Total curvature of complete hypersurfaces

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Definition: A complete (open) hypersurface in Euclidean $\mathbb{E}^{n+1}$ is an embedded hypersurface such that the embedding is proper (preimage of a compact set is compact) and such that the image is complete. Similarly we have complete open immersed hypersurfaces.

Theorem (Gauss-Bonnet)
For any compact and oriented Riemannian 2-manifold with boundary the equality

$$2\pi\chi(M) - \int_M K dA = \int_{\partial M} \kappa(s) ds$$

holds where κ denotes the geodesic curvature on the oriented boundary. In particular, if all boundary curves are geodesics, we obtain

$$2\pi\chi(M) - \int_M K dA = 0,$$

the same formula which holds for compact 2-manifolds without boundary.

In the case of non-compact 2-manifolds things are a little bit more complicated.

Theorem (Cohn-Vossen)
If (M, g) is a complete Riemannian 2-manifold of finite topological type and with absolutely integrable Gauss curvature K, then the inequality

$$2\pi\chi(M) - \int_M K dA \geq 0$$

holds. In particular, we have $\int_M K dA \leq 2\pi$ if M is non-compact.

There are more subtle versions for the case that M is not of finite topological type. Furthermore, there are a number of additional conditions under which the Gauss-Bonnet equality $2\pi\chi(M) - \int_M K dA = 0$ holds in the non-compact case.

Theorem (Osserman)
For embedded and complete minimal surfaces with finite total curvature the equation

$$2\pi\chi(M) - \int_M K dA = 2\pi k$$

holds where k is the number of ends. In the case of immersed minimal surfaces one has to take "multiplicities" at the ends into account.

Wintgen suggested that the curvature defect of a complete and properly immersed surface in Euclidean 3-space is the length of the set M_∞ of the so-called *limit directions* $\lim_{n\to\infty} f(x_n)/||f(x_n)||$. This is true under reasonable additional assumptions.

For a hypersurface in Euclidean space $\mathbb{E}^{n+1}$ we have the *Gauss-Kronecker curvature* $K = K_n$ which is defined as the determinant of the shape operator. It is well-known that K is intrinsic if n is even.

NOTATION: The constant c_n denotes the volume of the standard unit n-sphere. This can be expressed in terms of the Gamma function as follows: $c_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$. The symbol dV denotes the volume element of a submanifold, sometimes in the form dV_M for specifying the manifold on which it is defined.

Theorem (Gauss-Bonnet-Hopf)
Let $M^n \subset \mathbb{E}^{n+1}$ be an embedded compact hypersurface such that M is the boundary of its interior $M_{\mathrm{int}} \subset \mathbb{E}^{n+1}$, and let K denote the Gauss-Kronecker curvature of M with respect to the inner normal (pointing to M_{int}). Then the following hold:

(i) $\int_M K dV_M = c_n \cdot \chi(M_{\mathrm{int}})$.

(ii) If n is even, then $\chi(M) = 2\chi(M_{\mathrm{int}})$ and, consequently, $\int_M K dV_M = (c_n/2) \cdot \chi(M)$.

Moreover, this equality holds for arbitrary immersions

$$f\colon M \to \mathbb{E}^{n+1}$$

of a compact orientable n-manifold, even if M is not the boundary of any $(n+1)$-manifold.

As a matter of fact, for odd dimensions the total curvature does depend on the choice of M_{int}, i.e., on the choice of the embedding. Nevertheless, we have the following folklore result:

Proposition

Within the class of all immersions $f\colon M^n \to \mathbb{E}^{n+1}$ of a fixed compact manifold M, the total Gauss-Kronecker curvature $\int_M K dV_M$ depends only on the regular homotopy class of f.

This follows from the variational formula for the total curvature, see below. The gradient of the curvature functional $\int_M K dV$ is identically zero. The theorem on turning tangents (the "Hopf Umlaufsatz") can be regarded as the special case $n = 1$ in the Gauss-Bonnet-Hopf theorem.

