Simplicial Resolutions

Bachelor Thesis

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Chapter 0

Introduction

0.1 Example: groups and model categories

Consider the category Grp of groups. We have the full subcategory FreeGrp $\subseteq$ Grp of free groups. If we want to resolve a group by free groups, we do not have classical homological algebra at our disposal, for Grp is not additive. As a replacement, one can simplicially resolve a group using free groups.

Simplicial resolutions can also be applied in model categories such as the category of topological spaces, the category of simplicial sets and the category of simplicial groups.

0.2 Reduced limits

We will work with different types of category-theoretic limits. For a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$, a limit of F is given by an object $L \in \operatorname{Ob}\mathcal{C}$ and a tuple of morphisms $(L \xrightarrow{\omega_X} FX)_{X \in \operatorname{Ob}\mathcal{D}}$ such that for each morphism $(X \xrightarrow{\alpha} Y) \in \operatorname{Mor}\mathcal{D}$ we have $\omega_X \cdot F\alpha = \omega_Y$ and such that this tuple is universal with this property.

For the particular case that the category $\mathcal{D}$ is a finite poset it is easier to work with a slightly modified limit, which we will call reduced limit. The reduced limit comes only with morphisms $L \xrightarrow{\omega_X} FX$ such that X is a minimal element in the poset. This facilitates explicit constructions in some cases; cf. e.g. Example 39. The reduced limit yields the limit by defining the missing morphisms as composites of the given ones with the morphisms that appear in the image of the functor F.

0.3 Simplicial and semisimplicial objects

The simplex category Δ has as objects the sets $[0, n]$, containing the integers from 0 to n, for $n \geqslant 0$ and as morphisms the monotone maps between them.

A simplicial object in $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. A simplicial morphism is a transformation between such functors. The category of simplicial objects and simplicial morphisms is written $\operatorname{Simp}(\mathcal{C})$.

The category $\Delta_{\text{inj}} \subseteq \Delta$ is the subcategory containing only the injective monotone maps.

A semisimplicial object in $\mathcal{C}$ is a functor $X : \Delta_{\text{inj}}^{\text{op}} \to \mathcal{C}$. A semisimplicial morphism is a transformation between such functors. The category of semisimplicial objects and simplicial morphisms is written $\operatorname{SemiSimp}(\mathcal{C})$.

0.4 Semisimplicial resolutions

As an intermediate step towards the construction of simplicial resolutions, we construct semisimplicial resolutions following Myles Tierney and Wolfgang Vogel [**1**].

Classically, one builds a projective resolution of a module by choosing a projective module mapping onto it, taking the kernel, choosing a projective module mapping onto it, taking the kernel, etc. To build a semisimplicial resolution of an object, the kernel is replaced by the simplicial kernel and the projective modules are replaced by objects in a resolving subcategory, as we shall explain now.

Given a tuple of n morphisms $(X \xrightarrow{f_i} Y)_{i \in [1,n]}$ in a category $\mathcal{C}$, the simplicial kernel of this tuple is a tuple of $n+1$ morphisms $(K \xrightarrow{k_i} X)_{i \in [1,n+1]}$ satisfying $k_j f_i = k_i f_{j-1}$ for $1 \leqslant i < j \leqslant n+1$ and being universal with this property. We will see that simplicial kernels are just the reduced limits of certain functors, which we will construct in the proof of Proposition 23.

Instead of using projective objects as one does for a classical projective resolution, we choose a resolving subcategory $\mathcal{P}$ of $\mathcal{C}$, which is a full subcategory having properties resembling those of the subcategory of projective modules in all modules: for each $X \in \operatorname{Ob}\mathcal{C}$, there exists $P \xrightarrow{f} X$ such that $P \in \operatorname{Ob}\mathcal{P}$ and such that for each $Q \xrightarrow{g} X$ with $Q \in \operatorname{Ob}\mathcal{P}$ there exists $Q \xrightarrow{u} P$ with $uf = g$.

$$\begin{tikzcd} & Q \arrow[dl, dashed, "u"'] \arrow[d, "g"] \\ P \arrow[r, "f"] & X \end{tikzcd}$$

For instance, in the category $\mathcal{C} := \operatorname{Grp}$ of groups, we may let $\mathcal{P} := \operatorname{FreeGrp} \subseteq \operatorname{Grp} = \mathcal{C}$, making use of the fact that to every group, there exists a surjective group morphism from a free group.

Or, for instance, in a model category $\mathcal{C}$, we may let $\mathcal{P}$ be the full subcategory of cofibrant objects, making use of the fact that to every object of $\mathcal{C}$, there exists an acyclic fibration from a cofibrant object. Cf. Remark 34.

Or, for instance, in a model category $\mathcal{C}$, we may let $\mathcal{P}$ be the full subcategory of acyclic cofibrant objects, making use of the fact that to every object of $\mathcal{C}$, there exists a fibration from an acyclic cofibrant object. Cf. Remark 35.

Now suppose given an object X in $\mathcal{C}$, which we want to resolve semisimplicially. First, choose $P_0 \to X$ with $P_0 \in \operatorname{Ob}\mathcal{P}$ as described above. Let $K_1 \rightrightarrows P_0$ be its simplicial kernel. Choose $P_1 \to K_1$ with $P_1 \in \operatorname{Ob}\mathcal{P}$ as described above. Compose to the tuple $P_1 \rightrightarrows P_0$. Let $K_2 \Rrightarrow P_1$ be its simplicial kernel. Choose $P_2 \to K_2$ with $P_2 \in \operatorname{Ob}\mathcal{P}$ as described above. Compose to the tuple $P_2 \Rrightarrow P_1$. Etc. The objects P_n for $n \geqslant 0$, together with the morphism tuples between them, yield a semisimplicial object, which we define to be a semisimplicial resolution of X; cf. Definition 36, Remark 37, Proposition 43.

0.5 From semisimplicial to simplicial resolutions

A semisimplicial resolution of an object in $\mathcal{C}$ yields a semisimplicial object in $\mathcal{C}$. To turn this semisimplicial resolution into a simplicial resolution, we need to find an appropriate way to contruct a simplicial object out of a semisimplicial object. To this end we will construct a left adjoint functor

$$\mathcal{F}_{\mathcal{C}} \;:\; \mathrm{SemiSimp}(\mathcal{C}) \;\to\; \mathrm{Simp}(\mathcal{C})$$

to the forgetful functor

$$\mathcal{V}_{\mathcal{C}} \;:\; \mathrm{Simp}(\mathcal{C}) \;\to\; \mathrm{SemiSimp}(\mathcal{C})\;,$$

which restricts a given simplicial object X, i.e. a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$, from Δ^{op} to $\Delta^{\mathrm{op}}_{\mathrm{inj}}$.

We will first construct the functor $\mathcal{F}_{\mathcal{C}}$ in the case $\mathcal{C} = \mathrm{Set}$, because there the construction follows the intuition of adding formal degeneracy maps and because the general case is modelled on this particular case.

If one uses a Kan extension along the inclusion $\Delta^{\mathrm{op}}_{\mathrm{inj}} \to \Delta^{\mathrm{op}}$ to construct $\mathcal{F}_{\mathcal{C}}$, one usually works with a direct limit. In the case $\mathcal{C} = \mathrm{Set}$, this amounts to working with equivalence classes.

Here, to construct $\mathcal{F}_{\mathcal{C}}$ we use the fact that there exists a unique factorization of a monotone map into a surjective and an injective monotone map; cf. Remark 49. In this way, we avoid equivalence classes in case $\mathcal{C} = \mathrm{Set}$, and we may use mere coproducts instead of colimits in the general case.

So if our resolving subcategory $\mathcal{P}$ is closed under coproducts in $\mathcal{C}$, then we can resolve $X \in \mathrm{Ob}\,\mathcal{C}$ with a semisimplicial object in $\mathcal{P}$, which then yields a simplicial object in $\mathcal{P}$ by an application of $\mathcal{F}_{\mathcal{C}}$. This simplicial object is the simplicial resolution of X.

0.6 Conventions

- In case we write "for $x \in X$", it means "for all $x \in X$".
- Suppose given a finite set S. Then $|S|$ is the cardinality of S.
- Suppose given a map $\alpha : X \to Y$. We write the image of $x \in X$ under α as $x\alpha$. Moreover, $\mathrm{Im}(\alpha) \subseteq Y$ denotes the image of α.
- Given a set L and subsets $M, N \subseteq L$, we write $L = M \,\dot{\cup}\, N$ if $L = M \cup N$ and $M \cap N = \emptyset$.
- Suppose given sets I and X_i for $i \in I$. We write $\bigsqcup\limits_{i \in I} X_i = \{(i, x) : x \in X_i\}$ for the disjoint union of the X_i for $i \in I$.
- Suppose given sets A, B, C, D. Suppose that $A \subseteq B$ and $C \subseteq D$. Suppose given a map $f : B \to D$ such that $\mathrm{Im}(f) \subseteq C$.

  We write $f|_A^C : A \to C$, $a \mapsto af|_A^C := af$ for the restriction of f to A in the domain and to C in the range.

  If $C = D$, we also write $f|_A := f|_A^D$. If $A = B$, we also write $f|^C := f|_B^C$.
- For a set M we write $\mathbb{P}(M)$ for the power set of M.

- Suppose given a poset $(\mathcal{M}, \leqslant_{\mathcal{M}})$. We call a poset $(\mathcal{S}, \leqslant_{\mathcal{S}})$, where $\mathcal{S} \subseteq \mathcal{M}$, a *full subposet* of $(\mathcal{M}, \leqslant_{\mathcal{M}})$, if $(s_1 \leqslant_{\mathcal{M}} s_2) \Leftrightarrow (s_1 \leqslant_{\mathcal{S}} s_2)$ for $s_1, s_2 \in \mathcal{S}$. We often write $\mathcal{M}$ instead of $(\mathcal{M}, \leqslant)$.

- For $z_1, z_2 \in \mathbb{Z}$ let $[z_1, z_2] := \{z \in \mathbb{Z} : z_1 \leqslant z \leqslant z_2\}$. Let $\mathbb{Z}_{\geqslant 0} := \{z \in \mathbb{Z} : z \geqslant 0\}$. For $z \in \mathbb{Z}_{\geqslant 0}$ we often abbreviate $[z] := [0, z]$.

- We write Set for the category of sets.

- All categories $\mathcal{C}$ under consideration are small, which means that $\operatorname{Ob}\mathcal{C}$ and $\operatorname{Mor}\mathcal{C}$ are sets. If necessary, we choose a universe with respect to which the category under consideration is small. We call a category $\mathcal{C}$ a *finite category* if $\operatorname{Ob}\mathcal{C}$ and $\operatorname{Mor}\mathcal{C}$ are finite sets.

- We write composition on the right. That means, given morphisms $a \xrightarrow{\alpha} b$ and $b \xrightarrow{\beta} c$, the composite of these two morphisms is written $a \xrightarrow{\alpha \cdot \beta} c$ or $a \xrightarrow{\alpha\beta} c$.

  We write composition of functors on the left. That means, given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, we write $G \circ F$ or GF for their composite.

- Given a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$, we often write $F_x := Fx$ for $x \in \operatorname{Ob}\mathcal{D}$ and $F_\alpha := F\alpha$ for $\alpha \in \operatorname{Mor}\mathcal{D}$, e.g. if we consider F as a diagram with values in $\mathcal{C}$.

- Suppose given a category $\mathcal{C}$ and objects $x, y \in \operatorname{Ob}\mathcal{C}$. Then ${}_{\mathcal{C}}(x, y)$ is the set of all morphisms of $\mathcal{C}$ with source x and target y.

- Suppose given a category $\mathcal{C}$. Then $\mathcal{C}^{\mathrm{op}}$ is the opposite category of $\mathcal{C}$. For a morphism $(a \xrightarrow{\alpha} b) \in \operatorname{Mor}\mathcal{C}$, let $b \xrightarrow{\alpha^{\mathrm{op}}} a$ denote the corresponding morphism in $\mathcal{C}^{\mathrm{op}}$.

- Suppose given categories $\mathcal{D}$ and $\mathcal{C}$. We write $\mathcal{C}^{\mathcal{D}}$ for the functor category of functors from $\mathcal{D}$ to $\mathcal{C}$.

- Suppose given a category $\mathcal{C}$.

  Suppose given integers $a \leqslant b$ and morphisms $(X_{i-1} \xrightarrow{\alpha_i} X_i)$ in $\mathcal{C}$ for $i \in [a+1, b]$. We write
  $$\prod_{i \in \lceil a+1\,, b \rceil}^{X_a \quad X_b} \alpha_i := \alpha_{a+1} \cdots \alpha_b \text{ if } a < b$$
  and
  $$\prod_{i \in \lceil a+1\,, b \rceil}^{X_a \quad X_b} \alpha_i := \mathrm{id}_{X_a} \text{ if } a = b\,.$$

  Suppose given integers $a \leqslant b$ and morphisms $(X_{i-1} \xleftarrow{\alpha_i} X_i)$ in $\mathcal{C}$ for $i \in [a+1, b]$. We write
  $$\prod_{i \in \lfloor b\,, a+1 \rfloor}^{X_b \quad X_a} \alpha_i := \alpha_b \cdots \alpha_{a+1} \text{ if } a < b$$
  and
  $$\prod_{i \in \lfloor b\,, a+1 \rfloor}^{X_b \quad X_a} \alpha_i := \mathrm{id}_{X_b} \text{ if } a = b\,.$$

- Suppose given functors $F, F' : \mathcal{C} \to \mathcal{D}$. A tuple of morphisms $(FX \xrightarrow{t_X} F'X)_{X \in \mathrm{Ob}\,\mathcal{C}}$ in $\mathcal{D}$ is called natural, if $t_X \cdot F'u = Fu \cdot t_Y$ for all morphisms $X \xrightarrow{u} Y$ in $\mathcal{C}$. If such a tuple is natural, we call it a transformation.

- Suppose given a category $\mathcal{C}$. Suppose given a set I and $X_i \in \mathrm{Ob}\,\mathcal{C}$. We write $\coprod\limits_{i \in I} X_i$ for the coproduct of the objects X_i for $i \in I$.

- We write Grp for the category of groups and FreeGrp for the full subcategory of free groups.

- Groups are written multiplicatively. That includes that the neutral element of a group G is written 1_G. The inverse of $g \in G$ is often written g^-. The trivial group is written 1.

- Suppose given a set M. We write $\mathrm{Free}(M)$ for the free group generated by the elements of M. Elements of $\mathrm{Free}(M)$ are denotated as words in the alphabet given by the set $M \cup M^-$, where $M^- := \{m^- : m \in M\}$. The empty word, which is the neutral element, is written $1_{\mathrm{Free}(M)}$.

- Suppose given a group G. Suppose given $M \subseteq G$. We write $\langle\!\!| M |\!\!\rangle := \bigcap\limits_{N \trianglelefteq G,\, M \subseteq N} N$ for the normal subgroup generated by M.

- Suppose given a group G and $U \trianglelefteq G$. We write G/U for the factor group. For $g \in G$ we write gU for the image of the residue class morphism of g in G/U.

- Suppose given a set E. Let $R \subseteq \mathrm{Free}(E)$. Let $Q := \langle\!\!| R |\!\!\rangle$ be the normal subgroup generated by R. Then we define $\langle E | R \rangle := \mathrm{Free}(E)/Q$. Instead of $\langle \{e_1, \ldots, e_n\} | \{r_1, \ldots, r_m\} \rangle$ we often write $\langle e_1, \ldots, e_n | r_1, \ldots, r_m \rangle$.

Chapter 1

Preliminaries

1.1 Transformations

Remark 1. Suppose given functors $\mathcal{B} \xrightarrow{K} \mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D} \xrightarrow{H} \mathcal{E}$ and a transformation $\alpha = (\alpha_X)_{X \in \mathrm{Ob}\,\mathcal{C}} : F \to G$. Then

(i) $H\alpha := (H(\alpha_X))_{X \in \mathrm{Ob}\,\mathcal{C}}$ is a transformation from $H \circ F$ to $H \circ G$.

(ii) $\alpha K := (\alpha_{K_X})_{X \in \mathrm{Ob}\,\mathcal{B}}$ is a transformation from $F \circ K$ to $G \circ K$.

Proof. Ad (i). Suppose given $(X \xrightarrow{f} Y) \in \mathrm{Mor}\,\mathcal{C}$. We have to show commutativity of the following diagram.

$$\begin{tikzcd} (H \circ F)_X \arrow[r, "H(\alpha_X)"] \arrow[d, "(H\circ F)_f"'] & (H \circ G)_X \arrow[d, "(H\circ G)_f"] \\ (H \circ F)_Y \arrow[r, "H(\alpha_Y)"'] & (H \circ G)_Y \end{tikzcd}$$

We have

$H(\alpha_X) \cdot (H \circ G)_f = H(\alpha_X) \cdot H(G_f) = H(\alpha_X \cdot G_f) \overset{\alpha \text{ transformation}}{=} H(F_f \cdot \alpha_y) = H(F_f) \cdot H(\alpha_Y) = (H \circ G)_f \cdot H(\alpha_Y)$.

Ad (ii). Suppose given $(X \xrightarrow{f} Y) \in \mathrm{Mor}\,\mathcal{B}$. We have to show commutativity of the following diagram.

$$\begin{tikzcd} (F \circ K)_X \arrow[r, "\alpha_{K_X}"] \arrow[d, "(F\circ K)_f"'] & (G \circ K)_X \arrow[d, "(G\circ K)_f"] \\ (F \circ K)_Y \arrow[r, "\alpha_{K_Y}"'] & (G \circ K)_Y \end{tikzcd}$$

We have

$\alpha_{K_X} \cdot (G \circ K)_f = \alpha_{K_X} \cdot G_{K(f)} \overset{\alpha \text{ transformation}}{=} F_{K(f)} \cdot \alpha_{K_Y} = (F \circ K)_f \cdot \alpha_{K_Y}$. $\square$

1.2 Limits

Definition 2 (Limit)**.** Suppose given categories $\mathcal{D}$ and $\mathcal{C}$. Suppose given a functor $F : \mathcal{D} \to \mathcal{C}$. Suppose given $L \in \operatorname{Ob}\mathcal{C}$ and a tuple of morphisms $(L \xrightarrow{\omega_x} F_x)_{x \in \operatorname{Ob}\mathcal{D}}$ in $\mathcal{C}$. Then the pair $(L, (\omega_x)_{x \in \operatorname{Ob}\mathcal{D}})$ is a *limit* of F, if the following properties (i, ii) hold.

(i) For each morphism $x \xrightarrow{\alpha} y$ in $\mathcal{D}$ we have $\omega_x F_\alpha = \omega_y$.

$$\begin{tikzcd} & F_x \arrow[dd, "F_\alpha"] \\ L \arrow[ur, "\omega_x"] \arrow[dr, "\omega_y"'] & \\ & F_y \end{tikzcd}$$

(ii) Suppose given $(L', (L' \xrightarrow{\omega'_x} F_x)_{x \in \operatorname{Ob}\mathcal{D}})$ with the property that for each morphism $x \xrightarrow{\alpha} y$ in $\mathcal{D}$ we have $\omega'_x F_\alpha = \omega'_y$. Then there exists a unique morphism $L' \xrightarrow{\mu} L$ with $\mu\omega_x = \omega'_x$ for $x \in \operatorname{Ob}\mathcal{D}$.

$$\begin{tikzcd} & & F_x \\ L' \arrow[urr, "\omega'_x"] \arrow[r, dashed, "\mu"'] & L \arrow[ur, "\omega_x"'] & \end{tikzcd}$$

Remark 3. The universal property in Definition 2 (i, ii) yields uniqueness of limits up to isomorphism. Suppose that $(L, (L \xrightarrow{\omega_x} F_x)_{x \in \operatorname{Ob}\mathcal{D}})$ and $(L', (L' \xrightarrow{\omega'_x} F_x)_{x \in \operatorname{Ob}\mathcal{D}})$ are limits of $\mathcal{D} \xrightarrow{F} \mathcal{C}$. Then there exists an isomorphism $\mu : L \to L'$ with $\mu\omega'_x = \omega_x$ for $x \in \operatorname{Ob}\mathcal{D}$ and $\mu^{-1}\omega_x = \omega'_x$ for $x \in \operatorname{Ob}\mathcal{D}$.

Proof. Suppose that $(L, (L \xrightarrow{\omega_x} F_x)_{x \in \operatorname{Ob}\mathcal{D}})$ and $(L', (L' \xrightarrow{\omega'_x} F_x)_{x \in \operatorname{Ob}\mathcal{D}})$ are limits of $\mathcal{D} \xrightarrow{F} \mathcal{C}$. Then $\omega_x F_\alpha = \omega_y$ for $(x \xrightarrow{\alpha} y) \in \operatorname{Mor}\mathcal{D}$. Because of the Universal Property of $(L', (L' \xrightarrow{\omega'_x} F_x)_{x \in \operatorname{Ob}\mathcal{D}})$, there exists a (unique) morphism $L \xrightarrow{\mu} L'$ satisfying $\mu\omega'_x = \omega_x$ for $x \in \operatorname{Ob}\mathcal{D}$. Analogously there exists a morphism $L' \xrightarrow{\nu} L$ satisfying $\nu\omega_x = \omega'_x$ for $x \in \operatorname{Ob}\mathcal{D}$. It is $\mu\nu\omega_x = \mu\omega'_x = \omega_x$ for $x \in \operatorname{Ob}\mathcal{D}$. On the other hand, we have $\mathrm{id}_L\,\omega_x = \omega_x$ for $x \in \operatorname{Ob}\mathcal{D}$. Because of the Universal Property, which says that there exists only one such morphism with source L and target L, we get $\mu\nu = \mathrm{id}_L$. Analogously, we get $\nu\mu = \mathrm{id}_{L'}$. Hence μ is an isomorphism and ν its inverse. $\square$

Remark 4. The dual notion of a limit is that of a *colimit*. That means, given a category $\mathcal{C}$, a limit of $\mathcal{C}^{\mathrm{op}}$ is a colimit, when viewed in $\mathcal{C}$.

Remark 5. We regard a diagram in a category $\mathcal{C}$ as a functor from a suitable category $\mathcal{D}$, which determines the type of the diagram, to $\mathcal{C}$. This allows us to speak of a limit of a diagram, which is just a limit of the associated functor.

Remark 6. Suppose given a poset $(\mathcal{M}, \leqslant)$. We define a category $\mathcal{C}_\mathcal{M}$ as follows. $\operatorname{Ob}\mathcal{C}_\mathcal{M} := \mathcal{M}$. For $m_1, m_2 \in \operatorname{Ob}\mathcal{C}_\mathcal{M}$, let

$$c_\mathcal{M}(m_1, m_2) := \begin{cases} \{(m_1, m_2)\} & \text{if } m_1 \leqslant m_2 \\ \emptyset & \text{if } m_1 \not\leqslant m_2 \,. \end{cases}$$

Composition of morphisms is given by $(m_1 \xrightarrow{(m_1,m_2)} m_2 \xrightarrow{(m_2,m_3)} m_3) := (m_1 \xrightarrow{(m_1,m_3)} m_3)$. This is well-defined by transitivity of $(\leqslant)$. The identity on $m \in \operatorname{Ob}\mathcal{C}_\mathcal{M}$ is given by $\mathrm{id}_m = (m,m)$, which is possible by reflexivity of $(\leqslant)$.

