On closed Weingarten surfaces

Wolfgang Kühnel and Michael Steller

Abstract: We investigate closed surfaces in Euclidean 3-space satisfying certain functional relations $\kappa = F(\lambda)$ between the principal curvatures κ, λ. In particular we find analytic closed surfaces of genus zero where F is a quadratic polynomial or $F(\lambda) = c\lambda^{2n+1}$. This generalizes results by H.Hopf on the case where F is linear and the case of ellipsoids of revolution where $F(\lambda) = c\lambda^3$.

2000 MSC classification: 53A05, 53C40

Keywords: curvature diagram, surface of revolution, Hopf surface, Weingarten surface

Introduction

A surface in 3-space is called a *Weingarten surface* or a *W-surface* if the two principal curvatures κ and λ are not independent of one another or, equivalently, if a certain relation $\Phi(\kappa, \lambda) = 0$ is identically satisfied on the surface. The set of solutions of this equation is also called the *curvature diagram* or the *W-diagram* [4] of the surface. The study of Weingarten surfaces is a classical topic in differential geometry, as introduced by Weingarten in 1861 [11]. For applications in CAGD see [2]. If the curvature diagram degenerates to exactly one point then the surface has two constant principal curvatures which is possible only for a piece of a plane, a sphere or a circular cylinder. If the curvature diagram is contained in one of the coordinate axes through the origin then the surface is developable. If the curvature diagram is contained in the main diagonal $\kappa = \lambda$ then the surface is a piece of a plane or a sphere because every point is an umbilic. The curvature diagram is contained in a straight line parallel to the diagonal $\kappa = -\lambda$ if and only if the mean curvature is constant. It is contained in a standard hyperbola $\kappa = c/\lambda$ if and only the Gaussian curvature is constant. Locally there are the following five main classes of Weingarten surfaces:

1. surfaces of revolution,
2. tubes around curves where one principal curvature is constant,
3. helicoidal surfaces,
4. surfaces of constant Gaussian curvature,
5. surfaces of constant mean curvature (*cmc* surfaces).

This list is, of course, not exhausting. It is not difficult to obtain closed smooth Weingarten surfaces of arbitrary genus by glueing together pieces of spheres, other surfaces of revolution and tubes. The classical analytic examples are the closed surfaces of revolution of genus 0 or genus 1 on the one hand and tubes around closed curves on the other hand. Since the discovery of the Wente torus [12] analytic example of type 5 above have been investigated. N.Kapouleas [5] found closed surfaces of constant mean curvature for higher genus. However, K.Voss proved in [10] that a closed analytic Weingarten surface of genus zero is necessarily a surface of revolution. From the Delaunay surfaces one obtains as a corollary a classical result of H.Hopf [4] that a closed surface of genus zero with constant mean curvature must be a round sphere. It was the discovery of H.Hopf in the same article that there are closed analytic surfaces of genus zero with a linear curvature diagram. In the sequel the method of Hopf is further extended to a larger class of surfaces with a prescribed curvature diagram. As a result, there are explicit analytic solutions of genus zero also with self-intersections, see the section on quadratic Weingarten surfaces.

The case of a linear curvature diagram: Hopf surfaces

Unless one of the principal curvatures is constant, the curvature diagram in the (κ, λ)-plane is linear if and only if an equation

$$\kappa = c\lambda + d$$

is satisfied with two constants c, d where $c \neq 0$. It was observed by H.Hopf [4, p.238] that on a closed analytic surface of genus $g \geq 2$ such a relation is impossible unless $c = -1$ which is the case of a *cmc* surface. By that time it was unknown whether such a surface exists. As one of the main results, in the same article Hopf proved that in an umbilical point of an analytic surface either c or $1/c$ must be an odd integer. This implies that on a closed analytic surface of genus zero a relation $\kappa = c\lambda + d$ can hold only for these specific values of c. Furthermore, all these values are in fact realized by certain closed surfaces of revolution. The case of the standard sphere corresponds to the case $c = 1$ and (necessarily) $d = 0$.

NOTATION: *Whenever we talk about a surface of revolution then κ denotes the curvature of the profile curve and λ denotes the other principal curvature of the surface.*

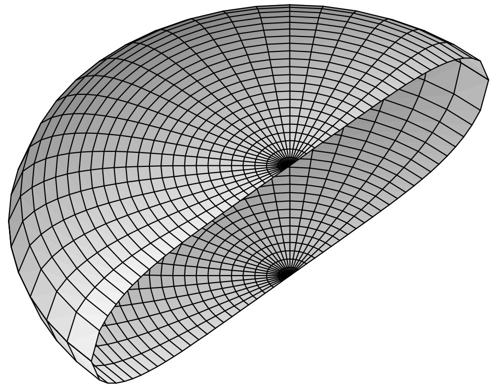

Figure 1: A W-surface with $\kappa = 5\lambda$ (*Hopf surface*).

1. Proposition (H.Hopf [4])
For any $c > 1$ and $d \leq 0$ there is a unique (up to scaling) closed convex C^2-surface of revolution satisfying the equation $\kappa = c\lambda + d$ and which is distinct from the standard sphere. This surface is analytic if and only if c is an odd integer.

