Conformally Einstein product spaces

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Notations

Question:
When does a Riemannian product admit a (local or global) conformal mapping onto an Einstein space ?

Notations:
(M^n, g) denotes an n-dimensional pseudo-Riemannian manifold (of arbitrary signature). $F\colon (M,g) \to (N,h)$ is *conformal* iff $F^*h = \varphi^{-2} g$ for a real function φ that is never zero:

$$h_{F(x)}\left(dF_x(X), dF_x(Y)\right) = \varphi^{-2}(x) \cdot g_x(X,Y)$$

$\nabla^2\varphi(X,Y) = g\left(\nabla_X \mathrm{grad}\varphi, Y\right)$ denotes the *Hessian* $(0,2)$-tensor
$\Delta\varphi = \mathrm{div}\;(\mathrm{grad}\;\varphi) \; = \mathrm{tr}(\nabla^2\varphi)$ is the *Laplacian* of φ, Ric denotes the *Ricci tensor*, (M^n, g) is *Einstein* iff $\mathrm{Ric} = \lambda \cdot g$.
$n \geq 3$: *Einstein constant* λ, $n = 2$: $\lambda = K$ (Gaussian curvature)
scalar curvature $\mathrm{tr}(\mathrm{Ric}) = S = n\lambda$,
normalized scalar curvature $k = \frac{S}{n(n-1)} = \frac{\lambda}{n-1}$ $\big($= 1 for $S^n(1)\big)$.

Basic formulas

Lemma 0.1

The following formula holds for any conformal change $g \mapsto \overline{g} = \varphi^{-2} g$ *of a metric on an* n*-dimensional manifold:*

(1) $\overline{\mathrm{Ric}} - \mathrm{Ric} = \varphi^{-2}\Big((n-2)\cdot\varphi\cdot\nabla^2\varphi + \Big[\varphi\cdot\Delta\varphi - (n-1)\cdot\|\mathrm{grad}\varphi\|^2\Big]\cdot g\Big).$

Consequently, the metric $\overline{g}$ *is Einstein if and only if the equation*

(2) $$\varphi \cdot \mathrm{Ric} + (n-2) \cdot \nabla^2\varphi = \theta \cdot g$$

holds for some function θ *or, equivalently,*

$$\varphi \cdot (\mathrm{Ric})^\circ + (\mathrm{n}-2) \cdot (\nabla^2\varphi)^\circ = 0$$

where $(\ )^\circ$ *denotes the trace-free part.*

Immediate consequences

Corollary 0.2

A metric g on a manifold M is (locally or globally) conformally Einstein if and only if there is a (local or global) positive solution φ of the equation

$$\varphi \cdot (\mathrm{Ric})^\circ + (\mathrm{n}-2) \cdot (\nabla^2 \varphi)^\circ = 0.$$

Corollary 0.3 (H.W.Brinkmann 1925)

If g is an Einstein metric then $\overline{g}$ is also an Einstein metric if and only if

$$(\nabla^2 \varphi)^\circ = 0.$$

Another approach

Theorem 0.4 (I.R.Miklashevskii 1987)

A metric g on an n-manifold M is conformally Einstein if and only if a certain vector bundle over M of rank $n+2$ admits a horizontal section. The connection is determined by the conformal structure.

Further results by H.Baum, R.Gover, F.Leitner and others *(conformal holonomy)*

Theorem 0.5 (A.Derdzinski 1983)

If (M, g) is a 4-dimensional Kähler manifold such that $\overline{g} = \varphi^{-2} g$ is Einstein, then φ coincides - up to a constant - with the scalar curvature of g.

