Tight polyhedral embeddings of simply connected 4-manifolds

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Definition:
An embedding $M \to \mathbb{E}^N$ of a compact manifold is called **tight**, if for any open half space $E_+ \subset \mathbb{E}^N$ the induced homomorphism

$$H_*(M \cap E_+) \longrightarrow H_*(M)$$

is injective where H_* denotes an appropriate homology theory with coefficients in a certain field.

In the smooth (and, with certain modifications, also in the polyhedral) case this is equivalent to the condition that almost all height functions on M are perfect functions, i.e., have the minimum number of critical points.

For 2-manifolds without boundary tightness is equivalent to the **Two-piece-property** (TPP) which states that the intersection of M with any open halfspace H_+ is **connected**.

One of the results is that any given closed surface admits a tight polyhedral embedding into some Euclidean space. For obtaining this, it is sufficient to start with the three cases of the sphere, the real projective plane and the Klein bottle and then to attach handles tightly.

For simply connected 4-manifolds without boundary the tightness is equivalent to the requirement that $M \cap E_+$ is always **connected and simply connected**.
An equivalent condition is that almost all height functions have critical points of even index only.

Existence and non-existence:

smooth immersions:

- few examples ($\mathbb{C}P^2, S^2 \times S^2 \sharp \cdots \sharp S^2 \times S^2$)
- codimension is ≤ 4
- diff. topological obstructions (G.Thorbergsson 1986)

polyhedral embeddings:

- many examples (including $\mathbb{C}P^2 \sharp (-\mathbb{C}P^2)$)
- codimension is unbounded
- no PL topological obstructions known

Theorem 1 (Cairns 1940)
The equivalence classes of smooth 4-manifolds and PL 4-manifolds are in (1−1)-correspondence. More precisely, every smooth 4-manifold induces precisely one PL manifold (up to PL-homeomorphism) and, vice versa, every PL 4-manifold admits exactly one smoothing (up to diffeomorphism).

Theorem 2 (Rohlin 1952)
The signature of any simply connected smooth or PL 4-manifold with an even intersection form is an integer multiple of 16.

$$rank(H_2(M;\mathbb{Z})) = \chi(M) - 2$$

(unimodular) intersection form:

$$Q\colon H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$signature :=$
$:= \sharp(neg.eigenvalues) - \sharp(pos.eigenvalues)$

Theorem 3 (Donaldson 1983)
If the intersection form of a simply connected PL 4-manifold is positive (or negative) definite then it is diagonalizable and, in particular, odd.

Theorem 4 (Freedman 1982)
The homeomorphism classes of simply connected PL 4-manifolds are uniquely classified by their intersection forms.

More precisely: Two such manifolds $M, \widetilde{M}$ are homeomorphic if and only if their intersection forms $Q, \widetilde{Q}$ are equivalent over the integers.

Theorem 5 (algebraic)

1. Any odd quadratic form over the integers is equivalent to $l(+1) \oplus k(-1)$.
2. Any indefinite and even quadratic form over the integers is equivalent to $n(\mp E_8) \oplus m\binom{0\ 1}{1\ 0}$.

The rank is $k+l$ or $8n+2m$, respectively, the signature is $k-l$ or $\pm 8n$, respectively. Vice versa, rank and signature of the quadratic form determine these numbers k, l, m, n uniquely. We have $\mathrm{Det}(E_8) = 1$ and

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

Corollary The manifolds

$$l(\mathbb{C}P^2)\sharp k(-\mathbb{C}P^2) \text{ with } k, l \geq 0$$

and

$$n(K3)\sharp m(S^2 \times S^2) \text{ with } m, n \geq 0$$

cover all homeomorphism classes of simply connected PL 4-manifolds with intersection form

$$l(+1) \oplus k(-1) \quad \text{or} \quad 2\nu(\mp E_8) \oplus \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $k, l \geq 0, \mu \geq 3\nu \geq 0$, respectively.

Remark The $\frac{11}{8}$-conjecture states that no other quadratic form can occur as an even intersection form of a simply connected PL 4-manifold. Recall that for the quadratic form $Q = 2\nu(\mp E_8) \oplus \mu\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we have

$$\frac{rank(Q)}{|sign(Q)|} = \frac{16\nu + 2\mu}{16\nu} \geq \frac{11}{8} \quad \text{whenever} \quad \mu \geq 3\nu.$$

Main Theorem

Let M be a simply connected PL 4-manifold with an indefinite intersection form Q.

Assume further that

$$\operatorname{rank}(Q) \geq \frac{11}{8}|\operatorname{sign}(Q)| + 44$$

in case that the intersection form is even and that $|\operatorname{sign}(Q)| \geq 32$.

Then there exists a PL 4-manifold $\widetilde{M}$ and a tight polyhedral embedding $\widetilde{M} \to \mathbb{E}^N$ for some N such that M and $\widetilde{M}$ are homeomorphic.

By a theorem of C.T.C.Wall 1964 there is always a number $k \geq 0$ such that the manifolds
$M \sharp k(S^2 \times S^2)$ and $\widetilde{M} \sharp k(S^2 \times S^2)$
are not only homeomorphic but PL homeomorphic. So in some sense in most of the cases we can not only prescribe the topological type but also the PL type.

