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PL MORSE THEORY IN LOW DIMENSIONS

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ABSTRACT. We discuss a PL analog of Morse theory for PL manifolds. There are several notions of regular and critical points. A point is homologically regular if the homology does not change when passing through its level; it is strongly regular if the function can serve as one coordinate in a chart. Several criteria for strong regularity are presented. In particular we show that in dimensions $d \leq 4$ a homologically regular point on a PL d-manifold is always strongly regular. Examples show that this fails in higher dimensions $d \geq 5$. One of our constructions involves an embedding of the dunce hat into 4-space and Mazur's contractible 4-manifold. Finally, decidability questions in this context are discussed.

1. Introduction

What is nowadays called *Morse theory* after its pioneer Marston Morse (1892–1977) is based on the following idea: A real function on a compact differential manifold has minima, maxima and possibly other critical points, i.e., points with gradient 0. Generically, a smooth real function has isolated critical points, and at each critical point the Hessian matrix is nondegenerate. The index of the Hessian is then taken as the *index* of the critical point. This leads to the Morse lemma and the Morse relations, as well as a handle decomposition of the manifold [34, 37].

Already in the early days of Morse theory, this approach was extended to non-smooth functions on suitable spaces [35, 36, 27], leading to several analogs of Morse theory for PL manifolds or for polyhedra in general.

As the gradient and the Hessian have no natural substitute in this setting, the notion of critical point has to be redefined. One possibility is to study the change of the topology when passing through the critical point. Especially in higher dimensions we have certain topological phenomena that have no analog in classical Morse theory like contractible but not collapsible polyhedra, homology points that are not homotopy points, and triangulations that don't induce a PL structure [4].

From an application viewpoint, piecewise linear functions arise in many fields, for example from simulation experiments or from measured data. One way to explore such a function that is defined, say, on a three-dimensional domain, is by the interactive visualization of level sets. In this context, it is interesting to know the topological changes between level sets, and critical points are precisely those points where such changes occur.

The first part of this article presents a survey on Morse theory for manifolds with or without boundary and, more generally, for polytopal complexes. Critical and regular (non-critical) points are introduced in several meanings.

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The new results are presented in the last sections. While in dimensions up to 4, the weaker notion of homological regularity is sufficient to guarantee strong regularity (Section 6), this is no longer true in higher dimensions. Sections 7 and 8 give various examples of phenomena that arise in high dimensions. Finally, in Section 9, we discuss the algorithmic questions regarding the concept of strong regularity, and we show undecidability results in high dimensions.

The results of Sections 4 and 9 are based on the Ph.D. thesis of the first author [19]. Preliminary approaches to these questions were sketched in [38]. Some further material and more examples are contained in the preprint [20].

2. Polyhedra and PL manifolds

Definition 2.1. (PL manifold, combinatorial manifold)
A topological manifold M (as usual Hausdorff and paracompact) is called a PL manifold *if it is equipped with a covering $(M_i)_{i \in I}$ of* charts M_i *such that all coordinate transformations between two overlapping charts are piecewise linear homeomorphisms of open parts of Euclidean space.*

From the practical point of view, a compact PL d-manifold M can be interpreted as a finite *polytopal complex* K built up from convex d-polytopes such that $|K|$ is homeomorphic to M and such that the star of each (relatively open) cell is piecewise linearly homeomorphic to an open ball in d-space. Since every polytope can be triangulated, any compact PL d-manifold can be triangulated such that the link of every k-simplex is a combinatorial $(d-k-1)$-sphere. Such a simplicial complex is often called a *combinatorial d-manifold* [29]. In the sequel we will often interpret PL manifolds in this way without explicitly talking about triangulations.

In Section 3 we will develop a Morse theory for PL functions defined on general polytopal complexes as well as on combinatorial manifolds. For a general outline and the terminology of PL topology we refer to [39] and [40].

We will repeatedly run into the Schoenflies problem:

The PL Schoenflies Conjecture. *A combinatorial $(d-1)$-sphere S^{d-1} embedded into a combinatorial d-sphere S^d decomposes S^d into two combinatorial d-balls.*

This conjecture is true for $d \leq 3$ and unknown in higher dimensions. Under the additional assumption that the closure of each component of $S^d \setminus S^{d-1}$ is a manifold with boundary, the conclusion holds for all $d \neq 4$ [39, Ch. 3].

3. Regular and critical points of PL functions

The simplest way to carry over the ideas of Morse theory to the PL world is to consider functions that are linear on each polytopal cell (or simplex in the simplicial case) and *generic* in the sense that the function values at all vertices are distinct. Such a theory was sketched by Brehm and Kühnel [11, 25] for obtaining lower bounds for the number of vertices of combinatorial manifolds of certain type.

PL functions will be defined on an abstract polytopal complex, see [44, Ch. 5] for a formal definition. The reader may think of the boundary complex of a convex d-polytope, or any subcomplex of it.

Definition 3.1. (generic PL function)

Let P be a finite (abstract) polytopal complex. A function $f\colon P \to \mathbb{R}$ is called generic PL *if it is linear on each polytopal cell separately and if $f(v) \neq f(w)$ for any two distinct vertices v, w of P. As a consequence, f is not constant on any edge or higher-dimensional cell.*

We denote by f_a and f^a the sublevel set and the superlevel set:

$$f_a := \{x \mid f(x) \leq a\}, \quad f^a := \{x \mid f(x) \geq a\}$$

Lemma 3.2. *If $f\colon P \to \mathbb{R}$ is generic PL and if $f^{-1}[a, b]$ contains no vertex of P, then f_a is a strong deformation retract of the sublevel f_b, and $f^{-1}[a, b]$ is homeomorphic to the cylinder $f^{-1}(a) \times [a, b]$.*

Proof. If P is a convex polytope then the assertion is obviously true. Therefore it holds for any single polytopal cell of P and — in combination — for the entire complex P. □

Lemma 3.2 tells us that all points $p \in P$ that are not vertices satisfy the following regularity condition in Morse theory: The topology of the sublevel does not change when passing through p.

