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Abstract Reduced basis methods allow efficient model reduction of parametrized partial differential equations. In the current paper, we consider a reduced basis scheme for homogeneous domain decomposition problems. The method is based on iterative Dirichlet-Neumann coupling. We prove convergence of the iterative reduced scheme, derive rigorous a-posteriori error bounds and provide a full offline/online decomposition. Different methods for basis generation are investigated, in particular a variant of the POD-Greedy procedure. Experiments confirm the rigor of the error estimators and identify beneficial basis construction procedures.

Keywords reduced basis method · model reduction, domain decomposition · iterative methods · Dirichlet-Neumann coupling

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AN ITERATIVE DOMAIN DECOMPOSITION PROCEDURE FOR THE REDUCED BASIS METHOD

I. MAIER[†] AND B. HAASDONK[†]

Abstract. Reduced basis methods allow efficient model reduction of parametrized partial differential equations. In the current paper, we consider a reduced basis scheme for homogeneous domain decomposition problems. The method is based on iterative Dirichlet-Neumann coupling. We prove convergence of the iterative reduced scheme, derive rigorous a-posteriori error bounds and provide a full offline/online decomposition. Different methods for basis generation are investigated, in particular a variant of the POD-Greedy procedure. Experiments confirm the rigor of the error estimators and identify beneficial basis construction procedures.

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1. Introduction. The reduced basis (RB) method has become a powerful approach for a fast and reliable treatment of parametrized partial differential equations (PDEs) in the last decades. Whenever solutions of such PDEs have to be approximated for plenty of parameters — the so called multi-query context — or simulations have to be done in real-time, the runtime demands of precise approximation methods like Finite Elements or Finite Volumes is a big drawback. The RB method dissolves this problem by precomputing some snapshots of the solution manifold to generate a low dimensional approximation space. Subsequently, a reduced solution can be found rapidly in this low-dimensional space. The resulting reduced models can then be used in various settings, e.g. optimization with PDE-constraints [15], parameter studies or smartphone applications.

RB methods, which already emerged in the last century [1], have been studied for both time independent, e.g. [16, 18], and time dependent problems such as [5, 7]. When computing a basis for the low dimensional reduced space, one has to choose sample parameters. This is mostly done in an adaptive manner with the Greedy-algorithm [20]. For this procedure the a-posteriori error estimation is an essential tool, which can be very effective and often completely relies on known and computable quantites. A counterpart for time dependent problems is the so called POD-Greedy method [7] and recently there has been much investigation in theoretical convergence analysis for the Greedy-algorithms [3, 2, 6].

The idea to combine RB methods with domain decomposition techniques led to the RB element method [10, 11, 12]. In a new approach [8] the RB element method deals with reference components, which are connected via suitable ports. Especially, seperate constructions for basis functions on interfaces and the domain interior are proposed. Domain decomposition techniques in general [17, 19] are motivated by different purposes, such as parallelization of numerical simulations by assigning subdomains to different cores, or multiphysics, where different PDEs have to be solved on neighbouring domains. The resulting schemes are mostly of iterative nature.

In the current paper we develop an iterative RB method based on initial results of [13]. We will use the Dirichlet-Neumann method, which is well known for Finite Elements [14]. In contrast to [8] our RB scheme is exactly following the iterative nature of the detailed scheme, allowing new analytical statements as a basis for solutions to

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more complex problems. We propose suitable assumptions on the RB construction, which naturally arise from this context. In investigating different methods for basis generation, it is shown that these assumptions are realistic. To obtain computational efficiency we state a full offline/online decomposition of our procedure. The offline stage is parameter independent and involves all computations of high complexity. This allows for a very fast, parameter dependent online stage and is based on the affine parameter dependency of the data.

The structure of the paper is as follows: In Section 2 the neccessary notation and the problem formulation are specified. In Section 3 a full approximation scheme consisting of a detailed and a reduced procedure is presented. Results of analytical investigations follow in Section 4 and detailed considerations on computational aspects in Section 5. To complete the picture we present our numerical experiments in Section 6 and finally conclude with some closing remarks. In order to maintain readability of the main text, some proofs of analytical statements have been shifted to an appendix.

2. Notation. Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz–boundary $\partial\Omega$ and $x \in \overline{\Omega}$ the space variable. Let $\mathcal{P} \subset \mathbb{R}^P$, $P \in \mathbb{N}$ be the domain of the parameter $\mu \in \mathcal{P}$. We introduce a Hilbert space $X \subset H_0^1(\Omega)$ with the norm $\|v\|_X := \|v\|_{H^1(\Omega)}$ which can be either finite or infinite dimensional. We now consider a decomposition of Ω into 2 subdomains, i.e. $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The interface Γ is defined as $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$. We assume that Ω_1 and Ω_2 have Lipschitz–boundaries and that Γ, $\partial\Omega_1 \setminus \Gamma$ and $\partial\Omega_2 \setminus \Gamma$ have a nonvanishing $(n-1)$-dimensional measure. Several function spaces are defined according to the domain decomposition.

$$
\begin{aligned}
X_k &:= \left\{ v_{|\Omega_k} | v \in X \right\}, \\
X_k^0 &:= \left\{ v \in X_k | \gamma v = 0 \right\}, \\
X_\Gamma &:= \gamma(X_1) = \gamma(X_2), \\
\hat{X} &:= \left\{ v \in L^2(\Omega) | v_{|\Omega_1} \in X_1, v_{|\Omega_2} \in X_2 \right\},
\end{aligned}
$$

where $k = 1, 2$. The operator γ denotes the trace operator on Γ, where we do not notationally discriminate between the spaces X_1 or X_2, as it will always be clear from the context. It holds $X_1 \subset H^1(\Omega_1)$, $X_2 \subset H^1(\Omega_2)$ and $X_\Gamma \subset H_{00}^{1/2}(\Gamma)$. We equip the Hilbert spaces X_k, $k = 1, 2$ with the norms $\|v\|_{X_k} := \|v\|_{H^1(\Omega_k)}$, X_Γ with $\|g\|_{X_\Gamma} := \|g\|_{L_2(\Gamma)}$ and $\hat{X}$ with the norm $\|v\|_{\hat{X}} := (\|v_{|\Omega_1}\|_{X_1}^2 + \|v_{|\Omega_2}\|_{X_2}^2)^{1/2}$. On $X \subset \hat{X}$ the $\hat{X}$-norm coincides with the X-norm. We introduce the parametric elliptic variational problem

$$
\text{given } \mu \in \mathcal{P} \text{ find } u(\mu) \in X : \quad a(u(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X, \tag{2.1}
$$

with a parametric bilinear form $a : \hat{X} \times \hat{X} \times \mathcal{P} \to \mathbb{R}$ and a parametric linear form $f : \hat{X} \times \mathcal{P} \to \mathbb{R}$. The function $u(\mu)$ is called *exact solution*. We assume that a is continuous for all $\mu \in \mathcal{P}$ with continuity constant

$$
M(\mu) := \sup_{v \in \hat{X} \setminus \{0\}} \sup_{w \in \hat{X} \setminus \{0\}} \frac{a(v, w; \mu)}{\|v\|_{\hat{X}} \|w\|_{\hat{X}}} < \infty
$$

and coercive for all $\mu \in \mathcal{P}$ with coercivity constant

$$
\alpha(\mu) := \inf_{v \in \hat{X} \setminus \{0\}} \frac{a(v, v; \mu)}{\|v\|_{\hat{X}}^2} > 0.
$$

We also assume that f is continuous and that a and f are affinely parameter dependent, i.e. there exist decompositions

$$a(v,w;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^q(v,w), \quad \forall v,w \in \hat{X}, \mu \in \mathcal{P},$$
$$f(v;\mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) f^q(v), \quad \forall v \in \hat{X}, \mu \in \mathcal{P},$$

with preferably small integers Q_a, Q_f and μ–independent continuous bilinear forms a^q and continuous linear forms f^q, respectively.

Coercivity and continuity of a on $\hat{X}$ is inherited to the subspace X, hence the Lemma of Lax–Milgram states that (2.1) has a unique solution for all $\mu \in \mathcal{P}$. We assume that this exact solution is approximated with an iterative domain decomposition procedure. According to the domain decomposition bilinear and linear forms on the subdomains can be defined.

$$a_k(v,w;\mu) := a\left(\hat{v},\hat{w};\mu\right), \quad \forall v,w \in X_k, \mu \in \mathcal{P},$$
$$f_k(v;\mu) := f\left(\hat{v};\mu\right), \quad \forall v \in X_k, \mu \in \mathcal{P},$$

for $k = 1,2$ with the zero-extension $\hat{v} \in \hat{X}$ defined through $\hat{v}_{|\Omega_k} = v$, $\hat{v}_{|\Omega\setminus\Omega_k} = 0$. It follows that a_k, $k = 1,2$ is continuous and coercive on X_k and f_k, $k = 1,2$ is continuous. Let $M_k(\mu)$ be the continuity constant and $\alpha_k(\mu)$ be the coercivity constant of a_k, $k = 1,2$. It holds $\alpha_k(\mu) \geq \alpha(\mu)$ and $M_k(\mu) \leq M(\mu)$. The affine parameter dependency also transfers from above;

$$a_k(v,w;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a_k^q(v,w), \quad \forall v,w \in X_k, \mu \in \mathcal{P},$$
$$f_k(v;\mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) f_k^q(v), \quad \forall v \in X_k, \mu \in \mathcal{P},$$

with bilinear forms $a_k^q(v,w) := a^q(\hat{v},\hat{w})$ and linear forms $f_k^q(v) := f^q(\hat{v})$. The iterative procedure stated below involves the variational formulation of transmission conditions. To simplify notations we define a linear functional, which arises from this context and represents the conormal flux on the interface Γ [4]. With $R_1 : X_\Gamma \to X_1$ a linear, continuous extension operator, i.e. $\gamma(R_1 g) = g$ for $g \in X_\Gamma$, we define for an arbitrary $v \in X_1$:

$$b_v(w;\mu) := f_1\left(R_1\gamma w;\mu\right) - a_1\left(v, R_1\gamma w;\mu\right), \quad \forall w \in X_2, \mu \in \mathcal{P}. \tag{2.2}$$

The interpretation as a conormal flux can easily be illustrated: For example, if (2.1) is the weak form of the boundary value problem $-\Delta u^{\mathrm{bvp}}(\mu) = h(\mu)$ in Ω, $u^{\mathrm{bvp}}(\mu) = 0$ on $\partial\Omega$ for some $h(\mu) \in L^2(\Omega)$, an integration by parts in (2.2) with $v = u_1(\mu) := u(\mu)|_{\Omega_1}$ ($u(\mu)$ is the weak solution) yields

$$b_{u_1(\mu)}(w;\mu) = \int_\Gamma w\, \nabla u_1(\mu) \cdot n \, ds,$$

where n is the outer normal on $\partial\Omega_1$. We lastly introduce the energy norms for $\mu \in \mathcal{P}$

$$\|v\|_\mu := \sqrt{a(v,v;\mu)}, \quad v \in \hat{X},$$
$$\|v\|_{k,\mu} := \sqrt{a_k(v,v;\mu)}, \quad v \in X_k,$$

for $k = 1, 2$.

