

Equivalence of symmetric union diagrams

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ABSTRACT

Motivated by the study of ribbon knots we explore symmetric unions, a beautiful construction introduced by Kinoshita and Terasaka 50 years ago. It is easy to see that every symmetric union represents a ribbon knot, but the converse is still an open problem. Besides existence it is natural to consider the question of uniqueness. In order to attack this question we extend the usual Reidemeister moves to a family of moves respecting the symmetry, and consider the symmetric equivalence thus generated. This notion being in place, we discuss several situations in which a knot can have essentially distinct symmetric union representations. We exhibit an infinite family of ribbon two-bridge knots each of which allows two different symmetric union representations.

Keywords: ribbon knot, symmetric union presentation, equivalence of knot diagrams under generalized Reidemeister moves, knots with extra structure, constrained knot diagrams and constrained moves

Mathematics Subject Classification 2000: 57M25

Dedicated to Louis H. Kauffman on the occasion of his 60th birthday

1. Motivation and background

Given a ribbon knot K , Louis Kauffman emphasized in his course notes *On knots* [8, p. 214] that “in some algebraic sense K looks like a connected sum with a mirror image. Investigate this concept.” Symmetric unions are a promising geometric counterpart of this analogy, and in continuation of Kauffman’s advice, their investigation shall be advertised here. Algebraic properties, based on a refinement of the bracket polynomial, will be the subject of a forthcoming paper. *Happy Birthday, Lou!*

1.1. Symmetric unions

In this article we consider symmetric knot diagrams and study the equivalence relation generated by symmetric Reidemeister moves. Figure 1 shows two typical examples of such diagrams. Notice that we allow any number of crossings on the axis — they necessarily break the mirror symmetry, but this defect only concerns the crossing sign and is localized on the axis alone.

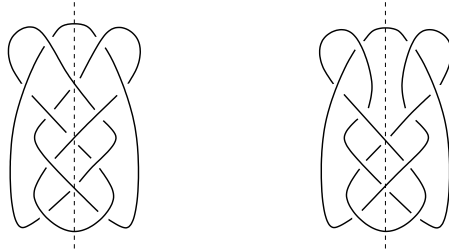


Fig. 1. Two symmetric union presentations of the knot 9_{27}

We are particularly interested in *symmetric unions*, where we require the diagram to represent a knot (that is, a one-component link) that traverses the axis in exactly two points that are not crossings. In other words, a symmetric union looks like the connected sum of a knot K_+ and its mirror image K_- , with additional crossings inserted on the symmetry axis.

Conversely, given a symmetric union, one can easily recover the two partial knots K_- and K_+ as follows: they are the knots on the left and on the right of the axis, respectively, obtained by cutting open each crossing on the axis, according to $\times \mapsto \rangle \langle$ or $\times \mapsto \rangle \langle$. The result is a connected sum, which can then be split by one final cut $\times \mapsto \rangle \langle$ to obtain the knots K_+ and K_- , as desired. (In Figure 1, for example, we find the partial knot 5_2 .)

1.2. Ribbon knots

Symmetric unions have been introduced in 1957 by Kinoshita and Terasaka [11]. Apart from their striking aesthetic appeal, symmetric unions appear naturally in the study of ribbon knots. We recall that a knot $K \subset \mathbb{R}^3$ is a *ribbon knot* if it bounds a smoothly immersed disk $\mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ whose only singularities are ribbon singularities as shown in Figure 2: two sheets intersecting in an arc whose preimage consists of a properly embedded arc in \mathbb{D}^2 and an embedded arc interior to \mathbb{D}^2 . Figure 3 displays two examples.

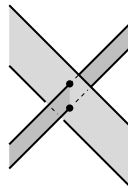


Fig. 2. An immersed disk with ribbon singularity

Put another way, a knot $K \subset \mathbb{R}^3$ is a ribbon knot if and only if it bounds a locally flat disk $\mathbb{D}^2 \hookrightarrow \mathbb{R}_+^4 = \{x \in \mathbb{R}^4 \mid x_4 \geq 0\}$ without local minima. More generally, if K bounds an arbitrary locally flat disk in \mathbb{R}_+^4 , then K is called a *slice knot*. It is a difficult open question whether every smoothly slice knot is a ribbon knot. For a general reference see [15].

For the rest of this article we will exclusively work in the smooth category.

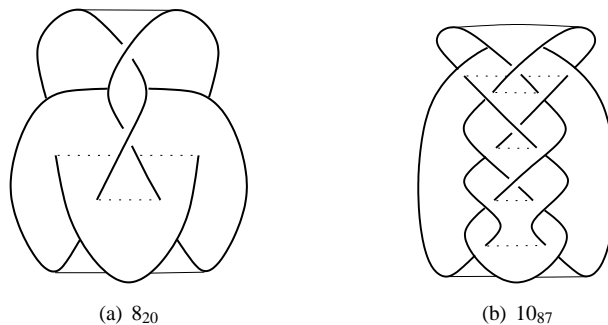


Fig. 3. The knots 8_{20} and 10_{87} represented as symmetric unions. The figure indicates the resulting symmetric ribbon with twists.

1.3. Which ribbon knots are symmetric unions?

While it is easy to see that every symmetric union represents a ribbon knot, as in Figure 3, the converse question is still open. The following partial answers are known:

- There are 21 non-trivial prime ribbon knots with at most 10 crossings. By patiently compiling an exhaustive list, Lamm [12,13] has shown that each of them can be presented as a symmetric union.
- In 1975, Casson and Gordon [4] exhibited three infinite families of two-bridge ribbon knots, and Lamm [13] has shown that each of them can be presented as a symmetric union. Recently, Lisca [14] has shown that the three Casson-Gordon families exhaust all two-bridge ribbon knots.