Folklore result

On an Einstein space with $n \geq 3$ the equation $(\nabla^2\varphi)^\circ = 0$ can be explicitly solved in the sense that g and φ can be determined. Roughly the results are the following:

As long as $g(\mathrm{grad}\varphi, \mathrm{grad}\varphi) \neq 0$, the metric is a warped product

$$g = \epsilon dt^2 + (\varphi'(t))^2 g_*$$

with an $(n-1)$-dimensional Einstein space (M_*, g_*), $\epsilon = \pm 1$, and where φ depends only on t and satifies the following equations:

(3) $$\varphi''' + \epsilon k \varphi' = 0, \quad (\varphi'')^2 + \epsilon k (\varphi')^2 = \epsilon k_*$$

If $g(\mathrm{grad}\varphi, \mathrm{grad}\varphi) = 0$ on an open subset then we have $\nabla^2\varphi = 0$ and $\mathrm{Ric} = \overline{\mathrm{Ric}} = 0$. This leads to a so-called *Brinkmann space*. Typical 4-dimensional examples are pp-waves

$$g = -2dudv - 2H(u,x,y)du^2 + dx^2 + dy^2$$

with a parallel gradient $\frac{\partial}{\partial v}$ of $\varphi = u$ and with $H_{xx} + H_{yy} = 0$.

A classical example: The generalized Mercator projection

Example (conformal cylinder)
Let M_* be an $(n-1)$-dimensional Einstein space with Einstein constant $\lambda_* = n-2$. Then the cylinder $M = \mathbb{R} \times M_*$ with the product metric $g = dt^2 + g_*$ is conformally Einstein:
The metric $\overline{g} = \cosh^{-2} t \cdot g$ is Einstein with $\overline{\lambda} = n-1$.
If $M_* = S^{n-1}(1)$ then (M, g) is a cylinder representing the (classical n-dimensional) Mercator projection from the n-sphere without north and south pole. We verify
$\varphi \cdot (\mathrm{Ric})^\circ + (\mathrm{n}-2) \cdot (\nabla^2 \varphi)^\circ = 0$ for $\varphi = \cosh t$ by the block matrix structure

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 \\ 0 & (n-2) g_* \end{pmatrix}, \ \nabla^2 \varphi = \begin{pmatrix} \varphi'' & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix},$$

$$(\mathrm{Ric})^\circ = \begin{pmatrix} -\frac{(n-1)(n-2)}{n} & 0 \\ 0 & \frac{n-2}{n} g_* \end{pmatrix}, \ (\nabla^2 \varphi)^\circ = \begin{pmatrix} \frac{(n-1)\varphi}{n} & 0 \\ 0 & -\frac{\varphi}{n} g_* \end{pmatrix}.$$

A classical example: The generalized Mercator projection

Remarks:

(1) In the special case of a compact Einstein space $M_* \ncong S^{n-1}$ this generalized Mercator projection is the result of a theorem by Moroianu and Ornea 2008. Here the assumption is a globally conformally Einstein product $\mathbb{R} \times M_*$ of strictly positive scalar curvature.

(2) The transition from the conformal cylinder $\overline{g} = \cosh^{-2} t\big(dt^2 + g_*\big)$ to the more familiar version $\overline{g} = ds^2 + \sin^2 s\ g_*$ in polar coordinates is achieved by the parameter transformation $t \mapsto s(t)$ with $ds/dt = \cosh^{-1} t$ leading to the *Gudermann function*

$$s(t) = \int_{-\infty}^{t} \cosh^{-1} \tau\ d\tau = 2 \arctan e^t.$$

The equation $\sin s = \cosh^{-1} t$ follows.

Einstein warped products revisited

Proposition (Conformally Einstein products of type $\mathbb{R} \times M_*$)

If f is a non-constant function only on the real parameter t then the metric

$$\overline{g} = f^{-2}(\epsilon dt^2 + g_*)$$

is Einstein if and only if (M_*, g_*) is an n-dimensional Einstein space and f satisfies the ODE

$$k_* f^2 - \epsilon(f')^2 = \overline{k}.$$

Compare the Einstein warped products

$$g = \epsilon ds^2 + (\varphi'(s))^2 g_*$$

above with the similar ODE

$$k(\varphi')^2 + \epsilon(\varphi'')^2 = k_*.$$

Special cases

Riemannian: $\epsilon = 1$.
$f(t) = \cosh t,\ \varphi'(s) = \sin s,\ \ k = \overline{k} = k_* = 1$
$\cosh^{-1} t = \sin s,\ t \in (-\infty, \infty),\ s \in (0, \pi)$ (Mercator)
For fixed s the M_*-slices behave like small spheres in the unit sphere parallel to the equatorial sphere (Fermi coordinates).