3. Reduced basis scheme. In this section a full scheme for reduced basis approximation of the exact solution is presented. The domain decomposition is preserved in all parts of the scheme. Further computational aspects are given in Section 5.

The idea of the reduced basis method is to approximate the set of solutions $\{u(\mu)|\mu \in \mathcal{P}\}$ through a low dimensional linear space. The linear space is spanned by a few snapshots $u(\mu_l)$ for selected μ_l. Once this is established, mostly a Galerkin projection of the solution is computed. We do this not directly but compute snapshots on the subdomains for the usage in an iterative procedure. It will turn out that in the limit we regain a standard reduced basis approximation as described above.

3.1. Detailed procedure. Firstly, we describe a common iterative domain decomposition procedure for approximating the exact solution $u(\mu)$ of (2.1). This is the so called Dirichlet-Neumann scheme and serves as a snapshot supplier.

DEFINITION 3.1 (detailed procedure). *Given $\mu \in \mathcal{P}$, let $g^0(\mu) = 0 \in X_\Gamma$, $n_{\text{stop}} \in \mathbb{N}$, $\epsilon_{\text{tol}} > 0$ and $\theta^n(\mu) \in [0, 1]$ for $n \geq 1$ be given. We now construct sequences $u_1^n(\mu) \in X_1$, $u_2^n(\mu) \in X_2$ and $g^n(\mu) \in X_\Gamma$ for $n \geq 1$ satisfying*

$$\begin{aligned} a_1(u_1^n(\mu), v; \mu) &= f_1(v; \mu), \quad \forall v \in X_1^0, \\ \gamma u_1^n(\mu) &= g^{n-1}(\mu), \\ a_2(u_2^n(\mu), v; \mu) &= f_2(v; \mu) + b_{u_1^n(\mu)}(v; \mu), \quad \forall v \in X_2, \\ g^n(\mu) &= (1 - \theta^n(\mu))\, g^{n-1}(\mu) + \theta^n(\mu)\gamma u_2^n(\mu). \end{aligned}$$

The iteration is terminated if

$$\|u_1^n(\mu) - u_1^{n-1}(\mu)\|_{1,\mu}^2 + \|u_2^n(\mu) - u_2^{n-1}(\mu)\|_{2,\mu}^2 \leq \epsilon_{\text{tol}},$$

or if $n = n_{\text{stop}}$. The number of actually accomplished iterations is denoted by $n(\mu)$.

Here, $\theta^n(\mu)$ is a relaxation parameter. If it is chosen properly, the constructed sequences converge to the exact solution. The proof of convergence can be found in [14] for $X = H_0^1(\Omega)$ and $X = V_h$, where V_h is a space of Finite Elements. As stated in Section 2, we do not decide on finite or infinite dimensional spaces. For analytical results, we can work with the proper infinite dimensional spaces. In practice the space X will be a Finite Element space.

3.2. Reduced procedure. To define appropriate RB spaces we consider a finite sample set $\mathcal{S} = \{\mu_1, \ldots, \mu_{N_\mathcal{S}}\} \subset \mathcal{P}$ of parameters. One can think of several ways to generate bases for the RB spaces by the snapshots $u_k^n(\mu_l)$, $k = 1, 2$, $n = 1, \ldots, n(\mu_l)$, $l = 1, \ldots, N_\mathcal{S}$. This topic will be discussed in detail in Section 5. For the moment we assume the reduced bases to be given. To be more precise, for $k = 1, 2$ let now N_k^0 and N_k^Γ be natural numbers and $N_k := N_k^0 + N_k^\Gamma$. We assume that we are given bases

$$\begin{aligned} \Phi_k^0 &= \left\{ \varphi_{N,k}^{(i)} \,\middle|\, i = 1, \ldots, N_k^0 \right\} \subset X_k^0, \\ \Phi_k^\Gamma &= \left\{ \varphi_{N,k}^{(i)} \,\middle|\, i = N_k^0 + 1, \ldots, N_k \right\} \subset X_k, \\ \Phi_k &= \Phi_k^0 \cup \Phi_k^\Gamma, \end{aligned}$$

where $\gamma(\Phi_k^\Gamma) := \{\gamma\varphi_{N,k}^{(i)} | i = N_k^0 + 1, \ldots, N_k\}$, $k = 1, 2$ are linearly independent and span the same linear space on Γ. We define appropriate RB spaces for $k = 1, 2$:

$$X_{N,k} := \operatorname{span}(\Phi_k),$$

$$
\begin{aligned}
X_{N,k}^0 &:= \operatorname{span}\left(\Phi_k^0\right), \\
X_{N,k}^\Gamma &:= \operatorname{span}\left(\Phi_k^\Gamma\right), \\
X_{N,\Gamma} &:= \operatorname{span}\left(\gamma\left(\Phi_1^\Gamma\right)\right) = \operatorname{span}\left(\gamma\left(\Phi_2^\Gamma\right)\right), \\
X_N &:= \left\{v \in X \middle| v_{|\Omega_1} \in X_{N,1}, v_{|\Omega_2} \in X_{N,2}\right\}.
\end{aligned}
$$

It obviously holds $X_{N,k} \subset X_k$, $X_{N,k}^0 \subset X_k^0$, $X_{N,\Gamma} \subset X_\Gamma$ and $X_N \subset X$. Note that $X_{N,k}^0 = X_{N,k} \cap H_0^1(\Omega_1)$ and $X_{N,\Gamma} = \gamma(X_{N,k}^\Gamma) = \gamma(X_{N,k})$ for $k = 1, 2$. As $X_{N,k} \cong X_{N,k}^0 \oplus X_{N,k}^\Gamma$, a function $v \in X_{N,k}$ can uniquely be written as $v = v^0 + v^\Gamma$ with $v^0 \in X_{N,k}^0$ and $v^\Gamma \in X_{N,k}^\Gamma$.

For a reduced iterative procedure we replace all steps of the detailed procedure with reduced basis approximations. Since $R_1(X_{N,\Gamma}) \subset X_{N,1}$ does not hold in general, let $R_{N,1} : X_{N,\Gamma} \to X_{N,1}$ be a linear, continuous extension operator on the RB spaces, i.e. $\gamma(R_{N,1}g) = g$ for all $g \in X_{N,\Gamma}$. The counterpart to the linear form b_v then is defined as

$$
b_{N,v}(w;\mu) := f_1\left(R_{N,1}\gamma w;\mu\right) - a_1\left(v, R_{N,1}\gamma w;\mu\right), \quad \forall v \in X_{N,1}, w \in X_{N,2}, \mu \in \mathcal{P}.
$$

Now, the reduced procedure can be stated.

Definition 3.2 (reduced procedure). *Given $\mu \in \mathcal{P}$, let $g_N^0(\mu) = 0 \in X_{N,\Gamma}$, $n_{N,\mathrm{stop}} \in \mathbb{N}$, $\epsilon_{N,\mathrm{tol}} > 0$ and $\theta_N^n(\mu) \in [0,1]$ for $n \geq 1$ be given. We now construct sequences $u_{N,1}^n(\mu) \in X_{N,1}$, $u_{N,2}^n(\mu) \in X_{N,2}$ and $g_N^n(\mu) \in X_{N,\Gamma}$ for $n \geq 1$ satisfying*

$$
a_1(u_{N,1}^n(\mu), v;\mu) = f_1(v;\mu), \quad \forall v \in X_{N,1}^0, \tag{3.1}
$$

$$
\gamma u_{N,1}^n(\mu) = g_N^{n-1}(\mu), \tag{3.2}
$$

$$
a_2(u_{N,2}^n(\mu), v;\mu) = f_2(v;\mu) + b_{N,u_{N,1}^n(\mu)}(v;\mu), \quad \forall v \in X_{N,2}, \tag{3.3}
$$

$$
g_N^n(\mu) = \left(1 - \theta_N^n(\mu)\right) g_N^{n-1}(\mu) + \theta_N^n(\mu)\gamma u_{N,2}^n(\mu). \tag{3.4}
$$

The iteration is terminated if

$$
\|u_{N,1}^n(\mu) - u_{N,1}^{n-1}(\mu)\|_{1,\mu}^2 + \|u_{N,2}^n(\mu) - u_{N,2}^{n-1}(\mu)\|_{2,\mu}^2 \leq \epsilon_{N,\mathrm{tol}},
$$

or if $n = n_{N,\mathrm{stop}}$. The number of actually accomplished iterations is denoted by $n_N(\mu)$.

By the carefully selected assumptions on the reduced bases the reduced and detailed procedure are perfectly corresponding, the only difference being the underlying Hilbert spaces.

3.3. Matrix formulation. In this subsection a matrix formulation of the reduced procedure is given. The elements of vectors and matrices will be denoted by superscripts $*^{(i)}$ and $*^{(i,j)}$. With the coefficient vectors $\underline{u}_{N,k}^n(\mu) \in \mathbb{R}^{N_k}$, $k = 1, 2$, $n \in \mathbb{N}$ and $\underline{g}_N^{n-1}(\mu) \in \mathbb{R}^{N_1^\Gamma}$, $n \in \mathbb{N}$ the functions $u_{N,k}^n(\mu)$, $k = 1, 2$, $n \in \mathbb{N}$ and $g_N^{n-1}(\mu)$, $n \in \mathbb{N}$ are given through

$$
u_{N,k}^n(\mu) = \sum_{i=1}^{N_k} \underline{u}_{N,k}^n(\mu)^{(i)} \varphi_{N,k}^{(i)}, \qquad g_N^{n-1}(\mu) = \sum_{i=1}^{N_1^\Gamma} \underline{g}_N^{n-1}(\mu)^{(i)} \gamma\varphi_{N,1}^{(N_1^0+i)}.
$$

We introduce the matrix $\underline{G}_N \in \mathbb{R}^{N_1^\Gamma \times N_2}$ for the computation of the basis coefficients in $X_{N,\Gamma}$, i.e.

$$
\sum_{i=1}^{N_1^\Gamma} \underline{G}_N^{(i,j)} \gamma\varphi_{N,1}^{(N_1^0+i)} = \gamma\varphi_{N,2}^{(j)}, \quad \forall j \in \{1, \ldots, N_2\}.
$$

It holds $\underline{G}_N^{(i,j)} = 0$ for $j = 1, \dots, N_2^0$. If there is a one-to-one correspondence of the bases on Γ the lower block is a diagonal matrix, else it can be computed via the orthogonal $L^2(\Gamma)$-projection. Let $\underline{R}_{N,1} \in \mathbb{R}^{N_1 \times (N_1 - N_1^0)}$ be the matrix of the linear map $R_{N,1}$, i.e.

