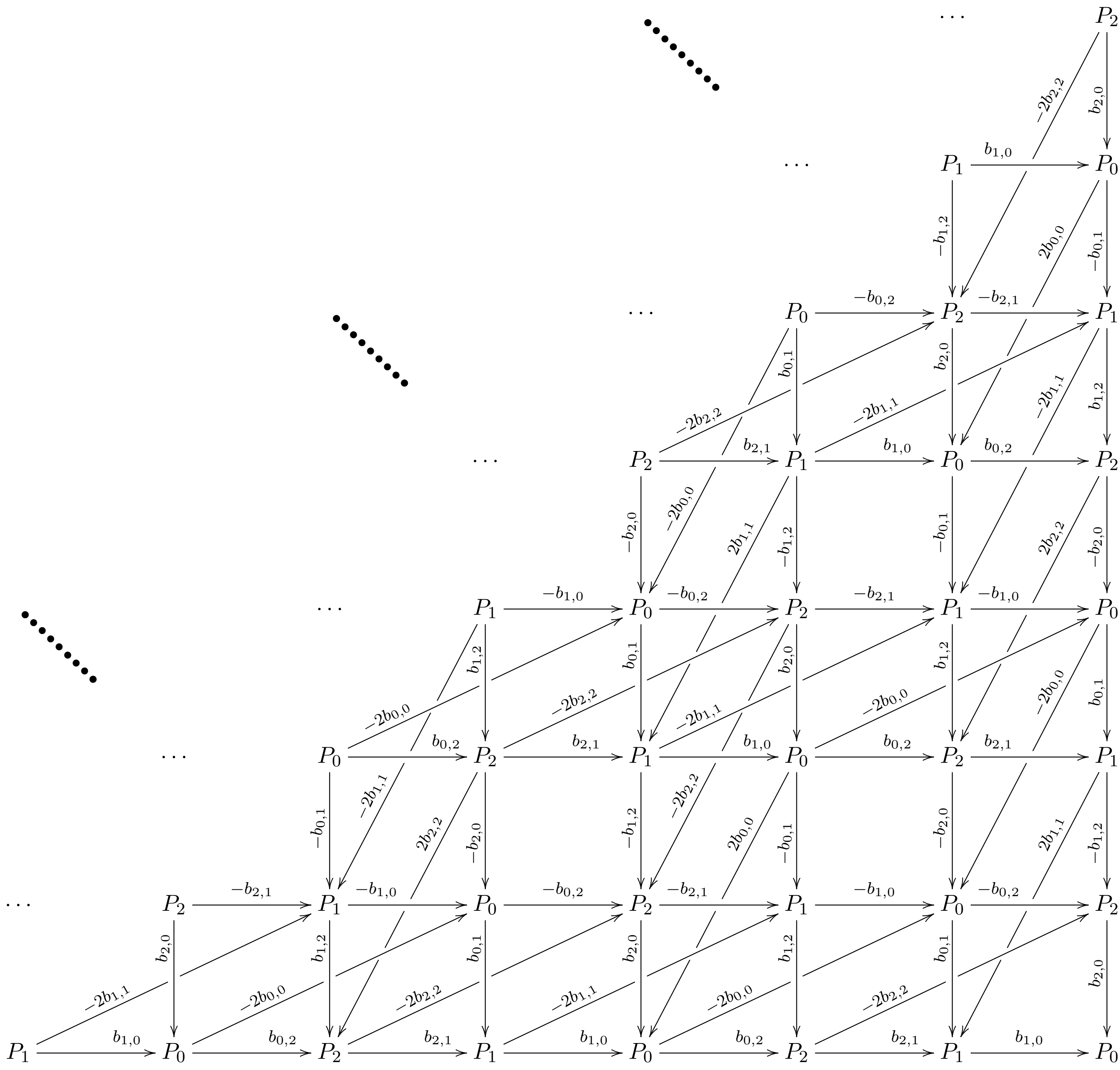
$$d_3 = \begin{pmatrix} b_{1,0} & -2b_{1,1} & 0 & 0 \\ b_{2,0} & -b_{2,1} & 0 & 0 \\ 2b_{0,0} & -b_{0,1} & b_{0,2} & -2b_{0,0} \\ 0 & 0 & b_{1,2} & -b_{1,0} \\ 0 & 0 & 2b_{2,2} & -b_{2,0} \end{pmatrix} :\ Q_4 = P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \longrightarrow P_0 \oplus P_1 \oplus P_2 \oplus P_0 = Q_3$$

$$d_4 = \begin{pmatrix} b_{2,1} & -2b_{2,2} & 0 & 0 & 0 \\ b_{0,1} & -b_{0,2} & 0 & 0 & 0 \\ 0 & -b_{1,2} & b_{1,0} & -2b_{1,1} & 0 \\ 0 & -2b_{2,2} & b_{2,0} & -b_{2,1} & 0 \\ 0 & 0 & 0 & -b_{0,1} & b_{0,2} \\ 0 & 0 & 0 & -2b_{1,1} & b_{1,2} \end{pmatrix} : \quad \begin{aligned} Q_5 = P_2 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_0 \oplus P_1 \\ \longrightarrow P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \end{aligned}$$

$$d_5 = \begin{pmatrix} b_{0,2} & -2b_{0,0} & 0 & 0 & 0 & 0 \\ b_{1,2} & -b_{1,0} & 0 & 0 & 0 & 0 \\ 2b_{2,2} & -b_{2,0} & b_{2,1} & -2b_{2,2} & 0 & 0 \\ 0 & 0 & b_{0,1} & -b_{0,2} & 0 & 0 \\ 0 & 0 & 2b_{1,1} & -b_{1,2} & b_{1,0} & -2b_{1,1} \\ 0 & 0 & 0 & 0 & b_{2,0} & -b_{2,1} \\ 0 & 0 & 0 & 0 & 2b_{0,0} & -b_{0,1} \end{pmatrix} : \quad \begin{aligned} Q_6 = P_0 \oplus P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_0 \\ \longrightarrow P_2 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_0 \oplus P_1 \end{aligned}$$

$$d_6 = \begin{pmatrix} b_{1,0} & -2b_{1,1} & 0 & 0 & 0 & 0 & 0 \\ b_{2,0} & -b_{2,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_{0,1} & b_{0,2} & -2b_{0,0} & 0 & 0 & 0 \\ 0 & -2b_{1,1} & b_{1,2} & -b_{1,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_{2,0} & b_{2,1} & -2b_{2,2} & 0 \\ 0 & 0 & 0 & -2b_{0,0} & b_{0,1} & -b_{0,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_{1,2} & b_{1,0} \\ 0 & 0 & 0 & 0 & 0 & -2b_{2,2} & b_{2,0} \end{pmatrix} : \begin{aligned} Q_7 &= P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \\ &\longrightarrow P_0 \oplus P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_0 \end{aligned}$$

These differentials can be visualized in the following diagram.



Theorem 28 Consider the following sequences of projective Λ-modules from Definition 25 with the differentials from Definition 26.

$$Q' := \left(\cdots \longrightarrow Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \xrightarrow{\varepsilon} S \longrightarrow 0 \longrightarrow \cdots \right)$$
$$Q := \left(\cdots \longrightarrow Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \longrightarrow 0 \longrightarrow \cdots \right)$$

Then Q' is an augmented minimal projective resolution of the trivial Λ-module S and Q the corresponding minimal projective resolution.

Proof. Let $l \geq 0$.

We verify that Q' is a complex. In Section 3.2.1 we show that the composition of two successive differentials is the zero map. For this we have to consider the three cases
$d_0\varepsilon = 0$, shown in §3.2.1.1,
$d_{2l+1} \cdot d_{2l} = 0$, shown in §3.2.1.2 and
$d_{2l+2} \cdot d_{2l+1} = 0$, shown in §3.2.1.3.

We verify that Q' is acyclic. In Section 3.2.2 we show that Q' is acyclic via a homotopy h defined in Definition 30. Again we have to distinguish several cases:
$h_{-1}\varepsilon = 1$, $h_0 d_0 + \varepsilon h_{-1} = 1$, shown in §3.2.2.1,
$h_{2l+1}d_{2l+1} + d_{2l}h_{2l} = 1$, shown in §3.2.2.2 and
$h_{2l+2}d_{2l+2} + d_{2l+1}h_{2l+1} = 1$, shown in §3.2.2.3.

Then Q' is an augmented projective resolution of S by Lemma 2.

We verify that Q is minimal. In Section 3.2.3 we show that the projective resolution Q is minimal.

3.2.1 Differential condition

3.2.1.1 Differential condition at 0

We show that $d_0\varepsilon = 0$. Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$ and $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$.

We obtain

$$\left(\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right), \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \right) d_0\varepsilon = \left(0, \begin{pmatrix} 2b \\ 2e \\ 2h \end{pmatrix}\right) \varepsilon + \left(0, \begin{pmatrix} 2a \\ 2d \\ 2g \end{pmatrix}\right) \varepsilon = 0 + 0 = 0.$$

3.2.1.2 Differential condition, odd-even case

We want to show that $d_{2l+1} \cdot d_{2l} \overset{!}{=} 0$ for $l \geq 0$. We have

$$d_{2l+1} \cdot d_{2l} = \left(\sum_{i \in [0,l]} \pi_{2l+2,i} \, B^{-}_{\overline{l+i}} \, \iota_{2l+1,i} \right) \left(\sum_{j \in [0,l-1]} \pi_{2l+1,j} \, B^{+}_{\overline{l+j}} \, \iota_{2l,j} \right)$$

$$
\begin{aligned}
&+ \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ B^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\pi_{2l+1,l}\ (B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right)\\
&+ \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ B^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\sum_{j\in[0,l-1]} \pi_{2l+1,j+1}\ C^+_{\overline{l+j}}\ \iota_{2l,j}\right)\\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ C^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\sum_{j\in[0,l-1]} \pi_{2l+1,j}\ B^+_{\overline{l+j}}\ \iota_{2l,j}\right)\\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ C^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\pi_{2l+1,l}\ (B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right)\\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ C^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\sum_{j\in[0,l-1]} \pi_{2l+1,j+1}\ C^+_{\overline{l+j}}\ \iota_{2l,j}\right)\\
&+ \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\left(\sum_{j\in[0,l-1]} \pi_{2l+1,j}\ B^+_{\overline{l+j}}\ \iota_{2l,j}\right)\\
&+ \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\left(\pi_{2l+1,l}\ (B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right)\\
&+ \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\left(\sum_{j\in[0,l-1]} \pi_{2l+1,j+1}\ C^+_{\overline{l+j}}\ \iota_{2l,j}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i}\ B^-_{\overline{l+i}}B^+_{\overline{l+i}}\ \iota_{2l,i}\right) + \left(\pi_{2l+2,l}\ (B^-_{\overline{2l}}B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right)\\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ B^-_{\overline{l+i+1}}C^+_{\overline{l+i}}\ \iota_{2l,i}\right) + \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ C^-_{\overline{l+i}}B^+_{\overline{l+i}}\ \iota_{2l,i}\right) + 0\\
&+ \left(\sum_{i\in[0,l-2]} \pi_{2l+2,i+2}\ C^-_{\overline{l+i+1}}C^+_{\overline{l+i}}\ \iota_{2l,i}\right) + 0\\
&+ \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}}B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right) + \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}}C^+_{\overline{2l-1}})\ \iota_{2l,l-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i}\ B^-_{\overline{l+i}}B^+_{\overline{l+i}}\ \iota_{2l,i}\right) + \left(\pi_{2l+2,l}\ (B^-_{\overline{2l}}B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right)\\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ (B^-_{\overline{l+i+1}}C^+_{\overline{l+i}} + C^-_{\overline{l+i}}B^+_{\overline{l+i}})\ \iota_{2l,i}\right)\\
&+ \left(\sum_{i\in[0,l-2]} \pi_{2l+2,i+2}\ C^-_{\overline{l+i+1}}C^+_{\overline{l+i}}\ \iota_{2l,i}\right)
\end{aligned}
$$

$$+\left(\pi'_{2l+2}\left(\iota_{P_{\overline{l+1}}} C^-_{\overline{2l}} B^+_{\overline{2l}} \pi_{P_{\bar{l}}}\right) \iota'_{2l}\right) + \left(\pi'_{2l+2}\left(\iota_{P_{\overline{l+1}}} C^-_{\overline{2l}} C^+_{\overline{2l-1}}\right) \iota_{2l,l-1}\right)$$

$$\overset{(*)}{=} \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i}\begin{pmatrix}0&0\\0&0\end{pmatrix}\iota_{2l,i}\right) + \left(\pi_{2l+2,l}\begin{pmatrix}0\\0\end{pmatrix}\iota'_{2l}\right)$$

$$+\left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\begin{pmatrix}0&0\\0&0\end{pmatrix}\iota_{2l,i}\right) + \left(\sum_{i\in[0,l-2]} \pi_{2l+2,i+2}\begin{pmatrix}0&0\\0&0\end{pmatrix}\iota_{2l,i}\right)$$

$$+\left(\pi'_{2l+2}(0)\iota'_{2l}\right) + \left(\pi'_{2l+2}\begin{pmatrix}0&0\end{pmatrix}\iota_{2l,l-1}\right)$$

$$= 0.$$

For $(*)$ we use the following auxiliary calculations; cf. Lemma 22 and Definition 26.

$$\begin{aligned}
B_0^- \cdot B_0^+ &= \begin{pmatrix} b_{2,1}b_{1,0} - 2b_{2,2}b_{2,0} & -2b_{2,1}b_{1,1} + 2b_{2,2}b_{2,1} \\ b_{0,1}b_{1,0} - b_{0,2}b_{2,0} & -2b_{0,1}b_{1,1} + b_{0,2}b_{2,1} \end{pmatrix} = \begin{pmatrix} 2b_{2,0} - 2b_{2,0} & -2b_{2,1} + 2b_{2,1} \\ \widetilde{b}_{0,0} - \widetilde{b}_{0,0} & -2b_{0,1} + 2b_{0,1} \end{pmatrix} \\
&= \begin{pmatrix}0&0\\0&0\end{pmatrix} \\
B_1^- \cdot B_1^+ &= \begin{pmatrix} b_{1,0}b_{0,2} - 2b_{1,1}b_{1,2} & -2b_{1,0}b_{0,0} + 2b_{1,1}b_{1,0} \\ b_{2,0}b_{0,2} - b_{2,1}b_{1,2} & -2b_{2,0}b_{0,0} + b_{2,1}b_{1,0} \end{pmatrix} = \begin{pmatrix} 2b_{1,2} - 2b_{1,2} & -2b_{1,0} + 2b_{1,0} \\ \widetilde{b}_{2,2} - \widetilde{b}_{2,2} & -2b_{2,0} + 2b_{2,0} \end{pmatrix} \\
&= \begin{pmatrix}0&0\\0&0\end{pmatrix} \\
B_2^- \cdot B_2^+ &= \begin{pmatrix} b_{0,2}b_{2,1} - 2b_{0,0}b_{0,1} & -2b_{0,2}b_{2,2} + 2b_{0,0}b_{0,2} \\ b_{1,2}b_{2,1} - b_{1,0}b_{0,1} & -2b_{1,2}b_{2,2} + b_{1,0}b_{0,2} \end{pmatrix} = \begin{pmatrix} 2b_{0,1} - 2b_{0,1} & -2b_{0,2} + 2b_{0,2} \\ \widetilde{b}_{1,1} - \widetilde{b}_{1,1} & -2b_{1,2} + 2b_{1,2} \end{pmatrix} \\
&= \begin{pmatrix}0&0\\0&0\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
B_1^- \cdot C_0^+ + C_0^- \cdot B_0^+ &= \begin{pmatrix} 0 & -b_{1,0}b_{0,1} + 4b_{1,1}b_{1,1} \\ 0 & -b_{2,0}b_{0,1} + 2b_{2,1}b_{1,1} \end{pmatrix} + \begin{pmatrix} 2b_{1,1}b_{1,0} - b_{1,2}b_{2,0} & -4b_{1,1}b_{1,1} + b_{1,2}b_{2,1} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\widetilde{b}_{1,1} + 4b_{1,1} \\ 0 & -2b_{2,1} + 2b_{2,1} \end{pmatrix} + \begin{pmatrix} 2b_{1,0} - 2b_{1,0} & -4b_{1,1} + \widetilde{b}_{1,1} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix}0&0\\0&0\end{pmatrix} \\
B_2^- \cdot C_1^+ + C_1^- \cdot B_1^+ &= \begin{pmatrix} 0 & -b_{0,2}b_{2,0} + 4b_{0,0}b_{0,0} \\ 0 & -b_{1,2}b_{2,0} + 2b_{1,0}b_{0,0} \end{pmatrix} + \begin{pmatrix} 2b_{0,0}b_{0,2} - b_{0,1}b_{1,2} & -4b_{0,0}b_{0,0} + b_{0,1}b_{1,0} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\widetilde{b}_{0,0} + 4b_{0,0} \\ 0 & -2b_{1,0} + 2b_{1,0} \end{pmatrix} + \begin{pmatrix} 2b_{0,2} - 2b_{0,2} & -4b_{0,0} + \widetilde{b}_{0,0} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix}0&0\\0&0\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
B_0^- \cdot C_2^+ + C_2^- \cdot B_2^+ &= \begin{pmatrix} 0 & -b_{2,1}b_{1,2} + 4b_{2,2}b_{2,2} \\ 0 & -b_{0,1}b_{1,2} + 2b_{0,2}b_{2,2} \end{pmatrix} + \begin{pmatrix} 2b_{2,2}b_{2,1} - b_{2,0}b_{0,1} & -4b_{2,2}b_{2,2} + b_{2,0}b_{0,2} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\widetilde{b}_{2,2} + 4b_{2,2} \\ 0 & -2b_{0,2} + 2b_{0,2} \end{pmatrix} + \begin{pmatrix} 2b_{2,1} - 2b_{2,1} & -4b_{2,2} + \widetilde{b}_{2,2} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
C_1^- \cdot C_0^+ &= \begin{pmatrix} 0 & -2b_{0,0}b_{0,1} + 2b_{0,1}b_{1,1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2b_{0,1} + 2b_{0,1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
C_2^- \cdot C_1^+ &= \begin{pmatrix} 0 & -2b_{2,2}b_{2,0} + 2b_{2,0}b_{0,0} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2b_{2,0} + 2b_{2,0} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
C_0^- \cdot C_2^+ &= \begin{pmatrix} 0 & -2b_{1,1}b_{1,2} + 2b_{1,2}b_{2,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2b_{1,2} + 2b_{1,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\iota_{P_1} C_0^- \cdot B_0^+ \pi_{P_0} &= \begin{pmatrix} 2b_{1,1} & -b_{1,2} \end{pmatrix} \begin{pmatrix} b_{1,0} \\ b_{2,0} \end{pmatrix} = (2b_{1,1}b_{1,0} - b_{1,2}b_{2,0}) = (2b_{1,0} - 2b_{1,0}) = (0) \\
\iota_{P_0} C_1^- \cdot B_1^+ \pi_{P_2} &= \begin{pmatrix} 2b_{0,0} & -b_{0,1} \end{pmatrix} \begin{pmatrix} b_{0,2} \\ b_{1,2} \end{pmatrix} = (2b_{0,0}b_{0,2} - b_{0,1}b_{1,2}) = (2b_{0,2} - 2b_{0,2}) = (0) \\
\iota_{P_2} C_2^- \cdot B_2^+ \pi_{P_1} &= \begin{pmatrix} 2b_{2,2} & -b_{2,0} \end{pmatrix} \begin{pmatrix} b_{2,1} \\ b_{0,1} \end{pmatrix} = (2b_{2,2}b_{2,1} - b_{2,0}b_{0,1}) = (2b_{2,1} - 2b_{2,1}) = (0)
\end{aligned}$$

3.2.1.3 Differential condition, even-odd case

We want to show that $d_{2l+2} \cdot d_{2l+1} \stackrel{!}{=} 0$ for $l \geq 0$. We have

$$\begin{aligned}
d_{2l+2} \cdot d_{2l+1} = &\left(\sum_{i \in [0,l]} \pi_{2l+3,i}\ B^+_{\overline{l+i+1}}\ \iota_{2l+2,i} \right) \left(\sum_{j \in [0,l]} \pi_{2l+2,j}\ B^-_{\overline{l+j}}\ \iota_{2l+1,j} \right) \\
&+ \left(\sum_{i \in [0,l]} \pi_{2l+3,i}\ B^+_{\overline{l+i+1}}\ \iota_{2l+2,i} \right) \left(\sum_{j \in [0,l-1]} \pi_{2l+2,j+1}\ C^-_{\overline{l+j}}\ \iota_{2l+1,j} \right) \\
&+ \left(\sum_{i \in [0,l]} \pi_{2l+3,i}\ B^+_{\overline{l+i+1}}\ \iota_{2l+2,i} \right) \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}} C^-_{\overline{2l}})\ \iota_{2l+1,l} \right) \\
&+ \left(\pi_{2l+3,l+1}\ (B^+_{\overline{2l+2}} \pi_{P_{\overline{l+1}}})\ \iota'_{2l+2} \right) \left(\sum_{j \in [0,l]} \pi_{2l+2,j}\ B^-_{\overline{l+j}}\ \iota_{2l+1,j} \right) \\
&+ \left(\pi_{2l+3,l+1}\ (B^+_{\overline{2l+2}} \pi_{P_{\overline{l+1}}})\ \iota'_{2l+2} \right) \left(\sum_{j \in [0,l-1]} \pi_{2l+2,j+1}\ C^-_{\overline{l+j}}\ \iota_{2l+1,j} \right) \\
&+ \left(\pi_{2l+3,l+1}\ (B^+_{\overline{2l+2}} \pi_{P_{\overline{l+1}}})\ \iota'_{2l+2} \right) \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}} C^-_{\overline{2l}})\ \iota_{2l+1,l} \right) \\
&+ \left(\sum_{i \in [0,l]} \pi_{2l+3,i+1}\ C^+_{\overline{l+i+1}}\ \iota_{2l+2,i} \right) \left(\sum_{j \in [0,l]} \pi_{2l+2,j}\ B^-_{\overline{l+j}}\ \iota_{2l+1,j} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\sum_{i\in[0,l]} \pi_{2l+3,i+1}\ C^{+}_{\overline{l+i+1}}\ \iota_{2l+2,i}\right)\left(\sum_{j\in[0,l-1]} \pi_{2l+2,j+1}\ C^{-}_{\overline{l+j}}\ \iota_{2l+1,j}\right)\\
&+\left(\sum_{i\in[0,l]} \pi_{2l+3,i+1}\ C^{+}_{\overline{l+i+1}}\ \iota_{2l+2,i}\right)\left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}})\ \iota_{2l+1,l}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\sum_{i\in[0,l]} \pi_{2l+3,i}\ B^{+}_{\overline{l+i+1}} B^{-}_{\overline{l+i}}\ \iota_{2l+1,i}\right)+\left(\sum_{i\in[0,l-1]} \pi_{2l+3,i+1}\ B^{+}_{\overline{l+i+2}} C^{-}_{\overline{l+i}}\ \iota_{2l+1,i}\right)+0\\
&+0+0+\left(\pi_{2l+3,l+1}\ (B^{+}_{\overline{2l+2}} \pi_{P_{\overline{l+1}}} \iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}})\ \iota_{2l+1,l}\right)\\
&+\left(\sum_{i\in[0,l]} \pi_{2l+3,i+1}\ C^{+}_{\overline{l+i+1}} B^{-}_{\overline{l+i}}\ \iota_{2l+1,i}\right)+\left(\sum_{i\in[0,l-1]} \pi_{2l+3,i+2}\ C^{+}_{\overline{l+i+2}} C^{-}_{\overline{l+i}}\ \iota_{2l+1,i}\right)+0
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\sum_{i\in[0,l]} \pi_{2l+3,i}\ B^{+}_{\overline{l+i+1}} B^{-}_{\overline{l+i}}\ \iota_{2l+1,i}\right)\\
&+\left(\sum_{i\in[0,l-1]} \pi_{2l+3,i+1}\ (B^{+}_{\overline{l+i+2}} C^{-}_{\overline{l+i}} + C^{+}_{\overline{l+i+1}} B^{-}_{\overline{l+i}})\ \iota_{2l+1,i}\right)\\
&+\left(\pi_{2l+3,l+1}\ (B^{+}_{\overline{2l+2}} \pi_{P_{\overline{l+1}}} \iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}} + C^{+}_{\overline{2l+1}} B^{-}_{\overline{2l}})\ \iota_{2l+1,l}\right)\\
&+\left(\sum_{i\in[0,l-1]} \pi_{2l+3,i+2}\ C^{+}_{\overline{l+i+2}} C^{-}_{\overline{l+i}}\ \iota_{2l+1,i}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(**)}{=}\left(\sum_{i\in[0,l]} \pi_{2l+3,i}\begin{pmatrix}0&0\\0&0\end{pmatrix}\iota_{2l+1,i}\right)\\
&+\left(\sum_{i\in[0,l-1]} \pi_{2l+3,i+1}\begin{pmatrix}0&0\\0&0\end{pmatrix}\iota_{2l+1,i}\right)\\
&+\left(\pi_{2l+3,l+1}\begin{pmatrix}0&0\\0&0\end{pmatrix}\iota_{2l+1,l}\right)\\
&+\left(\sum_{i\in[0,l-1]} \pi_{2l+3,i+2}\begin{pmatrix}0&0\\0&0\end{pmatrix}\iota_{2l+1,i}\right)
\end{aligned}
$$

$$= 0.$$

For $(**)$ we use the following auxiliary calculations; cf. Lemma 22 and Definition 26.