Remark 1.1. Presenting a given knot K as a symmetric union is one way of proving that K is a ribbon knot, and usually a rather efficient one, too. The explicit constructions presented here have mainly been a matter of patience, and it is fair to say that symmetry is a good guiding principle.

Example 1.2. When venturing to knots with 11 crossings, there still remain, at the time of writing, several knots that are possibly ribbon in the sense that their algebraic invariants do not obstruct this. It remains to explicitly construct a ribbon — or to refute this possibility by some refined argument. According to Cha and Livingston [5], as of March 2006, there remained eleven knots of which it was not known whether they were slice. Figure 4 solves this question for five of them by presenting them as symmetric unions. In the same vein, Figure 5 displays some 12-crossing knots (which all have identical partial knots).

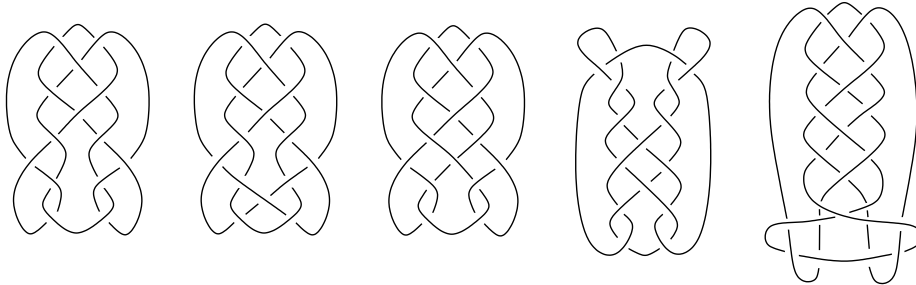


Fig. 4. Symmetric unions representing 11a28, 11a35, 11a36, 11a96, and 11a164. This proves that they are ribbon, hence smoothly slice.

Remark 1.3 (notation). For knots with up to 10 crossings we follow the traditional numbering of Rolfsen's tables [18] with the correction by Perko [17]. For knots with crossing number between 11 and 16 we use the numbering of the KnotScape library [7]. Finally, $C(a_1, a_2, \dots, a_n)$ is Conway's notation for two-bridge knots, see [10, §2.1].

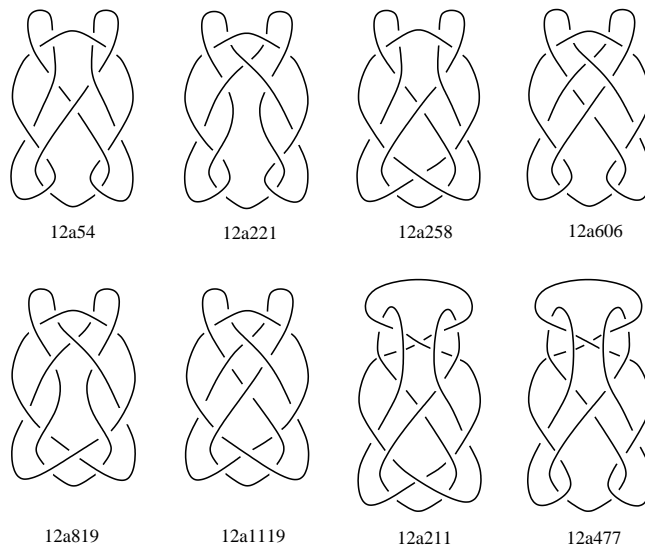


Fig. 5. Symmetric union presentations for some ribbon knots with 12 crossings, all with partial knot $C(2, 1, 1, 2)$.

Question 1.4. Can every ribbon knot be presented as a symmetric union? This would be very nice, but practical experience suggests that it is rather unlikely. A general construction, if it exists, must be very intricate.

Remark 1.5. The search for symmetric union presentations can be automated, and would constitute an interesting project at the intersection of computer science and knot theory. The idea is to produce symmetric union diagrams in a systematic yet efficient way, and then to apply KnotScape to identify the resulting knot. Roughly speaking, the first step is easy but the second usually takes a short while and could turn out to be too time-consuming. Library look-up should thus be used with care, and the production of candidates should avoid duplications as efficiently as possible. (This is the non-trivial part of the programming project.)

In this way one could hope to find symmetric union presentations for all remaining 11-crossing knots, and for many knots with higher crossing numbers as well. Such a census will yield further evidence of how large the family of symmetric unions is within the class of ribbon knots — and possibly exhibit ribbon knots that defy symmetrization. No such examples are known at the time of writing. Of course, once a candidate is at hand, a suitable obstruction has to be identified in order to prove that it cannot be represented as a symmetric union. (This is the non-trivial mathematical part.)

1.4. Symmetric equivalence

Besides the problem of *existence* it is natural to consider the question of *uniqueness* of symmetric union representations. Motivated by the task of tabulating symmetric union diagrams for ribbon knots, we are led to ask when two such diagrams should be regarded as equivalent. One way to answer this question is to extend the usual Reidemeister moves to a family of moves respecting the symmetry, as explained in §2.1.

Example 1.6. It may well be that two symmetric union representations are equivalent (via the usual Reidemeister moves), but that such an equivalence is not symmetric, that is, the transformation cannot be performed in a symmetric way. One possible cause for this phenomenon is the existence of *two* axes of symmetry. The simplest (prime) example of this type seems to be the knot $16n524794$ shown in Figure 6: the symmetric unions have partial knots 6_1 and 8_{20} , respectively, and thus cannot be symmetrically equivalent.