By a theorem of K.Voss [10] any closed analytic Weingarten surface of genus zero is necessarily a surface of revolution. This leads to the following Corollary:

2. Corollary *Any closed analytic surface of genus zero satisfying the same equation $\kappa = c\lambda + d$ is congruent to the standard sphere or to one of the Hopf surfaces in Proposition 1, up to scaling.*

We do not repeat the proof of Proposition 1 here since this is a special case of a more general construction described below. For $d = 0$ an elementary exposition is given in [7, 3.27] including a picture of typical profile curves. Figure 1 shows a solution for $c = 5$ and $d = 0$ which is, therefore, real analytic. As a matter of fact [6], any of these surfaces with $\kappa = c\lambda$ satisfy in addition the equation $K_{II} = H$ where K_{II} denotes the inner curvature of the second fundamental form regarded as a Riemannian metric. Similarly, for negative c one obtains complete surfaces of negative Gaussian curvature, including the catenoid. Locally surfaces satisfying $\kappa = c\lambda$ were rediscovered 30 years later in [9].

Constructing closed surfaces

For the construction of surfaces of revolution satisfying particular equations between the principal curvatures we choose a parametrization of the type

$$f(r,\phi) = (r\cos\phi, r\sin\phi, h(r))$$

where $r \geq 0$ and $0 \leq \phi \leq 2\pi$ and where h is a function only of the parameter r. This is possible except at points where the tangent of the profile curve is parallel to the axis of rotation. We call this a *vertical tangent*. In such exceptional points there is a zero of $1/h'$ (or a pole of h'). This has to be taken into account in all the calculations below. Then the principal curvatures κ, λ are given by the equations

$$\lambda = \frac{h'}{r(1+h'^2)^{1/2}} \quad , \qquad \kappa = (r\lambda)' = \frac{h''}{(1+h'^2)^{3/2}}$$

where $(\ \ )'$ denotes differentiation by r. Consequently any equation of the type

$$\kappa = F(\lambda)$$

with a given continuous function F leads to the ODE $(r\lambda)' = F(\lambda)$ or, equivalently,

$$\lambda'(r) \quad = \quad \frac{F(\lambda)-\lambda}{r}. \tag{1}$$

It follows that locally any equation of this type with an arbitrary continuous function F admits a solution. Apparently this was rediscovered in the paper [8]. In fact, this ODE is explicitly solvable by separation of variables. As long as $r^2\lambda^2(r) < 1$, any solution of (1) satisfies the equation

$$h'(r) \quad = \quad \pm\sqrt{\frac{r^2\lambda^2(r)}{1-r^2\lambda^2(r)}}. \tag{2}$$

Consequently the function h admits the following representation

$$h(r) \quad = \quad \pm\int \frac{r\lambda(r)}{\sqrt{1-\lambda^2(r)r^2}}dr. \tag{3}$$

Example: In the case of Hopf surfaces with $F(\lambda) = c\lambda + d$ one obtains the solution $\lambda(r) = \frac{1}{c-1}(r^{c-1} - d)$ which is essentially unique. Then equation (3) leads to an explicit expression for $h(r)$. The other principal curvature is $\kappa(r) = \frac{1}{c-1}(cr^{c-1} - d)$.

THE CASE OF GENUS ZERO

If a connected component of the profile curve starts on the axis of rotation with a horizontal tangent and ends at a point p with a vertical tangent, then we obtain a closed surface of genus zero by glueing together two mirror symmetric copies of the surface. Under some additional conditions this closed surface is of class C^2 (resp. C^∞).

3. Lemma Let λ be a solution of (1). Then the following conditions on λ and the resulting h are necessary and sufficient for obtaining a compact surface of revolution of class C^2:

1. λ is defined on an interval $(0, \varepsilon)$, with a finite limit $\lim_{r \searrow 0} \lambda(r) =: \lambda(0)$.
2. For $r = 0$ there is an umbilic, i.e. $\lambda(0) = F(\lambda(0))$.
3. There is a maximal interval $[0, r_0)$ on which λ and h' are defined as differentiable solutions, necessarily with a vertical tangent at r_0 i.e., necessarily with a pole of h' at r_0 and with $\lambda(r_0) = \pm\frac{1}{r_0}$.
4. h is a continuous function on the interval $[0, r_0]$ including the endpoint r_0 (i.e., the corresponding improper integral in (3) converges).

Furthermore, the surface is analytic everywhere if and only if the following three conditions are satisfied:

5. λ is an even function at $r = 0$
6. F is analytic
7. $\frac{dF(\lambda)}{d\lambda}|_{r=0}$ is a positive and odd integer (see [4, p.236]).

Remark: If the condition 1. above is satisfied, then the condition 2. follows from (1).

In the case of the Hopf surfaces conditions 1. – 4. are easily checked. This leads to the construction in Proposition 1.