$$\sum_{i=1}^{N_1} \underline{R}_{N,1}^{(i,j)} \varphi_{N,1}^{(i)} = R_{N,1} \gamma \varphi_{N,1}^{(N_1^0 + j)}, \quad \forall j \in \{1, \dots, N_1 - N_1^0\}.$$

We further introduce the matrices $\underline{A}_{N,k}(\mu)$ and $\underline{F}_{N,k}(\mu)$, $k = 1, 2$:

$$\begin{aligned} \underline{A}_{N,k}(\mu) &:= \left(a_k \left(\varphi_{N,k}^{(j)}, \varphi_{N,k}^{(i)} \right) \right)_{i,j=1}^{N_k} \in \mathbb{R}^{N_k \times N_k}, \\ \underline{F}_{N,k}(\mu) &:= \left(f_k \left(\varphi_{N,k}^{(i)} \right) \right)_{i=1}^{N_k} \in \mathbb{R}^{N_k}. \end{aligned}$$

For a matrix $\underline{v} \in \mathbb{R}^{N_1 \times m}$, where $m \in \{1, N_1\}$ we introduce the notations

$$\begin{aligned} \underline{v}^0 &:= \left(\underline{v}^{(i,j)} \right)_{i=1,j=1}^{N_1^0, m} \in \mathbb{R}^{N_1^0 \times m}, \\ \underline{v}^\Gamma &:= \left(\underline{v}^{(N_1^0 + i, j)} \right)_{i=1,j=1}^{N_1^\Gamma, m} \in \mathbb{R}^{N_1^\Gamma \times m} \end{aligned}$$

and for the case $m = N_1$

$$\begin{aligned} \underline{v}^{00} &:= \left(\underline{v}^{(i,j)} \right)_{i,j=1}^{N_1^0} \in \mathbb{R}^{N_1^0 \times N_1^0}, \\ \underline{v}^{0\Gamma} &:= \left(\underline{v}^{(i, N_1^0 + j)} \right)_{i=1,j=1}^{N_1^0, N_1^\Gamma} \in \mathbb{R}^{N_1^0 \times N_1^\Gamma}. \end{aligned}$$

We now can rewrite the reduced procedure of Definition 3.2 in terms of matrices and vectors.

$$\begin{aligned} \underline{A}_{N,1}(\mu)^{00} \underline{u}_{N,1}^n(\mu)^0 &= \underline{F}_{N,1}(\mu)^0 - \underline{A}_{N,1}(\mu)^{0\Gamma} \underline{g}_N^{n-1}(\mu), \\ \underline{u}_{N,1}^n(\mu)^\Gamma &= \underline{g}_N^{n-1}(\mu), \\ \underline{B}_{N,2}^n(\mu) &= \underline{G}_N^T \underline{R}_{N,1}^T \left(\underline{F}_{N,1}(\mu)^\Gamma - \underline{A}_{N,1}(\mu)^\Gamma \underline{u}_{N,1}^n(\mu) \right), \\ \underline{A}_{N,2}(\mu) \underline{u}_{N,2}^n(\mu) &= \underline{F}_{N,2}(\mu) + \underline{B}_{N,2}^n(\mu), \\ \underline{g}_N^n(\mu) &= (1 - \theta_N^n(\mu)) \, \underline{g}_N^{n-1}(\mu) + \theta_N^n(\mu) \underline{G}_N \underline{u}_{N,2}^n(\mu). \end{aligned}$$

The stopping criterion then reads

$$\sum_{k=1}^{2} \left(\underline{u}_{N,k}^n(\mu) - \underline{u}_{N,k}^{n-1}(\mu) \right)^T \underline{A}_{N,k}(\mu) \left(\underline{u}_{N,k}^n(\mu) - \underline{u}_{N,k}^{n-1}(\mu) \right) \leq \epsilon_{N,\mathrm{tol}}.$$

4. Analytical results. There are three aspects, in which we are interested in this section. The well-posedness of the reduced procedure, the convergence of the iterative scheme and the error quantification of the gained approximative solution.

4.1. Well-posedness. We shortly explain the well-posedness of the reduced procedure. We adopt the notation for matrices to functions:

$$u_{N,1}^n(\mu)^0 := \sum_{i=1}^{N_1^0} \underline{u}_{N,1}^n(\mu)^{(i)} \varphi_{N,1}^{(i)},$$
$$u_{N,1}^n(\mu)^\Gamma := \sum_{i=1}^{N_1^\Gamma} \underline{u}_{N,1}^n(\mu)^{(N_1^0+i)} \varphi_{N,1}^{(N_1^0+i)}.$$

Since $\gamma|_{X_{N,1}^\Gamma}$ is bijective, $u_{N,1}^n(\mu)^\Gamma$ is given through Equation (3.2) in a unique way. Then we can rewrite Equation (3.1) as $a_1(u_{N,1}^n(\mu)^0, v; \mu) = f_1(v;\mu) - a_1(u_{N,1}^n(\mu)^\Gamma, v; \mu)$ and therefore have existence and uniqueness of $u_{N,1}^n(\mu)^0$ due to the Lemma of Lax-Milgram. Again in Equation (3.3) we can apply Lax-Milgram because $b_{N,v}(\cdot;\mu)$ is linear and continuous for all $v \in X_{N,1}$ and $\mu \in \mathcal{P}$. The well-posedness of Equation (3.4) requires the coincidence $\gamma(X_{N,1}) = \gamma(X_{N,2})$, which was an assumption on the RB spaces. To gain good conditions of the system matrices $\underline{A}_{N,1}(\mu)^{00}$ and $\underline{A}_{N,2}(\mu)$ the bases Φ_1^0 and Φ_2 should be orthonormalized. As will be seen later, this is always possible for Φ_1^0, in contrast to Φ_2.

4.2. Convergence. Similar as for the detailed procedure, the convergence of the reduced procedure is determined by the choice of the corresponding relaxation parameter. The following theorem describes this more precisely.

THEOREM 4.1. *Provided $\theta_N^n(\mu) \geq \theta_{N,\min}(\mu)$ for all $n \geq 1$ and an arbitrary $\theta_{N,\min}(\mu) > 0$ there exists $\theta_N^*(\mu) > 0$ such that the sequences $\{u_{N,1}^n(\mu)\}_{n\in\mathbb{N}}$ and $\{u_{N,2}^n(\mu)\}_{n\in\mathbb{N}}$ of the reduced procedure converge in $X_{N,1}$ and $X_{N,2}$, respectively, if $\theta_N^n(\mu) < \theta_N^*(\mu)$ for all $n \geq 1$. Then especially holds $\lim_{n\to\infty} u_{N,1}^n(\mu) = u_N(\mu)_{|\Omega_1}$ and $\lim_{n\to\infty} u_{N,2}^n(\mu) = u_N(\mu)_{|\Omega_2}$ where $u_N(\mu)$ is the solution of the following problem.*

$$\text{find } u_N(\mu) \in X_N: \quad a(u_N(\mu), v; \mu) = f(v;\mu), \quad \forall v \in X_N. \tag{4.1}$$

For the sake of completeness we give the proof of Theorem 4.1 in the appendix. The solution $u_N(\mu)$ of problem (4.1) is the standard RB approximation of the exact solution $u(\mu)$ from (2.1). This already indicates that we obtain a good approximation in the reduced procedure, because the iteration sequences converge to a function, which is known to be a good approximation. But the performance of the procedure also relies on the rate of convergence, which is influenced by the relaxation parameters $\theta_N^n(\mu)$. The optimal relaxation parameter — the one with a fastest convergent iteration — can be approximated by a routine which is reported in [14] for Finite Elements. We note that an analogue routine can be applied in the RB scheme. Within this routine we need a separate basis Φ_2^0 of $X_{N,2}^0$, which was not used in the RB scheme so far.

4.3. A-posteriori error estimation. In this subsection we consider the error $\|u_N^n(\mu) - u(\mu)\|_{\hat{X}}$ of the iterative solution $u_N^n(\mu)$ which is the assembly of the iteration functions of the reduced procedure, i.e. $u_N^n(\mu)_{|\Omega_1} = u_{N,1}^n(\mu)$ and $u_N^n(\mu)_{|\Omega_2} = u_{N,2}^n(\mu)$. An a-posteriori error estimator, which is a strict upper bound for the error, is presented. This error estimator especially can be used later in an adaptive basis generation algorithm.

DEFINITION 4.2 (residuals). *For $n \geq 1$ and $\mu \in \mathcal{P}$ we define the residual $r^n(\cdot;\mu) : X \to \mathbb{R}$ through*

$$r^n(v;\mu) := a(u_N^n(\mu), v; \mu) - f(v;\mu), \quad \forall v \in X,$$

Let $\nu^n(\mu) \in X$ be the Riesz-representative of $r^n(\cdot;\mu)$.

THEOREM 4.3 (a-posteriori error estimator). *Let $\tilde{R}_1 : X_\Gamma \to X_1$ be an arbitrary linear extension operator. For $n \geq 1$ and $\mu \in \mathcal{P}$ we can estimate the error $u_N^n(\mu) - u(\mu)$ in the $\hat{X}$-Norm:*

$$\|u_N^n(\mu) - u(\mu)\|_{\hat{X}} \leq \Delta_N^n(\mu),$$

where

$$\Delta_N^n(\mu) := \frac{1}{\alpha(\mu)}\|\nu^n(\mu)\|_X + \left(1 + \sqrt{\frac{M(\mu)}{\alpha(\mu)}}\right) \left\|\tilde{R}_1\left(\gamma u_{N,1}^n(\mu) - \gamma u_{N,2}^n(\mu)\right)\right\|_{X_1}. \tag{4.2}$$

From the definition of the estimator we get a criterion for the choice of the extension operator $\tilde{R}_1$. It is convenient to choose $\tilde{R}_1$ such that $\sup_{v \in X_\Gamma \setminus \{0\}} \|\tilde{R}_1 v\|_{X_1} / \|v\|_{X_\Gamma}$ is small. In the experiments in Section 6 we will decide for the trivial Finite Element extension.

