Discrete models of isoparametric families in spheres

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By a theorem of E.Cartan (1939) all isoparametric families of hypersurfaces in the sphere with at most three principal curvatures are given by the following list:

1. tubes around a point in S^n
2. tubes around a great sphere $S^k \subset S^n$ where $1 \leq k \leq n-2$
3. tubes around any of the Veronese-type standard embeddings of the projective planes $\mathbb{R}P^2 \to S^4$, $\mathbb{C}P^2 \to S^7$, $\mathbb{H}P^2 \to S^{13}$, or $\mathbb{O}P^2 \to S^{25}$.

In these three cases we have 1, 2 or 3 constant principal curvatures, respectively.

Topologically, the hypersurfaces in Case 3 are total spaces of S^k-bundles over the projective plane over $\mathbb{F}(k)$ where the real dimension of $\mathbb{F}$ is k.

In particular, the dimension of the total space is $3k$ in each of the cases.

In addition isoparametric hypersurfaces have the geometric property of tightness and tautness.

Definition An embedding $M \to \mathbb{E}^N$ of a compact manifold is called **tight**, if for any open half space $E_+ \subset \mathbb{E}^N$ the induced homomorphism

$$H_*(M \cap E_+) \longrightarrow H_*(M)$$

is injective where H_* denotes an appropriate homology theory with coefficients in a certain field. The notion of k-**tightness** refers to the injectivity in the low dimensions $H_i(M \cap E_+) \to H_i(M)$, $i = 0, \ldots, k$. The notion of tightness is projectively invariant. Tightness of a subset means that it is embedded *as convexly as possible.*

In the smooth case (and, with certain modifications, also in the polyhedral case) this is equivalent to the condition that almost all height functions on M are perfect functions, i.e., have the minimum number of critical points. The similar notion of **tautness** refers to the condition that almost all distance functions are perfect functions.

It is well known that the ε-tube around any taut submanifold or around any embedded tight submanifold is again taut or tight, respectively. The reason is that the cohomology ring of the total space of the unit normal bundle is isomorphic to the tensor product of the cohomology rings of the base and the fibre.

Theorem 1 *In each of the cases of an isoparametric hypersurface of S^n mentioned above (except possibly for the octonion case) there is a simplicial n-sphere in Euclidean space such that the following conditions are satisfied:*

1. *It contains two disjoint simplicial subcomplexes triangulating the two focal sets of the isoparametric family as a kind of "top" and "bottom" of the simplicial n-sphere (for the case of $\mathbb{H}P^2$ see below),*
2. *each member of the isoparametric family corresponds to a slice through this n-sphere between top and bottom,*
3. *each member of the family (including the focal sets) is a tight polyhedral sub-*

manifold in the boundary complex of a certain convex $(n+1)$-polytope. So in particular the real Cartan hypersurface is tight in the boundary complex of a 5-polytope, the complex Cartan hypersurface is tight in the boundary complex of an 8-polytope.

In the case of the quaternionic Cartan hypersurface these polyhedral models exist, but a complete proof of their topological properties is not available. In the case of the octonion Cartan hypersurface an appropriate triangulation of the focal set is still missing. If there exists a tight 27-vertex triangulation of $\mathbb{O}P^2$ then this case is included as well.

The construction will make use of the following three ingredients:

1. Higher-dimensional octahedra (duals of the cube),

2. Tight triangulations of the projective planes over $\mathbb{R}$ and $\mathbb{C}$,

3. Sarkaria's deleted join of a simplicial complex with itself, and the Bier sphere.

The $(n+1)$-dimensional cross polytope β_{n+1} (also called $(n+1)$-dimensional octahedron) is defined as the convex hull of the points

$$(0, \ldots, 0, \underbrace{\pm 1}_{i}, 0, \ldots, 0), \quad i = 0, 1, \ldots, n$$

in $(n+1)$-space.

CASE 1: In the first case we pick two antipodal vertices, say, $(\pm 1, 0, \ldots, 0)$. Then the polyhedral model of the isoparametric family with one principal curvature is just given by all the slices through $\partial\beta_{n+1}$ by hyperplanes orthogonal to $(1, 0, \ldots, 0)$. Each member of the family is a convex polyhedron in n-space and is therefore tight.