However, there are an infinite number of undecided cases left. In particular we do not have any example of a tight polyhedral realization of a manifold homeomorphic to $K3 \sharp K3 \sharp \cdots \sharp K3$.
Such an example could remove the number 44 from the extra assumption which then would just transform into the hypothesis of the $\frac{11}{8}$-conjecture.

Definition

A *simple polyhedral sphere* Σ^{k-1} is a triangulation of the sphere S^{k-1} with $k+1$ vertices. This is nothing but the boundary complex of a k-dimensional simplex.
A *short link* of a certain $(n-k)$-simplex in a triangulated n-manifold is a link which is combinatorially equivalent to a simple polyhedral sphere Σ^{k-1}.
Notice that the link of a codimension-1-face is always short, the link of a codimension-2-face is short if and only if it has exactly 3 vertices and 3 edges.

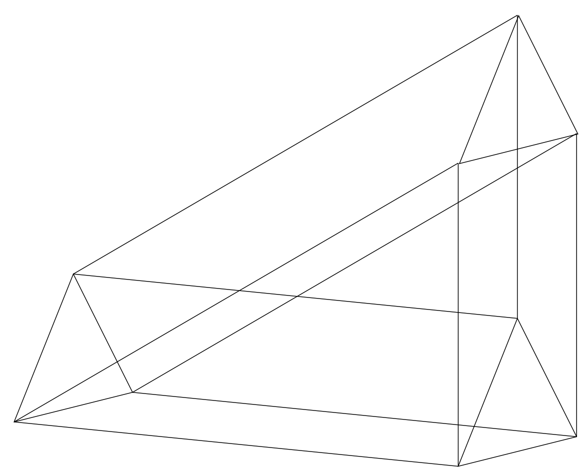

Figure 1: Attaching a 2-handle tightly; the case $k = 2, n = 3$

Lemma

Assume that $M^n \subset \mathbb{E}^N$ is a simplicial submanifold containing a simplex $\triangle^{n-k}$ with a short link Σ^{k-1} such that all vertices of the star of $\triangle^{n-k}$ are in general position. Then there is a polyhedral "solid torus" of type $S^{k-1} \times B^{n-k+1}$ within the open star of $\triangle^{n-k}$ which is a tight submanifold-with-boundary in the subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^N$ which is spanned by the $n+2$ vertices of the star of $\triangle^{n-k}$. Moreover, it can be arranged that the convex hull of the short link does not hit M except for its boundary. Therefore, we can choose the tight solid torus in such a way that its convex hull does not hit M either except for the solid torus itself.

Corollary

Whenever we have a tight triangulation of an n-manifold M with a short link of one $(n-k)$-simplex $(k \leq n/2)$ then we can attach arbitrarily many handles tightly of type $S^k \times S^{n-k}$.

Proof of the Main Theorem:

starting examples:

Tight triangulations $\mathbb{C}P^2_9$ and $(K3)_{16}$ (K.-Casella)

odd intersection form $l(+1) \oplus k(-1)$.
By assumption it is indefinite, so $k \geq l \geq 1$.
Equivalent description:
$(+1) \oplus (k-l+1)(-1) \oplus (l-1)\binom{0\ 1}{1\ 0}$.
Tight realization:

Start with the tight $\mathbb{C}P^2_9$ in 8-space, truncate vertices and glue in $k-l+1$ combinatorially equivalent copies of $-\mathbb{C}P^2_9$ with an open vertex star removed (see K., LNM 1612). Finally attach $l-1$ handles of type $S^2 \times S^2$.
Crucial: $\mathbb{C}P^2_9$ contains triangles with a short link.

even intersection form with signature 0, 16 or $16n \geq 32$. Can be written as $2n(-E_8) \oplus m\binom{0\ 1}{1\ 0}$ or $2E_8 \oplus 2(-E_8) \oplus 2n(-E_8) \oplus (m-16)\binom{0\ 1}{1\ 0}$.
The rank is $16n+32+(2m-32)$. If the signature is zero we just take the standard ladder construction of tight connected sums of $S^2 \times S^2$ (J.Hebda 1984). The case of the 4-sphere is trivial: take any convex polyhedron.

If the signature is 16 we start with the tight $K3$ surface in 15-space and attach handles of type $S^2 \times S^2$ tightly. Crucial: this triangulation contains a triangle with a short link.

If the signature is $16n \geq 32$ we first build a tight $(-K3)\sharp(K3)\sharp n(K3)$ by the truncation process above and then attach $m-3n-22$ handles of type $S^2 \times S^2$ tightly. By the extra assumption

$$\operatorname{rank}(Q) \geq \tfrac{11}{8}\operatorname{sign}(Q) + 44$$

we have $m-3n-22 \geq 0$ handles to attach.
(The $\frac{11}{8}$ conjecture states $m-3n \geq 0$)
In any case the resulting tightly embedded 4-manifold has the same intersection form as M and is, therefore, homeomorphic to M by Theorem 4. $\square$

REMARK: The cases which are not covered by the Main Theorem are

- $\mathbb{C}P^2 \sharp k(\mathbb{C}P^2)$ where $k \geq 1$ and
- $K3 \sharp m(K3) \sharp l \binom{0\ 1}{1\ 0}$ where $m \geq 1$ and $l < 22$.

Examples of that kind would imply that – modulo the validity of the $\frac{11}{8}$-conjecture – *every* simply connected PL 4-manifold admit a tight embedding into some Euclidean space, up to homeomorphism.