

The integral double Burnside ring of the symmetric group S_3

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The double Burnside R -algebra $B_R(G, G)$ of a finite group G with coefficients in a commutative ring R has been introduced by S. Bouc. It is R -linearly generated by finite (G, G) -bisets, modulo a relation identifying disjoint union and sum. Its multiplication is induced by the tensor product. In his thesis at NUI Galway, B. Masterson described $B_{\mathbf{Q}}(S_3, S_3)$ as a subalgebra of $\mathbf{Q}^{8 \times 8}$. We give a variant of this description and continue to describe $B_R(S_3, S_3)$ for $R \in \{\mathbf{Z}, \mathbf{Z}_{(2)}, \mathbf{F}_2, \mathbf{Z}_{(3)}, \mathbf{F}_3\}$ via congruences as suborders of certain R -orders respectively via path algebras over R .

0 Introduction

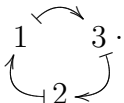
0.1 Groups

Groups describe symmetries of objects. That is to say, any mathematical object X has a symmetry group, called automorphism group $\text{Aut}(X)$, consisting of isomorphisms from X to X . For instance, for a natural number n , the set $\{1, 2, \dots, n\}$ has as automorphism group the symmetric group $\text{Aut}(\{1, 2, \dots, n\}) = S_n$. This group consists of all bijections from $\{1, 2, \dots, n\}$ to itself. For example, we obtain

$$\begin{aligned} S_3 &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\} \\ &= \{\text{id}, (1, 2), (1, 3), (2, 3), (1, 2, 3), (1, 3, 2)\} . \end{aligned}$$

In the first row, $\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$ is the map sending $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$.

In the second row, we have used the cycle notation, e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1, 3, 2)$, the latter

meaning 

We multiply by composition, e.g. $(1, 2) \bullet (1, 3) = (1, 2, 3)$.

By a theorem of Cayley, any finite group is isomorphic to a subgroup of S_n for some n .

0.2 The Biset category and biset functors

Suppose given finite groups H and G . An (H, G) -biset X is a finite set X together with a multiplication with elements of H on the left and a multiplication with elements of G on the right that commute with each other, i.e.

$$(h \cdot x) \cdot g = h \cdot (x \cdot g) =: h \cdot x \cdot g$$

for $h \in H$, $g \in G$ and $x \in X$.

As a first example, $M_1 := S_3$ is a (S_2, S_3) -biset via multiplication in S_3 . So for $h \in S_2 = \{\text{id}, (1, 2)\}$, $g \in S_3$ and $x \in M_1$ we let $h \cdot x \cdot g := h \bullet x \bullet g$.

As a second example, consider the cyclic group $C_3 = \{\text{id}, (1, 2, 3), (1, 3, 2)\}$ and the group isomorphism $\alpha : C_3 \rightarrow C_3$, $x \mapsto x^2$. Then the set $M_2 := C_3$ is a (C_3, C_3) -biset, on the left via multiplication, on the right via application of α and then multiplication. E.g.

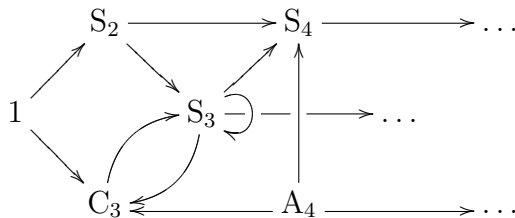
$$\begin{aligned} (1, 2, 3) \cdot (1, 3, 2) \cdot (1, 3, 2) &= (1, 2, 3) \bullet (1, 3, 2) \bullet \alpha((1, 3, 2)) \\ &= (1, 2, 3) \bullet (1, 3, 2) \bullet (1, 2, 3) = (1, 2, 3) . \end{aligned}$$

Suppose given a commutative ring R . S. Bouc introduced the *biset category* Biset_R , see [5, §3.1], see also the historical comments in [5, §1.4]. As objects, the category Biset_R has finite groups. The R -module of morphisms between two finite groups H and G is given by the double Burnside R -module $\text{Biset}_R(H, G) = B_R(H, G)$, which is R -linearly generated by finite (H, G) -bisets, modulo a relation identifying disjoint union and sum.

In particular, each (H, G) -biset M yields a morphism $H \xrightarrow{[M]} G$ in Biset_R . Composition of morphisms in Biset_R is given by a tensor product operation on bisets that is similar to the tensor product of bimodules. Given an (H, G) -biset M and an (G, K) -biset N , we write $M \times_G N$ for their tensor product, which is an (H, K) -biset. So in Biset_R , we have the commutative triangle

$$\begin{array}{ccc} H & \xrightarrow{[M \times_G N]} & K \\ & \searrow [M] & \nearrow [N] \\ & G & \end{array} .$$

The category Biset_R may roughly be imagined by a picture like this.



Here, A_4 is the alternating group on 4 elements. Each biset yields an arrow, and so does each R -linear combination of bisets. Of course, there are many more objects in Biset_R – each finite group is an object there – and many more arrows between them that are not in our picture.

0.3 Biset functors

Let \mathcal{X} and \mathcal{Y} be classes of finite groups closed under forming subgroups, factor groups and extensions. Following Bouc [3, §3.4.1], we say that an (H, G) -biset M is $(\mathcal{X}, \mathcal{Y})$ -free if for each $m \in M$ the left stabilizer of m in H is in \mathcal{X} and the right stabilizer of m in G is in \mathcal{Y} . We have the subcategory $\text{Biset}_R^{\mathcal{X}, \mathcal{Y}}$ of Biset_R : As objects, it has finite groups. The R -module of morphisms in $\text{Biset}_R^{\mathcal{X}, \mathcal{Y}}$ between two finite groups H and G is given by the submodule of $B_R(H, G)$ generated by the images of $(\mathcal{X}, \mathcal{Y})$ -free (H, G) -bisets, cf. [3, Lemme 4].

Certain classical theories may now be formulated as contravariant functors from $\text{Biset}_R^{\mathcal{X}, \mathcal{Y}}$ to the category of R -modules, called *biset functors* over R .

Consider a prime number p . Let \mathcal{X} be the class of all finite groups. Let \mathcal{Y} be the class of finite groups whose orders are not divisible by p . Then e.g. the (S_2, S_3) -biset M_1 and the (C_3, C_3) -biset M_2 from §0.2 yield morphisms in $\text{Biset}_{\mathbf{Z}}^{\mathcal{X}, \mathcal{Y}}$.

Suppose given an object of $\text{Biset}_{\mathbf{Z}}^{\mathcal{X}, \mathcal{Y}}$, i.e. a finite group G . Let

$$\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z} = \{0, \dots, p-1\},$$

where we agree to calculate modulo p . An \mathbf{F}_p -representation of G is a finite dimensional \mathbf{F}_p -vector space V , together with a left multiplication with elements of G . Such a representation is called simple if it does not have a nontrivial subrepresentation. Each representation has a sequence of subrepresentations with simple steps, called composition factors.

Let $\text{Rep}_{\mathbf{F}_p}(G)$ be the free abelian group on the set of isoclasses of simple representations. Each \mathbf{F}_p -representation V of G yields an element $[V]$ in $\text{Rep}_{\mathbf{F}_p}(G)$, namely the formal sum of its composition factors. Given finite groups H and G and an (H, G) -biset M , we obtain the map

$$\begin{array}{ccc} \text{Rep}_{\mathbf{F}_p}(G) & \xrightarrow{\text{Rep}_{\mathbf{F}_p}([M])} & \text{Rep}_{\mathbf{F}_p}(H) \\ [V] & \mapsto & [\mathbf{F}_p M \otimes_{\mathbf{F}_p G} V], \end{array}$$

using the usual tensor product over rings.

These constructions furnish a contravariant \mathbf{Z} -linear functor $\text{Rep}_{\mathbf{F}_p}$ from $\text{Biset}_{\mathbf{Z}}^{\mathcal{X}, \mathcal{Y}}$ to the category of \mathbf{Z} -modules, i.e. to the category of abelian groups. In particular, using the bisets M_1 and M_2 from §0.2, we obtain the maps

$$\begin{array}{ccc} \text{Rep}_{\mathbf{F}_p}(S_3) & \xrightarrow{\text{Rep}_{\mathbf{F}_p}([M_1])} & \text{Rep}_{\mathbf{F}_p}(S_2) \\ [V] & \mapsto & [\text{restriction of } V \text{ to } S_2] \end{array}$$

and

$$\begin{array}{ccc} \text{Rep}_{\mathbf{F}_p}(C_3) & \xrightarrow{\text{Rep}_{\mathbf{F}_p}([M_2])} & \text{Rep}_{\mathbf{F}_p}(C_3) \\ [V] & \mapsto & [\text{twist of } V \text{ with } \alpha]. \end{array}$$

Note that, if $p \leq n$, even the simple \mathbf{F}_p -representations of S_n are not entirely known: One knows a construction, due to James [9], but one does not know their \mathbf{F}_p -dimensions. Biset functors do not directly aim to solve this problem, but at any rate they are a tool to work with these representations.

0.4 Globally-defined Mackey functors

There is an equivalence of categories between the category of biset functors over R and the category of *globally-defined Mackey functors* $\text{Mack}_R^{\mathcal{X}, \mathcal{Y}}$ [6, §8]. Here, a globally-defined Mackey functor, with respect to \mathcal{X} and \mathcal{Y} , maps groups to R -modules and each group morphism α covariantly to an R -module morphism α_* , provided $\text{kern}(\alpha) \in \mathcal{Y}$, and contravariantly to α^* , provided $\text{kern}(\alpha) \in \mathcal{X}$. It is required that these morphisms satisfy a list of compatibilities, amongst which a Mackey formula, see e.g. [6, §8]. By that equivalence, these requirements on a Mackey functor can now be viewed as properties that result from being a contravariant functor from $\text{Biset}_R^{\mathcal{X}, \mathcal{Y}}$ to $R\text{-Mod}$.

0.5 Further examples

We list two examples of biset functors, [6, §8].

- Let $\mathcal{X} = \{1\}$ and let \mathcal{Y} consist of all finite groups. Let $n \geq 0$. Consider the biset functor $\text{Biset}_{\mathbf{Z}}^{\mathcal{X}, \mathcal{Y}} \rightarrow \mathbf{Z}\text{-Mod}$ that maps a finite group G to the algebraic K-theory $K_n(\mathbf{Z}G)$ of $\mathbf{Z}G$.
- Let \mathcal{X} consist of all finite groups and let $\mathcal{Y} = \{1\}$. Let $n \geq 0$. Consider the biset functor $\text{Biset}_R^{\mathcal{X}, \mathcal{Y}} \rightarrow R\text{-Mod}$ that maps a finite group G to the cohomology $H^n(G, R)$ of G with trivial coefficients.

For some more examples, see [6, §8]. The example of the classical Burnside ring, depending on a group G , is also explained in [4, §6.1].

0.6 The double Burnside algebra

Suppose given a finite group G , i.e an object of Biset_R . Its endomorphism ring $B_R(G, G)$ in the category Biset_R is called *double Burnside algebra* of G .

The isomorphism classes of finite transitive (G, G) -bisets form an R -linear basis of $B_R(G, G)$. In particular, if we choose a system $\mathcal{L}_{G \times G}$ of representatives for the conjugacy classes of subgroups of $G \times G$, we have the R -linear basis $([(G \times G)/U] : U \in \mathcal{L}_{G \times G})$.

If G is cyclic and if R is a field in which $|G|$ and $\varphi(|G|)$ are invertible, where φ denotes Euler's totient function, then the double Burnside algebra $B_R(G, G)$ is semisimple. This is shown in [7, Theorem 8.11, Remark 8.12(a)].