CASE 2: In the case of two principal curvatures we start with a β_{k+1} as the subcomplex of β_{n+1} given by all vertices above where $0 \leq i \leq k$ and a complementary β_{n-k} given by all vertices with $k+1 \leq i \leq n$. As a matter of fact, the boundary $\partial\beta_{n+1}$ is just the join complex $\partial\beta_{k+1} * \partial\beta_{n-k}$ where, as usual, the **join** $\triangle^k * \triangle^{n-k-1}$ of two simplices $\triangle^k = \langle v_0, v_1, \ldots, v_k \rangle$ and $\triangle^{n-k-1} = \langle v_{k+1}, \ldots, v_n \rangle$ is defined as the simplex $\triangle^n = \langle v_0, v_1, \ldots, v_k, v_{k+1}, \ldots, v_n \rangle$. Since each of the vertices of β_{n+1} is either in β_{k+1} or in the complementary β_{n-k}, we can define a simplexwise linear function f on the boundary complex of β_{n+1} which is 0 on $\partial\beta_{k+1}$ and 1 on $\partial\beta_{n-k}$. More precisely, f is assumed to be affine linear in the barycentric coordinates on each simplex, i.e., $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$ where $\sum_i \lambda_i = 1$. Now the polyhedral analogue of the isoparametric family is given by the levels of the function f. Clearly each level set $f^{-1}(t)$ defines a polyhedral manifold, for $0 < t < 1$ the level set is a polyhedral decomposition of $S^k \times S^{n-k-1}$ als a subset of $\partial\beta_{n+1} \cong S^n$.

Each of these level sets $f^{-1}(t)$ is tightly embedded into $(n+1)$-space.

Case 3: In the third case of three principal curvature we have to consider the tubes around two antipodal real or complex Veronese-type embeddings $\mathbb{R}P^2 \to S^4$ or $\mathbb{C}P^2 \to S^7$, respectively. The quaternionic case will be discussed later.

First of all, there are tight polyhedral analogues of these Veronese-type embeddings themselves. These are the canonical embeddings of the unique 6-vertex triangulation $\mathbb{R}P^2_6$ of $\mathbb{R}P^2$ into the 5-simplex and of the unique 9-vertex triangulation $\mathbb{C}P^2_9$ of $\mathbb{C}P^2$ into the 8-simplex.

For our purpose we have to find an appropriate triangulation of the 4-sphere or 7-sphere, respectively, which contains two antipodal copies of them, linking one another as required by the Cartan decomposition.

Definition

(1) The **deleted join** $K *_{\Delta} K$ of a simplicial complex K with itself is a part of the ordinary join of two disjoint copies K_1 and K_2 of K where we take the join of only those two simplices in K_1 and K_2, respectively, which are disjoint in K. So in particular, each vertex of K leads to a missing edge (a *diagonal*) in $K *_{\Delta} K$.

(2) Similarly we have the **deleted join** $K *_{\Delta} K^*$ of an n-vertex simplicial complex K with its combinatorial Alexander dual K^* where the **combinatorial Alexander dual** is defined as the set of the complements of the non-faces of K. Here we think of a face as a subset of $\{1, 2, \ldots, n\}$ and its complement as the set-theoretic complement. Accordingly, a **non-face** is a subset that does

not correspond to a face in the complex. The vertex set of the deleted join will be denoted by $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ with diagonals $1\overline{1}, 2\overline{2}, \ldots, n\overline{n}$.

The notion of the deleted join is due to K.Sarkaria.

Theorem 2 *For any given simplicial complex K with n vertices the deleted join of K with its combinatorial Alexander dual K^* is a triangulated $(n-2)$-sphere with at most $2n$ vertices. It is called the* Bier sphere *$Bier_n(K)$ because it was discovered by Thomas Bier in* 1992. *After subdivision, the Bier sphere coincides with the first barycentric subdivision of an $(n-1)$-simplex*
(M. de Longueville 2004).

Theorem 3

1. *Any n-vertex triangulation of a combinatorial $2k$-manifold M satisfies $n \geq 3k+3$ unless M is a sphere. In case of equality $n = 3k+3$ we have necessarily $k = 0, 1, 2, 4, 8$, and for $k \geq 1$ M has the same cohomology ring as the projective plane over*

   $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. Moreover, for $k = 1, 2$ the triangulation is combinatorially isomorphic with $\mathbb{R}P^2_6$ or $\mathbb{C}P^2_9$, respectively.

2. *Any combinatorial $2k$-manifold with $n = 3k+3$ vertices (which is not a sphere) satisfies the following*

combinatorial complementarity condition*:*

- Any subset of vertices spans a simplex in the triangulation if and only if the complementary subset does not.