It remains to look at the vertices. Measuring the topology by the rank of the homology leads to the following definition:

Definition 3.3. (H-critical points, [26, Sect. 3B])

Let $f\colon P \to \mathbb{R}$ be generic PL and let v be a vertex with the level $f(v) = a$. Then v is called homologically critical for f *or* H-critical *for short if $H_*(f_a, f_a \setminus \{v\}; \mathbb{F}) \neq 0$ where H_* denotes an appropriate homology theory with coefficients in a field $\mathbb{F}$. The total rank of $H_*(f_a, f_a \setminus \{v\})$ is called the* total multiplicity *of v with respect to f. If*

$$H_k(f_a, f_a \setminus \{v\}) \neq 0$$

then we say that v is H-critical of index k, *and the rank of $H_k(f_a, f_a \setminus \{v\})$ is referred to as the corresponding* multiplicity *of v restricted to the index k.*

By excision and the long exact sequence for the reduced homology $\widetilde{H}$ in a simplicial complex P we can detect H-criticality in the link $lk(v)$ and the star $st(v)$ of a vertex v:

$$\widetilde{H}_k(f_a, f_a \backslash \{v\}) \cong \widetilde{H}_k(f_a \cap st(v), f_a \cap lk(v)) \cong \widetilde{H}_{k-1}(f_a \cap lk(v)) \cong \widetilde{H}_{k-1}(lk^-(v))$$

for $k \geq 1$ where $lk^-(v)$ denotes

$$lk^-(v) := \{x \in lk(v) \mid f(x) \leq f(v)\} = lk(v) \cap f_a.$$

The homology of $lk^-(v)$ is the same as that of the full span of those vertices in the link of v whose level lies below $f(v)$. Similarly we will use the notation

$$lk^+(v) := \{x \in lk(v) \mid f(x) \geq f(v)\} = lk(v) \cap f^a.$$

This definition is also applicable to classical smooth Morse functions on a smooth manifold. Then a critical point of index k is also critical with respect to Definition 3.3 with the same index, and the total multiplicity is always 1. However, for polyhedral surfaces the case of higher total multiplicity occurs, as the example of a polyhedral monkey saddle shows. It is also easy to construct polyhedra such that there are critical vertices of several indices simultaneously: Take the 1-point union of a 1-sphere with a 2-sphere.

It remains to discuss the possible case of $H_*(f_a, f_a \setminus \{v\}) = 0$ for some vertex v. Since homology does not detect it as critical, we would like to call it *non-critical* or *regular*. However, we have to be careful since regularity in the sense of the conclusion of Lemma 3.2 is different. The question is: Can $f_{a+\epsilon}$ and $f_{a-\epsilon}$ be topologically distinct in this case?

Definition 3.4. *A vertex v with $f(v) = a$ is called* homologically regular for f *or* H-regular *for short if $H_*(f_a, f_a \setminus \{v\}; \mathbb{F}) = 0$ for an arbitrary field $\mathbb{F}$.*

In classical Morse theory any H-regular point is actually regular in a stronger sense (compare Section 4). We will see in Section 6 that for generic PL functions on PL manifolds, this remains true only in dimensions $d \leq 4$.

Theorem 3.5. (Morse relations, duality [37, 27, 25])

Let $f\colon M \to \mathbb{R}$ be a generic PL function on a compact PL d-manifold M without boundary, and let $v_1, \ldots, v_n$ be the vertices. By a_i we denote the level $a_i = f(v_i)$. Then the Morse inequality

$$\sum_i \mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F}) \geq \mathrm{rk} H_k(M; \mathbb{F}) \tag{1}$$

and, in particular, the critical point theorem

$$\sum_k (-1)^k \sum_i \mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F}) = \sum_k (-1)^k \mathrm{rk} H_k(M; \mathbb{F}) = \chi(M) \tag{2}$$

hold for any k and any field $\mathbb{F}$.

The expression $\mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F})$ is nothing but the multiplicity of v_i restricted to the index k, and $\sum_i \mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F})$ is the number $\mu_k(f)$ of critical points of index k, weighted by their multiplicities. Therefore the Morse relations can also be written in the form

$$\mu_k(f) \geq \mathrm{rk} H_k(M; \mathbb{F}), \quad \sum_k (-1)^k \mu_k(f) = \chi(M)$$

and the duality *in the form*

$$\mu_k(-f) = \mu_{d-k}(f).$$

We have formulated the theorem only for PL manifolds since the duality statement is not true for general PL complexes.

Proof of the duality. By Alexander duality in the link of a vertex v one has $\widetilde{H}_{d-k-1}(lk^+(v)) \cong \widetilde{H}_{k-1}(lk^-(v))$ for $1 \leq k \leq d-1$ and consequently

$$\widetilde{H}_{d-k}(f^a, f^a \setminus v) \cong \widetilde{H}_k(f_a, f_a \setminus v).$$

Clearly a local minimum of f (with $k = 0$) is a local maximum (with $k = d$) for $-f$ and conversely. This means that the number of critical points of f of index k coincides with the number of critical points of $-f$ of index $d-k$, weighted with multiplicities. □

Alternative notions of critical vertices.

The idea behind our notion is that the homological type of the sublevel set changes when passing through an H-critical point. Since no two vertices have the same level under f, the homology of $f_a \setminus \{v\}$ is the same as that of the open sublevel $(f_a)^\circ = \{x \mid f(x) < a\}$.

There are several related notions that are concurrently referred to as PL Morse theory. They differ in the strength by which they distinguish topological differences. The approach of Banchoff [1, 2] is based on a *local Euler characteristic* in the neighborhood of a vertex. In this context the critical point theorem in [1] appears as the weaker statement $\sum_v (1 - \chi(lk^-(v)) = \chi(M)$ for a fixed generic PL function on a simplicial complex where the sum ranges over all vertices. Here a vertex is tacitly regarded as critical if and only if $1 - \chi(lk^-(v)) \neq 0$. So if $lk^-(v)$ consists of a point and a 1-sphere (this is possible on 3-manifolds) or of $\mathbb{R}P^2$ (see Example 7.5), the vertex v is not detected as critical.

More recently, a variant of PL Morse theory based on the changes of the *homotopy type* at a critical level was proposed and applied to problems in geometric group theory [6, 7]. Our Definition 3.3 above with a classification of the vertices according to the induced change in *homology* was sketched already in [11, 25]. The strongest version of PL Morse theory that considers any kind of topological change between level sets as critical, was historically also the first one to be studied [15, 23]. We will present it in Definition 4.1.

A homological definition due to [13] compares the homology of the $(a-\epsilon)$-level with that of the $(a+\epsilon)$-level if a is the critical level. This is equivalent to our definition for the case of polyhedra, but not for general topological spaces, as pointed out in [18]. The problem with the incorrect *Critical Value Lemma* in [13] is that a nested sequence of closed intervals can converge to a common boundary point. Then no open ϵ-neighborhood around the critical level can fit into any of the closed intervals. Instead of the definition above one could compare the open sublevel $(f_a)^\circ = f_a \setminus f^{-1}(a)$ to the closed sublevel f_a. For polytopal complexes (with closed polytopal faces) this will lead to the same definition.

4. PL Morse functions

In classical Morse theory, there is a particular normal form for the neighborhood of critical points, with the consequence that passing through a critical level attaches one cell to the sublevel [34, Thm.3.2]. Moreover, the dimension of the cell coincides with the index of the critical point, and — regarded as an H-critical point — the total multiplicity is always 1 in this case. One can adapt this to the PL case as follows:

Definition 4.1. (regular and critical points for PL Morse functions)

Let M be a PL d-manifold and $f\colon M \to \mathbb{R}$ a generic PL function.