Definition (unit normal space, total curvature)

For a compact submanifold $M^n \subset \mathbb{E}^{m+1}$ with boundary ∂M we define the *unit normal space* N^1 by

$$N^1 = \perp^1 (M) \cup \perp^1_+ (\partial M).$$

It carries a canonical volume form dV_{can} as in the case of a submanifold without boundary. Then the *total curvature* of M is defined as the sum of the total curvatures of the two parts from $\perp^1 (M \setminus \partial M)$ and from $\perp^1_+ (\partial M)$:

$$\mathrm{TC}(M, \partial M) := \int_{N^1} K dV_{can}$$

$$= \int_{\xi \in \perp^1 (M \setminus \partial M)} K_n(\xi) dV_{can} + \int_{\xi \in \perp^1_+ (\partial M)} K_{n-1}(-\xi) dV_{can}.$$

Theorem

For a compact submanifold $M^n \subset \mathbb{E}^{m+1}$ with boundary ∂M (or an immersion of M) the Gauss-Bonnet formula holds as follows:

$$\mathrm{TC}(M, \partial M) = c_m \cdot \chi(M).$$

Moreover, if m is even, then we have $\chi(N^1) = 2\chi(M)$.

The Gauss-Bonnet difference term

$$c_m \chi(M) - \int_{\perp^1(M \setminus \partial M)} K_n dV_{can}$$

can be expressed as the integral of K_{n-1} over the set of outer unit normals at ∂M. Obviously, any $\widetilde{\xi} \in (\perp^1_+)_p$ can be uniquely written as

$$\widetilde{\xi} = \cos\varphi \cdot \nu_{out} + \sin\varphi \cdot \xi,$$

where $0 \leq \varphi \leq \pi/2$ and ξ is a unit normal vector to M at $p \in \partial M$. Vice versa, any such ξ leads to a $\widetilde{\xi}$ in $(\perp^1_+)_p$ for any φ with $0 \leq \varphi \leq \pi/2$. This enables us to compute this integral by Fubini's theorem, pointwise evaluated for the normal sphere S^{m-n} on the one hand and half the normal sphere S^{m-n+1} on the other hand.

Theorem (Gauss-Bonnet theorem for submanifolds in the closed unit ball)
Let $(M^n, \partial M^n) \subset (B^{m+1}, S^m)$ be a compact submanifold which is orthogonal at the boundary, i.e., the outer normal ν_{out} of M at each boundary point coincides with the outer normal of S^m. Then for the Gauss-Bonnet defect the equation

$$c_m \chi(M) - \int_{\perp^1(M \setminus \partial M)} K dV_{can}$$

$$= \sum_{0 \leq 2i \leq n-1} \frac{c_m}{c_{m-n+2i} c_{n-1-2i}} \int_{\perp^1(\partial M)} K_{2i} \, dV_{can}$$

holds, where K_j denotes the j^{th} elementary symmetric function of the eigenvalues of the shape operator of the embedding $\partial M \to S^m$.

Proof: At each boundary point $p \in \partial M$ we compute the boundary term as follows: $\int_{\widetilde{\xi} \in (\perp^1_+)_p} K_{n-1}(-\widetilde{\xi}) \; dV_{S^{m-n+1}} = \int_{\xi \in \perp^1_p, \; 0 \leq \varphi \leq \pi/2} K_{n-1}(\sin\varphi \cdot \xi - \cos\varphi \cdot \nu_{out}) \; dV_{S^{m-n+1}}$
$= \int_{\perp^1_p} \int_0^{\pi/2} \det(\sin\varphi \cdot A_\xi - \cos\varphi \cdot A_{\nu_{out}}) dV_{S^{m-n}} \wedge \sin^{m-n}\varphi \; d\varphi$
$= \int_{\perp^1_p} \int_0^{\pi/2} \sin^{m-1}\varphi \det(A_\xi + \cot\varphi \cdot \mathrm{Id}) \; dV_{S^{m-n}} \wedge d\varphi$
$= \int_{\perp^1_p} \int_0^{\pi/2} \sin^{m-1}\varphi \sum_{j=0}^{n-1} K_j(\xi) \cot^{n-1-j}\varphi \; dV_{S^{m-n}} \wedge d\varphi$
$= \sum_{j=0}^{n-1} \int_0^{\pi/2} \sin^{m-n+j}\varphi \cos^{n-1-j}\varphi \; d\varphi \int_{\xi \in \perp^1_p} K_j(\xi) dV_{S^{m-n}}$.
Note that in our case the shape operator A_ξ of ∂M in the ambient Euclidean space coincides with the shape operator of ∂M in S^m and that $A_{\nu_{out}}$ is nothing but the negative identity, namely, the shape operator of $S^m \subset \mathbb{E}^{m+1}$. The last integral vanishes for odd j, and so we obtain the sum over all even $j = 2i$. The proof is completed by the equation

$$\int_0^{\pi/2} \sin^{m-n+j}\varphi \cos^{n-1-j}\varphi d\varphi = \frac{c_m}{c_{m-n+j} c_{n-1-j}}.$$