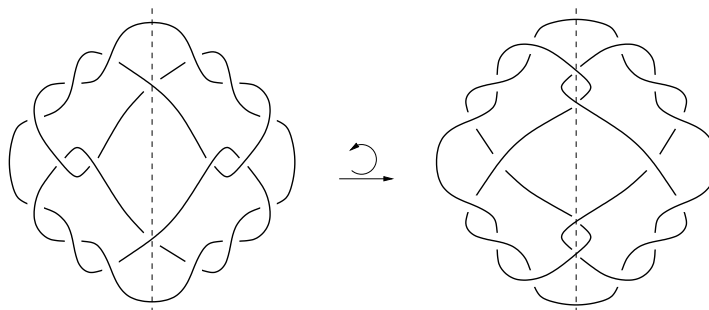


Fig. 6. A symmetric union with two symmetry axes

Remark 1.7. For a symmetric union representing a knot K with partial knots K_+ and K_- the determinant satisfies the product formula $\det(K) = \det(K_+) \det(K_-)$ and is thus a square. This was already noticed by Kinoshita and Terasaka [11] in the special case that they considered; for the general case see [12]. For a symmetric union with two symmetry axes this means that the determinant is necessarily a fourth power.

Example 1.8. It is easy to see that symmetric Reidemeister moves do not change the partial knots (see §2.3). Consequently, if a knot K can be represented by two symmetric unions with distinct pairs of partial knots, then the two representations cannot be equivalent under symmetric Reidemeister moves. Two examples of this type are depicted in Figure 7. The smallest known examples are the knots 8_8 and 8_9 : for each of them we found two symmetric unions with partial knots 4_1 and 5_1 , respectively. This shows that partial knots need not be unique even for the class of two-bridge ribbon knots.

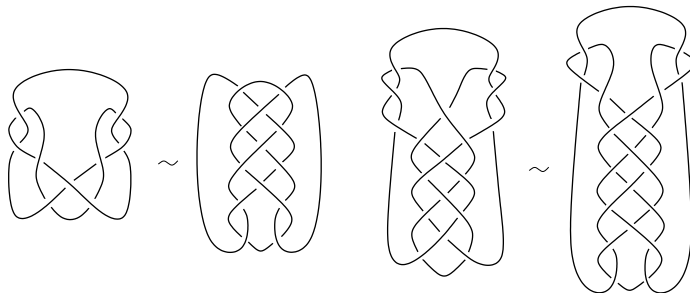


Fig. 7. Two symmetric union presentations of the knot 8_8 with partial knots 4_1 and 5_1 , respectively, and the knot $12a3$ with distinct partial knots $C(3,1,2)$ and $C(2,5)$.

Partial knots form an obvious obstruction for symmetric equivalence, but there also exist examples of symmetric union representations with the same partial knots, but which are not symmetrically equivalent. Figure 8 shows a transformation between the two symmetric union representations of the knot 9_{27} displayed in Figure 1 at the beginning of this article.

Theorem 1.9. *The two symmetric union diagrams shown in Figure 1 both represent the knot 9_{27} and both have 5_2 as partial knot. They are, however, not equivalent under symmetric Reidemeister moves as defined in §2. \square*

While some experimentation might convince you that this result is plausible, it is not so easy to prove. We will give the proof in a forthcoming article [6], based on a two-variable refinement of the Jones polynomial for symmetric unions. The diagrams displayed here are the first pair of an infinite family of two-bridge knots exhibited in §3.

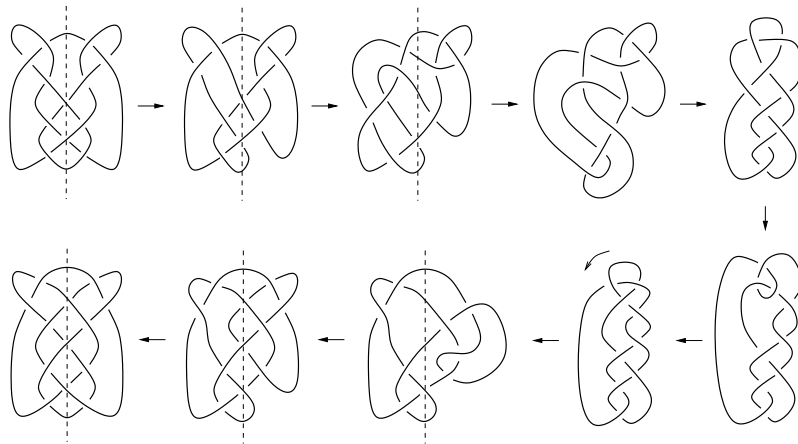


Fig. 8. A transformation between two symmetric union representations of the knot 9_{27} . The intermediate stages are not symmetric.

1.5. Knots with extra structure

The study of symmetric diagrams and symmetric equivalence is meaningful also for other types of symmetries, or even more general constraints. It can thus be seen as an instance of a very general principle, which could be called *knots with extra structure*, and which seems worthwhile to be made explicit.

Generally speaking, we are given a class of diagrams satisfying some constraint and a set of (generalized) Reidemeister moves respecting the constraint. It is then a natural question to ask whether the equivalence classes under constrained moves are strictly smaller than those under usual Reidemeister moves (ignoring the constraint, e.g. breaking the symmetry). If this is the case then two opposing interpretations are possible:

- (a) We might have missed some natural but less obvious move that respects the constraint. Such a move should be included to complete our list.
- (b) The constraint introduces some substantial obstructions that cannot be easily circumvented. The induced equivalence is an object in its own right.

In order to illustrate the point, let us cite some prominent examples, which have developed out of certain quite natural constraints.

- Perhaps the most classical example of diagrams and moves under constraints is provided by alternating diagrams and Tait's flype moves, cf. [16].
- Braids form another important and intensely studied case. Here one considers link diagrams in the form of a closed braid and Markov moves, cf. [1,2].

In these two settings the fundamental result is that constrained moves generate the same equivalence as unconstrained moves. In the following two examples, however, new classes of knots have emerged:

- Given a contact structure, one can consider knots that are everywhere transverse (resp. tangent) to the plane field, thus defining the class of transverse (resp. legendrian) knots. Again one can define equivalence by isotopies respecting this constraint, and it is a natural question to what extent this equivalence is a refinement of the usual equivalence, cf. [3].
- Virtual knots can also be placed in this context: here one introduces a new type of crossing, called virtual crossing, and allows suitably generalized Reidemeister moves, cf. [9]. Strictly speaking, this is an extension rather than a constraint, and classical knots inject into the larger class of virtual knots.