In case of $G = S_3$, we have 22 conjugacy classes of subgroups of $S_3 \times S_3$ and thus $\text{rk}_R(B_R(S_3, S_3)) = 22$. The double Burnside \mathbf{Q} -algebra $B_{\mathbf{Q}}(S_3, S_3)$ has been described by B. Masterson [1, §8] and then by B. Masterson and G. Pfeiffer [2, §7]. We describe $B_{\mathbf{Q}}(S_3, S_3)$ independently, using a direct Magma-supported calculation [10], with the aim of being able to pass from $B_{\mathbf{Q}}(S_3, S_3)$ to $B_{\mathbf{Z}}(S_3, S_3)$ in the sequel.

In order to do that, we first restate some preliminaries on bisets and the double Burnside ring in §1 and construct a \mathbf{Z} -linear basis of $B_{\mathbf{Z}}(S_3, S_3)$ in §2.

In §3 we tackle the problem that the double Burnside \mathbf{Q} -algebra $B_{\mathbf{Q}}(S_3, S_3)$ is not semisimple [5, Proposition 6.1.5], thus not isomorphic to a direct product of matrix rings. As a substitute, we use a suitable isomorphic copy A of $B_{\mathbf{Q}}(S_3, S_3)$. We obtain this copy using

a Peirce decomposition of $B_{\mathbf{Q}}(S_3, S_3)$. In addition, we give a description of $B_{\mathbf{Q}}(S_3, S_3)$ as path algebra modulo relations.

The next step, in §4, is to pass from $B_{\mathbf{Q}}(S_3, S_3)$ to $B_{\mathbf{Z}}(S_3, S_3)$. We find a \mathbf{Z} -order $A_{\mathbf{Z}}$ inside A such that $A_{\mathbf{Z}}$ contains an isomorphic copy of $B_{\mathbf{Z}}(S_3, S_3)$, which we describe via congruences, cf. Proposition 5, Theorem 8.

$$\begin{array}{ccc} B_{\mathbf{Q}}(S_3, S_3) & \xrightarrow{\sim} & A \\ \uparrow & & \uparrow \\ B_{\mathbf{Z}}(S_3, S_3) & \xrightarrow{\text{injective}} & A_{\mathbf{Z}} \end{array}$$

We calculate a path algebra for $B_{\mathbf{Z}_{(2)}}(S_3, S_3)$, cf. Proposition 11. We deduce that $B_{\mathbf{F}_2}(S_3, S_3)$ is Morita equivalent to the path algebra

$$\mathbf{F}_2 \left[\begin{array}{ccccc} & & \tilde{\tau}_4 & & \\ & \tilde{\tau}_2 & \curvearrowright & & \\ \tilde{e}_3 & \rightleftarrows & \tilde{e}_5 & \rightleftarrows & \tilde{e}_4 \\ & \tilde{\tau}_1 & \curvearrowleft & & \\ & & \tilde{\tau}_3 & & \end{array} \right] / \left(\begin{array}{ccc} \tilde{\tau}_2 \tilde{\tau}_1 & , & \tilde{\tau}_2 \tilde{\tau}_3 & , & \tilde{\tau}_2 \tilde{\tau}_7 \\ \tilde{\tau}_4 \tilde{\tau}_1 & , & \tilde{\tau}_4 \tilde{\tau}_3 & , & \tilde{\tau}_4 \tilde{\tau}_7 \\ \tilde{\tau}_7 \tilde{\tau}_1 & , & \tilde{\tau}_7 \tilde{\tau}_3 & , & \tilde{\tau}_7^2 - \tilde{\tau}_1 \tilde{\tau}_2 \end{array} \right) ,$$

cf. Corollary 12.

We calculate a path algebra for $B_{\mathbf{Z}_{(3)}}(S_3, S_3)$, cf. Proposition 15. We deduce that $B_{\mathbf{F}_3}(S_3, S_3)$ is Morita equivalent to the path algebra

$$\mathbf{F}_3 \left[\begin{array}{ccccccc} & & \tilde{\tau}_2 & & \tilde{\tau}_4 & & \\ & & \curvearrowright & & \curvearrowright & & \\ \tilde{e}_5 & & \tilde{e}_3 & & \tilde{e}_6 & & \tilde{e}_4 \\ & & \tilde{\tau}_1 & & \tilde{\tau}_3 & & \end{array} \right] / (\tilde{\tau}_4 \tilde{\tau}_3, \tilde{\tau}_4 \tilde{\tau}_1, \tilde{\tau}_2 \tilde{\tau}_1, \tilde{\tau}_2 \tilde{\tau}_3) ,$$

cf. Corollary 16.

1 Preliminaries on bisets and the double Burnside algebra

Bisets. Recall that an (G, G) -biset X is a finite set X together with a left G and a right G -action that commute with each other, i.e. $(h \cdot x) \cdot g = h \cdot (x \cdot g) =: h \cdot x \cdot g$ for $h, g \in G$ and $x \in X$.

Every (G, G) -biset X can be regarded as a left $(G \times G)$ -set by setting $(h, g)x := hxg^{-1}$ for $(h, g) \in G \times G$ and $x \in X$. Likewise, every left $(G \times G)$ -set Y can be regarded as an (G, G) -biset by setting $h \cdot y \cdot g := (h, g^{-1})y$ for $h, g \in G$ and $y \in Y$. We freely use this identification.

Tensor product. Let M be an (G, G) -biset and let N be a (G, G) -biset. The cartesian product $M \times N$ is a (G, G) -biset via $h(m, n)p = (hm, np)$ for $h, p \in G$ and $(m, n) \in M \times N$. It becomes a left G -set via $g(m, n) = (mg^{-1}, gn)$ for $g \in G$ and $(m, n) \in M \times N$. We call the set of G -orbits on $M \times N$ the *tensor product* $M \times_G N$ of M and N . This also is an (G, G) -biset. The G -orbit of the element $(m, n) \in M \times N$ is denoted by $m \times_G n \in M \times_G N$. Moreover,

let L be a (G, G) -biset. Then $M \times_G (N \times_G L) \xrightarrow{\sim} (M \times_G N) \times_G L$, $m \times_G (n \times_G \ell) \mapsto (m \times_G n) \times_G \ell$ as (G, G) -bisets.

Double Burnside R -algebra. We denote by $B_R(G, G)$ the double Burnside R -algebra of G . Recall that $B_R(G, G)$ is the R -module freely generated by the isomorphism classes of finite (G, G) -bisets, modulo the relations $[M \sqcup N] = [M] + [N]$ for (G, G) -bisets M, N . Multiplication is defined by $[M] \cdot [N] = [M \times_G N]$ for (G, G) -bisets M, N . An R -linear basis of $B_R(G, G)$ is given by $([(G \times G)/U] : U \in \mathcal{L}_{G \times G})$, where we choose a system $\mathcal{L}_{G \times G}$ of representatives for the conjugacy classes of subgroups of $G \times G$. Moreover, $1_{B_{\mathbf{Z}}(G, G)} = [G]$.
Abbreviation. In case of $G = S_3$, we often abbreviate $B_R := B_R(S_3, S_3)$.

2 \mathbf{Z} -linear basis of $B_{\mathbf{Z}}(S_3, S_3)$

The following calculations were done using the computer algebra system Magma [10].

The group S_3 has the subgroups

$$V_0 := \{\text{id}\}, V_1 := \langle(1, 2)\rangle, V_2 := \langle(1, 3)\rangle, V_3 := \langle(2, 3)\rangle, V_4 := \langle(1, 2, 3)\rangle, V_5 := S_3.$$

The set $\{V_0, V_1, V_4, V_5\}$ is a system of representatives for the conjugacy classes of subgroups of S_3 . In S_3 , we write $a := (1, 2)$, $b := (1, 2, 3)$ and $1 := \text{id}$. So $V_1 = \langle a \rangle$, $V_4 = \langle b \rangle$ and $V_5 = \langle a, b \rangle$.

A system of representatives for the conjugacy classes of subgroups of $S_3 \times S_3$ is given by

$$\begin{aligned} U_{0,0} &:= V_0 \times V_0 = \{(1, 1)\}, & U_{4,1} &:= V_4 \times V_1 = \langle(b, 1), (1, a)\rangle, \\ U_{1,0} &:= V_1 \times V_0 = \langle(a, 1)\rangle, & U_{1,4} &:= V_1 \times V_4 = \langle(a, 1), (1, b)\rangle, \\ U_{0,1} &:= V_0 \times V_1 = \langle(1, a)\rangle, & U_7 &:= \langle(a, a), (b, 1)\rangle, \\ \Delta(V_1) &= \langle(a, a)\rangle, & \Delta(V_5) &= \langle(a, a), (b, b)\rangle, \\ U_{4,0} &:= V_4 \times V_0 = \langle(b, 1)\rangle, & U_{4,4} &:= V_4 \times V_4 = \langle(b, 1), (1, b)\rangle, \\ U_{0,4} &:= V_0 \times V_4 = \langle(1, b)\rangle, & U_{1,5} &:= V_1 \times V_5 = \langle(a, 1), (1, a), (1, b)\rangle, \\ \Delta(V_4) &= \langle(b, b)\rangle, & U_{5,1} &:= V_5 \times V_1 = \langle(a, 1), (b, 1), (1, a)\rangle, \\ U_{1,1} &:= V_1 \times V_1 = \langle(a, 1), (1, a)\rangle, & U_{4,5} &:= V_4 \times V_5 = \langle(b, 1), (1, a), (1, b)\rangle, \\ U_{5,0} &:= V_5 \times V_0 = \langle(a, 1), (b, 1)\rangle, & U_{5,4} &:= V_5 \times V_4 = \langle(a, 1), (b, 1), (1, b)\rangle, \\ U_{0,5} &:= V_0 \times V_5 = \langle(1, a), (1, b)\rangle, & U_8 &:= \langle(a, a), (b, 1), (1, b)\rangle, \\ U_6 &:= \langle(a, a), (1, b)\rangle, & U_{5,5} &:= V_5 \times V_5 = \langle(a, 1), (1, a), (b, 1), (1, b)\rangle. \end{aligned}$$

Let $H_{i,j} := [(S_3 \times S_3)/U_{i,j}]$ for $i, j \in \{0, 1, 4, 5\}$, $H_s := [(S_3 \times S_3)/U_s]$ for $s \in [6, 8]$ and $H_t^\Delta := [(S_3 \times S_3)/\Delta(V_t)]$ for $t \in \{1, 4, 5\}$.

So we obtain the \mathbf{Z} -linear basis

$$\mathcal{H} := (H_{0,0}, H_{1,0}, H_{0,1}, H_1^\Delta, H_{4,0}, H_{0,4}, H_4^\Delta, H_{1,1}, H_{5,0}, H_{0,5}, H_6, \\ H_{4,1}, H_{1,4}, H_7, H_5^\Delta, H_{4,4}, H_{1,5}, H_{5,1}, H_{4,5}, H_{5,4}, H_8, H_{5,5})$$

of $B_{\mathbf{Z}}(S_3, S_3)$. Of course, \mathcal{H} is also a \mathbf{Q} -linear basis of $B_{\mathbf{Q}}(S_3, S_3)$.

3 $B_{\mathbf{Q}}(S_3, S_3)$

3.1 Peirce decomposition of $B_{\mathbf{Q}}(S_3, S_3)$

Using Magma [10] we obtain an orthogonal decomposition of $1_{B_{\mathbf{Q}}}$ into the following idempotents of $B_{\mathbf{Q}} = B_{\mathbf{Q}}(S_3, S_3)$.

$$\begin{aligned} e &:= -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0} & \varepsilon_2 &:= -H_{0,0} + H_{1,0} + H_{0,1} + H_1^\Delta - 2H_{1,1} \\ g &:= \frac{4}{3}H_{0,0} - 2H_{1,0} - \frac{4}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1} & \varepsilon_3 &:= -\frac{1}{4}H_{0,0} + \frac{1}{4}H_{4,0} + \frac{1}{4}H_{0,4} + \frac{1}{2}H_4^\Delta - \frac{3}{4}H_{4,4} \\ h &:= -\frac{1}{12}H_{0,0} + \frac{1}{3}H_{0,1} + \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{4}H_{4,4} - H_{4,1} & \varepsilon_4 &:= \frac{1}{2}H_{0,0} - H_1^\Delta - \frac{1}{2}H_4^\Delta + H_5^\Delta \end{aligned}$$

Write $\varepsilon_1 := e + g + h$. In Remark 1 and Remark 3, we shall see that these idempotents are primitive.