Usually, we write $\mathcal{M}$ also for the category $\mathcal{C}_\mathcal{M}$ by abuse of notation.

Example 7. Let $\mathcal{C}$ be a category. Consider the following diagram in $\mathcal{C}$.

$$\begin{tikzcd} & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & X \end{tikzcd}$$

This can be regarded as the functor from the poset $(\{\{1\},\{2\},\{1,2\}\},\subseteq)$ to $\mathcal{C}$ that maps $\{1\}$ to A, $\{2\}$ to B, $\{1,2\}$ to X, $(\{1\},\{1,2\})$ to α and $(\{2\},\{1,2\})$ to β.

$$\begin{tikzcd} & F_{\{1\}} \arrow[d, "F_{(\{1\},\{1,2\})}"] \\ F_{\{2\}} \arrow[r, "F_{(\{2\},\{1,2\})}"'] & F_{\{1,2\}} \end{tikzcd}$$

A limit of such a diagram completes it to a *pullback*. It can be illustrated by adding the respective pair of an object and a tuple of morphisms to the diagram, which then is commutative.

$$\begin{tikzcd} L \arrow[r, "\omega_{\{1\}}"] \arrow[d, "\omega_{\{2\}}"'] \arrow[dr, "\omega_{\{1,2\}}" description] & F_{\{1\}} \arrow[d, "F_{(\{1\},\{1,2\})}"] \\ F_{\{2\}} \arrow[r, "F_{(\{2\},\{1,2\})}"'] & F_{\{1,2\}} \end{tikzcd}$$

We observe that the morphism $\omega_{\{1,2\}}$ is redundant. It would be sufficient, if we only had the two morphisms $\omega_{\{1\}}$ and $\omega_{\{2\}}$ together with the requirement of a universal commutative quadrangle to have a pullback, so it looks as follows.

$$\begin{tikzcd} L \arrow[r, "\omega_{\{1\}}"] \arrow[d, "\omega_{\{2\}}"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & X \end{tikzcd}$$

This will be made precise in Lemma 11 below.

Definition 8. Suppose given a poset $(\mathcal{M},\leqslant)$. We define

$$\check{\mathcal{M}} := \{x \in \mathcal{M} : \{y \in \mathcal{M} : y \leqslant x\} = \{x\}\}$$

as the full subposet of *minimal elements* of $\mathcal{M}$.

Definition 9 (Reduced limit)**.** Suppose given a finite poset $(\mathcal{M},\leqslant)$ and a category $\mathcal{C}$. Suppose given a functor $F : \mathcal{M} \to \mathcal{C}$. Suppose given $L \in \operatorname{Ob}\mathcal{C}$ and a tuple of morphisms $(L \xrightarrow{\omega_x} F_x)_{x\in\check{\mathcal{M}}}$. Then the pair $(L,(\omega_x)_{x\in\check{\mathcal{M}}})$ is a *reduced limit* of F, if the following conditions ($\mathrm{i}_{\mathrm{red}}$, $\mathrm{ii}_{\mathrm{red}}$) hold.

($\mathrm{i}_{\mathrm{red}}$) For $x, y \in \check{M}$ and $z \in \mathcal{M}$ such that $x \leqslant z$ and $y \leqslant z$, we have $\omega_x F_{(x,z)} = \omega_y F_{(y,z)}$.

($\mathrm{ii}_{\mathrm{red}}$) Suppose given $(L', (L' \xrightarrow{\omega'_x} F_x)_{x \in \check{\mathcal{M}}})$ with the property that for $x, y \in \check{\mathcal{M}}$ and $z \in \mathcal{M}$ such that $x \leqslant z$ and $y \leqslant z$, we have $\omega'_x F_{(x,z)} = \omega'_y F_{(y,z)}$. Then there exists a unique morphism $L' \xrightarrow{\mu} L$ with $\mu\omega_x = \omega'_x$ for $x \in \check{\mathcal{M}}$. (Universal Property)

Remark 10. Suppose given a finite poset $(\mathcal{M}, \leqslant)$. For every $m \in \mathcal{M}$, we can choose $\check{m} \in \check{\mathcal{M}}$ such that $\check{m} \leqslant m$. This defines a map $\mathcal{M} \to \check{\mathcal{M}}, m \mapsto \check{m}$.

Proof. *Assume* there is an element $m \in \mathcal{M}$ for which there is no element $n \in \check{\mathcal{M}}$ with $n \leqslant m$. The element m is not minimal, hence there exists $m_1 \in \mathcal{M}$ with and $m_1 < m$. Also m_1 is not minimal. Again there exists an $m_2 \in \mathcal{M}$ with $m_2 < m_1$. Continuing this way we can now contruct an infinite chain of elements $\cdots < m_i < \cdots < m_2 < m_1 < m$. Then $\{m_i : i \in \mathbb{N}\}$ is an infinite subset of $\mathcal{M}$, which *contradicts* $\mathcal{M}$ being finite. $\square$

Lemma 11. Suppose given a finite poset $(\mathcal{M}, \leqslant)$ and a category $\mathcal{C}$. Suppose given a functor $F : \mathcal{M} \to \mathcal{C}$. We make use of the map $\mathcal{M} \to \check{\mathcal{M}}, m \mapsto \check{m}$ from Remark 10, so that $\check{m} \leqslant m$.

(1) Let $(L, (\omega_x)_{x \in \mathcal{M}})$ be a limit of F. Then $(L, (\omega_x)_{x \in \check{\mathcal{M}}})$ is a reduced limit.

(2) Let $(\tilde{L}, (\tilde{\omega}_x)_{x \in \check{\mathcal{M}}})$ be a reduced limit. Then $(\tilde{L}, (\tilde{\omega}_{\check{x}} F_{(\check{x},x)})_{x \in \mathcal{M}})$ is a limit.

Note that for $x' \in \check{\mathcal{M}}$ with $x' \leqslant x$, we have $\tilde{\omega}_{x'} F_{(x',x)} = \tilde{\omega}_{\check{x}} F_{(\check{x},x)}$ by Definition 9 ($\mathrm{i}_{\mathrm{red}}$). Hence (2) is independent of the choice made in Remark 10.

Proof.

Ad (1). We have to show conditions ($\mathrm{i}_{\mathrm{red}}$) and ($\mathrm{ii}_{\mathrm{red}}$) of Definition 9.

Ad ($\mathrm{i}_{\mathrm{red}}$). Let $x, y \in \check{\mathcal{M}}$ and $z \in \mathcal{M}$ with $x \leqslant z$ and $y \leqslant z$. We get $\omega_x F_{(x,z)} = \omega_z = \omega_y F_{(y,z)}$ by Definition 2 (i).

Ad ($\mathrm{ii}_{\mathrm{red}}$). Suppose given $(L', (L' \xrightarrow{\omega'_x} F_x)_{x \in \check{\mathcal{M}}})$ with the property that for $x, y \in \check{\mathcal{M}}$ and $z \in \mathcal{M}$ such that $x \leqslant z$ and $y \leqslant z$, we have $\omega'_x F_{(x,z)} = \omega'_y F_{(y,z)}$. Then $(L', (\omega'_{\check{x}} F_{(\check{x},x)})_{x \in \mathcal{M}})$ satisfies $\omega'_{\check{x}} F_{(\check{x},x)} F_{(x,y)} = \omega'_{\check{x}} F_{(\check{x},y)} = \omega'_{\check{y}} F_{(\check{y},y)}$ for $x, y \in \mathcal{M}$ such that $x \leqslant y$. Hence there exists a unique morphism $L' \xrightarrow{\mu} L$ with $\mu\omega_x = \omega'_{\check{x}} F_{(\check{x},x)}$ for $x \in \mathcal{M}$ by Definition 1 (ii). In particular, μ satisfies $\mu\omega_x = \omega'_{\check{x}} F_{(\check{x},x)} = \omega'_x$ for $x \in \check{\mathcal{M}}$, since $x = \check{x}$. Suppose given $\nu : L' \to L$ with $\nu\omega_x = \omega'_x$ for $x \in \check{\mathcal{M}}$. Then $\nu\omega_x = \nu\omega_{\check{x}} F_{(\check{x},x)} = \omega'_{\check{x}} F_{(\check{x},x)}$ for $x \in \mathcal{M}$. Hence $\nu = \mu$.

Ad (2). We have to show conditions (i) and (ii) of Definition 2.

Ad (i). Let $x, y \in \mathcal{M}$ with $x \leqslant y$. We get $\tilde{\omega}_{\check{x}} F_{(\check{x},x)} F_{(x,y)} = \tilde{\omega}_{\check{x}} F_{(\check{x},y)} = \tilde{\omega}_{\check{y}} F_{(\check{y},y)}$ by Definition 9 ($\mathrm{i}_{\mathrm{red}}$).

Ad (ii). Suppose given $(L', (L' \xrightarrow{\omega'_x} F_x)_{x \in \mathrm{Ob}\,\mathcal{D}})$ with the property that for $x, y \in \mathcal{M}$ such that $x \leqslant y$, we have $\omega'_x F_{(x,y)} = \omega'_y$. Let $x, y \in \check{\mathcal{M}}$ and $z \in \mathcal{M}$ such that $x \leqslant z$ and $y \leqslant z$. Then we get $\omega'_x F_{(x,z)} = \omega'_z = \omega'_y F_{(y,z)}$. Hence there exists a unique morphism $\mu : L' \to \tilde{L}$ with $\mu\tilde{\omega}_x = \omega'_x$ for $x \in \check{\mathcal{M}}$ by Definition 9 ($\mathrm{ii}_{\mathrm{red}}$). It follows that $\mu\tilde{\omega}_{\check{x}} F_{(\check{x},x)} = \omega'_{\check{x}} F_{(\check{x},x)} = \omega'_x$ for $x \in \mathcal{M}$. Suppose given $\nu : L' \to L$ with $\nu\tilde{\omega}_{\check{x}} F_{(\check{x},x)} = \omega'_x$ for $x \in \mathcal{M}$. Then, in particular, we have $\nu\tilde{\omega}_x = \omega'_x$ for $x \in \check{\mathcal{M}}$, since $x = \check{x}$. Hence $\nu = \mu$.

□

Definition 12. Let $\mathcal{C}$ be a category.

We say that $\mathcal{C}$ *has limits*, if for every category $\mathcal{D}$ and every functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ a limit of F exists.

We say that $\mathcal{C}$ *has finite limits*, if for every finite category $\mathcal{D}$ and every functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ a limit of F exists.

We say that $\mathcal{C}$ *has colimits*, if for every category $\mathcal{D}$ and every functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ a colimit of F exists.

We say that $\mathcal{C}$ *has finite colimits*, if for every finite category $\mathcal{D}$ and every functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ a colimit of F exists.

Remark 13.

(i) Suppose a category $\mathcal{C}$ has finite limits. Then $\mathcal{C}$ contains a terminal object.

(ii) Suppose a category $\mathcal{C}$ has finite colimits. Then $\mathcal{C}$ contains an initial object.

Proof.

Ad (i). Let $\mathcal{D}$ be the empty category, which means $\operatorname{Ob}\mathcal{D} = \emptyset$ and $\operatorname{Mor}\mathcal{D} = \emptyset$. Let $(T, ())$ be a limit of $F : \mathcal{D} \to \mathcal{C}$. Then T is a terminal object, since for $X \in \operatorname{Ob}\mathcal{C}$, the pair $(X, ())$ satisfies the condition in (ii) in Definition 2 and so there exists a unique morphism $\mu : X \to T$ satisfying an empty condition. Hence μ is unique. So T is a terminal object.

Ad (ii). This dual to (i). □

Example 14. The category of groups Grp has limits.

In fact, we can construct a limit of a functor $\mathcal{D} \xrightarrow{F} \mathrm{Grp}$ as follows.

Let $P := \prod_{x\in\operatorname{Ob}\mathcal{D}} F_x$ be the direct product the groups F_x, where $x \in \operatorname{Ob}\mathcal{D}$. Let

$$L := \{(g_x)_{x\in\operatorname{Ob}\mathcal{D}} \in P :\ (g_a)F_\alpha = g_b \text{ for } (a \xrightarrow{\alpha} b) \in \operatorname{Mor}\mathcal{D}\}\,.$$

Then L is a subgroup of P, as we shall see now.

The neutral element $(1_{F_x})_{x\in\operatorname{Ob}\mathcal{D}}$ is contained in L, because the image F_α of every morphism $(a \xrightarrow{\alpha} b)$ is a group morphism, which sends 1_{F_a} to 1_{F_b}.

Suppose given $(g_x)_{x\in\operatorname{Ob}\mathcal{D}}\,, (h_x)_{x\in\operatorname{Ob}\mathcal{D}} \in L$ and $(a \xrightarrow{\alpha} b) \in \operatorname{Mor}\mathcal{D}$. Then F_α is a group morphism, so that we get $((g_a)^- h_a)F_\alpha = ((g_a)F_\alpha)^-(h_a)F_\alpha = (g_b)^- h_b$. Hence $((g_x)_{x\in\operatorname{Ob}\mathcal{D}})^-(h_x)_{x\in\operatorname{Ob}\mathcal{D}} = ((g_x)^- h_x)_{x\in\operatorname{Ob}\mathcal{D}} \in L$.

Given $a \in \operatorname{Ob}\mathcal{D}$, let $\omega_a : L \to F_a, (g_x)_{x\in\operatorname{Ob}\mathcal{D}} \mapsto g_a$. It follows from the definition of L that $\omega_a F_\alpha = \omega_b$ for $(a \xrightarrow{\alpha} b) \in \operatorname{Mor}\mathcal{D}$.

It remains to prove the universal property. Suppose given a group L' and a tuple of group morphisms $(L' \xrightarrow{\omega'_d} F_d)_{d\in\operatorname{Ob}\mathcal{D}}$ that satisfy $\omega'_a F_\alpha = \omega'_b$ for $(a \xrightarrow{\alpha} b) \in \operatorname{Mor}\mathcal{D}$. We can define the map $\mu : L' \to L, l \mapsto ((l)\omega'_d)_{d\in\operatorname{Ob}\mathcal{D}}$. The image of μ is contained in L, because $((g)\omega'_a)F_\alpha = (g)\omega'_b$ for $g \in L'$ and $(a \xrightarrow{\alpha} b) \in \operatorname{Mor}\mathcal{D}$, so μ is well defined. Also μ is a group morphism. We have $(l)\mu\omega_d = (((l)\omega'_d)_{d\in\operatorname{Ob}\mathcal{D}})\omega_x = (l)\omega'_x$ for $l \in L'$ and $x \in \operatorname{Ob}\mathcal{D}$, for short $\mu\omega_x = \omega'_x$ for $x \in \operatorname{Ob}\mathcal{D}$. Suppose given a group morphism $\nu : L' \to L$ such that $\nu\omega_x = \omega'_x$. Then $(l)\nu\omega_x = (l)\omega'_x = (l)\mu\omega_x$ for $l \in L'$ and $x \in \operatorname{Ob}\mathcal{D}$, hence $(l)\nu = ((l)\omega'_x)_{x\in Ob\mathcal{D}} = (l)\mu$, hence $\nu = \mu$.

Example 15. Let $\mathcal{M}$ be a finite poset.

We can construct a reduced limit of a functor $\mathcal{M} \xrightarrow{F} \mathrm{Grp}$ as follows.

Let $P := \prod_{m \in \check{\mathcal{M}}} F_m$. Let

$L := \{(g_m)_{m \in \check{\mathcal{M}}} : \ (n_1)F_{(n_1,m)} = (n_2)F_{(n_2,m)}$ for $m \in \mathcal{M}$ and $n_1, n_2 \in \check{\mathcal{M}}$ such that $n_1 \leqslant m, n_2 \leqslant m\}$.

Then L is a subgroup of P, as we shall see now.

The neutral element $(1_{F_m})_{m \in \check{\mathcal{M}}}$ is contained in L, because a group morphism maps the neutral elements to neutral elements.

Suppose given $(g_m)_{m \in \check{\mathcal{M}}}$, $(h_m)_{m \in \check{\mathcal{M}}} \in L$, $m \in \mathcal{M}$ and $n_1, n_2 \in \check{\mathcal{M}}$ such that $n_1 \leqslant m, n_2 \leqslant m$. We then have

$((g_{n_1})^{-} h_{n_1})F_{(n_1,m)} = ((g_{n_1})F_{(n_1,m)})^{-}(h_{n_1})F_{(n_1,m)} = ((g_{n_2})F_{(n_2,m)})^{-}(h_{n_2})F_{(n_2,m)} = ((g_{n_2})^{-} h_{n_2})F_{(n_2,m)}$.

Hence $((g_m)_{m \in \check{\mathcal{M}}})^{-}(h_m)_{m \in \check{\mathcal{M}}} = ((g_m)^{-} h_m)_{m \in \check{\mathcal{M}}} \in L$.

Given $n \in \check{\mathcal{M}}$, let $\omega_n : L \to F_n, (g_m)_{m \in \check{\mathcal{M}}} \mapsto g_n$. It follows from the definition of L that $\omega_{n_1} F_{(n_1,m)} = \omega_{n_2} F_{(n_2,m)}$ for $m \in \mathcal{M}$ and $n_1, n_2 \in \check{\mathcal{M}}$ such that $n_1 \leqslant m, n_2 \leqslant m$.

It remains to prove the universal property.

Suppose given a group L' and a tuple of group morphisms $(L' \xrightarrow{\omega'_d} F_d)_{d \in \check{\mathcal{M}}}$ that satisfy $\omega'_{n_1} F_{(n_1,m)} = \omega'_{n_2} F_{(n_2,m)}$ for $m \in \mathcal{M}$ and $n_1, n_2 \in \check{\mathcal{M}}$ such that $n_1 \leqslant m, n_2 \leqslant m$. We can define the map $\mu : L' \to L, l \mapsto ((l)\omega'_d)_{d \in \check{\mathcal{M}}}$. The image of μ is contained in L, because $((g)\omega'_{n_1})F_{(n_1,m)} = ((g)\omega'_{n_2})F_{(n_2,m)}$ for $g \in L'$ and $m \in \mathcal{M}$ and $n_1, n_2 \in \check{\mathcal{M}}$ such that $n_1 \leqslant m, n_2 \leqslant m$, so μ is well defined. Also μ is a group morphism. We have $(l)\mu\omega_m = (((l)\omega'_d)_{d \in \mathrm{Ob}\,\mathcal{D}})\omega_m = (l)\omega'_m$ for $l \in L'$ and $m \in \check{\mathcal{M}}$, for short $\mu\omega_m = \omega'_m$ for $m \in \check{\mathcal{M}}$. Suppose given a group morphism $\nu : L' \to L$ such that $\nu\omega_m = \omega'_m$ for $m \in \check{\mathcal{M}}$. Then $(l)\nu\omega_m = (l)\omega'_m = (l)\mu\omega_m$ for $l \in L'$ and $m \in \check{\mathcal{M}}$, hence $(l)\nu = ((l)\omega'_m)_{m \in \check{\mathcal{M}}} = (l)\mu$, hence $\nu = \mu$.

Example 16. Suppose given a set M. Then M yields the discrete poset $(M, \leqslant)$ by defining

$$m_1 \leqslant m_2 \ :\Leftrightarrow \ m_1 = m_2 \ .$$

So we obtain a category $\tilde{M}$ with $\mathrm{Ob}\,\tilde{M} = M$ and $\mathrm{Mor}\,\tilde{M} = \{\mathrm{id}_m : m \in M\}$.

Suppose given a category $\mathcal{C}$ and a tuple $(X_m)_{m \in M}$ of objects in $\mathcal{C}$. We define the following functor.

$$\begin{array}{rccl} F: & \tilde{M} & \to & \mathcal{C} \\ & m & \mapsto & X_m \quad \text{for } m \in M \\ & \mathrm{id}_m & \mapsto & \mathrm{id}_{X_m} \quad \text{for } m \in M \end{array}$$

Then we choose a colimit of F and write it $\coprod\limits_{m \in M} X_m$, the *coproduct* of the tuple $(X_m)_{m \in M}$.

1.3 Free products of groups

Remark 17 (Representation of groups by generators and relations)**.** Suppose given a group G. Let $\tilde{G}$ be the set of all elements of G. Let $\mathrm{Free}(\tilde{G})$ be the free group generated by $\tilde{G}$. The map $\mathrm{id}_{\tilde{G}}$ extends to a unique group morphism $f : \mathrm{Free}(\tilde{G}) \to G$, $g \mapsto g$, which is surjective. Let K_G be the kernel of f. So we have the short exact sequence

$$K_G \hookrightarrow \mathrm{Free}(\tilde{G}) \xrightarrow{f} G\,.$$

Let $\tilde{K}_G$ be the set of elements in K_G. So the normal subgroup generated by $\tilde{K}_G$ is just K_G. Altogether, we have $G \simeq \mathrm{Free}(\tilde{G})/K_G = \langle \tilde{G} | \tilde{K}_G \rangle$.

Remark 18. Suppose given sets M, N and a map $M \xrightarrow{f} N$. Then we have the following induced group morphism.

$$\begin{array}{ccc} \mathrm{Free}(M) & \xrightarrow{\mathrm{Free}(f)} & \mathrm{Free}(N) \\ m & \longmapsto & (m)f \end{array}$$

Definition 19 (Free product)**.** Suppose given a set I. Suppose given a tuple of groups $(G_i)_{i\in I}$. Suppose given $i \in I$. Let $\tilde{G}_i$ be the set of elements of G_i. The map $\mathrm{id}_{\tilde{G}_i}$ extends to a unique group morphism $f_i : \mathrm{Free}(\tilde{G}_i) \to G_i$, $x \mapsto x$. Let K_{G_i} be the kernel of f_i. Let $\tilde{K}_{G_i}$ be the set of elements in K_{G_i}. Let $\tau_i : \tilde{G}_i \to \bigsqcup_{j\in I} \tilde{G}_j$, $x \mapsto (i, x)$ and $\mathrm{Free}(\tilde{G}_i) \xrightarrow{\sigma_i := \mathrm{Free}(\tau_i)} \mathrm{Free}(\bigsqcup_{j\in I} \tilde{G}_j)$.