Corollary (Special cases)

1. *For a compact surface* $(M^2, \partial M^2) \subset (B^3, S^2)$ *of this type we have*

$$4\pi\chi(M) - 2\int_M K dV_M = 2 \cdot \text{length}(\partial M).$$

2. *For a compact hypersurface* $(M^4, \partial M^4) \subset (B^5, S^4)$ *of this type we have*

$$\frac{8}{3}\pi^2\chi(M) - 2\int_M K_4 \; dV_M = \frac{1}{3}\int_{\partial M} (S-2) \; dV_{\partial M},$$

where S *denotes the scalar curvature of* ∂M^4*.*

Proof: From the formula above we obtain

$$c_2\chi(M) - \text{TC}(M \setminus \partial M) = \frac{c_2}{c_0 c_1}\int_{\partial M} 2K_0 \; dV_{\partial M} = 2\int_{\partial M} dV_{\partial M}$$

in the case (i). For (ii) we have

$$\frac{8}{3}\pi^2\chi(M) - \int_{\perp^1(M)} K_4 \; dV_{can} = \frac{8}{3}\pi^2 \int_{\partial M} (\frac{2}{4\pi^2} + \frac{2K_2}{8\pi^2}) \; dV_{\partial M}$$

$$= \frac{1}{3}\int_{\partial M} (4 + 2K_2) \; dV_{\partial M} = \frac{1}{3}\int_{\partial M} (S-2) \; dV_{\partial M},$$

where $S = 6 + 2K_2$ is the scalar curvature of ∂M.

For a compact 3-dimensional hypersurface we obtain

$$c_3\chi(M) - 0 = \frac{c_3}{2c_2}\int_{\partial M} (1 + K_2) = \frac{c_3}{2c_2}\int_{\partial M} K = \frac{c_3}{2}\chi(\partial M).$$

Definition (Cone-like end) An end E of a complete submanifold $M^n \subset \mathbb{E}^{m+1}$ with associated component M_∞^E (which is assumed to be either a smooth submanifold or a point) in the set of limit directions is said to be (*asymptotically*) *cone-like* if the following conditions are satisfied:

1. There is a point q such that for sufficiently large R the intersection $E \cap S^m(R;q)$ is an $(n-1)$-dimensional submanifold of the sphere of radius R around q, and
$$\lim_{R\to\infty} \frac{1}{R}(E \cap S^m(R;q)) = M_\infty^E$$
(in the C^2-topology if it is a manifold). This property is actually independent of the choice of q, so that we may assume that q is the origin 0.

2. For every ϵ there is a number R_0 such that for each $R > R_0$ the angle between outer unit normal of the submanifold $E \cap B^{m+1}(R;0)$ at any point $p \in E$, $||p|| = R$, and the position vector p is at most ϵ.

Theorem
For a complete submanifold $M^n \subset \mathbb{E}^{m+1}$ with finitely many cone-like ends the Gauss-Bonnet defect is given by the same formula for $M_\infty \subset S^m$ as above
$$c_m\chi(M) - \int_{\perp^1} K dV_{can} = \sum_{0\le 2i\le n-1} \frac{c_m}{c_{m-n+2i}c_{n-1-2i}} \mathbf{K}_{2i}(M_\infty).$$

Corollary

1. If in addition all curvatures K_{2i} of M_∞ are nonnegative, then the 'Cohn-Vossen inequality' $c_m\chi(M) - \int_{\perp^1} K dV_{can} \geq 0$ holds.

2. If in addition for each end M_∞^E is totally geodesic in S^m, then we have
$$\chi(M) - \frac{1}{c_m}\int_{\perp^1} K dV_{can} = k,$$
where k denotes the number of ends.

3. For a 2-dimensional open surface $M^2 \subset \mathbb{E}^3$ with cone-like ends the Gauss-Bonnet defect equals the total length of $M_\infty \subset S^2$ (counted with multiplicity, i.e., for each end separately):
$$2\pi\chi(M) - \int_M K dA = \mathrm{length}(M_\infty) \geq 0,$$
where K is the Gauss curvature. This implies the Cohn-Vossen inequality.