$f(t) = \sin t,\ \varphi'(s) = \cosh s,\ \ k = \overline{k} = k_* = -1$
$\sin^{-1} t = \cosh s,\ t \in (0, \pi),\ s \in (-\infty, \infty)$ (hyperbolic Mercator)
For fixed s the M_*-slices behave like parallel hypersurfaces to a hyperbolic hyperplane (Fermi coordinates).

$f(t) = t,\ \varphi'(s) = e^{-s},\ \ k = \overline{k} = -1,\ k_* = 0$ (Poincaré half space)
If g_* is Ricci flat but not flat we obtain a *generalized Poincaré half space* of type $(0, \infty) \times M_*$ with the metric
$t^{-2}(dt^2 + g_*) = ds^2 + e^{-2s} g_*$ and $s = \log t$.
For fixed s the M_*-slices behave like horospheres.
Special case if M_* is a Ricci flat K3 surface.

Main Theorem on conformally Einstein products

Let $(M^n, \widetilde{g})$ and $(M_*^{n_*}, g_*)$ be pseudo-riemannian manifolds with $n + n_* \geq 3$. If $f(y, x)$ is a non-constant function depending on $y \in M$ and $x \in M_*$ and if the metric $\overline{g} = f^{-2}(\widetilde{g} + g_*)$ on $M \times M_*$ is Einstein then one of the following cases occurs:

(1) $\overline{g}$ is a warped product, i.e., f depends only on one of the factors M or M_*. Moreover the fibre is an Einstein space.

(2) $f(y, x) = a(y) + b(x)$ with non-constant a and non-constant b, and both (M, g) and (M_*, g_*) are Einstein spaces, and a satisfies the equation $(\widetilde{\nabla}^2 a)^\circ = 0$ and, simultaneously, b satisfies the equation $(\nabla_*^2 b)^\circ = 0$.

Main theorem (continued)

If $n \geq 3$ or $n_* \geq 3$ then we have necessarily

$$\widetilde{\nabla}^2 a = (-\widetilde{k} a + c) g$$

$$\nabla_*^2 b = (-k_* b + c) g_*$$

with a constant c and with normalized scalar curvatures $\widetilde{k} = -k_*$. Such Einstein spaces can be (locally and globally) classified.

If $n = n_* = 2$ then either the Gaussian curvatures are constant and satisfy $\widetilde{K} = -K_*$, or both are non-constant and satisfy the equations $\widetilde{\nabla}^2 K = \frac{\Delta \widetilde{K}}{2} \widetilde{g}$ and $\nabla_*^2 K_* = \frac{\Delta K_*}{2} g_*$. Such metrics are also called *extremal*.

REMARK: A complete classification of Einstein warped products in (1) is not known. However, Einstein warped products with a 1-dimensional base are easy to classify by the folklore result above. For the case of a 2-dimensional base see the book by A.Besse.

Main theorem (converse direction)

Conversely, any Einstein warped product in Case (1) is conformally equivalent with a product space, and any two Einstein metrics $\widetilde{g}, g_*$ with constant $\widetilde{k} = -k_*$ and with solutions $a(y), b(x)$ of the equations $\widetilde{\nabla}^2 a = (-\widetilde{k} a + c) g$ and $\nabla_*^2 b = (-k_* b + c) g_*$ lead to an Einstein metric

$$\overline{g} = (a+b)^{-2}(\widetilde{g} + g_*)$$

on $M \times M_*$ in Case (2).