Let $\tilde{K} := \bigcup_{i\in I} (\tilde{K}_{G_i})\sigma_i \subseteq \mathrm{Free}(\bigsqcup_{i\in I} \tilde{G}_i)$. Then we define

$$\mathop{*}_{i\in I} G_i := \langle \bigsqcup_{i\in I} \tilde{G}_i \mid \tilde{K} \rangle\,,$$

which we call the *free product* of the tuple $(G_i)_{i\in I}$. If $I = [1, k]$ for some $k \geqslant 1$, we also write $\mathop{*}_{i\in I} G_i = G_1 * \cdots * G_k\,.$

Lemma 20. Suppose given a set I. Suppose given a tuple of groups $(G_i)_{i\in I}$. Suppose given tuples of sets $(E_i)_{i\in I}$ and $(R_i)_{i\in I}$, where $R_i \subseteq \mathrm{Free}(E_i)$ for $i \in I$, and a tuple of isomorphisms $(\langle E_i | R_i \rangle \xrightarrow[\sim]{\alpha_i} G_i)_{i\in I}$. Cf. e.g. Remark 17.

Let $\tau_i : E_i \to \bigsqcup_{j\in I} E_j$, $x \mapsto (i, x)$ and $\mathrm{Free}(E_i) \xrightarrow{\sigma_i := \mathrm{Free}(\tau_i)} \mathrm{Free}(\bigsqcup_{j\in I} E_j)$.

Let $R := \bigcup_{i\in I} (R_i)\sigma_i \subseteq \mathrm{Free}(\bigsqcup_{i\in I} E_i)$

Let $P := \langle \bigsqcup_{i\in I} E_i | R \rangle$.

(1) We have the group morphism

$$\begin{array}{ccc} G_i & \xrightarrow{\iota_{i,P}} & P \\ (x \langle\!\langle R_i \rangle\!\rangle)\alpha_i & \mapsto & (x)\sigma_i \langle\!\langle R \rangle\!\rangle \end{array}$$

for $i \in I$, where $x \in \mathrm{Free}(E_i)$.

(2) Suppose given a group H. Suppose given a tuple of group morphisms $(G_i \xrightarrow{f_i} H)_{i\in I}$. Then there exists a unique group morphism $\mu : P \to H$ satisfying $\iota_{i,P}\mu = f_i$ for $i \in I$.

So $(P, (\iota_{i,P})_{i\in I})$ is a coproduct of $(G_i)_{i\in I}$, cf. Example 16.

(3) The pair $(\underset{i\in I}{*} G_i, (\iota_{i,\underset{i\in I}{*} G_i})_{i\in I})$ is a coproduct of $(G_i)_{i\in I}$; cf. Definition 19 and (1).

We have a group isomorphism $P \overset{\varphi}{\xrightarrow{\sim}} \underset{i\in I}{*} G_i$ satisfying $\iota_{i,P}\varphi = \iota_{i,\underset{i\in I}{*} G_i}$ for $i \in I$.

Proof.

We will write $\iota_i := \iota_{i,P}$ for $i \in I$.

Let $q_i : \mathrm{Free}(E_i) \to \mathrm{Free}(E_i)/\langle\!\langle R_i \rangle\!\rangle = \langle E_i | R_i \rangle$, $x \mapsto x \langle\!\langle R_i \rangle\!\rangle$ be the residue class morphism for $i \in I$.

Let $q : \mathrm{Free}(\bigsqcup_{i\in I} E_i) \to \mathrm{Free}(\bigsqcup_{i\in I} E_i)/\langle\!\langle R \rangle\!\rangle = P$, $x \mapsto x \langle\!\langle R \rangle\!\rangle$ be the residue class morphism.

Ad (1). Let $i \in I$. We have $(R_i)\sigma_i \subseteq R$. Hence R_i is contained in the kernel of $\sigma_i q$, and so is $\langle\!\langle R_i \rangle\!\rangle$. Therefore there exists a unique morphism $\gamma_i : \langle E_i | R_i \rangle \to P$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{Free}(E_i) \arrow[r, "\sigma_i q"] \arrow[d, two heads, "q_i"] & P \\ \langle E_i | R_i \rangle \arrow[ur, "\gamma_i"'] & \end{tikzcd}$$

We may define $\iota_i := \alpha_i^{-1}\gamma_i$, since for $x \in \mathrm{Free}(E_i)$ we get

$$((x \langle\!\langle R_i \rangle\!\rangle)\alpha_i)\alpha_i^{-1}\gamma_i = (x \langle\!\langle R_i \rangle\!\rangle)\gamma_i = (x)q_i\gamma_i = (x)\sigma_i q = (x)\sigma_i \langle\!\langle R \rangle\!\rangle \ .$$

Ad (2).

Uniqueness.

We can write every $x \in P = \mathrm{Free}(\bigsqcup_{i\in I} E_i)/\langle\!\langle R \rangle\!\rangle$ in the form

$$x \;=\; (i_1, e_{i_1})^{\varepsilon_1} \cdot (i_2, e_{i_2})^{\varepsilon_2} \cdots (i_n, e_{i_n})^{\varepsilon_n} \langle\!\langle R \rangle\!\rangle \ ,$$

where $n \geqslant 0$ and where $i_k \in I$, $e_{i_k} \in E_{i_k}$ and $\varepsilon_k \in \{-1, +1\}$ for $k \in [1, n]$. We can rewrite a factor in this product as $(i_k, e_{i_k})^{\varepsilon_k} \langle\!\langle R \rangle\!\rangle = (e_{i_k}^{\varepsilon_k} \langle\!\langle R_{i_k} \rangle\!\rangle)\alpha_{i_k}\iota_{i_k}$.

So we can write every $x \in P$ as a product $x = (g_{i_1})\iota_{i_1} \cdots (g_{i_n})\iota_{i_n}$, where $n \geqslant 0$ and $g_{i_j} \in G_{i_j}$ for $j \in [1, n]$. Thus

$$(x)\mu = ((g_{i_1})\iota_{i_1} \cdots (g_{i_n})\iota_{i_n})\mu = (g_{i_1})\iota_{i_1}\mu \cdots (g_{i_n})\iota_{i_n}\mu = (g_{i_1})f_{i_1} \cdots (g_{i_n})f_{i_n} \ .$$

Existence.

Let $\tilde{H}$ be the set of elements of H. The map $d : \bigsqcup_{i\in I} E_i \to \tilde{H}$, $(i, e_i) \mapsto (e_i \langle\!\langle R_i \rangle\!\rangle)\alpha_i f_i$ extends to a unique group morphism $\delta : \mathrm{Free}(\bigsqcup_{i\in I} E_i) \to H$. We have $\sigma_i\delta = q_i\alpha_i f_i$ for $i \in I$.

Suppose given $x \in R$. We can write $x = (r_i)\sigma_i$ for some $i \in I$ and some $r_i \in R_i$. Then $(x)\delta = (r_i)\sigma_i\delta = (r_i \langle\!\langle R_i \rangle\!\rangle)\alpha_i f_i = (1_{\langle E_i | R_i \rangle})\alpha_i f_i = 1_H$. Hence R is contained in the kernel of δ, and so is $\langle\!\langle R \rangle\!\rangle$. Therefore there exists a unique morphism $\mu : P \to H$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{Free}(\bigsqcup\limits_{i\in I} E_i) \arrow[r, "\delta"] \arrow[d, two heads, "q"] & P' \\ P \arrow[ur, "\mu"'] & \end{tikzcd}$$

For $i \in I$ and $x \in \mathrm{Free}(E_i)$ we get

$$((x\langle\!\langle R_i \rangle\!\rangle)\alpha_i)\iota_i\mu = ((x)\sigma_i\langle\!\langle R \rangle\!\rangle)\mu = ((x)\sigma_i)q\mu = ((x)\sigma_i)\delta = ((x\langle\!\langle R_i \rangle\!\rangle)\alpha_i)f_i\ .$$

So $\iota_i\mu = f_i$.

Ad (3). By (2) and by Remark 17, $(\mathop{*}\limits_{i\in I} G_i, (\iota_{i,\mathop{*}\limits_{i\in I} G_i})_{i\in I})$ is a coproduct of $(G_i)_{i\in I}$.

By the dual assertion to Remark 3, φ is an isomorphism. $\square$

Example 21. Suppose given a set I. Suppose given a free group G_i for $i \in I$. Then the free product $\mathop{*}\limits_{i\in I} G_i$ is a free group.

We use the notation of Lemma 20.

We may let $R_i := \emptyset$ for $i \in I$. Then also $R = \bigcup\limits_{i\in I} (R_i)\sigma_i = \emptyset$. So the group P is free. By Lemma 20.(3), P is isomorphic to $\mathop{*}\limits_{i\in I} G_i$.

Chapter 2

Simplicial Kernels

Definition 22 (Simplicial kernel)**.** Let $\mathcal{C}$ be a category and $X, Y \in \operatorname{Ob}\mathcal{C}$. Suppose given $n \geqslant 0$ and a tuple $(X \xrightarrow{f_i} Y)_{i\in[0,n]}$ of morphisms in $\mathcal{C}$. Suppose given $K \in \operatorname{Ob}\mathcal{C}$ and a tuple of morphisms $(K \xrightarrow{k_i} X)_{i\in[0,n+1]}$. Then $(K, (K \xrightarrow{k_i} X)_{i\in[0,n+1]})$ is a *simplicial kernel* or *n-equalizer* of $(f_i)_{i\in[0,n]}$ if the following conditions (i, ii) hold.

(i) We have $k_j f_i = k_i f_{j-1}$ for $i, j \in [0, n+1]$ such that $i < j$.

(ii) Suppose given $(Z \xrightarrow{h_i} X)_{i\in[0,n+1]}$ satisfying $h_j f_i = h_i f_{j-1}$ for $i, j \in [0, n+1]$ such that $i < j$, then there exists a unique morphism $\mu : Z \to K$ with $\mu k_i = h_i$ for $i \in [0, n+1]$.

Proposition 23. Let $\mathcal{C}$ be a category. Suppose that $\mathcal{C}$ has finite limits. Suppose given a natural number $n \geqslant 0$ and a tuple $(X \xrightarrow{f_i} Y)_{i\in[0,n]}$ of morphisms in $\mathcal{C}$. Then a simplicial kernel of $(f_i)_{i\in[0,n]}$ exists.

Proof. We construct a diagram, whose limit is a simplicial kernel. We define the following poset. Let

$$\mathcal{M}_1 := \{\{z\} : z \in [0, n+1]\}$$

and

$$\mathcal{M}_2 := \{\{i, j\} : i, j \in [0, n+1] \text{ such that } i < j\}$$

and

$$\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2 \,.$$

Then $(\mathcal{M}, \subseteq)$ is a poset as full subposet of $\mathbb{P}([0, n+1])$. We define the functor $F : \mathcal{M} \to \mathcal{C}$ by

$$F(m) := \begin{cases} X & \text{if } m \in \mathcal{M}_1 \\ Y & \text{if } m \in \mathcal{M}_2 \end{cases} \qquad \text{and} \qquad F_{(\{i\},\{i,j\})} := \begin{cases} f_{j-1} & \text{if } i < j \\ f_j & \text{if } i > j \end{cases} .$$

Let $(K, (K \xrightarrow{(k_i)} F_{\{i\}})_{i\in[0,n+1]})$ be a reduced limit of F, which exists by Lemma 11. We have to show that $(K, (k_i)_{i\in[0,n+1]})$ is a simplicial kernel of $(f_i)_{i\in[0,n]}$ by checking (i) and (ii) in Definition 22.

Ad (i). Let $0 \leqslant i < j \leqslant n+1$. Then $k_i f_{j-1} = k_i F_{(\{i\},\{i,j\})} = k_j F_{(\{j\},\{i,j\})} = k_j f_i \,.$

Ad (ii). Suppose given $(Z \xrightarrow{h_i} X)_{i \in [0,n+1]}$ satisfying $h_j f_i = h_i f_{j-1}$ for $0 \leqslant i < j \leqslant n+1$. Suppose given $i, j \in [0, n+1]$.

Case $i \neq j$. Without loss of generality we can suppose that $i < j$. So we get

$$h_i F_{(\{i\},\{i,j\})} = h_i f_{j-1} = h_j f_i = h_j F_{(\{j\},\{i,j\})} .$$

Case $i = j$. Then

$$h_i F_{(\{i\},\{i,j\})} = h_i F_{(\{i\},\{i\})} = h_i = h_j = h_j F_{(\{j\},\{j\})} = h_j F_{(\{j\},\{i,j\})} .$$

Hence $(h_i)_{i \in [0,n+1]}$ satisfies $(\mathrm{i}_{\mathrm{red}})$ in Definition 9. Thus there exists a unique morphism $Z \xrightarrow{\mu} K$ with $\mu k_i = h_i$ for $i \in [0, n]$. $\square$

Example 24. The poset defined in the proof of Proposition 23 takes the following shape on the first two cases.

$n = 0$:

$$\begin{tikzcd}
& \{0,1\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{0\} & & \{1\}
\end{tikzcd}$$

$n = 1$:

$$\begin{tikzcd}
& \{0,1\} \arrow[dl, dash] \arrow[dr, dash] & \{0,2\} \arrow[dll, dash] \arrow[drr, dash] & \{1,2\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{0\} & & \{1\} & & \{2\}
\end{tikzcd}$$

Chapter 3

Semisimplicial Resolutions

3.1 Resolving subcategories

Definition 25. Suppose given a category $\mathcal{C}$. Let $\mathcal{P}$ be a full subcategory of $\mathcal{C}$. Suppose given $(X \xrightarrow{\varphi} Y) \in \operatorname{Mor}\mathcal{C}$. We say that φ is $\mathcal{P}$-*epic* or a $\mathcal{P}$-*epimorphism* if for $P \in \operatorname{Ob}\mathcal{P}$ and $(P \xrightarrow{\alpha} Y) \in \operatorname{Mor}\mathcal{C}$, there exists a morphism $P \xrightarrow{\beta} X$ such that $\beta\varphi = \alpha$. That means the map ${}_{\mathcal{C}}(P, X) \xrightarrow{(-)\varphi} {}_{\mathcal{C}}(P, Y), \beta \mapsto \beta\varphi$ is a surjection for $P \in \operatorname{Ob}\mathcal{P}$.

$$\begin{tikzcd}
& P \arrow[dl, dashed, "\beta"'] \arrow[d, "\alpha"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

Remark 26. Suppose given a category $\mathcal{C}$ and a full subcategory $\mathcal{P}$ of $\mathcal{C}$.

- Let $X \in \operatorname{Ob}\mathcal{C}$. Then id_X is a $\mathcal{P}$-epimorphism, since for $P \in \operatorname{Ob}\mathcal{P}$ and $(P \xrightarrow{\alpha} X) \in \operatorname{Mor}\mathcal{C}$, we have $\alpha\,\mathrm{id}_X = \alpha$.

- Suppose given $\mathcal{P}$-epimorphisms $X \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\beta} Z$. Then the composite $\alpha\beta$ is a $\mathcal{P}$-epimorphism. In fact, for $P \in \operatorname{Ob}\mathcal{P}$ and $(P \xrightarrow{\gamma} Z) \in \operatorname{Mor}\mathcal{C}$, there exists $P \xrightarrow{\delta} Y$ such that $\delta\beta = \gamma$, and then there exists $P \xrightarrow{\varepsilon} X$ such that $\varepsilon\alpha = \delta$. Hence we get $\varepsilon\alpha\beta = \delta\beta = \gamma$.

$$\begin{tikzcd}
& & P \arrow[dll, dashed, "\varepsilon"'] \arrow[dl, dashed, "\delta"] \arrow[d, "\gamma"] \\
X \arrow[r, "\alpha"'] & Y \arrow[r, "\beta"'] & Z
\end{tikzcd}$$

Definition 27 (Resolving subcategory)**.** Let $\mathcal{C}$ be a category. Suppose given a full subcategory $\mathcal{P}$ of $\mathcal{C}$. We call $\mathcal{P}$ a *resolving subcategory* in $\mathcal{C}$ if for every $X \in \operatorname{Ob}\mathcal{C}$, there exists an object $P \in \operatorname{Ob}\mathcal{P}$ and a morphism $\varphi : P \to X$ that is $\mathcal{P}$-epic.

Remark 28. The class of objects contained in a resolving subcategory is referred to as a projective class by Tierney and Vogel [**1**].

Example 29. Suppose given a category $\mathcal{C}$. Then $\mathcal{C}$ is a resolving subcategory in $\mathcal{C}$. In fact, for every object $X \in \operatorname{Ob}\mathcal{C}$, we have $X \xrightarrow{\mathrm{id}} X$ as a $\mathcal{P}$-epimorphism.

Example 30. Suppose a category $\mathcal{C}$ has an initial object I. Then the full subcategory $\mathcal{I}$ with $\operatorname{Ob}\mathcal{I} = \{I\}$ is a resolving subcategory in $\mathcal{C}$, since for a given object $X \in \operatorname{Ob}\mathcal{C}$ we have exactly one morphism $I \xrightarrow{\varphi} X$. Then φ is $\mathcal{I}$-epic, because for $\alpha \in {}_{\mathcal{C}}(I, X) = \{\varphi\}$ there exists id_I, which satisfies $\mathrm{id}_I\,\alpha = \alpha = \varphi$.

Remark 31. Consider the category of groups Grp. We have a full subcategory FreeGrp of free groups. Let $\varphi \in \operatorname{Mor}\mathrm{Grp}$. We have

$$\varphi \text{ is FreeGrp-epic} \iff \varphi \text{ is surjective}\,.$$

Proof.

Ad $\Rightarrow$. Suppose that $\varphi : X \to Y$ is FreeGrp-epic. Let $\tilde{Y}$ be the set of all elements of Y. Let $\mathrm{Free}(\tilde{Y})$ be the free group generated by $\tilde{Y}$. The map $\mathrm{id}_{\tilde{Y}}$ extends to a unique group morphism $\alpha : \mathrm{Free}(\tilde{Y}) \to Y$. Since φ is FreeGrp-epic there exists a group morphism $\beta : \mathrm{Free}(\tilde{Y}) \to X$ such that $\beta\varphi = \alpha$. Since α is surjective φ must be surjective.

$$\begin{tikzcd} & \mathrm{Free}(\tilde{Y}) \arrow[dl, dashed, "\beta"'] \arrow[d, "\alpha"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

Ad $\Leftarrow$. Suppose that $\varphi : X \to Y$ is surjective. Let $\tilde{X}$ be set of elements in X and $\tilde{Y}$ be the set of elements in Y. Choose a map $\tilde{X} \xleftarrow{c} \tilde{Y}$ such that $c\varphi = \mathrm{id}_{\tilde{Y}}$. Let M be a set and $\mathrm{Free}(M)$ be the free group generated by M. Suppose given a qroup morphism $\alpha : \mathrm{Free}(M) \to Y$. This group morphism retracts to a map $a : M \to \tilde{Y}$. We define a map $(M \xrightarrow{b} \tilde{X}) := (M \xrightarrow{a} \tilde{Y}) \cdot (\tilde{Y} \xrightarrow{c} \tilde{X})$. Then b extends to a unique group morphism $\beta : \mathrm{Free}(M) \to X$. We have to show that $\beta\varphi = \alpha$. Let $m \in M$. Then $((m)\beta)\varphi = ((m)b)\varphi = (((m)a)c)\varphi = (m)a = (m)\alpha$.

$$\begin{tikzcd} & M \arrow[dl, dashed, "b"'] \arrow[d, "a"] \\ \tilde{X} & \tilde{Y} \arrow[l, "c"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Free}(M) \arrow[dl, dashed, "\beta"'] \arrow[d, "\alpha"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

$\square$

Remark 32. The category FreeGrp is a resolving subcategory in Grp.

Proof. Suppose given a group G. Let $\tilde{G}$ be the set of all elements of G. Let $\mathrm{Free}(\tilde{G})$ be the free group generated by $\tilde{G}$. The map $\mathrm{id}_{\tilde{G}}$ extends to a unique group morphism $\gamma : \mathrm{Free}(\tilde{G}) \to G$, which is surjective. $\square$

Definition 33 (Model category)**.** Suppose given a category $\mathcal{C}$.

Suppose given $\mathrm{Fib}(\mathcal{C}), \mathrm{Cof}(\mathcal{C}), \mathrm{Wke}(\mathcal{C}) \subseteq \operatorname{Mor}\mathcal{C}$.

We call $\mathcal{C}$ a *model category* if the following properties (i, ii, iii, iv, v, vi, vii, viii) hold.

(i) The category $\mathcal{C}$ has finite limits and finite colimits.

(ii) Suppose given composable morphisms $X \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\beta} Z$. Let $\gamma := \alpha\beta$.

If $\alpha, \beta \in \mathrm{Wke}(\mathcal{C})$, then $\gamma \in \mathrm{Wke}(\mathcal{C})$.

If $\beta, \gamma \in \mathrm{Wke}(\mathcal{C})$, then $\alpha \in \mathrm{Wke}(\mathcal{C})$.

If $\alpha, \gamma \in \mathrm{Wke}(\mathcal{C})$, then $\beta \in \mathrm{Wke}(\mathcal{C})$.

(iii) Suppose given $\alpha \in \mathrm{Mor}\,\mathcal{C}$. If α is an isomorphism, then $\alpha \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{Cof}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$.

(iv) Suppose given morphisms $X \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\beta} Z$ in $\mathrm{Fib}(\mathcal{C})$. Then $\alpha\beta \in \mathrm{Fib}(\mathcal{C})$.

Suppose given morphisms $X' \xrightarrow{\alpha'} Y'$ and $Y' \xrightarrow{\beta'} Z'$ in $\mathrm{Cof}(\mathcal{C})$. Then $\alpha'\beta' \in \mathrm{Cof}(\mathcal{C})$.

(v) Suppose given a pullback in $\mathcal{C}$

$$\begin{tikzcd} Y \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & X \end{tikzcd} .$$

If $\alpha \in \mathrm{Fib}(\mathcal{C})$, then $\delta \in \mathrm{Fib}(\mathcal{C})$.

If $\alpha \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$, then $\delta \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$.

(vi) Suppose given a pushout in $\mathcal{C}$

$$\begin{tikzcd} Y & A \arrow[l, "\gamma"'] \\ B \arrow[u, "\delta"] & X \arrow[l, "\beta"] \arrow[u, "\alpha"'] \end{tikzcd} .$$

If $\alpha \in \mathrm{Cof}(\mathcal{C})$, then $\delta \in \mathrm{Cof}(\mathcal{C})$.

If $\alpha \in \mathrm{Cof}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$, then $\delta \in \mathrm{Cof}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$.

(vii) Suppose given a commutative quadrangle

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & C \arrow[d, "\beta"] \\ B \arrow[r, "\delta"'] & D \end{tikzcd}$$

in $\mathcal{C}$. Suppose that $\gamma \in \mathrm{Cof}(\mathcal{C})$ and $\beta \in \mathrm{Fib}(\mathcal{C})$. Suppose that $\{\gamma, \beta\} \cap \mathrm{Wke}(\mathcal{C}) \neq \emptyset$. Then there exists $B \xrightarrow{\mu} C \in \mathrm{Mor}\,\mathcal{C}$ such that $\gamma\mu = \alpha$ and $\mu\beta = \delta$.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & C \arrow[d, "\beta"] \\ B \arrow[r, "\delta"'] \arrow[ur, dashed, "\mu"] & D \end{tikzcd}$$

(viii) For each $\alpha \in \mathrm{Mor}\,\mathcal{C}$, there exist $\gamma \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$ and $\beta \in \mathrm{Cof}(\mathcal{C})$ such that $\alpha = \beta\gamma$.