In a next step, we fix \mathbf{Q} -linear bases of the Peirce components.

Peirce component	\mathbf{Q} -linear basis
$e B_{\mathbf{Q}} e$	$e = -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0}$
$e B_{\mathbf{Q}} g$	$b_{e,g} := \frac{1}{2}H_{0,0} - H_{1,0} - \frac{1}{2}H_{0,1} - \frac{1}{2}H_{4,0} + H_{1,1} + \frac{1}{2}H_{4,1}$
$e B_{\mathbf{Q}} h$	$b_{e,h} := -\frac{1}{8}H_{0,0} + \frac{1}{4}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{8}H_{4,0} - \frac{3}{8}H_{0,4} - H_{1,1} - \frac{1}{2}H_{4,1} + \frac{3}{4}H_{1,4} + \frac{3}{8}H_{4,4}$
$g B_{\mathbf{Q}} e$	$b_{g,e} := -\frac{4}{3}H_{0,0} + 2H_{1,0} + H_{4,0}$
$g B_{\mathbf{Q}} g$	$g = \frac{4}{3}H_{0,0} - 2H_{1,0} - \frac{4}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1}$
$g B_{\mathbf{Q}} h$	$b_{g,h} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{4}{3}H_{0,1} + \frac{1}{4}H_{4,0} - H_{0,4} - 2H_{1,1} - H_{4,1} + \frac{3}{2}H_{1,4} + \frac{3}{4}H_{4,4}$
$h B_{\mathbf{Q}} e$	$b_{h,e} := -\frac{1}{3}H_{0,0} + H_{4,0}$
$h B_{\mathbf{Q}} g$	$b_{h,g} := \frac{1}{3}H_{0,0} - \frac{1}{3}H_{0,1} - H_{4,0} + H_{4,1}$
$h B_{\mathbf{Q}} h$	$h = -\frac{1}{12}H_{0,0} + \frac{1}{3}H_{0,1} + \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{4}H_{4,4} - H_{4,1}$
$e B_{\mathbf{Q}} \varepsilon_4$	$b_{e,\varepsilon_4} := -\frac{1}{8}H_{0,0} + \frac{1}{4}H_{1,0} + \frac{1}{4}H_{0,1} + \frac{1}{8}H_{4,0} + \frac{1}{8}H_{0,4} - \frac{1}{2}H_{1,1} - \frac{1}{4}H_{0,5} - \frac{1}{4}H_{4,1} - \frac{1}{4}H_{1,4} - \frac{1}{8}H_{4,4} + \frac{1}{2}H_{1,5} + \frac{1}{4}H_{4,5}$
$g B_{\mathbf{Q}} \varepsilon_4$	$b_{g,\varepsilon_4} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{2}{3}H_{0,1} + \frac{1}{4}H_{4,0} + \frac{1}{3}H_{0,4} - H_{1,1} - \frac{2}{3}H_{0,5} - \frac{1}{2}H_{4,1} - \frac{1}{2}H_{1,4} - \frac{1}{4}H_{4,4} + H_{1,5} + \frac{1}{2}H_{4,5}$
$h B_{\mathbf{Q}} \varepsilon_4$	$b_{h,\varepsilon_4} := -\frac{1}{12}H_{0,0} + \frac{1}{6}H_{0,1} + \frac{1}{4}H_{4,0} + \frac{1}{12}H_{0,4} - \frac{1}{6}H_{0,5} - \frac{1}{2}H_{4,1} - \frac{1}{4}H_{4,4} + \frac{1}{2}H_{4,5}$
$\varepsilon_2 B_{\mathbf{Q}} \varepsilon_2$	$\varepsilon_2 = -H_{0,0} + H_{1,0} + H_{0,1} + H_1^\Delta - 2H_{1,1}$
$\varepsilon_2 B_{\mathbf{Q}} \varepsilon_4$	$b_{\varepsilon_2,\varepsilon_4} := -\frac{1}{2}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_1^\Delta + \frac{1}{2}H_{0,4} - H_{1,1} - \frac{1}{2}H_{0,5} - \frac{1}{2}H_6 - \frac{1}{2}H_{1,4} + H_{1,5}$
$\varepsilon_3 B_{\mathbf{Q}} \varepsilon_3$	$\varepsilon_3 = -\frac{1}{4}H_{0,0} + \frac{1}{4}H_{4,0} + \frac{1}{4}H_{0,4} + \frac{1}{2}H_4^\Delta - \frac{3}{4}H_{4,4}$
$\varepsilon_4 B_{\mathbf{Q}} e$	$b_{\varepsilon_4,e} := \frac{1}{6}H_{0,0} - \frac{1}{3}H_{1,0} - \frac{1}{6}H_{4,0} + \frac{1}{3}H_{5,0}$
$\varepsilon_4 B_{\mathbf{Q}} g$	$b_{\varepsilon_4,g} := -\frac{1}{6}H_{0,0} + \frac{1}{3}H_{1,0} + \frac{1}{6}H_{0,1} + \frac{1}{6}H_{4,0} - \frac{1}{3}H_{1,1} - \frac{1}{3}H_{5,0} - \frac{1}{6}H_{4,1} + \frac{1}{3}H_{5,1}$
$\varepsilon_4 B_{\mathbf{Q}} h$	$b_{\varepsilon_4,h} := \frac{1}{24}H_{0,0} - \frac{1}{12}H_{1,0} - \frac{1}{6}H_{0,1} - \frac{1}{24}H_{4,0} + \frac{1}{8}H_{0,4} + \frac{1}{3}H_{1,1} + \frac{1}{12}H_{5,0} + \frac{1}{6}H_{4,1} - \frac{1}{4}H_{1,4} - \frac{1}{8}H_{4,4} - \frac{1}{3}H_{5,1} + \frac{1}{4}H_{5,4}$
$\varepsilon_4 B_{\mathbf{Q}} \varepsilon_2$	$b_{\varepsilon_4,\varepsilon_2} := -\frac{1}{2}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_1^\Delta + \frac{1}{2}H_{4,0} - H_{1,1} - \frac{1}{2}H_{5,0} - \frac{1}{2}H_{4,1} - \frac{1}{2}H_7 + H_{5,1}$
$\varepsilon_4 B_{\mathbf{Q}} \varepsilon_4$	$\varepsilon_4 = \frac{1}{2}H_{0,0} - H_1^\Delta - \frac{1}{2}H_4^\Delta + H_5^\Delta,$ $b'_{\varepsilon_4,\varepsilon_4} := \frac{1}{24}H_{0,0} - \frac{1}{12}H_{1,0} - \frac{1}{12}H_{0,1} - \frac{1}{24}H_{4,0} - \frac{1}{24}H_{0,4} + \frac{1}{6}H_{1,1} + \frac{1}{12}H_{5,0} + \frac{1}{12}H_{0,5} + \frac{1}{12}H_{4,1} + \frac{1}{12}H_{1,4} + \frac{1}{24}H_{4,4} - \frac{1}{6}H_{1,5} - \frac{1}{6}H_{5,1} - \frac{1}{12}H_{4,5} - \frac{1}{12}H_{5,4} + \frac{1}{6}H_{5,5},$ $b''_{\varepsilon_4,\varepsilon_4} := \frac{1}{4}H_{0,0} - \frac{3}{4}H_{1,0} - \frac{3}{4}H_{0,1} + \frac{1}{4}H_1^\Delta - \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{2}H_{1,1} + \frac{3}{4}H_{5,0} + \frac{3}{4}H_{0,5} - \frac{1}{4}H_6 + \frac{3}{4}H_{4,1} + \frac{3}{4}H_{1,4} - \frac{1}{4}H_7 + \frac{1}{4}H_{4,4} - \frac{3}{2}H_{1,5} - \frac{3}{2}H_{5,1} - \frac{3}{4}H_{4,5} - \frac{3}{4}H_{5,4} + \frac{1}{4}H_8 + \frac{3}{2}H_{5,5}$

Remark 1. The idempotents $e, g, h, \varepsilon_2, \varepsilon_3$ are primitive, as $eB_{\mathbf{Q}}e \cong \mathbf{Q}$, $gB_{\mathbf{Q}}g \cong \mathbf{Q}$, $hB_{\mathbf{Q}}h \cong \mathbf{Q}$, $\varepsilon_2B_{\mathbf{Q}}\varepsilon_2 \cong \mathbf{Q}$ and $\varepsilon_3B_{\mathbf{Q}}\varepsilon_3 \cong \mathbf{Q}$.

We have the following multiplication table for the basis elements of $B_{\mathbf{Q}} = B_{\mathbf{Q}}(S_3, S_3)$.

(\cdot)	e	$b_{e,g}$	$b_{e,h}$	$b_{g,e}$	g	$b_{g,h}$	$b_{h,e}$	$b_{h,g}$	h	b_{e,ε_4}	b_{g,ε_4}	b_{h,ε_4}	ε_2	$b_{\varepsilon_2,\varepsilon_4}$	ε_3	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	$b_{\varepsilon_4,\varepsilon_2}$	ε_4	$b'_{\varepsilon_4,\varepsilon_4}$	$b''_{\varepsilon_4,\varepsilon_4}$
e	e	$b_{e,g}$	$b_{e,h}$	0	0	0	0	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
$b_{e,g}$	0	0	0	e	$b_{e,g}$	$b_{e,h}$	0	0	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{e,h}$	0	0	0	0	0	e	$b_{e,g}$	$b_{e,h}$	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{g,e}$	$b_{g,e}$	g	$b_{g,h}$	0	0	0	0	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
g	0	0	0	$b_{g,e}$	g	$b_{g,h}$	0	0	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{g,h}$	0	0	0	0	0	$b_{g,e}$	g	$b_{g,h}$	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{h,e}$	$b_{h,e}$	$b_{h,g}$	h	0	0	0	0	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
$b_{h,g}$	0	0	0	$b_{h,e}$	$b_{h,g}$	h	0	0	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0
h	0	0	0	0	0	$b_{h,e}$	$b_{h,g}$	h	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0
b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{e,ε_4}	0	0
b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{g,ε_4}	0	0
b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{h,ε_4}	0	0
ε_2	0	0	0	0	0	0	0	0	0	0	0	0	ε_2	$b_{\varepsilon_2,\varepsilon_4}$	0	0	0	0	0	0	0	0
$b_{\varepsilon_2,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_2,\varepsilon_4}$	0	0
ε_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ε_3	0	0	0	0	0	0	0
$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,g}$	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,h}$	0	0	0	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,\varepsilon_2}$	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_4,\varepsilon_2}$	$b''_{\varepsilon_4,\varepsilon_4} - 12b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0
ε_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	$b_{\varepsilon_4,\varepsilon_2}$	ε_4	$b'_{\varepsilon_4,\varepsilon_4}$	$b''_{\varepsilon_4,\varepsilon_4}$
$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0
$b''_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b''_{\varepsilon_4,\varepsilon_4}$	0	0

We see that ε_3 is even a central element.

Lemma 2. Consider $\mathbf{Q}[\eta, \xi]/(\eta^2, \eta\xi, \xi^2) = \mathbf{Q}[\bar{\eta}, \bar{\xi}]$, where we let $\bar{\xi} := \xi + (\eta^2, \eta\xi, \xi^2)$ and $\bar{\eta} := \eta + (\eta^2, \eta\xi, \xi^2)$.