In particular, if K denotes the simplicial complex triangulating the manifold, then we have $K^* = K$, and K is $(k+1)$-neighborly which means that any $(k+1)$-tuple of vertices spans a simplex in K.

3. *In the cases $k = 0, 1, 2, 4$ there exists such a combinatorial manifold with $3, 6, 9, 15$ vertices, respectively. It is unique for $k = 0, 1, 2$ and not unique for $k = 4$. For $k = 8$ the existence is still open.*

Corollary 4 *If K denotes any simplicial complex triangulating a combinatorial $2k$-manifold with $n = 3k + 3$ vertices which is not a sphere, then the deleted join $Bier_n(K) = K *_\Delta K$ is a combinatorial sphere of dimension $n - 2$ with $2n$ vertices. It can be regarded as a subcomplex of the cross polytope β_n.*

In particular, this applies to the triangulations $K = \mathbb{R}P^2_6$ and $K = \mathbb{C}P^2_9$. In these cases the deleted join coincides with the Bier sphere $Bier_6(\mathbb{R}P^2_6)$ or $Bier_9(\mathbb{C}P^2_9)$, respectively. Recall that the Veronese-type standard embedding maps each of the four projectives planes into the $(n - 2)$-sphere where $n = 3k + 3$.

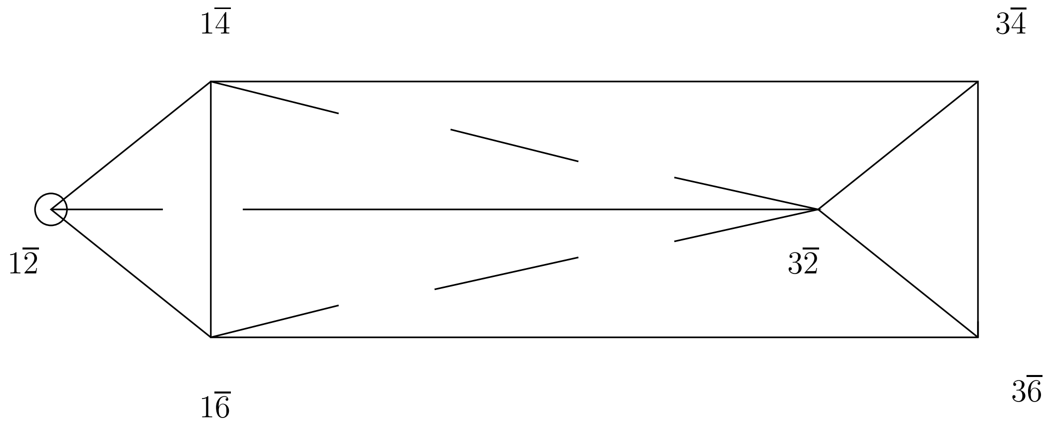

Figure 1: slice through the simplex $\langle 13\bar{2}\bar{4}\bar{6}\rangle$ together with the link of $1\overline{2}$

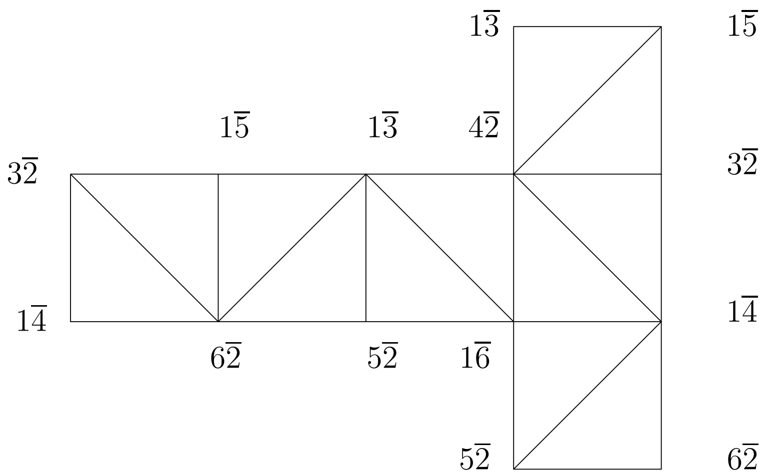

Figure 2: The link of the vertex $1\overline{2}$ in the slice through $\mathbb{R}P_6^2 *_\Delta \mathbb{R}P_6^2$

