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Sheet 14

Problem 32 Let R be an integral domain. Let $p \in R^\times \smallsetminus \mathrm{U}(R)$.
Consider the grading category $\mathbf{Z}$.
Define a $\mathbf{Z}$-graded module A by letting

$$\begin{array}{lcl} A^{-1} & := & \{ \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right) \in R^{2\times 2} \,:\, c \in R \} \\ A^0 & := & \{ \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in R^{2\times 2} \,:\, a,\, d \,\in\, R \} \\ A^1 & := & \{ \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in R^{2\times 2} \,:\, b \in R \} \\ A^i & := & 0 \qquad \text{for } i \in \mathbf{Z} \smallsetminus \{-1, 0, 1\} \end{array}$$

Define the shift-graded linear map $m_1 : A \to A$ of degree 1 as follows.

$$\begin{array}{rcl} A^{-1} & \xrightarrow{m_1} & A^0 \\ \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right) & \mapsto & p \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \\ A^0 & \xrightarrow{m_1} & A^1 \\ \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) & \mapsto & p \left(\begin{smallmatrix} 0 & a-d \\ 0 & 0 \end{smallmatrix}\right) \end{array}$$

Define the shift-graded linear map $m_2 : A^{\otimes 2} \to A$ of degree 0 as follows. Suppose given $i,\, j \,\in\, \{-1, 0, 1\}$. Let

$$\begin{array}{ccccc} A^i & \otimes & A^j & \xrightarrow{m_2} & A^{i+j} \\ S & \otimes & T & \mapsto & ST\,. \end{array}$$

(1) Show that A is a classical differential graded algebra.

(2) Calculate the $\mathbf{Z}$-graded modules $\mathrm{Z}A$, $\mathrm{B}A$ and $\mathrm{H}A$.

(3) Show that there is no structure $(\tilde{m}_k)_k$ of a minimal A_∞-algebra on $\mathrm{H}A$ over $\mathbf{Z}$ such that there exists a quasiisomorphism from $\mathrm{H}A$ to A.

(4) Show that there is no structure $(\tilde{m}_k)_k$ of a minimal A_∞-algebra on $\mathrm{H}A$ over $\mathbf{Z}$ such that there exists a quasiisomorphism from A to $\mathrm{H}A$.

(5) Let C be the complex having at positions 0 and 1 the differential $R \xrightarrow{p} R$, and zero objects elsewhere.

    Show that A is *strictly isomorphic* to the regular differential graded category $\tilde{A}$ on (C), which is in fact a classical differential graded algebra. By this we mean that there is a piecewise bijective shift-graded linear map $g : A \to \tilde{A}$ of degree 0 such that $\mathrm{strict}_\infty(g) : A \to \tilde{A}$ is a morphism of A_∞-algebras; cf. Problem 23.(4).