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A_∞-categories, WS 16/17

Sheet 12

Problem 29 Let $A = (A, (m_1))$ be an A_1-algebra over the grading category $\mathbf{Z}$.
Suppose that $A^z = 0$ for $z \in \mathbf{Z} \smallsetminus \{0, 1\}$.
Suppose given augmented projective resolutions

$$\cdots \to \tilde{A}^{\langle 3\rangle,-2} \xrightarrow{d^{\langle 3\rangle,-2}} \tilde{A}^{\langle 2\rangle,-1} \xrightarrow{d^{\langle 2\rangle,-1}} \tilde{A}^{\langle 1\rangle,0} \xrightarrow{d^{\langle 1\rangle,0}} \tilde{A}^{\langle 0\rangle,1} \xrightarrow{\varepsilon^1} (\mathrm{H}A)^1$$

and

$$\cdots \to \tilde{A}^{\langle 3\rangle,-3} \xrightarrow{d^{\langle 3\rangle,-3}} \tilde{A}^{\langle 2\rangle,-2} \xrightarrow{d^{\langle 2\rangle,-2}} \tilde{A}^{\langle 1\rangle,-1} \xrightarrow{d^{\langle 1\rangle,-1}} \tilde{A}^{\langle 0\rangle,0} \xrightarrow{\varepsilon^0} (\mathrm{H}A)^0 \ .$$

(1) Construct a minimal eA_1-algebra $\tilde{A} = (\tilde{A}, (\tilde{m}_1), (\tilde{A}^{\langle i\rangle})_i)$ over $\mathbf{Z}$ and a quasiisomorphism $(q_1) : \tilde{A} \to A$ such that the following holds.

For $z \in \mathbf{Z}$, we have

$$(\tilde{A}^{-z} \xrightarrow{\tilde{m}_1^{-z}} \tilde{A}^{-z+1}) \;=\; (\tilde{A}^{\langle z+1\rangle,-z} \oplus \tilde{A}^{\langle z\rangle,-z} \xrightarrow{\begin{pmatrix} d^{\langle z+1\rangle,-z} & e^{\langle z+1\rangle,-z} \\ 0 & d^{\langle z\rangle,-z+1} \end{pmatrix}} \tilde{A}^{\langle z\rangle,-z+1} \oplus \tilde{A}^{\langle z-1\rangle,-z+1})$$

for some linear maps $e^{\langle z+1\rangle,-z} : \tilde{A}^{\langle z+1\rangle,-z} \to \tilde{A}^{\langle z-1\rangle,-z+1}$. Cf. proof of Proposition 60.

(2) In the course of the construction in (1), write

$$(\tilde{A}^{-z} \xrightarrow{q_1^{-z}} A^{-z}) \;=:\; (\tilde{A}^{\langle z+1\rangle,-z} \oplus \tilde{A}^{\langle z\rangle,-z} \xrightarrow{\begin{pmatrix} q^{\langle z+1\rangle,-z} \\ q^{\langle z\rangle,-z} \end{pmatrix}} A^{-z})$$

for $z \in \mathbf{Z}$.

Show that we may choose $q^{\langle 1\rangle,0} = 0$ if and only if the residue class map $A^1 \to A^1/(A^0)m_1^0$ is a retraction.

(3) Suppose that $q^{\langle 1\rangle,0} = 0$; cf. (2).

When is it possible to choose $e^{\langle 2\rangle,-1} = 0$?