4. Lemma If the conditions 1. and 3. above are satisfied by a particular solution λ with $\lambda(r_0) = \pm\frac{1}{r_0}$ then the condition 4. is equivalent to each of the following equivalent conditions

(a) $(h')^2$ has a simple pole at $r = r_0$

(b) $F(\pm\frac{1}{r_0}) \neq 0$

(c) $\lambda'(r_0) \neq \mp r_0^{-2}$

Consequently, Conditions 1. and 3. are sufficient conditions for obtaining a compact C^2-surface of genus zero whenever $F(\pm\frac{1}{r_0}) \neq 0$.

The case of genus one

If a connected component of the profile curve $(r, h(r))$ starts at a point p_0 and ends at a point p_1 with vertical tangents in opposite directions and on the same h-level, then we obtain a closed surface of genus one by glueing together two mirror symmetric copies of it. The condition that the start- and endpoint are on the same h-level is crucial and might be difficult to decide because the levels of those points are represented by improper integrals. Under some additional conditions this closed surface is of class C^2 (resp. C^∞).

5. Lemma Let λ be a solution of (1). Then the following conditions on λ and the resulting h are sufficient for obtaining a compact surface of revolution of class C^2 of genus one:

1* There is a maximal interval $(r_0, r_1) \subset (0, \infty)$ on which λ and h' are defined as differentiable solutions, necessarily with vertical tangents at r_0 and r_1 (in opposite directions), i.e., necessarily with a pole of h' at $r_{0/1}$ and with $\lambda(r_0) = \pm\frac{1}{r_0}$ and $\lambda(r_1) = \mp\frac{1}{r_1}$.

2* h is a continuous function on the interval $[r_0, r_1]$ including the endpoints $r_{0/1}$ (i.e., we have convergence of the corresponding improper integral in (3))

3* At the two endpoints we have $h(r_0) = h(r_1)$.

Furthermore, the surface is analytic everywhere if and only if F is analytic

6. Lemma If the condition 1* above are satisfied by a particular solution λ with $\lambda(r_0) = \pm\frac{1}{r_0}$ and $\lambda(r_1) = \mp\frac{1}{r_1}$ then the condition 2* is equivalent to each of the following equivalent conditions

(a*) $(h')^2$ has a simple pole at $r = r_0$ and $r = r_1$

(b*) $F(\pm\frac{1}{r_0}) \neq 0$ and $F(\mp\frac{1}{r_1}) \neq 0$

(c*) $\lambda'(r_0) \neq \mp r_0^{-2}$ and $\lambda'(r_1) \neq \pm r_1^{-2}$

Consequently, Condition 1* is sufficient for obtaining a compact C^2-surface of genus one whenever $F(\pm\frac{1}{r_0}) \neq 0$, $F(\mp\frac{1}{r_1}) \neq 0$ and $h(r_0) = h(r_1)$.

For a compact surface of genus one the following necessary conditions must be satisfied:

1. There is a maximal interval (r_0, r_2) on which λ and h' are defined as differentiable solutions, necessarily with a vertical tangent at r_0 and r_2, i.e., necessarily with a pole of h' at r_0 and r_2 and with $\lambda(r_0) = \pm\frac{1}{r_0}$, $\lambda(r_2) = \mp\frac{1}{r_2}$. In between there is a zero of λ at some point $r_1 \in (r_0, r_2)$ which corresponds to a point with a horizontal tangent.

2. h is a differentiable function on the interval $[r_0, r_2]$.

Quadratic Weingarten surfaces: a quadric as diagram

Closed surfaces satisfying a linear relation $aK + bH + c = 0$ between the mean curvature and the Gaussian curvature with constants a, b, c were studied by Chern in [3]. The result is that any closed and convex surface of this type with $K > 0$ and $a^2 + b^2 + c^2 \neq 0$ is a standard sphere. This can be seen as follows: For $b \neq 0$ the curvature diagram corresponding to this equation is nothing but the ordinary rectangular hyperbola

$$\left(\kappa + \frac{a}{2b}\right)\left(\lambda + \frac{a}{2b}\right) = \frac{a^2 - 4bc}{4b^2}.$$

At an umbilical point we have necessarily $a^2 - 4bc \geq 0$. Obviously, within the class of convex surfaces the cases $b = 0$ and $a^2 - 4bc = 0$ are possible only for the standard sphere. It turns out that the case $a^2 - 4bc > 0$ also leads to the standard sphere by the following argument which is due to Hilbert: *The larger principal curvature cannot attain its maximum if simultaneously the other principal curvature attains its minimum.*

In this section we study the case that the curvature diagram is a standard parabola with the corresponding quadratic function

$$F(\lambda) = c(\lambda - \lambda_*)^2 + \lambda - a$$

where a, c, λ_* are constants, $c \neq 0$. Since any closed surface of genus zero has an umbilic, we have necessarily $ac \geq 0$. The standard sphere is the particular solution where λ is constant. In all other cases the ODE (1) can be solved as follows, with the notation $y = \lambda(r)$ for a variable and $\lambda_b(r)$ for the various solutions:

$$\int \frac{1}{c(y-\lambda_*)^2 - a} dy = \log r + b \quad \text{with a constant } b \in \mathbb{R} \tag{4}$$

<u>Case 1: $a = 0$.</u> Then the equation above becomes

$$\begin{aligned} -\frac{1}{c(\lambda_b(r) - \lambda_*)} &= \log r + b \qquad (5) \\ \text{or, equivalently,} \qquad \lambda_b(r) &= \lambda_* - \frac{1}{c(\log r + b)} \end{aligned}$$

We have to check the conditions according to Lemma 3 above:

1. $r = 0$ corresponds to $\lambda = \lambda_*$.