Any compact factor M or M_* in Case (2) is necessarily a standard sphere with a positive or negative definite metric. However, $(M \times M_*, \overline{g})$ cannot be compact since then $a+b$ has a zero.

If $n = n_* = 2$ then there are also examples $M \times M_*$ with two surfaces M, M_* that are not of constant curvature. However, by a theorem of Calabi (quoted in Derdzinski's handbook article) there are no compact examples of this kind.

Method of proof

We use Equation (1) in the first Lemma:

$$f^2\big(\overline{\mathrm{Ric}} - \mathrm{Ric}\big) = (N-1)f \cdot \nabla^2 f + \Big[f \cdot \Delta f - N \cdot \|\mathrm{grad} f\|^2\Big] \cdot g$$

If $\overline{g}$ is Einstein with $f^2\overline{\mathrm{Ric}} = f^2\overline{\lambda}\overline{g} = \overline{\lambda} g$
then $\nabla^2 f$ admits a block matrix decomposition.
This implies $\frac{\partial^2 f}{dy_j dx_i} = 0$ for any coordinate y_j on the first factor and x_i on the second.
Therefore f splits as

$$f(y,x) = a(y) + b(x)$$

with functions a on M and b of M_*, and we have

$$\nabla^2 f = \begin{pmatrix} \widetilde{\nabla}^2 a & 0 \\ 0 & \nabla_*^2 b \end{pmatrix}$$

The equation to be solved

$$\overline{\lambda}\begin{pmatrix} \widetilde{g} & 0 \\ 0 & g_* \end{pmatrix} - f^2\begin{pmatrix} \widetilde{\mathrm{Ric}} & 0 \\ 0 & \mathrm{Ric}_* \end{pmatrix} \qquad (**)$$

$$= (N-1)f\begin{pmatrix} \widetilde{\nabla}^2 a & 0 \\ 0 & \nabla_*^2 b \end{pmatrix} + \left[f\cdot\Delta f - N\cdot\|\mathrm{grad} f\|^2\right]\cdot\begin{pmatrix} \widetilde{g} & 0 \\ 0 & g_* \end{pmatrix}$$

From this equation it is obvious that a constant function a implies that $\widetilde{g}$ is Einstein and a constant function b implies that g_* is Einstein.
In each of these cases $\overline{g}$ is a warped product metric with an Einstein fibre. This is case (1) in the Main theorem.

The mixed case

What happens if a and b both are non-constant ?
In this case the system of equations (**) is coupled. We differentiate in a direction X tangent to M_* and Y tangent to M such that $\nabla_Y a \neq 0,\ \nabla_X b \neq 0$:

$$0 = 2f\nabla_X b \cdot \widetilde{\mathrm{Ric}} + (N-1)\nabla_X b \cdot \widetilde{\nabla}^2 a$$
$$+\Big[\nabla_X b \cdot \Delta f + f\nabla_X \Delta_* b - N \cdot \nabla_X \|\mathrm{grad} f\|^2\Big] \cdot \widetilde{g},$$
$$0 = 2f\nabla_Y a \cdot \mathrm{Ric}_* + (N-1)\nabla_Y a \cdot \nabla_*^2 b$$
$$+\Big[\nabla_Y a \cdot \Delta f + f\nabla_Y \widetilde{\Delta} a - N \cdot \nabla_Y \|\mathrm{grad} f\|^2\Big] \cdot g_*$$

Dividing through by $\nabla_X b$ or $\nabla_Y a$ and differentiating once more leads to

$$0 = 2\nabla_X f \cdot \widetilde{\mathrm{Ric}} + \nabla_X\Big[\Delta f + (\nabla_X b)^{-1}\Big(f\nabla_X \Delta_* b - N \cdot \nabla_X \|\mathrm{grad} f\|^2\Big)\Big] \cdot \widetilde{g},$$

$$0 = 2\nabla_Y f \cdot \mathrm{Ric}_* + \nabla_Y\Big[\Delta f + (\nabla_Y a)^{-1}\Big(f\nabla_Y \widetilde{\Delta} a - N \cdot \nabla_Y \|\mathrm{grad} f\|^2\Big)\Big] \cdot g_*.$$

General conclusion

A direct consequence:

1. $\widetilde{g}$ and g_* are Einstein metrics.