4. For an open hypersurface $M^4 \subset \mathbb{E}^5$ with cone-like ends the Gauss-Bonnet defect is
$$\frac{4}{3}\pi^2\chi(M) - \int_M K_4 dV_M = \frac{1}{6}\int_{M_\infty}(S-2)dV_{M_\infty},$$
where the integral has to be taken for each end separately.

Notice that the value 2 for the scalar curvature has a special meaning by the following gap theorem: It is known that a compact hypersurface of $S^4(1)$ with constant mean curvature and constant scalar curvature can satisfy $S \leq 2$ only if it is a member of Cartan's isoparametric family of hypersurface with $S = 0$.

Corollary *The Cohn-Vossen inequality does not hold in general for complete open* 4*-dimensional hypersurfaces in Euclidean* 5*-space.*

A Key Example is a 4-manifold with one end which is (asymptotically) a cone over Cartan's hypersurface. Here for M_∞ we have three principal curvatures $\sqrt{3}, 0, -\sqrt{3}$, and hence

$$K_1 = 0,\ K_2 = -3,\ S = 6 + 2K_2 = 0$$

with vanishing scalar curvature and non-vanishing volume. This implies that the Gauss-Bonnet defect is strictly negative.

The variational problem for the total curvature

The variation of the extrinsic higher mean curvature functionals

$$\mathbf{K}_i(M) = \int_M K_i dV_M$$

where K_i denotes the i^{th} elementary symmetric function of the eigenvalues of the shape operator A of a hypersurface. The normalization is chosen such that the characteristic polynomial is $\det(A + \lambda \cdot \mathrm{Id}) = \sum_i K_i \lambda^{n-i}$ if M is n-dimensional. In terms of the principal curvatures κ_i one has $K_i = \sum_{j_1 < \cdots < j_i} \kappa_{j_1} \kappa_{j_2} \cdots \kappa_{j_i}$.

Theorem (Pinl–Trapp, K. Voss, R. Reilly)
For any hypersurface in Euclidean space the gradient of the i^{th} curvature functional $\mathbf{K_i} = \int K_i dV$ is the function

$$-(i+1)K_{i+1}.$$

Theorem (R. Reilly)
For a hypersurface in the unit n-sphere the gradient of the curvature functional $\mathbf{K_i} = \int K_i dV$ is the function

$$-(i+1)K_{i+1} + (n-i)K_{i-1}.$$

Theorem

For even n the gradient of the total outer curvature functional (= the right hand side of the formula for M_∞) of a hypersurface in S^n is the negative Gauss-Kronecker curvature $-K_{n-1}$ of this hypersurface.

Proof If δ denotes the gradient, then we have

$$\delta\mathbf{K}_i = -(i+1)K_{i+1} + (n-i)K_{i-1}$$

by the theorem above. If n is even, this implies

$$\delta\Big(\sum_{0\le 2i\le n-1} \frac{c_n}{c_{2i}c_{n-1-2i}}\mathbf{K}_{2i}\Big)$$

$$= \sum_{0\le 2i\le n-2} \frac{c_n}{c_{2i}c_{n-1-2i}}\big(-(2i+1)K_{2i+1} + (n-2i)K_{2i-1}\big)$$

$$= -\frac{c_n}{c_{n-2}c_1}(n-1)K_{n-1} + \sum_{0\le 2i\le n-4} c_n\Big(\frac{n-2i-2}{c_{2i+2}c_{n-3-2i}} - \frac{2i+1}{c_{2i}c_{n-1-2i}}\Big)K_{2i+1}$$

$$= -K_{n-1}.$$

In the last step we used the equation

$$(j-1)c_j = c_1 c_{j-2},$$

which holds for arbitrary j.

Remark If n is odd, then the same calculation shows that the gradient vanishes identically because the leading term K_n vanishes on the $(n-1)$-dimensional boundary. This is not surprising, since we know that in this case the total curvature is constant, namely, the Euler characteristic.

Corollary

The total curvature $\int_M K_n dV$ *of an even-dimensional open hypersurface* $M \subset \mathbb{E}^{n+1}$ *with cone-like ends (as submanifolds of* S^n*) is stationary (within the class of such hypersurfaces having cone-like ends) if and only if each component of* M_∞ *has vanishing Gauss-Kronecker curvature in the sphere "at infinity" or, equivalently, if it has one vanishing principal curvature at each point.*

Notice that for $n = 2$ the Gauss-Kronecker curvature of M_∞ is nothing but the geodesic curvature of the boundary curve. Thus in the stationary 2-dimensional case we have the same behavior as in Osserman's formula for minimal surfaces: The Gauss-Bonnet defect equals 2π times the number of ends.