2. For $r = 0$ we have an umbilic since $\kappa = F(\lambda) = \lambda$ there.

3. In the region $\lambda > \lambda_*$ the function $F(\lambda)$ tends to $\pm\infty$ for $\lambda \to \infty$ depending on the sign of c. Moreover, λ has a pole at $r = e^{-b}$. Therefore there is an intersection point of the graph of $\lambda(r)$ with that of $\pm\frac{1}{r}$, i.e., a point with $\lambda^2(r_0) = \frac{1}{r_0^2}$, as required.

4. The extra condition $F(\pm\frac{1}{r_0}) \neq 0$ is satisfied because of

$$F\Big(\pm\frac{1}{r_0}\Big) = c\Big(\pm\frac{1}{r_0} - \lambda_*\Big)^2 \pm \frac{1}{r_0} = c\Big(\pm\frac{1}{r_0} - \lambda_*\Big)^2 + \frac{c}{|c|}\cdot\frac{1}{r_0}.$$

Consequently, there is a closed convex surface satisfying this equation. This surface is never of class C^3 because λ_b is not differentiable at $r = 0$ since $\lim\limits_{r\to 0} \lambda_b'(r) = \pm\infty$.

<u>Case 2: $ac > 0$.</u> Then the equation (4) above becomes

$$\int \frac{1}{c(y-\lambda_*)^2 - a} dy = \frac{1}{2\sqrt{ac}} \log \pm \frac{\lambda - \lambda_* - \sqrt{a/c}}{\lambda - \lambda_* + \sqrt{a/c}} \quad = \quad \log r + b \tag{6}$$

or

$$\int \frac{1}{c(y-\lambda_*)^2 - a} dy = \frac{\operatorname{artanh}\frac{\pm c(\lambda_* - \lambda)}{\sqrt{ac}}}{\sqrt{ac}} \quad = \quad \log r + b \tag{7}$$

We can choose the positive sign in (6) and (7) which corresponds to just one of the branches of the solution. Nevertheless, the considerations below can be adapted also to the negative sign. This leads to the following two analytic expressions for λ_b:

$$\lambda_b(r) \quad = \quad \lambda_* - \frac{\sqrt{ac}}{c} \tanh\left(\sqrt{ac}(\log r + b)\right) \tag{8}$$

and

$$\lambda_b(r) \quad = \quad \lambda_* + \sqrt{\frac{a}{c}} \cdot \frac{1 + r^{2c\sqrt{a/c}} \cdot e^{2bc\sqrt{a/c}}}{1 - r^{2c\sqrt{a/c}} \cdot e^{2bc\sqrt{a/c}}} \tag{9}$$

Again we have to check the conditions as above:

1. $r = 0$ corresponds to $\lambda = \lambda_* + \frac{\sqrt{ac}}{c}$.
2. For $r = 0$ we have an umbilic since $\kappa = F(\lambda) = \lambda$ there.
3. Because λ_b is continuous on $[0, \infty)$ we have to show that there exist some constants b leading to at least one intersection point of the graphs of the functions $\lambda_b(r)$ and $\pm\frac{1}{r}$ (we always take the intersection point r_0 nearest to the axis of rotation). If we combine the equation $\lambda_b(r) = \pm\frac{1}{r}$ with (8) and resolve it for $b = b(r)$ then we get:

$$b(r) = \frac{1}{\sqrt{ac}} \operatorname{artanh}\Big(\frac{c(\lambda_* \mp 1/r)}{\sqrt{ac}}\Big) - \log r$$

Therefore $b(r)$ is well defined if and only if the following inequality holds

$$\Big|\frac{c(\lambda_* \mp 1/r)}{\sqrt{ac}}\Big| < 1 \quad \Longleftrightarrow \quad \Big|\lambda_* \mp \frac{1}{r}\Big| < \frac{\sqrt{ac}}{|c|}$$

Depending on the sign, there is a solution r in a union of at most two intervals $I = I_1 \cup I_2 \subset \mathbb{R}^+$ where one of them is not empty. It follows that we always get a 1-parameter family of solutions λ_b depending on constants b ranging in I_1 or in I_2, respectively, such that there are intersection points with $\pm\frac{1}{r}$ because $b(r)$ is continuous and not constant (i.e., for one $i \in \{1,2\}$ holds $\mid b(I_i) \mid > 1$).