(i) A point p is called strongly regular *if there is a chart around p such that the function f can be used as one of the coordinates, i.e., if in those coordinates*

$$f(x_1, \dots, x_d) = f(p) + x_d. \tag{3}$$

If, in a concrete polyhedral decomposition of M, distinct vertices have distinct values of f, then f is also generic PL, and moreover all points are strongly regular except possibly the vertices.

(ii) A vertex v is called non-degenerate critical *if there is a PL chart around v such that in those coordinates $x_1, \ldots, x_d$ the function f can be expressed as*

$$f(x_1, \ldots, x_d) = f(v) - |x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_d|. \tag{4}$$

The number k is then uniquely determined and coincides with the index of v. The multiplicity is always 1 in this case: $H_k(f_a, f_a \setminus \{v\}; \mathbb{F}) \cong \mathbb{F}$ and $H_j(f_a, f_a \setminus \{v\}) = 0$ for any $j \neq k$. The change by passing through the critical level can be either $H_k(f_{a+\epsilon}) \cong H_k(f_{a-\epsilon}) \oplus \mathbb{F}$ or $H_{k-1}(f_{a-\epsilon}) \cong H_{k-1}(f_{a+\epsilon}) \oplus \mathbb{F}$. A function such that the second case never occurs is called a perfect function.

(iii) The function f is called a PL Morse function *if all vertices are either non-degenerate critical or strongly regular.*

In the terminology of Morse [35], non-degenerate critical and strongly regular vertices are called *topologically critical* and *topologically ordinary*, respectively, and a PL Morse function is called *topologically non-degenerate*. On the other hand, the notion *PL Morse function* is sometimes used in a more general sense,see the end of Section 3: In [6, 7], it denotes what we call a *generic PL function* (Definition 3.1).

Lemma 4.2. (strongly regular points) [19, Theorem 3.11]

Let f be a generic PL function on a combinatorial d-manifold. Then a vertex v is strongly regular for f if and only if $lk^-(v)$ is a PL $(d-1)$-ball.

We have a similar characterization for non-degenerate critical points:

Lemma 4.3. (non-degenerate critical points)

Let f be a generic PL function on a combinatorial d-manifold. Then a vertex v is non-degenerate critical for f with index k if and only if $lk^-(v)$ is a tubular neighborhood of an unknotted $(k-1)$-sphere embedded into the $(d-1)$-sphere $lk(v)$.

In particular this criterion is applicable in higher codimension: A $(k-1)$-sphere embedded into the $(d-1)$-sphere is always unknotted if $d - k \geq 3$ [42], [39, Thm.7.1].

Lemma 4.4. (Morse Lemma, [19, Thm.5.1])

Let $f \colon M \to \mathbb{R}$ be a PL Morse function and assume that there are no critical points with f-values in the interval $[a, b]$. Then f_a and f_b are PL homeomorphic to each other, and $f^{-1}([a, b])$ is PL homeomorphic to the "collar" $f^{-1}(a) \times [a, b]$.

The following corollary follows from Theorem 3.5.

Corollary 4.5. (Morse relations, duality)

Let $f \colon M \to \mathbb{R}$ be a PL Morse function on a compact PL manifold M without boundary, and let $\mu_k(f)$ be the number of critical vertices of index k, then the Morse inequality

$$\mu_k(f) \geq \mathrm{rk} H_k(M; \mathbb{F}) \tag{5}$$

holds for any k and any field $\mathbb{F}$. Moreover we have the critical point theorem

$$\sum_k (-1)^k \mu_k(f) = \chi(M)$$

and the duality
$$\mu_k(-f) = \mu_{d-k}(f).$$

Corollary 4.6. (Reeb theorem [23]) *Let M be a compact PL d-manifold and $f\colon M \to \mathbb{R}$ be a PL Morse function with exactly two critical vertices. Then M is PL homeomorphic to the sphere S^d.*

Proof. Since the minimum p and maximum q of f are always critical the assumption can be reformulated by saying that any point between minimum and maximum is strongly regular. Let us consider the restriction
$$f_| \colon M \setminus \{p, q\} \to \mathbb{R}$$
without critical points. For any level $f^{-1}(c)$ with $f(p) < c < f(q)$ the Morse lemma tells us that there is an $\epsilon > 0$ such that $f^{-1}(c-\epsilon, c+\epsilon)$ is PL homeomorphic to the cylinder $f^{-1}(c) \times (-\epsilon, \epsilon)$. Furthermore there is a $\delta > 0$ such that $f^{-1}[f(p), f(p)+\delta]$ and $f^{-1}[f(q)-\delta, f(q)]$ are PL homeomorphic to d-balls. Consequently $f^{-1}(f(p)+\delta)$ and $f^{-1}(f(p)-\delta)$ are PL homeomorphic to the $(d-1)$-sphere. This implies that $f^{-1}[f(p)+\delta, f(q)-\delta]$ is PL homeomorphic to the cylinder
$$f^{-1}(c) \times [p+\delta, q-\delta] \cong S^{d-1} \times [p+\delta, q-\delta].$$
Putting the three parts together we see that M is PL homeomorphic to the (standard) d-sphere S^d. □

Remarks. (1) In the smooth theory the same kind of proof leads only to a homeomorphism to the standard S^d but not to a diffeomorphism. There are exotic 7-spheres admitting a Morse function with two critical points, thus providing a counterexample. By contrast it is well known that the d-sphere ($d \neq 4$) admits a unique PL structure [28, Thm. 7]. Therefore this problem could occur only for $d = 4$. But glueing together two standard 4-balls along their boundaries leads to the standard 4-sphere. Therefore the proof above gives a PL homeomorphism even for $d = 4$.

(2) For the case of compact PL manifolds admitting a PL Morse function with exactly three critical points, see [15]. The only possibilities occur in dimensions $d = 2, 4, 8, 16$ with an intermediate critical point of index $k = 1, 2, 4, 8$, respectively.

Corollary 4.7. *(i) If there is an exotic PL 4-sphere then any PL Morse function on it must have at least four critical points.*

(ii) If M is a homology sphere that is not a homotopy sphere, then any PL Morse function f on M has at least six critical points. The same holds for any generic PL function if the critical points are counted with multiplicity. Consequently, it cannot admit a perfect function.

Proof of (ii). If a PL Morse function on M has no critical point of index 1 then M is simply connected. This follows — as in the classical setting — from attaching a k-cell when passing through a critical point of index k. By assumption M has a non-abelian fundamental group with a trivial commutator factor group. Therefore f must have a critical point of index 1. If there is only one of them then this leads to a free fundamental group in the critical sublevel f_a. If a critical point of index ≥ 2 introduces a relation in that group, the quotient will be abelian. A non-abelian group requires a second generator, and this requires a second critical point of index 1. Since the fundamental group is not free, there must be a critical point of index ≥ 2 introducing a relation between the generators. By the Euler relation the number of critical points must be even, so there are two critical points of index 1, minimum and maximum, and two others.