REMARK 1. *Note that the first term in (4.2) is the standard elliptic RB estimator [18]. Hence our error bound additionaly covers the discontinuity of $u_N^n(\mu)$ across the interface Γ.*

Proof of Theorem 4.3. We split up the function $u_N^n(\mu)$ in $\hat{u}_N^n(\mu) \in \hat{X}$ defined through

$$\begin{aligned} \hat{u}_N^n(\mu)_{|\Omega_1} &:= \tilde{R}_1\left(\gamma u_{N,1}^n(\mu) - \gamma u_{N,2}^n(\mu)\right), \\ \hat{u}_N^n(\mu)_{|\Omega_2} &:= 0 \end{aligned}$$

and $\bar{u}_N^n(\mu) := u_N^n(\mu) - \hat{u}_N^n(\mu) \in X$. By application of the triangle inequality we get

$$\|u_N^n(\mu) - u(\mu)\|_{\hat{X}} \leq \|\bar{u}_N^n(\mu) - u(\mu)\|_{\hat{X}} + \|\hat{u}_N^n(\mu)\|_{\hat{X}}. \tag{4.3}$$

For $\hat{u}_N^n(\mu)$ holds

$$\|\hat{u}_N^n(\mu)\|_{\hat{X}} = \left\|\tilde{R}_1\left(\gamma u_{N,1}^n(\mu) - \gamma u_{N,2}^n(\mu)\right)\right\|_{X_1}.$$

We now consider the error $\bar{u}_N^n(\mu) - u(\mu) \in X$ in the energy norm:

$$\begin{aligned} \|\bar{u}_N^n(\mu) - u(\mu)\|_\mu^2 &= a\left(\bar{u}_N^n(\mu), \bar{u}_N^n(\mu) - u(\mu); \mu\right) - a\left(u(\mu), \bar{u}_N^n(\mu) - u(\mu); \mu\right) \\ &= a\left(u_N^n(\mu), \bar{u}_N^n(\mu) - u(\mu); \mu\right) - a\left(\hat{u}_N^n(\mu), \bar{u}_N^n(\mu) - u(\mu); \mu\right) \\ &\quad - f\left(\bar{u}_N^n(\mu) - u(\mu); \mu\right) \\ &= r^n\left(\bar{u}_N^n(\mu) - u(\mu); \mu\right) - a\left(\hat{u}_N^n(\mu), \bar{u}_N^n(\mu) - u(\mu); \mu\right) \\ &= \left(\nu^n(\mu), \bar{u}_N^n(\mu) - u(\mu)\right)_X - a\left(\hat{u}_N^n(\mu), \bar{u}_N^n(\mu) - u(\mu); \mu\right). \end{aligned}$$

With the Cauchy-Schwarz inequality, coercivity and continuity of $a(\cdot,\cdot;\mu)$ follows

$$\begin{aligned} \|\bar{u}_N^n(\mu) - u(\mu)\|_\mu^2 &\leq \|\nu^n(\mu)\|_X \|\bar{u}_N^n(\mu) - u(\mu)\|_X + \|\hat{u}_N^n(\mu)\|_\mu \|\bar{u}_N^n(\mu) - u(\mu)\|_\mu \\ &\leq \frac{1}{\sqrt{\alpha(\mu)}} \|\nu^n(\mu)\|_X \|\bar{u}_N^n(\mu) - u(\mu)\|_\mu \\ &\quad + \sqrt{M(\mu)} \|\hat{u}_N^n(\mu)\|_{\hat{X}} \|\bar{u}_N^n(\mu) - u(\mu)\|_\mu \\ \Rightarrow \quad \|\bar{u}_N^n(\mu) - u(\mu)\|_\mu &\leq \frac{1}{\sqrt{\alpha(\mu)}} \|\nu^n(\mu)\|_X + \sqrt{M(\mu)} \|\hat{u}_N^n(\mu)\|_{\hat{X}}. \end{aligned}$$

The latter estimation is inserted into Equation (4.3) using the coercivity inequality $\|\bar{u}_N^n(\mu) - u(\mu)\|_{\hat{X}} \leq \alpha(\mu)^{-1/2} \|\bar{u}_N^n(\mu) - u(\mu)\|_\mu$ to obtain the statement. ∎

5. Computational aspects. We now want to go into some computational aspects of the proposed method. The space X now is considered to be a high-dimensional space, e.g. Finite Element space, with dimension $\dim(X) = \mathcal{N}$. This leads to high, finite dimensions $\dim(X_k) = \mathcal{N}_k$, $k = 1, 2$ and $\dim(X_\Gamma) = \mathcal{N}_\Gamma$.

In Section 5.1 several possibilities of generating the reduced bases are presented. The accuracy of the approximation can be strongly influenced by a proper basis generation. It is shown how this can be achieved with a Greedy-algorithm while meeting the special requirements of our method. We assume that this leads to RB spaces with dimensions $N_k \ll \mathcal{N}_k$ for $k = 1, 2$ and $N_\Gamma \ll \mathcal{N}_\Gamma$, where $N_\Gamma = N_1^\Gamma = N_2^\Gamma$.

In Section 5.2 a full offline/online decomposition is provided. In the offline phase all high-dimensional computings are managed, so that the computations in the online phase do not depend on high dimensions. This garantuees the efficiency of the reduced basis approach and is based on the affine parameter dependency of the involved data.

5.1. Basis generation. We concentrate now on the missing step of generating the bases Φ_k^0 and Φ_k^Γ, $k = 1, 2$ out of the snapshots $u_k^n(\mu_l)$, $k = 1, 2$, $n = 1, \dots, n(\mu_l)$, $l = 1, \dots, N_\mathcal{S}$ and on how to choose the sample parameters μ_l. The bases strongly influence the approximation quality of the reduced procedure. Our goal is to generate bases such that the error can be bounded by a given tolerance for all parameters in $\mathcal{P}$. This can be done approximately with the Greedy-algorithm [20]. Since $\mathcal{P}$ is an infinite set, we have to weaken the goal by considering only a finite training set $\mathcal{M}_{\text{train}} \subset \mathcal{P}$. It is sufficient to replace the error by the error estimator $\Delta_N^n(\mu)$ in this context, since $\Delta_N^n(\mu)$ is an upper bound for the error. In the following variant of the Greedy procedure a choice of a basis extension method is kept open and can be filled by method A or B explained later.

DEFINITION 5.1 (Greedy-algorithm). *Given initial bases Φ_k^0, Φ_k^Γ, $k = 1, 2$ with the properties of Section 3.2, an error tolerance ϵ_{gre} and $N_{\text{stop},k} \in \mathbb{N}$, $k = 1, 2$, the Greedy-algorithm is given by*

- *while* $\max_{\mu \in \mathcal{M}_{\text{train}}} \Delta_N^{n_N(\mu)}(\mu) > \epsilon_{\text{gre}}$
  - (i) $\mu^* := \arg\max_{\mu \in \mathcal{M}_{\text{train}}} \Delta_N^{n_N(\mu)}(\mu)$
  - (ii) *compute* $\Xi_k := \{u_k^n(\mu^*) | n = 1, \dots, n(\mu^*)\}$, $k = 1, 2$ *by the detailed procedure from Definition 3.1*
  - (iii) *extend the bases* Φ_k^0, Φ_k^Γ, $k = 1, 2$ *by method A or B as long as* $|\Phi_k^0| + |\Phi_k^\Gamma| \leq N_{\text{stop},k}$, $k = 1, 2$.

It remains to specify how the bases are extended in step (iii). For a fixed $\bar{\mu} \in \mathcal{P}$ and $k = 1, 2$ we define extension operators $R_{k,\bar{\mu}} : X_\Gamma \to X_k$ through

$$\begin{aligned} a_k\left(R_{k,\bar{\mu}} g, v; \bar{\mu}\right) &= 0, \qquad \forall v \in X_k^0, \\ R_{k,\bar{\mu}} g &= g \qquad \text{on } \Gamma. \end{aligned} \tag{5.1}$$

We also involve the orthogonal projection, which we denote by $\mathbb{P}_Y$ for an arbitrary subspace Y and the first mode of the proper orthogonal decomposition (POD) [9, 21] for an arbitrary set $\Xi \subset X_k$, $k = 1$ or 2, which is defined as

$$\text{POD}(\Xi) := \arg \min_{v \in X_k, \|v\|_{X_k} = 1} \sum_{w \in \Xi} \|w - \mathbb{P}_{\text{span}(v)} w\|_{X_k}^2.$$

We now can propose our two methods, which get Ξ_k, $k = 1, 2$ as input and extend the bases Φ_k^0, Φ_k^Γ for $k = 1, 2$. Both methods are extended variants of the POD-Greedy

procedure [7]. Method A is motivated by the idea that an arbitrary function $v \in X_k$, $k = 1, 2$ is the sum of $v - R_{k,\bar{\mu}}\gamma v \in X_k^0$ and $R_{k,\bar{\mu}}\gamma v$ in X_k. With the help of this split-up both Φ_k^0 and Φ_k^Γ are extended simultaneously. Furthermore the usage of the POD yields orthonormal bases Φ_k^0 and $\{\gamma\varphi \,|\, \varphi \in \Phi_k^\Gamma\}$, if the initial bases already have these properties.

5.1.1. Method A.

1. extension of Φ_k^0, $k = 1, 2$:

(i) $\Xi_k^0 := \{\xi - R_{k,\bar{\mu}}\gamma\xi \,|\, \xi \in \Xi_k\}$

(ii) $\xi_k^0 := \mathrm{POD}(\{\xi - \mathbb{P}_{X_{N,k}^0}\xi \,|\, \xi \in \Xi_k^0\})$

(iii) $\Phi_k^0 \leftarrow \Phi_k^0 \cup \{\xi_k^0\}$

2. extension of Φ_k^Γ, $k = 1, 2$:

(i) $\xi^\Gamma := \mathrm{POD}(\{\gamma\xi - \mathbb{P}_{X_{N,\Gamma}}\gamma\xi \,|\, \xi \in \Xi_1 \cup \Xi_2\})$

(ii) $\Phi_k^\Gamma \leftarrow \Phi_k^\Gamma \cup \{R_{k,\bar{\mu}}\xi^\Gamma\}$

The POD is not defined when the input set is $\{0\}$. So it depends on steps 1.(ii) and 2.(i) whether the bases are extended at all. With method A we do not take into account, that for the reduced procedure (Definition 3.2) we do not necessarily have to generate functions for Φ_2^0. Method B consists in a different processing of Ξ_2. In each step of the Greedy-algorithm the basis Φ_2 gains at most one vector, which is assigned to Φ_2^Γ as long as $\gamma(\Phi_2^\Gamma)$ is linearly independent, which was an assumption in our bases framework for the reduced procedure. In the other case a new vector for Φ_2^0 is generated. Hence, potentially method B allows some savings in the overall final basis size N.