2. In addition $\widetilde{\nabla}^2 a$ is a scalar multiple of $\widetilde{g}$ and that $\nabla^2_* b$ is a scalar multiple of g_*. This is precisely the equation in the Folklore result. In combination we have Case (2) in the Main theorem.

3. Thus, if $n \geq 3$ and $n_* \geq 3$ we obtain warped product Einstein metrics $\widetilde{g}, g_*$ with warping functions a, b satisfying $\widetilde{\nabla}^2 a = (-\widetilde{k}a + c)g$ and simultaneously $\nabla^2_* b = (-k_* b + c_*)g_*$ with constants c, c_*.

4. Furthermore Equation (**) implies $\widetilde{k} = -k_*$ and $c = c_*$.

Special conclusion if $n = n_* = 2$

If $n = n_* = 2$ then we have warped product metrics

$$\widetilde{g} = \epsilon dt^2 \pm a'(t)^2 dx^2,\ g_* = \epsilon_* ds^2 \pm b'(s)^2 dy^2$$

$$\text{with} \quad \nabla^2 a = \epsilon a'' \widetilde{g} \quad \text{and} \quad \nabla_*^2 b = \epsilon_* b'' g_*.$$

The Gaussian curvatures are $K = -\epsilon a'''/a', K_* = -\epsilon_* b'''/b'$.

Then Equation $(**)$ reads as follows:

$$\overline{\lambda} = (a+b)^2 K + 2(a+b)\epsilon a'' + (a+b)(2\epsilon a'' + 2\epsilon_* b'') - 3(\epsilon a'^2 + \epsilon_* b'^2)$$

$$\overline{\lambda} = (a+b)^2 K_* + 2(a+b)\epsilon_* b'' + (a+b)(2\epsilon a'' + 2\epsilon_* b'') - 3(\epsilon a'^2 + \epsilon_* b'^2).$$

In particular we have

$$\begin{aligned} 0 &= (a+b)(K - K_*) + 2(\epsilon a'' - \epsilon_* b'') \\ &= aK + 2\epsilon a'' - (bK_* + 2\epsilon_* b'') + bK - aK_* \end{aligned}$$

This leads to

$$\frac{K'}{a'} = \frac{K'_*}{b'} = c \text{ (constant)}$$
$$\text{and} \quad K = ca, K_* = cb.$$

Inserting this into the last equation leads to the ODEs

$$ca^2 + 2\epsilon a'' = d$$

and

$$cb^2 + 2\epsilon_* b'' = d$$

with a constant d.

These equations have non-constant solutions.

A non-periodic solution with $d = 0, c = 1, \epsilon = \epsilon_* = 1$:

$$a(t) = K(t) = -12t^{-2},\ b(s) = K_*(s) = -12s^{-2}.$$

This leads to the Ricci flat (and non-flat) metric

$$\overline{g} = \frac{t^4 s^4}{(t^2 + s^2)^2}\Big(dt^2 + \frac{576}{t^6}dx^2 + ds^2 + \frac{576}{s^6}dy^2\Big).$$

Are there compact solutions ?

Candidates of compact Einstein manifolds in Case (2) that are globally conformal to products:

(1) In any dimension: Products $S^n(1) \times (-S^{n_*}(1))$ with functions

$$a(t) = \cos t,\ b(s) = \cos s$$

that are globally defined.
Unfortunately the conformal image is not compact since the function $a + b$ has a zero (necessarily).