For each $\alpha \in \mathrm{Mor}\,\mathcal{C}$, there exist $\gamma' \in \mathrm{Fib}(\mathcal{C})$ and $\beta' \in \mathrm{Cof}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$ such that $\alpha = \beta'\gamma'$.

Morphisms in $\mathrm{Fib}(\mathcal{C})$ are called *fibrations*, morphisms in $\mathrm{Cof}(\mathcal{C})$ are called *cofibrations* and morphisms in $\mathrm{Wke}(\mathcal{C})$ are called *weak equivalences*.

Remark 34. Suppose given a model category $\mathcal{C}$.

Choose an initial object I in $\mathcal{C}$, cf. Remark 13.

Let $\mathcal{C}_{\mathrm{cof}}$ be the full subcategory of $\mathcal{C}$ of *cofibrant objects*, i.e.

$$\mathrm{Ob}\,\mathcal{C}_{\mathrm{cof}} \;:=\; \{X \in \mathrm{Ob}\,\mathcal{C} : (I \to X) \in \mathrm{Cof}(\mathcal{C})\}\ .$$

(1) Morphisms in $\mathrm{Fib}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$ are $\mathcal{C}_{\mathrm{cof}}$-epic.

(2) The subcategory $\mathcal{C}_{\mathrm{cof}}$ is a resolving subcategory of $\mathcal{C}$.

Proof. Ad (1). Let $(X \xrightarrow{\varphi} Y) \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$. Let $Z \in \mathrm{Ob}\,\mathcal{C}_{\mathrm{cof}}$ and $Z \xrightarrow{\alpha} Y \in \mathrm{Mor}\,\mathcal{C}$. Let $I \xrightarrow{\gamma} Z$ be the unique morphism form I to Z and let $I \xrightarrow{\delta} X$ be the unique morphism form I to X. We have a commutative quadrangle

$$\begin{tikzcd} I \arrow[r, "\delta"] \arrow[d, "\gamma"'] & X \arrow[d, "\varphi"] \\ Z \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

and $(I \xrightarrow{\gamma} Z) \in \mathrm{Cof}(\mathcal{C})$. From Definition 33 (vii) it follows that there exists a morphism $Z \xrightarrow{\mu} X$ with $\mu\varphi = \alpha$. Hence φ is $\mathcal{C}_{\mathrm{cof}}$-epic.

Ad(2). Suppose given $Y \in \mathrm{Ob}\,\mathcal{C}$. We have a unique morphism $I \xrightarrow{\alpha} Y$. By Definition 33 (viii), we have $\alpha = \beta\gamma$, where $(X \xrightarrow{\gamma} Y) \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})$ and $(I \xrightarrow{\beta} X) \in \mathrm{Cof}(\mathcal{C})$. So $X \in \mathrm{Ob}\,\mathcal{C}_{\mathrm{cof}}$ and γ is a $\mathcal{C}_{\mathrm{cof}}$-epimorphism using (1). □

Remark 35. Suppose given a model category $\mathcal{C}$.

Choose an initial object I in $\mathcal{C}$, cf. Remark 13.

Let $\mathcal{C}_{\mathrm{cof,ac}}$ be the full subcategory of $\mathcal{C}$ of *acyclic cofibrant objects*, i.e.

$$\mathrm{Ob}\,\mathcal{C}_{\mathrm{cof,ac}} \;:=\; \{X \in \mathrm{Ob}\,\mathcal{C} : (I \to X) \in \mathrm{Cof}(\mathcal{C}) \cap \mathrm{Wke}(\mathcal{C})\}\ .$$

(1) Morphisms in $\mathrm{Fib}(\mathcal{C})$ are $\mathcal{C}_{\mathrm{cof,ac}}$-epic.

(2) The subcategory $\mathcal{C}_{\mathrm{cof,ac}}$ is a resolving subcategory of $\mathcal{C}$.

Proof. Ad (1). Let $(X \xrightarrow{\varphi} Y) \in \mathrm{Fib}(\mathcal{C})$. Let $Z \in \mathrm{Ob}\,\mathcal{C}_{\mathrm{cof,ac}}$ and $Z \xrightarrow{\alpha} Y \in \mathrm{Mor}\,\mathcal{C}$. Let $I \xrightarrow{\gamma} Z$ be the unique morphism form I to Z and let $I \xrightarrow{\delta} X$ be the unique morphism form I to X. We have a commutative quadrangle

$$\begin{tikzcd} I \arrow[r, "\delta"] \arrow[d, "\gamma"'] & X \arrow[d, "\varphi"] \\ Z \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

and $(I \xrightarrow{\gamma} Z) \in \operatorname{Cof}(\mathcal{C}) \cap \operatorname{Wke}(\mathcal{C})$. From Definition 33 (vii) it follows that there exists a morphism $Z \xrightarrow{\mu} X$ with $\mu\varphi = \alpha$. Hence φ is $\mathcal{C}_{\text{cof,ac}}$-epic.

Ad(2). Suppose given $Y \in \operatorname{Ob}\mathcal{C}$. We have a unique morphism $I \xrightarrow{\alpha} Y$. By Definition 33 (viii), we have $\alpha = \beta\gamma$, where $(X \xrightarrow{\gamma} Y) \in \operatorname{Fib}(\mathcal{C})$ and $(I \xrightarrow{\beta} X) \in \operatorname{Cof}(\mathcal{C}) \cap \operatorname{Wke}(\mathcal{C})$. So $X \in \operatorname{Ob}\mathcal{C}_{\text{cof,ac}}$ and γ is a $\mathcal{C}_{\text{cof,ac}}$-epimorphism using (1). $\square$

3.2 Construction of semisimplicial resolutions

Definition 36 (Semisimplicial resolution)**.** Suppose given a category $\mathcal{C}$. Suppose given $X \in \operatorname{Ob}\mathcal{C}$. Suppose that $\mathcal{C}$ has finite limits. Suppose given a resolving subcategory $\mathcal{P}$. We can now choose $P_1 \in \operatorname{Ob}\mathcal{P}$ and a $\mathcal{P}$-epimorphism $P_0 \xrightarrow{f_0} X$. We get the following diagram.

$$P_0 \xrightarrow{f_0} X$$

We can now choose a simplicial kernel $(K_1, (k_0^1, k_1^1))$ of f_0, which exists by Proposition 23. So we get

$$\begin{tikzcd}
 & P_0 \arrow[r, "f_0"] & X \\
K_1 \arrow[ur, bend left, "k_0^1"] \arrow[ur, bend right, "k_1^1"'] & &
\end{tikzcd}$$

Again we can choose $P_1 \in \operatorname{Ob}\mathcal{P}$ and a $\mathcal{P}$-epimorphism $P_1 \xrightarrow{f_1} K_1$. Let $d_0^1 := f_1 k_0^1$ and $d_1^1 := f_1 k_1^1$. We get the following diagram.

$$\begin{tikzcd}
P_1 \arrow[rr, bend left, "d_0^1"] \arrow[rr, bend right, "d_1^1"'] \arrow[dr, "f_1"'] & & P_0 \arrow[r, "f_0"] & X \\
 & K_1 \arrow[ur, bend left, "k_0^1"] \arrow[ur, bend right, "k_1^1"'] & &
\end{tikzcd}$$

We can now choose a simplicial kernel $(K_2, (k_0^2, k_1^2, k_2^2))$ of (d_0^1, d_1^1) by Proposition 23. We choose $P_2 \in \operatorname{Ob}\mathcal{P}$ and a $\mathcal{P}$-epimorphism $P_2 \xrightarrow{f_2} K_2$. Let $d_0^2 := f_2 k_0^2$ and $d_1^2 := f_2 k_1^2$ and $d_2^2 := f_2 k_2^2$. Our diagram then looks like this.

$$\begin{tikzcd}
P_2 \arrow[rr, bend left, "d_0^2"] \arrow[rr, "d_1^2"] \arrow[rr, bend right, "d_2^2"'] \arrow[dr, "f_2"'] & & P_1 \arrow[rr, bend left, "d_0^1"] \arrow[rr, bend right, "d_1^1"'] \arrow[dr, "f_1"'] & & P_0 \arrow[r, "f_0"] & X \\
 & K_2 \arrow[ur, bend left, "k_0^2"] \arrow[ur, "k_1^2"] \arrow[ur, bend right, "k_2^2"'] & & K_1 \arrow[ur, bend left, "k_0^1"] \arrow[ur, bend right, "k_1^1"'] & &
\end{tikzcd}$$

This process can be continued. Let $n \geqslant 3$. Suppose we already have constructed

$$(P_{n-1} \xrightarrow{d_i^{n-1}} P_{n-2})_{i \in [0,n-1]} \,.$$

We choose a simplicial kernel $(K_n, (k_i^n)_{i\in[0,n]})$ of $(d_i^{n-1})_{i\in[0,n-1]}$, cf. Proposition 23. We choose $P_n \in \operatorname{Ob}\mathcal{P}$ and a $\mathcal{P}$-epimorphism $P_n \xrightarrow{f_{n-1}} K_n$. Let $d_i^n := f_n k_i^n$ for $i \in [0,n]$.

Write $d_0^0 := f_0$ and $P_{-1} := X$. Note that P_{-1} is *not* contained in $\operatorname{Ob}\mathcal{P}$ in general. The diagram $((P_n)_{n\geqslant 0}, ((d_i^n)_{i\in[0,n]})_{n\geqslant 1})$ resulting from this construction is called a *semisimplicial resolution* of X. The diagram $((P_n)_{n\geqslant -1}, ((d_i^n)_{i\in[0,n]})_{n\geqslant 0})$ resulting from this construction is called an *augmented semisimplicial resolution* of X.

Remark 37. The maps d_i^n in Definition 36 satisfy $d_j^n d_i^{n-1} = d_i^n d_{j-1}^{n-1}$ for $n \geqslant 2$ and $0 \leqslant i < j \leqslant n$.

Proof. By construction we have $d_j^n d_i^{n-1} = f_n k_j^n d_i^{n-1} \overset{\text{Definition 22 (i)}}{=} f_n k_i^n d_{j-1}^{n-1} = d_i^n d_{j-1}^{n-1}$. $\square$

Example 38. Let $\mathcal{C}$ be a category and $\mathcal{P}$ a resolving subcategory in $\mathcal{C}$. Let $P \in \operatorname{Ob}\mathcal{P}$.

Suppose given $n \geqslant 0$. We *claim* that a simplicial kernel of the tuple

$$(X \xrightarrow{f_i} Y)_{i\in[0,n]} = (P \xrightarrow{\mathrm{id}_P} P)_{i\in[0,n]}$$

is given by the tuple

$$(K \xrightarrow{k_i} X)_{i\in[0,n+1]} = (P \xrightarrow{\mathrm{id}_P} P)_{i\in[0,n+1]}\,.$$

We have $k_j f_i = \mathrm{id}_P\,\mathrm{id}_P = k_i f_{j-1}$ for $i, j \in [0, n+1]$ such that $i < j$. Hence (i) in Defintion 22 holds.

Suppose given a tuple $(K' \xrightarrow{k_i'} X)_{i\in[0,n+1]}$ such that $k_j' f_i = k_i' f_{j-1}$ for $i, j \in [0, n+1]$, $i < j$. We then have $k_i'\,\mathrm{id}_P = k_{i+1}'\,\mathrm{id}_P$ for $i \in [0,n]$ and inductively we derive $k_i' = k_j' =: k'$ for $i, j \in [0, n+1]$. Then $K' \xrightarrow{k'} P$ is the unique morphism satisfying $k' k_i = k'\,\mathrm{id}_P = k_i'$ for $i \in [0, n+1]$. Hence (i) in Defintion 22 holds.

The morphism id_P is $\mathcal{P}$-epic. So it follows that a semisimplicial resolution of P is given by $((P_n)_{n\geqslant 0}, ((d_i^n)_{i\in[0,n]})_{n\geqslant 1}) = ((P)_{n\geqslant 0}, ((\mathrm{id}_P)_{i\in[0,n]})_{n\geqslant 1})\,.$

$$\begin{tikzcd}
\cdots\ P \arrow[r, bend left, "\mathrm{id}_P"] \arrow[r, "\mathrm{id}_P"] \arrow[r, bend right, "\mathrm{id}_P"'] & P \arrow[r, shift left, "\mathrm{id}_P"] \arrow[r, shift right, "\mathrm{id}_P"'] & P \arrow[r, "\mathrm{id}_P"] & P
\end{tikzcd}$$

Example 39. Let $n \geqslant 2$. We want to semisimplicially resolve the cyclic group $\mathbb{Z}/n\mathbb{Z}$ by free groups. We temporarly write groups additive. We choose the quotient group morphism $q : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}, z \mapsto z + n\mathbb{Z}$, which is surjective and hence FreeGrp-epic by Remark 31. A simplicial kernel of $\mathbb{Z} \xrightarrow{q} \mathbb{Z}/n\mathbb{Z}$ is given by a reduced limit of the following diagram.

$$\begin{tikzcd}
& \mathbb{Z}/n\mathbb{Z} & \\
\mathbb{Z} \arrow[ur, "q"] & & \mathbb{Z} \arrow[ul, "q"']
\end{tikzcd}$$

A reduced limit of such a diagram is a pullback, which we obtain by considering the subgroup $U := \{(x,y) \in \mathbb{Z}\times\mathbb{Z} : x - y \in n\mathbb{Z}\}$ of $\mathbb{Z}\times\mathbb{Z}$. Let $p_1 : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}, (x,y) \mapsto x$ and $p_2 : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}, (x,y) \mapsto y$ be the projections. Then the pullback is given by the following diagram.

$$\begin{tikzcd}
& \mathbb{Z} \arrow[dr, "q"] & \\
U \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & \mathbb{Z}/n\mathbb{Z} \\
& \mathbb{Z} \arrow[ur, "q"'] &
\end{tikzcd}$$

We have an injective group morphism $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, (x, y) \mapsto (x, x + ny)$, whose image is U. Hence U is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and we can write the pullback the following way.

$$\begin{tikzcd}
& \mathbb{Z} \arrow[dr, "q"] & \\
\mathbb{Z} \times \mathbb{Z} \arrow[ur, "(x,y)\mapsto x"] \arrow[dr, "(x,y)\mapsto x+ny"'] & & \mathbb{Z}/n\mathbb{Z} \\
& \mathbb{Z} \arrow[ur, "q"'] &
\end{tikzcd}$$

Up to this step we are only dealing with abelian groups. But for $\mathbb{Z} \times \mathbb{Z}$ is not cyclic, we have to involve the nonabelian free group generated by two elements to continue, since we need a surjective map from a free group to $\mathbb{Z} \times \mathbb{Z}$.

Let $F_2 := \mathrm{Free}(\{a, b\})$ be the free group generated by the elements a and b. We define the group morphism $h : F_2 \to \mathbb{Z} \times \mathbb{Z}, a \mapsto (1, 0), b \mapsto (0, 1)$, which is FreeGrp-epic. Let $f_1^1 := hgp_1$ and $f_2^1 := hgp_2$. We have

$$\begin{array}{rccc} f_1^1 : & F_2 & \to & \mathbb{Z} \\ & a & \mapsto & 1 \\ & b & \mapsto & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rccc} f_2^1 : & F_2 & \to & \mathbb{Z} \\ & a & \mapsto & 1 \\ & b & \mapsto & n \end{array} .$$

We obtain the following diagram.

$$\begin{tikzcd}
F_2 \arrow[r, shift left, "f_1^1"] \arrow[r, shift right, "f_2^1"'] & \mathbb{Z} \arrow[r, "q"] & \mathbb{Z}/n\mathbb{Z}
\end{tikzcd}$$

A simplicial kernel of $F_2 \overset{f_1^1}{\underset{f_2^1}{\rightrightarrows}} \mathbb{Z}$ is a reduced limit of the following diagram.

$$\begin{tikzcd}
& & \mathbb{Z} & & \\
& F_2 \arrow[ur, "f_1^1"] \arrow[dl, "f_2^1"'] & & F_2 \arrow[ul, "f_2^1"'] \arrow[dr, "f_1^1"] & \\
\mathbb{Z} & & F_2 \arrow[ll, "f_1^1"] \arrow[rr, "f_2^1"'] & & \mathbb{Z}
\end{tikzcd}$$

The reduced limit can be constructed following Example 15.

So let $L := \{(x, y, z) \in F_2 \times F_2 \times F_2 : (x)f_1^1 = (z)f_2^1, (y)f_1^1 = (x)f_2^1, (z)f_1^1 = (y)f_2^1\}$.

Let

$$\begin{array}{ccc} L & \xrightarrow{t_1} & F_2 \\ (a,b,c) & \mapsto & a \end{array}, \qquad \begin{array}{ccc} L & \xrightarrow{t_2} & F_2 \\ (a,b,c) & \mapsto & b \end{array}, \qquad \begin{array}{ccc} L & \xrightarrow{t_3} & F_2 \\ (a,b,c) & \mapsto & c \end{array}$$

be the projections, restricted to L.

Let $\mathrm{Free}(L) \xrightarrow{j} L$ be a surjective group morphism, where $\mathrm{Free}(L)$ is the free group on the underlying set of L.

Let $f_1^2 := jt_1$, $f_2^2 := jt_2$ and $f_3^2 := jt_3$. So we observe that from the first three steps in resolving $\mathbb{Z}/n\mathbb{Z}$, we obtain the following diagram.

$$\begin{tikzcd} \mathrm{Free}(L) \arrow[r, bend left, "f_1^2"] \arrow[r, "f_2^2"] \arrow[r, bend right, "f_3^2"'] & F_2 \arrow[r, bend left, "f_1^1"] \arrow[r, bend right, "f_2^1"'] & \mathbb{Z} \arrow[r, "q"] & \mathbb{Z}/n\mathbb{Z} \end{tikzcd}$$

Chapter 4

Semisimplicial and Simplicial Objects

Definition 40.

(i) We define the category Δ as subcategory of Set as follows. Let

$$\mathrm{Ob}\,\Delta := \{[i] : i \in \mathbb{Z}_{\geqslant 0}\}$$

For $[a], [b] \in \mathrm{Ob}\,\Delta$, let

$$_{\Delta}([a],[b]) := \{f \in {}_{\mathrm{Set}}([a],[b]) : x \geqslant y \Rightarrow xf \geqslant yf \text{ for } x, y \in [a]\}$$

be the set of all monotone maps from $[a]$ to $[b]$.

(ii) We define the category Δ_{inj} as subcategory of Δ by setting

$$\mathrm{Ob}\,\Delta_{\mathrm{inj}} := \mathrm{Ob}\,\Delta$$

and

$$_{\Delta_{\mathrm{inj}}}([a],[b]) := \{f \in {}_{\Delta}([a],[b]) : f \text{ is injective }\} \ .$$

(iii) We define the category Δ_{surj} as subcategory of Δ by setting

$$\mathrm{Ob}\,\Delta_{\mathrm{surj}} := \mathrm{Ob}\,\Delta$$

and

$$_{\Delta_{\mathrm{surj}}}([a],[b]) := \{f \in {}_{\Delta}([a],[b]) : f \text{ is surjective }\} \ .$$

We often abreviate $\mathrm{surj} := \mathrm{Mor}\,\Delta_{\mathrm{surj}}$.

(iv) For $n \geqslant 0$ and $i \in [n+1]$, let

$$\begin{array}{rcl} \partial_i^{n+1} \ : \ [n] & \longrightarrow & [n+1] \\ x & \longmapsto & \begin{cases} x & \text{for } x < i \\ x+1 & \text{for } x \geqslant i \, . \end{cases} \end{array}$$

Remark 41. The maps ∂_i^{n+1} in Definition 40 (iv) satisfy $\partial_i^{n+1}\partial_j^{n+2} = \partial_{j-1}^{n+1}\partial_i^{n+2}$ for $0 \leqslant i < j \leqslant n+2$ and $n \geqslant 0$.

Proof. Suppose given $x \in [n]$.

Case $x \in [0, i-1]$. Then

$$(x)\partial_i^{n+1}\partial_j^{n+2} \overset{(x<i)}{=} (x)\partial_j^{n+2} \overset{(x<i<j)}{=} x \qquad \text{and}$$
$$(x)\partial_{j-1}^{n+1}\partial_i^{n+2} \overset{(x\leqslant i-1<j-1)}{=} (x)\partial_i^{n+2} \overset{(x<i)}{=} x\,.$$

Case $x \in [i, j-2]$. Then

$$(x)\partial_i^{n+1}\partial_j^{n+2} \overset{(x\geqslant i)}{=} (x+1)\partial_j^{n+2} \overset{(x+1<j)}{=} x+1 \quad \text{and}$$
$$(x)\partial_{j-1}^{n+1}\partial_i^{n+2} \overset{(x<j-1)}{=} (x)\partial_i^{n+2} \overset{(x\geqslant i)}{=} x+1\,.$$

Case $x \in [j-1, n]$. Then

$$(x)\partial_i^{n+1}\partial_j^{n+2} \overset{(x\geqslant j-1\geqslant i)}{=} (x+1)\partial_j^{n+2} \overset{(x+1\geqslant j)}{=} x+2 \qquad \text{and}$$
$$(x)\partial_{j-1}^{n+1}\partial_i^{n+2} \overset{(x\geqslant j-1)}{=} (x+1)\partial_i^{n+2} \overset{(x+1\geqslant j>i)}{=} x+2\,.$$

So in every case we find $(x)\partial_i^{n+1}\partial_j^{n+2} = (x)\partial_{j-1}^{n+1}\partial_i^{n+2}$. Hence $\partial_i^{n+1}\partial_j^{n+2}$ and $\partial_{j-1}^{n+1}\partial_i^{n+2}$ are equal. □

Lemma 42. Suppose given $m \geqslant n \geqslant 0$ and an injective monotone map $f : [n] \to [m]$. Let $k_1 < \ldots < k_{m-n}$ denote the elements of $[m]$ that do not appear in the image of f. So $[m] = ([n])f \,\dot\cup\, \{k_1, \ldots, k_{m-n}\}$.

Then $f = \partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$; cf. Definition 40 (iv).

Note that in the case $m = n$, we set id_m to be the empty composite.

Proof. The map f is injective and monotone. The map $\partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$ is injective and monotone as a composite of injective monotone maps. Hence it is sufficient to show that their images are equal, i.e. $([n])f \overset{!}{=} ([n])\partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$.

We have to show that $\{k_1, \ldots, k_{m-n}\} \overset{!}{=} [m] \setminus ([n])\partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$.

We only have to prove $\{k_1, \ldots, k_{m-n}\} \overset{!}{\subseteq} [m] \setminus ([n])\partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$, because $\partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$ is injective.