$$
\begin{aligned}
B_1^+ \cdot B_0^- &= \begin{pmatrix} b_{0,2}b_{2,1} - 2b_{0,0}b_{0,1} & -2b_{0,2}b_{2,2} + 2b_{0,0}b_{0,2} \\ b_{1,2}b_{2,1} - b_{1,0}b_{0,1} & -2b_{1,2}b_{2,2} + b_{1,0}b_{0,2} \end{pmatrix} = \begin{pmatrix} 2b_{0,1} - 2b_{0,1} & -2b_{0,2} + 2b_{0,2} \\ \widetilde{b}_{1,1} - \widetilde{b}_{1,1} & -2b_{1,2} + 2b_{1,2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_2^+ \cdot B_1^- &= \begin{pmatrix} b_{2,1}b_{1,0} - 2b_{2,2}b_{2,0} & -2b_{2,1}b_{1,1} + 2b_{2,2}b_{2,1} \\ b_{0,1}b_{1,0} - b_{0,2}b_{2,0} & -2b_{0,1}b_{1,1} + b_{0,2}b_{2,1} \end{pmatrix} = \begin{pmatrix} 2b_{2,0} - 2b_{2,0} & -2b_{2,1} + 2b_{2,1} \\ \widetilde{b}_{0,0} - \widetilde{b}_{0,0} & -2b_{0,1} + 2b_{0,1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_0^+ \cdot B_2^- &= \begin{pmatrix} b_{1,0}b_{0,2} - 2b_{1,1}b_{1,2} & -2b_{1,0}b_{0,0} + 2b_{1,1}b_{1,0} \\ b_{2,0}b_{0,2} - b_{2,1}b_{1,2} & -2b_{2,0}b_{0,0} + b_{2,1}b_{1,0} \end{pmatrix} = \begin{pmatrix} 2b_{1,2} - 2b_{1,2} & -2b_{1,0} + 2b_{1,0} \\ \widetilde{b}_{2,2} - \widetilde{b}_{2,2} & -2b_{2,0} + 2b_{2,0} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
B_2^+ \cdot C_0^- + C_1^+ \cdot B_0^- &= \begin{pmatrix} 2b_{2,1}b_{1,1} & -b_{2,1}b_{1,2} \\ 2b_{0,1}b_{1,1} & -b_{0,1}b_{1,2} \end{pmatrix} + \begin{pmatrix} -b_{2,0}b_{0,1} & b_{2,0}b_{0,2} \\ -2b_{0,0}b_{0,1} & 2b_{0,0}b_{0,2} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{2,1} & -\widetilde{b}_{2,2} \\ 2b_{0,1} & -2b_{0,2} \end{pmatrix} + \begin{pmatrix} -2b_{2,1} & \widetilde{b}_{2,2} \\ -2b_{0,1} & 2b_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_0^+ \cdot C_1^- + C_2^+ \cdot B_1^- &= \begin{pmatrix} 2b_{1,0}b_{0,0} & -b_{1,0}b_{0,1} \\ 2b_{2,0}b_{0,0} & -b_{2,0}b_{0,1} \end{pmatrix} + \begin{pmatrix} -b_{1,2}b_{2,0} & b_{1,2}b_{2,1} \\ -2b_{2,2}b_{2,0} & 2b_{2,2}b_{2,1} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{1,0} & -\widetilde{b}_{1,1} \\ 2b_{2,0} & -2b_{2,1} \end{pmatrix} + \begin{pmatrix} -2b_{1,0} & \widetilde{b}_{1,1} \\ -2b_{2,0} & 2b_{2,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_1^+ \cdot C_2^- + C_0^+ \cdot B_2^- &= \begin{pmatrix} 2b_{0,2}b_{2,2} & -b_{0,2}b_{2,0} \\ 2b_{1,2}b_{2,2} & -b_{1,2}b_{2,0} \end{pmatrix} + \begin{pmatrix} -b_{0,1}b_{1,2} & b_{0,1}b_{1,0} \\ -2b_{1,1}b_{1,2} & 2b_{1,1}b_{1,0} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{0,2} & -\widetilde{b}_{0,0} \\ 2b_{1,2} & -2b_{1,0} \end{pmatrix} + \begin{pmatrix} -2b_{0,2} & \widetilde{b}_{0,0} \\ -2b_{1,2} & 2b_{1,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
B_2^+ \pi_{P_1} \cdot \iota_{P_1} C_0^- + C_1^+ \cdot B_0^- &= \begin{pmatrix} b_{2,1} \\ b_{0,1} \end{pmatrix} \begin{pmatrix} 2b_{1,1} & -b_{1,2} \end{pmatrix} + \begin{pmatrix} 0 & -b_{2,0} \\ 0 & -2b_{0,0} \end{pmatrix} \begin{pmatrix} b_{2,1} & -2b_{2,2} \\ b_{0,1} & -b_{0,2} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{2,1}b_{1,1} & -b_{2,1}b_{1,2} \\ 2b_{0,1}b_{1,1} & -b_{0,1}b_{1,2} \end{pmatrix} + \begin{pmatrix} -b_{2,0}b_{0,1} & b_{2,0}b_{0,2} \\ -2b_{0,0}b_{0,1} & 2b_{0,0}b_{0,2} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{2,1} & -\widetilde{b}_{2,2} \\ 2b_{0,1} & -2b_{0,2} \end{pmatrix} + \begin{pmatrix} -2b_{2,1} & \widetilde{b}_{2,2} \\ -2b_{0,1} & 2b_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_0^+ \pi_{P_0} \cdot \iota_{P_0} C_1^- + C_2^+ \cdot B_1^- &= \begin{pmatrix} b_{1,0} \\ b_{2,0} \end{pmatrix} \begin{pmatrix} 2b_{0,0} & -b_{0,1} \end{pmatrix} + \begin{pmatrix} 0 & -b_{1,2} \\ 0 & -2b_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,0} & -2b_{1,1} \\ b_{2,0} & -b_{2,1} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{1,0}b_{0,0} & -b_{1,0}b_{0,1} \\ 2b_{2,0}b_{0,0} & -b_{2,0}b_{0,1} \end{pmatrix} + \begin{pmatrix} -b_{1,2}b_{2,0} & b_{1,2}b_{2,1} \\ -2b_{2,2}b_{2,0} & 2b_{2,2}b_{2,1} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{1,0} & -\widetilde{b}_{1,1} \\ 2b_{2,0} & -2b_{2,1} \end{pmatrix} + \begin{pmatrix} -2b_{1,0} & \widetilde{b}_{1,1} \\ -2b_{2,0} & 2b_{2,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_1^+ \pi_{P_2} \cdot \iota_{P_2} C_2^- + C_0^+ \cdot B_2^- &= \begin{pmatrix} b_{0,2} \\ b_{1,2} \end{pmatrix} \begin{pmatrix} 2b_{2,2} & -b_{2,0} \end{pmatrix} + \begin{pmatrix} 0 & -b_{0,1} \\ 0 & -2b_{1,1} \end{pmatrix} \begin{pmatrix} b_{0,2} & -2b_{0,0} \\ b_{1,2} & -b_{1,0} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{0,2}b_{2,2} & -b_{0,2}b_{2,0} \\ 2b_{1,2}b_{2,2} & -b_{1,2}b_{2,0} \end{pmatrix} + \begin{pmatrix} -b_{0,1}b_{1,2} & b_{0,1}b_{1,0} \\ -2b_{1,1}b_{1,2} & 2b_{1,1}b_{1,0} \end{pmatrix} \\
&= \begin{pmatrix} 2b_{0,2} & -\widetilde{b}_{0,0} \\ 2b_{1,2} & -2b_{1,0} \end{pmatrix} + \begin{pmatrix} -2b_{0,2} & \widetilde{b}_{0,0} \\ -2b_{1,2} & 2b_{1,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
C_2^+ \cdot C_0^- \;=\; C_0^+ \cdot C_1^- \;=\; C_1^+ \cdot C_2^- \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

3.2.2 Acyclicity

Definition 29 We define the following S-linear maps.

$$\alpha^0_{0,1}: P_0 \longrightarrow P_1, \quad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\alpha^0_{0,1} := \frac{1}{2}\left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right)$$

$$\alpha^0_{0,2}: P_0 \longrightarrow P_2, \quad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\alpha^0_{0,2} := \frac{1}{2}\left(c, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right)$$

$$\alpha^0_{1,1}: P_1 \longrightarrow P_1, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\alpha^0_{1,1} := \frac{1}{2}\left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right)$$

$$\alpha^0_{1,2}: P_1 \longrightarrow P_2, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\alpha^0_{1,2} := \frac{1}{2}\left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right)$$

$$\alpha^1_{2,0}: P_2 \longrightarrow P_0, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\alpha^1_{2,0} := \frac{1}{2}\left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right)$$

$$\alpha^1_{2,1}: P_2 \longrightarrow P_1, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\alpha^1_{2,1} := \frac{1}{2}\left(d, \begin{pmatrix} a-s \\ d \\ 0 \end{pmatrix}\right)$$

$$\alpha^1_{0,0}: P_0 \longrightarrow P_0, \quad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\alpha^1_{0,0} := \frac{1}{2}\left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right)$$

$$\alpha^1_{0,1}: P_0 \longrightarrow P_1, \quad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\alpha^1_{0,1} := \frac{1}{2}\left(0, \begin{pmatrix} 0 \\ 0 \\ i-u \end{pmatrix}\right)$$

$$\alpha^2_{1,2}: P_1 \longrightarrow P_2, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\alpha^2_{1,2} := \frac{1}{2}\left(b, \begin{pmatrix} b \\ e-r \\ 0 \end{pmatrix}\right)$$

$$\alpha^2_{1,0}: P_1 \longrightarrow P_0, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\alpha^2_{1,0} := \frac{1}{2}\left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right)$$

$$\alpha^2_{2,2}: P_2 \longrightarrow P_2, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\alpha^2_{2,2} := \frac{1}{2}\left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\alpha^2_{2,0}: P_2 \longrightarrow P_0, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\alpha^2_{2,0} := \frac{1}{2}\left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\beta^0_{1,2}: P_1 \longrightarrow P_2, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\beta^0_{1,2} := \frac{1}{2}\left(b, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\beta^0_{1,0}: \ P_1 \longrightarrow P_0, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\beta^0_{1,0} := \frac{1}{2}\left(h, \begin{pmatrix} 0 \\ e-r \\ h \end{pmatrix}\right)$$

$$\beta^0_{1,1}: \ P_1 \longrightarrow P_1, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\beta^0_{1,1} := \frac{1}{2}\left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right)$$

$$\beta^0_{2,2}: \ P_2 \longrightarrow P_2, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\beta^0_{2,2} := \frac{1}{2}\left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\beta^0_{2,0}: \ P_2 \longrightarrow P_0, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\beta^0_{2,0} := \frac{1}{2}\left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\beta^0_{2,1}: \ P_2 \longrightarrow P_1, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\beta^0_{2,1} := \frac{1}{2}\left(d, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right)$$

$$\beta^1_{0,1}: \ P_0 \longrightarrow P_1, \quad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\beta^1_{0,1} := \frac{1}{2}\left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right)$$

$$\beta^1_{0,2}: \ P_0 \longrightarrow P_2, \quad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\beta^1_{0,2} := \frac{1}{2}\left(c, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right)$$

$$\beta^1_{0,0}: \ P_0 \longrightarrow P_0, \quad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\beta^1_{0,0} := \frac{1}{2}\left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right)$$

$$\beta^1_{1,1}: \ P_1 \longrightarrow P_1, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\beta^1_{1,1} := \frac{1}{2}\left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right)$$

$$\beta^1_{1,2}: \ P_1 \longrightarrow P_2, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\beta^1_{1,2} := \frac{1}{2}\left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right)$$

$$\beta^1_{1,0}: \ P_1 \longrightarrow P_0, \quad \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \longmapsto \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)\beta^1_{1,0} := \frac{1}{2}\left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right)$$

$$\beta^2_{2,0}: \ P_2 \longrightarrow P_0, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\beta^2_{2,0} := \frac{1}{2}\left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right)$$

$$\beta^2_{2,1}: \ P_2 \longrightarrow P_1, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\beta^2_{2,1} := \frac{1}{2}\left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\beta^2_{2,2}: \ P_2 \longrightarrow P_2, \quad \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \longmapsto \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)\beta^2_{2,2} := \frac{1}{2}\left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\beta^2_{0,0}: \quad P_0 \longrightarrow P_0, \qquad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\beta^2_{0,0} := \frac{1}{2}\left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right)$$

$$\beta^2_{0,1}: \quad P_0 \longrightarrow P_1, \qquad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\beta^2_{0,1} := \frac{1}{2}\left(f, \begin{pmatrix} 0 \\ f \\ i-u \end{pmatrix}\right)$$

$$\beta^2_{0,2}: \quad P_0 \longrightarrow P_2, \qquad \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \longmapsto \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)\beta^2_{0,2} := \frac{1}{2}\left(c, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right)$$

These maps can also be written as S-linear matrices; cf. Appendix, Remark 56.

Definition 30

$$\begin{tikzcd}
{} \arrow[r, dotted] & Q_3 \arrow[r, "d_2"] \arrow[d, "1"] & Q_2 \arrow[r, "d_1"] \arrow[d, "1"] \arrow[dl, "h_2"] & Q_1 \arrow[r, "d_0"] \arrow[d, "1"] \arrow[dl, "h_1"] & Q_0 \arrow[r, "\varepsilon"] \arrow[d, "1"] \arrow[dl, "h_0"] & S \arrow[d, "1"] \arrow[dl, "h_{-1}"] \\
{} \arrow[r, dotted] & Q_3 \arrow[r, "d_2"] & Q_2 \arrow[r, "d_1"] & Q_1 \arrow[r, "d_0"] & Q_0 \arrow[r, "\varepsilon"] & S
\end{tikzcd}$$

Let

$$H_0^+ := \begin{pmatrix} \alpha^0_{0,1} & \alpha^0_{0,2} \\ -\alpha^0_{1,1} & -\alpha^0_{1,2} \end{pmatrix}: \ P_0 \oplus P_1 \longrightarrow P_1 \oplus P_2, \quad H_0^- := \begin{pmatrix} \beta^0_{1,2} & \beta^0_{1,0} \\ -\beta^0_{2,2} & -\beta^0_{2,0} \end{pmatrix}: \ P_1 \oplus P_2 \longrightarrow P_2 \oplus P_0,$$

$$H_1^+ := \begin{pmatrix} \alpha^1_{2,0} & \alpha^1_{2,1} \\ -\alpha^1_{0,0} & -\alpha^1_{0,1} \end{pmatrix}: \ P_2 \oplus P_0 \longrightarrow P_0 \oplus P_1, \quad H_1^- := \begin{pmatrix} \beta^1_{0,1} & \beta^1_{0,2} \\ -\beta^1_{1,1} & -\beta^1_{1,2} \end{pmatrix}: \ P_0 \oplus P_1 \longrightarrow P_1 \oplus P_2,$$

$$H_2^+ := \begin{pmatrix} \alpha^2_{1,2} & \alpha^2_{1,0} \\ -\alpha^2_{2,2} & -\alpha^2_{2,0} \end{pmatrix}: \ P_1 \oplus P_2 \longrightarrow P_2 \oplus P_0, \quad H_2^- := \begin{pmatrix} \beta^2_{2,0} & \beta^2_{2,1} \\ -\beta^2_{0,0} & -\beta^2_{0,1} \end{pmatrix}: \ P_2 \oplus P_0 \longrightarrow P_0 \oplus P_1,$$

$$G_0^+ := \begin{pmatrix} \alpha^0_{0,1} & \alpha^0_{0,2} \end{pmatrix}: \ P_0 \longrightarrow P_1 \oplus P_2, \qquad G_0^- := \begin{pmatrix} \beta^0_{1,1} \\ -\beta^0_{2,1} \end{pmatrix}: \ P_1 \oplus P_2 \longrightarrow P_1,$$

$$G_1^+ := \begin{pmatrix} \alpha^1_{2,0} & \alpha^1_{2,1} \end{pmatrix}: \ P_2 \longrightarrow P_0 \oplus P_1, \qquad G_1^- := \begin{pmatrix} \beta^1_{0,0} \\ -\beta^1_{1,0} \end{pmatrix}: \ P_0 \oplus P_1 \longrightarrow P_0,$$

$$G_2^+ := \begin{pmatrix} \alpha^2_{1,2} & \alpha^2_{1,0} \end{pmatrix}: \ P_1 \longrightarrow P_2 \oplus P_0, \qquad G_2^- := \begin{pmatrix} \beta^2_{2,2} \\ -\beta^2_{0,2} \end{pmatrix}: \ P_2 \oplus P_0 \longrightarrow P_2.$$

So for $i \in \{0, 1, 2\}$ we have

$$H_i^+ = \begin{pmatrix} \alpha^i_{\overline{-i},\overline{-i+1}} & \alpha^i_{\overline{-i},\overline{-i+2}} \\ -\alpha^i_{\overline{-i+1},\overline{-i+1}} & -\alpha^i_{\overline{-i+1},\overline{-i+2}} \end{pmatrix}: \ P_{\overline{-i}} \oplus P_{\overline{-i+1}} \longrightarrow P_{\overline{-i+1}} \oplus P_{\overline{-i+2}},$$

$$H_i^- = \begin{pmatrix} \beta^i_{\overline{-i+1},\overline{-i+2}} & \beta^i_{\overline{-i+1},\overline{-i}} \\ -\beta^i_{\overline{-i+2},\overline{-i+2}} & -\beta^i_{\overline{-i+2},\overline{-i}} \end{pmatrix}: \ P_{\overline{-i+1}} \oplus P_{\overline{-i+2}} \longrightarrow P_{\overline{-i+2}} \oplus P_{\overline{-i}},$$

$$G_i^+ = \begin{pmatrix} \alpha^i_{\overline{-i},\overline{-i+1}} & \alpha^i_{\overline{-i},\overline{-i+2}} \end{pmatrix}: \ P_{\overline{-i}} \longrightarrow P_{\overline{-i+1}} \oplus P_{\overline{-i+2}},$$

$$G_i^i = \begin{pmatrix} \beta^i_{\overline{-i+1},\overline{-i+1}} \\ -\beta^i_{\overline{-i+2},\overline{-i+1}} \end{pmatrix}: \ P_{\overline{-i+1}} \oplus P_{\overline{-i+2}} \longrightarrow P_{\overline{-i+1}}.$$

Using this we define the following S-linear maps for $l \geq 0$.

$$h_{-1} : S \to Q_0,\ r \mapsto \left(r, \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \right)$$

$$h_{2l} := \left(\sum_{i \in [0,l-1]} \pi_{2l,i}\ H^{+}_{\overline{l+i}}\ \iota_{2l+1,i} \right) + \left(\pi'_{2l}\ G^{+}_{\overline{2l}}\ \iota_{2l+1,l} \right) :\ Q_{2l} \to Q_{2l+1}$$

$$h_{2l+1} := \left(\sum_{i \in [0,l]} \pi_{2l+1,i}\ H^{-}_{\overline{l+i}}\ \iota_{2l+2,i} \right) + \left(\pi_{2l+1,l}\ G^{-}_{\overline{2l}}\ \iota'_{2l+2} \right) :\ Q_{2l+1} \to Q_{2l+2}$$

3.2.2.1 Homotopy condition at (-1) and 0

We show that $h_{-1}\varepsilon = 1$ and $h_0 d_0 + \varepsilon h_{-1} = 1$. Suppose given $r \in S$, $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right) \in P_0$.

We obtain

$$\begin{aligned}
r \cdot h_{-1}\varepsilon &= \left(r, \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \right) \varepsilon = r. \\
\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right) \left(G^{+}_{0}(B^{+}_{0}\pi_{P_0}) + \varepsilon h_{-1} \right) &= \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right) \left(\alpha^{0}_{0,1} b_{1,0} + \alpha^{0}_{0,2} b_{2,0} \right) + \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right) \varepsilon h_{-1} \\
&= \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix} \right) + \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix} \right) + \left(u, \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \right) \\
&= \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right).
\end{aligned}$$

3.2.2.2 Homotopy condition at odd positions

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & Q_{2l+2} \arrow[r, "d_{2l+1}"] \arrow[d, "1"] & Q_{2l+1} \arrow[r, "d_{2l}"] \arrow[d, "1"] \arrow[dl, "h_{2l+1}"'] & Q_{2l} \arrow[r, dotted] \arrow[d, "1"] \arrow[dl, "h_{2l}"'] & \cdots \\
\cdots \arrow[r, dotted] & Q_{2l+2} \arrow[r, "d_{2l+1}"] & Q_{2l+1} \arrow[r, "d_{2l}"] & Q_{2l} \arrow[r, dotted] & \cdots
\end{tikzcd}$$

We want to show that $h_{2l+1}d_{2l+1} + d_{2l}h_{2l} \overset{!}{=} 1$ for $l \geq 0$. We have

$$h_{2l+1}d_{2l+1} + d_{2l}h_{2l} = \left(\sum_{i \in [0,l]} \pi_{2l+1,i}\ H^{-}_{\overline{l+i}}\ \iota_{2l+2,i} \right) \left(\sum_{j \in [0,l]} \pi_{2l+2,j}\ B^{-}_{\overline{l+j}}\ \iota_{2l+1,j} \right)$$

$$
\begin{aligned}
&+\left(\sum_{i\in[0,l]}\pi_{2l+1,i}\ H^-_{\overline{l+i}}\ \iota_{2l+2,i}\right)\left(\sum_{j\in[0,l-1]}\pi_{2l+2,j+1}\ C^-_{\overline{l+j}}\ \iota_{2l+1,j}\right)\\
&+\left(\sum_{i\in[0,l]}\pi_{2l+1,i}\ H^-_{\overline{l+i}}\ \iota_{2l+2,i}\right)\left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\\
&+\left(\pi_{2l+1,l}\ G^-_{\overline{2l}}\ \iota'_{2l+2}\right)\left(\sum_{j\in[0,l]}\pi_{2l+2,j}\ B^-_{\overline{l+j}}\ \iota_{2l+1,j}\right)\\
&+\left(\pi_{2l+1,l}\ G^-_{\overline{2l}}\ \iota'_{2l+2}\right)\left(\sum_{j\in[0,l-1]}\pi_{2l+2,j+1}\ C^-_{\overline{l+j}}\ \iota_{2l+1,j}\right)\\
&+\left(\pi_{2l+1,l}\ G^-_{\overline{2l}}\ \iota'_{2l+2}\right)\left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\\
&\\
&+\left(\sum_{i\in[0,l-1]}\pi_{2l+1,i}\ B^+_{\overline{l+i}}\ \iota_{2l,i}\right)\left(\sum_{j\in[0,l-1]}\pi_{2l,j}\ H^+_{\overline{l+j}}\ \iota_{2l+1,j}\right)\\
&+\left(\sum_{i\in[0,l-1]}\pi_{2l+1,i}\ B^+_{\overline{l+i}}\ \iota_{2l,i}\right)\left(\pi'_{2l}\ G^+_{\overline{2l}}\ \iota_{2l+1,l}\right)\\
&+\left(\pi_{2l+1,l}\ (B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right)\left(\sum_{j\in[0,l-1]}\pi_{2l,j}\ H^+_{\overline{l+j}}\ \iota_{2l+1,j}\right)\\
&+\left(\pi_{2l+1,l}\ (B^+_{\overline{2l}}\pi_{P_{\bar{l}}})\ \iota'_{2l}\right)\left(\pi'_{2l}\ G^+_{\overline{2l}}\ \iota_{2l+1,l}\right)\\
&+\left(\sum_{i\in[0,l-1]}\pi_{2l+1,i+1}\ C^+_{\overline{l+i}}\ \iota_{2l,i}\right)\left(\sum_{j\in[0,l-1]}\pi_{2l,j}\ H^+_{\overline{l+j}}\ \iota_{2l+1,j}\right)\\
&+\left(\sum_{i\in[0,l-1]}\pi_{2l+1,i+1}\ C^+_{\overline{l+i}}\ \iota_{2l,i}\right)\left(\pi'_{2l}\ G^+_{\overline{2l}}\ \iota_{2l+1,l}\right)\\
&\\
&=\left(\sum_{i\in[0,l]}\pi_{2l+1,i}\ H^-_{\overline{l+i}}B^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)+\left(\sum_{i\in[0,l-1]}\pi_{2l+1,i+1}\ H^-_{\overline{l+i+1}}C^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\\
&+0+0+0+\left(\pi_{2l+1,l}\ G^-_{\overline{2l}}(\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\\
&+\left(\sum_{i\in[0,l-1]}\pi_{2l+1,i}\ B^+_{\overline{l+i}}H^+_{\overline{l+i}}\ \iota_{2l+1,i}\right)+0+0+\left(\pi_{2l+1,l}\ (B^+_{\overline{2l}}\pi_{P_{\bar{l}}})G^+_{\overline{2l}}\ \iota_{2l+1,l}\right)\\
&+\left(\sum_{i\in[0,l-1]}\pi_{2l+1,i+1}\ C^+_{\overline{l+i}}H^+_{\overline{l+i}}\ \iota_{2l+1,i}\right)+0
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i}\, (H^-_{\overline{l+i}} B^-_{\overline{l+i}} + B^+_{\overline{l+i}} H^+_{\overline{l+i}})\, \iota_{2l+1,i} \right) \\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i+1}\, (H^-_{\overline{l+i+1}} C^-_{\overline{l+i}} + C^+_{\overline{l+i}} H^+_{\overline{l+i}})\, \iota_{2l+1,i} \right) \\
&+ \Big(\pi_{2l+1,l}\, \Big(H^-_{\overline{2l}} B^-_{\overline{2l}} + G^-_{\overline{2l}} (\iota_{P_{\overline{l+1}}} C^-_{\overline{2l}}) + (B^+_{\overline{2l}} \pi_{P_{\overline{l}}}) G^+_{\overline{2l}} \Big)\, \iota_{2l+1,l} \Big) \\
&\overset{(*)}{=} \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iota_{2l+1,i} \right) \\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i+1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iota_{2l+1,i} \right) \\
&+ \left(\pi_{2l+1,l} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iota_{2l+1,l} \right) \\
&= I_{2l}\ .
\end{aligned}
$$

For $(*)$ we use Calculations 1 to 3; cf. Lemma 22, Definition 26 and Definition 30.

Calculation 1

We want to show

$$
\begin{aligned}
H_0^- B_0^- + B_0^+ H_0^+ \;&=\; \begin{pmatrix} \beta^0_{1,2} b_{2,1} + \beta^0_{1,0} b_{0,1} & -2\beta^0_{1,2} b_{2,2} - \beta^0_{1,0} b_{0,2} \\ -\beta^0_{2,2} b_{2,1} - \beta^0_{2,0} b_{0,1} & 2\beta^0_{2,2} b_{2,2} + \beta^0_{2,0} b_{0,2} \end{pmatrix} \\
&+\; \begin{pmatrix} b_{1,0}\alpha^0_{0,1} + 2b_{1,1}\alpha^0_{1,1} & b_{1,0}\alpha^0_{0,2} + 2b_{1,1}\alpha^0_{1,2} \\ b_{2,0}\alpha^0_{0,1} + b_{2,1}\alpha^0_{1,1} & b_{2,0}\alpha^0_{0,2} + b_{2,1}\alpha^0_{1,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix} \right) \in P_1$ and $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix} \right) \in P_2$. We obtain

$$
\begin{aligned}
&\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix} \right) \left(\beta^0_{1,2} b_{2,1} + \beta^0_{1,0} b_{0,1} + b_{1,0}\alpha^0_{0,1} + 2b_{1,1}\alpha^0_{1,1} \right) \\
&\quad = \left(0, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} \right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ h \end{pmatrix} \right) + \left(e, \begin{pmatrix} 0 \\ e \\ 0 \end{pmatrix} \right) + \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix} \right) = \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix} \right) \\
&\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix} \right) \left(-2\beta^0_{1,2} b_{2,2} - \beta^0_{1,0} b_{0,2} + b_{1,0}\alpha^0_{0,2} + 2b_{1,1}\alpha^0_{1,2} \right) \\
&\quad = -\left(b, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} \right) - \left(0, \begin{pmatrix} 0 \\ e-r \\ h \end{pmatrix} \right) + \left(b, \begin{pmatrix} b \\ 0 \\ h \end{pmatrix} \right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix} \right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)
\end{aligned}
$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-\beta^0_{2,2} b_{2,1} - \beta^0_{2,0} b_{0,1} + b_{2,0}\alpha^0_{0,1} + b_{2,1}\alpha^0_{1,1}\right)$$
$$= -\left(0, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(d, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) + \left(-d, \begin{pmatrix} 0 \\ -d \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(2\beta^0_{2,2} b_{2,2} + \beta^0_{2,0} b_{0,2} + b_{2,0}\alpha^0_{0,2} + b_{2,1}\alpha^0_{1,2}\right)$$
$$= \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(a, \begin{pmatrix} a \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) = \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right).$$

We want to show

$$\begin{aligned} H_1^- B_1^- + B_1^+ H_1^+ &= \begin{pmatrix} \beta^1_{0,1} b_{1,0} + \beta^1_{0,2} b_{2,0} & -2\beta^1_{0,1} b_{1,1} - \beta^1_{0,2} b_{2,1} \\ -\beta^1_{1,1} b_{1,0} - \beta^1_{1,2} b_{2,0} & 2\beta^1_{1,1} b_{1,1} + \beta^1_{1,2} b_{2,1} \end{pmatrix} \\ &+ \begin{pmatrix} b_{0,2}\alpha^1_{2,0} + 2b_{0,0}\alpha^1_{0,0} & b_{0,2}\alpha^1_{2,1} + 2b_{0,0}\alpha^1_{0,1} \\ b_{1,2}\alpha^1_{2,0} + b_{1,0}\alpha^1_{0,0} & b_{1,2}\alpha^1_{2,1} + b_{1,0}\alpha^1_{0,1} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$ and $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(\beta^1_{0,1} b_{1,0} + \beta^1_{0,2} b_{2,0} + b_{0,2}\alpha^1_{2,0} + 2b_{0,0}\alpha^1_{0,0}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(i, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}\right) + \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) = \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-2\beta^1_{0,1} b_{1,1} - \beta^1_{0,2} b_{2,1} + b_{0,2}\alpha^1_{2,1} + 2b_{0,0}\alpha^1_{0,1}\right)$$
$$= -\left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(f, \begin{pmatrix} c \\ f \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ i-u \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-\beta^1_{1,1} b_{1,0} - \beta^1_{1,2} b_{2,0} + b_{1,2}\alpha^1_{2,0} + b_{1,0}\alpha^1_{0,0}\right)$$
$$= -\left(0, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) + \left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(-h, \begin{pmatrix} 0 \\ 0 \\ -h \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(2\beta^1_{1,1}b_{1,1} + \beta^1_{1,2}b_{2,1} + b_{1,2}\alpha^1_{2,1} + b_{1,0}\alpha^1_{0,1}\right)$$
$$= \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) + \left(e, \begin{pmatrix} b \\ e \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) = \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right).$$