For a generic PL function the argument is analogous: there is one critical point of index 1 and multiplicity ≥ 2 or there are at least two critical points of index 1. □

Example 4.8. *An example with three critical points:* For the unique 9-vertex triangulation of the complex projective plane [26, Sect. 4B] any generic PL function assigning distinct levels to the 9 vertices is a PL Morse function with three critical points: minimum, maximum and a saddle point of index 2 in between. Since 123 is a 2-face of the triangulation, for the special case $f(1) < f(2) < f(3) < f(4) < \cdots < f(9)$ the sublevel f_a will be a 4-ball for $f(1) < a < f(4)$ and the complement of a 4-ball for $f(4) < a < f(9)$. Since 1234 is not a 3-face of the triangulation, the critical sublevel $f_{f(4)}$ consists of the boundary of the tetrahedron spanned by 1234 extended by sections through all 4-simplices except 56789.

5. Manifolds with boundary

Classical Morse theory was extended to smooth manifolds with boundary $(M, \partial M)$ in [10]. Here a *Morse function* is defined as a smooth function having only non-degenerate critical points in $M \setminus \partial M$ and no critical points on ∂M, i.e., $\mathrm{grad} f \neq 0$ on ∂M. Furthermore the restriction $f|_{\partial M}$ is assumed to be a Morse function on ∂M.

Definition 5.1. *A critical point p of $f|_{\partial M}$ is called* (+)-critical for f *if* $\mathrm{grad} f|_p$ *is an interior vector on M (pointing into M), and* (−)-critical for f *if* $\mathrm{grad} f|_p$ *is an exterior vector on M (pointing away from M).*

Proposition 5.2. (Braess [10]) *Let M be a compact smooth manifold with boundary, let $\mu(f|_{M\setminus\partial M})$ denote the number of critical points and let $\mu^+(f)$ and $\mu^-(f)$ denote the number of $(+)$- and $(-)$-critical points. The* index *for boundary points is the same as the index for $f|_{\partial M}$, so we have $\mu_k^+(f)$ and $\mu_k^-(f)$ for index k. Then all $(+)$-critical points are H-critical and change the sublevel by attaching a cell, the $(-)$-critical points are H-regular. Moreover $f_{a-\epsilon}$ is a deformation retract of $f_{a+\epsilon}$ if $f^{-1}[a-\epsilon, a+\epsilon]$ contains only a $(-)$-critical point on ∂M and no critical point in $M \setminus \partial M$. Then the Morse inequality reads as*

$$\mu(f|_{M\setminus\partial M}) + \mu^+(f) \geq rkH_*(M),$$

on the boundary one has

$$\mu^+(f) + \mu^-(f) = \mu(f|_{\partial M}) \geq \mathrm{rk} H_*(\partial M).$$

The critical point theorem $\sum_k (-1)^k \big(\mu_k(f|_{M\setminus\partial M}) + \mu_k^+(f)\big) = \chi(M)$ follows also, but there is no duality on M since $\mu^+_{d-k-1}(f) = \mu_k^-(-f)$ holds.

For a proof see [10, Satz 4.1 and Satz 7.1]. In Satz 4.1 the assumption should be that the interval contains no critical point in the interior and no (+)-critical point on the boundary.

In the case of a generic PL function we can directly apply Definition 3.3 with the following result for a vertex $v \in \partial M$ with $f(v) = a$ [24]:

$$\mathrm{rk} H_*(f_a, f_a \setminus \{v\}) + \mathrm{rk} H_*(f^a, f^a \setminus \{v\}) \geq \mathrm{rk} H_*((f|_{\partial M})_a, (f|_{\partial M})_a \setminus \{v\})$$

In general the last inequality is not always an equality since it can happen that a boundary point is H-critical for f but H-regular for $f|_{\partial M}$. Consider a 2-dimensional polyhedral surface in 3-space with a straight line segment in the boundary. Let v be a vertex on this segment and assume that the faces of the star of v are creased into three independent directions. Then for certain height functions v contributes some positive term to the left hand side whereas the right hand side at v vanishes. The average of the number of critical points over all directions of height functions is half the average of the boundary separately in the smooth case and greater or equal to half this average in the PL case [24].

By combining the definitions for PL Morse functions in Section 4 with the ideas of Definition 5.1 above we can formulate a theory of PL Morse functions on manifolds with boundary as follows.

Definition 5.3. *Let M be a compact PL d-manifold with boundary and $f\colon M \to \mathbb{R}$ a generic PL function. Then f is called a* PL Morse function *if all interior vertices are either non-degenerate critical or strongly regular in the sense of Definition 4.1 and all vertices on ∂M are either $(+)$-critical or $(-)$-critical or strongly regular.*

A point $p \in \partial M$ is called strongly regular *if there is a chart around p such that M is described by $x_1 \leq 0$ and the function f can be used as the coordinates x_d in ∂M, i.e., if in those coordinates*

$$f(x_1, \ldots, x_d) = f(p) + x_d \tag{6}$$

for $x_1 \leq 0$. If in a concrete polyhedral decomposition of M distinct vertices have distinct f-values, then f is also generic PL, and moreover all points are strongly regular except possibly the vertices.

A vertex $v \in \partial M$ is called non-degenerate $(+)$-critical *(or* $(-)$-critical, *respectively) of index k, if there is a PL chart with coordinates $x_1, \ldots, x_d$ around v for which the set M is described by the constraint*

$$x_d \geq -|x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}|$$
$$(\textit{or } x_d \leq -|x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}|, \ \textit{respectively})$$

and the function f can be expressed as

$$f(x_1, \ldots, x_d) = f(v) + x_d. \tag{7}$$

See Figure 1 for an illustration. In this case the boundary is represented by the equation

$$x_d = -|x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}|,$$

and the restriction $f|_{\partial M}$ can be written as

$$f(x_1, \ldots, x_{d-1}) = f(v) - |x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}|, \tag{8}$$

so v is non-degenerate critical of index k for $f|_{\partial M}$.

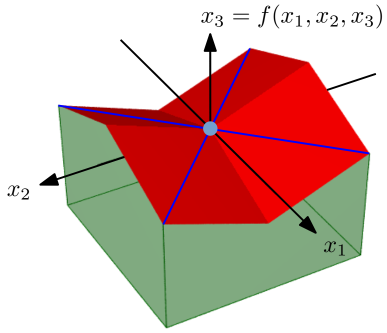

FIGURE 1. A non-degenerate critical point (blue) of index 1 on the boundary (red) of a 3-manifold M. If M consists of the volume below the red roof, as indicated by the green walls, then the point is $(-)$-critical; if M lies above the red surface, it is $(+)$-critical. The blue cross is the level set at the critical value.