Considering symmetric unions, two nearby generalizations also seem promising:

- Analogous to diagrams that are symmetric with respect to reflection, one can consider strongly amphichiral diagrams. Here the symmetry is a rotation of 180° about a point, which maps the diagram to itself reversing all crossings. Again there are some obvious moves respecting the symmetry, leading to a natural equivalence relation on the set of strongly amphichiral diagrams.
- Since ribbon knots are in general not known to be representable as symmetric unions, one could consider band presentations of ribbon knots and Reidemeister moves respecting the band presentation. The equivalence classes will thus correspond to ribbons modulo isotopy. For a given knot K the existence and uniqueness questions can be subsumed by asking how many ribbons are there for K .

Of course, the paradigm of “knots with extra structure” cannot be expected to produce any general answers; the questions are too diverse and often rather deep. Nevertheless, we think of it as a good generic starting point and a unifying perspective. Its main merit is that it leads to interesting questions. In the present article we will begin investigating this approach in the special case of symmetric unions.

2. Symmetric diagrams and symmetric moves

Having seen some examples of symmetric unions that are equivalent by asymmetric Reidemeister moves, we wish to make precise what we mean by *symmetric equivalence*. As can be suspected, this will be the equivalence relation generated by symmetric Reidemeister moves, but the details require some attention.

2.1. Symmetric Reidemeister moves

We consider the euclidian plane \mathbb{R}^2 with the reflection $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto (-x, y)$. The map ρ reverses orientation and its fix-point set is the vertical axis $\{0\} \times \mathbb{R}$. A link diagram $D \subset \mathbb{R}^2$ is *symmetric* with respect to this axis if and only if $\rho(D) = D$ except for crossings on the axis, which are necessarily reversed.

By convention we will not distinguish two symmetric diagrams D and D' if they differ only by an orientation preserving diffeomorphism $h: \mathbb{R}^2 \xrightarrow{\sim} \mathbb{R}^2$ respecting the symmetry, in the sense that $h \circ \rho = \rho \circ h$.

Definition 2.1. Given a symmetric diagram, a *symmetric Reidemeister move* with respect to the reflection ρ is a move of the following type:

- A symmetric Reidemeister move off the axis, that is, an ordinary Reidemeister move, R1–R3 as depicted in Figure 9, carried out simultaneously with its mirror-symmetric counterpart with respect to the reflection ρ .
- A symmetric Reidemeister move on the axis, of type S1–S3 as depicted in Figure 10, or a generalized Reidemeister move on the axis, of type S2(\pm) as depicted in Figure 11, or of type S4 as depicted in Figure 12.

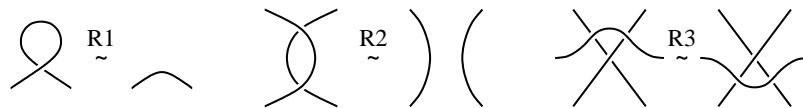


Fig. 9. The classical Reidemeister moves (off the axis)

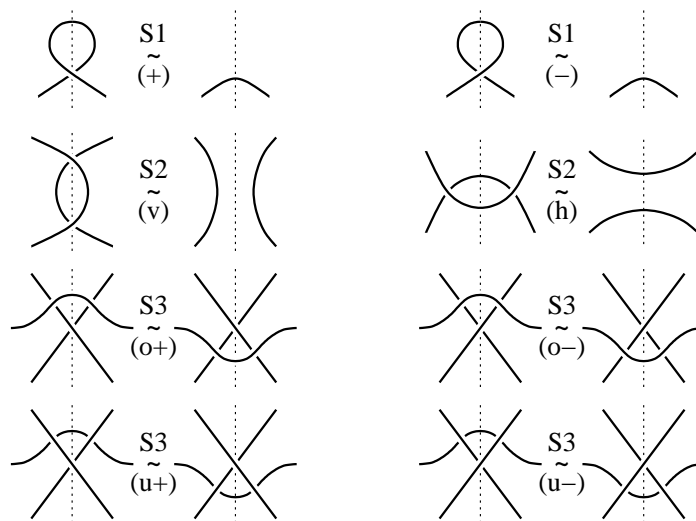


Fig. 10. Symmetric Reidemeister moves on the axis

Remark 2.2. By convention the axis is not oriented in the local pictures, so that we can turn Figures 10, 11, 12 upside-down. This adds one variant for each S1-, S2-, and S4-move shown here; the four S3-moves are invariant under this rotation. Moreover, the S4-move comes in four variants, obtained by changing the over- and under-crossings on the axis.



Fig. 11. A symmetric move on two strands with three crossings

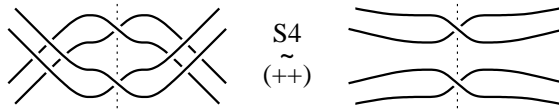


Fig. 12. A symmetric move on four strands across the axis

2.2. Are these moves necessary?

The emergence of the somewhat unusual moves $S2(\pm)$ and $S4$ may be surprising at first sight. One might wonder whether they are necessary or already generated by the other, simpler moves:

Theorem 2.3. *The four oriented link diagrams shown in Figure 13 all represent the Hopf link with linking number +1. The pairs $D_1 \sim D_2$ and $D_3 \sim D_4$ are equivalent via symmetric Reidemeister moves, but D_1, D_2 are only asymmetrically equivalent to D_3, D_4 . Moreover, the symmetric equivalence $D_1 \sim D_2$ cannot be established without using $S2(\pm)$ -moves, and the symmetric equivalence $D_3 \sim D_4$ cannot be established without using $S4$ -moves.*

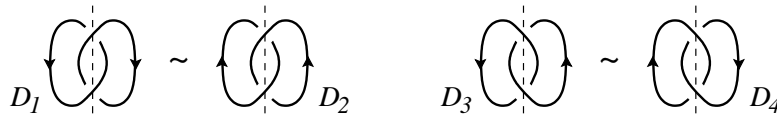


Fig. 13. $S2(\pm)$ -moves and $S4$ -moves are necessary.