We want to show

$$\begin{aligned} H_2^- B_2^- + B_2^+ H_2^+ &= \begin{pmatrix} \beta^2_{2,0}b_{0,2} + \beta^2_{2,1}b_{1,2} & -2\beta^2_{2,0}b_{0,0} - \beta^2_{2,1}b_{1,0} \\ -\beta^2_{0,0}b_{0,2} - \beta^2_{0,1}b_{1,2} & 2\beta^2_{0,0}b_{0,0} + \beta^2_{0,1}b_{1,0} \end{pmatrix} \\ &+ \begin{pmatrix} b_{2,1}\alpha^2_{1,2} + 2b_{2,2}\alpha^2_{2,2} & b_{2,1}\alpha^2_{1,0} + 2b_{2,2}\alpha^2_{2,0} \\ b_{0,1}\alpha^2_{1,2} + b_{0,2}\alpha^2_{2,2} & b_{0,1}\alpha^2_{1,0} + b_{0,2}\alpha^2_{2,0} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$ and $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(\beta^2_{2,0}b_{0,2} + \beta^2_{2,1}b_{1,2} + b_{2,1}\alpha^2_{1,2} + 2b_{2,2}\alpha^2_{2,2}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(a, \begin{pmatrix} a \\ d \\ 0 \end{pmatrix}\right) + \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) = \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-2\beta^2_{2,0}b_{0,0} - \beta^2_{2,1}b_{1,0} + b_{2,1}\alpha^2_{1,0} + 2b_{2,2}\alpha^2_{2,0}\right)$$
$$= -\left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) - \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-\beta^2_{0,0}b_{0,2} - \beta^2_{0,1}b_{1,2} + b_{0,1}\alpha^2_{1,2} + b_{0,2}\alpha^2_{2,2}\right)$$
$$= -\left(0, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ f \\ i-u \end{pmatrix}\right) + \left(c, \begin{pmatrix} c \\ f \\ 0 \end{pmatrix}\right) + \left(-c, \begin{pmatrix} -c \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(2\beta^2_{0,0}b_{0,0} + \beta^2_{0,1}b_{1,0} + b_{0,1}\alpha^2_{1,0} + b_{0,2}\alpha^2_{2,0}\right)$$
$$= \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ f \\ i-u \end{pmatrix}\right) + \left(i, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right) = \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right).$$

Calculation 2

We want to show

$$H_1^- C_0^- + C_0^+ H_0^+ \quad = \quad \begin{pmatrix} 2\beta^1_{0,1}b_{1,1} & -\beta^1_{0,1}b_{1,2} \\ -2\beta^1_{1,1}b_{1,1} & \beta^1_{1,1}b_{1,2} \end{pmatrix} + \begin{pmatrix} b_{0,1}\alpha^0_{1,1} & b_{0,1}\alpha^0_{1,2} \\ 2b_{1,1}\alpha^0_{1,1} & 2b_{1,1}\alpha^0_{1,2} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$ and $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(2\beta^1_{0,1}b_{1,1} + b_{0,1}\alpha^0_{1,1}\right) = \left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(-f, \begin{pmatrix} 0 \\ -f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-\beta^1_{0,1}b_{1,2} + b_{0,1}\alpha^0_{1,2}\right) = -\left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-2\beta^1_{1,1}b_{1,1} + 2b_{1,1}\alpha^0_{1,1}\right) = -\left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(\beta^1_{1,1}b_{1,2} + 2b_{1,1}\alpha^0_{1,2}\right) = \left(0, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

We want to show

$$H_2^- C_1^- + C_1^+ H_1^+ \quad = \quad \begin{pmatrix} 2\beta^2_{2,0}b_{0,0} & -\beta^2_{2,0}b_{0,1} \\ -2\beta^2_{0,0}b_{0,0} & \beta^2_{0,0}b_{0,1} \end{pmatrix} + \begin{pmatrix} b_{2,0}\alpha^1_{0,0} & b_{2,0}\alpha^1_{0,1} \\ 2b_{0,0}\alpha^1_{0,0} & 2b_{0,0}\alpha^1_{0,1} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$ and $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(2\beta^2_{2,0}b_{0,0} + b_{2,0}\alpha^1_{0,0}\right) = \left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(-g, \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-\beta^2_{2,0}b_{0,1} + b_{2,0}\alpha^1_{0,1}\right) = -\left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-2\beta^2_{0,0}b_{0,0} + 2b_{0,0}\alpha^1_{0,0}\right) = -\left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(\beta^2_{0,0}b_{0,1} + 2b_{0,0}\alpha^1_{0,1}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ i-u \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

We want to show

$$H_0^- C_2^- + C_2^+ H_2^+ = \begin{pmatrix} 2\beta^0_{1,2} b_{2,2} & -\beta^0_{1,2} b_{2,0} \\ -2\beta^0_{2,2} b_{2,2} & \beta^0_{2,2} b_{2,0} \end{pmatrix} + \begin{pmatrix} b_{1,2}\alpha^2_{2,2} & b_{1,2}\alpha^2_{2,0} \\ 2b_{2,2}\alpha^2_{2,2} & 2b_{2,2}\alpha^2_{2,0} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$ and $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$. We obtain

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(2\beta^0_{1,2} b_{2,2} + b_{1,2}\alpha^2_{2,2}\right) = \left(b, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) + \left(-b, \begin{pmatrix} -b \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-\beta^0_{1,2} b_{2,0} + b_{1,2}\alpha^2_{2,0}\right) = -\left(0, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-2\beta^0_{2,2} b_{2,2} + 2b_{2,2}\alpha^2_{2,2}\right) = -\left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(\beta^0_{2,2} b_{2,0} + 2b_{2,2}\alpha^2_{2,0}\right) = \left(0, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

Calculation 3

We want to show

$$H_0^- B_0^- + G_0^-(\iota_{P_1} C_0^-) + (B_0^+ \pi_{P_0}) G_0^+ = \begin{pmatrix} \beta^0_{1,2} b_{2,1} + \beta^0_{1,0} b_{0,1} & -2\beta^0_{1,2} b_{2,2} - \beta^0_{1,0} b_{0,2} \\ -\beta^0_{2,2} b_{2,1} - \beta^0_{2,0} b_{0,1} & 2\beta^0_{2,2} b_{2,2} + \beta^0_{2,0} b_{0,2} \end{pmatrix}$$
$$+ \begin{pmatrix} 2\beta^0_{1,1} b_{1,1} & -\beta^0_{1,1} b_{1,2} \\ -2\beta^0_{2,1} b_{1,1} & \beta^0_{2,1} b_{1,2} \end{pmatrix} + \begin{pmatrix} b_{1,0}\alpha^0_{0,1} & b_{1,0}\alpha^0_{0,2} \\ b_{2,0}\alpha^0_{0,1} & b_{2,0}\alpha^0_{0,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$ and $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$. We obtain

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(\beta^0_{1,2} b_{2,1} + \beta^0_{1,0} b_{0,1} + 2\beta^0_{1,1} b_{1,1} + b_{1,0}\alpha^0_{0,1}\right)$$
$$= \left(0, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ h \end{pmatrix}\right) + \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(e, \begin{pmatrix} 0 \\ e \\ 0 \end{pmatrix}\right) = \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-2\beta^0_{1,2} b_{2,2} - \beta^0_{1,0} b_{0,2} - \beta^0_{1,1} b_{1,2} + b_{1,0}\alpha^0_{0,2}\right)$$
$$= -\left(b, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ e-r \\ h \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(b, \begin{pmatrix} b \\ 0 \\ h \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-\beta^0_{2,2} b_{2,1} - \beta^0_{2,0} b_{0,1} - 2\beta^0_{2,1} b_{1,1} + b_{2,0}\alpha^0_{0,1}\right)$$
$$= -\left(0, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) - \left(d, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) + \left(d, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(2\beta^0_{2,2} b_{2,2} + \beta^0_{2,0} b_{0,2} + \beta^0_{2,1} b_{1,2} + b_{2,0}\alpha^0_{0,2}\right)$$
$$= \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) + \left(a, \begin{pmatrix} a \\ 0 \\ g \end{pmatrix}\right) = \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right).$$

We want to show

$$H_1^- B_1^- + G_1^-(\iota_{P_0} C_1^-) + (B_1^+ \pi_{P_2}) G_1^+ = \begin{pmatrix} \beta^1_{0,1} b_{1,0} + \beta^1_{0,2} b_{2,0} & -2\beta^1_{0,1} b_{1,1} - \beta^1_{0,2} b_{2,1} \\ -\beta^1_{1,1} b_{1,0} - \beta^1_{1,2} b_{2,0} & 2\beta^1_{1,1} b_{1,1} + \beta^1_{1,2} b_{2,1} \end{pmatrix}$$
$$+ \begin{pmatrix} 2\beta^1_{0,0} b_{0,0} & -\beta^1_{0,0} b_{0,1} \\ -2\beta^1_{1,0} b_{0,0} & \beta^1_{1,0} b_{0,1} \end{pmatrix} + \begin{pmatrix} b_{0,2}\alpha^1_{2,0} & b_{0,2}\alpha^1_{2,1} \\ b_{1,2}\alpha^1_{2,0} & b_{1,2}\alpha^1_{2,1} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$ and $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(\beta^1_{0,1} b_{1,0} + \beta^1_{0,2} b_{2,0} + 2\beta^1_{0,0} b_{0,0} + b_{0,2}\alpha^1_{2,0}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(i, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}\right) = \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-2\beta^1_{0,1} b_{1,1} - \beta^1_{0,2} b_{2,1} - \beta^1_{0,0} b_{0,1} + b_{0,2}\alpha^1_{2,1}\right)$$
$$= -\left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(f, \begin{pmatrix} c \\ f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-\beta^1_{1,1} b_{1,0} - \beta^1_{1,2} b_{2,0} - 2\beta^1_{1,0} b_{0,0} + b_{1,2}\alpha^1_{2,0}\right)$$
$$= -\left(0, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) - \left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(2\beta^1_{1,1}b_{1,1} + \beta^1_{1,2}b_{2,1} + \beta^1_{1,0}b_{0,1} + b_{1,2}\alpha^1_{2,1}\right)$$
$$= \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(e, \begin{pmatrix} b \\ e \\ 0 \end{pmatrix}\right) = \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right).$$

We want to show

$$H_2^- B_2^- + G_2^-(\iota_{P_2} C_2^-) + (B_2^+ \pi_{P_1}) G_2^+ = \begin{pmatrix} \beta^2_{2,0}b_{0,2} + \beta^2_{2,1}b_{1,2} & -2\beta^2_{2,0}b_{0,0} - \beta^2_{2,1}b_{1,0} \\ -\beta^2_{0,0}b_{0,2} - \beta^2_{0,1}b_{1,2} & 2\beta^2_{0,0}b_{0,0} + \beta^2_{0,1}b_{1,0} \end{pmatrix}$$
$$+ \begin{pmatrix} 2\beta^2_{2,2}b_{2,2} & -\beta^2_{2,2}b_{2,0} \\ -2\beta^2_{0,2}b_{2,2} & \beta^2_{0,2}b_{2,0} \end{pmatrix} + \begin{pmatrix} b_{2,1}\alpha^2_{1,2} & b_{2,1}\alpha^2_{1,0} \\ b_{0,1}\alpha^2_{1,2} & b_{0,1}\alpha^2_{1,0} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$ and $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(\beta^2_{2,0}b_{0,2} + \beta^2_{2,1}b_{1,2} + 2\beta^2_{2,2}b_{2,2} + b_{2,1}\alpha^2_{1,2}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(a, \begin{pmatrix} a \\ d \\ 0 \end{pmatrix}\right) = \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-2\beta^2_{2,0}b_{0,0} - \beta^2_{2,1}b_{1,0} - \beta^2_{2,2}b_{2,0} + b_{2,1}\alpha^2_{1,0}\right)$$
$$= -\left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) - \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-\beta^2_{0,0}b_{0,2} - \beta^2_{0,1}b_{1,2} - 2\beta^2_{0,2}b_{2,2} + b_{0,1}\alpha^2_{1,2}\right)$$
$$= -\left(0, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ f \\ i-u \end{pmatrix}\right) - \left(c, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right) + \left(c, \begin{pmatrix} c \\ f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(2\beta^2_{0,0}b_{0,0} + \beta^2_{0,1}b_{1,0} + \beta^2_{0,2}b_{2,0} + b_{0,1}\alpha^2_{1,0}\right)$$
$$= \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ f \\ i-u \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right) + \left(i, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}\right) = \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right).$$

3.2.2.3 Homotopy condition at even positions

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & Q_{2l+3} \arrow[r, "d_{2l+2}"] \arrow[d, "1"'] & Q_{2l+2} \arrow[r, "d_{2l+1}"] \arrow[d, "1"] \arrow[dl, "h_{2l+2}"'] & Q_{2l+1} \arrow[r, dotted] \arrow[d, "1"] \arrow[dl, "h_{2l+1}"'] & \cdots \\
\cdots \arrow[r, dotted] & Q_{2l+3} \arrow[r, "d_{2l+2}"] & Q_{2l+2} \arrow[r, "d_{2l+1}"] & Q_{2l+1} \arrow[r, dotted] & \cdots
\end{tikzcd}$$

We want to show that $h_{2l+2}d_{2l+2} + d_{2l+1}h_{2l+1} \overset{!}{=} 1$ for $l \geq 0$. We have

$$\begin{aligned}
h_{2l+2}d_{2l+2} + d_{2l+1}h_{2l+1} = {} & \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ H^+_{\overline{l+i+1}}\ \iota_{2l+3,i}\right)\left(\sum_{j\in[0,l]} \pi_{2l+3,j}\ B^+_{\overline{l+j+1}}\ \iota_{2l+2,j}\right) \\
& + \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ H^+_{\overline{l+i+1}}\ \iota_{2l+3,i}\right)\left(\pi_{2l+3,l+1}\ (B^+_{\overline{2l+2}}\pi_{P_{\overline{l+1}}})\ \iota'_{2l+2}\right) \\
& + \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ H^+_{\overline{l+i+1}}\ \iota_{2l+3,i}\right)\left(\sum_{j\in[0,l]} \pi_{2l+3,j+1}\ C^+_{\overline{l+j+1}}\ \iota_{2l+2,j}\right) \\
& + \left(\pi'_{2l+2}\ G^+_{\overline{2l+2}}\ \iota_{2l+3,l+1}\right)\left(\sum_{j\in[0,l]} \pi_{2l+3,j}\ B^+_{\overline{l+j+1}}\ \iota_{2l+2,j}\right) \\
& + \left(\pi'_{2l+2}\ G^+_{\overline{2l+2}}\ \iota_{2l+3,l+1}\right)\left(\pi_{2l+3,l+1}\ (B^+_{\overline{2l+2}}\pi_{P_{\overline{l+1}}})\ \iota'_{2l+2}\right) \\
& + \left(\pi'_{2l+2}\ G^+_{\overline{2l+2}}\ \iota_{2l+3,l+1}\right)\left(\sum_{j\in[0,l]} \pi_{2l+3,j+1}\ C^+_{\overline{l+j+1}}\ \iota_{2l+2,j}\right) \\
& \\
& + \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ B^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\sum_{j\in[0,l]} \pi_{2l+1,j}\ H^-_{\overline{l+j}}\ \iota_{2l+2,j}\right) \\
& + \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ B^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\pi_{2l+1,l}\ G^-_{\overline{2l}}\ \iota'_{2l+2}\right) \\
& + \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ C^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\sum_{j\in[0,l]} \pi_{2l+1,j}\ H^-_{\overline{l+j}}\ \iota_{2l+2,j}\right) \\
& + \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ C^-_{\overline{l+i}}\ \iota_{2l+1,i}\right)\left(\pi_{2l+1,l}\ G^-_{\overline{2l}}\ \iota'_{2l+2}\right) \\
& + \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\left(\sum_{j\in[0,l]} \pi_{2l+1,j}\ H^-_{\overline{l+j}}\ \iota_{2l+2,j}\right) \\
& + \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}}C^-_{\overline{2l}})\ \iota_{2l+1,l}\right)\left(\pi_{2l+1,l}\ G^-_{\overline{2l}}\ \iota'_{2l+2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ H^{+}_{\overline{l+i+1}} B^{+}_{\overline{l+i+1}}\ \iota_{2l+2,i} \right) + 0 \\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ H^{+}_{\overline{l+i+2}} C^{+}_{\overline{l+i+1}}\ \iota_{2l+2,i} \right) + 0 \\
&+ \left(\pi'_{2l+2}\ G^{+}_{\overline{2l+2}} (B^{+}_{\overline{2l+2}} \pi_{P_{\overline{l+1}}})\ \iota'_{2l+2} \right) + \left(\pi'_{2l+2}\ G^{+}_{\overline{2l+2}} C^{+}_{\overline{2l+1}}\ \iota_{2l+2,l} \right) \\
&+ \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ B^{-}_{\overline{l+i}} H^{-}_{\overline{l+i}}\ \iota_{2l+2,i} \right) + \left(\pi_{2l+2,l}\ B^{-}_{\overline{2l}} G^{-}_{\overline{2l}}\ \iota'_{2l+2} \right) \\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ C^{-}_{\overline{l+i}} H^{-}_{\overline{l+i}}\ \iota_{2l+2,i} \right) + 0 \\
&+ \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}}) H^{-}_{\overline{2l}}\ \iota_{2l+2,l} \right) + \left(\pi'_{2l+2}\ (\iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}}) G^{-}_{\overline{2l}}\ \iota'_{2l+2} \right) \\
\\
&= \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\ (H^{+}_{\overline{l+i+1}} B^{+}_{\overline{l+i+1}} + B^{-}_{\overline{l+i}} H^{-}_{\overline{l+i}})\ \iota_{2l+2,i} \right) \\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\ (H^{+}_{\overline{l+i+2}} C^{+}_{\overline{l+i+1}} + C^{-}_{\overline{l+i}} H^{-}_{\overline{l+i}})\ \iota_{2l+2,i} \right) \\
&+ \left(\pi'_{2l+2}\ \left(G^{+}_{\overline{2l+2}} (B^{+}_{\overline{2l+2}} \pi_{P_{\overline{l+1}}}) + (\iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}}) G^{-}_{\overline{2l}} \right)\ \iota'_{2l+2} \right) \\
&+ \left(\pi'_{2l+2}\ \left(G^{+}_{\overline{2l+2}} C^{+}_{\overline{2l+1}} + (\iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}}) H^{-}_{\overline{2l}} \right)\ \iota_{2l+2,l} \right) \\
&+ \left(\pi_{2l+2,l}\ B^{-}_{\overline{2l}} G^{-}_{\overline{2l}}\ \iota'_{2l+2} \right) \\
\\
&\overset{(**)}{=} \left(\sum_{i\in[0,l]} \pi_{2l+2,i} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iota_{2l+2,i} \right) \\
&+ \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iota_{2l+2,i} \right) \\
&+ \left(\pi'_{2l+2} \begin{pmatrix} 1 \end{pmatrix} \iota'_{2l+2} \right) + \left(\pi'_{2l+2} \begin{pmatrix} 0 & 0 \end{pmatrix} \iota_{2l+2,l} \right) + \left(\pi_{2l+2,l} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iota'_{2l+2} \right) \\
\\
&= I_{2l+1}\ .
\end{aligned}
$$

For $(**)$ we use Calculations 1 to 5; cf. Lemma 22, Definition 26 and Definition 30.

Calculation 1

We want to show

$$\begin{aligned} H_1^+ B_1^+ + B_0^- H_0^- &= \begin{pmatrix} \alpha^1_{2,0} b_{0,2} + \alpha^1_{2,1} b_{1,2} & -2\alpha^1_{2,0} b_{0,0} - \alpha^1_{2,1} b_{1,0} \\ -\alpha^1_{0,0} b_{0,2} - \alpha^1_{0,1} b_{1,2} & 2\alpha^1_{0,0} b_{0,0} + \alpha^1_{0,1} b_{1,0} \end{pmatrix} \\ &+ \begin{pmatrix} b_{2,1}\beta^0_{1,2} + 2b_{2,2}\beta^0_{2,2} & b_{2,1}\beta^0_{1,0} + 2b_{2,2}\beta^0_{2,0} \\ b_{0,1}\beta^0_{1,2} + b_{0,2}\beta^0_{2,2} & b_{0,1}\beta^0_{1,0} + b_{0,2}\beta^0_{2,0} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$ and $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\begin{aligned} &\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(\alpha^1_{2,0} b_{0,2} + \alpha^1_{2,1} b_{1,2} + b_{2,1}\beta^0_{1,2} + 2b_{2,2}\beta^0_{2,2}\right) \\ &\quad = \left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ d \\ 0 \end{pmatrix}\right) + \left(a, \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}\right) + \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) = \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \end{aligned}$$

$$\begin{aligned} &\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-2\alpha^1_{2,0} b_{0,0} - \alpha^1_{2,1} b_{1,0} + b_{2,1}\beta^0_{1,0} + 2b_{2,2}\beta^0_{2,0}\right) \\ &\quad = -\left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) - \left(0, \begin{pmatrix} a-s \\ d \\ 0 \end{pmatrix}\right) + \left(g, \begin{pmatrix} 0 \\ d \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \end{aligned}$$

$$\begin{aligned} &\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-\alpha^1_{0,0} b_{0,2} - \alpha^1_{0,1} b_{1,2} + b_{0,1}\beta^0_{1,2} + b_{0,2}\beta^0_{2,2}\right) \\ &\quad = -\left(0, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ 0 \\ i-u \end{pmatrix}\right) + \left(c, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right) + \left(-c, \begin{pmatrix} -c \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \end{aligned}$$

$$\begin{aligned} &\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(2\alpha^1_{0,0} b_{0,0} + \alpha^1_{0,1} b_{1,0} + b_{0,1}\beta^0_{1,0} + b_{0,2}\beta^0_{2,0}\right) \\ &\quad = \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ i-u \end{pmatrix}\right) + \left(i, \begin{pmatrix} 0 \\ f \\ i \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right) = \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right). \end{aligned}$$

We want to show

$$\begin{aligned} H_2^+ B_2^+ + B_1^- H_1^- &= \begin{pmatrix} \alpha^2_{1,2} b_{2,1} + \alpha^2_{1,0} b_{0,1} & -2\alpha^2_{1,2} b_{2,2} - \alpha^2_{1,0} b_{0,2} \\ -\alpha^2_{2,2} b_{2,1} - \alpha^2_{2,0} b_{0,1} & 2\alpha^2_{2,2} b_{2,2} + \alpha^2_{2,0} b_{0,2} \end{pmatrix} \\ &+ \begin{pmatrix} b_{1,0}\beta^1_{0,1} + 2b_{1,1}\beta^1_{1,1} & b_{1,0}\beta^1_{0,2} + 2b_{1,1}\beta^1_{1,2} \\ b_{2,0}\beta^1_{0,1} + b_{2,1}\beta^1_{1,1} & b_{2,0}\beta^1_{0,2} + b_{2,1}\beta^1_{1,2} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$ and $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$. We obtain

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(\alpha^2_{1,2} b_{2,1} + \alpha^2_{1,0} b_{0,1} + b_{1,0}\beta^1_{0,1} + 2b_{1,1}\beta^1_{1,1}\right)$$
$$= \left(0, \begin{pmatrix} b \\ e-r \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(e, \begin{pmatrix} 0 \\ e \\ 0 \end{pmatrix}\right) + \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) = \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-2\alpha^2_{1,2} b_{2,2} - \alpha^2_{1,0} b_{0,2} + b_{1,0}\beta^1_{0,2} + 2b_{1,1}\beta^1_{1,2}\right)$$
$$= -\left(b, \begin{pmatrix} b \\ e-r \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(b, \begin{pmatrix} b \\ 0 \\ h \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-\alpha^2_{2,2} b_{2,1} - \alpha^2_{2,0} b_{0,1} + b_{2,0}\beta^1_{0,1} + b_{2,1}\beta^1_{1,1}\right)$$
$$= -\left(0, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(d, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) + \left(-d, \begin{pmatrix} 0 \\ -d \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(2\alpha^2_{2,2} b_{2,2} + \alpha^2_{2,0} b_{0,2} + b_{2,0}\beta^1_{0,2} + b_{2,1}\beta^1_{1,2}\right)$$
$$= \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(a, \begin{pmatrix} a \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) = \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right).$$

We want to show

$$\begin{aligned} H_0^+ B_0^+ + B_2^- H_2^- &= \begin{pmatrix} \alpha^0_{0,1} b_{1,0} + \alpha^0_{0,2} b_{2,0} & -2\alpha^0_{0,1} b_{1,1} - \alpha^0_{0,2} b_{2,1} \\ -\alpha^0_{1,1} b_{1,0} - \alpha^0_{1,2} b_{2,0} & 2\alpha^0_{1,1} b_{1,1} + \alpha^0_{1,2} b_{2,1} \end{pmatrix} \\ &+ \begin{pmatrix} b_{0,2}\beta^2_{2,0} + 2b_{0,0}\beta^2_{0,0} & b_{0,2}\beta^2_{2,1} + 2b_{0,0}\beta^2_{0,1} \\ b_{1,2}\beta^2_{2,0} + b_{1,0}\beta^2_{0,0} & b_{1,2}\beta^2_{2,1} + b_{1,0}\beta^2_{0,1} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$ and $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(\alpha^0_{0,1} b_{1,0} + \alpha^0_{0,2} b_{2,0} + b_{0,2}\beta^2_{2,0} + 2b_{0,0}\beta^2_{0,0}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(i, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}\right) + \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) = \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-2\alpha^0_{0,1}b_{1,1} - \alpha^0_{0,2}b_{2,1} + b_{0,2}\beta^2_{2,1} + 2b_{0,0}\beta^2_{0,1}\right)$$
$$= -\left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right) + \left(f, \begin{pmatrix} 0 \\ f \\ i-u \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-\alpha^0_{1,1}b_{1,0} - \alpha^0_{1,2}b_{2,0} + b_{1,2}\beta^2_{2,0} + b_{1,0}\beta^2_{0,0}\right)$$
$$= -\left(0, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) - \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) + \left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(-h, \begin{pmatrix} 0 \\ 0 \\ -h \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(2\alpha^0_{1,1}b_{1,1} + \alpha^0_{1,2}b_{2,1} + b_{1,2}\beta^2_{2,1} + b_{1,0}\beta^2_{0,1}\right)$$
$$= \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) + \left(e, \begin{pmatrix} 0 \\ e \\ h \end{pmatrix}\right) = \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right).$$

Calculation 2

We want to show

$$\begin{aligned} H_2^+C_1^+ + C_0^-H_0^- &= \begin{pmatrix} 0 & -\alpha^2_{1,2}b_{2,0} - 2\alpha^2_{1,0}b_{0,0} \\ 0 & \alpha^2_{2,2}b_{2,0} + 2\alpha^2_{2,0}b_{0,0} \end{pmatrix} \\ &+ \begin{pmatrix} 2b_{1,1}\beta^0_{1,2} + b_{1,2}\beta^0_{2,2} & 2b_{1,1}\beta^0_{1,0} + b_{1,2}\beta^0_{2,0} \\ 0 & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$ and $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$. We obtain

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(0 + 2b_{1,1}\beta^0_{1,2} + b_{1,2}\beta^0_{2,2}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) + \left(b, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) + \left(-b, \begin{pmatrix} -b \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-\alpha^2_{1,2}b_{2,0} - 2\alpha^2_{1,0}b_{0,0} + 2b_{1,1}\beta^0_{1,0} + b_{1,2}\beta^0_{2,0}\right)$$
$$= -\left(0, \begin{pmatrix} b \\ e-r \\ 0 \end{pmatrix}\right) - \left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(h, \begin{pmatrix} 0 \\ e-r \\ h \end{pmatrix}\right) + \left(0, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) (0 + 0) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\begin{aligned}
&\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(\alpha^2_{2,2} b_{2,0} + 2\alpha^2_{2,0} b_{0,0} + 0\right) \\
&\quad = \left(0, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).
\end{aligned}$$