Corollary 5.4. *In the situation of Definition 5.3 all* $(+)$*-critical points on the boundary are* H*-critical, necessarily with multiplicity 1 and index* k*. Any* $(-)$*-critical point on the boundary is* H*-regular.*

Proof. For a $(+)$-critical point $v \in \partial M$ the number k in Definition 5.3 is uniquely determined and coincides with the index of v, and v is H-critical since the multiplicity is always 1 in this case: $H_k(f_a, f_a \setminus \{v\}; \mathbb{F}) \cong \mathbb{F}$ and $H_j(f_a, f_a \setminus \{v\}) = 0$ for any $j \neq k$. The change by passing through the critical level can be either $H_k(f_{a+\epsilon}) \cong H_k(f_{a-\epsilon}) \oplus \mathbb{F}$ or $H_{k-1}(f_{a-\epsilon}) \cong H_{k-1}(f_{a+\epsilon}) \oplus \mathbb{F}$. For a $(-)$-critical vertex $v \in \partial M$ the homotopy types of f_a and $f_a \setminus \{v\}$ coincide, so it is H-regular. □

Corollary 5.5. *Proposition 5.2 remains valid for PL Morse functions on PL manifolds with boundary.*

Example 5.6. Let us consider the solid torus defined by the convex hull of the vertices $(\pm 2, \pm 2, \pm 1)$ in 3-space with the subset $(-1, 1) \times (-1, 1) \times [-1, 1]$ removed. Any height function with distinct levels for distinct vertices is a PL Morse function, on the boundary as well as on the solid torus. On the boundary we have a minimum, a maximum and two nondegenerate critical points of index 1. On the solid torus the minimum is $(+)$-critial, the maximum is $(-)$-critical, and the saddle point with the higher level is $(+)$-critical, the other one $(-)$-critical. All other points are strongly regular.

6. THE SPECIAL CASE OF LOW DIMENSIONS

The critical vertices play the role of the critical points in classical Morse theory, either in the version of non-degenerate points or — more generally — of generic PL functions where higher multiplicities are admitted. However, the H-regular vertices that are not strongly regular do not fit this analogy: They do not contribute to the Morse inequalities, and they have no analog in the classical theory since they do not allow the cylindrical decomposition in a neighborhood with an isotopy between the upper and the lower sublevel. In some sense they are the most exotic objects to be considered here. Therefore the question is whether they can occur or not. In low dimensions $d \leq 4$ this is indeed not the case.

The one-dimensional case is easy: For a generic PL function on a PL 1-manifold (a union of curves), every vertex is a minimum of multiplicity 1 (index 0), a maximum of multiplicity 1 (index 1), or strongly regular.

We proceed to two dimensions.

Proposition 6.1. *Let M be a PL 2-manifold (a surface) with a generic PL function $f\colon M \to \mathbb{R}$. The critical points (vertices) are of the following types:*

1. *Local minima (index 0, multiplicity 1),*
2. *local maxima (index 2, multiplicity 1),*
3. *saddle points (index 1, multiplicity arbitrary).*

Any H-regular vertex is also strongly regular, and any saddle point is non-degenerate critical in the sense of Definition 4.1 if its (total) multiplicity is 1 in the sense of Definition 3.3.

Proof. The link of a vertex v is a closed circuit of edges. If $lk^-(v)$ is empty we have a minimum, if $lk^-(v) = lk(v)$ we have a maximum ($lk^+(v)$ is empty), in all other cases $lk^-(v)$ and $lk^+(v)$ have the same number of components, say r components. Then v is critical of index 1 and multiplicity $r-1$. An ordinary (non-degenerate) saddle point has $r = 2$, a monkey saddle $r = 3$.

An H-regular vertex corresponds to the case $r = 1$. Since $st(v)$ is a topological disc, this implies that both $st^-(v)$ and $st^+(v)$ are discs, fitting together along the $f(v)$-level, which is an interval. Then we can apply Lemma 4.2.

An ordinary saddle point corresponds to the case $r = 2$. The two components in $lk^-(v)$ and $lk^+(v)$ determine one coordinate line each such that the function f is linearly decreasing or increasing, respectively. The $f(v)$-level in between is the cross of the two diagonals in that coordinate system. □

Corollary 6.2. *Any generic PL function on a PL 2-manifold is a PL Morse function if the multiplicity of every saddle point is 1.*

A splitting process of saddle points with higher multiplicity into ordinary saddle points is described in [16, p. 93].

Theorem 6.3. *Let M be a PL 3-manifold with a generic PL function $f\colon M \to \mathbb{R}$. The critical points (vertices) are only of the following types:*

1. *Local minima (index 0, multiplicity 1),*
2. *local maxima (index 3, multiplicity 1),*
3. *mixed saddle points (index 1 or 2 or both, multiplicity arbitrary).*

Any H-regular vertex is also strongly regular, and any saddle point is non-degenerate critical in the sense of Definition 4.1 if its (total) multiplicity is 1.

Proof. Let v be a H-regular vertex (not a local minimum) with

$$H_0(lk^-(v); \mathbb{F}) \cong \mathbb{F}, \quad H_1(lk^-(v)) = 0 \quad \text{and} \quad H_2(lk^-(v)) = 0.$$

Therefore $lk^-(v) = f_a \cap lk(v)$ is a subset of $lk(v) \cong S^2$ which is a homology point. This implies that it is a homotopy point also, hence contractible. Consequently, $lk^-(v) \subset S^2$ is a disc since it is also a compact 2-manifold with boundary. Its complement is a disc also. Then we can apply Lemma 4.2.

Now let v be a saddle point with total multiplicity 1. This means that $lk^-(v)$ and $lk^+(v)$ are subsets of a 2-sphere with homology of a 0-sphere and a 1-sphere, respectively (in any order). So there are two discs in $lk^-(v)$ and a cylinder in $lk^+(v)$ or vice versa. Let us pick one point in each disc and a circle in the cylinder as "souls". Then the cones from v determine one coordinate direction with decreasing f and two directions with increasing f (or vice versa). This defines the chart according to Definition 4.1. □

Theorem 6.4. *Let M be a PL 4-manifold with a generic PL function $f: M \to \mathbb{R}$. Then any H-regular vertex is also strongly regular.*