Proof. The symmetric equivalences $D_1 \sim D_2$ and $D_3 \sim D_4$ are easily established, and will be left as an amusing exercise. Less obvious is the necessity of moves $S2(\pm)$ and $S4$. Likewise, D_2 and D_3 are asymmetrically equivalent, but we need an obstruction to show that they cannot be symmetrically equivalent. Given an oriented diagram D , we consider the points on the axis where two distinct components cross. To each such crossing we associate an element in the free group $F = \langle s, t, u, v \rangle$ as follows:

$$\begin{array}{cccc}
 \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \mapsto s^{+1} & \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} \mapsto t^{+1} & \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} \mapsto u^{+1} & \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} \mapsto v^{+1} \\
 \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} \mapsto s^{-1} & \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} \mapsto t^{-1} & \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} \mapsto u^{-1} & \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} \mapsto v^{-1}
 \end{array}$$

Traversing the axis from top to bottom we read a word on the alphabet $\{s^\pm, t^\pm, u^\pm, v^\pm\}$, which defines an element $w(D) \in F$. It is an easy matter to verify how symmetric Reidemeister moves affect $w(D)$. Moves off the axis have no influence. S1-moves are neglected by construction. S2(v)-moves change the word by a trivial relation so that $w(D) \in F$ remains unchanged. S2(h)-moves and S3-moves have no influence. An S2(\pm)-move can change one factor $u \leftrightarrow v$, but leaves factors s and t unchanged. An S4-move, finally, interchanges two adjacent factors.

In our example we have $w(D_1) = u^2$ and $w(D_2) = v^2$, so at least two S2(\pm)-moves are necessary in the transformation. Furthermore, $w(D_3) = st$ and $w(D_4) = ts$, so at least one S4-move is necessary in the transformation. Finally, no symmetric transformation can change D_1 or D_2 into D_3 or D_4 . \square

Remark 2.4 (orientation). One might object that the preceding proof introduces the orientation of strands as an artificial subtlety. Denoting for each oriented diagram D the underlying unoriented diagram by \bar{D} , we see that $\bar{D}_1 = \bar{D}_2$ and $\bar{D}_3 = \bar{D}_4$ are identical unoriented diagrams. Orientations obviously simplify the argument, but it is worth noting that the phenomenon persists for unoriented knot diagrams as well:

Corollary 2.5. *The unoriented diagrams \bar{D}_2 and \bar{D}_3 are not symmetrically equivalent.*

Proof. If \bar{D}_2 were symmetrically equivalent to \bar{D}_3 , then we could equip \bar{D}_2 with an orientation, say D_2 , and carry it along the transformation to end up with some orientation for \bar{D}_3 . Since the linking number must be $+1$, we necessarily obtain D_3 or D_4 . But $w(D_2) = v^2$ can be transformed neither into $w(D_3) = st$ nor $w(D_4) = ts$. This is a contradiction. \square

Corollary 2.6. *The moves S2(\pm) and S4 are also necessary for the symmetric equivalence of unoriented diagrams.*

Proof. The following trick allows us to apply the above argument to unoriented links: we take the diagrams of Figure 13 and tie a non-invertible knot into each component, symmetrically on the left and on the right. This introduces an intrinsic orientation. \square

Remark 2.7 (linking numbers). The proof of the theorem shows that the composition $\bar{w}: \{\text{oriented diagrams}\} \rightarrow F \rightarrow \mathbb{Z}^3$ defined by $s \mapsto (1, 0, 0)$, $t \mapsto (0, 1, 0)$, $u, v \mapsto (0, 0, 1)$ is invariant under *all* symmetric Reidemeister moves. For example $\bar{w}(D_1) = \bar{w}(D_2) = (0, 0, 2)$ and $\bar{w}(D_3) = \bar{w}(D_4) = (1, 1, 0)$ yields the obstruction to symmetric equivalence used above. The invariant \bar{w} can be interpreted as a refined *linking number* for crossings on the axis. This already indicates that the symmetry constraint may have surprising consequences.

Remark 2.8 (symmetric unions). While refined linking numbers may be useful for symmetric diagrams in general, such invariants become useless when applied to *symmetric unions*, which are our main interest. In this more restrictive setting we only have *one* component. When trying to imitate the above construction, S1-moves force the relation $s = t = 1$. Moreover, orientations are such that a crossing on the axis always points “left” or “right” but never “up” or “down”, so factors u^\pm and v^\pm never occur.

2.3. Invariance of partial knots

Recall that for every symmetric union diagram D we can define partial diagrams D_- and D_+ as follows: first, we resolve each crossing on the axis by cutting it open according to $\times \mapsto \rangle($ or $\times \mapsto \rangle($. The result is a diagram \hat{D} without any crossings on the axis. If we suppose that D is a symmetric union, then \hat{D} is a connected sum, which can then be split by a final cut $\times \mapsto \rangle($. We thus obtain two disjoint diagrams: D_- in the half-space $H_- = \{(x,y) \mid x < 0\}$, and D_+ in the half-space $H_+ = \{(x,y) \mid x > 0\}$. The knots K_- and K_+ represented by D_- and D_+ , respectively, are called the *partial knots* of D . Since D was assumed symmetric, K_+ and K_- are mirror images of each other.