We have the \mathbf{Q} -algebra isomorphism

$$\begin{aligned} \mu : \mathbf{Q}[\bar{\eta}, \bar{\xi}] &\rightarrow \varepsilon_4 B_{\mathbf{Q}} \varepsilon_4 \\ \bar{\eta} &\mapsto b'_{\varepsilon_4,\varepsilon_4} \\ \bar{\xi} &\mapsto b''_{\varepsilon_4,\varepsilon_4}. \end{aligned}$$

Proof. Since $\varepsilon_4 B_{\mathbf{Q}} \varepsilon_4 = \mathbf{Q}\langle \varepsilon_4, b'_{\varepsilon_4,\varepsilon_4}, b''_{\varepsilon_4,\varepsilon_4} \rangle$ is commutative and $(b'_{\varepsilon_4,\varepsilon_4})^2 = 0$, $(b''_{\varepsilon_4,\varepsilon_4})^2 = 0$ and $b'_{\varepsilon_4,\varepsilon_4} b''_{\varepsilon_4,\varepsilon_4} = 0$, the map μ is a well-defined \mathbf{Q} -algebra morphism.

As the \mathbf{Q} -linear basis $(1, \bar{\eta}, \bar{\xi})$ is mapped to the \mathbf{Q} -linear basis $(\varepsilon_4, b'_{\varepsilon_4,\varepsilon_4}, b''_{\varepsilon_4,\varepsilon_4})$, it is bijective. \square

Remark 3. The ring $\mathbf{Q}[\bar{\eta}, \bar{\xi}]$ is local. In particular, ε_4 is a primitive idempotent of $B_{\mathbf{Q}}$.

Proof. We have $U(\mathbf{Q}[\bar{\eta}, \bar{\xi}]) = \mathbf{Q}[\bar{\eta}, \bar{\xi}] \setminus (\bar{\eta}, \bar{\xi})$, as for $u := a + b\bar{\eta} + c\bar{\xi}$ the inverse is given by $u^{-1} = a^{-1} - a^{-2}b\bar{\eta} - a^{-2}c\bar{\xi}$ for $a, b, c \in \mathbf{Q}$, with $a \neq 0$. Thus the nonunits of $\mathbf{Q}[\bar{\eta}, \bar{\xi}]$ form an ideal and so $\mathbf{Q}[\bar{\eta}, \bar{\xi}]$ is a local ring. \square

To standardize notation, we aim to construct a \mathbf{Q} -algebra $A := \bigoplus_{i,j} A_{i,j}$ with $A \cong \mathbf{B}_{\mathbf{Q}}(\mathbf{S}_3, \mathbf{S}_3)$.

In a first step to do so, we choose \mathbf{Q} -vector spaces $A_{i,j}$ and \mathbf{Q} -linear isomorphisms $\gamma_{i,j} : A_{i,j} \xrightarrow{\sim} \varepsilon_i \mathbf{B}_{\mathbf{Q}} \varepsilon_j$ for $i, j \in [1, 4]$. We define the tuple of \mathbf{Q} -vector spaces

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} := \begin{pmatrix} \mathbf{Q}^{3 \times 3} & 0 & 0 & \mathbf{Q}^{3 \times 1} \\ 0 & \mathbf{Q} & 0 & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} & 0 \\ \mathbf{Q}^{1 \times 3} & \mathbf{Q} & 0 & \mathbf{Q}[\bar{\eta}, \bar{\xi}] \end{pmatrix}, \text{ cf. Lemma 2.}$$

We have $\gamma_{s,t} = 0$ for $(s, t) \in \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 3)\}$.

Let

$$\begin{aligned} \gamma_{1,1} : A_{1,1} &\xrightarrow{\sim} \varepsilon_1 \mathbf{B}_{\mathbf{Q}} \varepsilon_1 & \gamma_{1,4} : A_{1,4} &\xrightarrow{\sim} \varepsilon_1 \mathbf{B}_{\mathbf{Q}} \varepsilon_4 \\ \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} &\mapsto \begin{pmatrix} r_{1,1}e & + r_{1,2}b_{e,g} & + r_{1,3}b_{e,h} \\ + r_{2,1}b_{g,e} & + r_{2,2}g & + r_{2,3}b_{g,h} \\ + r_{3,1}b_{h,e} & + r_{3,2}b_{h,g} & + r_{3,3}h \end{pmatrix}, & \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &\mapsto \begin{pmatrix} u_1b_{e,\varepsilon_4} \\ + u_2b_{g,\varepsilon_4} \\ + u_3b_{h,\varepsilon_4} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \gamma_{2,2} : A_{2,2} &\xrightarrow{\sim} \varepsilon_2 \mathbf{B}_{\mathbf{Q}} \varepsilon_2 & \gamma_{2,4} : A_{2,4} &\xrightarrow{\sim} \varepsilon_2 \mathbf{B}_{\mathbf{Q}} \varepsilon_4 \\ u &\mapsto u\varepsilon_2 & u &\mapsto ub_{\varepsilon_2, \varepsilon_4}, \end{aligned}$$

$$\begin{aligned} \gamma_{3,3} : A_{3,3} &\xrightarrow{\sim} \varepsilon_3 \mathbf{B}_{\mathbf{Q}} \varepsilon_3 & \gamma_{4,1} : A_{4,1} &\xrightarrow{\sim} \varepsilon_4 \mathbf{B}_{\mathbf{Q}} \varepsilon_1 \\ u &\mapsto u\varepsilon_3 & \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} &\mapsto v_1b_{\varepsilon_4, e} + v_2b_{\varepsilon_4, g} + v_3b_{\varepsilon_4, h}, \end{aligned}$$

$$\begin{aligned} \gamma_{4,2} : A_{4,2} &\xrightarrow{\sim} \varepsilon_4 \mathbf{B}_{\mathbf{Q}} \varepsilon_2 & \gamma_{4,4} \stackrel{\text{L.2}}{:=} \mu : A_{4,4} &\xrightarrow{\sim} \varepsilon_4 \mathbf{B}_{\mathbf{Q}} \varepsilon_4 \\ u &\mapsto ub_{\varepsilon_4, \varepsilon_2} & a + b\bar{\eta} + c\bar{\xi} &\mapsto a\varepsilon_4 + bb'_{\varepsilon_4, \varepsilon_4} + cb''_{\varepsilon_4, \varepsilon_4}. \end{aligned}$$

Let $\beta : \mathbf{B}_{\mathbf{Q}} \times \mathbf{B}_{\mathbf{Q}} \rightarrow \mathbf{B}_{\mathbf{Q}}$ be the multiplication map on $\mathbf{B}_{\mathbf{Q}}$. Write

$$\beta_{i,j,k} := \beta|_{\varepsilon_i \mathbf{B}_{\mathbf{Q}} \varepsilon_j \times \varepsilon_j \mathbf{B}_{\mathbf{Q}} \varepsilon_k}^{\varepsilon_i \mathbf{B}_{\mathbf{Q}} \varepsilon_k} : \varepsilon_i \mathbf{B}_{\mathbf{Q}} \varepsilon_j \times \varepsilon_j \mathbf{B}_{\mathbf{Q}} \varepsilon_k \rightarrow \varepsilon_i \mathbf{B}_{\mathbf{Q}} \varepsilon_k.$$

Now, we construct \mathbf{Q} -bilinear multiplication maps $\alpha_{i,j,k}$ for $i, j, k \in [1, 4]$ such that the following quadrangle of maps commutes.

$$\begin{array}{ccc} A_{i,j} \times A_{j,k} & \xrightarrow{\alpha_{i,j,k}} & A_{i,k} \\ \gamma_{i,j} \times \gamma_{j,k} \downarrow & & \downarrow \gamma_{i,k} \\ \varepsilon_i \mathbf{B}_{\mathbf{Q}} \varepsilon_j \times \varepsilon_j \mathbf{B}_{\mathbf{Q}} \varepsilon_k & \xrightarrow{\beta_{i,j,k}} & \varepsilon_i \mathbf{B}_{\mathbf{Q}} \varepsilon_k \end{array}$$

I.e. we set $\alpha_{i,j,k} := \gamma_{i,k}^{-1} \circ \beta_{i,j,k} \circ (\gamma_{i,j} \times \gamma_{j,k})$. This leads to

- $\alpha_{i,j,k} = 0$
if (i, j) , (j, k) or (i, k) is contained in $\{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 3)\}$
- $\alpha_{1,1,1} : A_{1,1} \times A_{1,1} \rightarrow A_{1,1}, (X, Y) \mapsto XY$

- $\alpha_{1,1,4} : A_{1,1} \times A_{1,4} \rightarrow A_{1,4}, (X, u) \mapsto Xu$
- $\alpha_{1,4,1} = 0$
- $\alpha_{1,4,4} : A_{1,4} \times A_{4,4} \rightarrow A_{1,4}, (u, a + b\bar{\eta} + c\bar{\xi}) \mapsto ua$
- $\alpha_{2,2,2} : A_{2,2} \times A_{2,2} \rightarrow A_{2,2}, (u, v) \mapsto uv$
- $\alpha_{2,2,4} : A_{2,2} \times A_{2,4} \rightarrow A_{2,4}, (u, v) \mapsto uv$
- $\alpha_{2,4,2} = 0$
- $\alpha_{2,4,4} : A_{2,4} \times A_{4,4} \rightarrow A_{2,4}, (u, a + b\bar{\eta} + c\bar{\xi}) \mapsto ua$
- $\alpha_{3,3,3} : A_{3,3} \times A_{3,3} \rightarrow A_{3,3}, (u, v) \mapsto uv$
- $\alpha_{4,1,1} : A_{4,1} \times A_{1,1} \rightarrow A_{4,1}, (v, X) \mapsto vX$
- $\alpha_{4,1,4} : A_{4,1} \times A_{1,4} \rightarrow A_{4,4}, (v, u) \mapsto v\bar{u}\eta$
- $\alpha_{4,2,2} : A_{4,2} \times A_{2,2} \rightarrow A_{4,2}, (u, v) \mapsto uv$
- $\alpha_{4,2,4} : A_{4,2} \times A_{2,4} \rightarrow A_{4,4}, (u, v) \mapsto uv(\bar{\xi} - 12\bar{\eta})$
- $\alpha_{4,4,1} : A_{4,4} \times A_{4,1} \rightarrow A_{4,1}, (a + b\bar{\eta} + c\bar{\xi}, v) \mapsto av$
- $\alpha_{4,4,2} : A_{4,4} \times A_{4,2} \rightarrow A_{4,2}, (a + b\bar{\eta} + c\bar{\xi}, v) \mapsto av$
- $\alpha_{4,4,4} : A_{4,4} \times A_{4,4} \rightarrow A_{4,4}, (a + b\bar{\eta} + c\bar{\xi}, \tilde{a} + \tilde{b}\bar{\eta} + \tilde{c}\bar{\xi}) \mapsto (a + b\bar{\eta} + c\bar{\xi}) \cdot (\tilde{a} + \tilde{b}\bar{\eta} + \tilde{c}\bar{\xi})$
where $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbf{Q}$

For convenience, we fix a notation similar to matrices and matrix multiplication.

Notation 4. Suppose given $r \in \mathbf{Z}_{\geq 0}$. Suppose given R -modules $M_{i,j}$ for $i, j \in [1, r]$. We write

$$\bigoplus_{i,j \in [1,r]} M_{i,j} =: \begin{bmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,r} \\ M_{2,1} & M_{2,2} & \dots & M_{2,r} \\ \vdots & \vdots & \dots & \vdots \\ M_{r,1} & M_{r,2} & \dots & M_{r,r} \end{bmatrix}.$$

Accordingly, elements of this direct sum are written as matrices with entries in the respective summands, i.e. in the form $[m_{i,j}]_{i,j}$ with $m_{i,j} \in M_{i,j}$ for $i, j \in [1, r]$.