4. From 3. we know that for every $r \in I$ we have an intersection point of of the graphs of $\lambda_b(r)$ and $\pm\frac{1}{r}$. Because F has at most two zeros it follows that there are at most two elements $\overline{r}_0, \overline{r}_1 \in I$ with $F(\pm\frac{1}{\overline{r}_0}) = F(\pm\frac{1}{\overline{r}_1}) = 0$ (for at least one of the signs). For these $\overline{r}_{0/1}$ we know that the constant b have to be $b(\overline{r}_0)$ resp. $b(\overline{r}_1)$ where we have to regard the sign in the definition of $b(r)$. By analyticity of the solutions in (8) there are always a finite number of constants $b \in b(I)$ so that $\overline{r}_{0/1}$ is an intersection point of λ_b and $\pm\frac{1}{r}$. Combining this with Lemma 4 (b) we see that the extra condition $F(\pm\frac{1}{r_0}) \neq 0$ is satisfied for all constants b ranging in the interval above after removing a finite number of constants. Furthermore we can choose a smaller interval $B \subset b(I)$ within the remaining part of $b(I)$. With respect to this interval B we finally obtain a 1-parameter family of solutions λ_b depending on $b \in B$ and satisfying 3. and 4.

The surface is analytic if and only if $\sqrt{ac}$ is an integer since we have

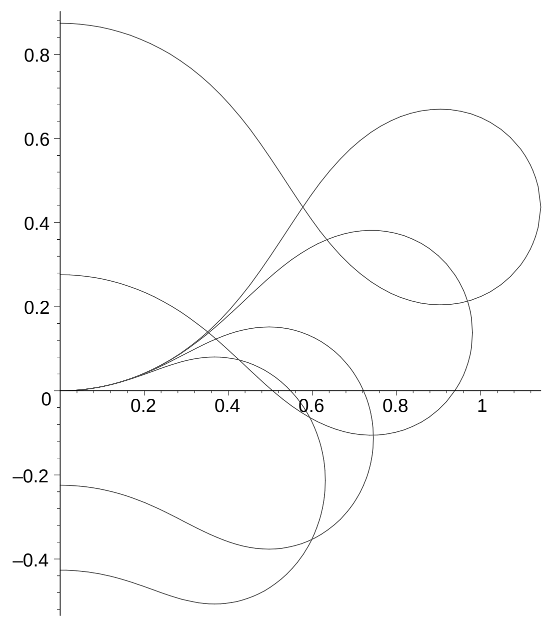

Figure 2: Profile curves of W-surfaces with $\kappa = \lambda^2 + \lambda - 4$.

$$\frac{dF(\lambda)}{d\lambda}\Big|_{r=0} = 2c(\lambda_* + \frac{\sqrt{ac}}{c} - \lambda_*) + 1 = 2\sqrt{ac} + 1$$

and because the expression in (9) for λ_b is an even function if $\sqrt{ac}$ is an integer.

This proves the following Propositions 7 and 8:

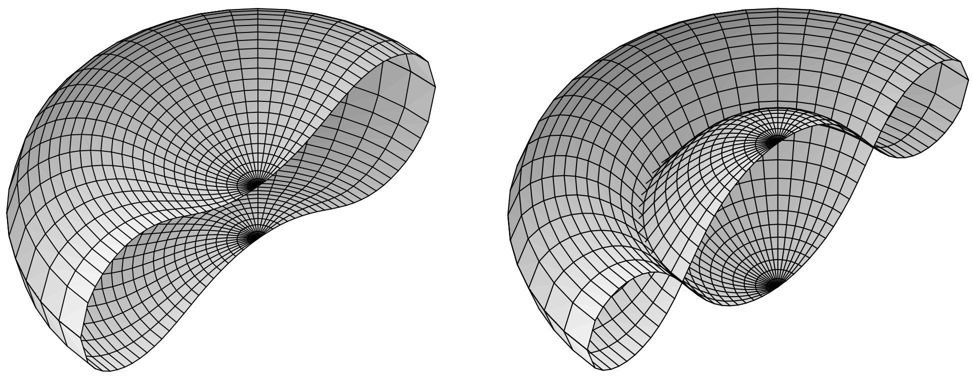

Figure 3: W-surfaces satisfying $\kappa = \lambda^2 + \lambda - 4$ with $b = 0.7$ and $b = 0.1$.

7. Proposition (quadratic W-surfaces)
For any constants a, c and λ_* with $ac > 0$ there exists a 1-parameter family (depending on the choice of a parameter b) of closed surfaces of class C^2 satisfying the equation

$$\kappa = c(\lambda - \lambda_*)^2 + \lambda - a.$$

These are analytic if and only if $\sqrt{ac}$ is an integer.

Figure 2 shows several profile curves for $c = 1$, $a = 4$, and $\lambda_* = 0$, Figure 3 shows two particular surfaces which are analytic since $\sqrt{ac} = 2$ in this case.

8. Proposition *For any constants $c \neq 0$ and λ_* there exists a 1-parameter family (depending on the choice of a parameter b) of closed C^2-surfaces satisfying the equation $\kappa = c(\lambda - \lambda_*)^2 + \lambda$. None of them is of class C^3, except for the standard sphere.*

Generalized Hopf surfaces: the case $\kappa = \lambda^\alpha$.