For $i \in [1, m-n]$ consider k_i. We *claim* that k_i is not contained in the image of $\partial_{k_i}^{n+i} \cdots \partial_{k_j}^{n+j}$ for $j \in [i, m-n]$. This we do by induction over j. For the base clause, we get from Definition 40 (iv) that k_i is not contained in the image of $\partial_{k_i}^{n+i}$. Now suppose that we already have proved the claim for a certain $j \geqslant i$ and now we want to prove it for $j+1$. We observe that $\partial_{k_{j+1}}^{n+j+1}$ maps only the element k_i to k_i since $k_{j+1} > k_i$, cf. Definition 40 (iv). By induction hypothesis k_i does not appear in the image of $\partial_{k_i}^{n+i} \cdots \partial_{k_j}^{n+j}$, hence it does not appear in the image of $\partial_{k_i}^{n+i} \cdots \partial_{k_{j+1}}^{n+j+1}$. This proves the *claim*.

Hence k_i does not appear in the image of $\partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$ for $i \in [1, m-n]$.

So $\{k_1, \ldots, k_{m-n}\} \subseteq [m] \setminus ([n])\partial_{k_1}^{n+1} \cdots \partial_{k_{m-n}}^{m}$ as required. □

Proposition 43. Let $\mathcal{C}$ be a category. Suppose given $U_n \in \operatorname{Ob}\mathcal{C}$ for $n \in \mathbb{Z}_{\geqslant 0}$. Suppose for $n \geqslant 1$ and $i \in [0,n]$ we have morphisms $U_n \xrightarrow{d_i^n} U_{n-1}$. Suppose we have

$$d_j^n d_i^{n-1} = d_i^n d_{j-1}^{n-1} \quad \text{for } n \geqslant 2 \text{ and } 0 \leqslant i < j \leqslant n\,. \qquad (*)$$

Then there exists exactly one functor $X : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$ such that $X[n] = U_n$ for $n \in \mathbb{Z}_{\geqslant 0}$ and $X((\partial_i^n)^{\mathrm{op}}) = d_i^n$ for $n \in \mathbb{Z}_{\geqslant 1}$ and $i \in [0,n]$.

Proof.

Uniqueness. Suppose given a functor $\tilde{X} : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$ with $\tilde{X}[n] = U_n$ for $n \in \mathbb{Z}_{\geqslant 0}$ and $\tilde{X}(\partial_i^n)^{\mathrm{op}} = d_i^n$ for $n \in \mathbb{Z}_{\geqslant 1}$ and $i \in [0,n]$. Then we have $\tilde{X}[n] = U_n = X[n]$ for $[n] \in \operatorname{Ob}\Delta_{\mathrm{inj}}^{\mathrm{op}}$.

For $([n] \xrightarrow{f} [m]) \in \operatorname{Mor}\Delta_{\mathrm{inj}}$ we have $f = {}^{[n]}\prod^{[m]}_{i \in \lceil 1, m-n \rceil} \partial_{k_i}^{n+i}$ with $k_1 < \ldots < k_{m-n}$ by Lemma 42 and thus $f^{\mathrm{op}} = {}^{[m]}\prod^{[n]}_{i \in \lfloor m-n, 1 \rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}}$.

Therefore $\tilde{X} f^{\mathrm{op}} = \tilde{X}\big({}^{[m]}\prod^{[n]}_{i \in \lfloor m-n, 1 \rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}}\big) = {}^{U_m}\prod^{U_n}_{i \in \lfloor m-n, 1 \rfloor} \tilde{X}((\partial_{k_i}^{n+i})^{\mathrm{op}}) = {}^{U_m}\prod^{U_n}_{i \in \lfloor m-n, 1 \rfloor} d_{k_i}^{n+i} = {}^{U_m}\prod^{U_n}_{i \in \lfloor m-n, 1 \rfloor} (X(\partial_{k_i}^{n+i})^{\mathrm{op}}) = X\big({}^{[m]}\prod^{[n]}_{i \in \lfloor m-n, 1 \rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}}\big) = X f^{\mathrm{op}}$.

Hence $\tilde{X} = X$.

Existence. For a morphism $(f^{\mathrm{op}} : [m] \to [n]) \in \operatorname{Mor}\Delta_{\mathrm{inj}}^{\mathrm{op}}$ we have $f^{\mathrm{op}} = {}^{[m]}\prod^{[n]}_{i \in \lfloor m-n, 1 \rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}}$ with $0 \leqslant k_1 < \ldots < k_{m-n} \leqslant m$ and $\{k_1, \ldots, k_{m-n}\} = [m] \setminus ([n])f$, cf. Lemma 42. We want to define a functor

$$\begin{array}{rcll} X : \quad \Delta_{\mathrm{inj}}^{\mathrm{op}} & \longrightarrow & \mathcal{C} & \\ [n] & \longmapsto & U_n & \text{for} \quad [n] \in \operatorname{Ob}\Delta_{\mathrm{inj}}^{\mathrm{op}} \\ f^{\mathrm{op}} = {}^{[m]}\prod^{[n]}_{i \in \lfloor m-n, 1 \rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}} & \longmapsto & {}^{U_m}\prod^{U_n}_{i \in \lfloor m-n, 1 \rfloor} d_{k_i}^{n+i} & \text{for} \quad (f^{\mathrm{op}} : [m] \to [n]) \in \operatorname{Mor}\Delta_{\mathrm{inj}}^{\mathrm{op}}\,. \end{array}$$

We have to show that X is a functor. We remark that X maps identities to identities.

So suppose given composable morphisms $(f^{\mathrm{op}} : [m] \to [n]), (g^{\mathrm{op}} : [n] \to [p]) \in \operatorname{Mor}\Delta_{\mathrm{inj}}^{\mathrm{op}}$. We have to show that $X(f^{\mathrm{op}} \cdot g^{\mathrm{op}}) \stackrel{!}{=} X f^{\mathrm{op}} \cdot X g^{\mathrm{op}}$.

Let $l_1 < \ldots < l_{n-p}$ denote the elements of $[n]$ that do not appear in the image of g.

Let $k_1 < \ldots < k_{m-n}$ denote the elements of $[m]$ that do not appear in the image of f.

We have $g = {}^{[p]}\prod^{[n]}_{i \in \lceil 1, n-p \rceil} \partial_{l_i}^{p+i}$ and $f = {}^{[n]}\prod^{[m]}_{i \in \lceil 1, m-n \rceil} \partial_{k_i}^{n+i}$, cf. Lemma 42.

Hence we get $g^{\mathrm{op}} = {}^{[n]}\prod^{[p]}_{i \in \lfloor n-p, 1 \rfloor} (\partial_{l_i}^{p+i})^{\mathrm{op}}$ and $f^{\mathrm{op}} = {}^{[m]}\prod^{[n]}_{i \in \lfloor m-n, 1 \rfloor} (\partial_{k_i}^{n+i})^{\mathrm{op}}$.

Let $h_1 < \ldots < h_{m-p}$ denote the elements of $[m]$ that do not appear in the image of gf.

We have $gf = {}^{[p]}\prod\limits_{i\in\lceil 1\,,m-p\rceil}^{[m]} \partial_{h_i}^{p+i}$, cf. Lemma 42.

Hence we get $f^{\mathrm{op}} g^{\mathrm{op}} = {}^{[m]}\prod\limits_{i\in\lfloor m-p\,,1\rfloor}^{[p]} (\partial_{h_i}^{p+i})^{\mathrm{op}}$.

So $Xf^{\mathrm{op}} \cdot Xg^{\mathrm{op}} = ({}^{U_m}\prod\limits_{i\in\lfloor m-n\,,1\rfloor}^{U_n} d_{k_i}^{n+i}) \cdot ({}^{U_n}\prod\limits_{i\in\lfloor n-p\,,1\rfloor}^{U_p} d_{l_i}^{p+i})$ and $X(f^{\mathrm{op}} g^{\mathrm{op}}) = {}^{U_m}\prod\limits_{i\in\lfloor m-p\,,1\rfloor}^{U_p} d_{h_i}^{p+i}$.

So we have to show that $({}^{U_m}\prod\limits_{i\in\lfloor m-n\,,1\rfloor}^{U_n} d_{k_i}^{n+i}) \cdot ({}^{U_n}\prod\limits_{i\in\lfloor n-p\,,1\rfloor}^{U_p} d_{l_i}^{p+i}) \overset{!}{=} {}^{U_m}\prod\limits_{i\in\lfloor m-p\,,1\rfloor}^{U_p} d_{h_i}^{p+i}$.

This we do by induction over $n-p$.

Base of the induction. Suppose that $n-p=0$.

In this case we have $g = \mathrm{id}_{[n]}$ and therefore $k_i = h_i$ for $i \in [0, m-n]$. Hence
$({}^{U_m}\prod\limits_{i\in\lfloor m-n\,,1\rfloor}^{U_n} d_{k_i}^{n+i}) \cdot \mathrm{id}_{U_n} = {}^{U_m}\prod\limits_{i\in\lfloor m-p\,,1\rfloor}^{U_p} d_{h_i}^{p+i}$.

Induction step. Suppose that $n-p \geqslant 1$.

Let $\tilde{g} := {}^{[p+1]}\prod\limits_{i\in\lceil 2\,,n-p\rceil}^{[n]} \partial_{l_i}^{p+i}$, so we have $g = \partial_{l_1}^{p+1} \cdot \tilde{g}$. Let $\tilde{h}_2 < \ldots < \tilde{h}_{m-p}$ denote the elements of $[m]$ that do not appear in the image of $\tilde{g}f$. So $\tilde{g}f = {}^{[p+1]}\prod\limits_{i\in\lceil 2\,,m-p\rceil}^{[m]} \partial_{\tilde{h}_i}^{[p+i]}$.

Hence $gf = \partial_{l_1}^{p+1} \cdot ({}^{[p+1]}\prod\limits_{i\in\lceil 2\,,m-p\rceil}^{[m]} \partial_{\tilde{h}_i}^{[p+i]})$

By induction hypothesis we have

$$({}^{U_m}\prod_{i\in\lfloor m-n\,,1\rfloor}^{U_n} d_{k_i}^{n+i}) \cdot ({}^{U_n}\prod_{i\in\lfloor n-p\,,2\rfloor}^{U_{p+1}} d_{l_i}^{p+i}) = {}^{U_m}\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d_{\tilde{h}_i}^{p+i} .$$

So we have to show that

$$({}^{U_m}\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d_{\tilde{h}_i}^{p+i}) \cdot d_{l_1}^{p+1} \overset{!}{=} {}^{U_m}\prod_{i\in\lfloor m-p\,,1\rfloor}^{U_p} d_{h_i}^{p+i} .$$

Let $A := \{i \in [1, m-p-1] : \tilde{h}_{i+1} - i + 1 \leqslant l_1\} \cup \{0\}$. Let $a := \max A \in [0, m-p-1]$. So $A = [0, a]$.

We *claim* to have

$$({}^{U_m}\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d_{\tilde{h}_i}^{p+i}) \cdot d_{l_1}^{p+1} \overset{!}{=} ({}^{U_m}\prod_{i\in\lfloor m-p\,,b+2\rfloor}^{U_{p+b+1}} d_{\tilde{h}_i}^{p+i}) \cdot d_{l_1+b}^{p+b+1} \cdot ({}^{U_{p+b}}\prod_{i\in\lfloor b\,,1\rfloor}^{U_p} d_{\tilde{h}_{i+1}}^{p+i})$$

for $b \in [0, a]$. We prove this by induction over b.

Base of the induction. Suppose that $b=0$.

Then

$$\Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+1}_{l_1} = \Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,0+2\rfloor}^{U_{p+0+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+0+1}_{l_1+0} = \Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,b+2\rfloor}^{U_{p+b+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+b+1}_{l_1+b}\cdot\Big(\,{}^{U_{p+b}}\!\!\prod_{i\in\lfloor b\,,1\rfloor}^{U_p} d^{p+i}_{\tilde h_{i+1}}\Big)\,,$$

since ${}^{U_{p+b}}\!\prod_{i\in\lfloor b\,,1\rfloor}^{U_p} d^{p+i}_{\tilde h_{i+1}} = \mathrm{id}_{U_p}$.

Step of the induction. Suppose that $b \geqslant 1$.

By induction hypothesis we have

$$\Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+1}_{l_1} = \Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,b+1\rfloor}^{U_{p+b}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+b}_{l_1+b-1}\cdot\Big(\,{}^{U_{p+b-1}}\!\!\prod_{i\in\lfloor b-1\,,1\rfloor}^{U_p} d^{p+i}_{\tilde h_{i+1}}\Big)\,.$$

Since $b \leqslant a$, we have $b \in A\setminus\{0\}$, hence $\tilde h_{b+1} \leqslant l_1+b-1$.

Hence we can use $(*)$ to get $d^{p+b+1}_{\tilde h_{b+1}} d^{p+b}_{l_1+b-1} = d^{p+b+1}_{l_1+b} d^{p+b}_{\tilde h_{b+1}}$.

Hence we have

$$\begin{aligned}
\Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+1}_{l_1} &= \Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,b+2\rfloor}^{U_{p+b+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+b+1}_{\tilde h_{b+1}}\cdot d^{p+b}_{l_1+b-1}\cdot\Big(\,{}^{U_{p+b-1}}\!\!\prod_{i\in\lfloor b-1\,,1\rfloor}^{U_p} d^{p+i}_{\tilde h_{i+1}}\Big)\\
&= \Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,b+2\rfloor}^{U_{p+b+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+b+1}_{l_1+b}\cdot d^{p+b}_{\tilde h_{b+1}}\cdot\Big(\,{}^{U_{p+b-1}}\!\!\prod_{i\in\lfloor b-1\,,1\rfloor}^{U_p} d^{p+i}_{\tilde h_{i+1}}\Big)\\
&= \Big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,b+2\rfloor}^{U_{p+b+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+b+1}_{l_1+b}\cdot\Big(\,{}^{U_{p+b}}\!\!\prod_{i\in\lfloor b\,,1\rfloor}^{U_p} d^{p+i}_{\tilde h_{i+1}}\Big)\,.
\end{aligned}$$

This proves the *claim*.

Hence we have $\Big(\,{}^{U_m}\!\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+1}_{l_1} = \Big(\,{}^{U_m}\!\prod_{i\in\lfloor m-p\,,a+2\rfloor}^{U_{a+b+1}} d^{p+i}_{\tilde h_i}\Big)\cdot d^{p+a+1}_{l_1+a}\cdot\Big(\,{}^{U_{p+b}}\!\prod_{i\in\lfloor a\,,1\rfloor}^{U_p} d^{p+i}_{\tilde h_{i+1}}\Big)$.

We *claim* that

$$\partial^{p+1}_{l_1}\cdot\Big(\,{}^{[p+1]}\!\!\prod_{i\in\lceil 2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde h_i}\Big) \overset{!}{=} \Big(\,{}^{[p]}\!\!\prod_{i\in\lceil 1\,,b\rceil}^{[p+b]} \partial^{p+i}_{\tilde h_{i+1}}\Big)\cdot\partial^{p+b+1}_{l_1+b}\cdot\Big(\,{}^{[p+b+1]}\!\!\prod_{i\in\lceil b+2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde h_i}\Big)$$

for $b \in [0, a]$. We prove this by induction over b.

Base of the induction. Suppose that $b = 0$.

Then

$$\partial^{p+1}_{l_1}\cdot\Big(\,{}^{[p+1]}\!\!\prod_{i\in\lceil 2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde h_i}\Big) = \partial^{p+0+1}_{l_1+0}\cdot\Big(\,{}^{[p+0+1]}\!\!\prod_{i\in\lceil 0+2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde h_i}\Big) = \Big(\,{}^{[p]}\!\!\prod_{i\in\lceil 1\,,b\rceil}^{[p+b]} \partial^{p+i}_{\tilde h_{i+1}}\Big)\cdot\partial^{p+b+1}_{l_1+b}\cdot\Big(\,{}^{[p+b+1]}\!\!\prod_{i\in\lceil b+2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde h_i}\Big)\,,$$

since $\Big(\,{}^{[p]}\!\prod_{i\in\lceil 1\,,b\rceil}^{[p+b]} \partial^{p+i}_{\tilde h_{i+1}}\Big) = \mathrm{id}_{[p]}$.

Step of the induction. Suppose that $b \geqslant 1$.

By induction hypothesis we have

$$\partial^{p+1}_{l_1}\cdot\Big(\,{}^{[p+1]}\!\!\prod_{i\in\lceil 2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde h_i}\Big) = \Big(\,{}^{[p]}\!\!\prod_{i\in\lceil 1\,,b-1\rceil}^{[p+b-1]} \partial^{p+i}_{\tilde h_{i+1}}\Big)\cdot\partial^{p+b}_{l_1+b-1}\cdot\Big(\,{}^{[p+b]}\!\!\prod_{i\in\lceil b+1\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde h_i}\Big)\,.$$

Since $b \leqslant a$, we have $b \in A \setminus \{0\}$, hence $\tilde{h}_{b+1} \leqslant l_1 + b - 1$.

Hence we can use Remark 41 to get $\partial^{p+b}_{l_1+b-1}\partial^{p+b+1}_{\tilde{h}_{b+1}} = \partial^{p+b}_{\tilde{h}_{b+1}}\partial^{p+b+1}_{l_1+b}$.

Hence we have

$$\begin{aligned}
\partial^{p+1}_{l_1} \cdot \big(\,{}^{[p+1]}\!\!\prod_{i\in\lceil 2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde{h}_i}\big) &= \big(\,{}^{[p]}\!\!\prod_{i\in\lceil 1\,,b-1\rceil}^{[p+b-1]} \partial^{p+i}_{\tilde{h}_{i+1}}\big)\cdot \partial^{p+b}_{l_1+b-1}\cdot \partial^{p+b+1}_{\tilde{h}_{b+1}} \cdot \big(\,{}^{[p+b+1]}\!\!\prod_{i\in\lceil b+2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde{h}_i}\big)\\
&= \big(\,{}^{[p]}\!\!\prod_{i\in\lceil 1\,,b-1\rceil}^{[p+b-1]} \partial^{p+i}_{\tilde{h}_{i+1}}\big)\cdot \partial^{p+b}_{\tilde{h}_{b+1}}\cdot \partial^{p+b+1}_{l_1+b} \cdot \big(\,{}^{[p+b+1]}\!\!\prod_{i\in\lceil b+2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde{h}_i}\big)\\
&= \big(\,{}^{[p]}\!\!\prod_{i\in\lceil 1\,,b\rceil}^{[p+b]} \partial^{p+i}_{\tilde{h}_{i+1}}\big)\cdot \partial^{p+b+1}_{l_1+b} \cdot \big(\,{}^{[p+b+1]}\!\!\prod_{i\in\lceil b+2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde{h}_i}\big)\,.
\end{aligned}$$

This proves the *claim*.

Hence we have $gf = \partial^{p+1}_{l_1} \cdot \big(\,{}^{[p+1]}\prod_{i\in\lceil 2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde{h}_i}\big) = \big(\,{}^{[p]}\prod_{i\in\lceil 1\,,a\rceil}^{[p+a]} \partial^{p+i}_{\tilde{h}_{i+1}}\big)\cdot \partial^{p+a+1}_{l_1+a} \cdot \big(\,{}^{[p+a+1]}\prod_{i\in\lceil a+2\,,m-p\rceil}^{[m]} \partial^{p+i}_{\tilde{h}_i}\big)\,.$

We have $\tilde{h}_2 < \ldots < \tilde{h}_{a+1} < l_1 + a < \tilde{h}_{a+2} < \ldots < \tilde{h}_{m-p}$, which do not appear in the image of gf.

So we find that $\tilde{h}_i = h_i$ for $i \in [a+2, m-p]$ and $l_1 + a = h_{a+1}$ and $\tilde{h}_{i+1} = h_i$ for $i \in [1, a]$.

Therefore we have

$$\big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,a+2\rfloor}^{U_{p+a+1}} d^{p+i}_{\tilde{h}_i}\big)\cdot d^{p+a+1}_{l_1+a}\cdot \big(\,{}^{U_{p+a}}\!\!\prod_{i\in\lfloor a\,,1\rfloor}^{U_p} d^{p+i}_{\tilde{h}_{i+1}}\big) = {}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,1\rfloor}^{U_p} d^{p+i}_{h_i}$$

and hence

$$\big(\,{}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,2\rfloor}^{U_{p+1}} d^{p+i}_{\tilde{h}_i}\big)\cdot d^{p+1}_{l_1} = {}^{U_m}\!\!\prod_{i\in\lfloor m-p\,,1\rfloor}^{U_p} d^{p+i}_{h_i}\,,$$

which we had to show. □

Definition 44 (Simplicial Object)**.** Suppose given a category $\mathcal{C}$.

A *simplicial object* in $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$.

A *semisimplicial object* in $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{C}$.

We write

$$X[n] =: X_n$$

and

$$X\partial^{n+1}_i =: d^{X,n+1}_i \text{ or short } X\partial^{n+1}_i =: d_i\,.$$

For a morphism $f : [m] \to [n]$ we often write $Xf^{\mathrm{op}} =: X_f$.

Remark 45. Historically, simplicial sets were first called "complete semi-simplicial complexes" by Eilenberg and Zilber [**2,** p. 508]. Later, this has been abbreviated to "simplicial sets" and generalized to "simplicial objects" by May [**3,** Def. 2.1]. We allow ourselves to reuse the word "semisimplicial" with a different meaning.

Definition 46.

(i) Suppose given simplicial objects $X : \Delta^{\mathrm{op}} \to \mathcal{C}$, $Y : \Delta^{\mathrm{op}} \to \mathcal{C}$ and $Z : \Delta^{\mathrm{op}} \to \mathcal{C}$. Suppose given a tuple $\alpha = (\alpha_n)_{n\geqslant 0}$ of morphisms $(X_n \xrightarrow{\alpha_n} Y_n) \in \operatorname{Mor}\mathcal{C}$. We call α a *simplicial morphism* from X to Y if

$$X_f \cdot \alpha_m = \alpha_n \cdot Y_f \qquad \text{for } ([m] \xrightarrow{f^{\mathrm{op}}} [n]) \in \operatorname{Mor}\Delta^{\mathrm{op}}\,.$$

This means that a simplicial morphism is a transformation between simplicial objects. Composition of simplicial morphisms $X \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\beta} Z$ is given by $\alpha \cdot \beta := (\alpha_n \cdot \beta_n)_{n \geqslant 0}$ and the identity morphism is given by $\mathrm{id}_X = (\mathrm{id}_{X_n})_{n \geqslant 0}$.