We want to show

$$\begin{aligned}
H^+_0 C^+_2 + C^-_1 H^-_1 &= \begin{pmatrix} 0 & -\alpha^0_{0,1} b_{1,2} - 2\alpha^0_{0,2} b_{2,2} \\ 0 & \alpha^0_{1,1} b_{1,2} + 2\alpha^0_{1,2} b_{2,2} \end{pmatrix} \\
&+ \begin{pmatrix} 2b_{0,0}\beta^1_{0,1} + b_{0,1}\beta^1_{1,1} & 2b_{0,0}\beta^1_{0,2} + b_{0,1}\beta^1_{1,2} \\ 0 & 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$ and $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\begin{aligned}
&\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(0 + 2b_{0,0}\beta^1_{0,1} + b_{0,1}\beta^1_{1,1}\right) \\
&\quad = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) + \left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(-f, \begin{pmatrix} 0 \\ -f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \\
&\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-\alpha^0_{0,1} b_{1,2} - 2\alpha^0_{0,2} b_{2,2} + 2b_{0,0}\beta^1_{0,2} + b_{0,1}\beta^1_{1,2}\right) \\
&\quad = -\left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) - \left(c, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(c, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \\
&\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) (0+0) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \\
&\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(\alpha^0_{1,1} b_{1,2} + 2\alpha^0_{1,2} b_{2,2} + 0\right) \\
&\quad = \left(0, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ e-r \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).
\end{aligned}$$

We want to show

$$\begin{aligned}
H^+_1 C^+_0 + C^-_2 H^-_2 &= \begin{pmatrix} 0 & -\alpha^1_{2,0} b_{0,1} - 2\alpha^1_{2,1} b_{1,1} \\ 0 & \alpha^1_{0,0} b_{0,1} + 2\alpha^1_{0,1} b_{1,1} \end{pmatrix} \\
&+ \begin{pmatrix} 2b_{2,2}\beta^2_{2,0} + b_{2,0}\beta^2_{0,0} & 2b_{2,2}\beta^2_{2,1} + b_{2,0}\beta^2_{0,1} \\ 0 & 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$ and $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\begin{aligned}
&\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(0 + 2b_{2,2}\beta^2_{2,0} + b_{2,0}\beta^2_{0,0}\right) \\
&\quad = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) + \left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(-g, \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \\
&\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-\alpha^1_{2,0}b_{0,1} - 2\alpha^1_{2,1}b_{1,1} + 2b_{2,2}\beta^2_{2,1} + b_{2,0}\beta^2_{0,1}\right) \\
&\quad = -\left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) - \left(d, \begin{pmatrix} a-s \\ d \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(d, \begin{pmatrix} 0 \\ d \\ g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \\
&\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) (0+0) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \\
&\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(\alpha^1_{0,0}b_{0,1} + 2\alpha^1_{0,1}b_{1,1} + 0\right) \\
&\quad = \left(0, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ i-u \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).
\end{aligned}$$

Calculation 3

We want to show

$$G^+_2(B^+_2\pi_{P_1}) + (\iota_{P_1}C^-_0)G^-_0 \quad = \quad \left(\ \alpha^2_{1,2}b_{2,1} + \alpha^2_{1,0}b_{0,1}\ \right) + \left(\ 2b_{1,1}\beta^0_{1,1} + b_{1,2}\beta^0_{2,1}\ \right) \overset{!}{=} (1)\,.$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\begin{aligned}
&\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(\alpha^2_{1,2}b_{2,1} + \alpha^2_{1,0}b_{0,1} + 2b_{1,1}\beta^0_{1,1} + b_{1,2}\beta^0_{2,1}\right) \\
&\quad = \left(0, \begin{pmatrix} b \\ e-r \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(r-e, \begin{pmatrix} 0 \\ r-e \\ 0 \end{pmatrix}\right) + \left(e, \begin{pmatrix} 0 \\ e \\ 0 \end{pmatrix}\right) = \left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right).
\end{aligned}$$

We want to show

$$G^+_0(B^+_0\pi_{P_0}) + (\iota_{P_0}C^-_1)G^-_1 \quad = \quad \left(\ \alpha^0_{0,1}b_{1,0} + \alpha^0_{0,2}b_{2,0}\ \right) + \left(\ 2b_{0,0}\beta^1_{0,0} + b_{0,1}\beta^1_{1,0}\ \right) \overset{!}{=} (1)\,.$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(\alpha^0_{0,1} b_{1,0} + \alpha^0_{0,2} b_{2,0} + 2b_{0,0}\beta^1_{0,0} + b_{0,1}\beta^1_{1,0}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(u-i, \begin{pmatrix} 0 \\ 0 \\ u-i \end{pmatrix}\right) + \left(i, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}\right) = \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right).$$

We want to show

$$G_1^+(B_1^+\pi_{P_2}) + (\iota_{P_2} C_2^-)G_2^- \;=\; \left(\ \alpha^1_{2,0} b_{0,2} + \alpha^1_{2,1} b_{1,2}\ \right) + \left(\ 2b_{2,2}\beta^2_{2,2} + b_{2,0}\beta^2_{0,2}\ \right) \stackrel{!}{=} (1)\,.$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$. We obtain

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(\alpha^1_{2,0} b_{0,2} + \alpha^1_{2,1} b_{1,2} + 2b_{2,2}\beta^2_{2,2} + b_{2,0}\beta^2_{0,2}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ d \\ 0 \end{pmatrix}\right) + \left(s-a, \begin{pmatrix} s-a \\ 0 \\ 0 \end{pmatrix}\right) + \left(a, \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}\right) = \left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right).$$

Calculation 4

We want to show

$$\begin{aligned} G_2^+ C_1^+ + (\iota_{P_1} C_0^-)H_0^- \;&=\; \left(\ 0 \quad -\alpha^2_{1,2} b_{2,0} - 2\alpha^2_{1,0} b_{0,0}\ \right) \\ &+\; \left(\ 2b_{1,1}\beta^0_{1,2} + b_{1,2}\beta^0_{2,2} \quad 2b_{1,1}\beta^0_{1,0} + b_{1,2}\beta^0_{2,0}\ \right) \stackrel{!}{=} (0 \quad 0)\,. \end{aligned}$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(0 + 2b_{1,1}\beta^0_{1,2} + b_{1,2}\beta^0_{2,2}\right)$$
$$= \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) + \left(b, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) + \left(-b, \begin{pmatrix} -b \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(-\alpha^2_{1,2} b_{2,0} - 2\alpha^2_{1,0} b_{0,0} + 2b_{1,1}\beta^0_{1,0} + b_{1,2}\beta^0_{2,0}\right)$$
$$= -\left(0, \begin{pmatrix} b \\ e-r \\ 0 \end{pmatrix}\right) - \left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) + \left(h, \begin{pmatrix} 0 \\ e-r \\ h \end{pmatrix}\right) + \left(0, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

We want to show

$$\begin{aligned} G_0^+ C_2^+ + (\iota_{P_0} C_1^-) H_1^- &= \begin{pmatrix} 0 & -\alpha^0_{0,1} b_{1,2} - 2\alpha^0_{0,2} b_{2,2} \end{pmatrix} \\ &+ \begin{pmatrix} 2b_{0,0}\beta^1_{0,1} + b_{0,1}\beta^1_{1,1} & 2b_{0,0}\beta^1_{0,2} + b_{0,1}\beta^1_{1,2} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\begin{aligned} &\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(0 + 2b_{0,0}\beta^1_{0,1} + b_{0,1}\beta^1_{1,1}\right) \\ &= \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) + \left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) + \left(-f, \begin{pmatrix} 0 \\ -f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \end{aligned}$$

$$\begin{aligned} &\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(-\alpha^0_{0,1} b_{1,2} - 2\alpha^0_{0,2} b_{2,2} + 2b_{0,0}\beta^1_{0,2} + b_{0,1}\beta^1_{1,2}\right) \\ &= -\left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) - \left(c, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(c, \begin{pmatrix} c \\ 0 \\ i-u \end{pmatrix}\right) + \left(0, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right). \end{aligned}$$

We want to show

$$\begin{aligned} G_1^+ C_0^+ + (\iota_{P_2} C_2^-) H_2^- &= \begin{pmatrix} 0 & -\alpha^1_{2,0} b_{0,1} - 2\alpha^1_{2,1} b_{1,1} \end{pmatrix} \\ &+ \begin{pmatrix} 2b_{2,2}\beta^2_{2,0} + b_{2,0}\beta^2_{0,0} & 2b_{2,2}\beta^2_{2,1} + b_{2,0}\beta^2_{0,1} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 \end{pmatrix}. \end{aligned}$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$. We obtain

$$\begin{aligned} &\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(0 + 2b_{2,2}\beta^2_{2,0} + b_{2,0}\beta^2_{0,0}\right) \\ &= \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) + \left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) + \left(-g, \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \end{aligned}$$

$$\begin{aligned} &\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(-\alpha^1_{2,0} b_{0,1} - 2\alpha^1_{2,1} b_{1,1} + 2b_{2,2}\beta^2_{2,1} + b_{2,0}\beta^2_{0,1}\right) \\ &= -\left(0, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) - \left(d, \begin{pmatrix} a-s \\ d \\ 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} a-s \\ 0 \\ 0 \end{pmatrix}\right) + \left(d, \begin{pmatrix} 0 \\ d \\ g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right). \end{aligned}$$

Calculation 5

We want to show

$$B_0^- G_0^- = \begin{pmatrix} b_{2,1}\beta^0_{1,1} + 2b_{2,2}\beta^0_{2,1} \\ b_{0,1}\beta^0_{1,1} + b_{0,2}\beta^0_{2,1} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Suppose given $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$ and $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$. We obtain

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(b_{2,1}\beta^0_{1,1} + 2b_{2,2}\beta^0_{2,1}\right) = \left(-d, \begin{pmatrix} 0 \\ -d \\ 0 \end{pmatrix}\right) + \left(d, \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(b_{0,1}\beta^0_{1,1} + b_{0,2}\beta^0_{2,1}\right) = \left(-f, \begin{pmatrix} 0 \\ -f \\ 0 \end{pmatrix}\right) + \left(f, \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

We want to show

$$B_1^- G_1^- \quad = \quad \begin{pmatrix} b_{1,0}\beta^1_{0,0} + 2b_{1,1}\beta^1_{1,0} \\ b_{2,0}\beta^1_{0,0} + b_{2,1}\beta^1_{1,0} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Suppose given $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$ and $\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \in P_2$. We obtain

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(b_{1,0}\beta^1_{0,0} + 2b_{1,1}\beta^1_{1,0}\right) = \left(-h, \begin{pmatrix} 0 \\ 0 \\ -h \end{pmatrix}\right) + \left(h, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(s, \begin{pmatrix} a \\ d \\ g \end{pmatrix}\right) \left(b_{2,0}\beta^1_{0,0} + b_{2,1}\beta^1_{1,0}\right) = \left(-g, \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}\right) + \left(g, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

We want to show

$$B_2^- G_2^- \quad = \quad \begin{pmatrix} b_{0,2}\beta^2_{2,2} + 2b_{0,0}\beta^2_{0,2} \\ b_{1,2}\beta^2_{2,2} + b_{1,0}\beta^2_{0,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Suppose given $\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \in P_0$ and $\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \in P_1$. We obtain

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \left(b_{0,2}\beta^2_{2,2} + 2b_{0,0}\beta^2_{0,2}\right) = \left(-c, \begin{pmatrix} -c \\ 0 \\ 0 \end{pmatrix}\right) + \left(c, \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(r, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right) \left(b_{1,2}\beta^2_{2,2} + b_{1,0}\beta^2_{0,2}\right) = \left(-b, \begin{pmatrix} -b \\ 0 \\ 0 \end{pmatrix}\right) + \left(b, \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

3.2.3 Minimality

We aim to prove that the projective resolution Q of the trivial Λ-module S is minimal. By [**3,** Proposition 9, §3.6] we have to show that $Q_{n+1}d_n \subseteq \mathrm{rad}(\Lambda)Q_n$ for every $n \in \mathbb{Z}_{\geq 0}$.

To verify this we first have to calculate $\mathrm{rad}(\Lambda)$.

By Lemma 22 we have the following orthogonal decomposition of 1_Λ into primitive idempotent elements.

$$\begin{aligned} 1_\Lambda &= \left(1,1,1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right) = \mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2 \\ &= \left(1,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right) + \left(0,1,0,\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right) + \left(0,0,1,\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\right) \end{aligned}$$

We show that $\mathcal{E}_i\Lambda \cong \mathcal{E}_j\Lambda$ if and only if $i = j$ for $i,j \in \{0,1,2\}$.

Suppose $\mathcal{E}_i\Lambda \cong \mathcal{E}_j\Lambda$ for $i,j \in \{0,1,2\}$ with $i \neq j$. Let $r_{\mathcal{E}_i} : \Lambda \to \Lambda,\ \lambda \mapsto \lambda\mathcal{E}_i$ be the right multiplication with $\mathcal{E}_i$. Then the images of $\mathcal{E}_i\Lambda$ and $\mathcal{E}_j\Lambda$ under $r_{\mathcal{E}_i}$ must have the same rank over S, since $\mathcal{E}_i\Lambda \cong \mathcal{E}_j\Lambda$. However $\mathrm{rank}(\mathcal{E}_i\Lambda\mathcal{E}_i) = 2$ whereas $\mathrm{rank}(\mathcal{E}_j\Lambda\mathcal{E}_i) = 1$, cf. Lemma 24.

We want to use [**7,** Proposition 217] to calculate $\mathrm{rad}(\Lambda)$. Therefore we have to check two requirements.

By Lemma 21, the ground ring S is a discrete valuation ring with maximal ideal $(2) \subseteq S$.

The S-order Λ must be stable, which means that for every primitive idempotent $\mathcal{E} \in \Lambda$ the idempotent $\mathcal{E} + 2\Lambda \in \Lambda/2\Lambda$ must also be primitive, cf. [**7,** Definition 207]. However, by [**7,** Remark 208] the S-order Λ fulfills this condition.

So we obtain by [**7,** Proposition 217] that

$$(*) \qquad \begin{aligned} \mathrm{rad}(\Lambda) = {} & \mathrm{rad}(\mathcal{E}_0\Lambda\mathcal{E}_0) \oplus \mathrm{rad}(\mathcal{E}_1\Lambda\mathcal{E}_1) \oplus \mathrm{rad}(\mathcal{E}_2\Lambda\mathcal{E}_2) \oplus \\ & (\mathcal{E}_0\Lambda\mathcal{E}_1) \oplus (\mathcal{E}_0\Lambda\mathcal{E}_2) \oplus (\mathcal{E}_1\Lambda\mathcal{E}_0) \oplus (\mathcal{E}_1\Lambda\mathcal{E}_2) \oplus (\mathcal{E}_2\Lambda\mathcal{E}_0) \oplus (\mathcal{E}_2\Lambda\mathcal{E}_1). \end{aligned}$$

Let $\Xi := \{(x,y) \in S \times S : x \equiv_4 y\} \subseteq S \times S$. We have isomorphisms of S-algebras

$$\begin{aligned} \theta_0 &: \Xi \xrightarrow{\sim} \mathcal{E}_0\Lambda\mathcal{E}_0,\ (x,y) \mapsto \left(x,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&y\end{pmatrix}\right) \\ \theta_1 &: \Xi \xrightarrow{\sim} \mathcal{E}_1\Lambda\mathcal{E}_1,\ (x,y) \mapsto \left(0,x,0,\begin{pmatrix}0&0&0\\0&y&0\\0&0&0\end{pmatrix}\right) \\ \theta_2 &: \Xi \xrightarrow{\sim} \mathcal{E}_2\Lambda\mathcal{E}_2,\ (x,y) \mapsto \left(0,0,x,\begin{pmatrix}y&0&0\\0&0&0\\0&0&0\end{pmatrix}\right). \end{aligned}$$

Let $K := \mathrm{frac}(S)$. Then $1_{K\Xi} = (1,1) = (1,0) + (0,1)$ is the orthogonal decomposition into central primitive idempotent elements $\xi_1 := (1,0)$ and $\xi_2 := (0,1)$ of $K\Xi$.

We determine the radical of $\xi_1\Xi \subseteq S \times S$ and $\xi_2\Xi \subseteq S \times S$.

$$\begin{aligned}\mathrm{rad}(\xi_1\Xi) &= \mathrm{rad}(\{(x,0) : x \in S\}) = \{(x,0) : x \in S, x \equiv_2 0\}\\ \mathrm{rad}(\xi_2\Xi) &= \mathrm{rad}(\{(0,y) : y \in S\}) = \{(0,y) : y \in S, y \equiv_2 0\}\end{aligned}$$

By [**7,** Remark 208], the S-order Ξ is stable as well so that we can use [**7,** Proposition 222] to calculate

$$\begin{aligned}\mathrm{rad}(\Xi) &= \Xi \cap (\mathrm{rad}(\xi_1\Xi) \oplus \mathrm{rad}(\xi_2\Xi))\\ &= \{(x,y) \in S \times S : x \equiv_4 y\} \cap (\{(x,0) : x \in S, x \equiv_2 0\} \oplus \{(0,y) : y \in S, y \equiv_2 0\})\\ &= \{(x,y) \in S \times S : x \equiv_4 y\} \cap \{(x,y) \in S \times S : x \equiv_2 0 \equiv_2 y\}\\ &= \{(x,y) \in S \times S : x \equiv_4 y \equiv_2 0\}.\end{aligned}$$

By means of θ_0, θ_1 and θ_2 we obtain

$$\begin{aligned}\mathrm{rad}(\mathcal{E}_0\Lambda\mathcal{E}_0) &= \left\{\left(u,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&i\end{pmatrix}\right) : u,i \in S, u \equiv_4 i \equiv_2 0\right\}\\ \mathrm{rad}(\mathcal{E}_1\Lambda\mathcal{E}_1) &= \left\{\left(0,r,0,\begin{pmatrix}0&0&0\\0&e&0\\0&0&0\end{pmatrix}\right) : r,e \in S, r \equiv_4 e \equiv_2 0\right\}\\ \mathrm{rad}(\mathcal{E}_2\Lambda\mathcal{E}_2) &= \left\{\left(0,0,s,\begin{pmatrix}a&0&0\\0&0&0\\0&0&0\end{pmatrix}\right) : s,a \in S, s \equiv_4 a \equiv_2 0\right\}.\end{aligned}$$

Together with equation $(*)$ we have

$$\mathrm{rad}(\Lambda) = \left\{\left(u,r,s,\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\right) \in S \times S \times S \times S^{3\times 3} : \begin{matrix} u \equiv_4 i \equiv_2 0,\ r \equiv_4 e \equiv_2 0,\ s \equiv_4 a \equiv_2 0,\\ b \equiv_2 c \equiv_2 d \equiv_2 f \equiv_2 g \equiv_2 h \equiv_2 0\end{matrix}\right\}$$

so that

$$\begin{aligned}\mathrm{rad}(\Lambda)P_0 = \mathrm{rad}(\Lambda)\mathcal{E}_0 &= \left\{\left(u,0,0,\begin{pmatrix}0&0&c\\0&0&f\\0&0&i\end{pmatrix}\right) : \begin{matrix} u,c,f,i \in S\\ u \equiv_4 i \equiv_2 0,\ c \equiv_2 0 \equiv_2 f\end{matrix}\right\}\\ \mathrm{rad}(\Lambda)P_1 = \mathrm{rad}(\Lambda)\mathcal{E}_1 &= \left\{\left(0,r,0,\begin{pmatrix}0&b&0\\0&e&0\\0&h&0\end{pmatrix}\right) : \begin{matrix} r,b,e,h \in S\\ r \equiv_4 e \equiv_2 0,\ b \equiv_2 0 \equiv_2 h\end{matrix}\right\}\\ \mathrm{rad}(\Lambda)P_2 = \mathrm{rad}(\Lambda)\mathcal{E}_2 &= \left\{\left(0,0,s,\begin{pmatrix}a&0&0\\d&0&0\\g&0&0\end{pmatrix}\right) : \begin{matrix} s,a,d,g \in S\\ s \equiv_4 a \equiv_2 0,\ d \equiv_2 0 \equiv_2 g\end{matrix}\right\}.\end{aligned}$$

By [**4,** Proposition 6, §9.3] we have $\mathrm{rad}(\Lambda)P = \mathrm{rad}(P)$ for a projective Λ-module P so that we will often write $\mathrm{rad}(P_i) = \mathrm{rad}(\Lambda)P_i$ for $i \in \{0,1,2\}$.

Now we verify that $Q_{n+1}d_n \subseteq \mathrm{rad}(\Lambda)Q_n$ for $n \geq 0$, cf. Definition 25 and Definition 26.

First we note that $\mathrm{Im}(b_{i,j}) \subseteq \mathrm{rad}(P_j)$ for all $i,j \in \{0,1,2\}$ with $i \neq j$ and $\mathrm{Im}(2b_{i,i}) \subseteq \mathrm{rad}(P_i)$ for all $i \in \{0,1,2\}$; cf. Lemma 24.

Let $i \in \{0,1,2\}$ and $x \in P_{\overline{-i+1}}$, $y \in P_{\overline{-i+2}}$, $z \in P_{\overline{-i}}$. Then we have

$$(x,y)B_i^+ = \left(\underbrace{xb_{\overline{-i+1},\overline{-i}}}_{\in \operatorname{rad}(P_{\overline{-i}})} + \underbrace{yb_{\overline{-i+2},\overline{-i}}}_{\in \operatorname{rad}(P_{\overline{-i}})}, - \underbrace{x2b_{\overline{-i+1},\overline{-i+1}}}_{\in \operatorname{rad}(P_{\overline{-i+1}})} - \underbrace{yb_{\overline{-i+2},\overline{-i+1}}}_{\in \operatorname{rad}(P_{\overline{-i+1}})} \right)$$

and therefore $\operatorname{Im}(B_i^+) \subseteq \operatorname{rad}(P_{\overline{-i}}) \oplus \operatorname{rad}(P_{\overline{-i+1}})$.
Similarly we have

$$\begin{aligned}
(y,z)B_i^- &= \left(yb_{\overline{-i+2},\overline{-i+1}} + zb_{\overline{-i},\overline{-i+1}}, -y2b_{\overline{-i+2},\overline{-i+2}} - zb_{\overline{-i},\overline{-i+2}}\right)\\
(z,x)C_i^+ &= \left(0, -zb_{\overline{-i},\overline{-i+1}} - x2b_{\overline{-i+1},\overline{-i+1}}\right)\\
(x,y)C_i^- &= \left(x2b_{\overline{-i+1},\overline{-i+1}}, -xb_{\overline{-i+1},\overline{-i+2}}\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
\operatorname{Im}(B_i^-) &\subseteq \operatorname{rad}(P_{\overline{-i+1}}) \oplus \operatorname{rad}(P_{\overline{-i+2}})\\
\operatorname{Im}(C_i^+) &\subseteq \operatorname{rad}(P_{\overline{-i}}) \oplus \operatorname{rad}(P_{\overline{-i+1}})\\
\operatorname{Im}(C_i^-) &\subseteq \operatorname{rad}(P_{\overline{-i+1}}) \oplus \operatorname{rad}(P_{\overline{-i+2}}).
\end{aligned}$$

Let $l \geq 0$.
Recall the formula for the differentials; cf. Definition 26.

$$\begin{aligned}
d_{2l} &= \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i}\, B^+_{\overline{l+i}}\, \iota_{2l,i}\right) + \left(\pi_{2l+1,l}\, (B^+_{\overline{2l}}\pi_{P_{\overline{l}}})\, \iota'_{2l}\right) + \left(\sum_{i\in[0,l-1]} \pi_{2l+1,i+1}\, C^+_{\overline{l+i}}\, \iota_{2l,i}\right)\\
d_{2l+1} &= \left(\sum_{i\in[0,l]} \pi_{2l+2,i}\, B^-_{\overline{l+i}}\, \iota_{2l+1,i}\right) + \left(\sum_{i\in[0,l-1]} \pi_{2l+2,i+1}\, C^-_{\overline{l+i}}\, \iota_{2l+1,i}\right) + \left(\pi'_{2l+2}\, (\iota_{P_{\overline{l+1}}} C^-_{\overline{2l}})\, \iota_{2l+1,l}\right)
\end{aligned}$$

By [**4,** Corollary 2, §9.1] and Definition 25 we obtain

$$\begin{aligned}
\operatorname{rad}(\Lambda)Q_{2l} = \operatorname{rad}(Q_{2l}) &= \left(\bigoplus_{i\in[0,l-1]} \operatorname{rad}(P_{\overline{2l+2i}}) \oplus \operatorname{rad}(P_{\overline{2l+2i+1}})\right) \oplus \operatorname{rad}(P_{\overline{l}})\\
\operatorname{rad}(\Lambda)Q_{2l+1} = \operatorname{rad}(Q_{2l+1}) &= \left(\bigoplus_{i\in[0,l]} \operatorname{rad}(P_{\overline{2l+2i+1}}) \oplus \operatorname{rad}(P_{\overline{2l+2i+2}})\right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
Q_{2l+1}d_{2l} &= \left(\bigoplus_{i\in[0,l]} (P_{\overline{2l+2i+1}} \oplus P_{\overline{2l+2i+2}})\right) d_{2l}\\
&\subseteq \left(\sum_{i\in[0,l-1]} (P_{\overline{2l+2i+1}} \oplus P_{\overline{2l+2i+2}})\ B^+_{\overline{l+i}}\, \iota_{2l,i}\right) + \left((P_{\overline{l+1}} \oplus P_{\overline{l+2}})\ (B^+_{\overline{2l}}\pi_{P_{\overline{l}}})\, \iota'_{2l}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\sum_{i\in[0,l-1]} \left(P_{\overline{2l+2i}} \oplus P_{\overline{2l+2i+1}}\right)\ C^{+}_{\overline{l+i}}\ \iota_{2l,i} \right) \\
&\subseteq \left(\sum_{i\in[0,l-1]} \left(\mathrm{rad}(P_{\overline{2l+2i}}) \oplus \mathrm{rad}(P_{\overline{2l+2i+1}})\right)\ \iota_{2l,i} \right) + \left(\mathrm{rad}(P_{\bar{l}})\ \iota'_{2l}\right) \\
&+ \left(\sum_{i\in[0,l-1]} \left(\mathrm{rad}(P_{\overline{2l+2i}}) \oplus \mathrm{rad}(P_{\overline{2l+2i+1}})\right)\ \iota_{2l,i} \right) \\
&= \left(\bigoplus_{i\in[0,l-1]} \mathrm{rad}(P_{\overline{2l+2i}}) \oplus \mathrm{rad}(P_{\overline{2l+2i+1}}) \right) \oplus \mathrm{rad}(P_{\bar{l}}) \\
&= \mathrm{rad}(Q_{2l}) = \mathrm{rad}(\Lambda)Q_{2l}
\end{aligned}
$$

$$
\begin{aligned}
Q_{2l+2}d_{2l+1} &= \left(\left(\bigoplus_{i\in[0,l]} \left(P_{\overline{2l+2i+2}} \oplus P_{\overline{2l+2i}}\right) \right) \oplus P_{\overline{l+1}} \right) d_{2l+1} \\
&\subseteq \left(\sum_{i\in[0,l]} \left(P_{\overline{2l+2i+2}} \oplus P_{\overline{2l+2i}}\right)\ B^{-}_{\overline{l+i}}\ \iota_{2l+1,i} \right) \\
&+ \left(\sum_{i\in[0,l-1]} \left(P_{\overline{2l+2i+1}} \oplus P_{\overline{2l+2i+2}}\right)\ C^{-}_{\overline{l+i}}\ \iota_{2l+1,i} \right) + \left(P_{\overline{l+1}}\ (\iota_{P_{\overline{l+1}}} C^{-}_{\overline{2l}})\ \iota_{2l+1,l} \right) \\
&\subseteq \left(\sum_{i\in[0,l]} \left(\mathrm{rad}(P_{\overline{2l+2i+1}}) \oplus \mathrm{rad}(P_{\overline{2l+2i+2}})\right)\ \iota_{2l+1,i} \right) \\
&+ \left(\sum_{i\in[0,l-1]} \left(\mathrm{rad}(P_{\overline{2l+2i+1}}) \oplus \mathrm{rad}(P_{\overline{2l+2i+2}})\right)\ \iota_{2l+1,i} \right) \\
&+ \left(\left(\mathrm{rad}(P_{\overline{l+1}}) \oplus \mathrm{rad}(P_{\overline{l+2}})\right)\ \iota_{2l+1,l}\right) \\
&= \left(\bigoplus_{i\in[0,l]} \mathrm{rad}(P_{\overline{2l+2i+1}}) \oplus \mathrm{rad}(P_{\overline{2l+2i+2}}) \right) \\
&= \mathrm{rad}(Q_{2l+1}) = \mathrm{rad}(\Lambda)Q_{2l+1}\,.
\end{aligned}
$$

□

Chapter 4

Projective resolution over $\mathbb{Z}_{(2)}\mathrm{A}_4$

Let $\zeta := \zeta_3$ be a third primitive root of unity over $\mathbb{Q}$.