(2) For $n = n_* = 2$ periodic solutions of the equations above leading to products $S^2 \times S^2$, each factor being a surface of revolution. Can that work ?

History

(A) There are periodic solutions of the ODE $ca^2 + 2\epsilon a'' = d$.

(B) Tashiro (JDG 1981): *There are compact surfaces satisfying the requirements above.*

(C) Tashiro (JDG 1983): *Corrigendum: These compact surfaces have one singularity each. They look like a drop.*

The error was pointed out by Derdzinski.

Correct statement: *There is a compact product* $(S^2, g) \times (S^2, g)$ *of two drop surfaces that is conformally Einstein.*

(D) **Theorem** (Calabi 1982, Derdzinski): *The only compact conformally Einstein products* $M^2 \times M^2_*$ *are Einstein products themselves with a constant conformal factor.*

There are no compact examples in Case (2)

(E) SIMPLE PROOF: On a compact surface the metric must be positive or negative definite. Assume that there is a periodic solution $a(t)$ of $ca^2 + 2a'' = d$ with a minimum at $t = 0$ such that the metric $dt^2 + a'^2 dx^2$ is smooth in a neighborhood. Here dx^2 describes the metric of a circle, like $x = e^{i\theta}$.
Necessarily we have $a''(0) = A > 0$ *(like polar coordinates)*.

If a hat a maximum at $t = t_0$ then we have $a''(t_0) = -A$. Otherwise the metric has a singularity there.

This implies:

$$ca^2 = d - 2A \text{ at the minimum}$$

$$ca^2 = d + 2A \text{ at the maximum.}$$

Integrating the ODE leads to $a'^2 = da - \frac{c}{3}a^3 + e$ with a constant e. At the minimum and maximum we get

$$0 = \sqrt{\frac{d-2A}{c}}\Big(d - \frac{d-2A}{3}\Big) + e$$

$$0 = \sqrt{\frac{d+2A}{c}}\Big(d - \frac{d+2A}{3}\Big) + e$$

In combination:

$$\sqrt{d-2A}\big(d+A\big) = \sqrt{d+2A}\big(d-A\big).$$

There is no real solution d unless $A = 0$. Contradiction. $\square$

Corollary 0.6 (Moroianu & Ornea 2008)

If M_* is compact Riemannian (but not a round sphere) and if $\overline{g} = f^{-2}(dt^2 + g_*)$ is Einstein with $\overline{S} > 0$ and with a non-constant function $f(t, x)$ that is globally defined and never zero on $\mathbb{R} \times M_*$, then the following holds:
(M_*, g_*) is an Einstein space with $S_* > 0$, and the function $f(t, x)$ is the $\cosh$-function on the real t-axis, up to constants. In particular f does not depend on $x \in M_*$. This is precisely the generalized Mercator projection above.

Proof. If $f(t,x)$ depends on t and on x then (M_*, g_*) is a round sphere by our main theorem in combination with the well known theorem that the only compact pseudo-Riemannian Einstein space admitting a non-constant solution of the equation $(\nabla_*^2 b)^\circ = 0$ is the round sphere.

If f depends only on t then g_* is Einstein. Furthermore the case $\overline{S} > 0$ is only possible for a function of $\cosh$-type (up to additive or multiplicative constants). This is a consequence of the ODE $ff'' - (f')^2 = \overline{k} > 0$. Moreover from $f'' = k_* f$ we get $k_* > 0$.