(ii) Suppose given semisimplicial objects $X : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$, $Y : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$ and $Z : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$. Suppose given a tuple $\alpha = (\alpha_n)_{n \geqslant 0}$ of morphisms $(X_n \xrightarrow{\alpha_n} Y_n) \in \operatorname{Mor} \mathcal{C}$. We call α a *semisimplicial morphism* from X to Y if

$$X_f \cdot \alpha_m = \alpha_n \cdot Y_f \qquad \text{for } ([m] \xrightarrow{f^{\mathrm{op}}} [n]) \in \operatorname{Mor} \Delta_{\mathrm{inj}}^{\mathrm{op}} .$$

This means that a semisimplicial morphism is a transformation between semisimplicial objects. Composition of semisimplicial morphisms $X \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\beta} Z$ is given by $\alpha \cdot \beta := (\alpha_n \cdot \beta_n)_{n \geqslant 0}$ and the identity morphism is given by $\mathrm{id}_X = (\mathrm{id}_{X_n})_{n \geqslant 0}$.

Definition 47. Suppose given a category $\mathcal{C}$.

(1) Let $\operatorname{Simp}(\mathcal{C}) := \mathcal{C}^{\Delta^{\mathrm{op}}}$ be the category of *simplicial objects* and *simplicial morphisms* in $\mathcal{C}$.

(2) Let $\operatorname{SemiSimp}(\mathcal{C}) := \mathcal{C}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}}$ be the category of *semisimplicial objects* and *semisimplicial morphisms* in $\mathcal{C}$.

Chapter 5

From Semisimplicial to Simplicial Objects

5.1 The forgetful functor from $\mathrm{Simp}(\mathcal{C})$ to $\mathrm{SemiSimp}(\mathcal{C})$

The category $\Delta^{\mathrm{op}}_{\mathrm{inj}}$ is a subcategory of Δ^{op} with $\mathrm{Ob}\,\Delta^{\mathrm{op}} = \mathrm{Ob}\,\Delta^{\mathrm{op}}_{\mathrm{inj}} = \{[n] : n \in \mathbb{Z}_{\geqslant 0}\}$.

Let

$$\begin{array}{rccc} I : & \Delta^{\mathrm{op}}_{\mathrm{inj}} & \longrightarrow & \Delta^{\mathrm{op}} \\ & ([m] \xrightarrow{f} [n]) & \longmapsto & ([m] \xrightarrow{f} [n]) \end{array}$$

be the inclusion functor.

Suppose given a category $\mathcal{C}$ and a simplicial object $X : \Delta \to \mathcal{C}$ in $\mathcal{C}$. Then $X \circ I : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{C}$ is a semisimplicial object in $\mathcal{C}$. Hence every simplicial object gives us a semisimplicial object by restriction along I.

Definition 48. Suppose given a category $\mathcal{C}$. We define a *forgetful* functor

$$\begin{array}{rcclll} \mathcal{V}_{\mathcal{C}} : & \mathrm{Simp}(\mathcal{C}) & \longrightarrow & \mathrm{SemiSimp}(\mathcal{C}) & & \\ & X & \longmapsto & X \circ I & \text{for} & X \in \mathrm{Ob}\,\mathrm{Simp}(\mathcal{C}) \\ & \alpha & \longmapsto & \alpha & \text{for} & \alpha \in \mathrm{Mor}\,\mathrm{Simp}(\mathcal{C})\,. \end{array}$$

Our goal in this chapter is to find a left adjoint functor to $\mathcal{V}_{\mathcal{C}}$, provided $\mathcal{C}$ has finite coproducts.

5.2 The adjoint in case $\mathcal{C} = \mathrm{Set}$

Remark 49. Suppose given $(g : [n] \to [m]) \in \mathrm{Mor}\,\Delta$. Then there exist unique monotone maps $\bar{g} : [n] \to [l]$ and $\dot{g} : [l] \to [m]$ such that $\bar{g}\dot{g} = g$, such that $\bar{g}$ is surjective and such that $\dot{g}$ is injective.

$$\begin{tikzcd} & {[l]} \arrow[dr, hook, "\dot{g}"] & \\ {[n]} \arrow[ur, two heads, "\bar{g}"] \arrow[rr, "g"'] & & {[m]} \end{tikzcd}$$

In particular, we have $l = |\operatorname{Im}(g)| - 1$. Alternatively, we write $\dot{g} = g^\bullet$.

Lemma 50 (Construction of a simplicial set out of a semisimplicial set)**.** Suppose given a semisimplicial set

$$X : \Delta_{\text{inj}}^{\text{op}} \to \text{Set} .$$

We want to construct a simplicial set, i.e. a functor

$$\tilde{X} : \Delta^{\text{op}} \to \text{Set} .$$

(i) (Construction of $\tilde{X}_n$). Let $n \geqslant 0$. We define

$$\tilde{X}_n := \{(x, f) : \text{there exists } k \in [0, n] \text{ such that } x \in X_k,\ (f : [n] \to [k]) \in \operatorname{Mor} \Delta \text{ is surjective}\} .$$

(ii) (Construction of $\tilde{X}_g$).

Let $([m] \xrightarrow{g^{\text{op}}} [n]) \in \operatorname{Mor} \Delta^{\text{op}}$.

We define

$$\begin{array}{rrcl} \tilde{X}_g : & \tilde{X}_m & \longrightarrow & \tilde{X}_n \\ & (x, ([m] \xrightarrow{f} [k])) & \longmapsto & ((x)X_{(gf)^\bullet}\, , \overline{gf})\, . \end{array}$$

Note that for $[n] \xrightarrow{g} [m] \xrightarrow{f} [k]$ with $l = |\operatorname{Im}(gf)| - 1$, we have the injective monotone map $(gf)^\bullet : [l] \to [k]$, so that $X_{(gf)^\bullet}$ is defined. We have $(x)X_{(gf)^\bullet} \in X_l$ and $\overline{gf} : [n] \to [l]$ is surjective. Hence $\tilde{X}_g$ is welldefined.

The assignment

$$\begin{array}{rcll} [n] & \mapsto & \tilde{X}_n & \text{for } [n] \in \operatorname{Ob} \Delta^{\text{op}} \\ g^{\text{op}} & \mapsto & \tilde{X}_g & \text{for } g^{\text{op}} \in \operatorname{Mor} \Delta^{\text{op}} \end{array}$$

defines a simplicial set $\tilde{X}$.

Proof. We have to show that $\tilde{X} : \Delta^{\text{op}} \to \text{Set}$ is a functor.

Note that if $([n] \xrightarrow{f} [k]) \in \operatorname{Mor} \Delta$ is surjective, we have $\dot{f} = \mathrm{id}_{[k]}$ and $\bar{f} = f$. Hence we get $((x, f))\tilde{X}_{\mathrm{id}_{[n]}} = ((x)X_{\mathrm{id}_{[k]}}, f) = ((x)\,\mathrm{id}_{X_k}, f) = (x, f)$ for $n \geqslant 0$ and $(x, ([n] \xrightarrow{f} [k])) \in \tilde{X}_n$.

Hence $\tilde{X}_{\mathrm{id}_{[n]}} = \mathrm{id}_{\tilde{X}_n}$ for $[n] \in \operatorname{Ob} \Delta^{\text{op}}$.

Now suppose given $([p] \xrightarrow{h} [n]), ([n] \xrightarrow{g} [m]) \in \operatorname{Mor} \Delta$. We have to show $\tilde{X}_{hg} = \tilde{X}_g \tilde{X}_h$.

So let $(x, ([m] \xrightarrow{f} [k])) \in \tilde{X}_m$.

Then

$((x, f))\tilde{X}_g \cdot \tilde{X}_h = (((x)X_{(gf)^\bullet}, \overline{gf}))\tilde{X}_h = (((x)X_{(gf)^\bullet})X_{(h \cdot \overline{gf})^\bullet}, \overline{h \cdot \overline{gf}}) = ((x)X_{(h \cdot \overline{gf})^\bullet \cdot (gf)^\bullet}, \overline{h \cdot \overline{gf}})$

and

$((x, f))\tilde{X}_{hg} = ((x)X_{(hgf)^\bullet}\, , \overline{hgf})$.

So we have to show that $X_{(h \cdot \overline{gf})^\bullet \cdot (gf)^\bullet} \overset{!}{=} X_{(hgf)^\bullet}$ and $\overline{h \cdot \overline{gf}} \overset{!}{=} \overline{hgf}$.

Consider the following commutative diagram.

$$\begin{tikzcd}
 & {[l_2]} \arrow[dr, "(h\cdot\overline{gf})^\bullet"] & & \\
 & & {[l_1]} \arrow[dr, "(gf)^\bullet"] & \\
{[p]} \arrow[uur, "\overline{h\cdot\overline{gf}}"] \arrow[r, "h"] \arrow[dr, "\overline{hgf}"'] & {[n]} \arrow[r, "g"] \arrow[ur, "\overline{gf}"] & {[m]} \arrow[r, "f"] & {[k]} \\
 & {[l_3]} \arrow[urr, "(hgf)^\bullet"'] & &
\end{tikzcd}$$

We get

$$hgf = h \cdot \overline{gf} \cdot (gf)^\bullet = \overline{h \cdot \overline{gf}} \cdot (h \cdot \overline{gf})^\bullet \cdot (gf)^\bullet$$

We also have

$$hgf = \overline{hgf} \cdot (hgf)^\bullet$$

Since $\overline{h \cdot \overline{gf}}$ is surjective and $(h \cdot \overline{gf})^\bullet \cdot (gf)^\bullet$ is injective, we find that $l_2 = l_3$ and $\overline{h \cdot \overline{gf}} = \overline{hgf}$ and $(h \cdot \overline{gf})^\bullet \cdot (gf)^\bullet = (hgf)^\bullet$.

Hence $X_{(h\cdot\overline{gf})^\bullet\cdot(gf)^\bullet} = X_{(hgf)^\bullet}$ and $\overline{h \cdot \overline{gf}} = \overline{hgf}$, which we had to show. $\square$

Remark 51. For $n \geqslant 0$ we have an injective map $\iota_{X,n} : X_n \to \tilde{X}_n\, , x \mapsto (x, \mathrm{id}_n)$. The tuple $\iota_X := (\iota_{X,n})_{n\geqslant 0}$ is a semisimplicial morphism from X to $\mathcal{V}_{\mathrm{Set}}\tilde{X}$.

Proof. Suppose given $([n] \xrightarrow{g} [m]) \in \operatorname{Mor}\Delta_{\mathrm{inj}}$. We have to show that the following diagram is commutative.

$$\begin{tikzcd}
X_m \arrow[r, "\iota_{X,m}"] \arrow[d, "X_g"'] & \tilde{X}_m \arrow[d, "\tilde{X}_g"] \\
X_n \arrow[r, "\iota_{X,n}"'] & \tilde{X}_n
\end{tikzcd}$$

Note that we have $\tilde{X}_n = \mathcal{V}_{\mathrm{Set}}\tilde{X}_n$, $\tilde{X}_m = \mathcal{V}_{\mathrm{Set}}\tilde{X}_m$ and $\tilde{X}_g = \mathcal{V}_{\mathrm{Set}}\tilde{X}_g$.

So let $x \in X_m$.

We have $(x)\iota_{X,m} \cdot \tilde{X}_g = (x, \mathrm{id}_{[m]})\tilde{X}_g = ((x)\tilde{X}_{(g\,\mathrm{id}_{[m]})^\bullet}, \overline{g\,\mathrm{id}_{[m]}}) = ((x)X_g, \mathrm{id}_{[n]}) = (x)X_g \cdot \iota_{X,n}\,.$ $\square$

Lemma 52 (Construction of a simplicial map out of a semisimplicial map)**.** Suppose given a semisimplicial morphism $\alpha = (\alpha_n)_{n\geqslant 0} : X \to Y$. For $n \geqslant 0$ we define

$$\begin{array}{rccc}
\tilde{\alpha}_n : & \tilde{X}_n & \longrightarrow & \tilde{Y}_n \\
 & (x, ([n] \xrightarrow{f} [k])) & \longmapsto & ((x)\alpha_k\, , ([n] \xrightarrow{f} [k]))\, .
\end{array}$$

Then $\tilde{\alpha} = (\tilde{\alpha}_n)_{n\geqslant 0}$ is a simplicial morphism from $\tilde{X}$ to $\tilde{Y}$.

Proof. Suppose given $([n] \xrightarrow{g} [m]) \in \operatorname{Mor} \Delta$. We have to show commutativity of the following diagram.

$$\begin{tikzcd} \tilde{X}_m \arrow[r, "\tilde{\alpha}_m"] \arrow[d, "\tilde{X}_g"'] & \tilde{Y}_m \arrow[d, "\tilde{Y}_g"] \\ \tilde{X}_n \arrow[r, "\tilde{\alpha}_n"] & \tilde{Y}_n \end{tikzcd}$$

Let $(x, ([m] \xrightarrow{f} [k])) \in \tilde{X}_m$.

We have $((x,f))(\tilde{\alpha}_m \cdot \tilde{Y}_g) = ((x)\alpha_k, f)\tilde{Y}_g = (((x)\alpha_k)Y_{(gf)^\bullet}, \overline{gf}) = ((x)(\alpha_k \cdot Y_{(gf)^\bullet}), \overline{gf})$.

We have $((x,f))(\tilde{X}_g \cdot \tilde{\alpha}_n) = (((x)X_{(gf)^\bullet}, \overline{gf}))\tilde{\alpha}_n = (((x)X_{(gf)^\bullet})\alpha_l, \overline{gf}) = ((x)(X_{(gf)^\bullet} \cdot \alpha_l), \overline{gf})$.

Since the diagram

$$\begin{tikzcd} X_k \arrow[r, "\alpha_k"] \arrow[d, "X_{(gf)^\bullet}"'] & Y_k \arrow[d, "Y_{(gf)^\bullet}"] \\ X_l \arrow[r, "\alpha_l"] & Y_l \end{tikzcd}$$

is commutative, we find that $(x)\alpha_k \cdot Y_{(gf)^\bullet} = (x)X_{(gf)^\bullet} \cdot \alpha_l$.

Hence $\tilde{\alpha}_m \cdot \tilde{Y}_g = \tilde{X}_g \cdot \tilde{\alpha}_n$. $\square$

Lemma 53. The assignment

$$\begin{array}{rrcll} \mathcal{F}_{\mathrm{Set}}: & \mathrm{SemiSimp(Set)} & \longrightarrow & \mathrm{Simp(Set)} & \\ & X & \longmapsto & \tilde{X} & \text{for } X \in \operatorname{Ob} \mathrm{SemiSimp(Set)} \\ & \alpha & \longmapsto & \tilde{\alpha} & \text{for } \alpha \in \operatorname{Mor} \mathrm{SemiSimp(Set)} \end{array}$$

defines a functor.

Proof. Suppose given $X \in \operatorname{Ob} \mathrm{SemiSimp(Set)}$. We have $\mathrm{id}_X = (\mathrm{id}_{X_n})_{n \geqslant 0}$. Let $n \geqslant 0$. Let $(x, ([n] \xrightarrow{f} [k])) \in \tilde{X}_k$. Then $((x,f))\widetilde{\mathrm{id}}_{X_n} = ((x)\,\mathrm{id}_k, f) = (x,f)$. Thus $\widetilde{\mathrm{id}}_X = \mathrm{id}_{\tilde{X}}$. So $\mathcal{F}_{\mathrm{Set}}$ maps identities to identities.

Suppose given $(X \xrightarrow{\alpha} Y), (Y \xrightarrow{\beta} Z) \in \operatorname{Mor} \mathrm{SemiSimp(Set)}$. Let $(x, ([n] \xrightarrow{f} [k])) \in \tilde{X}_k$. Then $(x,f)(\widetilde{\alpha \cdot \beta})_n = ((x)(\alpha \cdot \beta)_k, f) = ((x)\alpha_k \cdot \beta_k, f) = (((x)\alpha_k, f))\tilde{\beta}_n = (((x,f))\tilde{\alpha}_n)\tilde{\beta}_n$ $= (x,f)(\tilde{\alpha} \cdot \tilde{\beta})_n$. So $\tilde{\alpha} \cdot \tilde{\beta} = (\widetilde{\alpha \cdot \beta})$.

Hence $\mathcal{F}_{\mathrm{Set}}$ is a functor. $\square$

Remark 54. The tuple $\iota := (\iota_X)_{X \in \operatorname{Ob} \mathrm{SemiSimp(Set)}}$, cf. Remark 51, is a transformation from $\mathrm{id}_{\mathrm{SemiSimp(Set)}}$ to $\mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}$.

Proof. Suppose given $X, Y : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathrm{Set}$ and $(X \xrightarrow{\alpha} Y) \in \operatorname{Mor} \mathrm{SemiSimp(Set)}$. We have to show commutativity of the following diagram.

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[d, "\alpha"'] & \mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}X \arrow[d, "\mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}\alpha"] \\ Y \arrow[r, "\iota_Y"'] & \mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}Y \end{tikzcd}$$

This means that we have to show commutatity of the diagram

$$\begin{tikzcd}
X_n \arrow[r, "\iota_{X,n}"] \arrow[d, "\alpha_n"'] & (\mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}X)_n \arrow[d, "(\mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}\alpha)_n"] \\
Y \arrow[r, "\iota_{Y,n}"'] & (\mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}Y)_n
\end{tikzcd}$$

for $n \geqslant 0$.

So suppose given $x \in X_n$.

Then we have

$(x)(\iota_{X,n} \cdot (\mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}\alpha)_n) = ((x, \mathrm{id}_n))(\mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}\alpha)_n = ((x)\alpha_n, \mathrm{id}_n) = ((x)\alpha_n)\iota_{Y,n} = (x)\alpha_n \cdot \iota_{Y,n}$. □

Remark 55. Suppose given $X \in \mathrm{Simp(Set)}$. For $n \geqslant 0$ we define a map

$$\begin{array}{rrcl}
\eta_{X,n} : & (\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_n & \to & X_n \\
& (x, f : [n] \to [k]) & \mapsto & (x)X_f \, .
\end{array}$$

Then $\eta_X := (\eta_{X,n})_{n \geqslant 0}$ is a simplicial map from $\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X$ to X.

Proof. Suppose given $([n] \xrightarrow{g} [m]) \in \mathrm{Mor}\,\Delta$. We have to show commutativity of the following diagram.

$$\begin{tikzcd}
(\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_m \arrow[r, "\eta_{X,m}"] \arrow[d, "(\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_g"'] & X_m \arrow[d, "X_g"] \\
(\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_n \arrow[r, "\eta_{X,n}"'] & X_n
\end{tikzcd}$$

So let $(x, f : [m] \to [k]) \in (\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_n$.

We have $((x, f))((\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_g \cdot \eta_{X,n}) = (((x)X_{(gf)^\bullet}\, , \overline{gf}))\eta_{X,n} = ((x)X_{(gf)^\bullet})X_{\overline{gf}} = (x)X_{\overline{gf}\cdot(gf)^\bullet} = (x)X_{gf} = ((x)X_f)X_g = ((x, f))(\eta_{X,n} \cdot X_g)$. □

Remark 56. The tuple $\eta := (\eta_X)_{X \in \mathrm{Ob}\,\mathrm{Simp(Set)}}$ is a transformation from $\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}$ to $\mathrm{id}_{\mathrm{Simp(Set)}}$.

Proof. Suppose given $X, Y : \Delta^{\mathrm{op}} \to \mathrm{Set}$ and $(X \xrightarrow{\alpha} Y) \in \mathrm{Mor}\,\mathrm{Simp(Set)}$. We have to show commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X \arrow[r, "\eta_X"] \arrow[d, "\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}\alpha"'] & X \arrow[d, "\alpha"] \\
\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}Y \arrow[r, "\eta_Y"'] & Y
\end{tikzcd}$$

This means that we have to show commutatity of the diagram

$$\begin{tikzcd}
(\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_n \arrow[r, "\eta_{X,n}"] \arrow[d, "(\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}\alpha)_n"'] & X_n \arrow[d, "\alpha_n"] \\
(\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}Y)_n \arrow[r, "\eta_{Y,n}"'] & Y_n
\end{tikzcd}$$

for $n \geqslant 0$.

So suppose given $(x, f : [n] \to [k]) \in (\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}X)_n$.

Then we have

$((x, f))((\mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}}\alpha)_n \cdot \eta_{Y,n}) = (((x)\alpha_k, f))\eta_{Y,n} = ((x)\alpha_k)Y_f = (x)(\alpha_k \cdot Y_f)$

and

$((x, f))(\eta_{X,n} \cdot \alpha_n) = ((x)X_f)\alpha_n = (x)(X_f \cdot \alpha_n)$.

Since α is a simplicial morphism we find that $X_f \cdot \alpha_n = \alpha_k \cdot Y_f$. So the diagram above is commutative. □

Proposition 57. The functor

$$\mathcal{F}_{\mathrm{Set}} : \mathrm{SemiSimp(Set)} \to \mathrm{Simp(Set)}$$

is left adjoint to the functor

$$\mathcal{V}_{\mathrm{Set}} : \mathrm{Simp(Set)} \to \mathrm{SemiSimp(Set)}\,,$$

i.e. $\mathcal{F}_{\mathrm{Set}} \dashv \mathcal{V}_{\mathrm{Set}}$. The transformation $\eta : \mathcal{F}_{\mathrm{Set}}\mathcal{V}_{\mathrm{Set}} \to \mathrm{id}_{\mathrm{Simp(Set)}}$ is a counit and the transformation $\iota : \mathrm{id}_{\mathrm{SemiSimp(Set)}} \to \mathcal{V}_{\mathrm{Set}}\mathcal{F}_{\mathrm{Set}}$ is a unit of this adjunction.

Proof. We write $\mathcal{F} := \mathcal{F}_{\mathrm{Set}}$ and $\mathcal{V} := \mathcal{V}_{\mathrm{Set}}$. We have to show commutativity of the following diagrams.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathcal{F}\iota"] \arrow[dr, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F}\mathcal{V}\mathcal{F} \arrow[d, "\eta\mathcal{F}"] \\
& \mathcal{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V} \arrow[r, "\iota\mathcal{V}"] \arrow[dr, "\mathrm{id}_{\mathcal{V}}"'] & \mathcal{V}\mathcal{F}\mathcal{V} \arrow[d, "\mathcal{V}\eta"] \\
& \mathcal{V}
\end{tikzcd}$$

At first we show commutativity of the left diagram. This means that we have to show commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\mathcal{F}\iota_X"] \arrow[dr, "\mathrm{id}_{\mathcal{F}X}"'] & \mathcal{F}\mathcal{V}\mathcal{F}X \arrow[d, "\eta_{\mathcal{F}X}"] \\
& \mathcal{F}X
\end{tikzcd}$$

for $X \in \mathrm{Ob}\,\mathrm{SemiSimp(Set)}$.

This means that we have to show commutativity of the diagram

$$\begin{tikzcd}
(\mathcal{F}X)_n \arrow[r, "(\mathcal{F}\iota_X)_n"] \arrow[dr, "\mathrm{id}_{(\mathcal{F}X)_n}"'] & (\mathcal{F}\mathcal{V}\mathcal{F}X)_n \arrow[d, "\eta_{\mathcal{F}X,n}"] \\
& (\mathcal{F}X)_n
\end{tikzcd}$$

for $X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathrm{Set})$ and $n \geqslant 0$.

So let $(x, f : [n] \to [k]) \in (\mathcal{F}X)_n$.