Let $R := \mathbb{Z}_{(2)}$ and $S := \mathbb{Z}_{(2)}[\zeta]$. Let $\otimes := \underset{R}{\otimes}$.

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$ with $\zeta\sigma = \zeta^2$. Given $\xi \in S$, we write $\bar{\xi} := \xi\sigma$.

We attempt to projectively resolve the trivial $R\mathrm{A}_4$-module R.

We arrive at some terms of a projective resolution of R over $R\mathrm{A}_4$ via two methods and at an entire projective resolution of $R \oplus R$ over $R\mathrm{A}_4$.

4.1 Indecomposable projective modules

Recall that

$$\check{\Lambda} := \left\{ \left(u, r + s\zeta, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) \in R \times R[\zeta] \times R^{3\times 3} : \begin{array}{c} u \equiv_4 i,\ g \equiv_4 0,\ h \equiv_4 0, \\ e - a \equiv_4 s \equiv_2 d,\ r - e \equiv_4 b + d \equiv_2 0 \end{array} \right\};$$

cf. Corollary 7.

Definition 31 We define the following idempotent elements of $\check{\Lambda}$.

$$\check{\mathcal{E}}_0 := \left(1, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \qquad \check{\mathcal{E}}_1 := \left(0, 1, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

Lemma 32 Let

$$b_1 := \left(1, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \qquad b_2 := \left(0, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right)$$

$$b_3 := \left(0, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}\right) \qquad b_4 := \left(0, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}\right)$$

$$b_5 := \left(0, 0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \qquad b_6 := \left(0, 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$b_7 := \left(0, 1 + 0\zeta, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \qquad b_8 := \left(0, 1 + 1\zeta, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$b_9 := \left(0, 2 + 0\zeta, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \qquad b_{10} := \left(0, 2 + 2\zeta, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$b_{11} := \left(0, 4 + 0\zeta, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \qquad b_{12} := \left(0, 0 + 4\zeta, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

Then (b_1, b_2) is an R-linear basis of $\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0$, (b_3, b_4) is an R-linear basis of $\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_1$, (b_5, b_6) is an R-linear basis of $\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_0$ and $(b_7, b_8, b_9, b_{10}, b_{11}, b_{12})$ is an R-linear basis of $\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1$.

So altogether, $(b_i : i \in [1, 12])$ is an R-linear basis of $\check{\Lambda}$.

We have the following multiplication table for the basis elements.

$(\cdot)$	b_1	b_2	b_3	b_4	b_5	b_6
b_1	b_1	b_2	b_3	b_4	0	0
b_2	b_2	$4b_2$	$4b_3$	$4b_4$	0	0
b_3	0	0	0	0	b_2	0
b_4	0	0	0	0	0	b_2
b_5	b_5	$4b_5$	$4b_7 - 2b_{10} + b_{12}$	$4b_8 + 2b_9 - 2b_{10} - b_{11}$	0	0
b_6	b_6	$4b_6$	$2b_9 - b_{11}$	$2b_{10} - b_{11} - b_{12}$	0	0
b_7	0	0	0	0	b_5	b_6
b_8	0	0	0	0	$-b_6$	$b_5 + b_6$
b_9	0	0	0	0	$2b_6$	0
b_{10}	0	0	0	0	0	$2b_6$
b_{11}	0	0	0	0	0	0
b_{12}	0	0	0	0	0	0

$(\cdot)$	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
b_1	0	0	0	0	0	0
b_2	0	0	0	0	0	0
b_3	b_3	b_4	0	0	0	0
b_4	b_4	$b_4 - b_3$	$2b_3$	$2b_4$	0	0
b_5	0	0	0	0	0	0
b_6	0	0	0	0	0	0
b_7	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
b_8	b_8	$b_8 - b_7$	$2b_7 + b_9 - b_{10} + b_{12}$	$2b_8 + b_9 - b_{11}$	$b_{11} + b_{12}$	$-b_{11}$
b_9	b_9	b_{10}	b_{11}	$b_{11} + b_{12}$	$2b_{11}$	$2b_{12}$
b_{10}	b_{10}	$-b_9 + b_{10}$	$2b_9 + b_{12}$	$2b_{10} - b_{11}$	$2(b_{11} + b_{12})$	$-2b_{11}$
b_{11}	b_{11}	$b_{11} + b_{12}$	$2b_{11}$	$2(b_{11} + b_{12})$	$4b_{11}$	$4b_{12}$
b_{12}	b_{12}	$-b_{11}$	$2b_{12}$	$-2b_{11}$	$4b_{12}$	$-4(b_{11} + b_{12})$

Lemma 33 We have

$$\begin{aligned}\operatorname{rad}(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0) &= {}_R\langle 2b_1, b_2\rangle,\\ \operatorname{rad}(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1) &= {}_R\langle 2b_7, 2b_8, b_9, b_{10}, b_{11}, b_{12}\rangle\end{aligned}$$

with

$$(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0)/\operatorname{rad}(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0) \cong \mathbb{F}_2 \qquad\qquad (\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1)/\operatorname{rad}(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1) \cong \mathbb{F}_4.$$

Proof. Let $I_0 := {}_R\langle 2b_1, b_2\rangle \subseteq \check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0$ and $I_1 := {}_R\langle 2b_7, 2b_8, b_9, b_{10}, b_{11}, b_{12}\rangle \subseteq \check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1$. Note that both I_0 and I_1 are ideals, as we verify using the multiplication table of Lemma 32.

First we show that $(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0)/I_0$ and $(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1)/I_1$ are semi-simple.

We have

$$(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0)/I_0 = {}_R\langle b_1 + I_0\rangle \cong \mathbb{F}_2$$

with ${}_R\langle b_1 + I_0\rangle \to \mathbb{F}_2 \;:\; (b_1 + I_0) \mapsto 1_{\mathbb{F}_2}$.
By Lemma 32 we have

$$(b_8 + I_1)^2 = (b_8 + I_1) - (b_7 + I_1) = (b_8 + I_1) - 1$$

so that $(b_8 + I_1)$ has the minimal polynomial $x^2 + x + 1 \in \mathbb{F}_2[x]$. So we obtain

$$(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1)/I_1 = {}_R\langle b_7 + I_1, b_8 + I_1\rangle \cong \mathbb{F}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_4$$

with $\mathbb{F}_2[x]/(x^2 + x + 1) \to {}_R\langle b_7 + I_1, b_8 + I_1\rangle \;:\; x + (x^2 + x + 1) \mapsto (b_8 + I_1)$.

Since they are isomorphic to fields, $(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0)/I_0$ and $(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1)/I_1$ are semi-simple and we obtain that $\operatorname{rad}(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0) \subseteq I_0$ and $\operatorname{rad}(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1) \subseteq I_1$ by [**8,** Theorem (6.10)].

Using the multiplication table of Lemma 32 we see that $I_0^2 \subseteq 2\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0$ and $I_1^3 \subseteq 2\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1$. So we obtain $I_0 \subseteq \operatorname{rad}(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0)$ and $I_1 \subseteq \operatorname{rad}(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1)$ by [**7,** Lemma 213].

In conclusion we have $I_0 = \operatorname{rad}(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0)$ and $I_1 = \operatorname{rad}(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1)$. ▫

Lemma 34 We have the orthogonal decomposition $1_{\check{\Lambda}} = \check{\mathcal{E}}_0 + \check{\mathcal{E}}_1$ into primitive idempotent elements of $\check{\Lambda}$. We obtain the Peirce decomposition

$$\check{\Lambda} = \check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0 \oplus \check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_1 \oplus \check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_0 \oplus \check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1.$$

Proof. We show that $\check{\mathcal{E}}_0$ and $\check{\mathcal{E}}_1$ are primitive. Let $j \in \{0,1\}$.

Suppose $\mathcal{E}_j = \alpha_j + \beta_j$ with $\alpha_j, \beta_j \in \check{\Lambda}\backslash\{0\}$ orthogonal idempotent elements. We have

$$\alpha_j = (\alpha_j + \beta_j)\alpha_j(\alpha_j + \beta_j) = \check{\mathcal{E}}_j\alpha_j\check{\mathcal{E}}_j \in \check{\mathcal{E}}_j\check{\Lambda}\check{\mathcal{E}}_j\,.$$

By Lemma 33 we have that $(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0)/\operatorname{rad}(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0) \cong \mathbb{F}_2$ and $(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1)/\operatorname{rad}(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1) \cong \mathbb{F}_4$. Hence, both $\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0$ and $\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1$ are local rings by [**7,** Remark 192].

Since local rings have only 0 and 1 as idempotent elements, we obtain $\alpha_j = 0$ or $\alpha_j = 1$.

Therefore there exists no nontrivial orthogonal decomposition of $\check{\mathcal{E}}_0$ or $\check{\mathcal{E}}_1$ into idempotent elements. □

Definition 35 We denote the indecomposable projective modules belonging to the idempotent elements from Definition 31 by

$$\check{P}_0 := \check{\Lambda}\check{\mathcal{E}}_0 = \left\{ \left(u, 0, \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & i \end{pmatrix} \right) : \begin{array}{c} u, c, f, i \in R \\ u \equiv_4 i \end{array} \right\}$$

$$\check{P}_1 := \check{\Lambda}\check{\mathcal{E}}_1 = \left\{ \left(0, r + s\zeta, \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{pmatrix} \right) : \begin{array}{c} r, s, a, b, d, e, g, h \in R \\ g \equiv_4 0 \equiv_4 h \\ a - e \equiv_4 s \equiv_2 d,\ r - e \equiv_4 b + d \equiv_2 0 \end{array} \right\};$$

cf. Corollary 7.

By abuse of notation we often write

$$\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right) := \left(u, 0, \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & i \end{pmatrix} \right) \in \check{P}_0$$

$$\left(r + s\zeta, \begin{pmatrix} a & b \\ d & e \\ g & h \end{pmatrix} \right) := \left(0, r + s\zeta, \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{pmatrix} \right) \in \check{P}_1.$$

So given $\lambda := \left(u', r' + s'\zeta, \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} \right) \in \check{\Lambda}$,

$x := \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right) \in \check{P}_0$ and $y := \left(r + s\zeta, \begin{pmatrix} a & b \\ d & e \\ g & h \end{pmatrix} \right) \in \check{P}_1$ we have

$$\lambda \cdot x = \left(u'u, \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right) \in \check{P}_0,$$

$$\lambda \cdot y = \left((r' + s'\zeta)(r + s\zeta), \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} \begin{pmatrix} a & b \\ d & e \\ g & h \end{pmatrix} \right) \in \check{P}_1.$$

Remark 36

We identify along

$$\begin{aligned} \mathrm{Hom}_{\check{\Lambda}}(\check{P}_i, \check{P}_j) &\xrightarrow{\sim} \check{\mathcal{E}}_i \Lambda \check{\mathcal{E}}_j \\ \varphi &\longmapsto \check{\mathcal{E}}_i \varphi \\ (\mu\check{\mathcal{E}}_i \mapsto \mu\check{\mathcal{E}}_i \lambda \check{\mathcal{E}}_j) &\longleftarrow\!\shortmid \check{\mathcal{E}}_i \lambda \check{\mathcal{E}}_j \end{aligned}$$

for $i, j \in \{0, 1\}$.

Let $i \in \{1, 2\}$, $j \in \{3, 4\}$, $k \in \{5, 6\}$ and $l \in \{7, 8, 9, 10, 11, 12\}$. The above identification yields the following maps; cf. Lemma 32.

$$\begin{array}{cccc} \check{P}_0 \xrightarrow{b_i} \check{P}_0 & \check{P}_0 \xrightarrow{b_j} \check{P}_1 & \check{P}_1 \xrightarrow{b_k} \check{P}_0 & \check{P}_1 \xrightarrow{b_l} \check{P}_1 \\ \xi \mapsto \xi b_i & \xi \mapsto \xi b_j & \xi \mapsto \xi b_k & \xi \mapsto \xi b_l \end{array}$$

4.2 Attempt to projectively resolve $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{A}_4$

Recall that $R = \mathbb{Z}_{(2)}$ and $S = R[\zeta] = \mathbb{Z}_{(2)}[\zeta]$.

We attempt to construct a projective resolution

$$\check{Q} = \left(\cdots \longrightarrow \check{Q}_3 \xrightarrow{\check{d}_2} \check{Q}_2 \xrightarrow{\check{d}_1} Q_1 \xrightarrow{\check{d}_0} \check{Q}_0 \right)$$

of the trivial module R over $\check{\Lambda}$ such that $R[\zeta] \otimes \check{Q} \cong Q$; cf. Remark 8 and Theorem 28.

Remark 37 We have the isomorphism of S-algebras

$$\begin{aligned} S\mathrm{A}_4 &\underset{\sim}{\xrightarrow{\widetilde{\omega}}} \Lambda'_{(2)} = \Lambda \\ (1,2)(3,4) &\mapsto \left(1, 1, 1, \begin{pmatrix} -3 & 2\zeta^2 & -2\zeta \\ -2\zeta & 1 & -2\zeta^2 \\ 2\zeta^2 & -2\zeta & 1 \end{pmatrix}\right) \\ (1,2,3) &\mapsto \left(1, \zeta, \zeta^2, \begin{pmatrix} \zeta^2 & -2\zeta & -2\zeta \\ 0 & \zeta & -2\zeta^2 \\ 0 & 0 & 1 \end{pmatrix}\right) \end{aligned}$$

as a consequence of Lemma 13.

Recall the isomorphism of S-algebras

$$\varphi \;:\; S \otimes \check{\Lambda} \xrightarrow{\sim} \Lambda,\ \xi \otimes (u, v, N) \longmapsto \left(\xi u, \xi v, \xi \bar{v}, \xi N^T\right),$$

with $T = \begin{pmatrix} 1 & \zeta & 0 \\ \zeta & 1 & 0 \\ 0 & 0 & 2\zeta^2 \end{pmatrix}$; cf. Lemma 10.

Let $x := \left(\zeta^2, 1, 3(\zeta^2 - 1), \begin{pmatrix} 1-\zeta^2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}\right) \in \mathrm{U}(\Lambda)$; cf. Corollary 15.

We have the isomorphism of S-algebras

$$\widetilde{\varphi}\;:\;S\otimes\check{\Lambda}\xrightarrow{\sim}\Lambda,\;\xi\otimes(u,v,N)\longmapsto x\big(\,(\xi\otimes(u,v,N))\,\varphi\big)x^{-1}=\Big(\xi u,\xi v,\xi\bar{v},\xi N^{\widetilde{T}}\Big),$$

with $\widetilde{T}=\begin{pmatrix}-\zeta^2&\zeta&0\\1&1&0\\0&0&2\end{pmatrix}$. In the notation of Remark 11 this means, we only conjugate with the first three matrix factors.

Experiments have shown that isomorphic replacement via $\widetilde{\varphi}$ is better suited than via φ in order to find the differentials of our projective resolution; cf. Corollary 40.

Remark 38 Let $\check{X}$ be a $\check{\Lambda}$-module.

(1) $S\otimes\check{X}$ is an $S\otimes\check{\Lambda}$-module via

$$(\xi\otimes\lambda)\cdot(\eta\otimes x)=\xi\eta\otimes\lambda x$$

for $\xi,\eta\in S,\lambda\in\check{\Lambda}$ and $x\in\check{X}$.

(2) $S\otimes\check{X}$ is also a Λ-module by identifying along $\widetilde{\varphi}^{-1}$.

(3) Given $\xi\in S\otimes\check{\Lambda}$ and $\lambda\in\Lambda$ we have

$$(\lambda\cdot\xi)\widetilde{\varphi}=\big((\lambda\widetilde{\varphi}^{-1})\xi\big)\,\widetilde{\varphi}=\lambda\widetilde{\varphi}^{-1}\widetilde{\varphi}\cdot\xi\widetilde{\varphi}=\lambda\cdot(\xi\widetilde{\varphi})$$

so that $\widetilde{\varphi}$ is Λ-linear.

Lemma 39 We have $(1\otimes\check{\mathcal{E}}_0)\widetilde{\varphi}=\mathcal{E}_0$ and $(1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}=(\mathcal{E}_1+\mathcal{E}_2)$.

Proof. Recall that $\check{\mathcal{E}}_0=\left(1,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right)$ and $\check{\mathcal{E}}_1=\left(0,1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)$.

$$(1\otimes\check{\mathcal{E}}_0)\widetilde{\varphi}=\left(1,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}^{\widetilde{T}}\right)=\left(1,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right)=\mathcal{E}_0$$

$$(1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}=\left(0,1,1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}^{\widetilde{T}}\right)=\left(0,1,1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)=\mathcal{E}_1+\mathcal{E}_2$$

□

Corollary 40 We have the isomorphisms of Λ-modules $\widetilde{\varphi}_0:=\widetilde{\varphi}|^{P_0}_{S\otimes\check{P}_0}$ and $\widetilde{\varphi}_1:=\widetilde{\varphi}|^{P_1\oplus P_2}_{S\otimes\check{P}_1}$ so that

$$S\otimes\check{P}_0\xrightarrow[\sim]{\widetilde{\varphi}_0}P_0$$

$$\xi\otimes\left(u,0,\begin{pmatrix}0&0&c\\0&0&f\\0&0&i\end{pmatrix}\right)\mapsto\left(\xi u,0,0,\xi\begin{pmatrix}0&0&c\\0&0&f\\0&0&i\end{pmatrix}^{\widetilde{T}}\right)$$

$$S \otimes \check{P}_1 \xrightarrow[\sim]{\widetilde{\varphi}_1} P_1 \oplus P_2$$

$$\xi \otimes \left(0, r+s\zeta, \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{pmatrix}\right) \mapsto \left(0, \xi(r+s\zeta), \xi(r+s\zeta^2), \xi \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{pmatrix}^{\widetilde{T}}\right)$$

where $u, r, s, a, b, c, d, e, f, g, h, i \in R$ and $\xi \in S$.

Proof. We show $(S \otimes \check{P}_0)\widetilde{\varphi}_0 = P_0$ and $(S \otimes \check{P}_1)\widetilde{\varphi}_1 = P_1 \oplus P_2$ using Lemma 39.

$$(S \otimes \check{P}_0)\widetilde{\varphi}_0 = (S \otimes \check{\Lambda}\check{\mathcal{E}}_0)\widetilde{\varphi}_0 = (S \otimes \check{\Lambda})\widetilde{\varphi} \cdot (1 \otimes \check{\mathcal{E}}_0)\widetilde{\varphi}_0 = \Lambda\mathcal{E}_0 = P_0$$
$$(S \otimes \check{P}_1)\widetilde{\varphi}_1 = (S \otimes \check{\Lambda}\check{\mathcal{E}}_1)\widetilde{\varphi}_1 = (S \otimes \check{\Lambda})\widetilde{\varphi} \cdot (1 \otimes \check{\mathcal{E}}_1)\widetilde{\varphi}_1 = \Lambda(\mathcal{E}_1 + \mathcal{E}_2) = P_1 \oplus P_2$$

Hence $\widetilde{\varphi}_0 = \widetilde{\varphi}|_{S\otimes\check{P}_0}^{P_0}$ and $\widetilde{\varphi}_1 = \widetilde{\varphi}|_{S\otimes\check{P}_1}^{P_1\oplus P_2}$ are isomorphisms of Λ-modules since $\widetilde{\varphi}$ is an isomorphism of Λ-modules; cf. Remark 38, (3). ▫

Remark 41 When considering $P_1 \oplus P_2$ as an external direct sum, we obtain

$$S \otimes \check{P}_0 \xrightarrow[\sim]{\widetilde{\varphi}_0} P_0$$

$$\xi \otimes \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \mapsto \left(\xi u, \xi \begin{pmatrix} 2(c - f\zeta) \\ -2(c + f\zeta^2) \\ i \end{pmatrix}\right)$$

$$S \otimes \check{P}_1 \xrightarrow[\sim]{\widetilde{\varphi}_1} P_1 \oplus P_2$$

$$\xi \otimes \left(r+s\zeta, \begin{pmatrix} a & b \\ d & e \\ g & h \end{pmatrix}\right) \mapsto \left(\left(\xi(r+s\zeta), \xi \begin{pmatrix} a\zeta + b - d\zeta^2 - e\zeta \\ -a\zeta - b - d - e\zeta^2 \\ \frac{1}{2}(g\zeta + h) \end{pmatrix}\right), \left(\xi(r+s\zeta^2),\ \xi \begin{pmatrix} -a\zeta^2 + b + d - e\zeta \\ a\zeta^2 - b + d\zeta - e\zeta^2 \\ \frac{1}{2}(-g\zeta^2 + h) \end{pmatrix}\right)\right)$$

where $u, r, s, a, b, c, d, e, f, g, h, i \in R$ and $\xi \in S$.

Corollary 42 We have the following commutative diagrams; cf. Lemma 24 and Lemma 32.

$$\begin{array}{ccc} S \otimes \check{P}_0 & \xrightarrow[\sim]{\widetilde{\varphi}_0} & P_0 \\ \downarrow{\scriptstyle S\otimes b_1} & & \downarrow{\scriptstyle b_{0,0}} \\ S \otimes \check{P}_0 & \xrightarrow[\sim]{\widetilde{\varphi}_0} & P_0 \end{array} \qquad \begin{array}{ccc} S \otimes \check{P}_0 & \xrightarrow[\sim]{\widetilde{\varphi}_0} & P_0 \\ \downarrow{\scriptstyle S\otimes b_2} & & \downarrow{\scriptstyle \widetilde{b}_{0,0}} \\ S \otimes \check{P}_0 & \xrightarrow[\sim]{\widetilde{\varphi}_0} & P_0 \end{array}$$

$$\begin{array}{ccc} S \otimes \check{P}_0 & \xrightarrow[\sim]{\widetilde{\varphi}_0} & P_0 \\ \downarrow{\scriptstyle S\otimes b_3} & & \downarrow{\scriptstyle (\zeta b_{0,1} \quad -\zeta^2 b_{0,2})} \\ S \otimes \check{P}_1 & \xrightarrow[\sim]{\widetilde{\varphi}_1} & P_1 \oplus P_2 \end{array} \qquad \begin{array}{ccc} S \otimes \check{P}_0 & \xrightarrow[\sim]{\widetilde{\varphi}_0} & P_0 \\ \downarrow{\scriptstyle S\otimes b_4} & & \downarrow{\scriptstyle (b_{0,1} \quad b_{0,2})} \\ S \otimes \check{P}_1 & \xrightarrow[\sim]{\widetilde{\varphi}_1} & P_1 \oplus P_2 \end{array}$$

$$\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_5"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}-b_{1,0}\\ b_{2,0}\end{pmatrix}"] \\ S\otimes\check{P}_0 \arrow[r, "\widetilde{\varphi}_0", "\sim"'] & P_0 \end{tikzcd}
\qquad
\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_6"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}-\zeta^2 b_{1,0}\\ -\zeta b_{2,0}\end{pmatrix}"] \\ S\otimes\check{P}_0 \arrow[r, "\widetilde{\varphi}_0", "\sim"'] & P_0 \end{tikzcd}$$

$$\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_7"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}b_{1,1} & 0\\ 0 & b_{2,2}\end{pmatrix}"] \\ S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] & P_1\oplus P_2 \end{tikzcd}
\qquad
\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_8"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}-\zeta^2 b_{1,1} & 0\\ -\zeta b_{2,1} & -\zeta b_{2,2}\end{pmatrix}"] \\ S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] & P_1\oplus P_2 \end{tikzcd}$$

$$\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_9"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}2b_{1,1}-\widetilde{b}_{1,1} & \zeta b_{1,2}\\ -\zeta^2 b_{2,1} & 2b_{2,2}\end{pmatrix}"] \\ S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] & P_1\oplus P_2 \end{tikzcd}
\qquad
\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_{10}"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}-2\zeta^2 b_{1,1} & -\zeta^2 b_{1,2}\\ -\zeta b_{2,1} & -2\zeta b_{2,2}\end{pmatrix}"] \\ S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] & P_1\oplus P_2 \end{tikzcd}$$

$$\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_{11}"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}4b_{1,1}-\widetilde{b}_{1,1} & 0\\ 0 & 4b_{2,2}-\widetilde{b}_{2,2}\end{pmatrix}"] \\ S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] & P_1\oplus P_2 \end{tikzcd}$$

$$\begin{tikzcd} S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] \arrow[d, "S\otimes b_{12}"'] & P_1\oplus P_2 \arrow[d, "\begin{pmatrix}\zeta(4b_{1,1}-\widetilde{b}_{1,1}) & 0\\ 0 & \zeta^2(4b_{2,2}-\widetilde{b}_{2,2})\end{pmatrix}"] \\ S\otimes\check{P}_1 \arrow[r, "\widetilde{\varphi}_1", "\sim"'] & P_1\oplus P_2 \end{tikzcd}$$

Proof. Since all appearing maps are Λ-linear, we only have to consider the Λ-linear generators $(1\otimes\check{\mathcal{E}}_0)$ of $S\otimes\check{P}_0$ and $(1\otimes\check{\mathcal{E}}_1)$ of $S\otimes\check{P}_1$ to show that the diagrams commute.