Examples: The total curvature is stationary if each end is of one of the following types:

1. a point $p \in S^2$ (follows from the Cohn-Vossen inequality),
2. a totally geodesic great sphere $S^{n-1} \subset S^n$,
3. Cartan's isoparametric hypersurface in S^4.

Moreover, if every end is a point then we have the Gauss-Bonnet equation $(c_n/2)\chi(M) - \int_M K dV = 0$. If every end is a totally geodesic sphere then we have $(c_n/2)\chi(M) - \int_M K dV = (c_n/2) \cdot k$ where k denotes the number of ends.

Hypersurfaces of S^{n+1} with vanishing Gauss-Kronecker curvature

It seems that not too much is known about compact hypersurfaces of the standard sphere with vanishing Gauss-Kronecker curvature.

Theorem

Let M^3 be a compact hypersurface of $S^4(1)$ with vanishing Gauss-Kronecker curvature. Assume that the rank of the shape operator is constant. Then

$$\frac{1}{8\pi^2}\int_M (S-2)dV \in \mathbb{Z}.$$

Proof If M^3 is totally geodesic, then $S = 6$, $volM = 2\pi^2$ and the proof is finished. By a theorem of Ferus the rank of the shape operator cannot be 1, so that we can assume that the rank is 2. Then M^3 is a tube over an immersed surface N and we can apply the formulas $K_2 dV = dV_{can} = dN \wedge ds_{S^1}$ and $dV(p,\xi) = \det(A_\xi)dV_{can}(p,\xi)$. The Gauss equation for M implies $S = 6 + 2K_2$, so that $(S-2)dV = 4dV + 2K_2dV = 4dV + 2dV_{can}$. Under the assumptions we know that K_2 is nowhere zero. Let ϵ be the sign of K_2. Then $\int_M (S-2)dV = 4\int_{\perp^1(N)} dV + 2\int_{\perp^1(N)} dV_{can}$
$= 4\int_{\perp^1(N)} \det A_\xi dV_{can} + 4\pi\epsilon volN$
$= 4\epsilon \int_N \pi(K-1)dN + 4\pi\epsilon volN$
$= 8\pi^2\epsilon\chi(N)$.

Remark Under the assumptions of the theorem above the topology of the 3-dimensional hypersurface is essentially unique: Either it is totally geodesic and thus an equatorial 3-sphere or it must be diffeomorphic to Cartan's isoparametric hypersurface, according to a theorem of R.Miyaoka et al.(1999). However, the geometry is quite flexible in this case. One can slightly perturb the Veronese surface and then consider the tube around it of radius $\pi/2$.

Theorem (Quantization of the total curvature)
Let M^4 be a complete open hypersurface of $\mathbb{E}^5$ with finitely many cone-like ends and with stationary total curvature. Assume that for each end the rank of the shape operator in the sphere "at infinity" is constant. Then the normalized total curvature takes values in the integers:

$$\frac{3}{4\pi^2} \int_M K_4 dV \in \mathbb{Z}.$$

This theorem can be considered as a kind of quantization of the total curvature for hypersurfaces with cone-like ends, under the additional condition that the total curvature is stationary (or, equivalently, that the Gauss-Kronecker curvature at infinity vanishes) and a condition on the rank of the shape operator, which we conjecture to be superfluous.

Conjecture

For any complete open hypersurface $M^n \subset \mathbb{E}^{n+1}$ (n even) with cone-like ends and with stationary total curvature $\int_M K dV$, the normalized total curvature $(2/c_n) \int_M K dV$ is an integer.

We remark that the conjecture holds for $n = 2$. Indeed, in that case each end is a great circle or a point, such that the length of M_∞ is a multiple of 2π. It follows that $(1/2\pi) \int_M K dV$ is an integer.

QUESTIONS: 1. One of the open questions is whether or not every compact hypersurface in the sphere with vanishing Gauss-Kronecker curvature is a $\pi/2$-tube around some other submanifold. If yes, then this would provide a strategy for proving the conjecture on the quantization of the total curvature.

2. Since the Gauss-Bonnet difference term can be expressed by intrinsic curvatures K_{2i} of M_∞, the question arises whether this difference can be described purely intrinsically in the original manifold M. For 4-dimensional complete Riemannian manifolds one would have to introduce a volume and an appropriate version of a scalar curvature of the ideal boundary "at infinity".