Proof. Let v be a H-regular vertex (not a local minimum) with

$$H_0(lk^-(v);\mathbb{F}) \cong \mathbb{F}, \ \ H_1(lk^-(v)) = 0, \ \ H_2(lk^-(v)) = 0 \ \text{ and } \ H_3(lk^-(v)) = 0$$

for any field $\mathbb{F}$. Therefore $lk^-(v)$ is a subset of $lk(v) \cong S^3$ which is a homology point for arbitrary $\mathbb{F}$, hence it is also a homology point for $\mathbb{Z}$, in other words: it is $\mathbb{Z}$-acyclic. The following argument is taken from [30]: $lk^-(v)$ is a compact 3-manifold which is $\mathbb{Z}$-acyclic, so the Euler characteristic is $\chi(lk^-(v)) = 1$. The Euler characteristic of the boundary is twice the Euler characteristic of the entire manifold, so $\chi = 2$ for the boundary which therefore contains a 2-sphere as one connected component, tamely (or locally flat) embedded into a polyhedral S^3. Then by the 3-dimensional Schoenflies theorem in PL [28] it bounds a 3-ball in S^3 on either side. This in turn shows that in our case there is no other component of the boundary since it would contradict the assumption that $lk^-(v)$ is acyclic. Then we can apply Lemma 4.2. □

Remark. In higher dimensions $d \geq 5$ one obstruction is that a homology point contained in a vertex link is not necessarily a homotopy point, see Section 7 below. In particular there are acyclic 2-complexes in the 4-sphere that are not contractible [30], moreover there are particular embeddings of the contractible dunce hat into the 4-sphere with regular neighborhoods that are again contractible but not 4-balls [43]. These phenomena make it impossible to carry over the proofs above to dimensions higher than $d = 4$.

7. Counterexamples in higher dimensions

By Definition 3.3 one can detect H-criticality by the homology $H_*(f_a, f_a \setminus v)$ or, equivalently, by $H_*(lk^-(v))$. Therefore, in higher dimensions d, we can prescribe a subcomplex of $lk(v) \cong S^{d-1}$ and arrange a function f such that this subcomplex coincides with $lk^-(v)$. This leads to many examples with unexpected or even pathological properties of critical points.

Example 7.1. (Critical point of total multiplicity 1 containing a knot)

We start with an ordinary knot built up by edges in a combinatorial 3-sphere. A concrete example is the 6-vertex trefoil knot in the 1-skeleton of the Brückner-Grünbaum sphere with 8 vertices, see [25, Fig.4]. After a barycentric subdivision the knot coincides with the full subcomplex spanned by its vertices. This combinatorial 3-sphere can be the link of a vertex v in a 4-manifold. Define a generic PL function f with $f(v) = 0, f(x) < 0$ for all vertices x on the knot, and $f(y) > 0$ for all the other vertices y in the 3-sphere. This vertex v will be critical for f of index 2 and multiplicity 1, so homologically it behaves like a non-degenerate critical point of index 2 of a PL Morse function. However, the critical level will be a cone from v to a knotted torus in $lk(v)$. Therefore v is not a non-degenerate critical point in the sense of Definition 4.1.

Example 7.2. (H-regular point that is not strongly regular)

There are homology spheres that are not homotopy spheres. The most prominent example is the Poincaré sphere Σ^3, which admits a simplicial triangulation with only 16 vertices [8]. By removing an open 3-ball we obtain a space that is a homology point but not a homotopy point since its fundamental group does not vanish. By removing one open vertex star we find an example with 15 vertices $v_1, \ldots, v_{15}$. This simplicial complex C can be embedded into a high-dimensional combinatorial sphere S^d_k with vertices $v_1, \ldots, v_k$, $k > 15$, such that C is the full complex spanned by those 15 vertices $v_1, \ldots, v_{15}$. Then we can build a combinatorial $(d+1)$-manifold M such that the star of one vertex v_0 is this combinatorial sphere S^d_k. The simplest example seems to be the suspension $S(S^d_k)$ of this combinatorial sphere S^d_k with altogether $k+2$ vertices. Next we define a simplexwise linear function f on M in such a way that

$$f(v_1) < f(v_2) < \cdots < f(v_{15}) < f(v_0) < f(v_{16}) < f(v_{17}) < \cdots < f(v_k)$$

and with arbitrary but distinct values for all the other vertices of M. Then the vertex v_0 is H-regular for f since in the link below the level and above the level the homology is trivial. However, it is not strongly regular since in the open vertex star the sublevel of v_0 is not contractible and is therefore not an open ball. In other words: Homology is unable to detect that v_0 is a non-regular point. It behaves exactly like any of the points in the interior of a top-dimensional simplex (which of course is strongly regular).

Example 7.3. (H-regular point that is not strongly regular)

There is a $\mathbb{Z}$-acyclic but not contractible 2-dimensional simplicial complex K with 23 vertices polyhedrally embedded into a polyhedral 4-sphere [30]. This can be extended to a triangulation of the 4-sphere with additional vertices outside K such that K coincides with the full subcomplex spanned by the 23 original vertices. As in Example 7.2, one can define a generic PL function f on some PL 5-manifold such that in the link of a vertex v_0 the sublevel is spanned by those 23 vertices. Consequently $lk^-(v_0)$ is acyclic, so v_0 is H-regular for f. It is not strongly regular since $lk^-(v_0)$ is not contractible, so it cannot be a 4-ball and $f_a \cap st(v_0)$ cannot be a 5-ball.

By further embedding of K into higher dimensional spheres it follows that a regular neighborhood of K is always homologically trivial but not contractible. Consequently, for any $d \geq 5$ there is an example of a generic PL function on a PL d-manifold with a H-regular critical point that is not strongly regular. This bound is optimal by the results of Section 6.

Example 7.4. (Degenerate critical point of total multiplicity 1)

It is well known that the double suspension $S(S(\Sigma^3))$ of the Poincaré sphere Σ^3 in Example 7.2 is homeomorphic to the sphere S^5 (the so-called *Edwards sphere* [29]). However, since the link of certain edges is Σ^3, the triangulation is not combinatorial and does not induce a PL structure. Nevertheless, we can define generic PL functions adapted to this 20-vertex triangulation of $S(S(\Sigma^3))$. If this 5-sphere occurs as the link of a vertex v in a 6-manifold M, then we can find a generic PL function such that $f(v) = 0$, $f(x) < 0$ for all vertices of Σ^3 and $f(x) > 0$ for the others. Then v is a H-critical point that homologically behaves like a non-degenerate critical point of index 4 and multiplicity 1 but it is degenerate, M is not a PL manifold, so f cannot be a PL Morse function.

Example 7.5. (Critical point that cannot be detected by the Euler characteristic)

The *Bier sphere* [3] is defined as the deleted join of two copies of the 6-vertex triangulation of $\mathbb{R}P^2$. It is a combinatorial triangulation of the 4-sphere with 12 vertices. If this appears as the link of a vertex v in a 5-manifold then we can define a generic PL function f such that $lk^-(v)$ consists precisely of one of the triangulated real projective planes. Consequently, $\chi(lk^-(v)) = \chi(lk^+(v)) = 1$, and the definition in [1] cannot detect v as a critical point of f or $-f$, see the end of Section 3.