It is well known [1, Ex.3] that the ellipsoid of revolution

$$\Big\{(x, y, z) \;\Big|\; \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1\Big\}$$

satisfies the relation

$$\kappa = \frac{a^4}{b^2}\lambda^3$$

between the two principal curvatures (note our convention that κ denotes the curvature of the profile curve). Conversely, any closed surface of revolution satisfying $\kappa = c\lambda^3$ with a certain constant c is congruent to some ellipsoid of revolution. More generally, in this section we study equations of the type $\kappa = c\lambda^\alpha$ or, equivalently,

$$F(\lambda) = c\lambda^\alpha$$

with some constants $c > 0$ and $\alpha \neq 1, \alpha \neq 0$. Recall that the case $\alpha = 1$ leads to a linear curvature diagram and was studied above. Hence the ODE (1) takes the particular form

$$\lambda'(r) = \frac{c}{r} \cdot \Big(\lambda(r)\Big)^\alpha - \frac{1}{r} \cdot \lambda(r)$$

which is known as *Bernoulli's differential equation*. Any solution is of the form

$$\lambda_b(r) = \Big(c + \frac{b}{r^{1-\alpha}}\Big)^{\frac{1}{1-\alpha}}$$

with a constant of integration $b \in \mathbb{R}$. The particular case $b = 0$ leads to $\lambda_0(r) = c^{\frac{1}{1-\alpha}}$ and, consequently, to constant principal curvatures $\kappa_0 = F(\lambda_0) = c\lambda_0^\alpha = \lambda_0$. This is the case of the round sphere of radius $R = 1/\lambda_0$. Therefore in the sequel we assume $b \neq 0$. There are two cases:

<u>Case 1: $\alpha < 1$.</u>
In this case we have

$$\lim_{r \searrow 0} |\lambda_b(r)| = \lim_{r \searrow 0} \Big(|b| r^{\alpha-1}\Big)^{1/(1-\alpha)} = \infty.$$

Therefore, no regular surface of genus zero can satisfy this equation since any point on the axis of rotation is a singularity. However, the solution leads to a convex surface with two isolated singularities on the axis, just as the classical Hopf surfaces for $0 < c < 1$.

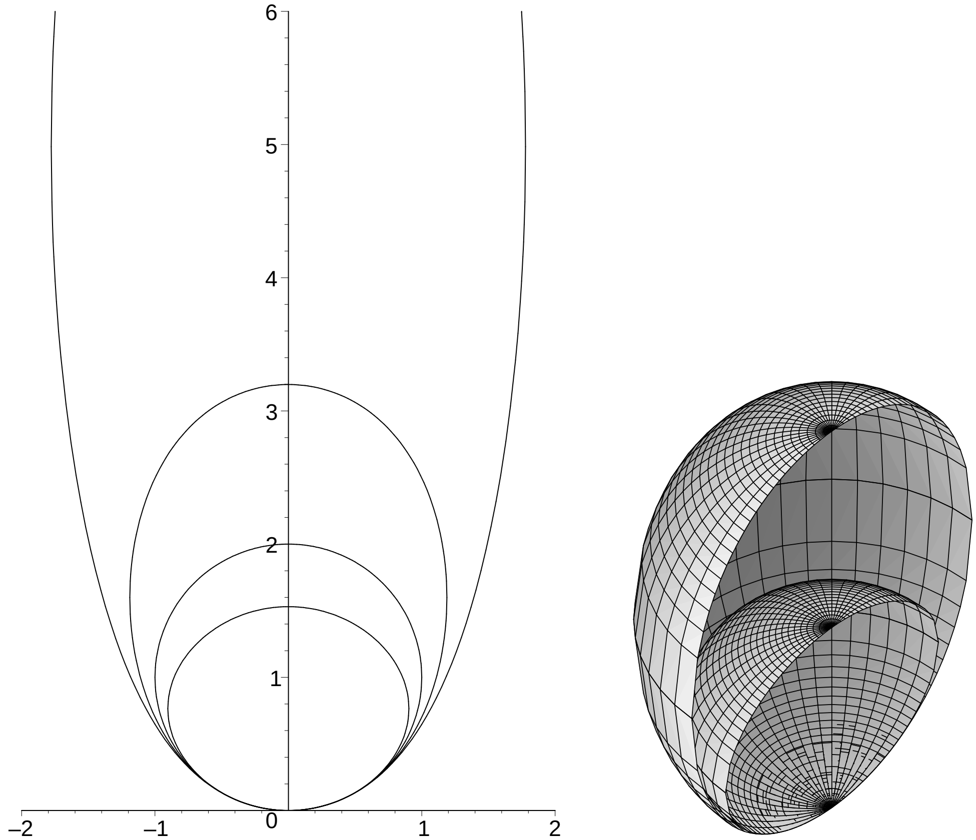

Figure 4: Generalized Hopf surfaces with $\kappa = \lambda^5$.

Case 2: $\alpha > 1$.

In this case we can verify that all the conditions in Lemma 3 are satisfied:

1. We have

$$\lim_{r \searrow 0} \lambda_b(r) = c^{1/(1-\alpha)}$$

for arbitrary choice of b. For $b > 0$ the solution λ_b is defined on $[0, \infty)$, for $b < 0$ the solution has a pole at $r = (-\frac{c}{b})^{1/(\alpha-1)}$.