Proposition 5. Let

$$A := \bigoplus_{i,j \in [1,4]} A_{i,j} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^{3 \times 3} & 0 & 0 & \mathbf{Q}^{3 \times 1} \\ 0 & \mathbf{Q} & 0 & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} & 0 \\ \mathbf{Q}^{1 \times 3} & \mathbf{Q} & 0 & \mathbf{Q}[\bar{\eta}, \bar{\xi}] \end{bmatrix}.$$

Define the multiplication

$$\begin{aligned} A \times A &\rightarrow A \\ ([a_{i,j}]_{i,j}, [a'_{s,t}]_{s,t}) &\mapsto \left[\sum_{r \in [1,4]} \alpha_{i,r,j}(a_{i,r}, a'_{r,j}) \right]_{i,j}. \end{aligned}$$

We obtain a \mathbf{Q} -algebra isomorphism

$$A \xrightarrow{\gamma} B_{\mathbf{Q}}(S_3, S_3)$$

$$[a_{i,j}]_{i,j \in [1,4]} \mapsto \sum_{i,j \in [1,4]} \gamma_{i,j}(a_{i,j}) .$$

3.2 $B_{\mathbf{Q}}(S_3, S_3)$ as path algebra modulo relations

We aim to write $B_{\mathbf{Q}} = B_{\mathbf{Q}}(S_3, S_3) \cong A$, up to Morita equivalence, as a path algebra modulo relations.

We denote by $e_{i,j} \in A_{1,1} = \mathbf{Q}^{3 \times 3}$ the elements that have a single non-zero entry 1 at position (i, j) . We have $a_{1,1} := \gamma^{-1}(e) = e_{1,1} \in \mathbf{Q}^{3 \times 3} \subseteq A$, $\gamma^{-1}(g) = e_{2,2} \in \mathbf{Q}^{3 \times 3} \subseteq A$, $\gamma^{-1}(e) = e_{3,3} \in \mathbf{Q}^{3 \times 3} \subseteq A$ and $a_{k,k} := \gamma^{-1}(\varepsilon_k)$ for $k \in [2, 4]$, cf. Proposition 5.

We have $Aa_{1,1} \cong Ae_{2,2}$ as A -modules, using multiplication with $e_{1,2}$ from the right from $Aa_{1,1}$ to $Ae_{2,2}$ and multiplication with $e_{2,1}$ from the right from $Ae_{2,2}$ to $Aa_{1,1}$. Note that $e_{1,2}e_{2,1} = a_{1,1}$ and $e_{2,1}e_{1,2} = e_{2,2}$. Similarly $Aa_{1,1} \cong Ae_{3,3}$.

Therefore, A is Morita equivalent to

$$A' := \left(\sum_{i \in [1,4]} a_{i,i} \right) A \left(\sum_{i \in [1,4]} a_{i,i} \right) = \bigoplus_{i,j \in [1,4]} a_{i,i} A a_{j,j} = \bigoplus_{i,j \in [1,4]} a_{i,i} A_{i,j} a_{j,j} .$$

Write $A'_{i,j} := a_{i,i} A_{i,j} a_{j,j} = A_{i,j}$ for $i, j \in [2, 4]$.

Identify $A'_{1,1} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = a_{1,1} A_{1,1} a_{1,1} \subseteq A_{1,1} = \mathbf{Q}^{3 \times 3}$.

Identify $A'_{1,4} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q} \\ 0 \\ 0 \end{pmatrix} = a_{1,1} A_{1,4} a_{4,4} \subseteq A_{1,4} = \mathbf{Q}^{3 \times 1}$.

Identify $A'_{4,1} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q} & 0 & 0 \end{pmatrix} = a_{4,4} A_{4,1} a_{1,1} \subseteq A_{4,1} = \mathbf{Q}^{1 \times 3}$. Let $A'_{1,j} := 0$ and $A'_{j,1} := 0$ for $j \in [2, 3]$.

We have the \mathbf{Q} -linear basis of A'

$$\begin{array}{l} a_{1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad a_{2,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad a_{3,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ a_{4,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad a_{1,4} := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad a_{4,1} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ a_{2,4} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad a_{4,2} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad a'_{4,4} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\eta} \end{bmatrix} \\ a''_{4,4} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\xi} \end{bmatrix} \end{array}$$

We have the following multiplication table for the basis elements.

(\cdot)	$a_{1,1}$	$a_{1,4}$	$a_{2,2}$	$a_{2,4}$	$a_{3,3}$	$a_{4,1}$	$a_{4,2}$	$a_{4,4}$	$a'_{4,4}$	$a''_{4,4}$
$a_{1,1}$	$a_{1,1}$	$a_{1,4}$	0	0	0	0	0	0	0	0
$a_{1,4}$	0	0	0	0	0	0	0	$a_{1,4}$	0	0
$a_{2,2}$	0	0	$a_{2,2}$	$a_{2,4}$	0	0	0	0	0	0
$a_{2,4}$	0	0	0	0	0	0	0	$a_{2,4}$	0	0
$a_{3,3}$	0	0	0	0	$a_{3,3}$	0	0	0	0	0
$a_{4,1}$	$a_{4,1}$	$a'_{4,4}$	0	0	0	0	0	0	0	0
$a_{4,2}$	0	0	$a_{4,2}$	$a''_{4,4} - 12a'_{4,4}$	0	0	0	0	0	0
$a_{4,4}$	0	0	0	0	0	$a_{4,1}$	$a_{4,2}$	$a_{4,4}$	$a'_{4,4}$	$a''_{4,4}$
$a'_{4,4}$	0	0	0	0	0	0	0	$a'_{4,4}$	0	0
$a''_{4,4}$	0	0	0	0	0	0	0	$a''_{4,4}$	0	0

We have $a'_{4,4} = a_{4,1} \cdot a_{1,4}$ and $a''_{4,4} = a_{4,2} \cdot a_{2,4} + 12a_{4,1} \cdot a_{1,4}$. Hence, as a \mathbf{Q} -algebra A' is generated by $a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, a_{1,4}, a_{4,1}, a_{2,4}, a_{4,2}$.

Consider the quiver $\Psi := \left[\begin{array}{ccccccc} & & & \xrightarrow{\sigma} & & \xleftarrow{\pi} & \\ \tilde{a}_{3,3} & & \tilde{a}_{2,2} & & \tilde{a}_{4,4} & & \tilde{a}_{1,1} \\ & & \xleftarrow{\vartheta} & & \xrightarrow{\rho} & & \end{array} \right]$.

We have a surjective \mathbf{Q} -algebra morphism $\varphi : \mathbf{Q}\Psi \rightarrow A'$ by sending

$$\begin{aligned} \tilde{a}_{1,1} &\mapsto a_{1,1} \quad , \quad \tilde{a}_{2,2} \mapsto a_{2,2} \quad , \quad \tilde{a}_{3,3} \mapsto a_{3,3} \quad , \quad \tilde{a}_{4,4} \mapsto a_{4,4} \quad , \\ \rho &\mapsto a_{4,1} \quad , \quad \pi \mapsto a_{1,4} \quad , \quad \vartheta \mapsto a_{4,2} \quad , \quad \sigma \mapsto a_{2,4} \quad . \end{aligned}$$

We establish the following multiplication trees, where we underline the elements that are not in a \mathbf{Q} -linear relation with previously underlined elements.

$$\begin{array}{ccc} \underline{a_{1,1}} \xrightarrow{a_{1,4}} \underline{a_{1,4}} \xrightarrow{a_{4,1}} a_{1,4}a_{4,1} = 0 & & \underline{a_{2,2}} \xrightarrow{a_{2,4}} \underline{a_{2,4}} \xrightarrow{a_{4,2}} a_{2,4}a_{4,2} = 0 \\ & \begin{array}{c} a_{4,2} \downarrow \\ a_{1,4}a_{4,2} = 0 \end{array} & & \begin{array}{c} a_{4,1} \downarrow \\ a_{2,4}a_{4,1} = 0 \end{array} \\ \\ & \begin{array}{c} \begin{array}{c} \underline{a_{4,4}} \\ \swarrow^{a_{4,2}} \quad \searrow^{a_{4,1}} \\ \underline{a_{4,2}} \quad \underline{a_{4,1}} \end{array} \\ \begin{array}{c} \swarrow^{a_{2,4}} \quad \searrow^{a_{1,4}} \\ \underline{a_{4,2}a_{2,4}} \quad \underline{a_{4,1}a_{1,4}} \end{array} \\ \begin{array}{c} \downarrow^{a_{4,1}} \\ \underline{a_{4,2}a_{2,4}} \\ \downarrow^{a_{4,2}} \\ \underline{(a''_{4,4} - 12a'_{4,4})a_{4,2}} = 0 \end{array} \end{array} & & \begin{array}{c} \begin{array}{c} \underline{a_{4,1}a_{1,4}} \xrightarrow{a_{4,1}} \underline{a'_{4,4}a_{4,1}} = 0 \\ \downarrow^{a_{4,2}} \\ \underline{a'_{4,4}a_{4,2}} = 0 \end{array} \end{array} \end{array}$$

The multiplication tree of the idempotent $a_{3,3}$ consists only of the element $a_{3,3}$. So the kernel of φ contains the elements:

$$\begin{aligned} \pi\rho \quad , \quad \sigma\vartheta \quad , \quad \rho\pi\rho \quad , \quad \vartheta\sigma\rho \quad , \\ \pi\vartheta \quad , \quad \sigma\rho \quad , \quad \rho\pi\vartheta \quad , \quad \vartheta\sigma\vartheta \quad . \end{aligned}$$

Let I be the ideal in $\mathbf{Q}\Psi$ generated by those elements. So $I \subseteq \ker(\varphi)$. Therefore, φ induces a surjective \mathbf{Q} -algebra morphism from $\mathbf{Q}\Psi/I$ to A' .

Note that $\mathbf{Q}\Psi/I$ is \mathbf{Q} -linearly generated by

$$\mathcal{N} := \{\tilde{a}_{3,3} + I, \tilde{a}_{2,2} + I, \tilde{a}_{4,4} + I, \tilde{a}_{1,1} + I, \sigma + I, \pi + I, \vartheta + I, \rho + I, \vartheta\sigma + I, \rho\pi + I\},$$

cf. the underlined elements above. To see that, note that a product ξ of k generators may be written as a product in \mathcal{N} of k' generators and a product of k'' generators, where $k = k' + k''$ and where k' is chosen maximal. We call k'' the excess of ξ . If $k'' \geq 1$ then, using the trees above, we may write ξ as an \mathbf{Q} -linear combination of products of generators that have excess $\leq k'' - 1$. In the present case, we even have $\xi = 0$.

Moreover, note that $|\mathcal{N}| = 10 = \dim_{\mathbf{Q}}(A')$.

Since we have a surjective \mathbf{Q} -algebra morphism from $\mathbf{Q}\Psi/I$ to A' , this dimension argument shows this morphism to be bijective. In particular, $I = \ker(\varphi)$.

We may reduce this list to obtain $\ker(\varphi) = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$. So we obtain the

Proposition 6. *Recall that $I = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$. We have the isomorphism of \mathbf{Q} -algebras*

$$A' \xrightarrow{\sim} \mathbf{Q} \left[\begin{array}{cccc} & & \sigma & \\ & \tilde{a}_{3,3} & \searrow & \pi \\ \tilde{a}_{2,2} & & \tilde{a}_{4,4} & \swarrow \\ & \vartheta & \nearrow & \rho \\ & & & \tilde{a}_{1,1} \end{array} \right] / I = \mathbf{Q}\Psi/I$$

$$\begin{aligned} a_{1,1} &\mapsto \tilde{a}_{1,1} + I \\ a_{2,2} &\mapsto \tilde{a}_{2,2} + I \\ a_{3,3} &\mapsto \tilde{a}_{3,3} + I \\ a_{4,4} &\mapsto \tilde{a}_{4,4} + I \\ a_{4,1} &\mapsto \rho + I \\ a_{1,4} &\mapsto \pi + I \\ a_{4,2} &\mapsto \vartheta + I \\ a_{2,4} &\mapsto \sigma + I . \end{aligned}$$

In particular, $\mathbf{Q}\Psi/I$ is Morita equivalent to $A \cong \mathbf{B}_{\mathbf{Q}}(\mathbf{S}_3, \mathbf{S}_3)$.