5.1.2. Method B.

1. extension of Φ_1^0:

(i) $\Xi_1^0 := \{\xi - R_{1,\bar{\mu}}\gamma\xi \,|\, \xi \in \Xi_1\}$

(ii) $\xi_1^0 := \mathrm{POD}(\{\xi - \mathbb{P}_{X_{N,1}^0}\xi \,|\, \xi \in \Xi_1^0\})$

(iii) $\Phi_1^0 \leftarrow \Phi_1^0 \cup \{\xi_1^0\}$

2. extension of Φ_1^Γ and Φ_2^Γ:

(i) $\xi_2 := \mathrm{POD}(\{\xi - \mathbb{P}_{X_{N,2}}\xi \,|\, \xi \in \Xi_2\})$

(ii) $\xi_\Gamma := \gamma\xi_2 - \mathbb{P}_{X_{N,\Gamma}}\gamma\xi_2$

(iii) if $\xi_\Gamma \neq 0$

$\Phi_1^\Gamma \leftarrow \Phi_1^\Gamma \cup \{R_{1,\bar{\mu}}\gamma\xi_2\}$

$\Phi_2^\Gamma \leftarrow \Phi_2^\Gamma \cup \{\xi_2\}$

3. extension of Φ_2^0:

(i) if $\xi_\Gamma = 0$

$\xi_2^0 := \xi_2 - (\gamma|_{X_{N,2}^\Gamma})^{-1}\mathbb{P}_{X_{N,\Gamma}}\gamma\xi_2$

$\xi_2^0 \leftarrow \xi_2^0 - \mathbb{P}_{X_{N,2}^0}\xi_2^0$

$\xi_2^0 \leftarrow \frac{1}{\|\xi_2^0\|_{X_2}}\xi_2^0$

$\Phi_2^0 \leftarrow \Phi_2^0 \cup \{\xi_2^0\}$

By this procedure the bases Φ_1^0, Φ_2^Γ and Φ_2^0 are orthonormalized. With method A

all bases Φ_k^0 and Φ_k^Γ, $k = 1, 2$ are orthonormalized. So the system matrix $\underline{A}_{N,1}^{00}(\mu)$ will be well conditioned in both cases. However this is not ensured for $\underline{A}_{N,2}(\mu)$, because $\Phi_2^0 \perp \Phi_2^\Gamma$ is not ensured.

REMARK 2. *The assumption* $\operatorname{span}(\gamma(\Phi_1^\Gamma)) = \operatorname{span}(\gamma(\Phi_2^\Gamma))$ *given in Subsection 3.2 is fullfilled by construction when using both method A or method B, if again the initial bases have this property.*

5.1.3. Initial bases. The computation of initial bases according to method A or B can be done with the methods themselves and $\Xi_k := \{u_k^n(\mu_{\text{init}}) \,|\, n = 1, \ldots, n(\mu_{\text{init}})\}$ where $\mu_{\text{init}} \in \mathcal{P}$ is a random parameter and $\Phi_k^0, \Phi_k^\Gamma = \emptyset$ for $k = 1, 2$.

5.1.4. The sequence mode. In step (ii) of the Greedy-algorithm the whole sequences of approximates are included. In our numerical experiments we also consider the option $\Xi_k = \{u_k^{n(\mu^*)}(\mu^*)\}$, $k = 1, 2$ where only the last approximates of the iteration are included. The "POD-steps" in method A and B then degenerate to a normalization of one vector. The numerical experiments in Section 6 will show that this also is sufficient to build up bases that bring forward the approximation with the reduced procedure. We append -SEQ, if using the whole sequences and -NSEQ, if only using the final snapshot, to the method name A or B.

5.2. Offline/online decomposition. We assume that there is a need of computing an approximation and an error estimate for plenty of parameters $\mu \in \mathcal{P}$, as it is the case e.g. in an optimization problem. To perform the reduced procedure we need to compute the matrices and vectors $\underline{A}_{N,k}(\mu)$, $\underline{F}_{N,k}(\mu)$ for $k = 1, 2$, $\underline{G}_N$ and $\underline{R}_{N,1}$. The last two do not depend on μ, so we have to compute them once only. But the μ-dependent matrices have to be computed for every parameter and this has to be done in an efficient way, otherwise the computation time for the RB scheme will deteriorate. In the case of $\underline{A}_{N,1}(\mu)$ the assumed affine parameter dependency of a_1 leads to the following decomposition.

$$\underline{A}_{N,1}(\mu) := \sum_{q=1}^{Q_a} \Theta_a^q(\mu) \underline{A}_{N,1}^q, \tag{5.2}$$

where the component-matrices $\underline{A}_{N,1}^q$ are given by

$$\underline{A}_{N,1}^q := \left(a_k^q \left(\varphi_{N,1}^{(j)}, \varphi_{N,1}^{(i)} \right) \right)_{i,j=1}^{N_1}.$$

With precomputed matrices $\underline{A}_{N,1}^q$ the computation of $\underline{A}_{N,1}(\mu)$ for an arbitrary parameter μ can be done in $\mathcal{O}(N_1^2)$ via Equation (5.2). This is a huge acceleration, since $N_1 \ll \mathcal{N}_1$. With this insight the offline/online decomposition for the reduced procedure can be stated.

Offline stage (reduced procedure)

1. generate the bases Φ_k, $k = 1, 2$ with the method of your choice
2. compute the component-matrices

$$\underline{A}_{N,k}^q = (a_k^q(\varphi_{N,k}^{(j)}, \varphi_{N,k}^{(i)}))_{i,j=1,}^{N_k} \text{ for } q = 1, \ldots, Q_a, \ k = 1, 2$$
$$\underline{F}_{N,k}^q = (f_k^q(\varphi_{N,k}^{(i)}))_{i=1}^{N_k} \text{ for } q = 1, \ldots, Q_f, \ k = 1, 2$$

3. compute the matrices $\underline{G}_N$ and $\underline{R}_{N,1}$

Online stage (reduced procedure)

Given $\mu \in \mathcal{P}$, $\epsilon_{N,\mathrm{tol}} > 0$ and an integer $n_{N,\mathrm{stop}} \geq 1$

1. compute the matrices

$$\underline{A}_{N,k}(\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) \underline{A}_{N,k}^q \text{ for } k = 1, 2$$

$$\underline{F}_{N,k}(\mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) \underline{F}_{N,k}^q \text{ for } k = 1, 2$$

2. perform the reduced procedure according to Definition 3.2.

Since the matrices in the reduced procedure are small-sized, we benefit by saving the Cholesky factors of $\underline{A}_{N,1}(\mu)^{00}$ and $\underline{A}_{N,2}(\mu)$ or even $(\underline{A}_{N,1}(\mu)^{00})^{-1}\underline{F}_{N,1}(\mu)^0$, etc. in the first iteration.

For an efficient computing of the a-posteriori error estimator we derive the affine parameter dependency of $\nu^n(\mu)$. It holds for all $v \in X$

$$\begin{aligned}
(\nu^n(\mu), v)_X &= f(v;\mu) - a\left(u_N^n(\mu), v; \mu\right) \\
&= \sum_{k=1}^{2} \left(f_k(v;\mu) - a_k\left(u_{N,k}^n(\mu), v; \mu\right)\right) \\
&= \sum_{k=1}^{2} \left(\sum_{q=1}^{Q_f} \Theta_f^q(\mu) f_k^q(v) - \sum_{q=1}^{Q_a} \sum_{\bar{q}=1}^{N_k} \Theta_a^q(\mu) \underline{u}_{N,k}^n{}^{(\bar{q})}(\mu) a_k^q\left(\varphi_{N,k}^{(\bar{q})}, v\right) \right).
\end{aligned}$$

So, an affine decomposition $\nu^n(\mu) = \sum_{q=1}^{Q_\nu} \Theta_{\nu^n}^q \nu^q$ is given with an n-independent number $Q_\nu := \sum_{k=1}^{2}(Q_f + N_k Q_a)$ by the coefficient-vector

$$\Theta_{\nu^n}(\mu) = \left(\Theta_{\nu^n}^1(\mu), \ldots, \Theta_{\nu^n}^{Q_\nu}(\mu)\right) = \left(\Theta_{\nu^n}^{p1}(\mu), \Theta_{\nu^n}^{p2}(\mu)\right),$$

where for $k = 1, 2$

$$\begin{aligned}
\Theta_{\nu^n}^{pk}(\mu) = \Big(&\Theta_f^1(\mu), \ldots, \Theta_f^{Q_f}(\mu), \\
&-\Theta_a^1(\mu)\underline{u}_{N,k}^n{}^{(1)}(\mu), \ldots, -\Theta_a^{Q_a}(\mu)\underline{u}_{N,k}^n{}^{(1)}(\mu), \\
&\qquad\qquad \vdots \\
&-\Theta_a^1(\mu)\underline{u}_{N,k}^n{}^{(N_k)}(\mu), \ldots, -\Theta_a^{Q_a}(\mu)\underline{u}_{N,k}^n{}^{(N_k)}(\mu)\Big)
\end{aligned}$$

and the n-independent components $(\nu^n)^q = \nu^q$, $q = 1, \ldots, Q_\nu$ which are given by

$$\left((\nu^n)^q, v\right)_X = f_1^q(v), \forall v \in X,$$

for $q = 1, \ldots, Q_{f_1}$,

$$\left(\nu^q, v\right)_X = a_1^i\left(\varphi_{N,1}^{(j)}, v\right), \forall v \in X,$$

for $q = Q_f + Q_a(j-1) + i$, $i = 1, \ldots, Q_a$, $j = 1, \ldots, N_1$,

$$\left(\nu^q, v\right)_X = f_2^i(v), \forall v \in X,$$

for $q = Q_f + N_1 Q_a + i$, $i = 1, \dots, Q_f$ and

$$(\nu^q, v)_X = a_2^i \left(\varphi_{N,2}^{(j)}, v\right), \forall v \in X,$$

for $q = Q_f + N_1 Q_a + Q_f + Q_a(j-1) + i$, $i = 1, \dots, Q_a$, $j = 1, \dots, N_2$. With the help of this decomposition we can decompose the computation of the error estimator as follows.

Offline stage (error estimator)

Given bases Φ_k, $k = 1, 2$

1. compute the components ν^q for $q = 1, \dots, Q_\nu$
2. compute the matrix

$$\underline{G}_\nu = \left(\left(\nu^j, \nu^i\right)_X\right)_{i,j=1}^{Q_\nu}.$$

3. compute the matrices

$$\underline{\tilde{R}}_{N,kk} = \left(\left(\tilde{R}_1 \gamma \varphi_{N,k}^{(j)}, \tilde{R}_1 \gamma \varphi_{N,k}^{(i)}\right)_{X_1}\right)_{i,j=1}^{N_k} \text{ for } k = 1, 2$$

$$\underline{\tilde{R}}_{N,12} = \left(\left(\tilde{R}_1 \gamma \varphi_{N,1}^{(j)}, \tilde{R}_1 \gamma \varphi_{N,2}^{(i)}\right)_{X_1}\right)_{i,j=1}^{N_2,N_1}$$

Online stage (error estimator)

Given $u_{N,k}^n(\mu)$, $k = 1, 2$ for some $\mu \in \mathcal{P}$, $n \geq 1$

1. compute

$$\|\nu^n(\mu)\|_X^2 = \underline{\Theta}_{\nu^n}(\mu)^T \underline{G}_\nu \underline{\Theta}_{\nu^n}(\mu)$$

2. compute

$$\left\|\tilde{R}_1\left(\gamma u_{N,1}^n(\mu) - \gamma u_{N,2}^n(\mu)\right)\right\|_{X_1}^2 = \sum_{k=1}^{2} \underline{u}_{N,k}^n(\mu)^T \underline{\tilde{R}}_{N,kk} \underline{u}_{N,k}^n(\mu) - 2\underline{u}_{N,1}^n(\mu)^T \underline{\tilde{R}}_{N,12} \underline{u}_{N,2}^n(\mu)$$

3. compute $\Delta_N^n(\mu)$ according to (4.2).