$$(1\otimes\check{\mathcal{E}}_0)(S\otimes b_1)\widetilde{\varphi}_0 = (1\otimes b_1)\widetilde{\varphi}_0 = \left(1, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \mathcal{E}_0 b_{0,0} = (1\otimes\check{\mathcal{E}}_0)\widetilde{\varphi}_0 b_{0,0}$$

$$(1\otimes\check{\mathcal{E}}_0)(S\otimes b_2)\widetilde{\varphi}_0 = (1\otimes b_2)\widetilde{\varphi}_0 = \left(0, \begin{pmatrix}0\\0\\4\end{pmatrix}\right) = \mathcal{E}_0 \widetilde{b}_{0,0} = (1\otimes\check{\mathcal{E}}_0)\widetilde{\varphi}_0 \widetilde{b}_{0,0}$$

$$(1\otimes\check{\mathcal{E}}_0)(S\otimes b_3)\widetilde{\varphi}_1 = (1\otimes b_3)\widetilde{\varphi}_1 = \left(\left(0, \begin{pmatrix}0\\0\\2\zeta\end{pmatrix}\right), \left(0, \begin{pmatrix}0\\0\\-2\zeta^2\end{pmatrix}\right)\right) = \mathcal{E}_0\begin{pmatrix}\zeta b_{0,1} & -\zeta^2 b_{0,2}\end{pmatrix}$$

$$\begin{aligned}
&= (1\otimes\check{\mathcal{E}}_0)\widetilde{\varphi}_0 \begin{pmatrix}\zeta b_{0,1} & -\zeta^2 b_{0,2}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_0)(S\otimes b_4)\widetilde{\varphi}_1 &= (1\otimes b_4)\widetilde{\varphi}_1 = \left(\left(0,\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(0,\begin{pmatrix}0\\0\\2\end{pmatrix}\right)\right) = \mathcal{E}_0 \begin{pmatrix}b_{0,1} & b_{0,2}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_0)\widetilde{\varphi}_0 \begin{pmatrix}b_{0,1} & b_{0,2}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_5)\widetilde{\varphi}_0 &= (1\otimes b_5)\widetilde{\varphi}_0 = \left(0,\begin{pmatrix}2\\-2\\0\end{pmatrix}\right) = (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}-b_{1,0}\\ b_{2,0}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}-b_{1,0}\\ b_{2,0}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_6)\widetilde{\varphi}_0 &= (1\otimes b_6)\widetilde{\varphi}_0 = \left(0,\begin{pmatrix}-2\zeta\\-2\zeta^2\\0\end{pmatrix}\right) = (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}-\zeta^2 b_{1,0}\\ -\zeta b_{2,0}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}-\zeta^2 b_{1,0}\\ -\zeta b_{2,0}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_7)\widetilde{\varphi}_1 &= (1\otimes b_7)\widetilde{\varphi}_1 = \left(\left(1,\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(1,\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\right) = (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}b_{1,1} & 0\\ 0 & b_{2,2}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}b_{1,1} & 0\\ 0 & b_{2,2}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_8)\widetilde{\varphi}_1 &= (1\otimes b_8)\widetilde{\varphi}_1 = \left(\left(1+\zeta,\begin{pmatrix}-2\zeta\\-\zeta^2\\0\end{pmatrix}\right),\left(1+\zeta^2,\begin{pmatrix}-\zeta\\0\\0\end{pmatrix}\right)\right) = (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}-\zeta^2 b_{1,1} & 0\\ -\zeta b_{2,1} & -\zeta b_{2,2}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}-\zeta^2 b_{1,1} & 0\\ -\zeta b_{2,1} & -\zeta b_{2,2}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_9)\widetilde{\varphi}_1 &= (1\otimes b_9)\widetilde{\varphi}_1 = \left(\left(2,\begin{pmatrix}-2\zeta^2\\-2\\0\end{pmatrix}\right),\left(2,\begin{pmatrix}2\\2\zeta\\0\end{pmatrix}\right)\right) = (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}2b_{1,1}-\widetilde{b}_{1,1} & \zeta b_{1,2}\\ -\zeta^2 b_{2,1} & 2b_{2,2}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}2b_{1,1}-\widetilde{b}_{1,1} & \zeta b_{1,2}\\ -\zeta^2 b_{2,1} & 2b_{2,2}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_{10})\widetilde{\varphi}_1 &= (1\otimes b_{10})\widetilde{\varphi}_1 = \left(\left(2+2\zeta,\begin{pmatrix}-2\zeta\\-2\zeta^2\\0\end{pmatrix}\right),\left(2+2\zeta^2,\begin{pmatrix}-2\zeta\\-2\zeta^2\\0\end{pmatrix}\right)\right)\\
&= (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}-2\zeta^2 b_{1,1} & -\zeta^2 b_{1,2}\\ -\zeta b_{2,1} & -2\zeta b_{2,2}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}-2\zeta^2 b_{1,1} & -\zeta^2 b_{1,2}\\ -\zeta b_{2,1} & -2\zeta b_{2,2}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_{11})\widetilde{\varphi}_1 &= (1\otimes b_{11})\widetilde{\varphi}_1 = \left(\left(4,\begin{pmatrix}0\\0\\0\end{pmatrix}\right),\left(4,\begin{pmatrix}0\\0\\0\end{pmatrix}\right)\right) = (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}4b_{1,1}-\widetilde{b}_{1,1} & 0\\ 0 & 4b_{2,2}-\widetilde{b}_{2,2}\end{pmatrix}\\
&= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}4b_{1,1}-\widetilde{b}_{1,1} & 0\\ 0 & 4b_{2,2}-\widetilde{b}_{2,2}\end{pmatrix}\\
(1\otimes\check{\mathcal{E}}_1)(S\otimes b_{12})\widetilde{\varphi}_1 &= (1\otimes b_{12})\widetilde{\varphi}_1 = \left(\left(4\zeta,\begin{pmatrix}0\\0\\0\end{pmatrix}\right),\left(4\zeta^2,\begin{pmatrix}0\\0\\0\end{pmatrix}\right)\right)\\
&= (\mathcal{E}_1,\mathcal{E}_2)\begin{pmatrix}\zeta(4b_{1,1}-\widetilde{b}_{1,1}) & 0\\ 0 & \zeta^2(4b_{2,2}-\widetilde{b}_{2,2})\end{pmatrix}
\end{aligned}$$

$$= (1\otimes\check{\mathcal{E}}_1)\widetilde{\varphi}_1\begin{pmatrix}\zeta(4b_{1,1}-\widetilde{b}_{1,1}) & 0\\ 0 & \zeta^2(4b_{2,2}-\widetilde{b}_{2,2})\end{pmatrix}.$$

□

Definition 43 We define the following $\check{\Lambda}$-linear maps.

$$\check{\varepsilon}: \check{P}_0 \to R,\ \left(u, \begin{pmatrix}c\\ f\\ i\end{pmatrix}\right) \mapsto u$$

$$\check{d}_0 := \begin{pmatrix}b_5\end{pmatrix}\ :\ \check{P}_1 \to \check{P}_0$$

$$\check{d}_1 := \begin{pmatrix}b_{10}\\ b_4\end{pmatrix}\ :\ \check{P}_1\oplus\check{P}_0 \to \check{P}_1$$

$$\check{d}_2 := \begin{pmatrix}-b_4 & 2b_1\\ b_3 & 0\\ b_{10}-2b_8 & b_5\end{pmatrix}\ :\ \check{P}_0\oplus\check{P}_0\oplus\check{P}_1 \to \check{P}_1\oplus\check{P}_0$$

$$\check{d}_3 := \begin{pmatrix}-b_5 & 0 & 2b_7-b_{10}\\ 0 & -b_5 & 2b_8+b_9-b_{10}\\ 0 & -2b_1 & b_4\end{pmatrix}\ :\ \check{P}_1\oplus\check{P}_1\oplus\check{P}_0 \to \check{P}_0\oplus\check{P}_0\oplus\check{P}_1$$

$$\check{d}_4 := \begin{pmatrix}-b_{10} & 2b_7 & -b_5\\ b_4 & 0 & 0\\ 0 & b_4 & 0\\ 0 & -2b_7+2b_8+b_9-b_{10} & b_5\end{pmatrix}\ :\ \check{P}_1\oplus\check{P}_0\oplus\check{P}_0\oplus\check{P}_1 \to \check{P}_1\oplus\check{P}_1\oplus\check{P}_0$$

$$\check{d}_5 := \begin{pmatrix}b_3 & 0 & 0 & b_3+b_4\\ b_4 & 2b_1 & 0 & b_4\\ 2b_8+2b_9-b_{10}-b_{11} & b_5 & 0 & 2b_8+2b_9-b_{11}\\ 0 & 0 & b_5 & 2b_8+b_9-b_{10}\\ 0 & 0 & 2b_1 & b_4\end{pmatrix}\ :\ \begin{array}{l}\check{P}_0\oplus\check{P}_0\oplus\check{P}_1\oplus\check{P}_1\oplus\check{P}_0\\ \quad\to \check{P}_1\oplus\check{P}_0\oplus\check{P}_0\oplus\check{P}_1\end{array}$$

Remark 44 Recall the trivial $\check{\Lambda}$-module R; cf. Remark 8.

Consider the following sequence of $\check{\Lambda}$-modules with the maps from Definition 43.

$$\check{Q} := \left(\check{P}_0\oplus\check{P}_0\oplus\check{P}_1\oplus\check{P}_1\oplus\check{P}_0 \xrightarrow{\check{d}_5} \check{P}_1\oplus\check{P}_0\oplus\check{P}_0\oplus\check{P}_1 \xrightarrow{\check{d}_4} \check{P}_1\oplus\check{P}_1\oplus\check{P}_0 \xrightarrow{\check{d}_3} \check{P}_0\oplus\check{P}_0\oplus\check{P}_1 \xrightarrow{\check{d}_2} \check{P}_1\oplus\check{P}_0 \xrightarrow{\check{d}_1} \check{P}_1 \xrightarrow{\check{d}_0} \check{P}_0 \xrightarrow{\check{\varepsilon}} R\right)$$

Then $\check{Q}$ is an exact sequence and $\check{\varepsilon}$ surjective.

Proof. Since $S = R[\zeta]$ is free of rank 2, hence faithfully flat over R, it suffices to verify that $S\otimes\check{Q}$ is an exact sequence and that $S\otimes\check{\varepsilon}$ is surjetcive.

With regard to Corollary 42 we define the following Λ-linear maps.

$$\widetilde{d}_0 := \begin{pmatrix}-b_{1,0}\\ b_{2,0}\end{pmatrix}\ :\ P_1\oplus P_2 \to P_0$$

$$\widetilde{d}_1 := \begin{pmatrix}-2\zeta^2 b_{1,1} & -\zeta^2 b_{1,2}\\ -\zeta b_{2,1} & -2\zeta b_{2,2}\\ b_{0,1} & b_{0,2}\end{pmatrix}\ :\ P_1\oplus P_2\oplus P_0 \to P_1\oplus P_2$$

$$\widetilde{d}_2 := \begin{pmatrix}-b_{0,1} & -b_{0,2} & 2b_{0,0}\\ \zeta b_{0,1} & -\zeta^2 b_{0,2} & 0\\ 0 & -\zeta^2 b_{1,2} & -b_{1,0}\\ \zeta b_{2,1} & 0 & b_{2,0}\end{pmatrix}\ :\ P_0\oplus P_0\oplus P_1\oplus P_2 \to P_1\oplus P_2\oplus P_0$$

$$\widetilde{d_3} := \begin{pmatrix} b_{1,0} & 0 & -2\zeta b_{1,1} & \zeta^2 b_{1,2} \\ -b_{2,0} & 0 & \zeta b_{2,1} & -2\zeta^2 b_{2,2} \\ 0 & b_{1,0} & 2b_{1,1} - \widetilde{b}_{1,1} & -b_{1,2} \\ 0 & -b_{2,0} & b_{2,1} & 2b_{2,2} \\ 0 & -2b_{0,0} & b_{0,1} & b_{0,2} \end{pmatrix} : P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \to P_0 \oplus P_0 \oplus P_1 \oplus P_2$$

$$\widetilde{d_4} := \begin{pmatrix} 2\zeta^2 b_{1,1} & \zeta^2 b_{1,2} & 2b_{1,1} & 0 & b_{1,0} \\ \zeta b_{2,1} & 2\zeta b_{2,2} & 0 & 2b_{2,2} & -b_{2,0} \\ b_{0,1} & b_{0,2} & 0 & 0 & 0 \\ 0 & 0 & b_{0,1} & b_{0,2} & 0 \\ 0 & 0 & -\widetilde{b}_{1,1} & -b_{1,2} & -b_{1,0} \\ 0 & 0 & b_{2,1} & 0 & b_{2,0} \end{pmatrix} : \begin{array}{l} P_1 \oplus P_2 \oplus P_0 \oplus P_0 \oplus P_1 \oplus P_2 \\ \quad \to P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \end{array}$$

$$\widetilde{d_5} := \begin{pmatrix} \zeta b_{0,1} & -\zeta^2 b_{0,2} & 0 & 0 & -\zeta^2 b_{0,1} & (1-\zeta^2) b_{0,2} \\ b_{0,1} & b_{0,2} & 2b_{0,0} & 0 & b_{0,1} & b_{0,2} \\ -\widetilde{b}_{1,1} & (\zeta - 1) b_{1,2} & -b_{1,0} & 0 & -2\zeta^2 b_{1,1} - \widetilde{b}_{1,1} & 2\zeta b_{1,2} \\ (1-\zeta^2) b_{2,1} & \widetilde{b}_{2,2} & b_{2,0} & 0 & 2b_{2,1} & -2\zeta b_{2,2} + \widetilde{b}_{2,2} \\ 0 & 0 & 0 & -b_{1,0} & 2b_{1,1} - \widetilde{b}_{1,1} & -b_{1,2} \\ 0 & 0 & 0 & b_{2,0} & b_{2,1} & 2b_{2,2} \\ 0 & 0 & 0 & 2b_{0,0} & b_{0,1} & b_{0,2} \end{pmatrix} :$$
$$P_0 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \to P_1 \oplus P_2 \oplus P_0 \oplus P_0 \oplus P_1 \oplus P_2$$

Let $\widetilde{Q} :=$

$$\left(\begin{array}{l} P_0 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \xrightarrow{\widetilde{d_5}} P_1 \oplus P_2 \oplus P_0 \oplus P_0 \oplus P_1 \oplus P_2 \xrightarrow{\widetilde{d_4}} P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \xrightarrow{\widetilde{d_3}} P_0 \oplus P_0 \oplus P_1 \oplus P_2 \cdots \\ \qquad \cdots \xrightarrow{\widetilde{d_2}} P_1 \oplus P_2 \oplus P_0 \xrightarrow{\widetilde{d_1}} P_1 \oplus P_2 \xrightarrow{\widetilde{d_0}} P_0 \xrightarrow{\varepsilon} S \end{array} \right).$$

Using the projective Λ-modules from Definition 25 and the differentials from Definition 26 we define

$$Q_{[-1,6]} := \left(Q_6 \xrightarrow{d_5} Q_5 \xrightarrow{d_4} Q_4 \xrightarrow{d_3} Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \xrightarrow{\varepsilon} S \right).$$

By Theorem 28 the sequence $Q_{[-1,6]}$ is exact and ε surjective. We show that $S \otimes \check{Q} \cong \widetilde{Q} \cong Q_{[-1,6]}$ as complexes over Λ.

For this we define the following Λ-linear maps.

$$T_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : P_1 \oplus P_2 \to P_1 \oplus P_2$$

$$T_2 := \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & -1 \end{pmatrix} : P_1 \oplus P_2 \oplus P_0 \to P_1 \oplus P_2 \oplus P_0$$

$$T_3 := \begin{pmatrix} 1 & \zeta^2 & 0 & 0 \\ 1 & -\zeta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : P_0 \oplus P_0 \oplus P_1 \oplus P_2 \to P_0 \oplus P_0 \oplus P_1 \oplus P_2$$

$$T_4 := \begin{pmatrix} 1 & 0 & -\zeta^2 & 0 & -\zeta^2 b_{1,0} \\ 0 & 1 & 0 & -\zeta & 0 \\ -1 & 0 & -\zeta & 0 & -\zeta b_{1,0} \\ 0 & -1 & 0 & -\zeta^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \to P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0$$

$$T_5 := \begin{pmatrix} \zeta & 0 & 0 & -\zeta b_{1,0} & -\zeta & 0 \\ 0 & \zeta^2 & 0 & 0 & 0 & -\zeta^2 \\ 0 & 0 & 1 & \zeta^2(\widetilde{b}_{0,0}-1) & \zeta^2 b_{0,1} & 0 \\ 0 & 0 & 1 & -\zeta(\widetilde{b}_{0,0}-1) & -\zeta b_{0,1} & 0 \\ \zeta & 0 & 0 & \zeta^2 b_{1,0} & \zeta^2 & 0 \\ 0 & \zeta^2 & 0 & 0 & 0 & \zeta \end{pmatrix} : \begin{array}{l} P_1 \oplus P_2 \oplus P_0 \oplus P_0 \oplus P_1 \oplus P_2 \\ \quad \to P_1 \oplus P_2 \oplus P_0 \oplus P_0 \oplus P_1 \oplus P_2 \end{array}$$

$$T_6 := \begin{pmatrix} \zeta^2 & -1 & 0 & 0 & 2\zeta^2 b_{0,1} & 0 & \zeta^2 \\ \zeta & 1 & 0 & 0 & \zeta b_{0,1} & 0 & \zeta \\ \zeta^2 b_{1,0} & 0 & 1 & 0 & \zeta^2 & 0 & -\zeta^2 b_{1,0} \\ \zeta b_{2,0} & 0 & 0 & 1 & 0 & \zeta & -\zeta b_{2,0} \\ -\zeta^2 b_{1,0} & 0 & -1 & 0 & \zeta & 0 & \zeta^2 b_{1,0} \\ -\zeta b_{2,0} & 0 & 0 & -1 & 0 & 1-\zeta & \zeta b_{2,0} \\ 1 & 0 & 0 & 0 & -\zeta^2 b_{0,1} & 0 & 0 \end{pmatrix} : \begin{array}{l} P_0 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \\ \quad \to P_0 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \end{array}$$

Row and column operations show that $\det(T_i) \in \{1, -1\} \subseteq \mathrm{U}(S)$ for $i \in [1, 6]$, so that the above maps are all isomorphisms.

Recall the trivial Λ-module S and the Λ-linear map

$$\varepsilon : P_0 \to S, \ \left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right) \mapsto u\,;$$

cf. Remark 17 and Definition 26.

We have the trivial $\check{\Lambda}$-module R, so that $S \otimes R$ becomes a Λ-module via $\widetilde{\varphi}^{-1}$; cf. Remark 38.

Let $\psi : S \otimes R \to S,\ s \otimes r \mapsto sr$, which is S-linear. We show that ψ is Λ-linear. Given $s \in S, r \in R$ and $\lambda = (1 \otimes \check{\lambda})\widetilde{\varphi} \in \Lambda$ for some $\check{\lambda} =: (u, v, N) \in \check{\Lambda}$, it suffices to verify that

$$(\lambda \cdot (s \otimes r))\,\psi = \lambda\,((s \otimes r)\psi)\,.$$

In fact, we have $\lambda = (u, v, \bar{v}, N^{\widetilde{T}})$ and so

$$\begin{aligned} (\lambda \cdot (s \otimes r))\,\psi &= \big(\lambda\widetilde{\varphi}^{-1}(s \otimes r)\big)\,\psi = \big((1 \otimes \check{\lambda})(s \otimes r)\big)\,\psi = (s \otimes \check{\lambda} \cdot r)\psi = (s \otimes ur)\psi \\ &= s\,u\,r = \lambda \cdot sr = \lambda\,((s \otimes r)\psi) \end{aligned}$$

Hence ψ is Λ-linear.

We show that the following diagram commutes.

$$
\begin{tikzcd}[column sep=small]
 & P_0 \oplus P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r, "d_5"] \arrow[d, "{U_6:=\begin{pmatrix}1&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&0&1&0&0&0\\0&1&0&0&0&0&0\end{pmatrix}}"] & P_2 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_0 \oplus P_1 \arrow[r, "d_4"] \arrow[d, "{U_5:=\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\1&0&0&0&0&0\end{pmatrix}}"] & P_1 \oplus P_2 \oplus P_0 \oplus P_1 \oplus P_2 \arrow[r, "d_3"] \arrow[d, "{U_4:=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix}}"] & P_0 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r, "d_2"] \arrow[d, "{U_3:=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}}"] & P_2 \oplus P_0 \oplus P_1 \arrow[r, "d_1"] \arrow[d, "{U_2:=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}}"] & P_1 \oplus P_2 \arrow[r, "d_0"] \arrow[d, "{U_1:=\begin{pmatrix}1&0\\0&1\end{pmatrix}}"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\begin{pmatrix}1\end{pmatrix}"] & S \arrow[d, "1"] \\
\text{(I)} & & & & & & & & \\
 & P_0 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r] \arrow[d, "{\begin{pmatrix}\zeta^2&-1&0&0&2\zeta^2 b_{0,1}&0&\zeta^2\\\zeta&1&0&0&\zeta b_{0,1}&0&\zeta\\\zeta^2 b_{1,0}&0&1&0&\zeta^2&0&-\zeta^2 b_{1,0}\\\zeta b_{2,0}&0&0&1&0&\zeta&-\zeta b_{2,0}\\-\zeta^2 b_{1,0}&0&-1&0&\zeta&0&\zeta^2 b_{1,0}\\-\zeta b_{2,0}&0&0&-1&0&1-\zeta&\zeta b_{2,0}\\1&0&0&0&-\zeta^2 b_{0,1}&0&0\end{pmatrix}}"] & P_1 \oplus P_2 \oplus P_0 \oplus P_0 \oplus P_1 \oplus P_2 \arrow[r] \arrow[d, "{\begin{pmatrix}\zeta&0&0&-\zeta b_{1,0}&-\zeta&0\\0&\zeta^2&0&0&0&-\zeta^2\\0&0&1&\zeta^2(\widetilde{b}_{0,0}-1)&\zeta^2 b_{0,1}&0\\0&0&1&-\zeta(\widetilde{b}_{0,0}-1)&-\zeta b_{0,1}&0\\\zeta&0&0&\zeta^2 b_{1,0}&\zeta^2&0\\0&\zeta^2&0&0&0&\zeta\end{pmatrix}}"] & P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r] \arrow[d, "{\begin{pmatrix}1&0&-\zeta^2&0&-\zeta^2 b_{1,0}\\0&1&0&-\zeta&0\\-1&0&-\zeta&0&-\zeta b_{1,0}\\0&-1&0&-\zeta^2&0\\0&0&0&0&1\end{pmatrix}}"] & P_0 \oplus P_0 \oplus P_1 \oplus P_2 \arrow[r] \arrow[d, "{\begin{pmatrix}1&\zeta^2&0&0\\1&-\zeta&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}}"] & P_1 \oplus P_2 \oplus P_0 \arrow[r] \arrow[d, "{\begin{pmatrix}\zeta&0&0\\0&\zeta^2&0\\0&0&-1\end{pmatrix}}"] & P_1 \oplus P_2 \arrow[r] \arrow[d, "{\begin{pmatrix}-1&0\\0&1\end{pmatrix}}"] & P_0 \arrow[r] \arrow[d, "\begin{pmatrix}1\end{pmatrix}"] & S \arrow[d, "1"] \\
\text{(II)} & & & & & & & & \\
 & P_0 \oplus P_0 \oplus P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r, "\widetilde{d}_5"] & P_1 \oplus P_2 \oplus P_0 \oplus P_0 \oplus P_1 \oplus P_2 \arrow[r, "\widetilde{d}_4"] & P_1 \oplus P_2 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r, "\widetilde{d}_3"] & P_0 \oplus P_0 \oplus P_1 \oplus P_2 \arrow[r, "\widetilde{d}_2"] & P_1 \oplus P_2 \oplus P_0 \arrow[r, "\widetilde{d}_1"] & P_1 \oplus P_2 \arrow[r, "\widetilde{d}_0"] & P_0 \arrow[r, "\varepsilon"] & S \\
\text{(III)} & & & & & & & & \\
 & (S\otimes\check{P}_0) \oplus (S\otimes\check{P}_0) \oplus (S\otimes\check{P}_1) \oplus (S\otimes\check{P}_1) \oplus (S\otimes\check{P}_0) \arrow[r, "S\otimes\check{d}_5"] \arrow[u, "{\begin{pmatrix}\widetilde{\varphi}_0&0&0&0&0\\0&\widetilde{\varphi}_0&0&0&0\\0&0&\widetilde{\varphi}_1&0&0\\0&0&0&\widetilde{\varphi}_1&0\\0&0&0&0&\widetilde{\varphi}_0\end{pmatrix}}"'] & (S\otimes\check{P}_1) \oplus (S\otimes\check{P}_0) \oplus (S\otimes\check{P}_0) \oplus (S\otimes\check{P}_1) \arrow[r, "S\otimes\check{d}_4"] \arrow[u, "{\begin{pmatrix}\widetilde{\varphi}_1&0&0&0\\0&\widetilde{\varphi}_0&0&0\\0&0&\widetilde{\varphi}_0&0\\0&0&0&\widetilde{\varphi}_1\end{pmatrix}}"'] & (S\otimes\check{P}_1) \oplus (S\otimes\check{P}_1) \oplus (S\otimes\check{P}_0) \arrow[r, "S\otimes\check{d}_3"] \arrow[u, "{\begin{pmatrix}\widetilde{\varphi}_1&0&0\\0&\widetilde{\varphi}_1&0\\0&0&\widetilde{\varphi}_0\end{pmatrix}}"'] & (S\otimes\check{P}_0) \oplus (S\otimes\check{P}_0) \oplus (S\otimes\check{P}_1) \arrow[r, "S\otimes\check{d}_2"] \arrow[u, "{\begin{pmatrix}\widetilde{\varphi}_0&0&0\\0&\widetilde{\varphi}_0&0\\0&0&\widetilde{\varphi}_1\end{pmatrix}}"'] & (S\otimes\check{P}_1) \oplus (S\otimes\check{P}_0) \arrow[r, "S\otimes\check{d}_1"] \arrow[u, "{\begin{pmatrix}\widetilde{\varphi}_1&0\\0&\widetilde{\varphi}_0\end{pmatrix}}"'] & (S\otimes\check{P}_1) \arrow[r, "S\otimes\check{d}_0"] \arrow[u, "\begin{pmatrix}\widetilde{\varphi}_1\end{pmatrix}"'] & (S\otimes\check{P}_0) \arrow[r, "S\otimes\check{\varepsilon}"] \arrow[u, "\begin{pmatrix}\widetilde{\varphi}_0\end{pmatrix}"'] & S \otimes R \arrow[u, "\psi"']
\end{tikzcd}
$$

$$
\Big\uparrow\; S \underset{R}{\otimes} -
$$

$$
\check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \xrightarrow{\check{d}_5} \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \xrightarrow{\check{d}_4} \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \xrightarrow{\check{d}_3} \check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \xrightarrow{\check{d}_2} \check{P}_1 \oplus \check{P}_0 \xrightarrow{\check{d}_1} \check{P}_1 \xrightarrow{\check{d}_0} \check{P}_0 \xrightarrow{\check{\varepsilon}} R
$$

We show that Part (III) of the diagram commutes.
We shall verify the equality $\widetilde{\varphi}_0\varepsilon = (S \otimes \varepsilon)\psi$. All other quadrangles commute because of Corollary 42. We have

$$(1 \otimes \check{\mathcal{E}}_0)\widetilde{\varphi}_0\varepsilon \overset{\text{L.37}}{=} \mathcal{E}_0\varepsilon = 1 = (1 \otimes 1)\psi = (1 \otimes \check{\mathcal{E}}_0)(S \otimes \check{\varepsilon})\psi\,.$$

Since ε, $\widetilde{\varphi}_0$ and ψ are Λ-linear, the equality $\widetilde{\varphi}_0\varepsilon = (S \otimes \varepsilon)\psi$ holds.

We show that part (I) and (II) commute. For this we verify

$$U_{i+1}T_{i+1}\check{d}_i = d_iU_iT_i$$

for $i \in [0,5]$. Recall the differentials d_i for $i \in [0,5]$; cf. Remark 27. We have

$$U_1T_1\widetilde{d}_0 = \begin{pmatrix} b_{1,0} \\ b_{2,0} \end{pmatrix} = d_0U_0T_0$$

$$U_2T_2\widetilde{d}_1 = \begin{pmatrix} -b_{2,1} & -2b_{2,2} \\ -b_{0,1} & -b_{0,2} \\ -2b_{1,1} & -b_{1,2} \end{pmatrix} = d_1U_1T_1$$

$$U_3T_3\widetilde{d}_2 = \begin{pmatrix} 0 & \zeta^2 b_{0,2} & 2b_{0,0} \\ 0 & \zeta^2 b_{1,2} & b_{1,0} \\ \zeta b_{2,1} & 0 & b_{2,0} \\ \zeta b_{0,1} & 0 & 2b_{0,0} \end{pmatrix} = d_2U_2T_2$$

$$U_4T_4\widetilde{d}_3 = \begin{pmatrix} b_{1\cdot 0} & \zeta^2 b_{1,0} & 2b_{1,1} & 0 \\ b_{2,0} & \zeta^2 b_{2,0} & b_{2,1} & 0 \\ 0 & -2b_{0,0} & b_{0,1} & b_{0,2} \\ -b_{1,0} & \zeta b_{1,0} & 0 & b_{1,2} \\ -b_{2,0} & \zeta b_{2,0} & 0 & 2b_{2,2} \end{pmatrix} = d_3U_3T_3$$

$$U_5T_5\widetilde{d}_4 = \begin{pmatrix} b_{2,1} & 2b_{2,2} & -\zeta^2 b_{2,1} & 2\zeta^2 b_{2,2} & -2\zeta^2 b_{2,0} \\ b_{0,1} & b_{0,2} & -\zeta^2 b_{0,1} & \zeta^2 b_{0,2} & -\zeta^2 \widetilde{b}_{0,0} \\ 2b_{1,1} & b_{1,2} & 2\zeta b_{1,1} & \zeta^2 b_{1,2} & (\zeta - \zeta^2)b_{1,0} \\ b_{2,1} & 2b_{2,2} & \zeta b_{2,1} & 2\zeta^2 b_{2,2} & (\zeta - \zeta^2)b_{2,0} \\ b_{0,1} & b_{0,2} & \zeta b_{0,1} & -\zeta b_{0,2} & \zeta\widetilde{b}_{0,0} \\ 2b_{1,1} & b_{1,2} & 2\zeta b_{1,1} & -\zeta b_{1,2} & 2\zeta b_{1,0} \end{pmatrix} = d_4U_4T_4$$

$$U_6T_6\widetilde{d}_5 = \begin{pmatrix} 0 & \zeta^2 b_{0,2} & -2b_{0,0} & -2\zeta^2(\widetilde{b}_{0,0} - b_{0,0}) & -2\zeta^2 b_{0,1} & -\zeta^2 b_{0,2} \\ 0 & \zeta^2 b_{1,2} & -b_{1,0} & -3\zeta^2 b_{1,0} & -\zeta^2\widetilde{b}_{1,1} & -\zeta^2 b_{1,2} \\ \zeta b_{2,1} & 0 & -b_{2,0} & -\zeta^2 b_{2,0} & -\zeta^2 b_{2,1} & 2b_{2,2} \\ \zeta b_{0,1} & -\zeta^2 b_{0,2} & 0 & \zeta^2\widetilde{b}_{0,0} & \zeta^2 b_{0,1} & -\zeta b_{0,2} \\ 0 & -\zeta^2 b_{1,2} & b_{1,0} & (2\zeta^2 - \zeta)b_{1,0} & -\zeta\widetilde{b}_{1,1} - 2b_{1,1} & -\zeta b_{1,2} \\ -\zeta b_{2,1} & 0 & b_{2,0} & -\zeta b_{2,0} & -\zeta b_{2,1} & 0 \\ -\zeta b_{0,1} & 0 & 2b_{0,0} & \zeta(2b_{0,0} - \widetilde{b}_{0,0}) & -\zeta b_{0,1} & 0 \end{pmatrix} = d_5U_5T_5.$$

Hence the diagram is commutative.