8. A special obstruction: the dunce hat

Homotopy is a stronger concept than homology. So one might hope that a vertex v is strongly regular whenever both $lk^-(v)$ and $lk^+(v)$ are contractible, so that no homotopy group would detect anything critical (one might call this *homotopically regular*). The results of Section 6 imply that this is true for generic PL functions on d-manifolds with $d \leq 4$. We are going to show that this no longer holds in dimensions $d \geq 5$.

The dunce hat is a 2-dimensional space that is known to be contractible [43]. Thus, if a triangulated dunce hat occurs as the spanning full subcomplex of $lk^-(v)$ for some generic PL function f then neither homology nor homotopy will detect that v is a critical point of f. However, v will not be strongly regular provided that $lk^-(v)$, which is a tubular neighborhood of the embedded dunce hat, is not a d-ball. If we embed the dunce hat in the 3-sphere S^3, we cannot construct a counterexample in this way because its tubular neighborhood is always a 3-ball [5]. However, there are embeddings of the dunce hat into S^4 for which a tubular neighborhood is not a 4-ball, but Mazur's contractible 4-manifold with boundary [32, 43]. We present a simple model for this situation based on the triangulation D in Figure 2, which is equivalent to the triangulation used in [5]. Any triangle contains either 1 or 8 or two vertices with consecutive labels $j, j+1$. By Gale's evenness condition [44], this implies that D can be embedded into the boundary complex of the cyclic 5-polytope $C_5(8)$ with vertices $1, 2, \ldots, 8$ in this order.

We shall investigate a tubular neighborhood M of D in the 4-dimensional boundary complex of the cyclic 5-polytope $C_5(8)$. M is contractible since the dunce hat is. This leaves the possibility that M is a 4-ball, but we will show that it is not. We constructed M using the Sage[1] mathematics software system and checked the fundamental group of its boundary ∂M. The fundamental group turned out to have a presentation with two generators

[1] `http://www.sagemath.org/`

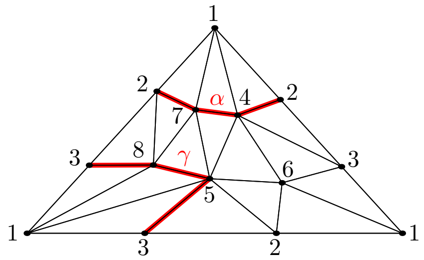

FIGURE 2. A triangulated dunce hat D, and two cycles α and γ in the link of vertex 1.

u, v and the relations $uvu^{-4}v = 1 = (v^2u^{-1}v^{-1}u^{-1})^2v$. By introducing the extra relation $u^5 = 1$ we obtain $uv = (uv)^{-1} = v^{-1}u^{-1}$ and consequently

$$u^5 = v^7 = (uv)^2 = 1.$$

This group is known to be infinite [14, Sect. 5.3]. It coincides with the group of orientation-preserving automorphisms of the regular $(7, 5)$-tessellation of the hyperbolic plane, in accordance with [32].

As an independent confirmation, Benjamin Burton (private communication) analyzed M with the REGINA software for low-dimensional topology[2]. REGINA could simplify ∂M to 9 tetrahedra. Then from the built-in census database of REGINA, ∂M was recognized as a Seifert fibred space, SFS [S2: (2,1) (5,1) (7,−5)]. In summary, the result was in both cases that the boundary ∂M of the tubular neighborhood is not a 3-sphere.

We remark that an unproved conjecture of Zeeman would allow an easier proof without computation: Zeeman [43] identifies two cycles $\alpha = 247$ and $\gamma = 358$ that are linked inside the link of vertex 1 in D (see Figure 2), and by Conjecture 3 of Zeeman [43], this would imply that M is not a 4-ball.

Corollary 8.1. *A regular neighborhood of the* 8*-vertex dunce hat above in the boundary complex of the cyclic polytope* $C_5(8)$ *is a contractible* 4*-manifold with boundary but not a* 4*-ball since its boundary is not a sphere.*

Corollary 8.2. (explicit triangulation of a contractible 4-manifold) *The second barycentric subdivision of the cyclic polytope* $C_5(8)$ *contains a triangulation of a contractible* 4*-manifold with boundary which is not a* 4*-ball.*

For the construction one just has to take the closed subcomplex of all simplices that meet the embedded dunce hat in $C_5(8)$ above. According to [4] this triangulation is not locally constructible.

Corollary 8.3. *There is a generic PL function on a* 5*-manifold with a vertex* v *that is H-regular but not strongly regular and — in addition — with the special property that both* $lk^-(v)$ *and* $lk^+(v)$ *are contractible. There are examples of this kind in every dimension* $d \geq 6$ [22][3].

For the construction we start with a combinatorial 5-manifold containing a vertex v whose link is the boundary of the cyclic polytope $C_5(8)$; a concrete example is the cyclic polytope $C_6(9)$. Then we define a generic PL

[2] `https://regina-normal.github.io/`

[3] see `https://en.wikipedia.org/wiki/Mazur_manifold`

function f on the second barycentric subdivision such that the open regular neighborhood of the embedded dunce hat lies below $f(v)$ and its open complement lies above. Then the level of v itself in $lk(v)$ is a homology sphere but not a sphere, in contrast with the characterization of Lemma 4.2.

9. Computational aspects: Is regularity decidable?

The first problem is the *manifold recognition problem*: Given a pure simplicial complex of dimension d, can we algorithmically decide whether it is the triangulation of a combinatorial manifold? More precisely, can we algorithmically decide whether all vertex links are $(d-1)$-dimensional combinatorial spheres? This is trivial for $d = 1$ and fairly easy for $d = 2$. For $d = 3$ we can decide whether a vertex link is a connected 2-manifold, and then the Euler characteristic $\chi = 2$ is a sufficient criterion for being a 2-sphere. For $d = 4$ we can first decide whether a certain vertex link is a connected 3-manifold. Then we can apply the sphere recognition algorithm of A. Mijatović [33] and obtain:

Corollary 9.1. *It is algorithmically decidable whether a given simplicial complex of dimension d is a combinatorial d-manifold whenever $d \leq 4$.*

For a generic PL function on a PL manifold it is clearly decidable whether a vertex v is H-regular: One just has to compute the integral homology of $lk^-(v)$. There are even software packages to do so. It is a much more delicate question to decide whether a vertex v is strongly regular. Since H-regularity is a sufficient criterion in low dimensions (see Section 6), we can state:

Corollary 9.2. *(i) For a PL manifold M of dimension $d \leq 4$ and a generic PL function f on M it is decidable whether a given vertex is strongly regular.*

(ii) Moreover, for $d \leq 3$, it is decidable whether f is a PL Morse function.