2. For $r = 0$ we have an umbilic since $F(\lambda(0)) = \lambda(0)$.

3. Under the assumption $b < 0$ the solution λ_b has a pole. Therefore there is an intersection point of its graph with the graph of the function $\pm\frac{1}{r}$, i.e., there is a number r_0 such that $\lambda(r_0) = \pm\frac{1}{r_0}$. For $0 < b < 1$ the resulting function $h'(r)$ has a pole where necessarily $r^2\lambda^2 = 1$ is satisfied, compare equation (2).

4. The additional condition $F(\pm\frac{1}{r_0}) \neq 0$ is trivially satisfied.

The surface is analytic if and only if α is an odd integer since we have

$$\frac{dF(\lambda)}{d\lambda}\bigg|_{r=0} = c\alpha(\lambda(0))^{\alpha-1} = \alpha$$

on the one hand and since λ_b is an even function for odd α on the other.

Therefore we obtain the following generalization of Proposition 1 above:

9. Proposition (Generalized Hopf surfaces)
For any constant $\alpha > 1$ and any $c > 0$ there exists a 1-parameter family (depending on the choice of a parameter $b < 1$) of closed convex surfaces of class C^2 satisfying the equation $\kappa = c\lambda^\alpha$. These are analytic if and only if α is an odd integer. Up to scaling, these surfaces form a 2-parameter family. The particular case $\alpha = 3$ corresponds to the classical ellipsoids of revolution.

In the case $\alpha = 3$ the solution is

$$\lambda_b(r) = \left(c + br^2\right)^{-\frac{1}{2}}$$

and, consequently,

$$h = \pm \int \frac{r}{\sqrt{c + (b-1)r^2}} dr = \frac{\pm 1}{b-1} \sqrt{c + (b-1)r^2} + const$$

which describes an ellipsoid for $b < 1$ and a hyperboloid for $b > 1$. The particular case $b = 1$ leads to a paraboloid, for $b = 0$ one obtains a sphere.

Similarly, each of the cases $\alpha = 5, 7, 9, \ldots$ leads to a family of *generalized ellipsoids* in the sense that these are analytic convex surfaces with a shape which is quite similar to that of an ellipsoid. In any case the standard sphere is a particular member of this family. Similarly, for $b = 1$ we obtain generalized paraboloids and for $b > 1$ generalized hyperboloids. Figure 4 shows generalized ellipsoids with $\alpha = 5$, $c = 1$ and four cases $b = 0.9,\ 0.5,\ 0,\ -0.5$.

In constrast to this situation, we have the following:

10. Proposition *For any constant $\alpha < 1$ and any $c > 0$ there is no closed genus zero C^2-surface of revolution satisfying the equation $\kappa = c\lambda^\alpha$, except for the standard sphere. Therefore, if there is an analytic surface of genus zero (distinct from the standard sphere) satisfying $\kappa = c\lambda^\alpha$ with $\alpha < 1$ and $c > 0$ then there is no analytic surface of genus zero satisfying $\lambda = \tilde{c}\kappa^\alpha$ with the same α and any constant $\tilde{c}$. In particular, the ellipsoids of revolution are the only analytic surfaces of genus zero where the principal curvatures κ_1, κ_2 satisfy $\kappa_1 = c(\kappa_2)^3$.*

Closed Weingarten surfaces of genus one: An example

For obtaining solutions λ which lead to a closed surface of genus one we have to check the conditions in Lemma 5 above. The crucial condition 3^* is necessary for glueing together two mirror symmetric copies of the surface. The trivial case is $\kappa = c$ (constant) leading to the standard torus of revolution. Unfortunately, for the functions F studied in the

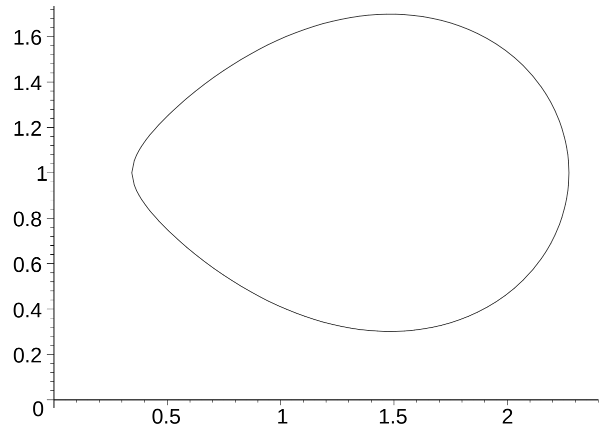

Figure 5: Profile curve of a W-surface with $\kappa = \cosh \lambda + \lambda$ for $c_0 = 1.177$.

sections above we could not verify condition 3*. Therefore, as an example we study the case of a curvature diagram associated with the function

$$F(\lambda) = \cosh(\lambda) + \lambda.$$

Then the ODE (1) can be solved by

$$\lambda(r) = \log\Big(\tan\big(\frac{1}{2}(\log r + c)\big)\Big) \tag{10}$$

where $c \in \mathbb{R}$ and $r > 0$. This solution is defined on any interval of the form $(r_k^0, r_k^1) := (e^{2k\pi - c}, e^{(2k+1)\pi - c})$ for arbitrary $k \in \mathbb{Z}$.