4 The double Burnside R -algebra $\mathbf{B}_R(\mathbf{S}_3, \mathbf{S}_3)$ for $R \in \{\mathbf{Z}, \mathbf{Z}_{(2)}, \mathbf{F}_2, \mathbf{Z}_{(3)}, \mathbf{F}_3\}$

4.1 $\mathbf{B}_{\mathbf{Z}}(\mathbf{S}_3, \mathbf{S}_3)$ via congruences

Recall that $A = \bigoplus_{i,j \in [1,4]} A_{i,j} \xrightarrow[\gamma]{\sim} \mathbf{B}_{\mathbf{Q}}$, cf. Proposition 5. In the \mathbf{Q} -algebra A , we define the \mathbf{Z} -order

$$A_{\mathbf{Z}} := \begin{bmatrix} A_{\mathbf{Z},1,1} & A_{\mathbf{Z},1,2} & A_{\mathbf{Z},1,3} & A_{\mathbf{Z},1,4} \\ A_{\mathbf{Z},2,1} & A_{\mathbf{Z},2,2} & A_{\mathbf{Z},2,3} & A_{\mathbf{Z},2,4} \\ A_{\mathbf{Z},3,1} & A_{\mathbf{Z},3,2} & A_{\mathbf{Z},3,3} & A_{\mathbf{Z},3,4} \\ A_{\mathbf{Z},4,1} & A_{\mathbf{Z},4,2} & A_{\mathbf{Z},4,3} & A_{\mathbf{Z},4,4} \end{bmatrix} := \begin{bmatrix} \mathbf{Z}^{3 \times 3} & 0 & 0 & \mathbf{Z}^{3 \times 1} \\ 0 & \mathbf{Z} & 0 & \mathbf{Z} \\ 0 & 0 & \mathbf{Z} & 0 \\ \mathbf{Z}^{1 \times 3} & \mathbf{Z} & 0 & \mathbf{Z}[\bar{\eta}, \bar{\xi}] \end{bmatrix} \subseteq A .$$

In fact, $A_{\mathbf{Z}}$ is a subring of A , as $\alpha_{i,j,k}(A_{\mathbf{Z},i,j} \times A_{\mathbf{Z},j,k}) \subseteq A_{\mathbf{Z},i,k}$ for $i, j, k \in [1, 4]$.

Remark 7. As $A \cong B_{\mathbf{Q}}$ is not semisimple, there are no maximal \mathbf{Z} -orders in A , [8, §10]. So $A_{\mathbf{Z}}$ is not a canonical choice of a \mathbf{Z} -order in A , but it nonetheless enables us to describe Λ inside $A_{\mathbf{Z}}$ via congruences.

Consider the following elements of $U(A)$.

$$x_1 := \begin{bmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ 6 & 6 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_3 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{bmatrix}.$$

We define the injective ring morphism $\delta : B_{\mathbf{Z}} \rightarrow A$, $y \mapsto x_3^{-1} \cdot x_2^{-1} \cdot x_1^{-1} \cdot \gamma^{-1}(y) \cdot x_1 \cdot x_2 \cdot x_3$. The conjugating element x_1 was constructed such that its image lies in $A_{\mathbf{Z}}$. The elements x_2, x_3 serve the purpose of simplifying the congruences of $\delta(B_{\mathbf{Z}})$.

Theorem 8. *The image $\delta(B_{\mathbf{Z}})$ in $A_{\mathbf{Z}}$ is given by*

$$\Lambda := \delta(B_{\mathbf{Z}}) = \left\{ \begin{array}{l} \left[\begin{array}{cccccc} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\bar{\eta} + z_3\bar{\xi} \end{array} \right] \in A_{\mathbf{Z}} : \\ \begin{array}{l} 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ x_1 \equiv_4 0 \\ x_2 \equiv_4 0 \\ x_3 \equiv_4 0 \\ y \equiv_2 0 \\ t_1 \equiv_2 0 \\ t_2 \equiv_2 0 \\ t_3 \equiv_2 0 \\ v \equiv_2 0 \\ \\ x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{array} \end{array} \right\}.$$

In particular, we have $B_{\mathbf{Z}} = B_{\mathbf{Z}}(S_3, S_3) \cong \Lambda$ as rings.

More symbolically written, we have

$$\Lambda = \left[\begin{array}{cccccc} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) \\ 0 & 0 & 0 & \mathbf{Z} & 0 & (2) \\ 0 & 0 & 0 & 0 & \mathbf{Z} & 0 \\ (12) & (12) & (12) & (2) & 0 & \mathbf{Z} \end{array} \right] \begin{array}{l} \xrightarrow{-2} \\ \xrightarrow{2} \\ \xrightarrow{1} \end{array} \textcircled{8} \begin{array}{l} \\ \\ \\ \\ \\ + (12)\bar{\eta} \\ + (4)\bar{\xi} \end{array}$$

Proof. We identify $\mathbf{Z}^{22 \times 1}$ and $A_{\mathbf{Z}}$ along the isomorphism

$$\left(\begin{array}{c} s_{1,1}, s_{2,1}, s_{3,1}, s_{1,2}, s_{2,2}, s_{3,2}, s_{1,3}, s_{2,3}, s_{3,3}, \\ x_1, x_2, x_3, u, y, w, t_1, t_2, t_3, v, z_1, z_2, z_3 \end{array} \right)^t \mapsto \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\bar{\eta} + z_3\bar{\xi} \end{bmatrix}.$$

Let M be the representation matrix of δ , with respect to the bases $\tilde{\mathcal{H}} = (H_{0,0}, H_{0,1}, H_{1,0}, H_1^\Delta, H_{0,4}, H_{4,0}, H_4^\Delta, H_{1,1}, H_{0,5}, H_{5,0}, H_7, H_{1,4}, H_{4,1}, H_6, H_5^\Delta, H_{4,4}, H_{5,1}, H_{1,5}, H_{5,4}, H_{4,5}, H_8, H_{5,5})$ of $B_{\mathbf{Z}}$ and the standard basis of $A_{\mathbf{Z}}$. We obtain

$$M = \begin{pmatrix} 0 & 0 & 15 & -3 & 0 & 20 & 8 & 6 & 0 & 25 & 7 & 9 & 8 & -3 & 1 & 12 & 10 & 3 & 15 & 4 & 3 & 5 \\ 0 & 0 & -18 & 0 & 0 & -24 & 0 & -9 & 0 & -30 & -12 & -6 & -12 & 0 & 0 & -8 & -15 & -3 & -10 & -4 & -4 & -5 \\ 0 & 0 & 126 & -6 & 0 & 168 & 12 & 60 & 0 & 210 & 78 & 48 & 80 & -6 & 0 & 64 & 100 & 21 & 80 & 28 & 26 & 35 \\ -5 & -2 & -60 & 9 & -3 & -55 & -23 & -24 & -1 & -85 & -16 & -36 & -22 & 10 & 0 & -33 & -34 & -12 & -51 & -11 & -5 & -17 \\ 6 & 3 & 72 & 3 & 2 & 66 & 2 & 36 & 1 & 102 & 33 & 24 & 33 & 1 & 1 & 22 & 51 & 12 & 34 & 11 & 11 & 17 \\ -42 & -20 & -504 & 2 & -16 & -462 & -46 & -240 & -7 & -714 & -208 & -192 & -220 & 15 & 0 & -176 & -340 & -84 & -272 & -77 & -65 & -119 \\ 0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -15 & -3 & -6 & -4 & 2 & 0 & -6 & -6 & -2 & -9 & -2 & -1 & -3 \\ 0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 18 & 6 & 4 & 6 & 0 & 0 & 4 & 9 & 2 & 6 & 2 & 2 & 3 \\ 0 & 0 & -84 & 4 & 0 & -84 & -6 & -40 & 0 & -126 & -38 & -32 & -40 & 4 & 1 & -32 & -60 & -14 & -48 & -14 & -12 & -21 \\ 0 & 0 & -756 & 36 & 0 & -1008 & -72 & -360 & 0 & -1260 & -468 & -288 & -480 & 72 & 0 & -384 & -600 & -108 & -480 & -144 & -120 & -180 \\ 252 & 120 & 3024 & -12 & 96 & 2772 & 276 & 1440 & 36 & 4284 & 1248 & 1152 & 1320 & -228 & 0 & 1056 & 2040 & 432 & 1632 & 396 & 252 & 612 \\ 0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 756 & 228 & 192 & 240 & -48 & 0 & 192 & 360 & 72 & 288 & 72 & 48 & 108 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -10 & -4 & -6 & -4 & 2 & 0 & -6 & -4 & -2 & -6 & -2 & -2 & -2 \\ 0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 12 & 6 & 4 & 6 & 0 & 0 & 4 & 6 & 2 & 4 & 2 & 2 & 2 \\ 0 & 0 & -84 & 4 & 0 & -84 & -6 & -40 & 0 & -84 & -40 & -32 & -40 & 4 & 0 & -32 & -40 & -14 & -32 & -14 & -14 & -14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 504 & 240 & 192 & 240 & -48 & 0 & 192 & 240 & 72 & 192 & 72 & 24 & 72 \\ 0 & 4 & 0 \end{pmatrix}.$$

$$\text{Let } \lambda := \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\bar{\eta} + z_3\bar{\xi} \end{bmatrix} \in A_{\mathbf{Z}}, \text{ identified with } \lambda \in \mathbf{Z}^{22 \times 1}.$$

We have

$$\begin{aligned} \lambda \in \Lambda &\Leftrightarrow \exists q \in \mathbf{Z}^{22 \times 1} \text{ such that } \lambda = Mq \\ &\Leftrightarrow \exists q \in \mathbf{Z}^{22 \times 1} \text{ such that } M^{-1} \cdot \lambda = q \\ &\Leftrightarrow 24M^{-1} \cdot \lambda \in 24\mathbf{Z}^{22 \times 1} \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 18 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} s_{1,1} \\ s_{2,1} \\ s_{3,1} \\ \vdots \\ s_{3,3} \\ x_1 \\ x_2 \\ x_3 \\ u \\ y \\ w \\ t_1 \\ t_2 \\ t_3 \\ v \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \in 24\mathbf{Z}^{11 \times 1} \\
&\Leftrightarrow \left\{ \begin{array}{l} 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ x_1 \equiv_4 0 \\ x_2 \equiv_4 0 \\ x_3 \equiv_4 0 \\ y \equiv_2 0 \\ t_1 \equiv_2 0 \\ t_2 \equiv_2 0 \\ t_3 \equiv_2 0 \\ v \equiv_2 0 \\ \\ x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{array} \right\}.
\end{aligned}$$

□

4.2 Localisation at 2: $B_{\mathbf{Z}_{(2)}}(\mathcal{S}_3, \mathcal{S}_3)$ via congruences

Write $R := \mathbf{Z}_{(2)}$. In the \mathbf{Q} -algebra A , cf. Proposition 5, we have the R -order

$$A_R := \begin{bmatrix} A_{R,1,1} & A_{R,1,2} & A_{R,1,3} & A_{R,1,4} \\ A_{R,2,1} & A_{R,2,2} & A_{R,2,3} & A_{R,2,4} \\ A_{R,3,1} & A_{R,3,2} & A_{R,3,3} & A_{R,3,4} \\ A_{R,4,1} & A_{R,4,2} & A_{R,4,3} & A_{R,4,4} \end{bmatrix} := \begin{bmatrix} R^{3 \times 3} & 0 & 0 & R^{3 \times 1} \\ 0 & R & 0 & R \\ 0 & 0 & R & 0 \\ R^{1 \times 3} & R & 0 & R[\bar{\eta}, \bar{\xi}] \end{bmatrix} \subseteq A.$$

Corollary 9. *We have*

$$\Lambda_{(2)} = \left\{ \begin{array}{l} \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\bar{\eta} + z_3\bar{\xi} \end{bmatrix} \\ \in A_R : \begin{array}{l} 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ x_1 \equiv_4 0 \\ x_2 \equiv_4 0 \\ x_3 \equiv_4 0 \\ y \equiv_2 0 \\ t_1 \equiv_2 0 \\ t_2 \equiv_2 0 \\ t_3 \equiv_2 0 \\ v \equiv_2 0 \end{array} \end{array} \right\} \subseteq A_R.$$

In particular, we have $B_R = B_R(\mathcal{S}_3, \mathcal{S}_3) \cong \Lambda_{(2)}$ as R -algebras.