Proposition 2.9. *For every symmetric union diagram D the partial knots K_- and K_+ are invariant under symmetric Reidemeister moves.*

Proof. This is easily seen by a straightforward case-by-case verification. □

2.4. Horizontal and vertical flypes

The symmetric Reidemeister moves displayed above give a satisfactory answer to the local equivalence question. There are also some semi-local moves that merit attention, most notably flype moves.

Proposition 2.10. *Every horizontal flype across the axis, as depicted in Figure 14, can be decomposed into a finite sequence of symmetric Reidemeister moves.* □

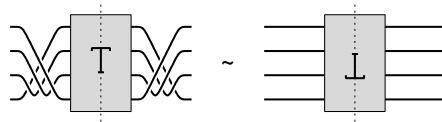


Fig. 14. A horizontal flype (across the axis)

Definition 2.11. *A vertical flype along the axis is a move as depicted in Figure 15, where the tangle F can contain an arbitrary diagram that is symmetric with respect to the axis.*

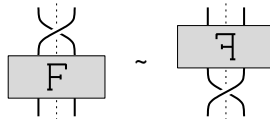


Fig. 15. A vertical flype (along the axis)

Example 2.12. Strictly speaking a flype is not a local move, because the tangle F can contain an arbitrarily complicated diagram. Such a flype allows us, for example, to realize a rotation of the entire diagram around the axis, as depicted in Figure 16.

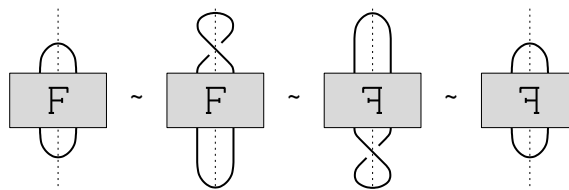


Fig. 16. A flype rotating the entire diagram

While a horizontal flype can be achieved by symmetric Reidemeister moves, this is in general not possible for a vertical flype: when decomposed into Reidemeister moves, the intermediate stages are in general no longer symmetric. This is also manifested in the following observation:

Proposition 2.13. *A vertical flype changes the partial knots in a well-controlled way, from $K_- \# L_-$ and $K_+ \# L_+$ to $K_- \# L_+$ and $K_+ \# L_-$, where (K_-, K_+) and (L_-, L_+) are pairs of mirror images. In general this cannot be realized by symmetric Reidemeister moves. \square*

2.5. Connected sum

As a test-case for symmetric equivalence, we wish to construct a connected sum for symmetric unions and show that it shares some properties with the usual connected sum. This is by no means obvious, and the first problem will be the very definition: is the connected sum well-defined on equivalence classes? The fact that the answer is affirmative can be seen as a confirmation of our chosen set of Reidemeister moves.

In order to define a connected sum of diagrams we have to specify which strands will be joined. To this end we consider pointed diagrams as follows.

Definition 2.14. Each symmetric union diagram D traverses the axis at exactly two points that are not crossings. We mark one of them as the basepoint of D . The result will be called a *pointed diagram*. Given two symmetric union diagrams D and D' that are pointed and oriented, we can define their *connected sum* $D \# D'$ as indicated in Figure 17. The result is again a symmetric union diagram that is pointed and oriented.

More explicitly, we start with the distant union $D \sqcup D'$ by putting D above D' along the axis. Both diagrams intersect the axis transversally at two points each. We choose the unmarked traversal of D and the marked traversal of D' and join the strands to form the connected sum. If other strands (with crossings on the axis) are between them, we pass over all of them by convention. If the orientations do not match, we perform an $S1+$ move on one of the strands before joining them.

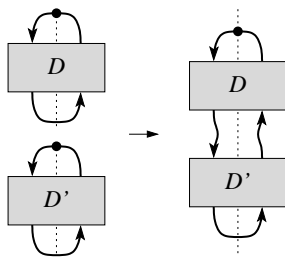


Fig. 17. Connected sum $D \# D'$ of two symmetric union diagrams D and D' , each equipped with a basepoint and an orientation

All symmetric moves discussed previously generalize to pointed diagrams: in each instance the basepoint is transported in the obvious way. The upshot is the following result:

Theorem 2.15. *The connected sum induces a well-defined operation on equivalence classes modulo symmetric Reidemeister moves and flypes. More explicitly, this means that $D_1 \sim D_2$ and $D'_1 \sim D'_2$ imply $D_1 \# D'_1 \sim D_2 \# D'_2$. The connected sum operation is associative and has the class of the trivial diagram as two-sided unit element.*

Proof. Consider a symmetric Reidemeister move performed on the diagram D' . If the basepoint is not concerned then the same move can be carried out in $D \# D'$. Only two special cases need clarification: an S1-move (or more generally a flype move) on D' affecting the basepoint translates into a flype move on $D \# D'$. An S3-move on D' affecting the basepoint can be translated to a sequence of symmetric Reidemeister moves on $D \# D'$. The situation is analogous for D concerning the unmarked traversal of the axis; the verifications are straightforward. \square

2.6. Open questions

The connected sum of symmetric unions, as defined above, is associative but presumably not commutative. The usual trick is to shrink D' and to slide it along D so as to move from $D \# D'$ to $D' \# D$, but this transformation is not compatible with our symmetry constraint. Even though non-commutativity is a plausible consequence, this does not seem easy to prove.

Question 2.16. Is the connected sum operation on symmetric unions non-commutative, as it seems plausible? How can we prove it? Does this mean that we have missed some less obvious but natural move? Or is it an essential feature of symmetric unions?

Remark 2.17. On the one hand non-commutativity may come as a surprise for a connected sum operation of knots. On the other hand, the connected sum of symmetric unions is halfway between knots and two-string tangles, and the latter are highly non-commutative. The theory of symmetric unions retains some of this two-string behaviour.