Obviously, the time complexities in the online stage for both the reduced procedure and the error estimator do not depend on the high dimensions $\mathcal{N}_1, \mathcal{N}_2$ or $\mathcal{N}_\Gamma$ any more. To be more precise, if we assume $Q_a, Q_f \ll N_1, N_2$ and $N_1^\Gamma \lessapprox N_1^0$ the online stage for a single iteration in the reduced procedure can be performed in $\mathcal{O}((N_1^0)^3 + N_2^3)$ and the online stage for the error estimator in $\mathcal{O}(N_1^2 + N_2^2)$.

6. Experiments. The introduced methods were implemented in the environment of the MATLAB software package RBmatlab[2], which provides a general framework for reduced basis simulations. All computations are done on an Intel Core 2 Quad CPU (2.83 GHz).

[2] http://www.morepas.org/software/

6.1. Model. We consider a "thermal block" example, i.e. the simple, static heat equation on the unit square $\Omega = (0,1)^2 \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions,

$$\begin{aligned} -\operatorname{div}(\kappa(x;\mu)\nabla u(x;\mu)) &= h(x;\mu), & x &\in \Omega, \\ u(x;\mu) &= 0, & x &\in \partial\Omega. \end{aligned}$$

The domain Ω is seperated into 4 blocks $B_1 = (0,0.5)^2$, $B_2 = (0.5,1)\times(0,0.5)$, $B_3 = (0,0.5)\times(0.5,1)$ and $B_4 = (0.5,1)^2$ with corresponding heat coefficients contributing to the function κ as follows.

$$\kappa(x;\mu) = \sum_{i=1}^{3} \mu_i \chi_{B_i}(x) + \chi_{B_4}(x),$$

where $\mu_i \in [0.1, 10]$, $i = 1,2,3$ and χ_{B_i} the characteristic functions, $\chi_{B_i}(x) = 1$ if $x \in B_i$ and 0 if $x \notin B_i$, $i = 1,\dots,4$. The fourth entry of the parameter vector is a weight in the source function h,

$$h(x;\mu) = 2\mu_4 \exp\left(-\beta_1 |x - z_1|^2\right) + 2(1-\mu_4)\exp\left(-\beta_2 |x - z_2|^2\right),$$

and is restricted to $\mu_4 \in [0,1]$. We further set $\beta_1 = \beta_2 = 20$, and the midpoints of the exponential bubbles $z_1 = (0.5, 0.5)^T$, $z_2 = (0.875, 0.875)^T$. Two realizations of this function are shown in Figure 6.1. A domain decomposition is set according to Figure 6.2. We note that $z_1 \in \Omega_1$ and $z_2 \in \Omega_2$.

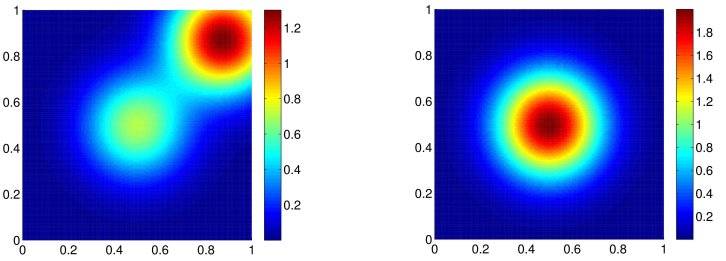

FIG. 6.1. *source function $h(\mu)$, left-hand side $\mu_4 = 0.35$, right-hand side $\mu_4 = 1$*

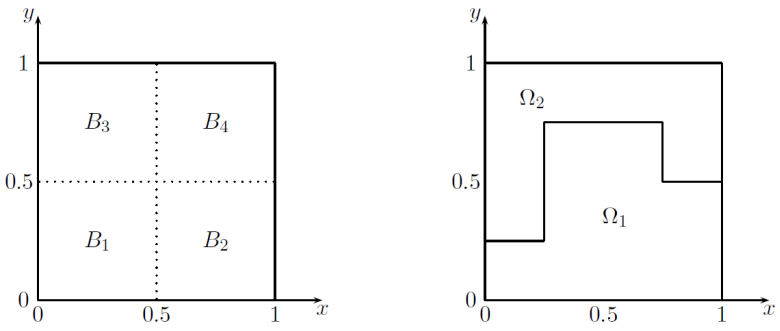

FIG. 6.2. *thermal blocks and domain decomposition of $\Omega = (0,1)^2$*

We use a triangular grid with a Finite Element discretization of order 1 for the detailed procedure. A uniform triangulation with mesh size $h = 0.011$ and a total of 16641 degrees of freedom on $\overline{\Omega}$ is used. The grid is consistent to the domain decomposition in the sense that the intersection of a triangle with the interface Γ is either an edge or a vertex of the grid.

6.2. Basis generation. With the Greedy-algorithm described in Section 5.1 a reduced basis is generated. The Greedy-algorithm involves both the detailed and reduced procedure. We set the stopping tolerances $\epsilon_{\text{tol}} = \epsilon_{N,\text{tol}} = 1 \cdot 10^{-12}$ and the maximal number of iterations $n_{\text{stop}} = n_{N,\text{stop}} = 10000$. For the Greedy-algorithm we set the error estimate tolerance $\epsilon_{\text{gre}} = 1 \cdot 10^{-6}$ and the maximal basis sizes $N_{\text{stop},k} = 100$, $k = 1, 2$. The training set of parameters $\mathcal{M}_{\text{train}}$ is given by 5^4 equidistantly distributed points in the parameter domain. For the computation of the error estimate $\Delta_N^n(\mu)$ defined in (4.2) we have to specify which extension operator $\tilde{R}_1$ we use. It is the trivial Finite Element extension, where all degrees of freedom not lying on Γ are set to zero. For the harmonic extensions in (5.1) we set $\bar{\mu} = (5.05, 5.05, 5.05, 0.5)$.

First tests showed that the reduced simulation may perform badly, if one of the bases is empty. To achieve a reliable start of the Greedy-algorithm, initial bases are computed with a random parameter and method A-*, *=SEQ, NSEQ, despite the fact that we may use method B-* in the Greedy-algorithm. So all bases are non-empty in the beginning.

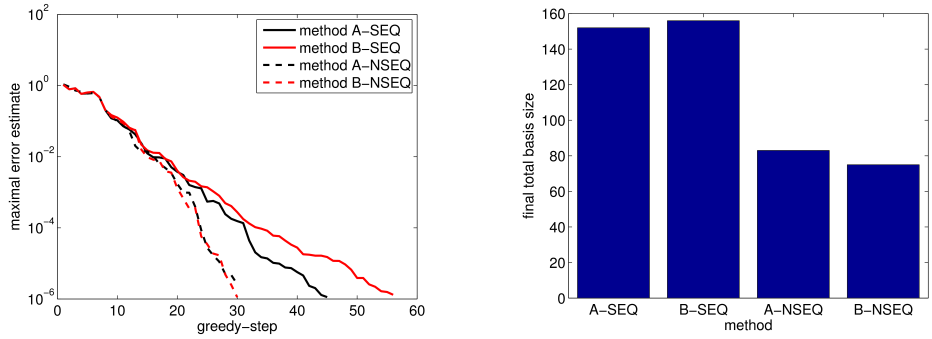

FIG. 6.3. *Progress of the maximal error estimate on $\mathcal{M}_{\text{train}}$ in the Greedy-algorithm on the left hand side, sum of bases dimensions on the right hand side*

Figure 6.3 shows that a bases extension with NSEQ is not only sufficient but even better than one with SEQ regarding the error decay. Taking more POD-modes from the sequences to better approximate the detailed iteration would result in even larger bases. Method B-NSEQ yields slightly smaller dimensions than method A-NSEQ, while the error decay shows no difference. In all cases convergence of the Greedy-algorithm is achieved.

We mentioned before that the orthogonality of Φ_2 and so the stability of the corresponding system is not ensured by construction. We have a look at the 2-condition numbers of the corresponding matrix $\underline{A}_{N,2}(\mu)$ in Figure 6.4.

Still there is no advantage in performing SEQ. The difference between method A and B is reasonable, since by definition method B — in contrast to method A — yields an orthonormal basis extension of Φ_2^Γ, while Φ_2^0 only is extended if the extension of Φ_2^Γ is exhausted. Regarding method A-NSEQ, the condition of $\underline{A}_{N,2}(\mu)$ can effectivily be improved by diagonal preconditioning. Figure 6.4 also indicates that the condition

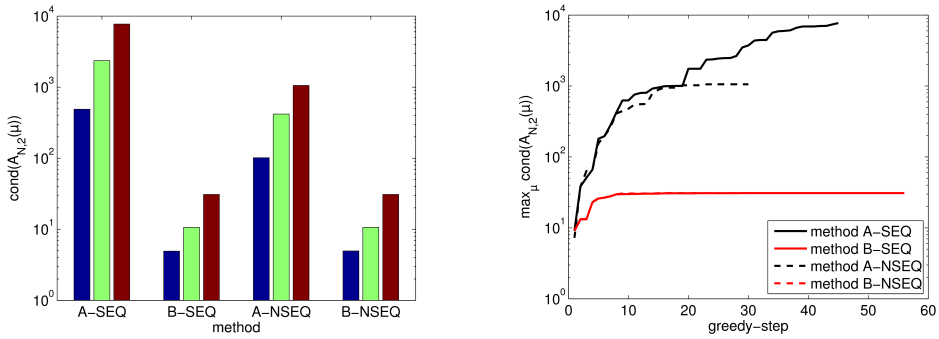

FIG. 6.4. *Minimum (blue), mean (green) and maximum (red) 2-condition of $\underline{A}_{N,2}(\mu)$ out of 100 randomly chosen parameter with full bases on the left hand side, progress of the maximum in the Greedy-algorithm on the right hand side*

numbers of B-SEQ, A-NSEQ and B-NSEQ remain bounded, so we assume that the stability in this context is given independent of the given basis dimensions.