We have to check the conditions according to Lemma 5 above:

1*. We have $\lim_{r \searrow e^{2k\pi - c}} \lambda(r) = -\infty$ and $\lim_{r \nearrow e^{(2k+1)\pi - c}} \lambda(r) = \infty$. Hence there is a maximal interval (r_0, r_1) satisfying the conditions in 1*, i.e., at the endpoints of the interval we have an intersection of the graph of λ with that of $1/r$ and $-1/r$, respectively.

2*. From $F(x) = \cosh(x) + x > 0$ for $x \in \mathbb{R}$ we obtain $F(\pm\frac{1}{r_1}) \neq 0$ and $F(\mp\frac{1}{r_1}) \neq 0$. Hence 2* follows from Lemma 6.

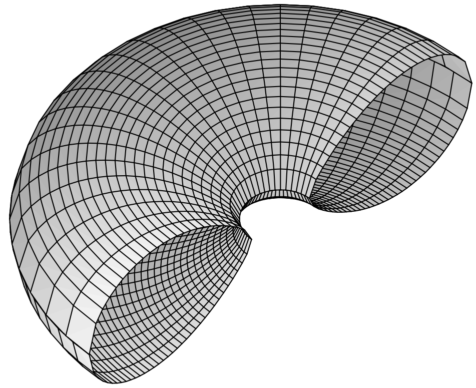

Figure 6: A W-torus satisfying $\kappa = \cosh \lambda + \lambda$.

3*. In order to verify 3* we have to evaluate $h(r_0)$ and $h(r_1)$ as improper integrals. We now choose $k = 0$ and $1/2 < c < \pi/2$ so we have exactly one point $r_0(c)$ with $\lambda(r_0(c)) = -1/r_0(c)$ and one point $r_1(c)$ with $\lambda(r_1(c)) = 1/r_1(c)$. The difference $d(c) := h(r_0(c)) - h(r_1(c))$ is a continuous function of c. By a numerical calculation we obtain $d(1) \approx 0.3418424004$ and $d(1.2) \approx -0.2427157496$. Because of continuity of d there is a $c_0 \in (1, 1.2)$ with $d(c_0) = 0$. A reasonable approximation is $c_0 \approx 1.177$ with $r_0(c_0) \approx 0.3437410829$ and $r_1(c_0) \approx 2.271531739$. Figure 5 shows the profile curve for $c_0 = 1.177$ where the vertical coordinate axis is the axis of rotation, and Figure 6 shows the W-surface of genus one itself. The surface is analytic because F is analytic.

This implies the following:

11. Proposition *There is a closed analytic surface of revolution of genus one satisfying the equation* $\kappa = \cosh(\lambda) + \lambda$.

Presumably, by the same method it is possible to find many other examples of W-surfaces of genus one with prescribed relations $\kappa = F(\lambda)$.

References

[1] B. van–Brunt and K. Grant, Hyperbolic Weingarten surfaces, Math. Proc. Camb. Philos. Soc. **116**, 489–504 (1994)

[2] B. van–Brunt and K. Grant, Potential applications of Weingarten surfaces in CAGD. I: Weingarten surfaces and surface shape investigation, Comput. Aided Geom. Des. **13**, 569–582 (1996)

[3] S.-s. Chern, Some new characterizations of the Euclidean sphere, Duke Math. J. **12**, 279–290 (1945)

[4] H. Hopf, Über Flächen mit einer Relation zwischen den Hauptkrümmungen, Math. Nachr. **4**, 232–249 (1951)

[5] N. Kapouleas, Compact constant mean curvature surfaces in Euclidean three-space, J. Differ. Geom. **33**, 683–715 (1991)

[6] W. Kühnel, Zur inneren Krümmung der zweiten Grundform, Monatsh. Math. **91**, 241–251 (1981)

[7] W. Kühnel, Differential Geometry, Curves – Surfaces – Manifolds, AMS, Providence, R.I. 2002 (German edition: Differentialgeometrie, Vieweg, Wiesbaden 2003)

[8] X. Huang, Weingarten surfaces in three-dimensional spaces (Chinese), Acta Math. Sin. **31**, 332–340 (1988), see Zbl. 678.53002

[9] B. Papantoniou, Classification of the surfaces of revolution whose principal curvatures are connected by the relation $Ak_1 + Bk_2 = 0$ where A or B is different from zero, Bull. Calcutta Math. Soc. **76**, 49–56 (1984)

[10] K. Voss, Über geschlossene Weingartensche Flächen, Math. Annalen **138**, 42–54 (1959)

[11] J. Weingarten, Ueber eine Klasse auf einander abwickelbarer Flächen, J. Reine Angew. Math. **59**, 382-393 (1861)

[12] H. Wente, Counterexample to a conjecture of H. Hopf, Pacific J. Math. **121**, 193–243 (1986)

Fachbereich Mathematik, Universität Stuttgart, 70550 Stuttgart, Germany

kuehnel@mathematik.uni-stuttgart.de
steller@mathematik.uni-stuttgart.de