Although only loosely related, we should also like to point out that similar phenomena appear for virtual knots. There the connected sum is well-defined only for long knots, corresponding to a suitable marking how to join strands. Moreover, the connected sum for long virtual knots is not commutative [19].

Question 2.18. Symmetric unions of the form $K_+ \# K_-$ belong to the centre: the usual trick of shrinking and sliding K_{\pm} along the strand still works in the symmetric setting. Are there any other central elements?

Question 2.19. What are the invertible elements, satisfying $D \# D' = \textcircled{1}$? An invertible symmetric union diagram necessarily represents the unknot. It is not clear, however, if it is equivalent to the unknot by symmetric moves.

Question 2.20. Do we have unique decomposition into prime elements?

Question 2.21. Some geometric invariants such as bridge index, braid index, and genus, can be generalized to the setting of symmetric unions, leading to the corresponding notions of symmetric bridge index, symmetric braid index, and symmetric genus. Do they have similar properties as in the classical case, i.e. is the unknot detected and does connected sum translate into addition?

Remark 2.22. If we had some additive invariant v and a non-trivial symmetric union representation U of the unknot with $v(U) > 0$, then every symmetric union diagram D would yield an infinite family $D \# U^{\#k}$ of distinct diagrams representing the same knot.

3. Inequivalent symmetric union representations

3.1. An infinite family

In this section we exhibit an infinite family of symmetric unions which extend the phenomenon observed for the diagrams of 9₂₇. Notice that we will be dealing with prime knots, so this non-uniqueness phenomenon is essentially different from the non-uniqueness caused by the non-commutativity of the connected sum operation.

Definition 3.1. For each integer $n \geq 2$ we define two symmetric union diagrams $D_1(n)$ and $D_2(n)$ as follows. We begin with the connected sum $C(2, n) \# C(2, n)^*$ and insert crossings on the axis as indicated in Fig. 18, distinguishing the odd case $n = 2k + 1$ and the even case $n = 2k$.

Theorem 3.2. For each $n \geq 2$ the diagrams $D_1(n)$ and $D_2(n)$ can be transformed one into another by a sequence of Reidemeister moves, not respecting the symmetry:

$$D_1(n) \sim D_2(n) \sim \begin{cases} S((2n+1)^2, 2n^2) & \text{if } n \text{ is odd,} \\ S((2n+1)^2, 2n^2 - 1) & \text{if } n \text{ is even.} \end{cases}$$

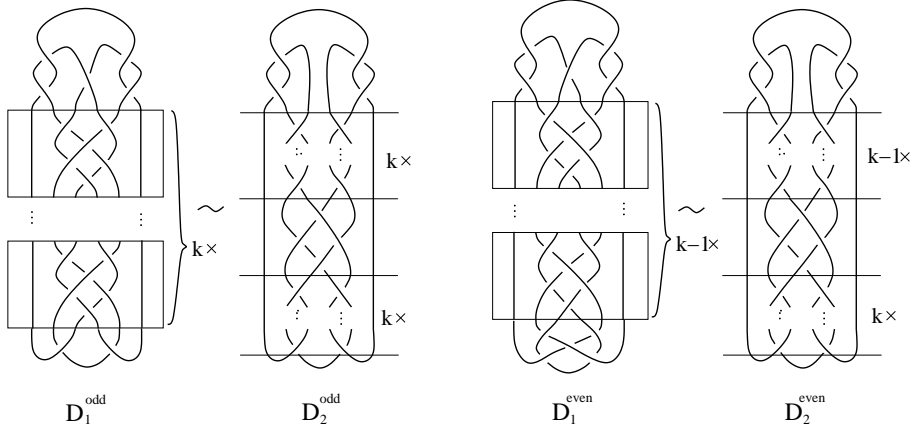


Fig. 18. Statement of the theorem

Here $S(p, q)$ is Schubert's notation for two-bridge knots, see [10, §2.1].

Example 3.3. For $n = 2$ we obtain two mirror-symmetric diagrams $D_1(2)$ and $D_2(2)$ of the knot 8_9 , which turn out to be symmetrically equivalent. For $n = 3$ we obtain the two symmetric union representations of 9_{27} depicted in Fig. 1.

These and the following cases yield two symmetric union representations of the two-bridge knots $K(a, b) = C(2a, 2, 2b, -2, -2a, 2b)$ with $b = \pm 1$, up to mirror images: more explicitly, we so obtain the knots $8_9 = K(-1, -1)$ for $n = 2$, $9_{27} = K(-1, 1)$ for $n = 3$, $10_{42} = K(1, 1)$ for $n = 4$, $11a_{96} = K(1, -1)$ for $n = 5$, $12a_{715} = K(-2, -1)$ for $n = 6$, $13a_{2836} = K(-2, 1)$ for $n = 7$. They all have genus 3 and their crossing number is $6 + n$.

After some experimentation you might find it plausible that $D_1(n)$ and $D_2(n)$ are not symmetrically equivalent for $n \geq 3$. Notice, however, that the obvious obstruction fails: by construction, both have the same partial knots $C(2, n)$ and $C(2, n)^*$. Their non-equivalence will be studied in [6] where we develop the necessary tools.

Proof. We first analyze the braid β_1 that is shown boxed in diagram D_1^{odd} . Using the braid relations we have

$$\begin{aligned} \beta_1 &= \sigma_2^{-1} \sigma_4 \sigma_3^{-1} \sigma_2^{-1} \sigma_4 \sigma_3 = \sigma_4 \sigma_2^{-1} \sigma_3^{-1} \sigma_2^{-1} \sigma_4 \sigma_3 \\ &= \sigma_4 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} \sigma_4 \sigma_3 = \sigma_4 \sigma_3^{-1} \sigma_2^{-1} \sigma_4 \sigma_3 \sigma_4^{-1} \end{aligned}$$

Therefore $\beta_1^k = \sigma_4 \sigma_3^{-1} \sigma_2^{-k} \sigma_4^k \sigma_3 \sigma_4^{-1}$. For the braid β_2 , shown boxed in diagram D_1^{even} , we have similarly

$$\beta_2 = \sigma_2^{-1} \sigma_4 \sigma_3 \sigma_2^{-1} \sigma_4 \sigma_3^{-1} = \sigma_2^{-1} \sigma_3 \sigma_2^{-1} \sigma_4 \sigma_3^{-1} \sigma_2$$

and $\beta_2^k = \sigma_2^{-1} \sigma_3 \sigma_2^{-k} \sigma_4^k \sigma_3^{-1} \sigma_2$. With this information at hand, we pursue the odd and even cases separately.