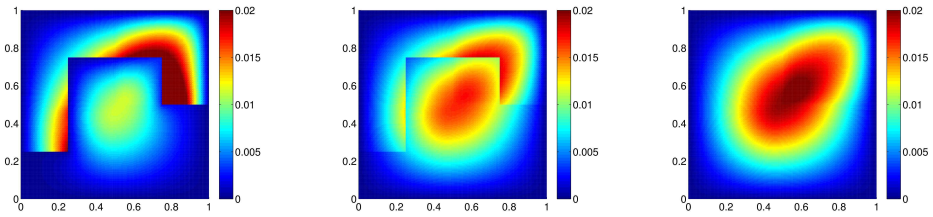

FIG. 6.5. *The solution $u_N^n(\mu)$ in three steps of the reduced iteration for one sample parameter. To the left $n = 1$, in the middle $n = 5$ and to the right $n = n_N(\mu) = 96$.*

6.3. Reduced simulation. Figure 6.5 provides a nice insight into how the reduced procedure works. Here and in what follows the stopping tolerances $\epsilon_{\text{tol}} = \epsilon_{N,\text{tol}} = 1 \cdot 10^{-10}$ are used. To investigate the procedure more precisely, we exemplarily show in Figure 6.6 the approximating quality of the bases generated with method B-NSEQ for one random parameter. One can see that the error decay in the reduced procedure is similar to the error decay in the detailed procedure and that the exact solution is approximated with an accuracy of approximately 10^{-6}, as was intended with the Greedy-algorithm. To measure the effectivity of the error estimate, which already can be observed on the left hand side of Figure 6.6, we further compute $\Delta_N^{n_N(\mu)}(\mu)/\|u_N^{n_N(\mu)}(\mu) - u(\mu)\|_{\hat{X}}$ for 100 randomly chosen parameters and show a histogram of the values on the right hand side. The overestimation seems to be only one order of magnitude, hence the effectivity is very satisfactory.

Last but not least the efficiency of the reduction technique, which mainly relies on the affine parameter dependency, is verified by comparing the simulation times of the detailed and the reduced procedure; again we randomly choose 100 test parameters. While the simulation times of the detailed procedure range from $1.435s$ to $50.535s$, those of the reduced procedure range from $0.015s$ to $0.244s$. The mean speed-up factor is 141.700. It further increases if the Finite Element discretization is refined. These results meet the expectations on the efficiency of the reduction method.

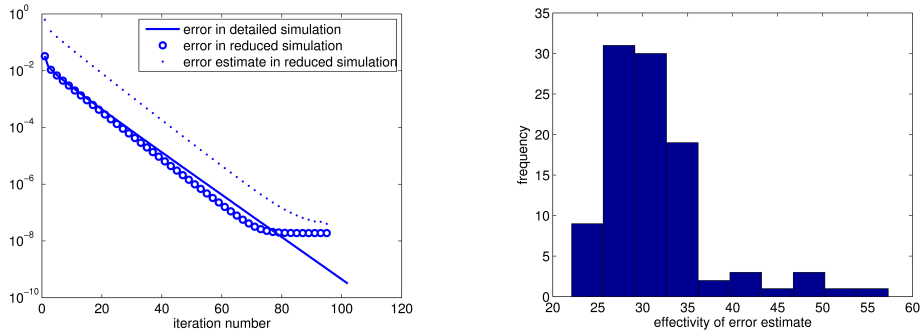

FIG. 6.6. *Progress of the error of the detailed and reduced simulation and error estimator in the iterative procedures for one random parameter on the left hand side, spreading of the error estimate effectivity for* 100 *random test parameters on the right hand side.*

7. Conclusion. To summarize, an efficient model reduction method for homogeneous domain decomposition problems was developed. A specific selection of the parameters for the bases was investigated and demonstrated by numerical experiments. Further an a-posteriori estimation of the reduction error was realized in a reliable fashion.

The formulation of an iterative procedure aims at parallelization of the computations. Future works will focus on parallelizable domain decomposition procedures. An extension of the methology to more general settings like heterogeneous domain decomposition problems is striven for.

Appendix. Proof of convergence. The proof of Theorem 4.1 follows the same lines as the proof of convergence in [14]. To start, we have to define appropriate extension operators. An explicit definition of the extension operator $R_{N,1}$ was not done because this has no impact on the reduced procedure. Considering another arbitrary linear and continuous extension operator $\tilde{R}_{N,1} : X_N^\Gamma \to X_{N,1}$ it holds $(R_{N,1}g)^\Gamma = (\tilde{R}_{N,1}g)^\Gamma$ for all $g \in X_N^\Gamma$ since γ is bijective on $X_{N,1}^\Gamma$. Furthermore, for $n \geq 1$ and $v \in X_{N,2}$, due to Equation (3.1):

$$\begin{aligned} b_{N,u_{N,1}^n(\mu)}(v;\mu) &= f_1\left(R_{N,1}\gamma v;\mu\right) - a_1\left(u_{N,1}^n(\mu), R_{N,1}\gamma v;\mu\right) \\ &= f_1\left(\left(R_{N,1}\gamma v\right)^\Gamma;\mu\right) - a_1\left(u_{N,1}^n(\mu), \left(R_{N,1}\gamma v\right)^\Gamma;\mu\right) \\ &= f_1\left(\left(\tilde{R}_{N,1}\gamma v\right)^\Gamma;\mu\right) - a_1\left(u_{N,1}^n(\mu), \left(\tilde{R}_{N,1}\gamma v\right)^\Gamma;\mu\right) \\ &= f_1\left(\tilde{R}_{N,1}\gamma v;\mu\right) - a_1\left(u_{N,1}^n(\mu), \tilde{R}_{N,1}\gamma v;\mu\right). \end{aligned}$$

This justifies that we define now some specific extension operator to prove the convergence of the reduced iteration.

DEFINITION A.1. *Let* $\tilde{R}_{N,k,\mu} : X_{N,\Gamma} \to X_{N,k}$ *be defined by*

$$\begin{aligned} \tilde{R}_{N,k,\mu}g : a_k\left(\tilde{R}_{N,k,\mu}g, v;\mu\right) = 0, \qquad & \forall v \in X_{N,k}^0, \\ \tilde{R}_{N,k,\mu}g = g, \qquad & \text{on } \Gamma. \end{aligned}$$

Additionally, a norm on X_N^Γ *is given through*

$$g \in X_N^\Gamma : \|g\|_{\Gamma,\mu} := \|\tilde{R}_{N,1,\mu}g\|_{1,\mu}$$

and induced by the following inner product.

$$(g, \hat{g})_{\Gamma,\mu} = a_1\left(\tilde{R}_{N,1,\mu} g, \tilde{R}_{N,1,\mu}\hat{g}; \mu\right).$$

We replace $R_{N,1}$ by $\tilde{R}_{N,1,\mu}$ in the reduced procedure and state the following equivalence.

LEMMA A.2. *$u_N(\mu) \in X_N$ solves the reduced problem (4.1) for a given $\mu \in \mathcal{P}$ if and only if $u_{N,1}(\mu) = u_N(\mu)|_{\Omega_1}$ and $u_{N,2}(\mu) = u_N(\mu)|_{\Omega_2}$ are the solutions of the following equations.*

$$a_1\left(u_{N,1}(\mu), v; \mu\right) = f_1(v; \mu), \qquad \forall v \in X^0_{N,1}, \tag{A.1}$$

$$\begin{aligned} a_2\left(u_{N,2}(\mu), v; \mu\right) = {} & f_2(v; \mu) + f_1\big(\tilde{R}_{N,1,\mu}\gamma v; \mu\big) \\ & - a_1\big(u_{N,1}(\mu), \tilde{R}_{N,1,\mu}\gamma v; \mu\big), \qquad \forall v \in X_{N,2}. \end{aligned} \tag{A.2}$$

Proof. Let $u_N(\mu)$ be the solution of problem (4.1). For $v \in X^0_{N,1}$ let $\hat{v}$ be the extension of v on Ω_2 by zero. It holds $\hat{v} \in X_N$ and

$$a_1\left(u_{N,1}(\mu), v; \mu\right) = a\left(u_N(\mu), \hat{v}; \mu\right) = f(\hat{v}, \mu) = f_1(v; \mu).$$

For $v \in X_{N,2}$ let $\hat{v} \in X_N$ be the extension of v on Ω_1 by $\tilde{R}_{N,1,\mu}\gamma v$. It holds $\hat{v} \in X_N$ and

$$\begin{aligned} a_2\left(u_{N,2}(\mu), v; \mu\right) &= a\left(u_N(\mu), \hat{v}; \mu\right) - a_1\left(u_{N,1}(\mu), \tilde{R}_{N,1,\mu}\gamma v; \mu\right) \\ &= f(\hat{v}; \mu) - a_1\left(u_{N,1}(\mu), \tilde{R}_{N,1,\mu}\gamma v; \mu\right) \\ &= f_2(v; \mu) + f_1\left(\tilde{R}_{N,1,\mu}\gamma v; \mu\right) - a_1\left(u_{N,1}(\mu), \tilde{R}_{N,1,\mu}\gamma v; \mu\right). \end{aligned}$$

Vice versa, let $u_N(\mu) \in X_N$ be given by (A.1) and (A.2) for $u_{N,1}(\mu)$ and $u_{N,2}(\mu)$. For $v \in X_N$ set $v_k := v|_{\Omega_k}$, $k = 1, 2$. It holds

$$\begin{aligned} a\left(u_N(\mu), v; \mu\right) &= a_1\left(u_{N,1}(\mu), v_1; \mu\right) + a_2\left(u_{N,2}(\mu), v_2; \mu\right) \\ &= a_1\left(u_{N,1}(\mu), v_1 - \tilde{R}_{N,1,\mu}\gamma v_2; \mu\right) + a_1\left(u_{N,1}(\mu), \tilde{R}_{N,1,\mu}\gamma v_2; \mu\right) \\ &\quad + a_2\left(u_{N,2}(\mu), v_2; \mu\right). \end{aligned}$$

We now can apply Equation (A.2) and since $v_1 - \tilde{R}_{N,1,\mu}\gamma v_2 \in X^0_{N,1}$ also Equation (A.1).

$$\begin{aligned} a\left(u_N(\mu), v; \mu\right) &= f_1\left(v_1 - \tilde{R}_{N,1,\mu}\gamma v_2; \mu\right) + f_2(v_2; \mu) + f_1\left(\tilde{R}_{N,1,\mu}\gamma v_2; \mu\right) \\ &= f(v; \mu). \end{aligned}$$

□

LEMMA A.3. *We assume that for $\mu \in \mathcal{P}$ there exists a $\theta_{N,\min}(\mu) > 0$ with $\theta^n_N(\mu) \geq \theta_{N,\min}(\mu)$ for all $n \geq 1$. If the sequence $\{g^n_N(\mu)\}_{n\geq 1}$ converges in X^Γ_N, the sequences $\{u^n_{N,1}(\mu)\}_{n\geq 1}$ and $\{u^n_{N,2}(\mu)\}_{n\geq 1}$ converge in $X_{N,1}$ and $X_{N,2}$, respectively.*

Proof. We assume that $\{g^n_N(\mu)\}_{n\geq 1}$ converges on above assumptions and so is a Cauchy sequence. It is sufficient to show that $\{u^n_{N,k}(\mu)\}_{n\geq 1}$, $k = 1, 2$ are