More symbolically written, we have

$$\Lambda_{(2)} = \begin{bmatrix} R & R & R & 0 & 0 & (2) \\ R & R & R & 0 & 0 & (2) \\ R & R & R & 0 & 0 & (2) \\ 0 & 0 & 0 & R & 0 & (2) \\ 0 & 0 & 0 & 0 & R & 0 \\ (4) & (4) & (4) & (2) & 0 & R \end{bmatrix} \cdot$$

Remark 10. We claim that $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3 + e_4 + e_5$ is an orthogonal decomposition into primitive idempotents, where

$$e_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof. We have $e_1 \Lambda_{(2)} e_1 \cong R$, $e_2 \Lambda_{(2)} e_2 \cong R$, $e_3 \Lambda_{(2)} e_3 \cong R$ and $e_4 \Lambda_{(2)} e_4 \cong R$. So, it follows that e_1, e_2, e_3, e_4 are primitive.

As R -algebras, we have

$$e_5 \Lambda_{(2)} e_5 \cong \left\{ \left(w, z_1 + z_2 \bar{\eta} + z_3 \bar{\xi} \right) \in R \times R[\bar{\eta}, \bar{\xi}] : 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \right\} =: \Gamma$$

$$\subseteq R \times R[\bar{\eta}, \bar{\xi}].$$

To show that e_5 is primitive, we show that Γ is local.

We have the R -linear basis (b_1, b_2, b_3, b_4) of Γ , where

$$b_1 = \left(1, 1 \right), \quad b_2 = \left(0, 2 + 4\bar{\eta} \right),$$

$$b_3 = \left(0, 8\bar{\eta} \right), \quad b_4 = \left(0, 4\bar{\xi} \right).$$

We claim that the Jacobson radical of Γ is given by $J := {}_R \langle 2b_1, b_2, b_3, b_4 \rangle$, that $\Gamma/J \cong \mathbf{F}_2$ and that Γ is local.

In fact, the multiplication table for the basis elements is given by

$$\begin{array}{c|c|c|c|c}
(\cdot) & b_1 & b_2 & b_3 & b_4 \\
\hline
b_1 & b_1 & b_2 & b_3 & b_4 \\
\hline
b_2 & b_2 & 2b_2 + b_3 & 2b_3 & 2b_4 \\
\hline
b_3 & b_3 & 2b_3 & 0 & 0 \\
\hline
b_4 & b_4 & 2b_4 & 0 & 0
\end{array} .$$

This shows that J is an ideal. Moreover, J is topologically nilpotent as

$$J^3 = {}_R\langle 8b_1, 4b_2, 2b_3, 4b_4 \rangle \subseteq 2e_5 \Lambda_{(2)} e_5 .$$

Since $\Gamma/J \cong \mathbf{F}_2$, the *claim* follows. \square

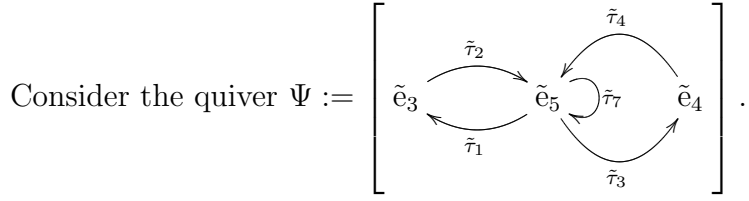
4.3 $B_{\mathbf{Z}_{(2)}}(S_3, S_3)$ and $B_{\mathbf{F}_2}(S_3, S_3)$ as path algebras modulo relations

Write $R := \mathbf{Z}_{(2)}$. We aim to write $\Lambda_{(2)}$, up to Morita equivalence, as path algebra modulo relations. The R -algebra $\Lambda_{(2)}$ is Morita equivalent to $\Lambda'_{(2)} := (e_3 + e_4 + e_5)\Lambda_{(2)}(e_3 + e_4 + e_5)$ since $\Lambda_{(2)}e_1 \cong \Lambda_{(2)}e_2 \cong \Lambda_{(2)}e_3$ using multiplication with elements of $\Lambda_{(2)}$ with a single nonzero entry 1 in the upper (3×3) -corner.

We have the R -linear basis of $\Lambda'_{(2)}$ consisting of

$$\begin{array}{l}
e_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
\tau_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix}, \quad \tau_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tau_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}, \\
\tau_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tau_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8\bar{\eta} \end{bmatrix}, \quad \tau_6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\bar{\xi} \end{bmatrix}, \\
\tau_7 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 + 4\bar{\eta} \end{bmatrix}
\end{array}$$

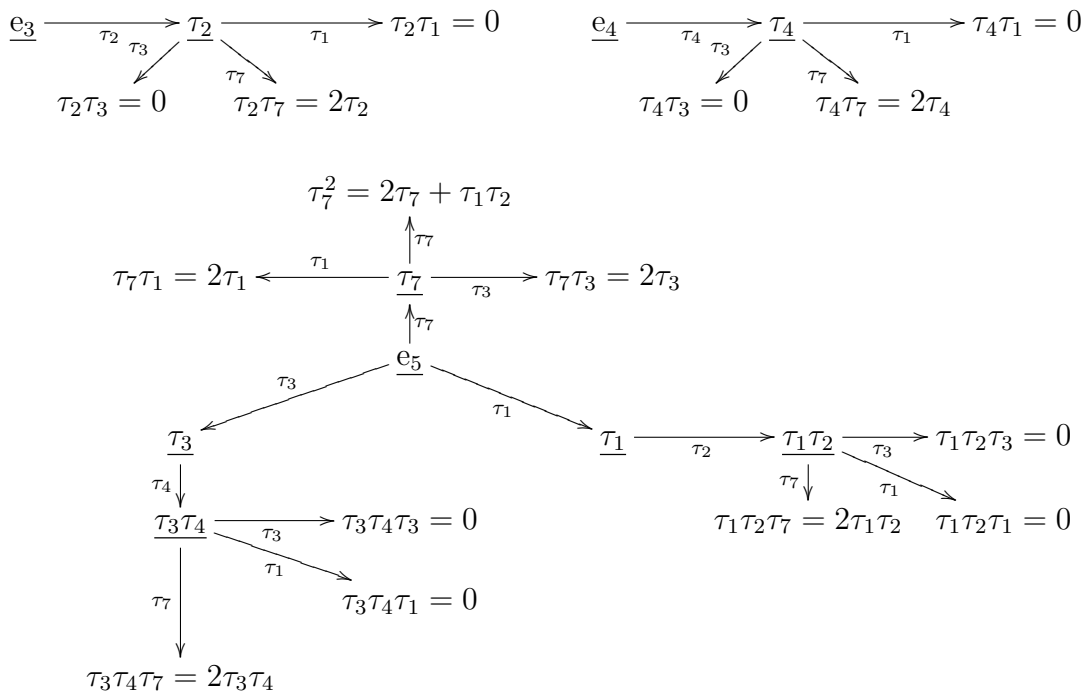
We have $\tau_5 = \tau_1\tau_2$ and $\tau_6 = \tau_3\tau_4 + 6\tau_1\tau_2$. Hence, as an R -algebra $\Lambda'_{(2)}$ is generated by $e_3, e_4, e_5, \tau_1, \tau_2, \tau_3, \tau_4, \tau_7$.



We have a surjective R -algebra morphism $\varphi : R\Psi \rightarrow \Lambda'_{(2)}$ by sending

$$\begin{aligned} \tilde{e}_3 &\mapsto e_3, & \tilde{e}_4 &\mapsto e_4, & \tilde{e}_5 &\mapsto e_5, & \tilde{\tau}_1 &\mapsto \tau_1, \\ \tilde{\tau}_2 &\mapsto \tau_2, & \tilde{\tau}_3 &\mapsto \tau_3, & \tilde{\tau}_4 &\mapsto \tau_4, & \tilde{\tau}_7 &\mapsto \tau_7. \end{aligned}$$

We establish the following multiplication trees, where we underline the elements that are not in an R -linear relation with previous elements.



So, the kernel of φ contains the elements:

$$\begin{aligned} &\tilde{\tau}_2\tilde{\tau}_1, & \tilde{\tau}_4\tilde{\tau}_1, & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_1, & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_1, & \tilde{\tau}_7\tilde{\tau}_1 - 2\tilde{\tau}_1, \\ &\tilde{\tau}_2\tilde{\tau}_3, & \tilde{\tau}_4\tilde{\tau}_3, & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3, & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_3, & \tilde{\tau}_7\tilde{\tau}_3 - 2\tilde{\tau}_3, \\ &\tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_2, & \tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_4, & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_1\tilde{\tau}_2, & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_3\tilde{\tau}_4, & \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1\tilde{\tau}_2. \end{aligned}$$

Let I be the ideal generated by these elements. So $I \subseteq \ker(\varphi)$. Therefore, φ induces a surjective R -algebra morphism from $R\Psi/I$ to $\Lambda'_{(2)}$. We may reduce the list of generators to obtain

$$I = (\tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_7\tilde{\tau}_1 - 2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_7\tilde{\tau}_3 - 2\tilde{\tau}_3, \tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_2, \tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_4, \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1\tilde{\tau}_2).$$

Note that $R\Psi/I$ is R -linearly generated by

$$\mathcal{N} := \{\tilde{e}_3 + I, \tilde{e}_4 + I, \tilde{e}_5 + I, \tilde{\tau}_1 + I, \tilde{\tau}_2 + I, \tilde{\tau}_3 + I, \tilde{\tau}_4 + I, \tilde{\tau}_7 + I, \tilde{\tau}_3\tilde{\tau}_4 + I, \tilde{\tau}_1\tilde{\tau}_2 + I\},$$

cf. the underlined elements above. To see that, note that a product ξ of k generators may be written as a product in \mathcal{N} of k' generators and a product of k'' generators, where $k = k' + k''$ and where k' is chosen maximal. We call k'' the excess of ξ . If $k'' \geq 1$ then, using the trees above, we may write ξ as an R -linear combination of products of generators that have excess $\leq k'' - 1$. Moreover, note that $|\mathcal{N}| = 10 = \text{rk}_R(\Lambda'_{(2)})$.

Since we have a surjective R -algebra morphism from $R\Psi/I$ to $\Lambda'_{(2)}$, this rank argument shows this morphism to be bijective. In particular, $I = \text{kern}(\varphi)$.

So, we obtain the

Proposition 11. Recall that $I = \left(\begin{array}{ccc} \tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_2, \\ \tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_4, \\ \tilde{\tau}_7\tilde{\tau}_1 - 2\tilde{\tau}_1 & , & \tilde{\tau}_7\tilde{\tau}_3 - 2\tilde{\tau}_3 & , & \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1\tilde{\tau}_2 \end{array} \right)$.