First case: n is odd. The simplification of D_1^{odd} done by computing β_1^k is shown in diagram D_1^{odd} in Fig. 19. This diagram can be further transformed, yielding diagram D_1^{odd} which is in two-bridge form. Its Conway notation is $C(2, k, 2, 1, 2, -k - 1)$.

Diagram D_2^{odd} in Fig. 18 simplifies to D_2^{odd} in Fig. 19 because certain crossings are cancelled. Further transformation gives its two-bridge form, shown in diagram D_2^{odd} . Its Conway notation is $C(2, k, 2, 2, -2, -k)$. The continued fractions for both knots evaluate to $\frac{(4k+3)^2}{8k^2+8k+2} = \frac{(2n+1)^2}{2n^2}$, so both knots are equal.

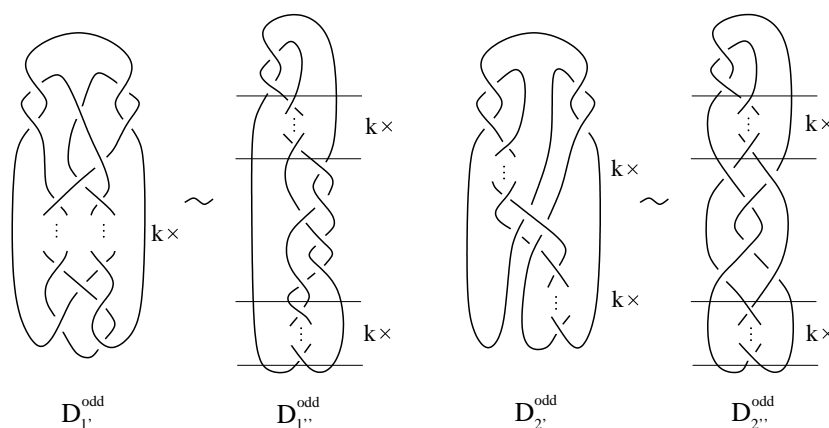


Fig. 19. Proof in the odd case

Second case: n is even. We simplify the braid $\beta_2^{k-1} \sigma_2^{-1} \sigma_4 \sigma_3 \sigma_2^{-1} \sigma_4$ occurring in diagram D_1^{even} of Fig. 18: using the formula for β_2^{k-1} and applying braid relations we get

$$\sigma_2^{-1} \sigma_3 (\sigma_2^{-1} \sigma_4)^{k-1} \sigma_4 \sigma_3 \sigma_2^{-1}$$

which is depicted in diagram D_1^{even} in Fig. 20.

The transformation to two-bridge form is similar to the odd case and we get the knot $C(2, k, -2, -1, -2, -k)$ shown in diagram D_1^{even} . The simplification of diagram D_2^{even} in Fig. 18 to diagram D_2^{even} in Fig. 20 is straightforward, and diagram D_2^{even} allows us to read off its two-bridge form $C(2, k - 1, 2, -2, -2, -k)$. The continued fractions for both knots evaluate to $\frac{(4k+1)^2}{8k^2-1} = \frac{(2n+1)^2}{2n^2-1}$. \square

3.2. Open questions

As we have seen, certain ribbon knots have more than one symmetric representation. We have not succeeded in finding such an ambiguity for the two smallest ribbon knots:

Question 3.4. Can the unknot be represented by symmetric union diagrams belonging to more than one equivalence class? It is known that the partial knots of a symmetric union representation of the unknot are necessarily trivial, see [12, Theorem 3.5].

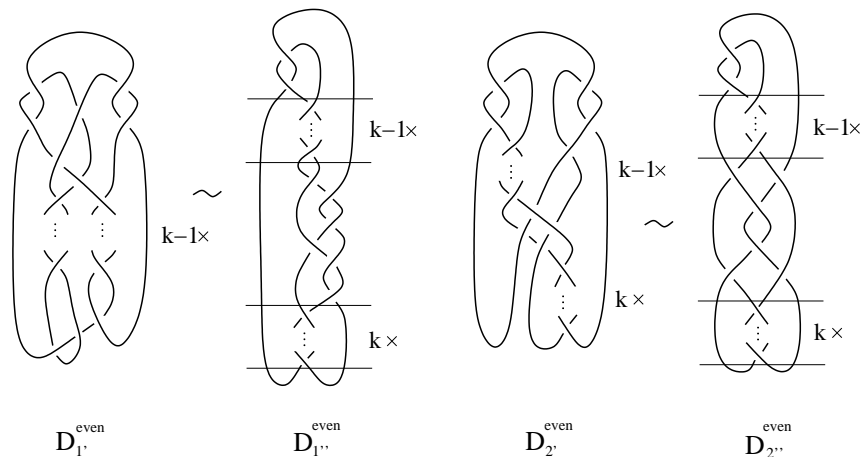


Fig. 20. Proof in the even case

Question 3.5. Can the knot 6_1 be represented by symmetric union diagrams belonging to more than one equivalence class?

Question 3.6. Is the number of equivalence classes of symmetric unions representing a given knot K always finite? Does non-uniqueness have some geometric meaning? For example, do the associated ribbon bands differ in some essential way?

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