Cauchy sequences because they live in Hilbert spaces. Due to Equation (3.1) it holds $a_1(u^m_{N,1}(\mu) - u^n_{N,1}(\mu), v) = 0$, $\forall v \in X^0_{N,1}$ and therefore

$$\begin{aligned}\|u^m_{N,1}(\mu) - u^n_{N,1}(\mu)\|_{1,\mu} &= \|\tilde{R}_{N,1}(g^{m-1}_N(\mu) - g^{n-1}_N(\mu))\|_{1,\mu} \\ &\overset{def}{=} \|g^{m-1}_N(\mu) - g^{n-1}_N(\mu)\|_{\Gamma,\mu}.\end{aligned}$$

The latter expression converges to zero for $m, n \to \infty$ and so $\{u^n_{N,1}(\mu)\}_{n\geq 1}$ is a Cauchy sequence. Rewriting (3.4) for $\gamma u^n_{N,2}(\mu)$ gives

$$\gamma u^n_{N,2}(\mu) = \frac{1}{\theta^n_N(\mu)}(g^n_N(\mu) - g^{n-1}_N(\mu)) + g^{n-1}_N(\mu).$$

Since $\{1/\theta^n_N(\mu)\}_{n\geq 1}$ is assumed to be bounded, we can determine the limits of both sides and get

$$\lim_{n\to\infty} \gamma u^n_{N,2}(\mu) = \lim_{n\to\infty} g^{n-1}_N(\mu). \tag{A.3}$$

Then, with (3.1), (3.3) and the Cauchy-Schwarz inequality we state

$$\begin{aligned}\left\|u^m_{N,2}(\mu) - u^n_{N,2}(\mu)\right\|_{2,\mu} &= -a_1\Big(u^m_{N,1}(\mu) - u^n_{N,1}(\mu), \tilde{R}_{N,1,\mu}\left(\gamma u^m_{N,2}(\mu) - \gamma u^n_{N,2}(\mu)\right); \mu\Big) \\ &\leq \left\|u^m_{N,1}(\mu) - u^n_{N,1}(\mu)\right\|_{1,\mu} \left\|\gamma u^m_{N,2}(\mu) - \gamma u^n_{N,2}(\mu)\right\|_{\Gamma,\mu}\end{aligned}$$

Because $\{u^n_{N,1}(\mu)\}_{n\geq 1}$ and $\{\gamma u^n_{N,2}(\mu)\}_{n\geq 1}$ are Cauchy sequences, the latter term tends to zero for $n \to \infty$. □

LEMMA A.4. *In the case of convergence in Lemma A.3, the limits of the sequences* $\{u^n_{N,k}(\mu)\}_{n\in\mathbb{N}}$, $k = 1, 2$ *are* $u_{N,k}(\mu)$, $k = 1, 2$ *of Lemma A.2.*

Proof. To derive (A.1) and (A.2) take the limit $n \to \infty$ in (3.1) and (3.3), respectively. With (3.2) and (A.3), it follows

$$\gamma u_{N,1}(\mu) = \lim_{n\to\infty} \gamma u^n_{N,1}(\mu) = \lim_{n\to\infty} g^{n-1}_N(\mu) = \lim_{n\to\infty} \gamma u^n_{N,2}(\mu) = \gamma u_{N,2}.$$

So $u_N(\mu) \in X_N$. □

For the convergence of the reduced iteration it remains to show, that the sequence $\{g^n_N(\mu)\}_{n\geq 1}$ is convergent in X^Γ_N. We define an operator $T : X^\Gamma_N \to X^\Gamma_N$ as follows.

$$\begin{aligned} & X^\Gamma_N \ni g \mapsto Tg = \gamma w_2, \\ & w_2 \in X_{N,2} : a_2(w_2, v; \mu) = -a_1\left(w_1, \tilde{R}_{N,1,\mu}\gamma v; \mu\right), \qquad \forall v \in X_{N,2}, \\ & w_1 \in X_{N,1} : a_1(w_1, v; \mu) = 0, \qquad \forall v \in X^0_{N,1}, \\ & \gamma w_1 = g. \end{aligned} \tag{A.4}$$

By definition, it holds

$$\begin{aligned} w_1 &= \tilde{R}_{N,1,\mu} g, \\ w_2 &= \tilde{R}_{N,2,\mu}\gamma w_2 = \tilde{R}_{N,2,\mu} Tg. \end{aligned} \tag{A.5}$$

We now define for a $\theta > 0$

$$T_\theta : X^\Gamma_N \to X^\Gamma_N : T_\theta g = \theta Tg + (1-\theta)g.$$

Then, let $g_N(\mu) := \gamma u_{N,1}(\mu)$. With Lemma A.2 it is easy to see that

$$g_N^{n+1}(\mu) - g_N(\mu) = T_{\theta_N^n(\mu)}(g_N^n(\mu) - g_N(\mu)). \tag{A.6}$$

If $T_{\theta_N^n}$ is a contraction, g_N^n converges towards g_N. This is the last past of the proof of Theorem 4.1.

LEMMA A.5. *There exists a $\theta_N^*(\mu) \in (0,1]$, such that $\{g_N^n(\mu)\}_{n\geq 1}$ converges towards $g_N(\mu)$ in X_N^Γ, if $0 < \theta_N^n(\mu) < \theta_N^*(\mu)$ for all $n \geq 1$.*

Proof. We show that T_θ is a contraction, if θ is chosen properly. For this we consider

$$\|T_\theta g\|_{\Gamma,\mu}^2 = \theta^2 \|Tg\|_{\Gamma,\mu}^2 + 2\theta(1-\theta)(g, Tg)_{\Gamma,\mu} + (1-\theta)^2 \|g\|_{\Gamma,\mu}^2. \tag{A.7}$$

By setting $v := \tilde{R}_{N,2,\mu} Tg$ in Equation (A.4), we get together with (A.5)

$$(g, Tg)_{\Gamma,\mu} = a_1\left(\tilde{R}_{N,1,\mu} g, \tilde{R}_{N,1,\mu} Tg; \mu\right) \tag{A.8}$$

$$= -a_2\left(\tilde{R}_{N,2,\mu} Tg, \tilde{R}_{N,2,\mu} Tg; \mu\right) = -\|\tilde{R}_{N,2,\mu} Tg\|_{2,\mu}^2. \tag{A.9}$$

Inserting this in (A.7), we get

$$\|T_\theta g\|_{\Gamma,\mu}^2 = \theta^2 \|Tg\|_{\Gamma,\mu}^2 - 2\theta(1-\theta)\left\|\tilde{R}_{N,2,\mu} Tg\right\|_{2,\mu}^2 + (1-\theta)^2 \|g\|_{\Gamma,\mu}^2. \tag{A.10}$$

The norms $\|\tilde{R}_{N,1,\mu}(\cdot)\|_{1,\mu}$ and $\|\tilde{R}_{N,2,\mu}(\cdot)\|_{2,\mu}$ on X_N^Γ are equivalent, because X_N^Γ is finite dimensional. So there exist constants $\sigma_N(\mu)$ and $\tau_N(\mu)$, such that

$$\left\|\tilde{R}_{N,1,\mu} g\right\|_{1,\mu}^2 \leq \sigma_N(\mu)\left\|\tilde{R}_{N,2,\mu} g\right\|_{2,\mu}^2,$$
$$\left\|\tilde{R}_{N,2,\mu} g\right\|_{2,\mu}^2 \leq \tau_N(\mu)\left\|\tilde{R}_{N,1,\mu} g\right\|_{1,\mu}^2.$$

With Equation (A.9) and the Cauchy-Schwarz inequality we can estimate

$$\begin{aligned}
\|Tg\|_{\Gamma,\mu}^2 &= \left\|\tilde{R}_{N,1,\mu} Tg\right\|_{1,\mu}^2 \leq \sigma_N(\mu)\left\|\tilde{R}_{N,2,\mu} Tg\right\|_{2,\mu}^2 \\
&= -\sigma_N(\mu) a_1\left(\tilde{R}_{N,1,\mu} g, \tilde{R}_{N,1,\mu} Tg; \mu\right) \leq \sigma_N(\mu)\left\|\tilde{R}_{N,1,\mu} g\right\|_{1,\mu}\left\|\tilde{R}_{N,1,\mu} Tg\right\|_{1,\mu} \\
&= \sigma_N(\mu)\|g\|_{\Gamma,\mu}\|Tg\|_{\Gamma,\mu}.
\end{aligned}$$

Dividing by $\|Tg\|_{\Gamma,\mu}$ yields

$$\|Tg\|_{\Gamma,\mu} \leq \sigma_N(\mu)\|g\|_{\Gamma,\mu}.$$

We still have to estimate the second term in (A.10). Equation (A.4) with $v := \tilde{R}_{N,2,\mu} g$ yields $a_1(\tilde{R}_{N,1,\mu} g, \tilde{R}_{N,1,\mu} g; \mu) = -a_2(\tilde{R}_{N,2,\mu} Tg, \tilde{R}_{N,2,\mu} g; \mu)$ and together with the Cauchy-Schwarz inequality

$$\begin{aligned}
\left\|\tilde{R}_{N,1,\mu} g\right\|_{1,\mu}^2 &= a_1\left(\tilde{R}_{N,1,\mu} g, \tilde{R}_{N,1,\mu} g; \mu\right) = -a_2\left(\tilde{R}_{N,2,\mu} Tg, \tilde{R}_{N,2,\mu} g; \mu\right) \\
&\leq \left\|\tilde{R}_{N,2,\mu} Tg\right\|_{2,\mu}\left\|\tilde{R}_{N,2,\mu} g\right\|_{2,\mu} \\
&\leq \sqrt{\tau_N(\mu)}\left\|\tilde{R}_{N,2,\mu} Tg\right\|_{2,\mu}\left\|\tilde{R}_{N,1,\mu} g\right\|_{1,\mu}.
\end{aligned}$$

Dividing by $\|\tilde{R}_{N,1,\mu}g\|_{1,\mu}$ yields

$$\left\|\tilde{R}_{N,2,\mu}Tg\right\|_{2,\mu} \geq \frac{1}{\sqrt{\tau_N(\mu)}}\left\|\tilde{R}_{N,1,\mu}g\right\|_{1,\mu} = \frac{1}{\sqrt{\tau_N(\mu)}}\|g\|_{\Gamma,\mu}.$$

We now insert the obtained estimates in (A.10) and get

$$\|T_\theta g\|_{\Gamma,\mu} \leq \kappa_N(\theta,\mu)\|g\|_{\Gamma,\mu},$$

with

$$\kappa_N(\theta,\mu) := \left(\theta^2\sigma_N(\mu)^2 + (1-\theta)^2 - \frac{2\theta(1-\theta)}{\tau_N(\mu)}\right)^{1/2}.$$

A simple calculation shows that $\kappa_N(\theta,\mu) < 1$, if $0 < \theta < \theta_N^*(\mu)$ with

$$\theta_N^*(\mu) := \min\left(1, \frac{2(\tau_N(\mu)+1)}{\sigma_N(\mu)^2\tau_N(\mu) + \tau_N(\mu) + 2}\right).$$

□

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