Therefore, since $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ as well as T_i and U_i are isomorphisms for $i \in [1,6]$, we have $S \otimes \check{Q} \cong \widetilde{Q} \cong Q_{[-1,6]}$ as complexes over Λ.

As a result, $S \otimes \check{Q}$ is an exact sequence with $S \otimes \check{\varepsilon}$ being surjective. Hence, $\check{Q}$ is an exact sequence and $\check{\varepsilon}$ surjective. ▫

Remark 45 We have ended our experimental calculation with d_5 and $\check{d}_5$. The further differentials seem to be out of reach with our methods.

In particular, there is no regular behaviour in sight of the differentials calculated to this point. This might be caused by automorphisms of projective modules, which we cannot control.

4.3 Projective resolution of $\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{A}_4$

Recall that $R = \mathbb{Z}_{(2)}$ and $S = \mathbb{Z}_{(2)}[\zeta]$. Recall that $\otimes = \underset{R}{\otimes}$.

> We attempt to construct a projective resolution of R over RA_4 by restricting the projective resolution Q of S over $S\mathrm{A}_4$ to a projective resolution of $R \oplus R$ over $R\mathrm{A}_4$; cf. Theorem 28.

Write $T := \begin{pmatrix} 1 & \zeta & 0 \\ \zeta & 1 & 0 \\ 0 & 0 & 2\zeta^2 \end{pmatrix}$.

Recall the isomorphism of S-algebras

$$\varphi \ : \ S \otimes \check{\Lambda} \xrightarrow{\sim} \Lambda, \ \xi \otimes (u, v, N) \longmapsto \left(\xi u, \xi v, \xi \bar{v}, \xi N^T\right);$$

cf. Lemma 10

Recall the injective ring homomorphism

$$\iota \ : \ \check{\Lambda} \to \Lambda, \ (u, v, N) \mapsto (u, v, \bar{v}, N^T)$$

and let

$$\tau : \check{\Lambda} \to S \otimes \check{\Lambda}, \ \lambda \mapsto 1 \otimes \lambda.$$

Then $\iota = \tau\varphi$; cf. Corollary 12.

Remark 46

- Given a Λ-module X, we denote its restriction to a $\check{\Lambda}$-module via ι by $X\!\downarrow_\iota$.

  In this way, we obtain the $\check{\Lambda}$-modules $P_0\!\downarrow_\iota$, $P_1\!\downarrow_\iota$, $P_2\!\downarrow_\iota$ and $S\!\downarrow_\iota$; cf. Definition 23 and Remark 17.

- We have the isomorphism of $\check{\Lambda}$-modules

$$\rho_S : R \oplus R \xrightarrow{\sim} S\!\downarrow_\iota, \ (x, y) \mapsto x\zeta + y\zeta^2.$$

Definition 47 Let

$$\begin{aligned} \pi_1 \ &: \ R[\zeta] \longrightarrow R, \ x\zeta + y\zeta^2 \longmapsto x \\ \pi_2 \ &: \ R[\zeta] \longrightarrow R, \ x\zeta + y\zeta^2 \longmapsto y \end{aligned}$$

where $x, y \in \mathbb{R}$.

For $N := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in S^{3\times 3}$ and $i \in \{1,2\}$ we write

$$N\pi_i^{3\times 3} := \begin{pmatrix} a\pi_i & b\pi_i & c\pi_i \\ d\pi_i & e\pi_i & f\pi_i \\ g\pi_i & h\pi_i & i\pi_i \end{pmatrix} \in R^{3\times 3}.$$

Define the R-linear maps

$$\begin{array}{rcl} R[\zeta] \otimes \check{\Lambda} \xrightarrow{\nu_1} \check{\Lambda} & \qquad & R[\zeta] \otimes \check{\Lambda} \xrightarrow{\nu_2} \check{\Lambda} \\ (\zeta \otimes t) + (\zeta^2 \otimes w) \mapsto t & \qquad & (\zeta \otimes t) + (\zeta^2 \otimes w) \mapsto w. \end{array}$$

where $t, w \in \check{\Lambda}$.

We shall construct isomorphisms

$$\check{P}_0 \oplus \check{P}_0 \xrightarrow{\sim} P_0{\downarrow_\iota} \oplus P_0{\downarrow_\iota} \qquad \check{P}_1 \xrightarrow{\sim} P_1{\downarrow_\iota} \qquad \check{P}_1 \xrightarrow{\sim} P_2{\downarrow_\iota}$$

and their inverses.

Lemma 48 The inverse S-algebra isomorphism of

$$\varphi \;:\; S \otimes \check{\Lambda} \xrightarrow{\sim} \Lambda,\ \xi \otimes (u, v, N) \longmapsto \left(\xi u, \xi v, \xi \bar{v}, \xi N^T\right)$$

is given by

$$\begin{aligned} \varphi^{-1} \;:\; \Lambda &\longrightarrow S \otimes \check{\Lambda} \\ (u, r, s, N) &\longmapsto \quad \zeta \otimes \left(u\pi_1, \frac{1}{3}\left((\zeta^2 - 1)r + (\zeta - 1)\bar{s}\right), N^{T^{-1}}\pi_1^{3\times 3}\right) \\ &\qquad + \zeta^2 \otimes \left(u\pi_2, \frac{1}{3}\left((\zeta - 1)r + (\zeta^2 - 1)\bar{s}\right), N^{T^{-1}}\pi_2^{3\times 3}\right) \end{aligned}$$

with $T^{-1} = \dfrac{1}{3}\begin{pmatrix} 1-\zeta & \zeta^2 - \zeta & 0 \\ \zeta^2 - \zeta & 1 - \zeta & 0 \\ 0 & 0 & \frac{3}{2}\zeta \end{pmatrix}$.

Proof. Let $\check{\Gamma} := R \times R[\zeta] \times R^{3\times 3}$ and

$$\begin{aligned} \varphi' \;:\; \Lambda &\longrightarrow S \otimes \check{\Gamma} \\ (u, r, s, N) &\longmapsto \quad \zeta \otimes \left(u\pi_1, \frac{1}{3}\left((\zeta^2 - 1)r + (\zeta - 1)\bar{s}\right), N^{T^{-1}}\pi_1^{3\times 3}\right) \\ &\qquad + \zeta^2 \otimes \left(u\pi_2, \frac{1}{3}\left((\zeta - 1)r + (\zeta^2 - 1)\bar{s}\right), N^{T^{-1}}\pi_2^{3\times 3}\right). \end{aligned}$$

Given $\xi := x\zeta + y\zeta^2 \in R[\zeta]$ and $\lambda := (u, v, N) \in \check{\Lambda}$ we have

$$(\xi \otimes \lambda)\varphi\varphi' = \left(\xi u,\ \xi v,\ \xi \bar{v},\ \xi N^T\right)\varphi'$$

$$
\begin{aligned}
&= \zeta \otimes \left(xu, \frac{1}{3}\left((\zeta^2-1)(x\zeta + y\zeta^2)v + (\zeta-1)(x\zeta^2+y\zeta)v \right), xN \right) \\
&+ \zeta^2 \otimes \left(yu, \frac{1}{3}\left((\zeta-1)(x\zeta + y\zeta^2)v + (\zeta^2-1)(x\zeta^2+y\zeta)v \right), yN \right) \\
&= \zeta \otimes \left(xu, \frac{1}{3}\left(x - x\zeta + x - x\zeta^2 + y\zeta - y\zeta^2 + y\zeta^2 - y\zeta \right) v, xN \right) \\
&+ \zeta^2 \otimes \left(yu, \frac{1}{3}\left(x\zeta^2 - x\zeta + x\zeta - x\zeta^2 + y - y\zeta^2 + y - y\zeta \right) v, yN \right) \\
&= x\zeta \otimes (u,v,N) + y\zeta^2 \otimes (u,v,N) = \xi \otimes (u,v,N) = \xi \otimes \lambda.
\end{aligned}
$$

Hence $\Lambda\varphi' = \big((S\otimes\check{\Lambda})\varphi\big)\varphi' \subseteq S\otimes\check{\Lambda}$ and we have $\varphi\varphi'|^{S\otimes\check{\Lambda}} = \mathrm{id}_{S\otimes\check{\Lambda}}$. Since Λ and $S\otimes\check{\Lambda}$ are free S-modules of the same rank, we obtain $\varphi'|^{S\otimes\check{\Lambda}} = \varphi^{-1}$. ▫

Corollary 49 Recall the following idempotent elements of Λ.

$$
\mathcal{E}_0 = \left(1,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right) \quad
\mathcal{E}_1 = \left(0,1,0,\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right) \quad
\mathcal{E}_2 = \left(0,0,1,\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\right)
$$

We have

$$
\begin{aligned}
\check{\mathcal{E}}_0\iota &= \left(1,0,\bar{0},\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}^T\right) = \left(1,0,0,\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right) = \mathcal{E}_0 \\
\check{\mathcal{E}}_1\iota &= \left(0,1,\bar{1},\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}^T\right) = \left(0,1,1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right) = \mathcal{E}_1 + \mathcal{E}_2\,.
\end{aligned}
$$

Let

$$
\begin{aligned}
p &:= \left(0, \frac{1}{3}(\zeta^2-1), \frac{1}{3}\begin{pmatrix}-1&1&0\\-1&-2&0\\0&0&0\end{pmatrix}\right) \in \check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1 \subseteq \check{\Lambda} \\
q &:= \left(0, \frac{1}{3}(\zeta-1), \frac{1}{3}\begin{pmatrix}-2&-1&0\\1&-1&0\\0&0&0\end{pmatrix}\right) \in \check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1 \subseteq \check{\Lambda}\,.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\mathcal{E}_1\varphi^{-1}\nu_1 &= p = \mathcal{E}_2\varphi^{-1}\nu_2 \\
\mathcal{E}_1\varphi^{-1}\nu_2 &= q = \mathcal{E}_2\varphi^{-1}\nu_1
\end{aligned}
$$

and

$$
\begin{aligned}
\check{\mathcal{E}}_1 &= \mathcal{E}_1\varphi^{-1}(-\nu_1-\nu_2) = \mathcal{E}_2\varphi^{-1}(-\nu_1-\nu_2) \\
&= -p-q\,.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\mathcal{E}_1\varphi^{-1} &= \zeta\otimes p + \zeta^2\otimes q \\
\mathcal{E}_2\varphi^{-1} &= \zeta\otimes q + \zeta^2\otimes p\,.
\end{aligned}
$$

Moreover, $\mathcal{E}_0\varphi^{-1} = 1\otimes\check{\mathcal{E}}_0$.

Remark 50 Given $a, b, c, d, e, f, g, h, i \in R$ we have

$$\begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{pmatrix}^T = \frac{1}{3}\begin{pmatrix} a(1-\zeta)+b(\zeta-\zeta^2)+d(\zeta^2-\zeta)+e(1-\zeta^2) & a(\zeta-\zeta^2)+b(1-\zeta)+d(1-\zeta^2)+e(\zeta^2-\zeta) & 0 \\ a(\zeta^2-\zeta)+b(1-\zeta^2)+d(1-\zeta)+e(\zeta-\zeta^2) & a(1-\zeta^2)+b(\zeta^2-\zeta)+d(\zeta-\zeta^2)+e(1-\zeta) & 0 \\ \frac{3}{2}(g\zeta+h\zeta^2) & \frac{3}{2}(g\zeta^2+h\zeta) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & i \end{pmatrix}^T = \frac{1}{3}\begin{pmatrix} 0 & 0 & 2c(\zeta^2-1)+2f(\zeta-1) \\ 0 & 0 & 2c(\zeta-1)+2f(\zeta^2-1) \\ 0 & 0 & 3i \end{pmatrix}.$$

Given $a, b, c, d, e, f, g, h, i \in S$ we have

$$\begin{pmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{pmatrix}^{T^{-1}} = \frac{1}{3}\begin{pmatrix} a(1-\zeta)+d(\zeta-\zeta^2) & a(\zeta^2-\zeta)+d(1-\zeta^2) & 0 \\ a(\zeta-\zeta^2)+d(1-\zeta) & a(1-\zeta^2)+d(\zeta^2-\zeta) & 0 \\ 2g(\zeta^2-1) & 2g(\zeta-1) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & b & 0 \\ 0 & e & 0 \\ 0 & h & 0 \end{pmatrix}^{T^{-1}} = \frac{1}{3}\begin{pmatrix} b(\zeta^2-\zeta)+e(1-\zeta^2) & b(1-\zeta)+e(\zeta-\zeta^2) & 0 \\ b(1-\zeta^2)+e(\zeta^2-\zeta) & b(\zeta-\zeta^2)+e(1-\zeta) & 0 \\ 2h(\zeta-1) & 2h(\zeta^2-1) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & i \end{pmatrix}^{T^{-1}} = \begin{pmatrix} 0 & 0 & \frac{1}{2}(c\zeta+f\zeta^2) \\ 0 & 0 & \frac{1}{2}(c\zeta^2+f\zeta) \\ 0 & 0 & i \end{pmatrix}.$$

Lemma 51 We have the following isomorphisms of $\check{\Lambda}$-modules.

$$\rho_0 \;:\; \check{P}_0 \oplus \check{P}_0 \xrightarrow{\sim} P_0{\downarrow_\iota},\; (\lambda,\mu) \mapsto \zeta(\lambda\iota) + \zeta^2(\mu\,\iota)$$

$$\rho_{1,1} \;:\; \check{P}_1 \xrightarrow{\sim} P_1{\downarrow_\iota},\; \lambda \mapsto (\lambda\iota)\mathcal{E}_1$$

$$\rho_{1,2} \;:\; \check{P}_1 \xrightarrow{\sim} P_2{\downarrow_\iota},\; \lambda \mapsto (\lambda\iota)\mathcal{E}_2$$

The inverse maps are given by

$$\rho_0^{-1} \;:\; P_0{\downarrow_\iota} \to \check{P}_0 \oplus \check{P}_0,\; \lambda \mapsto \left(\lambda\varphi^{-1}\nu_1, \lambda\varphi^{-1}\nu_2\right)$$

$$\rho_{1,1}^{-1} \;:\; P_1{\downarrow_\iota} \to \check{P}_1,\; \lambda \mapsto -\left(\lambda\varphi^{-1}(\nu_1+\nu_2)\right)$$

$$\rho_{1,2}^{-1} \;:\; P_2{\downarrow_\iota} \to \check{P}_1,\; \lambda \mapsto -\left(\lambda\varphi^{-1}(\nu_1+\nu_2)\right).$$

In particular, $P_0{\downarrow_\iota}, P_1{\downarrow_\iota}$ and $P_2{\downarrow_\iota}$ are projective $\check{\Lambda}$-modules; cf. also Corollary 12.

Proof. Note that for $\lambda \in \check{P}_0$ we have $\lambda\iota \in P_0$ so that ρ_0 in fact maps to $P_0{\downarrow_\iota}$; cf. Remark 50.

We show that ρ_0, $\rho_{1,1}$ and $\rho_{1,2}$ are $\check{\Lambda}$-linear.

Given $\lambda, \mu, \mu' \in \check{\Lambda}$ we have

$$\begin{aligned} \left(\lambda(\mu,\mu')\right)\rho_0 &= \zeta(\lambda\mu\iota) + \zeta^2(\lambda\mu'\iota) = (\lambda\iota)\left(\zeta(\mu\iota)+\zeta^2(\mu'\iota)\right) = \lambda\cdot\left(\zeta(\mu\iota)+\zeta^2(\mu'\iota)\right) = \lambda\cdot(\mu,\mu')\rho_0 \\ (\lambda\mu)\,\rho_{1,1} &= (\lambda\mu\iota)\mathcal{E}_1 = (\lambda\iota)(\mu\iota)\mathcal{E}_1 = \lambda\cdot(\mu\iota)\mathcal{E}_1 = \lambda\cdot(\mu\rho_{1,1}) \\ (\lambda\mu)\,\rho_{1,2} &= (\lambda\mu\iota)\mathcal{E}_2 = (\lambda\iota)(\mu\iota)\mathcal{E}_2 = \lambda\cdot(\mu\iota)\mathcal{E}_2 = \lambda\cdot(\mu\rho_{1,2})\,; \end{aligned}$$

cf. Remark 46.

We show that the map already denoted by ρ_0^{-1} is the both-sided inverse of ρ_0.

Note that for $\lambda_0 \in P_0\!\downarrow_\iota$ we have $\lambda_0\varphi^{-1} \in S \otimes \check{P}_0$ so that ρ_0^{-1} in fact maps to $\check{P}_0 \oplus \check{P}_0$; cf. Remark 50.

Given $\lambda \in P_0\!\downarrow_\iota$, we have $\lambda\varphi^{-1} = \zeta \otimes t + \zeta^2 \otimes w$ for some $t, w \in \check{\Lambda}$ so that

$$\begin{aligned}\lambda\rho_0^{-1}\rho_0 &= \left(\lambda\varphi^{-1}\nu_1, \lambda\varphi^{-1}\nu_2\right)\rho_0 = \zeta(\lambda\varphi^{-1}\nu_1\iota) + \zeta^2(\lambda\varphi^{-1}\nu_2\iota)\\ &= \zeta(\zeta \otimes t + \zeta^2 \otimes w)\nu_1\iota + \zeta^2(\zeta \otimes t + \zeta^2 \otimes w)\nu_2\iota\\ &= \zeta(t\tau\varphi) + \zeta^2(w\tau\varphi) = (\zeta \otimes t + \zeta^2 \otimes w)\varphi = \lambda\,.\end{aligned}$$

Hence, $\rho_0^{-1}\rho_0 = \mathrm{id}_{P_0\downarrow_\iota}$. In particular, ρ_0 is surjective. Since $P_0\!\downarrow_\iota$ and $\check{P}_0 \oplus \check{P}_0$ are free R-modules of the same rank, ρ_0 must be bijective and therefore ρ_0^{-1} the both-sided inverse of ρ_0.

We show that the maps already denoted by $\rho_{1,1}^{-1}$ and $\rho_{1,2}^{-1}$ are the both-sided inverses of $\rho_{1,1}$ and $\rho_{1,2}$ respectively.

Let $i \in \{1, 2\}$. Note that for $\lambda_i \in P_i\!\downarrow_\iota$ we have $\lambda_i\varphi^{-1} \in S \otimes \check{P}_1$ so that $\rho_{1,i}^{-1}$ in fact maps to $\check{P}_1$; cf. Remark 50.

Given $\lambda \in \check{P}_1$ we have with Corollary 49 that

$$\begin{aligned}\lambda\rho_{1,1}\rho_{1,1}^{-1} &= ((\lambda\iota)\mathcal{E}_1)\,\rho_{1,1}^{-1} = -\,((\lambda\iota)\mathcal{E}_1)\,\varphi^{-1}(\nu_1 + \nu_2) = -\,((\lambda\tau\varphi)\mathcal{E}_1)\,\varphi^{-1}(\nu_1 + \nu_2)\\ &= -\left((\lambda\tau\varphi\varphi^{-1}) \cdot (\mathcal{E}_1\varphi^{-1})\right)(\nu_1 + \nu_2) = -\left((1 \otimes \lambda) \cdot (\zeta \otimes p + \zeta^2 \otimes q)\right)(\nu_1 + \nu_2)\\ &= -(\zeta \otimes \lambda p + \zeta^2 \otimes \lambda q)(\nu_1 + \nu_2) = \lambda(-p - q) = \lambda\check{\mathcal{E}}_1 = \lambda\,.\end{aligned}$$

In the same way we obtain

$$\begin{aligned}\lambda\rho_{1,2}\rho_{1,2}^{-1} &= ((\lambda\iota)\mathcal{E}_2)\,\rho_{1,2}^{-1} = -\,((\lambda\iota)\mathcal{E}_2)\,\varphi^{-1}(\nu_1 + \nu_2) = -\,((\lambda\tau\varphi)\mathcal{E}_2)\,\varphi^{-1}(\nu_1 + \nu_2)\\ &= -\left((\lambda\tau\varphi\varphi^{-1}) \cdot (\mathcal{E}_2\varphi^{-1})\right)(\nu_1 + \nu_2) = -\left((1 \otimes \lambda) \cdot (\zeta \otimes q + \zeta^2 \otimes p)\right)(\nu_1 + \nu_2)\\ &= -(\zeta \otimes \lambda q + \zeta^2 \otimes \lambda p)(\nu_1 + \nu_2) = \lambda(-p - q) = \lambda\check{\mathcal{E}}_1 = \lambda\,.\end{aligned}$$

Since $P_1\!\downarrow_\iota$, $P_2\!\downarrow_\iota$ and $\check{P}_1$ are free R-modules of the same rank, the map $\rho_{1,1}^{-1}$ must be the both-sided inverse of $\rho_{1,1}$ and $\rho_{1,2}^{-1}$ the both-sided inverse of $\rho_{1,2}$. ▫

Corollary 52 Recall the projective resolution Q of the trivial Λ-module S; cf. Theorem 28.

We have the augmented projective resolution

$$\cdots \longrightarrow Q_4\!\downarrow_\iota \xrightarrow{d_3} Q_3\!\downarrow_\iota \xrightarrow{d_2} Q_2\!\downarrow_\iota \xrightarrow{d_1} Q_1\!\downarrow_\iota \xrightarrow{d_0} Q_0\!\downarrow_\iota \xrightarrow{\varepsilon\rho_S^{-1}} R \oplus R$$

of the $\check{\Lambda}$-module $R \oplus R$ over $R\mathrm{A}_4$.

Remark 53 We define the $\check{\Lambda}$-linear map

$$\begin{aligned}\check{P}_1 \oplus \check{P}_1 &\xrightarrow{\rho_1} P_1\!\downarrow_\iota \oplus P_2\!\downarrow_\iota\\ (\lambda, \mu) &\longmapsto \left(\left(\zeta(\lambda\iota) + \zeta^2(\mu\iota)\right)\mathcal{E}_1, \left(\zeta(\lambda\iota) + \zeta^2(\mu\iota)\right)\mathcal{E}_2\right).\end{aligned}$$

Let

$$C := \begin{pmatrix} 2q-p & 2p-q \\ 2p-q & 2q-p \end{pmatrix} \;:\; \check{P}_1 \oplus \check{P}_1 \to \check{P}_1 \oplus \check{P}_1\,;$$

cf. Corollary 49.

The inverse of C is given by

$$C^{-1} := \begin{pmatrix} p & q \\ q & p \end{pmatrix} \;:\; \check{P}_1 \oplus \check{P}_1 \to \check{P}_1 \oplus \check{P}_1$$

so that C is a $\check{\Lambda}$-linear automorphism of $\check{P}_1 \oplus \check{P}_1$.

We have $\rho_1 = C \begin{pmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{1,2} \end{pmatrix}$ and ρ_1 is an isomorphism of $\check{\Lambda}$ modules.

Proof. Recall that

$$p = \left(0, \frac{1}{3}(\zeta^2 - 1), \frac{1}{3}\begin{pmatrix} -1 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \qquad q = \left(0, \frac{1}{3}(\zeta - 1), \frac{1}{3}\begin{pmatrix} -2 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right);$$

cf. Corollary 49. We have

$$p^2 = \left(0, -\frac{1}{3}\zeta^2, \frac{1}{3}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right), \quad q^2 = \left(0, -\frac{1}{3}\zeta, \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right), \quad pq = \left(0, \frac{1}{3}, \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = qp\,.$$

Therefore we obtain

$$C^{-1}C = \begin{pmatrix} 4pq - p^2 - q^2 & 2p^2 - 2pq + 2q^2 \\ 2q^2 - 2pq + 2p^2 & 4pq - q^2 - p^2 \end{pmatrix} = \begin{pmatrix} \check{\mathcal{E}}_1 & 0 \\ 0 & \check{\mathcal{E}}_1 \end{pmatrix} = \mathrm{id}_{\check{P}_1 \oplus \check{P}_1}\ .$$

We show that $\begin{pmatrix} 2q-p & 2p-q \\ 2p-q & 2q-p \end{pmatrix} \begin{pmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{1,2} \end{pmatrix} = \rho_1$.

Given $\lambda, \mu \in \check{P}_1$ we have with Corollary 49 that

$$\begin{aligned}
&(\lambda, \mu) \begin{pmatrix} 2q-p & 2p-q \\ 2p-q & 2q-p \end{pmatrix} \begin{pmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{1,2} \end{pmatrix} \\
&= (\lambda(2q-p) + \mu(2p-q), \lambda(2p-q) + \mu(2q-p)) \begin{pmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{1,2} \end{pmatrix} \\
&= (((\lambda\iota)(2q-p)\iota + (\mu\iota)(2p-q)\iota)\,\mathcal{E}_1, ((\lambda\iota)(2p-q)\iota + (\mu\iota)(2q-p)\iota)\,\mathcal{E}_2) \\
&= (((\lambda\iota)(2 \otimes q - 1 \otimes p)\varphi + (\mu\iota)(2 \otimes p - 1 \otimes q)\varphi)\,\mathcal{E}_1, ((\lambda\iota)(2 \otimes p - 1 \otimes q)\varphi + (\mu\iota)(2 \otimes q - 1 \otimes p)\varphi)\,\mathcal{E}_2) \\
&= \big(\zeta(\lambda\iota)\big((\zeta \otimes p + \zeta^2 \otimes q)\varphi + \zeta(\zeta^2 \otimes p + \zeta \otimes q)\varphi\big)\,\mathcal{E}_1 + \zeta^2(\mu\iota)\big((\zeta \otimes p + \zeta^2 \otimes q)\varphi + \zeta^2(\zeta^2 \otimes p + \zeta \otimes q)\varphi\big)\,\mathcal{E}_1, \\
&\quad\ \zeta(\lambda\iota)\big((\zeta \otimes q + \zeta^2 \otimes p)\varphi + \zeta(\zeta^2 \otimes q + \zeta \otimes p)\varphi\big)\,\mathcal{E}_2 + \zeta^2(\mu\iota)\big((\zeta \otimes q + \zeta^2 \otimes p)\varphi + \zeta^2(\zeta^2 \otimes q + \zeta \otimes p)\varphi\big)\,\mathcal{E}_2\big) \\
&= \big(\zeta(\lambda\iota)(\mathcal{E}_1 + \zeta\mathcal{E}_2)\mathcal{E}_1 + \zeta^2(\mu\iota)(\mathcal{E}_1 + \zeta^2\mathcal{E}_2)\mathcal{E}_1, \zeta(\lambda\iota)(\mathcal{E}_2 + \zeta\mathcal{E}_1)\mathcal{E}_2 + \zeta^2(\mu\iota)(\mathcal{E}_2 + \zeta^2\mathcal{E}_1)\mathcal{E}_2\big) \\
&= \big(\big(\zeta(\lambda\iota) + \zeta^2(\mu\iota)\big)\,\mathcal{E}_1, \big(\zeta(\lambda\iota) + \zeta^2(\mu\iota)\big)\,\mathcal{E}_2\big) \\
&= (\lambda, \mu)\rho_1\,.
\end{aligned}$$

$\square$

In the following remark we calculate the isomorphic replacement of

$$Q_3{\downarrow_\iota} \xrightarrow{d_2} Q_2{\downarrow_\iota} \xrightarrow{d_1} Q_1{\downarrow_\iota} \xrightarrow{d_0} Q_0{\downarrow_\iota} \xrightarrow{\varepsilon\rho_S^{-1}} R \oplus R\,,$$

which is the first part of the augmented projective resolution of Corollary 52, via the isomorphisms ρ_0 and ρ_1; cf. Lemma 51 and Remark 53.