We have the isomorphism of $\mathbf{Z}_{(2)}$ -algebras

$$\Lambda'_{(2)} \xrightarrow{\sim} R \left[\begin{array}{c} \tilde{\tau}_2 \\ \tilde{\tau}_1 \\ \tilde{\tau}_4 \\ \tilde{\tau}_3 \\ \tilde{\tau}_7 \end{array} \right] / I$$

$$e_i \mapsto \tilde{e}_i + I \text{ for } i \in [3, 5]$$

$$\tau_j \mapsto \tilde{\tau}_j + I \text{ for } j \in [1, 7] \setminus \{5, 6\} .$$

Recall that $B_{\mathbf{Z}_{(2)}}(\mathbf{S}_3, \mathbf{S}_3)$ is Morita equivalent to $\Lambda'_{(2)}$.

Corollary 12. As \mathbf{F}_2 -algebras, we have

$$\Lambda'_{(2)}/2\Lambda'_{(2)} \cong \mathbf{F}_2 \left[\begin{array}{c} \tilde{\tau}_2 \\ \tilde{\tau}_1 \\ \tilde{\tau}_4 \\ \tilde{\tau}_3 \\ \tilde{\tau}_7 \end{array} \right] / \left(\begin{array}{ccc} \tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_2\tilde{\tau}_7, \\ \tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_4\tilde{\tau}_7, \\ \tilde{\tau}_7\tilde{\tau}_1 & , & \tilde{\tau}_7\tilde{\tau}_3 & , & \tilde{\tau}_7^2 - \tilde{\tau}_1\tilde{\tau}_2 \end{array} \right) .$$

Recall that $B_{\mathbf{F}_2}(\mathbf{S}_3, \mathbf{S}_3)$ is Morita equivalent to $\Lambda'_{(2)}/2\Lambda'_{(2)}$.

4.4 Localisation at 3: $B_{\mathbf{Z}_{(3)}}(\mathbf{S}_3, \mathbf{S}_3)$ via congruences

Write $R = \mathbf{Z}_{(3)}$. In the \mathbf{Q} -algebra A , cf. Proposition 5, we have the R -order

$$A_R := \begin{bmatrix} A_{R,1,1} & A_{R,1,2} & A_{R,1,3} & A_{R,1,4} \\ A_{R,2,1} & A_{R,2,2} & A_{R,2,3} & A_{R,2,4} \\ A_{R,3,1} & A_{R,3,2} & A_{R,3,3} & A_{R,3,4} \\ A_{R,4,1} & A_{R,4,2} & A_{R,4,3} & A_{R,4,4} \end{bmatrix} := \begin{bmatrix} R^{3 \times 3} & 0 & 0 & R^{3 \times 1} \\ 0 & R & 0 & R \\ 0 & 0 & R & 0 \\ R^{1 \times 3} & R & 0 & R[\bar{\eta}, \bar{\xi}] \end{bmatrix} \subseteq A .$$

Corollary 13. *We have*

$$\Lambda_{(3)} = \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\bar{\eta} + z_3\bar{\xi} \end{bmatrix} \in A_R : \begin{matrix} x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{matrix} \right\} \subseteq A_R.$$

In particular, we have $B_R = B_R(S_3, S_3) \cong \Lambda_{(3)}$ as R -algebras.

More symbolically written, we have

$$\Lambda_{(3)} = \begin{bmatrix} R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ 0 & 0 & 0 & R & 0 & R \\ 0 & 0 & 0 & 0 & R & 0 \\ (3) & (3) & (3) & R & 0 & R & +(3)\bar{\eta} & +R\bar{\xi} \end{bmatrix}.$$

Remark 14. *We claim that $1_{\Lambda_{(3)}} = e_1 + e_2 + e_3 + e_4 + e_5 + e_6$ is an orthogonal decomposition into primitive idempotents, where*

$$e_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof. We have $e_s \Lambda_{(3)} e_s \cong R$ for $s \in [1, 5]$. Therefore it follows that e_1, e_2, e_3, e_4, e_5 are primitive.

To show that e_6 is primitive, we *claim* that the ring $e_6 \Lambda_{(3)} e_6 \cong R[\bar{\eta}, \bar{\xi}]$ is local.

We have $U(R[\bar{\eta}, \bar{\xi}]) = R[\bar{\eta}, \bar{\xi}] \setminus (3, \bar{\eta}, \bar{\xi})$. In fact, for $u := a + b\bar{\eta} + c\bar{\xi}$ with $a \in R \setminus (3)$ and $b, c \in R$, the inverse is given by $u^{-1} = a^{-1} - a^{-2}b\bar{\eta} - a^{-2}c\bar{\xi}$ as

$$uu^{-1} = aa^{-1} + (-a^{-1}b + a^{-1}b)\bar{\eta} + (-a^{-1}c + a^{-1}c)\bar{\xi} = 1.$$

Thus the nonunits of $R[\bar{\eta}, \bar{\xi}]$ form an ideal and so $R[\bar{\eta}, \bar{\xi}]$ is a local ring. This proves the *claim*. \square

4.5 $B_{\mathbf{Z}(3)}(S_3, S_3)$ and $B_{\mathbf{F}_3}(S_3, S_3)$ as path algebras modulo relations

Write $R := \mathbf{Z}(3)$. We aim to write $\Lambda_{(3)}$, up to Morita equivalence, as path algebra modulo relations.

The R -algebra $\Lambda_{(3)}$ is Morita equivalent to $\Lambda'_{(3)} := (e_3 + e_4 + e_5 + e_6)\Lambda_{(3)}(e_3 + e_4 + e_5 + e_6)$ since $\Lambda_{(3)}e_1 \cong \Lambda_{(3)}e_2 \cong \Lambda_{(3)}e_3$ using multiplication with elements of $\Lambda_{(3)}$ with a single nonzero entry 1 in the upper (3×3) -corner. We have the R -linear basis of $\Lambda'_{(3)}$ consisting of

$$\begin{aligned}
e_3 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & e_4 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & e_5 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
e_6 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, & \tau_1 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}, & \tau_2 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\tau_3 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, & \tau_4 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \tau_5 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3\bar{\eta} \end{bmatrix}, \\
\tau_6 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\xi} \end{bmatrix}.
\end{aligned}$$

We have $\tau_5 = \tau_1\tau_2$ and $\tau_6 = \tau_3\tau_4 + 4\tau_1\tau_2$. Hence, as an R -algebra $\Lambda'_{(3)}$ is generated by $e_3, e_4, e_5, e_6, \tau_1, \tau_2, \tau_3, \tau_4$.

Consider the quiver $\Psi := \left[\begin{array}{cccc} \tilde{e}_5 & & & \\ & \tilde{e}_3 & \xrightarrow{\tilde{\tau}_2} & \tilde{e}_6 \\ & & \xleftarrow{\tilde{\tau}_1} & \\ & & & \tilde{e}_4 \\ & & & \xleftarrow{\tilde{\tau}_3} & \end{array} \right]$. We have a surjective R -algebra morphism $\varphi : R\Psi \rightarrow \Lambda'_{(3)}$ by sending

$$\begin{aligned}
\tilde{e}_3 &\mapsto e_3, & \tilde{e}_4 &\mapsto e_4, & \tilde{e}_5 &\mapsto e_5, & \tilde{e}_6 &\mapsto e_6, \\
\tilde{\tau}_1 &\mapsto \tau_1, & \tilde{\tau}_2 &\mapsto \tau_2, & \tilde{\tau}_3 &\mapsto \tau_3, & \tilde{\tau}_4 &\mapsto \tau_4.
\end{aligned}$$

We establish the following multiplication trees, where we underline the elements that are not in an R -linear relation with previous elements.

The multiplication tree of the idempotent e_5 consists only of the element e_5 .

$$\begin{array}{ccc}
\underline{e_4} & \xrightarrow{\tau_4} & \underline{\tau_4} \xrightarrow{\tau_3} \tau_4\tau_3 = 0 \\
& & \tau_1 \downarrow \\
& & \tau_4\tau_1 = 0
\end{array}
\qquad
\begin{array}{ccc}
\underline{e_3} & \xrightarrow{\tau_2} & \underline{\tau_2} \xrightarrow{\tau_1} \tau_2\tau_1 = 0 \\
& & \tau_3 \downarrow \\
& & \tau_2\tau_3 = 0
\end{array}$$

$$\begin{array}{c}
\tau_3\tau_4\tau_3 = 0 \\
\uparrow \tau_3 \quad \tau_4 \\
\underline{\tau_3\tau_4} \\
\tau_1 \downarrow \\
\tau_3\tau_4\tau_1 = 0
\end{array}
\quad
\begin{array}{c}
\tau_3 \\
\swarrow \\
\underline{\tau_3}
\end{array}
\quad
\begin{array}{c}
\tau_1 \\
\searrow \\
\underline{\tau_1}
\end{array}
\quad
\begin{array}{c}
\tau_2 \\
\rightarrow \\
\underline{\tau_1\tau_2} \xrightarrow{\tau_1} \tau_1\tau_2\tau_1 = 0 \\
\tau_3 \downarrow \\
\tau_1\tau_2\tau_3 = 0
\end{array}$$

So the kernel of φ contains the elements: $\tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_1$. Let I be the ideal generated by these elements. So, $I \subseteq \ker(\varphi)$. Therefore, φ induces a surjective R -algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$. We may reduce the list of generators to obtain $I = (\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3)$.

Note that $R\Psi/I$ is R -linearly generated by

$$\mathcal{N} := \{\tilde{e}_3 + I, \tilde{e}_4 + I, \tilde{e}_5 + I, \tilde{e}_6 + I, \tilde{\tau}_1 + I, \tilde{\tau}_2 + I, \tilde{\tau}_3 + I, \tilde{\tau}_4 + I, \tilde{\tau}_3\tilde{\tau}_4 + I, \tilde{\tau}_1\tilde{\tau}_2 + I\},$$

cf. the underlined elements above.

To see that, note that a product ξ of k generators may be written as a product in \mathcal{N} of k' generators and a product of k'' generators, where $k = k' + k''$ and where k' is chosen maximal. If $k'' \geq 1$ then, using the trees above, we have $\xi = 0$. Moreover, note that $|\mathcal{N}| = 10 = \text{rk}_R(\Lambda'_{(3)})$.

Since we have an surjective algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$, this rank argument shows this morphism to be bijective. In particular, $I = \ker(\varphi)$.

So, we obtain the

Proposition 15. *Recall that $I = (\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3)$.*

We have the isomorphisms of R -algebras

$$\Lambda'_{(3)} \xrightarrow{\sim} R \left[\begin{array}{ccccc} & & \tilde{\tau}_2 & & \tilde{\tau}_4 \\ & & \curvearrowright & & \curvearrowleft \\ \tilde{e}_5 & \tilde{e}_3 & & \tilde{e}_6 & \tilde{e}_4 \\ & & \tilde{\tau}_1 & & \tilde{\tau}_3 \\ & & \curvearrowleft & & \curvearrowright \end{array} \right] / I$$

$$\begin{array}{l}
e_i \mapsto \tilde{e}_i + I \text{ for } i \in [3, 6] \\
\tau_i \mapsto \tilde{\tau}_i + I \text{ for } i \in [1, 4]
\end{array}$$

Recall that $B_{\mathbf{Z}_{(3)}}(\mathbb{S}_3, \mathbb{S}_3)$ is Morita equivalent to $\Lambda'_{(3)}$.

Corollary 16. *As \mathbf{F}_3 -algebras, we have*

$$\Lambda'_{(3)}/3\Lambda'_{(3)} \cong \mathbf{F}_3 \left[\begin{array}{c} \tilde{e}_5 \\ \tilde{e}_3 \quad \tilde{e}_6 \quad \tilde{e}_4 \\ \tilde{e}_3 \quad \tilde{e}_6 \quad \tilde{e}_4 \end{array} \right] / (\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3) .$$

Recall that $B_{\mathbf{F}_3}(S_3, S_3)$ is Morita equivalent to $\Lambda'_{(3)}/3\Lambda'_{(3)}$.

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