We have $((x,f))((\mathcal{F}\iota_X)_n \cdot \eta_{\mathcal{F}X,n}) = (((x, \mathrm{id}_{[k]}), f))\eta_{\mathcal{F}X,n} = ((x, \mathrm{id}_{[k]}))(\mathcal{F}X)_f = ((x)X_{(f\,\mathrm{id}_{[k]})^\bullet}, \overline{f\,\mathrm{id}_{[k]}}) = ((x)X_{\mathrm{id}_{[k]}}, f) = (x, f) = ((x,f))\,\mathrm{id}_{(\mathcal{F}X)_n}$.

Hence the left diagram is commutative.

Now we show that the right diagram is commutative. As in the previous case this means that we have to show commutativity of the diagram

$$\begin{tikzcd}
(\mathcal{V}X)_n \arrow[r, "\iota_{\mathcal{V}X,n}"] \arrow[dr, "\mathrm{id}_{(\mathcal{V}X)_n}"'] & (\mathcal{V}\mathcal{F}\mathcal{V}X)_n \arrow[d, "(\mathcal{V}\eta_X)_n"] \\
& (\mathcal{V}X)_n
\end{tikzcd}$$

for $X \in \operatorname{Ob}\operatorname{Simp}(\mathrm{Set})$ and $n \geqslant 0$.

So let $x \in (\mathcal{V}X)_n$.

We have $(x)(\iota_{\mathcal{V}X,n} \cdot (\mathcal{V}\eta_X)_n) = ((x, \mathrm{id}_{[n]}))(\mathcal{V}\eta_X)_n = ((x, \mathrm{id}_{[n]}))\eta_{X,n} = (x)X_{\mathrm{id}_{[n]}} = x = (x)\,\mathrm{id}_{(\mathcal{V}X)_n}$.

Hence the right diagram is commutative. □

5.3 The adjoint for general $\mathcal{C}$

In the previous section 5.2 we had

$$\mathcal{F}_{\mathrm{Set}}(X_n) = \bigsqcup_{(f:[n]\to[k])\in\mathrm{surj}} X_k$$

for $n \geqslant 0$ and $X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathrm{Set})$. This can be generalized by using coproducts, cf. Example 16.

Suppose given a category $\mathcal{C}$ that has finite coproducts.

Lemma 58 (Construction of a simplicial object out of a semisimplicial object)**.** Suppose given a semisimplicial object

$$X : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{C}$$

in $\mathcal{C}$. We want to construct a simplicial object, i.e. a functor

$$\tilde{X} : \Delta^{\mathrm{op}} \to \mathcal{C}\,.$$

(i) (Construction of $\tilde{X}_n$). Let $n \geqslant 0$.

We define $\tilde{X}_n := \coprod_{([n]\xrightarrow{f}[k])\in\text{surj}} X_k$.

Then for $([n] \xrightarrow{g} [j]) \in \text{surj}$ we have morphisms $i_{X,n,g} : X_j \to \tilde{X}_n = \coprod_{([n]\xrightarrow{f}[k])\in\text{surj}} X_k$ such that the following universal property holds.

Given $C \in \operatorname{Ob}\mathcal{C}$ and morphisms $(X_k \xrightarrow{\mu_f} C)_{(f:[n]\to[k])\in\text{surj}}$ there exists a unique morphism $\tilde{X}_n = \coprod_{([n]\xrightarrow{f}[k])\in\text{surj}} X_k \xrightarrow{\mu} C$ such that for $([n] \xrightarrow{g} [j]) \in \text{surj}$ the diagram

$$\begin{tikzcd} \tilde{X}_n = \coprod_{([n]\xrightarrow{f}[k])\in\text{surj}} X_k \arrow[r, dashed, "\mu"] & C \\ X_j \arrow[u, "i_{X,n,g}"] \arrow[ur, "\mu_g"'] & \end{tikzcd}$$

commutes.

(ii) (Construction of $\tilde{X}_g$).

Let $([m] \xrightarrow{g^{\text{op}}} [n]) \in \operatorname{Mor}\Delta^{\text{op}}$. Let $([m] \xrightarrow{f} [k]) \in \text{surj}$.

We have $([n] \xrightarrow{gf} [k]) = ([n] \xrightarrow{\overline{gf}} [l])\cdot([l] \xrightarrow{(gf)^\bullet} [k])$, where $[l] = |\operatorname{Im}(gf)-1|$, cf. Remark 49.

Then we have a unique morphism $\tilde{X}_g : \tilde{X}_m \to \tilde{X}_n$ that makes the following diagram commute for $([m] \xrightarrow{f} [k]) \in \text{surj}$.

$$\begin{tikzcd} \tilde{X}_m \arrow[r, dashed, "\tilde{X}_g"] & \tilde{X}_n \\ X_k \arrow[u, "i_{X,m,f}"] \arrow[r, "X_{(gf)^\bullet}"] & X_l \arrow[u, "i_{X,n,\overline{gf}}"'] \end{tikzcd}$$

The assignment

$$\begin{array}{rcll} [n] & \mapsto & \tilde{X}_n & \text{for } [n] \in \operatorname{Ob}\Delta^{\text{op}} \\ g^{\text{op}} & \mapsto & \tilde{X}_g & \text{for } g^{\text{op}} \in \operatorname{Mor}\Delta^{\text{op}} \end{array}$$

defines a simplicial object $\tilde{X}$ in $\mathcal{C}$.

Proof. We have to show that $\tilde{X} : \Delta^{\text{op}} \to \mathcal{C}$ is a functor.

Consider the case $g = \mathrm{id}_{[n]}$. For $([n] \xrightarrow{f} [k]) \in \text{surj}$ we have $(\mathrm{id}_{[n]}\cdot f)^\bullet = \mathrm{id}_{[k]}$ and $\overline{\mathrm{id}_{[n]}\cdot f} = f$. Hence we have $X_{(\mathrm{id}_{[n]}\cdot f)^\bullet} = X_{\mathrm{id}_{[k]}} = \mathrm{id}_{X_k}$ and $i_{X,n,\overline{\mathrm{id}_{[n]}\cdot f}} = i_{X,n,f}$. The following diagram commutes.

$$\begin{tikzcd} \tilde{X}_n \arrow[r, dashed, "\mathrm{id}_{\tilde{X}_n}"] & \tilde{X}_n \\ X_k \arrow[u, "i_{X,n,f}"] \arrow[r, "\mathrm{id}_{X_k}"] & X_k \arrow[u, "i_{X,n,f}"'] \end{tikzcd}$$

Using the universal property of the coproduct we find that $\tilde{X}_{\mathrm{id}_{[n]}} = \mathrm{id}_{\tilde{X}_n}$ for $[n] \in \mathrm{Ob}\,\Delta^{\mathrm{op}}$.

Now suppose given $([p] \xrightarrow{h} [n]), ([n] \xrightarrow{g} [m]) \in \mathrm{Mor}\,\Delta$. We have to show $\tilde{X}_{hg} = \tilde{X}_g \tilde{X}_h$.

For an arbitrary morphism $([m] \xrightarrow{f} [k]) \in \mathrm{surj}$ we have commutativity of the following diagram.

$$\begin{tikzcd}
 & {[l_2]} \arrow[dr, "(h\cdot\overline{gf})^\bullet"] & & \\
 & & {[l_1]} \arrow[dr, "(gf)^\bullet"] & \\
{[p]} \arrow[uur, "\overline{h\cdot\overline{gf}}"] \arrow[r, "h"] \arrow[dr, "\overline{hgf}"'] & {[n]} \arrow[r, "g"] \arrow[ur, "\overline{gf}"] & {[m]} \arrow[r, "f"] & {[k]} \\
 & {[l_3]} \arrow[urr, "(hgf)^\bullet"'] & &
\end{tikzcd}$$

So we get $X_{(gf)^\bullet} \cdot X_{(h\cdot\overline{gf})^\bullet} = X_{(h\cdot\overline{gf})^\bullet\cdot(gf)^\bullet} = X_{(hgf)^\bullet}$ and $\overline{h\cdot\overline{gf}} = \overline{hgf}$.

Also the diagram

$$\begin{tikzcd}
\tilde{X}_m \arrow[r, "\tilde{X}_g"] & \tilde{X}_n \arrow[r, "\tilde{X}_h"] & \tilde{X}_p \\
X_k \arrow[u, "i_{X,m,f}"] \arrow[r, "X_{(gf)^\bullet}"'] & X_{l_1} \arrow[u, "i_{X,n,\overline{gf}}"] \arrow[r, "X_{(h\cdot\overline{gf})^\bullet}"'] & X_{l_2} \arrow[u, "i_{X,p,\overline{h\cdot\overline{gf}}}"']
\end{tikzcd}$$

is commutative.

So we find that the following diagrams are both commutative.

$$\begin{tikzcd}
\tilde{X}_m \arrow[r, "\tilde{X}_g\cdot\tilde{X}_h"] & \tilde{X}_p \\
X_k \arrow[u, "i_{X,m,f}"] \arrow[r, "X_{(hgf)^\bullet}"'] & X_{l_3} \arrow[u, "i_{X,p,\overline{hgf}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{X}_m \arrow[r, "\tilde{X}_{hg}"] & \tilde{X}_p \\
X_k \arrow[u, "i_{X,m,f}"] \arrow[r, "X_{(hgf)^\bullet}"'] & X_{l_3} \arrow[u, "i_{X,p,\overline{hgf}}"']
\end{tikzcd}$$

Using the universal property of the coproduct we find that $\tilde{X}_{hg} = \tilde{X}_g \cdot \tilde{X}_h$. □

Lemma 59 (Construction of a simplicial morphism out of a semisimplicial morphism)**.** Suppose given a semisimplicial morphism $\alpha = (\alpha_n)_{n\geqslant 0} : X \to Y$. Let $n \geqslant 0$. We define $\tilde{\alpha}_{n,f} := \alpha_k \cdot i_{n,f}$. There exists a unique morphism $\tilde{\alpha}_n : \tilde{X}_n \to \tilde{Y}_n$ such that for $(f : [n] \to [k]) \in \mathrm{surj}$ the following diagram is commutative.

$$\begin{tikzcd}
\tilde{X}_n \arrow[r, "\tilde{\alpha}_n"] & \tilde{Y}_n \\
X_k \arrow[u, "i_{X,n,f}"] \arrow[r, "\alpha_k"'] & Y_k \arrow[u, "i_{Y,n,f}"']
\end{tikzcd}$$

We define $\tilde{\alpha} := (\tilde{\alpha}_n)_{n\geqslant 0}$. Then $\tilde{\alpha}$ is a simplicial morphism from $\tilde{X}$ to $\tilde{Y}$.

Proof. Suppose given $([n] \xrightarrow{g} [m]) \in \operatorname{Mor}\Delta$. We have to show commutativity of the following diagram.

$$\begin{tikzcd}
\tilde{X}_m \arrow[r, "\tilde{\alpha}_m"] \arrow[d, "\tilde{X}_g"'] & \tilde{Y}_m \arrow[d, "\tilde{Y}_g"] \\
\tilde{X}_n \arrow[r, "\tilde{\alpha}_n"] & \tilde{Y}_n
\end{tikzcd}$$

Let $([m] \xrightarrow{f} [k]) \in \text{surj}$. We have $([n] \xrightarrow{gf} [k]) = ([n] \xrightarrow{\overline{gf}} [l]) \cdot ([l] \xrightarrow{(gf)^\bullet} [k])$.

Except for $\begin{tikzcd}
\tilde{X}_m \arrow[r, "\tilde{\alpha}_m"] \arrow[d, "\tilde{X}_g"'] & \tilde{Y}_m \arrow[d, "\tilde{Y}_g"] \\
\tilde{X}_n \arrow[r, "\tilde{\alpha}_n"] & \tilde{Y}_n
\end{tikzcd}$, we have commutativity of every quadrangle in the following diagram

$$\begin{tikzcd}
& \tilde{Y}_m \arrow[rr, "\tilde{Y}_g"] & & \tilde{Y}_n \\
\tilde{X}_m \arrow[ur, "\tilde{\alpha}_m"] \arrow[rr, "\tilde{X}_g"] & & \tilde{X}_n \arrow[ur, "\tilde{\alpha}_n"] & \\
& Y_k \arrow[uu, "i_{Y,m,f}"'] \arrow[rr, "Y_{(gf)^\bullet}"] & & Y_l \arrow[uu, "i_{Y,n,\overline{gf}}"'] \\
X_k \arrow[uu, "i_{X,m,f}"] \arrow[ur, "\alpha_k"] \arrow[rr, "X_{(gf)^\bullet}"'] & & X_l \arrow[uu, "i_{X,n,\overline{gf}}"'] \arrow[ur, "\alpha_l"] &
\end{tikzcd}$$

So we get $i_{X,m,f} \cdot \tilde{\alpha}_m \cdot \tilde{Y}_g = i_{X,m,f} \cdot \tilde{X}_g \cdot \tilde{\alpha}_n$.

So the diagrams

$$\begin{tikzcd}
\tilde{X}_m \arrow[r, "\tilde{\alpha}_m \cdot \tilde{Y}_g"] & \tilde{Y}_n \\
X_k \arrow[u, "i_{X,m,f}"] \arrow[ur, "i_{X,m,f} \cdot \tilde{\alpha}_m \cdot \tilde{Y}_g"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\tilde{X}_m \arrow[r, "\tilde{X}_g \cdot \tilde{\alpha}_n"] & \tilde{Y}_n \\
X_k \arrow[u, "i_{X,m,f}"] \arrow[ur, "i_{X,m,f} \cdot \tilde{\alpha}_m \cdot \tilde{Y}_g"'] &
\end{tikzcd}$$

are both commutative for $([m] \xrightarrow{f} [k]) \in \text{surj}$.

Using the universal property of the coproduct we get $\tilde{\alpha}_m \cdot \tilde{Y}_g = \tilde{X}_g \cdot \tilde{\alpha}_n$. $\square$

Lemma 60. The assignment

$$\begin{array}{rrcll}
\mathcal{F}_{\mathcal{C}}: & \operatorname{SemiSimp}(\mathcal{C}) & \longrightarrow & \operatorname{Simp}(\mathcal{C}) & \\
& X & \longmapsto & \tilde{X} & \text{for } X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathcal{C}) \\
& \alpha & \longmapsto & \tilde{\alpha} & \text{for } \alpha \in \operatorname{Mor}\operatorname{SemiSimp}(\mathcal{C})
\end{array}$$

defines a functor.

Proof. Suppose given $X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathcal{C})$. We have $\mathrm{id}_X = (\mathrm{id}_{X_n})_{n \geqslant 0}$. Let $n \geqslant 0$. The diagram

$$\begin{tikzcd} \tilde{X}_n \arrow[r, "\mathrm{id}_{\tilde{X}_n}"] & \tilde{X}_n \\ X_k \arrow[u, "i_{X,n,f}"] \arrow[r, "\mathrm{id}_{X_k}"'] & X_k \arrow[u, "i_{X,n,f}"'] \end{tikzcd}$$

is commutative for $([n] \xrightarrow{f} [k]) \in \mathrm{surj}$. Thus $\widetilde{\mathrm{id}_X} = \mathrm{id}_{\tilde{X}}$. So $\mathcal{F}_{\mathcal{C}}$ maps identities to identities.

Suppose given $(X \xrightarrow{\alpha} Y), (Y \xrightarrow{\beta} Z) \in \operatorname{Mor}\operatorname{SemiSimp}(\mathcal{C})$.

We have commutativity of the following diagrams for $([n] \xrightarrow{f} [k]) \in \mathrm{surj}$.

$$\begin{tikzcd} \tilde{X}_n \arrow[r, "\tilde{\alpha}_n"] & \tilde{Y}_n \arrow[r, "\tilde{\beta}_n"] & \tilde{Z}_n \\ X_k \arrow[u, "i_{X,n,f}"] \arrow[r, "\alpha_k"'] & Y_k \arrow[u, "i_{Y,n,f}"] \arrow[r, "\beta_k"'] & Z_k \arrow[u, "i_{Z,n,f}"] \end{tikzcd} \qquad \begin{tikzcd} \tilde{X}_n \arrow[r, "\widetilde{(\alpha \cdot \beta)}_n"] & \tilde{Z}_n \\ X_k \arrow[u, "i_{X,n,f}"] \arrow[r, "\alpha_k \cdot \beta_k"'] & Z_k \arrow[u, "i_{Z,n,f}"'] \end{tikzcd}$$

Using the universal property of the coproduct we find that $\tilde{\alpha}_n \cdot \tilde{\beta}_n = \widetilde{(\alpha \cdot \beta)}_n$. So altogether $\tilde{\alpha} \cdot \tilde{\beta} = \widetilde{(\alpha \cdot \beta)}$.

Hence $\mathcal{F}_{\mathcal{C}}$ is a functor. □

Remark 61. Suppose given $X \in \operatorname{Ob}\operatorname{Simp}(\mathcal{C})$. Let $\iota_{X,n} := i_{X,n,\mathrm{id}_{[n]}} : X_n \to \tilde{X}_n = (\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X)_n$ for $n \geqslant 0$. The tuple $\iota_X := (\iota_{X,n,})_{n \geqslant 0}$ is a semisimplicial morphism from X to $\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X$.

Proof. Suppose given $([n] \xrightarrow{g} [m]) \in \operatorname{Mor}\Delta_{\mathrm{inj}}$. Note that we have $\tilde{X}_n = (\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X)_n$, $\tilde{X}_m = (\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X)_m$ and $\tilde{X}_g = (\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X)_g$. We have to show that the following diagram is commutative.

$$\begin{tikzcd} X_m \arrow[r, "\iota_{X,m}"] \arrow[d, "X_g"'] & \tilde{X}_m \arrow[d, "\tilde{X}_g"] \\ X_n \arrow[r, "\iota_{X,n}"'] & \tilde{X}_n \end{tikzcd}$$

But by Lemma 58 (ii), we have commutativity of the diagram

$$\begin{tikzcd} X_m \arrow[r, "i_{X,m,\mathrm{id}_{[m]}}"] \arrow[d, "X_{(g \cdot \mathrm{id}_{[m]})^\bullet}"'] & \tilde{X}_m \arrow[d, "\tilde{X}_g"] \\ X_n \arrow[r, "i_{X,n,\overline{g \cdot \mathrm{id}_{[m]}}}"'] & \tilde{X}_n \end{tikzcd}$$

and we have $\overline{g \cdot \mathrm{id}_{[m]}} = \mathrm{id}_{[n]}$ and $(g \cdot \mathrm{id}_{[m]})^\bullet = g$ and $\iota_{X,m} = i_{X,m,\mathrm{id}_{[m]}}$ and $\iota_{X,n} = i_{X,n,\mathrm{id}_{[n]}}$. □

Remark 62. The tuple $\iota := (\iota_X)_{X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathcal{C})}$ is a transformation from $\mathrm{id}_{\operatorname{SemiSimp}(\mathcal{C})}$ to $\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}$.

Proof. Suppose given functors $X, Y : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{C}$ and $(X \xrightarrow{\alpha} Y) \in \operatorname{Mor}\operatorname{SemiSimp}(\mathcal{C})$. We have to show commutativity of the following diagram.

$$\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[d, "\alpha"'] & \mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X \arrow[d, "\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}\alpha"] \\
Y \arrow[r, "\iota_Y"'] & \mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}Y
\end{tikzcd}$$

This means that we have to show commutatity of the diagram

$$\begin{tikzcd}
X_n \arrow[r, "\iota_{X,n}"] \arrow[d, "\alpha_n"'] & (\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X)_n \arrow[d, "(\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}\alpha)_n"] \\
Y_n \arrow[r, "\iota_{Y,n}"'] & (\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}Y)_n
\end{tikzcd}$$

for $n \geqslant 0$.

Since $\mathrm{id}_{[n]}$ is surjective, Lemma 58 gives the commutativity of the diagram

$$\begin{tikzcd}
X_n \arrow[r, "i_{X,n,\mathrm{id}_{[n]}}"] \arrow[d, "\alpha_n"'] & \tilde{X}_n \arrow[d, "\tilde{\alpha}_n"] \\
Y_n \arrow[r, "i_{Y,n,\mathrm{id}_{[n]}}"'] & \tilde{Y}_n
\end{tikzcd}$$

and also we have $(\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}X)_n = \tilde{X}_n$, $(\mathcal{V}_{\mathcal{C}}\mathcal{F}_{\mathcal{C}}Y)_n = \tilde{Y}_n$, $\iota_{X,n} = i_{X,n,\mathrm{id}_{[n]}}$ and $\iota_{Y,n} = i_{Y,n,\mathrm{id}_{[n]}}$. □

Remark 63. Suppose given $X \in \operatorname{Ob}\operatorname{Simp}(\mathcal{C})$. For $n \geqslant 0$ there exists a unique morphism $\eta_{X,n} : (\mathcal{F}_{\mathcal{C}}\mathcal{V}_{\mathcal{C}}X)_n \to X_n$ that makes the diagram

$$\begin{tikzcd}
(\mathcal{F}_{\mathcal{C}}\mathcal{V}_{\mathcal{C}}X)_n \arrow[r, "\eta_{X,n}"] & X_n \\
X_k \arrow[u, "i_{\mathcal{V}_{\mathcal{C}}X,n,f}"] \arrow[ur, "X_f"'] &
\end{tikzcd}$$

commutative for $([n] \xrightarrow{f} [k]) \in \mathrm{surj}$, cf. Lemma 58 (i). Note that we used that $(\mathcal{V}_{\mathcal{C}}X)_k = X_k$.

Then $\eta_X := (\eta_{X,n})_{n \geqslant 0}$ is a simplicial map from $\mathcal{F}_{\mathcal{C}}\mathcal{V}_{\mathcal{C}}X$ to X.

Proof. Suppose given $([n] \xrightarrow{g} [m]) \in \operatorname{Mor}\Delta$. We have to show commutativity of the following diagram.

$$\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_m \arrow[r, "\eta_{X,m}"] \arrow[d, "(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_g"'] & X_m \arrow[d, "X_g"] \\
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_n \arrow[r, "\eta_{X,n}"'] & X_n
\end{tikzcd}$$

Suppose given $([m] \xrightarrow{f} [k]) \in \text{surj}$. We have $([n] \xrightarrow{gf} [k]) = ([n] \xrightarrow{\overline{gf}} [l]) \cdot ([l] \xrightarrow{(gf)^\bullet} [k])$ for some $l \geqslant 0$.