We then replace isomorphically such that every 2×2-block of the replaced differentials obtains a diagonal form where both entries are the same.

Remark 54 We define

$$\widetilde{\varepsilon'} \;:\; \check{P}_0 \oplus \check{P}_0 \longrightarrow R \oplus R,\; \left(\left(u, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\right), \left(u', \begin{pmatrix} c' \\ f' \\ i' \end{pmatrix}\right)\right) \longmapsto (u, u')$$

and

$$\widetilde{d'_0} := \begin{pmatrix} -b_5 - b_6 & 0 \\ 0 & -b_5 - b_6 \end{pmatrix} \;:\; \check{P}_1 \oplus \check{P}_1 \longrightarrow \check{P}_0 \oplus \check{P}_0$$

$$\widetilde{d'_1} := \frac{1}{3}\begin{pmatrix} -2b_7 + 2b_8 + b_{10} - b_{11} - 2b_{12} & 2(-2b_7 + 2b_8 + b_{10} - b_{11} - 2b_{12}) \\ -2(-2b_7 + 2b_8 + b_{10} - b_{11} - 2b_{12}) & -(-2b_7 + 2b_8 + b_{10} - b_{11} - 2b_{12}) \\ b_3 - b_4 & 2b_3 - 2b_4 \\ -2b_3 + 2b_4 & -b_3 + b_4 \end{pmatrix} \;:\; \begin{array}{r} \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \\ \longrightarrow \check{P}_1 \oplus \check{P}_1 \end{array}$$

$$\widetilde{d'_2} := \begin{pmatrix} \frac{1}{3}(-2b_3 - b_4) & \frac{1}{3}(-b_3 + b_4) & -2b_1 & 0 \\ \frac{1}{3}(b_3 - b_4) & \frac{1}{3}(-b_3 - 2b_4) & 0 & -2b_1 \\ 2b_8 + 2b_9 - b_{10} - b_{11} & 0 & b_5 + b_6 & 0 \\ 0 & 2b_8 + 2b_9 - b_{10} - b_{11} & 0 & b_5 + b_6 \\ \frac{1}{3}(-b_3 - 2b_4) & \frac{1}{3}(b_3 - b_4) & -2b_1 & 0 \\ \frac{1}{3}(-b_3 + b_4) & \frac{1}{3}(-2b_3 - b_4) & 0 & -2b_1 \end{pmatrix} \;:\; \begin{array}{r} \check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \\ \longrightarrow \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0\,. \end{array}$$

Let

$$T_0 := \mathrm{id}_{\check{P}_0 \oplus \check{P}_0}$$

$$T_1 := \begin{pmatrix} b_8 - b_7 & 0 \\ 0 & b_8 - b_7 \end{pmatrix} \;:\; \check{P}_1 \oplus \check{P}_1 \to \check{P}_1 \oplus \check{P}_1$$

$$T_2 := \frac{1}{3}\begin{pmatrix} b_9 - b_8 & 2(b_9 - b_8) & 0 & 0 \\ -2(b_9 - b_8) & -(b_9 - b_8) & 0 & 0 \\ 0 & 0 & b_1 & 2b_1 \\ 0 & 0 & -2b_1 & -b_1 \end{pmatrix} \;:\; \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \to \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0$$

$$T_3 := \frac{1}{3}\begin{pmatrix} -3b_1 & -3b_1 & 0 & 0 & -b_1 & -2b_1 \\ 3b_1 & 0 & 0 & 0 & 2b_1 & b_1 \\ 0 & 0 & b_7 + 3b_8 + 2b_9 - 2b_{10} & 2(b_7 + 3b_8 + 2b_9 - 2b_{10}) & 0 & 0 \\ 0 & 0 & -2(b_7 + 3b_8 + 2b_9 - 2b_{10}) & -(b_7 + 3b_8 + 2b_9 - 2b_{10}) & 0 & 0 \\ 0 & -3b_1 & 0 & 0 & -b_1 & -2b_1 \\ 3b_1 & 3b_1 & 0 & 0 & 2b_1 & b_1 \end{pmatrix} \;:$$

$$\check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \to \check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0\,.$$

We define

$$\check{\varepsilon}' := \begin{pmatrix} \check{\varepsilon} & 0 \\ 0 & \check{\varepsilon} \end{pmatrix} \;:\; \check{P}_0 \oplus \check{P}_0 \to R \oplus R\,; \quad \text{cf. Definition 43}$$

$$\check{d}'_0 := \begin{pmatrix} b_5 & 0 \\ 0 & b_5 \end{pmatrix} \; : \; \check{P}_1 \oplus \check{P}_1 \to \check{P}_0 \oplus \check{P}_0$$

$$\check{d}'_1 := \begin{pmatrix} b_{10} & 0 \\ 0 & b_{10} \\ b_4 & 0 \\ 0 & b_4 \end{pmatrix} \; : \; \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \to \check{P}_1 \oplus \check{P}_1$$

$$\check{d}'_2 := \begin{pmatrix} b_3 & 0 & 0 & 0 \\ 0 & b_3 & 0 & 0 \\ b_{10} - 2b_8 & 0 & b_5 & 0 \\ 0 & b_{10} - 2b_8 & 0 & b_5 \\ -b_4 & 0 & 2b_1 & 0 \\ 0 & -b_4 & 0 & 2b_1 \end{pmatrix} \; : \quad \begin{aligned} &\check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \\ &\qquad\qquad \to \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \,. \end{aligned}$$

We assert that the following diagram commutes. We omit to denote the restriction arrows.

$$\begin{tikzcd}
\arrow[d, phantom, "\text{(I)}"] & P_0 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r, "d_2"] & P_2 \oplus P_0 \oplus P_1 \arrow[r, "d_1"] & P_1 \oplus P_2 \arrow[r, "d_0"] & P_0 \arrow[r, "\varepsilon"] & S \\
\arrow[d, phantom, "\text{(II)}"] & P_0 \oplus P_1 \oplus P_2 \oplus P_0 \arrow[r] \arrow[u, "\wr", "{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}"'] & P_1 \oplus P_2 \oplus P_0 \arrow[r] \arrow[u, "\wr", "{U := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}"'] & P_1 \oplus P_2 \arrow[r] \arrow[u, "\wr", "{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}"'] & P_0 \arrow[r] \arrow[u, "\wr", "{\begin{pmatrix} 1 \end{pmatrix}}"'] & S \arrow[u, "\wr", "1"'] \\
\arrow[d, phantom, "\text{(III)}"] & \check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \arrow[r, "\tilde{d}'_2"] \arrow[u, "\wr", "{\begin{pmatrix} \rho_0 & 0 & 0 \\ 0 & \rho_1 & 0 \\ 0 & 0 & \rho_0 \end{pmatrix}}"'] \arrow[d, "\wr"', "T_3"] & \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \arrow[r, "\tilde{d}'_1"] \arrow[u, "\wr", "{\begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_0 \end{pmatrix}}"'] \arrow[d, "\wr"', "T_2"] & \check{P}_1 \oplus \check{P}_1 \arrow[r, "\tilde{d}'_0"] \arrow[u, "\wr", "{\begin{pmatrix} \rho_1 \end{pmatrix}}"'] \arrow[d, "\wr"', "T_1"] & \check{P}_0 \oplus \check{P}_0 \arrow[r, "\tilde{\varepsilon}'"] \arrow[u, "\wr", "{\begin{pmatrix} \rho_0 \end{pmatrix}}"'] \arrow[d, "\wr"', "1 = T_0"] & R \oplus R \arrow[u, "\wr", "\rho_S"'] \arrow[d, "\wr"', "1"] \\
& \check{P}_0 \oplus \check{P}_0 \oplus \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \arrow[r, "\check{d}'_2"] & \check{P}_1 \oplus \check{P}_1 \oplus \check{P}_0 \oplus \check{P}_0 \arrow[r, "\check{d}'_1"] & \check{P}_1 \oplus \check{P}_1 \arrow[r, "\check{d}'_0"] & \check{P}_0 \oplus \check{P}_0 \arrow[r, "\check{\varepsilon}'"] & R \oplus R
\end{tikzcd}$$

Proof. We verify that part (I) and part (II) of the diagram commute.

Using Corollary 49, we obtain

$$\begin{aligned}
(\check{\mathcal{E}}_0, \check{\mathcal{E}}_0)\rho_0 &= (\zeta \mathcal{E}_0 + \zeta^2 \mathcal{E}_0) = -\mathcal{E}_0 \\
(\check{\mathcal{E}}_1, \check{\mathcal{E}}_1)\rho_1 &= \big((\zeta + \zeta^2)(\mathcal{E}_1 + \mathcal{E}_2)\mathcal{E}_1, (\zeta + \zeta^2)(\mathcal{E}_1 + \mathcal{E}_2)\mathcal{E}_2\big) = (-\mathcal{E}_1, -\mathcal{E}_2)\,.
\end{aligned}$$

We have

$$(\check{\mathcal{E}}_0,\check{\mathcal{E}}_0)\rho_0\varepsilon = -\mathcal{E}_0\varepsilon = -1 = \zeta+\zeta^2 = (1,1)\rho_S = (\check{\mathcal{E}}_0,\check{\mathcal{E}}_0)\widetilde{\varepsilon'}\rho_S\,.$$

We have

$$\begin{aligned}
(\check{\mathcal{E}}_1,\check{\mathcal{E}}_1)\rho_1 d_0 &= (-\mathcal{E}_1,-\mathcal{E}_2)\begin{pmatrix} b_{1,0}\\ b_{2,0}\end{pmatrix} = \left(0,0,0,\begin{pmatrix}0&0&-2\\0&0&-2\\0&0&0\end{pmatrix}\right)\\
&= \zeta\left(0,0,0,\begin{pmatrix}0&0&2\\0&0&2\\0&0&0\end{pmatrix}\right)+\zeta^2\left(0,0,0,\begin{pmatrix}0&0&2\\0&0&2\\0&0&0\end{pmatrix}\right)\\
&= \left(\left(0,0,0,\begin{pmatrix}0&0&-1\\0&0&-1\\0&0&0\end{pmatrix}\right),\left(0,0,0,\begin{pmatrix}0&0&-1\\0&0&-1\\0&0&0\end{pmatrix}\right)\right)\rho_0\\
&= (-b_5-b_6,-b_5-b_6)\rho_0 = (\check{\mathcal{E}}_1,\check{\mathcal{E}}_1)\widetilde{d_0'}\rho_0\,.
\end{aligned}$$

We have

$$\begin{aligned}
&(\check{\mathcal{E}}_1,\check{\mathcal{E}}_1,\check{\mathcal{E}}_0,\check{\mathcal{E}}_0)\begin{pmatrix}\rho_1&0\\0&\rho_0\end{pmatrix}Ud_1\\
&= (-\mathcal{E}_1,-\mathcal{E}_2,-\mathcal{E}_0)Ud_1 = (-\mathcal{E}_2,-\mathcal{E}_0,-\mathcal{E}_1)\begin{pmatrix}b_{2,1}&-2b_{2,2}\\ b_{0,1}&-b_{0,2}\\ 2b_{1,1}&-b_{1,2}\end{pmatrix}\\
&= (-b_{2,1}-2b_{1,1}-b_{0,1},\ 2b_{2,2}+b_{0,2}+b_{1,2})\\
&= \left(\left(0,-2,0,\begin{pmatrix}0&-2&0\\0&-2&0\\0&-2&0\end{pmatrix}\right),\left(0,0,2,\begin{pmatrix}2&0&0\\2&0&0\\2&0&0\end{pmatrix}\right)\right)\\
&= \frac{1}{3}\left(\left(0,2+4\zeta,2+4\zeta^2,\begin{pmatrix}-(2+4\zeta)&2+4\zeta&0\\-(2+4\zeta)&2+4\zeta&0\\-(2+4\zeta)&2+4\zeta&0\end{pmatrix}\right)(\zeta-\zeta^2)\mathcal{E}_1\ ,\right.\\
&\qquad\left.\left(0,2+4\zeta,2+4\zeta^2,\begin{pmatrix}-(2+4\zeta)&2+4\zeta&0\\-(2+4\zeta)&2+4\zeta&0\\-(2+4\zeta)&2+4\zeta&0\end{pmatrix}\right)(\zeta-\zeta^2)\mathcal{E}_2\right)\\
&= \frac{1}{3}\left(\left(0,2+4\zeta,\begin{pmatrix}2&-2&0\\2&-2&0\\-4&4&0\end{pmatrix}\right)\left(0,-2-4\zeta,\begin{pmatrix}-2&2&0\\-2&2&0\\4&-4&0\end{pmatrix}\right)\right)\rho_1\\
&= \frac{1}{3}(+2b_7-2b_8-b_{10}+b_{11}+2b_{12}-b_3+b_4,\ -2b_7+2b_8+b_{10}-b_{11}-2b_{12}+b_3-b_4)\rho_1\\
&= (\check{\mathcal{E}}_1,\check{\mathcal{E}}_1,\check{\mathcal{E}}_0,\check{\mathcal{E}}_0)\frac{1}{3}\begin{pmatrix}-2b_7+2b_8+b_{10}-b_{11}-2b_{12} & 2(-2b_7+2b_8+b_{10}-b_{11}-2b_{12})\\ -2(-2b_7+2b_8+b_{10}-b_{11}-2b_{12}) & -(-2b_7+2b_8+b_{10}-b_{11}-2b_{12})\\ b_3-b_4 & 2b_3-2b_4\\ -2b_3+2b_4 & -b_3+b_4\end{pmatrix}\rho_1\\
&= (\check{\mathcal{E}}_1,\check{\mathcal{E}}_1,\check{\mathcal{E}}_0,\check{\mathcal{E}}_0)\widetilde{d_1'}\rho_1\,.
\end{aligned}$$

We have

$$\begin{aligned}
&(\check{\mathcal{E}}_0,\check{\mathcal{E}}_0,\check{\mathcal{E}}_1,\check{\mathcal{E}}_1,\check{\mathcal{E}}_0,\check{\mathcal{E}}_0)\begin{pmatrix}\rho_0&0&0\\0&\rho_1&0\\0&0&\rho_0\end{pmatrix}d_2\\
&= (-\mathcal{E}_0,-\mathcal{E}_1,-\mathcal{E}_2,-\mathcal{E}_0)\begin{pmatrix}b_{0,2}&-2b_{0,0}&0\\ b_{1,2}&-b_{1,0}&0\\ 0&-b_{2,0}&b_{2,1}\\ 0&-2b_{0,0}&b_{0,1}\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= (-b_{0,2} - b_{1,2},\ 4b_{0,0} + b_{1,0} + b_{2,0},\ -b_{2,1} - b_{0,1}) \\
&= \left(\left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\right), \left(4,0,0, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix}\right), \left(0,0,0, \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}\right) \right) \\
&= \left(\left(0,0,0, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\right)(\zeta+\zeta^2), \left(0,0,0, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\right)(\zeta+\zeta^2), \left(-4,0,0, \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \end{pmatrix}\right)(\zeta+\zeta^2) \right) U \\
&= \left(\left(0,0, \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ -4 & -4 & 0 \end{pmatrix}\right), \left(0,0, \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ -4 & -4 & 0 \end{pmatrix}\right), \right. \\
&\qquad \left. \left(-4,0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{pmatrix}\right), \left(-4,0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{pmatrix}\right) \right) \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_0 \end{pmatrix} U \\
&= (-b_3 - b_4 + 2b_8 + 2b_9 - b_{10} - b_{11},\ -b_3 - b_4 + 2b_8 + 2b_9 - b_{10} - b_{11}, \\
&\qquad\qquad -4b_1 + b_5 + b_6,\ -4b_1 + b_5 + b_6) \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_0 \end{pmatrix} U \\
&= (\check{\mathcal{E}}_0, \check{\mathcal{E}}_0, \check{\mathcal{E}}_1, \check{\mathcal{E}}_1, \check{\mathcal{E}}_0, \check{\mathcal{E}}_0) \begin{pmatrix} \frac{1}{3}(-2b_3 - b_4) & \frac{1}{3}(-b_3 + b_4) & -2b_1 & 0 \\ \frac{1}{3}(b_3 - b_4) & \frac{1}{3}(-b_3 - 2b_4) & 0 & -2b_1 \\ 2b_8 + 2b_9 - b_{10} - b_{11} & 0 & b_5 + b_6 & 0 \\ 0 & 2b_8 + 2b_9 - b_{10} - b_{11} & 0 & b_5 + b_6 \\ \frac{1}{3}(-b_3 - 2b_4) & \frac{1}{3}(b_3 - b_4) & -2b_1 & 0 \\ \frac{1}{3}(-b_3 + b_4) & \frac{1}{3}(-2b_3 - b_4) & 0 & -2b_1 \end{pmatrix} \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_0 \end{pmatrix} U \\
&= (\check{\mathcal{E}}_0, \check{\mathcal{E}}_0, \check{\mathcal{E}}_1, \check{\mathcal{E}}_1, \check{\mathcal{E}}_0, \check{\mathcal{E}}_0)\, \widetilde{d}_2' \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_0 \end{pmatrix} U\,.
\end{aligned}
$$

Since all appearing maps are $\check{\Lambda}$-linear, part (I) and part (II) of the diagram commute.

We verify that part (III) of the diagram commutes.
Note that $\widetilde{\varepsilon}' = \check{\varepsilon}'$; cf. Definition 43. We have

$$
\begin{aligned}
T_1 \check{d}_0' &= \begin{pmatrix} -b_5 - b_6 & 0 \\ 0 & -b_5 - b_6 \end{pmatrix} = \widetilde{d}_0' \\
T_2 \check{d}_1' &= \frac{1}{3} \begin{pmatrix} -2b_8 - b_9 + 2b_{11} + b_{12} & 2(-2b_8 - b_9 + 2b_{11} + b_{12}) \\ -2(-2b_8 - b_9 + 2b_{11} + b_{12}) & -(-2b_8 - b_9 + 2b_{11} + b_{12}) \\ b_4 & 2b_4 \\ -2b_4 & -b_4 \end{pmatrix} = \widetilde{d}_1' T_1 \\
T_3 \check{d}_2' &= \frac{1}{3} \begin{pmatrix} b_4 - 3b_3 & 2b_4 - 3b_3 & -2b_1 & -4b_1 \\ 3b_3 - 2b_4 & -b_4 & 4b_1 & 2b_1 \\ \widetilde{b} & 2\widetilde{b} & b_5 + b_6 & 2(b_5 + b_6) \\ -2\widetilde{b} & -\widetilde{b} & -2(b_5 + b_6) & -(b_5 + b_6) \\ b_4 & 2b_4 - 3b_3 & -2b_1 & -4b_1 \\ 3b_3 - 2b_4 & 3b_3 - b_4 & 4b_1 & 2b_1 \end{pmatrix} = \widetilde{d}_2' T_2
\end{aligned}
$$

with $\widetilde{b} := 6b_7 - 2b_8 - b_9 - 3b_{10} + b_{11} + 2b_{12}$.

We verify that T_1, T_2 and T_3 are isomorphisms.

Recall that

$$\begin{aligned}\mathrm{rad}(\check{\mathcal{E}}_0\check{\Lambda}\check{\mathcal{E}}_0) &= {}_R\langle 2b_1, b_2\rangle,\\ \mathrm{rad}(\check{\mathcal{E}}_1\check{\Lambda}\check{\mathcal{E}}_1) &= {}_R\langle 2b_7, 2b_8, b_9, b_{10}, b_{11}, b_{12}\rangle\,;\end{aligned}$$

cf. Lemma 33. Modulo the radical $\mathfrak{r} := \mathrm{rad}(\check{\Lambda})$ we obtain

$$b_8(b_7 + b_8) \equiv_{\mathfrak{r}} b_7 = \check{\mathcal{E}}_1\,;$$

cf. Lemma 32. Hence, T_1 and T_2 are invertible modulo $\mathfrak{r}$, so that T_1 and T_2 are isomorphisms. To show that T_3 is invertible mod $\mathfrak{r}$, we additionally use for the matrix consisting of the outer 2×2-blocks of T_3, that

$$\begin{pmatrix}1&1&1&0\\1&0&0&1\\0&1&1&0\\1&1&0&1\end{pmatrix} \in \mathrm{GL}_4(\mathbb{F}_2).$$

Hence also T_3 is an isomorphism. □

Remark 55 Note that the entry of every 2×2-block of the maps $\check{\varepsilon}'$ and $\check{d}_i'$ for $i \in \{0, 1, 2\}$ is the same as in the differentials of Definition 43, up to row permutation.

Chapter 5

Appendix

Remark 56 Let $\zeta := \zeta_3$ be a third primitive root of unity over $\mathbb{Q}$. Let $S := \mathbb{Z}_{(2)}[\zeta]$.

The maps defined in Lemma 24 and Definition 29 used for the construction of the differentials and homotopies can be written as S-linear matrices.

We have the following S-linear bases for the projective modules of Λ; cf. Lemma 24.

$P_0 : \left(b_{0,0}, b_{1,0}, b_{2,0}, \widetilde{b}_{0,0}\right)$
$P_1 : \left(b_{0,1}, b_{1,1}, b_{2,1}, \widetilde{b}_{1,1}\right)$
$P_2 : \left(b_{0,2}, b_{1,2}, b_{2,2}, \widetilde{b}_{2,2}\right)$

Using for each map the respective bases we obtain the following matrices, which alternatively can be used for direct calculations.

$$b_{0,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : P_0 \to P_0 \qquad b_{1,0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix} : P_1 \to P_0$$

$$b_{2,0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} : P_2 \to P_0 \qquad \widetilde{b}_{0,0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} : P_0 \to P_0$$

$$b_{0,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_1 \qquad b_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : P_1 \to P_1$$

$$b_{2,1} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} : P_2 \to P_1 \qquad \widetilde{b}_{1,1} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} : P_1 \to P_1$$

$$b_{0,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_2 \qquad b_{1,2} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 \end{pmatrix} : P_1 \to P_2$$

$$b_{2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : P_2 \to P_2 \qquad \widetilde{b}_{2,2} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} : P_2 \to P_2$$

$$\alpha^0_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_1 \qquad \alpha^0_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_2$$

$$\alpha^0_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} : P_1 \to P_1 \qquad \alpha^0_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : P_1 \to P_2$$

$$\alpha^1_{2,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_2 \to P_0 \qquad \alpha^1_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} : P_2 \to P_1$$

$$\alpha^1_{0,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_0 \qquad \alpha^1_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_1$$

$$\alpha^2_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : P_1 \to P_2 \qquad \alpha^2_{1,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_1 \to P_0$$

$$\alpha^2_{2,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} : P_2 \to P_2 \qquad \alpha^2_{2,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} : P_2 \to P_0$$

$$\beta^0_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_1 \to P_2 \qquad \beta^0_{1,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : P_1 \to P_0$$

$$\beta^0_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} : P_1 \to P_1 \qquad \beta^0_{2,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} : P_2 \to P_2$$

$$\beta^0_{2,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} : P_2 \to P_0 \qquad \beta^0_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_2 \to P_1$$

$$\beta^1_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_1 \qquad \beta^1_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_2$$

$$\beta^1_{0,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_0 \qquad \beta^1_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} : P_1 \to P_1$$

$$\beta^1_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : P_1 \to P_2 \qquad \beta^1_{1,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_1 \to P_0$$

$$\beta^2_{2,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_2 \to P_0 \qquad \beta^2_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} : P_2 \to P_1$$

$$\beta^2_{2,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} : P_2 \to P_2 \qquad \beta^2_{0,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_0$$

$$\beta^2_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_1 \qquad \beta^2_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : P_0 \to P_2$$

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Zusammenfassung

Sei $S := \mathbb{Z}_{(2)}[\zeta_3]$ und A_4 die alternierende Gruppe auf vier Elementen.

Wir konstruieren die minimale projektive Auflösung Q von S über $S\mathrm{A}_4$.

$$\cdots \longrightarrow Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \longrightarrow 0$$

Dazu finden wir die projektiv unzerlegbaren Moduln P_0, P_1 und P_2 von $S\mathrm{A}_4$ als projektiv unzerlegbare Moduln des Wedderburn-Bildes Λ von $S\mathrm{A}_4$ und definieren zwischen diesen $S\mathrm{A}_4$- lineare Abbildungen $b_{k,l}$ für $k, l \in \{0, 1, 2\}$ als Multiplikation mit Basiselementen von Λ. Daraus konstruieren wir für $i \geq 0$ die projektiven Moduln Q_i als direkte Summe der P_0, P_1 und P_2 und die Differentiale d_i als Matrizen, mit Vielfachen der $b_{k,l}$ als Einträgen.

Die so konstruierte minimale projektive Auflösung von S über $S\mathrm{A}_4$ zeigt ein regelmäßiges Verhalten. Um zu zeigen, dass Q tatsächlich eine projektive Auflösung ist, konstruieren wir S-lineare Homotopien, die ebenfalls ein regelmäßiges Verhalten zeigen.

Sei $R := \mathbb{Z}_{(2)}$ und $\otimes := \underset{R}{\otimes}$.
Wir versuchen, R über $R\mathrm{A}_4$ mit Hilfe der projektiven Auflösung Q von S über $S\mathrm{A}_4$ projektiv aufzulösen. Dazu geben wir zunächst die projektiv unzerlegbaren Moduln $\check{P}_0$ und $\check{P}_1$ von $R\mathrm{A}_4$ als projektiv unzerlegbare Moduln des Wedderburn-Bildes $\check{\Lambda}$ von $R\mathrm{A}_4$ an.

Wir finden die ersten Terme $\check{Q}$ einer augmentierten projektiven Auflösung von R über $R\mathrm{A}_4$ so, dass $S \otimes \check{Q}$ isomorph zu

$$Q_6 \xrightarrow{d_5} Q_5 \xrightarrow{d_4} Q_4 \xrightarrow{d_3} Q_3 \xrightarrow{d_2} Q_2 \xrightarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \xrightarrow{\varepsilon} S$$

ist.

Wir finden weiterhin eine projektive Auflösung von $R \oplus R$ über $R\mathrm{A}_4$, indem wir die Moduln von Λ entlang eines injektiven Ringhomomorphismus $\iota : \check{\Lambda} \to \Lambda$ zu Moduln über R einschränken. Mittels Isomorphismen $\check{P}_0 \xrightarrow{\sim} P_0{\downarrow_\iota}$, $\check{P}_1 \xrightarrow{\sim} P_1{\downarrow_\iota}$ und $\check{P}_1 \xrightarrow{\sim} P_2{\downarrow_\iota}$ zeigen wir

$$\left(\check{Q}_3 \xrightarrow{\check{d}_2} \check{Q}_2 \xrightarrow{\check{d}_1} \check{Q}_1 \xrightarrow{\check{d}_0} \check{Q}_0 \xrightarrow{\check{\varepsilon}} \mathbb{Z}_{(2)} \right)^{\oplus 2} \cong \left(Q_3{\downarrow_\iota} \xrightarrow{d_2} Q_2{\downarrow_\iota} \xrightarrow{d_1} Q_1{\downarrow_\iota} \xrightarrow{d_0} Q_0{\downarrow_\iota} \longrightarrow \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)} \right).$$

Um obige Konstruktionen durchführen zu können, geben wir zu Beginn Wedderburn-Isomorphismen

$$R\mathrm{A}_4 \underset{\sim}{\xrightarrow{\check{\omega}}} \check{\Lambda} \quad \text{und} \quad S\mathrm{A}_4 \underset{\sim}{\xrightarrow{\omega}} \Lambda$$

an, deren Bilder $\check{\Lambda}$ und Λ jeweils Teilringe von direkten Produkten von Matrixringen sind, die sich durch Kongruenzen beschreiben lassen. Neben einem direkten Beweis geben wir auch eine Konstruktion von ω ausgehend von $\check{\omega}$.

Hiermit versichere ich,

(1) dass ich meine Arbeit selbstständig verfasst habe,

(2) dass ich keine anderen als die angegebenen Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,

(3) dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und

(4) dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, Januar 2016

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