We believe that the decidability statement (ii) should also hold for 4-manifolds. Here, the treatment of a critical point v of index 1 (and by symmetry, index 3) is straightforward: $lk^-(v)$ consists of two homology points, and $lk^+(v)$ consists of a homology 2-sphere, embedded into $lk(v) \cong S^3$. By the argument used in Theorem 6.4, each homology point is a 3-ball, and the homology 2-sphere is a regular neighborhood of an embedded PL 2-sphere. From this situation one can reconstruct a chart with 1 direction with decreasing f and 3 directions with increasing f.

The difficult case is a critical point v of index 2 and total multiplicity 1. Both $lk^-(v)$ and $lk^+(v)$ are connected homology 1-spheres that are linked in $lk(v) \cong S^3$. In particular, the critical level $f_a \cap f^a \cap lk(v)$ is connected. Since it is a closed surface with $\chi = 0$, it is a torus. This torus may be knotted, which means that it bounds a knotted solid torus on one side, see Example 7.1. In this case, f is not a PL Morse function. While the knottedness of closed polyhedral *curves* is known to be decidable [21], we don't know whether this can be leveraged to decide whether a *torus* is unknotted.

For 5-manifolds we run into several problems: The Schoenflies problem is unsolved for embeddings of the 3-sphere into the 4-sphere, the Hauptvermutung (uniqueness of the PL structure) is unknown for the 4-sphere, and an algorithm for recognizing 4-spheres (and hence 5-manifolds) is not available.

For d-manifolds of higher dimension $d \geq 6$, we even obtain undecidability results for the recognition of strongly regular points. Novikov proved [41, 12, 31] that recognition of spheres in dimension $d \geq 5$ is undecidable. Hence the manifold recognition problem is undecidable for d-manifolds with $d \geq 6$.

Theorem 9.3. *It is undecidable whether a given vertex for a generic PL function is strongly regular*

(1) *for arbitrary simplicial d-complexes with dimension $d \geq 6$,*
(2) *for d-manifolds embedded in Euclidean d-space with $d \geq 12$.*

Proof. We perform a reduction from the sphere recognition problem. The input for this undecidable problem is a d-dimensional simplicial homology sphere K for $d \geq 5$. K is either a PL sphere, or it has a non-trivial fundamental group [31, Theorem 3.1].

(1) The cone $C(K)$ of K is a $(d+1)$-complex with an extra vertex v. Define f on $C(K)$ by setting $f(v) = 0$, $f(w) = -1$ for a single vertex w of K, and choosing distinct positive f-values for the remaining vertices. If K is a sphere, then this construction yields a strongly regular vertex v, because $lk^{-}(v)$ is a regular neighborhood of the vertex w in $lk(v) = K$, hence a ball. If K is not a sphere the vertex v is not strongly regular. Moreover, its link K witnesses that $C(K)$ is not even a manifold.

(2) We start with a 5-dimensional simplicial homology sphere K with n vertices. Remove a maximal simplex from K and denote the result by K'. Now we take the boundary sphere S' of the cyclic d-polytope $C_d(n)$ for $d \geq 12$ with n vertices. This polytope is 6-neighborly: Every 6-tuple of vertices forms a face of S', and hence we can embed K' as a subcomplex.

Subdivide the $(d-1)$-complex S' to obtain an embedding of K' as a full subcomplex. Denote the subdivided complex by S'' and the full subcomplex representing K' by K''. Take the cone on S'' with an additional vertex v in the interior of the cyclic d-polytope $C_d(n)$. The result is a d-ball embedded in d-space, with S'' being the link of v.

Define f by setting $f(v) = 0$, choosing distinct negative f-values for the vertices from K'' and distinct positive f-values for the remaining vertices. Then $lk^{-}(v)$ is a regular neighborhood of K'' embedded in S''.

If K is a sphere, then K' and consequently K'' and its regular neighborhood $lk^{-}(v)$ are balls. Hence v is a strongly regular vertex. On the other hand, if K has a non-trivial fundamental group, then, by the Seifert–van Kampen theorem, K' (and K'') has the same non-trivial fundamental group. Since K'' and $lk^{-}(v)$ are homotopy equivalent, the latter is not a ball, thus v is not strongly regular. □

The reduction for part (1) is somewhat unsatisfactory because it produces non-manifold complexes from negative instances. To get undecidability when the input is guaranteed to be a manifold, we had to go to 12 dimensions. It is conceivable that the recognition of strongly regular vertices is undecidable already for embedded 5-dimensional complexes. (This would imply that 4-sphere recognition is undecidable.)

10. Further results and extensions

10.1. **An isotopy between level sets.** The notion that the "topology does not change" as the level passes over a strongly regular vertex can be interpreted as the requirement that any two level sets in the vicinity of the vertex are homeomorphic. A stronger interpretation requires an isotopy between all level sets in a range $[a, b]$, i.e., a function

$$\phi\colon f^{-1}(a) \times [a, b] \to f^{-1}[a, b]$$

such that $f(\phi(x, t)) = t$ holds for all arguments [38]. Such an isotopy can be used for visualization, by putting some texture on the level sets in order to show the correspondence between different level sets.

In fact, such an isotopy ϕ can be constructed whenever all vertices in the interval $[a, b]$ are strongly regular, and it is piecewise linear even when considered as a function of all variables, including the interpolation parameter $t \in [a, b]$, see [19, Sect. 4.2.3, Lemma 4.13 and Theorem 4.20] or [20, Sect. 5].

From an application viewpoint, there are also quantitative aspects that play a role here. Isotopies that do not deform the level sets strongly and that use few additional vertices are preferable. Some results in this direction are given in [19, Sect. 6.2].

10.2. **Discrete Morse functions induce PL Morse functions.** Forman's *Discrete Morse Theory* [17] gives a combinatorial abstraction of Morse functions f, which associate values $f(S)$ to the *faces* S of various dimensions in a complex M. This notion classifies certain faces of M as critical cells.

Bloch [9] established a connection between such *discrete Morse functions* and PL functions. He gave a construction that starts with a generic discrete Morse function f on a combinatorial manifold M and constructs a generic PL Morse function $\hat{f}$ which is linear on cells of a derived subdivision $\hat{M}$ of M. Then he considers for each vertex in $\hat{M}$ its *index* with respect to $\hat{f}$ in the sense introduced by Banchoff [1, 2]. Bloch showed that a vertex corresponding to a non-critical cell has index 0 and a vertex corresponding to a critical cell of dimension k has index $(-1)^k$.

One can show the stronger statement that the PL function $\hat{f}$ has a nondegenerate vertex of index k (in the sense of Definitions 3.3 and 4.1) corresponding to each critical cell of f of dimension k, and all other vertices are strongly regular. For more Details see [19, Section 3.2.4] or [20, Section 7].

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