Lemma 6 *For any* $0 \leq t \leq 1$ *the two embeddings*

$$f^{-1}(t) \cap \mathbb{R}P_6^2 *_\Delta \mathbb{R}P_6^2 \longrightarrow f^{-1}(t) \cap \partial\beta_6 \longrightarrow \mathbb{E}^5$$

$$f^{-1}(t) \cap \mathbb{C}P_9^2 *_\Delta \mathbb{C}P_9^2 \longrightarrow f^{-1}(t) \cap \partial\beta_9 \longrightarrow \mathbb{E}^8$$

are tight with respect to $\mathbb{Z}_2$. *These are polyhedral analogues of the family of isoparametric tubes around the real or complex Veronese-type surface, respectively.*

Branched simplicial coverings related to $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$

It was pointed out by Massey in 1973 that incidentally a number of interesting 4-manifolds (among them the complex projective plane) are (branched or non-branched) quotients of $S^2 \times S^2$. In particular, $\mathbb{C}P^2$ is the quotient of $S^2 \times S^2$ by the involution $\tau(x, y) = (y, x)$, and the 4-sphere is the quotient of $\mathbb{C}P^2$ modulo complex conjugation σ where $\sigma[z_0, z_1, z_2] = [\overline{z_0}, \overline{z_1}, \overline{z_2}]$. In the latter case the branch locus consists precisely of the real part which is a real projective plane. Opposite to it we find the complex quadric $z_0^2 + z_1^2 + z_2^2 = 0$ on which the involution σ acts freely.

By a branched simplicial k-sheeted covering between two d-manifolds we mean a simplicial mapping which is simultaneously a branched k-sheeted covering. In particular, it is required that the preimage of any (open) d-simplex consists of k disjoint (open) d-simplices and that there is no collapsing of lower-dimensional simplices. Then the branch locus is a simplicial subcomplex of each of the two triangulated d-manifolds.

Proposition 7 *There is a branched simplicial* 2*-sheeted covering from a triangulated* $\mathbb{C}P^2$ *onto a triangulated* 4*-sphere which is branched along a subcomplex isomorphic to* $\mathbb{R}P^2_6$*. We can denote it – by slight abuse of notation – as follows:*

$$\mathbb{C}P^2_{18} := S^2_{12} *_\Delta \mathbb{R}P^2_6 \longrightarrow \mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6.$$

Here S^2_{12} *denotes the icosahedral triangulation of the* 2*-sphere with its* 2*-fold simplicial covering* $S^2_{12} \longrightarrow \mathbb{R}P^2_6$*. The complex* $S^2_{12} *_\Delta \mathbb{R}P^2_6$ *does not literally denote the deleted join but the join where each simplex is deleted which involves one vertex of* $\mathbb{R}P^2_6$ *and any of the two corresponding antipodal vertices of the icosahedron* S^2_{12}*.*

Corollary 8 *The polyhedral Cartan hypersurface halfway between the two copies of* $\mathbb{R}P^2_6$ *in the Bier sphere lifts to a* 2*-fold covering halfway between* S^2_{12} *and* $\mathbb{R}P^2_6$*. This is a polyhedral decomposition of the lens space* $L(4,1)$ *which occurs as a tubular neighborhood of the real projective plane in the complex projective plane. Combinatorially, it consists of* 120 *triangular prisms.*

PROOF: Since the intermediate levels do not intersect the branch locus, this defines a twofold non-branched covering of some 3-manifold onto the Cartan isoparametric hypersurface in S^4. Topologically the latter is the quaternion space S^3/Q where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ denotes the quaternion group of order 8. Any twofold covering in between is a quotient of S^3 by a group of order 4 which is a subgroup of Q. This is possible only for the cyclic group C_4, e.g., for $\{\pm 1, \pm i\} \subset Q$. Consequently, the twofold covering of the Cartan hypersurface is a lens space $L(4, 1)$ with fundamental group C_4. Its combinatorial automorphism group of order 240 acts transitively on the 120 prisms.

Proposition 9 *There is a branched simplicial 2-sheeted covering from a triangulated $S^2 \times S^2$ onto a triangulated $\mathbb{C}P^2$ which is branched along a subcomplex isomorphic to the icosahedral triangulation of S^2. We can denote it – wth the same remark as in Proposition 7 above – as follows:*

$$(S^2 \times S^2)_{24} := S^2_{12} *_\Delta S^2_{12} \longrightarrow S^2_{12} *_\Delta \mathbb{R}P^2_6$$

where $S^2_{12} \longrightarrow \mathbb{R}P^2_6$ denotes the same 2-fold simplicial covering as above.