The following diagrams are commutative, cf. Lemma 58 (ii).

$$\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_m \arrow[r, "(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_g"] & (\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_n \arrow[r, "\eta_{X,n}"] & X_n \\
X_k \arrow[u, "i_{\mathcal{V}_\mathcal{C}X,m,f}"] \arrow[r, "X_{(gf)^\bullet}"'] & X_l \arrow[u, "i_{\mathcal{V}_\mathcal{C}X,n,\overline{gf}}"] \arrow[ur, "X_{\overline{gf}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_m \arrow[r, "\eta_{X,m}"] & X_m \\
X_k \arrow[u, "i_{\mathcal{V}_\mathcal{C}X,m,f}"] \arrow[ur, "X_f"'] &
\end{tikzcd}$$

So we have

$$i_{\mathcal{V}_\mathcal{C}X,m,f} \cdot (\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_g \cdot \eta_{X,n} = X_{(gf)^\bullet} \cdot X_{\overline{gf}} = X_{\overline{gf} \cdot (gf)^\bullet} = X_{gf} = X_f \cdot X_g = i_{\mathcal{V}_\mathcal{C}X,m,f} \cdot \eta_{X,m} \cdot X_g$$

for $([m] \xrightarrow{f} [k]) \in \text{surj}$.

So the diagrams

$$\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_m \arrow[r, "(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_g \cdot \eta_{X,n}"] & X_n \\
X_k \arrow[u, "i_{\mathcal{V}_\mathcal{C}X,m,f}"] \arrow[ur, "X_{gf}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_m \arrow[r, "\eta_{X,m} \cdot X_g"] & X_n \\
X_k \arrow[u, "i_{\mathcal{V}_\mathcal{C}X,m,f}"] \arrow[ur, "X_{gf}"'] &
\end{tikzcd}$$

are both commutative for $([m] \xrightarrow{f} [k]) \in \text{surj}$.

Using the universal property of the coproduct we find that $(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_g \cdot \eta_{X,n} = \eta_{X,m} \cdot X_g$. $\square$

Remark 64. The tuple $\eta := (\eta_X)_{X \in \operatorname{Ob}\operatorname{Simp}(\mathcal{C})}$ is a transformation from $\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}$ to $\mathrm{id}_{\operatorname{Simp}(\mathcal{C})}$.

Proof. Suppose given functors $X, Y : \Delta^{\text{op}} \to \mathcal{C}$ and $(X \xrightarrow{\alpha} Y) \in \operatorname{Mor}\operatorname{Simp}(\mathcal{C})$. We have to show commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X \arrow[r, "\eta_X"] \arrow[d, "\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}\alpha"'] & X \arrow[d, "\alpha"] \\
\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}Y \arrow[r, "\eta_Y"'] & Y
\end{tikzcd}$$

This means that we have to show commutatity of the diagram

$$\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_n \arrow[r, "\eta_{X,n}"] \arrow[d, "(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}\alpha)_n"'] & X_n \arrow[d, "\alpha_n"] \\
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}Y)_n \arrow[r, "\eta_{Y,n}"'] & Y_n
\end{tikzcd}$$

for $n \geqslant 0$.

The following diagram commutes for $([n] \xrightarrow{f} [k]) \in \text{surj}$, cf. Remark 63, Lemma 59.

$$\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_n \arrow[rr, "\eta_{X,n}"] \arrow[ddd, "(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}\alpha)_n"'] & & X_n \arrow[ddd, "\alpha_n"] \\
& X_k \arrow[ul, "i_{\mathcal{V}_\mathcal{C}X,n,f}"'] \arrow[ur, "X_f"] \arrow[d, "\alpha_k"] & \\
& Y_k \arrow[dl, "i_{\mathcal{V}_\mathcal{C}Y,n,f}"] \arrow[dr, "Y_f"'] & \\
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}Y)_n \arrow[rr, "\eta_{Y,n}"'] & & Y_n
\end{tikzcd}$$

So we get $i_{\mathcal{V}_\mathcal{C}X,n,f} \cdot (\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}\alpha)_n \cdot \eta_{Y,n} = \alpha_k \cdot Y_f = i_{\mathcal{V}_\mathcal{C}X,n,f} \cdot \eta_{X,n} \cdot \alpha_n$ for $([n] \xrightarrow{f} [k]) \in \text{surj}$.

So the diagrams

$$\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_n \arrow[r, "(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}\alpha)_n \cdot \eta_{Y,n}"] & Y_n \\
X_k \arrow[u, "i_{\mathcal{V}_\mathcal{C}X,n,f}"] \arrow[ur, "\alpha_k \cdot Y_f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}X)_n \arrow[r, "\eta_{X,n} \cdot \alpha_n"] & Y_n \\
X_k \arrow[u, "i_{\mathcal{V}_\mathcal{C}X,n,f}"] \arrow[ur, "\alpha_k \cdot Y_f"'] &
\end{tikzcd}$$

are both commutative for $([m] \xrightarrow{f} [k]) \in \text{surj}$.

Using the universal property of the coproduct we find that $(\mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C}\alpha)_n \cdot \eta_{Y,n} = \eta_{X,n} \cdot \alpha_n$. □

Proposition 65. The functor

$$\mathcal{F}_\mathcal{C} : \text{SemiSimp}(\mathcal{C}) \to \text{Simp}(\mathcal{C})$$

is left adjoint to the functor

$$\mathcal{V}_\mathcal{C} : \text{Simp}(\mathcal{C}) \to \text{SemiSimp}(\mathcal{C}) \,,$$

i.e. $\mathcal{F}_\mathcal{C} \dashv \mathcal{V}_\mathcal{C}$. The transformation $\eta : \mathcal{F}_\mathcal{C}\mathcal{V}_\mathcal{C} \to \mathrm{id}_{\mathrm{Simp}(\mathcal{C})}$ is a counit and the transformation $\iota : \mathrm{id}_{\mathrm{SemiSimp}(\mathcal{C})} \to \mathcal{V}_\mathcal{C}\mathcal{F}_\mathcal{C}$ is a unit of this adjunction.

Proof. We write $\mathcal{F} := \mathcal{F}_\mathcal{C}$ and $\mathcal{V} := \mathcal{V}_\mathcal{C}$. We have to show commutativity of the following diagrams.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathcal{F}\iota"] \arrow[dr, "\mathrm{id}_\mathcal{F}"'] & \mathcal{F}\mathcal{V}\mathcal{F} \arrow[d, "\eta\mathcal{F}"] \\ & \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{V} \arrow[r, "\iota\mathcal{V}"] \arrow[dr, "\mathrm{id}_\mathcal{V}"'] & \mathcal{V}\mathcal{F}\mathcal{V} \arrow[d, "\mathcal{V}\eta"] \\ & \mathcal{V} \end{tikzcd}$$

At first we show commutativity of the left diagram. This means that we have to show commutativity of the diagram

$$\begin{tikzcd} \mathcal{F}X \arrow[r, "\mathcal{F}\iota_X"] \arrow[dr, "\mathrm{id}_{\mathcal{F}X}"'] & \mathcal{F}\mathcal{V}\mathcal{F}X \arrow[d, "\eta_{\mathcal{F}X}"] \\ & \mathcal{F}X \end{tikzcd}$$

for $X \in \mathrm{Ob}\,\mathrm{SemiSimp}(\mathcal{C})$.

This means that we have to show commutativity of the diagram

$$\begin{tikzcd} (\mathcal{F}X)_n \arrow[r, "(\mathcal{F}\iota_X)_n"] \arrow[dr, "\mathrm{id}_{(\mathcal{F}X)_n}"'] & (\mathcal{F}\mathcal{V}\mathcal{F}X)_n \arrow[d, "\eta_{\mathcal{F}X,n}"] \\ & (\mathcal{F}X)_n \end{tikzcd}$$

for $X \in \mathrm{Ob}\,\mathrm{SemiSimp}(\mathcal{C})$ and $n \geqslant 0$.

We have commutativity of the following diagram for $([n] \xrightarrow{f} [k]) \in \mathrm{surj}$.

$$\begin{tikzcd} & X_k \arrow[r, "X_{(f\cdot\mathrm{id}_{[k]})^\bullet}"] \arrow[dl, "i_{X,n,f}"'] \arrow[d, "i_{X,k,\mathrm{id}_{[k]}}"] & X_k \arrow[d, "i_{X,n,\overline{f\cdot\mathrm{id}_{[k]}}}"] \\ (\mathcal{F}X)_n \arrow[dr, "(\mathcal{F}\iota_X)_n"'] & (\mathcal{V}\mathcal{F}X)_k \arrow[r, "(\mathcal{F}X)_f"] \arrow[d, "i_{\mathcal{V}\mathcal{F}X,n,f}"] & (\mathcal{F}X)_n \\ & (\mathcal{F}\mathcal{V}\mathcal{F}X)_n \arrow[ur, "\eta_{\mathcal{F}X,n}"'] & \end{tikzcd}$$

Therefore we have $i_{X,n,f} \cdot (\mathcal{F}\iota_X)_n \cdot \eta_{\mathcal{F}X,n} = X_{(f\cdot\mathrm{id}_{[k]})^\bullet} \cdot i_{X,n,\overline{f\cdot\mathrm{id}_{[k]}}} = X_{\mathrm{id}_{[k]}} \cdot i_{X,n,f} = \mathrm{id}_{X_k} \cdot i_{X,n,f}$ for $([n] \xrightarrow{f} [k]) \in \mathrm{surj}$.

So we find that the diagrams

$$
\begin{tikzcd}
(\mathcal{F}X)_n \arrow[r, "(\mathcal{F}\iota_X)_n \cdot \eta_{\mathcal{F}X,n}"] & (\mathcal{F}X)_n \\
X_k \arrow[u, "i_{X,n,f}"] \arrow[r, "\mathrm{id}_{X_k}"'] & X_k \arrow[u, "i_{X,n,f}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{F}X)_n \arrow[r, "\mathrm{id}_{(\mathcal{F}X)_n}"] & (\mathcal{F}X)_n \\
X_k \arrow[u, "i_{X,n,f}"] \arrow[r, "\mathrm{id}_{X_k}"'] & X_k \arrow[u, "i_{X,n,f}"']
\end{tikzcd}
$$

are both commutative for $([n] \xrightarrow{f} [k]) \in \mathrm{surj}$.

Using the universal property of the coproduct we find that $(\mathcal{F}\iota_X)_n \cdot \eta_{\mathcal{F}X,n} = \mathrm{id}_{(\mathcal{F}X)_n}$.

Hence the left diagram is commutative.

Now we show that the right diagram is commutative. As in the previous case this means that we have to show commutativity of the diagram

$$
\begin{tikzcd}
(\mathcal{V}X)_n \arrow[r, "\iota_{\mathcal{V}X,n}"] \arrow[dr, "\mathrm{id}_{(\mathcal{V}X)_n}"'] & (\mathcal{V}\mathcal{F}\mathcal{V}X)_n \arrow[d, "(\mathcal{V}\eta_X)_n"] \\
& (\mathcal{V}X)_n
\end{tikzcd}
$$

for $X \in \mathrm{Ob}\,\mathrm{Simp}(\mathcal{C})$ and $n \geqslant 0$.

But we have commutativity of the following diagram.

$$
\begin{tikzcd}
X_n \arrow[r, "i_{\mathcal{V}X,n,\mathrm{id}_{[n]}}"] \arrow[dr, "X_{\mathrm{id}_{[n]}}"'] & (\mathcal{F}\mathcal{V}X)_n \arrow[d, "\eta_{X,n}"] \\
& X_n
\end{tikzcd}
$$

Since $\iota_{\mathcal{V}X,n} = i_{\mathcal{V}X,n,\mathrm{id}_{[n]}}$, $X_{\mathrm{id}_{[n]}} = \mathrm{id}_{X_n} = \mathrm{id}_{(\mathcal{V}X)_n}$ and $\eta_{X,n} = (\mathcal{V}\eta_X)_n$ we find that $\mathrm{id}_{(\mathcal{V}X)_n} = \iota_{\mathcal{V}X,n} \cdot (\mathcal{V}\eta_X)_n$.

Hence the right diagram is commutative. $\square$

Chapter 6

Simplicial Resolutions

Suppose given a category $\mathcal{C}$ that has finite limits and finite coproducts. Suppose given a resolving subcategory $\mathcal{P}$ in $\mathcal{C}$, closed under finite coproducts.

Definition 66. Let $X \in \operatorname{Ob}\mathcal{C}$.

Let $((P_n)_{n\geqslant 0}, ((d_i^n)_{i\in[0,n]})_{n\geqslant 1})$ be a semisimplicial resolution of X, cf. Definition 36.

We have $d_j^n d_i^{n-1} = d_i^n d_{j-1}^{n-1}$ for $n \geqslant 2$ and $0 \leqslant i < j \leqslant n$, cf. Remark 37.

Thus there exists a unique functor

$$R : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$$

such that $R[n] = P_n$ for $n \in \mathbb{Z}_{\geqslant 0}$ and $R((\partial_i^n)^{\mathrm{op}}) = d_i^n$ for $n \in \mathbb{Z}_{\geqslant 1}$ and $i \in [0,n]$, cf. Proposition 43. So

$$R \in \operatorname{SemiSimp}(\mathcal{C}),$$

i.e. R is a semisimplicial object in $\mathcal{C}$.

We apply the functor

$$\mathcal{F}_{\mathcal{C}} : \operatorname{SemiSimp}(\mathcal{C}) \to \operatorname{Simp}(\mathcal{C}),$$

cf. Lemma 60. We call

$$\mathcal{F}_{\mathcal{C}} R \in \operatorname{Simp}(\mathcal{C})$$

a *simplicial resolution* of X.

Example 67. Let $P \in \operatorname{Ob}\mathcal{P}$. A semisimplicial resolution of P is given by

$$((P_n)_{n\geqslant 0}, ((d_i^n)_{i\in[0,n]})_{n\geqslant 1}) = ((P)_{n\geqslant 0}, ((\mathrm{id}_P)_{i\in[0,n]})_{n\geqslant 1}),$$

cf. Example 38. So the semisimplicial resolution yields in this case the constant functor

$$\begin{array}{rlcll} X : & \Delta_{\mathrm{inj}}^{\mathrm{op}} & \to & \mathcal{C} & \\ & [n] & \mapsto & P & \text{for } [n] \in \operatorname{Ob}\Delta_{\mathrm{inj}}^{\mathrm{op}} \\ & f & \mapsto & \mathrm{id}_P & \text{for } f \in \operatorname{Mor}\Delta_{\mathrm{inj}}^{\mathrm{op}} \end{array}$$

as semisimplicial object, i.e. $X \in \operatorname{Ob}\operatorname{SemiSimp}(\mathcal{C})$. Let $\tilde{X} := \mathcal{F}_{\mathcal{C}} X \in \operatorname{Ob}\operatorname{Simp}(\mathcal{C})$.

Then for $[n] \in \operatorname{Ob}\Delta^{\mathrm{op}}$ we have $\tilde{X}_n = \coprod_{([n]\xrightarrow{f}[k])\in\mathrm{surj}} P$, cf. Lemma 58 (i).

Claim. For $n \geqslant 0$ there are exactly 2^n morphisms in surj having source $[n]$. This we can show by induction over n.

Base of the induction. Suppose that $n = 0$. The only surjective morphism in Δ with source $[0]$ is $\mathrm{id}_{[0]}$.

Step of the induction. Suppose that $n \geqslant 1$.

Let $S_{[n]}$ be the set of all morphisms in Δ starting in $[n]$. Let $S'_{[n]} := \{f \in S_{[n]} : (n-1)f = (n)f\}$ and $S''_{[n]} := \{f \in S_{[n]} : (n-1)f = (n)f - 1\}$.

Then $S_{[n]} = S'_{[n]} \dot\cup S''_{[n]}$.

We have the following bijections.

$$\begin{array}{ccc} S'_{[n]} & \to & S_{[n-1]} \\ f & \mapsto & f|_{[n-1]} \\ \left\{ \begin{array}{lll} i & \mapsto & (i)g \quad \text{for } i \in [n-1] \\ n & \mapsto & (n-1)g \end{array} \right. & \leftarrow\!\shortmid & g \end{array}$$

$$\begin{array}{ccc} S''_{[n]} & \to & S_{[n-1]} \\ f & \mapsto & f|_{[n-1]}^{[|\operatorname{Im}(f)|-2]} \\ \left\{ \begin{array}{lll} i & \mapsto & (i)g \qquad \text{for } i \in [n-1] \\ n & \mapsto & (n-1)g + 1 \end{array} \right. & \leftarrow\!\shortmid & g \end{array}$$

So we find that $|S_{[n]}| = |S'_{[n]}| + |S''_{[n]}| = 2|S_{[n-1]}| \overset{\text{ind. hyp.}}{=} 2 \cdot 2^{n-1} = 2^n$. This proves the *claim.*

So we have $\tilde{X}_n = \coprod\limits_{i \in [1,2^n]} P$ for $n \geqslant 0$.

Example 68. Consider the case $\mathcal{C} = \mathrm{Grp}$ and $\mathcal{P} = \mathrm{FreeGrp}$; cf. Remark 32. The subcategory $\mathcal{P}$ is closed under finite coproducts in $\mathcal{C}$; cf. Example 21.

Recall that in Example 39 we regarded the first steps in the semisimplicial resolution of the group $\mathbb{Z}/n\mathbb{Z}$. We got a functor $X : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathrm{Grp}$ with $X_0 = \mathbb{Z}$, $X_1 = F_2$ and $X_2 = \mathrm{Free}(L)$.

Let $\tilde{X} := \mathcal{F}_{\mathrm{Grp}} X$, which is a simplicial resolution of $\mathbb{Z}/n\mathbb{Z}$; cf. Definition 66.

We want to calculate $\tilde{X}_0$, $\tilde{X}_1$ and $\tilde{X}_2$ up to isomorphism. We remark that by Lemma 20 the coproduct in the category of groups is the free product, written $(*)$.

X_0: The only surjective morphism in Δ starting in $[0]$ is $\mathrm{id}_{[0]}$. So we get $\tilde{X}_0 = \mathbb{Z}$.

X_1: We have the surjective morphisms $c_1 : [1] \to [0]$ and $\mathrm{id}_{[1]}$. So we get $\tilde{X}_1 = \mathbb{Z} * F_2$, which is isomorphic to the free group generated by three elements.

X_2: We have the surjective morphisms $c_2 : [2] \to [0]$, $s_0 : [2] \to [1]$, $1 \mapsto 0$, $s_1 : [2] \to [1]$, $1 \mapsto 1$ and $\mathrm{id}_{[2]}$. So we get $\tilde{X}_2 = \mathbb{Z} * F_2 * F_2 * \mathrm{Free}(L)$.

Example 69. We may let $\mathcal{C}$ be a model category and $\mathcal{P} := \mathcal{C}_{\mathrm{cof}}$; cf. Definition 33, Remark 34.

In fact, given X, $Y \in \operatorname{Ob} \mathcal{P}$, the pushout

$$\begin{tikzcd} I \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \sqcup Y \end{tikzcd}$$

shows that the coproduct $X \sqcup Y$ is a cofibrant object.

So $\mathcal{P}$ is closed under finite coproducts in $\mathcal{C}$.

In conclusion, each object of $\mathcal{C}$ has a simplicial object in $\mathcal{P} = \mathcal{C}_{\text{cof}}$ as simplicial resolution; cf. Definition 66.

Example 70. We may let $\mathcal{C}$ be a model category and $\mathcal{P} := \mathcal{C}_{\text{ac,cof}}$; cf. Definition 33, Remark 35.

In fact, given $X,\ Y\ \in\ \operatorname{Ob}\mathcal{P}$, the pushout

$$\begin{tikzcd} I \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \sqcup Y \end{tikzcd}$$

shows that the coproduct $X \sqcup Y$ is an acyclic cofibrant object.

So $\mathcal{P}$ is closed under finite coproducts in $\mathcal{C}$.

In conclusion, each object of $\mathcal{C}$ has a simplicial object in $\mathcal{P} = \mathcal{C}_{\text{ac,cof}}$ as simplicial resolution; cf. Definition 66.

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Zusammenfassung

Sei $\mathcal{C}$ eine Kategorie mit

(1) endlichen Limiten und

(2) einer auflösenden Unterkategorie $\mathcal{P}$.

Dabei ist eine auflösende Unterkategorie eine volle Teilkategorie, deren Objekte Eigenschaften haben, die die Eigenschaften der projektiven Moduln in einer Modulkategorie verallgemeinern.

Beispiele für $\mathcal{P} \subseteq \mathcal{C}$ sind die freien Gruppen in den Gruppen und kofasernde Objekte in Modellkategorien.

Wir zeigen, dass aufgrund von (1) in $\mathcal{C}$ simpliziale Kerne existieren, d.h. zu einem Tupel von Morphismen $(X \xrightarrow{f_i} Y)_{i \in [0,n]}$ existiert ein universelles Tupel $(K \xrightarrow{k_i} X)_{i \in [0,n+1]}$ so, dass für $0 \leqslant i < j \leqslant n+1$ gilt:

$$k_j f_i = k_i f_{j-1} \tag{3}$$

Wir können ein Objekt $X \in \operatorname{Ob} \mathcal{C}$ *semisimplizial auflösen* durch schrittweise Konstruktion eines Diagramms der Form

$$\begin{tikzcd}
\dots P_2 \arrow[rr, shift left=2, "d_0^2"] \arrow[rr, "d_1^2"] \arrow[rr, shift right=2, "d_2^2"'] \arrow[dr, "f_2"'] & & P_1 \arrow[rr, shift left, "d_0^1"] \arrow[rr, shift right, "d_1^1"'] \arrow[dr, "f_1"'] & & P_0 \arrow[r, "f_0"] & X \,, \\
& K_2 \arrow[ur, bend left, "k_0^2"] \arrow[ur, "k_1^2"] \arrow[ur, bend right, "k_2^2"'] & & K_1 \arrow[ur, bend left, "k_0^1"] \arrow[ur, bend right, "k_1^1"'] & &
\end{tikzcd}$$

wobei $P_i \xrightarrow{f_i} K_i$ eine Auflösung ist und wobei K_i ein simplizialer Kern des Diagramms aus P_i, P_{i-1} und den zwischenliegenden Morphismen ist.

Sei nun Δ die Simplexkategorie, die als Objekte endliche Intervalle und als Morphismen monotone Abbildungen hat. Sei darin Δ_{inj} die Teilkategorie aus injektiven monotonen Abbildungen. Ein kontravarianter Funktor von Δ_{inj} nach $\mathcal{C}$ heißt dann semisimpliziales Objekt in $\mathcal{C}$. Ein kontravarianter Funktor von Δ nach $\mathcal{C}$ heißt dann simpliziales Objekt in $\mathcal{C}$.

Die Relation (3) führt dazu, dass wir aus dem konstruierten Diagramm ein semisimpliziales Objekt erhalten, eine *semisimpliziale Auflösung* von X.

Um daraus ein simpliziales Objekt zu erhalten, konstruieren wir den zum Vergissfunktor linksadjungierten Funktor, der aus einem semisimplizialen ein simpliziales Objekt macht.

Anwendung dieses Funktors auf diese semisimpliziale Auflösung von X liefert eine *simpliziale Auflösung* von X wie gesucht.

Hiermit versichere ich,

(1) dass ich meine Arbeit selbstständig verfasst habe,

(2) dass ich keine anderen als die angegebenen Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,

(3) dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und

(4) dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, April 2017

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