If f depends only on x then we obtain a contradiction: S_* is negative, and by Corvino's result $\Delta_* f$ is positive at a positive maximum of f, and it is negative at a negative minimum of f. Here we use that f never vanishes on M_*. This is a contradiction on a compact manifold M_* because one of these cases must occur. $\square$

Non-compact examples

I. Here is an example of a complete Riemannian manifold

$$M = \mathbb{R} \times \mathbb{R} \times \widehat{M} = \mathbb{R} \times M_*$$

admitting a global solution $f = a(t) + b(x)$ where both a, b are non-constant and never zero. Unfortunately f has zeros. Let $(\widehat{M}, \widehat{g})$ be a complete Ricci flat manifold of dimension $n-1$ and

$$a(t) = \cos t - 2, \quad b(s) = e^s + 2.$$

Then on M the function $f(t, s, x) = a(t) + b(s)$ satisfies all conditions above: The metric $g_* = ds^2 + e^{2s}\widehat{g}$ on $M_* = \mathbb{R} \times \widehat{M}$ is a complete Einstein space with $k_* = -1$, and with $c = -2$ we have

$$a'' = k_* a + c, \quad b'' = -k_* b + c, \quad \nabla_*^2 b = (-k_* b + c) g_*.$$

By the results above $\overline{g} = f^{-2}\big(dt^2 + ds^2 + e^{2s}\widetilde{g}\big)$ on $M = \mathbb{R} \times M_*$ is Einstein with $\overline{k} = -1$ whenever $f(t, s) = \cos t + e^s \neq 0$.
If $\widetilde{g}$ is not flat then $\overline{g}$ is not of constant curvature.

Non-compact examples

II. A similar example starts with an $(n-1)$-dimensional Einstein space $(\widetilde{M}, \widetilde{g})$ with $\widetilde{k} = -1$. Then $M = \mathbb{R} \times \widetilde{M}$ with $g = dt^2 + \cosh^2(t)\widetilde{g}$ is also Einstein with $k = -1$ and satisfies $\nabla^2 \cosh(t) = \cosh(t) g$. If $(\widehat{M}, \widehat{g})$ is $(m-1)$-dimensional Einstein with $\widehat{k} = 1$ then $M_* = \mathbb{R} \times \widehat{M}$ with $g_* = ds^2 + \cos^2(s)\widehat{g}$ is also Einstein with $k_* = 1$ and satisfies $\nabla_*^2 \cos(s) = -\cos(s) g_*$. Let

$$a(t) = \cosh t, \quad b(s) = \cos s.$$

Then on $M \times M_*$ the function $f(t, x, s, y) = a(t) + b(s)$ satisfies all conditions above with $c = 0$. By the results above the metric

$$\overline{g} = f^{-2}\big(dt^2 + \cosh^2(t)\widetilde{g} + ds^2 + \cos^2(s)\widehat{g}\big)$$

on $M \times M*$ is Ricci flat whenever $f(t, s) \neq 0$. This is the case at least if $\cos s \neq -1$. If $\widetilde{g}$ is not hyperbolic then g is not of constant curvature and not conformally flat. Consequently, $\overline{g}$ is not flat.

A global pseudo-Riemannian example

III. Let $(\widetilde{M}, \widetilde{g})$ be complete, Ricci flat and Riemannian. Then $M = \mathbb{R} \times \widetilde{M}$ with $g = dt^2 + \exp(2t)\widetilde{g}$ is complete Einstein with $k = -1$ and satisfies $\nabla^2 \exp = \exp g$. Then $-g$ is complete Einstein with $k = 1$. Let $a(t) = \exp t, \quad b(s) = \exp s$. Then on $(M, g) \times (M, -g)$ the function $f(t, x, s, y) = \exp t + \exp s$ satisfies all conditions above with $c = 0$ and $f \neq 0$ everywhere. By the results above the metric

$$\overline{g} = (\exp t + \exp s)^{-2} \big(dt^2 + \exp(2t)\widetilde{g} - ds^2 - \exp(2s)\widetilde{g}\big)$$

on $M \times M$ is Ricci flat everywhere. If $\widetilde{g}$ is not flat then g is not of constant curvature and not conformally flat.
Consequently, $\overline{g}$ is not flat.

In other words: $(M \times M, \overline{g})$ *is Einstein (Ricci flat) and globally conformal with a complete product metric.*

The end